Math 136.382 Final Exam Solutions
Summer 2006

① $$N = C(1 - be^{-kt}) \qquad (k > 0)$$

Assume C is known

$$\frac{N}{C} = 1 - be^{-kt}$$

$$\left(1 - \frac{N}{C}\right) = be^{-kt} \quad 1$$

$$\ln\left(1 - \frac{N}{C}\right) = \ln b - kt. \quad 1$$

let $Y = \ln\left(1 - \frac{N}{C}\right)$ 1, so $Y = -kt + c$
$c = \ln b$ 1

For least-squares

$$\left.\begin{aligned}\left(\sum t_i^2\right)(-k) + \left(\sum t_i\right)c &= \sum t_i Y_i \\ \left(\sum t_i\right)(-k) + nc &= \sum Y_i\end{aligned}\right\} \quad 3$$

i.e.
$$\left.\begin{aligned}\left(\sum_{i=1}^{n} t_i^2\right)(-k) + \left(\sum_{i=1}^{n} t_i\right)\ln b &= \sum_{i=1}^{n} t_i \ln\left(1 - \frac{N_i}{C}\right) \\ \left(\sum_{i=1}^{n} t_i\right)(-k) + n\ln b &= \sum_{i=1}^{n} \ln\left(1 - \frac{N_i}{C}\right)\end{aligned}\right\} \quad 3$$

② (a) Note: $\Delta^2 y_i$ should be constant & has ①

average value $\dfrac{4+4+3+6+3+4+5+2}{8}$

$= \dfrac{31}{8} \simeq 4.$ ①

Thus, assume $\Delta^2 y_i = 4$ ① (since the polynomial has integer coefficients & hence must give integer values only) ①

Corrected table

x_i	y_i	Δy_i	$\Delta^2 y_i$
1	2	5	
2	7	9	4
3	16	13	4
4	29	17	4
5	46	21	4
6	67	25	4
7	92	29	4
8	121	33	4
9	154	37	4
10	191		

Corrections shown.

⑥ corrections

⑩

(b) need only 3 data pts — use first three

$y = ax^2 + bx + c$

$$\rightarrow \left.\begin{aligned} 2 &= a+b+c \\ 7 &= 4a+2b+c \\ 16 &= 9a+3b+c \end{aligned}\right\} ①$$

$5 = 3a + b$

$9 = 5a + b$ $\quad 4 = 2a$

$$\left.\begin{aligned} a &= 2 \\ b &= -1 \\ c &= 1 \end{aligned}\right\} ③$$

⑤

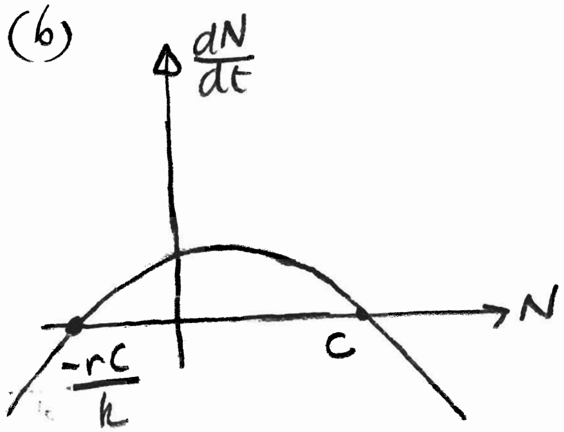
$y = 2x^2 - x + 1$ ①

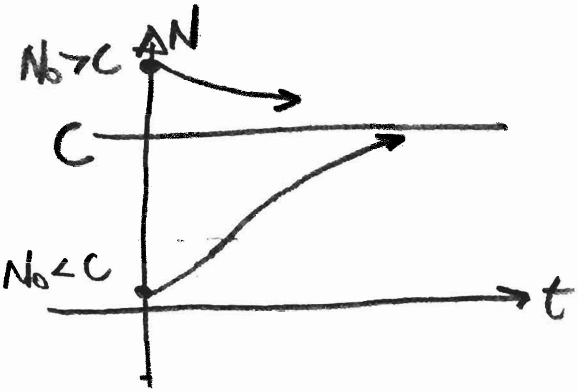
③ (a) $\dfrac{dN}{dt} = kN\left(1-\dfrac{N}{C}\right) + i$

$i = r(C-N)$ $\qquad$ $r>0$ so $i>0$ when $N<C$.

$$\frac{dN}{dt} = kN\left(1-\frac{N}{C}\right) + rC\left(1-\frac{N}{C}\right)$$
$$= (kN + rC)\left(1-\frac{N}{C}\right)$$

(b)





note: $\dfrac{-rC}{k} < 0$

(c) $M = N + \dfrac{rC}{k} \Longrightarrow \dfrac{dM}{dt} = \dfrac{dN}{dt}$

$\Downarrow$

$N = M - \dfrac{rC}{k}$

$$= \left(k\left(M - \frac{rC}{k}\right) + rC\right)\left(1 - \frac{M - \frac{rC}{k}}{C}\right)$$
$$= kM\left(\frac{C - M + \frac{rC}{k}}{C}\right)$$
$$= kM\left[\frac{C\left(1+\frac{r}{k}\right) - M}{C}\right]$$
$$= kM\left(1+\frac{r}{k}\right)\left[1 - \frac{M}{C\left(1+\frac{r}{k}\right)}\right]$$
$$= k^* M\left[1 - \frac{M}{C^*}\right]$$

with $$k^* = k\left(1+\frac{r}{k}\right) > k$$
$$C^* = C\left(1+\frac{r}{k}\right) > C$$

(d) $$\eta = \frac{C^*}{1+Fe^{-k^* t}}$$

(e) But $\eta = N + \frac{rC}{k}$, $\quad k^* = k\left(1+\frac{r}{k}\right)$
$\quad C^* = C\left(1+\frac{r}{k}\right)$

so $$N + \frac{rC}{k} = \frac{C(1+r/k)}{1+Fe^{-k(1+r/k)t}}$$

$$\Rightarrow N = \frac{C(1+r/k)}{1+Fe^{-k(1+r/k)t}} - \frac{rC}{k}.$$

4

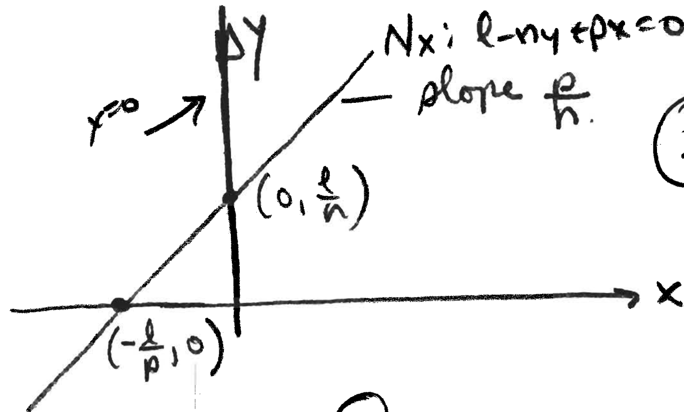
$$\frac{dx}{dt} = x(l - ny) + px^2$$

$$\frac{dy}{dt} = y(mx - k) + qy^2$$

$\left.\right\}$ $\frac{k}{q} > \frac{l}{n}$

(a) nullclines: for x: N_x : $x = 0$ 1 or $l - ny + px = 0$ 1

$\longrightarrow y = \frac{l}{n} + \frac{p}{n}x$.



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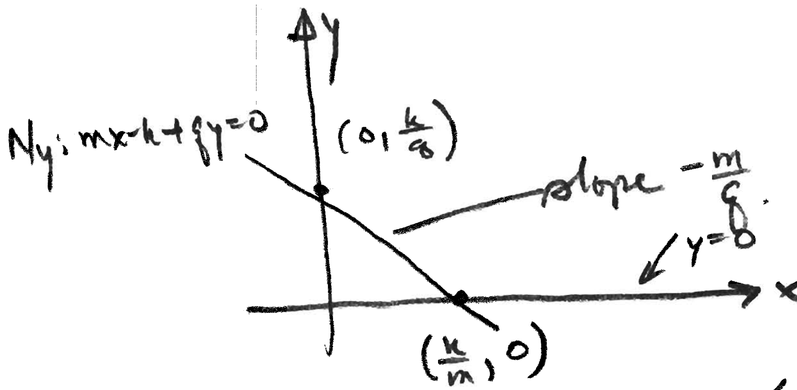
for y: N_y : $y = 0$ 1 or $mx - k + qy = 0$ 1

$y = \frac{k}{q} - \frac{m}{q}x$.

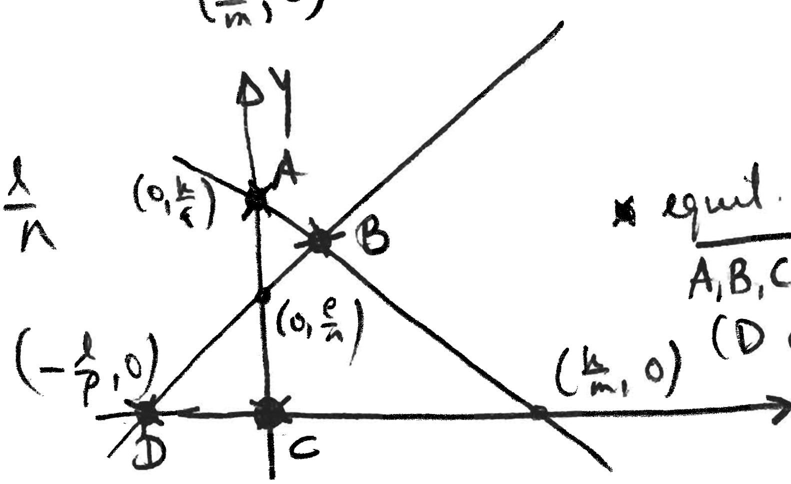


2

~~8~~

(b) superimpose with $\frac{k}{q} > \frac{l}{n}$

~~3~~



$\times$ equil. pts.

A, B, C, D

(D not physically acceptable)

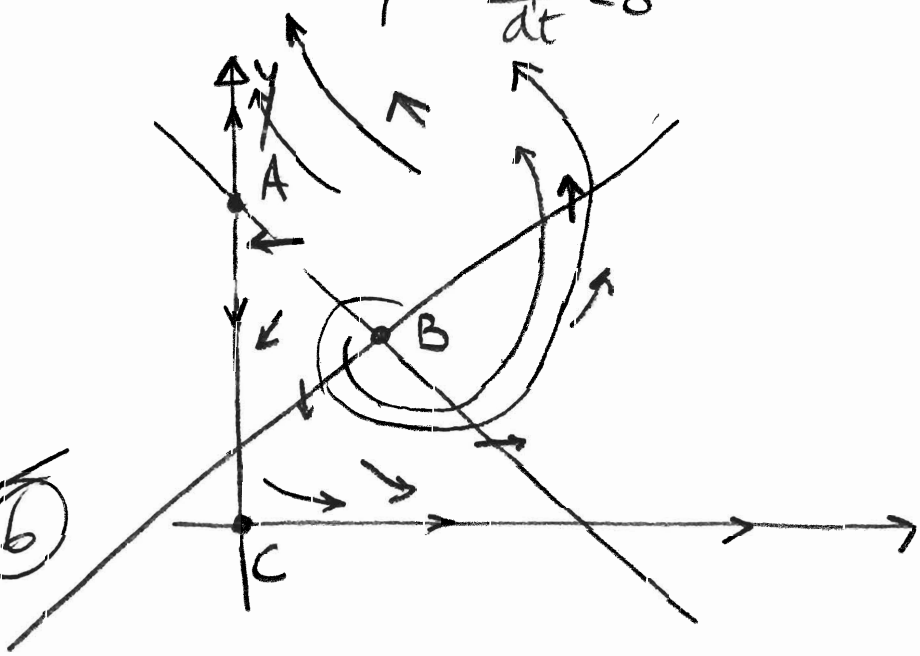
(c)(a) restrict consideration to $x \geq 0, y \geq 0$:

above N_x: $y > \frac{l}{n} + \frac{p}{n}x \Rightarrow \quad 0 > l - ny + px$
$\rightarrow \frac{dx}{dt} < 0$ } 2

below N_x: $\frac{dx}{dt} > 0$

above N_y: $y > \frac{k}{q} - \frac{m}{q}x \rightarrow \quad mx - k + qy > 0 \rightarrow \frac{dy}{dt} > 0$ } 1

below N_y: $\frac{dy}{dt} < 0$



direction fields 2

anticipated trajectories 2

(b)

(c) It appears that all three of the physically meaningful equil. pts are unstable (although it might be possible to reach pt A).

3

(5) $\frac{dx}{dt} = lx - nxy$ with $l = 0.2$, $n = 0.001$

$\frac{dy}{dt} = ky - mxy$ with $k = 0.4$, $m = 0.002$

(a) $x_c = \frac{k}{m} = \frac{0.4}{0.002} = 200$ (2)

$y_c = \frac{l}{n} = \frac{0.2}{0.001} = 200$ (2)

~~(4)~~

(b) $y^l e^{-ny} = K x^k e^{-mx}$. (2)

$y^{0.2} e^{-0.001y} = K x^{0.4} e^{-0.002x}$.

$(100, 150)$ on $\overset{(1)}{\Longrightarrow}$ $K = \frac{(150)^{0.2} e^{-0.001(150)}}{(100)^{0.4} e^{-0.002(100)}}$

$= \frac{(150)^{0.2} e^{-0.15}}{(100)^{0.4} e^{-.2}} \simeq 0.453871566$ (2)

$\therefore$ trajectory is given by

~~(5)~~

$y^{0.2} e^{-0.001y} = 0.453871566\, x^{0.4} e^{-0.002x}$. (1)

(c) for max value of x we must have $y = y_c = 200$

$\rightarrow$ x_{max} must satisfy

$(200)^{0.2} e^{-0.001(200)} = 0.453871566\, x^{0.4} e^{-0.002x}$

$\Rightarrow x^{0.4} e^{-0.002x} = 5.204920814$. (2)

~~(3)~~

(1)

(d) let $f(x) = x^{0.4} e^{-0.002x} - 5.204920814$

To solve $f(x) = 0$ let

$x_0 =$ any number greater than x_c $(x_0 > 200)$ (2)

& consider the recursively defined sequence

(5) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad n \geq 1.$$ [Newton's method.]

other methods OK if explained fully!

6 (a) $(P_N)_{MAX} = P_N((t_M)_N) = P_N\left(\frac{1}{b}\ln\left(\frac{N}{N_0}\right)\right)$

$$= \binom{N-1}{N_0-1} e^{-N_0 \ln\left(\frac{N}{N_0}\right)} \left[1-e^{-\ln\left(\frac{N}{N_0}\right)}\right]^{(N-N_0)}$$

$$= \binom{N-1}{N_0-1} \left(\frac{N}{N_0}\right)^{-N_0} \left[1-\left(\frac{N}{N_0}\right)^{-1}\right]^{(N-N_0)}$$

$$= \binom{N-1}{N_0-1} \left(\frac{N_0}{N}\right)^{N_0} \left(\frac{N-N_0}{N}\right)^{(N-N_0)}$$

$$= \frac{(N-1)!}{(N_0-1)!\,(N-N_0)!} \, \frac{N_0^{\,N_0}\,(N-N_0)^{N-N_0}}{N^N}.$$

(b) $$\frac{(P_{N+1})_{MAX}}{(P_N)_{MAX}} = \frac{\dfrac{N!}{\cancel{(N_0-1)!}\,(N+1-N_0)!}\,\dfrac{\cancel{N_0^{N_0}}\,(N+1-N_0)^{N+1-N_0}}{(N+1)^{N+1}}}{\dfrac{(N-1)!}{\cancel{(N_0-1)!}\,(N-N_0)!}\,\dfrac{\cancel{N_0^{N_0}}\,(N-N_0)^{N-N_0}}{N^N}}$$

$$= \left(\frac{N}{N+1-N_0}\right) \frac{(N+1-N_0)^{N+1-N_0}}{(N+1)^{N+1}} \frac{N^N}{(N-N_0)^{N-N_0}}$$

$$= \frac{(N+1-N_0)^{N-N_0}}{(N-N_0)^{N-N_0}} \frac{N^{N+1}}{(N+1)^{N+1}}$$

$$= \left(1+\frac{1}{N-N_0}\right)^{N-N_0} \Big/ \left(1+\frac{1}{N}\right)^N.$$

But $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is increasing ↑

$N-N_0 < N$ so $\left(1+\frac{1}{N-N_0}\right)^{N-N_0} < \left(1+\frac{1}{N}\right)^N$

$$\Longrightarrow \frac{(P_{N+1})_{MAX}}{(P_N)_{MAX}} < 1$$

(c) Since a population must reach size N before reaching size $N+1$, $(P_N)_{MAX}$ must be greater that $(P_{N+1})_{MAX}$.

7) $\frac{dS}{dt} = -\beta S I$, $\frac{dR}{dt} = rI$, $S+I+R=N$.

(a) $\frac{dI}{dt} = -\frac{dS}{dt} - \frac{dR}{dt} = \beta S I - rI = I(\beta S - r)$

$\frac{dI}{dt} = 0 \iff I = 0$ or $S = r/\beta$.

clearly this provides a minimum value for I.

(b) along a trajectory : $\frac{dI}{dS} = \frac{\beta S I - rI}{-\beta S I}$

$= \frac{r}{\beta S} - 1$

seperate: $dI = \left(\frac{r}{\beta S} - 1\right) dS$

$\Rightarrow I = \frac{r}{\beta} \ln S - S + C$

But $I = 10$ when $S = 9990$

$\Rightarrow 10 = \frac{r}{\beta} \ln 9990 - 9990 + C$

$\Rightarrow C = 10000 - \frac{r}{\beta} \ln(9990)$

$\Rightarrow I = 10000 + \frac{r}{\beta} \ln\left(\frac{S}{9990}\right) - S$. along a trajectory.

(c) For I_{max} we must have $\frac{dI}{dt} = 0$

$\Rightarrow$ for I_{max}, $S = \frac{r}{\beta}$

$\Rightarrow I_{max} = 10000 + \frac{r}{\beta} \ln\left(\frac{r}{9990\beta}\right) - \frac{r}{\beta}$

But $\frac{r}{\beta} = \frac{0.9}{0.0002} = 4500$.

$$\therefore I_{max} = 10000 + 4500 \ln\left(\frac{4500}{9990}\right) - 4500$$

$$= 5500 + 4500 \ln\left(\frac{4500}{9990}\right)$$

$$= 1911.21761852.$$