Instructions:

This is an OPEN-BOOK examination.
Any literature may be consulted.
Electronic calculators are permitted.
Attempt any combination of problems.
The total number of marks available is 130.
However, a score of 95 (or more) will be regarded as "full marks".

[10] 1. Assume that a given set of data

$$\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$$

may be approximated by a function of the form

$$y = Cxe^{-Dx},$$

where C and D are unknown constants to be determined.

Introduce a transformation of variables which will allow you to rewrite this function in the form of a polynomial, and thus obtain a *linear system of equations* which may be solved to provide *least-squares estimates* for the constants C and D appearing in the assumed approximating function.

It is **not** necessary to determine the numerical values for either the coefficients of the linear system or the solutions of this system.

However, the resulting system **must** be expressed in terms of the original variables and parameters.

VALUES

[10] 2. Consider the set of data given in the following table:

x	e^y
0	2.1
1	5.1
2	10.2
3	17.1
4	26.2
5	36.9
6	50.0
7	65.1
8	81.9
9	100.9

Assume that the data in the second column may be approximated by a polynomial in x of unknown degree.

(a) Illustrate how a difference table may be used to <u>suggest</u> the most appropriate choice for the degree of this polynomial.
[NOTE: It is <u>not</u> necessary to determine the values of the coefficients of this polynomial.]

(b) Based on your results of part (a), indicate the form of the functional dependence of y as an **explicit** function of x .

VALUES

[20] 3. In the absence of human intervention it is found that a given fish population experiences growth described by the Logistic Law

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right) \quad \text{(with } k > 0 \text{).}$$

In effort to make a profit from this resource, a local fishing company institutes a policy of "harvesting" this population at a constant rate "h" $(h > 0)$, and thus uses the modified law

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right) - h$$

subject to the initial condition $N(0) = N_0$, as its model to describe this situation.
DO NOT ATTEMPT TO SOLVE THIS MODEL.

(a) Find the equilibrium points of this model.

(b) **Assuming that the quadratic equation**

$$\frac{k}{C}N^2 - kN + h = 0$$

has a positive discriminant, plot $\frac{dN}{dt}$ vs. N, and thus sketch a graph of a typical solution (i.e., N vs. t) in each of the three cases:

(i) $N_2^* < N_0$,

(ii) $N_1^* < N_0 < N_2^*$

(iii) $N_0 < N_1^*$

where

$$N_2^* = \frac{C}{2k}\left(k + \left(k^2 - \frac{4kh}{C}\right)^{\frac{1}{2}}\right), \quad N_1^* = \frac{C}{2k}\left(k - \left(k^2 - \frac{4kh}{C}\right)^{\frac{1}{2}}\right)$$

(c) How is the situation **altered** in the case when $h > \frac{kC}{4}$?

Interpret your conclusion physically (i.e., explain you conclusion in terms of the assumed model.).

VALUES

[10] 4. The differential equation

$$\frac{dN}{dt} = 1 - e^{-k\left(1 - \frac{N}{C}\right)}$$

is sometimes used as an ***alternative*** to the logistic law for single-species population growth with "carrying capacity" C, since $\frac{dN}{dt} > 0$ for $N < C$ and $\frac{dN}{dt} \to 0$ as $N \to C^-$.

Solve this equation for the instantaneous population size $N = N(t)$ at time $t > 0$, given that $N(0) = N_0$.

[Note: it is **not** necessary to write this solution explicitly as a function of time; an implicit relationship between N and t is acceptable.]

VALUES

[20] 5. The standard Lotka-Volterra predator-prey model is

$$\left.\begin{aligned}\frac{dx}{dt} &= x(\ell - ny)\\ \frac{dy}{dt} &= y(mx - k)\end{aligned}\right\} \quad k, \ell, m, n > 0 \ .$$

However, certain species derive benefits from living in a "large" population, and hence the growth rate for each such species increases as its population size increases. (For example, wolves are 'pack' animals which hunt most effectively as a team, while bison are 'herd' animals deriving security from their herding instinct.) A model which has been proposed to study the interaction of such species is the so-called Lotka-Volterra model "with increasing returns", namely

$$\left.\begin{aligned}\frac{dx}{dt} &= x(\ell - ny) + px^2\\ \frac{dy}{dt} &= y(mx - k) + qy^2\end{aligned}\right\} \quad k, \ell, m, n, p, q > 0 \ .$$

Throughout the remainder of this problem, assume that $\dfrac{k}{q} > \dfrac{\ell}{n}$.

(a) Identify, and sketch on a phase-plane diagram, the nullclines of this model.

(b) Determine the equilibrium solutions of this model.

(c) In each of the regions into which the phase-plane is divided by the nullclines, indicate the direction to be followed by the trajectories of this model.

(d) Sketch anticipated trajectories of this model.

(e) Based on the above information, predict whether each of the equilibrium solutions of this model is "stable" or "unstable".

VALUES

[25] 6. Consider the following ***special case*** of the ***logistic competitive-hunters model***

$$\frac{dx}{dt} = x(\ell - ny - px)$$
$$\frac{dy}{dt} = ay(\ell - ny - px)$$

with a, ℓ, n and p positive constants.

(a) Identify the equilibrium point(s) of the model.

(b) On a phase-plane diagram sketch anticipated trajectories of this model.

(c) Does this model support or violate the "principle of competitive exclusion"?

(d) Find the equation of the trajectory which passes through the "initial" point (x_0, y_0) .

(e) If $a = \frac{1}{2}$, $\ell = 60$, $p = \frac{3}{1000}$ and $n = \frac{1}{500}$, and the trajectory begins at the initial point $(40000, 30000)$, determine the "ultimate outcome" of the competition between these two competing species.

VALUES

[20] 7. Consider the following ***special case*** of ***the "mutual grievance" version of the Richardson's Arms Race model*** for the total expenditures $x = x(t)$ and $y = y(t)$ for countries X and Y:

$$\frac{dx}{dt} = ky - mx + r$$
$$\frac{dy}{dt} = kx - my + s$$

with k, m, r and s positive constants.
Note: ***In this version of the model the "escalation coefficients" are identical for the two countries, as are the "braking coefficients".***

(a) If $z = z(t)$ denotes the total expenditure of the two countries

i.e., $z(t) = x(t) + y(t)$,

show that it must satisfy the differential equation

$$\frac{dz}{dt} = (k - m)z + (r + s).$$

(b) If $z_0 = z(0)$ denotes the initial value of $z(t)$ at time $t = 0$, show that

$$z(t) = \begin{cases} \left(\frac{r+s}{m-k}\right) + \left(z_0 - \frac{r+s}{m-k}\right) e^{(k-m)t}, & \text{for } k \neq m \\ (r+s)t + z_0, & \text{for } k = m. \end{cases}$$

(c) Evaluate the limit of $z(t)$ as $t \to \infty$.

(d) Explain the significance of the results of parts (b) and (c), making reference to the phase-plane diagrams discussed in lectures.

To help you understand this, it might be useful to show the appropriate phase-plane diagrams for this modified model, which may easily be obtained from the corresponding diagrams discussed in lectures by making the changes required to obtain the modified model.

THE UNIVERSITY OF MANITOBA

VALUES

[15] 8. Consider the probabilistic single-species population dynamics model given by the set of differential-difference equations

$$\frac{dP_N(t)}{dt} + bNP_N(t) = b(N-1)P_{N-1}(t) \text{ for } N \ge N_0 \text{ and } t \ge 0,$$

with initial conditions $P_N(0) = \begin{cases} 1 & \text{for } N = N_0 \\ 0 & \text{for } N \ne N_0 \end{cases}$.

We have shown that, for

$$P_{N_0}(t) = e^{-bN_0 t},$$

$$P_{N_0+1}(t) = N_0 e^{-bN_0 t}\left[1 - e^{-bt}\right],$$

and

$$P_{N_0+2}(t) = \frac{N_0(N_0+1)}{2} e^{-bN_0 t}\left[1 - e^{-bt}\right]^2,$$

which lead us to make the conjecture that

$$P_N(t) = \binom{N-1}{N_0-1} e^{-bN_0 t}\left[1 - e^{-bt}\right]^{(N-N_o)} \text{ for } N \ge N_0 .$$

Verify that this conjecture is correct.

THE END

HAVE A GREAT SUMMER!!