UNIVERSITY OF MANITOBA

DATE: April 23, 2007 — FINAL EXAMINATION
PAPER NO.: 641 —
DEPARTMENT & COURSE NO.: MATH 3820 — TIME: 3 hours
EXAMINATION: Intro. Math. Model. — EXAMINER: J. Arino

NAME: ____________________

STUDENT NUMBER: ____________________ SEAT NUMBER: ____________________

SIGNATURE: ____________________

(I understand that cheating is a serious offense)

This is a 3 hours exam. Cell phones are not allowed. Calculators and notes are allowed; books are **not** allowed.

This exam has a title page and 2 pages of questions.

PLEASE SHOW YOUR WORK CLEARLY. A correct but unclear answer will not get full marks, nor will a correct answer that does not give some detail of the method used to obtain the solution.

Question	Points	Score
1	20	
2	10	
3	15	
4	20	
Total:	65	

1. (20 points) We want to model the survival of a population of whales. We assume that if the number of whales falls below a minimum survival level m, then the species becomes extinct. In addition, we assume that the population is limited by the carrying capacity M of the environment, that is, if the population is above M, it will experience a decline because the environment cannot sustain that large a population level.
   (a) Let a_t represent the whale population after t years. Discuss the model
   $$a_{t+1} = a_t + k(M - a_t)(a_t - m),$$
   where $k > 0$. Does it make sense in terms of the description above?
   (b) Find the fixed points of the model, and determine their stability.
   (c) Assume that $M = 5000$, $m = 100$ and $k = 0.0001$. Perform a graphical stability analysis.
   (d) The model has two serious shortcomings. What are they? [Hint: Consider what happens when $a_0 < m$, and when $a_0 \gg M$.]

2. (10 points) We consider the partial differential equation
   $$u_t = Du_{xx}, \tag{1}$$
   with $D > 0$.
   (a) Show that the function
   $$g(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}}$$
   is a solution to the diffusion equation (1).
   (b) Verify that $g(x,t) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. Investigate the limits as $x \to \pm\infty$ and $t \to \infty$.

3. (15 points) A forest ecosystem is observed for several years. When a tree dies, the type of tree that replaces it is recorded. The following table is obtained:

	RO	HI	TU	RM	BE
Red oak (RO)	0.12	0.12	0.12	0.42	0.22
Hickory (HI)	0.14	0.05	0.10	0.53	0.18
Tulip tree (TU)	0.12	0.08	0.10	0.32	0.38
Red maple (RM)	0.12	0.28	0.05	0.20	0.35
Beech (BE)	0.13	0.27	0.08	0.19	0.33

(a) Is this a Markov chain?

(b) Assume that we start with a forest comprising only red oaks. What are the first 2 iterates of the process.

(c) Is this Markov chain regular?

(d) How would you obtain the equilibrium value of the distribution of trees? Do not do the computations.

4. (20 points) We consider models of a fishery. In the absence of fishing, the fish population grows logistically,

$$N' = rN\left(1 - \frac{N}{K}\right),$$

with $r, K > 0$. We add terms to this equation to represent the harvesting of the population by fishing. This gives us the following three models:

$$N' = rN\left(1 - \frac{N}{K}\right) - H_1, \tag{2}$$

$$N' = rN\left(1 - \frac{N}{K}\right) - H_2 N, \tag{3}$$

and

$$N' = rN\left(1 - \frac{N}{K}\right) - H_3\frac{N}{A+N}, \tag{4}$$

where $H_1, H_2, H_3, A > 0$.

(a) For each model, give an interpretation of the fishing term. How do these terms differ? How do you interpret the parameters H_1, H_2, H_3, A?

(b) Why is (2) not realistic?

(c) Which of (3) and (4) do you think is best?

[Hint: In this exercise, although it is not asked explicitly, you may want to study the models (2), (3) and (4) more in detail (fixed points, stability, conditions leading to the extinction of the population, etc.).]