6.337 Mid-Term Test - 2 March 1995

Time : 75 Minutes T. G. Berry

Note: The total number of "marks" available on this test is 60. However, a score of 50 (or more) will be regarded as "full marks". Thus, the student may do any combination of problems, or parts thereof, totalling 50 (or more) marks, bearing in mind that in multi-part problems each part may depend on previous parts.

10 1. The monomolecular law for single-species population growth, namely

$$\frac{dN}{dt} = kN\frac{be^{-kt}}{1-be^{-kt}} \quad (k>0,\ \ 1>b>0)$$

has solution

$$N(t) = C\left(1-be^{-kt}\right) . \qquad (*)$$

Since $N \to C$ as $t \to \infty$, the parameter C is interpreted as the "carrying capacity" for the model.

Assume that a given set of data $\{(t_i, N_i) \text{ with } i = 1,2,\ldots,n\}$ can be approximated by the monomolecular function (*) with known carrying capacity $C = 100000$.

Introduce a transformation of variables which will allow you to rewrite (*) in the form of a polynomial in t, and thus obtain a *linear system of equations* which can be solved to provide *least-squares estimates* for the parameters k and b appearing in (*).

10 2. You are given a set of data $\{ (x_i, y_i) \mid i = 0,1,2,\ldots,n \}$ with $x_i = x_0 + i\Delta x$, $\Delta x > 0$. Suppose that

$$Y_i = \ell n(y_i)$$

and that $\Delta^2 Y_i = \varepsilon$ (a constant).

(a) Find y_i as a function of x_i .

(b) Show that recursively one may find y_i in terms of y_{i-1} and y_{i-2} by the relation

$$y_i = K\frac{y_{i-1}^2}{y_{i-2}} \quad \left(K = e^{\varepsilon}\right) .$$

10 3. The solution of the logistic model for single-species population growth in a limited environment, namely

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right), \quad N(t_0) = N_0 \quad (k > 0)$$

may be written in the form

$$N(t) = \frac{C}{1 + e^{-k(t-t^*)}} \qquad (+)$$

where

$$t^* = t_0 - \frac{1}{k}\ell n\left(\frac{N_0}{C - N_0}\right). \qquad (++)$$

(a) Verify that when $N_0 < \frac{C}{2}$, this solution has a point of inflection at time $t = t^*$.

(b) Consider the population data shown in the accompanying table:

t	$N(t)$
0	10
1	15
2	22
3	31
4	42
5	55
6	67
7	79
8	90
9	98
10	103
11	106
12	108
13	109

Assume that this set of data may be modelled by the logistic function (+), with carrying capacity C being estimated as 110.

Obtain estimates for the remaining model parameters t^* and k as follows:

(i) use the results of part (a) and the above table to estimate t^* ,

(ii) use equation (++) to estimate k .

15 4. One motivation that we discussed for the logistic law for population growth involved the introduction of a <u>variable</u> relative growth rate g(N) into the Malthusian model to yield

$$\frac{dN}{dt} = N\, g(N), \qquad (**)$$

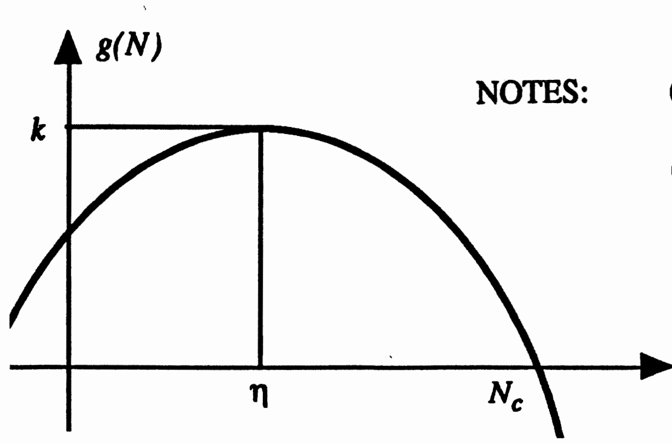
with $g(N)$ being chosen to be

$$g(N) = k\left(1 - \frac{N}{C}\right),$$

in which $k > 0$ is interpreted as the initial relative growth rate, and C is interpreted as the logistic carrying capacity. The above choice of $g(N)$ is made to guarantee that $g(N) \to 0$ as N increases (from its initial value $N_0 = N(0)$), so that $N = C$ becomes a stable equilibrium point of the logistic model.

The above development has been criticized in that it does not recognize the so-called <u>Allee effect</u> which requires that "*the relative growth rate is small when the population is small, reaches a maximum value at some intermediate population size* η *, and then decreases toward zero as* N *continues to increase* . "

In an effort to incorporate the Allee effect into a single-species population model, let us adopt equation (**) as the basis of the model, and furthermore suppose that the graph of $g(N)$ is shown below:



NOTES: (i) $g(\eta) = k$ is the maximum value of $g(N)$.

(ii) $N = N_c$ is the only positive root of $g(N) = 0$.

(a) Assume that $g(N)$ is a quadratic function of N of the form

$$g(N) = k - \alpha(N - \eta)^2 \qquad (for\ N \geq 0)$$

and show that

$$\alpha < \frac{k}{\eta^2}, \quad N_c = \eta + \sqrt{\frac{k}{\alpha}}.$$

(b) Construct a phase diagram for the resulting model, and use this information to sketch anticipated graphs of solutions of this model.

(c) Compare and contrast this model with the logistic model

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right).$$

15 5. Suppose that the concentration $C = C(t)$ of a drug administered directly into the bloostream is governed by the Malthusian Law

$$\frac{dC}{dt} = -\,k\,C\,,$$

in which k is a given positive "elimination" constant, characteristic of the particular drug being administered. It is well-known that the solution of this model is

$$C(t) = C^{*}e^{-k(t-t_0)}, \quad \text{with} \quad C^{*} = C(t_0)\,.$$

Assume that, with each successive "dose" of the drug, the concentration in the bloodstream is raised by a fixed amount C_0.

Moreover, assume that the drug is administered repeatedly at regular times $t = 0\,, T\,, 2T\,, 3T\,, \ldots$, where the length T of the "dosage time interval " is some (unknown) constant.

Let C_n denote the concentation in the bloodstream immediately after the n^{th} dose has been given, and let R_n denote the "residual" concentration immediately preceding the administration of the $(n+1)^{st}$ dose.

(a) Find C_n and R_n as functions of n.

(b) If it is assumed that the drug has a known "tolerance" concentration level S, above which unacceptable side effects occur, find the minimum dosage interval T which can be used, if the drug is to be used "safely for an indefinite amount of time".