SIAM J. NUMER. ANAL.
Vol. 17, No. 1, February 1980

0036-1429/80/1701-0006 $01.00/0

THE INVERSES OF SOME MATRICES DEVIATING SLIGHTLY FROM A SYMMETRIC, TRIDIAGONAL, TOEPLITZ FORM*

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Abstract. In some problems in numerical analysis one is faced with solving a linear system of equations in which the matrix of the linear system is symmetric, tridiagonal and Toeplitz, except for elements at or near the corners. The anomalous elements usually arise from boundary conditions in the original problem. This paper is concerned with expressing the inverse of this type of matrix in such a way that some of its theoretical properties may be obtained.

1. Introduction. Symmetric, tridiagonal, Toeplitz matrices arise as the matrix of a linear system of equations in several problems in numerical analysis. The equations for interpolating cubic splines with equi-spaced knots and certain boundary conditions [1, p. 11 and p. 13] have this special form. Symmetric, tridiagonal, Toeplitz matrices are used as comparison matrices in the error analysis of second order boundary value problems with certain boundary conditions [4, p. 362]. The explicit inverse of the symmetric, tridiagonal, Toeplitz matrix has been found by several authors [8], [3] and [9].

This paper is concerned with matrices which differ slightly from the symmetric, tridiagonal, Toeplitz pattern in that elements at or near the corners of the matrix do not fit that pattern. Such matrices arise in interpolation with cubic splines if the boundary conditions are natural conditions [2, p. 133] or periodic boundary conditions [1, p. 12], or a variety of other conditions [6]. The error analysis of the finite difference solution of second order boundary value problems with certain boundary conditions requires a comparison matrix of this type [7].

Kershaw [5] has found the inverses of two matrices which are slight variations of symmetric, tridiagonal, Toeplitz matrices. In this paper more extensive variations of the symmetric, tridiagonal, Toeplitz form are considered. The theory of difference equations is used to develop formulae for the inverses of the matrices considered.

2. Matrices with four anomalous corner elements. The first matrix to examine, B, is a symmetric, tridiagonal, Toeplitz matrix which has had the four corner elements changed. Let B be the $N\times N$ matrix

$$B=\begin{pmatrix} a & -1 & & & c \\ -1 & x & -1 & & 0 \\ & -1 & x & \ddots & \\ & 0 & \ddots & x & -1 \\ b & & & -1 & d \end{pmatrix}_{N\times N}$$

and supposing that B is nonsingular, let $S=(s_{i,j})_{N\times N}$ be the inverse of B. The identity $BS=I$ is expressed by the equations

$$as_{1,j}-s_{2,j}+cs_{N,j}=\delta_{1,j}, \tag{1}$$

$$-s_{i-1,j}+xs_{i,j}-s_{i+1,j}=\delta_{i,j}, \qquad i=2,3,\cdots,N-1, \tag{2}$$

and

$$bs_{1,j}-s_{N-1,j}+ds_{N,j}=\delta_{N,j} \tag{3}$$

for each $j=1,2,\cdots,N$, where $\delta_{i,j}$ is the Kronecker delta.

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* Received by the editors August 16, 1978.

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The elements $s_{i,j}$ can be written as a linear sum of the solutions to two difference equations which have been solved explicitly. The two difference equations are:

(4) $\quad -e_{i-1}+xe_i-e_{i+1}=0, \qquad i=2,3,\cdots,N-1, \qquad e_1=0 \quad \text{and } e_N=1,$

and

(5) $\quad -t_{i-1,j}+xt_{i,j}-t_{i+1,j}=\delta_{i,j}, \qquad i=2,3,\cdots,N-1, \qquad t_{1,j}=0 \quad \text{and} \quad t_{N,j}=0,$

for $j=2,3,\cdots,N-1$, while $t_{i,1}=t_{i,N}=0$.

The solutions to (4) and (5) are (see [3])

(6)
$$e_i=\begin{cases}\sinh(i-1)\theta/\sinh(N-1)\theta, & \text{if } x>2 \text{ and } 2\cosh\theta=x,\\ (i-1)/(N-1) & \text{if } x=2,\\ \sin(i-1)\theta/\sin(N-1)\theta, & \text{if } x<2 \text{ and } 2\cos\theta=x,\end{cases}$$

and

(7)
$$t_{i,j}=\begin{cases}e_je_{N-i+1}/e_2 & \text{if } i\geqq j,\\ e_ie_{N-j+1}/e_2 & \text{if } i<j.\end{cases}$$

Suppose $s_{i,j}$ is expressed as

(8)
$$s_{i,j}=t_{i,j}+A_je_{N-i+1}+B_je_i,$$

where A_j and B_j have yet to be determined. It can be seen that the above expression for $s_{i,j}$ will always satisfy the equations (2), and that in general A_j and B_j can be determined so that it satisfies equations (1) and (3). If the expression (8) is substituted into equation (1), then

(9)
$$(a-e_{N-1})A_j+(c-e_2)B_j=t_{2,j}+\delta_{1,j}$$

and if the expression (8) is substituted into equation (3), then

(10)
$$(b-e_2)A_j+(d-e_{N-1})B_j=t_{N-1,j}+\delta_{N,j}.$$

The quantity $t_{2,j}+\delta_{1,j}$ is equal to e_{N-j+1}, and the quantity $t_{N-1,j}+\delta_{N,j}$ is equal to e_j (see equation (7)), thus the equations for A_j and B_j become

(11)
$$\begin{pmatrix}a-e_{N-1} & c-e_2\\ b-e_2 & d-e_{N-1}\end{pmatrix}\begin{pmatrix}A_j\\ B_j\end{pmatrix}=\begin{pmatrix}e_{N-j+1}\\ e_j\end{pmatrix}.$$

If equation (11) has a unique solution, then the inverse of matrix B is $S=(s_{i,j})_{N\times N}$ where $s_{i,j}$ is expressed in the form (8), e_i and $t_{i,j}$ are given by (4) and (5), and A_j and B_j are given by (11).

As an example of the use of the above formulae, sufficient conditions on a, b, c, d and x so that $B^{-1}\geqq 0$ will now be stated. Firstly, take $x\geqq 2$ so that $e_i\geqq 0$ for $i=1,2,\cdots,N$, and $t_{i,j}\geqq 0$ for all pairs (i,j), $i=1,2,\cdots,N$, $j=1,2,\cdots,N$. From expression (8), it is clear that A_j and B_j should be nonnegative, thus

(12)
$$\begin{pmatrix}a-e_{N-1} & c-e_2\\ b-e_2 & d-e_{N-1}\end{pmatrix}^{-1}\geqq 0$$

is required. This can be simplified somewhat, but weakened, by the requirement

(13)
$$\begin{pmatrix}a-r & c\\ b & d-r\end{pmatrix}^{-1}>0,$$

where $r=x/2-\sqrt{(x/2)^2-1}$. If N is sufficiently large, then condition (13) implies condition (12). Thus $x\geqq 2$ and condition (13) ensure that $B^{-1}\geqq 0$ for all N above some value. For example,

$$\begin{pmatrix} a & -1 & & 0 & c \\ -1 & 5.2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 5.2 & -1 \\ b & 0 & & -1 & d \end{pmatrix}_{N\times N}^{-1} \geqq 0$$

for N sufficiently large if $a \geqq 0.2$, $d \geqq 0.2$, $c \leqq 0$, $b \leqq 0$ and $(a-0.2)(d-0.2) > bc$.

It should be noted that if matrix B does have a positive inverse, then the row sums of B^{-1} can be obtained fairly easily since formulae for $\sum_{j=1}^{N} t_{ij}$ and $\sum_{i=1}^{N} e_i$ are known [3].

3. Matrices with sixteen anomalous corner elements. The second matrix to examine is a symmetric, tridiagonal Toeplitz matrix in which sixteen corner elements have been changed. Let B be the $N \times N$ matrix

$$B = \begin{pmatrix} a_{11} & a_{12} & & & & c_{11} & c_{12} \\ a_{21} & a_{22} & -1 & & & c_{21} & c_{22} \\ & -1 & x & -1 & 0 & & \\ & & -1 & \ddots & \ddots & & \\ & & 0 & \ddots & x & -1 & \\ b_{11} & b_{12} & & & -1 & d_{11} & d_{12} \\ b_{21} & b_{22} & & & & d_{21} & d_{22} \end{pmatrix}_{N\times N}$$

and supposing B is nonsingular, let $S = (s_{i,j})_{N\times N}$ be the inverse of B. The development of formulae for $s_{i,j}$ is similar to that used in the previous section and will be sketched here. The identity $BS = I$ can be written:

$$a_{11}s_{1,j} + a_{12}s_{2,j} + c_{11}s_{N-1,j} + c_{12}s_{N,j} = \delta_{1,j},$$

$$a_{21}s_{1,j} + a_{22}s_{2,j} - s_{3,j} + c_{21}s_{N-1,j} + c_{22}s_{N,j} = \delta_{2,j},$$

$$-s_{i-1,j} + xs_{i,j} - s_{i+1,j} = \delta_{i,j}, \qquad i = 3, 4, \cdots, N-2,$$

$$b_{11}s_{1,j} + b_{12}s_{2,j} - s_{N-2,j} + d_{11}s_{N-1,j} + d_{12}s_{N,j} = \delta_{N-1,j},$$

$$b_{21}s_{1,j} + b_{22}s_{2,j} + d_{21}s_{N-1,j} + d_{22}s_{N,j} = \delta_{N,j}.$$

If e_i and $t_{i,j}$ are the solutions to the following difference equations

$$-e_{i-1} + xe_i - e_{i+1} = 0, \qquad i = 3, 4, \cdots, N-2, \qquad e_2 = 0 \quad \text{and} \quad e_{N-1} = 1$$

and

$$-t_{i-1,j} + xt_{i,j} - t_{i+1,j} = \delta_{i,j}, \qquad i = 3, 4, \cdots, N-2, \qquad t_{2,j} = 0 \quad \text{and} \quad t_{N-1,j} = 0,$$

for $j = 3, 4, \cdots, N-2$, while $t_{i,2} = t_{i,N-1} = 0$; then $s_{i,j}$ can in general be expressed as

$$s_{i,j} = t_{i,j} + A_j e_{N-i+1} + B_j e_i, \qquad i = 2, 3, \cdots, N-1.$$

The quantities $s_{1,j}$, $s_{N,j}$, A_j and B_j must satisfy the equation

$$\begin{pmatrix} a_{11} & a_{12} & c_{11} & c_{12} \\ a_{21} & a_{22} - e_{N-2} & c_{21} - e_3 & c_{22} \\ b_{11} & b_{12} - e_3 & d_{11} - e_{N-2} & d_{12} \\ b_{21} & b_{22} & d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} s_{1,j} \\ A_j \\ B_j \\ s_{N,j} \end{pmatrix} = \begin{pmatrix} \delta_{1,j} \\ e_{N-j+1} \\ e_j \\ \delta_{N,j} \end{pmatrix}.$$

and it will be assumed that this equation has a unique solution.

As an example of the usefulness of the above formulae, consider sufficient conditions that B^{-1} be nonnegative. The choice $x \geqq 2$ and the choice of corner elements so that

$$\begin{pmatrix} a_{11} & a_{12} & c_{11} & c_{21} \\ a_{21} & a_{22}-e_{N-2} & c_{21}-e_3 & c_{22} \\ b_{11} & b_{12}-e_3 & d_{11}-e_{N-2} & d_{12} \\ b_{21} & b_{22} & d_{21} & d_{22} \end{pmatrix}^{-1} \geqq 0$$

ensures that $B^{-1} \geqq 0$. As before, the above condition may be replaced by

$$\begin{pmatrix} a_{11} & a_{12} & c_{11} & c_{21} \\ a_{21} & a_{22}-r & c_{21} & c_{22} \\ b_{11} & b_{12} & d_{11}-r & d_{12} \\ b_{12} & b_{22} & d_{21} & d_{22} \end{pmatrix}^{-1} > 0,$$

where $r = x/2 - \sqrt{(x/2)^2 - 1}$ and then one can say that $B^{-1} \geqq 0$ for sufficiently large N.

Since

$$\begin{pmatrix} 8 & -4 & 2 & -1 \\ -4 & 7 & -4 & 2 \\ 2 & -4 & 7 & -4 \\ -1 & 2 & -4 & 8 \end{pmatrix}^{-1} > 0,$$

$$\begin{pmatrix} 8 & -4 & & & & 2 & -1 \\ -4 & 8 & -1 & & & -4 & 2 \\ & -1 & 2 & -1 & 0 & & \\ & & -1 & \ddots & \ddots & & \\ & & & \ddots & 2 & -1 & \\ 2 & -4 & 0 & & -1 & 8 & -4 \\ -1 & 2 & & & & -4 & 8 \end{pmatrix}^{-1}_{N \times N} \geqq 0$$

for all N above some value

Acknowledgment. The author acknowledges the financial support of the National Research Council of Canada.

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