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Source: *Biometrics*, Vol. 25, No. 2 (Jun., 1969), pp. 309-315
Published by: International Biometric Society
Stable URL: http://www.jstor.org/stable/2528791
Accessed: 08/02/2009 01:48

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A MATRIX MODEL FOR FOREST MANAGEMENT

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SUMMARY

A model for the management of renewable resources is developed, a variant of a model previously given by Usher [1966] which posed one question—is the manager faced with one or several solutions? The mathematical development here shows that there is only one solution of the model that is biologically meaningful. This is associated with the only latent root of the matrix which is greater than unity. This dominant latent root has a latent vector such that all the elements can be chosen as positive. The result is based on a study of the derivative of the model, and from this a rapid method of solving the model is described. Data for a Scots pine forest are given.

INTRODUCTION

A model of the age structure of animal populations was formulated by Leslie [1945; 1948]. His model is deterministic, and has been described by Williamson [1959; 1967], Pollard [1966], Lefkovitch [1967], and Pennycuick *et al.* [1968]. The model is concerned with animals grouped into age classes. Thus, p_i $(i = 0, 1, \cdots, n-1)$ represents the probability that an animal aged i will be alive at time $i + 1$; and f_i $(i = 0, 1, \cdots, n)$ represents the number of female offspring born per female of age i during the period of time t to $t + 1$. The maximum age at which reproduction takes place is n. Leslie's matrix equation is

$$\mathbf{A}\mathbf{a}_t = \mathbf{a}_{t+1} ,$$

where

$$\mathbf{A} = \begin{bmatrix} f_0 & f_1 & f_2 \cdots & f_{n-1} & f_n \\ p_0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & p_1 & \cdot & \cdot & \cdot \\ \vdots & & & & \\ \cdot & \cdot & \cdot & p_{n-1} & \cdot \end{bmatrix}, \qquad \mathbf{a}_t = \begin{bmatrix} a_{t,0} \\ a_{t,1} \\ \vdots \\ a_{t,n} \end{bmatrix}.$$

Here $\mathbf{a}_t$ and $\mathbf{a}_{t+1}$ are vectors representing the population at times t and $t + 1$, respectively.

This approach deals with animals of different age classes, and the development to a stochastic model is given by Pollard [1966].

In many biological cases the age of an organism is difficult or impossible to measure. Since ease of management is a practical criterion when dealing with biological resources, it is often simpler to use size or developmental classes instead of age. Lefkovitch [1965] considers populations that are grouped by stages, and his work is often linked with the four developmental stages in the life of an insect: egg, larva, pupa, and adult (Lefkovitch [1966]). In forest management, tree size is a suitable criterion, and Usher [1966] has developed Leslie's model for such cases.

The model described by Usher [1966] predicted the stable structure of a renewable resource categorised by size classes, and it showed that, given various conditions, at least one biologically meaningful structure could always be determined. This structure is associated with a real rate of increase of the resource, and thus the dominant latent root of the matrix is greater than unity. Associated with this latent root is a latent vector such that all its elements can be chosen as positive. However, the model did not show that there was only one biologically meaningful solution, and left unanswered the question: if there is more than one solution, which of the solutions should the resource manager adopt?

THE MODEL

Consider a set of trees grouped into $n + 2$ classes, which can be termed 'classes 0, 1, $\cdots$, n, too small to be measured.' The classes are such that class i ($i = 0, 1, \cdots, n - 1$) contains trees in the size interval $[(i - \frac{1}{2})h, (i + \frac{1}{2})h]$, where h is the interval between successive classes. Class n will contain trees that are larger than size $(n - \frac{1}{2})h$, and there will be a reservoir of small trees of a size smaller than $(0 - \frac{1}{2})h$.

Let the probability that a tree enlarges from class i to class $i + 1$ during a defined period of time be b_i ($i = 0, 1, \cdots, n - 1$). Let the probability that a tree remains in the same class be a_i. If all the trees are enumerated, whether alive or dead, and the dead trees subsequently removed as part of the harvest, then

$$a_i + b_i = 1. \tag{1}$$

Also, since if $a_i = 1$ all classes greater than i must contain no trees, and it is assumed that all classes up to n are attainable,

$$0 \leq a_i < 1 \qquad (0 \leq i < n). \tag{2}$$

Let the number of trees of class 0 that develop in the gap caused by the exploitation of a tree of class i be c_i ($i = 0, 1, \cdots, n$). c_i will not be just a ratio of the crown sizes of the trees, since the crowns of trees surrounding the gap will enlarge. Hence, for small i, c_i will be zero or near zero, whilst for large i the value of c_i will be greater than unity. In particular,

$$c_{i+1} \geq c_i \qquad (0 \leq i \leq n - 1) \tag{3}$$

since, on average, gaps caused by felling larger trees are larger than gaps caused by felling smaller trees. Regeneration can occur only when a tree is felled, and hence the terms in the first row of the matrix have to be multiplied by a factor depending upon the intensity of the exploitation. If, during a period of time, the number of organisms has increased from N to λN, then for stability by numbers $(\lambda - 1)N$ organisms must be harvested. When there is stability both in terms of numbers and of proportions between the numbers in various classes, λ is a latent root. With such harvesting occurring over all the classes, and with regeneration occurring wherever a tree is felled, the stable model for the forest resource is

$$\mathbf{Qq} = \lambda \mathbf{q}, \tag{4}$$

where

$$\mathbf{Q} = \begin{bmatrix} a_0 + c_0(\lambda - 1) & c_1(\lambda - 1) & c_2(\lambda - 1) & \cdots & c_{n-1}(\lambda - 1) & c_n(\lambda - a_n) \\ b_0 & a_1 & \cdot & & \cdot & \cdot \\ \cdot & b_1 & a_2 & & \cdot & \cdot \\ \vdots & & & & & \\ \cdot & \cdot & \cdot & & a_{n-1} & \cdot \\ \cdot & \cdot & \cdot & & b_{n-1} & a_n \end{bmatrix}$$

and the vector $\mathbf{q}$ represents the stable structure of the resource.

The term a_n has to be specially defined. The constants a_i, b_i, and c_i $(i = 0, 1, \cdots, n - 1)$ and c_n can all be measured as properties of the growing forest, but a_n depends on the forest manager. If he wishes to exploit all timber in this largest class the value of a_n is zero. If, however, he wishes to maintain a forest reserve of large timber, the value of a_n is the proportion that he wishes to keep. Thus, if his reserve is a quarter of the large timber, a_n is 0.25. b_n, which is needed in the equations that follow, is similarly defined as $1 - a_n$.

From equation (4) it can be seen that

$$q_1 = b_0 q_0 / (\lambda - a_1)$$

or more generally

$$q_i = q_0 \frac{b_0}{b_i} \prod_{j=1}^{i} \left(\frac{b_j}{\lambda - a_j} \right) \qquad i = 1, 2, \cdots, n \tag{5}$$

and also

$$(a_0 - \lambda) q_0 + (\lambda - 1) \sum_{i=0}^{n-1} c_i q_i + c_n(\lambda - a_n) q_n = 0. \tag{6}$$

Substituting (5) in (6) and cancelling q_0 which can be chosen as non-zero

$$f(\lambda) = a_0 + c_0(\lambda - 1) - \lambda + \sum_{i=1}^{n-1} c_i(\lambda - 1) \frac{b_0}{b_i} \prod_{j=1}^{i} \left(\frac{b_j}{\lambda - a_j} \right) + c_n(\lambda - a_n) \frac{b_0}{b_n} \prod_{j=1}^{n} \left(\frac{b_j}{\lambda - a_j} \right) = 0 \tag{7}$$

from which it can be shown that when $\lambda = 1$

$$\left.\begin{array}{l} f(1) = b_0(c_n - 1) > 0 \quad \text{since} \quad c_n > 1 \quad \text{and} \quad b_0 > 0 \\ \text{and, as } \lambda \text{ tends to infinity} \\ f(\lambda) \to \lambda(c_0 - 1) + a_0 - c_0 + b_0 c_1 < 0 \quad \text{since} \quad c_0 < 1. \end{array}\right\} \tag{8}$$

Thus, there is at least one value of λ greater than unity for which $f(\lambda) = 0$. Such values of λ are the required solutions for the forest manager. Firstly they show that the resource is increasing, i.e. that there is an increment during the period of time. Secondly, due to the form of equations (2) and (5), any value of λ greater than or equal to unity determines a structure, $\mathbf{q}$, such that all the elements, corresponding to numbers of trees, are positive.

UNIQUENESS OF THE SOLUTION

Using a similar model, Usher [1966] states: 'Thus, any latent root greater than unity of the matrix $\mathbf{Q}$ determines a structure that is biologically meaningful. It cannot yet be proved that there is only one latent root greater than unity which satisfies the matrix $\mathbf{Q}$. There is, however, an optimal solution corresponding to the largest latent root.'

A resource manager is interested in the number of possible solutions. If there is more than one, how would he choose which solution to employ? The form of inequalities (8), by the mean value theorem, shows that there can be any odd number of solutions less than n. Intuitively, it was felt that there must be a unique solution, and all applications of the similar model so far worked out have shown a single value of λ greater than unity. If this is the case, then there are no problems of decision for the manager. A proof that there is only one solution can be obtained by studying the derivative of equation (7). Differentiating (7) term by term with respect to λ gives

$$f'(\lambda) = c_0 - 1 + \sum_{i=1}^{n-1} \left\{ \frac{b_0 c_i}{b_i} \left\{ 1 - \sum_{k=1}^{i} \left(\frac{\lambda - 1}{\lambda - a_k} \right) \right\} \prod_{j=1}^{i} \left(\frac{b_j}{\lambda - a_j} \right) \right\} + \frac{b_0 c_n}{b_n} \left\{ 1 - \sum_{k=1}^{n} \left(\frac{\lambda - a_n}{\lambda - a_k} \right) \right\} \prod_{j=1}^{n} \left(\frac{b_j}{\lambda - a_j} \right). \tag{9}$$

Putting $\lambda = 1$ in (9) gives

$$\begin{aligned} f'(1) &= c_0 - 1 + \sum_{i=1}^{n-1} \frac{b_0 c_i}{b_i} + \frac{b_0 c_n}{b_n} \left(1 - \sum_{i=1}^{n} \frac{b_n}{b_i} \right) && \text{by } (1) \\ &= c_0 - 1 + b_0 \sum_{i=1}^{n} (c_i - c_n)/b_i \\ &\le c_0 - 1 && \text{by } (3) \\ &< 0 \qquad \text{since } c_0 < 1. \end{aligned}$$

Similarly, letting λ tend to infinity, $f'(\lambda) \to c_0 - 1$ since the product term tends to zero.

The slope of $f(\lambda)$ is thus negative both when $\lambda = 1$ and as $\lambda \to \infty$. When $\lambda = 1$, $f(\lambda)$ is positive, whilst, as $\lambda \to \infty$, $f(\lambda)$ is negative (equation (8)). Thus, to prove that $f(\lambda) = 0$ once and only once for all real values of $\lambda \ge 1$ it has to be shown that $f'(\lambda) < 0$ for all values of $\lambda \ge 1$, thus demonstrating that there are no maxima or minima in the graph of $f(\lambda)$.

Since, by definition, $0 \le a_n < 1$, it follows that

$$\sum_i \left(\frac{\lambda - a_n}{\lambda - a_i}\right) > \sum_i \left(\frac{\lambda - 1}{\lambda - a_i}\right).$$

Thus, from (9)

$$\begin{aligned} f'(\lambda) &< c_0 - 1 + \sum_{i=1}^{n} \left\{\frac{b_0 c_i}{b_i} \left\{1 - \sum_{k=1}^{i} \left(\frac{\lambda - 1}{\lambda - a_k}\right)\right\} \prod_{j=1}^{i} \left(\frac{b_j}{\lambda - a_j}\right)\right\} \\ &\equiv c_0 - 1 + \sum_{i=1}^{n} \left\{\frac{b_0 c_i}{b_i} g(\lambda) h(\lambda)\right\}, \end{aligned} \tag{10}$$

where

$$g(\lambda) = 1 - \sum_{k=1}^{i} \left(\frac{\lambda - 1}{\lambda - a_k}\right) \quad \text{and} \quad h(\lambda) = \prod_{j=1}^{i} \left(\frac{b_j}{\lambda - a_j}\right).$$

Now,

$$\sum_{i=1}^{r} \left(\frac{\lambda - 1}{\lambda - a_i}\right) < \sum_i \left(\frac{\lambda - 1}{\lambda - 1}\right) = r \qquad \text{since} \quad a_i < 1$$

and

$$\sum_{i=1}^{r} \left(\frac{\lambda - 1}{\lambda - a_i}\right) \geq \sum_i \left(\frac{\lambda - 1}{\lambda}\right) = r - \frac{r}{\lambda} \qquad \text{since} \quad a_i \geq 0.$$

Thus, this implies that,

$$1 - r + r/\lambda \geq g(\lambda) > 1 - r.$$

Hence, for $\lambda > 1$, $g(\lambda) < 1$ since r takes values $1, 2, \cdots, n$. Similarly, for $\lambda = 1$, $g(\lambda) \leq 1$. Thus, $g(\lambda)$ is a decreasing function, becoming negative for sufficiently large λ and r.

Also, $h(\lambda) \to 0$ as $\lambda \to \infty$, whilst, for $\lambda = 1$, $h(\lambda) = 1$, and the function $h(\lambda)$ decreases with increasing λ.

Thus, for increasing values of λ greater than unity the function $G(\lambda) \equiv g(\lambda)h(\lambda)$ is decreasing, and has its maximum value, for all real values of $\lambda \geq 1$, when $\lambda = 1$. Also, $G(\lambda)$ tends to zero as λ tends to infinity.

It has already been shown that $f'(1) \leq c_0 - 1$, and since

$$f'(\lambda) < \sum \left\{\frac{b_0 c_i}{b_i} G(\lambda)\right\} - 1 + c_0$$

this derivative can take only values less than or equal to $c_0 - 1$. Thus, the derivative is negative for all values of $\lambda \geq 1$, and hence there is only one real positive solution greater than unity of the equation $f(\lambda) = 0$.

This is the required result, and it shows that there is a unique solution to the model. A resource manager, applying the model, will never find more than one solution that is biologically meaningful.

APPLICATION OF THE MODEL

In an earlier paper (Usher [1966]) data was derived from a Scots pine (*Pinus sylvestris*) forest at Corrour, Inverness-shire, Scotland. The area is managed on a 6-year cycle, and hence all the trees are enumerated and measured every six years. The matrix Q, not allowing for any reserve of large timber, is

$$\mathbf{Q} = \begin{bmatrix} 0.72 & 0 & 0 & 3.6(\lambda - 1) & 5.1(\lambda - 1) & 7.5\lambda \\ 0.28 & 0.69 & 0 & 0 & 0 & 0 \\ 0 & 0.31 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.77 & 0 & 0 \\ 0 & 0 & 0 & 0.23 & 0.63 & 0 \\ 0 & 0 & 0 & 0 & 0.37 & 0 \end{bmatrix}$$

The methods of calculation of λ and the vector $\mathbf{q}$ given by Usher [1966] are slow, particularly if a poor initial estimate of λ is taken. A quicker method is by the Newton-Raphson iterative process. If λ_n is an estimate of the latent root, then a better approximation to the root is given by

$$\lambda_{n+1} = \lambda_n - f(\lambda_n)/f'(\lambda_n),$$

where $f(\lambda_n)$ and $f'(\lambda_n)$ are given by equations (7) and (9), respectively. The final value of λ can be used in (5) to generate the elements of the latent vector.

The iterative procedure has been programmed in Atlas Autocode and ALGOL. Using the example of the Corrour Scots pine forest, and with $\lambda_0 = 1.1$ as the initial estimate of the latent root, the iterations calculated by computer are given in Table 1. It will be seen that the process converged quickly, since only five iterations were required to evaluate λ correct to six decimal places.

Using the values of $\lambda = 1.2043$ and $q_0 = 1000$, equations (5) give the latent vector as {1000, 544, 372, 214, 86, 26}.

TABLE 1

THE VALUES OF λ_n AND λ_{n+1} DERIVED BY USING THE NEWTON-RAPHSON PROCESS TO FIND THE LARGEST REAL LATENT ROOT OF THE MATRIX Q

n	λ_n	λ_{n+1}
0	1.100000	1.160275
1	1.160275	1.195605
2	1.195605	1.203910
3	1.203910	1.204266
4	1.204266	1.204266

UN MODELE MATRICIEL POUR L'EXPLOITATION DES FORETS

RESUME

Un modèle pour l'exploitation de ressources renouvelables est développé comme variante d'un modèle antérieurement donné par Usher [1966] qui posait une question—le responsable de l'exploitation dispose-t-il d'une solution ou de plusieurs?

Le développement mathématique donné ici montre qu'il n'existe qu'une solution du modèle qui ait un sens biologique. La solution est associée à la seule valeur propre de la matrice qui dépasse l'unité. Cette valeur propre est associée à un vecteur propre tel que tous ses éléments puissent être choisis positifs. Le résultat s'appuie sur la dérivation du modèle qui sert de point de départ à une méthode rapide de résolution du modèle. On donne un exemple pour une forêt de pins écossais.

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Received April 1968, *Revised November* 1968