April 16 19 82 Final EXAMINATION

PAPER NO.: 375 PAGE NO.: 1 of 2

DEPARTMENT & COURSE NO.: Applied Mathematics 6.250 TIME: 3 HOURS

EXAMINATION: Mathematical Modelling B EXAMINER: D. W. Trim

Values

10 1. You are required to fit a curve of the form

$$y = \frac{C}{1 + B\ln(x+5)}$$

to n pairs of observations $(x_1,\bar{y}_1)$, $(x_2,\bar{y}_2)$, ... , $(x_n,\bar{y}_n)$. Explain how you would use least squares to find values for B and C.

20 2. A population with specific growth rate k begins at time t = 0 with N_0 individuals.

(a) If it is assumed that the population experiences Malthusian growth, find a formula for the time taken for the population to reach $2N_0$ individuals.

(b) Repeat part (a) but assume that the population experiences logistic growth with carrying capacity C.

(c) Show that if N_0 is very much less than C, then the formula in (b) is approximately the same as that in (a).

15 3. The logistic model for population growth states that

$$\frac{dN}{dt} = kN(1 - N/C)$$

where k is a constant, N is the number of members in the population, C is the carrying capacity of the environment, and t is time.

(a) If N(t) represents the population of a country, modify the model to incorporate both an immigration rate which is proportional to the difference between the present size of the population and the carrying capacity, and a constant emigration rate.

(b) Suppose N(t) represents the population size of a rare species which may face extinction if the population ever drops below m. Revise the model to incorporate this minimum viable population.

(c) Suppose N(t) represents the population size of a country. If 50% of the population is suddenly destroyed by some natural disaster, how do you incorporate this catastrophe into the model?

20 4. A competitive-hunter situation is described by the system of differential equations

$$\frac{dx}{dt} = 0.2x - 0.001xy \quad ,$$

$$\frac{dy}{dt} = 0.4y - 0.002xy \quad .$$

(a) If the initial populations of the species are $x_0 = 100$ and $y_0 = 150$, show that their numbers are related by the equation

$$y^{0.2}e^{-0.001y} = 0.45387x^{0.4}e^{-0.002x} \quad .$$

(b) What are the critical values of x(t) and y(t)?

(c) Show that the maximum size of x(t) is given by the equation

$$5.205 = x^{0.4}e^{-0.002x} \quad .$$

(d) Use any method you wish to find an approximate solution to the equation in (c).

.... see over ....

Values

20 5.(a) In developing our probabilistic model for a pure birth process, we assumed that there were two possibilities for attaining a population of N individuals at time $t+\Delta t$:

(i) At time t there were N individuals and no births took place in Δt.

(ii) At time t there were N-1 individuals and exactly one birth took place in Δt.

Modify the development to take into account a third possibility:

(iii) At time t there were N-2 individuals and exactly two births took place in Δt.

In particular show that $P_N(t)$ must still satisfy the differential equation

$$\frac{dP_N(t)}{dt} + bNP_N(t) = b(N-1)P_{N-1}(t) \quad .$$

(b) Assuming that the population begins with N_o at time $t = 0$, and that

$$P_{N_o+1}(t) = N_o e^{-bN_o t}(1 - e^{-bt}) \quad ,$$

solve the differential equation in (a) for $P_{N_o+2}(t)$.

15 6. An epidemic is modelled by the equations

$$L(t) = 0 \quad ,$$

$$\frac{dS}{dt} = -\beta SI \quad , \quad \beta > 0 \quad ,$$

$$\frac{dR}{dt} = rI \quad , \quad r > 0 \quad ,$$

$$S + R + I = N = \text{constant} \quad ,$$

where

$$r = 0.9 \quad , \quad \beta = 0.0002 \quad , \quad N = 10{,}000 \quad , \quad S(0) = 9990 \quad , \quad I(0) = 10 \; .$$

What is the maximum value of I(t)?