6.337 EXAM SOLUTIONS - 1993

1. $y = Cxe^{-Dx}$

$\frac{y}{x} = Ce^{-Dx}$

Alternative: $\ln y = \ln C + \ln x - Dx$

$\ln y - \ln x = \ln C - Dx$

$\ln\left(\frac{y}{x}\right) = \ln C - Dx.$

$\ln\left(\frac{y}{x}\right) = \ln C - Dx = Ax + B \quad \left(\begin{array}{l} A = -D \\ B = \ln C \end{array}\right).$

let $Y = \ln\left(\frac{y}{x}\right)$ — linear!

4

let $Y_i = \ln\left(\frac{y_i}{x_i}\right)$

$$\left(\sum_{i=1}^{n} x_i^2\right) A + \left(\sum_{i=1}^{n} x_i\right) B = \sum_{i=1}^{n} x_i Y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) A + nB = \sum_{i=1}^{n} Y_i$$

or equivalently.

$$\left(\sum_{i=1}^{n} x_i^2\right)(-D) + \left(\sum_{i=1}^{n} x_i\right)\ln C = \sum_{i=1}^{n} x_i \ln\left(\frac{y_i}{x_i}\right)$$

6

$$\left(\sum_{i=1}^{n} x_i\right)(-D) + n\ln C = \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)$$

form 4
correct variables 2

(2)

x_i	y_i	Δy_i	$\Delta^2 y_i$
1	2	5	
2	7	9	4
3	16	13	4
4	29	~~16~~ 17	~~3~~ 4
5	~~45~~ 46	~~22~~ 21	~~6~~ 4
6	67	25	~~3~~ 4
7	92	29	4
8	121	~~34~~ 33	~~5~~ 4
9	~~155~~ 154	~~36~~ 37	~~2~~ 4
10	191		

Alternative: expect $\Delta^3 y_i = 0$ & then work backwards

(a) The average value of the $\Delta^2 y_i$ is

$$\frac{4+4+3+6+3+4+5+2}{8} = \frac{31}{8}.$$

(2) reasons
(2) conclusion

But since the polynomial has integer coefficients its values must be integers (at integer values of x) $\therefore$ $\Delta^2 y_i$ must be integers. Thus, assume $\boxed{\Delta^2 y_i = 4.}$ Work backwards to correct Δy_i & y_i as shown.

(4)

(4)

Total (8)

(b) $y = ax^2 + bx + c$

– use any 3 data pts:

(1)

$y(1) = 2$
$y(2) = 7$
$y(3) = 16$

$$\Longrightarrow \quad \begin{aligned} 2 &= a + b + c \\ 7 &= 4a + 2b + c \\ 16 &= 9a + 3b + c \end{aligned} \qquad \begin{aligned} 5 &= 3a + b \\ 9 &= 5a + b \end{aligned}$$

(3)

(3) $\Rightarrow 4 = 2a \Rightarrow$ $\boxed{a = 2, \; b = 5 - 3(2) = -1, \; c = 2 - 2 - (-1) = 1}$

(7) $\Rightarrow y(x) = 2x^2 - x + 1$

3

$$N(t) = \frac{C}{1+e^{-k(t-t^*)}}$$

$$t^* = t_0 - \frac{1}{k}\ln\left(\frac{N_0}{C-N_0}\right).$$

(a) there is always a point of inflection at $N = \frac{C}{2}$ 2 $\Rightarrow$ $\frac{C}{2} = \frac{C}{1+e^{-k(t-t^*)}}$

$$\Rightarrow \quad 2 = 1+e^{-k(t-t^*)}$$

$$\Rightarrow \quad 1 = e^{-k(t-t^*)}$$

Total 4

$$\Rightarrow \quad -k(t-t^*) = 0 \rightarrow t = t^*. \quad 2$$

(b) If $C = 110$.

then $\boxed{\frac{C}{2} = 55 \Rightarrow t = t^* = 6}$ — 2

Also $t_0 = 0$ & $N(t_0) = N_0 = 10$, so (**) — 2

becomes

$$5 = 0 - \frac{1}{k}\ln\left(\frac{10}{110-10}\right) \text{ ——— } 1$$

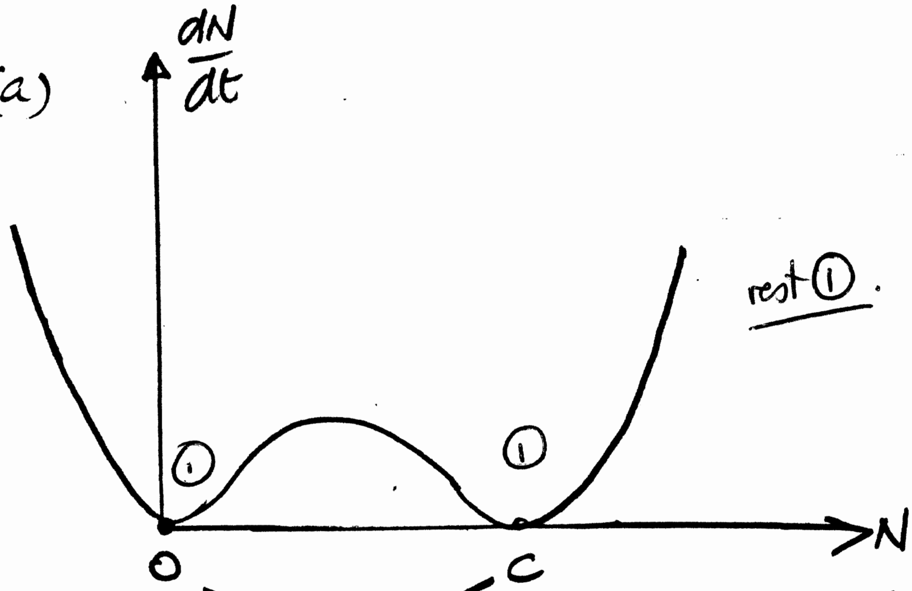
Total 6

$$\Rightarrow \quad k = -\frac{1}{5}\ln(.1) = \frac{1}{5}\ln 10$$

$$\simeq 0.4605. \text{ ——— } 1$$

10

4 (a)



rest 1.

5

equilibrium pts: $N=0$, $N=C$ } 2

(b) $$\frac{d^2N}{dt^2} = k\,2N\frac{dN}{dt}\left(1-\frac{N}{C}\right)^2 + kN^2\,2\left(1-\frac{N}{C}\right)\left(-\frac{1}{C}\frac{dN}{dt}\right)$$ 2

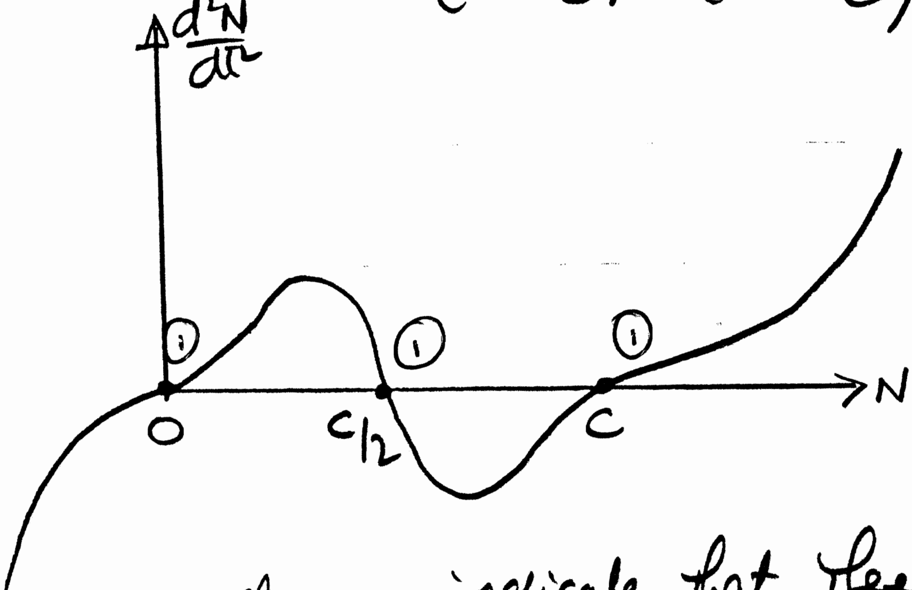
$$= 2kN\left(1-\frac{N}{C}\right)\frac{dN}{dt}\left[\left(1-\frac{N}{C}\right)-\frac{N}{C}\right]$$ 1

4

$$= 2kN\left(1-\frac{N}{C}\right)\frac{dN}{dt}\left(1-\frac{2N}{C}\right)$$

$$= 2k^2N^3\left(1-\frac{N}{C}\right)^3\left(1-\frac{2N}{C}\right)$$ 1

(c)

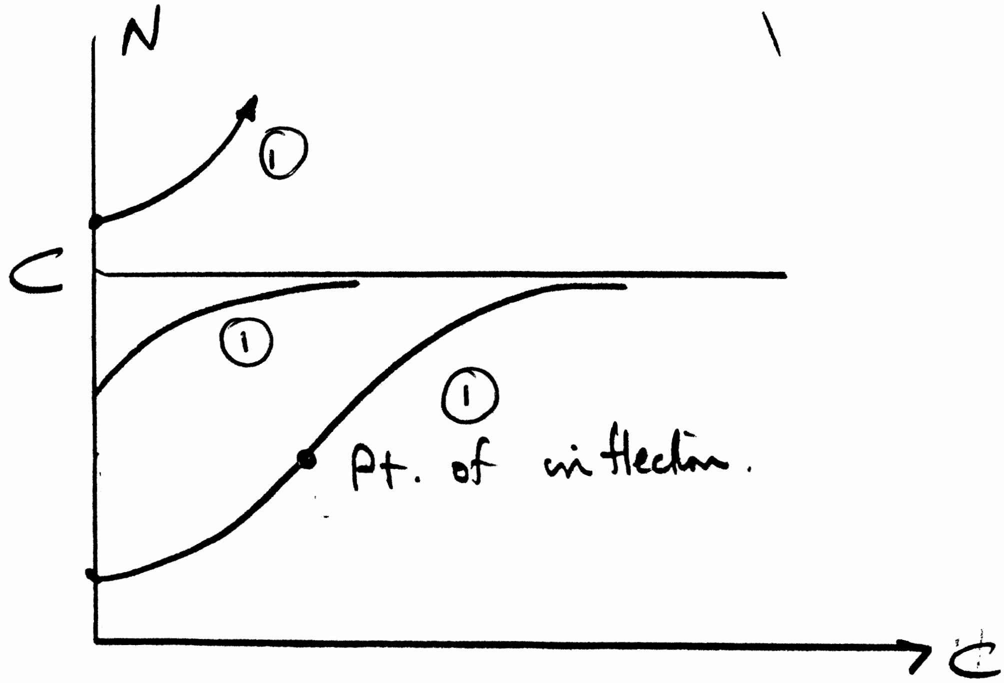


sign changes indicate that there are possible points of inflection at $N=0$, $N=C/2$ & $N=C$.

However $N=0$ & $N=C$ are equilibrium points & $\therefore$ cannot be points of inflection.

4 1 – only point of inflection occurs at $N = C/2$.

(d)



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(e) for small N, $\frac{dN}{dt}$ is smaller than in the

1 logistic case. However as $N \to \frac{C}{2}$, $\frac{dN}{dt}$

1 $\to \frac{kC^2}{4}\left(\frac{1}{4}\right) = \frac{kC^2}{16}$ while for the logistic

1 case as $N \to \frac{C}{2}$, $\frac{dN}{dt} \to k\left(\frac{C}{2}\right)\left(\frac{1}{2}\right) = \frac{kC}{4}$

1 Thus, at least for $C > 4$, $\frac{dN}{dt}$ is greater in this case than it is for the logistic case.

4

Total 20

(5) $$\frac{dx}{dt} = x(l - ny)$$

$$\frac{dy}{dt} = y\left(k - \lambda \frac{y}{x}\right).$$

(a) for $y \ll x$: $\frac{dy}{dt} \simeq ky$ so we expect essential Malthusian growth when the prey is (relatively) abundant.

(2)

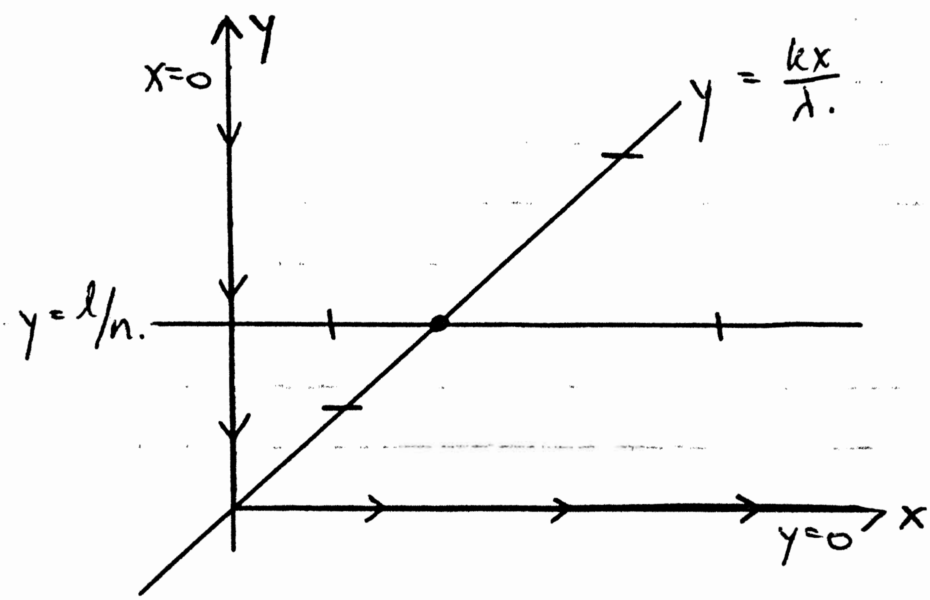
for $y \gg x$: $\frac{dy}{dt} < 0$ as a result of the competition in the predator population for (relatively) small # of prey.

(2)

(4)

(b) nullclines: for x : $x(l-ny) = 0 \to x = 0$ or $y = l/n$. (1)

for y : $y\left(k - \lambda\frac{y}{x}\right) = 0 \to y = 0$ or $y = \frac{kx}{\lambda}$. (1)



graph (2)

(4)

(c) : Equilibrium Pts: (i) $x=0$, $y=0$ $\longrightarrow$ $(0,0)$

(ii) $x=0$, $y=\frac{kx}{\lambda}$ $\longrightarrow$ $(0,0)$

(iii) $y=\frac{l}{n}$, $y=0$ $\longleftarrow$ inconsistent

(iv) $y=\frac{l}{n}$, $y=\frac{kx}{\lambda}$ $\Rightarrow x=\frac{\lambda y}{k}=\frac{\lambda l}{kn}$ $\longrightarrow \left(\frac{\lambda l}{kn},\frac{l}{n}\right)$

3

(d) : assume $x>0, y>0$:

above $y=\frac{l}{n} \Rightarrow y>\frac{l}{n} \Rightarrow ny>l \Rightarrow 0>l-ny$

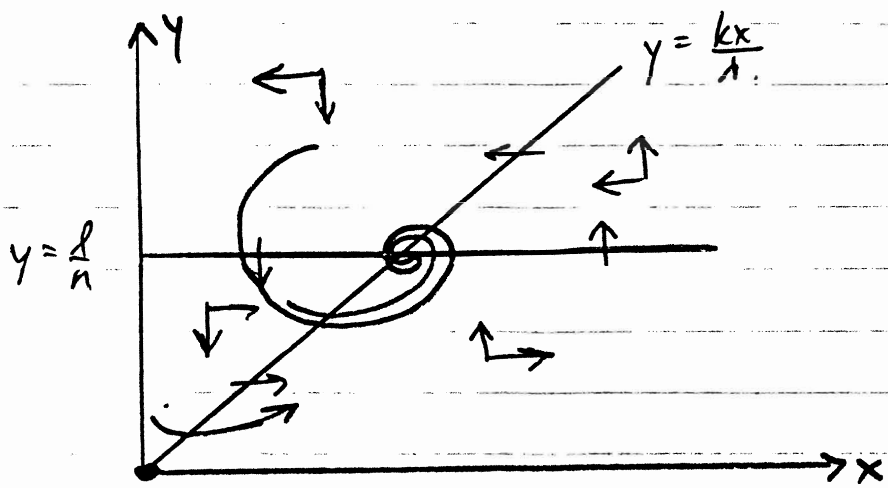
$\Rightarrow \frac{dx}{dt}<0$

below: $y=\frac{l}{n} \Rightarrow \frac{dx}{dt}>0$

above $y=\frac{kx}{\lambda} \Rightarrow y>\frac{kx}{\lambda} \Rightarrow \frac{\lambda y}{x}>k \Rightarrow 0>k-\frac{\lambda y}{x}$

$\Rightarrow \frac{dy}{dt}<0$

below $y=\frac{kx}{\lambda} \Rightarrow \frac{dy}{dt}>0$.



2

1 (e) $(0,0)$ unstable

1 $\left(\frac{\lambda l}{kn},\frac{l}{n}\right)$ stable.

Total 15

(6)

$$\left.\begin{aligned}\frac{dx}{dt} &= ky - mx + r\\ \frac{dy}{dt} &= kx - my + s\end{aligned}\right\}\quad r, s > 0.$$

$$z = x + y$$

(a) (2)

$$\frac{dz}{dt} = (k-m)y + (k-m)x + r + s$$

$$= (k-m)z + (r+s)$$

(b) (6)

$$\frac{dz}{dt} + (m-k)z = (r+s) \longleftarrow \underline{\text{linear}}.$$

integrating factor: $\mu = e^{(m-k)t}$ (1) $(m \neq k)$.

$$\frac{d}{dt}\left(e^{(m-k)t} z\right) = (r+s)\, e^{(m-k)t} \quad (m \neq k) \ (1)$$

$$e^{(m-k)t} z = \frac{r+s}{m-k} e^{(m-k)t} + c. \ (1) \ (m \neq k)$$

$$z(0) = z_0 \Longrightarrow z_0 = \frac{r+s}{m-k} + c \longrightarrow c = z_0 - \frac{r+s}{m-k} \ (1)$$

$$\therefore \boxed{z(t) = \frac{r+s}{m-k} + \left(z_0 - \frac{r+s}{m-k}\right) e^{(k-m)t} \ (2) \ (m \neq k)}$$

or (3) when $m = k$: $\frac{dz}{dt} = r + s$ (1)

$$z(t) = (r+s)t + c_1 \ (1)$$

$$z(0) = z_0 \Longrightarrow c_1 = z_0 \Longrightarrow \boxed{z(t) = (r+s)t + z_0 \ (1) \ (m = k)}$$

(c) (4) If $k = m$, $z(t) \longrightarrow \infty$ as $t \to \infty$. (1)

If $k > m$, $z(t) \longrightarrow \infty$ as $t \to \infty$ (1)

If $k < m$, $z(t) \longrightarrow \left(\frac{r+s}{m-k}\right)$ as $t \to \infty$. (1)

If $k \geq m$ the arms race is uncontrolled, while if $k < m$ it is stable. (1)

7 $$\frac{dP_I(t)}{dt} + \beta I(N-I)P_I(t) = \beta(I-1)(N-I+1)P_{I-1}(t)$$

(a): $$\left.\begin{aligned} P_1(0) &= 1 \\ P_I(0) &= 0 \quad \forall I > 0 \end{aligned}\right) \; 2$$

$I=1$: $$\frac{dP_1(t)}{dt} + \beta(1)(N-1)P_1(t) = 0, \quad — 1$$

$$\frac{dP_1(t)}{dt} = -\beta(N-1)P_1(t)$$

$$\Rightarrow P_1(t) = Ke^{-\beta(N-1)t} \quad — 2$$

$$\hookrightarrow P_1(0) = 1 \Rightarrow K = 1 \quad — 1$$

$$\boxed{P_1(t) = e^{-\beta(N-1)t}} \quad — 1$$

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$I=2$: $$\frac{dP_2(t)}{dt} + \beta(2)(N-2)P_2(t) = \beta(1)(N-1)P_1(t)$$

$$\frac{dP_2(t)}{dt} + 2\beta(N-2)P_2(t) = \beta(N-1)e^{-\beta(N-1)t} \quad 1$$

$$\mu = e^{2\beta(N-2)t} \quad 1$$

$$\frac{d}{dt}\left[e^{2\beta(N-2)t}P_2(t)\right] = \beta(N-1)e^{2\beta(N-2)t}e^{-\beta(N-1)t}$$

$$= \beta(N-1)e^{\beta(N-3)t} \quad 2$$

$$P_2(t)\, e^{2\beta(N-2)t} = \frac{\cancel{\beta}(N-1)e^{\beta(N-3)t}}{\cancel{\beta}(N-3)} + c. \quad 1$$

8

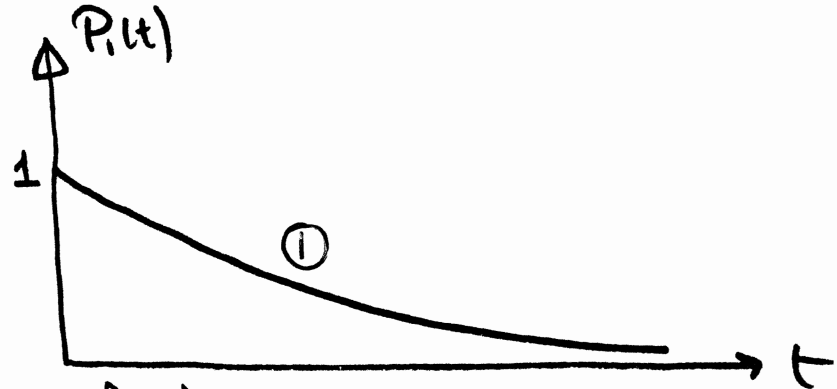
But $P_2(0)=0 \rightarrow \quad 0 = \frac{N-1}{N-3} + c$

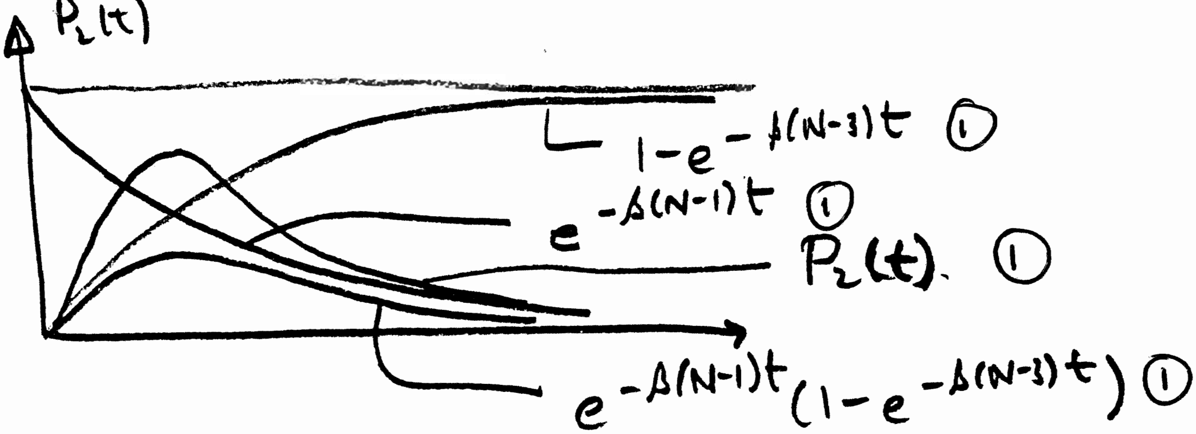
$\Rightarrow c = -\frac{N-1}{N-3}$ 1

$$P_2(t) = \frac{N-1}{N-3}\left[e^{\beta(N-3)t}-1\right]e^{-2\beta(N-2)t}$$ 2

$$= \left(\frac{N-1}{N-3}\right) e^{-\beta(N-1)t}\left[1-e^{-\beta(N-3)t}\right]$$

(b)





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Total 20