# A PROJECT ON MATHEMATICAL MODELLING FOR FISH POPULATION

## Description of Problem.

An effort is made to present a model on the population of fishes.Fishes are being harvested by fishermen and others and this reduces the fish population.On the other hand the population of fishes grows on their own.Fishes are living species and their growth and death depends on a number of factors .They may have natural death or may be killed (by harvesting and/or by fishes themselves,e.g.small fishes being eaten by large fishes).Harvesting of fishes also depends on a ~~number of factors~~ such as economic reasons,dietry needs , human population .It would be nice to have a mathematical model ~~which can predict~~ the rate of growth of fish population and the population at a given time.Based on the principle of simplicity rule a simple model will be considered.The following technique is used to present the model: ?

At initial stages the model describes the growth of population based on Logistic- Principle(subect to certain assumptions explained later in report).Then the effects of various harvesting schemes are considered.Each harvesting scheme is explained in detail with their assumptions.Since harvesting affects the carrying capacity it ~~is~~ desireable to find out the relationship between harvesting and a minimum viable population.This aspect is considered in the model and explained both mathematically and by phase-plane analysis.The conclusions and refinements are dealt at the end.The model is supported with various examples(calculations). ?

NB

## Review of Logistic Model

Suppose that a population consists of single species,fishes, and the rate of growth is constant and that this population has unlimited resources and is thus allowed to grow without any hinderances(e,g,no deaths,no decrease,) relative growth rate constant

Let N(t) represent population at a time 't' and 'k' denote ~~the~~ relative growth rate(constant).It is clear under the assumptions that the growth rate will be proportional to the population.This relationship can be described by the following equation: (Malthusian Law) ?

(1) $dN/dt = kN$

The solution of above equtation is :

$$(2) \qquad N(t) = De^{kt}$$

As can be seen from above equation that the population growth will be exponetial for k>0 and will continue to rise without limits.This seems unrealistic as the population growth is goverened by several factors.To remove the unlimited growth by removing the unlimited resources,the model can be modified by introducing the 'encounter-principle (a term used to describe the competition for resoureces by species).Thus the population will be limited to a number 'C' described as maximum supportable population.This can be presented by following equatoin (Logistic-Law):

$$(3) \qquad \frac{dN}{dt} = KN\left(\frac{C-N}{C}\right), \qquad K>0,\ C>0,\ N<C.$$
$$= KN\left(1-\frac{N}{C}\right)$$

When N(t) appoaches C, dN/dt approaches '0' and thus it reduces to Malthusian growth,dN/dt=kN. ?
"k"is now potential growth rate.Based on above assumptions the Logistic-Model will look like the following equation:

$$(4) \qquad \frac{dN}{dt} = kN\left(\frac{C-N}{C}\right) = kN(1-\frac{N}{C})$$

This model could be best understood by phase-plane analysis:

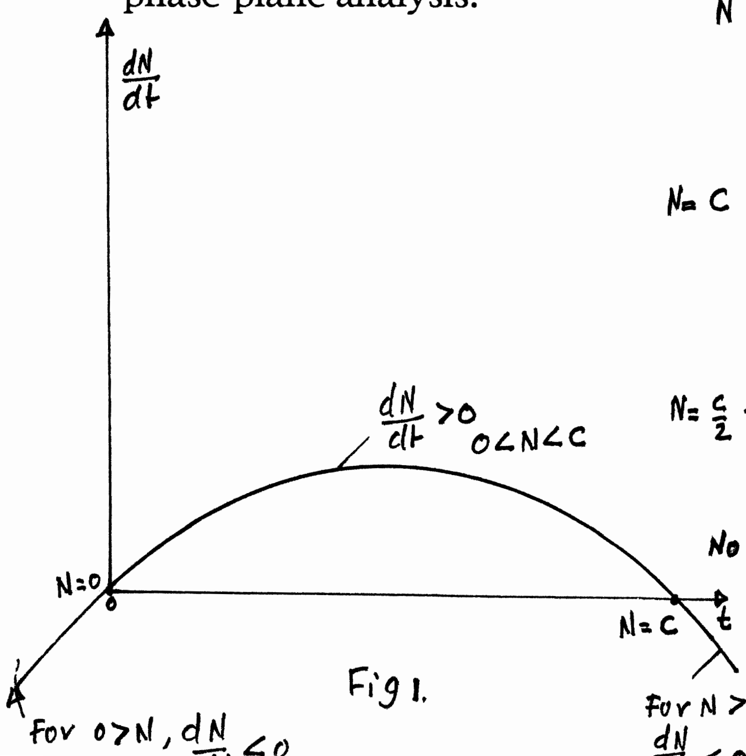



Fig 1.

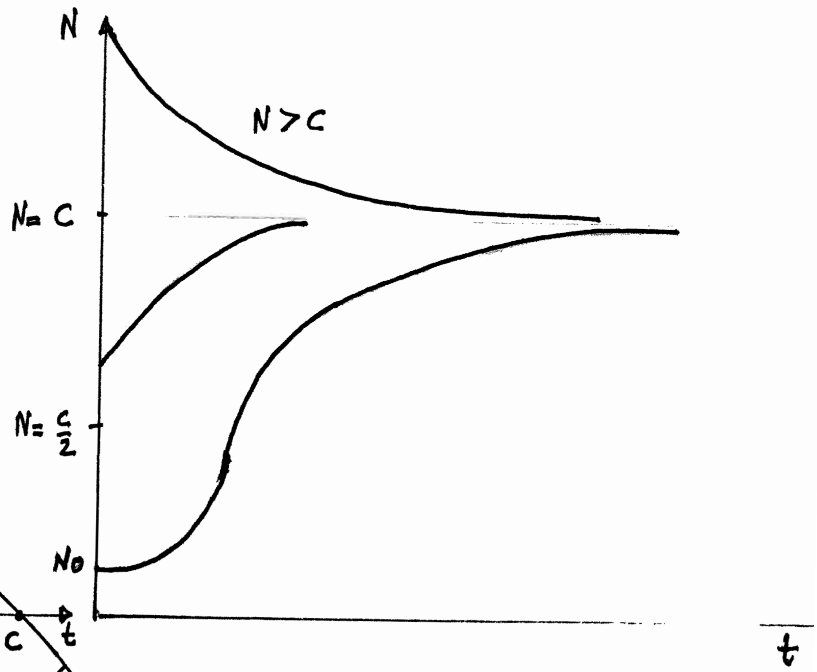



Fig 2.

Logistic-Model will be used to predict the fish population.Since it was dealt in class and well known to us therefore it is not explained in detail.Only phase-plane analysis diagrams are shown.

**Identification of variables and parameters**

The various variables and parameters used in this model are as follows:(these are explained further with their properties and meanings when actually used in equation and/or analysis.)

- Time is denoted by "t' and is a variable parameter.
- Population is denoted by 'N', which is an integer.
- Relative growth rate,denoted by'k'
- Harvesting rate denoted by 'h'. h>0
- Carrying capacity,maximum supportable population,denoted by "C"
- Minimum viable population denoted by 'M'
- New ~~carrying~~ capacity denoted by C*
- Time delay, td

Finally

- Various constants of integeration e.g.F,L,D,C1 etc. are explained as they appear in report

**Assumptions including reasons and descriptions.**

In order to make this model simple several assumptions are made.These are described as:

- The population of fishes is homogeneous and thus all fishes are considered as single species.This is done so that mathematical model will be simplified.

- Only fish numbers matter i.e. sizes,weights length,types are not considered.

- The relative groth rate of fishes is given by 'k'.The death rate birth rate,and other factors affecting growth rate may vary but their resultant will always be equal to k.

- The fishes have contetition for resources e.g. food, amongst themseleves and thus their population is limited to a maximum supportable population C.

- The relative harvesying rate is given by h>0.The rate at which people harvest may vary,the resources to harvest e.g. boats may vary,the other factors affecting harvesting may vary but the resultant of all these rates (and factors) is assumed to lead to harvesting rate h.

- Time is physcially continous and variable while N the number of population is just a step function (piece wise continous),not continous,not differential.
- Catastrophic events which may affect the population growth adversely have not been considered.
- Although it is not possible to calculate the population of fishes at any time it will be nice to predict about their growth. The growth can be repesented by differential equations.
- The growth can be described by Logistic-Model.

**Mathematical Analysis.**

The Logistic - Model as given by eq.(4 )

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right)$$

and its solution is,

$$N(t) = \frac{C}{1 + Fe^{-kt}} .$$

c = carrying capacity
F = Constant ( constant of integeration)
k > 0.

**Harvesting Schemes:**

A) The harvesting 'h' is proportion to population N. $(h(N) = hN)$

To take into account the harvesting the above eq.( 4 ) can be written as:

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right) - hN \qquad (k>0,\ h>0) \tag{5}$$

$$\therefore \frac{dN}{dt} = N\left[k - \frac{kN}{C} - h\right]$$

$$= N\left[(k-h) - \frac{kN}{C}\right]$$

$$(6) \quad \frac{dN}{dt} = (K-h)N\left(1-\frac{KN}{c(K-h)}\right)$$

Comparing above eq.(6) with Logistic eq.(4),it can be concluded that both equtations are similar.In fact above equation can be written as:

$$(7) \quad \frac{dN}{dt} = K^*N\left[1-\frac{N}{c^*}\right],$$ where $K^* = K-h$, similar to K. $c^* = \frac{c(K-h)}{K}$, new carrying capacity

$$\frac{dN}{dt} = K^*N\left(\frac{c^*-N}{c^*}\right),$$ separating the variables and integerating,

$$\int \frac{c^*dN}{N(c-N)} = \int K^*dt \quad \therefore \quad \int \frac{c^*dN}{N(c^*-N)} = K^*t + c_1$$ $c_1$ = constant of integeration

L.H.S. of equation can be integerated using partial fractions

$$\frac{c^*}{N(c^*-N)} = \frac{A}{N} + \frac{B}{(c^*-N)}$$ for some constants A and B.

solving for A and B, $A = 1 = B$.

$$\int \frac{c^*dN}{N(c^*-N)} = \int \frac{dN}{N} + \frac{dN}{(c^*-N)} = \ln|N| - \ln|c^*-N|$$

$$\therefore \ln|N| - \ln|c^*-N| = K^*t + c_1$$

$$\left|\frac{N}{c^*-N}\right| = e^{c_1}e^{K^*t} = De^{K^*t} \qquad D = e^{c_1} > 0$$

$$\left(\frac{N}{c-N}\right) = Le^{K^*t} \qquad L = \pm D.$$

$$(8) \quad \therefore N(t) = \frac{c^*}{1+Fe^{-K^*t}}, \qquad F = \frac{1}{L}$$

Since the model for this harvesting is similar to Logistic Model therefore the phase- plane diagrams are as:

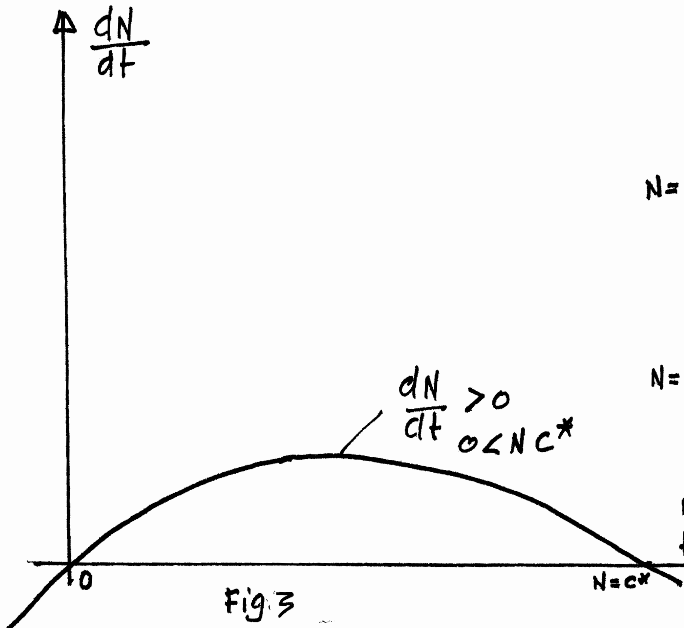


Fig 3

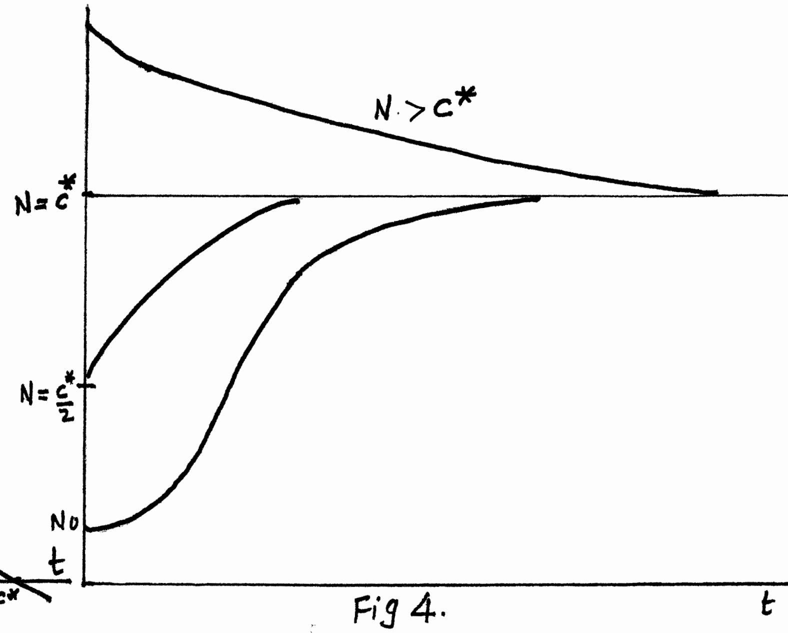
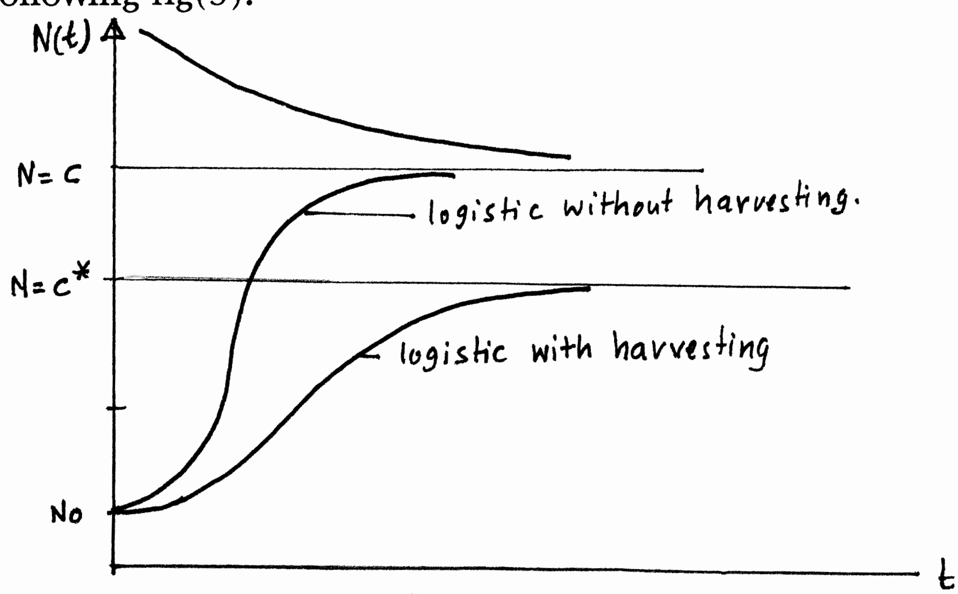


Fig 4.

When N=0, dN/dt=0. and thus the graph of N vs t is a straight line.This is equilibrium state. ~~When~~ N=C*, dN/dt=0,and thus the graph is a straight line parallel to x axis and starting at N=C* at y axis,This line represents ~~another~~ er equilibrium state.

As can be seen that as N increases a bit from N=0 i.e. equilibrium, then dN/dt starts to rise and approaches a maximum value and then starts to decrease, and when N=C*,dN/dt =0.The equilibrium represented by N=0 is unstable equilibrium while that of at N=C* is a stable.This is shown by a curve on the graph.

Since C*= C(k-h)/k and k*=(k-h) therefore C* relative will be lower than C and also the growth rate will be slower.The Logistic curves with and without harvesting are shown in the following fig(5).



As can be seen from fig.( 5) that the new carrying capacity is now lower than the original carrying capacity.The reduction depends on rate of harvesting.Also the relative growth rate is reduced when compared to the Logistic curve without harvesting.

Interpretion of Model

i) When k>0 and k>h

In this case k*=(k-h), is positive and k* is less than k

C*= C(k-h)/k therefore C* is positive but less than C.

The phase plane diagram for this case is similar to that of fig.(5).

Examples:

(a) Assume C=100,000,000.
k=.05
h=.02

$$N(t) = \frac{C}{1+Fe^{-kt}}$$ logistic solution without harvesting.

Assume at $t=0$, $N(0) = 50,000$.
With above assumptions, the solution will be

$$N(t) = \frac{100,000,000}{1+1999\,e^{-0.05t}}$$

With harvesting, $k^* = k-h = 0.05-0.02 = .03$

$c^* = C(1-\frac{h}{k}) = 0.6C.$ $\therefore c^* = 0.6 \times 100,000,000.$

Also assume $N(t^*) = 50,000$ at $t=0$.

Therefore, $$N^*(t) = \frac{60,000,000}{1+1199\,e^{-0.03t}}$$

(b) When $h = 0.045$, close to value of $k = 0.05$. Based on above values of $c = 100,000,000$ and $N(0) = 50,000$.

$$N(t) = \frac{100,000,000}{1+1999\,e^{-0.05t}}$$

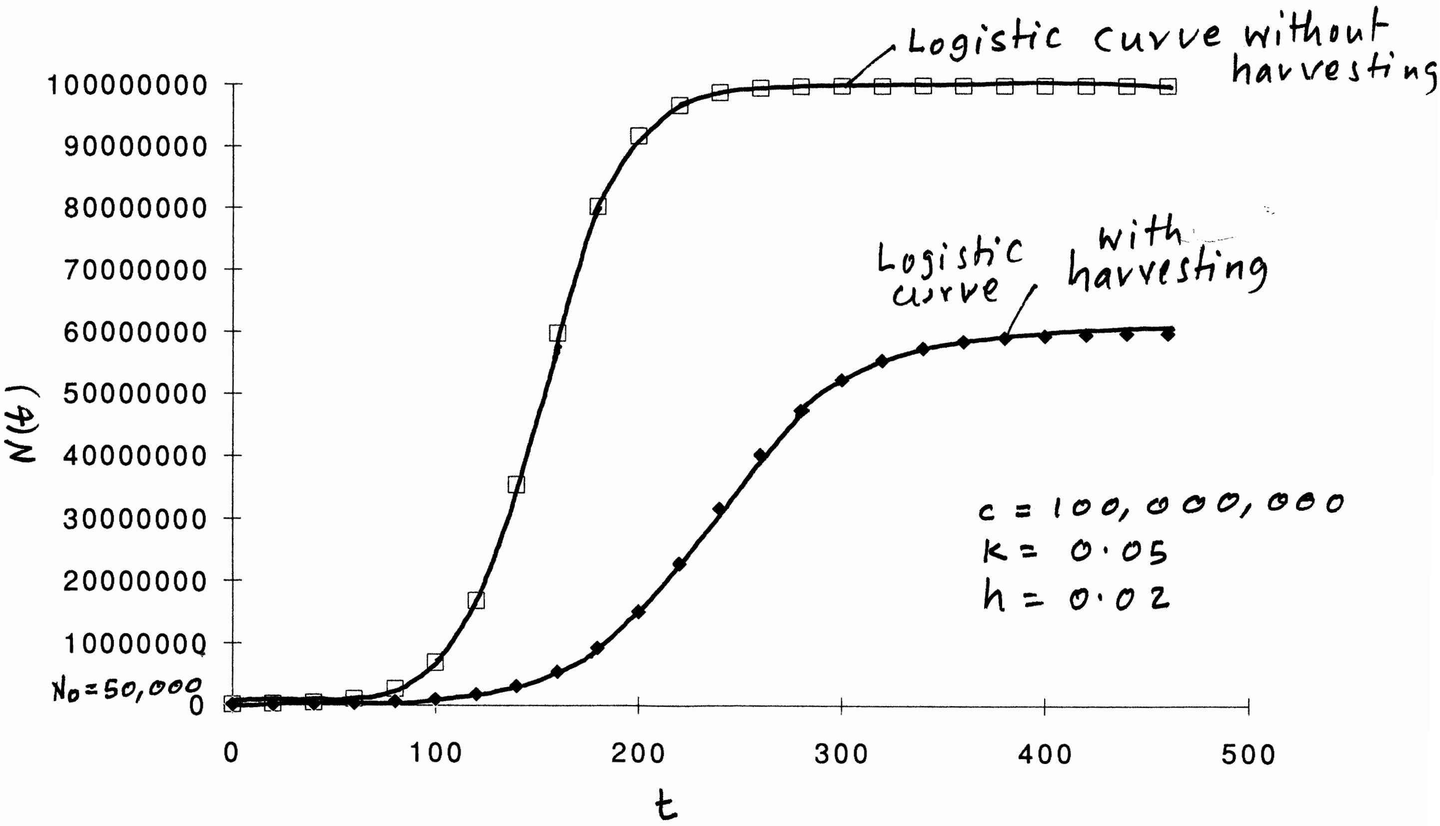
NB

$$N(t^*) = \frac{10,000,000}{1+199\,e^{-0.005t}}$$

$c^* = c(1-\frac{.045}{.05}) = 0.1C$

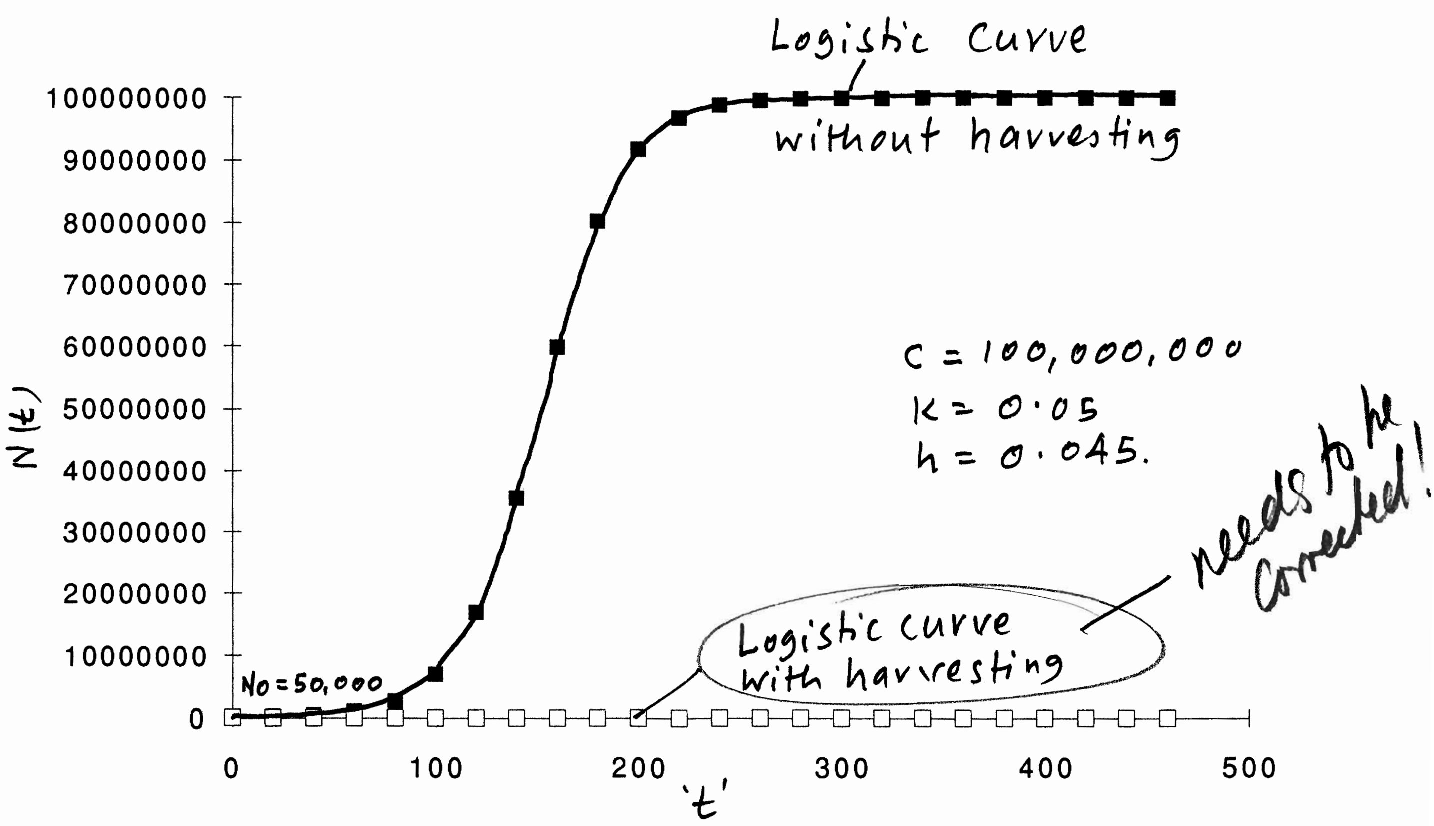
$K^* = K-h = .05-0.045 = .005.$

The logistic curves with and without harvesting are plotted and shown in figures 6, and 7 and table 1.




case (a) Fig. 6.

Note: because of the large difference between initial value $N(0)$ and C. the curves seem to start at '0', which could not be shown more clearly because of scale of graph. The curves start at different value i.e. 50000 as can be confirmed from table 1.




case (b)    Fig. 7

Note: As N(t)* values are very small and under the present scale almost result in a st-line. It is difficult to present ~~both~~ curves very clearly due to choxsen scale and choxsen values. For actual values see ~~the table~~ table 1.

TABLE # 1.

| | | | |
|---|---|---|---|
| 0 | 50000 | 50000 | 50000 |
| 20 | 135797.422 | 91043.5662 | 134.756345 |
| 40 | 368276.336 | 165685.505 | 358.015887 |
| 60 | 994783.854 | 301214.849 | 916.789725 |
| 80 | 2658657.93 | 546593.318 | 2152.9396 |
| 100 | 6911253.13 | 988554.272 | 4271.94984 |
| 120 | 16792539.1 | 1777191 | 6696.70504 |
| 140 | 35425165.1 | 3161272.87 | 8464.07142 |
| 160 | 59859099.1 | 5521065.91 | 9374.20557 |
| 180 | 80212003.6 | 9352523.91 | 9760.30115 |
| 200 | 91679661.8 | 15105649.4 | 9910.46307 |
| 220 | 96769195.9 | 22804312.6 | 9966.87372 |
| 240 | 98786674.4 | 31659656.4 | 9987.78795 |
| 260 | 99550192.6 | 40234102.4 | 9995.50397 |
| 280 | 99834053.3 | 47258386.6 | 9998.34553 |
| 300 | 99938887.5 | 52266249.1 | 9999.39129 |
| 320 | 99977509.3 | 55493549 | 9999.77606 |
| 340 | 99991724.9 | 57440056.8 | 9999.91762 |
| 360 | 99996955.6 | 58567497.3 | 9999.96969 |
| 380 | 99998880 | 59205264.9 | 9999.98885 |
| 400 | 99999588 | 59561217.8 | 9999.9959 |
| 420 | 99999848.4 | 59758394 | 9999.99849 |
| 440 | 99999944.2 | 59867162.5 | 9999.99944 |
| 460 | 99999979.5 | 59927024.3 | 9999.9998 |
| t | N(t) | N(t*a) | N(t*b) |

C* should be 10,000,000 in this case

CALCULATED VALUES OF N(t) FOR CASES (a) and (b)

Note: N(t) = population at time 't' without harvesting

N(t*a) = population at time 't' with harvesting for case (a)

N(t*b) = population at time 't' with harvesting for case (b)

Fig 6 represents the graph between $N(t)$ and 't' and between $N(t^*)$ and $t$. Fig 7 is a similar graph. Fig. 6 is for case (a) and fig 7. is for case (b).

In both cases carrying capacities are reduced as expected.

ii) When $k>0$, $k<h$

explain why?

$k^*=(k-h)$ is negative
$C^*=C(1-h/k)$ is negative.

The new growth rate is negative. The negative carrying capacity is meaningless. The population of fishes therefore decreases at a rate $k^*$. This is represented as follows:

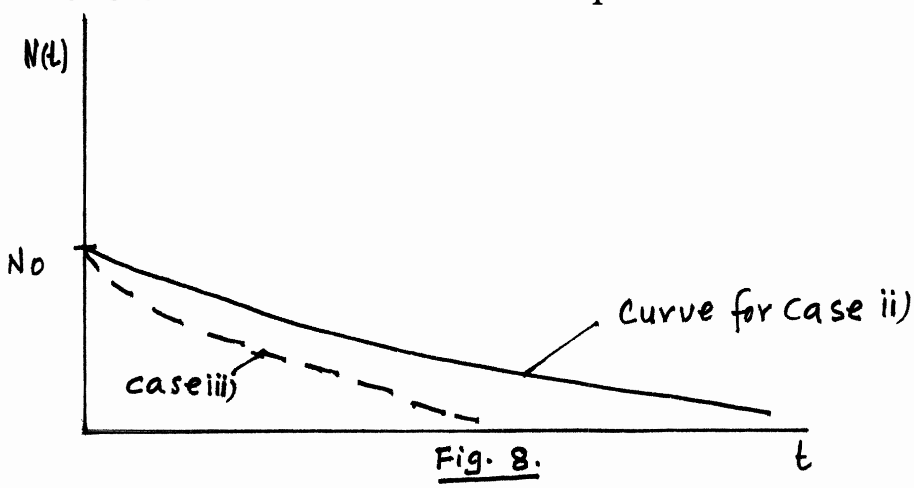



Fig. 8.

iii) When $k<0$, $h>0$

as $h>k$, therefore, $k^*=(k-h)$ is negative
$C^*=C(1-h/k)$ is negative.

ok

When $k<0$, meaning that growth of fishes is decaying and if at the same time harvesting is done then net effect would result in faster decrease when compared to case ii) above. This is also shown in fig.( 8 ) above.

iv) When k=h

$k^*=(k-h)=0$
$C^*=C(1-h/k)=0$

In this case the new carrying capicity is $C^*=0$ i.e.at equilibrium and $k^*=0$ therefore no growth

meaning? consequence?

B. Harvesting rate 'h' is proportional to $N^2$ i.e, $h(N) = hN^2$

The Logistic Model for this case would be:

$$\frac{dN}{dt} = kN\left(1-\frac{N}{c}\right) - hN^2 \tag{9}$$

Mathematical analysis:

$$\frac{dN}{dt} = kN\left[1-\frac{N}{c}-\frac{hN}{k}\right] = kN\left[1-\left(\frac{N}{c}+\frac{hN}{k}\right)\right]$$

$$\frac{dN}{dt} = kN\left[1-N\left(\frac{1}{c}+\frac{h}{k}\right)\right] = kN\left[1-N\left(\frac{k+ch}{ck}\right)\right] \tag{10}$$

comparing this eq. with Logistic equation without harvesting,

$$\frac{dN}{dt} = kN\left[1-\frac{N}{c}\right]$$

$$\therefore \frac{dN(t)^*}{dt} = k^*N\left[1-\frac{N}{c^*}\right]$$

we can see the similarlity between two equations and note that the eq.(10) is Logistic equation with different parameters.

$k^* = k.$ , similar to k.

$c^* = c\,\frac{k}{k+ch}$ . , new carrying capacity

<u>Interpretion of the model</u>
<u>1a when k>0, h>0, k>h</u>

$k^*=k$
$C^*=C(k/(k+Ch))$, therefore $C^*< C$ ✓

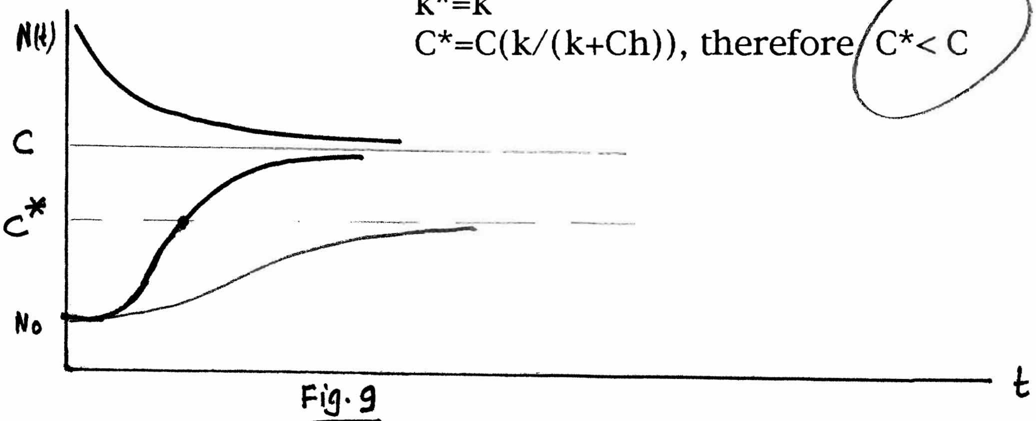



Fig. 9

diagram incomplete

Same growth rate as for Logistic without harvesting but C* is reduced therefore population levels off earlier.

To illustrate how this model works some examples for various values of h and k are shown for following conditions.(In each case the phase plane analysis is also done).
i) Assume

C=100,000,000.
k=.05
h=.02

) same values as used previously!

$$N(t) = \frac{C}{1+Fe^{-kt}} \quad \text{without harvesting}$$

Assume at $t=0$, $N(0) = 50,000$

$\therefore F = 1999$

$$\therefore N(t) = \frac{100,000,000}{1+1999e^{-0.05t}}$$

With harvesting

$$N(t)^*_a = \frac{C^*}{1+Fe^{-k^*t}}, \quad k^* = k = 0.05$$

$$C^* = C\frac{k}{k+Ch} = 100,000,000\frac{0.05}{0.05 \times 0.02 \times 10^8}$$
$$= 2.5 ✓$$

Carrying capacity reduced drastically. and $F = -0.99995$

$$N(t)^* = \frac{2.5}{1+(-0.99995)e^{0.05t}}$$

The values of $N(t)$ and $N(t^*a)$ with respect to time 'I' are plotted and shown in fig 10 and table 2.

ii) when $h = 0.045$, close to $k = 0.05$.

$$N(t) = \frac{100{,}000{,}000}{1 + Fe^{-0.05t}} \quad \text{and for } N(0) = 50{,}000, \quad F = 1999$$

$$\therefore N(t) = \frac{100{,}000{,}000}{1 + 1999e^{-0.05t}} \quad \text{without harvesting.}$$

$$N(t)^*_b = \frac{c^*}{1 + Fe^{-k^*t}} \quad \text{with harvesting}$$

$k^* = k = 0.05$

$$c^* = c\frac{k}{c + kh} = 100{,}000{,}000 \frac{0.05}{0.05 + 10^{8} \times 0.045}$$

$= 1.11$, very much reduced.

For $N(0) = 50{,}000$

$F = -0.9999778$

$$N(t)^*_b = \frac{1.11}{1 + (-0.9999778)e^{-0.05t}}$$

With the above equations obtained, the graphs for N(t) vs 't' are plotted and tabulated values of N(t) for time 't' are also shown. These are shown in fig.10,11 and table 2.

<u>1b When k>0, h>0, k<h</u>

k*=k

C*=C((k/(k+ch) therfore C*< C

Similar graph as in 1a. but C* will be different depending on values of k and h.

Example: Assume C=100,000,000

k=.02

h=.05

$N(0) = 50{,}000$ at $t = 0$.

$$N(t) = \frac{c}{1 + Fe^{-kt}} \quad \text{without harvesting}$$

$$\therefore N(t) = \frac{100{,}000{,}000}{1 + 1999e^{-0.02t}}.$$

GRAPH OF 'N(t) Vs 't'

case a) $K = 0\cdot05$
$h = 0\cdot02.$

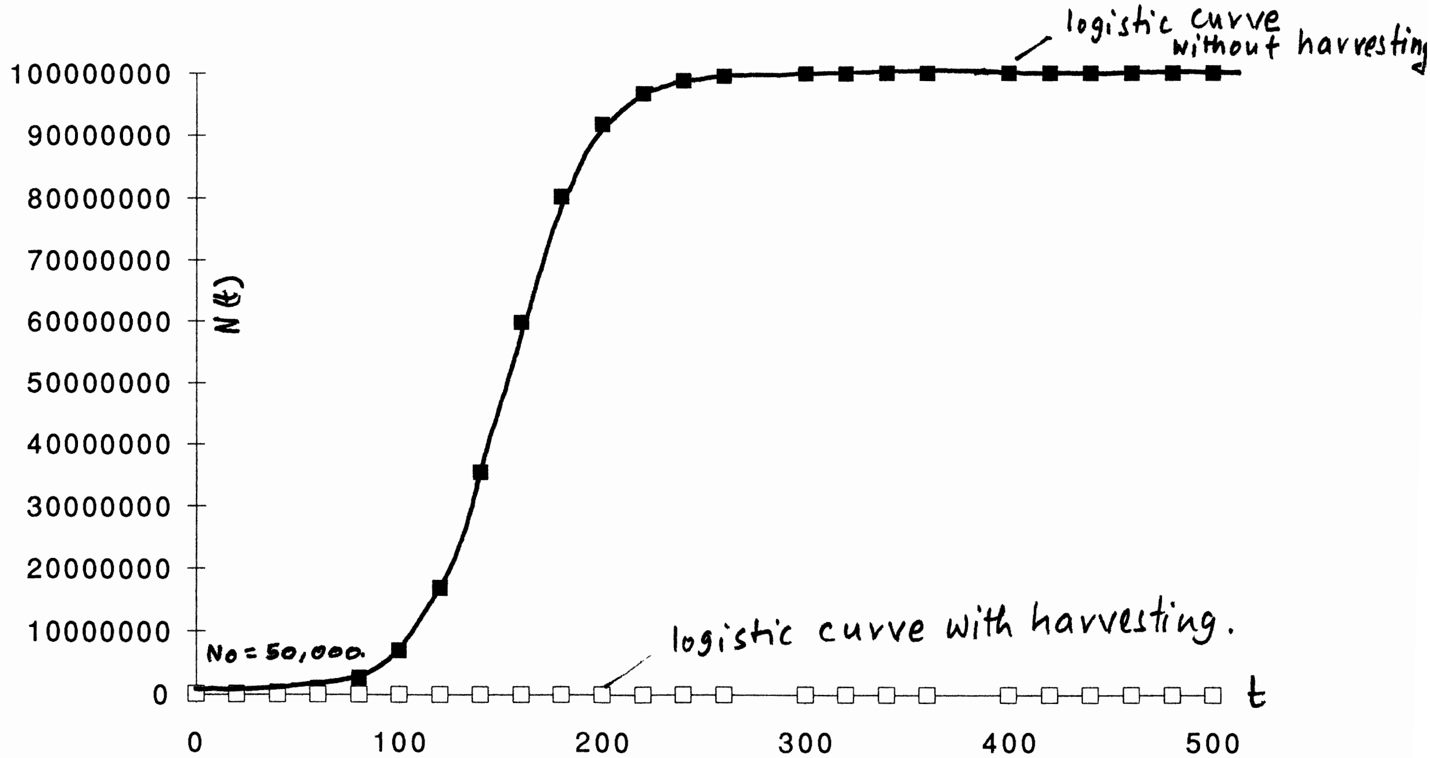



Fig 10

Note: Because of ~~chosen~~ values and scale it is not possible to show both curves more clearly. The curve 'with harvesting' seems like a straight line. which is not true (for details on values see table 2.) Also for same reasons the curves seems to start at zero, actually they start at $N(0) = 50000$ (due to scale

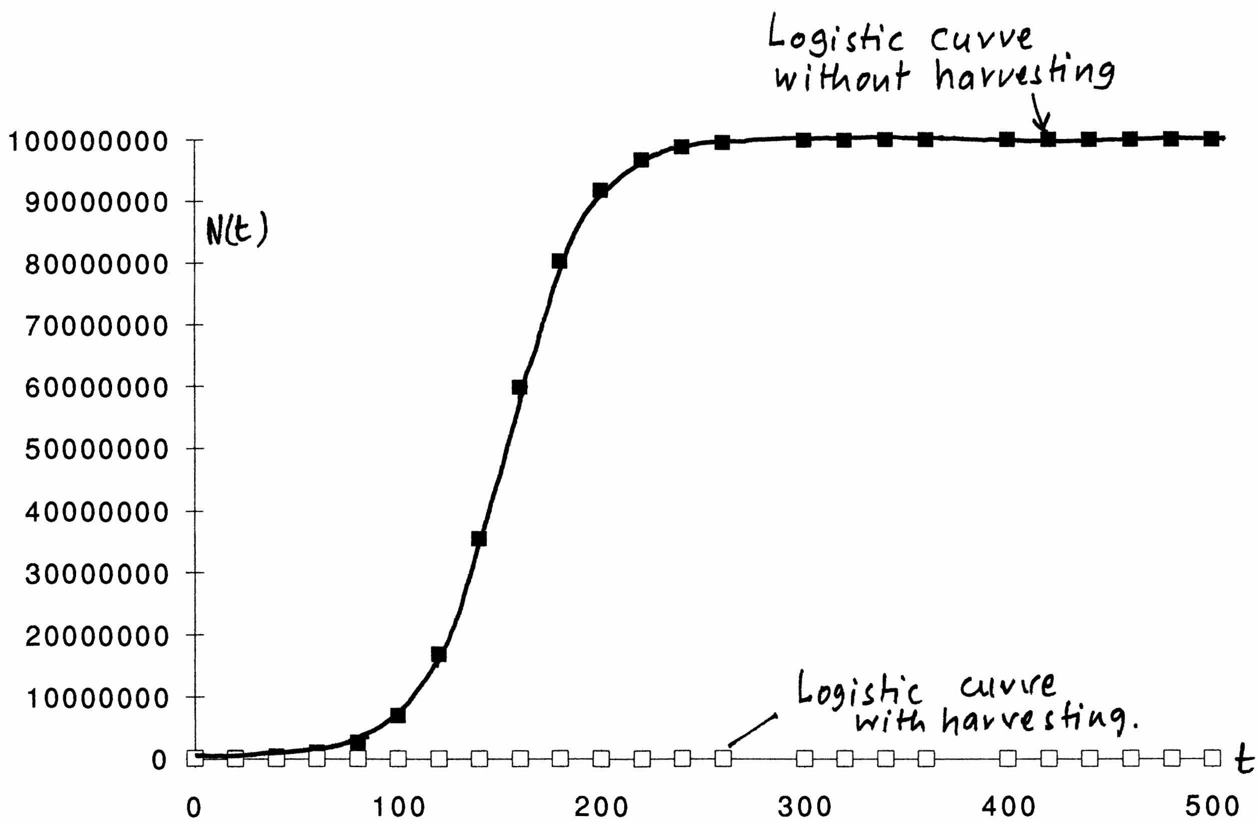



Fig. 11

**Note:** Because of ~~choosen~~ chosen values it is not possible to show the Logistic curve with harvesting and without harvesting on the same graph. The latter curve seems like a st. line but is not. For detail (values) see table 2. Also $N(0)$ seems to start at '0', actually it starts at $5000 = N(0)$ but due to scale (seem to start at 0.)

TABLE 2.

| t | N(t) | N(t*a) | N(t*b) |
|---|---|---|---|
| 0 | 50000 | 50000 | 50000 |
| 20 | 135797.422 | 3.95482669 | 1.75597146 |
| 40 | 368276.336 | 2.89127148 | 1.28373012 |
| 60 | 994783.854 | 2.63098235 | 1.16815786 |
| 80 | 2658657.93 | 2.54664103 | 1.1307092 |
| 100 | 6911253.13 | 2.51695828 | 1.11752969 |
| 120 | 16792539.1 | 2.50621197 | 1.11275819 |
| 140 | 35425165.1 | 2.50228167 | 1.11101309 |
| 160 | 59859099.1 | 2.5008389 | 1.11037248 |
| 180 | 80212003.6 | 2.50030855 | 1.110137 |
| 200 | 91679661.8 | 2.5001135 | 1.1100504 |
| 220 | 96769195.9 | 2.50004175 | 1.11001854 |
| 240 | 98786674.4 | 2.50001536 | 1.11000682 |
| 260 | 99550192.6 | 2.50000565 | 1.11000251 |
| 300 | 99938887.5 | 2.50000076 | 1.11000034 |
| 320 | 99977509.3 | 2.50000028 | 1.11000012 |
| 340 | 99991724.9 | 2.5000001 | 1.11000005 |
| 360 | 99996955.6 | 2.50000004 | 1.11000002 |
| 400 | 99999588 | 2.50000001 | 1.11 |
| 420 | 99999848.4 | 2.5 | 1.11 |
| 440 | 99999944.2 | 2.5 | 1.11 |
| 460 | 99999979.5 | 2.5 | 1.11 |
| 480 | 99999992.5 | 2.5 | 1.11 |
| 500 | 99999997.2 | 2.5 | 1.11 |

calculated values of N(t) for cases (a) and (b)

N(t) - logistic curve without harvesting

N(t*a) and N(t*b) values are for harvesting.

with harvesting: $N(t^*) = \frac{C^*}{1+F e^{-K^* t}}$ $K^* = k = 0.02.$

$C^* = C\frac{k}{Ch+k} = 0.4.$
Very much reduced.

with $N(0) = 50,000.$
we can obtain the solution i.e.
equation for $N(t^*)$.

$$N(t^*) = \frac{0.4}{1+(0.999992)e^{-0.02t}}$$

Similar graphs a for 1.a but C* is very much reduced.

usually not considered for logistic law

.

2 When k<0, h>0, h>k

$C^* = C\left(\frac{k}{k+hC}\right)$

k*=k is negative
C*=C(k/(k+h)) is meaningless.
The population decays from its value to zero.

negative & hence has no physical interpretation

Example:

C=100,000,000
k=-.05
h=.02

$\therefore K^* = K = -0.05$

$$c^* = C\frac{K}{K+Ch} = \frac{100,000(-0.05)}{(-0.05)+(100,000,000\times 0.02)}$$

-ve population.

Sketch & consider

Negative population meaningless.

3 When k=h

k*=k
C*=C(k/(k+Ch)) $= C\frac{K}{K+KC} = C\frac{1}{1+C}$

It is interesting to note that in this case new carrying capacity is independent of k and h. and is $\sim 1$.

Example:

Assume, $c = 100,000,000$

$N_0 = 50,000$ at $t = 0$

$h = 0.05$.

$K = 0.05$.

$$N(t) = \frac{c}{1 + 1999 e^{-0.05t}}$$ without harvesting.

with harvesting,

$$K^* = K = 0.05$$

$$c^* = \frac{c}{1+c} = \frac{100,000,000}{100,000,000+1} \simeq 1$$

The new carrying capacity reduced almost to 1. from 100 million. i.e. rapid decay in population.

## MODIFIED HARVESTING MODEL

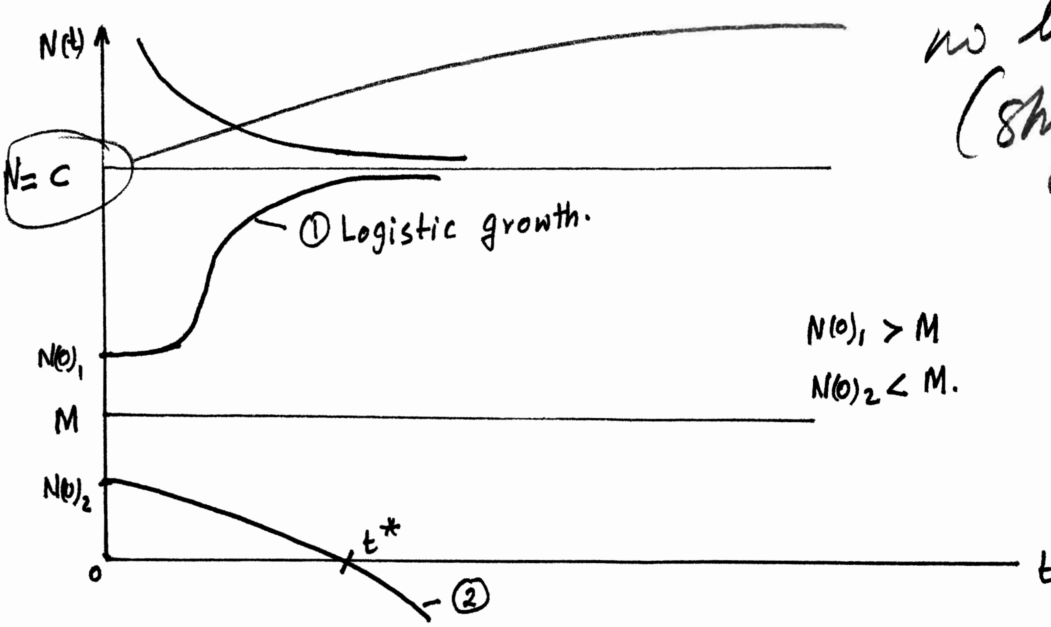
As can be seen that with different values of 'h' the carrying capacity decreases. The above models do not tell, as to how the carrying capacity should be reduced i.e. by what amount the fishes should be harvested. For example if all fishes are harvested at one time then the economic gain for fishermen would be high but no fishes would be left. This might affect future economics of fishermen. Therefore it would be nice to have a model which can show the relation between the harvesting rate and the number of fishes left (balance) so that the growth should be sustainable. Let us assume that for a sustainable growth the minimum population should be M. Also, assume that the growth rate is proportional to (N-M) and so is harvesting rate h. The differential equation for this case will be:

Surely we could use $c^*$ to determine h.

$$\frac{dN}{dt} = k(N-M)\left(1 - \frac{N}{c}\right) - h(N-M) \tag{11}$$

Not explained well.

Thus, when population of fishes is equal to minimum viable population then dN/dt =0, It would be better to draw a phase-plane diagram for above equation.




no longer $N = C$ (should be $C^{*} = \frac{C(k-h)}{k} = C\left(1-\frac{h}{k}\right)$ justification?

Fig 12.

When N=M, dN/dt =0
when initial population is less than M then N(t) follows curve 2.
when initial population is more than M then the growth of N(t) is Logistic.

In order to understand this model better let us solve the differential equation:

$$\frac{dN}{dt} = (N-M)\left[k\left(1-\frac{N}{C}\right) - h\right]$$

$$= (N-M)\,k\left[1-\frac{N}{C}-\frac{h}{k}\right]$$

$$= k(N-M)\left[\frac{k-h}{k}-\frac{N}{C}\right]$$

$$= \frac{(k-h)\,k(N-M)}{k}\left[1-\frac{N}{C}\,\frac{k}{(k-h)}\right]$$

$$= (k-h)\,(N-M)\left[1-\frac{kN}{(k-h)c}\right]$$

$$= (k-h)N\left(1-\frac{M}{N}\right)\left[1-\frac{Nk}{c(k-h)}\right]$$

This can be written as;

$$\frac{dN}{dt} = Nk^*\left(1-\frac{N}{c^*}\right)\left(1-\frac{M}{N}\right)$$

where $k^* = k-h$

$c^* = \dfrac{c(k-h)}{k}$. as in the first harvesting model!

Let us solve the above equation.

$$\frac{dN}{dt} = Nk^*\left(\frac{c^*-N}{c^*}\right)\left(\frac{N-M}{N}\right)$$

$$= k^*\frac{(c^*-N)}{c^*}(N-M)$$

separating the variables.

$$\therefore \frac{c^*\,dN}{(c^*-N)(N-M)} = k^*\,dt$$

Integ~~r~~rating,

$$\int \frac{c^*\,dN}{(c^*-N)(N-M)} = k^*t + c_1 \quad , \quad c_1 = \text{constant of integeration.}$$

Right hand side of equation can be solved by using partial fractions.

$$\frac{c^*}{(c^*-N)(N-M)} = \frac{A}{(c^*-N)} + \frac{B}{(N-M)} = \frac{AN-AM+c^*B-NB}{(c^*-N)(N-M)}$$

$$= \frac{N(A-B)+c^*B-AM}{(c^*-N)(N-M)}$$

When $(A-B)=0$

$$\therefore A = B$$

then $Bc^* - AM = c^*$

$$\therefore Bc^* - BM = c^* \quad \therefore B = \frac{c^*}{c^* - M} = A.$$

$$\frac{c^*}{(c^*-N)(N-M)} = \frac{c^*}{(c^*-M)(c^*-N)} + \frac{c^*}{(c^*-M)(N-M)}$$

$$\frac{c^*}{c^*-M}\left[\int \frac{dN}{(c^*-N)} + \frac{dN}{(N-M)}\right] = k^*t + c_1$$

$$\ln\left|\frac{N-M}{c^*-N}\right| = \frac{c^*-M}{c^*}\left(k^*t + c_1\right)$$

$$\left|\frac{N-M}{c^*-N}\right| = e^{\frac{c^*-M}{c^*}(k^*t + c_1)} = e^{Rc_1} e^{Rk^*t}$$

where, $R = \dfrac{c^*-M}{c^*}$

$$\left|\frac{N-M}{c^*-N}\right| = De^{k^*Rt} \qquad D = e^{Rc_1} > 0$$

$$\left(\frac{N-M}{c^*-N}\right) = \pm De^{k^*Rt} = L\,e^{k^*Rt}, \quad L = \pm D.$$

$$(N-M) = (c^*-N)\,Le^{k^*Rt}$$

$$N\left(1 + Le^{k^*Rt}\right) = c^*Le^{Rkt} + M$$

$$\therefore N(t) = \frac{M + c^*Le^{k^*Rt}}{1 + Le^{k^*Rt}}$$

When $t=0$, $N(0) = N_0$ is initial population.
For this condition the equation becomes,

$$N(0) = N_0 = \frac{M + c^* L e^0}{1 + L e^0} = \frac{M + c^* L}{1 + L}$$

$$M + Lc^* = N_0 + N_0 L$$

$$L(c^* - N_0) = N_0 - M$$

$\therefore \quad L = \dfrac{N_0 - M}{c^* - N_0}$, thus constant of integration is now known.

Therefore

$$N(t) = \frac{M + c^* \left(\frac{N_0 - M}{c^* - N_0}\right) e^{K^* R t}}{1 + \left(\frac{N_0 - M}{c^* - N_0}\right) e^{K^* R t}}$$

Dividing top & bottom by $\dfrac{N_0 - M}{c^* - N_0} e^{RK^* t}$

we get,

$$N(t) = \frac{M\left(\frac{c^* - N_0}{N_0 - M}\right) e^{-K^* R t} + c^*}{\left(\frac{c^* - N_0}{N_0 - M}\right) e^{-K^* R t} + 1}$$

$$= \frac{c^* + M S e^{-K^* R t}}{1 + S e^{-K^* R t}}$$

where, $S = \dfrac{c^* - N_0}{N_0 - M}$ and. $R = \dfrac{c^* - M}{c^*}$

M

If the initial population is less than 'No' then the extinction must occur in a definite time. Let us prove this, when $N_0 < M$.

For extinction to occur $N(t) = 0$ at a time 't' to be determined.

$$\therefore \quad 0 = \frac{c^* + MSe^{-k^* Rt}}{1 + Se^{-Rk^* t}}$$

$$\therefore \quad c^* + Mse^{-k^* Rt} = 0$$

and thus

$$Mse^{-k^* Rt} = -c^*$$

$$e^{-k^* Rt} = \frac{-c^*}{Ms}$$

$$-k^* Rt = \ln\left(\frac{-c^*}{Ms}\right)$$

$$\therefore t = \frac{1}{Rk^*} \ln\left(-\frac{Ms}{c^*}\right) = \frac{1}{\left(\frac{c^*-M}{c^*}\right)k^*} \ln\left[-\frac{M}{c}\left(\frac{c^*-N_0}{N_0-M}\right)\right]$$

$$= \frac{1}{\left(\frac{c^*-M}{c^*}\right)k^*} \ln\left[+\frac{M}{c^*}\frac{(c^*-N_0)}{(M-N_0)}\right]$$

As $M > N_0$ and $c^* > \cancel{M} N_0$

Therefore 't' is both positive and finite.

need $c^* > M > N_0 > 0$

so $\frac{M}{c^*}\frac{(c^*-N_0)}{(M-N_0)} > 1$

This can also be explained by following graph.

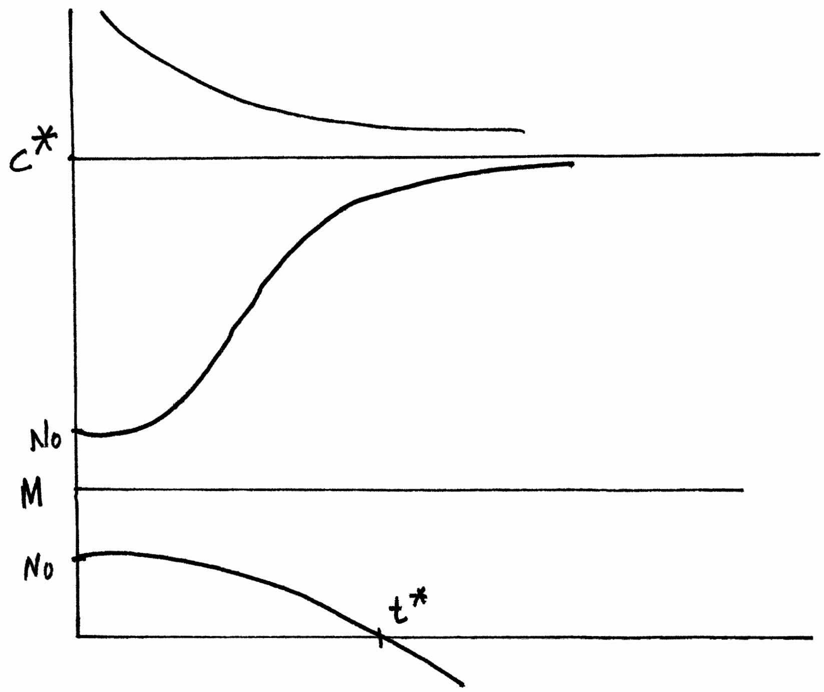



Fig. 13

Justification? (see earlier)

As can be seen from above graph that when initial population is less than M (i.e. $N_0 < M$) then extinction of population occurs at time $t^*$ shown in fig. 13. when the initial population is more than M (i.e. $N_0 > M$) then the growth in population is similar to Logistic-growth.

$$N(t) = \frac{c^* + Ms e^{-k^* R t}}{1 + s e^{-k^* R t}}$$

when $t \to \infty$, $N(t) \longrightarrow c^*$

when time approaches to infinity then $N(t)$ approaches the carrying capacity. (in our case $c^*$). This is also shown by examples.

true even when $N_0 < M$ !!

Examples:

To show how this model works, let us consider some examples.

(1). Assume,

$$C = 100{,}000{,}000$$
$$k = 0.05$$
$$h = 0.02.$$
$$M = 80{,}000$$
$$N_1(0) = 50{,}000 < M$$
$$N_2(0) = 100{,}000 > M.$$

logistic solution without harvesting is

$$N(t) = \frac{C}{1+Fe^{-kt}}$$

For $N_1(0) = 50{,}000$ at $t=0$

$$N(t) = \frac{100{,}000{,}000}{1+1999\,e^{-0.05t}}$$

For $N_2(0) = 100{,}000$, at $t=0$

$$N(t) = \frac{100{,}000{,}000}{1+999\,e^{-0.05t}}$$

With harvesting,

$$k^* = k-h = 0.05-0.02 = 0.03$$

$$c^* = \frac{c(k-h)}{k} = \frac{100{,}000{,}000(0.03)}{0.05} = 60{,}000{,}000$$

For $N_1(0) = 50{,}000$

$$R = \frac{c^*-M}{c^*} = \frac{60{,}000{,}000-80{,}000}{60{,}000{,}000} = 0.99866$$

$$\delta = \frac{c^*-N_0}{N_0-M} = \frac{60{,}000{,}000-50{,}000}{50{,}000-80{,}000} = -1998.3333$$

$$\therefore\ N(t)^*_1 = \frac{60{,}000{,}000+80{,}000(-1998.3333\,e^{-0.03\times .99866.t}}{1+(-1998.3333)\,e^{-0.99866(.03)t}.}$$

when $N_2(0) = 100{,}000 > M$

$$R = 0.99866$$

$$S = \frac{60{,}000{,}000 - 100{,}000}{100{,}000 - 80{,}000} = 2995$$

$$\therefore N(t) = \frac{60{,}000{,}000 + 80{,}000(2995)\, e^{-0.03(0.99866)t}}{1 + 2995\, e^{-0.99866(0.03).t}}$$

The curves for $N(t)$ vs 't' (without harvesting) and $N(t)_1$ vs 't' with harvesting are shown in fig. 14. and table 3. (for values of $N_0 = 50{,}000$ and $M = 80{,}000$)

The Curves for $N(t)$ vs 't' (without harvesting) and $N(t)_2$ vs 't' with harvesting are shown in fig 15 and table 3.

(2) Example

same values of c, h and k.

$k^* = k - h = 0.05 - 0.03 = 0.02$

$c = 100{,}000{,}000 \quad \therefore c^* = 60{,}000{,}000$

$M = 15{,}000{,}000$

$N_1(0) = 5{,}000{,}000 < M$

$N_2(0) = 20{,}000{,}000 > M$

For $N_1(0) = 5{,}000{,}000$, $\quad R = 0.75 = \dfrac{60{,}000{,}000 - 15{,}000{,}000}{60{,}000{,}000}$

$$S = \frac{60{,}000{,}000 - 5{,}000{,}000}{5{,}000{,}000 - 15{,}000{,}000} = -5.5$$

$$\therefore N(t) = \frac{60{,}000{,}000 + 15{,}000{,}000(-5.5)\, e^{-0.03(0.75)t}}{1 + (-5.5\, e^{-(0.03)\, 0.75t}}$$

For $N_2(0) = 20{,}000{,}000$, $\quad R = 0.75$

$S = 8.0$

$$\therefore N(t) = \frac{60{,}000{,}000 + 120{,}000{,}000\, e^{-0.0225t}}{1 + 8\, e^{-0.0255t}}$$

...lotted and shown in fig 16 ...

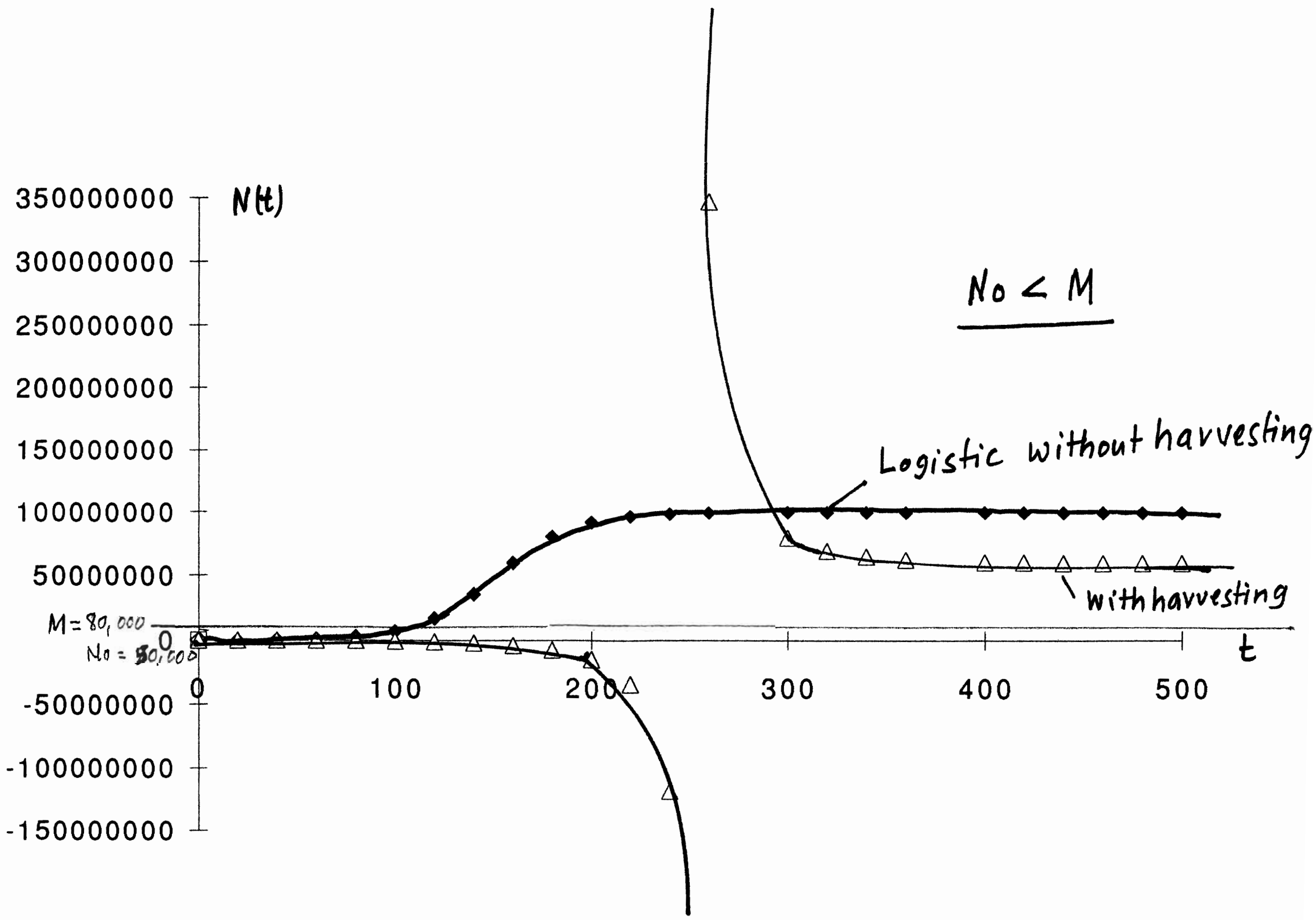



Fig 14.

Note: Because of the chosen values which are smaller compared to maximum value the curves seem to start at zero. Actually they start at $N(0) = 50,000$ below minimum 'M' = 80,000. For details of values please see table 3.

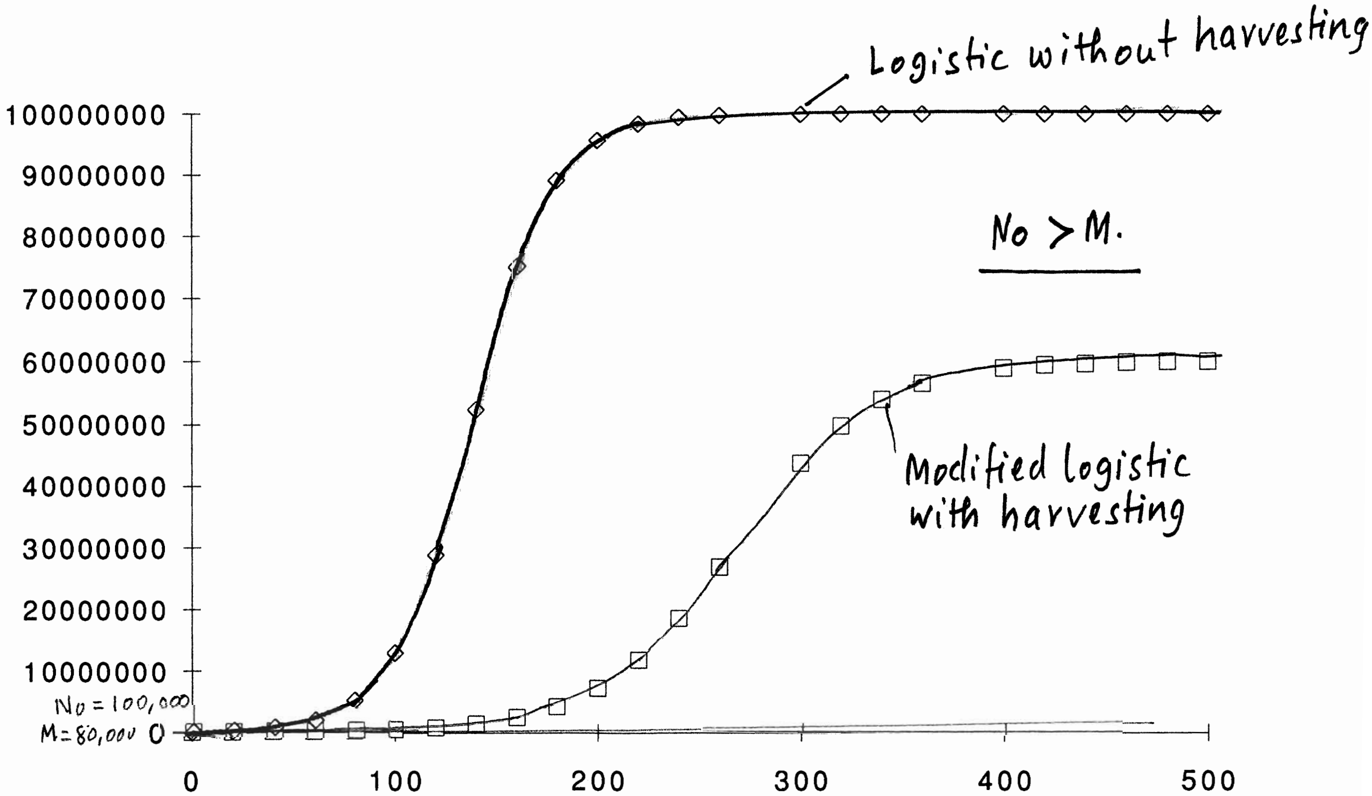



Fig 15.

Note: same comments as in fig 14 (for scale).

TABLE 3

| t | N(t) | N(t)1 | N(t)2 | N(t) |
|---|---|---|---|---|
| 0 | 50000 | 49999.9995 | 100000 | 100000 |
| 20 | 135797.422 | 25357.9159 | 116403.117 | 271361.907 |
| 40 | 368276.336 | -19558.858 | 146244.467 | 734214.671 |
| 60 | 994783.854 | -101509.77 | 200498.956 | 1970937.3 |
| 80 | 2658657.93 | -251290.125 | 299025.489 | 5182065.86 |
| 100 | 6911253.13 | -525914.036 | 477577.095 | 12934587.5 |
| 120 | 16792539.1 | -1032391.24 | 799930.361 | 28766436.9 |
| 140 | 35425165.1 | -1976612.76 | 1377946.57 | 52329443 |
| 160 | 59859099.1 | -3772905.99 | 2401838.07 | 74899232.5 |
| 180 | 80212003.6 | -7325594.32 | 4176982.68 | 89024491.4 |
| 200 | 91679661.8 | -14924913.2 | 7142880.08 | 95661325.6 |
| 220 | 96769195.9 | -34305055.5 | 11804891.1 | 98358882.1 |
| 240 | 98786674.4 | -118247524 | 18473333.3 | 99389937.8 |
| 260 | 99550192.6 | 347220743 | 26829405.7 | 99774701.8 |
| 300 | 99938887.5 | 79931394.5 | 43686823.6 | 99969449.7 |
| 320 | 99977509.3 | 69520010.9 | 49786594.5 | 99988759 |
| 340 | 99991724.9 | 64879457.2 | 53923389 | 99995864.4 |
| 360 | 99996955.6 | 62585166.7 | 56502529.3 | 99998478.5 |
| 400 | 99999588 | 60757080.9 | 58900052.6 | 99999794.1 |
| 420 | 99999848.4 | 60413474.2 | 59390809.9 | 99999924.3 |
| 440 | 99999944.2 | 60226397.8 | 59663860.1 | 99999972.1 |
| 460 | 99999979.5 | 60124138.3 | 59814906.1 | 99999989.7 |
| 480 | 99999992.5 | 60068119.7 | 59898194.9 | 99999996.2 |
| 500 | 99999997.2 | 60037395.8 | 59944040.4 | 99999998.6 |

Note : 1) N(t) is Logistic value without harvesting while $N(t)_1$ and $N(t)_2$ are values of N(t) with harvesting for examples 1) and for values of $N_0 = 50{,}000$, and $M = 80{,}000$ and $N_0 = 100{,}000$ and $M = 80{,}000$.

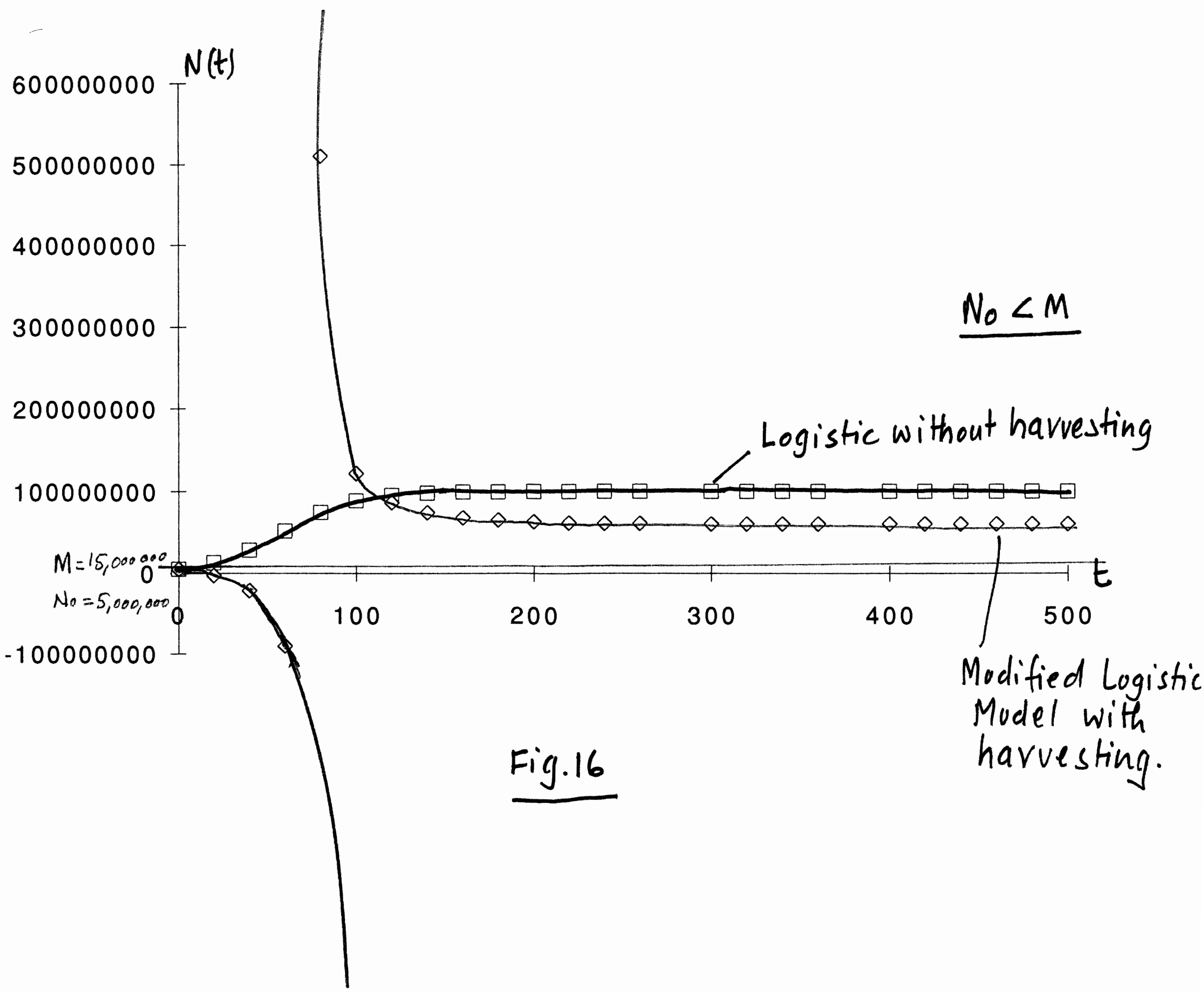

N(t)
600000000
500000000
400000000
300000000
200000000
100000000
0
-100000000
M=15,000,000
No=5,000,000
0
100
200
300
400
500
t
No < M
Logistic without harvesting
Modified Logistic Model with harvesting.

Fig.16

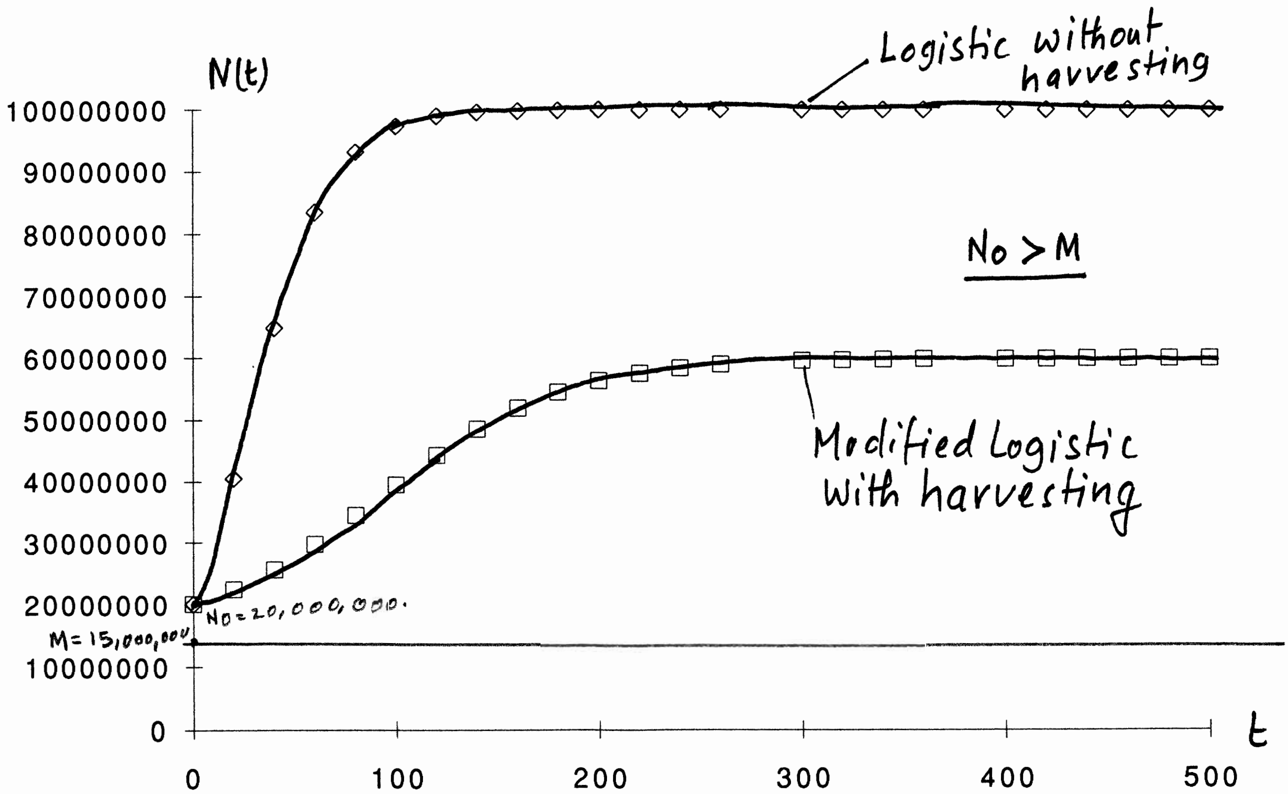

N(t)
Logistic without harvesting
100000000
90000000
80000000
70000000
60000000
50000000
40000000
30000000
20000000
10000000
0
No > M
Modified Logistic with harvesting
No = 20,000,000.
M = 15,000,000
0
100
200
300
400
500
t

Fig. 17.

TABLE:4

| t | N(t) | N(t)1 | N(t)2 | N(t) |
|---|---|---|---|---|
| 0 | 5000000 | 5000000 | 20000000 | 20000000 |
| 20 | 12516099.8 | -2950064.11 | 22375809.5 | 40460967.5 |
| 40 | 28000456.2 | -21403846 | 25581868.1 | 64878564.4 |
| 60 | 51388668.3 | -90678099.9 | 29639278 | 83392523 |
| 80 | 74184133.7 | 510288623 | 34376581.3 | 93173845.9 |
| 100 | 88650832.4 | 122065295 | 39414144.7 | 97375554.7 |
| 120 | 95502200.5 | 86386682.2 | 44265549.8 | 99018233.4 |
| 140 | 98296931.2 | 73876379.4 | 48511639.7 | 99636572.8 |
| 160 | 99366657.8 | 67958651.2 | 51927931.6 | 99865994.8 |
| 180 | 99766069.9 | 64769019.8 | 54495198.8 | 99950660.4 |
| 200 | 99913814.5 | 62928400.4 | 56327172.7 | 99981843.3 |
| 220 | 99968276.8 | 61824212.8 | 57586726.4 | 99993319.8 |
| 240 | 99988327.4 | 61146330 | 58430732.6 | 99997542.4 |
| 260 | 99995705.6 | 60724246.7 | 58986584.6 | 99999095.9 |
| 300 | 99999418.8 | 60291671 | 59582395.1 | 99999877.6 |
| 320 | 99999786.2 | 60185541.9 | 59732824.9 | 99999955 |
| 340 | 99999921.3 | 60118130.2 | 59829274.3 | 99999983.4 |
| 360 | 99999971.1 | 60075251.6 | 59890990.6 | 99999993.9 |
| 400 | 99999996.1 | 60030564.7 | 59955616.3 | 99999999.2 |
| 420 | 99999998.6 | 60019484.1 | 59971689.6 | 99999999.7 |
| 440 | 99999999.5 | 60012421.7 | 59981944.4 | 99999999.9 |
| 460 | 99999999.8 | 60007919.6 | 59988485.5 | 100000000 |
| 480 | 99999999.9 | 60005049.4 | 59992657.4 | 100000000 |
| 500 | 100000000 | 60003219.5 | 59995317.9 | 100000000 |

Note:- $N(t)$ is logistic value without harvesting

$N(t)_1$ and $N(t)_2$ are values with harvesting for example 2. with $N_0 = 5{,}000{,}000$ and $M = 15{,}000{,}000$ and for $N_0 = 20{,}000{,}000$ and $M = 15{,}000{,}000$. respectively.

## CONCLUSIONS AND COMMENTS ON MODIFIED MODEL

In order to comment on the model let us draw the phase-plane diagram.

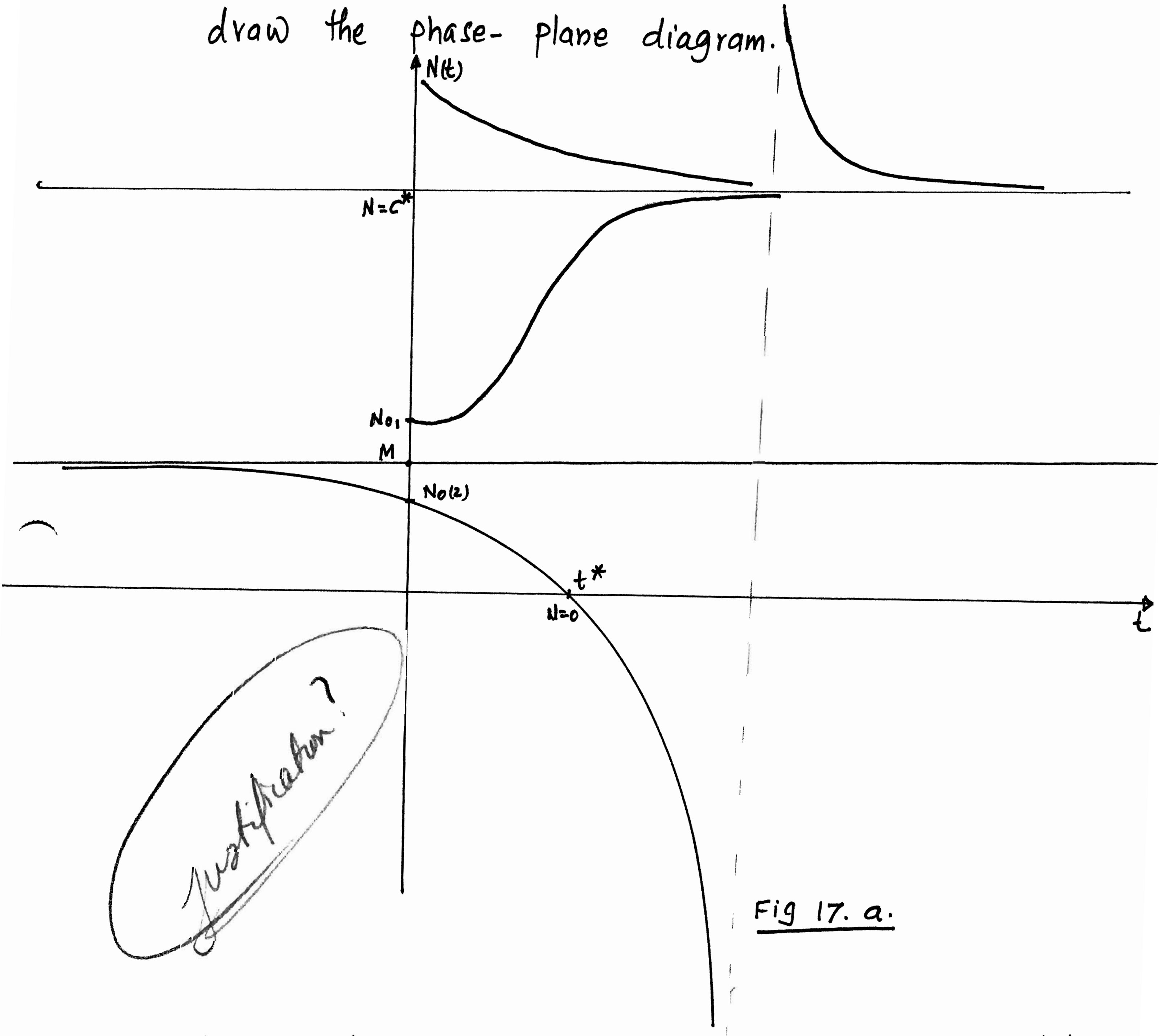



Fig 17. a.

As can be seen from above that when initial population is less than minimum population (supportable or viable) then the population reaches zero at time $t^*$. In fact in this case the curve becomes asymtotic, reaching a very-very low, negative value and then jumping to a high positive value and later on levelling off to $c^*$ (carrying capacity). From our point of view any values after $t^*$ is of no meaning (as population has already reached zero at $t^*$).

From equation,

$$N(t) = \frac{c^* + Mse^{-k^*Rt}}{1 + se^{-k^*Rt}}$$

it is easy to see that when $t \to \infty$ the population $N(t) \to c^*$.

Also when $t \to -\infty$ then population reaches 'M'. Meaning that our 't' axis has moved to 'M'.

When initial population is more than minimum viable population then the growth is similar to Logistic.

When denomenator in above equation is zero then $N(t) = \infty$.

$$\therefore 1 + se^{-k^*Rt} = 0$$

$$\therefore se^{-k^*Rt} = -1 \quad \therefore e^{-k^*Rt} = -1/s$$

$$\therefore e^{-k^*Rt} = -1/s =$$

$$\therefore \frac{1}{e^{k^*Rt}} = -\frac{1}{s}, \Rightarrow e^{k^*Rt} = -s$$

$$k^*Rt = \ln(-s), \quad t = \frac{1}{k^*R}\ln(-s)$$

when $s = -ve$, as we have seen in case $N_0 < M$ then above situation occurs.

<u>Conclusion and suggestion for improvement</u>.

The above models can not be substantiated experimentally as data on fish population are not available.Therefore for each model some assumed values were used to show the performance of models under various harvesting conditions.The results seem in agreement with theory. ?

The above model may be further modified to take into account the time-delay effects of environment on fish population.

Some experiments on environmentally limited population growth have shown some good qualitative and quantitative agreement with Logistic growth model,other experiments have shown that the carrying capacity of environment is not always reached monotonically.Instead some of thes experiments show the types of behaviour sketched in following figure. references?

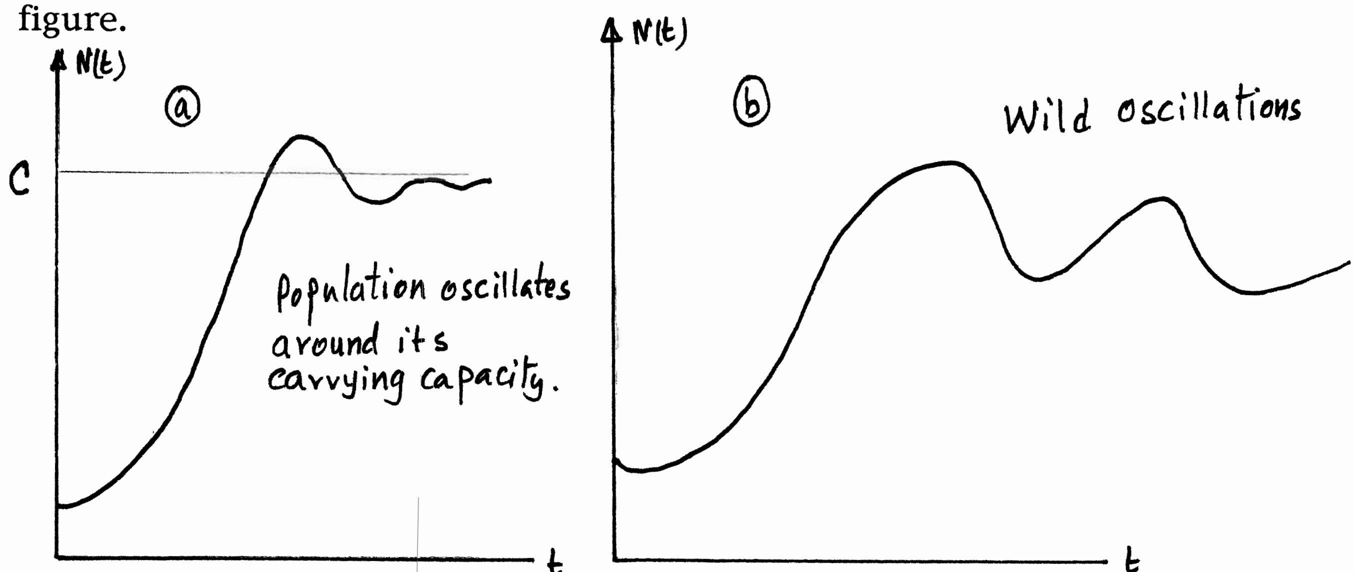



Fig. 18

We know that

$$\frac{1}{N}\frac{dN}{dt} = K \quad \text{or} \quad \frac{dN}{dt} = f(N)$$

The rate of change of population depends in an arbitary way.To take into account the oscillation around equilibrium population we will consider the situation in which population may oscillate about its carrying capacity. The basic idea is that population does not respond immediately to its environment.Instead there is a delay.The rate of change of population is not a function of present population N(t) but Of a past population N(t-td),where td is time delay.

Two different model of situation are:

$$\left.\begin{aligned} \frac{dN(t)}{dt} &= f(N(t-td)) \\ \frac{1}{N}\frac{dN(t)}{dt} &= K(N-(t-td.)) \end{aligned}\right\} \text{Time delayed differential equations}$$

For fishes the delay time may be the time it takes for an egg to develop into fertile adult fish.
Suppose growth rate is costant but occurs with a delayed 'td'

$$\therefore \frac{dN(t)}{dt} = kN\left(t - t_d\right)$$

If we apply the ideas behind logistic growth to the delay mechanism,then population growth is

$$\frac{dN(t)}{dt} = kN(t)\left[1 - \frac{N(t-t_d)}{c}\right]$$

In general time differential equations are difficult to solve than ordinary differential equations,therefore we will try to describe the effect of 'td' on logistic model by graph.Suppose a species,fishes, grows in logistic manner without a delay;for population less than equilibrium the population grows and vice-versa.If there is a delayed mechanism for example described by eq. then the fishes would grow at a rate determined by population at a previous time.At a time when population has almost reached equilibrium indicated by * the population would continue to grow at a rate for population not near the eqilibrium (if 'td' is sufficiently large)

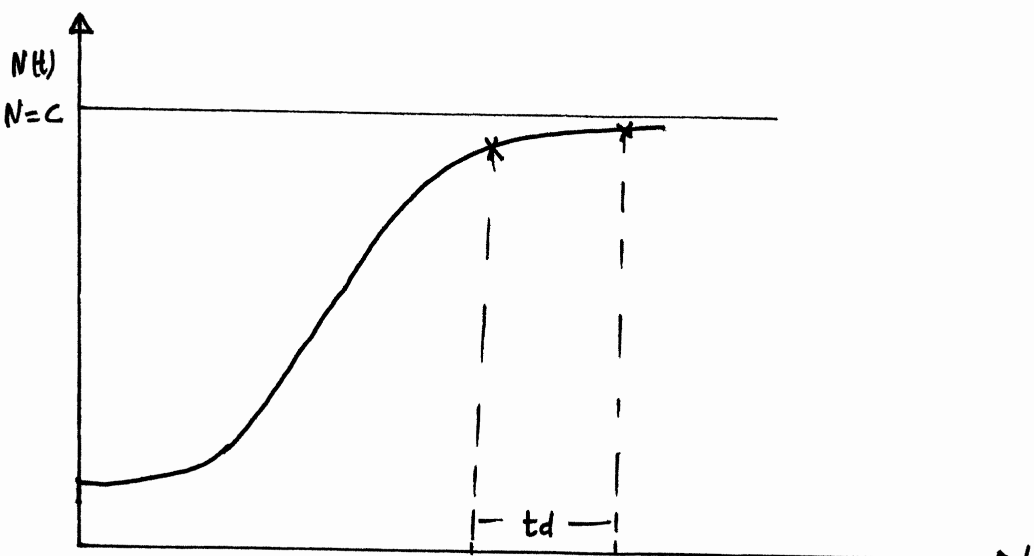



Fig 19

In this way the population could go beyond its eqilibrium value.Furhermore if the delay is sufficiently large the population oscillate around equilibrium as shown in (a) or perhaps the delay wold establish the equilibrium population as shown in (b).

(2) Instead of numbers representing the population, it would be more realistic to modify the model on basis of bio-mass of fishes. ok

(3) Accordig to present law in Manitoba only certain sizes of fishes can be harvested to allow for juvenile population to grow.
(4) Model could be modified to take into account the harvesting methods.
(5)Model may be modified to take into account the sudden and large decay in population due to catastrophic events e.g. spillage of ship oil can kill a large number of fishes.

## REFERENCES

(1)Charles,A.T. and White,G.N.(editors): Natural rsource modelling and analysis.-proceedings of conference held at Saint Mary's University and Bedford Institute of Oceanography.Halifax.,Canada.

(2)Deriso,R.B. 'Harvesting strategies and parameter estimation for age-structured model' Canadian Journal of Fisheries and Aquatic Sciences.

(3)Leah-Edelstein-Keshet 'Mathematical models in biology'. Random House, Nework.

(4)Belender,E.A. 'An introduction to Mathematical Modelling' A Wiley Enterprise Publication.

(5)Roberts,H.M. 'Life science models' Springer-Verlag.

(6)Lennes,N.J. 'Differential and integral calculus' Harper & Brothers.

(7)Kapur,J.N.' Insight into mathematical modelling vol.1 Indian Sciences Academy.

(8)Smith,J.H. 'Models in ecology' Cambrige University Press.1974.

(9)Meyer,W.J. 'Concepts of mathematical modelling'.McGraw hill 1984

(10)Pielou,E.C. 'An introduction to mathematical ecology". Wiley 1969

(11)Olinick,M 'An introduction to mathematical models in social and life sciences. Addison Wesely.1978