The Cooling of a Cup of Coffee

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Janice Morgan
5206992

THE COOLING OF COFFEE

The purpose of this paper is to investigate the cooling and heating of liquids, specifically coffee and cream, both individually and as a mixture, in some different circumstances. Suppose two people in a restaurant are served identical cups of coffee. For these purposes, it is sufficient that the temperature and volume (and consequently the mass) of the coffee is the same and the cups are of the same size and shape and made of the same material. One person immediately adds cream to his coffee. As it happens, there is no more cream available at the table and the waiter is asked to bring more. The person who took the last of the cream feels badly about it and does not start to drink his coffee. Ten minutes later, more cream arrives and the second person adds some to his coffee. The question which may arise is this: are the two cups of coffee the same temperature and, if not, which one is hotter?

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Experiment has shown that Newton's Law of Cooling can be used to obtain an estimate of the rate of change of temperature of a body as it cools. Newton's Law states that the rate of change of temperature of an immersed body is directly proportional to the difference in temperature between the ambient medium and the immersed body itself. In this case, the immersed body is the cup of coffee and the ambient medium is the air surrounding it.

Implicit in this law is the assumption that the immersed body and the ambient medium both have uniform temperature distributions. This is a reasonable assumption if the coffee is stirred constantly. Such nervous behaviour might occur if our cream hog felt sufficiently guilty and his unfortunate friend was in real need of a caffeine fix. The assumption may also be reasonable for the surrounding air if there is good ventilation in the room. However, some localized heating of the air around the coffee will undoubtedly occur. The table on which the coffee is sitting also forms part of the ambient medium. Even if we assume that the temperature distribution of the table is uniform, it does not seem likely that the table's temperature distribution is the same as that of the air. However, we will invoke the simplicity principle and assume that variation from a uniform temperature distribution in either the coffee or the surrounding air and underlying table is not signifigant.

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We will further assume that the temperature of the ambient medium is constant. This is a reasonable assumption for a room which is temperature controlled. Even if we consider an outdoor restaurant, we will further assume that any change in temperature of the ambient medium is insignifigant. This is plausible given the relatively short time frame we are considering. Finally, any heat energy transferred from the coffee to the air is assumed to be negligible.

Newton's Law of Cooling is given by the relation

$$\frac{dT}{dt} = k [A - T(t)] \qquad (1)$$

where T(t) is the temperature of coffee as a function of time. A is the temperature of the air, a constant, and k is an unknown constant. Clearly, if the coffee is hotter than the room, the rate of change of temperature is negative and $A - T(t)$ is also negative. Therefore k is always greater than 0.

The solution of equation (1), for $A < T$, is

$$\frac{dT}{T - A} = -k\ dt$$

$$\log (T - A) = -(kt + c)$$

$$T = A + e^{-(kt + c)} \qquad k > 0$$
$$c \text{ is a constant of integration}$$

$$T = A + Le^{-kt} \qquad k > 0,\quad L = e^{c} > 0$$

As t approaches infinity, e^{-kt} approaches 0 and therefore, as we would expect, T approaches A.

Let us now consider the parameters L and k. As indicated, $L > 0$. At time t = 0, $T(0) = A + L$. Therefore, $L = T(0) - A$. The physical interpretation of L is the initial temperature difference between the coffee and the air. Again, the parameter $k > 0$. This seems to relate to some inate physical or chemical property of a cup of coffee which governs the rate at which it cools. If A and two data points on T(t) are known, k can be estimated.

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It would appear that the size and shape of the cup and the material of which it is made might all have an effect on the value of k but these factors are assumed to be identical for each cup of coffee. Different liquids, such as coffee and cream, may have different values of k when all other factors are equal. For the time being, we will apply the simplicity principle and assume that k is the same for cream, coffee, and any mixture of the two.

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In order to incorporate cream into the model, all of the assumptions made about coffee, such as uniform temperature distribution, should also be applied to cream. Considering the case where the temperature of the cream is greater than that of the air, we will assume that equation (1) still holds. This could be considered to be a corollary to Newton's Law of Cooling. Here, the rate of change of temperature is positive and the quantity $A - T$ is also positive, since $A > T$. Therefore, we have $k > 0$ again. The solution of equation (1) is slightly different, however, when $A > T$.

?

$$\frac{dT}{A - T} = -\frac{dT}{T - A} = k\ dt$$

$$-\log (A - T) = kt + c$$

$$T = A - Le^{-kt} \qquad \text{with} \qquad k > 0,\quad L > 0$$

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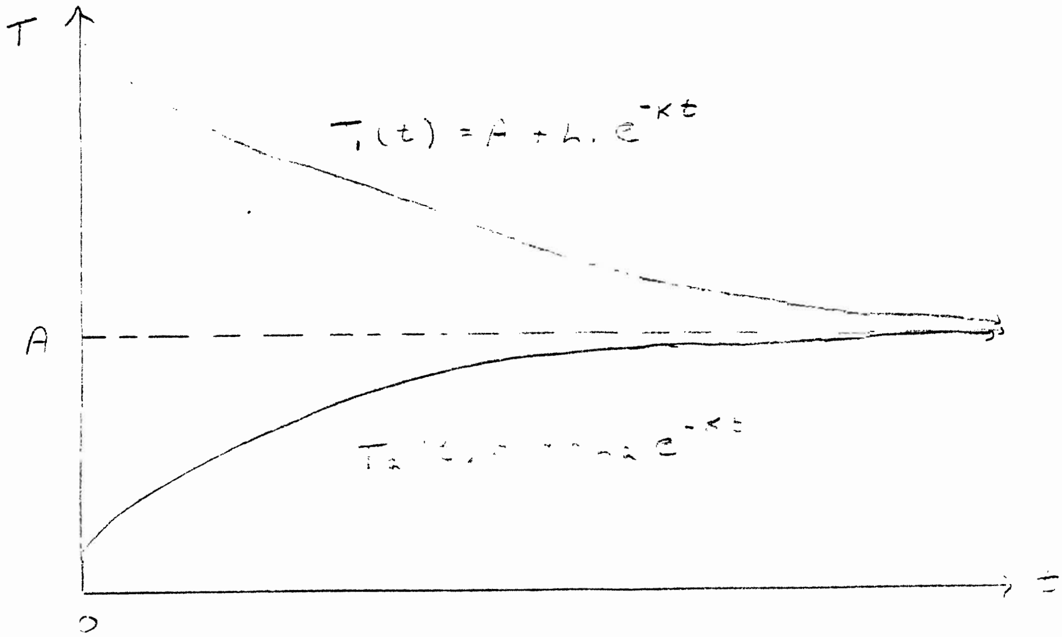
When the two solution curves are graphed together, it is obvious that a stable eqilibrium occurs at $T = A$. If $T > A$, T approaches A from above. If $T < A$, T approaches A from below.



Let us now apply the model to an example. We will assume that when cream is added to coffee, an instantaneous change in temperature occurs and the new temperature of the mixture is T^* where

$$T^* = \frac{T_1 M_1 + T_2 M_2}{M_1 + M_2}$$

The underlying assumption here is that there is only mixing of two liquids occuring and not some chemical reaction which might be exothermic or endothermic.

Suppose the restaurant is at a constant 20 degrees Celsius. The coffee is 90 degrees when it arrives and suppose that there is experimental data which says that, under these circumstances, black coffee will cool to 75 degrees in ten minutes. The cream has been stored at room temperature. We have, for black coffee

$$T(0) = 90 = 20 + L \qquad L = 70$$

$$T(10) = 75 = 20 + 70\, e^{-10 \times k}$$

$$\log\left\{\frac{55}{70}\right\} = -10 \times k$$

$$k = .0241162056$$

Once the black coffee has cooled for 10 minutes, the cream at 20 degrees is added. Suppose we have 200 grams of coffee and 15 grams of cream. The temperature of black coffee allowed to cool and with cream added after 10 minutes is $T^*(10)$ which is

$$\frac{200 \times 75 + 15 \times 20}{200 + 15} = 71.1627907$$

The temperature of the coffee with cream added immediately ,at time $t = 0$, is $T^*(0)$

$$\frac{200 \times 90 + 15 \times 20}{200 + 15} = 85.11627907$$

$$L = 65.11627907$$

$$T^*(10) = 20 + 65.11627907\, e^{-10\,k}$$

$$= 71.1627907$$

The time that the cream is added has no effect at all on the final temperature of the coffee when the cream is at room temperature.

Suppose now that the cream which was added to the first cup of coffee at time $t = 0$ was at 5 degrees celsius and that the cream added to the second cup of coffee was also at 5 degrees at time $t = 0$. In this case, the second cup of coffee will cool and the cream will warm independently, until they are mixed together at time t. Let t, the temperature of the coffee and the mass of coffee and cream be the same as in the previous example. So for black coffee we again have

$$T(0) = 90 \qquad T(10) = 75$$
$$L = 70 \qquad k = .0241162056$$

Now we must consider what is happening to the cream. We have assumed k to be equal for both coffee and cream. Therefore

$$T(0) = 5 = A - L\, e^{-k \times 0} \qquad L = 15$$

$$T(10) = 20 - 15\, e^{-k \times 10} = 8.214285705$$

When the cream is added to the coffee we have

$$T^*(10) = \frac{200 \times 75 + 15 \times 8.214285705}{200 + 15} = 70.3405316$$

For coffee with cream added immediately we have

$$T^*(0) = \frac{200 \times 90 + 15 \times 5}{200 + 15} = 84.06976744$$

$$T^*(0) = A + L \qquad L = 64.06976744$$

$$T^*(10) = 20 + L\, e^{-k \times 10} = 70.3405316$$

Again, the time the cream is added has no effect on the temperature.

Suppose now that the cream was refrigerated to 5 degrees in each case and that the rest of the circumstances remain the same. Again, for black coffee $T(10) = 75$. After the cream is added

$$T^*(10) = \frac{200 \times 75 + 15 \times 5}{200 + 15} = 69.97674419$$

For coffee with cream added immediately and then left to cool, the situation is identical to the previous example.

$$T^*(10) = 20 + 64.06976744\, e^{-10k} = 70.3405316$$

So the coffee with cream at 5 degrees added immediately is very slightly hotter after ten minutes than the coffee with cream at 5 degrees added ten minutes later. This makes sense on an intuitive level because each cup of coffee has cooled for ten minutes but, in the first case, the cream has been warming up for the same length of time while, in the second case, the cream has not had an opportunity to warm up at all.

In the first two examples, where both the coffee and the cream are immersed in the ambient medium for the same length of time, the model predicts that the two cups of coffee will be the same temperature. In the last example, the coffee with the cream which had not been exposed to the ambient medium was cooler. In fact, this model predicts that regardless of the initial temperature of either the coffee or the cream, the time at which they are mixed has no effect on the temperature of the mixture, so long as they have both been immersed in the ambient medium, the air, for the same length of time. Similarly, any component of a mixture which has not been immersed in the ambient medium will cause the temperature of the mixture to be lower if the temperature of the component is lower than that of the ambient medium and will cause the temperature of the mixture to be higher if the component's temperature is higher.

It appears that this relatively simple model, Newton's Law of Cooling behaves quite well. It does not predict anything which is contrary to experience or intuition. One of the difficulties is that in circumstances where it does predict a difference in temperature, this difference is too small to measure without spending more time, energy and money than any rational person would on a problem of this importance. That is, is would be virtually impossible to test the model. Perhaps a more signifigant criticism of the model is that there is no reason to assume, other than the simplicity principle, that k is constant for cream, coffee, and a mixture of the two.

We will consider an improvement to the model. Assume k in equation (1) has the value k_1 for coffee and k_2 for cream and some distinct value when cream is added to coffee. The rate of change of temperature increases as k increases. Therefore if k is relatively large, the curve T(t) will be steep and T will approach A fairly quickly, whereas if k is relatively small, the curve will be more flat and T will take a longer time to approach A. Let us further assume that when coffee and cream are mixed

$$k_3 = \frac{M_1 k_1 + M_2 k_2}{M_1 + M_2}$$

The rationale for this assumption is the simplicity principle. The effect of mixing coffee and cream on the value of k is assumed to be the same as the effect of mixing on T. Suppose $T_1(t)$ is the temperature function for coffee and $T_2(t)$ is the temperature function for cream.

$$T_1(t) = A + (T_1(0) - A)e^{-k_1 t} \qquad A < T_1(0)$$

$$T_2(t) = A - (A - T_2(0))e^{-k_2 t} \qquad A > T_1(0)$$

$$= A + (T_2(0) - A)e^{-k_2 t}$$

The formula can be generalized so that sign of $T(0) - A$ is irrelevant. Let $T_3(0)$ be the temperature function of the mixture mac at time $t = 0$. For the sake of simplifyin calculation, we will assume our restauran patrons have decided to make their own café au lait, that is, they are mixing equal parts of coffee and crear

$$T_3(t) = A + \left(\frac{T_1(0) + T_2(0)}{2} - A\right)e^{-\frac{k_1 + k_2}{2}t}$$ good

The temperature function of the mixture made at time $t = t$ is $T_*(t)$

$$T_*(t) = \frac{A + (T_1(0) - A)e^{-k_1 t} + A + (T_2(0) - A)e^{-k}}{2}$$

$$= A + \frac{1}{2}\left((T_1(0) - A)e^{-k_1 t} + (T_2(0) - A)e^{-k_2 t}\right)$$ ✓

Clearly $T_3(t)$ and $T*(t)$ are equal when $K_1 = K_2$ or when $t = 0$. ✓

example

Suppose $K_1 < K_2$ and that $K_1 = K$ and $K_2 = 3K$

$$T_3(t) = A + \tfrac{1}{2}\left(T_1(0) + T_2(0) - 2A\right)e^{-2Kt}$$

$$T*(t) = A + \tfrac{1}{2}\left((T_1(0) - A)e^{-Kt} + (T_2(0) - A)e^{-3Kt}\right)$$
$$= A + \tfrac{1}{2}e^{-Kt}\left(T_1(0) - A + (T_2(0) - A)e^{-2Kt}\right)$$

but $T_3(t)$ can be written as

$$= A + \tfrac{1}{2}e^{-Kt}\left(T_1(0)e^{-Kt} + T_2(0)e^{-Kt} - 2Ae^{-Kt}\right)$$

So if $T_1(0) - A + (T_2(0) - A)e^{-2Kt} >$ (2
$T_1(0)e^{-Kt} + T_2(0)e^{-Kt} - 2Ae^{-Kt}$

then $T*(t) > T_3(t)$

Let $T_1(0) = 90$, $T_2(0) = 5$ $A = 20$

(2) then becomes

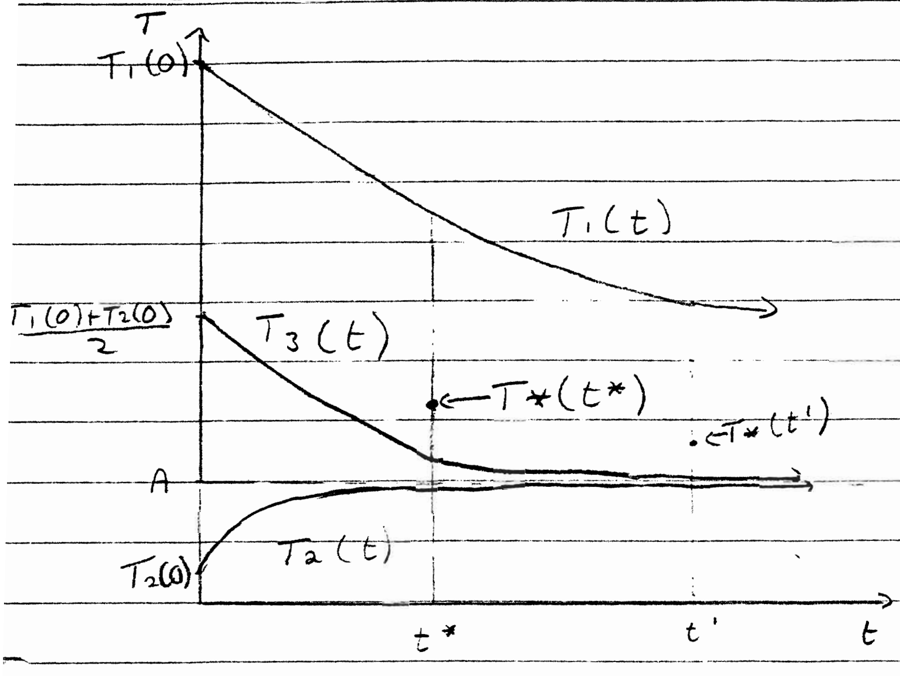
$$90 - 20 + (5 - 20)e^{-2Kt} >$$
$$90e^{-Kt} + 5e^{-Kt} - 2 \times 20e^{-Kt}$$

$\Rightarrow$

$$70 - 15e^{-2Kt} > 55e^{-Kt}$$

This inequality holds for $t > 0$ and is an equality at $t = 0$. So $T*(t)$, the mixture made later is always hotter than $T_3(t)$, the mixture made at time $t = 0$ when $K_1 < K_2$. ——— not quite this general

This example is represented by the following sketch



Considering the general case again we are comparing

for $T*(t)$ $\quad (T_1(0)-A)e^{-K_1 t}+(T_2(0)-A)e^{-K_2 t}$ $\quad (2)$

and

for $T_3(t)$ $\quad (T_1(0)+T_2(0)-2A)e^{-\frac{K_1+K_2}{2}t}$ $\quad (3)$

It is not clear which expression would be larger if only $K_1 < K_2$ is given. If we assume that $T_1(0) > A$ and $T_2(0) < A$ then for $T*(t)$ the term $(T_2(0)-A)e^{-K_2 t}$ is negative, and the term $(T_1(0)-A)e^{-K_1 t}$ is positive but this is not sufficient to determine the sign of expression (2). Similarly we do not have enough information to determine the sign of expression (3). However, if we are able to determine $T_1(0)$ and $T_1(\bar{t})$ where $\bar{t}$ is any fixed time we can estimate K_1 and with the same information about T_2 we can estimate K_2. From these parameters we can calculate K_3, $T_3(t)$ and $T^*(t)$, for any set of data. That is to say, one one piece of information, $T_2(\bar{t})$, the temperature of cream at $t = \bar{t}$, is all that is required to improve the model. This is obviously a significant improvement over the assumption $K_1 = K_2 = K_3$ with very little increase in complexity for calculating specific values of $T^*(t)$ and $T_3(t)$. Unfortunately there is some loss in the power of model to predict in general terms which cup of coffee will be hotter. It can be said, however, that if $K_1 \neq K_2$ and $t > 0$, the two cups of coffee will not generally be the same temperature. The factors affecting the answer to this question are the initial temperature difference between coffee and the room and between the cream and the room and of course the value of K_1 and K_2. The relationship between these factors warrants further investigation but time constraints prohibit this at this time.

There are a number of ways in which the model might be improved. Firstly, we could consider the situation where A(t) is not constant but some function of time. It does not seem necessary in the circumstances discussed here but an improvement of this type would be more important for cases where a longer time is considered or where the ambient medium is being changed during the course of the experiment. We could also reconsider Newton's Law of Cooling itself. It may be that the rate of change of temperature may be proportional to the difference in temperature of the immersed body and the ambient medium, raised to some exponent. That is

$$\frac{dT}{dt} = k\{ A(t) - T(t)\}^{a}$$

This model would require more information than simply two data points and some knowledge of A(t) in order to estimate the parameters. Since experimental data is required to proceed further, it might be appropriate to re-evaluate the signifigance of the problem before pursuing this avenue.

Another improvement which might not be as complex would be to consider an insulation factor. The assumption was made that the type of cup containing the coffee would effect the value of k. It seems more appropriate to assume that k is a property of the liquid itself and that the effect of the container should be accounted for in some other way. Perhaps

$$\frac{dT}{dt} = k\{ A(t) - T(t)\} + I(T)$$

probably should be/included with k.

where I(T) is a measure of the insulating value of the cup. This would make the model more general because the cream and the coffee would not have to be assumed to be in the same type of container, if it were possible to make some inferences about I(T) for both the coffee container and the cream container. Again, it would be necessary to have more initial information than was required by the simpler model in order to estimate the parameters of this model.

Finally, perhaps the most unreasonable physical assumption in the model was that of uniform temperature distribution for both the coffee and the ambient medium. There have been a number of studies of heat transfer and diffusion problems which might provide some insight to an interested reader as to how this problem might be overcome. However, the complexity of these models places them beyond the scope of this paper.

In conclusion, Newton's Law of Cooling would appear to be a nice simple model that behaves very well in the context of this simple problem, the cooling of coffee, before and after adding cream. It is unlikely that it could be improved upon signifigantly without experimentation and increased compexity which may be unwarranted in the circumstances. In addition, it makes reasonably good coffee table conversation.

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