I

A second-order linear, homogeneous differential equation with constant coefficients has two solutions $\{x_1, x_2\}$ which may take one of three forms:

$$\{e^{\lambda_1 t}, e^{\lambda_2 t}\}, \quad \{e^{\lambda_1 t}, te^{\lambda_1 t}\}, \quad \text{or} \quad \{e^{ut}\cos(vt), e^{ut}\sin(vt)\}.$$

In each case, compute the Wronskian and show that the solutions are linearly independent; the Wronskian does not equal zero.

II

For the following differential equations, find the equilibria; then graph the phase line diagrams. Use the phase line diagrams to determine the stability of the equilibria.

(a) $dx/dt = \sin(x)\cos(x)$

(b) $dx/dt = x(a - x)(x - b)^2, \quad 0 < a < b$

(c) $dx/dt = x(e^x - x - 2)$

III

A spatially implicit model for the proportion of islands occupied by a species was modeled by Levins (1969, 1970) and Hanski (1999). The model takes the following form:

$$\frac{dp}{dt} = (m + cp)(1 - p) - ep = f(p),$$

where $p(t)$ is the fraction of occupied islands at time t, $m \geq 0, c > 0$, and $e > 0$. The constants m, c, and e are the rates of immigration from the mainland, colonization, and extinction, respectively.

(a) Suppose there is no immigration from the mainland, $m = 0$, and the colonization rate is greater than the extinction rate, $c > e$. In Levins original model, $m = 0$. For this model, find the nonnegative equilibria and determine their stability. What happens if the colonization rate is less than the extinction rate, $c < e$?

(b) Suppose there is immigration from the mainland, $m > 0$. For this model, show that there exists a unique positive equilibrium which is asymptotically stable for all initial values $0 \leq p(0) \leq 1$. Note that $f(p)$ is quadratic in p. Use a phase line diagram.

IV

Find the general solution to the following differential equations, $x' = dx/dt$, $x'' = d^2x/dt^2$, and $x''' = d^3x/dt^3$.

(a) $x'' - x' - 6x = 0$

(b) $x'' - 4x' + 5x = 0$

(c) $x''' - 5x'' + 3x' + 9x = 0$

(d) $x''' + 16x' = 0$

(e) $x'' + 2ax' + (a^2 + b^2)x = 0, \ a, b \neq 0$

V

Use an integrating factor to find the unique solution to the following initial value problems.

(a) $\dfrac{dx}{dt} - 3t^2x = 4te^{-t^3}, \quad x(0) = 1$

(b) $\dfrac{dx}{dt} + \dfrac{2}{t}x = 2t + 5, \quad x(1) = 1$

VI

Growth of a population is modeled by the following differential equation:

$$\frac{dN}{dt} = \frac{\alpha n_2 + \beta n_1}{N} - (\alpha + \beta),$$

where α, β, n_1, and n_2 are positive constants.

(a) Find the equilibrium solution for this model; then draw a phase line diagram.

(b) If $N(0) > 0$, find $\lim_{t\to\infty} N(t)$.

VII

A mathematical model for the growth of a population is

$$\frac{dx}{dt} = \frac{2x^2}{1 + x^4} - x = f(x), \quad x(0) \geq 0,$$

where x is the population density. Find the equilibria and determine their stability. Sketch $f(x)$.

VIII. Classify the following ordinary differential equations by determining whether they are linear, what their order is, whether they are homogeneous, and whether their coefficients are constant.

(a) $(\sin x)y'' + \cos x = 0.$

(b) $y'' + y^2 = 2y'.$

(c) $\dfrac{d^3y}{dt^3} + \dfrac{2dy}{dt} = \sin y.$

(d) $\dfrac{d}{dt}(y^2 + 2y) = y.$

(e) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = e^t + e^{-t}.$

(f) $\dfrac{dy}{dt} = \dfrac{1}{1 + y}.$

(g) $\dfrac{dy}{dx} = \dfrac{1}{1 + x}.$

(h) $\dfrac{d^5y}{dx^5} = x^6 + 5x + 6.$

(i) $t\dfrac{dy}{dt} + ty = 1.$

IX. Find the steady states of the following systems of equations, and determine the Jacobian of the system for these steady states:

(a) $\dfrac{dx}{dt} = x^2 - y^2,$
$\dfrac{dy}{dt} = x(1 - y).$

(b) $\dfrac{dx}{dt} = y - xy,$
$\dfrac{dy}{dt} = xy.$

(c) $\dfrac{dx}{dt} = x - x^2 - xy,$
$\dfrac{dy}{dt} = y(1 - y).$

(d) $\dfrac{dx}{dt} = x - xy,$
$\dfrac{dy}{dt} = xy - y.$

X. Consider the equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0.$$

(a) Show that $x_1(t) = e^{3t}$ and $x_2(t) = e^{-t}$ are two solutions.
(b) Show that $x(t) = c_1x_1(t) + c_2x_2(t)$ is also a solution.

XI. The differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

has the general solution

$$x(t) = c_1e^{-t} + c_2e^{-2t}.$$

If we are told that, when $t = 0$, $x(0) = 1$ and its derivative $x'(0) = 1$, we can determine c_1 and c_2 by solving the equations