Math 3820 – Winter 2007
Sample midterm

$\boxed{1.}$ **The monomolecular law for single-species population growth, namely**

$$\frac{dN}{dt} = kN\frac{be^{-kt}}{1-be^{-kt}} \qquad (k>0\,, \quad 1>b>0)$$

has solution

$$N(t) = C\left(1-be^{-kt}\right) \,. \qquad (*)$$

Since $N \to C$ as $t \to \infty$, the parameter C is interpreted as the "carrying capacity" for the model.

Assume that a given set of data $\{(t_i\,, N_i) \text{ with } i=1,2,\ldots,n\}$ can be approximated by the monomolecular function (*) with known carrying capacity $C = 100000$.

Introduce a transformation of variables which will allow you to rewrite (*) in the form of a polynomial in t, and thus obtain a *linear system of equations* which can be solved to provide *least-squares estimates* for the parameters k and b appearing in (*) .

$\boxed{2.}$ Find a function $f:[0,1]\to[0,1]$ with exactly 3 fixed points, and draw its graph.

$\boxed{3.}$ Find a function $f:[0,1]\to[0,1]$ with no fixed points, and draw its graph.

4. One motivation that we discussed for the logistic law for population growth involved the introduction of a variable relative growth rate $g(N)$ into the Malthusian model to yield

$$\frac{dN}{dt} = N\, g(N), \qquad (**)$$

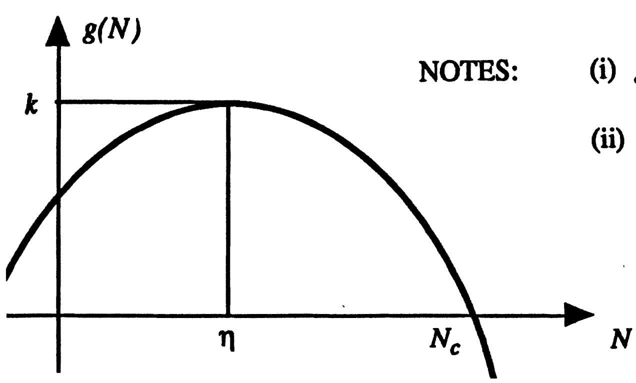
with $g(N)$ being chosen to be

$$g(N) = k\left(1 - \frac{N}{C}\right),$$

in which $k > 0$ is interpreted as the initial relative growth rate, and C is interpreted as the logistic carrying capacity. The above choice of $g(N)$ is made to guarantee that $g(N) \to 0$ as N increases (from its initial value $N_0 = N(0)$), so that $N = C$ becomes a stable equilibrium point of the logistic model.

The above development has been criticized in that it does not recognize the so-called Allee effect which requires that "*the relative growth rate is small when the population is small, reaches a maximum value at some intermediate population size* η *, and then decreases toward zero as* N *continues to increase* . "

In an effort to incorporate the Allee effect into a single-species population model, let us adopt equation (**) as the basis of the model, and furthermore suppose that the graph of $g(N)$ is shown below:



NOTES:

(i) $g(\eta) = k$ is the maximum value of $g(N)$.

(ii) $N = N_c$ is the only positive root of $g(N) = 0$.

Assume that $g(N)$ is a quadratic function of N of the form

$$g(N) = k - \alpha(N - \eta)^2 \qquad (\textit{for } N \geq 0\,)$$

and show that

$$\alpha < \frac{k}{\eta^2}, \quad N_c = \eta + \sqrt{\frac{k}{\alpha}}.$$

Construct a phase diagram for the resulting model, and use this information to sketch anticipated graphs of solutions of this model.

Compare and contrast this model with the logistic model

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right).$$

5. An experimental laboratory population with **known** (constant) relative growth rate $k > 0$ is established at time $t = t_0$ with exactly N_0 individuals.

If it is assumed that the population growth is governed by the **Malthusian law**

$$\frac{dN}{dt} = kN ,$$

show that the population size is given by

$$N(t) = N_0 e^{k(t-t_0)} .$$

Similarly, show that, under the assumption that the population growth is governed by the **logistic law**

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right)$$

with **specified** (constant) carrying capacity C, the population size at time t is given by

$$N(t) = \frac{C}{1 + \left(\dfrac{C}{N_0} - 1\right)e^{-k(t-t_0)}} .$$

The "doubling time" for a population is defined to be the length T of the time interval, measured from the initial time t_0, for the population to double its initial value [i.e., $N(t_0 + T) = 2N_0$].

In each of the two cases discussed in parts (a) and (b), find a formula for the "doubling time" for the given population, it being assumed in the case of the logistic law of part (b) that the initial population size N_0 is less than $\dfrac{C}{2}$.

Show that if N_0 is **very much smaller** than $\dfrac{C}{2}$, then the "doubling time" for the logistic law is approximately the same as the "doubling time" for the Malthusian law.