
This is a 120 minutes exam, with 4 questions for a total of 50 marks. Lecture Notes are allowed. PLEASE SHOW YOUR WORK CLEARLY. A correct answer without explanation will not get full marks.

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1. (10 points) The dynamics of a population of birds (measured in thousand) is described by the equation
$$\frac{dP}{dt} = 4P(1-8P^3).$$

   (a) Find the equilibria.

   Solution: This is a continuous time system. We write
   $$P' = f(P) := 4P(1-8P^3).$$
   Equilibria satisfy $f(x) = 0$. Clearly, $4P = 0$, i.e., $P = 0$, or $1 - 8P^3 = 0$, that is, $P = 1/2$.

   (b) Determine the local stability of each equilibrium.

   Solution: The local stability depends on the sign of f' at the equilibria. We have $f'(P) = 4(1-8P^3) + 4P(-8 \times 3P^2) = 4 - 128P^3$. Thus $f'(0) = 4 > 0$, so 0 is unstable, and $f'(1/2) = -12 < 0$, so 1/2 is locally asymptotically stable.

2. (10 points) The population (measured in billions) of insects in generation t is described as follows
$$P_{t+1} = P_t e^{4(1-3P_t)}$$

   (a) Find all fixed points.

   Solution: We define
   $$P_{t+1} = f(P_t) := P_t e^{4(1-3P_t)}.$$
   This is a discrete-time system, so equilibria satisfy $f(p) = p$. Thus we solve
   $$p = pe^{4(1-3p)}.$$

We find $p = 0$ and $1 = e^{4(1-3p)}$. The latter gives $\ln 1 = 4(1-3p)$, that is, $1 - 3p = 0$. Thus there are two equilibria, $p = 0$ and $p = 1/3$.

(b) Determine the local stability of each fixed point.

Solution: This is a discrete-time system, so the local stability of the equilibria is determined by whether $|f'(p)| < 1$ or not.
We have

$$f'(p) = (1 - 12p)e^{4(1-3p)}.$$

So, at $p = 0$, we have $|f'(0)| = e^4 > 1$, 0 is an unstable equilibrium. At $p = 1/3$, $|f'(1/3)| = |-3e^0| = 3 > 1$, so $1/3$ is also unstable.

3. (20 points) Assume that an insect population, $x(t)$, is controlled by a natural predator population, $y(t)$. We make the following assumptions:

- In the absence of predators, the dynamics of the insects is governed by a logistic equation.
- Preys and predators meet at a rate that is of mass action type.
- When a contact takes place, the probability per contact that a prey dies is k_1. [Hint: think of mass action contact in an epidemic model.]
- These contacts lead to an increase of the predator population with rate k_2.
- The predators are subject to natural death at the per capita rate d.

(a) Write a model describing the interaction of the 2 populations.

Solution: The most obvious model:

$$x' = rx\left(1 - \frac{x}{K}\right) - k_1 xy \tag{1a}$$
$$y' = k_2 xy - dy. \tag{1b}$$

This is considered with initial conditions $x(0), y(0) \geq 0$.

(b) Study the model you have written: is it well-posed, what are its equilibria (and when are they realistic), can you determine the stability of these equilibria?

Solution: First of all, the right hand side in (1) is a C^1 function (it is actually C^∞, as it consists of two multivariate polynomials). This implies that solutions exist and are unique.

Suppose $x(t) = 0$ for some t. Then $x' = 0$ for all subsequent t. By uniqueness of solutions, this means that $x(t) = 0$ for all $t \geq 0$, that is, $x(0) = 0$. Similarly, $y(t) = 0$ for some t implies that $y(0) = 0$. As a consequence, the solutions can never become negative if they start nonnegative.

Equilibria are found by solving

$$0 = rx\left(1 - \frac{x}{K}\right) - k_1xy \tag{2a}$$

$$0 = k_2xy - dy. \tag{2b}$$

From (2b), $y = 0$ or $k_2x = d$, that is, $x = d/k_2$. Subsituting $y = 0$ into (2a) gives the classical logistic equation, which we know to have $x = 0$ and $x = K$ for equilibria. So we have two equilibria, $(x, y) = (0, 0)$ and $(x, y) = (K, 0)$, which we call, respectively, E_1 and E_2. Now, substituting $x = d/k_2$ into (2a) gives

$$r\frac{d}{k_2}\left(1 - \frac{\frac{d}{k_2}}{K}\right) - k_1\frac{d}{k_2}y = 0,$$

that is,

$$\frac{rd}{k_2}\left(1 - \frac{d}{k_2K}\right) = \frac{dk_1}{k_2}y,$$

which gives

$$y = \frac{k_2}{dk_1}\frac{rd}{k_2}\left(1 - \frac{d}{k_2K}\right) = \frac{r}{k_1}\left(1 - \frac{d}{k_2K}\right).$$

So the last equilibrium is E_3, given by

$$(x, y) = \left(\frac{d}{k_2}, \frac{r}{k_1}\left(1 - \frac{d}{k_2K}\right)\right).$$

This equilibrium is (biologically) relevant only if $1 - d/(k_2K) > 0$, or, in other words, if $k_2K - d > 0$.

To determine the stability of the equilibria, we need to determine if all (the two, here) eigenvalues of the Jacobian have negative real parts (local asymptotic stability) or if there is an eigenvalue with a positive real part. The Jacobian, evaluated at an arbitrary point (x, y), takes the form

$$J = \begin{pmatrix} r - k_1 y - 2\dfrac{r}{K}x & -k_1 x \\ k_2 y & k_2 x - d \end{pmatrix}.$$

At $(x, y) = (0, 0)$, we have

$$J = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix},$$

thus eigenvalues are $-d$ and $r > 0$ and the equilibrium $(0, 0)$ is unstable. At $(x, y) = (K, 0)$,

$$J = \begin{pmatrix} -r & -k_1 K \\ 0 & k_2 K - d \end{pmatrix},$$

and eigenvalues are $-r$ and $k_2 K - d$. So the equilibrium $(K, 0)$ is locally asymptotically stable if $k_2 K - d < 0$, and unstable otherwise. [Remark that this means that if E_3 exists, E_2 is unstable, and if E_2 is locally asymptotically stable, E_3 does not exist.] Finally, at the last equilibrium,

$$J_{E_3} = \begin{pmatrix} -\dfrac{rd}{k_2 K} & -\dfrac{k_1 d}{k_2} \\ \dfrac{r}{k_1 K}(k_2 K - d) & 0 \end{pmatrix}.$$

Eigenvalues of J_{E_3} satisfy

$$|J_{E_3} - \lambda \mathbb{I}| = \begin{vmatrix} -\dfrac{rd}{k_2 K} - \lambda & -\dfrac{k_1 d}{k_2} \\ \dfrac{r}{k_1 K}(k_2 K - d) & -\lambda \end{vmatrix} = 0.$$

This gives the characteristic polynomial

$$P(z) = z^2 + \frac{rd}{k_2 K} z + \frac{rd}{k_2 K}(k_2 K - d).$$

The roots of $P(z)$ are the eigenvalues of J_{E_3}. We compute the discriminant

$$\Delta := \frac{rd}{k_2 K} - 4\frac{rd}{k_2 K}(k_2 K - d) = \frac{rd}{k_2 K}(1 - 4(k_2 K - d)).$$

Roots of P take the form
$$\frac{-\frac{rd}{k_2K} \pm \sqrt{\Delta}}{2}.$$
If $\Delta < 0$, then the roots are complex with real part given by the part outside of the square root (thus the real parts are negative). In that case, the equilibrium is locally asymptotically stable.
Now consider the case where $\Delta \geq 0$. The root taking the form
$$\frac{-\frac{rd}{k_2K} - \sqrt{\Delta}}{2}$$
is negative. On the other hand, the root taking the form
$$\frac{-\frac{rd}{k_2K} + \sqrt{\Delta}}{2}$$
is positive if
$$-\frac{rd}{k_2K} + \sqrt{\Delta} > 0.$$
This is equivalent to
$$\frac{rd}{k_2K} < \sqrt{\Delta},$$
that is,
$$\left(\frac{rd}{k_2K}\right)^2 < \Delta.$$
In other words, if
$$\left(\frac{rd}{k_2K}\right)^2 < \frac{rd}{k_2K}(1 - 4(k_2K - d)),$$
that is, if
$$\frac{rd}{k_2K} < 1 - 4(k_2K - d),$$
then E_3 is unstable. Otherwise, E_3 is locally asymptotically stable.

(c) Assume that an insecticide is used to reduce the population of insects, but it is also toxic to the predators; hence, the poison kills both preys and predators at rates proportional to their respective populations. Modify your model from (a). [For bonus marks, you may want to see how this modifies the analysis you carried out in (b).]

Solution: The most obvious model:

$$x' = rx\left(1 - \frac{x}{K}\right) - k_1xy - \delta_1 x \tag{3a}$$

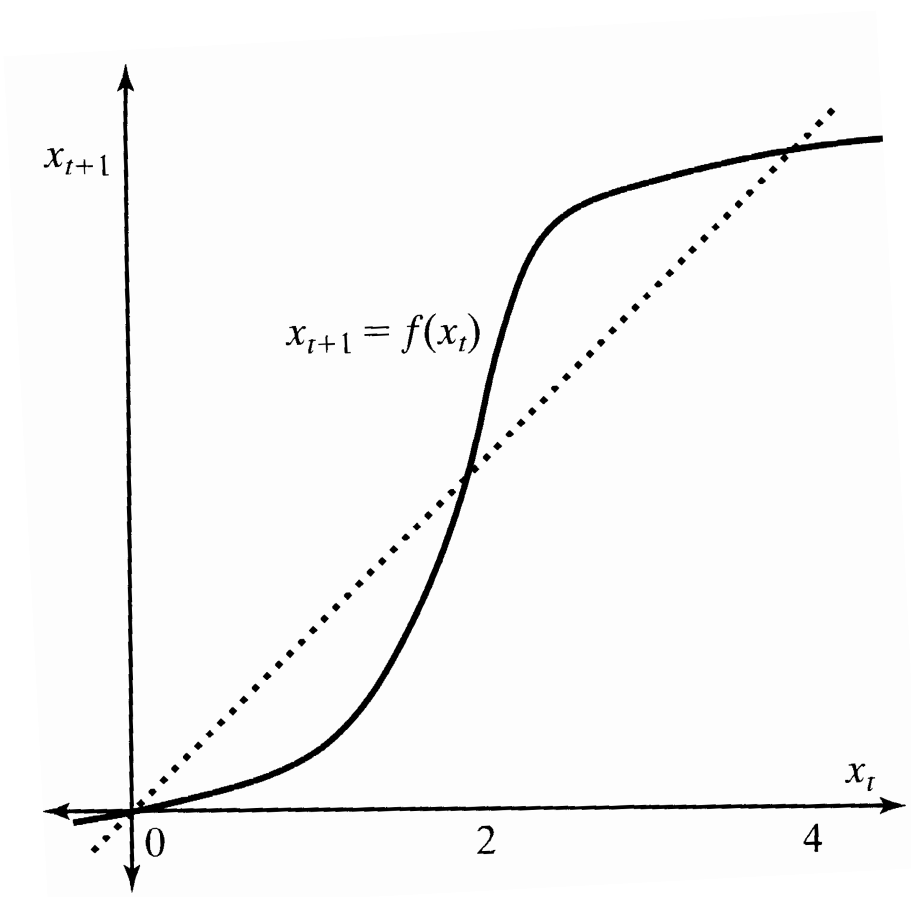
$$y' = k_2xy - dy - \delta_2 y. \tag{3b}$$

This is considered with initial conditions $x(0), y(0) \geq 0$.

4. (10 points) Consider the difference equation

$$x_{t+1} = f(x_t)$$

with graph shown in the figure below



(a) Find all fixed points.

> **Solution:** Fixed points are found at the intersections of f with the line $y = x$. So the equilibria are 0, 2 and 4.

(b) Determine the local stability of each fixed point. [Hint: what does the stability condition imply, in terms of the slope of f?]

> **Solution:** Saying that $|f'(p)| < 1$ is equivalent to saying that the slope of f at the point p is less than 1 (in absolute value). So if the slope is between that of the line $y = x$ (at the top) and the line $y = -x$ (at the bottom), the equilibrium is locally stable, whereas if the slope is above or below these lines, the equilibrium is unstable.
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> So the equilibria 0 and 4 are locally stable, and the equilibrium 2 is unstable.