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*SIAM Review*, Vol. 40, No. 4. (Dec., 1998), pp. 965-971.

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## ON THE COMPUTATION OF $A^{N^*}$

SABER N. ELAYDI<sup>†</sup> AND WILLIAM A. HARRIS, JR.<sup>‡</sup>

*This paper is dedicated to Professor Hugh L. Turrittin on his 90th birthday.*

**Abstract.** Methods, which are based on the Cayley–Hamilton theorem, for the computation of  $A^n$  for nonsingular  $A$  are presented.

**Key words.** Cayley–Hamilton, difference equation, Casorati matrix

**AMS subject classifications.** 15-01, 15A21, 39-01, 39A10

**PII.** S0036144597319235

Let  $A$  be a  $k \times k$  nonsingular matrix of constants; then a fundamental matrix for the linear system of difference equations  $x(n+1) = Ax(n)$  is  $X(n) = A^n$ .

$$X(n+1) = AX(n), \quad X(0) = I,$$

and the solution of the difference initial value problem

$$x(n+1) = Ax(n) + b(n), \quad x(0) = x_0,$$

is given by

$$\begin{aligned} x(n) &= A^n x_0 + \sum_{j=0}^{n-1} A^{n-j-1} b(j) \\ &= X(n) x_0 + X(n) \sum_{j=0}^{n-1} X^{-1}(j+1) b(j) \end{aligned}$$

which is analogous to

$$x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-As} b(s) ds$$

as the solution of the corresponding differential initial value problem. Hence, for linear systems of difference equations, the computation of  $A^n$  is the analogous problem to the computation of  $e^{At}$  for linear systems of differential equations.

One could use transformation methods, writing  $A^n = P(P^{-1}AP)^n P^{-1}$  for nonsingular  $P$  to compute  $(P^{-1}AP)^n$  and hence  $A^n$  by using Schur's canonical form (triangular) or the Jordan canonical form, e.g., for  $J = (\lambda I + N)$ , with  $N^s = 0$  we have  $J^n = \lambda^n I + \binom{n}{1} \lambda^{n-1} N + \cdots + \binom{n}{s-1} \lambda^{n-s+1} N^{s-1}$ . Such transformation methods require the determination of eigenvectors and generalized eigenvectors and can be computationally involved and tedious.

We present in this note alternative methods for the computation of  $A^n$  for nonsingular  $A$  which are pedagogically simpler, based on the Cayley–Hamilton theorem

\*Received by the editors March 31, 1997; accepted for publication December 12, 1997.  
<http://www.siam.org/journals/sirev/40-4/31923.html>

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and the exact determination of the eigenvalues of  $A$ , which are analogous to methods of Putzer [9], Fulmer [2], Kirchner [6], Hirsch and Smale [3], Leonard [8], and Hsieh, Kohno, and Sibuya [4] for computing  $e^{At}$ .

Let

$$(1) \quad \begin{aligned} c(\lambda) &= \det(\lambda I - A) = \lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_1\lambda + c_0 \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_k) \end{aligned}$$

be the characteristic polynomial for  $A$ . Then the Cayley–Hamilton theorem states that  $c(A) = 0$ , or

$$A^k + c_{k-1}A^{k-1} + \cdots + c_1A + c_0I = 0$$

and

$$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_k I) = 0.$$

**THEOREM 1.** *Let  $A$  be a  $k \times k$  nonsingular matrix with eigenvalues  $\lambda_1, \dots, \lambda_k$ , and let  $M(0) = I, M(j) = \prod_{i=1}^j (A - \lambda_i I), j \geq 1$ . Then, if  $u_j(n)$  satisfies the (recursive) linear difference system*

$$\begin{aligned} u_1(n+1) &= \lambda_1 u_1(n), & u_1(0) &= 1, \\ u_{j+1}(n+1) &= \lambda_{j+1} u_{j+1}(n) + u_j(n), & u_{j+1}(0) &= 0, j = 1, \dots, k-1, \end{aligned}$$

for  $n \geq k$ ,

$$A^n = \sum_{j=0}^{k-1} u_{j+1}(n) M(j).$$

We note that  $u_1(n) = \lambda_1^n$  and  $u_{j+1}(n) = \sum_{i=0}^{n-1} \lambda_{j+1}^{n-i-1} u_j(i), j = 1, \dots, k-1$ , and  $M(k) = \prod_{i=1}^k (A - \lambda_i I) = 0$  (Cayley–Hamilton). Since  $M(j)$  is the monic polynomial  $M(j) = A^j + \text{lower order terms}$ ,  $A^n, n \geq k$ , can also be written as a polynomial in  $A$  of degree at most  $k-1$ .

**THEOREM 2.** *Let  $A$  be a  $k \times k$  nonsingular matrix with characteristic polynomial  $c(\lambda) = \lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_1\lambda + c_0$ , and let  $z(n)$  be the solution of the scalar  $k$ th order difference equation.*

$$z(n+k) + c_{k-1}z(n+k-1) + \cdots + c_1z(n+1) + c_0z(n) = 0,$$

$$z(0) = z(1) = \cdots = z(k-2) = 0, \quad z(k-1) = 1,$$

and

$$\begin{pmatrix} q_1(n) \\ \vdots \\ q_k(n) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \cdots & \cdots & c_{k-1} & 1 \\ c_2 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{k-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} z(n) \\ \vdots \\ z(n+k-1) \end{pmatrix}.$$

Then, for  $n \geq k$ ,

$$A^n = \sum_{j=0}^{k-1} q_{j+1}(n) A^j.$$

We note that the difference equation  $z(n+k) + c_{k-1}z(n+k-1) + \cdots + c_0z(n) = 0$  can conveniently be written as  $c(E)z(n) = 0$  using the shift operator  $Ez(n) = z(n+1)$ . Further, if the matrix  $A$  is a companion matrix, the linear difference system  $x(n+1) = Ax(n)$  is equivalent to the  $k$ th order scalar difference equation  $c(E)z(n) = 0$ , and it is not surprising that solutions of  $c(E)z(n) = 0$  would play a role in computing  $A^n$ . That this is true in general follows from the observation

$$E(A^n) = A^{n+1} = A(A^n)$$

and hence if  $p(\lambda)$  is any polynomial,  $p(E)(A^n) = p(A)A^n$ , and thus for any annihilating polynomial  $p(\lambda)$  for  $A$ ,  $p(E)A^n = 0$ . Since the Cayley–Hamilton theorem simply states that  $c(\lambda)$  is an annihilating polynomial for  $A$ , every element of  $A^n$  satisfies the  $k$ th order scalar difference equation  $c(E)y(n) = 0$ . The minimal polynomial,  $m(\lambda)$ , is also an annihilating polynomial for  $A$ , but is easily determined only in special cases, e.g., when  $A$  is a companion matrix ( $m(\lambda) = c(\lambda)$ ) or  $A$  real symmetric,  $m(\lambda) = \prod_{i=1}^s (\lambda - \mu_j)$ , where  $\mu_j$  are the distinct eigenvalues of  $A$ .

**THEOREM 3.** *Let  $A$  be a  $k \times k$  nonsingular matrix with characteristic polynomial  $c(\lambda) = \lambda^k + c_{k-1}\lambda^{k+1} + \cdots + c_1\lambda + c_0$ , and let  $\{y_1(n), \dots, y_k(n)\}$  be a linearly independent set of solutions of the  $k$ th order scalar difference equation  $c(E)y(n) = 0$ . Then there exist constant matrices  $E_1, \dots, E_k$  such that*

$$A^n = y_1(n)E_1 + y_2(n)E_2 + \cdots + y_k(n)E_k.$$

We note that if the eigenvalues of  $A$  are distinct,  $\{\lambda_1^n, \dots, \lambda_k^n\}$  is a suitable linearly independent set of solutions for  $c(E)y(n) = 0$ , and the general form for  $A^n, n = 0, 1$  yields

$$\begin{aligned} I &= E_1 + E_2 + \cdots + E_k \\ A &= \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_k E_k. \end{aligned}$$

Further, it can be shown that  $E_i E_j = E_j E_i = 0, i \neq j$ , and hence  $E_i^2 = E_i$ . Thus we have a resolution of the identity and the spectral representation of  $A$ , and hence

$$A^n = (\lambda_1 E_1 + \cdots + \lambda_k E_k)^n = \lambda_1^n E_1 + \cdots + \lambda_k^n E_k.$$

The  $SN$  decomposition of a matrix is a natural generalization of the spectral representations of  $A$  when the eigenvalues of  $A$  are not distinct. *There exists a unique decomposition of the matrix  $A, A = S + N, SN = NS$  with  $S$  semisimple (similar to a diagonal matrix) and  $N$  nilpotent ( $N^k = 0$ ).*

**THEOREM 4.** *Let  $A$  be a  $k \times k$  nonsingular matrix, and let  $A = S + N$  be the unique  $SN$  decomposition,  $SN = NS, S$  semisimple,  $N$  nilpotent, then*

$$A^n = (S + N)^n = S^n + \binom{n}{1} S^{n-1}N + \binom{n}{2} S^{n-2}N^2 + \cdots + \binom{n}{k-1} S^{n-k+1}N^{k-1}.$$

*Proof of Theorem 1.* Let  $\Phi(n) = \sum_{j=0}^{k-1} u_{j+1}(n)M(j)$ . We need only show that  $\Phi(0) = I$  and  $\Phi(n+1) = A\Phi(n)$ . Clearly,  $\Phi(0) = \sum_{j=1}^{k-1} u_{j+1}(0)M(j) = u_1(0)M(0) = I$ . Using the fact that  $AM(j) = (A - \lambda_{j+1}I + \lambda_{j+1}I)M(j) = M(j+1) + \lambda_{j+1}M(j)$ , we have

$$\begin{aligned} \Phi(n+1) - A\Phi(n) &= \sum_{j=0}^{k-1} u_{j+1}(n+1)M(j) - \sum_{j=0}^{k-1} u_{j+1}(n)AM(j) \\ &= \sum_{j=0}^{k-1} [u_{j+1}(n+1) - \lambda_{j+1}u_{j+1}(n)]M(j) - \sum_{j=0}^{k-1} u_{j+1}(n)M(j+1). \end{aligned}$$

Writing  $\sum_{j=0}^{k-1} u_{j+1}(n)M(j+1) = \sum_{i=1}^k u_i(n)M(i) = \sum_{j=1}^{k-1} u_j(n)M(j)$ , since  $M(k) = 0$ . Thus

$$\Phi(n+1) - A\Phi(n) = [u_1(n+1) - \lambda_1 u_1(n)] + \sum_{j=1}^{k-1} [u_{j+1}(n+1) - \lambda_{j+1} u_{j+1}(n) - u_j(n)]M(j)$$

and the theorem is proved.  $\square$

*Remark.* Theorem 1 has appeared in LaSalle [7], Kelley and Peterson [5], and Elaydi [1].

*Proof of Theorem 2.* Let  $\Psi(n) = \sum_{j=0}^{k-1} q_{j+1}(n)A^j$ . Clearly,  $\Psi(0) = \sum_{j=0}^{k-1} q_{j+1}(0)A^j = q_1(0)I = I$ . Now

$$\begin{aligned}\Psi(n+1) - A\Psi(n) &= \sum_{j=0}^{k-1} q_{j+1}(n+1)A^j - \sum_{j=0}^{k-1} q_{j+1}(n)A^{j+1} \\ &= q_1(n+1)I + \sum_{j=1}^{k-1} [q_{j+1}(n+1) - q_j(n)]A^j - q_k(n)A^k,\end{aligned}$$

and using  $A^k + c_k A^{k-1} + \cdots + c_1 A + c_0 I = 0$ ,  $\Psi(n+1) - A\Psi(n) = 0$  requires that

$$\begin{aligned}q_1(n+1) + c_0 q_k(n) &= 0, \\ q_2(n+1) + c_1 q_k(n) &= q_1(n), \\ &\vdots \\ q_k(n+1) + c_{k-1} q_k(n) &= q_{k-1}(n).\end{aligned}$$

Defining  $q_k(n) = z(n)$  and applying  $E^{j-1}$  to the  $j$ th equation yields

$$\begin{aligned}q_1(n+1) + c_0 z(n) &= 0, \\ q_2(n+1) + c_1 z(n+1) &= q_1(n+1), \\ &\vdots \\ z(n+k) + c_{k-1} z(n+k-1) &= q_{k-1}(n+k-1).\end{aligned}$$

Adding these equations yields  $c(E)z(n) = 0$  and back substitution in the first set of equations completes the proof of the theorem.  $\square$

*Proof of Theorem 3.*  $A^n = y_1(n)E_1 + \cdots + y_k(n)E_k$  will be valid if the following system of equations has a unique solution:

$$\begin{aligned}I &= y_1(0)E_1 + \cdots + y_k(0)E_k, \\ A &= y_1(1)E_1 + \cdots + y_k(1)E_k, \\ &\vdots \\ A^{k-1} &= y_1(k-1)E_1 + \cdots + y_k(k-1)E_k.\end{aligned}$$

The coefficient matrix of this system is the Casorati matrix which is nonsingular for a linearly independent set of solutions  $(y_1(n), \dots, y_k(n))$  of the  $k$ th order scalar difference equations  $c(E)y(n) = 0$ , and thus the theorem is proved.  $\square$

*Remark.* If the eigenvalues of  $A$  are distinct, then  $y_1(n) = \lambda_1^n, \dots, y_k(n) = \lambda_k^n$  is a suitable linearly independent set of solutions of  $c(E)y(n) = 0$  and the Casorati matrix is a Vandermonde matrix and  $E_i = e_i(A)$ , where  $e_i(\lambda) = \prod_{j \neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}$  are the Lagrange interpolating polynomials. We note that the nonzero columns of  $E_j$  are eigenvectors if the  $\lambda_i$  are distinct and are generalized eigenvector chains in the general case.

*Proof of Theorem 4.* Following Hsieh, Kohno, and Sibuya [4], let  $\mu_j$  be the distinct eigenvalues of the matrix  $A$  and write the characteristic polynomial  $c(\lambda) = \prod_{i=1}^s (\lambda - \mu_i)^{n_i}$  with integers  $n_i \geq 1, n_1 + \dots + n_s = k$ . Let the partial fraction expression of  $c(\lambda)^{-1}$  be

$$\frac{1}{c(\lambda)} = \frac{c_1(\lambda)}{(\lambda - \mu_1)^{n_1}} + \frac{c_2(\lambda)}{(\lambda - \mu_2)^{n_2}} + \dots + \frac{c_s(\lambda)}{(\lambda - \mu_s)^{n_s}},$$

and define the polynomials  $f_i(\lambda) = c_i(\lambda) \prod_{j \neq i} (\lambda - \mu_j)^{n_j}$ . Then  $1 = f_1(\lambda) + f_2(\lambda) + \dots + f_s(\lambda)$ , and with  $F_i = f_i(A)$ , we have the resolution of the identity

$$I = F_1 + F_2 + \dots + F_s, \quad F_i F_j = F_j F_i = 0, \quad i \neq j \text{ and } F_i^2 = F_i,$$

using the Cayley–Hamilton theorem.

Writing  $S = \sum_{i=1}^s \mu_i F_i, N = A - S$ , we have  $A = S + N, SN = NS, N^k = 0, S$  semisimple, and the theorem is proved.  $\square$

*Remark.* The  $SN$  decomposition is essentially in Kirshner [6], and is explicitly stated and proved in Hirsch and Smale [3], but we prefer the construction of Hsieh, Kohno, and Sibuya [4].

*Remark.* The restriction to nonsingular  $A$  is natural if we wish  $A^0 = I$ . We note that for  $A \equiv 0$  and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have  $y(n) \equiv 0$  as a solution of  $y(n+1) = Ay(n)$ . A preliminary transformation will reduce the general case to the one studied in this note.

*Example.* Consider  $x(n+1) = Ax(n)$  where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -2 & 3 & 1 \\ -3 & 1 & 4 \end{pmatrix}.$$

The characteristic equation  $c(\lambda)$  is

$$c(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 2)^2(\lambda - 3).$$

$\bullet A^n = \sum_{j=0}^{k-1} u_{j+1}(n)M(j)$ , where

$$u_1(n) = 2^n, \quad u_2(n) = n2^{n-1}, \quad u_3(n) = -2^n + 3^n - n2^{n-1},$$

$$M(0) = I, \quad M(1) = \begin{pmatrix} -2 & 1 & 1 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix}, \quad M(2) = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{pmatrix}, \text{ and}$$

$$A^n = \begin{pmatrix} 2^{n-1} - 3^n - n2^{n-1} & n2^{n-1} & -2^n + 3^n \\ 2^n - 3^n - n2^{n-1} & (n+2)2^{n-1} & -2^n + 3^n \\ 2^{n+1} - 2 \cdot 3^n - n2^{n-1} & n2^{n-1} & -2^n + 2 \cdot 3^n \end{pmatrix}.$$

•  $A^n = \sum_{j=0}^{k-1} q_{j+1}(n)A^j$ , where

$$A^0 = I, \quad A^2 = \begin{pmatrix} -5 & 4 & 5 \\ -9 & 8 & 5 \\ -14 & 4 & 14 \end{pmatrix},$$

$$\begin{aligned} q_1(n) &= -3(1+n)2^n + 4 \cdot 3^n, \\ q_2(n) &= (8+5n)2^{n-1} - 4 \cdot 3^n, \\ q_3(n) &= -(2+n)2^{n-1} + 3^n, \end{aligned}$$

and  $z(n) = q_3(n)$  is a solution of

$$z(n+3) - 7z(n+2) + 16z(n+1) - 12z(n) = 0, \quad z(0) = z(1) = 0, \quad z(2) = 1.$$

•  $A^n = \sum_{j=1}^k y_j(n)E_j$ , where  $y_1(n) = 2^n$ ,  $y_2(n) = n2^{n-1}$ ,  $y_3(n) = 3^n$ ,

$$E_1 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{pmatrix}.$$

•  $A^n = (S + N)^n$ ,

$$\frac{1}{c(\lambda)} = \frac{1-\lambda}{(\lambda-2)^2} + \frac{1}{\lambda-3},$$

$$1 = (1-\lambda)(\lambda-3) + (\lambda-2)^2 = f_1(\lambda) + f_2(\lambda), \quad F_i = f_i(A),$$

$$F_1 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{pmatrix}$$

$$S = 2F_1 + 3F_2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ -2 & 0 & 4 \end{pmatrix}, \quad N = A - S = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

$$N^2 = 0, \quad SN = NS = 2N, \quad A^n = S^n + n2^{n-1}N.$$

Examination of  $F_1$  and  $F_2$  shows that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are eigenvectors of  $S$  corresponding to the eigenvalue 2, and

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

is an eigenvector of  $S$  corresponding to the eigenvalue 3.

Thus for

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad P^{-1}SP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\text{and } S^n = P(P^{-1}SP)^n P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

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