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A new numerical method for SDEs and its application in circuit simulation

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Abstract

In the context of computer-aided design of electric circuits, the modeling of circuits under the influence of electronic noise leads to stochastic differential-algebraic equations (SDAEs). A new numerical integration scheme for the pathwise approximative solution of stochastic differential equations (SDEs) is presented. The available numerical schemes for SDEs cannot solve these equations efficiently. The new discretization scheme is constructed by the means of Itô–Taylor expansions. A `Fortran77` implementation of the presented integration scheme reduces the simulation time for a ring oscillator to about 85% compared with common methods.

1. Introduction

The increasing integration density of electric circuits results in a signal-to-noise ratio close to 1. Among other disturbing signals electronic noise in the electronic devices is not avoidable. The noise behavior of the devices disturbs the desired electric behavior of integrated circuits. Also the electronic noise in a circuit is a lower limit for the device dimensions. In the domain of electric circuits we mainly can distinguish among thermal noise, shot noise and flicker noise. They describe different randomly disturbing phenomena in the semiconductor material [1,3,6,8].

Based on the theory of *stochastic differential equations* (SDEs) [9,12] randomly disturbed electric circuits can be modeled in the time domain. Up to now, the time domain analysis of randomly disturbed circuits, the so called transient simulation, is only possible for linearized stochastic circuit models [2].

The objective of this paper is the efficient simulation of circuits where the noise behavior of the circuit is included in the time analysis. We present a new numerical method which solves

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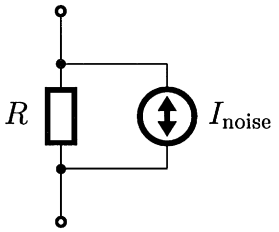

Fig. 1. Model of a resistor (resistance R) with thermal noise.

approximately a special class of circuit models, i.e. SDEs, and has strong convergence order 1. It is mean-square stable with respect to a linear test equation. The difference in numerical schemes for the approximative solution of SDEs, e.g. Milstein's scheme or Runge–Kutta-like schemes, is the reduced number of evaluations of the diffusion matrix which collects the randomly disturbing terms of the SDE. The reduction of the number of function evaluations is necessary due to complicated device models in circuit simulation. This makes the new scheme more efficient than other schemes.

2. Noise fundamentals

Noise phenomena in devices of electric circuits are thermal noise, shot noise and flicker noise, cf. [6]. Thermal noise is the irregular thermal motion of electrons in the semiconductor material. Shot noise is a random fluctuation of the drift and the diffusion of the charge carriers through the semiconductor surface (e.g. p–n-transition). Traps, i.e. impurities and crystal defects of the semiconductor, randomly catch or release charge carriers. This noise phenomenon is denoted by flicker noise.

To model the electronic noise of a device, a noise current source is in shunt to the deterministic device model. The current supply of the noise source is randomly distributed, the statistics describe one of the above noise phenomena. Fig. 1 shows the model of a resistor with resistance R and thermal noise.

The physical idealization of thermal and shot noise is white noise. A significant mathematical characterization of the noise current I_{noise} is its power spectral density $S_{I_{\text{noise}}}(f)$. In the case of thermal noise we have $S_{I_{\text{noise}}}(f) = 4kT/R$ and in the case of shot noise $S_{I_{\text{noise}}}(f) = 2q \cdot I_t$ (cf. [1]). Here k denotes Boltzmann's constant, T the operating temperature, q the elementary charge, f the frequency and I_t the deterministic current through the regarded terminals of the device. The process flicker noise cannot be modeled with a white noise source. One approach to simulate flicker noise is the approximation by a serial connection of white noise current sources [2,10]. Depending on the devices, a combination of the noise phenomena above is used.

The expansion of the deterministic circuit model with noise current sources where the forcing noise is white noise yields *stochastic differential-algebraic equations* (SDAEs). Using linear capacitors the SDAE has with Itô's interpretation the integral form

$$C_{\text{cap}}\,\mathrm{d}X_t = a(t,X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}B_t, \quad X_{t_0} = X_0. \tag{1}$$

Herein the matrix C_{cap} collects the capacitances of the circuit and can be singular; the vector function a describes the deterministic device models and independent voltage sources. The components of the multidimensional stochastic process B_t are Brownian motions (white noise is the distributional derivative of Brownian motion). The matrix function b collects the square roots of the power spectral densities of the noise current sources (e.g. $\sqrt{(4kT/R)}$ for the thermal noise). b has the dimension $d \times m$ where m is the number of stochastic independent noise sources, i.e., in the integral model the number of components of the Brownian motion vector process B_t, and d is the dimension of the circuit problem, i.e., of X_t. The components of the desired vector process X_t contain the nodal voltages of the circuit, eventually enhanced by currents through voltage defining devices.

For the approximative pathwise solution of SDEs we have mainly two convergence criteria for a numerical scheme: weak and strong convergence. In this paper we investigate methods of strong convergence order 1. Commonly used methods of strong convergence order 1 or higher orders, e.g. Milstein's scheme [7] or Runge–Kutta-like schemes [5], are not suited for circuit simulation. They need partial derivatives of $a(t,x)$ and $b(x)$ with respect to x and the diffusion matrix b has to be evaluated at least $m+1$ times at each integration step (m is the number of noise current sources of the circuit). For most problems in the circuit simulation m is larger than d.

Efficient circuit simulation has to avoid these problems, because the partial derivatives cannot be calculated analytically due to complicated device equations and multiple evaluations of the matrix b at each integration step increase the computing time enormously.

3. A new numerical method of strong convergence order 1

The deterministic model equations of integrated circuits have to be generated automatically. By means of the modified nodal analysis we obtain *differential algebraic equations* (DAEs). Theoretically, an index reduction of circuit DAEs is possible but can not be realized by software. In this paper we assume that the capacitance matrix C_{cap} in (1) is regular and constant. So we can transform (1) to an SDE. This paper presents a new numerical method for the calculation of pathwise solutions of an SDE of the form

$$\mathrm{d}X_t = a(t,X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}B_t, \quad X_{t_0} = X_0 \tag{2}$$

with a drift vector function a, a diffusion matrix function b and a vector process B_t. The dimensions are determined by $X_t \in \mathbb{R}^d$, $a \in \mathbb{R}^d$, $b \in \mathbb{R}^{d,m}$ and $B_t \in \mathbb{R}^m$. Further, we assume existence and uniqueness of a solution of the SDE.

The numerical scheme presented in this paper avoids these problems. The iteration formula for the desired vector X_t of an SDE (2) of dimension d with m white noise sources reads in the explicit case as

$$X_{n+1} = X_n + a_n h + b_n \begin{pmatrix} I_{(1)} \\ \vdots \\ I_{(m)} \end{pmatrix} + b_n \,\text{Itô}^{\top} b_n^{\top} \begin{pmatrix} \dfrac{1}{X_n^1 - X_{n-1}^1} \\ \vdots \\ \dfrac{1}{X_n^d - X_{n-1}^d} \end{pmatrix}$$

$$-(b_{1,n}\,\mathrm{It\hat{o}}^{\top}(b_n^{\top})^{\bullet 1},\ldots,b_{d,n}\,\mathrm{It\hat{o}}^{\top}(b_n^{\top})^{\bullet d})\begin{pmatrix}\dfrac{1}{X_n^1-X_{n-1}^1}\\ \vdots\\ \dfrac{1}{X_n^d-X_{n-1}^d}\end{pmatrix}. \tag{3}$$

Herein $t_0,\ldots,t_n,\ldots,t_{\mathrm{end}}$ is the discretization of the simulation time interval, X_n the numerical approximation for X_{t_n}, $a_n := a(t_n,X_n)$, $b_n := b(X_n)$, $(b_n^{\top})^{\bullet j}$ the jth column of $b_n^{\top}$, $b_{j,n} := b(X_n^1,\ldots,X_n^{j-1},X_{n-1}^j,X_n^{j+1},\ldots,X_n^d)$, $h := t_{n+1}-t_n$ the constant step size, $I_{(j)} := I_{(j),t_n,t_{n+1}} = B_{t_{n+1}}^j - B_{t_n}^j$ the Wiener increment $(j=1,\ldots,m)$ and $\mathrm{It\hat{o}} := (I_{(i,j),t_n,t_{n+1}})_{i,j=1,\ldots,m}$ is the matrix of double Itô integrals applied to the constant function 1 (cf. [5]). Components of vectors and matrices are indicated through an upper index, e.g. X_t^k is the kth component of the vector X_t.

Theorem 3.1. *The strong convergence order of the new numerical method* (3) *is* 1.

Proof. The following construction of the new method (3) indicates that the method has strong convergence order 1.

To simplify matters we consider the ith component $(i=1,\ldots,d)$ of the SDE (2) where $t_0=t_n$ and $t=t_{n+1}$, i.e.,

$$X_{t_{n+1}}^i = X_{t_n}^i + \int_{t_n}^{t_{n+1}} a^i(s,X_s)\,\mathrm{d}s + \sum_{j=1}^{m}\int_{t_n}^{t_{n+1}} b^{i,j}(X_s)\,\mathrm{d}B_s^j.$$

The drift components a^i and the diffusion components $b^{i,j}$ will be approximated by Itô–Taylor expansions for functions depending on the solution process X_t, cf. [5], which use only Wiener increments:

$$a^i(s,X_s) = a_n^i + \mathcal{O}_{\mathrm{strong}}((s-t_n)^{1/2}),$$

$$b^{i,j}(X_s) = b_n^{i,j} + \sum_{l=1}^{m}\sum_{k=1}^{d} b_n^{k,l}\frac{\partial b^{i,j}}{\partial x^k}(X_n)I_{(l),t_n,s} + \mathcal{O}_{\mathrm{strong}}(s-t_n),$$

where $\mathcal{O}_{\mathrm{strong}}((s-t_n)^p)$ denotes terms of strong convergence order p for the step size $s-t_n$. We have to calculate partial derivatives of $b^{i,j}$ with respect to x^k. They will be replaced by difference approximations

$$\frac{\partial b^{i,j}}{\partial x^k}(X_n) = \frac{b_n^{i,j} - b^{i,j}(X_n^1,\ldots,X_n^{k-1},X_{n-1}^k,X_n^{k+1},\ldots,X_n^d)}{X_n^k - X_{n-1}^k} + \mathcal{O}(X_n^k - X_{n-1}^k).$$

The approximation for a^i has to be Lebesgue integrated $(\mathrm{d}s)$, i.e.,

$$\begin{aligned}\int_{t_n}^{t_{n+1}} a^i(s,X_s)\,\mathrm{d}s &= \int_{t_n}^{t_{n+1}} (a_n^i + \mathcal{O}_{\mathrm{strong}}((s-t_n)^{1/2}))\,\mathrm{d}s\\ &= a_n^i(t_{n+1}-t_n) + \mathcal{O}_{\mathrm{strong}}((t_{n+1}-t_n)^{3/2}).\end{aligned}$$

The approximation for $b^{i,j}$ has to be Itô integrated ($\mathrm{d}B_s^j$), i.e.,

$$\int_{t_n}^{t_{n+1}} b^{i,j}(X_s)\,\mathrm{d}B_s^j = \int_{t_n}^{t_{n+1}} \left(b_n^{i,j} + \sum_{l=1}^{m}\sum_{k=1}^{d} b_n^{k,l} \left(\frac{b_n^{i,j} - b_{k,n}^{i,j}}{X_n^k - X_{n-1}^k} + \mathcal{O}(X_n^k - X_{n-1}^k) \right) I_{(l),t_n,s} + \mathcal{O}_{\text{strong}}(s - t_n) \right) \mathrm{d}B_s^j$$

$$= b_n^{i,j} I_{(j),t_n,t_{n+1}} + \sum_{l=1}^{m}\sum_{k=1}^{d} b_n^{k,l} \frac{b_n^{i,j} - b_{k,n}^{i,j}}{X_n^k - X_{n-1}^k} \cdot I_{(l,j),t_n,t_{n+1}} + \underbrace{\mathcal{O}(X_n^k - X_{n-1}^k)\mathcal{O}_{\text{strong}}(t_{n+1} - t_n)}_{\text{strong order 2, if } X_1^k - X_0^k \text{ is of strong order 1}} + \mathcal{O}_{\text{strong}}((t_{n+1} - t_n)^{3/2}).$$

This yields a strong order 1 approximation for the ith component of the form

$$X_{n+1}^i = X_n^i + a_n^i h + \sum_{j=1}^{m} b_n^{i,j} I_{(j),t_n,t_{n+1}} + \sum_{j=1}^{m}\sum_{l=1}^{m}\sum_{k=1}^{d} b_n^{k,l} \frac{b_n^{i,j} - b_{k,n}^{i,j}}{X_n^k - X_{n-1}^k} I_{(l,j),t_n,t_{n+1}} + \mathcal{O}_{\text{strong}}((t_{n+1} - t_n)^{3/2}).$$

Dropping the order term $\mathcal{O}_{\text{strong}}((t_{n+1} - t_n)^{3/2})$ and rewriting the whole system of these equations ($i = 1,\ldots,d$) we yield the numerical scheme (3) which is of strong convergence order 1. □

Remark. In the case $X_n^k - X_{n-1}^k = 0$ the implementation of method (3) sets $(\partial b^{i,j}/\partial x^k)(X_n) := 0$. In circuit simulation this yields good results because b is either constant or nonlinear.

To confirm the strong order of convergence method (3) has been compared to a method of the same order (cf. [5, p. 376]):

$$X_{n+1} = X_n + a_n h + b_n \begin{pmatrix} I_{(1)} \\ \vdots \\ I_{(m)} \end{pmatrix} + \frac{1}{\sqrt{h}} \sum_{j=1}^{m} b(t_n, Y_j) \begin{pmatrix} I_{(j,1)} \\ \vdots \\ I_{(j,m)} \end{pmatrix} - \frac{1}{\sqrt{h}} b_n \begin{pmatrix} \sum_{j=1}^{m} I_{(j,1)} \\ \vdots \\ \sum_{j=1}^{m} I_{(j,m)} \end{pmatrix} \tag{4}$$

with $Y_j = X_n + a_n h + b_n^{\bullet,j}\sqrt{h}$ for $j = 1,\ldots,m$, where $b_n^{\bullet,j}$ is the jth column of b_n.

Method (4) is a commonly used numerical scheme for solving SDEs, and so it will be called a "standard" scheme. Scheme (4) is similar to a deterministic Runge–Kutta scheme.

To compare the accuracy of the new method (3) and the standard method (4) these have been tested with a certain multidimensional SDE ($d = 2$ and $m = 3$) where the solution is a geometric Brownian motion, cf. [5]. The SDE which describes this multidimensional geometric Brownian motion has the form

$$\mathrm{d}\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \alpha \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} \mathrm{d}t + \beta_1 \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} \mathrm{d}B_t^1 + \beta_2 \begin{pmatrix} X_t^2 \\ X_t^1 \end{pmatrix} \mathrm{d}B_t^2 + \beta_3 \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} \mathrm{d}B_t^3, \tag{5}$$

Table 1
Absolute error ε of the standard scheme (4) and the new method (3) with regard to the analytical solution of the multidimensional stochastic differential equation (5)

Step size	h	2^{-10}	2^{-11}	2^{-12}	2^{-13}	2^{-14}
Standard method	ε	0.9933	0.5814	0.2737	0.0185	0.0073
New method	ε	0.7486	0.2319	0.0700	0.0142	0.0008

with $\alpha, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and $X_{t_0} = (1,1)^\top$. The computed results for given parameters and initial values are given in Table 1. We calculated the absolute difference between the analytical and the approximative solution and ε is the maximum of these absolute differences. We see that the standard and the new method have the same error behavior when we decrease the step size.

An important property for numerical schemes which can solve SDEs is the *mean-square stability* with respect to certain test equations. As a reference SDE of dimension 1 we consider a linear test equation with multiplicative noise

$$\mathrm{d}X_t = \lambda X_t \,\mathrm{d}t + \mu X_t \,\mathrm{d}B_t \tag{6}$$

with $\lambda, \mu \in \mathbb{C}$ and $X_0 = 1$. $X_t = X_0 \exp((\lambda - \frac{1}{2}\mu^2)t + \mu B_t)$ is the exact solution of Eq. (6).

We say that a numerical method is mean-square stable for test equation (6), if the following inequality holds for arbitrary step size $h > 0$:

$$E[|X_{n+1} - \tilde{X}_{n+1}|^2]^{1/2} < E[|X_0 - \tilde{X}_0|^2]^{1/2},$$

where X_{n+1}, resp. $\tilde{X}_{n+1}$ is the numerical approximation starting at the initial value X_0, resp. $\tilde{X}_0$. $E[*]$ is the expectation on an appropriate probability space. A numerical method is mean-square stable if a mean-square error in the initial value is an upper bound for the mean-square error of the numerical solution at the end of the simulation interval. The mean-square stability of the implicit version (replace a_n by a_{n+1}) of the new method will be stated by the following theorem.

Theorem 3.2. *Assume that* $\Re(\lambda) < 0$, $h > 0$ *and* $\sqrt{2}\Re(\lambda) + |\mu|^2 \leqslant 0$ ($\Re$ *denotes the real and* $\Im$ *the imaginary part of a complex number*). *The implicit version of* (3) *is mean-square stable with respect to test equation* (6).

Proof. The implicit version of (3) applied to test equation (6) results in

$$X_{n+1} = X_n \frac{1 + \mu I_{(1),t_n,t_{n+1}} + \mu^2 I_{(1,1),t_n,t_{n+1}}}{1 - \lambda h}.$$

We apply the method on two different initial values X_0 and $\tilde{X}_0$. For the difference between the two approximations we have

$$\begin{aligned} X_{n+1} - \tilde{X}_{n+1} &= (X_n - \tilde{X}_n)\frac{1 + \mu I_{(1),t_n,t_{n+1}} + \mu^2 I_{(1,1),t_n,t_{n+1}}}{1 - h\lambda} = \cdots \\ &= (X_0 - \tilde{X}_0)\prod_{j=0}^{n} \frac{1 + \mu I_{(1),t_j,t_{j+1}} + \mu^2 I_{(1,1),t_j,t_{j+1}}}{1 - h\lambda}. \end{aligned}$$

The random numbers $|X_0-\tilde{X}_0|$, $|1+\mu I_{(1),t_0,t_1}+\mu^2 I_{(1,1),t_0,t_1}|,\ldots,|1+\mu I_{(1),t_n,t_{n+1}}+\mu^2 I_{(1,1),t_n,t_{n+1}}|$ are stochastic independent (follows from the properties of the Wiener increments). Further, we have for the expectation of the Itô integrals for $j=0,\ldots,n$, cf. [5, p. 222]: $E[I_{(1),t_j,t_{j+1}}]=E[I_{(1,1),t_j,t_{j+1}}]=E[I_{(1),t_j,t_{j+1}}I_{(1,1),t_j,t_{j+1}}]=0$, $E[I^2_{(1),t_j,t_{j+1}}]=h$ and $E[I^2_{(1,1),t_j,t_{j+1}}]=\frac{1}{2}h^2$. We can estimate the quadratic mean-square of the deviation $|X_{n+1}-\tilde{X}_{n+1}|$:

$$E[|X_{n+1}-\tilde{X}_{n+1}|^2] \overset{\text{cf.[11,p. 22]}}{=} E[|X_0-\tilde{X}_0|^2]\prod_{j=0}^{n}\frac{1+|\mu|^2h+\frac{1}{2}|\mu|^4h^2}{|1-h\lambda|^2}.$$

For arbitrary $\lambda\in\mathbb{C}$, with $\Re(\lambda)<0$, the parameter $\mu\in\mathbb{C}$ has to fulfill the assumption $|\mu|^2<-\sqrt{2}\Re(\lambda)$ which yields further conditions: $|\mu|^2<-2\Re(\lambda)$ and $|\mu|^4<2\Re(\lambda)^2\leqslant 2(\Re(\lambda)^2+\Im(\lambda)^2)=2|\lambda|^2$. For the quadratic mean-square deviation above we need

$$\frac{1+|\mu|^2h+\frac{1}{2}|\mu|^4h^2}{|1-h\lambda|^2}<\frac{1-2\Re(\lambda)h+\frac{1}{2}2|\lambda|^2h^2}{1-2h\Re(\lambda)+h^2|\lambda|^2}=1,$$

where $h>0$. This estimation results for the mean-square deviation in

$$E[|X_{n+1}-\tilde{X}_{n+1}|^2]^{1/2}<E[|X_0-\tilde{X}_0|^2]^{1/2}. \tag{7}$$

Inequality (7) shows that the mean-square error of the numerical scheme is bounded by the mean-square error of the initial value. □

Remark. The often used stability condition $2\Re(\lambda)+|\mu|^2<0$ does not hold for the new method, so we have adjusted the condition on the parameters λ and μ.

Besides reliability, efficiency is an important property of numerical methods for industrial applications. Commonly used integration methods need at each integration step at least $m+1$ evaluations of the matrix b. When all noise sources in an electric circuit are considered, b has considerably more columns (complying with the number of noise sources m) than rows (complying with the number of nodes d). So in circuit simulation we usually have high-dimensional matrices b. Therefore, we should avoid many evaluations of b at each integration step. The new scheme can perform this, at each integration step "only" $d+1$ evaluations of b are necessary. This indicates the efficiency of the new scheme which will be shown in the next paragraph.

4. Circuit simulation

Circuit simulation is a standard task for the computer-aided design of electric circuits. The mathematical model of an electric circuit consists of the characteristic equations for the devices (resistor, MOSFET, etc.) and the device parameters. By means of the modified nodal analysis (MNA) [4] these device equations are directly inserted into Kirchhoff's current law which is applied for every node except ground. Additionally, the equations for the branch currents which have no simple admittance form have to be considered. We have applied the new method on different benchmark circuits like a single inverter, an inverter chain, an operational amplifier, an *RLC*-circuit and a ring oscillator. Applying the MNA to the circuit diagram of a ring oscillator with d inverters, cf. Fig. 2,

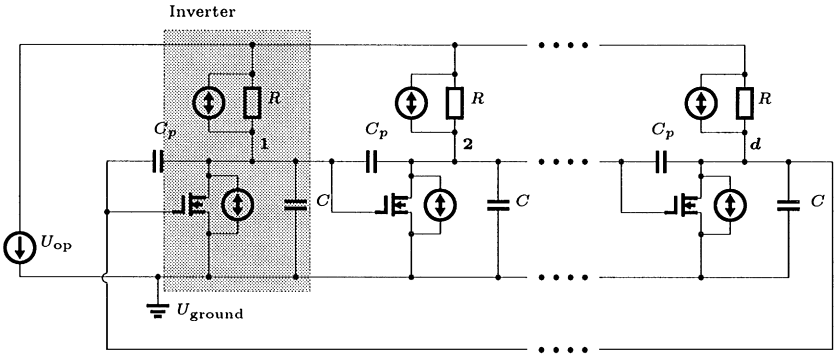

Fig. 2. Circuit diagram of a ring oscillator with d inverters and $m = 2d$ noise sources.

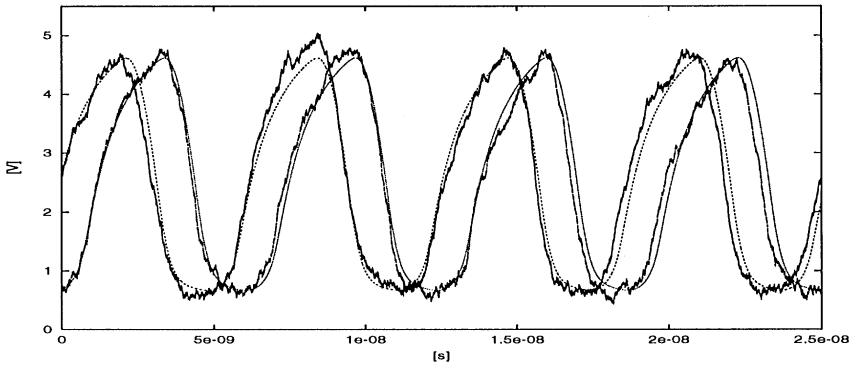

Fig. 3. Noisy (jagged) and ideal (smooth) voltage course at the nodes 3 and 5 of a ring oscillator with five inverters. The ploted part of the simulation interval is $[0, 2.5 \times 10^{-8}]$. The voltage course at node 5 has a phase delay in comparison to the voltage course at node 3 (delay to the right-hand side).

we get a mathematical circuit model of dimension d of the form

$$C_{\text{cap}}\,\mathrm{d}U_t = a(t, U_t)\,\mathrm{d}t + b(U_t)\,\mathrm{d}B_t, \qquad U_{t_0} = U_0, \tag{8}$$

where U_t is the vector of the desired nodal voltages of dimension d and the capacitance matrix C_{cap} is constant and regular.

SDE (8) has been approximatively solved with the standard scheme (4) and the new scheme (3) for $d = 5, 51$ and 101 inverters. $d = 5$ inverters yield the voltage-versus-time graphs at node 3 and 5 as shown in Fig. 3.

The computed results are given in Table 2. The computations have been made on a `Pentium-II`-processor machine (300 MHz), the numerical methods have been implemented in `Fortran77`. The simulation interval has been set to $[0, 2.5 \times 10^{-7}]$ and for the step size we used $h = 10^{-11}$. The results show the speed up of the computing time with the new method. The new method (3) has a speed up factor of 1.15 for $d = 101$ inverters.

Table 2
Comparison of the standard scheme (4) and the new scheme (3) for a ring oscillator with d inverters

Number of inverters d	5		51		101	
Method	Standard	New	Standard	New	Standard	New
a evaluations	24,999	24,999	24,999	24,999	24,999	24,999
b evaluations	274,989	149,999	2,574,897	1,299,999	5,074,797	2,549,999
.....	.....	.....	.....	.....	.....	.....
CPU time (s)	15	13	3,013	2,474	**19,797**	**17,141**

5. Conclusion

We have presented a new numerical integration scheme for the solution of stochastic differential equations of type (2). The efficiency and reliability of the new method has been stated by several circuit benchmarks and mathematical examples (geometric Brownian motion). In this paper we used a ring oscillator to show the speed up of the new method.

We have the intention to extend the new method to solve stochastic differential-algebraic equations. Approaches similar to Schein and Denk [13] are possible.

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