Traffic flow

Linear cascades
Linear systems
Delay differential equations
Laplace transform

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

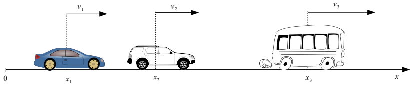
Problem formulation

Want to model

- N cars
- on a straight road
- no overtaking
- adjustment of speed on driver in front

Hypotheses

- N cars in total.
- Road is the x-axis.
- $x_n(t)$ position of the nth car at time t.
- $v_n(t) \triangleq x_n'(t)$ velocity of the nth car at time t.



- All cars start with the same initial speed v_0 before time $t = 0$.

Moving frame coordinates

To make computations easier, express velocity of cars in a reference frame moving at speed u_0.

Remark that here, speed=velocity, since movement is 1-dimensional.

Let

$$u_n(t) = v_n(t) - u_0.$$

Then $u_n(t) = 0$ for $t \leq 0$, and u_n is the speed of the nth car in the moving frame coordinates.

Modeling driver behavior

Assume that

- Driver adjusts his/her speed according to relative speed between his/her car and the car in front.
- This adjustment is a linear term, equal to λ for all drivers.

- First car: evolution of speed remains to be determined.
- Second car:
$$u_2' = \lambda(u_1 - u_2).$$
- Third car:
$$u_3' = \lambda(u_2 - u_3)$$
- Thus, for $n = 1, \ldots, N-1$,
$$u_{n+1}' = \lambda(u_n - u_{n+1}). \tag{1}$$

This can be solved using *linear cascades*: if $u_1(t)$ is known, then

$$u_2' = \lambda(u_1(t) - u_2)$$

is a linear first-order nonhomogeneous equation. Solution (integrating factors, or variation of constants) is

$$u_2(t) = \lambda e^{-\lambda t} \int_0^t u_1(s) e^{\lambda s} ds$$

Then use this function $u_2(t)$ in u_3' to get $u_3(t)$,

$$u_3(t) = \lambda e^{-\lambda t} \int_0^t u_2(s) e^{\lambda s} ds$$

Thus

$$\begin{aligned} u_3(t) &= \lambda e^{-\lambda t} \int_0^t u_2(s) e^{\lambda s} ds \\ &= \lambda e^{-\lambda t} \int_0^t \left(\lambda e^{-\lambda s} \int_0^s u_1(q) e^{\lambda q} dq \right) ds \\ &= \lambda^3 e^{-\lambda t} \int_0^t e^{-\lambda s} \int_0^s u_1(q) e^{\lambda q} dq ds \end{aligned}$$

Continue the process to get the solution.

Example

Suppose driver of car 1 follows this function

$$u_1(t) = \alpha \sin(\omega t)$$

that is, ω-periodic, 0 at $t = 0$ (we want all cars to start with speed relative to the moving reference equal to 0), and with amplitude α.

Then

$$\begin{aligned} u_2(t) &= \lambda\alpha e^{-\lambda t} \int_0^t \sin(\omega s) e^{\lambda s} ds \\ &= \lambda\alpha e^{-\lambda t} \left(\frac{\omega - \omega e^{\lambda t}\cos(\omega t) + \lambda e^{\lambda t}\sin(\omega t)}{\lambda^2 + \omega^2} \right) \\ &= \frac{\lambda\alpha}{\lambda^2+\omega^2} \left(\omega e^{-\lambda t} + \lambda \sin(\omega t) - \omega\cos(\omega t) \right). \end{aligned}$$

When $t \to \infty$, first term goes to 0, we are left with a ω-periodic term.

Continuing the process,

$$u_3(t) = \frac{\lambda^2\alpha}{\lambda^2+\omega^2} e^{-\lambda t} \times \int_0^t \left(\omega e^{-\lambda s} + \lambda\sin(\omega s) - \omega\cos(\omega s) \right) e^{\lambda s} ds$$

that is,

$$\begin{aligned} u_3(t) &= \frac{\lambda^2\alpha}{\lambda^2+\omega^2} e^{-\lambda t} \left(\omega t + \int_0^t (\lambda\sin(\omega s) - \omega\cos(\omega s)) e^{\lambda s} ds \right) \\ &= \frac{\lambda^2\alpha}{\lambda^2+\omega^2} \left(\omega t + \frac{2\lambda\omega}{\lambda^2+\omega^2} \right) e^{-\lambda t} \\ &\quad - \frac{\lambda^2\alpha}{(\lambda^2+\omega^2)^2} \left(2\lambda\omega\cos(\omega t) - \lambda^2\sin(\omega t) + \omega^2\sin(\omega t) \right) \end{aligned}$$

Once again, the terms in $e^{-\lambda t}$ vanishes for large t, and we are left with 3 ω-periodic terms.

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

Linear ODEs

Definition (Linear ODE)

A *linear* ODE is a differential equation taking the form

$$\frac{d}{dt}x = A(t)x + B(t), \tag{LNH}$$

where $A(t) \in \mathcal{M}_n(\mathbb{R})$ with continuous entries, $B(t) \in \mathbb{R}^n$ with real valued, continuous coefficients, and $x \in \mathbb{R}^n$. The associated IVP takes the form

$$\begin{aligned} \frac{d}{dt}x &= A(t)x + B(t) \\ x(t_0) &= x_0. \end{aligned} \tag{2}$$

Types of systems

- $x' = A(t)x + B(t)$ is linear nonautonomous ($A(t)$ depends on t) nonhomogeneous (also called *affine* system).
- $x' = A(t)x$ is linear nonautonomous homogeneous.
- $x' = Ax + B$, that is, $A(t) \equiv A$ and $B(t) \equiv B$, is linear autonomous nonhomogeneous (or affine autonomous).
- $x' = Ax$ is linear autonomous homogeneous.

- If $A(t+T) = A(t)$ for some $T > 0$ and all t, then linear periodic.

Existence and uniqueness of solutions

Theorem (Existence and Uniqueness)

Solutions to (2) *exist and are unique on the whole interval over which A and B are continuous.*
In particular, if A, B are constant, then solutions exist on $\mathbb{R}$.

Autonomous linear systems

Consider the autonomous affine system

$$\frac{d}{dt}x = Ax + B, \tag{A}$$

and the associated homogeneous autonomous system

$$\frac{d}{dt}x = Ax. \tag{L}$$

Exponential of a matrix

Definition (Matrix exponential)

Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of A, denoted e^{At}, is a matrix in $\mathcal{M}_n(\mathbb{K})$, defined by

$$e^{At} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k,$$

where $\mathbb{I}$ is the identity matrix in $\mathcal{M}_n(\mathbb{K})$.

Properties of the matrix exponential

- $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \in \mathbb{R}$. 1
- $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.
- $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.
- The unique solution ϕ of (L) with $\phi(t_0) = x_0$ is given by

$$\phi(t) = e^{A(t-t_0)}x_0.$$

Computing the matrix exponential

Let P be a nonsingular matrix in $\mathcal{M}_n(\mathbb{R})$. We transform the IVP

$$\begin{aligned}\frac{d}{dt}x &= Ax\\ x(t_0) &= x_0\end{aligned} \tag{L\_IVP}$$

using the transformation $x = Py$ or $y = P^{-1}x$.

The dynamics of y is

$$\begin{aligned}y' &= (P^{-1}x)'\\ &= P^{-1}x'\\ &= P^{-1}Ax\\ &= P^{-1}APy\end{aligned}$$

The initial condition is $y_0 = P^{-1}x_0$.

We have thus transformed IVP (L_IVP) into

$$\begin{aligned}\frac{d}{dt}y &= P^{-1}APy\\ y(t_0) &= P^{-1}x_0\end{aligned} \tag{L\_IVP\_y}$$

From the earlier result, we then know that the solution of (L_IVP_y) is given by

$$\psi(t) = e^{P^{-1}AP(t-t_0)}P^{-1}x_0,$$

and since $x = Py$, the solution to (L_IVP) is given by

$$\phi(t) = Pe^{P^{-1}AP(t-t_0)}P^{-1}x_0.$$

So everything depends on $P^{-1}AP$.

Diagonalizable case

Assume P nonsingular in $\mathcal{M}_n(\mathbb{R})$ such that

$$P^{-1}AP = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

with all eigenvalues $\lambda_1, \ldots, \lambda_n$ different.

We have

$$e^{P^{-1}AP} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k$$

For a (block) diagonal matrix M of the form

$$M = \begin{pmatrix} m_{11} & & 0 \\ & \ddots & \\ 0 & & m_{nn} \end{pmatrix}$$

there holds

$$M^k = \begin{pmatrix} m_{11}^k & & 0 \\ & \ddots & \\ 0 & & m_{nn}^k \end{pmatrix}$$

Therefore,

$$\begin{aligned} e^{P^{-1}AP} &= \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_n^k \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \end{aligned}$$

And so the solution to (L_IVP) is given by

$$\phi(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0.$$

Nondiagonalizable case

The Jordan canonical form is

$$P^{-1}AP = \begin{pmatrix} J_0 & & 0 \\ & \ddots & \\ 0 & & J_s \end{pmatrix}$$

so we use the same property as before (but with block matrices now), and

$$e^{P^{-1}APt} = \begin{pmatrix} e^{J_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_s t} \end{pmatrix}$$

The first block in the Jordan canonical form takes the form

$$J_0 = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

and thus, as before,

$$e^{J_0 t} = \begin{pmatrix} e^{\lambda_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}$$

Other blocks J_i are written as

$$J_i = \lambda_{k+i}\mathbb{I} + N_i$$

with $\mathbb{I}$ the $n_i \times n_i$ identity and N_i the $n_i \times n_i$ nilpotent matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$\lambda_{k+i}\mathbb{I}$ and N_i commute, and thus

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}$$

Since N_i is nilpotent, $N_i^k = 0$ for all $k \geq n_i$, and the series $e^{N_i t}$ terminates, and

$$e^{J_i t} = e^{\lambda_{k+i} t} \begin{pmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & & \\ 0 & & & 1 \end{pmatrix}$$

Theorem

For all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there is a unique solution $x(t)$ to (L_IVP) defined for all $t \in \mathbb{R}$. Each coordinate function of $x(t)$ is a linear combination of functions of the form

$$t^k e^{\alpha t} \cos(\beta t) \quad and \quad t^k e^{\alpha t} \sin(\beta t)$$

where $\alpha + i\beta$ is an eigenvalue of A and k is less than the algebraic multiplicity of the eigenvalue.

Generalized eigenvectors

Definition (Generalized eigenvectors)

Let $A \in \mathcal{M}_r(\mathbb{R})$. Suppose λ is an eigenvalue of A with multiplicity $m \leq n$. Then, for $k = 1, \ldots, m$, any nonzero solution v of

$$(A - \lambda \mathbb{I})^k v = 0$$

is called a *generalized eigenvector* of A.

Nilpotent matrix

Definition (Nilpotent matrix)

Let $A \in \mathcal{M}_n(\mathbb{R})$. A is *nilpotent* (of order k) if $A^j \neq 0$ for $j = 1, \ldots, k-1$, and $A^k = 0$.

Jordan normal form

Theorem (Jordan normal form)

Let $A \in \mathcal{M}_n(\mathbb{R})$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to their multiplicities.

- *Then there exists a basis of generalized eigenvectors for $\mathbb{R}^n$.*
- *And if $\{v_1, \ldots, v_n\}$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, then the matrix $P = [v_1 \cdots v_n]$ is invertible, and A can be written as*

  $$A = S + N,$$

  where

  $$P^{-1}SP = \text{diag}(\lambda_j),$$

  the matrix $N = A - S$ is nilpotent of order $k \leq n$, and S and N commute, i.e., $SN = NS$.

Theorem

Under conditions of the Jordan normal form Theorem, the linear system $x' = Ax$ with initial condition $x(0) = x_0$, has solution

$$x(t) = P \operatorname{diag}\left(e^{\lambda_j t}\right) P^{-1} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

The result is particularly easy to apply in the following case.

Theorem (Case of an eigenvalue of multiplicity n)

Suppose that λ is an eigenvalue of multiplicity n of $A \in \mathcal{M}_n(\mathbb{R})$. Then $S = \operatorname{diag}(\lambda)$, and the solution of $x' = Ax$ with initial value x_0 is given by

$$x(t) = e^{\lambda t} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

In the simplified case, we do not need the matrix P (the basis of generalized eigenvectors).

A variation of constants formula

Theorem (Variation of constants formula)

Consider the IVP

$$x' = Ax + B(t) \tag{3a}$$
$$x(t_0) = x_0, \tag{3b}$$

where $B : \mathbb{R} \to \mathbb{R}^n$ a smooth function on $\mathbb{R}$, and let $e^{A(t-t_0)}$ be matrix exponential associated to the homogeneous system $x' = Ax$. Then the solution ϕ of (3) is given by

$$\phi(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-s)} B(s) ds. \tag{4}$$

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

Computation in our case

Consider the case of 3 cars. Let

$$X = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

Then the system can be written as

$$X' = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} U + \begin{pmatrix} \lambda u_1(t) \\ 0 \end{pmatrix}$$

which we write for short as $X' = AX + B(t)$.

The matrix A has the eigenvalue $-\lambda$ with multiplicity 2. Its Jordan form is obtained by using the maple function `JordanForm`:

```
> with(LinearAlgebra)
> A := <<-lambda, lambda> | <0, -lambda>>:
> J := JordanForm(A)
```

giving

$$J = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

To get the matrix of change of basis,

```
> P := JordanForm(A,output='Q')
```

giving

$$P = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

which is such that $P^{-1}AP = J$.

Because $-\lambda$ is an eigenvalue with multiplicity 2 (same as the size of the matrix), we can use the simplified theorem, and only need N.

We have

$$\begin{aligned} N &= A - S \\ &= \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \end{aligned}$$

Clearly, $N^2 = 0$, so, by the theorem in the simplified case,

$$x(t) = e^{-\lambda t}\left(\mathbb{I} + Nt\right) x_0$$

But we know that solutions are unique, and that the solution to the differential equation is given by $x(t) = e^{At}x_0$. This means that

$$\begin{aligned} e^{At} &= e^{-\lambda t}\left(\mathbb{I} + Nt\right) \\ &= e^{-\lambda t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda t & 0 \end{pmatrix}\right) \\ &= e^{-\lambda t}\begin{pmatrix} 1 & 0 \\ \lambda t & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-\lambda t} & 0 \\ \lambda t e^{-\lambda t} & e^{-\lambda t} \end{pmatrix} \end{aligned}$$

Now notice that the solution to

$$X' = AX$$

is trivially established here, since

$$X(0) = \begin{pmatrix} u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus

$$X(t) = e^{At}0 = 0.$$

e^{At} does however play a role in the solution (fortunately), since it is involved in the variation of constants formula:

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B(s)ds$$

Thus we need to compute $e^{A(t-s)}B(s)$, and then the integral.

$$e^{A(t-s)}B(s) = \begin{pmatrix} e^{-\lambda(t-s)} & 0 \\ \lambda(t-s)e^{-\lambda(t-s)} & e^{-\lambda(t-s)} \end{pmatrix} \begin{pmatrix} \lambda u_1(s) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda e^{-\lambda(t-s)} u_1(s) \\ \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s) \end{pmatrix}$$

and thus

$$\int_0^t e^{A(t-s)}B(s)ds = \int_0^t \begin{pmatrix} \lambda e^{-\lambda(t-s)} u_1(s) \\ \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s) \end{pmatrix} ds$$
$$= \begin{pmatrix} \int_0^t \lambda e^{-\lambda(t-s)} u_1(s)ds \\ \int_0^t \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s)ds \end{pmatrix}$$
$$= \begin{pmatrix} \lambda e^{-\lambda t} \int_0^t e^{\lambda s} u_1(s)ds \\ \lambda^2 e^{-\lambda t} \int_0^t e^{\lambda s}(t-s)u_1(s)ds \end{pmatrix}$$
$$= \begin{pmatrix} \lambda e^{-\lambda t} \int_0^t e^{\lambda s} u_1(s)ds \\ \lambda^2 e^{-\lambda t} \left(t \int_0^t e^{\lambda s} u_1(s)ds - \int_0^t s e^{\lambda s} u_1(s)ds \right) \end{pmatrix}$$

Let

$$\Psi(t) = \int_0^t e^{\lambda s} u_1(s)ds$$

and

$$\Phi(t) = \int_0^t s e^{\lambda s} u_1(s)ds$$

These can be computed when we choose a function $u_1(t)$. Then, finally, we have

$$X(t) = \int_0^t e^{A(t-s)}B(s)ds$$
$$= \begin{pmatrix} \lambda e^{-\lambda t}\Psi(t) \\ \lambda^2 e^{-\lambda t}\left(t\Psi(t) - \Phi(t)\right) \end{pmatrix}$$

Case of the $\alpha \sin(\omega t)$ driver

We set

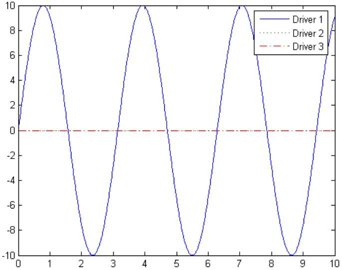
$$u_1(t) = \alpha \sin(\omega t).$$

Then

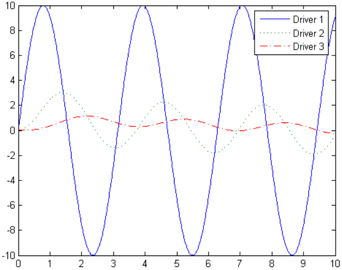
$$\Psi(t) = \frac{\alpha(\omega - \omega e^{\lambda t}\cos(\omega t) + \lambda e^{\lambda t}\sin(\omega t))}{\lambda^2 + \omega^2}$$

and

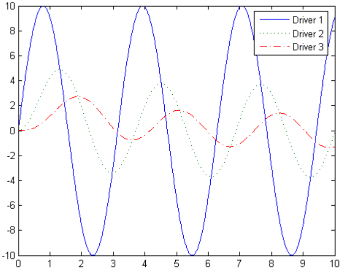
$$\Phi(t) = \frac{\alpha(\lambda^3 t + \lambda t\omega^2 - \lambda^2 + \omega^2)\sin(\omega t)e^{\lambda t}}{(\lambda^2+\omega^2)^2} - \frac{\alpha\omega\cos(\omega t)(t\lambda^2 + t\omega^2 - 2\lambda)e^{\lambda t} + 2\alpha\lambda\omega}{(\lambda^2+\omega^2)^2}$$



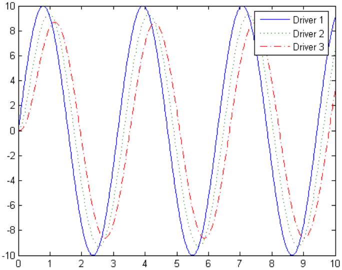
$\lambda = 0$



$\lambda = 0.4$



$\lambda = 0.8$



$\lambda = 5$

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

A delay differential equations model

- In the previous model, reaction time is instantaneous.
- In practice, this is known to be incorrect: reflexes and psychology play a role.
- It takes at least a few instants to acknowledge a change of speed in the car in front.
- If the change of speed is not threatening, then you may not want to react right away.
- When you press the accelerator or the brake, there is a delay between the action and the reaction..

A delayed model of traffic flow

We consider the same setting as previously, except that now, for $t > 0$,

$$u'_{n+1}(t) = \lambda(u_n(t-\tau) - u_{n+1}(t-\tau)), \tag{5}$$

for $n = 1, \ldots, N-1$. Here, $\tau \geq 0$ is called the *time delay* (or *time lag*), or for short, *delay* (or *lag*).

If $\tau = 0$, we are back to the previous model.

Initial data

For a delay equation such as (5), the initial conditions become *initial data*. This initial data must be specified on an interval of length τ, left of zero.

This is easy to see by looking at the terms: $u(t-\tau)$ involves, at time t, the state of u at time $t-\tau$. So if $t < \tau$, we need to know what happened for $t \in [-\tau, 0]$.

So, normally, we specify initial data as

$$u_n(t) = \phi(t) \text{ for } t \in [-\tau, 0],$$

where ϕ is some function, that we assume to be continuous. We assume $u_1(t)$ is known.

Here, we assume, for $n = 1, \ldots, N$,

$$u_n(t) = 0, \qquad t \leq (n-1)\tau$$

Important remark

Although (5) looks very similar to (1), you must keep in mind that it is in fact much more complicated.

- A solution to (1) is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (or to $\mathbb{R}^n$ if we consider the system).
- A solution to (5) is a continuous function in the space of continuous functions.
- The space $\mathbb{R}^n$ has dimension n. The space of continuous functions has dimension ∞.

We can use the Laplace transform to get some understanding of the nature of the solutions.

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

The Laplace transform

Definition (Laplace transform)

Let $f(t)$ be a function defined for $t \geq 0$. The *Laplace transform* of f is the function $F(s)$ defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The Laplace transform is a linear operator:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Rules of transformation

t-domain	s-domain
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
f'	$sF(s) - f(0)$
f''	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$\frac{f(t)}{t}$	$\int_s^\infty F(u) du$
$\int_0^t f(u) du = u(t) * f(t)$	$\frac{1}{s} F(s)$
$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
$e^{at} f(t)$	$F(s-a)$
$f(t-a)u(t-a)$	$e^{-as} F(s)$
$f(t) * g(t)$	$F(s)G(s)$

Here $f^{(n)}$ represents the nth derivative, not the nth iterate. $*$ is the convolution product.

Dirac delta – Heaviside function

In the table on the following slide,

- $\delta(t)$ is the Dirac delta,

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0. \end{cases}$$

- $H(t)$ is the Heaviside function,

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Note that $H(t) = \int_{-\infty}^{t} \delta(s) ds$.

Transforms of common functions

t-domain	s-domain
$\delta(t)$	1
$\delta(t-\tau)$	$e^{-\tau s}$
$H(t)$	$\frac{1}{s}$
$H(t-\tau)$	$\frac{e^{-\tau s}}{s}$
$\frac{t^n}{n!}H(t)$	$\frac{1}{s^{n+1}}$
$e^{-\alpha t}H(t)$	$\frac{1}{s+\alpha}$
$\sin(\omega t)H(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)H(t)$	$\frac{s}{s^2+\omega^2}$

Inverse Laplace transform

Definition

Given a function $F(s)$, if there exists $f(t)$, continuous on $[0, \infty)$ and such that

$$\mathcal{L}\{f\} = F,$$

then $f(t)$ is the *inverse Laplace transform* of $F(s)$, and is denoted $f = \mathcal{L}^{-1}\{F\}$.

Theorem

The inverse Laplace transform is a linear operator. Assume that $\mathcal{L}^{-1}\{F_1\}$ *and* $\mathcal{L}^{-1}\{F_2\}$ *exist, then*

$$\mathcal{L}^{-1}\{aF_1 + bF_2\} = a\mathcal{L}^{-1}\{F_1\} + b\mathcal{L}^{-1}\{F_2\}.$$

Solving differential equations using the Laplace transform

1. Take the Laplace transform of both sides of the equation.
2. Using the initial conditions, deduce an algebraic system of equations in s-space.
3. Solve the algebraic system in s-space.
4. Take the inverse Laplace transform of the solution in s-space, to obtain the solution of the differential equation in t-space.

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

Let

$$U_{k+1}(s) = \mathcal{L}\{u_{k+1}(t)\} = \int_0^\infty e^{-st} u_{k+1}(t) dt.$$

Since we have assumed initial data of the form

$$u_n(t) = 0 \qquad \text{for } t \leq (n-1)\tau,$$

we have

$$U_{k+1}(s) = \int_{k\tau}^\infty e^{-st} u_{k+1}(t) ds.$$

Since $u_{n+1}(t) = 0$ for $t \leq n\tau$,

$$\begin{aligned}\int_0^\infty e^{-st} u'_{n+1}(t) dt &= \left[u_{k+1}(t) e^{-st}\right]_{k\tau}^\infty + s\int_{k\tau}^\infty e^{-st} u_{k+1}(t) dt \\ &= sU_{k+1}(s)\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-st} u_{k+1}(t-\tau) dt &= \int_{(k-1)\tau}^\infty e^{-st} u_{k+1}(t-\tau) dt \\ &= \int_{(k-2)\tau}^\infty e^{-s(t+\tau)} u_k(\tau) d\tau \\ &= e^{-s\tau} U_k(s),\end{aligned}$$

since $e^{-st} u_{k+1}(t) \to 0$ for the improper integral to exist.
Note that we could have obtained this directly using the properties of the Laplace transform.

Multiply

$$u'_{n+1}(t) = \lambda(u_n(t-\tau) - u_{n+1}(t-\tau))$$

by e^{-st},

$$e^{-st} u'_{n+1}(t) = \lambda e^{-st}(u_n(t-\tau) - u_{n+1}(t-\tau))$$

integrate over $(0, \infty)$ (using the expressions found above),

$$sU_{n+1}(s) = \lambda(e^{-s\tau} U_n(s) - e^{-s\tau} U_{n+1}(s))$$

which is equivalent to

$$U_{n+1}(s) = \frac{\lambda U_n(s)}{\lambda + se^{s\tau}}$$

Thus, when $U_1(s)$ is known, we can deduce the values for all U_n.

Suppose

$$u_1(t) = \alpha \sin(\omega t)$$

From the table of Laplace transforms, it follows that

$$U_1(s) = \alpha \frac{\omega}{s^2 + \omega^2}$$

Therefore,

$$U_2 = \frac{\lambda U_1(s)}{\lambda + s e^{st}} = \alpha \frac{\lambda}{\lambda + s e^{st}} \, \frac{\omega}{s^2 + \omega^2}$$

and we can continue..

However, even though we know the solution in s-space, it is difficult to get the behavior in t-space, by hand, and maple does not help us either.