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A MATHEMATICAL MODEL RELATING COHORT AND PERIOD MORTALITY

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Abstract—This paper presents a mathematical model for changing mortality in functional form. This model may be used to obtain cohort forces of mortality and cohort survivorship functions from a period force of mortality and a period life table under conditions of gradually changing mortality if an estimate of the amount of change in mortality is available. An example is given to show how the cohort functions are derived from the period functions.

INTRODUCTION

An important problem in demography is relating cohort mortality to period mortality. The assumption of stable conditions (constant fertility and mortality over time) is a simplification often employed which forces the cohort and period mortality to be identical. Once the assumption of constant mortality over time is dropped (a condition Coale [1963] termed quasi-stable) cohort mortality is no longer the same as period mortality. One way of deriving cohort mortality from a series observations of period mortality is to follow one individual moving from one period table to the next. This method is cumbersome because the descriptions of both cohort and period mortality are entirely numerical. This procedure could be simplified by a suitable mathematical model of mortality as a function of both age and time.

There have been many attempts to find a mathematical description of mortality as a function of age. Any formula of mortality as a function of age should capture the "bathtub" shape with its high infant mortality, low adolescent mortality, and gradually increasing adult mortality. Two early attempts by Gompertz and Makeham both describe the situation where mortality is low at the younger ages and increases with age. They do not reflect one feature of the "bathtub"—the high infant mortality. Later, Lazarus introduced a formula for the force of mortality $\mu(x)$ as a function of age x. This was an adaptation of Makeham's modification of the Gompertz curve. The formula is given by:

$$\mu(x) = -\ln s - c^x \ln g \ln c - n^x \ln m \ln n \qquad (1)$$

where s, g, c, m, and n are constants. The corresponding survivorship function, $l(x)$, as a function of age x is given by:

$$l(x) = l(0)\, s^x\, g^{(c^x-1)}\, m^{(n^x-1)} \qquad (2)$$

(Elston, 1923). This formula describes childhood mortality as well as adult mortality if c and m are greater than one while all the other constants are between zero and one.

Coale (1963) introduced a model of mortality as a function of both age x and time t as shown below:

$$\mu(x, t) = \mu(x, 0) - t \cdot r(x). \qquad (3)$$

In this paper, $r(x)$ has a "bathtub" shape, but $\mu(x)$ and $r(x)$ are not given specific mathematical formulations.

Keyfitz (1977) proposed another model of mortality as a function of age and time where the change in mortality for any age at any time was roughly proportional to the original mortality at that age. This is shown as follows:

$$\mu_t(x) = \mu_0(x)[1 + At] \tag{4}$$

and

$$l_t(x) = [l_0(x)]^{[1+At]} \tag{5}$$

with A being the amount of change in the force of mortality. This was later modified (Eng, 1977) so that the change in mortality was proportional to mortality in the previous time period:

$$\mu_t(x) = \mu_0(x)[1 + A]^t = \mu_{t-1}(x)[1 + A] \tag{6}$$

and

$$l_t(x) = [l_0(x)]^{[1+A]^t} = [l_{t-1}(x)]^{[1+A]} \tag{7}$$

These two models do not give specific mathematical formulations for the force of mortality, but utilize numerical examples. Any model for mortality as a function of age and time should capture the "bathtub" shape for both cohort and period mortality. While this may be done numerically (Keyfitz, 1977; Eng, 1977), it would be easier to relate the period and cohort mortality if a mathematical model were available.

This paper presents a mathematical model of changing mortality utilizing the Lazarus formula. It is used to relate cohort and period mortality under conditions of steadily changing mortality.

MODEL FOR CHANGING MORTALITY

The mathematical formulation for the force of mortality employed in this paper is given by:

$$\mu(x) = \nu e^{-\nu x} \ln k - \ln s - c^x \ln g \ln c \tag{8}$$

where ν, k, s, g and c are constants. This is identical to the Lazarus formula if $n = e^{-\nu}$ and $m = k$. The corresponding survivorship function is given by:

$$l(x) = l(0)\, k^{(e^{-\nu x}-1)}\, s^x\, g^{(c^x-1)}. \tag{9}$$

In this paper, the radix of the life table $l(0)$ is defined to be one, so that $l(x)$ is the probability of surviving from age 0 to age x under the associated force of mortality. This life table may be substituted into the model for proportional change in mortality to yield:

$$l_t(x) = [l_0(x)]^{(1+A)^t} = [k^{(e^{-\nu x}-1)}\, s^x\, g^{(c^x-1)}]^{(1+A)^t}$$
$$= k^{(e^{-\nu x}-1)(1+A)^t}\, s^{x(1+A)^t}\, g^{(c^x-1)(1+A)^t} \tag{10}$$

where ν, k, s, g and c are values of the constants at time 0. Using this model of changing mortality, the life table at time t has exactly the same form as at time 0, but the values of k, s, and g are raised to the power $(1 + A)^t$. This model for changing mortality implies that c and ν remain constant over time. This applies whether t is positive or negative, so that one may look forward or backward in time. It holds for any value of A greater than -1. Presumably, this model will be used for values of A close to zero since changes in mortality are rather gradual. This model yields the following force of mortality at time t given the force of mortality at time 0:

$$\mu_t(x) = \mu_0(x)\,(1 + A)^t$$
$$= \nu e^{-\nu x}(1 + A)^t \ln k - (1 + A)^t \ln s$$
$$- c^x(1 + A)^t \ln g \ln c. \tag{11}$$

In order to guarantee that the survivorship function is a decreasing function of age, the force of mortality must be greater than zero for all ages. One way is to insure this is to restrict the values of the constants so that k and c are greater than one while $e^{-\nu}$, g and s are between zero and one. These conditions also guarantee the "bathtub" shape of the mortality curve. This model applies to either cohort or period mortality. If the original mortality is for a cohort born at time zero, then the mortality at time t is for a cohort born at time t. In this case, the survivorship function $l(x)$'s are actually probabilities of survival. If one begins with period mortality, then the resultant mortality is for a period. In this case, the survivorship function $l(x)$'s are not probabilities because no one individual actually follows a period schedule of mortality.

The method of determining a particular cohort life table from a period table is simple. Suppose that the available period survivorship function and force of mortality applies to time zero, and the desired cohort was born at time zero. Then the force of mortality for that cohort would be:

$$
\begin{aligned}
{}_c\mu_0(x) &= \mu_0(x)(1 + A)^x \\
&= \nu e^{-\nu x} (1 + A)^x \ln k - (1 + A)^x \\
&\quad \ln s - c^x (1 + A)^x \ln g \ln c. \quad (12)
\end{aligned}
$$

The corresponding cohort survivorship function would be:

$$ {}_cl_0(x) = k^{Q_1} s^{Q_2} g^{Q_3} \quad (13) $$

where

$$ Q_1 = \left[\frac{e^{-\nu x}(1 + A)^x - 1}{\nu - \ln(1 + A)}\right], $$

$$ Q_2 = \left[\frac{(1 + A)^x - 1}{\ln(1 + A)}\right] $$

and

$$ Q_3 = \left[\frac{\ln c\,[c^x(1 + A)^x - 1]}{\ln c + \ln(1 + A)}\right]. $$

If one were to consider a cohort born at time t, the force of mortality and survivorship function would be as follows:

$$
\begin{aligned}
{}_c\mu_t(x) &= \mu_0(x)[1 + A]^{x+t} \\
&= \nu e^{\nu x}[1 + A]^{x+t} \ln k \\
&\quad - [1 + A]^{x+t} \ln s \\
&\quad - c^x[1 + A]^{x+t} \ln g \ln c \quad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_cl_t(x) &= [{}_cl_0(x)]^{(1+A)^t} \quad (15) \\
&= k^{[1+A]^t Q_1} s^{[1+A]^t Q_2} g^{[1+A]^t Q_3}
\end{aligned}
$$

where

$$ Q_1 = \left[\frac{e^{-\nu x}(1 + A)^x - 1}{\nu - \ln[1 + A]}\right] $$

$$ Q_2 = \left[\frac{(1 + A)^x - 1}{\ln(1 + A)}\right] $$

and

$$ Q_3 = \ln c\left[\frac{c^x(1 + A)^x - 1}{\ln c + \ln(1 + A)}\right]. $$

The Lazarus formula was fitted to the West model life tables (Coale and Demeny, 1966) with expectations of life at birth ranging from 30 years to 75 years. The estimation of the constants was carried out using ${}_np_x$ the probability of surviving to age $x + n$ from age x because these are data points available for both period and cohort life tables.

$$
\begin{aligned}
{}_np_x &= \frac{l_{x+n}}{l_x} \\
&= \frac{k^{e^{-\nu(x+n)}-1} s^{x+n} g^{c^{x+n}-1}}{k^{e^{-\nu x}-1} s^x g^{c^x-1}} \\
&= k^{e^{-\nu x}(e^{-\nu n}-1)} s^n g^{c^x(c^n-1)} \quad (16)
\end{aligned}
$$

In equation (16), the constants k, s, g, ν and c are related to one another in a nonlinear form. However, by looking at

$$
\begin{aligned}
\ln {}_np_x &= e^{-\nu x}(e^{-\nu n} - 1) \ln k + n \ln s \\
&\quad + c^x (c^n - 1) \ln g, \quad (17)
\end{aligned}
$$

it is apparent that the values of $\ln k$, $\ln s$ and $\ln g$ may be estimated using linear regression techniques if ν and c are already known. In equation (8), the first term of $\mu(x)$ is $\nu e^{-\nu x} \ln k$ which may be thought of as $\delta e^{-\nu x}$ where δ is a constant. This term does not appear in the Makeham formulation but was included here to allow for high infant mortality. $e^{-\nu x}$ will decrease with increasing age x for $0 < e^{-\nu} < 1$, in other words any $\nu > 0$. In this paper, a value of $\nu = 1$ was chosen to simplify the estimation of the other constants and still permit high infant mortality which gradually decreases with age. For example $e^{-0} = 1$, $e^{-1} = .37$, $e^{-2} = .14$ and $e^{-5} = .01$. The level of infant mortality may be further adjusted by the value of $\ln k$.

According to Wolfenden (1942), the value of c is approximately 1.09 when the Gompertz or Makeham formula is used. Jordan (1975) also notes that $1.08 < c <$

1.12 for the Makeham formula. Since the Makeham model is mainly used for ages beyond childhood and $e^{-x} \ln k$ is approximately zero for large x, the range of values for c that Jordan mentioned would still apply. By setting $c = 1.09$, the remaining constants $\ln k$, $\ln g$ and $\ln s$ may be estimated easily by the method of weighted least squares using

$$\ln {}_np_x = e^{-x}(e^{-n} - 1)\ln k + n \ln s + c^x(c^n - 1)\ln g. \qquad (18)$$

The original observations ${}_np_x$ are binomial with unequal variances. Therefore, any weighted regression using ${}_np_x$ as the dependent variable should use weights $w(x) = 1/[{}_np_x(1 - {}_np_x)]$ to equalize the variances. Using the method of Carrier and Goh (1972), a weighted least squares regression with weights $w(x)\ {}_np_x$ and dependent variables $\ln {}_np_x$ is approximately the same as a weighted least squares regression with weights $w(x)$ and dependent variable ${}_np_x$. Therefore, a weighted least squares regression with weights

$$1/[1 - {}_np_x]$$

and dependent variable $\ln {}_np_x$ is roughly equivalent to a regression with dependent variable ${}_np_x$ and weights

$$1/[{}_np_x(1 - {}_np_x)].$$

This assumes that $\ln p = p - 1$, which is satisfactory when $p < 1$ but is close to 1. Since the equation with $\ln {}_np_x$ as the dependent variable is inherently linear, $\ln k$, $\ln s$ and $\ln g$ were estimated from equation (18) with weights $1/[1 - {}_np_x]$. The constants were constrained so that $\ln s$ and $\ln g$ must be negative and $\ln k$ must be positive. The resulting values of $\ln k$, $\ln s$ and $\ln g$ are then transformed to get k, s and g. The estimates for each constant are given in Table 1 along with two measures of how well the estimated life table approximated the actual model table. One measure is the difference in expectation of life ($\hat{e}_0 - e_0$) and the second measure is the average sum of squared deviations in ${}_np_x$. This measure is given by

$$\Sigma({}_n\hat{p}_x - {}_np_x)^2/[\# - 3]$$

where # is the number of age categories (in this case 17) and 3 is subtracted because three parameters (k, s and g) were estimated from this data.

According to the model, the constants k, s and g at time t are the same as at time 0 but raised to the $(1 + A)^t$ power. This means $k^t = k^{(1+A)^t}$. Using a Taylor series approximation $k^t \approx k^{(1+At)}$ which implies $[\ln k^t/\ln k] - 1 \simeq tA$. Similar computations may be made for s and g. Table 2

Table 1.—Estimates Based on the West Model Life Tables ($c = 1.09$, $v = 1.0$)

Expectation of Life e_0 (years)	Values of Constants: k	s	g	Difference in Expectation of Life $\hat{e}_0 - e_0$ (years)	Average Squared Difference $\Sigma({}_n\hat{p}_x - {}_np_x)^2/14$
30	1.55380	0.98948	0.99779	-0.03008	0.00008
35	1.42930	0.99144	0.99804	-0.07358	0.00005
40	1.33447	0.99314	0.99825	-0.10924	0.00004
45	1.25950	0.99462	0.99842	-0.13273	0.00004
50	1.19804	0.99592	0.99856	-0.15085	0.00004
55	1.14377	0.99710	0.99866	-0.22089	0.00005
60	1.10051	0.99815	0.99876	-0.22463	0.00006
65	1.06366	0.99904	0.99888	-0.21339	0.00009
70	1.03180	0.99976	0.99900	-0.17649	0.00012
75	1.00596	1.00000	0.99922	-0.02817	0.00023

Table 2.—Estimates of tA Based on Constants from Table 1

$\frac{\ln k'}{\ln k} - 1$	$\frac{\ln s'}{\ln s} - 1$	$\frac{\ln g'}{\ln g} - 1$
-0.18951	-0.18712	-0.11323
-0.19220	-0.19928	-0.10724
-0.20039	-0.21633	-0.09722
-0.21684	-0.24213	-0.08867
-0.25656	-0.28364	-0.06949
-0.28703	-0.36240	-0.07467
-0.35561	-0.48131	-0.09683
-0.49276	-0.75009	-0.10720
-0.81018	-1.00000	-0.22009

shows the values of

$$[\ln k'/\ln k] - 1, [\ln s'/\ln s] - 1$$

and $[\ln g'/\ln g] - 1$ using successive values of k, s and g from Table 1.

The ratio $[\ln s'/\ln s)-1]/[\ln k'/\ln k)-1]$ estimates the ratio of tA based on s to tA based on k. It ranges from a low of 0.80 to a high of 1.63. Similar ratios comparing g to s and g to k range from 0.13 to 0.69 and 0.20 to 0.55 respectively. Therefore, it appears that the change in k and s is centered around an amount A, while the change in g is centered about a smaller amount A'. Hence, the general model would be more flexible and permit all three constants k, s and g to change by different amounts A_1, A_2 and A_3. The general model then becomes:

$$\mu_t(x) = (1 + A_1)^t \nu e^{-\nu x} \ln k - (1 + A_2)^t \ln s - (1 + A_3)^t c^x \ln g \ln c \qquad (19)$$

and

$$l_t(x) = k^{[1+A_1]^t Q_1} s^{[1+A_2]^t Q_2} g^{[1+A_3]^t Q_3}$$

where $Q = e^{-\nu x} - 1$, $Q_2 = x$ and $Q_3 = c^x - 1$. (20)

The corresponding cohort force of mortality and life table for a person born at time t will then be:

$$_c\mu_t(x) = \nu e^{-\nu x}[1 + A_1]^{x+t} \ln k - [1 + A_2]^{x+t} \ln s - c^x[1 + A_3]^{x+t} \ln g \ln c \qquad (21)$$

and

$$_cl_t(x) = k^{[1+A_1]^t Q_1} s^{[1+A_2]^t Q_2} g^{[1+A_3]^t Q_3}$$

where $Q_1 = \left[\frac{e^{-\nu x}(1 + A_1)^x - 1}{\nu - \ln[1 + A_1]}\right]$,

$$Q_2 = \left[\frac{(1 + A_2)^x - 1}{\ln(1 + A_2)}\right]$$

and $Q_3 = \ln c\left[\frac{c^x(1 + A_3)^x - 1}{\ln c + \ln(1 + A_3)}\right]$. (22)

Using the formula in equation (19) for $\mu_t(x)$, the change in the force of mortality from time 1 to time 0 is given below:

$$\mu_0(x) - \mu_1(x) = -A_1\, \nu e^{-\nu x} \ln k + A_2 \ln s + A_3\, c^x \ln g \ln c \qquad (23)$$

The derivation of this change with respect to age gives the slope of the force of mortality:

$$\frac{\partial[\mu_0(x) - \mu_1(x)]}{\partial x} = A_1\, \nu^2 e^{-\nu x} \ln k + A_3\, c^x \ln g (\ln c)^2 \qquad (24)$$

At age $x = 0$, the slope is negative because ν, $\ln c$ and $\ln k$ are positive while A_1, A_3, and $\ln g$ are negative. For large age x, the second term on the right side of equation (24) predominates and the slope is positive. Therefore, though A_1, A_2 and A_3 may have different values, the change in the force of mortality remains "bathtub" shaped if A_1 and A_3 are negative. The specific model for these model life tables will use $A_1 = A_2 = -0.02$ and $A_3 = \frac{1}{2}A_1 = -0.01$. Also, at time 0, $k = 1.20$, $s = .966$, $c = 1.09$, $\nu = 1.0$ and $g = .9985$ (approximately equivalent to $\mathring{e}_0 = 50$ years). Table 3 shows the value of the force of mortality for periods and cohorts at various ages. By tracing the cohort values through a series of period values, the relationship between cohort and period forces of mortality may be seen (Figure 1). In both Table 3 and Figure 1, ${}_c\mu_0(20) = \mu_{20}(20)$, ${}_c\mu_{20}(20) = \mu_{40}(20)$ or more generally, ${}_c\mu_t(x) = \mu_{x+t}(x)$.

Table 4 shows the value of the survivorship function for periods and cohorts at various ages. Figure 2 contains period $l(x)$'s for age 0 to 100 at several points in time. It also traces the life table function for two cohorts. The relation between the cohort and period survivorship functions is not as obvious from the diagram, but ${}_cl_t(x)$ is between $l_t(x)$ and $l_{t+x}(x)$.

NUMERICAL EXAMPLE

Since a long series of period and cohort life tables are available for Sweden (Keyfitz and Flieger, 1968), this model was used to estimate the Swedish cohort life tables based on the period life tables for every tenth year. The constants k, s and g were estimated for the period tables with $c = 1.09$ and $\nu = 1.0$ using the same procedure as for the West model life tables. The estimation for each constant are given in Table 5 along with two measures of fit, $\hat{\mathring{e}}_0 - \mathring{e}_0$ and $\sum ({}_n\hat{p}_x - {}_np_x)^2/[\# - 3]$ where # is 18 in this case. For the Swedish tables, the time t between successive tables is known to be 10 years. Therefore, $k' = k^{(1+A)^{10}}$ implies $A = (\ln k'/\ln k)^{1/10} - 1$ and similarly for s and g. Table 6 shows values of $(\ln k'/\ln k)^{1/10} - 1$,

$$(\ln s'/\ln s)^{1/10} - 1$$

and $(\ln g'/\ln g)^{1/10} - 1$ using successive values of k, s and g from Table 5. Here, again, it appears that k and s change by some amount A while g changes by a smaller amount A'. It is also apparent that the values of A and A' vary with time, but since this model assumes A and A' constant over time, some compromise values for A and A' must be chosen for the entire time span. For this paper, the life tables for cohorts born between 1780 and 1925 were estimated using the 1850 period table as the base year ($t = 0$) with $k = 1.28247$, $s = 0.99407$ and $g = 0.99792$, while the change in mortality was $A_1 = A_2 = -0.01$ and $A_3 = -0.005$. 1850 was chosen as the base year because it is roughly midway between 1780 and 1925, the span of cohort tables being examined. Table 7 shows the difference in expectation of life at birth, $\hat{\mathring{e}}_0 - \mathring{e}_0$ and the average squared difference in ${}_np_x$, $\sum({}_n\hat{p}_x - {}_np_x)^2/\#$. In this case, # is 18 and this is the denominator since no parameters were estimated using the cohort life tables. The difference in $\mathring{e}_0$ is larger here than in Table 5 because estimating the cohort tables is akin to predicting the course of mortality based on the 1850 period table and an approximate measure of the change in mortality, while Table 5 shows the results when fitting a function to a set of points without any prediction. However, the difference in $\mathring{e}_0$

Table 3.—Period and Cohort Values for the Force of Mortality at Different Times

Age	Period Force of Mortality $\mu_t(x)$						Cohort Force of Mortality ${}_c\mu_t(x)$	
x	t = 0	20	40	60	80	100	t = 0	20
0	0.18646	0.12450	0.08313	0.05551	0.03707	0.02476	0.18646	0.12450
1	0.07122	0.04757	0.03177	0.02123	0.01418	0.00948	0.06980	0.04662
5	0.00544	0.00366	0.00247	0.00167	0.00113	0.00077	0.00492	0.00331
10	0.00432	0.00293	0.00199	0.00136	0.00093	0.00064	0.00356	0.00242
15	0.00448	0.00306	0.00210	0.00145	0.00101	0.00070	0.00337	0.00231
20	0.00473	0.00327	0.00227	0.00159	0.00112	0.00080	0.00327	0.00227
25	0.00512	0.00359	0.00253	0.00180	0.00130	0.00094	0.00329	0.00232
30	0.00572	0.00408	0.00293	0.00213	0.00156	0.00116	0.00346	0.00250
35	0.00665	0.00484	0.00355	0.00264	0100198	0.00150	0.00383	0.00284
40	0.00807	0.00600	0.00450	0.00342	0.00261	0.00202	0.00450	0.00342
45	0.01026	0.00779	0.00597	0.00461	0.00359	0.00282	0.00559	0.00433
50	0.01363	0.01054	0.00822	0.00646	0.00510	0.00405	0.00728	0.00573
55	0.01881	0.01478	0.01169	0.00929	0.00742	0.00595	0.00983	0.00785
60	0.02678	0.02130	0.01702	0.01365	0.01099	0.00887	0.01365	0.01099
65	0.03905	0.03133	0.02523	0.02036	0.01648	0.01336	0.01931	0.01563
70	0.05792	0.04677	0.03785	0.03069	0.02492	0.02026	0.02765	0.02247
75	0.08695	0.07052	0.05728	0.04658	0.03792	0.03089	0.03991	0.03251
80	0.13163	0.10706	0.08716	0.07102	0.05791	0.04725	0.05791	0.04725
85	0.20037	0.16328	0.13315	0.10864	0.08867	0.07241	0.08429	0.06883
90	0.30614	0.24979	0.20390	0.16651	0.13601	0.11112	0.12293	0.10045
95	0.46888	0.38289	0.31277	0.25555	0.20884	0.17069	0.17951	0.14674
100	0.71927	0.58769	0.48027	0.39255	0.32089	0.26234	0.26234	0.21449

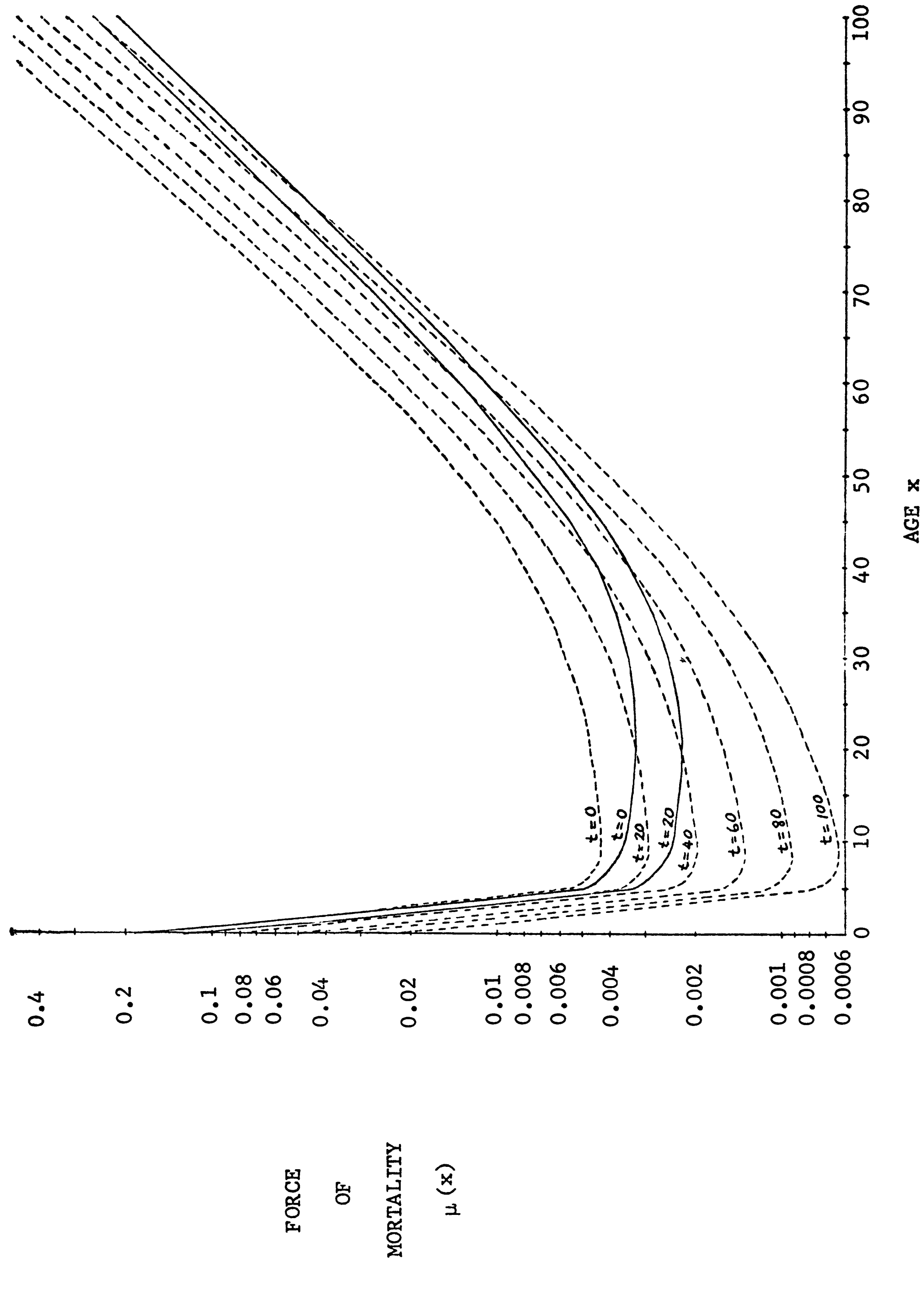

Figure 1.—Period(---) and Cohort (—) Forces of Mortality at Different Times

Table 4.—Period and Cohort Life Table Values at Different Times

Age	Period Life Tables $\ell_t(x)$						Cohort Life Tables ${}_c\ell_t(x)$	
x	t = 0	20	40	60	80	100	t = 0	20
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.88746	0.92337	0.94815	0.96507	0.97653	0.98426	0.88835	0.92399
5	0.81714	0.87377	0.91376	0.94149	0.96049	0.97339	0.82080	0.87638
10	0.79896	0.86058	0.90439	0.93492	0.95591	0.97021	0.80498	0.86492
15	0.78160	0.84781	0.89520	0.92838	0.95129	0.96695	0.79120	0.85479
20	0.76384	0.83453	0.88549	0.92138	0.94626	0.96335	0.77822	0.84509
25	0.74530	0.82039	0.87496	0.91363	0.94059	0.95920	0.76561	0.83547
30	0.72544	0.80489	0.86315	0.90474	0.93394	0.95422	0.75287	0.82551
35	0.70346	0.78725	0.84935	0.89410	0.92577	0.94795	0.73935	0.81463
40	0.67822	0.76637	0.83256	0.88080	0.91531	0.93974	0.72420	0.80209
45	0.64809	0.74064	0.81126	0.86348	0.90137	0.92858	0.70630	0.78683
50	0.61089	0.70782	0.78330	0.84019	0.88223	0.91296	0.68413	0.76747
55	0.56383	0.66490	0.74573	0.80817	0.85540	0.89072	0.65576	0.74211
60	0.50382	0.60826	0.69475	0.76371	0.81745	0.85878	0.61876	0.70834
65	0.42832	0.53422	0.62604	0.70230	0.76396	0.81303	0.57034	0.66324
70	0.33726	0.44069	0.53592	0.61930	0.68990	0.74841	0.50784	0.60364
75	0.23601	0.33010	0.42397	0.51197	0.59098	0.65983	0.42975	0.52696
80	0.13775	0.21315	0.29707	0.38324	0.46676	0.54434	0.33748	0.43271
85	0.06081	0.10954	0.17268	0.24624	0.32534	0.40544	0.23751	0.32483
90	0.01747	0.03961	0.07531	0.12507	0.18711	0.25804	0.14235	0.21381
95	0.00259	0.00834	0.02110	0.04425	0.08006	0.12894	0.06743	0.11610
100	0.00014	0.00076	0.00300	0.00897	0.02173	0.04440	0.02264	0.04757

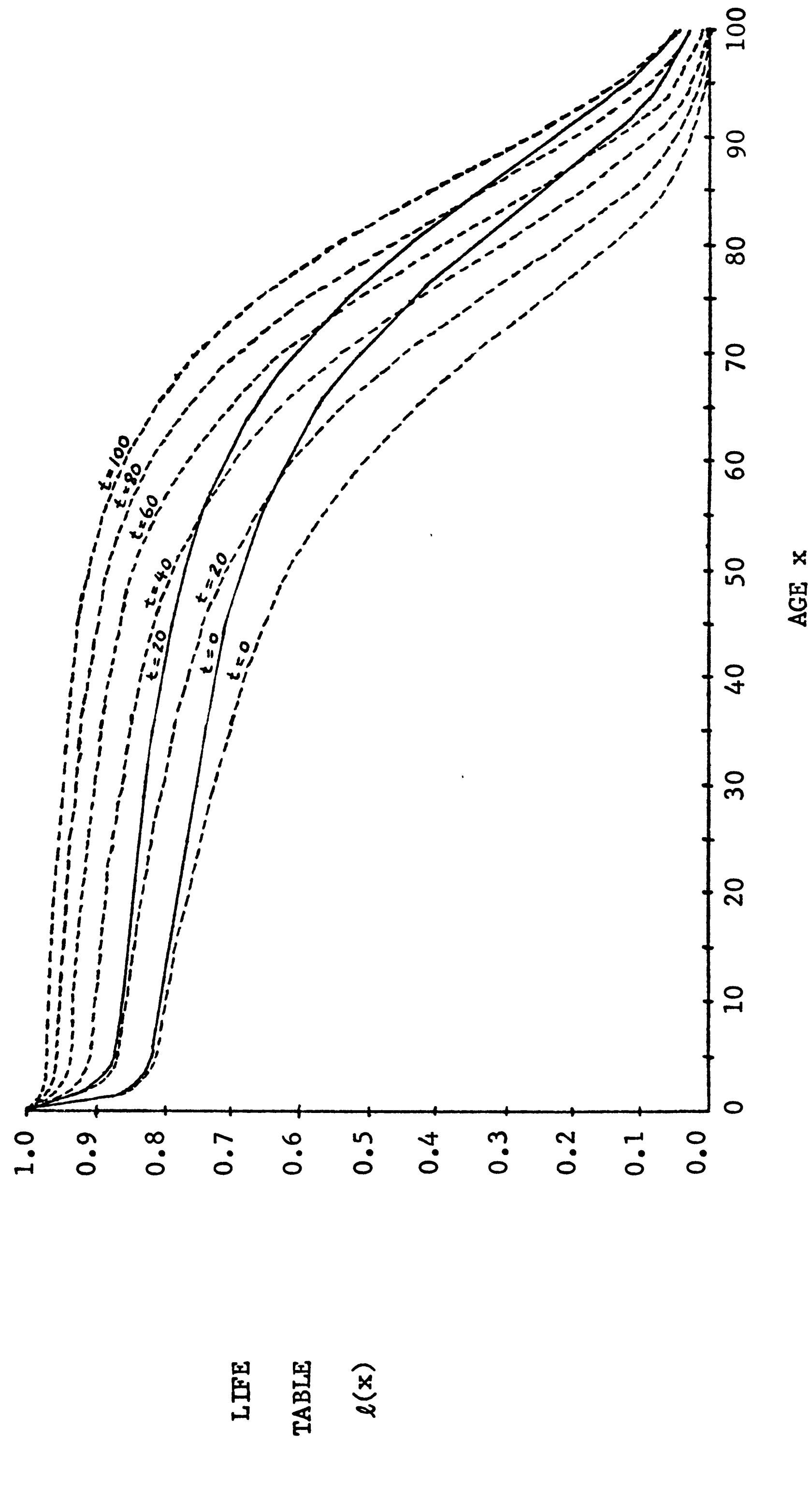

Figure 2.—Period (--) and Cohort (—) Survivorship Functions at Different Times

Table 5.—Estimates Based on Swedish Male Life Tables ($c = 1.09$, $\nu = 1.0$)

Year	Values of Constants			Difference in Expectation of Life $\hat{\overset{o}{e}}_0 - \overset{o}{e}_0$ (Years)	Average Squared Difference $\frac{\Sigma(_n\hat{p}_x - {}_np_x)^2}{15}$
	k	s	g		
1780	1.42655	0.99211	0.99832	-0.59277	0.00043
1790	1.43918	0.99019	0.99806	-0.25208	0.00015
1800	1.45562	0.99418	0.99760	-0.45879	0.00019
1810	1.49242	0.98763	0.99747	-0.10850	0.00014
1820	1.37526	0.99286	0.99788	-0.22404	0.00007
1830	1.35703	0.99268	0.99760	-0.50792	0.00020
1840	1.29842	0.99357	0.99782	-0.57280	0.00017
1850	1.28247	0.99407	0.99792	-0.44260	0.00009
1860	1.28681	0.99444	0.99826	-0.34652	0.00005
1870	1.29005	0.99492	0.99809	-0.29591	0.00003
1880	1.24955	0.99554	0.99844	-0.48302	0.00018
1890	1.20346	0.99622	0.99845	-0.23892	0.00036
1900	1.17506	0.99623	0.99857	-0.29078	0.00033
1910	1.11582	0.99690	0.99869	-0.19533	0.00020
1920	1.08865	0.99629	0.99873	-0.92782	0.00041
1930	1.07651	0.99841	0.99871	0.12228	0.00027
1940	1.05326	0.99928	0.99870	0.40142	0.00029
1950	1.02829	0.99997	0.99880	0.71442	0.00031
1960	1.01154	1.00000	0.99887	0.40790	0.00025

Table 6.—Estimates of A Based on Constants from Table 5

$\sqrt[10]{\frac{\ln k'}{\ln k}}-1$	$\sqrt[10]{\frac{\ln s'}{\ln s}}-1$	$\sqrt[10]{\frac{\ln g'}{\ln g}}-1$
0.00245	0.02212	0.01451
0.00308	-0.05106	0.02153
0.00646	0.07867	0.00530
-0.02258	-0.05372	-0.01755
-0.00427	0.00250	0.01250
-0.01550	-0.01292	-0.00958
-0.00484	-0.00809	-0.00469
0.00135	-0.00644	-0.01771
0.00099	-0.00901	0.00937
-0.01329	-0.01296	-0.02006
-0.01831	-0.01644	-0.00064
-0.01371	-0.00027	-0.00803
-0.03793	-0.01941	-0.00873
-0.02516	0.01816	-0.00310
-0.01406	-0.08134	0.00156
-0.03451	-0.07621	0.00077
-0.06027	-0.27228	-0.00798
-0.08501	0.00000	-0.00600

Table 7.—Comparison of the Actual Swedish Male Cohort Life Tables with Estimated Tables Based on the 1850 Period Life Table ($A_1 = A_2 = -.01$, $A_3 = -.005$)

Cohort birth year	Difference in Expectation of Life $\hat{\mathring{e}}_0 - \mathring{e}_0$ (years)	Average Squared Difference $\frac{\Sigma({}_n\hat{p}_x - {}_np_x)^2}{18}$	Difference in Expectation of Life if Assume Stability
1780	5.40253	0.00074	-7.14167
1785	4.04621	0.00047	-7.27240
1790	3.36296	0.00040	-6.73119
1795	3.20010	0.00037	-5.67274
1800	1.64415	0.00025	-6.01244
1805	1.74636	0.00045	-4.70081
1810	-1.49682	0.00030	-6.74308
1815	1.56737	0.00050	-2.48801
1820	2.04636	0.00053	-0.82963
1825	4.31973	0.00079	2.61036

Table 7.—(Continued)

1830	1.07805	0.00059	0.52133
1835	2.22614	0.00061	2.80702
1840	0.88055	0.00048	2.58302
1845	0.41292	0.00050	3.22012
1850	-0.62542	0.00051	3.26887
1855	-1.86733	0.00051	3.09578
1860	-1.15053	0.00045	4.86254
1865	-1.60916	0.00040	5.43455
1870	-2.69296	0.00039	5.36167
1875	-3.00213	0.00043	6.04338
1880	-1.98667	0.00037	8.02944
1885	-1.46162	0.00034	9.50463
1890	-0.88243	0.00036	11.01339
1895	-0.11283	0.00039	12.69193
1900	0.16177	0.00043	13.85483
1905	2.55775	0.00047	17.11853
1910	3.96909	0.00053	19.37710
1915	4.55909	0.00063	20.79374
1920	4.91594	0.00072	21.95743
1925	6.21119	0.00084	24.03931

is smaller than would be found if one assumed stability (i.e., no change in mortality) as seen in the last column of Table 7.

This paper introduces a mathematical model for mortality as a function of time and age. The formula in the model enables one to relate cohort and period mortality. The method of obtaining cohort mortality given period mortality is derived using mathematical formulas and a numerical example is presented to illustrate the procedure.

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