7-day test in tension were 210 pounds the predicted value for the 28-day test would be 275 pounds or 50 per cent of the 28-day tests should be above 275 pounds. Seventy-five per cent of them should be above 275 pounds minus 29 pounds or above 246 pounds. In like manner the limit necessary to include 90 per cent or 99 per cent of the tests may be found. Thus in the illustrated case 90 per cent would be above 275 pounds minus 55 pounds or 230 pounds, 99 per cent of the 28-day tests should be above 275 − 100 pounds or 175 pounds. The results for tensile ratios, compression and compressive ratios are similarly obtained.

TABLE III

28-DAY TEST PREDICTIONS NECESSARY TO HAVE THE GIVEN PERCENTAGE OF ACTUAL 28-DAY TEST RESULTS ABOVE THE PREDICTED VALUE, THE PREDICTION BEING MADE FROM THE 7-DAY TEST RESULTS

	Percentage of actual 28-day tests above predicted value			
KIND OF TEST	50	75	90	99
Tension (pounds)	Predicted value	—29 pounds	—55 pounds	—100 pounds
Tensile ratio	Predicted value	—11.7%	—22.6%	—40.8%
Compression (pounds)	Predicted value	—416 pounds	—801 pounds	—1447 pounds
Compressive ratio	Predicted value	—13.5%	—26.1%	—47.1%

In way of conclusion it may be said that the results of this study indicate that the factors determining the strength of mortar at the end of the 7-day period of curing compared with those determining strength at the end of the 28-day period of curing are sufficiently alike to make possible the prediction of the 28-day breaking strength from the 7-day breaking strength. The error of such a prediction is on the whole not excessive.

SKEW-GROWTH CURVES[1]

BY RAYMOND PEARL AND LOWELL J. REED

SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS UNIVERSITY

Read before the Academy November 12, 1924

In an earlier paper[2] we have pointed out that if, in the general growth curve

$$y = \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3 + \dots\dots + a_nx^n}}, \qquad (1)$$

the exponent of e be cut off at the cubic term, appropriate values of the constants will give a single cycle growth curve which is unsymmetrical or skew about the point of inflection. We gave by way of illustration a theoretical case of such a skew-growth curve, but have not hitherto published any example of the actual fitting of this curve to observational data on growth asymmetrically distributed about the point of inflection. The wide applicability of the general curve (1) to symmetrical population growth has been demonstrated.[3] It seems desirable now to show the ability of the curve to deal with phenomena of skew growth. Hitherto observational growth series which were plainly skew have been graduated (if graduated at all) either by a process of fitting two or more curves to different parts of the observed

TABLE I

Growth of Male Albino Rat. (Donaldson's Data)

AGE IN DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION I	AGE IN DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION I
10	13.5	14.1	70	106.3	103.8
11	13.3	14.5	73	113.8	110.7
12	14.8	15.0	76	121.3	117.6
13	15.3	15.5	79	128.2	124.3
14	15.2	16.1	82	135.0	130.9
15	16.5	16.7	85	143.8	137.4
17	17.8	17.9	88	148.4	143.7
19	19.5	19.3	92	152.3	151.7
21	21.2	20.8	97	160.0	161.2
23	22.9	22.4	102	168.8	170.0
25	25.3	24.2	107	177.6	178.1
27	27.4	26.1	112	183.8	185.5
29	29.5	28.2	117	191.4	192.2
31	31.8	30.5	124	197.3	200.6
34	34.9	33.2	131	202.5	208.1
37	37.8	38.3	138	209.7	214.5
40	42.2	42.7	143	218.3	218.6
43	46.3	48.6	150	225.4	223.7
46	50.5	52.8	157	227.0	228.2
49	56.7	58.3	164	231.4	232.1
52	62.5	64.2	171	235.8	235.7
55	68.5	70.4	178	239.4	238.9
58	73.9	76.8	185	239.8	241.9
61	81.7	83.4	216	252.9	252.7
64	89.1	90.1	256	265.4	264.4
67	99.3	97.0	365	279.0	279.6
			Root mean square deviation	...	4.96

series and then welding the separate curves together where they overlap, or by attempting to make a symmetrical curve fit obviously skew material. One or the other of these plans has been followed by Hatai on Donaldson's data,[4] and by Robertson[5] variously. Aside from the mathematical inelegance of such procedures, in the particular case of skew-growth data the fits obtained have in some cases been grotesquely poor.

We present here four examples of skew-growth curves fitted with the cubic form of (1). These are: (*a*) The growth in weight of male albino rats from 10 days of age on (data from Donaldson, *loc. cit.*, pp. 108–109); (*b*) the growth of female albino rats from 10 days of age on (data from

TABLE II

GROWTH OF FEMALE ALBINO RAT. (DONALDSON'S DATA)

AGE IN DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION II	AGE IN DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION II
10	13.0	14.0	70	99.8	99.8
11	12.8	14.5	73	105.6	105.6
12	15.1	15.0	76	110.4	111.2
13	15.1	15.5	79	118.8	116.7
14	15.6	16.1	82	124.7	122.0
15	17.7	16.7	85	131.5	127.0
17	19.2	18.0	88	136.0	131.8
19	20.6	19.4	92	139.8	137.9
21	22.6	21.0	97	146.3	144.8
23	24.9	22.7	102	153.1	151.2
25	27.4	24.5	107	155.8	156.9
27	30.0	26.5	112	161.4	162.0
29	31.4	28.7	117	168.0	166.6
31	32.9	31.0	124	172.6	172.2
34	35.7	34.8	131	181.0	177.0
37	39.5	38.9	138	185.0	181.1
40	43.7	43.4	143	186.6	183.8
43	47.9	48.3	150	188.2	187.0
46	52.0	53.4	157	188.0	189.9
49	57.7	58.8	164	189.5	192.4
52	62.9	64.4	171	192.2	194.7
55	68.4	70.1	178	197.0	196.7
58	74.6	76.0	185	200.0	198.6
61	78.4	82.0	192	202.2	200.5
64	85.8	88.0	365	226.4	226.9
67	96.0	93.9	Root mean square deviation	...	2.022

Donaldson, *loc. cit.*, pp. 110–111); (*c*) the growth in weight of *Cucurbita pepo* (Anderson's data from Robertson, *loc. cit.*, p. 74), (*d*) the growth in length of the regenerating tail of the tadpole (data from Durbin[6])

The equations in the several cases are:

I. *Male Rats.* $$y = 7 + \frac{273}{1 + e^{4.3204 - 7.2196x + 30.0878x^2 - 0.5291x^3}},$$

where y = weight in grams,
and x = age in hundred day units.

II. *Female Rats.* $y = 7 + \dfrac{220}{1 + e^{4.1426 - 7.6161x + 3.5065x^2 - 0.6262x^3}}$,

y and x in same units as above.

III. *Cucurbita pepo.* $y = 174 + \dfrac{5190}{1 + e^{10.3148 - 16.3399x + 8.1028x^2 - 1.6667x^3}}$,

where y = weight in grams,
and x = age in ten day units.

IV. *Tadpole tails.* $y = \dfrac{9.60}{1 + e^{4.4715 - 7.2829x + 3.3833x^2 - 0.6370x^3}}$,

where y = length of regenerated tail in number and x = age in ten day units.

The observed values and those calculated from the equations above set forth are exhibited in tables I to IV inclusive.

TABLE III

GROWTH OF CUCURBITA PEPO. (DATA FROM ROBERTSON)

DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION III	DAYS	OBSERVED WEIGHT IN GRAMS	CALCULATED WEIGHT BY OUR EQUATION III
5	267	267	16	4720	4680
6	443	399	17	4864	4850
7	658	645	18	4980	4984
8	961	1044	19	5114	5089
9	1498	1586	20	5176	5172
10	2200	2210	21	5242	5236
11	2920	2829	22	5298	5282
12	3366	3378	23	5352	5315
13	3758	3829	24	5360	5337
14	4092	4186	25	5366	5350
15	4488	4464	Root mean square deviation	...	46.1

The observations and fitted curves are shown graphically in figures 1 to 4 inclusive.

To give a concrete idea of the degree of skewness in these curves table V is presented.

The figures in the last column of the table are the significant ones. If the curves were all symmetrical their values would all be 50 per cent. In fact the highest is 38.4 and the lowest 32.5. They indicate that the departure of these curves from the symmetrical condition is substantial.

The results exhibited in the tables and graphs require little comment. All of the curves are obviously skew. The graduations are all excellent. They demonstrate the ability of our generalized growth curve to deal

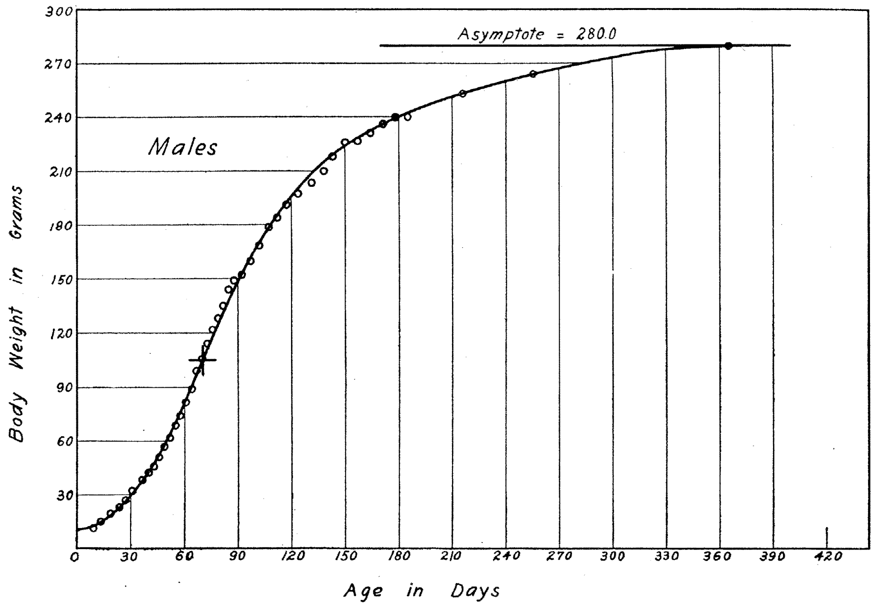

FIGURE 1

Growth of male albino rats (Donaldson's data). The circles give the observations in this and the following diagrams. The smooth curve is the graph of our equation I. The x in this and the following curves denotes the point of inflection.

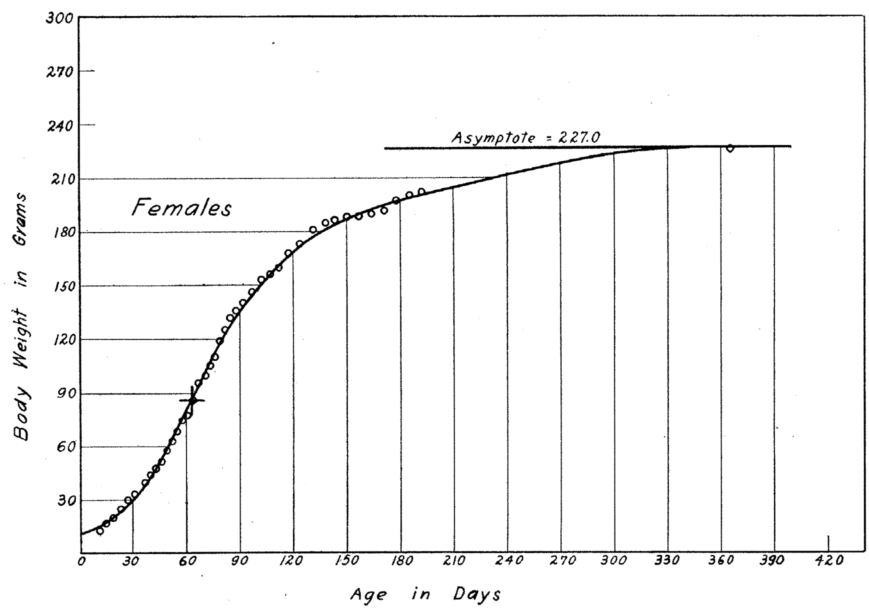

FIGURE 2

Growth of female albino rats (Donaldson's data). The smooth curve is the graph of our equation II.

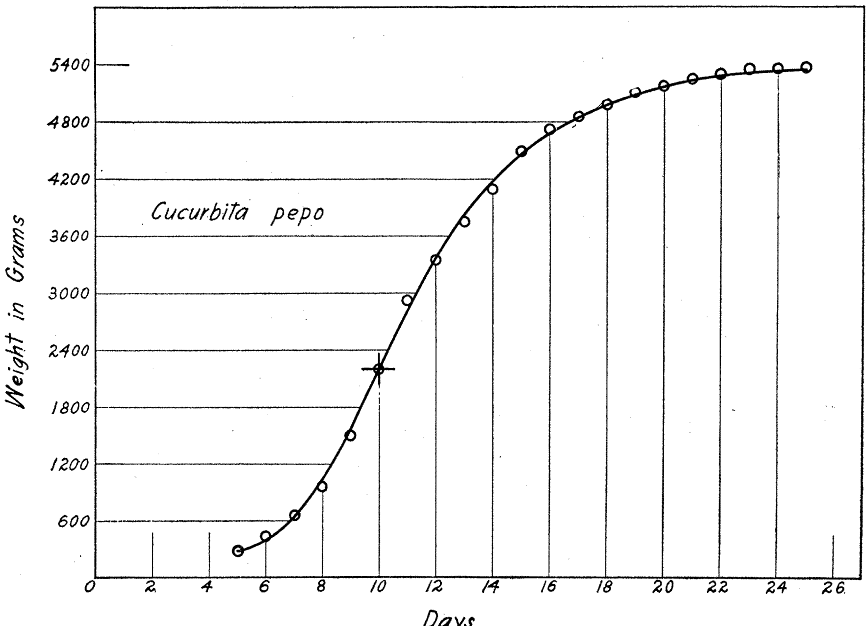

FIGURE 3
Growth of *Cucurbita pepo* (data from Robertson). The smooth curve is the graph of our equation III.

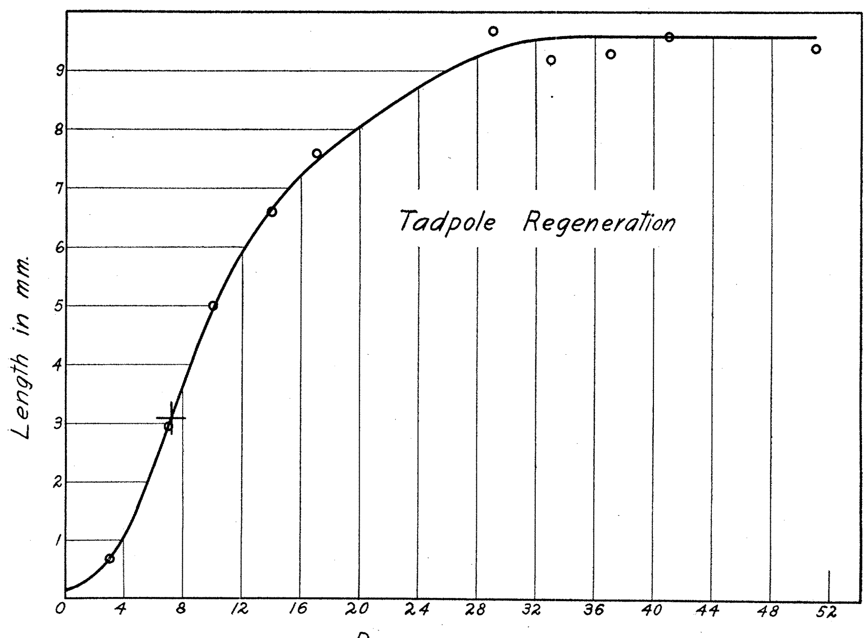

FIGURE 4
Regeneration of tadpole tail (Durbin's data). The smooth curve is the graph of our equation IV.

adequately with unsymmetrical growth phenomena as well as with symmetrical. This is all that we desire to demonstrate in this paper.

TABLE IV

REGENERATION OF TADPOLE TAIL. (DURBIN'S DATA)

DAYS	OBSERVED REGENERATION IN NUMBER	CALCULATED REGENERATION FROM OUR EQUATION IV	DAYS	OBSERVED REGENERATION IN NUMBER	CALCULATED REGENERATION FROM OUR EQUATION IV
0	...	0.11	13	...	6.37
1	...	0.22	14	6.60	6.71
2	...	0.40	15	...	7.00
3	0.68	0.68	17	7.60	7.48
4	...	1.09	19	...	7.88
5	...	1.62	23	...	8.58
6	...	2.25	27	...	9.17
7	2.95	2.95	29	9.68	9.38
8	...	3.66	31	...	9.50
9	...	4.34	33	9.20	9.57
10	5.00	4.96	37	9.30	9.60
11	...	5.50	41	9.60	9.60
12	...	5.97	51	9.40	9.60
			Root mean square deviation	...	0.196

TABLE V

SKEWNESS OF GROWTH CURVES

CURVE	POSITION OF POINT OF INFLECTION		
	ON x AXIS	ON y AXIS	PER CENT OF DISTANCE FROM LOWER TO UPPER ASYMPTOTE
I. Male rats	70.0	103.5	35.3
II. Female rats	62.3	84.5	35.2
III. *Cucurbita pepo*	9.9	2168.0	38.4
IV. Regeneration	7.3	3.12	32.5

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 111.

[2] Pearl, R., and Reed, L. J. On the mathematical theory of population growth. *Metron*, **3**, pp. 6-19, 1923.

[3] Pearl, R. *Studies in Human Biology*. Baltimore (Williams & Wilkins Co.), 1924.

[4] Donaldson, H. H. *The Rat*. Philadelphia (Wistar Institute), 1915.

[5] Robertson, T. B. *The Chemical Basis of Growth and Senescence*. Philadelphia (J. B. Lippincott Co.), 1923.

[6] Durbin, M. L. An analysis of the rate of regeneration throughout the regenerative process. *J. Expt. Zool.* 7, pp. 397–420, 1909.