University of Manitoba

MATH 2720
Multivariable Calculus

Course material

Winter 2012
Julien Arino

Introduction

This booklet contains some material helpful for the course MATH 2720, Multivariable calculus:

- a (tentative) detailed program;
- the course syllabus;
- assignments that had been handed out during the Fall 2006 and 2007 terms, with solutions;
- some midterm and final examinations from previous terms, with solutions for some;
- slides that are used during class.

A word of warning: **this course is not easy!** A good amount of material will be covered, some of which is not trivial.

You should work throughout the term, this is not a course that you will be able to cram for at the end of term. There are six assignments. You should do these seriously. You should also work on the suggested problems (posted on the course page `http://server.math.umanitoba.ca/~jarino/courses/math2720/`). You should prepare for the tutorial sessions by working on the posted exercises before you attend the tutorial.

When working on past midterms and finals, set aside the same amount of time as had been used for that examination and do not use notes and solutions (if available). Note that there is absolutely no guarantee that our examinations will in any way resemble the ones made available here.

I have tried very hard to make this document typo-free, but the probability that some typos remain is very high. If you find errors, let me know. These errors, as well as the ones I find, will be posted on the course page and a revised version of this booklet will be posted on jump.

Contents

Chapter 1

Program

Detailed course program

The following is an updated program. Program up to and including the underlined week is **up to date**, the rest is **tentative**. For each section, refer to Suggested exercises (in the booklet and online) to find recommended practice problems. Section numbering is indicated relative to the Marsden & Tromba (MT), 5th edition.

Numbering is also indicated in the Stewart (S), 6th edition. Note that the reference textbook for the course is the Marsden & Tromba. If you are using the Stewart, you are responsible for the definitions, theorems, etc., as they are presented in the Marsden & Tromba (see the MT book or the lecture slides), which is usually a little more general.

Week 0 (5/1/2012)

Review/introduction of logic and sets (extra material). Equation of a sphere (extra material; see also S 12.1). Spherical and cylindrical coordinates (MT 1.4 – S 15.7 & 15.8 but no integrals at this point). **Assignment 1 handed out.**

Week 1 (10&12/1/2012)

Types of functions (extra material). Geometry of real-valued functions (MT 2.1 – S 14.1). Limits and continuity (MT 2.2 – S 14.2). Differentiation (MT 2.3 – S 14.3 & 14.4).

Week 2 (17&19/1/2012)

Differentiation (continued). Paths and curves (MT 2.4 – S 13.1 & derivatives in 13.2). Properties of the derivative (MT 2.5 – S 14.5). Gradients and directional derivatives (MT 2.6 – S 14.6). **Assignment 2 handed out.**

Week 3 (24&26/1/2012)

Iterated partial derivatives (MT 3.1 – S 14.3). Extrema of real-valued functions (MT 3.3 – S 14.7).

Week 4 (31/1&2/2/2012)

Constrained extrema and Lagrange multipliers (MT 3.4 – S 14.8). Implicit function theorem (MT 3.5 – S 14.5 but only a particular case here so see slides). **Assignment 3 handed out.**

Week 5 (7&9/2/2012)

Differentiation of paths (in MT 4.1 – S 13.2). Arc length (4.2 – S 13.3). Differential geometry (MT Exercises 4.2.12-4.2.17– S 13.3). Vector fields (MT 4.3 – S 16.1). Divergence and curl (MT 4.4 – S 16.5). **Test 1 this week, Wednesday February 8 at 17:30 (1 hour duration)**. Program covers up to end of previous week.

Week 6 (14&16/2/2012)

Introduction to multiple integrals (MT 5.1). Double integral over a rectangle (MT 5.2 – S 15.1 & 15.2). Double integral over general regions (MT 5.3 – S 15.3). Changing order of integration (MT 5.4 – S 15.2). **Assignment 4 handed out.**

Week 7 (28/2&1/3/2012)

Triple integral (MT 5.5 – S 15.6). Geometry of maps from $\mathbb{R}^2 \to \mathbb{R}^2$ (MT 6.1 – S 15.9). Change of variables theorem (MT 6.2 – S15.9, 15.4, 15.7 & 15.8). Applications (MT 6.3 and extra material – S 15.5).

Week 8 (6&8/3/2012)

Improper integrals (MT 6.4 – S 7.8 for $f : \mathbb{R} \to \mathbb{R}$ but not done for $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^3 \to \mathbb{R}$). Path integral (MT 7.1 – S 13.2). Line integrals (MT 7.2 – S 16.2). **Test 2 this week, Wednesday March 7 at 17:30 (2 hours durations)**. Program covers up to end of previous week.

Week 9 (13/&15/3/2012)

Parametrized surfaces (MT 7.3 – S 16.6). Area of a surface (MT 7.4 – S 16.6). Integral of scalar functions over surfaces (MT 7.5 – S 16.7). **Assignment 5 handed out. March 16 is last day for Voluntary Withdrawal.**

Week 10 (20&22/3/2012)

Surface integrals of vector fields (MT 7.6 – S 16.7). Applications (MT 7.7).

Week 11 (27&29/3/2012)

Green's theorem (MT 8.1 – S 16.4). Stokes' theorem (MT 8.2 – S 16.8). **Assignment 6 handed out.**

Week 12 (3&5/4/2012)

Conservative fields (MT 8.3 – S 16.3). Gauss' theorem (MT 8.4 – S 16.5 & 16.9). **Classes end April 5. Final examination period April 9–23.**

MATHEMATICS DEPARTMENT
MATH 2720 – Multivariable Calculus

Winter 2012 Term

Instructor: Dr. J. Arino,
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Office: 436 Machray Hall
Phone: (204) 474-6927
Lectures: Tuesday & Thursday: 8:30 – 9:45 @ 111 Armes.
Office Hours: Tuesday & Thursday: 10:00 – 11:00.
Other times by appointment only.
Text: *Vector Calculus* 5th Ed. – J.E. Marsden and A.J. Tromba.
An additional booklet containing suggested exercises, old examinations and printouts of course slides should be available mid-January, probably from UMSU copy centre (for a fee) and/or online (free).
Course page: General information about the course, including past examinations, can be found at:
http://www.math.umanitoba.ca/courses/show_course.php?name=MATH_2720
Information specific to the Winter 2012 term will be posted on
http://server.math.umanitoba.ca/~jarino/courses/math2720/
or on Jump.
Topics: We will cover most of the material in the book.
Evaluation:

Test 1:	15%
Test 2:	20%
5 best out of 6 assignments:	20%
Final:	45%
Total:	100%

Tests: The first test will be held Wednesday, February 8th, from 17:30 to 18:30. The second test will be held Wednesday, March 7th, from 17:30 to 19:30.
Assignments: Assignments will be handed out on Thursdays: January 5th and 19th, February 2nd and 15th, March 15th and 29th. They are due one week later.
Final examination: 3 hours, scheduled by the Registrars Office during the final exam period.

Notes:
1. Notes, text(s), calculators and/or other reference materials or aids will **NOT** be allowed for the tests or the final examination.
2. You are responsible for the material in all class lectures and items posted on the Winter 2012 course page or Jump.
3. If you miss a test then you will automatically be given a zero mark unless reasons are provided together with evidence (e.g. letter from a medical doctor), in which case the final mark will be adjusted. No make-up tests will be given.
4. Late assignments will NOT be accepted.
5. Group work on assignments is **NOT** allowed. Solutions are to be produced independently; similarities in student assignments will be treated as a case of Academic Dishonesty.

Chapter 2

Past assignments

Contents

MATH 2720
Fall 2006
Assignment 1

This assignment is due **in class** Friday, October 6$^{\text{th}}$. Late assignments will **not** be accepted. Two remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.

2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you.

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1. Describe, i.e., write inequalities for, the region in 3-space bounded by the sphere centered at the origin with radius a, and the planes $z = 1/a$ and $z = -1/a$, for $a > 1$ a real number.

2. Find the point of intersection of the lines $r_1 = (1, 1, 0)+t(1, -1, 2)$ and $r_2 = (2, 0, 2)+t(-1, 1, 0)$, and an equation for the plane containing these two lines.

3. A surface consists of all points P such that the distance from P to the plane $z = 1$ is twice the distance from P to the point $(0, 0, -1)$. Find an equation for this surface and identify it.

4. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

5. Two particles travel along the space curves $r_1(t) = (t, t^2, t^3)$ and $r_2(t) = (1+2t, 1+6t, 1+14t)$. Do the particles collide? Do their paths intersect?

6. Let $r(t) = (e^{-2t}, e^t, te^{-3t})$. Find the curvature $\kappa(t)$.

MATH 2720
Fall 2006
Assignment 1
Solutions

Total marks: 95 + 5 marks for clarity.

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1. (10 points). Describe, i.e., write inequalities for, the region in 3-space bounded by the sphere centered at the origin with radius a, and the planes $z = 1/a$ and $z = -1/a$, for $a > 1$ a real number.

The sphere has equation $x^2 + y^2 + z^2 = a^2$, and thus points inside (bounded by) the sphere satisfy $x^2 + y^2 + z^2 \leq a^2$. This must hold at the same time that $|z| \leq 1/a$. So the region is

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq a^2 \text{ and } |z| \leq \frac{1}{a}\}.$$

This got full marks. To be more precise (although this was not required),

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq a^2 \text{ and } |z| \leq \frac{1}{a} \text{ and } a > 1\},$$

which means that the region described is always a "slice" of the sphere.

2. (15 points). Find the point of intersection of the lines $r_1 = (1, 1, 0) + t(1, -1, 2)$ and $r_2 = (2, 0, 2) + t(-1, 1, 0)$, and an equation for the plane containing these two lines.

For the lines to intersect, there must exist a value of, say, t for r_1 and s for r_2, such that $r_1(t) = r_2(s)$. That is, substitute s for t in r_2 and solve the system

$$\begin{aligned} 1 + t &= 2 - s \\ 1 - t &= s \\ 2t &= 2. \end{aligned}$$

From the third equation, $t = 1$, which substituted into the second equation gives $s = 0$. These values do satisfy the first equation, so the intersection occurs at $r_1(1)$, or, equivalently, at $r_2(0)$, i.e., the point $(2, 0, 2)$.

To find the equation of the plane, we need a vector normal to the plane, that is, orthogonal to b_1 and b_2, if we denote $r_1 = a_1 + b_1 t$ and $r_2 = a_2 + b_2 t$. For this, we compute

$$b_1 \times b_2 = (1, -1, 2) \times (-1, 1, 0) = (-2, -2, 0).$$

Therefore, an equation for the plane is given by $-2(x-2)-2(y-0)+0(z-2)=0$, that is, $x+y=2$.

3. (20 points). A surface consists of all points P such that the distance from P to the plane $z=1$ is twice the distance from P to the point $(0,0,-1)$. Find an equation for this surface and identify it.

Use the formula for the distance D_1 between the plane $ax+by+cz+d=0$ and the point $P(x_0,y_0,z_0)$,

$$D_1 = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Here, the plane has equation $z=1$, that is, $a=b=0$, $c=1$ and $d=-1$; Therefore, $D_1=|z_0-1|$. The distance D_2 between P and the point $(0,0,-1)$ is given by $D_2=\sqrt{x_0^2+y_0^2+(z_0+1)^2}$. Therefore, dropping the index 0, the set of points (x,y,z) that satisfy the conditions must satisfy the equation $D_1=2D_2$, that is,

$$|z-1| = 2\sqrt{x^2+y^2+(z+1)^2}.$$

Squaring both sides,

$$\begin{aligned}
(z-1)^2 = 4\left(x^2+y^2+(z+1)^2\right) &\Leftrightarrow 4\left(x^2+y^2\right)+3z^2+10z+3=0 \\
&\Leftrightarrow 4\left(x^2+y^2\right)+3\left(z^2+\frac{10}{3}z\right)+3=0 \\
&\Leftrightarrow 4x^2+4y^2+3\left(\left(z+\frac{5}{3}\right)^2-\frac{25}{9}\right)+3=0 \\
&\Leftrightarrow 4x^2+4y^2+3\left(z+\frac{5}{3}\right)^2=\frac{16}{3} \\
&\Leftrightarrow 4x^2+4y^2+3u^2=\frac{16}{3},
\end{aligned}$$

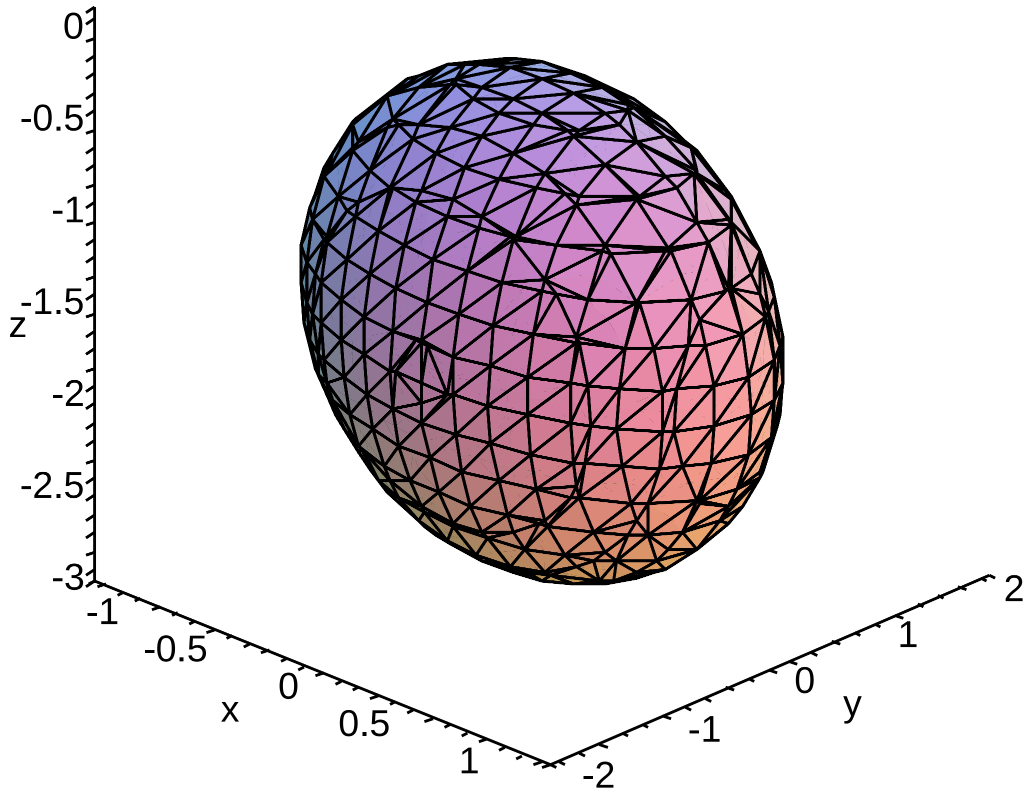
with $u=z+5/3$. This is the equation of an ellipse in the variables x,y,u in the coordinate system centered at $(x,y,u)=(0,0,0)$ or $(x,y,z)=(0,0,-5/3)$, defined by

$$\frac{3}{4}x^2+\frac{3}{4}y^2+\frac{9}{16}u^2=1.$$

With Maple, you can use the following command to generate the surface.

```
with(plots):
implicitplot3d(4*x^2+4*y^2+(z+3)*(3*z+1)=0,
x=-1.2..1.2, y=-2..2, z=-3..0, grid=[15,15,15]);
```

This generates a figure such as this one



4. (15 points). Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

First, we need to find an equation for the cylinder. For a given $z = k$, i.e., in the plane $z = k$, the cylinder is the circle with radius 2 centered at $(x, y) = (0, 0)$. Thus, a point is on the cylinder if $x = 2\cos t$ and $y = 2\sin t$, for $0 \leq t \leq 2\pi$.

But the curve must also lie on the plane $z = xy$, therefore $z = (2\cos t)(2\sin t) = 4\cos t \sin t$, which can also be written $z = 2\sin 2t$ (this last simplification was not needed).

Therefore, an equation for the curve is

$$r(t) = (2\cos t, 2\sin t, 4\cos t \sin t)\,, \quad 0 \leq t \leq 2\pi,$$

or

$$r(t) = (2\cos t, 2\sin t, 2\sin 2t)\,, \quad 0 \leq t \leq 2\pi,$$

depending on the final expression you obtained for z. It was not necessary to specify that t belong to $[0, 2\pi]$: t can belong to any interval of length at least 2π.

5. (15 points). Two particles travel along the space curves $r_1(t) = (t, t^2, t^3)$ and $r_2(t) = (1 + 2t, 1 + 6t, 1 + 14t)$. Do the particles collide? Do their paths intersect?

Collision means that there exists a value of t, equal for both curves, such that $r_1(t) = r_2(t)$. Equating components, such a t must satisfy

$$\begin{aligned} t &= 1 + 2t \\ t^2 &= 1 + 6t \\ t^3 &= 1 + 14t. \end{aligned}$$

The first equation gives $t = -1$, which does not satisfy any of the other two equations. Therefore, the particles do not collide.

To determine intersection of paths, we proceed as in exercise **2**. We replace t by s in one of the equations, say r_2, and seek values of t and s that are solutions to the system

$$\begin{aligned} t &= 1 + 2s \\ t^2 &= 1 + 6s \\ t^3 &= 1 + 14s. \end{aligned}$$

Substituting the value $t = 1 + 2s$ from the first equation into the second equation gives

$$(1 + 2s)^2 = 1 + 6s \Leftrightarrow 4s^2 - 2s = 0 \Leftrightarrow 2s(s - 1) = 0 \Leftrightarrow s = 0 \text{ or } s = \frac{1}{2}.$$

From the first equation, these values of s give the values $t = 1$ and $t = 2$, respectively. The pairs $(t, s) = (1, 0)$ and $(t, s) = (2, \frac{1}{2})$ both satisfy the third equation. Thus the paths intersect twice, at $r_1(1)$ and $r_1(2)$, or, equivalently, at $r_2(0)$ and $r_2(\frac{1}{2})$, that is, at the points $(1, 1, 1)$ and $(2, 4, 8)$, respectively.

6. Let $r(t) = (e^{-2t}, e^t, te^{-3t})$. Find the curvature $\kappa(t)$.

We find

$$r'(t) = \left(-2e^{-2t}, e^t, e^{-3t}(1 - 3t)\right)$$

and

$$r''(t) = \left(4e^{-2t}, e^t, 3e^{-3t}(3t - 2)\right).$$

Computing the cross-product,

$$\begin{aligned} r' \times r'' &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2e^{-2t} & e^t & e^{-3t}(1 - 3t) \\ 4e^{-2t} & e^t & 3e^{-3t}(3t - 2) \end{vmatrix} \\ &= \begin{vmatrix} e^t & e^{-3t}(1 - 3t) \\ e^t & 3e^{-3t}(3t - 2) \end{vmatrix} \vec{i} - \begin{vmatrix} -2e^{-2t} & e^{-3t}(1 - 3t) \\ 4e^{-2t} & 3e^{-3t}(3t - 2) \end{vmatrix} \vec{j} + \begin{vmatrix} -2e^{-2t} & e^t \\ 4e^{-2t} & e^t \end{vmatrix} \vec{k} \\ &= \left(e^t e^{-3t}(3(3t - 2) - (1 - 3t)), e^{-2t}e^{-3t}(6(3t - 2) + 4(1 - 3t)), e^t e^{-2t}(-2 - 4)\right) \\ &= \left(e^{-2t}(12t - 7), e^{-5t}(6t - 8), -6e^{-t}\right), \end{aligned}$$

and so

$$
\begin{aligned}
|r' \times r''| &= \sqrt{\left(e^{-2t}(12t-7)\right)^2 + \left(e^{-5t}(6t-8)\right)^2 + \left(-6e^{-t}\right)^2} \\
&= \sqrt{e^{-4t}(12t-7)^2 + e^{-10t}(6t-8)^2 + 36e^{-2t}} \\
&= \sqrt{\left(e^{-t}\right)^2 \left(e^{-2t}(12t-7)^2 + e^{-8t}(6t-8)^2 + 36\right)} \\
&= e^{-t}\sqrt{e^{-2t}(12t-7)^2 + e^{-8t}(6t-8)^2 + 36}.
\end{aligned}
$$

Also,

$$
|r'| = \sqrt{4e^{-4t} + e^{2t} + e^{-6t}(1-3t)^2}.
$$

We could simplify a little, for example by bringing out a e^{-t} from $|r'|$ as well, but this will not get us much further. In the end,

$$
\kappa(t) = \frac{e^{-t}\sqrt{e^{-2t}(12t-7)^2 + e^{-8t}(6t-8)^2 + 36}}{\left(4e^{-4t} + e^{2t} + e^{-6t}(1-3t)^2\right)^{\frac{3}{2}}}.
$$

MATH 2720
Fall 2006
Assignment 2

This assignment is due **in class** Monday, November 6^{th}. Late assignments will **not** be accepted. Two remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you.

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1. Find the first partial derivatives of the function $f(x,y)=\int_x^y \cos(t^2)dt$.

2. Find the domain of the function $f(x,y)=\ln(2x+3y)$, and find all the second partial derivatives of f.

3. Explain why the function $f(x,y)=\sin(2x+3y)$ is differentiable at the point $(-3,2)$, and find the linearization L of the function at that point.

4. Find the partial derivatives $\dfrac{\partial M}{\partial u}$ and $\dfrac{\partial M}{\partial v}$ when $u=3$, $v=-1$, for the function

$$M=xe^{y-z^2},\quad x=2uv,\quad y=u-v,\quad z=u+v.$$

5. Find $\partial z/\partial x$ and $\partial z/\partial y$ for $xyz=\cos(x+y+z)$.

6. Find the maximum rate of change of the function $f(x,y,z)=\tan(x+2y+3z)$ at the point $(-5,1,1)$.

MATH 2720
Fall 2006
Assignment 2

Total marks: 95 + 5 points for clarity.

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1. Find the first partial derivatives of the function $f(x,y) = \int_x^y \cos(t^2)dt$.

Write

$$f(x,y) = \int_y^x \cos(t^2)dt$$

Then

$$f_x(x,y) = \frac{\partial}{\partial x}\int_y^x \cos(t^2) = \cos(x^2),$$

by the Fundamental Theorem of Calculus. Similarly,

$$f_y(x,y) = \frac{\partial}{\partial y}\int_y^x \cos(t^2) = -\frac{\partial}{\partial y}\cos(t^2)dt = -\cos(y^2).$$

2. Find the domain of the function $f(x,y) = \ln(2x+3y)$, and find all the second partial derivatives of f.

The domain $\mathcal{D}$ is such that $2x+3y>0$, that is, $y > -\frac{2}{3}x$,

$$f_x(x,y) = \frac{2}{2x+3y}, \quad f_y(x,y) = \frac{3}{2x+3y},$$

and

$$f_{xx}(x,y) = -\frac{4}{(2x+3y)^2}, \quad f_{yy}(x,y) = -\frac{9}{(2x+3y)^2} \quad \text{and} \quad f_{xy}(x,y) = f_{yx}(x,y) = -\frac{6}{(2x+3y)^2},$$

where the last equality results from Clairaut's theorem, since f_x and f_y are continuous on $\mathcal{D}$.

3. Explain why the function $f(x,y) = \sin(2x+3y)$ is differentiable at the point $(-3,2)$, and find the linearization L of the function at that point.

f is defined on $\mathbb{R}^2$, with partial derivatives $f_x(x,y) = 2\cos(2x+3y)$ and $f_y(x,y) = 3\cos(2x+3y)$ continuous on $\mathbb{R}^2$. As a consequence, f is differentiable on $\mathbb{R}^2$, and in particular, at $(-3,2)$.

The linearization of f at $(-3,2)$ is given by

$$\begin{aligned} L(x,y) &= f(-3,2) + f_x(-3,2)(x+3) + f_y(-3,2)(y-2) \\ &= 2x + 3y. \end{aligned}$$

4. Find the partial derivatives $\dfrac{\partial M}{\partial u}$ and $\dfrac{\partial M}{\partial v}$ when $u = 3$, $v = -1$, for the function

$$M = xe^{y-z^2}, \quad x = 2uv, \quad y = u - v, \quad z = u + v.$$

We have

$$\begin{aligned} \frac{\partial M}{\partial u} &= \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u} \\ &= e^{y-z^2}(2v) + xe^{x-y^2}(1) + x(-2z)e^{y-z^2}(1) \\ &= e^{y-z^2}(2v - x - 2xz), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial M}{\partial v} &= \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v} \\ &= e^{y-z^2}(2u) + xe^{x-y^2}(-1) + x(-2z)e^{y-z^2}(1) \\ &= e^{y-z^2}(2u - x - 2xz). \end{aligned}$$

When $u = 3$, $v = -1$, we have $x = -6$, $y = 4$ and $z = 2$, so $\partial M/\partial u = 16$ and $\partial M/\partial v = 36$.

5. Find $\partial z/\partial x$ and $\partial z/\partial y$ for $xyz = \cos(x + y + z)$.

Let $F(x,y,z) = xyz - \cos(x+y+z)$. Then we have an equation of the form $F(x,y,z) = 0$, which implies that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}.$$

6. Find the maximum rate of change of the function $f(x,y,z) = \tan(x + 2y + 3z)$ at the point $(-5,1,1)$.

Note that $(\tan x') = 1 + \sin^2 x/\cos^2 x = \sec^2 x$. Then

$$\nabla f(x,y,z) = (\sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3y)(3)),$$

so at the point $(-5,1,1)$,

$$\nabla f(-5,1,1) = (\sec^2(0), 2\sec^2(0), 3\sec^2(0)) = (1,2,3)$$

is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5,1,1)| = \sqrt{14}$.

MATH 2720
Fall 2006
Assignment 3

This assignment is due **in class** Monday, November 27$^{\text{th}}$. Late assignments will **not** be accepted.
Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you.
3. Try to view this assignment as a practice final examination: set aside 2 hours, and do as much as you can of the assignment without looking at the textbook or your notes. This should help you identify the topics on which you need to do more work. Then return to the assignment, and complete those parts that you had not finished.

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1. Let a, b, c, d be vectors in V_3. Prove that

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

and that

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}.$$

2. Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x = 1$ and $2x^2+4y^2-2z^2-5y = 0$ lies in a plane.

3. If u, v, w are differentiable functions in V_3, find an expression for

$$\frac{d}{dt}[u(t) \cdot (v(t) \times w(t))].$$

4. Find a polynomial $P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1, \end{cases}$$

is continuous and has continuous slope and continuous curvature.

5. Find the differential of the function $u = r/(s+2t)$.

6. Suppose that $z = f(x,y)$, with $x = g(s,t)$ and $y = h(s,t)$. Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}.$$

7. Find the extreme values of

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5$$

on the region $x^2 + y^2 \leq 16$.

8. Sketch the region of integration and change the order of integration, for

$$\int_0^1 \int_{y^2}^{2-y} f(x,y)dxdy.$$

9. When computing a double integral over a region D, the following identity was obtained:

$$\iint_D f(x,y)dA = \int_0^1 \int_0^{2y} f(x,y)dxdy + \int_1^3 \int_0^{3-y} f(x,y)dxdy.$$

Sketch the region D, and change the order of integration.

10. Evaluate the integral

$$\iint_R ydA,$$

where R is the region of the first quadrant bounded by the circle $x^2 + y^2 = 9$ and the lines $x = 0$ and $y = 0$.

11. Find the area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

12. Compute

$$\iiint_E zdV,$$

where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$ and $z = 0$ in the first octant (the region in 3-space where x, y and z are nonnegative).

MATH 2720
Fall 2006
Assignment 3
Solutions

As usual, 5 points for clarity. Questions 4, 7, 8, 10 and 11 were marked (all were worth 20 marks except question 10, worth 15 marks).

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1. Let a, b, c, d be vectors in V_3. Prove that

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

and that

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}.$$

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$ and $d = (d_1, d_2, d_3)$.
First equality. We have

$$b \times c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1),$$

so

$$\begin{aligned} a \times (b \times c) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \\ &= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\vec{i} - [a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)]\vec{j} \\ &\quad + [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\vec{k} \\ &= (a_2b_1c_2 + a_3b_1c_3)\vec{i} + (a_1b_2c_1 + a_3b_2c_3)\vec{j} + (a_1b_3c_1 + a_2b_3c_2)\vec{k} \\ &\quad - (a_2b_2c_1 + a_3b_3c_1)\vec{i} - (a_1b_1c_2 + a_3b_3c_2)\vec{j} - (a_1b_1c_3 + a_2b_2c_3)\vec{k} \\ &= (a_2c_2 + a_3c_3)b_1\vec{i} + (a_1c_1 + a_3c_3)b_2\vec{j} + (a_1c_1 + a_2c_2)b_3\vec{k} \\ &\quad - (a_2b_2 + a_3b_3)c_1\vec{i} - (a_1b_1 + a_3b_3)c_2\vec{j} - (a_1b_1 + a_2b_2)c_3\vec{k}. \end{aligned}$$

At this point, we see that there are still a few terms missing to get the result. In the $\vec{i}$ term, we need to add $a_1b_1c_1$ to get the $\vec{i}$ component of $(a \cdot c)b$, and $a_1b_1c_1$ (with a negative sign) to get the $\vec{i}$ component of $(a \cdot b)c$. Thus, all that is needed is to add $a_1b_1c_1 - a_1b_1c_1$ (that is, zero) to the $\vec{i}$

component. Similarly, $a_2b_2c_2 - a_2b_2c_2$ must be added to the $\vec{j}$ component, and $a_3b_3c_3 - a_3b_3c_3$ to the $\vec{k}$ component. Doing this and factoring as previously, we have

$$\begin{aligned} a \times (b \times c) &= (a_1c_1 + a_2c_2 + a_3c_3)b_1\vec{i} + (a_1c_1 + a_2c_2 + a_3c_3)b_2\vec{j} + (a_1c_1 + a_2c_2 + a_3c_3)b_3\vec{k} \\ &\quad - (a_1b_1 + a_2b_2 + a_3b_3)c_1\vec{i} - (a_1b_1 + a_2b_2 + a_3b_3)c_2\vec{j} - (a_1b_1 + a_2b_2 + a_3b_3)c_3\vec{k} \\ &= (a \cdot c)b - (a \cdot b)c. \end{aligned}$$

Second equality. There are two methods. Short method: "I know some formulae".

$$\begin{aligned} (a \times b) \cdot (c \times d) &= (a \times b) \cdot v = a \cdot (b \times v) \\ &= a \cdot [b \times (c \times d)] \\ &= a \cdot [(b \cdot d)c - (b \cdot c)d] \\ &= (b \cdot d)(a \cdot c) - (b \cdot c)(a \cdot d) \\ &= \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}. \end{aligned}$$

The second method is more tedious, requires more time, but can be done with a very limited number of formula. We have

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1),$$

$$c \times d = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = (c_2d_3 - c_3d_2, c_3d_1 - c_1d_3, c_1d_2 - c_2d_1),$$

it follows that

$$\begin{aligned} (a \times b) \cdot (c \times d) &= (a_2b_3 - a_3b_2)(c_2d_3 - c_3d_2) + (a_3b_1 - a_1b_3)(c_3d_1 - c_1d_3) \\ &\quad + (a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1) \\ &= a_2b_3c_2d_3 + a_3b_2c_3d_2 + a_3b_1c_3d_1 + a_1b_3c_1d_3 + a_1b_2c_1d_2 + a_2b_1c_2d_1 \\ &\quad - a_2b_3c_3d_3 - a_3b_2c_2d_3 - a_3b_1c_1d_3 - a_1b_3c_3d_1 - a_1b_2c_2d_1 - a_2b_1c_1d_2 \\ &= (a_1c_1 + a_2c_2 + a_3c_3)(b_1d_1 + b_2d_2 + b_3d_3) \\ &\quad - (a_1d_1 + a_2d_2 + a_3d_3)(b_1c_1 + b_2c_2 + b_3c_3), \end{aligned}$$

since all products $a_ib_ic_id_i$, $i = 1, 2, 3$ appear twice, with a plus sign and with a minus sign; hence,

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}.$$

2. Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x = 1$ and $2x^2+4y^2-2z^2-5y = 0$ lies in a plane.

We write

$$F(x,y,z) = x^2 + 2y^2 - z^2 + 3x - 1$$

and

$$G(x,y,z) = 2x^2 + 4y^2 - 2z^2 - 5y.$$

The curve of intersection of the surfaces is such that

$$F(x,y,z) = G(x,y,z) = 0.$$

Note that $F(x,y,z) = 0$ is equivalent to $kF(x,y,z) = 0$, for $k \neq 0$. So we can seek x, y, z such that

$$2F(x,y,z) = G(x,y,z),$$

which is equivalent to

$$2F(x,y,z) - G(x,y,z) = 0,$$

finally leading to $6x - 5y - 2 = 0$. In xyz-space, this is a plane.

3. If u, v, w are differentiable functions in V_3, find an expression for

$$\frac{d}{dt}[u(t) \cdot (v(t) \times w(t))].$$

We know that

$$\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t),$$

and

$$\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t).$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}[u(t) \cdot (v(t) \times w(t))] &= u'(t) \cdot [v(t) \times w(t)] + u(t) \cdot [v'(t) \times w(t) + v(t) \times w'(t)] \\
&= u'(t) \cdot v(t) \times w(t) + u(t) \cdot v'(t) \times w(t) + u(t) \cdot v(t) \times w'(t) \\
&= u'(t) \times v(t) \cdot w(t) + u(t) \times v'(t) \cdot w(t) + u(t) \cdot v(t) \times w'(t) \\
&= [u'(t) \times v(t) + u(t) \times v'(t)] \cdot w(t) + u(t) \cdot v(t) \times w'(t).
\end{aligned}$$

This is just one of the expressions that could be obtained; several other expressions could be given.

4. Find a polynomial $P(x)$ of degree 5 such that the function F defined by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1, \end{cases}$$

is continuous and has continuous slope and continuous curvature.

We seek a polynomial of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5.$$

This polynomial must be continuous, that is, satisfy

$$P(0) = 0 \qquad P(1) = 1,$$

have continuous slope, that is, satisfy

$$P'(0) = 0 \qquad P'(1) = 0,$$

(the slope left of 0 and right of 1 is 0), and finally, have continuous curvature, that is, satisfy

$$\kappa_P(0) = 0 \qquad \kappa_P(1) = 0,$$

where $\kappa_P(x)$ is the curvature of P at the point x (the curvature of a line is 0). To compute the curvature, we use the formula

$$\kappa_P(x) = \frac{|P''(x)|}{[1 + (P'(x))^2]^{3/2}}.$$

We have

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4,$$

and

$$P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3,$$

so, finally,

$$\kappa_P(x) = \frac{|2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3|}{[1 + (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4)^2]^{3/2}}.$$

We should be careful about the sign in the numerator, because of the absolute value. We can, however, start by seeking nonnegative a_i and if we are not able to do so, consider the sign in more detail.

Write down the constraints:

$$\begin{aligned}
P(0) = 0 &\Leftrightarrow a_0 = 0 \\
P(1) = 1 &\Leftrightarrow a_1 + a_2 + a_3 + a_4 + a_5 = 1 \quad (\text{since } a_0 = 0) \\
P'(0) = 0 &\Leftrightarrow a_1 = 0 \\
P'(1) = 0 &\Leftrightarrow 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \quad (\text{since } a_1 = 0) \\
\kappa_P(0) = 0 &\Leftrightarrow |2a_2| = 0 \Leftrightarrow a_2 = 0 \\
\kappa_P(1) = 0 &\Leftrightarrow \frac{6a_3 + 12a_4 + 20a_5}{[1 + (3a_3 + 4a_4 + 5a_5)^2]^{3/2}} = 0 \\
&\Leftrightarrow 6a_3 + 12a_4 + 20a_5 = 0,
\end{aligned}$$

where the last equivalence results from the fact that $1 + (3a_3 + 4a_4 + 5a_5)^2 > 0$ for $a_3, a_4, a_5 \in \mathbb{R}$. We now collect all the remaining equations, simplify, giving the system

$$\begin{aligned} a_3 + a_4 + a_5 &= 1 \\ 3a_3 + 4a_4 + 5a_5 &= 0 \\ 6a_3 + 12a_4 + 20a_5 &= 0. \end{aligned}$$

This is the system $MA = B$, with

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to check that $\det M = 2$, hence M is invertible; the inverse is readily computed:

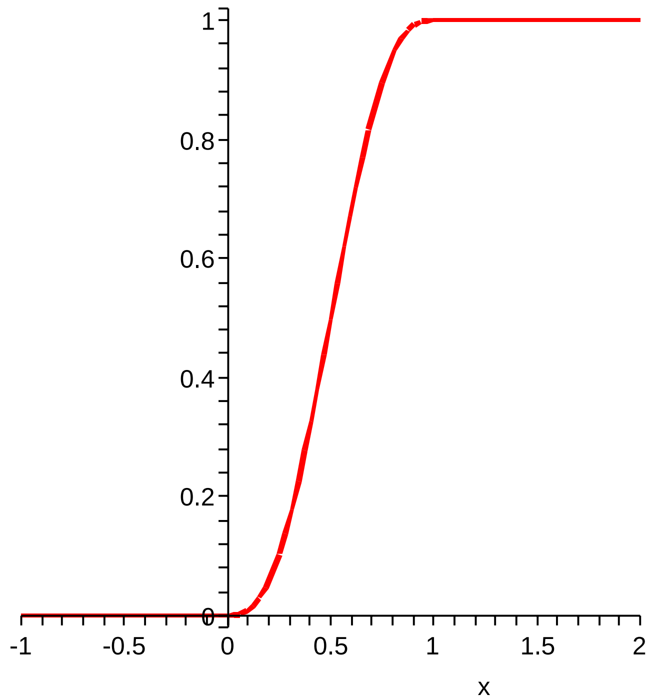
$$M^{-1} = \begin{pmatrix} 10 & -4 & \frac{1}{2} \\ -15 & 7 & -1 \\ 6 & -3 & \frac{1}{2} \end{pmatrix},$$

so that finally,

$$A = M^{-1}B = \begin{pmatrix} 10 \\ -15 \\ 6 \end{pmatrix}.$$

The following figure shows the result of using the function

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 10x^3 - 15x^4 + 6x^5 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$



5. Find the differential of the function $u = r/(s+2t)$.

u depends on three variables, so we have

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt,$$

with

$$\frac{\partial u}{\partial r} = \frac{1}{s+2t},$$

$$\frac{\partial u}{\partial s} = r(-1)(s+2t)^{-2} = -\frac{r}{(s+2t)^2}$$

and

$$\frac{\partial u}{\partial t} = r(-1)(s+2t)^{-2}(2) = -\frac{2r}{(s+2t)^2}.$$

So we find

$$du = \frac{1}{s+2t}dr - \frac{r}{(s+2t)^2}ds - \frac{2r}{(s+2t)^2}dt.$$

6. Suppose that $z = f(x,y)$, with $x = g(s,t)$ and $y = h(s,t)$. Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}.$$

We have

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t},$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right). \end{aligned} \tag{2.1}$$

We consider

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right),$$

since

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

can be obtained by substituting y for x. We have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) &= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\left(\frac{\partial}{\partial t}\frac{\partial x}{\partial t}\right) \\ &= \left(\frac{\partial}{\partial t}\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2},\end{aligned}$$

and so now need to evaluate

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial x}.$$

z is a function of x and y, and so, so is $\partial z/\partial x$. To avoid confusion, we write $\gamma(x,y) = \partial z/\partial x$. Then

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \gamma}{\partial y}\frac{\partial y}{\partial t},$$

or, in terms of $\partial z/\partial x$,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial z}{\partial x}\frac{\partial y}{\partial t} \\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t}.\end{aligned}$$

As a consequence,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) &= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} \\ &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2}.\end{aligned}$$

Substituting this value and the corresponding value in terms of y into (2.1), gives the desired result (assume Clairaut's theorem holds to get the 2).

7. Find the extreme values of

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the region $x^2 + y^2 \leq 16$.

We have

$$f_x = 4x - 4, \qquad f_y = 6y,$$

and therefore there is a unique critical point inside the disk $x^2 + y^2 < 16$, the point $(1, 0)$ (whereat $f(1, 0) = 7$). Now we consider the boundary $x^2 + y^2 = 16$. There are two approaches.

Parametrization: The disk $x^2 + y^2 = 16$ can be parametrized using $x = 4\cos t$, $y = 4\sin t$, for $t \in [0, 2\pi]$. We have

$$\begin{aligned} F(t) &= 2(4\cos t)^2 + 3(4\sin t)^2 - 4(4\cos t) - 5 \\ &= 32\cos^2 t + 48\sin^2 t - 16\cos t - 5 \\ &= 16(\sin^2 t - \cos t) + 27. \end{aligned}$$

We seek the maximum and minimum values of F for $t \in [0, 2\pi]$. We have

$$F'(t) = 32\cos t\sin t + 16\sin t = 16(2\cos t + 1)\sin t,$$

which is equal to 0 if $2\cos t + 1 = 0$ or $\sin t = 0$. In $[0, 2\pi]$, we have $\cos t = -1/2$ for $t = 2\pi/3$, and $\sin t = 0$ for $t = 0, \pi$ and 2π (we omit the latter from now, because of the periodicity of $\cos t$). So the critical points of F in $[0, 2\pi]$ are 0, $2\pi/3$, π, and $4\pi/3$, and

$$F(0) = 11, \quad F(\pi) = 43, \quad F(2\pi/3) = F(4\pi/3) = 16\left(\left(\pm\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\right) + 27 = 47.$$

Therefore, the minimum of f, -7, is at $(1, 0)$, and the maximum of f, 47, is at $(4\cos 2\pi/3, 4\sin 2\pi/3) = (-2, 2\sqrt{3})$ and $(4\cos 2\pi/3, 4\sin 4\pi/3) = (-2, -2\sqrt{3})$.

Lagrange multipliers: We write the system deduced from the gradients and the constraint.

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ x^2 + y^2 &= 16, \end{aligned}$$

that is,

$$4x - 4 = 2\lambda x \tag{2.2}$$

$$6y = 2\lambda y \tag{2.3}$$

$$x^2 + y^2 = 16, \tag{2.4}$$

From (2.3), $y = 0$ or $\lambda = 3$. Substituting $y = 0$ in (2.4) gives $x = \pm 4$, hence the points $(-4, 0)$ and $(4, 0)$. Substituting $\lambda = 3$ in (2.2) gives $x = 2$. To find the values of y corresponding to $x = 2$, we then use (2.4), giving $y^2 = 12$, i.e., $y = \pm 2\sqrt{3}$, and thus the points $(2, -2\sqrt{3})$ and $(2, 2\sqrt{3})$. We then have

$$f(2, -2\sqrt{3}) = f(2, 2\sqrt{3}) = 47, \quad f(-4, 0) = 43, \quad f(4, 0) = -5,$$

and the conclusion follows as in the parametrized case.

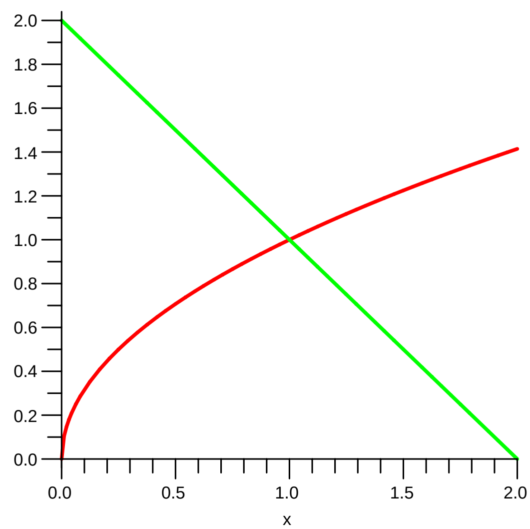
8. Sketch the region of integration and change the order of integration, for

$$\int_0^1 \int_{y^2}^{2-y} f(x, y) dx dy.$$

The region takes the form

$$E = \{(x, y) : 0 \leq y \leq 1, y^2 \leq x \leq 2 - y\},$$

and is represented here:



The curves $x = y^2$ and $x = 2 - y$ are expressed in terms of x as, respectively, $y = \sqrt{x}$ and $y = 2 - x$ (for the former, we use the positive part of $y = \pm\sqrt{x}$ since we are in a region where both x and y are nonnegative).

The region E can be written as

$$E = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}\} \cup \{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 2 - x\},$$

and therefore,

$$\int_0^1 \int_{y^2}^{2-y} f(x, y) dx dy = \int_0^1 \int_0^{\sqrt{x}} f(x, y) dy dx + \int_0^{2-x} f(x, y) dy dx.$$

9. When computing a double integral over a region D, the following identity was obtained:

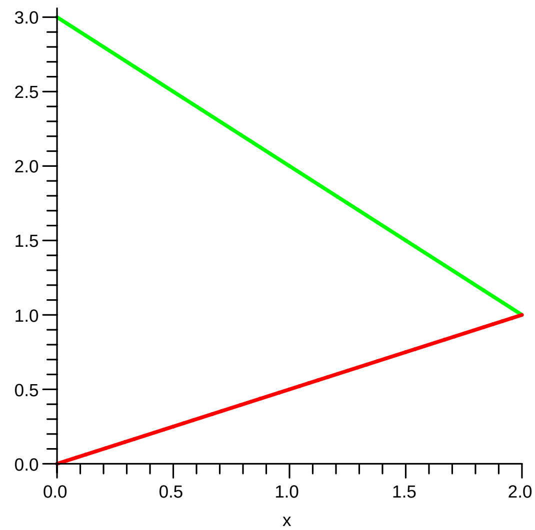
$$\iint_D f(x,y)dA = \int_0^1 \int_0^{2y} f(x,y)dxdy + \int_1^3 \int_0^{3-y} f(x,y)dxdy.$$

Sketch the region D, and change the order of integration.

It is clear that D takes the form

$$D = \{(x,y) : 0 \leq y \leq 1, 0 \leq x \leq 2y\} \cup \{(x,y) : 1 \leq y \leq 3, 0 \leq x \leq 3-y\}.$$

This region is represented here:



As a type I region, D is written

$$D = \{(x,y) : 0 \leq x \leq 2, \frac{x}{2} \leq y \leq 3-x\},$$

and therefore,

$$\iint_D f(x,y)dA = \int_0^2 \int_{\frac{x}{2}}^{3-x} f(x,y)dydx.$$

10. Evaluate the integral

$$\iint_R ydA,$$

where R is the region of the first quadrant bounded by the circle $x^2+y^2=9$ and the lines $x=0$ and $y=0$.

We switch to polar coordinates. Clearly, the region

$$R = \{(x,y) : x \geq 0, y \geq 0, x^2+y^2 \leq 9\}$$

takes the form

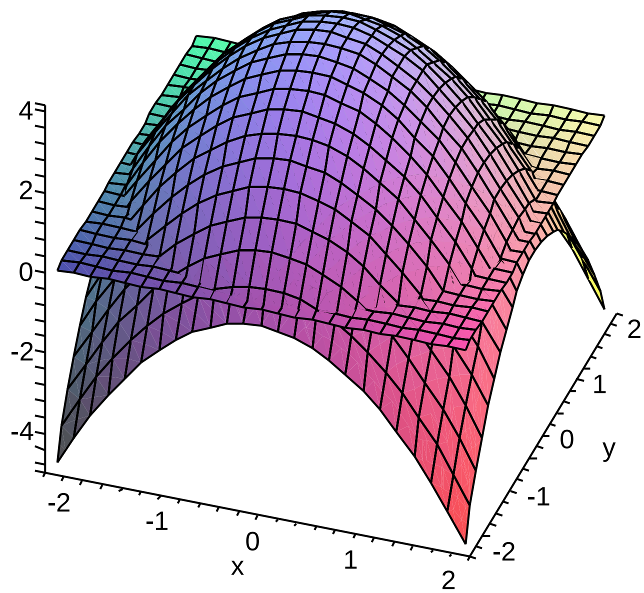
$$R = \{(r,\theta) : 0 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\}$$

in polar coordinates. Therefore, setting $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}\iint_R y dA &= \int_0^3 \int_0^{\frac{\pi}{2}} r\sin\theta\, r\, d\theta dr \\ &= \int_0^3 r^2\, dr \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \\ &= \left[\frac{r^3}{3}\right]_{r=0}^3 [-\cos\theta]_{\theta=0}^{\frac{\pi}{2}} \\ &= 9.\end{aligned}$$

11. Find the area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

The surface of which we want to compute the area is the following:



We use the formula, for $z = f(x,y)$,

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

We have

$$\left(\frac{\partial z}{\partial x}\right)^2 = (-2x)^2 = 4x^2$$

and, similarly,

$$\left(\frac{\partial z}{\partial y}\right)^2 = 4y^2,$$

so

$$A(S) = \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA.$$

To evaluate this integral, we switch to polar coordinates. But first, we need to determine D. D is the projection on the xy-plane of the surface, which takes the form $4 = x^2 + y^2$. (Note that it is here easy to determine this projection, since the surface is at its widest on the plane $z = 0$. If the surface were more balloon shaped, for example, we would have to determine the value of $k > 0$ for which the level set $f(x, y) = k$ is the largest.)

Therefore,

$$D = \{(x, y) : x^2 + y^2 \leq 4\} = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\},$$

and

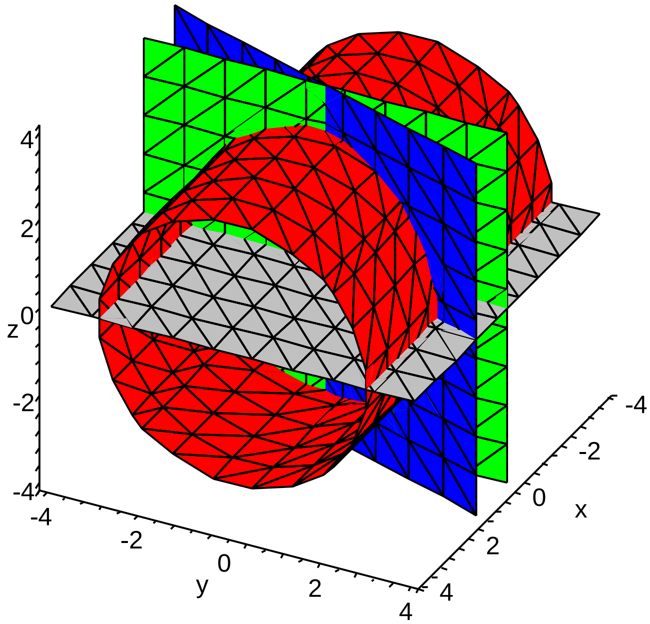
$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\; r\; dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{1 + 4r^2}\; r\; dr \\ &= [\theta]_{\theta=0}^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{3/2} \right]_{r=0}^2 \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 1\right). \end{aligned}$$

12. Compute

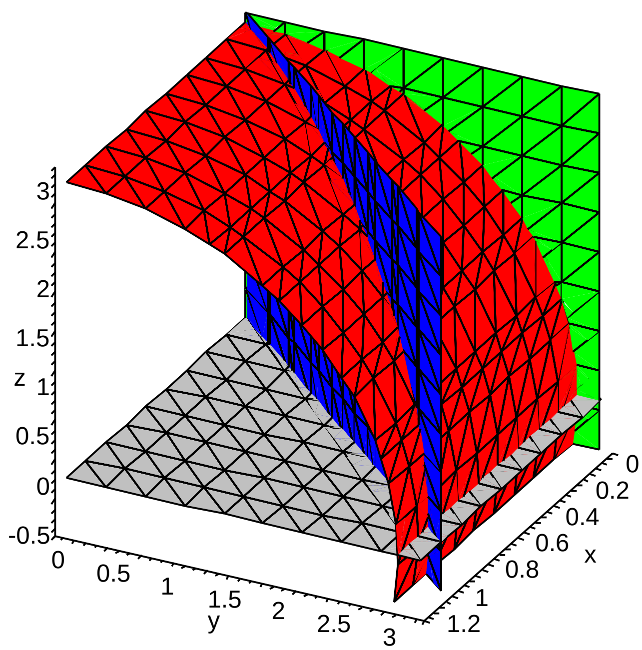
$$\iiint_E z dV,$$

where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$ and $z = 0$ in the first octant (the region in 3-space where x, y and z are nonnegative).

The general setting is the following:



Zooming in on the part of the figure that lies in the first octant,



E is the wedge between the green, red, blue and gray surfaces. Since the planes $y = 3x$ (blue) and $x = 0$ (green) are vertical, we can write E as a type I solid region,

$$E = \{(x, y, z) : (x, y) \in D, 0 \le z \le \sqrt{9 - y^2}\},$$

where the bounds for z are obtained by noting that z is above $z = 0$ (gray plane) and below the part of the cylinder $y^2 + z^2 = 9$ (red surface) that has both y and z nonnegative. We now need to express D. Looking at the figure, it is clear that we can write

$$D = \{(x,y) : 0 \leq x \leq 1, 3x \leq y \leq 3\}.$$

Therefore,

$$\begin{aligned}
\iiint_E z dV &= \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\ dzdydx \\
&= \int_0^1 \int_{3x}^3 \left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dydx \\
&= \frac{1}{2}\int_0^1 \int_{3x}^3 9 - y^2\ dydx \\
&= \frac{1}{2}\int_0^1 \left[9y - \frac{y^3}{3}\right]_{y=3x}^3 dx \\
&= \frac{1}{2}\int_0^1 27 - \frac{27}{3} - 27x + \frac{27x^3}{3}\ dx \\
&= \frac{1}{2}\left[18x - \frac{27}{2}x^2 + \frac{9}{4}x^4\right]_{x=0}^1 \\
&= \frac{1}{2}\left(18 - \frac{27}{2} + \frac{9}{4}\right) \\
&= \frac{1}{2}\ \frac{72-54+9}{4} = \frac{27}{8}.
\end{aligned}$$

MATH 2720
Fall 2006
Assignment 3

This assignment is due **in class** Monday, December 4th. Late assignments will **not** be accepted.
Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you.
3. Try to view this assignment as a practice final examination: set aside 2 hours, and do as much as you can of the assignment without looking at the textbook or your notes. This should help you identify the topics on which you need to do more work. Then return to the assignment, and complete those parts that you had not finished.

---

1. Reduce the equation
$$x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$$
to one of the standard forms, classify the surface, and sketch it. [If such a question were on the final, you would be provided with the Table of Graphs of quadric surfaces.]

2. Sketch the solid consisting of all points with spherical coordinates (ρ, θ, ϕ) such that
$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{6}, \quad 0 \leq \rho \leq 2\cos\phi.$$

3. A particle starts at the origin with initial velocity $\vec{i} - \vec{j} + 3\vec{k}$. Its acceleration is $a(t) = 6t\vec{i} + 12t^2\vec{j} - 6t\vec{k}$. Find its position function.

4. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

5. Find the absolute maximum and minimum values of
$$f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$$
on the disk $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4\}$.

6. Give five other integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\,dz dx dy.$$

7. Evaluate the line integral

$$\int_{\mathcal{C}} F \cdot dr,$$

where $F(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}$ and $\mathcal{C}$ is given by $r(t) = 2t\vec{i} + 3t\vec{j} - t^2\vec{k}$, $-1 \leq t \leq 1$.

8. Determine whether or not

$$F(x,y) = (1 + 2xy + \ln x)\vec{i} + x^2\vec{j}$$

is conservative. If it is, determine a function f such that $\nabla f = F$.

9. Let $F = \nabla f$, where $f(x,y) = \sin(x - 2y)$. Find curves $\mathcal{C}_1$ and $\mathcal{C}_2$ that are not closed and satisfy the equations

$$\int_{\mathcal{C}_1} F \cdot dr = 0, \qquad \int_{\mathcal{C}_2} F \cdot dr = 1.$$

MATH 2720
Fall 2006
Assignment 4
Solutions

Question 3 was marked out of 20, questions 5, 7 and 8 were marked out of 25, for a total of 95 marks. 5 additional points could be awarded for clarity.

---

1. Reduce the equation

$$x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$$

to one of the standard forms, classify the surface, and sketch it.

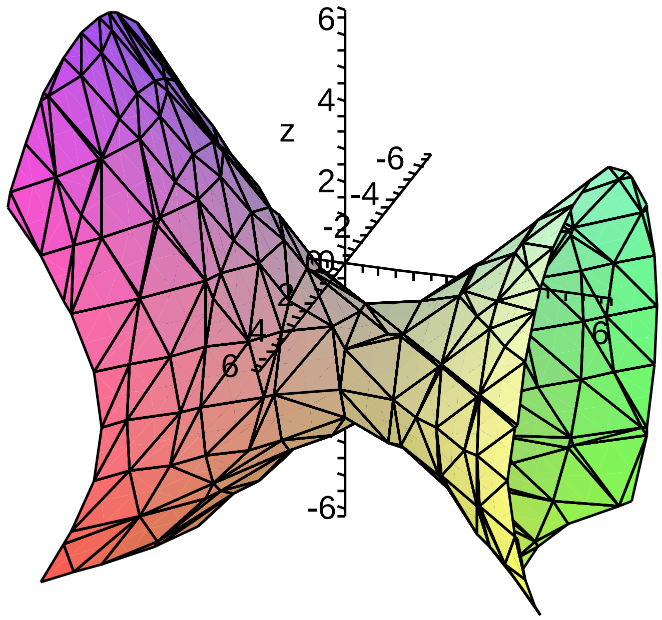
We have

$$\begin{aligned}
x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0 &\Leftrightarrow ((x-1)^2 - 1) - ((y-1)^2 - 1) + ((z+2)^2 - 4) = -2 \\
&\Leftrightarrow (x-1)^2 - (y-1)^2 + (z+2)^2 = 2 \\
&\Leftrightarrow \frac{(x-1)^2}{2} - \frac{(y-1)^2}{2} + \frac{(z+2)^2}{2} = 1.
\end{aligned}$$

Checking the table, we see that this is a hyperbola of one sheet, symmetric about the y-axis. To find the shift, reason as follows: consider the new variables X, Y and Z such that the previous equation can be written

$$\frac{X^2}{2} - \frac{Y^2}{2} + \frac{Z^2}{2} = 1.$$

In a domain centered at $(X, Y, Z) = (0, 0, 0)$, we have the figure shown in the table. To obtain these variables, we have to set $X = x - 1$, $Y = y - 1$ and $Z = z + 2$. To locate the point $(X, Y, Z) = (0, 0, 0)$ in the original domain, we write $x = X + 1$, $y = Y + 1$ and $z = Z - 2$, that is, $(X, Y, Z) = (0, 0, 0)$ is located at $(x, y, z) = (1, 1, -2)$. The figure below represents the surface.



To produce this figure, the following maple commands were used:

```
with(plots):
implicitplot3d((x-1)^2/2-(y-1)^2/2+(z+2)^2/2=1,x=-6..6,y=-6..6,z=-6..6);
```

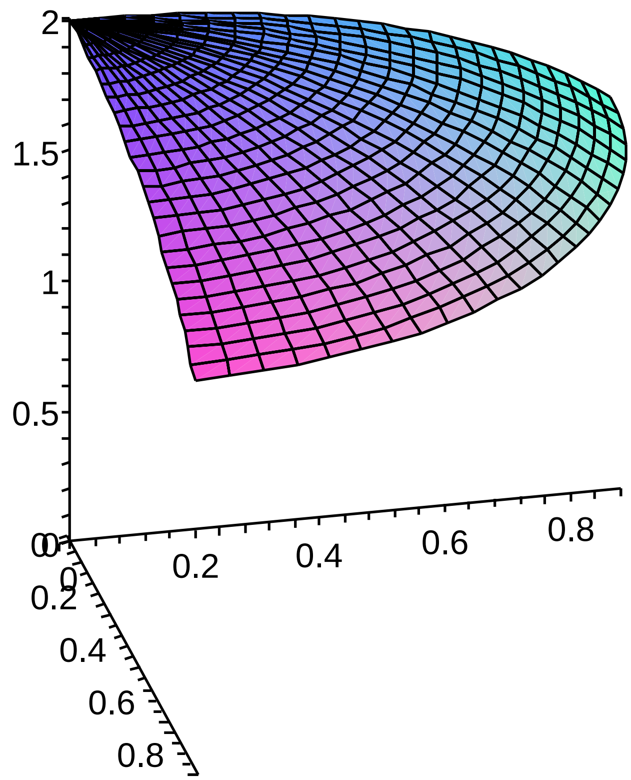
2. Sketch the solid consisting of all points with spherical coordinates (ρ, θ, ϕ) such that

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{6}, \quad 0 \leq \rho \leq 2\cos\phi.$$

The surface

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{6}, \quad \rho = 2\cos\phi,$$

is shown here:



The solid we are looking for consists of all the points that are between this surface and the origin (not the xy-plane).

The maple command used to produce the figure above is:

```
plot3d(2*cos(phi,theta=0..Pi/2,phi=0..Pi/6,coords=spherical,view=0..2,axes=normal);
```

3. A particle starts at the origin with initial velocity $\vec{i} - \vec{j} + 3\vec{k}$. Its acceleration is $a(t) = 6t\vec{i} + 12t^2\vec{j} - 6t\vec{k}$. Find its position function.

The acceleration is $a(t) = (6t, 12t^2, -6t)$, and therefore the velocity vector is of the form

$$\begin{aligned} v(t) &= \left(\int 6t\ dt, \int 12t^2\ dt, -\int 6t\ dt\right) \\ &= \left(3t^2 + K_1, 4t^3 + K_2, -3t^2 + K_3\right). \end{aligned}$$

But we know that the initial velocity (at $t = 0$) is $(1, -1, 3)$, so

$$\begin{aligned} v(t) = (1, -1, 3) &\Leftrightarrow \left(3t^2 + K_1, 4t^3 + K_2, -3t^2 + K_3\right) = (1, -1, 3) \\ &\Leftrightarrow K_1 = 1, \quad K_2 = -1, \quad K_3 = 3, \end{aligned}$$

which implies that the velocity is

$$v(t) = \left(3t^2 + 1, 4t^3 - 1, -3t^2 + 3\right).$$

Integrating, we find the position function

$$p(t) = \left(\int 3t^2 + 1\ dt, \int 4t^3 - 1\ dt, \int -3t^2 + 3\ dt\right)$$
$$= \left(t^3 + t + C_1, t^4 - t + C_2, -t^3 + 3t + C_3\right).$$

The particle starts at the origin at $t = 0$, so $C_1 = C_2 = C_3 = 0$, and, finally, the position vector is

$$p(t) = \left(t^3 + t, t^4 - t, -t^3 + 3t\right).$$

4. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

An equation for the tangent plane to the surface $z = 2x^2 - y^2$ at the point $(x_0, y_0, z_0) = (-2, 2, 4)$ is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

with $f_x = 4x$ and $f_y = -2y$; that is,

$$z - 4 = 4(-2)(x + 2) - 2(2)(y - 2),$$

or

$$8x + 4y + z + 4 = 0.$$

In this plane, we must select the line which has $z = 4$, which is accomplished by setting $z = 4$. This gives the equation for the line as

$$8x + 4y + 8 = 0.$$

To parametrize it, we can for example use the formula $r(t) = r_0(1 - t) + r_1 t$ with $t \in \mathbb{R}$, with $r_0 = (-2, 2, 4)$ and r_1 any other point on the line, obtained for example by setting $x = 0$, which, with the line equation, gives $y = -2$ (and, of course, $z = 4$), so $r_1 = (0, -2, 4)$. This gives

$$r(t) = (-2(1 - t), 2(1 - t) - 2t, 4(1 - t) - 4t) = (-2, 2, 4) + t(2, -4, 0).$$

There were plenty of other methods to obtain the same type of result. All produce results of the form

$$r(t) = (-2, 2, 4) + t(1, -2, 0).$$

5. Find the absolute maximum and minimum values of

$$f(x, y) = e^{-x^2 - y^2}(x^2 + 2y^2)$$

on the disk $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4\}$.

We first seek critical points inside the domain. We have

$$f_x = 2xe^{-x^2-y^2}(1-x^2-2y^2), \qquad f_y = 2ye^{-x^2-y^2}(2-x^2-2y^2).$$

These functions are defined on all the interior of $\mathcal{D}$, so the only critical points are points such that $f_x = f_y = 0$, if they exist within $\mathcal{D}$.

Since $e^{-x^2-y^2} > 0$ for $x, y \in \mathbb{R}$, we have $f_x = 0$ if, and only if, $x = 0$ or $x^2 + 2y^2 = 1$. In the case $x = 0$, substituting this value into f_y, we obtain $f_y = 2ye^{-y^2}(2-2y^2)$, which in turn is 0 if, and only if, $y = 0$ or $y^2 = 1$, that is, $y = \pm 1$. So $(0,0)$, $(0,-1)$ and $(0,1)$ are critical points of f.

Suppose now that $f_x = 0$ because we are on the ellipse $x^2 + 2y^2 = 1$ (which we call E). In that case, f_y can only be zero if $y = 0$, since the term $x^2 + 2y^2 = 2$ defines a larger ellipse than E, centered at the same point and that never intersects E. But $y = 0$ on the ellipse E implies that $x^2 = 1$, that is, $x = \pm 1$. So the points $(-1,0)$ and $(1,0)$ are also critical points of f.

Note that the latter two critical points could also be obtained by solving $f_y = 0$ in terms of y, and substituting the value $y = 0$ into f_x.

So, in the interior of $\mathcal{D}$, we find the critical points

$$(0,0), \quad (0,-1), \quad (0,1), \quad (-1,0) \quad \text{and} \quad (1,0),$$

with values

$$f(0,0) = 0, \quad f(0,-1) = f(0,1) = 2e^{-1} \quad \text{and} \quad f(-1,0) = f(1,0) = e^{-1}.$$

To find the critical points on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, we can proceed using a parametrized curve or using Lagrange multipliers.

Looking at f_x and f_y, we see that Lagrange multipliers will involve solving the system

$$\begin{aligned} 2xe^{-x^2-y^2}(1-x^2-2y^2) &= 2\lambda x \\ 2ye^{-x^2-y^2}(2-x^2-2y^2) &= 2\lambda y \\ x^2+y^2 &= 4, \end{aligned}$$

that is,

$$e^{-x^2-y^2}(1-x^2-2y^2) = \lambda \tag{2.5}$$

$$e^{-x^2-y^2}(2-x^2-2y^2) = \lambda \tag{2.6}$$

$$x^2+y^2 = 4. \tag{2.7}$$

Equations such as (2.5) and (2.6), which involve the unknown variables in an exponential multiplied by a polynomial, are called *transcendental* (or *quasi-polynomial*) equations. They are typically very hard to solve.

Therefore, we decide to use a parametrization rather than Lagrange multipliers. The boundary $\partial\mathcal{D}$ is easily parametrized with $x(t) = 2\cos t$ and $y(t) = 2\sin t$, for $t \in [0, 2\pi)$. We let

$$F(t) = e^{-4\cos^2 t - 4\sin^2 t}(4\cos^t + 8\sin^2 t) = 4e^{-4}(1+\sin^2 t).$$

Critical points of F on $[0, 2\pi)$ are found by considering $F'(t) = 0$. We have

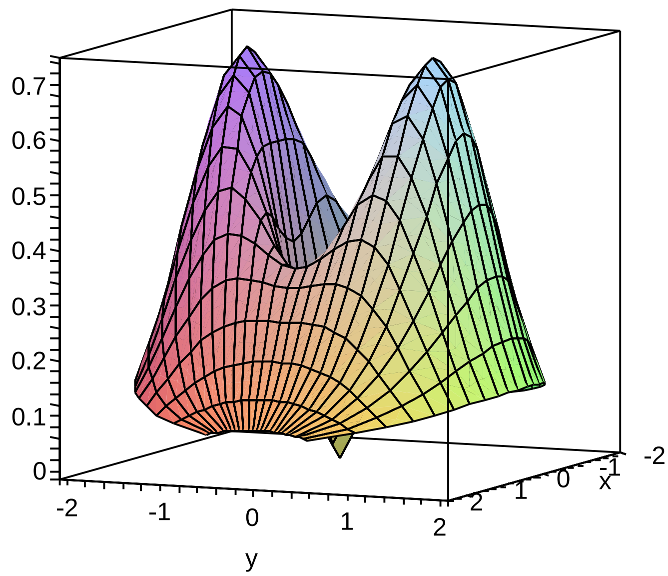
$$F'(t) = 8e^{-4} \cos t \sin t,$$

and thus critical points are $t = 0, \pi/2, \pi, 3\pi/2$. We have

$$F(0) = F(\pi) = 4e^{-4}, \quad F(\pi/2) = F(3\pi/2) = 8e^{-4}.$$

Finally, the minimum of f lies at $(0, 0)$ (taking the value 0), and the maxima of f lie at $(0, -1)$ and $(0, 1)$, with maximal value $2e^{-1}$.

The surface is shown here:



To obtain this figure, the following maple command was used:

```
plot3d(exp(-x^2-y^2)*(x^2+2*y^2),x=-2.1..2.1,y=-sqrt(4-x^2)..sqrt(4-x^2),
axes=boxed);
```

6. Give five other integrals that are equal to

$$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z) dz dx dy.$$

Let us call I_1 the given integral. The domain takes the form

$$E = \{(x, y, z) : 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\}.$$

Listing the possible orders of integration allows to determine how to proceed. For a function f of 3 variables x, y and z, we can compute the triple integrals $dxdydz$, $dxdzdy$, $dydxdz$, $dydzdx$, $dzdxdy$ and $dzdydx$.

The integral I_1 has $dzdxdy$. Since the bounds of x and of z do not depend on each other, we can of course interchange the two, obtaining

$$I_2 = \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x,y,z)dxdzdy.$$

The other 4 forms, $dxdydz$, $dydxdz$, $dydzdx$ and $dzdydx$ are obtained by writing the domain differently. We project E onto the xy, yz and xz-planes, giving the sets $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, respectively. We have

$$\mathcal{D}_1 = \{(x,y) : 0 \leq y \leq 2, 0 \leq x \leq y^3\} = \{(x,y) : 0 \leq x \leq 8, x^{1/3} \leq y \leq 2\},$$

$$\mathcal{D}_2 = \{(y,z) : 0 \leq z \leq 4, \sqrt{z} \leq y \leq 2\} = \{(y,z) : 0 \leq y \leq 2, 0 \leq z \leq y^2\},$$

and

$$\mathcal{D}_3 = \{(x,z) : 0 \leq x \leq 8, 0 \leq z \leq 4\}.$$

We get the following integrals:

$$\int_0^8 \int_{x^{1/3}}^2 \int_0^{y^2} f(x,y,z)dzdydx,$$

$$\int_0^4 \int_{\sqrt{x}}^2 \int_0^{y^3} f(x,y,z)dxdydz,$$

$$\int_0^8 \int_0^{x^{2/3}} \int_{x^{1/3}}^2 f(x,y,z)dydzdx + \int_0^8 \int_{x^{2/3}}^4 \int_{\sqrt{x}}^2 f(x,y,z)dydzdx,$$

$$\int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{x}}^2 f(x,y,z)dydxdz + \int_0^4 \int_{z^{3/2}}^8 \int_{x^{1/3}}^2 f(x,y,z)dydxdz.$$

7. Evaluate the line integral

$$\int_{\mathcal{C}} F \cdot dr,$$

where $F(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}$ and $\mathcal{C}$ is given by $r(t) = 2t\vec{i} + 3t\vec{j} - t^2\vec{k}$, $-1 \leq t \leq 1$.

We can either proceed directly, or try to use the extra result given with $\mathsf{curl}F$. Let us check if F is conservative. The component functions of F have continuous partial derivatives, and

$$\begin{aligned}
\mathsf{curl}F &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -z & y \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}y - \frac{\partial}{\partial z}(-z), \frac{\partial}{\partial z}x - \frac{\partial}{\partial x}y, \frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}x\right) \\
&= (2,0,0),
\end{aligned}$$

so, since $\mathsf{curl}F \neq 0$, we cannot use the fundamental theorem for line integrals. Writing $r(t) = (x(t), y(t), z(t))$, that is, $x(t) = 2t$, $y(t) = 3t$ and $z(t) = -t^2$, we have

$$\begin{aligned}
\int_C F \cdot dr &= \int_C x\ dx - z\ dy + y\ dz \\
&= \int_C x\ dx - \int_C z\ dy + \int_C y\ dz \\
&= \int_{-1}^1 x(t)x'(t)dt - \int_{-1}^1 z(t)y'(t)dt + \int_{-1}^1 y(t)z'(t)dt \\
&= \int_{-1}^1 (2t)(2)dt - \int_{-1}^1 (-t^2)(3)dt + \int_{-1}^1 (3t)(-2t)dt \\
&= \left[2t^2\right]_{t=-1}^1 + \left[t^3\right]_{t=-1}^1 - \left[2t^3\right]_{t=-1}^1 \\
&= 0 + 2 - 4 \\
&= -2.
\end{aligned}$$

8. Determine whether or not

$$F(x,y) = (1 + 2xy + \ln x)\vec{i} + x^2\vec{j}$$

is conservative. If it is, determine a function f such that $\nabla f = F$.

F is defined for $(x,y) \in \mathcal{D} = \mathbb{R}_+^* \times \mathbb{R}$, where $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$, which is open and simply connected. On $\mathcal{D}$,

$$\frac{\partial}{\partial y}(1 + 2xy + \ln x) = 2x$$

and

$$\frac{\partial}{\partial x}x^2 = 2x,$$

and therefore F is conservative. To find f, we note that $f_x = 1 + 2xy + \ln x$ and $f_y = x^2$. Taking (for example) the second term, integrating with respect to y gives

$$f(x,y) = \int x^2\ dy = x^2y + g(x),$$

where $g(x)$ does not depend on y. Differentiating this expression with respect to x gives

$$\frac{\partial}{\partial x} f(x,y) = 2xy + \frac{d}{dx} g(x).$$

This equals $f_x = 1 + 2xy + \ln x$ if and only if

$$\frac{d}{dx} g(x) = 1 + \ln x,$$

which gives

$$g(x) = \int 1 + \ln x \; dx = x + x\ln x - x + K = x \ln x + K.$$

K can be taken to be zero, so finally,

$$f(x,y) = x^2 y + x \ln x.$$

9. Let $F = \nabla f$, where $f(x,y) = \sin(x - 2y)$. Find curves $\mathcal{C}_1$ and $\mathcal{C}_2$ that are not closed and satisfy the equations

$$\int_{\mathcal{C}_1} F \cdot dr = 0, \qquad \int_{\mathcal{C}_2} F \cdot dr = 1.$$

Since $F = \nabla f$, F is conservative, and we can use the fundamental theorem of line integrals:

$$\int_{\mathcal{C}_1} F \cdot dr = f(r_1(b_1)) - f(r_1(a_1)),$$

and

$$\int_{\mathcal{C}_2} F \cdot dr = f(r_2(b_2)) - f(r_2(a_2)),$$

if $\mathcal{C}_1$ is the curve defined by $r_1(t)$ and going from $r_1(a_1)$ to $r_1(b_1)$, and $\mathcal{C}_2$ is the curve defined by $r_2(t)$ and going from $r_2(a_2)$ to $r_2(b_2)$, respectively.

So, if $r_1(a_1) = (x_1^1, y_1^1)$, $r_1(b_1) = (x_2^1, y_2^1)$ (remark that the superscripts here do not denote a power), we want

$$\int_{\mathcal{C}_1} F \cdot dr = f(x_2^1, y_2^1) - f(x_1^1, y_1^1) = 0,$$

and, if $r_2(a_2) = (x_1^2, y_1^2)$, $r_2(b_2) = (x_2^2, y_2^2)$, we want

$$\int_{\mathcal{C}_2} F \cdot dr = f(x_2^2, y_2^2) - f(x_1^2, y_1^2) = 1.$$

Let us start with $\mathcal{C}_1$. We seek (x_1^1, y_1^1) and (x_2^1, y_2^1) such that

$$\sin(x_2^1 - 2y_2^1) - \sin(x_1^1 - 2y_1^1) = 0,$$

that is,

$$\sin(x_2^1 - 2y_2^1) = \sin(x_1^1 - 2y_1^1).$$

For simplicity, we can assume $y_1^1 = y_2^1 = 0$, and thus, since sin is 2π-periodic, taking $x_1^1 = 0$ and $x_2^1 = 2\pi$ gives the desired result.

Now for $\mathcal{C}_2$. We seek (x_1^2, y_1^2) and (x_2^2, y_2^2) such that

$$\sin(x_2^2 - 2y_2^2) - \sin(x_1^2 - 2y_1^2) = 1,$$

that is,

$$\sin(x_2^2 - 2y_2^2) = 1 - \sin(x_1^2 - 2y_1^2).$$

Taking, for simplicity, $y_1^2 = y_2^2 = 0$, we look for x_1^2 and x_2^2 such that

$$\sin x_2^2 = 1 - \sin x_1^2.$$

Taking for example, $x_1^2 = \pi/6$, leading to $\sin x_1^2 = 1/2$, we can choose for example $x_2^2 = 5\pi/6$.

MATH 2720
Fall 2007
Assignment 1

This assignment is due **in class** Friday, September 21st. Late assignments will **not** be accepted.
Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you. Detail your answers, while trying to remain concise.
3. It is possible that all questions will not be marked.

---

1. Calculate $(3\vec{\imath} + 2\vec{\jmath} + \vec{k}) \cdot (\vec{\imath} + 2\vec{\jmath} - \vec{k})$.

2. Compute $\|u\|$, $\|v\|$, $u \cdot v$, $u \times v$ and normalize u and v, with $u = (15, -2, 4)$ and $v = (\pi, 3, -1)$.

3. Find two nonparallel vectors both orthogonal to $(1, 1, 1)$.

4. Compute $a \cdot (b \times c)$, where $a = \vec{\imath} - 2\vec{\jmath} + \vec{k}$, $b = 2\vec{\imath} + \vec{\jmath} + \vec{k}$ and $c = 3\vec{\imath} - \vec{\jmath} + 2\vec{k}$.

5. Find the intersection of the two planes with equations $3(x - 1) + 2y + (z + 1) = 0$ and $(x - 1) + 4y - (z + 1) = 0$.

6. Find an equation for the plane that contains the line $v = (-1, 1, 2) + t(3, 2, 4)$ and is perpendicular to the plane $2x + y - 3z + 4 = 0$.

7. Given vectors a and b, do the equations $x \times a = b$ and $x \cdot a = \|a\|$ determine a unique vector x? Argue both geometrically and analytically.

MATH 2720
Fall 2007
Assignment 1

The following questions were marked: 1 (20 points), 2 (25 points), 3 (25 points) and 6 (25 points), for a total of 95 points. A maximum of 5 additional marks could be awarded for clarity.

---

1. Calculate $(3\vec{i}+2\vec{j}+\vec{k})\cdot(\vec{i}+2\vec{j}-\vec{k})$.

We have

$$(3\vec{i}+2\vec{j}+\vec{k})\cdot(\vec{i}+2\vec{j}-\vec{k}) = 3+2.2-1 = 6.$$

2. Compute $\|u\|$, $\|v\|$, $u\cdot v$, $u\times v$ and normalize u and v, with $u=(15,-2,4)$ and $v=(\pi,3,-1)$.

We have $\|u\| = \sqrt{15^2+2^2+4^2} = \sqrt{245}$ and $\|v\| = \sqrt{\pi^2+3^2+1} = \sqrt{\pi^2+10}$. Also,

$$u\cdot v = 15\pi - 6 - 4 = 15\pi - 10$$

and

$$u\times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 15 & -2 & 4 \\ \pi & 3 & -1 \end{vmatrix} = (2-12, 4\pi+15, 45+2\pi) = (-10, 4\pi+15, 2\pi+45).$$

Finally, to *normalize* a vector means to make it a vector of length 1 (and in the same direction). We have

$$\tilde{u} = \frac{1}{\sqrt{245}}(15,-2,4)$$

and

$$\tilde{v} = \frac{1}{\sqrt{\pi^2+10}}(\pi,3,-1).$$

3. Find two nonparallel vectors both orthogonal to $(1,1,1)$.

Let $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$. a is orthogonal to $(1,1,1)$ iff $a\cdot(1,1,1) = a_1+a_2+a_3 = 0$, while b is orthogonal to $(1,1,1)$ iff $b\cdot(1,1,1) = b_1+b_2+b_3 = 0$. So we choose values for a satisfying this. For example, $a=(1,-1,0)$. Now b must not be parallel, so we choose b, for example, as $b=(0,1,-1)$. Clearly, they are not multiple of one another.

4. Compute $a\cdot(b\times c)$, where $a=\vec{i}-2\vec{j}+\vec{k}$, $b=2\vec{i}+\vec{j}+\vec{k}$ and $c=3\vec{i}-\vec{j}+2\vec{k}$.

We have

$$
\begin{aligned}
a\cdot(b\times c) &= \begin{vmatrix} 1 & -2 & 1\\ 2 & 1 & 1\\ 3 & -1 & 2\end{vmatrix}\\
&= 1\begin{vmatrix} 1 & 1\\ -1 & 2\end{vmatrix} + 2\begin{vmatrix} 2 & 1\\ 3 & 2\end{vmatrix} + 1\begin{vmatrix} 2 & 1\\ 3 & -1\end{vmatrix}\\
&= 2+1+2(4-3)+(-2-3)\\
&= 0
\end{aligned}
$$

5. Find the intersection of the two planes with equations $3(x-1)+2y+(z+1)=0$ and $(x-1)+4y-(z+1)=0$.

One way to proceed is to use methods from linear algebra. We write down the system

$$
\begin{aligned}
3(x-1)+2y+(z+1)&=0\\
(x-1)+4y-(z+1)&=0
\end{aligned}
$$

and simplify

$$
\begin{aligned}
3x+2y+z&=2\\
x+4y-z&=2.
\end{aligned}
$$

To solve this system, write the augmented matrix

$$
\begin{pmatrix} 3 & 2 & 1 & 2\\ 1 & 4 & -1 & 2\end{pmatrix}
$$

and obtain the reducted row echelon form,

$$
\begin{pmatrix} 1 & 0 & \frac{3}{5} & \frac{2}{5}\\ 0 & 1 & -\frac{2}{5} & \frac{2}{5}\end{pmatrix}.
$$

From this, we deduce that for a given $t\in\mathbb{R}$, the solution is $y-\frac{2}{5}t=\frac{2}{5}$, i.e., $y=\frac{2}{5}(t+1)$, and $x+\frac{3}{5}t=\frac{2}{5}$, i.e., $x=\frac{1}{5}(2-3t)$. So the two planes intersect along the line

$$
\left(\frac{1}{5}(2-3t),\frac{2}{5}(t+1),t\right),
$$

that is,

$$
\left(\frac{2}{5},\frac{2}{5},0\right)+\left(-\frac{3}{5},\frac{2}{5},1\right)t,\qquad t\in\mathbb{R}.
$$

6. Find an equation for the plane that contains the line $v=(-1,1,2)+t(3,2,4)$ and is perpendicular to the plane $2x+y-3z+4=0$.

Let $a = (3, 2, 4)$ be the vector giving the direction of the line $v = (-1, 1, 2) + t(3, 2, 4)$, and $n = (2, 1, -3)$ the normal vector to the plane $2x + y - 3z + 4 = 0$.

Then the vector $a \times n$ is perpendicular to the sought plane, i.e., is a normal vector to the sought plane. We have

$$a \times n = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 4 \\ 2 & 1 & -3 \end{vmatrix} = (-6 - 4, 9 + 8, 3 - 4) = (-10, 17, -1)$$

A point on the plane can be obtained by taking $t = 0$ in v, giving the point $r_0 = (-1, 1, 2)$. So an equation for the plane is

$$(-10, 17, -1) \cdot (r - (-1, 1, 2)) = 0.$$

7. Given vectors a and b, do the equations $x \times a = b$ and $x \cdot a = \|a\|$ determine a unique vector x? Argue both geometrically and analytically.

1) Following a false lead.. Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be given. We seek $x = (x_1, x_2, x_3)$ such that

$$x \times a = b \text{ and } x \cdot a = \|a\|.$$

Express this in terms of the components of a, b and x:

$$\begin{aligned} x \times a = b &\Leftrightarrow (x_2a_3 - x_3a_2, x_3a_1 - x_1a_3, x_1a_2 - x_2a_1) = (b_1, b_2, b_3) \\ &\Leftrightarrow \left| \begin{array}{l} x_2a_3 - x_3a_2 = b_1 \\ x_3a_1 - x_1a_3 = b_2 \\ x_1a_2 - x_2a_1 = b_3, \end{array} \right. \end{aligned}$$

and

$$x \cdot a = \|a\| \Leftrightarrow x_1a_1 + x_2a_2 + x_3a_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Since we seek x, we write this as a linear system where $x = (x_1, x_2, x_3)$ is the unknown:

$$\begin{array}{cccccc} & a_3x_2 & -a_2x_3 & = & b_1 \\ -a_3x_1 & & +a_1x_3 & = & b_2 \\ a_2x_1 & -a_1x_2 & & = & b_3 \\ a_1x_1 & +a_2x_2 & +a_3x_3 & = & \sqrt{a_1^2 + a_2^2 + a_3^2} \end{array}$$

that is,

$$\begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \sqrt{a_1^2 + a_2^2 + a_3^2} \end{pmatrix}$$

This system is *inconsistent*, as it has more equations (4) than unknowns (3). We say that it is an *overdetermined system.* So we have to proceed otherwise.

2) a and $x \times a$ are orthogonal, and since $x \times a = b$, a and b are also orthogonal. Therefore a and b must be orthogonal for x to exist that satisfies the first equation. So from now on, we assume $a \perp b$.

2.a) Suppose that $a = 0$. Then, since $b = x \times a$, $b = 0$ and any x is solution, meaning that it is not unique in this case.

2.b) Suppose now that $a \neq 0$, and for now, assume that $b \neq 0$. Since a and b are orthogonal, a, b and $a \times b$ form a basis of $\mathbb{R}^3$. Therefore, the vector x that we seek can be written as

$$x = \alpha a + \beta b + \gamma a \times b \tag{2.8}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Substitute this expression into $x \times a = b$, giving

$$(\alpha a + \beta b + \gamma a \times b) \times a = b$$

that is

$$\alpha a \times a + \beta b \times a + \gamma (a \times b) \times a = b$$

or

$$\beta b \times a + \gamma (a \times b) \times a = b \tag{2.9}$$

since $a \times a = 0$ for all a. We know that, for three vectors a, b, c, the vector triple product is

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b),$$

so, using $c = a$, we have

$$a \times (b \times a) = b(a \cdot a) - a(a \cdot b).$$

Since $a \perp b$, $a \cdot b = 0$ and therefore,

$$a \times (b \times a) = (a \cdot a)b = \|a\|^2 b. \tag{2.10}$$

Now remember that $a \times b = -b \times a$. Therefore $a \times (b \times a) = -(-a \times b) \times a = (a \times b) \times a$, and thus equation (2.10) can be written

$$(a \times b) \times a = \|a\|^2 b.$$

Substituting this expression into (2.12) gives

$$\beta b \times a + \gamma \|a\|^2 b = b.$$

Therefore $\beta = 0$, and $\gamma = 1/\|a\|^2$. From $x \cdot a = \|a\|$, $\alpha = 1/\|a\|$. Substituting these values into (2.8), we have

$$x = \frac{a}{\|a\|} + \frac{a \times b}{\|a\|^2}$$

which is unique, for given a and b.

In the case $b = 0$, we can find $x = a/\|a\|$, unique as well.

Note: This exercise was far from trivial.

MATH 2720
Fall 2007
Assignment 2

This assignment is due **in class** Friday, October 5th. Late assignments will **not** be accepted. Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you. Detail your answers, while trying to remain concise.
3. It is very likely that some questions will not be marked.

---

1 (F marks). ind the point(s) at which the curve

$$r(t) = (t, t^2, t^3)$$

intersects the plane $4x + 2y + z = 24$. What is the angle of intersection between the curve and the normal to the plane?

2 (L marks). et $f : \mathbb{R} \to \mathbb{R}^3$. Show that

$$\frac{d}{dt}\left(f(t) \times f'(t)\right) = f(t) \times f''(t). \tag{2.11}$$

Use (2.12), even if you have not managed to prove it, to show that if $f(t)$ is parallel to $f''(t)$ for all t, then $f \times f'$ is constant.

3 (L marks). et $g(t) = t\vec{i} + f(t)\vec{j}$. Calculate

$$g'(t_0), \quad \int_a^b g(t)dt, \quad \int_a^b g'(t)dt,$$

given that

$$f'(t_0) = m, \quad f(a) = c, \quad f(b) = d, \quad \int_a^b f(t)dt = A.$$

4 (F marks). ind the point at which the curves

$$r_1(t) = \left(e^t, 2\sin\left(t + \frac{\pi}{2}\right), t^2 - 2\right)$$

and

$$r_2(t) = t\vec{i} + 2\vec{j} + (t^2 - 3)\vec{k}$$

intersect, and find the angle of intersection.

5 (D marks). escribe the geometric meaning of the following mappings in spherical coordinates, i.e., given a point, what does the mapping do to it?

1. $(\rho, \theta, \phi) \mapsto (\rho, \theta + \pi, \phi)$.
2. $(\rho, \theta, \phi) \mapsto (\rho, \theta, \pi - \phi)$.
3. $(\rho, \theta, \phi) \mapsto (2\rho, \theta + \pi/2, \phi)$.

6 (F marks). ind the curve r such that $r(0) = (0, -5, 1)$ and $r'(t) = (t, e^t, t^2)$.

7 (L marks). et r be a curve in $\mathbb{R}^3$ with zero acceleration. Show that r is a straight line or a point.

8 (L marks). et r be the curve $r(t) = (t, t\sin t, t\cos t)$. Find the length of r between the points $(0, 0, 0)$ and $(\pi, 0, -\pi)$.

9 (L marks). et r be a smooth curve that is at least twice differentiable, and $T(t)$ be the unit tangent vector to $r(t)$ at t. Show that $T'(t) \cdot T(t) = 0$.
[Hint: Remember how we showed the result for r' and r in the case of constant $\|r\|$, and a property of unit vectors.]

MATH 2720

Fall 2007

Assignment 2 – Solutions

Marks: ______

1 (F marks). ind the point(s) at which the curve

$$r(t) = (t, t^2, t^3)$$

intersects the plane $4x + 2y + z = 24$. What is the angle of intersection between the curve and the normal to the plane?

Solution. We write $(x, y, z) = (t, t^2, t^3)$, so, substituting these values into the equation for the plane,

$$4t + 2t^2 + t^3 = 24.$$

Therefore the intersection occurs for t such that

$$4t + 2t^2 + t^3 - 24 = 0.$$

Here, we notice the obvious root $t = 2$. This means that the polynomial $4t + 2t^2 + t^3 - 24$ factors as $(at^2 + bt + c)(t - 2)$, where a, b, c must be found. The coefficient of t^3 in the original polynomial is 1, and therefore, $a = 1$. So we seek b and c such that

$$(t^2 + bt + c)(t - 2) = 4t + 2t^2 + t^3 - 24$$

Developping the left-hand side,

$$(t^2 + bt + c)(t - 2) = t^3 + bt^2 + ct - 2t^2 - 2bt - 2c = t^3 + (b - 2)t^2 + (c - 2b)t - 2c$$

and equating coefficients of monomials of the same degree, we have

$$b - 2 = 2, \qquad c - 2b = 4, \qquad -2c = -24,$$

which gives $b = 4$ and $c = 12$. Therefore,

$$4t + 2t^2 + t^3 - 24 = (t - 2)(t^2 + 4t + 12).$$

The polynomial $t^2 + 4t + 12$ has roots determined by the discriminant $\Delta = 16 - 48 < 0$, that is, has no real roots. Therefore, the curve r intersects the plane $4x + 2y + z = 24$ only when $t = 2$, which gives the point $r(4) = (2, 4, 8)$. (Check: it is indeed on the plane $4x + 2y + z = 24$.)

To find the angle at the intersection, we need the direction of the curve at the intersection, which is given by the derivative of r (the tangent vector). We have

$$r'(t) = (1, 2t, 3t^2)$$

and so the tangent vector to the curve at the intersection is

$$r'(2) = (1, 4, 12).$$

The normal to the plane $4x + 2y + z = 24$ is $n = (4, 2, 1)$, so the angle between r' and n satisfies

$$\cos\theta = \frac{r' \cdot n}{\|r'\|\,\|n\|} = \frac{(1, 4, 12) \cdot (4, 2, 1)}{\sqrt{1 + 4^2 + 12^2}\sqrt{4^2 + 2^2 + 1}} = \frac{24}{\sqrt{161}\sqrt{21}}$$

2 (L marks). et $f : \mathbb{R} \to \mathbb{R}^3$. Show that

$$\frac{d}{dt}\left(f(t) \times f'(t)\right) = f(t) \times f''(t). \tag{2.12}$$

Use (2.12), even if you have not managed to prove it, to show that if $f(t)$ is parallel to $f''(t)$ for all t, then $f \times f'$ is constant.

Solution. We use the rule for the derivative of the cross product of two vector functions u and v,

$$(u \times v)' = u' \times v + u \times v',$$

which, using $u = f$ and $v = f'$, gives

$$(f \times f')' = f' \times f' + f \times f''.$$

But for any vector a, $a \times a = 0$, so

$$(f \times f')' = f \times f''.$$

Now suppose that $f(t)$ is parallel to $f''(t)$ for all t. If two vectors a and b are parallel, we know that $a \times b = 0$. Therefore, we have

$$f \times f'' = 0.$$

From the first part, $(f \times f')' = f \times f''$. Since $f \times f'' = 0$, this means that $(f \times f')' = 0$. Integrating, we get that $f \times f'$ is constant.

3 (L marks). et $g(t) = t\vec{i} + f(t)\vec{j}$. Calculate

$$g'(t_0), \quad \int_a^b g(t)dt, \quad \int_a^b g'(t)dt,$$

given that

$$f'(t_0) = m, \quad f(a) = c, \quad f(b) = d, \quad \int_a^b f(t)dt = A.$$

Solution. We have

$$g'(t) = \vec{i} + f'(t)\vec{j},$$

and thus, since $f'(t_0) = m$,

$$g'(t_0) = \vec{i} + m\vec{j}.$$

Also,

$$\begin{aligned}\int_a^b g(t)dt &= \left(\int_a^b t dt\right)\vec{i} + \left(\int_a^b f(t)dt\right)\vec{j} \\ &= \left(\frac{b^2 - a^2}{2}\right)\vec{i} + A\vec{j}.\end{aligned}$$

Finally,

$$\begin{aligned}\int_a^b g'(t)dt &= \left(\int_a^b 1\ dt\right)\vec{i} + \left(\int_a^b f'(t)dt\right)\vec{j} \\ &= (b-a)\,\vec{i} + (d-c)\,\vec{j},\end{aligned}$$

where the second term in the last equation results from the fundamental theorem of calculus.

4 (F marks). ind the point at which the curves

$$r_1(t) = \left(e^t, 2\sin\left(t + \frac{\pi}{2}\right), t^2 - 2\right)$$

and

$$r_2(t) = t\vec{i} + 2\vec{j} + (t^2 - 3)\vec{k}$$

intersect, and find the angle of intersection.

Solution. We write both equations in the same form, but use a different name, say s, for the independent variable in one of the two. We thus seek s, t such that

$$\left(e^t, 2\sin\left(t + \frac{\pi}{2}\right), t^2 - 2\right) = (s, 2, s^2 - 3),$$

that is, s, t such that

$$e^t = s \tag{2.13}$$

$$2\sin\left(t + \frac{\pi}{2}\right) = 2 \tag{2.14}$$

$$t^2 - 2 = s^2 - 3. \tag{2.15}$$

From (2.14), $t = 2k\pi$, for $k \in \mathbb{Z}$. Substituting into (2.13) and (2.15), s must then satisfy, for $k \in \mathbb{Z}$,

$$\begin{aligned}e^{2k\pi} &= s \\ 4k^2\pi^2 - 2 &= s^2 - 3,\end{aligned}$$

that is,

$$s = e^{2k\pi} \tag{2.16}$$
$$s^2 = 4k^2\pi^2 + 1. \tag{2.17}$$

From (2.17), $s = \sqrt{4k^2\pi^2+1}$, which means that $k \in \mathbb{Z}$ must be such that

$$e^{2k\pi} = \sqrt{4k^2\pi^2+1}. \tag{2.18}$$

$k = 0$ is obviously a solution to (2.18). In fact, $k = 0$ is the only solution (you do not need to be able to show this, although you should be able to find the solution $k = 0$).

Therefore, the curves intersect for $t = 0$ (for $r_1(t)$) and $t = e^0 = 1$ (for $r_2(t)$), and the point of intersection is $(1, 2, -2)$. We have

$$r_1'(t) = \left(e^t, 2\cos\left(t + \frac{\pi}{2}\right), 2t\right)$$

and

$$r_2'(t) = \vec{i} + 2t\vec{k},$$

and thus

$$r_1'(0) = (1,0,0), \quad \|r_1'(0)\| = 1, \quad r_2'(1) = (1,0,2) \quad \text{and } \|r_2'(1)\| = \sqrt{5}.$$

Finally,

$$\cos\theta = \frac{r_1'(0)\cdot r_2'(1)}{\|r_1'(0)\|\,\|r_2'(1)\|} = \frac{1}{\sqrt{5}}$$

so $\theta \simeq 1.107$.

5 (D marks). escribe the geometric meaning of the following mappings in spherical coordinates, i.e., given a point, what does the mapping do to it?

1. $(\rho, \theta, \phi) \mapsto (\rho, \theta + \pi, \phi)$.
2. $(\rho, \theta, \phi) \mapsto (\rho, \theta, \pi - \phi)$.
3. $(\rho, \theta, \phi) \mapsto (2\rho, \theta + \pi/2, \phi)$.

Solution.

a) Gives the symmetric about the z-axis.

b) Gives the symmetric about the xy-plane.

c) Gives the point that is twice as far from the origin, and rotated an angle $\pi/2$ with respect to the xy-plane coordinates.

6 (F marks). ind the curve r such that $r(0) = (0, -5, 1)$ and $r'(t) = (t, e^t, t^2)$.

Solution. If $r'(t) = (t, e^t, t^2)$, then

$$r(t) = \int r'(t)dt = \left(\frac{t^2}{2} + C_1, e^t + C_2, \frac{t^3}{3} + C_3\right)$$

where C_1, C_2, C_3 are integration constants to be determined. To determine them, we note that the curve goes through the point $r(0) = (0, -5, 1)$. Using the expression found earlier,

$$r(0) = (C_1, 1 + C_2, C_3)$$

and thus, $C_1 = 0$, $1 + C_2 = -5$ (that is, $C_2 = -6$) and $C_3 = 1$. As a consequence, the curve has equation

$$r(t) = \left(\frac{t^2}{2}, e^t - 6, \frac{t^3}{3} + 1\right)$$

7 (L marks). et r be a curve in $\mathbb{R}^3$ with zero acceleration. Show that r is a straight line or a point.

Solution. We know that if $r(t)$ is the position, then $r''(t)$ is the acceleration. So assume that $r''(t) = (0, 0, 0)$. Then $r'(t) = \int r''(t)dt = (C_1, C_2, C_3)$, with $C_1, C_2, C_3 \in \mathbb{R}$ integration constants. And $r(t) = \int r'(t)dt = (C_1 t + K_1, C_2 t + K_2, C_3 t + K_3)$, where $K_1, K_2, K_3 \in \mathbb{R}$ integration constants.

Therefore, if C_1, C_2, C_3 are not all zero simultaneously, r is the equation of a straight line. If $C_1 = C_2 = C_3 = 0$, then r is a point.

8 (L marks). et r be the curve $r(t) = (t, t\sin t, t\cos t)$. Find the length of r between the points $(0, 0, 0)$ and $(\pi, 0, -\pi)$.

Solution. We need to evaluate

$$\kappa = \int_0^\pi \|r'(t)\| dt$$

(easy to see, since in r, the first component is t). We have

$$r' = (1, \sin t + t\cos t, \cos t - t\sin t)$$

and

$$\begin{aligned} \|r'\| &= \sqrt{1 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2} \\ &= \sqrt{1 + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + \cos^2 t + t^2\sin^2 t - 2t\cos t\sin t} \\ &= \sqrt{2 + t^2} \end{aligned}$$

Therefore, we want to evaluate

$$\int_0^\pi \sqrt{2 + t^2} dt.$$

In a table of integrals, we see that

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2 + u^2}\right| + C,$$

and therefore,

$$\begin{aligned}
\kappa &= \int_0^\pi \sqrt{2+t^2}dt \\
&= \frac{t}{2}\sqrt{\sqrt{2}^2+t^2}+\frac{\sqrt{2}^2}{2}\ln\left|t+\sqrt{\sqrt{2}^2+t^2}\right|\Bigg|_{t=0}^{t=\pi} \\
&= \frac{\pi}{2}\sqrt{2+\pi^2}+\ln\left|\pi+\sqrt{2+\pi^2}\right|-\ln\left|\sqrt{2}\right| \\
&\simeq 6.95
\end{aligned}$$

9 (L marks). et r be a smooth curve that is at least twice differentiable, and $T(t)$ be the unit tangent vector to $r(t)$ at t. Show that $T'(t)\cdot T(t)=0$.
[Hint: Remember how we showed the result for r' and r in the case of constant $\|r\|$, and a property of unit vectors.]

Solution. We know that $\|T(t)\|=1$ for all t. Therefore $\|T\|^2=1$, that is, $t\cdot T=1$. Differentiate both sides with respect to t, giving

$$\frac{d}{dt}(T\cdot T)=\frac{d}{dt}1,$$

that is,

$$T'\cdot T+T\cdot T'=0,$$

so, finally, $2T'\cdot T=0$, giving the result.

MATH 2720
Fall 2007
Assignment 3

This assignment is due **in class** Monday, October 22[nd]. Late assignments will **not** be accepted.
Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you. Detail your answers, while trying to remain concise.
3. It is very likely that some questions will not be marked.

---

1 (C marks). onsider the function

$$f(x,y)=\frac{\sin(x^2+y^2)}{x^2+y^2}.$$

1. Using polar coordinates, show that $\lim_{(x,y)\to(0,0)} f(x,y)=1$.
2. Try using the (ε,δ) approach to show the same result.

2 (U marks). se the (ε,δ) definition of the limit to show that

$$\lim_{(x,y)\to(0,0)} x+y=0.$$

3 (C marks). onsider the function

$$f(x,y)=\frac{x^2}{\sqrt{x^2+y^2}}.$$

Show, using the (ε,δ) approach, that $\lim_{(x,y)\to(0,0)} f(x,y)=0$.

4 (C marks). ompute the following limits, if they exist (if they do not, explain why).

1. $\lim_{(x,y)\to(0,0)} \dfrac{\cos x-1-x^2/2}{x^4+y^4}$.

2. $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{xy}$.

5 (C marks). an $\sin(x+y)/(x+y)$ be made continuous by suitably defining it at $(0,0)$?

6 (F marks). ind the first partial derivatives of the following functions.

1. $f(x,y) = \int_y^x \cos(t^2)dt$.
2. $f(x,y) = \exp(\sin(x/y))$.

7 (F marks). ind f_{xxy} if $f(x,y) = \ln\sin(x-y)$.

8 (I marks). f $u = \exp(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$, with $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u.$$

MATH 2720
Fall 2007
Assignment 3

Marking: 2 (25 points), 3 (25 points), 6 (25 points) and 8 (20 points).

---

1 (C marks). onsider the function

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}.$$

1. Using polar coordinates, show that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.
2. Try using the (ε,δ) approach to show the same result.

Solution. a) Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2+y^2 = r^2$, and

$$f(x,y) = g(r,\theta) = \frac{\sin r^2}{r^2}.$$

Considering $(x,y)\to(0,0)$ is equivalent to considering $r\to 0$ for any θ, so we are in fact considering

$$\lim_{r\to 0}\frac{\sin r^2}{r^2}.$$

You may remember from early calculus (or you can prove using L'Hospital's rule, for example), that this limit is equal to 1.
b) We use the form resulting from using polar coordinates, and thus want to show that

$$\lim_{r\to 0}\frac{\sin r}{r} = 1,$$

using the (ε,δ)-approach. This is far from trivial, and requires some serious work.

We start with geometric considerations: for a given r, the representations of $\cos r$, $\sin r$ and $\tan r$ are given in Figure 2.1 by the signed lengths of the segments OP, PB and AC. Suppose that $0 < r < \pi/2$. Then the area of the triangle OAB is given by

$$\frac{1}{2}\times\text{base}\times\text{height} = \frac{1}{2}\times OA\times PN = \frac{1}{2}\times 1\times\sin r = \frac{\sin r}{2}.$$

The area of the disk of the unit disk is π. Therefore, the area of the circular sector OAB is

$$\frac{r}{2\pi}\pi,\ \text{that is,}\ \frac{r}{2},$$

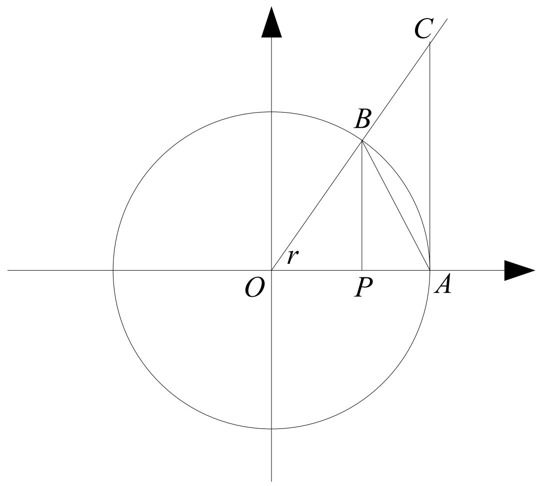


Figure 2.1: Lengths along the trigonometric circle, for an angle with the Ox-axis of r radians, we have $\cos r = OP$, $\sin r = BP$ and $\tan r = AC$.

since $r/(2\pi)$ is the fraction of the whole 2π represented by r. Finally, the area of the triangle OAC is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times OA \times AC = \frac{1}{2} \times 1 \times \tan r = \frac{\tan r}{2}.$$

Visually, these three regions have areas ordered, for $0 < r < \pi/2$, as

$$\text{area triangle OAB} < \text{area circular sector OAB} < \text{area triangle OAC}.$$

Therefore,

$$\frac{\sin r}{2} < \frac{r}{2} < \frac{\tan r}{2},$$

that is,

$$\sin r < r < \tan r.$$

Divide all terms by $\sin r$:

$$\frac{\sin r}{\sin r} < \frac{r}{\sin r} < \frac{\tan r}{\sin r},$$

that is,

$$1 < \frac{r}{\sin r} < \frac{1}{\cos r}.$$

Taking the inverse reverses inequalities, and thus, for $0 < r < \pi/2$,

$$\cos r < \frac{\sin r}{r} < 1. \tag{2.19}$$

Now, remark that we can write

$$1 - \cos r = (1 - \cos r)\frac{1 + \cos r}{1 + \cos r} = \frac{1 - \cos^2 r}{1 - \cos r} = \frac{\sin^2 r}{1 + \cos r}.$$

For $0 < r < \pi/2$, $\cos r > 0$, and therefore, $1 + \cos r > 1$, which implies that $\sin^2 r/(1 + \cos r) < \sin^2 r$. Therefore,

$$1 - \cos r < \sin^2 r.$$

But we have seen above in inequality (2.19) that $\sin r/r < 1$ for $0 < r < \pi/2$, which implies that $\sin r < r$, in turn implying that $\sin^2 r < r^2$ for $0 < r < \pi/2$. Therefore, we have

$$1 - \cos r < r^2,$$

that is,

$$\cos r > 1 - r^2,$$

when $0 < r < \pi/2$. Using this in (2.19) gives

$$1 - r^2 < \frac{\sin r}{r} < 1, \tag{2.20}$$

for $0 < r < \pi/2$.

It is easy to see that the above reasoning also applies for $-\pi/2 < r < 0$. As a consequence, if $0 < |r| < \pi/2$, then

$$-r^2 < \frac{\sin r}{r} - 1 < 0.$$

Now, assume $\varepsilon > 0$ is given, and choose $\delta = \min\{\pi/2, \sqrt{\varepsilon}\}$. If $0 < |r| < \delta$, then

$$\left| \frac{\sin r}{r} - 1 \right| < |r|^2 < \varepsilon.$$

The first inequality is obtained easily from $-r^2 < (\sin r)/r - 1 < 0$. Indeed, suppose for example $-a < b < 0$; then $|b| < |a|$. It follows that

$$\lim_{r \to 0} \frac{\sin r}{r} = 0.$$

2 (U marks). se the (ε, δ) definition of the limit to show that

$$\lim_{(x,y) \to (0,0)} x + y = 0.$$

Solution. We want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x + y| < \varepsilon$ whenever $0 < \sqrt{x^2 + y^2} < \delta$.

The easiest way here is to proceed using geometrical arguments. Suppose for now that ε and δ are known. The set

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : |x + y| < \varepsilon\}$$

is the square with vertices at the points $(0, \varepsilon)$, $(\varepsilon, 0)$, $(0, -\varepsilon)$ and $(-\varepsilon, 0)$ (see Figure 2.2). On the other hand,

$$\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : 0 < \sqrt{x^2 + y^2} < \delta\}$$

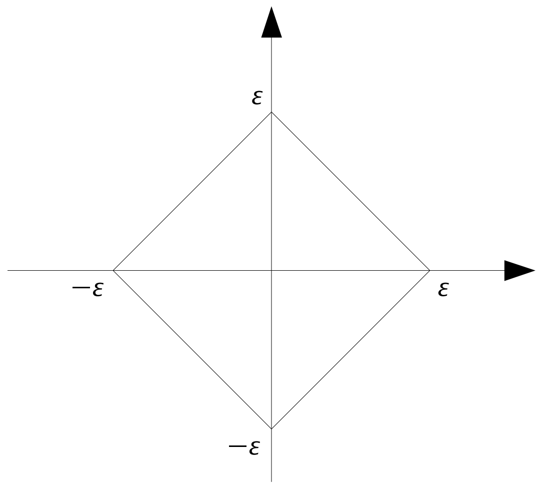


Figure 2.2: The set $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : |x + y| < \varepsilon\}$ in Exercise 2.

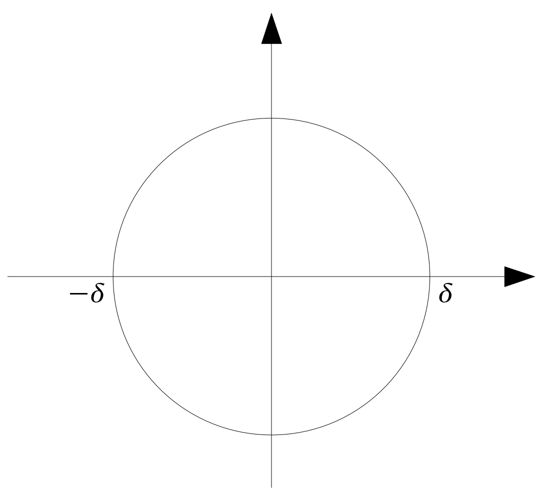
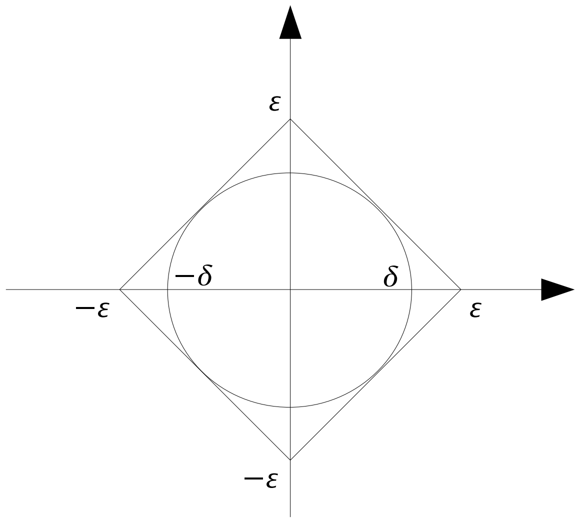


Figure 2.3: The set $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : 0 < \sqrt{x^2 + y^2} < \delta\}$ in Exercise 2.

is the open disk centrered at the origin, with radius δ, and not comprising the origin (see Figure 2.3).

So, what we want to show is that for a given $\varepsilon > 0$, we can find $\delta > 0$, such that $(x, y) \in \mathcal{D}$ whenever $(x, y) \in \mathcal{O}$. That is, in Figure 2.4, points must belong to the square whenever they are in the disk. The situation on the left is good: all points in the disk also belong to the square. The situation on the right, on the other hand, does not work: there are points in the disk that do not belong to the square. To obtain a figure such as the one on the left in Figure 2.4, we need



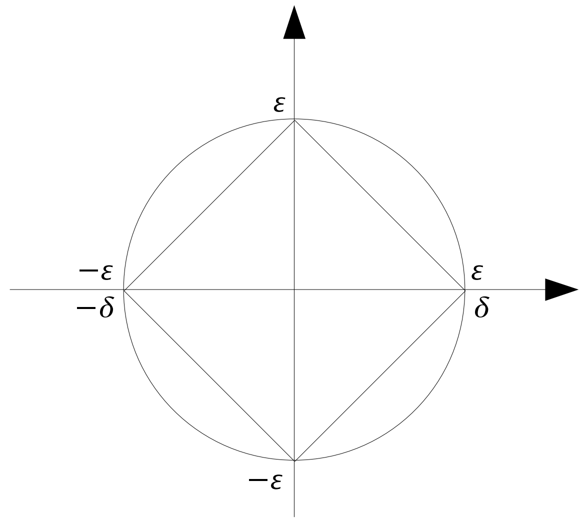


Figure 2.4: Intersection of sets $\mathcal{O}$ and $\mathcal{D}$ in Exercise 2. For the limit to exist, we need $\mathcal{O} \subset \mathcal{D}$, that is, the situation on the left. For the situation on the right, we have $\mathcal{O} \not\subset \mathcal{D}$, so it is not true that whenever $(x, y) \in \mathcal{O}$, $(x, y) \in \mathcal{D}$.

to inscribe the disk into the square. Therefore, the radius of the disk, δ, must be less than half of the length of the side of the square.

The square has diagonal of length 2ε. Using the Theorem of Pythagorus, the sides of the square have length ℓ satisfying $4\varepsilon^2 = 2\ell^2$, i.e., $\ell = \varepsilon\sqrt{2}$. Therefore, taking $\delta = \varepsilon/\sqrt{2}$, we have $\mathcal{O} \subset \mathcal{D}$. Remark that an obvious other choice for δ would have been $\delta = \varepsilon/2$, since $\varepsilon/2 < \varepsilon/\sqrt{2}$.

Therefore, for all $\varepsilon > 0$, there exists $\delta = \varepsilon/\sqrt{2}$ such that $(x, y) \in \mathcal{O}$ implies $(x, y) \in \mathcal{D}$, and thus $\lim_{(x,y)\to(0,0)} x + y = 0$.

3 (C marks). onsider the function

$$f(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}.$$

Show, using the (ε, δ) approach, that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Solution. We want to show that $\forall\varepsilon > 0$, $\exists\delta > 0$, such that

$$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow \left| \frac{x^2}{\sqrt{x^2 + y^2}} \right| < \varepsilon.$$

We start by considering

$$\left|\frac{x^2}{\sqrt{x^2+y^2}}\right| < \varepsilon.$$

Since $\sqrt{x^2+y^2} \geq \sqrt{x^2}$, $1/\sqrt{x^2+y^2} \leq 1/\sqrt{x^2}$, and thus

$$\left|\frac{x^2}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{x^2}{\sqrt{x^2}}\right| = |x|.$$

Now consider the inequality $\sqrt{x^2+y^2} < \delta$. As before, $\sqrt{x^2+y^2} \leq \sqrt{x^2} = |x|$.

Therefore, suppose $\varepsilon > 0$ given, let $\delta = \varepsilon$. Then $|x| < \delta$, that is, $\sqrt{x^2+y^2} \leq |x| < \delta$ implies that $|x| < \varepsilon$, but since $\left|\frac{x^2}{\sqrt{x^2+y^2}}\right| \leq |x|$, this in turn implies that $\left|\frac{x^2}{\sqrt{x^2+y^2}}\right| < \varepsilon$. It follows that $\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$.

4 (C marks). ompute the following limits, if they exist (if they do not, explain why).

1. $\lim_{(x,y)\to(0,0)} \dfrac{\cos x - 1 - x^2/2}{x^4+y^4}$.

2. $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{xy}$.

Solution. Will post these later. Hint for b): you can use easily the approach of Exercise 1.b).

5 (C marks). an $\sin(x+y)/(x+y)$ be made continuous by suitably defining it at $(0,0)$?

Solution. The function $f(x,y) = \sin(x+y)/(x+y)$ is continuous at $(0,0)$ if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Since f is not defined at $(0,0)$, we need to define f as follows:

$$f(x,y) = \begin{cases} \dfrac{\sin(x+y)}{x+y} & \text{if } (x,y) \neq (0,0) \\ \ell & \text{if } (x,y) = (0,0), \end{cases}$$

where $\ell = \lim_{(x,y)\to(0,0)} f(x,y)$. In fact, we should also worry about what happens along the line $x+y=0$, that is, $y=-x$, the second bisectrix, since f is not defined along this line. Let us treat the case $(0,0)$, the rest of the $y=-x$ line can be done similarly.

6 (F marks). ind the first partial derivatives of the following functions.

1. $f(x,y) = \int_y^x \cos(t^2)dt$.

2. $f(x,y) = \exp(\sin(x/y))$.

Solution. a) Do not try to integrate the function, but rather, use the fundamental theorem of calculus: if $A(t)$ is the antiderivative of $\cos(t^2)$, then

$$\int_x^y \cos(t^2) = A(y) - A(x),$$

where $A(x)$ depends only on x and $A(y)$ depends only on y. Therefore

$$\frac{\partial}{\partial x}\int_x^y \cos(t^2)dt = \frac{\partial}{\partial x}(A(y) - A(x)) = -\frac{d}{dx}A(x) = -\cos x^2,$$

and

$$\frac{\partial}{\partial y}\int_x^y \cos(t^2)dt = \frac{\partial}{\partial y}(A(y) - A(x)) = \frac{d}{dy}A(x) = \cos y^2.$$

b) Using the chain rule iteratively, we have

$$\begin{aligned}\frac{\partial}{\partial x}\exp(\sin(x/y)) &= \left(\frac{\partial}{\partial x}\sin(x/y)\right)\exp(\sin(x/y)) \\ &= \frac{1}{y}\cos(x/y)\exp(\sin(x/y))\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\exp(\sin(x/y)) &= \left(\frac{\partial}{\partial y}\sin(x/y)\right)\exp(\sin(x/y)) \\ &= -\frac{x}{y^2}\cos(x/y)\exp(\sin(x/y)).\end{aligned}$$

7 (F marks). ind f_{xxy} if $f(x,y) = \ln\sin(x-y)$.

Solution. We have, using the chain rule,

$$\begin{aligned}f_x &= \frac{\frac{\partial}{\partial x}\sin(x-y)}{\sin(x-y)} \\ &= \frac{\cos(x-y)}{\sin(x-y)}.\end{aligned}$$

Therefore, using the quotient rule and the chain rule

$$\begin{aligned}f_{xx} &= \frac{\sin(x-y)\left(\frac{\partial}{\partial x}\cos(x-y)\right) - \cos(x-y)\left(\frac{\partial}{\partial x}\sin(x-y)\right)}{\sin^2(x-y)} \\ &= \frac{-\sin(x-y)\sin(x-y) - \cos(x-y)\cos(x-y)}{\sin^2(x-y)} \\ &= -\frac{\sin^2(x-y) + \cos^2(x-y)}{\sin^2(x-y)} \\ &= -\frac{1}{\sin^2(x-y)}.\end{aligned}$$

Finally,

$$
\begin{aligned}
f_{xxy} &= \frac{\frac{\partial}{\partial y}\sin^2(x-y)}{\sin^4(x-y)} \\
&= \frac{-2\cos(x-y)\sin(x-y)}{\sin^4(x-y)} \\
&= \frac{-2\cos(x-y)}{\sin^3(x-y)}.
\end{aligned}
$$

8 (I marks). f $u = \exp(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$, with $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$, show that

$$
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = u.
$$

Solution. We have, for a given $k = 1, \dots, n$,

$$
\frac{\partial u}{\partial x_k} = a_k \exp(a_1x_1 + \cdots + a_nx_n)
$$

and

$$
\frac{\partial^2 u}{\partial x_k^2} = a_k^2 \exp(a_1x_1 + \cdots + a_nx_n).
$$

Therefore,

$$
\begin{aligned}
\sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2} &= \sum_{k=1}^n a_k^2 \exp(a_1x_1 + \cdots + a_nx_n) \\
&= \exp(a_1x_1 + \cdots + a_nx_n) \sum_{k=1}^n a_k^2
\end{aligned}
$$

and so, since $a_1^2 + \cdots + a_n^2 = 1$,

$$
\begin{aligned}
&= \exp(a_1x_1 + \cdots + a_nx_n) \\
&= u.
\end{aligned}
$$

MATH 2720
Fall 2007
Assignment 4

This assignment is due **in class** Friday, November 2nd. Late assignments will **not** be accepted. Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity. To get these 5 points, make sure that your reasoning is clear to the person reading (and marking) you. Detail your answers, while trying to remain concise.
3. It is very likely that some questions will not be marked.

---

1 (F marks). ind the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3, 1, 10)$.

2 (W marks). here does the plane tangent to $z = e^{x-y}$ at $(1, 1, 1)$ meet the z-axis?

3 (W marks). hy should the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ be called *tangent* at $(0, 0)$?

4 (C marks). ompute $\partial w/\partial x$ and $\partial w/\partial y$ for $w = e^{xy} \ln(x^2 + y^2)$ and $w = \cos(ye^{xy}) \sin x$.

5 (S marks). how that each of the following functions is differentiable at each point in its domain. Decide which of the functions are C^1.

1. $f(x, y) = \dfrac{2xy}{(x^2 + y^2)^2}$.
2. $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$.

6 (U marks). se the chain rule to find $\partial u/\partial s$ and $\partial u/\partial t$ at $(s, t) = (0, 1)$, when $u = xy + yz + zx$, $x = st$, $y = e^{st}$ and $z = t^2$.

7 (I marks). f $u = f(x, y)$ with $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right).$$

MATH 2720
Fall 2007
Assignment 4 – Solutions

Questions marked: 1 (25 marks), 4 (25 marks), 5, (25 marks) and 6 (20 marks) +5 Sohyun points

1 (F marks). ind the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3, 1, 10)$.

Solution. Use the formula

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

with $(x_0, y_0, z_0) = (3, 1, 10)$ and $f(x, y) = x^2 + y^3$ (check: $z_0 = 10$ is indeed $f(x_0, y_0) = f(3, 1) = 10$). We have

$$f_x = 2x, \quad f_x(3, 1) = 6, \quad f_y = 3y^2, \quad f_y(3, 1) = 3,$$

and thus the tangent plane has equation

$$z - 10 = 6(x - 3) + 3(y - 1) \Leftrightarrow z = 6x + 3y - 11.$$

2 (W marks). here does the plane tangent to $z = e^{x-y}$ at $(1, 1, 1)$ meet the z-axis?

Solution. We first seek the equation of the tangent plane. Proceeding as in Exercise 1, given that $f(x, y) = e^{x-y}$ and $(x_0, y_0, z_0) = (1, 1, 1)$, we have

$$f_x = e^{x-y}, \quad f_x(1, 1) = 1, \quad f_y = -e^{x-y}, \quad f_y(1, 1) = -1,$$

and the equation of the tangent plane is

$$z - 1 = (x - 1) - (y - 1) \Leftrightarrow z = x - y + 1.$$

This plane meets the z-axis at $x = y = 0$, that is, at $z = 1$.

3 (W marks). hy should the graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ be called *tangent* at $(0, 0)$?

Solution. We have $f(0, 0) = g(0, 0) = 0$, so the two surfaces have the point $(0, 0)$ in common. Now, the tangent plane to $z = f(x, y)$ at $(0, 0)$ has equation $z = 0$ (easy computation), and so does the tangent plane to the surface $z = g(x, y)$ at $(0, 0)$. So the two surfaces have the same tangent plane at $(0, 0)$, and thus, they are tangent at the point $(0, 0)$.

4 (C marks). ompute $\partial w/\partial x$ and $\partial w/\partial y$ for $w = e^{xy} \ln(x^2 + y^2)$ and $w = \cos(ye^{xy}) \sin x$.

Solution. We have

$$\begin{aligned}
\frac{\partial w}{\partial x} &= ye^{xy} \ln(x^2 + y^2) + e^{xy} \frac{2x}{x^2 + y^2} \\
&= e^{xy} \left(y \ln(x^2 + y^2) + \frac{2x}{x^2 + y^2} \right)
\end{aligned}$$

The role of x and y in w is exactly the same, so $\partial w/\partial y$ is obtained by simply interchanging y for x,

$$\frac{\partial w}{\partial y} = e^{xy}\left(x\ln(x^2+y^2) + \frac{2y}{x^2+y^2}\right)$$

For the second function,

$$\frac{\partial w}{\partial x} = \cos(ye^{xy})\cos x - y^2e^{xy}\sin(ye^{xy})\sin x$$

and

$$\frac{\partial w}{\partial y} = -(xye^{xy}+e^{xy})\sin(ye^{xy})\sin x = -e^{xy}(1+xy)\sin(ye^{xy})\sin x.$$

5 (S marks). how that each of the following functions is differentiable at each point in its domain. Decide which of the functions are C^1.

1. $f(x,y) = \dfrac{2xy}{(x^2+y^2)^2}$.
2. $f(x,y) = \dfrac{x^2y}{x^4+y^2}$.

Solution. a) The statement "at each point in its domain" indicates that you must first identify this domain. Here, it is clear that the domain is $\mathbb{R}^2 \setminus \{(0,0)\}$. We have

$$f_x = \frac{2y(x^2+y^2)^2 - 2xy(2x)}{(x^2+y^2)^4} \quad \text{and} \quad f_y = \frac{2x(x^2+y^2)^2 - 2xy(2y)}{(x^2+y^2)^4}.$$

Note that it is not necessary here to simplify at all: the only points of discontinuity, if any, are determined by the denominator. Clearly, f_x and f_y are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$, which is the same domain as that of f: f is therefore differentiable on its domain. Here, differentiability and C^1-ness agree, so f is also C^1. (A function can be differentiable and not C^1, since $f \in C^1$ is only sufficient for differentiability, not necessary. We will not meet this type of function often in this course.)
b) The domain is $\mathbb{R}^2 \setminus \{(0,0)\}$. We have

$$f_x = \frac{2xy(x^4+y^2) - 4x^5y}{(x^4+y^2)^2} \quad \text{and} \quad f_y = \frac{x^2(x^4+y^2) - 2x^2y^2}{(x^4+y^2)^2}.$$

Here again, C^1 and differentiability match, since discontinuities of the partials occur at the same point $(0,0)$ as those of f.

6 (U marks). se the chain rule to find $\partial u/\partial s$ and $\partial u/\partial t$ at $(s,t) = (0,1)$, when $u = xy + yz + zx$, $x = st$, $y = e^{st}$ and $z = t^2$.

Solution. At $(s,t)=(0,1)$, $x=0$, $y=1$ and $z=1$. We have

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}\\ &= (y+z)(t)+(x+z)(te^{st})+(y+x)(0),\end{aligned}$$

so, at $(s,t)=(0,1)$,

$$\frac{\partial u}{\partial s}(0,1)=(1+1)(1)+(0+1)(1)=3.$$

We have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}\\ &= (y+z)(s)+(x+z)(se^{st})+(y+x)(2t),\end{aligned}$$

so, at $(s,t)=(0,1)$,

$$\frac{\partial u}{\partial s}(0,1)=(1+1)(0)+(0+1)(0)+(1+0)(2)=2.$$

7 (I marks). f $u=f(x,y)$ with $x=e^s\cos t$ and $y=e^s\sin t$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{-2s}\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right).$$

MATH 2720
Fall 2007
Assignment 5

This assignment is due **in class** Wednesday, November 21st. Late assignments will **not** be accepted. Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity.
3. It is very likely that some questions will not be marked.

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1 (E marks). valuate $\iint_R |y| \cos\left(\frac{1}{4}\pi x\right) dydx$ when $R = [0,2] \times [-1,0]$.

2 (L marks). et f be continuous on $R = [a,b] \times [c,d]$. For $x \in (a,b)$ and $y \in (c,d)$, let

$$F(x,y) = \int_a^x \int_c^y f(u,v) dv du.$$

Show that $\partial^2 F/(\partial x \partial y) = \partial^2 F/(\partial y \partial x) = f(x,y)$.

3 (O marks). n a rectangle R, assume f continuous and $f \geq 0$. If $\iint_R f dA = 0$, show that $f = 0$ on R.

4 (L marks). et

$$D = \{(x,y) \in \mathbb{R}^2; a \leq x \leq b, -\phi(x) \leq y \leq \phi(x)\}$$

with $\phi(x) \geq 0$ and continuous on $[a,b]$. Suppose that $f(x,y)$ is defined on D and such that $f(x,y) = -f(x,-y)$ for all $(x,y) \in D$. Show that $\iint_D f(x,y) dA = 0$.

5 (F marks). ind the volume of the region between the planes $z = 0$ and $z = 12$ and the surface $z = x^2 + 2y^2$.

6 (D marks). escribe the region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$ as an elementary region.

7 (F marks). ind the area of the part of the plane $2x - 2y + 5z = 10$ that lies inside the ellipsoidal cylinder $x^2 + 3y^2 = 4$.

8 (D marks). efine the improper integral

$$\ell = \iint\limits_{\mathbb{R}^2} \exp(-(x^2+y^2)) dA = \lim_{a \to \infty} \iint\limits_{D_a} \exp(-(x^2+y^2)) dA,$$

where D_a is the disk with radius a centrered at the origin. Show that $\ell = \pi$. Now define ℓ as

$$\ell = \lim_{a\to\infty} \iint\limits_{\mathcal{S}_a} \exp(-(x^2+y^2))dA,$$

where S_a is the square with vertices $(\pm a, \pm a)$. Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi,$$

and deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

9 (R marks). ead the parts about centre of mass and moments of inertia in Sections 15.5 (Applications of double integrals) and 15.7 (Triple integrals). Then

1. Find the moment of inertia about the y axis for the ball $x^2+y^2+z^2 \leq R^2$ if the mass density is a constant δ.

2. Find the mass and centre of mass of the solid bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $y + z = 1$, $y = 0$, and $z = 0$, with density function given by $\rho(x, y, z) = 3$.

MATH 2720
Fall 2007
Assignment 5

Marks: 3 (10 marks), 4 (25 marks), 8 (30 marks) and 9 (30 marks).

1 (E marks). valuate $\iint_R |y| \cos\left(\frac{1}{4}\pi x\right) dydx$ when $R = [0,2] \times [-1,0]$.

Solution. We have

$$\iint_R |y| \cos\left(\frac{1}{4}\pi x\right) dydx = \int_0^2 \int_{-1}^0 |y| \cos\left(\frac{\pi}{4}x\right) dydx.$$

The "problem" is to deal with $|y|$. As y increases from -1 to 0, $|y| = -y$ decreases from 1 to 0, and so

$$\begin{aligned}\iint_R |y| \cos\left(\frac{\pi}{4}x\right) dydx &= \int_0^2 \int_1^0 -y \cos\left(\frac{\pi}{4}x\right) dydx \\ &= \int_0^2 \int_0^1 y \cos\left(\frac{\pi}{4}x\right) dydx \\ &= \int_0^2 \cos\left(\frac{\pi}{4}x\right) dx \int_0^1 ydy \\ &= \left(\frac{4}{\pi} \sin\left(\frac{\pi}{4}x\right)\Big|_0^2\right) \left(\frac{y^2}{2}\Big|_0^1\right) \\ &= \frac{4}{\pi}\,\frac{1}{2} \\ &= \frac{2}{\pi}.\end{aligned}$$

2 (L marks). et f be continuous on $R = [a,b] \times [c,d]$. For $x \in (a,b)$ and $y \in (c,d)$, let

$$F(x,y) = \int_a^x \int_c^y f(u,v)dvdu.$$

Show that $\partial^2 F/(\partial x \partial y) = \partial^2 F/(\partial y \partial x) = f(x,y)$.

Solution. Note that we are integrating on a rectangular domain, so

$$F(x,y) = \int_a^x \int_c^y f(u,v)dvdu = \int_c^y \int_a^x f(u,v)dudv,$$

where the second expression will be used when we compute $\partial F/\partial y$. From the fundamental theorem of calculus,

$$\frac{\partial F}{\partial x} = \int_c^y f(x,v)dv.$$

Applying the fundamental theorem of calculus again,

$$\frac{\partial^2 F}{\partial y \partial x} = f(x, y).$$

Similarly, from the fundamental theorem of calculus,

$$\frac{\partial F}{\partial y} = \int_a^x f(u, y) du,$$

and using the fundamental theorem of calculus again,

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y).$$

3 (O marks). n a rectangle R, assume f continuous and $f \geq 0$. If $\iint_R f dA = 0$, show that $f = 0$ on R.

Solution. An easy way to do this is to proceed by contradiction. Assume that $f \geq 0$, and that

$$\iint_R f dA = 0.$$

Now suppose that there exists a point $(\tilde{x}, \tilde{y}) \in R$ such that $f(\tilde{x}, \tilde{y}) > 0$. We have

$$\iint_R f dA = \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A,$$

where (x_{ij}^*, y_{ij}^*) is a sample point in the subrectangle R_{ij}. Whatever the values of m and n (that is, how fine a subdivision of R we are considering), we can always choose the sample point (x_{ij}^*, y_{ij}^*) to be $(\tilde{x}, \tilde{y})$ when we are in the rectangle $\tilde{R}_{ij}$ that contains the point $(\tilde{x}, \tilde{y})$. We denote $i = i_*$ and $j = j_*$ the values of i and j such that we are in $\tilde{R}_{ij}$. We can write

$$\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A = f(\tilde{x}, \tilde{y}) \Delta A + \sum_{i=1}^m \sum_{j=1}^n \tilde{\delta}_{ij} f(x_{ij}^*, y_{ij}^*) \Delta A,$$

where $\tilde{\delta}_{ij} = 0$ if $i = i_*$ and $j = j_*$ and $\tilde{\delta}_{ij} = 1$ otherwise, that is, if we are not in the rectangle $\tilde{R}_{ij}$ that contains $(\tilde{x}, \tilde{y})$. But this means that

$$\begin{aligned}\lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A &= \lim_{m,n \to \infty} \left(f(\tilde{x}, \tilde{y}) \Delta A + \sum_{i=1}^m \sum_{j=1}^n \tilde{\delta}_{ij} f(x_{ij}^*, y_{ij}^*) \Delta A \right) \\ &= f(\tilde{x}, \tilde{y}) \Delta A + \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n \tilde{\delta}_{ij} f(x_{ij}^*, y_{ij}^*) \Delta A\end{aligned}$$

and therefore,

$$\iint_R f dA \geq f(\tilde{x}, \tilde{y})\Delta A > 0,$$

a contradiction. Therefore, there cannot exist $(\tilde{x}, \tilde{y})$ such that $f(\tilde{x}, \tilde{y}) > 0$, and thus $f = 0$ on R.

4 (L marks). et

$$D = \{(x, y) \in \mathbb{R}^2; a \leq x \leq b, -\phi(x) \leq y \leq \phi(x)\}$$

with $\phi(x) \geq 0$ and continuous on $[a, b]$. Suppose that $f(x, y)$ is defined on D and such that $f(x, y) = -f(x, -y)$ for all $(x, y) \in D$. Show that $\iint_D f(x, y)dA = 0$.

Solution. The object here is to check that the same type of property holds true for multiple integrals as for simple integrals, that the integral of an odd function on a symmetric interval is zero. Here, however, "odd-ness" of the function must be defined with respect to one of the variables. Looking at the expression $f(x, y) = -f(x, -y)$, we see it is equivalent to $f(x, -y) = -f(x, y)$, that is, we can say that f is odd with respect to its second argument.

We then have

$$\begin{aligned}
\iint_D f(x, y)dA &= \int_a^b \int_{-\phi(x)}^{\phi(x)} f(x, y)dydx \\
&= \int_a^b \left(\int_{-\phi(x)}^{0} f(x, y)dy + \int_0^{\phi(x)} f(x, y)dy \right) dx
\end{aligned}$$

Let us focus now on the integral

$$\int_{-\phi(x)}^{0} f(x, y)dy.$$

We have

$$\int_{-\phi(x)}^{0} f(x, y)dy = -\int_0^{-\phi(x)} f(x, y)dy$$

which, using the change of variables $u = -y$, gives

$$\begin{aligned}
&= -\int_0^{\phi(x)} (-f(x, -y))dy \\
&= -\int_0^{\phi(x)} f(x, y)dy
\end{aligned}$$

since $f(x,-y) = -f(x,y)$. As a consequence,

$$\begin{aligned}
\iint_D f(x,y)dA &= \int_a^b \int_{-\phi(x)}^{\phi(x)} f(x,y)dydx \\
&= \int_a^b \left(\int_{-\phi(x)}^{0} f(x,y)dy + \int_0^{\phi(x)} f(x,y)dy \right) dx \\
&= \int_a^b \left(-\int_0^{\phi(x)} f(x,y)dy + \int_0^{\phi(x)} f(x,y)dy \right) dx \\
&= \int_a^b 0\ dx = 0.
\end{aligned}$$

5 (F marks). ind the volume of the region between the planes $z = 0$ and $z = 12$ and the surface $z = x^2 + 2y^2$.

Solution. Two approaches to this problem. **First**, using the formula

$$\text{Volume of } E = \iiint\limits_E dV.$$

The solid region E is defined, for each $z \in [0, 12]$, by the (x, y) inside the paraboloid $x^2 + 2y^2 = z$. That is, for a given value of z, (x, y) are inside the ellipse

$$\frac{x^2}{z} + \frac{y^2}{z/2} = 1,$$

that is, the ellipse with semimajor axis $\sqrt{z}$ and semiminor axis $\sqrt{z/2}$. This is easily written as an elementary region $\mathcal{D}$ for (x, y). For example,

$$\mathcal{D} = \left\{ (x,y) : -\sqrt{z} \leq x \leq \sqrt{z}, -\frac{1}{\sqrt{2}}\sqrt{z-x^2} \leq y \leq \frac{1}{\sqrt{2}}\sqrt{z-x^2} \right\}.$$

Therefore, the region E is written as an elementary region,

$$E = \left\{ (x,y,z) : 0 \leq z \leq 12, -\sqrt{z} \leq x \leq \sqrt{z}, -\frac{1}{\sqrt{2}}\sqrt{z-x^2} \leq y \leq \frac{1}{\sqrt{2}}\sqrt{z-x^2} \right\},$$

and the volume of E is given by

$$\begin{aligned}
\mathrm{Vol}(E) &= \iiint_E dV \\
&= \int_0^{12} \iint_D dA\, dz \\
&= \int_0^{12} \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\frac{1}{\sqrt{2}}\sqrt{z-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{z-x^2}} dydxdz \\
&= \sqrt{2} \int_0^{12} \int_{-\sqrt{z}}^{\sqrt{z}} \sqrt{z-x^2}\, dxdz \\
&= \sqrt{2} \int_0^{12} \left(\frac{x}{2}\sqrt{z-x^2} + \frac{z}{2}\arcsin\left(\frac{x}{\sqrt{z}}\right)\Big|_{x=-\sqrt{z}}^{x=\sqrt{z}} \right) dz. \\
&= \sqrt{2} \int_0^{12} \left(\frac{z}{2}\arcsin(1) - \frac{z}{2}\arcsin(-1) \right) dz \\
&= \sqrt{2} \int_0^{12} \frac{z}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) dz \\
&= \sqrt{2}\,\pi\, z^2\Big|_0^{12} \\
&= 144\sqrt{2}\,\pi.
\end{aligned}$$

Note that to obtain the antiderivative of $\sqrt{z-x^2}$, the formula

$$\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right) + C,$$

is used (which can be found in most calculus textbooks).

The **second** approach uses the fact that the volume of the solid is the difference between the volume of the cylinder x^2+2y^2 from the plane $z=0$ to the plane $z=12$, and the volume under the surface $z=x^2+2y^2$ inside that same cylinder. We thus need to consider

$$\iint_{\mathcal{D}} 12\, dA - \iint_{\mathcal{D}} x^2+2y^2\, dA,$$

where $\mathcal{D}$ is the region inside the ellipse $x^2+2y^2=12$, that is, the region found earlier but with $z=12$. Note that the resulting integral is more complicated than in the first approach.

6 (D marks). escribe the region between the cone $z=\sqrt{x^2+y^2}$ and the paraboloid $z=x^2+y^2$ as an elementary region.

Solution. Both the cone and the paraboloid are obviously symetric about the z-axis. Indeed, let $f(x,y)=\sqrt{x^2+y^2}$ and $g(x,y)=x^2+y^2$. Then we see that for any (x,y), $f(-x,-y)=f(x,y)$ and $g(-x,-y)=g(x,y)$.

To get an understanding of the situation, let us consider the situation on the xz-plane, that is, when $y = 0$. We have $f(x, 0) = |x|$ and $g(x, 0) = x^2$, so the situation is as shown in Figure 2.5. We can see that the cone lies below the paraboloid. So, the region takes the form

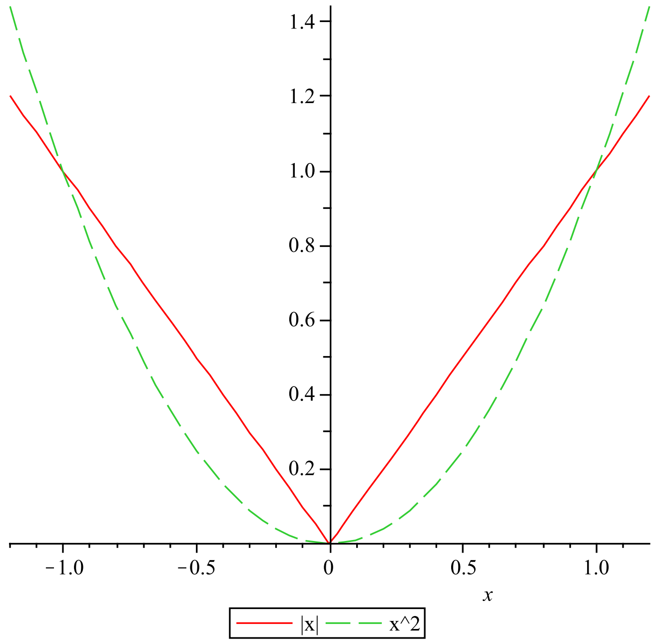


Figure 2.5: The situation for Exercise 6, when $y = 0$, showing the intersection of the xz-plane with the cone ($|x|$) and the paraboloid (x^2).

$$E = \{(x, y, z) : (x, y) \in D, x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2}\}.$$

To find D, we must find the intersection of the cone and the paraboloid. We have

$$x^2 + y^2 = \sqrt{x^2 + y^2} \Leftrightarrow x^2 + y^2 = 1 \text{ or } x^2 + y^2 = 0,$$

since $x^2 + y^2 \geq 0$. Therefore, D is the unit disk in $\mathbb{R}^2$.

7 (F marks). ind the area of the part of the plane $2x - 2y + 5z = 10$ that lies inside the ellipsoidal cylinder $x^2 + 3y^2 = 4$.

Solution. We use the formula that gives the area of the surface $z = f(x, y)$ for $(x, y) \in D$ as

$$\iint\limits_{\mathcal{D}} \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

Here, the surface is the plane $2x - 2y + 5z = 10$, which, when written in terms of x and y, is $z = f(x, y) = 2 - \frac{x}{5} + \frac{y}{5}$. The problem is to find D. The situation is shown in Figure 2.6.

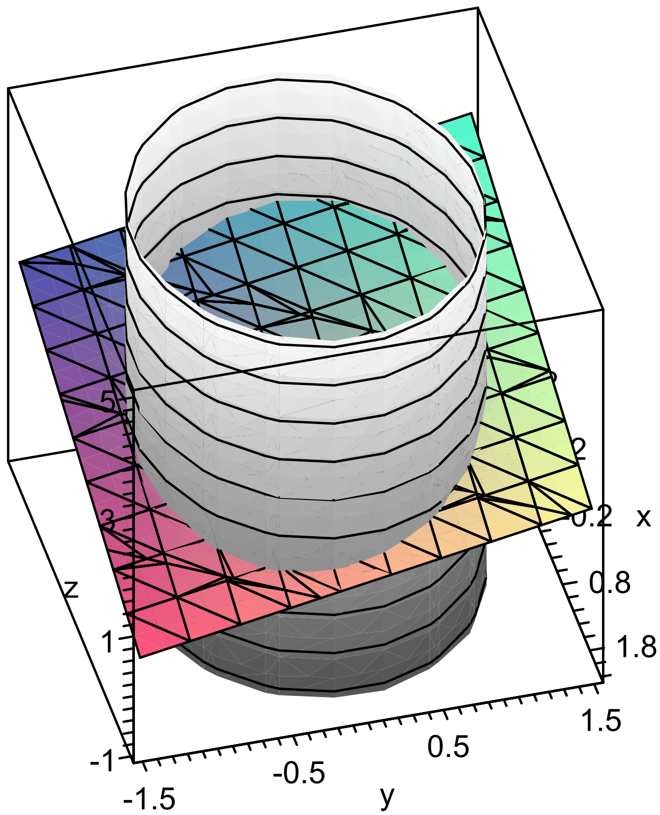


Figure 2.6: The situation for Exercise 7, showing the intersection of the plane $2x-2y+5z=10$ with the ellipsoidal cylinder $x^2+3y^2=4$.

The projection of the intersection of the plane $2x - 2y + 5z = 10$ with the ellipsoidal cylinder $x^2+3y^2 = 4$ onto the xy-plane coincides with the intersection of the ellipsoidal cylinder $x^2+3y^2 = 4$ onto the xy-plane, since the cylinder is vertical. This implies that

$$D = \{(x, y) : x^2 + 3y^2 = 4\},$$

which can also be written as

$$D = \{(x, y) : -\frac{2}{\sqrt{3}} \leq y \leq \frac{2}{\sqrt{3}}, -\sqrt{4 - 3y^2} \leq x \leq \sqrt{4 - 3y^2}\}$$

Note: to find the interval of variation of y, you simply need to look at the argument of the square root, $4 - 3y^2$, which must be nonnegative. We have

$$\begin{aligned} 4 - 3y^2 \geq 0 &\Leftrightarrow 4 \leq 3y^2 \\ &\Leftrightarrow y^2 \leq \frac{4}{3} \\ &\Leftrightarrow -\frac{2}{\sqrt{3}} \leq y \leq \frac{2}{\sqrt{3}}. \end{aligned}$$

We have $f_x = -1/5$, $f_y = 1/5$, and thus the area $A(S)$ of the plane inside the ellipsoidal cylinder is

$$\begin{aligned} A(S) &= \iint_{\mathcal{D}} \sqrt{f_x^2 + f_y^2 + 1}\, dA \\ &= \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} \int_{-\sqrt{4-3y^2}}^{\sqrt{4-3y^2}} \sqrt{1 + 1 - 1} dx dy \\ &= 2 \int_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} \sqrt{4 - 3y^2} dy \\ &= 2 \left(\frac{y}{2}\sqrt{4 - 3y^2} + \frac{2\sqrt{3}}{3} \arcsin\left(\frac{\sqrt{3}}{2} y\right) \Bigg|_{-\frac{2}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} \right) \\ &= \frac{4\sqrt{3}}{3}\pi, \end{aligned}$$

where the formula

$$\int \sqrt{a^2 - u^2} du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin\left(\frac{u}{a}\right) + C$$

was used to obtain the antiderivative of $\sqrt{4 - 3y^2}$.

8 (D marks). efine the improper integral

$$\ell = \iint_{\mathbb{R}^2} \exp(-(x^2 + y^2)) dA = \lim_{a\to\infty} \iint_{D_a} \exp(-(x^2 + y^2)) dA,$$

where D_a is the disk with radius a centrered at the origin. Show that $\ell = \pi$. Now define ℓ as

$$\ell = \lim_{a\to\infty} \iint_{S_a} \exp(-(x^2+y^2))dA,$$

where S_a is the square with vertices $(\pm a, \pm a)$. Show that

$$\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \pi,$$

and deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

Solution. We want to compute

$$\iint_{D_a} \exp(-(x^2+y^2))dA,$$

with D_a the disk with radius a centrered at the origin. For this, we switch to polar coordinates. This gives, with $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}
\iint_{D_a} \exp(-(x^2+y^2))dA &= \int_0^{2\pi}\int_0^a e^{-r^2} r\, dr d\theta \\
&= \int_0^{2\pi} \left.\frac{-e^{-r^2}}{2}\right|_{r=0}^{r=a} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} 1 - e^{-a^2} d\theta \\
&= \frac{1}{2}\left(1-e^{-a^2}\right)2\pi.
\end{aligned}$$

To find ℓ, we must then compute the limit as $a \to \infty$. This gives

$$\ell = \lim_{a\to\infty} \frac{1}{2}\left(1-e^{-a^2}\right)2\pi = \pi,$$

since as $a \to \infty$, $e^{-a^2} \to 0$.

But ℓ can also be defined as

$$\ell = \lim_{a\to\infty} \iint_{S_a} \exp(-(x^2+y^2))dA,$$

where S_a is the square with vertices $(\pm a, \pm a)$. Indeed, all we need is for the domain on which we compute the integral to tend to the whole plane, in the limit, no matter what shape the domain takes. We have

$$\begin{aligned}\iint_{S_a} \exp(-(x^2+y^2))dA &= \int_{-a}^{a}\int_{-a}^{a} \exp(-(x^2+y^2))dxdy \\ &= \int_{-a}^{a}\int_{-a}^{a} \exp(-x^2)\exp(-y^2)dxdy \\ &= \int_{-a}^{a} \exp(-x^2)dx \int_{-a}^{a} \exp(-y^2)dy,\end{aligned}$$

since the integrand is a separable function of x and y. Therefore,

$$\ell = \lim_{a\to\infty}\left(\int_{-a}^{a} \exp(-x^2)dx \int_{-a}^{a} \exp(-y^2)dy\right) = \int_{-\infty}^{\infty} \exp(-x^2)dx \int_{-\infty}^{\infty} \exp(-y^2)dy.$$

But note that the two integrals in that last expression are the same, save for the "dummy" variables x and y. Therefore, we can write that

$$\ell = \left(\int_{-\infty}^{\infty} \exp(-x^2)dx\right)^2.$$

From the first part, $\ell = \pi$, and as a consequence,

$$\left(\int_{-\infty}^{\infty} \exp(-x^2)dx\right)^2 = \pi,$$

that is,

$$\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}.$$

9 (R marks). ead the parts about centre of mass and moments of inertia in Sections 15.5 (Applications of double integrals) and 15.7 (Triple integrals). Then

1. Find the moment of inertia about the y axis for the ball $x^2+y^2+z^2 \leq R^2$ if the mass density is a constant δ.

2. Find the mass and centre of mass of the solid bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $y + z = 1$, $y = 0$, and $z = 0$, with density function given by $\rho(x, y, z) = 3$.

Solution. a. The moment of inertia about the y axis is given, for a solid with mass distribution $\rho(x, y, z)$ over a region E, by

$$I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)dV.$$

Here, E is the sphere of radius R, and

$$\begin{aligned}
I_y &= \iiint\limits_E (x^2+z^2)\rho \, dV \\
&= \rho \iiint\limits_E x^2+z^2 \, dV \\
&= \rho \iiint\limits_{\mathcal{S}} \left((r\cos\theta\sin\phi)^2 + (r\cos\phi)^2\right) r^2 \sin\phi \; drd\theta d\phi
\end{aligned}$$

where $\mathcal{S}$ is the sphere in spherical coordinates, and r has been used for ρ in the spherical coordinates to avoid confusion with the mass density,

$$\begin{aligned}
&= \rho \int_0^{\pi} \int_0^{2\pi} \int_0^R \left(r^2(\cos^2\theta+1)\sin^2\phi\right) r^2 \sin\phi \; drd\theta d\phi \\
&= \rho \int_0^{\pi} \int_0^{2\pi} \int_0^R r^4(\cos^2\theta+1)\sin^3\phi \; drd\theta d\phi \\
&= \rho \int_0^{\pi} \sin^3\phi \; d\phi \int_0^{2\pi} (\cos^2\theta+1) d\theta \int_0^R r^4 \; dr
\end{aligned}$$

since the equation is separable, and thus

$$\begin{aligned}
&= \rho \left(-\frac{1}{3}\sin^2\phi\cos\phi - \frac{2}{3}\cos\phi \Big|_0^{\pi} \right) \left(\frac{1}{2}\cos\theta\sin\theta + \frac{3}{2}\theta \Big|_0^{2\pi} \right) \left(\frac{r^5}{5} \Big|_0^R \right) \\
&= \rho \, \frac{4}{3} \, 3\pi \, \frac{R^5}{5} \\
&= \frac{4}{15}\pi \, R^2 \rho.
\end{aligned}$$

b. The first thing to do is to characterize the nature of the solid under consideration. For an assignment, it is a good thing to use a computer algebra system such as maple, which really helps in this case. Figure 2.7 was obtained using the commands:

```
with(plots):
implicitplot3d([z=1-x^2,y+z=1,y=0,z=0],x=-1.8..1.8,y=-.3..1.5,z=-1..2);
```

So we want to describe the region between the three planes and inside the paraboloid. The problem is made a little more complicated by the position of the $y+z=1$ plane (in green in Figure 2.7). Suppose, for example, that we take D to be on the xy-plane (the blue plane). For a given value of y between $y=0$ and $y=1$, the intersection of the blue plane with the red plane and the green plane, respectively, x varies between $x=-1$ and $x=1$, the intersection of the

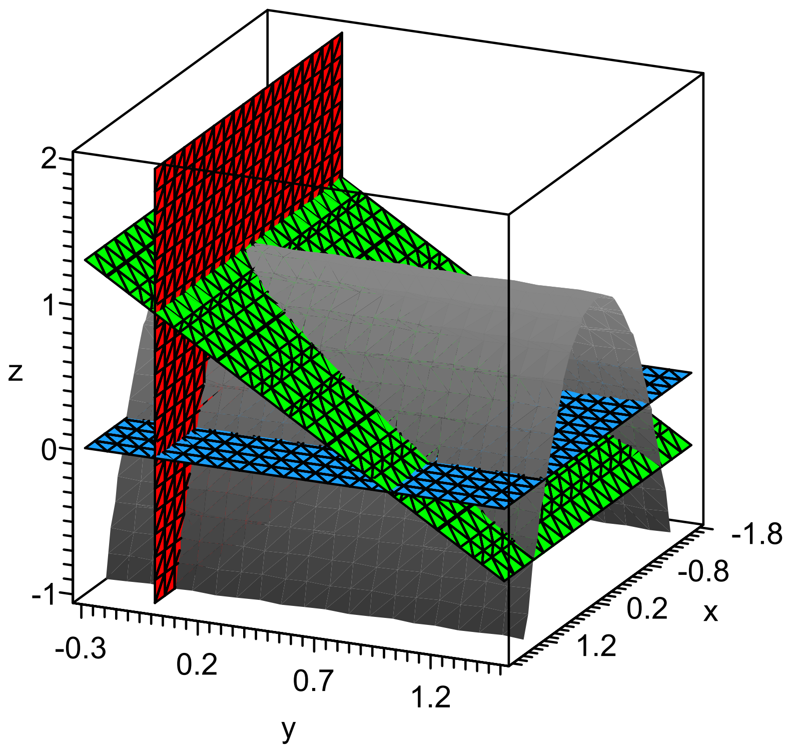


Figure 2.7: The situation for Exercise 9, showing the parabolic cylinder $z = 1 - x^2$ (gray) and the planes $y + z = 1$ (green), $y = 0$ (red), and $z = 0$ (blue).

paraboloid cylinder with the xy-plane, so D seems easy enough to define. But, for values of (x, y) in that set, z is bounded below by 0, and above by the paraboloid or the green plane, depending on (x, y).

To see this, take y fixed, say equal to 1/3. Then, the slice through the solid along the plane $y = 1/3$ takes the form of the curves on Figure 2.8.

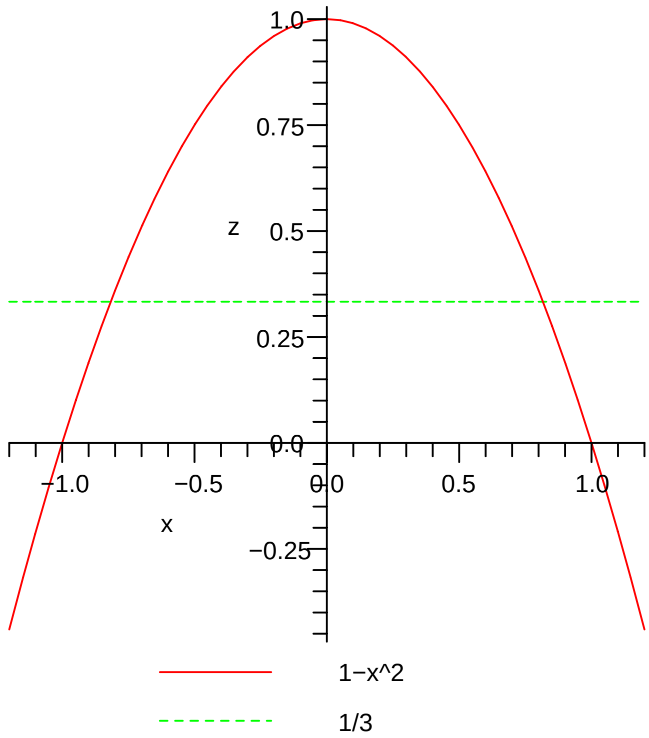


Figure 2.8: The situation for Exercise 9, showing the parabolic cylinder $z = 1 - x^2$ (red) and the plane $y + z = 1$ (green), when $y = 1/3$.

Therefore, for a given y, the domain must be decomposed: for $-1 \leq x \leq a$ and $b \leq x \leq 1$, z is bounded above by $1 - x^2$, while for $a \leq x \leq b$, z is bounded above by the plane $y + z = 1$, where a and b are the points of intersection of the plane $y + z = 1$ and the paraboloid $z = 1 - x^2$, for a given value of y. For y given, $z = 1 - y$ from the equation of the plane, which equals $1 - x^2$ if

$$x = \pm\sqrt{y},$$

that is, $a = -\sqrt{y}$ and $b = \sqrt{y}$. Therefore, the solid E can be described by

$$\begin{aligned} E =& \{(x, y, z) : 0 \leq y \leq 1, -1 \leq x \leq -\sqrt{y}, 0 \leq z \leq 1 - x^2\} \\ & \cup \{(x, y, z) : 0 \leq y \leq 1, -\sqrt{y} \leq x \leq \sqrt{y}, 0 \leq z \leq 1 - y\} \\ & \cup \{(x, y, z) : 0 \leq y \leq 1, \sqrt{y} \leq x \leq 1, 0 \leq z \leq 1 - x^2\}. \end{aligned}$$

We can now compute the mass and centre of mass. First of all, the mass is given by

$$\begin{aligned}
m &= \iiint_E \rho(x,y,z)dV \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} \int_0^{1-x^2} 3\ dzdx + \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} 3\ dzdx + \int_{\sqrt{y}}^{1} \int_0^{1-x^2} 3\ dzdx \right) dy \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} 3(1-x^2)\ dx + \int_{-\sqrt{y}}^{\sqrt{y}} 3(1-y)\ dx + \int_{\sqrt{y}}^{1} 3(1-x^2)\ dx \right) dy \\
&= \int_0^1 \left(\left(3x - x^3 \Big|_{-1}^{-\sqrt{y}} \right) + \left(3(1-y)x\Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} \right) + \left(3x - x^3 \Big|_{\sqrt{y}}^{1} \right) \right) dy \\
&= \int_0^1 \left(-3\sqrt{y} + \sqrt{y}^3 + 3 - 1 + 6(1-y)\sqrt{y} + 3 - 1 - 3\sqrt{y} + \sqrt{y}^3 \right) dy \\
&= \int_0^1 4 - 4\sqrt{y}^3\ dy \\
&= 4y - \frac{8}{5} y^{5/2} \Big|_0^1 \\
&= \frac{12}{5}.
\end{aligned}$$

We now compute the moments about the 3 coordinate planes.

$$\begin{aligned}
M_{yz} &= \iiint_E x\rho(x,y,z)dV \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} \int_0^{1-x^2} 3x\ dzdx + \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} 3x\ dzdx + \int_{\sqrt{y}}^{1} \int_0^{1-x^2} 3x\ dzdx \right) dy \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} 3x(1-x^2)\ dx + \int_{-\sqrt{y}}^{\sqrt{y}} 3x(1-y)\ dx + \int_{\sqrt{y}}^{1} 3x(1-x^2)\ dx \right) dy \\
&= \int_0^1 \left(\left(\frac{3}{2}x^2 - \frac{3}{4}x^4 \Big|_{-1}^{-\sqrt{y}} \right) + \left(\frac{3}{2}(1-y)x^2 \Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} \right) + \left(\frac{3}{2}x^2 - \frac{3}{4}x^4 \Big|_{\sqrt{y}}^{1} \right) \right) dy \\
&= \int_0^1 \left(\frac{3}{2}y - \frac{3}{4}y^2 - \frac{3}{2} + \frac{3}{4} + \frac{3}{2} - \frac{3}{4} - \frac{3}{2}y + \frac{3}{4}y^2 \right) dy \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
M_{xz} &= \iiint_E y\rho(x,y,z)dV \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} \int_0^{1-x^2} 3y\ dzdx + \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} 3y\ dzdx + \int_{\sqrt{y}}^{1} \int_0^{1-x^2} 3y\ dzdx \right) dy \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} 3y(1-x^2)\ dx + \int_{-\sqrt{y}}^{\sqrt{y}} 3y(1-y)\ dx + \int_{\sqrt{y}}^{1} 3y(1-x^2)\ dx \right) dy \\
&= \int_0^1 \left(\left(3yx - yx^3 \Big|_{x=-1}^{x=-\sqrt{y}} \right) + \left(3y(1-y)x\Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} \right) + \left(3yx - yx^3 \Big|_{x=\sqrt{y}}^{x=1} \right) \right) dy \\
&= \int_0^1 \left(-3y\sqrt{y} + y\sqrt{y}^3 + 3y - y + 6y(1-y)\sqrt{y} + 3y - y - 3y\sqrt{y} + y\sqrt{y}^3 \right) dy \\
&= \int_0^1 4y - 4y^{5/2} dy \\
&= 2y^2 - \frac{8}{7} y^{8/7} \Big|_0^1 \\
&= \frac{6}{7},
\end{aligned}
$$

and

$$
\begin{aligned}
M_{xy} &= \iiint_E z\rho(x,y,z)dV \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} \int_0^{1-x^2} 3z\ dzdx + \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} 3z\ dzdx + \int_{\sqrt{y}}^{1} \int_0^{1-x^2} 3z\ dzdx \right) dy \\
&= \int_0^1 \left(\int_{-1}^{-\sqrt{y}} 3z(1-x^2)\ dx + \int_{-\sqrt{y}}^{\sqrt{y}} 3z(1-y)\ dx + \int_{\sqrt{y}}^{1} 3z(1-x^2)\ dx \right) dy \\
&= \int_0^1 \left(\left(3zx - zx^3 \Big|_{x=-1}^{x=-\sqrt{y}} \right) + \left(3z(1-y)x\Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} \right) + \left(3zx - zx^3 \Big|_{x=\sqrt{y}}^{x=1} \right) \right) dy \\
&= \int_0^1 \left(-\frac{3}{2}\sqrt{y} + \frac{4}{5} - \frac{3}{10} y^{5/2} + y^{3/2} + 3(1-y)^2\sqrt{y} - \frac{3}{2}\sqrt{y} + \frac{4}{5} - \frac{3}{10} y^{5/2} + y^{3/2} \right) dy \\
&= \int_0^1 \frac{8}{5} + \frac{12}{5} y^{5/2} - 4y^{3/2} dy \\
&= \frac{8}{5} y + \frac{24}{35} y^{7/2} - \frac{8}{5} y^{5/2} \Big|_0^1 \\
&= \frac{24}{35}.
\end{aligned}
$$

Therefore, the centre of mass is located at

$$\begin{aligned}(\bar{x}, \bar{y}, \bar{z}) &= \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) \\ &= \frac{5}{12}\left(0, \frac{6}{7}, \frac{24}{35}\right) \\ &= \left(0, \frac{5}{14}, \frac{2}{7}\right).\end{aligned}$$

MATH 2720
Fall 2007
Assignment 6

This assignment is due **in class** Monday, December 3rd. Late assignments will **not** be accepted.
Remarks:

1. Do not hesitate to discuss this assignment with others. However, the solution that you hand in **must** be yours.
2. The assignment is out of 95 points and an additional 5 points for clarity.
3. It is very likely that some questions will not be marked.

---

1 (F marks). ind the mass of a wire formed by the intersection of the unit sphere in $\mathbb{R}^3$ and the plane $x+y+z=0$, if the density at (x,y,z) is given by $\rho(x,y,z)=x^2$ per unit length of wire.

2 (E marks). valuate

$$\int_{\mathcal{C}} x^2\,dx + xy^2\,dy + dz,$$

where $\mathcal{C}$ is parametrized by $r:[0,1]\to\mathbb{R}^3$, $r(t)=(t,t^2,1)$.

3 (E marks). valuate

$$\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz$$

where $\mathcal{C}$ consists of the straight-line segments joining $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$.

4 (L marks). et $\mathcal{C}$ be a smooth curve parametrized by $r(t)$.

1. Suppose F orthogonal to $r'(t)$ at the point $r(t)$, for all t. Show that

$$\int_{\mathcal{C}} F\cdot dr = 0.$$

2. If F is parallel to $r'(t)$ at $r(t)$ for all t, show that

$$\int_{\mathcal{C}} F\cdot dr = \int_{\mathcal{C}} \|F\| ds.$$

5 (E marks). valuate

$$\int_{\mathcal{C}} y\,dx + (3y^3-x)\,dy + z\,dz$$

for each of the curves $\mathcal{C}$ parametrized by $r(t)=(t,t^n,0)$, $0\leq t\leq 1$, for $n=1,2,3,\ldots$.

6 (E marks). valuate the line integral

$$\int_{\mathcal{C}} 2xyz\ dx + x^2z\ dy + x^2y\ dz$$

where $\mathcal{C}$ is an oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$.

MATH 2720
Fall 2007
Assignment 6

Marks: 2 (20 points), 3 (20 points), 5 (25 points), 6 (30 points).

1 (F marks). ind the mass of a wire formed by the intersection of the unit sphere in $\mathbb{R}^3$ and the plane $x+y+z=0$, if the density at (x,y,z) is given by $\rho(x,y,z)=x^2$ per unit length of wire.

Solution. We want to compute

$$\int_{\mathcal{C}} x^2\ ds,$$

where the curve $\mathcal{C}$ lies at the intersection of the unit sphere in $\mathbb{R}^3$ and the plane $x+y+z=0$ (see Figure 2.9).

2 (E marks). valuate

$$\int_{\mathcal{C}} x^2\ dx + xy^2\ dy + dz,$$

where $\mathcal{C}$ is parametrized by $r:[0,1]\to\mathbb{R}^3$, $r(t)=(t,t^2,1)$.

Solution. We have

$$\begin{aligned}\int_{\mathcal{C}} x^2\ dx + xy^2\ dy + dz &= \int_0^1 t^2 + t(t^2)^2(2t) + 0\ dt\\ &= \frac{t^2}{3} + \frac{2}{7}t^7\Big|_0^1\\ &= \frac{1}{3}+\frac{2}{7}=\frac{13}{21}.\end{aligned}$$

3 (E marks). valuate

$$\int_{\mathcal{C}} yz\ dx + xz\ dy + xy\ dz$$

where $\mathcal{C}$ consists of the straight-line segments joining $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$.

Solution. Note that the triangle is not closed, and thus, although tempting (since the vector field is conservative), you must not conclude directly that the integral is zero. It is, however, possible to use the conservativity of the field (we do that in a second approach).

First approach: we parametrize the segments as

$$\mathcal{C}_1 : r_1(t) = (1-t)(1,0,0) + t(0,1,0) = (1-t,t,0)$$

and

$$\mathcal{C}_2 : r_2(t) = (1-t)(0,1,0) + t(0,0,1) = (0,1-t,t),$$

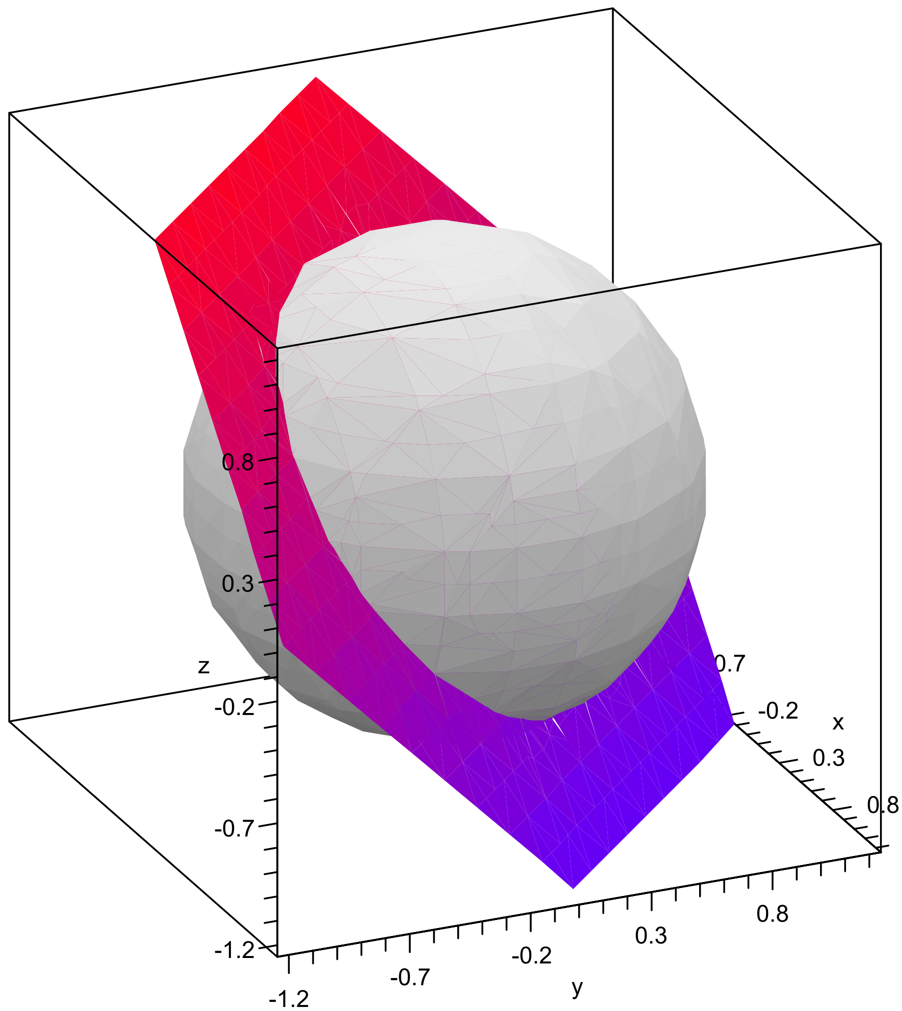


Figure 2.9: The intersection of the unit sphere in $\mathbb{R}^3$ and the plane $x+y+z=0$ in Exercise 1.

both defined for $t \in [0,1]$. When needed, we write $r_i(t) = (x_i(t), y_i(t), z_i(t))$, for $i = 1,2$. We then have

$$\begin{aligned}
\int_{\mathcal{C}} yz\ dx + xz\ dy + xy\ dz &= \int_0^1 y_1(t)z_1(t)x_1' + x_1(t)z_1(t)y_1'(t) + x_1(t)y_1(t)z_1'(t)\ dt \\
&\quad + \int_0^1 y_2(t)z_2(t)x_2' + x_2(t)z_2(t)y_2'(t) + x_2(t)y_2(t)z_2'(t)\ dt \\
&= \int_0^1 0\ dt + \int_0^1 0\ dt \\
&= 0.
\end{aligned}$$

Second approach: we have seen in class that the vector field $F = (yz, xz, xy)$ is conservative, the potential function is easily found to be $f(x,y,z) = xyz$. We divide the curve $\mathcal{C}$ into the curve from $(1,0,0)$ to $(0,1,0)$, and the curve from $(0,1,0)$ to $(0,0,1)$. Therefore,

$$\int_{\mathcal{C}} yz\ dx + xz\ dy + xy\ dz = (f(0,1,0) - f(1,0,0)) + (f(0,0,1) - f(0,1,0)) = 0.$$

4 (L marks). et $\mathcal{C}$ be a smooth curve parametrized by $r(t)$.

1. Suppose F orthogonal to $r'(t)$ at the point $r(t)$, for all t. Show that
$$\int_{\mathcal{C}} F \cdot dr = 0.$$
2. If F is parallel to $r'(t)$ at $r(t)$ for all t, show that
$$\int_{\mathcal{C}} F \cdot dr = \int_{\mathcal{C}} \|F\| ds.$$

Solution. a) We have

$$\begin{aligned}
\int_{\mathcal{C}} F \cdot dr &= \int F(r(t)) \cdot r'(t) dt \\
&= 0,
\end{aligned}$$

since $F(r(t)) \cdot r'(t) = 0$ for all t.
b) That F is parallel to $r'(t)$ at $r(t)$ for all t can be expressed by writing that

$$F(r(t)) = c(t)r'(t)$$

for all t, for some scalar function $c(t) \neq 0$. Note that $c(t)$ can vary with t, it must just not be zero.

5 (E marks). valuate

$$\int_{\mathcal{C}} y\ dx + (3y^3 - x)\ dy + z\ dz$$

for each of the curves $\mathcal{C}$ parametrized by $r(t) = (t, t^n, 0)$, $0 \leq t \leq 1$, for $n = 1, 2, 3, \ldots$.

Solution. We have

$$\begin{aligned}
\int_{\mathcal{C}} y\ dx + (3y^3 - x)\ dy + z\ dz &= \int_0^1 t^n + (3(t^n)^3 - t)nt^{n-1}\ dt \\
&= \int_0^1 (1-n)t^n + 3nt^{3n}t^{n-1}\ dt \\
&= \frac{1-n}{n+1}t^{n+1}\bigg|_0^1 + \int_0^1 3nt^{4n-1}\ dt \\
&= \frac{1-n}{n+1} + \left(\frac{3n}{4n}t^{4n}\bigg|_0^1\right) \\
&= \frac{1-n}{n+1} + \frac{3}{4}.
\end{aligned}$$

6 (E marks). valuate the line integral

$$\int_{\mathcal{C}} 2xyz\ dx + x^2z\ dy + x^2y\ dz$$

where $\mathcal{C}$ is an oriented simple curve connecting $(1, 1, 1)$ to $(1, 2, 4)$.

Solution. The way the question is formulated should be a strong hint that the vector field is conservative, since you are not given an expression for the curve $\mathcal{C}$. We check this first, by computing the $\nabla \times F$. We have

$$\begin{aligned}
\nabla \times F &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix} \\
&= \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & x^2y \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xyz & x^2y \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xyz & x^2z \end{vmatrix} \\
&= \left(x^2 - x^2, -2xy + 2xy, 2xz - 2xz\right) \\
&= (0, 0, 0),
\end{aligned}$$

and therefore, the vector field $F = (2xyz, x^2z, x^2y)$ is conservative. So we can look for a potential function f.

We have $f_x = 2xyz$, and therefore, integrating with respect to x,

$$f(x, y, z) = x^2yz + H(y, z), \tag{2.21}$$

where the function H does not depend on x. Taking the partial derivative of (2.21) with respect to y, we obtain

$$f_y = x^2z + \frac{\partial}{\partial y}H(y, z),$$

and since, from $F = (2xyz, x^2z, x^2z)$, $f_y = x^2z$, it follows that $\partial H/\partial y = 0$. This in turn implies that $H(y,z)$ does not, in fact, depend on y, so that we can write is as $H(y,z) = K(z)$, and equation (2.21) can be refined,

$$f(x,y,z) = x^2yz + K(z), \tag{2.22}$$

where K depends only on z, not on x, y. Taking the partial derivative of (2.22) with respect to z, we get

$$f_z = x^2y + K'(z),$$

which must equal x^2y if f is to be the potential function of F. Therefore, $K'(z) = 0$, implying that $K(z) = C$, a constant. We can take this constant to be zero, and thus finally,

$$f(x,y,z) = x^2yz$$

is the potential function for $F = (2xyz, x^2z, x^2z)$. Using the fundamental theorem for line integrals, we conclude that, for a curve $\mathcal{C}$ joining $(1,1,1)$ to $(1,2,4)$,

$$\begin{aligned}\int_{\mathcal{C}} 2xyz\; dx + x^2z\; dy + x^2y\; dz &= f(1,2,4) - f(1,1,1) \\ &= 1\cdot 2\cdot 4 - 1\cdot 1\cdot 1 \\ &= 7.\end{aligned}$$

Chapter 3

Past midterm examinations

Contents

Note that there is absolutely no guarantee that our examinations will in any way resemble the ones made available here.

3.1 October 22, 2001

Course	:	136.270
Examination	:	Midterm
Date	:	October 22, 2001
Duration	:	1 hour
Examiner	:	F. Ghahramani

1. For the curve C with the equation $\mathbf{r}(t) = \frac{2}{3}t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$.

a). (5 marks). Find the length of the arc between the points corresponding to $t = 0$ and $t = 1$. Note: $4t^4 + 4t^2 + 1 = (2t^2 + 1)^2$.

b). (5 marks). Find the unit tangent vector T at a general point. Simplify as much as possible.

c). Find the unit principal normal vector N at a general point. DO NOT SIMPLIFY.

2. (8 marks). Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy + x^2}{x^2 + y^2 + xy}$$

3. (13 marks). For the surface S given by the equation $z = x^2 - 3y^2 + xy$, find the equations of the tangent plane and the normal line at the point $(1, 1, -1)$ on S.

4. (12 marks). Let $f(x, y, z) = x^3e^y + z^2y$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ and $f_{3,2,2}(x, y, z)$.

5. (12 marks). Let $z = ye^x + xe^y$, $x = s + t$, and $y = s - t$. By using the chain rule, calculate $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial^2 z}{\partial s \partial t}$.

3.2 October 22, 2002

Course	:	136.270
Examination	:	Midterm
Date	:	October 22, 2002
Duration	:	1 hour
Examiner	:	F. Ghahramani

1. The curve C is given by the equation $\vec{r}(t) = 3\sin t\vec{i} + 4\sin t\vec{j} + 5\cos t\vec{k}$.

a). (4 marks). Find the length of the arc between the points corresponding to $t = 0$ and $t = 1$. [**Answer:** 5]

b). (8 marks). Find the Frenet frame $\{T, N, B\}$ at a point corresponding to the general parameter t.

2. Determine which one the following limits exists and which one does not exist. Find the value of the one that exists. **Justify your answers**.

a). (4 marks).

$$\lim_{(x,y)\to(0,0)} \frac{x + y^2}{x^2 + y^2}$$

b). (4 marks).

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + y^2 + xy^2}{2x^2 + y^2}$$

3. (14 marks). The surface S has the equation $z = 2x^2 - y^2 + 2xy$ in the three dimensional space. Find the equations of the tangent plane and the normal line to S at the point $(1, 2, 2)$ on S.

4. (13 marks). Let $f(x, y, z) = y^3e^x + xz^2$, find $\dfrac{\partial^2 f}{\partial z \partial x}$ and $f_{3,1,2}$.

5. (12 marks). Let $z = ye^x - x^2$, $x = s^2 + t$, and $y = s - t^2$. By using the **chain rule**, calculate $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial^2 z}{\partial s \partial t}$.

N.B. You will receive no credit if you attempt this question without using the chain rule.

3.3 October 27, 2003

Course	: 136.270
Examination	: Midterm
Date	: October 27, 2003
Duration	: 1 hour
Examiner	: S. Kalajdzievski

1. (12 marks). We are given the vector-values function $\mathbf{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ (In order to avoid bad initial errors that may lead to difficult computation, I emphasize that in the second coordinate t is outside the root).

a). Compute the unit tangent vector at the moment when $t = 0$.

b). Find the arc-length of that curve for $0 \leq t \leq 5$.

2. (12 marks). Evaluate the indicated limit or show it does not exist.

a). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

b). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$.

c). $\displaystyle\lim_{(x,y)\to(1,0)} \frac{e^y-1}{2x+y-2}$.

3. (12 marks). Consider the function $f(x,y) = \dfrac{x-y}{\sqrt{x^2+y}}$.

a). Sketch (in the $x-y$ plane) the domain of that function.

b). Find the equation of the tangent plane to the graph of that function at the point when $x = 1$ and $y = 1$.

4. (12 marks).

a). Use the chain rule to find $\frac{\partial f}{\partial u}$ if $f(x,y) = e^y + 3y\sin x$, $x = \ln(2u+3v)$, $y = \frac{v}{u}$.

b). The equation $xz^3 + 2yz - 3xy = 0$ defines z as a function of z and y. Compute $\frac{\partial z}{\partial x}$ at the point when $x = 1$, $y = 1$ and $z = 1$.

5. (12 marks).

a). State the definition of $\lim_{(x,y)\to(a,b)} f(x,y)$. That is, what does it mean to say that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to some number L?

b). Show **using the definition of limit** that $\lim_{(x,y)\to(0,0)} xy = 0$. No points will be given if other methods are used.

3.4 February 10, 2004

Course	:	136.270
Examination	:	Midterm 1
Date	:	February 10, 2004
Duration	:	1 hour
Examiner	:	Y. Zhang

2. (6 marks). Let $\vec{r}(t) = \sqrt{t+3}\vec{i} + \frac{t-1}{t^2-1}\vec{j} + \frac{\tan t}{2}\vec{k}$. Find the limits $\lim_{t \ to 1} \vec{r}(t)$ and $\lim_{t \to 0} \vec{r}(t)$.

3. (9 marks). Find the curvature of $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ at the point $(0, 1, 1)$.

4. (10 marks). Reparametrize the curve $\vec{r}(t) = (\frac{2}{t^2+1} - 1)\vec{i} + \frac{2t}{t^2+1}\vec{j}$ with respeet to arc length measured from the point where $t = 0$ in the direction of increasing t. You need to simpify your answer.

5. Write the equation $x^2 + y^2 = 2y$

a). (5 marks). In cylindrical coordinates.

b). (5 marks). In spherical coordinates.

6. (5 marks). Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leq z \leq 2$.

3.5 March 11, 2004

Course	:	136.270
Examination	:	Midterm 2
Date	:	March 11, 2004
Duration	:	1 hour
Examiner	:	Y. Zhang

1. (8 marks). Let $f(x, y, z) = xy^2z^3 + y \tan^{-1} z$, find f_{yz}.

2. (8 marks). Let $z = y^2 \tan x$, $x = t^2uv$, $y = u + tv^2$. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial u}$.

3. Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

a). (4 marks). Find the gradient of f.

b). (4 marks). Find the directional derivative of f at the point $(1, 2, 1)$ in the direction of $\vec{v} = \langle 2, 1, -2\rangle$.

4. (6 marks). Find $\frac{\partial x}{\partial y}$ if $xyz = \cos(x + y + 2z)$.

5. Consider the surface

$$x^2 + y^2 - z^2 - 2xy + 4xz = 4.$$

a). (5 marks). Find an equation of the normal plane to the surface at the point $(1, 0, 1)$.

b). (5 marks). Find equations of the normal line to the surface at the point $(1, 0, 1)$.

6. Find the limit if it exists or show that the limit does not exist.

a). (5 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y + x^2 + y^2}{x^2 + y^2}$.

b). (5 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2}$.

3.6 October 27, 2004

Course	:	136.270
Examination	:	Midterm
Date	:	October 27, 2004
Duration	:	1 hour
Examiner	:	S. Kalajdzievski

1. In all of a), b), c), d) and e) below, we consider the vector function $\mathbf{r}(t) = (2\cos t, t\sqrt{t}, 2\sin t)$.

a). (5 marks). Find the length of the arc between to points corresponding to $t = 0$ and $t = \pi$.

b). (4 marks). Sketch $\mathbf{r}(t) = (2\cos t, t\sqrt{5}, 2\sin t)$.

c). (5 marks). Find the unit tangent vector at the point $(2, 0, 0)$.

d). (4 marks). Find one point on the curve defined by the vector function $\mathbf{r}(t)$ where the normal plane is perpendicular to the unit tangent vector found in c) above.

e). Find the bi-normal vector at the point found in d) above.

2. Show that the following limits do not exist.

a). (6 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

b). (7 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$.

3.

a). (3 marks). State the definition of $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$.

b). (7 marks). Use the definition of limit (part a) above) to show that $\displaystyle\lim_{(x,y)\to(0,0)} 4x^2+4y^2+\frac{1}{3} = \frac{1}{3}$. No points will be given if other methods are used.

c). (4 marks). Use the Squeeze theorem and the result of part b) to show that Show **using the definition of limit** that $\displaystyle\lim_{(x,y)\to(0,0)} 3x^2+4y^2+\frac{1}{3} = \frac{1}{3}$. No points will be given if other methods are used.

4. Given that $f(x, y, z) = x(\sin z)e^{xy}$,

a). (5 marks). Find all three first partial derivatives.

b). (4 marks). Find $\dfrac{\partial^2 f}{\partial y \partial x}(1, 1\pi)$.

3.7 October 26, 2005

Course	:	136.272
Examination	:	Midterm
Date	:	October 26, 2005
Duration	:	1 hour
Examiner	:	S. Kalajdzievski

1. In all of a), b), c) and d) below, we consider the vector function $\mathbf{r}(t) = (\cos t + t\sin t, 0, \sin t - t\cos t)$ with $t \geq 0$.

a). (4 marks). Find the unit tangent vector $T(t)$.

b). (4 marks). Find the normal vector $N(t)$.

c). (4 marks). Compute the bi-normal vector $B(t)$ at the point found when $t = \pi$. Write the equation of the osculating plane at that point.

d). (4 marks). Find the curvature at the point when $t = \pi$.

2. (7 marks). It has been observed that the velocity of an asteroid could be approximated by the function $v(t) = (2t, 3t^2, 1 - 4t^3)$ (measured in millions of kilometers per day), where the time t is measured in days. The initial position of the asteroid (at $t = 0$ days) was at the point $(1, 2, 3)$ (measured in millions of kilometers). Find the position vector $r(t)$. If the planet X is viewed as a point and if it is positioned at $(101, 1002, -9997)$ (in millions of kilometers) will the asteroid hit it? If yes, how many days after the initial observation?

3. Show that the following limits do not exist.

a). (6 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^6 + y^6}$.

b). (7 marks). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^3}{x^2 + y^6}$.

4.

a). (7 marks). Use only the definition of of $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ to prove that

$$\lim_{(x,y)\to(0,0)} 2 + 3\sqrt{x^2 + y^2} = 2.$$

b). (7 marks). Use polar coordinates to evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^2}$. Justify your steps.

3.8 February 7, 2006

Course	:	136.272
Examination	:	Midterm 1
Date	:	February 7, 2006
Duration	:	1 hour 15 minutes
Examiner	:	S.H. Lui

1. (5 marks). Sketch the surface $x^2 - y^2 + z^2 + 2y = 0$. Label special point(s) of the surface. Restricted to the plane $z = 1$, what does this surface become and sketch it in the above diagram.

2. (5 marks). Find an equation of the plane which passes through the point $(1, 2, 3)$ and is parallel to the plane $x + y + z = 0$. Find also a parametric representation of the plane. (That is, find $r(t_1, t_2)$, the position vector to a point on the plane parametrized by t_1, t_2).

3. (4 marks). Given a nonzero vector-valued function $u = u(t)$ with the property that u and u' are perpendicular for all t. Here $'$ denotes derivatives with respect to t. Simplify $\dfrac{d}{dt}\dfrac{u \times u'}{\|u\|^2}$.

4. (4 marks). The trajectory of a particle $r(s) \equiv [x(s), y(s), z(s)] = [6s, 3s^2, s^3]$. Find the distance that the particle has traveled from the starting point $(0, 0, 0)$ to the point when it is at height $z = 1000$.

5. (4 marks). Let $g(x, y) = e^{\sin(2x+3y)}$. Compute g_x, g_y and g_{xy}.

6. (4 marks). Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y - y^2}{x^2 + y^2}$.

7. (6 marks). Given the curve $r(t) = [a\cos t, -\sin t, 5]$. Find the unit tangent vector, principal normal vector and the curvature at arbitrary t.

3.9 March 23, 2006

Course	:	136.272
Examination	:	Midterm 2
Date	:	March 23, 2006
Duration	:	1 hour 15 minutes
Examiner	:	S.H. Lui

1. (5 marks). Suppose $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$. Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

2. (6 marks). The temperature at a point (x, y, z) is given by $T(x, y, z) = e^{-x^2-3y^2-z^2}$. Find the rates of change of temperature at the point $P(2, -1, 2)$ *in the direction toward* the point $(3, -3, 3)$. In which direction does the temperature increase fastest at P and what is the maximum rate?

3. (7 marks). Let $f(x, y) = \dfrac{x^2y^2 - 8x + y}{xy}$. Find all critical points of f. Use the second derivative test to determine the type of each critical point. Determine whether each local extremum is also a global extremum.

4. (7 marks). Find the points on the surface $x^2y^2z = 1$ that are closest to the origin using the method of **Lagrange multipliers**.

5. (9 marks).

a). Find $\int_0^1 \int_0^x xy^2 dy dx$.

b). Find $\int_0^1 \int_{2y}^1 \sin(x^2) dx dy$.

c). Find $\int\int_R y^3 dx dy$ where R is the triangle with vertices $(0, 2)$, $(1, 1)$, $(3, 2)$.

6. (6 marks). **Using Polar Coordinates, set up the integral** for the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and outside the paraboloid $1 - z = x^2 + y^2$. Draw a picture. Find the value of this volume.

3.10 October 26, 2006

Course	:	Math 2720
Examination	:	Midterm 1
Date	:	October 26, 2006
Duration	:	1 hour
Examiner	:	J. Arino

1. (20 marks). Consider the function $r(t) = (e^{-t}, 2t, \cos(t))$.

a). Give an equation for the normal plane to the curve at the point $(1, 0, 1)$.

b). Give an expression for the arc length function from $t = 0$ (you may not be able to compute the integral explicitly).

c). Find the curvature $\kappa(t)$.

2. (15 marks). Find the limit, when it exists, or prove that it does not exist.

a). $\lim_{(x,y)\to(1,1)} \dfrac{x^3 + 2y^3}{x^3 + 4y^3}$.

b). $\lim_{(x,y)\to(0,0)} \dfrac{x - y}{x + y}$.

3. (15 marks). Consider the function

$$f(x, y) = \frac{\ln(xy)}{x^2 + y^3}.$$

a). Find the domain $\mathcal{D}$ of f.

b). Give an equation for the plane tangent to the surface $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \mathcal{D}, z = f(x, y)\}$ at a point $P_0(x_0, y_0) \in \mathcal{D}$.

4. (15 marks). Consider the function $f(x, y) = x/y$.

a). Explain why the function is differentiable at the point $(1, 1)$.

b). Find the tangent plane approximation of the function f at $(1, 1)$.

c). Find the maximum rate of change of f at the point $(1, 1)$ and the direction in which it occurs.

5. (10 marks). Find dy/dx if y is defined implicitly as a function of x by $(x + y)^2 = 2x$.

3.11 October 12, 2007

Course	:	Math 2720
Examination	:	Midterm 1
Date	:	October 12, 2007
Duration	:	50 minutes
Examiner	:	J. Arino

1. (5 marks). State the definition of a smooth curve.

2. (5 marks). Let $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Show that

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

[Hint: Use the Cauchy-Schwarz inequality with the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.]

3. (20 marks). Let $r(t) = (t^2 \cos t, t^2 \sin t, 2t)$. We consider the curve $\mathcal{C}$ defined by $r(t)$, starting at $t = 0$ and with $t \geq 0$.

a). Is $\mathcal{C}$ a smooth curve?

b). Find the length of the arc from $(0, 0, 0)$ to $(-\pi^2, 0, 2\pi)$.

c). Find the angle of intersection of $\mathcal{C}$ with the plane $z = 4\pi$. (You can give the angle in terms of its sine or cosine, no need to find it explicitly.)

d). Find the unit tangent vector to the curve at a point $r(t)$.

4. (5 marks). Express the plane $z = x$ in spherical coordinates.

5. (5 marks). Using the (ε, δ) approach, show that the function

$$f(t) = \left(2t - 1, \frac{t}{3}\right)$$

is continuous at $t = 3$.

3.12 November 7, 2007

Course	:	Math 2720
Examination	:	Midterm 2
Date	:	November 7, 2007
Duration	:	2 hours
Examiner	:	J. Arino

1. (5 marks). State the **definition** of the directional derivative of a function $f(x, y, z)$.

2. (8 marks). Let $e^{xz^2} + y^3 - z^2 = 0$. Find $\partial x/\partial z$.

3. (10 marks). Consider the function

$$F(x, y, z) = x^2 + 2y^2 + 3z^2.$$

Find the equation of the planes tangent to the surface $S = \{(x, y, z) \in \mathbb{R}^3; F(x, y, z) = 5\}$ at the points where $(x, y) = (0, 1)$.

4. (5 marks). Find the global extrema of the function $f(x, y, z) = x - y + z$ in the set

$$S = \{(x, y, z) : 0 < x^2 + y^2 + z^2 \leq 1\}.$$

5. (10 marks). Let $f(x, y) = x^2 + y^2 + kxy$, for $k \in \mathbb{R}$. Find the critical points and extrema, if any, of f, as k varies.

6. (10 marks). Use the chain rule to find a formula for

$$\frac{d}{dt}\left(f(t)^{g(t)}\right).$$

7. (5 marks). Suppose that $f(x, y, z) = g(z)$, where $g : \mathbb{R} \to \mathbb{R}$, that is, the function f is independent of the variables x and y. Compute ∇f in terms of g'.

8. (9 marks). Find the points on the surface $z = (x - 1)^2 + y^2 - 1$ that are closest to the origin.

9. (8 marks). Find the length of the curve $(\sin 3t, \cos 3t, 2t^{3/2})$ between the points $(0, 1, 0)$ and $(0, -1, 2\pi^{3/2})$.

10. (10 marks). Find values of a, b, c such that the function

$$f(t) = \begin{cases} (1 - 3t, 4\cos t, 4\sin t) & \text{if } t < 0 \\ (a + 3\sin t, b + 4t, c + 3\cos t) & \text{if } t \geq 0, \end{cases}$$

is a continuous function. Is f smooth? Where possible, compute the tangent vector $T(t)$ and the normal vector $N(t)$.

Chapter 4

Solutions of selected midterm examinations

Contents

4.1 October 26, 2006

Solution (Exercise 1). (a – 8 marks) The vector $r'(t)$ is orthogonal to the normal plane to the curve at the point $r(t)$. We have

$$r'(t) = (-e^{-t}, 2, -\sin t).$$

The point $(1,0,1)$ on the curve occurs for $t = 0$. Therefore, a vector orthogonal to the normal plane to the curve at the point $(1,0,1)$ is given by $r'(0) = (-1, 2, 0)$. In vector form, the normal plane has equation

$$(-1, 2, 0) \cdot (x - 1, y, z - 1) = 0.$$

In parametric form,

$$(x, y, z) = (1 - t, 2t, 1),$$

and in equation form

$$-x + 2y + 1 = 0 \text{ or } x - 2y - 1 = 0.$$

(b – 4 marks) The equation for the arc length function is

$$L = \int_a^t |r'(s)| ds.$$

So, evaluated from $t = 0$, this gives

$$L = \int_0^t |(-e^{-s}, 2, -\sin s)| ds = \int_0^t \sqrt{e^{-2s} + 4 + \sin^2 s} ds.$$

Note that it is important here to use a different upper bound for the integral and integration variable.

(c – 8 marks) We use the formula $\kappa(t) = |r' \times r''|/|r'|^3$ (but could also have used $\kappa(t) = |T'|/|r'|$, although this implied more complicated computations). We have

$$r''(t) = (e^{-t}, 0, -\cos t),$$

and thus

$$r' \times r'' = \begin{vmatrix} i & j & k \\ -e^{-t} & 2 & -\sin t \\ e^{-t} & 0 & -\cos t \end{vmatrix} = (-2\cos t, -e^{-t}(\cos t + \sin t), -2e^{-t}).$$

Therefore,

$$\begin{aligned} |r' \times r''| &= \sqrt{4\cos^2 t + e^{-2t}(\cos t + \sin t)^2 + 4e^{-2t}} \\ &= \sqrt{4\cos^2 t + e^{-2t}(\cos^2 t + \sin^2 t + 2\cos t \sin t) + 4e^{-2t}} \\ &= \sqrt{4\cos^2 t + e^{-2t}(5 + 2\cos t \sin t)}, \end{aligned}$$

and as a consequence,

$$\kappa(t) = \frac{\sqrt{4\cos^2 t + e^{-2t}(5 + 2\cos t \sin t)}}{\left(e^{-2t} + 4 + \sin^2 t\right)^{3/2}}.$$

Solution (Exercise 2). (a – 7 marks) The function is continuous, so the limit is evaluated by substituting $x = 1$ and $y = 1$ into the equation. This gives

$$\lim_{(x,y)\to(1,1)} \frac{x^3 + 2y^3}{x^3 + 4y^3} = \frac{1^3 + 2(1^3)}{1^3 + 4(1^3)} = \frac{3}{5}.$$

(b – 8 marks) The function is continuous, so the limit is evaluated by substituting $x = 0$ and $y = 0$ into the equation. But this gives an indeterminate form $0/0$.

So we use the method consisting of approaching the point $(0, 0)$ along a line $y = ax$, for $a \in \mathbb{R}$. This gives

$$\lim_{(x,y)\to(0,0)} \frac{x - y}{x + y} = \lim_{x\to 0} \frac{x - ax}{x + ax} = \frac{1 - a}{1 + a},$$

which depends on a. Therefore, the limit is different for each value of a, that is, every path approaching $(0, 0)$, and as a consequence, the limit does not exist.

Solution (Exercise 3). (a – 5 points) The domain $\mathcal{D}$ is such that $\ln(xy)$ is defined and $x^2 + y^3 \neq 0$ are defined. For $\ln(xy)$, we need $xy > 0$, that is, either $x, y > 0$ or $x, y < 0$. If $x, y > 0$, then $x^2 + y^3 \neq 0$ automatically. If $x, y < 0$, then we must further require that $x^2 + y^3 \neq 0$. Thus

$$\mathcal{D} = \mathbb{R}_+^{*2} \cup \{(x, y) \in \mathbb{R}_-^2 ; y^3 \neq -x^2\},$$

in which the notation $\mathbb{R}_+^*$ means $\mathbb{R}_+ \setminus \{0\}$. Note that $(0, 0)$ is excluded *de facto* from the set $\{(x, y) \in \mathbb{R}_-^2 ; y^3 \neq -x^2\}$, and thus we can use $\mathbb{R}_-$ instead of $\mathbb{R}_-^*$.

(b – 10 points) The equation of the tangent plane to the surface defined by the function $f(x, y)$ at the point $f(x_0, y_0)$ is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus we need f_x and f_y. We have

$$f_x(x, y) = \frac{x^2 + y^3 - 2\ln(xy)x^2}{x(x^2 + y^3)^2}$$

and

$$f_y(x, y) = \frac{x^2 + y^3 - 3\ln(xy)y^3}{y(x^2 + y^3)^2}.$$

Therefore, the equation of the tangent plane at (x_0, y_0) is given by

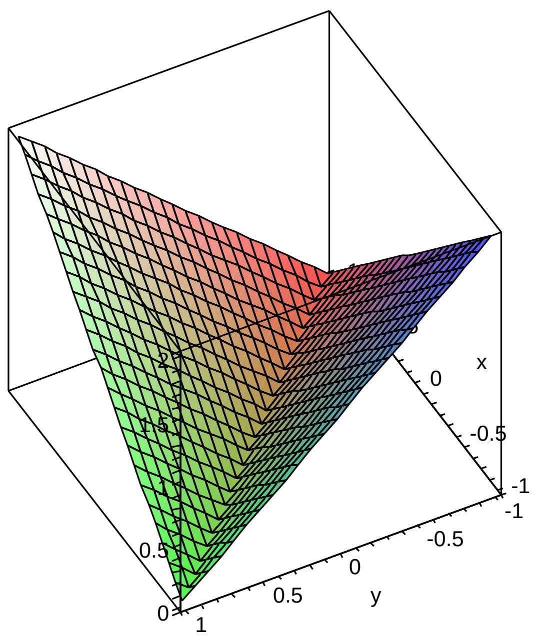
$$z - \frac{\ln(x_0 y_0)}{x_0^2 + y_0^3} = \frac{x_0^2 + y_0^3 - 2\ln(x_0 y_0)x_0^2}{x_0(x_0^2 + y_0^3)^2}(x - x_0) + \frac{x_0^2 + y_0^3 - 3\ln(x_0 y_0)y_0^3}{y_0(x_0^2 + y_0^3)^2}(y - y_0).$$

Solution (Exercise 4). (a – 6 marks) The partial derivatives are $f_x = 1/y$ and $f_y = -x/y^2$. Both of these functions are continuous at the point $(1, 1)$. By definition, f is thus differentiable at $(1, 1)$.

Be careful! As for functions of 1 variable, the following holds:

$$f(x, y) \text{ differentiable at } (x, y) \Rightarrow f(x, y) \text{ continuous at } (x, y),$$
$$\text{but}$$
$$f(x, y) \text{ continuous at } (x, y) \not\Rightarrow f(x, y) \text{ differentiable at } (x, y).$$

A classical example is the same as for functions of 1 variable: consider the function $g(x, y) = |x+y|$ (here, $|\cdot|$ represents the absolute value). The following figure shows a graph of g for $x, y \in [-1, 1]$; clearly, the line $x + y = 0$ poses a problem.



(b – 9 marks) We use the formula

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which here gives

$$f(x, y) \approx 1 + 1(x - 1) - 1(y - 1),$$

that is,

$$f(x, y) \approx x - y + 1.$$

(c – 0 marks) This was not on the program of the midterm, and thus was not mandatory. It was however very easy. We have seen that

$$D_u f(x, y) = \nabla f(x, y) \cdot u,$$

is maximal, at the point (x, y), in the direction of $\nabla f(x, y)$ and with magnitude (length) equal to $|\nabla f(x, y)|$.

We have

$$\nabla f = \left(\frac{1}{y}, -\frac{x}{y^2}\right).$$

Therefore the directional derivative of f is maximal, at the point $(1, 1)$, in the direction $(1, -1)$, with length $\sqrt{2}$.

Solution (Exercise 5). We use the formula

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y},$$

with the function $F(x, y)$ written as

$$F(x, y) = (x + y)^2 - 2x.$$

We have

$$F_x = 2(x + y) - 2$$

and

$$F_y = 2(x + y).$$

Therefore,

$$\frac{\partial y}{\partial x} = -\frac{2(x + y) - 2}{2(x + y)} = \frac{1}{x + y} - 1.$$

4.2 October 12, 2007

Solution (Exercise 1). Let $\mathcal{C}$ be a curve defined by the vector equation $r(t)$, for $t \in I$, where I is some interval in $\mathbb{R}$. $\mathcal{C}$ is smooth if r' continuous and $r'(t) \neq 0$ (the zero vector) for all $t \in I$, except maybe at the endpoints of I.

Solution (Exercise 2). Method 1 Let $x, y \in \mathbb{R}^n$. Then the Cauchy-Schwarz inequality states that

$$|x \cdot y| \leq \|x\| \, \|y\|.$$

We use $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$. This gives

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}.$$

Squaring both sides of the equation,

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right),$$

giving the desired result.
Method 2 Of course, this could also be done in "brute" force:

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 = a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2a_1 b_1 a_2 b_2 + 2a_1 b_1 a_3 b_3 + 2a_2 b_2 a_3 b_3$$

and

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2$$

The inequality

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

is equivalent to

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \geq 0. \tag{$\mathbb{I}$}$$

We substitute the values just found:

$$\begin{aligned}
(\mathbb{I}) \Leftrightarrow\ & a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2 \\
& - \left(a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2a_1 b_1 a_2 b_2 + 2a_1 b_1 a_3 b_3 + 2a_2 b_2 a_3 b_3\right) \geq 0 \\
\Leftrightarrow\ & a_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_3^2 + a_3^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_2^2 - (2a_1 b_2 a_2 b_1 + 2a_1 b_3 a_3 b_1 + 2a_2 b_3 a_3 b_2) \geq 0
\end{aligned}$$

(note how the terms have been reordered/rearranged in the previous line, to make the factorization more easy to see)

$$\Leftrightarrow (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 \geq 0.$$

Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, it follows that the last line is always true, and the result follows.

Method 3 This method uses in some sense the same ideas that was used to prove the Cauchy-Schwarz inequality, and hence does not require to remember it. Using the equality

$$x \cdot y = \|x\| \, \|y\| \cos\theta,$$

and squaring both sides, gives

$$(x \cdot y)^2 = \|x\|^2 \, \|y\|^2 \cos^2\theta.$$

Since for all θ, $\cos\theta \leq 1$, the latter equality implies that

$$(x \cdot y)^2 \leq \|x\|^2 \, \|y\|^2.$$

Using $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$ gives the result (using the fact that $\|x\|^2 = x \cdot x$ for all x).

Comment. There were also several other ways to prove this.

Solution (Exercise 3). a) We use the definition of a smooth curve that (hopefully) we gave earlier, and thus need to compute $r'(t)$. We have

$$r'(t) = \left(2t\cos t - t^2\sin t, 2t\sin t + t^2\cos t, 2\right).$$

Clearly, $r' \neq 0$ for all t (the third component is always 2), so $\mathcal{C}$ is smooth.

Comment. No need to look for complicated things here. Differentiate r and remark that the third component in r' is always nonzero. Many used $r'(0)$, which does not help: r' must be nonzero for all t, that it is nonzero at $t = 0$ does nothing for smoothness (in fact, since the curve is assumed to be defined for $t \geq 0$, r' could be zero for $t = 0$ and the curve would still be smooth, since $t = 0$ is an endpoint).

b) For the arc length, we need $\|r'\|$. We have

$$\begin{aligned}
\|r'\| &= \sqrt{(2t\cos t - t^2\sin t)^2 + (2t\sin t + t^2\cos t)^2 + 2^2} \\
&= \sqrt{4t^2(\sin^2 t + \cos^2 t) + t^4(\sin^2 t + \cos^2 t) + 4 - 4t^3\cos t\sin t + 4t^3\sin t\cos t} \\
&= \sqrt{t^4 + 4t^2 + 4} \\
&= \sqrt{(t^2+2)^2} \\
&= t^2 + 2.
\end{aligned}$$

The point $(-\pi^2, 0, 2\pi)$ corresponds to $t = \pi$. Therefore, the arc length is

$$L = \int_0^\pi \|r'(u)\| du = \int_0^\pi u^2 + 2 \; du = \frac{u^3}{3} + 2u\Big|_{u=0}^{u=\pi} = \frac{\pi^3}{3} + 2\pi.$$

Comment. You have to make it clear that you understand how to choose the integration bounds. It does not have to be long (see the sentence before the integral), but has to be there.

c) The curve intersects the plane $z = 4\pi$ when $t = 2\pi$. For $t = 2\pi$,

$$r'(2\pi) = \left(4\pi, 4\pi^2, 2\right).$$

The plane $z = 4\pi$ is horizontal, so a normal vector to this plane is vertical; for example, $n = (0, 0, 1)$. The angle θ of intersection then satisfies

$$\cos\theta = \frac{n \cdot r'(2\pi)}{\|n\| \, \|r'(2\pi)\|} = \frac{2}{\sqrt{16\pi^2 + 16\pi^4 + 4}} = \frac{1}{\sqrt{4\pi^2 + 4\pi^4 + 1}} = \frac{1}{2\pi^2 + 1}.$$

There was no need to try to evaluate that. (For information, we get $\theta \simeq 1.52$).

Comment. Here also, it has to be clear why you chose a given point. You could also evaluate the cosine in terms of t, then say that $t = 2\pi$. Many of you used $r(2\pi)$ instead of $r'(2\pi)$; remember, you want the angle of the curve with the plane, and the curve "goes" in the direction of r'.

d) We have computed r' and $\|r'\|$ already, we just need to put the two together:

$$T(t) = \left(\frac{2t\cos t - t^2 \sin t}{t^2 + 2}, \frac{2t\sin t + t^2 \cos t}{t^2 + 2}, \frac{2}{t^2 + 2}\right).$$

Comment. Do not forget to use parentheses or $\vec{i}, \vec{j}$ and $\vec{k}$ to indicate that you are dealing with vectors.

Solution (Exercise 4). The plane is as shown on Figure 4.1. We use $x = \rho\sin\phi\cos\theta$ and $z = \rho\cos\phi$ (knowing that $y = \rho\sin\phi\sin\theta$ can take any value). Then

$$\begin{aligned} x = z &\Leftrightarrow \rho\sin\phi\cos\theta = \rho\cos\phi \\ &\Leftrightarrow \sin\phi\cos\theta = \cos\phi \\ &\Leftrightarrow \tan\phi = \cos\frac{1}{\theta} \\ &\Leftrightarrow \phi = \arctan\left(\cos\frac{1}{\theta}\right) \end{aligned}$$

So the equation for the plane is

$$(\rho, \theta, \phi) = \left(\rho, \theta, \arctan\left(\cos\frac{1}{\theta}\right)\right)$$

for $\theta \in (0, 2\pi]$ and $\rho \in \mathbb{R}_+$. Note that we have to take θ between 0 (excluded) and 2π (included), because we are considering $\cos(1/\theta)$.

Comment. Nobody got full marks on this question. You must remember that if $x = z$ defines a plane, it is implicitly assumed that this is true for any value of y. Therefore, when computing $\rho^2 = \sqrt{x^2 + y^2 + z^2}$, y can take any value, and as a consequence, ρ takes any value.

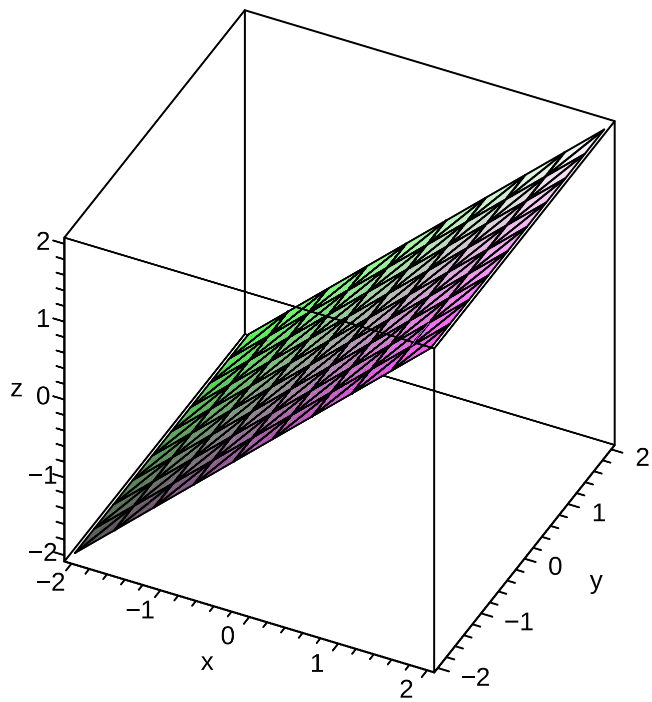


Figure 4.1: The plane $z = x$ in Exercise 4.

Solution (Exercise 5). Recall that $f(t)$ is continuous at $t = a$ iff $\lim_{t\to a} f(t) = f(a)$. Thus, we need to show that

$$\lim_{t\to 3}\left(2t-1, \frac{t}{3}\right) = (5,1) = f(3).$$

To compute this limit, we need to evaluate the limit of each component function.

We detail the first one. We want to check that for all $\varepsilon > 0$, there exists $\delta > 0$ (dependent on ε) such that $|2t-1-5| < \varepsilon$ whenever $0 < |t-3| < \delta$. Suppose $\varepsilon > 0$ is given, then

$$|2t-1-5| < \varepsilon \Leftrightarrow |2t-6| < \varepsilon \Leftrightarrow 2|t-3| < \varepsilon$$

Therefore, if, given $\varepsilon > 0$, we choose $\delta = \varepsilon/2$, we have that

$$|t-3| < \delta \Leftrightarrow |t-3| < \frac{\varepsilon}{2} \Leftrightarrow 2|t-3| < \varepsilon \Leftrightarrow |2t-6| < \varepsilon \Leftrightarrow |2t-1-5| < \varepsilon,$$

and thus $\lim_{t\to 3} 2t-1 = 5$.

For the second component function, we want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ (dependent on ε) such that $|\frac{t}{3} - 1| < \varepsilon$ whenever $0 < |t-3| < \delta$. Suppose $\varepsilon > 0$ is given, then

$$|\frac{t}{3} - 1| < \varepsilon \Leftrightarrow \frac{1}{3}|t-3| < \varepsilon$$

Therefore, if, given $\varepsilon > 0$, we choose $\delta = 3\varepsilon$, we have that

$$|t-3| < \delta \Leftrightarrow |t-3| < 3\varepsilon \Leftrightarrow \frac{1}{3}|t-3| < \varepsilon \Leftrightarrow |\frac{t}{3} - 1| < \varepsilon,$$

and thus $\lim_{t\to 3} \dfrac{t}{3} = 1$. Therefore, f is continuous at $t = 3$.

Comment. You **must** know the definition of continuity, which is quite straightforward: it simply states that the value of the limit is equal to the value of the function. We had treated this function as an example, in class, at $t = 2$ instead of $t = 3$ as it is done here.

4.3 November 7, 2007

1 (5 marks). State the **definition** of the directional derivative of a function $f(x, y, z)$.

Solution. Let f be a function from $\mathbb{R}^3 \to \mathbb{R}$, and u be a unit vector in $\mathbb{R}^3$. Then the directional derivative of f at the point $x \in \mathbb{R}^3$ in the direction of u is given by

$$D_u f(x) = \lim_{t\to 0} \frac{f(x+tu) - f(x)}{t},$$

when this limit exists.

Comment. Most of you did not give the definition of the directional derivative, but rather the characterization of $D_u f$ as $D_u f = \nabla f \cdot u$. Also, remember that u must be a unit vector (this allows to say that the extrema of the directional derivative is $\|\nabla f\|$).

2 (8 marks). Let $e^{xz^2} + y^3 - z^2 = 0$. Find $\partial x/\partial z$.

Solution. We have

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x}.$$

Now, $F_x = z^2 e^{xz^2}$ and $F_z = 2xze^{xz^2} - 2z = 2z(xe^{xz^2} - 1)$, so

$$\frac{\partial x}{\partial z} = -\frac{2z(xe^{xz^2} - 1)}{z^2 e^{xz^2}} = -\frac{2(xe^{xz^2} - 1)}{ze^{xz^2}}$$

3 (10 marks). Consider the function

$$F(x, y, z) = x^2 + 2y^2 + 3z^2.$$

Find the equation of the planes tangent to the surface $S = \{(x, y, z) \in \mathbb{R}^3; F(x, y, z) = 5\}$ at the points where $(x, y) = (0, 1)$.

Solution. We will use the formula $\nabla F(a) \cdot (\vec{x} - a) = 0$. We have

$$\nabla F(x, y, z) = (2x, 4y, 6z).$$

Now we need to find the points on the surface where the computation is to be done. We set $(x, y) = (0, 1)$, this gives

$$(0)^2 + 2(1)^2 + 3z^2 = 5 \Leftrightarrow z^2 = 1 \Leftrightarrow z = \pm 1.$$

So we need to find the tangent plane at $(0, 1, 1)$ and at $(0, 1, -1)$. At $(0, 1, 1)$, the equation of the tangent plane to S is given by

$$\nabla F(0, 1, 1) \cdot (\vec{x} - (0, 1, 1)) = 0 \Leftrightarrow 4(1)(y - 1) + 6(1)(z - 1) = 0 \Leftrightarrow 4y + 6z = 10.$$

At $(0,1,-1)$, the equation of the tangent plane to S is given by

$$\nabla F(0,1,-1)\cdot(\vec{x}-(0,1,-1))=0 \Leftrightarrow 4(1)(y-1)+6(-1)(z+1)=0 \Leftrightarrow 4y-6z=10.$$

Comment. The surface was defined using an implicit function, so the formula

$$z-f(x_0,y_0)=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

could not be used.

4 (5 marks). Find the global extrema of the function $f(x,y,z)=x-y+z$ in the set

$$S=\{(x,y,z):0<x^2+y^2+z^2\leq 1\}.$$

Solution. The set S is not a closed set, since the origin $(0,0,0)\notin S$ but the origin is a boundary point of S. Therefore the extreme value theorem cannot be applied. We have no way, *a priori*, to solve this problem.

Comment. Check that you can apply whatever result it is that you want to apply.

5 (10 marks). Let $f(x,y)=x^2+y^2+kxy$, for $k\in\mathbb{R}$. Find the critical points and extrema, if any, of f, as k varies.

Solution. We have

$$\nabla f=(2x+ky,2y+kx),$$

and therefore critical points, if any, are the solutions of

$$\begin{aligned}2x+ky&=0\\kx+2y&=0.\end{aligned}$$

The determinant of the matrix associated to this homogeneous linear system is $4-k^2$, so if $k\neq\pm 2$, the only critical point is $(0,0)$. If $k=2$, then $x=-y$ is solution, and if $k=-2$, $x=y$ is solution.

We have

$$f_x=2x+ky,\quad f_y=2y+kx,\quad f_{xx}=2,\quad f_{xy}=k,\quad f_{yx}=k,\quad f_{yy}=2,$$

and thus

$$D=\begin{vmatrix}f_{xx}&f_{xy}\\f_{yx}&f_{yy}\end{vmatrix}=\begin{vmatrix}2&k\\k&2\end{vmatrix}=4-k^2.$$

Therefore, $D>0$ for $k\in(-2,2)$, $D=0$ for $k=\pm 2$ and $D<0$ for $k\in(-\infty,-2)\cup(2,+\infty)$. From the second derivative test, in the first case, since $f_{xx}=2>0$, $(0,0)$ is a minimum. In the third case, $(0,0)$ is a saddle. The case $k=\pm 2$ is undetermined.

Comment. You were asked for critical points *and* extrema. This means you had to study the nature (minimum, maximum, saddle, undetermined) of $(0,0)$. The case of $k=\pm 2$ was more difficult, and you could get full marks without treating it.

6 (10 marks). Use the chain rule to find a formula for

$$\frac{d}{dt}\left(f(t)^{g(t)}\right).$$

Solution. Let $w = u^v$, with $u = f(t)$ and $v = g(t)$. We have, using the chain rule,

$$\begin{aligned}
\frac{d}{dt}w &= \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} \\
&= vu^{v-1}f'(t) + (\ln u)u^v g'(t) \\
&= u^v\left(\frac{v}{u}f'(t) + (\ln u)g'(t)\right) \\
&= f(t)^{g(t)}\left(\frac{g(t)}{f(t)}f'(t) + g'(t)\ln f(t)\right)
\end{aligned}$$

To find the partial derivative of w with respect to v, recall that for $c \in \mathbb{R}$, $c^x = \exp(x \ln c)$, and so

$$\frac{d}{dx}c^x = \ln c \exp(x \ln c).$$

7 (5 marks). Suppose that $f(x, y, z) = g(z)$, where $g : \mathbb{R} \to \mathbb{R}$, that is, the function f is independent of the variables x and y. Compute ∇f in terms of g'.

Solution. We have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

but here, $\partial f/\partial x = \partial f/\partial y = 0$, and $\partial f/\partial z = g'(t)$, so

$$\nabla f = (0, 0, g'(t)).$$

8 (9 marks). Find the points on the surface $z = (x-1)^2 + y^2 - 1$ that are closest to the origin.

Solution. Before going into berserk computation mode, look at the equation. We have

$$z = (x-1)^2 + y^2 - 1 = x^2 - 2x + 1 + y^2 - 1 = x(x-2) + y^2,$$

so, clearly, $(0, 0, 0)$ belongs to the surface, and therefore, the origin is the point that is closest to itself. Of course, it was also possible to use Lagrange multipliers or a direct approach, and provided you got the good

9 (8 marks). Find the length of the curve $(\sin 3t, \cos 3t, 2t^{3/2})$ between the points $(0, 1, 0)$ and $(0, -1, 2\pi^{3/2})$.

Solution. Let

$$r(t) = \left(\sin 3t, \cos 3t, 2t^{3/2}\right).$$

From the third entry in r, we must compute the length of the curve for t varying between 0 and π. We have
$$r'(t) = \left(3\cos 3t, -3\sin 3t, 3t^{1/2}\right)$$
and thus
$$\|r'(t)\| = \sqrt{3^2\cos^2 3t + 3^2\sin^2 3t + 9t} = \sqrt{9+9t} = 3\sqrt{1+t}.$$
As a consequence,
$$L = 3\int_0^{\pi} \sqrt{1+t}dt = 3\frac{2}{3}(1+t)^{3/2}\Big|_0^{\pi} = 2(1+\pi)^{3/2} - 2.$$

Comment. A lot of you forgot the -2 at the end.

10 (10 marks). Find values of a, b, c such that the function
$$f(t) = \begin{cases} (1-3t, 4\cos t, 4\sin t) & \text{if } t < 0 \\ (a + 3\sin t, b + 4t, c + 3\cos t) & \text{if } t \geq 0, \end{cases}$$
is a continuous function. Is f smooth? Where possible, compute the tangent vector $T(t)$ and the normal vector $N(t)$.

Solution. The curve is continuous if the limit when we approach $t = 0$ from the left is equal to the value of $f(0)$, which is given by the equation for $t \geq 0$. We have
$$\lim_{t\to 0^-} f(t) = (1, 4, 0)$$
and
$$f(0) = (a, b, c+3),$$
so the curve is continuous if $a = 1$, $b = 4$, and $c + 3 = 0$, i.e, $c = -3$. So we now consider the function
$$f(t) = \begin{cases} (1-3t, 4\cos t, 4\sin t) & \text{if } t < 0 \\ (1 + 3\sin t, 4 + 4t, -3 + 3\cos t) & \text{if } t \geq 0. \end{cases}$$
We have
$$f'(t) = \begin{cases} (-3, -4\sin t, 4\cos t) & \text{if } t < 0 \\ (3\cos t, 4, -3\sin t) & \text{if } t > 0. \end{cases}$$
Note that the derivative is not defined at $t = 0$, as the expressions for f' do not match at $t = 0$, since
$$\lim_{t\to 0^-} f'(t) = (-3, 0, 4)$$
and, using the expression for $t \geq 0$,
$$f(0) = (3, 4, 0).$$

As a consequence, f is not smooth, but is piecewise smooth. Then

$$\|f'(t)\| = \begin{cases} \sqrt{3^2+4^2} = 5 & \text{if } t < 0 \\ \sqrt{3^2+4^2} = 5 & \text{if } t > 0, \end{cases}$$

which implies that

$$T(t) = \begin{cases} \frac{1}{5}(-3, -4\sin t, 4\cos t) & \text{if } t < 0 \\ \frac{1}{5}(3\cos t, 4, -3\sin t) & \text{if } t > 0. \end{cases}$$

In turn,

$$T'(t) = \begin{cases} \frac{1}{5}(0, -4\cos t, -4\sin t) & \text{if } t < 0 \\ \frac{1}{5}(-3\sin t, 0, -3\cos t) & \text{if } t > 0, \end{cases}$$

so

$$\|T'(t)\| = \begin{cases} \frac{4}{5} & \text{if } t < 0 \\ \frac{3}{5} & \text{if } t > 0, \end{cases}$$

and thus finally,

$$N(t) = \begin{cases} (0, -\cos t, -\sin t) & \text{if } t < 0 \\ (-\sin t, 0, -\cos t) & \text{if } t > 0, \end{cases}$$

Comment. I was not stringent **this time** about the curve being piecewise-smooth rather than smooth, and the fact that the derivative, tangent vector and normal vector were not defined at $t = 0$. But be careful, I might not be that lenient next time..

Chapter 5

Past final examinations

Contents

Note that there is absolutely no guarantee that our examinations will in any way resemble the ones made available here.

5.1 December 8, 2003

Course	:	136.270
Examination	:	Final
Date	:	December 8, 2003
Duration	:	2 hours
Examiner	:	S. Kalajdzievski

1. Consider the curves $r_1(t) = (\cos t, \sin t, t)$ and $r_2(t) = (t+1, t^3+t^2, t^2)$.

a). (3 points). Find the intersection point of these two curves.

b). (3 points). Show that the tangent lines of these two curves at their intersecting point are perpendicular.

2. Evaluate the limit or show it does not exist.

a). (2 points). $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy^2+3x}{xy-1}$.

b). (3 points). $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy^2-x}{xy-1}$.

c). (3 points). $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

3. Problems a) and b) below are mutually independent.

a). (4 points). Suppose z is implicitly defined as a function of x and y by the equation $z^3x + zy - \sin(xy) = 0$. Evaluate $\frac{\partial z}{\partial y}$ at the point $(0,1,0)$.

b). (3 points). Use the Chain Rule to evaluate $\frac{\partial z}{\partial s}$ if $z = e^{x^2y}$, $x = st^2$ and $y = s^2 + \frac{1}{t}$.

4. (6 marks). Find the direction in which the function $g(x,y,z) = 3 - x^2z\cos(y^2z)$ increases the most rapidly at the point $(3,-2,0)$ and find the directional derivative in that direction.

5. (7 marks). Find all local extrema of the function $g(x,y) = x^3 + 5x^2 + 3y^2 - 6xy$. Then use the second derivative test to classify them.

6. (6 marks). Use Lagrange multipliers to find the absolute extrema of the function $f(x,y,z) = x$ subject to the constraint $x^2+y^2+z^2 = 1$.

7. (7 marks). Find the volume of the solid below the surface $z = \sqrt{x^3+1}$ and above the region D defined by $0 \le y \le 1$, $\sqrt{y} \le x \le 1$. (Hint: you may want to change the order of integration).

8. (7 marks). Evaluate

$$\iint_D (3y^2 - 4x)dA$$

where D is the closed region bounded by $y^2 = x + 6$ and $y = x$. Sketch D.

9. (6 marks). Evaluate $\int_C (x^2 + y^2 + z^2)ds$ where C is the curve defined by $\vec{r}(t) = (\cos \pi t, \sin \pi t, t)$, $0 \leq t \leq 6$.

5.2 April 23, 2004

Course	:	136.270
Examination	:	Final
Date	:	April 23, 2004
Duration	:	2 hours
Examiner	:	Y. Zhang

1. Let $f(x, y, z) = xy^2 + ye^{3z}$.

a). (2 points). Find the gradient of $f(x, y, z)$.

b). (4 points). Find the directional derivative of f at the point $(1, 1, 0)$ in the direction of $u = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

c). (2 points). In what direction does the function f increase most rapidly at the point $(1, 1, 0)$?

d). (4 points). Compute the line integral $\int_C \nabla f \cdot dr$, where C is the curve $r(t) = \sin(2t)\mathbf{i} + \tan t\mathbf{j} + \cos(2t)\mathbf{k}$, $0 \leq t \leq \pi/4$.

2.

a). (8 points). Find the mass of the thin wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq \pi$, if the density function $\rho(x, y, z) = x + y + z$.

b). (4 points). Find the length of the above wire. [**Answer:** $\sqrt{2}\pi$]

3. Consider the surface $x^2 + y^2 - z^2 - 2xy + 4xz = 4$.

a). (6 points). Find an equation of the tangent plane to the surface at the point $(1, 0, 1)$.

b). (4 points). Find an equation of the normal line to the surface at the point $(1, 0, 1)$.

4. (10 marks). Let $w = e^{x^2} + e^{xy}$, $x = e^{2s} \cos t$, $y = e^{-t} \sin s$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

5. (12 marks). Find the volume of the solid under the surface $z = 12(xy + y^3)$ and above the region bounded by $y = x^2$ and $x = y^2$.

6. (12 marks). Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

7. (10 marks). Compute the line integral

$$\int_C (y^2 - \tan^{-1} x)dx + (3x + \sin y)dy,$$

where C is the positive oriented boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.

8. (10 marks). Find and classify all the critical points of $f(x, y) = x^2 + y^2 + x^2y + 4$.

9. (12 marks). Find the absolute maximum and absolute minimum values of the function $f(x, y) = e^{-xy}$ on the region $x^2 + 4y^2 \leq 1$.

5.3 December 15, 2004

Course	:	136.270
Examination	:	Final
Date	:	December 15, 2004
Duration	:	2 hours
Examiner	:	S. Kalajdzievski

1. Consider the vector function $r(t) = (\sin t, \cos t, t)$.

a). (5 points). Find the tangent unit vector T at the point $(0, -1, \pi)$. [**Answer:** $(-1/\sqrt{2}, 0, 1/\sqrt{2})$]

b). (5 points). Find the curvature κ at the same point. [**Answer:** $1/2$]

c). (5 points). Find the length of the curve between the point $(0, 1, 0)$ and $(0, -1, \pi)$.

2.

a). (7 points). Find the equation of the tangent plane to the surface $f(x, y) = 4(x-1)^2+3(y+1)^2$ at the point $(2, 1, 16)$.

b). (8 points). Find the equation(s) of the horizontal tangent plane(s) to the surface $f(x, y) = 4(x - 1)^2 + 3(y + 1)^2$.

3.

a). (7 points). Use the chain rule to find $\frac{\partial w}{\partial r}$ if $w = e^{2x-y+2z^2}$, $x = r + s - t$, $y = 2r - 3s$, $z = \cos(rst)$.

b). (8 points). The equation $\ln(x^2 + y^2 + z^2) = xy^2z^3 + 1$ defines z as a function on x and y. Find $\frac{\partial z}{\partial x}$. Do NOT simplify your answer.

4.

a). (7 points). Find the directional derivative of the function $f(x, y) = x^3 + y^3$ at the point when $x = 3$ and $y = -3$, in the direction of the vector $u(3, 4)$.

b). (8 points). Find the direction from the point $P(3, -3)$ in which the function $f(x, y) = x^3+y^3$ increases the most rapidly and find the magnitude of the greatest rate of increase.

5.

a). (7 points). Find the critical points of the function $f(x, y) = 4x^2 + 3y^2 - 24x + 6y + 39$.

b). (8 points). Use the second derivative test to classify each critical point found in a) as local maximum, local minimum or a saddle point No mark will be given if the second derivative test is not used.

6. (15 marks). Use the method of Lagrange multipliers to maximize the function $f(x, y, z) = xyz$ subject to the condition $3x + 2y + z = 6$.

7. (15 marks). Find the volume of the solid below the plane $z = 2 - 3x - 5y$ and above the region bounded by $y = 0$, $y = x^2$ and the line segment from $(5, 0)$ to $(2, 4)$.

8. (The following two sub-questions are mutually independent.)

a). (7 points). Evaluate $\int_{\mathcal{C}} F \cdot dr$ where $F(x, y) = y^2x^3\mathbf{i} - x\sqrt{y}\mathbf{j}$ and $\mathcal{C}$ is given by the vector function $r(t) = -t^3\mathbf{i} - t^2\mathbf{j}$, $0 \leq t \leq 1$.

b). (8 points). Find f such that $\nabla f = F$, where $F(x, y) = (y - x^2)\mathbf{i} + (x + y^2)\mathbf{j}$. Then evaluate $\int_{\mathcal{C}} F \cdot dr$ where $\mathcal{C}$ is any smooth curve connecting the points $(0, 0)$ and $(1, 1)$.

5.4 April 26, 2005

Course	:	136.270
Examination	:	Final
Date	:	April 26, 2005
Duration	:	2 hours
Examiner	:	T. Holens

1.

a). (5 points). Show that $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ does not exist.

b). (5 points). Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2+2x^3y+y^2}{x^2+y^2}$ exists and find its value.

2. (8 marks). Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\,dxdy$.

3. Consider the curve given by the vector function $r(t) = \langle t, \ln(\sec t + \tan t), \ln(\sec t)\rangle$.

a). (6 points). Find the unit tangent vector T at the point on the curve where $t = \pi/6$. [**Answer:** $\sqrt{6}/4(1, 2\sqrt{3}/3, \sqrt{3}/3)$]

b). (4 points). Find the unit normal vector N at the point on the curve where $t = \pi/6$.

c). (4 points). Find the unit binormal vector B at the point on the curve where $t = \pi/6$.

d). (4 points). Find the curvature κ at the point on the curve where $t = \pi/6$.

4.

a). (6 points). Find an equation of the tangent plane to the ellipsoid $2x^2 + 3y^2 + z^2 = 25$ at the point $(3, 1, 2)$.

b). (4 points). Find a set of parametric equations for the normal line to the surface $2x^2 + 3y^2 + z^2 = 25$ at the point $(3, 1, 2)$.

5. A projectile is launched from the top of a hill that is 98m above the level ground below. The projectile is launched at an angle of 30° above the horizontal with an initial speed of 78.4 m/sec. If the point at ground level directly below where the projectile is launched is used as the origin, then the position vector $\mathbf{r}(t)$ of the projectile satisfies the differential equation $\mathbf{r}''(t) = -9.8\mathbf{j}\ m/sec^2$.

a). Find the equation giving the position vector $\mathbf{r}(t)$ of the projectile until it hits the ground. [**Answer:** $r(t) = (67.9t, 98 + 39.2t - 4.9t^2)$]

b). How long does it take until the projectile hits the ground? [**Answer:** 10 seconds]

c). What is the horizontal distance traveled by the projectile? [The horizontal distance is shown as x on the diagram above.] [**Answer:** $\simeq$ 679 metres]

6. (12 marks). Find the critical points of the function $f(x, y) = 2x^3 - 6xy + 3y^2$ and determine if the critical points are local maxima, local minima or saddle points.

7. (12 marks). Use the method of Lagrange multipliers to find the point on the circle $(x-3)^2 + (y-4)^2 = 1$ that is closest to the origin and the point on the circle that is farthest from the origin.

8. (8 marks). Find the volume of the solid region bounded below by the xy-plane and bounded above by the paraboloid $z = 16 - 4x^2 - 4y^2$.

9. (8 marks). Evaluate the line integral $\int_C (2x + y)ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$ from the point $(1, 0)$ to the point $(-1, 0)$.

10. (10 marks). Evaluate $\oint_C (y^2 + \sqrt{x + \sin x})dx + (2x + e^{y^2})dy$ where C is the circle $x^2 + y^2 = 25$.

5.5 December 14, 2005

Course	:	136.272
Examination	:	Final
Date	:	December 14, 2005
Duration	:	2 hours
Examiner	:	S. Kalajdzievski

1.

a). (7 points). Show that the angle between the tangent vector to the curve $r(t) = \left(t, \dfrac{t^2}{3}, \dfrac{2t^3}{27}\right)$ at any point and the vector $a = (1, 0, 1)$ is always a constant. Find that constant. [**Answer:** $\theta = \arccos(1/\sqrt{2})$]

b). (7 points). Find the curvature κ of $r(t) = \left(t, \dfrac{t^2}{3}, \dfrac{2t^3}{27}\right)$ at the point $(3, 3, 2)$.

2. Evaluate the following limits or show they do not exist:

a). (4 points). $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 + y^2 - 1}{x^2 - y^2 + 1}$.

b). (4 points). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 + x + y^2}$.

c). (6 points). $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\ln(x^2 + y^2 + 1)}{x^2 + y^2}$.

3.

a). (7 points). Find the equation of the tangent plane to the surface $f(x, y) = \sqrt{2xy}$ at the point $(2, 1, 2)$.

b). (7 points). Consider the level surface $2xy - z^2 = 0$ and the level surface $x^2 + y^2 + z^2 = 1$. Show that they are orthogonal at their points of intersection. (Recall that two surfaces are orthogonal if their normal lines are orthogonal.)

4.

a). (6 points). Use the chain rule to find $\frac{\partial z}{\partial u}$ at the point when $u = 0$ and $v = 1$ if $z = (\sin xy) + x \sin y$, $x = u^2 + v^2$ and $y = uv$.

b). (6 points). The equation $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0$ defines z as a function on x and y. Find $\frac{\partial z}{\partial x}$ at the point (π, π, π).

5. Consider the function $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$.

a). (4 points). Find the direction in which $f(x,y)$ increases most rapidly at $(1,1)$.

b). (4 points). Find the direction in which $f(x,y)$ decreases most rapidly at $(1,1)$.

c). (5 points). Find the directions at which the directional derivative of $f(x,y)$ at $(1,1)$ is 0.

6. (13 marks). Find the absolute minimum and the absolute maximum of the function $f(x,y) = 2+2x+2y-x^2-y^2$ over the triangular domain in the first quadrant bounded by the lines $x=0$, $y=0$ and $y=9-x$.

7. (12 marks). Use the method of Lagrange multipliers to find the extreme values of the function $f(x,y) = xy$ subject to the condition $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

8.

a). (7 points). Sketch the region of integration, reverse the order of integration, and evaluate $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$.

b). (7 points). Evaluate $\iint_R \frac{\sin x}{x} dx$ if R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

9. (12 marks). Sketch and find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

5.6 April 10, 2006

Course	:	136.272
Examination	:	Final
Date	:	April 10, 2006
Duration	:	2 hours
Examiner	:	S.H. Lui

1. (4 marks). Define the function

$$f(x,y) = \begin{cases} \sin(x-y)+3, & x > y \\ (x-y+\alpha)^2, & x \le y, \end{cases}$$

where α is a real number. How should α be defined so that f is a continuous function in $\mathbb{R}^2$?

2. (4 marks). Suppose a surface is defined by $x^2 + z = g(y,z)$ for some given function g. Find the equation of the tangent plane at the point (x_0, y_0, z_0) on the surface.

3. (8 marks). Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are positive. Parametrize this curve by θ, the angle that the position vector makes with the (positive) x-axis. Find $T(\theta)$ and $\kappa(\theta)$, the unit tangent vector and the curvature, respectively.

For 4 bonus marks, find $\int_0^L \kappa(s)ds$ where L is the length of the ellipse and κ is now parametrized by s, the arc length. Assume the following integral (with A, B positive):

$$\int_0^{2\pi} \frac{dx}{A\sin^2 x + B\cos^2 x} = \frac{2\pi}{\sqrt{AB}}$$

4. (10 marks). Find the maximum area of a triangle with vertices on a circle of radius one. Hint: Let a vertex of the triangle be at $(1,0)$. Suppose the other two vertices are at $(\cos\theta_j, \sin\theta_j)$, $j = 1,2$. Write down the expression of $A(\theta_1,\theta_2)$, the area of the triangle. (This is easiest done by first calculating a certain cross product.) Now find the absolute maximum of this function. Don't forget to write down the domain of A.

5. (7 marks). Suppose the function $f(u,v)$ satisfies $f_{uu} + f_{vv} = 0$ for all (u,v). Define $F(x,y) = f(x^2-y^2, 2xy)$. Find the value of $F_{xx} + F_{yy}$.

6. (5 marks). Evaluate $\iint_D \frac{\sin x}{x} dxdy$ where D is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

7. (6 marks). Write down the integral for the surface area of the cone $z = \sqrt{x^2+y^2}$ beneath the plane $z = 4$. Find this surface area.

8. (7 marks). Find the volume of the solid bounded by the surfaces $z = x^2+y^2$ and $18-z = x^2+y^2$. Draw a picture.

9. (7 marks). Show that the following line integral is independent of path and evaluate it

$$\int_C 2xydx + (x^2 + y)dy$$

with C as i) any path from $(0,1)$ to $(2,3)$, and ii) the path $\{(1+2\cos\theta, \sin\theta), 0 \leq \theta 2\pi\}$.

10. (6 marks). Evaluate the line integral

$$\oint_{\mathcal{C}} (\sin x + 3y^2)dx + (2x - e^{-y^2})dy,$$

where $\mathcal{C}$ is the boundary curve of the finite region bounded by the parabola $y = 1 - x^2$ and the line $y = 0$ with clockwise orientation.

5.7 December 9, 2006

Course	:	Math 2720
Examination	:	Final
Date	:	December 9, 2006
Duration	:	2 hours
Examiner	:	J. Arino

1. (20 marks). Consider the function

$$r(t) = \begin{cases} (a + \sin t, b + \cos t, t) & \text{if } t < 0, \\ (\cos t, \sin t, t) & \text{if } t \geq 0. \end{cases}$$

a). Find values of $a, b \in \mathbb{R}$ such that r is continuous on $\mathbb{R}$. For the remainder of this exercise, assume that a and b take the values found here.

b). Compute the length of the arc of $r(t)$ between the points $(1, -2, -\pi)$ and $(1, 0, 2\pi)$.

c). Is the function r differentiable at $t = 0$?

d). Is the curvature $\kappa(t)$ of $r(t)$ continuous at $t = 0$?

2. (15 marks). Consider the integral

$$I = \int_{-1}^{0} \int_{-\sqrt{1-y^2}}^{y+1} f(x, y)\, dxdy + \int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\, dxdy.$$

In this exercise, it is expected that precise expressions for the various regions will be given at each step.

a). Sketch the region of integration.

b). Reverse the order integration in I.

c). Express I using polar coordinates.

3. (10 marks). Find the critical points of the function $f(x, y) = x^2 - 3y^3 + 3xy$, and determine their nature.

4. (15 marks). Evaluate the line integral

$$\int_{\mathcal{C}} x + 2y + 3zds,$$

where $\mathcal{C}$ is the segment of the curve defined by

$$r(t) = \begin{cases} (1+\sin t, (\cos t) - 1, 2t) & \text{if } t < 0, \\ (\cos t, \sin t, t) & \text{if } t \geq 0 \end{cases}$$

which lies between the points $(1, -2, -2\pi)$ and $(1, 0, 2\pi)$.

5. (15 marks). Use Green's theorem to evaluate

$$\oint_{\mathcal{C}} F \cdot dr,$$

where $F(x, y) = (y^2 \cos x, x^2 + 2y \sin x)$ and $\mathcal{C}$ is the triangle with vertices $(0, 0)$, $(2, 6)$ and $(2, 0)$.

6. (15 marks). Show that the line integral

$$\int_{\mathcal{C}} (1 - ye^{-x})dx + e^{-x}dy$$

is independent of path, with $\mathcal{C}$ any path from $(0, 1)$ to $(1, 2)$, and evaluate the integral.

7. (10 marks). Consider the function

$$f(x, y) = \frac{1}{\cos(xy)}.$$

Determine the domain $\mathcal{D}$ and the range of f, sketch $\mathcal{D}$, and determine whether f is differentiable on $\mathcal{D}$.

5.8 December 13, 2007

Course	:	Math 2720
Examination	:	Final
Date	:	December 13, 2007
Duration	:	2 hours
Examiner	:	J. Arino

1. (35 marks). A spaceship S is on a trajectory with acceleration $a_S(t) = (-\cos t, -\sin t, 0)$, where the time t is measured in days.

a). At time $t = 0$, the spaceship has a velocity $v_S(0) = (0, 1, 1/4)$ and is located at $r_S(0) = (-2, 0, -3\pi/4)$. Show that the position of the spaceship at time t is given by

$$r_S(t) = \left(\cos t - 3, \sin t, \frac{t}{4} - \frac{3\pi}{4}\right).$$

b). The spaceship is heading toward the planet E, which has position at time t

$$r_E(t) = (4\cos t, \sin t, 0).$$

Compute the curvature $\kappa(t)$ of the trajectory of the planet at time t.

c). Will the spaceship "collide" with the planet, and if so, when and where?

d). If the planet and the spaceship "collide", then the *angle of collision* is given by the angle between the tangent vectors to both trajectories. If the angle of collision is greater than $|\pi/4|$, then the spaceship enters the atmosphere at too great an angle, and is pulverized. Does the spaceship make it safe to E? [Recall that $\cos\pi/4 = \sqrt{2}/2$.]

e). During its approach of the planet, the spaceship is subject to stellar winds emitted by the star around which E orbits. These winds are described by the vector field

$$F(x, y, z) = \left(yz^2 + 2xy, xz^2 + x^2, 2xyz\right).$$

Compute

$$\int_{\mathcal{C}} F \cdot dr,$$

where $\mathcal{C}$ is the curve representing the trajectory of the spaceship between the initial time $t = 0$ and the instant it lands on (or collides with) the planet. [Hint: Show that F is conservative.]

f). Find the length of $\mathcal{C}$, with $\mathcal{C}$ defined as in (e).

2. (25 marks). Consider the function

$$f(x, y, z) = xe^{yz} + x^2 z.$$

a). Find the gradient of f.

b). Find the directional derivative of f at a point (x_0, y_0, z_0) in the direction of $u = (u_1, u_2, u_3)$.

c). Find the direction in which f increases most rapidly at the point $(1, 1, 1)$.

d). Compute

$$\int_{\mathcal{C}} \nabla f \cdot dr,$$

if $\mathcal{C}$ is a curve from $(1, 0, 0)$ to $(0, 1, 0)$.

3. (15 marks). Evaluate the integral

$$\oint_{\mathcal{C}} \left(2x^3 + e^{-y^3}\right) dy - \left(\sqrt{x+1} + 2y^3\right) dx,$$

where $\mathcal{C}$ is the circle centrered at the origin and with radius 2.

4. (10 marks). Use the chain rule to find $\partial z/\partial v$ at the point where $(u, v) = (0, 1)$, if $z = (\sin xy) + y \sin x$, $x = u^3 + v^3$ and $y = uv$.

5. (5 marks). State the definition using (ε, δ) of the limit when $(x, y) \to (a, b)$ of the function $f(x, y)$.

6. (10 marks). Consider the function

$$f(x, y) = \sqrt{xy}.$$

Give the domain and range of f. Is f a C^1 function on its domain? Where appropriate, give the equation of the tangent plane to the surface $z = f(x, y)$.

7. (Bonus of 5 marks). Obtain conditions on the component functions $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ of the vector field $F = P\vec{i} + Q\vec{j} + R\vec{k}$ that guarantee that F is a conservative vector field. [Hint: compute $\mathsf{curl} F$.]

5.9 April 17, 2008

Course	:	Math 2720
Examination	:	Final
Date	:	April 17, 2008
Duration	:	2 hours
Examiner	:	F. Ghahramani

1. (20 marks). Let C be the curve given by the vector equation

$$\vec{r}(t) = \sqrt{2}\cos t\vec{i} + \sin t\vec{j} + \sin t\vec{k}.$$

a). Calculate the unit tangent vector $\vec{T(t)}$ and the unit normal vector $\vec{N(t)}$ at an arbitrary point.

b). Calculate the curvature $\kappa(t)$ for the above curve in terms of t at an arbitrary point.

2. (15 marks). For the function

$$f(x,y) = x^3y + x^2 - 4xy$$

determine all the critical points, and characterize at which points f has a local maximum, a local minimum or a saddle point.

3. (15 marks). Let $f(x,y) = x^2y + xy^2$.

a). Find $\nabla f(1,1)$. In which direction does f have the greatest rate of change at the point $(1,1)$? What is the value of the greatest rate of change at this point?

b). Let $\vec{u} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$. For the above function f find the directional derivative D_uf at the point $(1,1)$.

4. (10 marks). Find the shortest distance from the point $(1,1,0)$ to a point on the cone $z^2 = x^2+y^2$.

5. (10 marks). Let D be the closed triangular region in R^2 with vertices at the points $(0,0)$, $(0,1)$ and $(1,1)$. Evaluate $\iint_D \cos(y^2)dA$. [**Hint:** There are two ways for writing a double integral as a repeated (iterated) integral].

6. (10 marks).

a). Find the volume of the solid under the paraboloid $z = 4-(x^2+y^2)$, inside the cylinder $x^2+y^2 = 2$ and above the xy-plane [Sketching of this solid can help. Also, you may use polar coordinates in calculating the integrals].

b). Find the surface area of the part of the paraboloid $z = 4-(x^2+y^2)$ that lies above the xy-plane.

7. (20 marks).

a). Show that the vector field

$$F(x,y) = (3x^2y + 2xy^2 + y)\vec{i} + (x^3 + 2x^2y + x)\vec{j}$$

is conservative and find a potential function f for it, i.e., find an f satisfying $\nabla f(x,y) = F(x,y)$.

b). For the above vector field, evaluate

$$\int_C F(x,y)dr$$

where

$$C : \begin{cases} x = e^{\sin t} \\ y = e^{\cos t} \end{cases} \quad 0 \leq t \leq \frac{\pi}{2}.$$

8. (10 marks). Evaluate

$$\oint_C \left(3y - e^{\sin^2 x}\right) dx + \left(8x + \sqrt{y^4 + 2}\right) dy,$$

where C is the circle $x^2 + y^2 = 4$, oriented positively and transversed once. [**Hint:** There is a clever way of doing this question using a theorem of a mathematician whose name is a colour!]

Chapter 6

Solutions of selected final examinations

Contents

6.1 December 9, 2006

Solution (Exercise 1). **(a, 5 marks)** We want the two parts to connect to each other, and thus, we must have that

$$\lim_{t\to 0^-} r(t) = r(0),$$

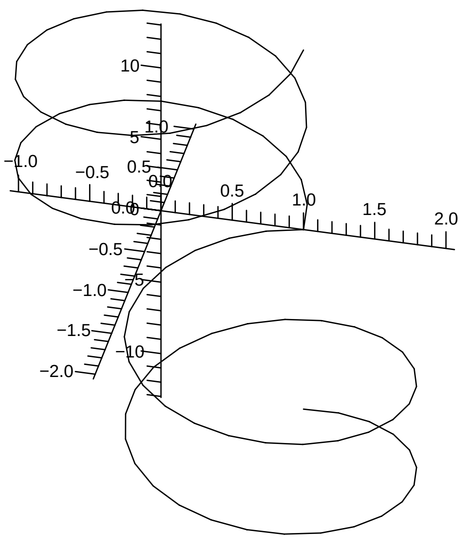
where $t \to 0^-$ means that t tends to 0 from the left, and where $r(0)$ is computed using the expression for $t \geq 0$. Thus, there must hold

$$(a, b+1, 0) = (1, 0, 0),$$

that is, $a = 1$ and $b = -1$. For the remainder of this exercise, we thus use the function

$$r(t) = \begin{cases} (1+\sin t, (\cos t) - 1, t) & \text{if } t < 0, \\ (\cos t, \sin t, t) & \text{if } t \geq 0. \end{cases}$$

We obtain a curve such as in the figure below.



(b, 5 marks) The length ℓ of the arc is given by

$$\begin{aligned}
\ell &= \int_{-\pi}^{2\pi} |r'(t)| dt \\
&= \int_{-\pi}^{0} |r'(t)| dt + \int_{0}^{2\pi} |r'(t)| dt \\
&= \int_{-\pi}^{0} \sqrt{\cos^2 t + \sin^2 t + 1}\; dt + \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\; dt \\
&= \int_{-\pi}^{0} \sqrt{2}\; dt + \int_{0}^{2\pi} \sqrt{2}\; dt \\
&= \pi\sqrt{2} + 2\pi\sqrt{2} \\
&= 3\pi\sqrt{2}
\end{aligned}$$

(c, 5 marks) For r to be differentiable at 0, its derivatives as 0 is approached from the left and from the right must be equal. Denote r_l' the former and r_r' the latter. We have

$$r_l'(t) = (-\cos t, \sin t, 1)$$

and

$$r_r'(t) = (\sin t, -\cos t, 1).$$

We have $\lim_{t\to 0^-} r_l'(t) = (-1, 0, 1)$, while $r_r'(0) = (0, -1, 1)$. Thus r is not differentiable at 0.

(d, 5 marks) We have

$$r'(t) = \begin{cases} (\cos t, -\sin t, 1) & \text{if } t < 0, \\ (-\sin t, \cos t, 1) & \text{if } t \geq 0 \end{cases}$$

and

$$r''(t) = \begin{cases} (-\sin t, -\cos t, 0) & \text{if } t < 0, \\ (-\cos t, -\sin t, 0) & \text{if } t \geq 0. \end{cases}$$

Therefore, for $t < 0$,

$$\begin{aligned} r' \times r'' &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} \\ &= (\cos t, \sin t, -\cos^2 t - \sin^2 t) \\ &= (\cos t, \sin t, -1), \end{aligned}$$

and thus

$$|r' \times r''| = \sqrt{\cos^2 t + \sin^2 + 1} = \sqrt{2}.$$

For $t \geq 0$,

$$\begin{aligned} r' \times r'' &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} \\ &= (\sin t, \cos t, \sin^2 t + \cos^2 t) \\ &= (\sin t, \cos t, 1), \end{aligned}$$

and thus

$$|r' \times r''| = \sqrt{\sin^2 t + \cos^2 + 1} = \sqrt{2}.$$

We have seen in **(b)** that $|r'(t)| = \sqrt{2}$ for all t. As a consequence,

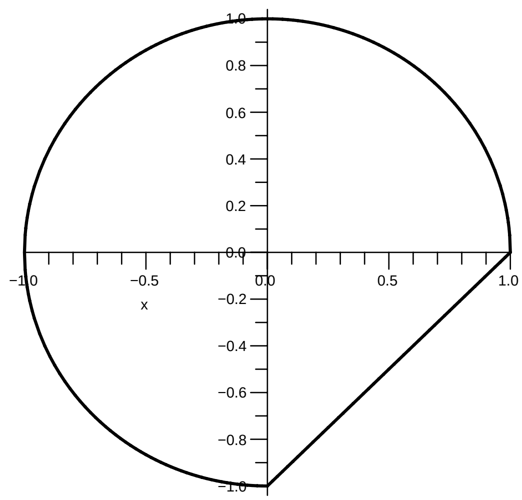
$$\kappa(t) = \frac{\sqrt{2}}{\left(\sqrt{2}\right)^3} = 2^{-1},$$

for all $t \in \mathbb{R}$. So the curvature is continuous on $\mathbb{R}$.

Solution (Exercise 2). **(a)** The region takes the form

$$\mathcal{D} = \{(x,y) : -1 \le y \le 0, -\sqrt{1-y^2} \le x \le y+1\}$$
$$\bigcup \{(x,y) : 0 \le y \le 1, -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\},$$

and is represented below.



(b) As a type I planar region, $\mathcal{D}$ is written

$$\mathcal{D} = \{(x,y) : -1 \le x \le 0, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$
$$\bigcup \{(x,y) : 0 \le x \le 1, x-1 \le y \le \sqrt{1-x^2}\},$$

and thus

$$I = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\, dydx + \int_{0}^{1} \int_{x-1}^{\sqrt{1-x^2}} f(x,y)\, dydx.$$

(c) In polar coordinates, $\mathcal{D}$ is expressed as

$$\mathcal{D} = \left\{(r,\theta) : 0 \le \theta \le \frac{3\pi}{2}, 0 \le r \le 1\right\}$$
$$\bigcup \left\{(r,\theta) : \frac{3\pi}{2} \le \theta \le 2\pi, 0 \le r \le \frac{1}{\cos\theta - \sin\theta}\right\}.$$

To get the polar equation of the segment from $(0,-1)$ to $(1,0)$, proceed as follows. For $x = r\cos\theta$ and $y = r\sin\theta$, a point is on the segment from $(0,-1)$ to $(1,0)$ if it satisfies $r\sin\theta = r\cos\theta - 1$ (the equation of the segment being $y = x-1$ in Cartesian coordinates). Therefore,

$$r = \frac{1}{\cos\theta - \sin\theta}.$$

Solution (Exercise 3). We have

$$f_x(x,y) = 2x + 3y, \quad f_y(x,y) = -9y^2 + 3x.$$

Therefore,

$$f_x = 0 \Leftrightarrow y = -\frac{2}{3}x,$$

which, when substituted into f_y gives

$$\begin{aligned} f_y = 0 &\Leftrightarrow -9\left(-\frac{2}{3}x\right)^2 + 3x = 0 \\ &\Leftrightarrow -4x^2 + 3x = 0 \\ &\Leftrightarrow (-4x+3)x = 0 \\ &\Leftrightarrow (x=0) \quad \text{or} \quad \left(x = \frac{3}{4}\right). \end{aligned}$$

Substituting these values into $f_x = 0$ gives the critical points $(0,0)$ and $(\frac{3}{4}, -\frac{1}{2})$. Now we evaluate

$$\begin{aligned} D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 3 & -18y \end{vmatrix} \\ &= -36y - 9 \\ &= -9(4y+1). \end{aligned}$$

Evaluating D at $(0,0)$ gives $D = -9$, so $(0,0)$ is a saddle. Evaluating D at $(\frac{3}{4}, -\frac{1}{2})$ gives $D = 9$; furthermore, $f_{xx}(\frac{3}{4}, -\frac{1}{2}) = 2$, so $(\frac{3}{4}, -\frac{1}{2})$ is a local minimum.

Solution (Exercise 4). We have

$$\begin{aligned} \int_{\mathcal{C}} x + 2y + 3z \, ds &= \int_{-\pi}^{2\pi} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\ &= \int_{-\pi}^{0} (1 + \sin t + 2(\cos t - 1) + 3(2t)) \sqrt{\cos^2 t + \sin^2 t + 4} \, dt \\ &\quad + \int_{0}^{2\pi} (\cos t + 2\sin t + 3t) \sqrt{\sin^2 t + \cos^2 t + 1} \, dt \\ &= \sqrt{5} \int_{-\pi}^{0} 6t - 1 + \sin t + 2\cos t \, dt \quad + \sqrt{2} \int_{0}^{2\pi} \cos t + 2\sin t + 3t \, dt \\ &= \sqrt{5} \left[3t^2 - t - \cos t + 2\sin t\right]_{-\pi}^{0} + \sqrt{2} \left[\sin t - 2\cos t + \frac{3}{2}t^2\right]_{0}^{2\pi} \\ &= \sqrt{5}(-1 - (3\pi^2 + \pi + 1)) + \sqrt{2}(-2 + 6\pi^2 + 2) \\ &= 6\pi^2\sqrt{2} - (3\pi^2 + \pi + 2)\sqrt{5}. \end{aligned}$$

Solution (Exercise 5). From Green's theorem,

$$\begin{aligned}
\oint_{\mathcal{C}} F \cdot dr &= \iint_{\mathcal{D}} \left(\frac{\partial}{\partial x}\left(x^2 + 2y\sin x\right) - \frac{\partial}{\partial y}\left(y^2 \cos x\right) \right) dA \\
&= \iint_{\mathcal{D}} 2x + 2y\cos x - 2y\cos x \; dA \\
&= \iint_{\mathcal{D}} 2x \; dA,
\end{aligned}$$

where $\mathcal{D}$ is the region inside the triangle with vertices $(0,0)$, $(2,6)$ and $(2,0)$. (As the orientation was not specified, assume for simplicity that it is positive –no marks were taken away if you assumed it was negative, provided the signs were correct–).

We need an equation for the line joining the vertices $(0,0)$ and $(2,6)$, in order to be able to write $\mathcal{D}$ as a type I planar region. Clearly, this is the line $y = 3x$, so

$$\mathcal{D} = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 3x\}$$

and

$$\begin{aligned}
\oint_{\mathcal{C}} F \cdot dr &= \int_0^2 \int_0^{3x} 2x \; dydx \\
&= \int_0^2 \left[2xy\right]_{y=0}^{3x} dx \\
&= \int_0^2 6x^2 dx \\
&= \left[2x^3\right]_0^2 \\
&= 16.
\end{aligned}$$

Solution (Exercise 6). The line integral is independent of path if we can find a potential function for the vector field $F(x,y) = (1 - ye^{-x}, e^{-x})$. Suppose $\nabla f = (f_x, f_y) = F$, then, integrating with respect to y,

$$f(x,y) = \int f_y dy = e^{-x}y + g(x),$$

where g does not depend on y. Differentiating with respect to x,

$$\frac{\partial}{\partial x}\left(e^{-x}y + g(x)\right) = -e^{-x}y + g'(x),$$

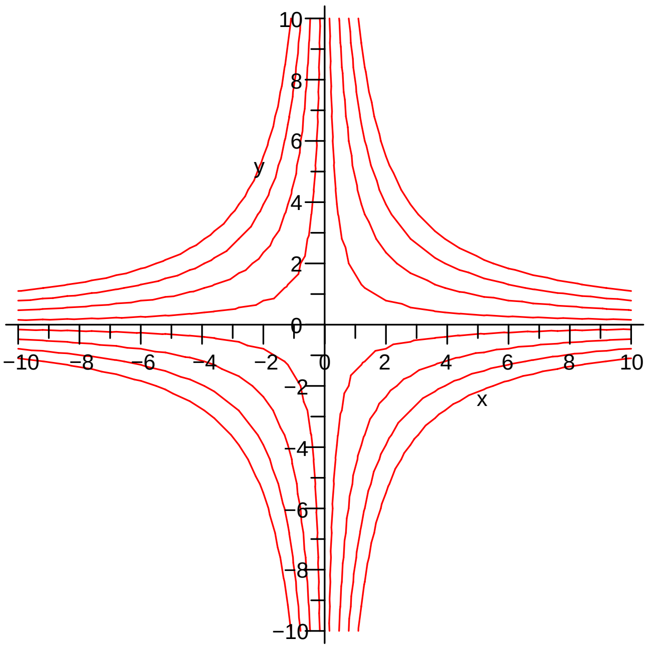
which is equal to $f_x = 1 - ye^{-x}$ if and only if $g'(x) = 1$. It follows, integrating g' with respect to x, that $g(x) = x + K$, where K is a constant that we can take equal to 0. As a consequence, the potential function is

$$f(x,y) = e^{-x}y + x,$$

and

$$\int_{\mathcal{C}} (1 - ye^{-x})dx + e^{-x}dy = f(1,2) - f(0,1) = 2e^{-1} + 1 - 1 = 2e^{-1}.$$

Solution (Exercise 7). We must have $\cos xy \neq 0$, that is, $xy \neq \dfrac{\pi}{2} + k\pi$, for $k \in \mathbb{Z}$. We get the following lines in the xy-plane at which $\cos xy$ is 0.



So $\mathcal{D}$ is the xy-plane from which these lines are removed, so

$$\mathcal{D} = \left\{ (x,y) : xy \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}.$$

The range of f is obtained by observing that $-1 \leq \cos xy \leq 1$, with $\cos xy \neq 0$, for all $x, y \in \mathcal{D}$. As a consequence,

$$\text{Range } (f) = (-\infty, -1] \cup [1, \infty).$$

To see this, consider for example $-1 \leq \cos xy < 0$, which has inverse f bounded above by $1/(-1) = -1$ and below by $-1/(0) = -\infty$. We have

$$f_x = \frac{\sin xy}{\cos^2 xy} y$$

and

$$f_y = \frac{\sin xy}{\cos^2 xy} x.$$

Both f_x and f_y are continuous for $x, y \in \mathcal{D}$, and as a consequence, f is differentiable on $\mathcal{D}$.

6.2 December 13, 2007

Solution (Exercise 1). (a, 5 marks) Recall that if $r(t)$ is the position of an object at time t, then $r'(t) = v(t)$ is its velocity and $r''(t) = v'(t) = a(t)$ is its acceleration, at time t. Here, we are given acceleration and thus must integrate twice to get $r(t)$.

We have

$$v_S(t) = \int a_S(t)dt = (-\sin t + K_1, \cos t + K_2, K_3),$$

and since $v_S(0) = (0, 1, 1/4)$ by assumption, there must hold that $v_S(0) = (K_1, 1 + K_2, K_3) = (0, 1, 1/4)$, that is, $K_1 = K_2 = 0$ and $K_3 = 1/4$. Thus the velocity is given by

$$v_S(t) = \left(-\sin t, \cos t, \frac{1}{4}\right).$$

Then

$$r_S(t) = \int v_S(t)dt = \left(\cos t + K_4, \sin t + K_5, \frac{t}{4} + K_6\right).$$

Therefore, using this expression, $r_S(0) = (1+K_4, K_5, K_6)$, which equals $(-2, 0, -3\pi/4)$ if and only if $K_4 = -3$, $K_5 = 0$ and $K_6 = -3\pi/4$. Therefore the position is given by

$$r_S(t) = \left(\cos t - 3, \sin t, \frac{t}{4} - \frac{3\pi}{4}\right).$$

(b, marks) We have $r'_E(t) = (-4\sin t, \cos t, 0)$, so

$$\begin{aligned}
\|r'_E\| &= \sqrt{16\sin^2 t + \cos^2 t} \\
&= \sqrt{(\cos^2 t + \sin^2 t) + 15\sin^2 t} \\
&= \sqrt{1 + 15\sin^2 t},
\end{aligned}$$

therefore,

$$T_E(t) = \frac{r'_E}{\|r'_E\|} = \left(\frac{-4\sin t}{\sqrt{1+15\sin^2 t}}, \frac{\cos t}{\sqrt{1+15\sin^2 t}}, 0\right).$$

Then

$$
\begin{aligned}
T_E'(t) &= \left(\frac{-4\cos t\sqrt{1+15\sin^2 t} + 4\sin t\frac{15\sin t\cos t}{\sqrt{1+15\sin^2 t}}}{1+15\sin^2 t}, \frac{-\sin t\sqrt{1+15\sin^2 t} + \cos t\frac{15\sin t\cos t}{\sqrt{1+15\sin^2 t}}}{1+15\sin^2 t}, 0 \right) \\
&= \left(\frac{(-4\cos t)(1+15\sin^2 t) + (4\sin t)(15\sin t\cos t)}{(1+15\sin^2 t)^{3/2}}, \frac{(-\sin t)(1+15\sin^2 t) + (\cos t)(15\sin t\cos t)}{(1+15\sin^2 t)^{3/2}}, 0 \right) \\
&= \left(\frac{-4\cos t - 60\cos t\sin^2 t + 60\cos t\sin^2 t}{(1+15\sin^2 t)^{3/2}}, \frac{-\sin t - 15\sin^3 t + 15\cos^2 t\sin t}{(1+15\sin^2 t)^{3/2}}, 0 \right) \\
&= \left(\frac{-4\cos t}{(1+15\sin^2 t)^{3/2}}, \frac{-\sin t - 15\sin t(\sin^2 t + \cos^2 t)}{(1+15\sin^2 t)^{3/2}}, 0 \right) \\
&= \left(\frac{-4\cos t}{(1+15\sin^2 t)^{3/2}}, \frac{-16\sin t}{(1+15\sin^2 t)^{3/2}}, 0 \right).
\end{aligned}
$$

The norm of T_E' is then given by

$$
\begin{aligned}
\|T_E'\| &= \sqrt{\left(\frac{-4\cos t}{(1+15\sin^2 t)^{3/2}} \right)^2 + \left(\frac{-16\sin t}{(1+15\sin^2 t)^{3/2}} \right)^2} \\
&= \frac{1}{(1+15\sin^2 t)^{3/2}} \sqrt{16\cos^2 t + 256\sin^2 t} \\
&= \frac{1}{(1+15\sin^2 t)^{3/2}} \sqrt{16(\cos^2 t + \sin^2 t) + 240\sin^2 t} \\
&= \frac{1}{(1+15\sin^2 t)^{3/2}} \sqrt{16(1+15\sin^2 t)} \\
&= \frac{4\sqrt{1+15\sin^2 t}}{(1+15\sin^2 t)^{3/2}} \\
&= \frac{4}{\sqrt{1+15\sin^2 t}}.
\end{aligned}
$$

Therefore, the curvature is

$$
\begin{aligned}
\kappa_E(t) &= \frac{\|T_E'\|}{\|r_E'\|} \\
&= \frac{4}{\sqrt{1+15\sin^2 t}} \frac{1}{\sqrt{1+15\sin^2 t}} \\
&= \frac{4}{1+15\sin^2 t}.
\end{aligned}
$$

Note: In practice, planetary orbits are usually much less eccentric than what is assumed here. The equation

$$
g(t) = (a\cos t, b\sin t),
$$

with $a \geq b$, describes an ellipse with semimajor axis a and semiminor axis b. For the orbit of a planet, a is called the *aphelion*, the point at which the planet is farthest from its sun, and b is the *perihelion*, the point at which the planet is closest to its sun. We assumed here a ratio of $a/b = 4$, whereas for earth, at aphelion (taking place on July 4$^{\text{th}}$), the earth is 152 million kilometres from the sun, and at perihelion (on January 3$^{\text{rd}}$), the earth is 147.5 million kilometres from the sun, giving a ratio of $152/147.5 \simeq 1.03$.

(c, marks) We want to solve

$$r_S(t) = r_E(s),$$

for s, t, that is

$$\begin{aligned} \cos t - 3 &= \cos s \\ \sin t &= \sin s \\ \frac{t}{4} - \frac{3\pi}{4} &= 0. \end{aligned}$$

From the third equation, it is clear that there is only one value of t, $t = 3\pi$. At $t = 3\pi$, $r_S(3\pi) = (-4, 0, 0)$, and any value of s such that $r_E(s) = (-4, 0, 0)$ can be taken, i.e., $s = \pi + 2k\pi$, for $k \in \mathbb{Z}$. So, at $t = 3\pi$, the spaceship S and the planet E collide at the point $(-4, 0, 0)$.

Note: You can think of s as the "time" of E, while t is the "time" of S. They do not have to coincide if no common frame of reference is chosen. The object under consideration here is S, so this is the timeframe with which we work.

(d, marks) We have

$$r_E'(3\pi) = (0, -1, 0)$$

and, since

$$r_S'(t) = \left(-\sin t, \cos t, \frac{1}{4}\right),$$

we have

$$r_S'(3\pi) = \left(0, -1, \frac{1}{4}\right).$$

The angle θ of collision then satisfies

$$\begin{aligned} \cos\theta &= \frac{r_S'(3\pi) \cdot r_E'(3\pi)}{\|r_S'(3\pi)\| \, \|r_E'(3\pi)\|} \\ &= \frac{(0, -1, 1/4) \cdot (0, -1, 0)}{\sqrt{1 + 1/16}} \\ &= \frac{1}{\sqrt{17/16}} \\ &= \frac{4}{\sqrt{17}}. \end{aligned}$$

Without doing any computation, we can already "feel" that this quantity is very close to 1. Indeed, $\sqrt{17}$ is just slightly larger than $\sqrt{16}$, which means that $4/\sqrt{17}$ is just smaller than $4/\sqrt{16} = 1$. If the cosine is close to 1, the vectors are almost parallel, and pointing in the same direction.

We confirm this intuition. We have

$$\frac{4}{\sqrt{17}} \geq \frac{\sqrt{2}}{2} \Leftrightarrow \frac{16}{17} \geq \frac{2}{4},$$

which is true, and therefore, $\theta \leq \pi/4$, and the spaceship is not pulverized upon entering the atmosphere of E.

(e, marks) As suggested, we check that the vector field is conservative. We have

$$\begin{aligned}
\mathsf{curl} F &= \nabla \times F \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (yz^2 + 2xy, xz^2 + x^2, 2xyz) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 + 2xy & xz^2 + x^2 & 2xyz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial z}(xz^2 + x^2)\right)\vec{i} - \left(\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial z}(yz^2 + 2xy)\right)\vec{j} \\
&\quad + \left(\frac{\partial}{\partial x}(xz^2 + x^2) - \frac{\partial}{\partial y}(yz^2 + 2xy)\right)\vec{k} \\
&= (2xz - (2xz))\,\vec{i} - (2yz - (2yz))\,\vec{j} + \left(z^2 + 2x - (z^2 + 2x)\right)\vec{k} \\
&= (0, 0, 0),
\end{aligned}$$

and thus F is conservative. So we now seek the potential function f such that $F = \nabla f = (f_x, f_y, f_z)$. Let us write $F = (F_1, F_2, F_3)$. Setting $f_x = F_1$, that is, $f_x = yz^2 + 2xy$, and integrating with respect to x,

$$f = xyz^2 + x^2y + H(y, z), \tag{6.1}$$

where $H(y, z)$ does not depend on x. Differentiating this expression with respect to y gives

$$f_y = xz^2 + x^2 + \frac{\partial H}{\partial y},$$

which must equal the term $F_2 = xz^2 + x^2$ in F, it follows that $\partial H/\partial y = 0$, which in turn implies that H does not depend on y, so we write it $H(y, z) = K(z)$. We thus can refine (6.1),

$$f = xyz^2 + x^2y + K(z).$$

Differentiating this expression with respect to z,

$$f_z = 2xyz + K'(z),$$

which must equal $F_3 = 2xyz$. It follows that $K'(z) = 0$, implying in turn that $K(z)$ is a constant, which we can take equal to 0. It follows that the potential function f of F takes the form

$$f = xyz^2 + x^2y.$$

We then have

$$\begin{aligned}\int_{\mathcal{C}} F\cdot dr &= f(r_S(3\pi)) - f(r_S(0))\\ &= f(-4,0,0) - f\left(-2,0,-\frac{3\pi}{4}\right)\\ &= 0.\end{aligned}$$

(f, marks) The arc length is given by

$$\begin{aligned}\int_0^{3\pi} \|r_S'(t)\| dt &= \int_0^{3\pi} \sqrt{\sin^2 t + \cos^2 t + \frac{1}{16}}\, dt\\ &= \int_0^{3\pi} \sqrt{\frac{17}{16}}\, dt\\ &= 3\pi\frac{\sqrt{17}}{4}.\end{aligned}$$

Solution (Exercise 2). **(a, marks)** We have

$$\nabla f = \left(e^{yz} + 2xz, xze^{yz}, xye^{yz} + x^2\right).$$

(b, marks) We have, for a unit vector $u = (u_1, u_2, u_3)$ and at the point (x_0, y_0, z_0),

$$\begin{aligned}D_u f(x_0,y_0,z_0) &= \nabla f(x_0,y_0,z_0)\cdot u\\ &= \left(e^{y_0z_0} + 2x_0z_0, x_0z_0e^{y_0z_0}, x_0y_0e^{y_0z_0} + x_0^2\right)\cdot(u_1,u_2,u_3)\\ &= \left(e^{y_0z_0} + 2x_0z_0\right)u_1 + \left(x_0z_0e^{y_0z_0}\right)u_2 + \left(x_0y_0e^{y_0z_0} + x_0^2\right)u_3.\end{aligned}$$

(c, marks) The direction of fastest increase is

$$\nabla f(1,1,1) = \left(e^1 + 2, 1e^1, 1e^1 + 1\right) = (2+e, e, 1+e).$$

(d, marks) By construction, a gradient vector field is conservative, and thus

$$\int_{\mathcal{C}} \nabla f\cdot dr = f(0,1,0) - f(1,0,0) = 0.$$

Solution (Exercise 3). We can write

$$\oint_{\mathcal{C}} \left(2x^3 + e^{-y^3}\right) dy - \left(\sqrt{x+1} + 2y^3\right) dx = \oint_{\mathcal{C}} P\ dx + Q\ dy,$$

with $P(x,y) = -\left(\sqrt{x+1} + 2y^3\right)$ and $Q(x,y) = 2x^3 + e^{-y^3}$. The domain $\mathcal{D}$ enclosed by $\mathcal{C}$ is simply connected, and $\mathcal{C}$ is a simple closed smooth curve, so we can apply Green's theorem, which gives

$$\oint_{\mathcal{C}} P\ dx + Q\ dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

We have

$$\frac{\partial Q}{\partial x} = 6x^2$$

and

$$\frac{\partial P}{\partial y} = -6y^2,$$

so

$$\begin{aligned}\oint_{\mathcal{C}} P\ dx + Q\ dy &= \iint_{\mathcal{D}} 6x^2 + 6y^2 dA \\ &= 6\iint_{\mathcal{D}} x^2 + y^2 dA \\ &= 6\iint_{\mathcal{D}} r^2 r\ dr d\theta,\end{aligned}$$

where $\mathcal{D}$ is the region

$$\{(r,\theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

in the polar plane. It follows that

$$\begin{aligned}\oint_{\mathcal{C}} P\ dx + Q\ dy &= \int_0^2 \int_0^{2\pi} r^3\ dr d\theta \\ &= \int_0^2 r^3\ dr \int_0^{2\pi} d\theta \\ &= \left.\frac{r^4}{4}\right|_0^2 (2\pi) \\ &= 8\pi.\end{aligned}$$

Solution (Exercise 4). From the chain rule, we have

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= (y\cos xy + y\cos x)(3v^2) + (x\cos xy + \sin x)u.\end{aligned}$$

At $(u,v) = (0,1)$, $x = 1$ and $y = 0$, so

$$\frac{\partial z}{\partial v} = (y\cos xy + y\cos x)(3v^2) + (x\cos xy + \sin x)u = (1+1)3 = 6.$$

Solution (Exercise 5). $\lim_{(x,y)\to(a,b)} f(x,y) = \ell$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - \ell| < \varepsilon$ whenever $0 < \|(x,y) - (a,b)\| < \delta$.

Solution (Exercise 6). The function f is defined for $xy \geq 0$, that is,

$$\mathcal{D} = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\} \cup \{(x,y) \in \mathbb{R}_- \times \mathbb{R}_-\}.$$

The range of f is $\mathbb{R}_+$. We have

$$\nabla f = \left(\frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}}\right),$$

and so f is C^1 on the set $\widetilde{\mathcal{D}}$ defined by

$$\widetilde{\mathcal{D}} = \{(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+^*\} \cup \{(x,y) \in \mathbb{R}_-^* \times \mathbb{R}_-^*\},$$

where $\mathbb{R}_+^*$ denotes $\mathbb{R}_+ \setminus \{0\}$, the positive real numbers, and $\mathbb{R}_-^*$ denotes $\mathbb{R}_- \setminus \{0\}$, the negative real numbers. Therefore, f is C^1 on $\widetilde{\mathcal{D}}$ but not on $\mathcal{D}$. On $\widetilde{\mathcal{D}}$, the equation of the plane tangent to the surface $z = xy$ at the point (x_0, y_0, x_0y_0) is given by

$$z - x_0y_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0),$$

that is,

$$z = x_0y_0 + \frac{y_0}{2\sqrt{x_0y_0}}(x - x_0) + \frac{x_0}{2\sqrt{x_0y_0}}(y - y_0),$$

or, collecting terms depending on x and y, and constant terms,

$$z = \frac{y_0}{2\sqrt{x_0y_0}}x + \frac{x_0}{2\sqrt{x_0y_0}}y + x_0y_0 - \frac{x_0y_0}{\sqrt{x_0y_0}}.$$

Solution (Exercise 7). We have

$$\begin{aligned}
\mathsf{curl} F &= \nabla \times F \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (P, Q, R) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).
\end{aligned}$$

Therefore, $\mathsf{curl} F = 0$, i.e., F is conservative, if and only if

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Chapter 7

Lecture slides

Please bear in mind that the likelihood that there are typos in these slides is high. So check the course website (`http://server.math.umanitoba.ca/~jarino/courses/math2720/`) from time to time, where a list of typos will be posted.

Reference to books in the slides

Material in these slides can be found in three books:

- Anton (MATH 1300 textbook)
- **MT**: Marsden & Tromba (our textbook)
- Stewart (textbook for & MATH 1500-1700-2730)

Topics are indicated only in reference to the Marsden & Tromba. For locations in the other books, see the Detailed Program posted online at
`http://server.math.umanitoba.ca/~jarino/courses/math2720`

Part I

Review and Introduction

Sets and elements

Definition (Set)

A **set** X is a collection of **elements**.

We write $x \in X$ or $x \notin X$ to indicate that the element x belongs to the set X or does not belong to the set X, respectively.

Definition (Subset)

Let X be a set. The set S is a **subset** of X, which is denoted $S \subset X$, if all its elements belong to X. S is a **proper subset** of X if it is a subset of X and not equal to X.

Quantifiers

A shorthand notation for "for all elements x belonging to X" is $\forall x \in X$. For example, if $X = \mathbb{R}$, the field of real numbers, then $\forall x \in \mathbb{R}$ means "for all real numbers x".

A shorthand notation for "there exists an element x in the set X" is $\exists x \in X$.

$\forall$ and $\exists$ are **quantifiers**.

Intersection and union of sets

Let X and Y be two sets.

Definition (Intersection)

The intersection of X and Y, $X \cap Y$, is the set of elements that belong to X **and** to Y,

$$X \cap Y = \{x : x \in X \textbf{ and } x \in Y\}$$

Definition (Union)

The union of X and Y, $X \cup Y$, is the set of elements that belong to X **or** to Y,

$$X \cup Y = \{x : x \in X \textbf{ or } x \in Y\}$$

A teeny bit of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. "The sky is blue" is also a proposition.

Let A be a proposition. We generally write

$$A$$

to mean that A is true, and

$$\textbf{not } A$$

to mean that A is false. **not** A is the **contraposition** of A (or **not** A is the contraposite of A).

A teeny bit of logic (cont.)

Let A, B be propositions. Then

- $A \Rightarrow B$ (read A implies B) means that whenever A is true, then so is B.
- $A \Leftrightarrow B$, also denoted A if and only if B (A iff B for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$. We also say that A and B are equivalent.

Let A and B be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\textbf{not } B \Rightarrow \textbf{not } A)$$

Necessary and/or sufficient conditions

Suppose we want to establish whether a given statement P is true, depending on the truth value of a statement H. Then we say that

- H is a **necessary condition** if $P \Rightarrow H$.
  (It is necessary that H be true for P to be true; so whenever P is true, so is H).

- H is a **sufficient condition** if $H \Rightarrow P$.
  (It suffices for H to be true for P to also be true).

- H is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., H and P are equivalent.

Playing with quantifiers

For the quantifiers $\forall$ (for all) and $\exists$ (there exists),

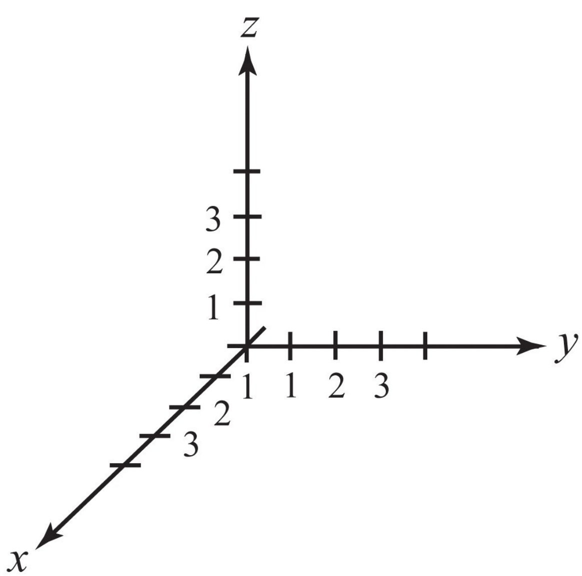
$$\exists \text{ is the contraposite of } \forall$$

Therefore, for example, the contraposite of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$



Cartesian coordinates.

Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A **vector** a is an ordered n-tuple of real numbers. For $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$$

is a vector. $a_1, \ldots, a_n$ are the **components** of a. Vectors are also denoted

$$\mathbf{a} = \langle a_1, \ldots, a_n \rangle$$

If unambiguous, a. Otherwise, $\mathbf{a}$ or $\vec{a}$.

General rule in this course

Know where your objects "live". If you do, then whether a is a scalar or a vector will be made clear by the context.

Vector space

Definition (Vector space)

A **vector space** over $\mathbb{R}$ is a set V together with two binary operations, **vector addition**, denoted $+$, and **scalar multiplication**, that satisfy the relations:

1. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2. $\forall \mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
3. $\exists \mathbf{0} \in V$, the zero vector, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
4. $\forall \mathbf{v} \in V$, there exists an element $\mathbf{w} \in V$, the additive inverse of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
5. $\forall \alpha \in \mathbb{R}$ and $\forall \mathbf{v}, \mathbf{w} \in V$, $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
6. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \mathbf{v} \in V$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \mathbf{v} \in V$, $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
8. $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$

Norms

Definition (Norm)

Let V be a vector space over $\mathbb{R}$, and $\mathbf{v} \in V$ be a vector. The **norm** of $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is a function from V to $\mathbb{R}_+$ that has the following properties:

1. For all $\mathbf{v} \in V$, $\|\mathbf{v}\| \geq 0$ with $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$
2. For all $\alpha \in \mathbb{R}$ and all $\mathbf{v} \in V$, $\|\alpha\mathbf{v}\| = |\alpha| \; \|\mathbf{v}\|$
3. For all $\mathbf{u}, \mathbf{v} \in V$, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Let V_n be a vector space (for example, $\mathbb{R}^2$ or $\mathbb{R}^3$).

The **zero element** (or **zero vector**) is the vector $\mathbf{0} = (0, \ldots, 0)$.

The **additive inverse** of $\mathbf{a} = (a_1, \ldots, a_n)$ is $-\mathbf{a} = (-a_1, \ldots, -a_n)$.

For $\mathbf{a} = (a_1, \ldots, a_n) \in V_n$, the length (or Euclidean norm) of $\mathbf{a}$ is the **scalar**

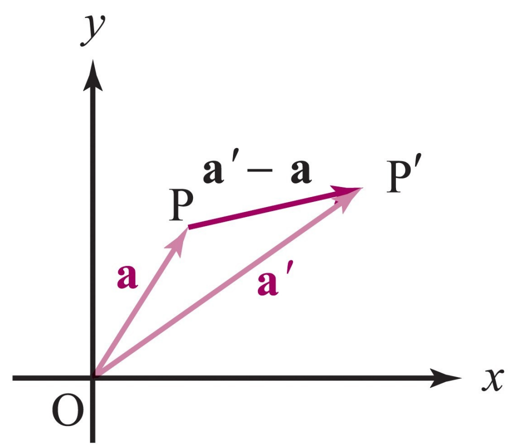
$$\|\mathbf{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$$

We also write $|\mathbf{a}|$.

To **normalize** the vector $\mathbf{a}$ consists in considering $\tilde{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\|$, i.e., the vector in the same direction as $\mathbf{a}$ that has unit length.

Let $P(x_1, y_1)$ and $P'(x_2, y_2)$. Then the vector $\overrightarrow{PP'}$ is

$$\overrightarrow{PP'} = (x_2 - x_1, y_2 - y_1)$$

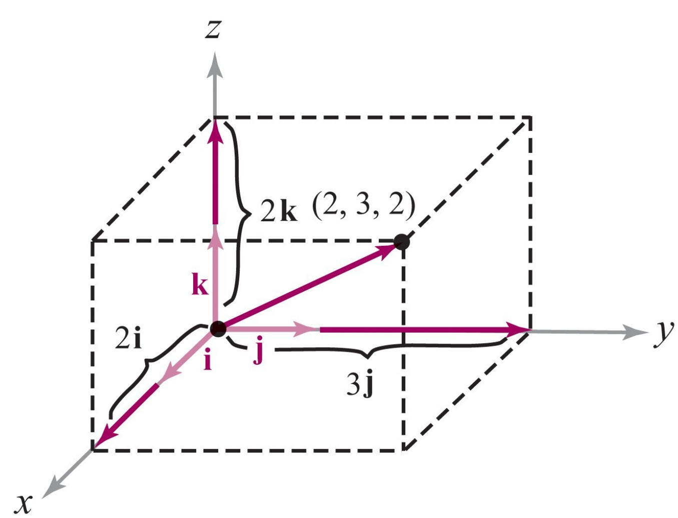


Note that a point with coordinates $P(x_1, y_1, z_1)$ can be interpreted as the vector $\overrightarrow{OP}$, from the origin to the point P.

Standard basis vectors

Vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are the **standard basis vectors** of $\mathbb{R}^3$. A vector $\mathbf{a} = (a_1, a_2, a_3)$ can then be written

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



For V_n ($\mathbb{R}^n$), the standard basis vectors are usually denoted $\mathbf{e}_1, \ldots, \mathbf{e}_n$, with

$$\mathbf{e}_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k+1})$$

Inner product

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n) \in V_n$, $\mathbf{b} = (b_1, \ldots, b_n) \in V_n$. The **inner product** (or **dot product**) of $\mathbf{a}$ and $\mathbf{b}$ is the **scalar**

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Properties of the inner product

Theorem

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_n$ *and* $\alpha \in \mathbb{R}$,

- $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$ *(so* $\mathbf{a} \bullet \mathbf{a} \geq 0$, *with* $\mathbf{a} \bullet \mathbf{a} = 0$ *iff* $\mathbf{a} = \mathbf{0}$*)*
- $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ *($\bullet$ is commutative)*
- $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ *($\bullet$ distributive over $+$)*
- $(\alpha \mathbf{a}) \bullet \mathbf{b} = \alpha(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (\alpha \mathbf{b})$
- $\mathbf{0} \bullet \mathbf{a} = 0$

Some results stemming from the inner product

Theorem

If θ is the angle between the vectors **a** *and* **b***, then*

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \; \|\mathbf{b}\| \; \cos\theta$$

Corollary (Cauchy-Schwarz inequality)

For any two vectors **a** *and* **b***, we have*

$$|\mathbf{a} \bullet \mathbf{b}| \leq \|\mathbf{a}\| \; \|\mathbf{b}\|$$

with equality if and only if **a** *is a scalar multiple of* **b***, or one of them is* **0***.*

Theorem

a *and* **b** *are orthogonal if and only if* $\mathbf{a} \bullet \mathbf{b} = 0$*.*

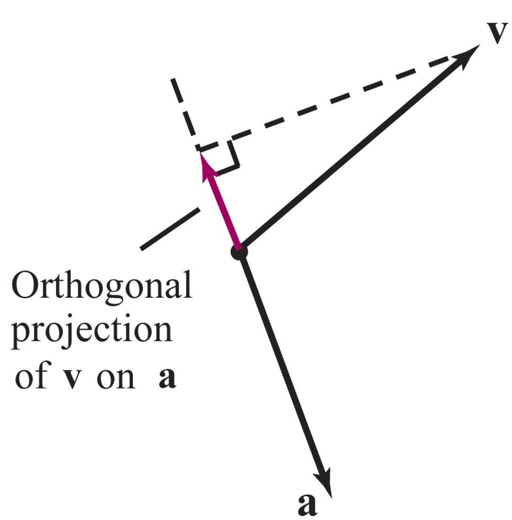
Scalar and vector projections

Scalar projection of **v** onto **a** (or component of **v** along **a**):

$$\mathrm{comp}_{\mathbf{a}}\mathbf{v} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|}$$

Vector (or orthogonal) projection of **v** onto **a**:

$$\mathrm{proj}_{\mathbf{a}}\mathbf{v} = \left(\frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|^2}\mathbf{a}$$



Cross product

Definition (Cross product)

Let $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in V_3$. Then the **cross product** of **a** and **b** is the **vector**

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

The cross product is defined **only** for 3-vectors.

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* **a** *and* **b**.

The orientation of $\mathbf{a} \times \mathbf{b}$ relative to **a** and **b**, i.e., the triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$, follows the right hand rule.

Thus, if taking $\mathbf{a} \times \mathbf{b}$ with standard basis vectors, we get

		b		
	$\times$	**i**	**j**	**k**
	i	0	**k**	$-\mathbf{j}$
a	**j**	$-\mathbf{k}$	0	**i**
	k	**j**	$-\mathbf{i}$	0

Theorem

If θ is the angle between $\mathbf{a}$ *and* $\mathbf{b}$*, with* $0 \leq \theta \leq \pi$*, then*

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \sin\theta$$

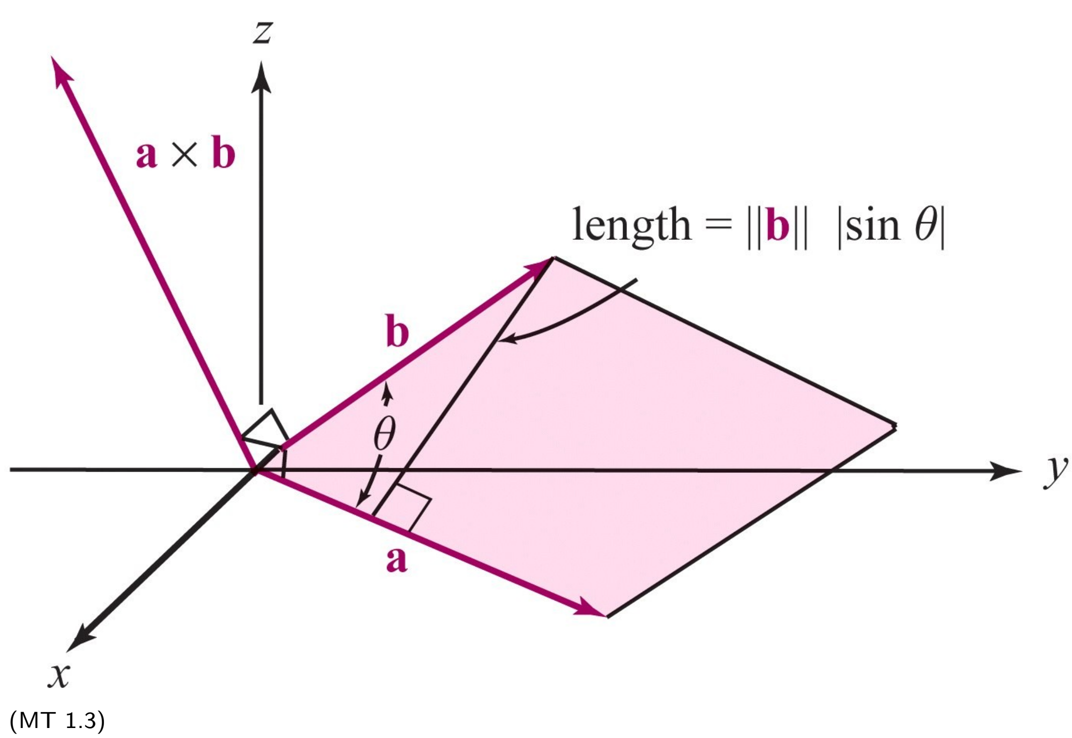
Theorem

Two nonzero vectors $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a} \times \mathbf{b} = 0$*.*

Geometric interpretation of the norm of the cross product

Theorem

The norm $\|\mathbf{a} \times \mathbf{b}\|$ *is equal to the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$*.*



Properties of the cross product

Theorem

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be 3-vectors,* $\alpha \in \mathbb{R}$. *Then*

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\alpha\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ *(scalar triple product)*
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$ *(vector triple product)*

Use of the scalar triple product

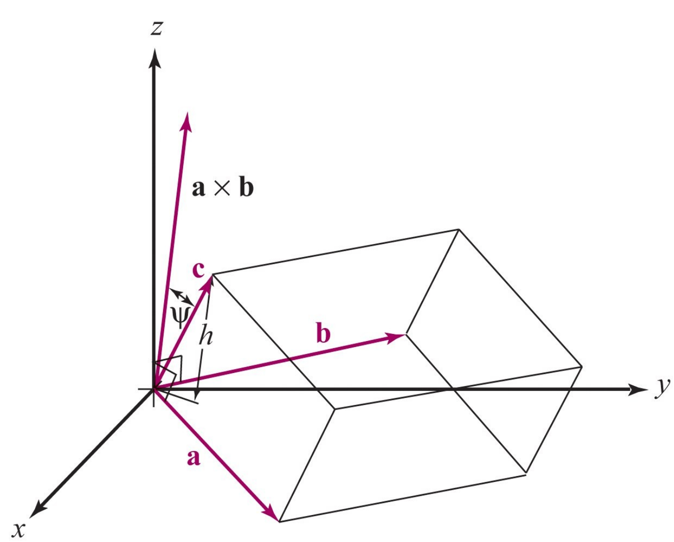
We have

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the absolute value of their scalar triple product,

$$V = |\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}|$$

"( )" not needed here, since the order of operations is important.



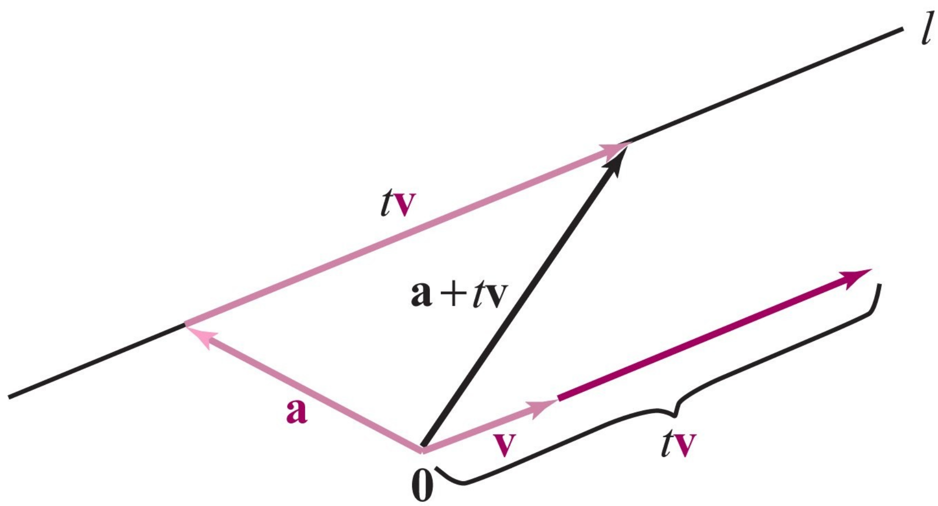
Vector equation of a line

Definition

Let $\mathbf{a}, \mathbf{v} \in V_n$. Then the **vector equation** of the line ℓ through the tip of $\mathbf{a}$ and in the direction of $\mathbf{v}$, is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{v},$$

with $t \in \mathbb{R}$.



Parametric equation of a line

Take for example the case $V_3 = \mathbb{R}^3$. If $\mathbf{a} = (x_0, y_0, z_0)$ and $\mathbf{v} = (v_1, v_2, v_3)$, we get

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3,$$

which is a **parametric equation** of ℓ.

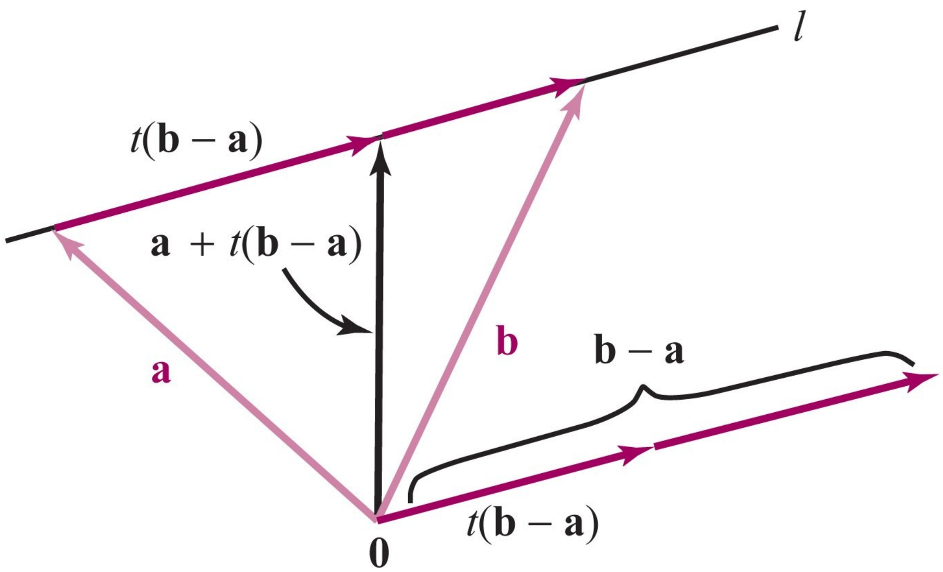
Equation of a line through two points

The equation for the line ℓ through the tips of vectors $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ is given in vector form by

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}), \quad t \in \mathbb{R}$$

or, in parametric form,

$$x = x_1 + (x_2 - x_1)t, \qquad y = y_1 + (y_2 - y_1)t, \qquad z = z_1 + (z_2 - z_1)t$$



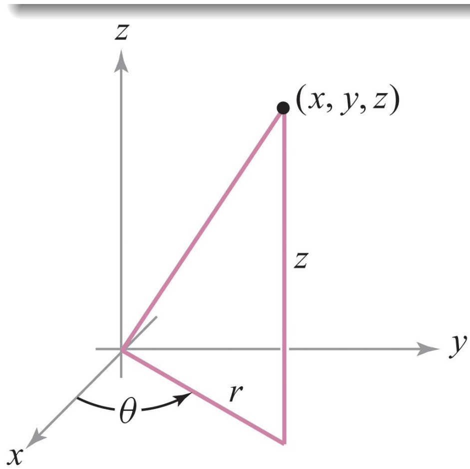
Cylindrical coordinates

Definition

The cylindrical coordinates (r, θ, z) of a point with cartesian coordinates (x, y, z) are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

with (usually) $r \geq 0$ and $0 \leq \theta \leq 2\pi$.



Given (x, y, z), to get (r, θ, z), set $r = \sqrt{x^2 + y^2}$, $\tan\theta = y/x$, i.e.,

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, y \geq 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \\ 2\pi + \tan^{-1}(y/x) & \text{if } x > 0, y \leq 0 \end{cases}$$

and $z = z$

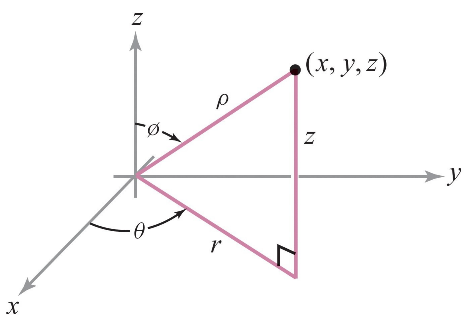
Spherical coordinates

Definition

The spherical coordinates (ρ, θ, ϕ) of a point with cartesian coordinates (x, y, z) are defined by

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi$$

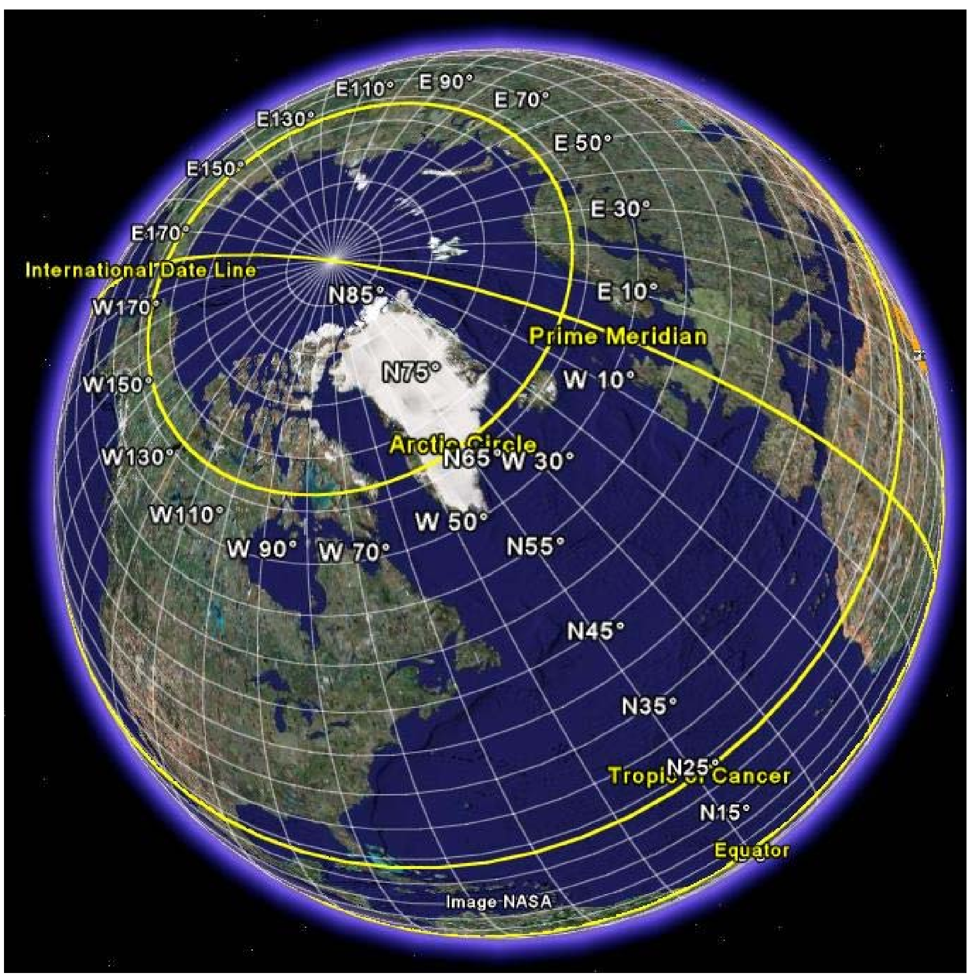
with $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

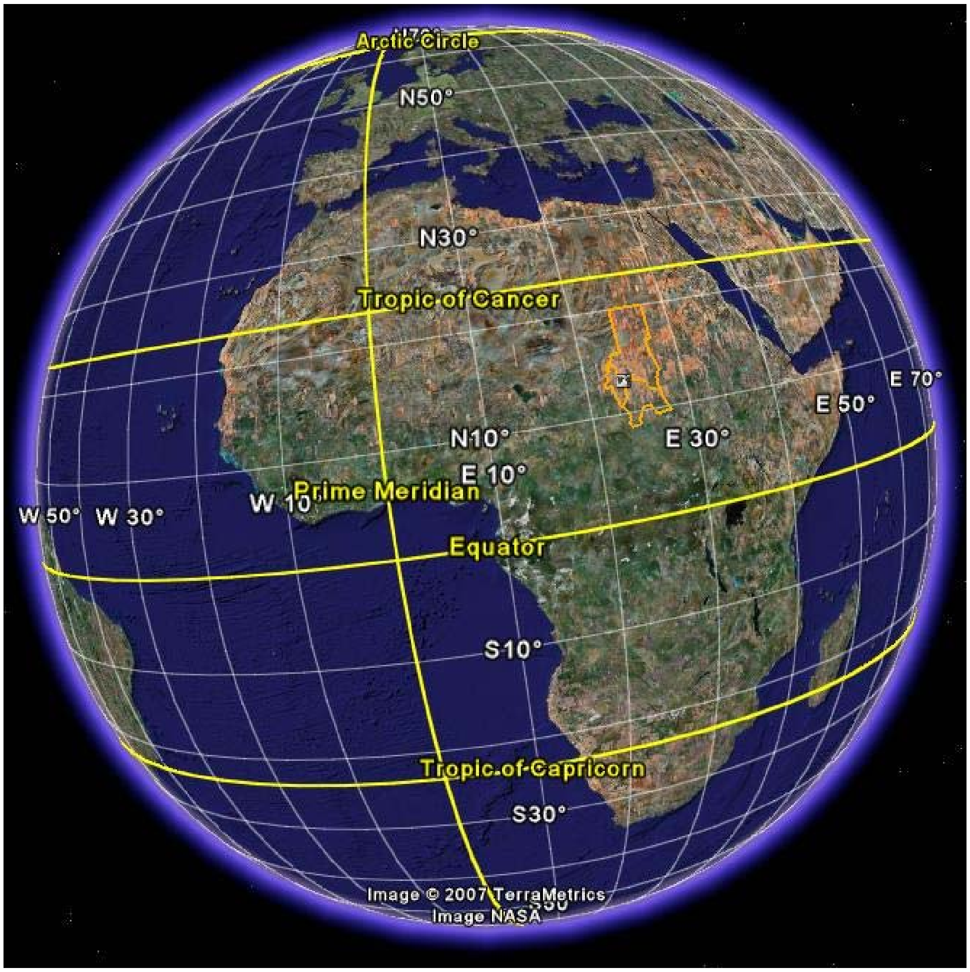


Given (x, y, z), to get (ρ, θ, ϕ), set $\rho = \sqrt{x^2 + y^2 + z^2}$, $\tan\theta = y/x$, i.e.,

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, y \geq 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \\ 2\pi + \tan^{-1}(y/x) & \text{if } x > 0, y \leq 0 \end{cases}$$

and $\phi = \cos^{-1}(z/\rho)$





So spherical coordinates are similar to latitude and longitude, with the following differences:

- **Latitude/Longitude:** Longitude is between 180° West and 180° East, that is, $[-\pi, \pi)$. Latitude is between 90° North and 90° South, that is, $[-\pi/2, \pi/2]$.
- **Spherical coordinates:** **Azimuth** is between 0 and 2π, and **zenith** is between 0 and π.

If, instead of using the previous convention for spherical coordinates, we used azimuth in $[-\pi, \pi)$ and zenith in $[-\pi/2, \pi/2]$ and with reference to the angle between the ray and the xy-plane (instead of the ray with the z axis, which is called co-latitude), the two systems would be equivalent.

Part II

Types of functions

Definitions

The different types of functions

Functions

Let A and B be two sets. A **function** $f : A \to B$ is a rule that assigns to each $a \in A$ a specific element $f(a)$ in B.

A is the **domain** of f, and the set

$$\{f(x) : x \in A\}$$

is the **range** of f.

Other names for functions are **maps** or **mappings**.

Notation

$$
\begin{aligned}
f : A &\to B \\
a &\mapsto f(a)
\end{aligned}
$$

f is a function/mapping from A to B, which maps an element $a \in A$ to an element $f(a) \in B$.

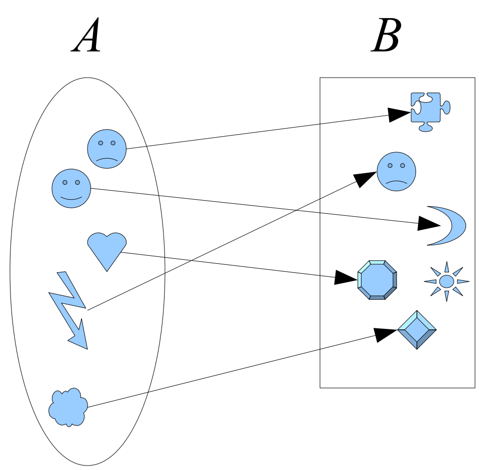
$$
\begin{aligned}
f : X \subset A &\to B \\
a &\mapsto f(a)
\end{aligned}
$$

f is a function/mapping from the subset X of A to B, which maps an element $a \in X$ to an element $f(a) \in B$.

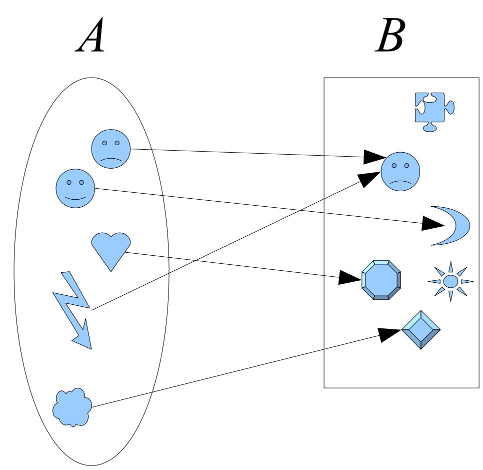
One-to-one functions

Definition (One-to-one function)

$f : A \to B$ is **one-to-one** (or **injective**, or an **injection**) if $\forall a, a' \in A,\ f(a) = f(a') \Rightarrow a = a'$.



one-to-one



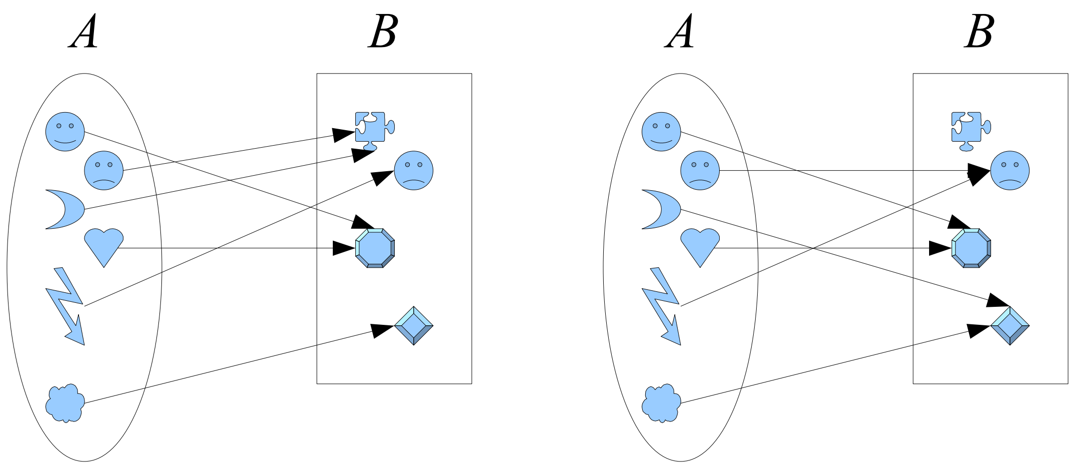
not one-to-one

Onto functions

Definition (Onto function)

$f : A \to B$ is **onto** (or **surjective**, or a **surjection**) if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

In other words, the range of f is B.



onto | not onto

Bijective functions

Definition (Bijection)

A function $f : A \to B$ that is both one-to-one and onto is called **bijective**, or a **bijection**.

If $f : A \to B$ is a bijection, then the inverse of f, denoted f^{-1}, and defined by

$$\begin{aligned} f^{-1} &: B \to A \\ & b \mapsto a \text{ such that } f(a) = b \end{aligned}$$

is a function (and a bijection). (This is not true if f is not a bijection. Consider for example $f(x) = \sin x$.)

Scalar real valued functions

The functions you have already studied:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) \end{aligned}$$

and subsets of the domain and range.

Examples: $f(x) = 2x$,

$$\begin{aligned} f : \mathbb{R} &\to [-1, 1] \\ x &\mapsto \cos x, \end{aligned}$$

$f(y) = e^y$,

$$g(z) = \frac{z^2 - 2z + 1}{z + 1}$$

etc.

Functions of several variables (MT 2.1)

In the most general sense, a function of several variables takes the form

$$\begin{aligned} f : \qquad \mathcal{A} \subset \mathbb{R}^n &\to \mathcal{B} \subset \mathbb{R}^m \\ (x_1, \ldots, x_n) &\mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \end{aligned}$$

or, for short, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f : \mathcal{A} \subset \mathbb{R}^n \to \mathcal{B} \subset \mathbb{R}^m$,

$$x \mapsto f(x)$$

Different types of functions of several variables

Vector function:

$$\begin{aligned} r : \quad & \mathbb{R} \to \mathbb{R}^n \\ & t \mapsto (r_1(t), \ldots, r_n(t)). \end{aligned}$$

(Real-valued) Function of several variables:

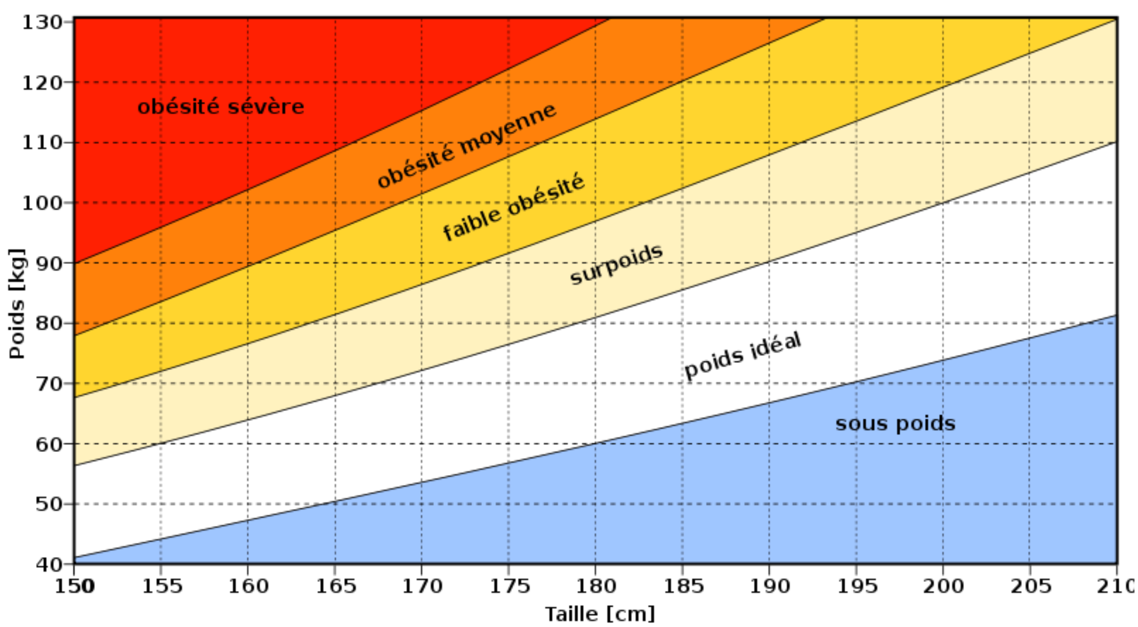
$$\begin{aligned} f : \quad & \mathbb{R}^n \to \mathbb{R} \\ & (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n). \end{aligned}$$

Vector field:

$$\begin{aligned} \mathbf{F} : \quad & \mathbb{R}^n \to \mathbb{R}^n \\ & (x_1, \ldots, x_n) \mapsto (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)). \end{aligned}$$

Example of $f : \mathbb{R}^2 \to \mathbb{R}$: the body mass index

If m is weight in kilos and h is height in metres, the body mass index $b(m, h) = \frac{m}{h^2}$



Part III

Differentiability

Graph of a function

Definition (Graph of a function)

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of several variables. The **graph** of f is the subset of $\mathbb{R}^{n+1}$ consisting of all the points

$$(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1},$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{U}$.

We can also write $(\mathbf{x}, f(\mathbf{x}))$.

Level curves and surfaces

Definition

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. The **level set** of c is defined by

$$\mathcal{L} = \{\mathbf{x} \in \mathcal{U};\ f(\mathbf{x}) = c\}\,.$$

Of course, $\mathcal{L} \subset \mathcal{U}$.

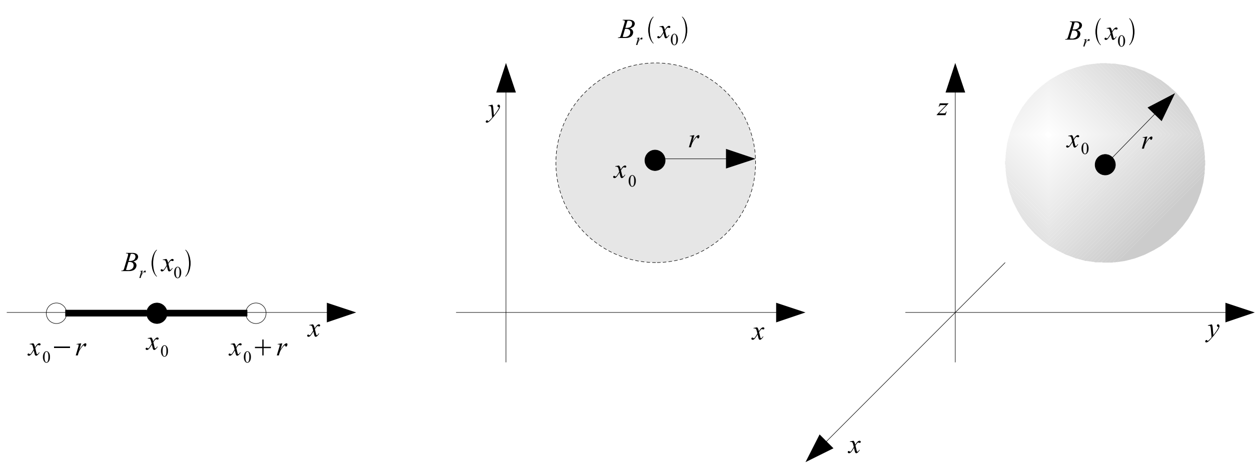
When $n = 2$ and $n = 3$, we speak of **level curves** and **level surfaces**, respectively.

Open balls

Definition (Open ball)

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $r > 0$. The **open ball** $\mathcal{B}_r(\mathbf{x}_0)$ of radius r and centre $\mathbf{x}_0$ is the set

$$\mathcal{B}_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n;\ \|\mathbf{x} - \mathbf{x}_0\| < r\}$$



Open sets

Definition (Open set)

The set $\mathcal{U} \subset \mathbb{R}^n$ is an **open set** if for every point $\mathbf{x}_0 \in \mathcal{U}$, there exists some $r > 0$ such that $\mathcal{B}_r(\mathbf{x}_0) \subset \mathcal{U}$.

By convention, the empty set $\emptyset$ is open.

Theorem

$\forall \mathbf{x}_0 \in \mathbb{R}$ and $\forall r > 0$, $\mathcal{B}_r(\mathbf{x}_0)$ is an open set.

Limit for $f : \mathbb{R} \to \mathbb{R}$

Definition

Let f be defined on some open interval containing x_0, except possibly at x_0. Then $\lim_{x \to x_0} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\text{if } 0 < |x - x_0| < \delta_\varepsilon \text{ then } |f(x) - L| < \varepsilon$$

In short:

$$\left(\lim_{x \to x_0} f(x) = L\right)$$
$$\Leftrightarrow (\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, 0 < |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon)$$

Notation δ_ε used to emphasize the fact that δ depends on the value of ε.

One-sided limits for $f : \mathbb{R} \to \mathbb{R}$

Definition (Left-hand limit)

$$\left(\lim_{x \to x_0^-} f(x) = L\right)$$
$$\Leftrightarrow (\forall \varepsilon > 0, \exists \delta_\varepsilon > 0,\ x_0 - \delta_\varepsilon < x < x_0 \Rightarrow |f(x) - L| < \varepsilon)$$

Definition (Right-hand limit)

$$\left(\lim_{x \to x_0^+} f(x) = L\right)$$
$$\Leftrightarrow (\forall \varepsilon > 0, \exists \delta_\varepsilon > 0,\ x_0 < x < x_0 + \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon)$$

Infinite limits for $f : \mathbb{R} \to \mathbb{R}$

Let f be defined on some open interval containing x_0, except possibly at x_0.

Definition

$$\left(\lim_{x \to x_0} f(x) = \infty\right)$$
$$\Leftrightarrow (\forall M > 0, \exists \delta_M > 0,\ 0 < |x - x_0| < \delta_M \Rightarrow f(x) > M)$$

Definition

$$\left(\lim_{x \to x_0} f(x) = -\infty\right)$$
$$\Leftrightarrow (\forall N < 0, \exists \delta_N > 0,\ 0 < |x - x_0| < \delta_N \Rightarrow f(x) < N)$$

Limit for $f : \mathbb{R}^n \to \mathbb{R}^m$

Definition

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with domain $\mathcal{U}$ containing points arbitrarily close to $\mathbf{x}_0 \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(x) = \mathbf{L}$$

if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$(\mathbf{x} \in \mathcal{U} \text{ and } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_\varepsilon) \Rightarrow (\|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon)$$

In other words,

$$\left(\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{L}\right)$$
$$\Leftrightarrow (\forall \varepsilon > 0, \exists \delta_\varepsilon > 0,\ 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_\varepsilon \Rightarrow \|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon)$$

Uniqueness

Theorem

If

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1 \ \textit{and}\ \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2$$

then $\mathbf{b}_1 = \mathbf{b}_2$.

Properties of limits

Theorem

Let $f, g : \mathcal{U} \in \mathbb{R}^n \to \mathbb{R}^m$, with $\mathcal{U}$ containing points arbitrarily close to $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^m$ and $c, c_1, c_2 \in \mathbb{R}$. Then

1. *If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} cf(\mathbf{x}) = c\mathbf{b}$.*
2. *If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (f+g)(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2$.*
3. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$. If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = c_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = c_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = c_1c_2$.*
4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$. If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = c \neq 0$ and $\forall \mathbf{x} \in \mathcal{U}$, $f(\mathbf{x}) \neq 0$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (1/f)(\mathbf{x}) = 1/c$.*
5. *If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $f_i : \mathcal{U} \to \mathbb{R}$ are the component functions of f, then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (b_1, \ldots, b_m)$ iff $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = b_i$ for $i = 1, \ldots, m$.*

Nonexistence criterion

Suppose that

- $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path $\mathbf{c}_1$,
- $f(x, y) \to L_2 \neq L_1$ as $(x, y) \to (a, b)$ along a path $\mathbf{c}_2 \neq \mathbf{c}_1$.

Then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

Continuity

Definition (Continuity at a point)

A function $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\mathbf{x}_0$ if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

If f is not continuous at $\mathbf{x}_0$, then f is **discontinuous** at $\mathbf{x}_0$.

Definition (Continuity)

f is continuous (on $\mathcal{U}$) if it is continuous at every point $\mathbf{x}_0 \in \mathcal{U}$. f is discontinuous if $\exists \mathbf{x}_0 \in \mathcal{U}$ where f is discontinuous.

Properties of continuous functions

Theorem

Let $f, g : \mathcal{U} \in \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x}_0 \in \mathcal{U}$ and $c \in \mathbb{R}$. Then

1. *If f continuous at $\mathbf{x}_0$, then cf is continuous at $\mathbf{x}_0$.*
2. *If f and g are continuous at $\mathbf{x}_0$, then $f + g$ is continuous at $\mathbf{x}_0$.*
3. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous at $\mathbf{x}_0$. Then fg is continuous at $\mathbf{x}_0$.*
4. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ and that $\forall \mathbf{x} \in \mathcal{U}$, $f(\mathbf{x}) \neq 0$. Then $1/f$ is continuous at $\mathbf{x}_0$.*
5. *If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $f_i : \mathcal{U} \to \mathbb{R}$ are the component functions of f, then f is continuous at $\mathbf{x}_0$ iff each of the real-valued component functions f_i is continuous at $\mathbf{x}_0$.*

Continuity of compositions

Theorem

Let $g : \mathcal{A} \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathcal{B} \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(A) \subset B$ so that $f \circ g = f(g(\mathbf{x}))$ is defined on $\mathcal{A}$. If g continuous at $\mathbf{x}_0 \in \mathcal{A}$ and f continuous at $\mathbf{y}_0 = g(\mathbf{x}_0)$, then $f \circ g$ continuous at $\mathbf{x}_0$.

Partial derivatives for a function of n variables

Definition (Partial derivatives)

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{U}$ open. The **partial derivative** of f with respect to the jth variable, $\partial f/\partial x_j$, is a function $\mathbb{R}^n \to \mathbb{R}$ defined, at the point $\mathbf{x} = (x_1, \ldots, x_n)$, by

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) &= \lim_{h \to 0} \frac{f(x_1, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} \end{aligned}$$

if the limit exists, where $\mathbf{e}_j$ is the jth standard basis vector. The domain of $\partial f/\partial x_j$ is the set of $\mathbf{x} \in \mathbb{R}^n$ for which this limit exists.

Notation

If $z = f(x,y)$, then

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Most frequently used are $\partial f/\partial x$ and f_x.

Partial derivatives for a function of 2 variables

Definition

Let $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$, $\mathcal{U}$ open. Then the partial derivatives of f with respect to x_1 and x_2 are defined, at the point (x_1, x_2), by

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = \lim_{h \to 0} \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}$$

and

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = \lim_{h \to 0} \frac{f(x_1, x_2 + h) - f(x_1, x_2)}{h}$$

respectively, if these limits exists. The domains of $\partial f/\partial x_1$ and $\partial f/\partial x_2$ are the sets $\mathbf{x} \in \mathbb{R}^2$ such that the limits exist.

It is possible that one partial would exist and not the other.

Differentiability

Definition (Differentiability)

Let $f : \mathbb{R}^2 \to \mathbb{R}$. We say f is **differentiable** at (x_0, y_0) if f_x and f_y exist at (x_0, y_0) and if

$$\frac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)}{\|(x,y)-(x_0,y_0)\|} \to 0$$

as $(x,y) \to (x_0,y_0)$.

Tangent plane to a surface

Definition (Tangent plane)

Let $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0). An equation of plane in $\mathbb{R}^3$ tangent to the graph of f at the point (x_0, y_0), called the **tangent plane** of the graph of f at (x_0, y_0), is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The linear function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

is the **linearization** of f at (x_0, y_0), and $f(x,y) \simeq L(x,y)$ is the **linear approximation** or **tangent plane approximation** of f at (x_0, y_0).

Differentiability – General case

Definition

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$, $\mathcal{U}$ open. f is **differentiable** at $\mathbf{x}_0 \in \mathcal{U}$ if all partial derivatives of f exist at $\mathbf{x}_0$ and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

where

$$\mathbf{D}f(\mathbf{x}_0) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ & & \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}$$

Condition of differentiability

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in \mathcal{U}$. Then f is continuous at $\mathbf{x}_0$.

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the partials $\partial f_i/\partial x_j$ of f all exist and are continuous in a neighbourhood of a point $\mathbf{x}_0 \in \mathcal{U}$. Then f is differentiable at $\mathbf{x}_0$.

Warning!!!

Never say "f is continuous at $\mathbf{x}_0$ so f differentiable at $\mathbf{x}_0$". This will result in an "auto-zero" (any question that contains this statement gets 0 automatically).

Terminology

Definition

A continuous function is C^0. A function with continuous partial derivatives is C^1. We sometimes write $f \in C^1$. Or $f \in C^1(A)$ to indicate that f is C^1 in a given set A.

So the differentiability condition can be reformulated as follows:

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that f is C^1 in a neighbourhood of a point $\mathbf{x}_0 \in \mathcal{U}$. Then f is differentiable at $\mathbf{x}_0$.

Paths and curves

Definition

A **path** in $\mathbb{R}^n$ is a map $\mathbf{c} : [a, b] \to \mathbb{R}^n$. The set of points

$$\mathcal{C} = \{\mathbf{c}(t),\ t \in [a, b]\}$$

obtained when t varies in $[a, b]$ is called a **curve** with **endpoints** $\mathbf{c}(a)$ and $\mathbf{c}(b)$. We say that the path **c parametrizes** the curve $\mathcal{C}$, or that $\mathbf{c}(t)$ **traces out** $\mathcal{C}$ as t varies.

We write

$$\mathbf{c}(t) = (x_1(t), \ldots, x_n(t)) = x_1(t)\mathbf{e}_1 + \cdots + x_n(t)\mathbf{e}_n,$$

with $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis vectors of $\mathbb{R}^n$ and $\mathbf{x}_i$ the **component functions**.

Limit and continuity of paths

Definition

Let $\mathbf{c}(t) = (x_1(t), \ldots, x_n(t))$. Then

$$\lim_{t \to t_0} \mathbf{c}(t) = \left(\lim_{t \to t_0} x_1(t), \ldots, \lim_{t \to t_0} x_n(t) \right)$$

provided the limits of the component functions exist.

Definition

The path $\mathbf{c}$ is continuous at the point a if

$$\lim_{t \to t_0} \mathbf{c}(t) = \mathbf{c}(t_0)$$

Velocity and speed

Definition

Let $\mathbf{c}$ be a differentiable path. The **velocity** of $\mathbf{c}$ at time t is

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h}$$

The **speed** of $\mathbf{c}(t)$ is $s = \|\mathbf{c}'(t)\|$.

For a path $\mathbf{c}(t) = (x(t), y(t))$ in $\mathbb{R}^2$, $\mathbf{c}'(t) = (x'(t), y'(t))$.

Remark

The velocity vector $\mathbf{c}'(t)$ is **tangent** to the path $\mathbf{c}(t)$ at time t. If $\mathcal{C}$ is a curve traced out by $\mathbf{c}$ and $\mathbf{c}'(t) \neq \mathbf{0}$, then $\mathbf{c}'(t)$ is a vector tangent to the curve $\mathcal{C}$ at the point $\mathbf{c}(t)$.

Tangent line to a path

Definition

Let $\mathbf{c}(t)$ be a path. Suppose that $\mathbf{c}'(t_0) \neq \mathbf{0}$, then the equation of the **tangent line** to $\mathbf{c}(t)$ at the point $\mathbf{c}(t_0)$ is

$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0)$$

If $\mathcal{C}$ is the curve traced out by $\mathbf{c}$, then the line traced out by ℓ is the tangent line to the curve $\mathcal{C}$ at $\mathbf{c}(t_0)$.

Constant multiple and sum of derivatives rules

Theorem

Constant multiple rule. *Let* $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ *be differentiable at* $\mathbf{x}_0$, $c \in \mathbb{R}$. *Then* $h(\mathbf{x}) = cf(\mathbf{x})$ *differentiable at* $\mathbf{x}_0$ *and*

$$\mathbf{D}h(\mathbf{x}_0) = c\mathbf{D}f(\mathbf{x}_0)$$

Sum rule. *Let* $f, g : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ *be differentiable at* $\mathbf{x}_0$. *Then* $h(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ *differentiable at* $\mathbf{x}_0$ *and*

$$\mathbf{D}h(\mathbf{x}_0) = \mathbf{D}f(\mathbf{x}_0) + \mathbf{D}g(\mathbf{x}_0)$$

Product and quotient of derivatives rules

Theorem

Product rule. *Let* $f, g : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* $\mathbf{x}_0$*. Then* $h(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ *is differentiable at* $\mathbf{x}_0$ *and*

$$\mathbf{D}h(\mathbf{x}_0) = g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) + f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)$$

Quotient rule. *Let* $f, g : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ *be differentiable at* $\mathbf{x}_0$*. Then, provided* $\forall \mathbf{x} \in \mathcal{U}$, $g(\mathbf{x}) \neq 0$, $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$ *is differentiable at* $\mathbf{x}_0$ *and*

$$\mathbf{D}h(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) - f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)}{[g(\mathbf{x}_0)]^2}$$

The Chain Rule – General version

Theorem

Let $\mathcal{U} \subset \mathbb{R}^n$ *and* $\mathcal{V} \subset \mathbb{R}^m$ *be open sets,* $g : \mathcal{U} \to \mathbb{R}^m$ *and* $f : \mathcal{V} \to \mathbb{R}^p$*. Assume that* g *maps* $\mathcal{U}$ *into* $\mathcal{V}$ *so that* $f \circ g$ *is defined. Suppose that* g *differentiable at* $\mathbf{x}_0$ *and* f *differentiable at* $\mathbf{y}_0 = g(\mathbf{x}_0)$*. Then* $f \circ g$ *differentiable and*

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0),$$

where the right hand side is the product of matrices $\mathbf{D}f(y_0)$ *and* $\mathbf{D}g(\mathbf{x}_0)$*.*

Chain rule – Special case 1

Theorem

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function of x, y and z, where $\mathbf{c}(t) = (x(t), y(t), z(t))$, $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$ is a differentiable path. Let $h(t) = f(\mathbf{c}(t))$. Then h is a differentiable function of t and

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Note that this can be written as

$$\nabla f(\mathbf{c}(t)) \bullet \mathbf{c}'(t) = \mathbf{D}f(\mathbf{c}(t))\mathbf{D}\mathbf{c}(t)$$

Chain rule – Special case 2

Theorem

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$. Write

$$g(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$$

and define $h : \mathbb{R}^3 \to \mathbb{R}$ by

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$$

Then

$$\begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

Gradient vector

Definition (Gradient vector)

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$. The **gradient vector** of f is the vector

$$\nabla f = \text{grad } f = (f_{x_1}, \ldots, f_{x_n}) = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Directional derivative

Definition (Directional derivative)

The **directional derivative** of $f : \mathbb{R}^3 \to \mathbb{R}$ at $\mathbf{x}$ along the vector $\mathbf{v}$ is

$$D_{\mathbf{v}} f(\mathbf{x}) = \frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\Big|_{t=0}$$

if this limit exists.

Usually, we use $\mathbf{v}$ such that $\|\mathbf{v}\| = 1$ and then speak of the **directional derivative of f in the direction $\mathbf{v}$**.

Can also define the directional derivative as

$$D_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

Theorem

If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then f all directional derivatives exist and are given by

$$D_{\mathbf{v}} f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \bullet \mathbf{v} = \left[\frac{\partial f}{\partial x}(\mathbf{x}_0)\right] v_1 + \left[\frac{\partial f}{\partial y}(\mathbf{x}_0)\right] v_2 + \left[\frac{\partial f}{\partial z}(\mathbf{x}_0)\right] v_3$$

where $\mathbf{v} = (v_1, v_2, v_3)$.

Directional derivative – Function of n variables

Definition

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$, $\mathbf{v} = (v_1, \ldots, v_n)$ be a unit vector and $\mathbf{x}_0 \in \mathcal{U}$. Then the directional derivative of f at $\mathbf{x}_0$ and in the direction of $\mathbf{v}$ is given by by

$$D_{\mathbf{v}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{v}) - f(\mathbf{x}_0)}{h}$$

Theorem

We have

$$D_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \bullet \mathbf{v}$$

The gradient points in the direction of fastest increase

Theorem

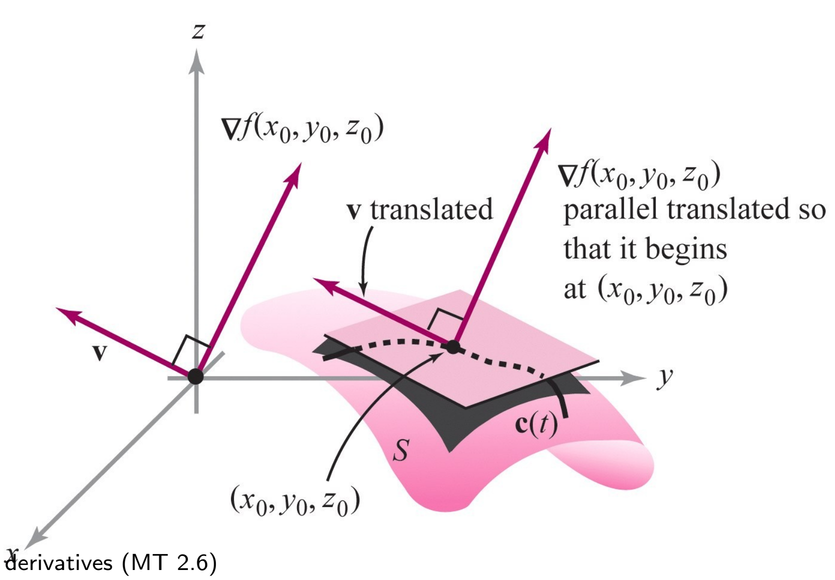
Assume that $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then $\nabla f(\mathbf{x})$ points in the direction along which f is increasing the fastest.

In other words, the maximum of the directional derivative $D_{\mathbf{v}} f(\mathbf{x}_0)$ occurs when $\mathbf{v} = k\nabla f(\mathbf{x}_0)$ for $k > 0$ such that $\|v\| = 1$, and equals $\|\nabla f(\mathbf{x}_0)\|$.

The gradient is normal to level surfaces

Theorem

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and $P(x_0, y_0, z_0)$ lie on the level surface $\mathcal{S}$ defined by $f(x, y, z) = k$, for $k \in \mathbb{R}$ a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: if $\mathbf{v}$ is the tangent vector at $t = 0$ of a path $\mathbf{c}(t)$ in $\mathcal{S}$ with $\mathbf{c}(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \bullet \mathbf{v} = 0$.



Tangent plane to a level surface

Theorem

Let $\mathcal{S}$ be the surface defined as the level set $f(x,y,z) = k$, for $k \in \mathbb{R}$, and (x_0, y_0, z_0) be a point on $\mathcal{S}$. Then the **tangent plane to the level surface** *$\mathcal{S}$ at the point (x_0, y_0, z_0) has equation*

$$\nabla f(x_0, y_0, z_0) \bullet (x - x_0, y - y_0, z - z_0) = 0$$

if $\nabla f(x_0, y_0, z_0) \neq 0$.

Part IV

Higher-order derivatives & extrema

Second-order partial derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$. The functions

$$\frac{\partial f}{\partial x_i}$$

are also $\mathbb{R}^n \to \mathbb{R}$, and so we can consider, for example,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

This is a **second-order** partial derivative or an **iterated** partial derivative. We write

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

and

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_i^2}$$

Order is important .. sometimes

Let $f(x, y)$ be a function $\mathbb{R}^2 \to \mathbb{R}$. Then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

is "partial derivative wrt x of partial derivative wrt y of f". It could be different from $\partial f/(\partial y \partial x)$, except when the following holds.

Theorem (Clairaut's theorem – Equality of mixed partials)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2, i.e., twice continuously differentiable (second-order partials are continuous). Then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Clairaut's theorem

Theorem (Clairaut)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^2. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for $1 \le i, j \le n$.

Local extrema

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$.

Definition (Local minimum)

f has a **local minimum** at $\mathbf{x}_0 \in \mathcal{U}$ if $f(\mathbf{x}) \ge f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$. The number $f(\mathbf{x}_0)$ is the **local minimum value** of f.

Definition (Local maximum)

f has a **local maximum** at $\mathbf{x}_0 \in \mathcal{U}$ if $f(\mathbf{x}) \le f(\mathbf{x}_0)$ for all $\mathbf{x}$ near $\mathbf{x}_0$. The number $f(\mathbf{x}_0)$ is the **local maximum value** of f.

Definition (Extremum)

A point $\mathbf{x}_0$ that is a local maximum or a local minimum of f is called a **local extremum point** of f.

Critical points and First derivative test

Definition (Critical point)

A point $\mathbf{x}_0$ is a **critical point** of f if either f is not differentiable at $\mathbf{x}_0$ or f is differentiable at $\mathbf{x}_0$ and $\mathbf{D}f(\mathbf{x}_0) = \mathbf{0}$.

Since $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{D}f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)$ (the gradient).

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, $\mathcal{U}$ open, and $\mathbf{x}_0 \in \mathcal{U}$. If $\mathbf{x}_0$ is a local extremum of f, then $\nabla f(\mathbf{x}_0) = \mathbf{0}$, i.e., $\mathbf{x}_0$ is a critical point of f.

Quadratic function

A quadratic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ that takes the form

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

where $A = [a_{ij}]$ is an $n \times n$ matrix. Another way to write a quadratic function is as

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{x}^T A \mathbf{x}$$

if $\mathbf{x} = (x_1, \ldots, x_n)^T$ is assumed to be a column vector.

Hessian

Definition (Hessian)

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be C^2 at $\mathbf{x}_0 \in \mathcal{U}$. The **Hessian** of f at $\mathbf{x}_0$ is the quadratic function

$$
\begin{aligned}
Hf(\mathbf{x}_0)(\mathbf{h}) &= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) h_i h_j \\
&= \frac{1}{2}(h_1, \ldots, h_n) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}
\end{aligned}
$$

Second derivative test

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function. f is **positive-definite** (resp. **negative-definite**) if $f(\mathbf{x}) \geq 0$ (resp. $f(\mathbf{x}) \leq 0$) for all $\mathbf{x} \in \mathbb{R}^n$, with $f(\mathbf{x}) = 0$ only when $\mathbf{x} = \mathbf{0}$.

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^3 and $\mathbf{x}_0 \in \mathcal{U}$ be a critical point of f.

- *If the Hessian $Hf(\mathbf{x}_0)$ is positive-definite, then $\mathbf{x}_0$ is a local minimum of f.*
- *If the Hessian $Hf(\mathbf{x}_0)$ is negative-definite, then $\mathbf{x}_0$ is a local maximum of f.*

Case of $f : \mathbb{R}^2 \to \mathbb{R}^2$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2. Then

$$Hf(x,y)(h) = \frac{1}{2}(h_1, h_2)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

In general, if

$$Hf(x,y)(h) = \frac{1}{2}(h_1, h_2)B\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

where

$$B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

then $Hf(x,y)(h)$ is positive-definite iff $a > 0$ and $\det(B) = ad - b^2 > 0$.

Second derivative test for $f : \mathbb{R}^2 \to \mathbb{R}$

Theorem

Let $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ be C^3 on $\mathcal{U}$ open. The point (x_0, y_0) is a strict local minimum [resp. maximum] of f if the following three conditions hold:

1. $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$
2. $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$ *[resp.* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$*]*
3. $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ *at* (x_0, y_0)

Critical point (x_0, y_0) such that $D \neq 0$ (resp. $D = 0$) is a **nondegenerate** (resp. **degenerate**) critical point.

Global extrema

Definition (Global extremum)

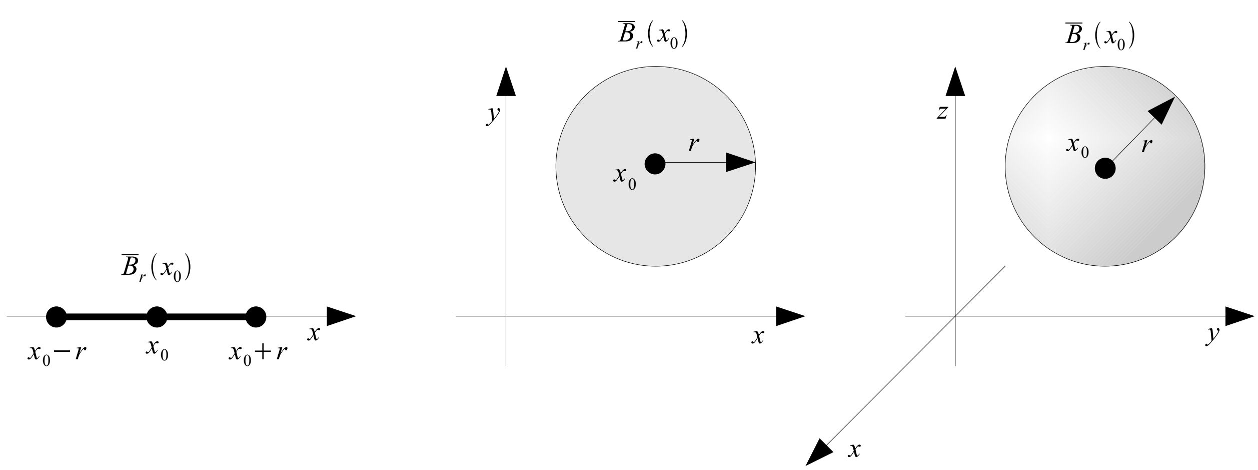
If the inequalities in the local definitions hold for all $\mathbf{x}$ in the domain $\mathcal{U}$ of f, then an extremum is called a **global extremum**.

Closed balls

Definition (Closed ball)

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $r > 0$. The **closed ball** $\overline{\mathcal{B}}_r$ of radius r and centre $\mathbf{x}_0$ is the set of all points $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{x}_0\| \le r$, i.e.,

$$\overline{\mathcal{B}}_r = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| \le r\}$$



Closed sets

Definition (Boundary point)

At point $\mathbf{x}_0 \in \mathcal{U} \subset \mathbb{R}^n$ is a **boundary point** of $\mathcal{U}$ if $\forall r > 0$, $B_r(\mathbf{x}_0)$ contains points in $\mathcal{U}$ and points not in $\mathcal{U}$.
In other words, denoting $\mathcal{U}^c$ the complement of $\mathcal{U}$ in $\mathbb{R}^n$, $\forall r > 0$, $B_r(\mathbf{x}_0) \cap \mathcal{U} \neq \emptyset$ and $B_r(\mathbf{x}_0) \cap \mathcal{U}^c \neq \emptyset$.

Definition (Closed set)

The set $\mathcal{U} \subset \mathbb{R}^n$ is a **closed set** if it contains all its boundary points.

Definition (Bounded set)

The set $\mathcal{U} \subset \mathbb{R}^n$ is a **bounded set** if it is contained within some disk.

Extreme value theorem and application

Theorem

If f is continuous on a closed, bounded set $\mathcal{D}$ in $\mathbb{R}^n$, then f attains a global maximum value $f(\mathbf{x}_M)$ and a global minimum value $f(\mathbf{x}_m)$ at some points $\mathbf{x}_M$ and $\mathbf{x}_m$ in $\mathcal{D}$, respectively.

To find the global maximum and minimum values of a continuous function $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ on a closed, bouded set $\mathcal{D}$:

1. Find all critical points of f in $\mathcal{D}$.
2. Find the values of f at the critical points of f in $\mathcal{D}$.
3. Find the extreme values of f on the boundary of $\mathcal{D}$, ∂D.
4. Largest value(s) obtained in previous steps is/are global maximum/maxima, smallest value(s) obtained in previous steps is/are global minimum/minima.

Lagrange multipliers

Theorem

Let $f, g : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. Let $\mathbf{x}_0 \in \mathcal{U}$, $g(\mathbf{x}_0) = c$ and $\mathcal{S}$ be the level set for g with value c.
Assume that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$.
If $f|\mathcal{S}$ (f restricted to $\mathcal{S}$) has a local extremum on $\mathcal{S}$ at $\mathbf{x}_0$, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

Method of Lagrange multipliers – One constraint

To find the extremum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, assuming that these extreme values exist,

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

   and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f, the smallest is the minimum value of f.

Method of Lagrange multipliers – Two constraints

To find the extremum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, assuming that these extreme values exist,

1. Find all values of x, y, z and λ, μ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

   and

$$g(x, y, z) = k_1 \qquad h(x, y, z) = k_2$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f, the smallest is the minimum value of f.

Global extrema

Definition (Smooth boundary)

Let $\mathcal{U} \subset \mathbb{R}^n$ be open with boundary $\partial\mathcal{U}$. The boundary $\partial\mathcal{U}$ is **smooth** if $\partial\mathcal{U}$ is the level set of a smooth function f whose gradient never vanishes ($\nabla f \neq 0$).

Strategy for locating global extrema

Let $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable on $\mathcal{D} = \mathcal{U} \cup \partial\mathcal{U}$, $\mathcal{U} \subset \mathbb{R}^n$ open, with $\partial\mathcal{U}$ smooth. To find global extrema on $\mathcal{D}$:

1. Locate all critical points of f in $\mathcal{U}$.
2. Use Lagrange multipliers to locate all critical points of $f|\partial\mathcal{U}$.
3. Compute value of f at all the critical points.
4. Select largest and smallest values.

Special Implicit Function Theorem

Theorem

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be C^1. Denote points in $\mathbb{R}^{n+1}$ by $(\mathbf{x}, z)$ with $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Assume that $(\mathbf{x}_0, z_0)$ s.t.

$$F(\mathbf{x}_0, z_0) = 0 \text{ and } \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$$

Then $\exists \mathcal{U} \subset \mathbb{R}^n$ ball containing $\mathbf{x}_0$ and neighbourhood V of z_0 in $\mathbb{R}$ such that there exists a unique $z = g(\mathbf{x})$ defined for $\mathbf{x} \in \mathcal{U}$ and $z \in V$ satisfying

$$F(\mathbf{x}, g(\mathbf{x})) = 0.$$

Moreover, if $\mathbf{x} \in \mathcal{U}$ and $z \in V$ satisfy $F(\mathbf{x}, z) = 0$, then $z = g(\mathbf{x})$.

Theorem (Continued)

Finally, $z = g(\mathbf{x})$ is C^1 and

$$\mathbf{D}g(\mathbf{x}) = -\left.\frac{1}{\frac{\partial F}{\partial z}(\mathbf{x}, z)} \mathbf{D}_{\mathbf{x}} F(\mathbf{x}, z)\right|_{z=g(\mathbf{x})}$$

where $\mathbf{D}_{\mathbf{x}} F$ is the (partial) derivative of F with respect to $\mathbf{x}$, $\mathbf{D}_{\mathbf{x}}\mathbf{F} = (\partial F/\partial x_1, \ldots, \partial F/\partial x_n)$. In other words,

$$\frac{\partial g}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

General Implicit Function Theorem

Consider the nonlinear system

$$\begin{aligned} F_1(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ &\vdots \\ F_m(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \end{aligned} \tag{1}$$

Let

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{vmatrix}$$

Theorem

If $\Delta \neq 0$, then near the point $(\mathbf{x}_0, \mathbf{z}_0)$, equation (1) defines unique smooth functions

$$z_i = k_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, m$$

whose derivatives can be computed by implicit differentiation.

Inverse function theorem

We want to *invert* the system

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= y_1 \\ &\vdots \\ f_n(x_1, \ldots, x_n) &= y_n \end{aligned} \tag{2}$$

i.e., solve it in terms for $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_n$.

Theorem

Let $f_i : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$, $\mathcal{U}$ open, and $f = (f_1, \ldots, f_n)$ ($f : \mathbb{R}^n \to \mathbb{R}^n$). Assume the f_i's are C^1. Consider (2) near a solution $\mathbf{x}_0, \mathbf{y}_0$. If

$$J(f)(\mathbf{x}_0) = det(\mathbf{D}f(\mathbf{x}_0)) \neq 0$$

then (2) can be solved uniquely as $\mathbf{x} = g(\mathbf{y})$ for $\mathbf{x}$ near $\mathbf{x}_0$ and $\mathbf{y}$ near $\mathbf{y}_0$. Moreover, g is C^1.

Part V

Vector functions

Position – Velocity – Acceleration

Let $\mathcal{C}$ be a curve described by the vector equation $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$.
Then

- $\mathbf{c}(t)$ is the **position** (of an object, for example) at time t
- $\mathbf{c}'(t)$ is the **velocity** at time t
- $\mathbf{c}''(t)$ is the **acceleration** at time t.

Most of the time, we will work with $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$.

Tangent vector – Reminder and additional properties

Let $\mathcal{C}$ be a curve described by the vector equation $\mathbf{c}$. Then

- $\mathbf{c}'(t)$ is the tangent vector to $\mathcal{C}$ at the tip of $\mathbf{c}(t)$.
- $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}$ is the **unit tangent vector** to $\mathcal{C}$ at the tip of $\mathbf{c}(t)$.

If $\|\mathbf{c}(t)\| = K$, a constant, for all t, then the tangent vector is orthogonal to $\mathbf{c}$, i.e., $\mathbf{c}'(t) \bullet \mathbf{c}(t) = \mathbf{T}(t) \bullet \mathbf{c}(t) = 0$ for all t in the domain of $\mathbf{c}'$.

Properties

Theorem

Let $\mathbf{c}$, $\mathbf{c}_1$ and $\mathbf{c}_2$ be differentiable paths $\mathbb{R} \to \mathbb{R}^3$, $k \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Then

- $\frac{d}{dt}\{\mathbf{c}_1(t) + \mathbf{c}_2(t)\} = \mathbf{c}_1'(t) + \mathbf{c}_2'(t)$
- $\frac{d}{dt}\{f(t)\mathbf{c}(t)\} = f'(t)\mathbf{c}(t) + f(t)\mathbf{c}'(t)$
- $\frac{d}{dt}\{\mathbf{c}_1(t) \bullet \mathbf{c}_2(t)\} = \mathbf{c}_1'(t) \bullet \mathbf{c}_2(t) + \mathbf{c}_1(t) \bullet \mathbf{c}_2'(t)$
- $\frac{d}{dt}\{\mathbf{c}_1(t) \times \mathbf{c}_2(t)\} = \mathbf{c}_1'(t) \times \mathbf{c}_2(t) + \mathbf{c}_1(t) \times \mathbf{c}_2'(t)$
- $\frac{d}{dt}\{\mathbf{c}(f(t))\} = f'(t)\mathbf{c}'(f(t))$ *(chain rule)*

Except for cross product, all these rules apply for paths $\mathbb{R} \to \mathbb{R}^n$.

Regular/smooth curves, piecewise smooth curves

Definition (Regular/smooth curve)

Let $\mathcal{C}$ be a curve defined by the vector equation $\mathbf{c}$. Then $\mathcal{C}$ is **regular** (or **smooth**) on the interval I if $\mathbf{c}'$ is continuous and $\mathbf{c}' \neq 0$ except maybe at any endpoint of I.

$\mathbf{c}'$ and $\mathbf{0}$ refer to vectors. So $\mathbf{c}' \neq \mathbf{0}$ means $(f', g', h') \neq (0, 0, 0)$: the three components must not be zero simultaneously.

Definition (Piecewise smooth curve)

A curve composed of a denumerable number of smooth curves is called **piecewise smooth** or **piecewise $\mathbf{C^1}$**.

Acceleration and Newton's Second Law

The acceleration of a path $\mathbf{c}(t) : \mathbb{R} \to \mathbb{R}^3$ is

$$a(t) = \mathbf{c}''(t)$$

If $\mathbf{F} \in \mathbb{R}^3$ is the force acting and $m \in \mathbb{R}_+$ is the mass of the particle, then

$$\mathbf{F} = m\mathbf{a}$$

Integral of a path (example with $c : \mathbb{R} \to \mathbb{R}^3$)

We have, for $\mathbf{c}(t) = (f(t), g(t), h(t))$,

$$\int_a^b \mathbf{c}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

Fundamental theorem of calculus

$$\int_a^b \mathbf{c}(t)dt = [R(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

with $\mathbf{R}$ an antiderivative of $\mathbf{c}$.

Works similarly for $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$.

Differential of arc length

Definition (Differential of arc length)

Let $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. An **infinitesimal displacement** of a particle following the path $\mathbf{c}(t)$ is

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)$$

and its length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

is the **differential of arc length**.

Arc length

Definition (Arc length)

Let $\mathbf{c}(t) = (x_1(t), \ldots, x_n(t))$ be piecewise C^1. Then the **length** of the path $\mathbf{c}(t)$ (or its **arc length**), between $t = t_0$ and $t = t_1$, is given by

$$\begin{aligned} L(\mathbf{c}) &= \int_{t_0}^{t_1} \|\mathbf{c}'(t)\| dt \\ &= \int_{t_0}^{t_1} \sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2} dt \end{aligned}$$

Arc length function

Definition (Arc length function)

Let $\mathcal{C}$ be piecewise C^1, defined by $\mathbf{c} = (f, g, h)$. Suppose that at least one of f, g, h is one-to-one on (t_0, t_1). Then the **arc length function** s is the function

$$\begin{aligned} s : I \subset (t_0, t_1) \subset \mathbb{R} &\to \mathbb{R}_+ \\ t &\mapsto \int_{t_0}^{t} \|\mathbf{c}'(u)\| du, \quad t \in [t_0, t_1] \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$s'(t) = \|\mathbf{c}'(t)\|$$

and, since $s(a) = 0$,

$$\int_a^b s'(t) dt = s(b) - s(a) = s(b)$$

Curvature

Definition (Curvature)

Let $\mathcal{C}$ be a smooth curve defined by the vector function $\mathbf{c}$. The **curvature** of $\mathcal{C}$ is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

where $\mathbf{T}$ is the unit tangent vector, and s is the arc length function, which satisfies $s'(t) = \|\mathbf{c}'(t)\|$.

Theorem

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{c}'(t)\|}$$

Some properties of curvature

Theorem

In 3-space, the curvature of the curve $\mathcal{C}$ defined by the vector function $\mathbf{c}$ is given by

$$\kappa(t) = \frac{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|}{\|\mathbf{c}'(t)\|^3}$$

Normal and binormal vectors

Let $\mathcal{C}$ be a smooth curve in 3-space defined by the function $\mathbf{c}$.

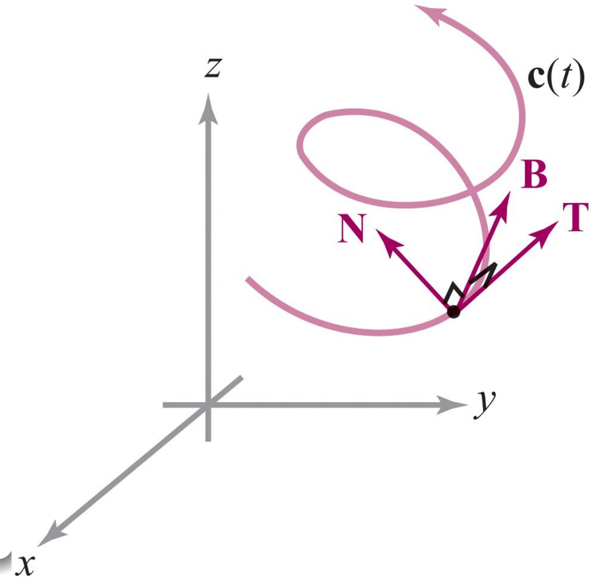
Definition

The (principal) **unit normal** (vector) $\mathbf{N}(t)$ is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

The **binormal vector** is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$



Normal and osculating planes

Let $\mathbf{c}(t)$ be a point on the smooth curve $\mathcal{C}$.

Definition (Normal plane)

The plane spanned by the vectors $\mathbf{N}(t)$ and $\mathbf{B}(t)$ and comprising the tip of $\mathbf{c}(t)$ is called the **normal plane** of $\mathcal{C}$ at $\mathbf{c}(t)$.

All lines on the normal plane are orthogonal to $\mathbf{T}(t)$.

Definition (Osculating plane)

The plane spanned by the vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is the **osculating plane** of $\mathcal{C}$ at $\mathbf{c}(t)$.

Vector fields

Definition (Vector field)

Let $\mathcal{A} \subset \mathbb{R}^n$. A vector field on $\mathbb{R}^n$ is a map $\mathbf{F} : \mathcal{A} \to \mathbb{R}^n$ that assigns to each point $\mathbf{x} \in \mathcal{A}$ a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$.

In $\mathbb{R}^2$, we speak of vector fields in the plane (or planar vector fields) and write $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j} = (F_1(x, y), F_2(x, y))$, or $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$.

Gradient field

Definition

If $f(x_1, \dots, x_n)$ is a function of n variables, then its gradient

$$\begin{aligned}\nabla f(x_1, \dots, x_n) &= (\frac{\partial}{\partial x_1} f(x_1, \dots, x_n), \dots, \frac{\partial}{\partial x_n} f(x_1, \dots, x_n)) \\ &= \frac{\partial}{\partial x_1} f(x_1, \dots, x_n)\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n} f(x_1, \dots, x_n)\mathbf{e}_n\end{aligned}$$

is a vector field on $\mathbb{R}^n$, with $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$.

Flow line

Definition (Flow line)

Let $\mathbf{F}$ be a vector field. A **flow line** for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$$

That is, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}(t)$.

del operator

We call **del operator** the operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

If $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by $f(x, y, z)$, applying ∇ to f gives

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

that is, the gradient of f.

Divergence

Definition (Divergence)

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. The **divergence** of $\mathbf{F}$ is the scalar field

$$\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Let $\mathbf{F} = \sum_{k=1}^{n} F_k \mathbf{e}_k$ be a vector field on $\mathbb{R}^n$. The divergence of $\mathbf{F}$ is

$$\text{div } \mathbf{F} = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Curl

Definition (Curl)

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. The **curl** of $\mathbf{F}$ is the vector field

$$\begin{aligned} \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

The gradient is curl free

Theorem

For any C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$\nabla \times (\nabla f) = 0$$

The curl of any gradient is the zero vector.

The curl is divergence free

Theorem

For any C^2 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\text{div curl } \mathbf{F} = \nabla \bullet (\nabla \times F) = 0.$$

The divergence of any curl is zero.

Laplacian

Definition (Laplacian)

The **Laplace operator** (or **Laplacian**), denoted ∇^2, is the divergence of the gradient. If $f : \mathbb{R}^n \to \mathbb{R}$, then

$$\nabla^2 f = \nabla \bullet (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$$

Properties

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$ for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ at points $\mathbf{x}$ where $g(x) \neq 0$
5. $\nabla \bullet (\mathbf{F}+\mathbf{G}) = \nabla \bullet \mathbf{F} + \nabla \bullet \mathbf{G}$ (div)
6. $\nabla \times (\mathbf{F}+\mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$ (curl)
7. $\nabla \bullet (f\mathbf{F}) = f\nabla \bullet \mathbf{F} + \mathbf{F} \bullet \nabla f$
8. $\nabla \bullet (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \bullet \nabla \times \mathbf{F} - \mathbf{F} \bullet \nabla \times \mathbf{G}$
9. $\nabla \bullet \nabla \times \mathbf{F} = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times \nabla f = 0$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \bullet \nabla g)$
13. $\nabla \bullet (\nabla f \times \nabla g) = 0$
14. $\nabla \bullet (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Part VI

Double and triple integrals

Theorem

If $f(x, y) \geq 0$, then

$$\iint_{\mathcal{R}} f(x, y)\, dA \text{ or } \iint_{\mathcal{R}} f(x, y)\, dx\, dy$$

is the volume of the solid lying between the rectangle $\mathcal{R}$ on the xy-plane and the surface $z = f(x, y)$.

Cavalieri's principle

Let S be a solid and, for $a \leq x \leq b$, P_x be the family of parallel planes such that

1. S lies between P_a and P_b
2. The area of the slice of S cut by P_x is $A(x)$.

Then the volume of S is

$$\int_a^b A(x)dx$$

Regular partition

Assume $f(x, y)$ defined on a domain $\mathcal{D} \subset \mathbb{R}^2$, and consider the rectangle $\mathcal{R} = [a, b] \times [c, d] \subset \mathcal{D}$.

Subdivide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$, $[c, d]$ into n subintervals of width $\Delta y = \frac{d-c}{n}$. The area of each box is $\Delta A = \Delta x \Delta y$. This is a **regular partition** of R of order n.

Definition of a double integral

In each rectangle, pick a **sample point** $\mathbf{c}_{ij} = (x_{ij}^*, y_{ij}^*)$.

Define the **Riemann sum** for f as

$$S_n = \sum_{i,j=0}^{n-1} f(\mathbf{c}_{ij})\Delta x \Delta y = \sum_{i,j=0}^{n-1} f(\mathbf{c}_{ij})\Delta A$$

Definition

If the sequence $\lim_{n\to\infty} S_n = S$, where S is the same regardless of the choice of sample point, then f is **integrable** over $\mathcal{R}$ and the limit S is denoted

$$\iint_{\mathcal{R}} f(x,y)\,dA, \qquad \iint_{\mathcal{R}} f(x,y)\,dx\,dy \text{ or } \iint_{\mathcal{R}} f\,dx\,dy$$

Theorem

Any continuous function defined on a closed rectangle $\mathcal{R}$ is integrable over $\mathcal{R}$.

Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be bounded on the rectangle $\mathcal{R}$ and suppose that the set of points where f is discontinuous lies on a finite union of graphs of continuous functions. Then f is integrable over $\mathcal{R}$.

Fubini's theorem

Theorem

Let $f(x, y)$ be continuous over $\mathcal{R} = [a, b] \times [c, d]$. Then

$$\begin{aligned}\iint_{\mathcal{R}} f(x, y)dA &= \int_a^b \int_c^d f(x, y)dydx \\ &= \int_c^d \int_a^b f(x, y)dxdy.\end{aligned}$$

Fubini's theorem (more general formulation)

Theorem

Let $f(x, y)$ be bounded over $\mathcal{R} = [a, b] \times [c, d]$ and suppose the discontinuities of f lie on a finite union of graphs of continuous functions. If $\int_c^d f(x, y)dy$ exists for all $x \in [a, b]$, then

$$\int_a^b \int_c^d f(x, y)dydx$$

exists and

$$\iint_{\mathcal{R}} f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx.$$

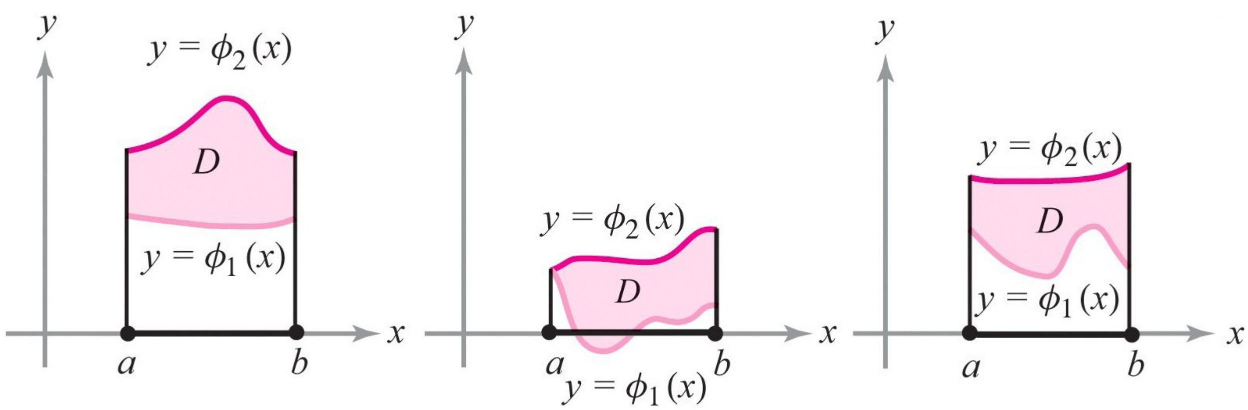
y simple region / region of type I

Definition

A region $\mathcal{D}$ in the plane is **y-simple**, or of **type I**, if it is bounded above and below by functions of x, that is,

$$\mathcal{D} = \{(x, y) : a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\},$$

with ϕ_1, ϕ_2 continuous on $[a, b]$.



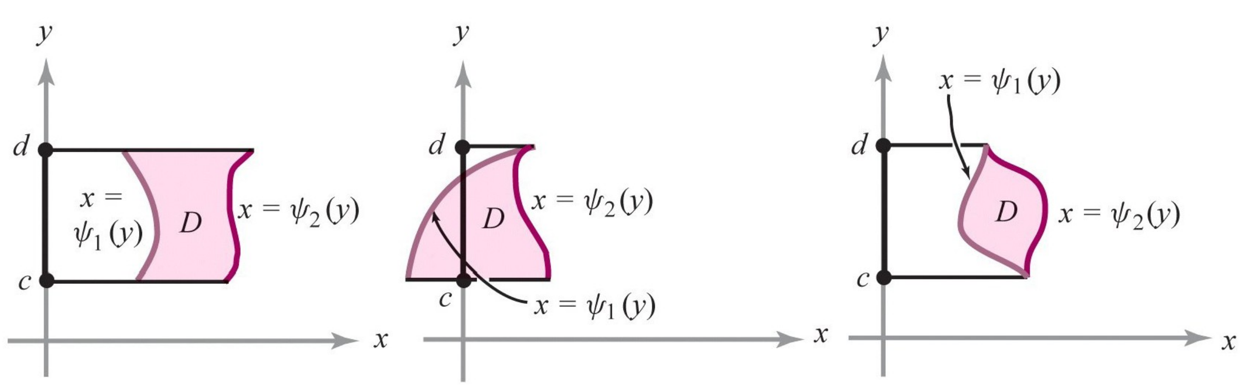
x simple region / region of type II

Definition

A region $\mathcal{D}$ in the plane is **x-simple**, or of **type II**, if it is bounded on the left and the right by functions of y, that is,

$$\mathcal{D} = \{(x, y) : c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\},$$

with ψ_1, ψ_2 continuous on $[c, d]$.



Double integral over a general region

Definition (Elementary region)

A region that is x-simple or y-simple is **elementary**.

Definition

Suppose $\mathcal{D}$ is an elementary region in the plane, such that $\mathcal{D} \subset \mathcal{R} = [a, b] \times [c, d]$. Consider a continuous function $f : \mathcal{D} \to \mathbb{R}$ (hence f bounded on $\mathcal{D}$). Let

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in \mathcal{D}, \\ 0 & \text{if } (x, y) \in \mathcal{R} \setminus \mathcal{D}. \end{cases}$$

Then

$$\iint_{\mathcal{D}} f(x, y) dA = \iint_{\mathcal{R}} F(x, y) dA.$$

Double integral over a y-simple region

Theorem

Suppose that f is continuous on y-simple region

$$\mathcal{D} = \{(x, y) : a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$$

with ϕ_1, ϕ_2 continuous on $[a, b]$. Then

$$\iint_{\mathcal{D}} f(x, y) dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy dx$$

Double integral over a x-simple region

Theorem

Suppose that f is continuous on a x-simple region

$$\mathcal{D} = \{(x, y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$$

with ψ_1, ψ_2 continuous on $[c, d]$. Then

$$\iint_{\mathcal{D}} f(x, y)dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dxdy$$

Properties

Theorem (Linearity)

The double integral is a **linear operator**, *that is, if $f, g : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then*

$$\iint_{\mathcal{D}} [\alpha f(x, y) + \beta g(x, y)]dA = \alpha \iint_{\mathcal{D}} f(x, y)dA + \beta \iint_{\mathcal{D}} g(x, y)dA$$

Theorem (Monotonicity)

If $f(x, y) \geq g(x, y)$ for all $x, y \in \mathcal{D}$, then

$$\iint_{\mathcal{D}} f(x, y)dA \geq \iint_{\mathcal{D}} g(x, y)dA$$

Theorem (Additivity)

Suppose that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, with $\mathcal{D}_1$ and $\mathcal{D}_2$ non overlapping except maybe on their boundary. Then

$$\iint_{\mathcal{D}} f(x,y)dA = \iint_{\mathcal{D}_1} f(x,y)dA + \iint_{\mathcal{D}_2} f(x,y)dA.$$

Theorem

$$\iint_{\mathcal{D}} dA = A(\mathcal{D}),$$

where $A(\mathcal{D})$ is the area of the region $\mathcal{D}$.

Theorem

Suppose that $m \leq f(x,y) \leq M$ for all $x, y \in \mathcal{D}$. Then

$$mA(\mathcal{D}) \leq \iint_{\mathcal{D}} f(x,y)dA \leq MA(\mathcal{D}).$$

Mean Value Theorem

Theorem (Mean value theorem)

Let $f : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous, with $\mathcal{D}$ an elementary region. Then there exists $(x_, y_0) \in \mathcal{D}$ such that

$$\iint_{\mathcal{D}} f(x,y)dA = f(x_0,y_0)A(\mathcal{D})$$

where $A(\mathcal{D})$ is the area of $\mathcal{D}$.

Riemman sums for a triple integral

Consider a function $f : \mathbb{R}^3 \to \mathbb{R}$ defined on a parallelepiped

$$\mathcal{B} = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$

Divide $\mathcal{B}$ into boxes of length Δx, Δy and Δz, with each box of the form

$$\mathcal{B}_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

for $i, j, k = 1, \ldots, n$. The triple Riemann sum is then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(c_{ijk}^*) \Delta V,$$

where $c_{ijk}^* = (x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in $\mathcal{B}_{ijk}$ and $\Delta V = \Delta x \Delta y \Delta z$.

Definition

The triple integral of f over $\mathcal{B}$ is

$$\iiint_{\mathcal{B}} f(x, y, z) dV = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(c_{ijk}^*) \Delta V,$$

if this limit exists. We then say that f is **integrable**.

Reduction to iterated integrals

Theorem

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be integrable on $\mathcal{B} = [a,b] \times [c,d] \times [r,s]$. Then

$$\begin{aligned}\iiint_{\mathcal{B}} f(x,y,z)dV &= \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz \\ &= \int_c^d \int_r^s \int_a^b f(x,y,z)\,dx\,dz\,dy \\ &= \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz \\ &= \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx\end{aligned}$$

(as well as $dz\,dy\,dx$ and $dz\,dx\,dy$)

Integrals over elementary regions for triple integrals

Let

$$E = \{(x,y,z) : (x,y) \in \mathcal{D},\ u_1(x,y) \le z \le u_2(x,y)\}$$

with $\mathcal{D}$ the projection of E onto the xy-plane. Then

$$\iiint_E f(x,y,z)dV = \iint_{\mathcal{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA$$

The nature of $\mathcal{D}$ is then determined: if $\mathcal{D}$ is, say, y-simple, then

$$\begin{aligned}E = \{(x,y,z) : a \le x \le b,& \\ g_1(x) \le y \le g_2(x),&\ u_1(x,y) \le z \le u_2(x,y)\}\end{aligned}$$

and

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

For

$$E = \{(x, y, z) : (x, y) \in \mathcal{D},\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

with $\mathcal{D}$ the projection of E onto the xy-plane, if $\mathcal{D}$ is x-simple,

$$E = \{(x, y, z) : c \leq y \leq d, \\ h_1(y) \leq x \leq h_2(y),\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

and

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy$$

Symmetric elementary regions

- $$E = \{(x, y, z) : (x, y) \in \mathcal{D},\ u_1(x, y) \leq z \leq u_2(x, y)\}$$
- $$E = \{(x, y, z) : (y, z) \in \mathcal{D},\ u_1(y, z) \leq x \leq u_2(y, z)\}$$
- $$E = \{(x, y, z) : (x, z) \in \mathcal{D},\ u_1(x, z) \leq y \leq u_2(x, z)\}$$

In Stewart, these are Type I, II and III solid regions, respectively.

If E can be described using any of these forms and that $\mathcal{D}$ can then be described using which combination of x-, y- and z-simple as are needed, then the region is a **symmetric elementary region**.

Volume of a region

If E is an elementary region, then

$$V(E) = \iiint_E dV$$

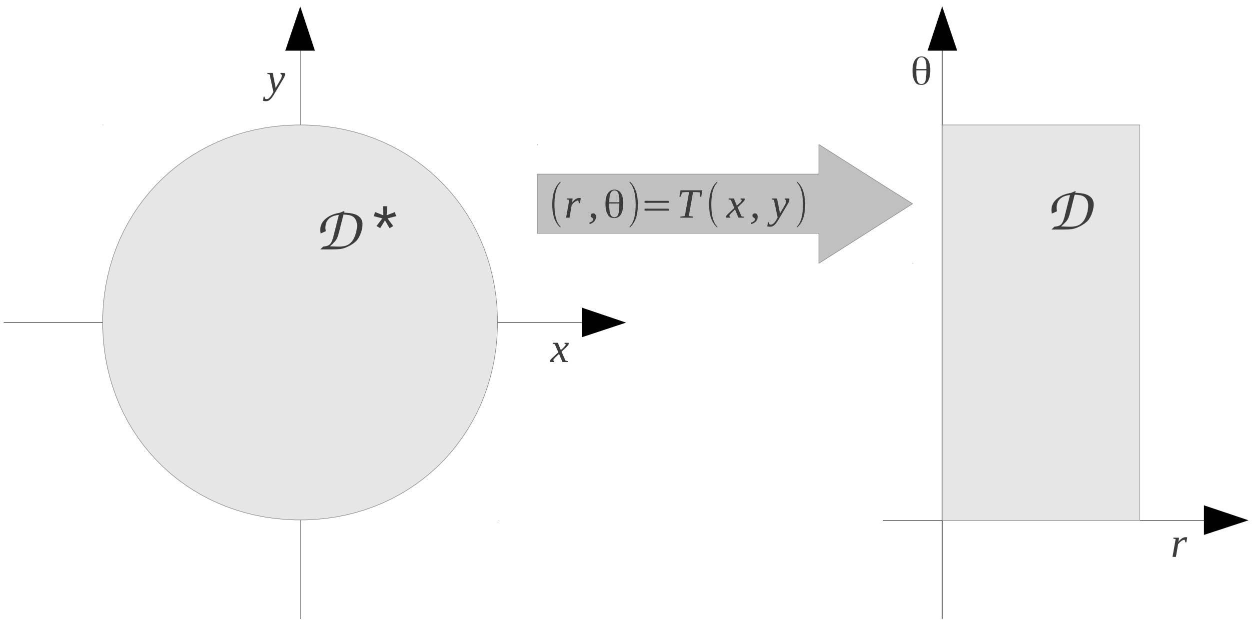
is the volume of the region (this means we integrate $f(x, y, z) = 1$).

Part VII

Change of variables formula

Transformations

Let $\mathcal{D}^* \subset \mathbb{R}^2$ and $T : \mathcal{D}^* \to \mathbb{R}^2$ be a C^1 map. Let us denote $\mathcal{D} = T(\mathcal{D}^*)$ the **image** of the set $\mathcal{D}^*$.



Objective of the transformation

Transform the point (u, v) in the uv-plane into the point (x, y) in the xy-plane, using

$$T: \quad uv\text{-plane} \to xy\text{-plane}$$
$$T(u, v) = (x, y), \; x = g(u, v), \; y = h(u, v)$$

with T a C^1 function.

If T is one-to-one (no two points have the same image through T), then the inverse T^{-1} exists and we can possibly find $u = G(x, y)$ and $v = H(x, y)$.

Linear maps/transformations

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, i.e., $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^2$ and A a 2×2 matrix. Suppose $\det(A) \neq 0$. Then T transforms parallelograms into parallelograms and vertices into vertices. Moreover, if $T(\mathcal{D}^*)$ is a parallelogram, then $\mathcal{D}^*$ must be a parallelogram.

Theorem

Let T be a linear transformation $T : \mathcal{D}^* \to \mathcal{D}$, with $\mathcal{D}^*, \mathcal{D} \subset \mathbb{R}^n$, defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

for $\mathbf{x} \in \mathbb{R}^n$, where A is an $n \times n$-matrix. Then T is one-to-one and onto (i.e., a bijection) if and only if $\det(A) \neq 0$.

Jacobian of a transformation

Definition

Let $T : \mathcal{D}^* \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 transformation given by

$$x_1 = x_1(u_1, \ldots, u_n), \ldots, x_n = x_n(u_1, \ldots, u_n)$$

The **Jacobian** (or **Jacobian matrix**) of T, denoted

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)}$$

is the matrix of partial derivatives $DT(u_1, \ldots, u_n)$ of T

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{pmatrix}$$

The **Jacobian determinant** of T is the determinant of DT.

Jacobian matrix ($T : \mathbb{R}^2 \to \mathbb{R}^2$ case)

Definition

Let T be the transformation $T(u,v) = (x,y)$ with $x = x(u,v)$ and $y = y(u,v)$. The Jacobian matrix of T is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

We then have

$$\Delta A = \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v,$$

if ΔA is the variation in $\mathcal{R}$, the image in the xy-plane of the box $\mathcal{S}$ in the uv-plane through T.

Change of variables for double integrals

Theorem

Let $\mathcal{D}^*$ and $\mathcal{D}$ be elementary regions in the plane and $T : \mathcal{D}^* \to \mathcal{D}$ be a C^1 transformation. Suppose that T is one-to-one on $\mathcal{D}^*$ and that $\mathcal{D} = T(\mathcal{D}^*)$. Then for any integrable function $f : \mathcal{D} \to \mathbb{R}$,

$$\iint_{\mathcal{D}} f(x,y)dA = \iint_{\mathcal{D}^*} f(x(u,v), y(u,v)) \left| det \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Polar coordinates for rectangular regions

Theorem

Suppose f continuous on the rectangle

$$\mathcal{R} = \{(r, \theta) : 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$$

with $0 \le \beta - \alpha \le 2\pi$*. Then*

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

Polar coordinates for elementary regions

Suppose that f is continuous on a polar region of the form

$$\mathcal{D}^* = \{(r, \theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\begin{aligned}\iint_{\mathcal{D}} f(x, y)\, dA &= \iint_{\mathcal{D}^*} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \\ &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta\end{aligned}$$

Case of triple integrals

Theorem

Let T be a C^1 transformation with Jacobian matrix

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

Suppose that T is one-to-one except maybe on a set that is the union of graphs of functions of two variables. If W^* is an elementary region mapped to W by T, then

$$\iiint_W f(x,y,z)dV =$$
$$\iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

Triple integrals in spherical coordinates

Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. Then, for the spherical wedge

$$E = \{(\rho,\theta,\phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\},$$

we have

$$\iiint_E f(x,y,z)dV =$$
$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

Fubini's theorem for improper integrals

Theorem

Let $\mathcal{D}$ be an elementary region in $\mathbb{R}^2$ and $f \geq 0$ be continuous except maybe at points on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$. If one of the integrals

- $\iint_{\mathcal{D}} f(x,y)\,dA$
- $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$ *(y-simple region)*
- $\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$ *(x-simple region)*

exists as an improper integral, then f is integrable and all these integrals are equal.

Part VIII

Integrals over paths and surfaces

Path integral in 3-space

Definition (Path integral)

Let $f(x, y, z)$ be a function $\mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{c} : I = [a, b] \to \mathbb{R}^3$ be a path with $\mathbf{c}(t) = (x(t), y(t), z(t))$.
The **path integral** or **integral of f along the path c** is defined when $\mathbf{c}$ is C^1 and the composite function $f(x(t), y(t), z(t))$ is continuous on I and is given by

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(x(t), y(t), z(t))\, \|\mathbf{c}'(t)\|\, dt$$

We also write

$$\int_{\mathbf{c}} f\, ds = \int_{\mathbf{c}} f(x, y, z)\, ds = \int_a^b f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt$$

If $\mathbf{c}$ piecewise C^1 or $f(\mathbf{c}(t))$ piecewise continuous, break down $[a, b]$ into subintervals where $f(\mathbf{c}(t))\|\mathbf{c}'(t)\|$ is continuous and sum resulting integrals:

Theorem

Suppose that $\mathbf{c}$ is piecewise C^1, i.e., $\mathbf{c} = \mathbf{c}_1 \cup \cdots \cup \mathbf{c}_k$, where the $\mathbf{c}_i$ are C^1. Then

$$\int_{\mathbf{c}} f\, ds = \int_{\mathbf{c}_1} f\, ds + \cdots + \int_{\mathbf{c}_k} f\, ds$$

Path integrals for planar curves

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{c} : I = [a, b] \to \mathbb{R}^2$, then

$$\int_{\mathbf{c}} f(x, y)\, ds = \int_a^b f(x(t), y(t))\, \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

General form of the path integral for $\mathbf{c} : \mathbb{R} \to \mathbb{R}^n$

Path integrals can be written in vector form

$$\int_a^b f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt$$

for $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

Warning – terminology !!!

Be very careful here: Stewart uses the name "line integral" for what Marsden & Tromba call "path integrals" and calls "line integrals of vector fields" what Marsden & Tromba call "line integrals".

The terminology used here is the one of Marsden & Tromba.

Line integrals

Definition (Line integral)

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field defined on a C^1 path $\mathbf{c} : [a, b] \to \mathbb{R}$. Then the **line integral** of $\mathbf{F}$ along $\mathcal{C}$ is

$$\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \bullet \mathbf{c}'(t)\, dt$$

If $\mathbf{F}(\mathbf{c}(t)) \bullet \mathbf{c}'(t)$ is piecewise continuous, we define the line integral by computing the integrals over subintervals and summing.

Theorem

Suppose that $\mathbf{c}'(t) \neq \mathbf{0}$ *for all* $t \in [a, b]$. *If*

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}$$

is the unit tangent vector to $\mathbf{c}$, *then we have*

$$\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} = \int_a^b [\mathbf{F}(\mathbf{c}(t)) \bullet \mathbf{T}(t)] \, \|\mathbf{c}'(t)\| \, dt$$

Integrals of differential forms along paths

Definition (Differential form)

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field $\mathbb{R}^3 \to \mathbb{R}^3$, i.e., each component function $F_i : \mathbb{R}^3 \to \mathbb{R}$. The expression

$$F_1 \, dx + F_2 \, dy + F_3 \, dz$$

is called a **differential form**.

The integral of a differential form along a path $\mathbf{c}$, with $\mathbf{c}(t) = (x(t), y(t), z(t))$, is

$$\begin{aligned} \int_{\mathbf{c}} F_1 \, dx + F_2 \, dy + F_3 \, dz &= \int_a^b \big(F_1 x'(t) + F_2 y'(t) + F_3 z'(t)\big) \, dt \\ &= \int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} \end{aligned}$$

Reparametrization

Definition (Reparametrization)

Let $h : I_1 \subset \mathbb{R} \to I_2 \subset \mathbb{R}$ be a one-to-one C^1 function, with $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. Let $\mathbf{c} : I_2 \to \mathbb{R}^3$ be a piecewise C^1 path. Then the composition

$$\mathbf{p} = \mathbf{c} \circ h : I \to \mathbb{R}^3$$

is a **reparametrization** of $\mathbf{c}$.

Definition

A reparametrization is **orientation-preserving** if $\mathbf{p}(a_1) = \mathbf{c}(a_2)$ and $\mathbf{p}(b_1) = \mathbf{c}(b_2)$ and **orientation-reversing** if $\mathbf{p}(a_1) = \mathbf{c}(b_1)$ and $\mathbf{p}(b_1) = \mathbf{c}(a_2)$.

Reparametrization for line integrals

Theorem

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field that is continuous on the C^1 path $\mathbf{c} : [a_1, b_1] \to \mathbb{R}^3$. Let $\mathbf{p} : [a_2, b_2] \to \mathbb{R}^3$ be a reparametrization of $\mathbf{c}$. Then

- $\int_{\mathbf{p}} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s}$ *if $\mathbf{p}$ is orientation-preserving*
- $\int_{\mathbf{p}} \mathbf{F} \bullet d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s}$ *if $\mathbf{p}$ is orientation-reversing*

Reparametrization for path integrals

Theorem

Let $f : \mathbb{R}^3 \to \mathbb{R}$ *be continuous on the* C^1 *path* $\mathbf{c} : [a_1, b_1] \to \mathbb{R}^3$*. Let* $\mathbf{p} : [a_2, b_2] \to \mathbb{R}^3$ *be a reparametrization of* $\mathbf{c}$*. Then*

$$\int_{\mathbf{c}} f \, ds = \int_{\mathbf{p}} f \, ds$$

Line integrals of gradient vector fields

Theorem

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ *is* C^1 *and* $\mathbf{c} : [a, b] \to \mathbb{R}^3$ *is piecewise* C^1*. Then*

$$\int_{\mathbf{c}} \nabla f \bullet d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Definition (Simple curve)

A **simple curve** $\mathcal{C}$ is the image of a piecewise C^1 map $\mathbf{c} : I \to \mathbb{R}$ that is one-to-one on the interval I; $\mathbf{c}$ is the parametrization of $\mathcal{C}$. A simple curve is a curve that does not intersect itself anywhere between its endpoints.

Definition (Closed curve)

A **closed curve** is the image of a piecewise C^1 map $\mathbf{c} : [a, b] \to \mathbb{R}^3$ that is such that $\mathbf{c}(a) = \mathbf{c}(b)$.

Definition (Simple closed curve)

A **simple closed curve** is the curve image of a piecewise C^1 map $\mathbf{c} : [a, b] \to \mathbb{R}^3$ that is simple and closed.

Theorem

If $\mathbf{c}$ is any orientation-preserving parametrization of $\mathcal{C}$, where $\mathcal{C}$ is a simple curve or a simple closed curve, then

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{c}} F \bullet d\mathbf{s} \quad and \quad \int_{\mathcal{C}} f \, ds = \int_{\mathbf{c}} f \, ds$$

Theorem

If $\mathcal{C}^-$ be the same curve as $\mathcal{C}$ but with opposite orientation, where $\mathcal{C}$ is a simple curve or a simple closed curve, then

$$\int_{\mathcal{C}^-} \mathbf{F} \bullet d\mathbf{s} = -\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s}$$

Theorem

Let $\mathcal{C}$ be an oriented curve consisting of several oriented curves $\mathcal{C}_i$, $i = 1, \ldots, k$. Then we write $\mathcal{C} = \mathcal{C}_1 + \cdots + \mathcal{C}_k$ or $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$ and we have

$$\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \bullet d\mathbf{s} + \cdots + \int_{\mathcal{C}_k} \mathbf{F} \bullet d\mathbf{s}$$

Parametrized surfaces

Definition (Parametrized surface)

A **parametrization of a surface** is a function $\Phi : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$, with $\mathcal{D}$ some domain in $\mathbb{R}^2$. The **surface** $\mathcal{S}$ corresponding to Φ is the image of $\mathcal{D}$, $\Phi(\mathcal{D})$, that we can write

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

If Φ is differentiable (or of class C^1), we say that $\mathcal{S}$ is a **differentiable surface** (or a $\mathbf{C^1}$ **surface**).

Tangent vectors to a surface

Let

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

be a differentiable parametrized surface. Fix $u = u_0$, then $t \mapsto \Phi(u_0, t)$ is a map $\mathbb{R} \to \mathbb{R}^3$ whose image is a curve on the surface.

The vector tangent to this curve at $\Phi(u_0, v_0)$ is

$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

$\mathbf{T}_u$ is defined similarly by computing $\partial\Phi/\partial u$.

Regular surfaces and Tangent planes

Definition (Regular surface)

The surface $\mathcal{S}$ is **regular** or **smooth at** $\Phi(u_0, v_0)$ if $\mathbf{T}_u \times \mathbf{T}_v \neq 0$ at (u_0, v_0). The surface is **regular** if it is regular at all $\Phi(u_0, v_0) \in \mathcal{S}$.

Definition (Normal to a regular surface)

If the surface $\mathcal{S}$ is regular, then the vector $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is **normal** to $\mathcal{S}$.

Definition (Tangent plane to a parametrized surface)

If Φ is regular at $\Phi(u_0, v_0)$, then an equation for the **tangent plane** at (x_0, y_0, z_0) on the surface is

$$(x - x_0, y - y_0, z - z_0) \bullet \mathbf{n} = 0$$

where $\mathbf{n}$ is evaluated at (u_0, v_0).

General context

We consider piecewise-regular surfaces that are unions of parametrized surfaces $\Phi_i : \mathcal{D}_i \to \mathbb{R}^3$ where

1. $\mathcal{D}_i$ elementary region in the plane
2. Φ_i is C^1 and one-to-one except possibly on the boundary $\partial\mathcal{D}_i$ of $\mathcal{D}_i$
3. $\mathcal{S}_i$, image of Φ_i, is regular, except possibly at a finite number of points

Area of a parametrized surface

Definition

The surface area $A(\mathcal{S})$ of the parametrized surface $\mathcal{S}$ is given by

$$A(\mathcal{S}) = \iint_{\mathcal{D}} \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$$

If $\mathcal{S}$ is a union of k surfaces $\mathcal{S}_i$, then $A(\mathcal{S}) = A(\mathcal{S}_1) + \cdots + A(\mathcal{S}_k)$.

Area of a graph

If the surface is $z = g(x, y)$, where $(x, y) \in \mathcal{D}$, then it admits the parametrization $x = u$, $y = v$, $z = g(u, v)$ for $(u, v) \in \mathcal{D}$ and therefore, if g is C^1,

$$A(\mathcal{S}) = \iint_{\mathcal{D}} \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial x}\right)^2 + 1}\, dA$$

Integral of a scalar valued function over a parametrized surface

Definition

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous on the parametrized surface $\mathcal{S}$. Then the **integral of f over $\mathcal{S}$** is

$$\iint_{\mathcal{S}} f(x, y, z)\, d\mathcal{S} = \iint_{\mathcal{S}} f\, d\mathcal{S} = \iint_{\mathcal{D}} f(\Phi(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

Surface integral of a vector field over a parametrized surface

Definition

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field defined on $\mathcal{S}$, the image of a parametrized surface Φ. The **surface integral of F over** Φ is

$$\iint_{\Phi} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F} \bullet (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

Oriented surfaces

Definition (Oriented surface)

An **oriented surface** has two sides: a **positive side** called the **outside** and a **negative side** called the **inside**. At each point $(x, y, z) \in \mathcal{S}$ there are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, with $\mathbf{n}_1 = -\mathbf{n}_2$; each is associated to one side of the surface.

To specify a side for $\mathcal{S}$, choose at each point a unit normal vector $\mathbf{n}$ that points away from the positive side of $\mathcal{S}$.

Orientation of surfaces

Definition

Let $\Phi : \mathcal{D} \to \mathbb{R}^3$ be a parametrization of an oriented surface $\mathcal{S}$. Assume $\mathcal{S}$ regular at $\Phi(u_0, v_0)$, $(u_0, v_0) \in \mathcal{D}$. If $\mathbf{n}(\Phi(u_0, v_0))$ is the unit normal to $\mathcal{S}$ at $\Phi(u_0, v_0)$, then

$$\frac{\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}}{\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\|} = \pm\mathbf{n}(\Phi(u_0, v_0))$$

The parametrization Φ is **orientation-preserving** if

$$\frac{\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}}{\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\|} = \mathbf{n}(\Phi(u_0, v_0))$$

and **orientation-reversing** if

$$\frac{\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}}{\|\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}\|} = -\mathbf{n}(\Phi(u_0, v_0))$$

Theorem

Let $\mathcal{S}$ be an oriented surface, $\mathbf{F}$ a continuous vector field defined on $\mathcal{S}$ and Φ_1, Φ_2 regular parametrizations of $\mathcal{S}$.

- *If Φ_1, Φ_2 are orientation-preserving, then*

$$\iint_{\Phi_1} \mathbf{F} \bullet d\mathbf{S} = \iint_{\Phi_2} \mathbf{F} \bullet d\mathbf{S}$$

- *If Φ_1 is orientation-preserving and Φ_2 is orientation-reversing, then*

$$\iint_{\Phi_1} \mathbf{F} \bullet d\mathbf{S} = -\iint_{\Phi_2} \mathbf{F} \bullet d\mathbf{S}$$

If $f : \mathbb{R}^3 \to \mathbb{R}$ is defined on $\mathcal{S}$ and Φ_1, Φ_2 are parametrizations of $\mathcal{S}$, then

$$\iint_{\Phi_1} f \, d\mathcal{S} = \iint_{\Phi_2} f \, d\mathcal{S}$$

Theorem

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \bullet \mathbf{n}\, d\mathcal{S}$$

i.e., the surface integral of $\mathbf{F}$ *over* $\mathcal{S}$ *equals the integral of the normal component of* $\mathbf{F}$ *over the surface.*

Theorem (Surface integral of a vector field over a graph $\mathcal{S}$)

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} &= \iint_{\mathcal{D}} \mathbf{F} \bullet (\mathbf{T}_x \times \mathbf{T}_y)\, dx\, dy \\ &= \iint_{\mathcal{D}} \left[F_1\left(-\frac{\partial g}{\partial x}\right) + F_2\left(-\frac{\partial g}{\partial y}\right) + F_3\right] dx\, dy\end{aligned}$$

Summary – Parametrized surfaces

Parametrized surface $\Phi(u, v)$

- Integral of scalar function f

$$\iint_{\mathcal{S}} f\, d\mathcal{S} = \iint_{\mathcal{D}} f(\Phi(u, v))\|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

- Scalar surface element

$$d\mathcal{S} = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

- Integral of vector field $\mathbf{F}$

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F} \bullet (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

- Vector surface element

$$d\mathbf{S} = (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv = \mathbf{n}\, d\mathcal{S}$$

Summary – Surface as graph of a function

Graph $z = g(x, y)$, $\cos\theta = \mathbf{n} \bullet \mathbf{k}$ ($\mathbf{n}$ unit normal to surface)

- Integral of scalar function f

$$\iint_{\mathcal{S}} f \, d\mathcal{S} = \iint_{\mathcal{D}} \frac{f(x, y, g(x, y))}{\cos\theta} \, dx \, dy$$

- Scalar surface element

$$d\mathcal{S} = \frac{dx \, dy}{\cos\theta} = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \, dx \, dy$$

- Integral of vector field $\mathbf{F}$

$$\iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{D}} \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3\right) dx \, dy$$

- Vector surface element

$$d\mathbf{S} = \mathbf{n} \bullet d\mathcal{S} = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right) dx \, dy$$

Part IX

Vector calculus

We denote $\mathcal{C}^+$ the positive (counterclockwise) orientation of a curve $\mathcal{C}$ in IR^2 and $\mathcal{C}^-$ its negative (clockwise) orientation.

Lemma

Let $\mathcal{D}$ be a y-simple region and $\mathcal{C}$ be its boundary. Suppose $P : \mathcal{D} \to \mathbb{R}$ is C^1. Then

$$\int_{\mathcal{C}^+} P\,dx = -\iint_{\mathcal{D}} \frac{\partial P}{\partial y}\,dx\,dy$$

Lemma

Let $\mathcal{D}$ be a x-simple region and $\mathcal{C}$ be its boundary. Suppose $Q : \mathcal{D} \to \mathbb{R}$ is C^1. Then

$$\int_{\mathcal{C}^+} Q\,dy = \iint_{\mathcal{D}} \frac{\partial Q}{\partial x}\,dx\,dy$$

Green's Theorem

Theorem (Green's Theorem)

Let $\mathcal{C}$ be a piecewise-smooth simple closed curve in the plane, and $\mathcal{D}$ be a simple region bounded by $\mathcal{C}$. If $P, Q : \mathcal{D} \to \mathbb{R}$ are C^1, then

$$\int_{\mathcal{C}^+} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

The notation $\oint$ is often used instead of $\int$, to indicate that the integral is computed along a closed curve. Another notation is

$$\int_{\partial\mathcal{D}} P\,dx + Q\,dy$$

where $\partial\mathcal{D}$ is the (positively oriented) boundary of $\mathcal{D}$.

Area of a region

Theorem

The area of the region $\mathcal{D}$ bounded by the curve $\mathcal{C}$ is given by

$$A(\mathcal{D}) = \oint_{\mathcal{C}^+} x\,dy = -\oint_{\mathcal{C}^+} y\,dx = \frac{1}{2}\oint_{\mathcal{C}^+} x\,dy - y\,dx.$$

Green's theorem can also be applied to regions that are not simply connected; the orientation has to be done correctly.

Vector form of Green's Theorem

Theorem

Let $\mathcal{D} \subset \mathbb{R}^2$ be a region where Green's theorem applies, $\partial\mathcal{D}$ be its positively oriented boundary and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be C^1 on $\mathcal{D}$. Then

$$\int_{\partial\mathcal{D}} \mathbf{F} \bullet d\mathbf{s} = \iint_{\mathcal{D}} (\text{curl } \mathbf{F}) \bullet \mathbf{k}\,dA = \iint_{\mathcal{D}} (\nabla \times \mathbf{F}) \bullet \mathbf{k}\,dA$$

Divergence theorem in the plane

Theorem

Let $\mathcal{D} \subset \mathbb{R}^2$ be a region where Green's theorem applies and $\partial\mathcal{D}$ be its boundary. Let $\mathbf{n}$ *be the outward unit normal to $\partial\mathcal{D}$. Let* $\mathbf{c} : [a, b] \to \mathbb{R}^2$ *be a positively oriented parametrization of $\partial\mathcal{D}$,* $\mathbf{c}(t) = (x(t), y(t))$. *Then* $\mathbf{n}$ *is given by*

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *be* C^1. *Then*

$$\int_{\partial\mathcal{D}} \mathbf{F} \bullet \mathbf{n} \, ds = \iint_{\mathcal{D}} \text{div } \mathbf{F} \, dA$$

Stokes' Theorem for graphs

Theorem

Let $\mathcal{S}$ be the oriented surface defined by a C^2 function $z = f(x, y)$, for $(x, y) \in \mathcal{D}$ where $\mathcal{D}$ is a region to which Green's theorem applies. Let $\mathbf{F}$ *be a C^1 vector field on $\mathcal{S}$. If $\partial\mathcal{S}$ is the oriented boundary curve of $\mathcal{S}$, then*

$$\iint_{\mathcal{S}} \text{curl } \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s}$$

Stokes' Theorem for parametrized surfaces

Theorem

Let $\mathcal{S}$ be the oriented surface defined by a one-to-one parametrization $\Phi : \mathcal{D} \subset \mathbb{R}^2 \to \mathcal{S}$, where $\mathcal{D}$ is a region to which Green's theorem applies. Let $\partial\mathcal{S}$ denote the oriented boundary of $\mathcal{S}$ and $\mathbf{F}$ be a C^1 vector field on $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \text{curl } \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F} \bullet d\mathbf{s}$$

If $\mathcal{S}$ has no boundary (e.g., sphere), then the integral on the left is zero.

Conservative vector field

Theorem

Let $\mathbf{F}$ be a C^1 vector field defined on $\mathbb{R}^3$ except maybe for a finite number of points. The following statements are equivalent:

1. $\forall \mathcal{C}$ *oriented simple closed curve,* $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{s} = 0$
2. $\forall \mathcal{C}_1, \mathcal{C}_2$ *oriented simple curves with same endpoints,*

$$\int_{\mathcal{C}_1} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \bullet d\mathbf{s}$$

3. $\mathbf{F}$ *is the gradient function of some function* f*, i.e.,* $\mathbf{F} = \nabla f$*.*
4. $\nabla \times \mathbf{F} = 0$

Definition (Conservative vector field)

A vector field $\mathbf{F}$ satisfying any (and thus all) of the conditions in the previous theorem is a **conservative vector field**.

Theorem

If $\mathbf{F}$ is a C^1 vector field on all of $\mathbb{R}^3$ with $\operatorname{div} \mathbf{F} = 0$, then there exists a C^1 field $\mathbf{G}$ with $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Theorem

If $\mathbf{F}$ is C^1 on $\mathbb{R}^2$, with $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ such that $\partial P/\partial y = \partial Q/\partial x$, then $\mathbf{F} = \nabla f$ for some f on $\mathbb{R}^2$.

Theorem

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ with component functions with continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Gauss' theorem a.k.a. the divergence theorem

Theorem (Gauss' theorem)

Let W be a symmetric elementary region in space. Denote by ∂W the oriented closed surface that bounds W. Let $\mathbf{F}$ be a smooth vector field defined on W. Then

$$\iiint_W \nabla \bullet \mathbf{F}\, dV = \iint_{\partial W} \mathbf{F} \bullet d\mathbf{S}$$

or, alternatively,

$$\iiint_W \operatorname{div} \mathbf{F}\, dV = \iint_{\partial W} \mathbf{F} \bullet \mathbf{n}\, dS$$