THE UNIVERSITY OF MANITOBA

Morning 14 August 2006 | FINAL EXAMINATION

PAPER NO.: | PAGE NO.: 1 of 7

DEPARTMENT & COURSE NO: Mathematics - 136.382 | Time: 3 hours

EXAMINATION: 136.382 - Introduction to Mathematical Modelling | EXAMINER: Dr. T. G. Berry

VALUES

Instructions:

This is an OPEN-BOOK examination.
Any literature may be consulted.
Electronic calculators are permitted.
Attempt any combination of problems.
The total number of marks available 110 .
However, a score of 80 (or more) will be regarded as "full marks".

[10] 1. The monomolecular law for single-species population growth, namely

$$\frac{dN}{dt} = kN\frac{be^{-kt}}{1-be^{-kt}} \qquad (k>0\,,\quad 1>b>0)$$

has solution

$$N(t) = C\left(1-be^{-kt}\right)$$

Since $N \to C$ as $t \to \infty$, the parameter C is interpreted as the "carrying capacity" for the model.

Assume that a given set of data $\left\{(t_i,N_i)\,\middle|\, N_i < C\,, i=1,2,\ldots,n\right\}$ can be approximated by the above monomolecular function with ***known*** carrying capacity $C = 100000$.

Introduce a transformation of variables which will allow you to rewrite $N(t)$ in the form of a polynomial in t , and thus obtain a *linear system of equations* which can be solved to provide *least-squares estimates* for the parameters k and b appearing in $N(t)$.

NOTE CAREFULLY: Your system must be expressed in terms of the original variables and parameters.

[15] 2. The following data represents a difference table for a function $y(x)$ which is *known to be a quadratic polynomial with integer coefficients* :

x_i	y_i	Δy_i	$\Delta^2 y_i$
1	2		
		5	
2	7		4
		9	
3	16		4
		13	
4	29		3
		16	
5	45		6
		22	
6	67		3
		25	
7	92		4
		29	
8	121		5
		34	
9	155		2
		36	
10	191		

Since $\Delta^2 y_i$ is not constant, it is clear that ***at least one*** of the recorded function values $y_i = y(x_i)$ is recorded incorrectly.

(a) Find and correct all errors in the table.
Explain in detail your reasons for making the corrections you propose.

(b) Use the corrected data to determine the coefficients of the quadratic function $y(x) = ax^2 + bx + c$ represented by this table.

[15] 3. The logistic Law for population growth in a limited environment states

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right)$$

in which $k > 0$ represents the instantaneous relative growth rate and $C >> k$ represents the carrying capacity (or maximum supportable population).

(a) Modify this model to incorporate an immigration term, under the assumption that the immigration rate is directly proportional (with factor of proportionality $r > 0$) to the difference between the carrying capacity and the existing population size.

(b) Plot $\frac{dN}{dt}$ vs. N , identify the equilibrium solutions, and use this information to sketch graphs of typical solutions $N = N(t)$ for various choices of initial population size $N_0 = N(0)$.

(c) Using the transformation of variables

$$\eta = N + \frac{rC}{k}$$

in order to rewrite the given differential equation in the form of a logistic equation, namely

$$\frac{d\eta}{dt} = k^* \, \eta\left(1 - \frac{\eta}{C^*}\right),$$

indicating clearly the relationship between the parameters k, k^*, C and C^*.

(d) Use the results of part (c) to find an analytic expression for the solution $\eta = \eta(t)$ of the differential equation of part (c).

(e) Use the result of part (d) to find an analytic expression for the solution $N = N(t)$ of the differential equation of part (a).

VALUES

[20] 4. The standard Lotka-Volterra predator-prey model is

$$\left.\begin{aligned} \frac{dx}{dt} &= x(\ell - ny) \\ \frac{dy}{dt} &= y(mx - k) \end{aligned}\right\} \quad k,\ell,m,n > 0 \ .$$

However, certain species derive benefits from living in a "large" population, and hence the growth rate for each such species increases as its population size increases. (For example, wolves are 'pack' animals which hunt most effectively as a team, while bison are 'herd' animals deriving security from their herding instinct.) A model which has been proposed to study the interaction of such species is the so-called Lotka-Volterra model "with increasing returns", namely

$$\left.\begin{aligned} \frac{dx}{dt} &= x(\ell - ny) + px^2 \\ \frac{dy}{dt} &= y(mx - k) + qy^2 \end{aligned}\right\} \quad k,\ell,m,n,p,q > 0 \ .$$

Throughout the remainder of this problem, assume that $\frac{k}{q} > \frac{\ell}{n}$.

(a) Identify, and sketch on a phase-plane diagram, the nullclines of this model.

(b) Determine the equilibrium solutions of this model.

(c) In each of the regions into which the phase-plane is divided by the nullclines, indicate the direction to be followed by the trajectories of this model.

(d) Sketch anticipated trajectories of this model.

(e) Based on the above information, predict whether each of the equilibrium solutions of this model is "stable" or "unstable".

VALUES

[20] 5. Consider the special case of the competitive-hunters model:

$$\begin{Bmatrix} \dfrac{dx}{dt} = \ell x - nxy \\ \dfrac{dy}{dt} = ky - mxy \end{Bmatrix}$$

with $\ell = 0.2, n = 0.001, k = 0.4, m = 0.002$.

(a) What are the critical values x_c and y_c of the two populations?

(b) Find an explicit formula for the trajectory through the initial point $(x_0, y_0) = (100,150)$.

(c) Find an equation which determines the maximum value x_{max} of the population whose size at time t is given by $x(t)$.

(d) Describe in detail (showing all relevant equations) a mathematical procedure for approximating the solution x_{max} of the equation of part (c).
Note: It is not necessary to implement this procedure; merely explain fully the mathematical procedure you would use, showing all relevant mathematical details.

VALUES

[15] 6. For the probabilistic single-species population dynamics model we discussed in lectures, the probability $P_N(t)$ that the population is of size $N \ge N_0$ at time $t \ge 0$ is given by

$$P_N(t) = \binom{N-1}{N_0-1} e^{-bN_0 t}\left[1 - e^{-bt}\right]^{(N-N_0)} .$$

For each $N \ge N_0$, $P_N(t)$ attains a single relative maximum value $(P_N)_{MAX}$ at time $(t_M)_N = \frac{1}{b}\ell n\left(\frac{N}{N_0}\right)$.

(a) Show that, as a function of N and N_0, $(P_N)_{MAX}$ is given by

$$(P_N)_{MAX} = \frac{(N-1)!}{(N_0-1)!(N-N_0)!} \frac{N_0^{N_0}(N-N_0)^{N-N_0}}{N^N} \quad \text{for } N \ge N_0 .$$

(b) Consider the sequence $\left\{(P_N)_{MAX}\right\}_{N=N_0}^{\infty}$ of maximum probabilities.

Verify the claim that this sequence is monotone decreasing. Show all your work and explain fully why you may draw this conclusion.

(c) Explain why one should intuitively expect the result of part (b).

VALUES

[15] 7. Consider a stable population of constant size N.

Suppose that an epidemic spreads through this population, in such a way that, at any instant in time, each member belongs to precisely one of the following "compartments":

"susceptibles" (size $S(t)$),
"infecteds" (size $I(t)$),
or "removeds" (size $R(t)$).

Assume that this situation is modelled by the system of equations:

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dR}{dt} = rI \\ S + I + R = N \end{cases}$$

in which β , r and N are positive constants.

Throughout the remainder of this problem, assume that these constants have the values

$$r = 0.9 \ , \ \beta = 0.0002 \ \text{ and } \ N = 10000 \ .$$

If $S(0) = 9990$ and $I(0) = 10$, find the maximum number of infected individuals I_{max} for this epidemic, using the following procedure:

(a) Determine condition(s) which guarantee that $\dfrac{dI}{dt} = 0$.

(b) Find a relationship between I and S along the trajectory of this system satisfying the given initial conditions.

(c) Use the results of parts (a) and (b) in order to find the maximum value I_{max} of $I(t)$ along the chosen trajectory.

THE END