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MATH3820 Assignment 2

5/5 good.

Problem

There are many situations where things fail. Living things die, cars breakdown, bread goes mouldly, and light bulbs burn out, just to name a few.

However it is not usually known how long these things will last before failure. Some organisms live longer than others, and some light bulbs burn out sooner than others, for example. So, we need some sort of a mathematical model that will help us understand how quickly things will fail.

Description of System

Suppose we are studying some particular system. The model which will be developed is quite general and so this system can be any number of things. It is such that it begins in a certain state and at some point in time, it switches to some other state and remains there. We want a mathematical model which describes the length of time that the system spends in the first state. However, this length of time we are interested in is not a known value. It is uncertain when the switch is going to happen.

The system under study can be made up of smaller parts. The state of each of these parts determines the state of the larger system.

It is helpful to think of an electrical component, although different situations can be thought of in just the same way. At any given time, this component is either still properly functioning, or has failed. It is not known ahead of time how long this electrical component will remain functioning for, and so it is useful to have a model stating how likely it is that the component will still be working after a certain period of time.

Many smaller parts make up this component. These parts themselves can fail. When any one part stops working, the entire component stops working. The chance that the component is still functioning is depends on how likely each individual part is to be functioning.

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Assumptions

Assume that the lifetime of a component can be described by a random variable called the time to failure. Also, assume that many smaller elements make up this component. The lifetime of each element can be described by its own time to failure random variable. It is assumed that each element will fail independently of every other element. Whether or not a particular element fails will have no effect on the chance that any other element fails.

An element can fail in a number of different ways. For example, it may wear out over time, or it may have a physical defect. Assume there are K different types of failure and that there are N_k elements coresponding to failure type k, for $k = 1,\ 2, \ldots, K$. The total number of elements is N, so

$$\sum_{k=1}^{K} N_k = N \ .$$

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The elements make up what is called a series system. This means that each element must be functioning for the entire system to be funtioning. In other words, as soon as one fails, the whole system fails. For this reason, the random variable describing the lifetime of the system takes on the minimum value of the random varables which correspond to each element.

The number of elements that make up a component are assumed to follow one of two distributions. There are N elements in total. The first model states that N has a Poisson distribution. Its probability mass function is

Poisson (like fish in French)

$$f_N(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \ x = 0,1,2,\ldots$$

The second model considers the case where the number of elements has the negative binomial distribution. Its probability mass function is

$$f_N(x) = \binom{x-1}{r-1}(1-p)^{x-r} p^r, \ x = 0,1,2,\ldots$$

Derivation

Let W be the random variable, time to failure, for the system. Its cumulative distribution function describes the probabilty that the system will fail by a cetain time, t. So,

$$F_W(t) = P(W \leq t) \quad .$$

We are interested in the reliability function, $R(t)$, which describes the probability that the system will still be functioning at time t. This function is

$$R(t) = 1 - F_W(t) = P(W > t) \quad .$$

This is also known as the survival function or soujourn time. It represents the time spent in one state before the system switches to another state.

The assumption was made that there are K different types of failures. Let X_k be the time to failure random variable coresponding to failure type k, for $k = 1,2,\ldots,K$.

Let X_i, $i = 1,2,\ldots,n$, be n independent random variables, each having cumulative distribution function, $F(x)$. Let the random variable Z_n be the minimum value of the $X_i s$. So, $Z_n = min(X_1, X_2, \ldots, X_n)$. Now,

$$P(Z_n \geq x) = P(X_1 > x, X_2 > x, \ldots, X_n > x) \quad ,$$

but since the $X_i s$ are independent, why? (quick explanation would be nice...)

$$P(Z_n \geq x) = [1 - F(x)]^n \quad ,$$

and so,

$$P(Z_n \leq x) = 1 - [1 - F(x)]^n \quad .$$

There exist a distribution function $\Phi(x)$ and sequences $\{a_n\}$ and $\{b_n\}$ such that

this will need a bit of explaining..

$$\lim_{n\to\infty} [1-[1-F(a_n x+b_n)]^n] = 1 - \Phi(x)$$

if and only if

$$\lim_{n\to\infty} nF(a_{nx}+b) = -\ln(1-\Phi(x)) \quad .$$

Given the total number of elements within a component, the conditional distribution of failure types is assumed to be multinomial. Suppose that

$P(N_k = n_k) = p_k$, where $p_k > 0$, for $k = 1,2,\ldots n$, and that $\sum_{k=1}^{K} p_k = 1$. The distribution of failure types is given by

$$P(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k \mid N=n) = \frac{n!}{n_1!n_2!\ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} .$$

This formula gives the probability that, given the total number of elements, there will be a specified number of elements for each failure type.

Let $y(s)$ be the probability (factorial moment) generating function for the random variable N. Let $Y_i,\ i = 1,\ldots,N$ be any subset of the random variables X_k and define $Z_n = \sum_{min}^{\infty} \{Y_1,\ldots,Y_N\}, N>0$. Then we can take the sum of the products of each p_k with its distribution function, subtracted from one, and plug this into the probability generating function to get $P(Z_n>t)$.

One model assumes that N follows a Poisson distribution. In this case we have that, given $\varepsilon > 0$,

$$\left| P(Z_N>t) - \prod_{k=1}^{K} \left\{ 1 - \Phi_k \left(\frac{t-b_\lambda(k)}{a_\lambda(k)} \right)^{p_k} \right\} \right| < \varepsilon$$

which is true for all t.

The second model assumes that N follows a negative binomial distribution with parameters m and p. Here we have the following result. For $\varepsilon > 0$, and large $\mu = \frac{p}{1-p}$,

$$\left| P(Z_n>t) - \left[1 - \sum_{k=1}^{K} p_k \ln \left(1 - \Phi_k \left(\frac{t-b_\mu(k)}{a_\mu(k)} \right) \right) \right]^{-m} \right|$$

$$= \left| P(Z_n>t) - \left[\prod_{k=1}^{K} \ln \left(1 - \Phi_k \left(\frac{t-b_\tau(k)}{a_\tau(k)} \right) \right)^{p_k} \right]^{-m} \right| < \varepsilon$$

Synthesis

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Two models have develped which describe the reliability of components. One is based on the number of elements within a component following a Poisson distribution. We end up with the following reliability function,

$$R(t) \simeq \exp\left\{-\sum_{k=1}^{K}\left(\frac{t}{\theta_k}\right)^{\alpha_k}\right\}$$

where $\theta_k = \dfrac{a_\lambda}{p_k^{1/\alpha_k}}$.

The other, based on the negative binomial distribution gives this reliability function,

$$R(t) \simeq \left[1 + \sum_{k=1}^{K}\left(\frac{t}{\theta_k}\right)^{\alpha_k}\right]^{-m}$$

where $\theta_k = \dfrac{a_\mu(k)}{p_k^{1/\alpha_k}}$.