THE UNIVERSITY OF MANITOBA

Afternoon 12 April 1993	FINAL EXAMINATION
PAPER NO. : 43	PAGE NO.: 1 OF 7
DEPARTMENT & COURSE NO: Applied Mathematics - 6.337	Time: 3 hours
EXAMINATION: 6.337 - Introduction to Mathematical Modelling	EXAMINER: T. G. Berry

VALUES

Instructions:

This is an OPEN BOOK examination.
Any literature may be consulted.
Electronic calculators are permitted.

Attempt ALL PROBLEMS

10 1. Assume that a given set of data

$$\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$$

may be approximated by a function of the form

$$y = Cxe^{-Dx}, \qquad (*)$$

where C and D are unknown constants to be determined.

Introduce a transformation of variables which will allow you to rewrite (*) in the form of a polynomial, and thus obtain a *linear system of equations* which may be solved to provide *least-squares estimates* for the constants C and D appearing in the assumed approximating function (*).

NOTE: You should rewrite the system of equations in terms of the original variables and parameters.

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15 2. The following data represents a difference table for a function $y(x)$ which is *known to be a quadratic polynomial with integer coefficients* :

x_i	y_i	Δy_i	$\Delta^2 y_i$
1	2		
		5	
2	7		4
		9	
3	16		4
		13	
4	29		3
		16	
5	45		6
		22	
6	67		3
		25	
7	92		4
		29	
8	121		5
		34	
9	155		2
		36	
10	191		

Since $\Delta^2 y_i$ is not constant, it is clear that at least one of the calculated function values $y_i = y(x_i)$ is recorded incorrectly.

(a) Find and correct all errors in the table.
Explain your reasons for making these corrections.

(b) Use the corrected data to determine the coefficients of the quadratic function $y(x) = ax^2 + bx + c$ represented by this table.

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10 3. The solution of the logistic model for single-species population growth in a limited environment , namely

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right), \quad N(t_0) = N_0 \quad (k > 0)$$

may be written in the form

$$N(t) = \frac{C}{1 + e^{-k(t-t^*)}} \qquad (+)$$

where

$$t^* = t_0 - \frac{1}{k}\ell n\left(\frac{N_0}{C - N_0}\right). \qquad (++)$$

(a) Verify that when $N_0 < \frac{C}{2}$, this solution has a point of inflection at time $t = t^*$.

(b) Consider the population data shown in the accompanying table:

t	$N(t)$
0	10
1	15
2	22
3	31
4	42
5	55
6	67
7	79
8	90
9	98
10	103
11	106
12	108
13	109

Assume that this set of data may be modelled by the logistic function (+) , with carrying capacity C being estimated as 110.

Obtain estimates for the remaining model parameters t^* and k as follows:

(i) use the results of part (a) and the above table to estimate t^* ,

(ii) use equation (++) to estimate k .

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20 4. As a modification of the logistic law for single-species population growth, consider the model based on the first order differential equation

$$\frac{dN}{dt} = kN^2\left(1-\frac{N}{C}\right)^2$$

in which k and C are positive constants and $k < C$.

(a) Sketch a graph of $\frac{dN}{dt}$ versus N , and identify all equilibrium points of this model.

(b) Show that $\frac{d^2N}{dt^2} = 2k^2N^3\left(1-\frac{N}{C}\right)^3\left(1-\frac{2N}{C}\right)$.

(c) Sketch a graph of $\frac{d^2N}{dt^2}$ versus N , and thus determine conditions under which a solution $N = N(t)$ of this model will have a point of inflection.

(d) Use the above information in order to sketch anticipated graphs of typical solutions $N = N(t)$ of this model in each of the following cases:

(i) $0 < N(0) < \frac{C}{2}$

(ii) $\frac{C}{2} < N(0) < C$

(iii) $C < N(0)$.

(e) In the case when $0 < N(0) < \frac{C}{2}$, compare the growth rate of the population as predicted by the above model with the corresponding growth rate as predicted by the logistic model, and explain your findings in terms of the graphs of the solutions of these two models.

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15 5. As an alternative to the standard Lotka-Volterra predator-prey model, the following model has been proposed:

$$\left.\begin{aligned} \frac{dx}{dt} &= x(\ell - ny) \\ \frac{dy}{dt} &= y\left(k - \lambda\frac{y}{x}\right) \end{aligned}\right\} \text{ for } k,\ \ell,\ n,\ \lambda > 0\ ,$$

in which $x = x(t)$ and $y = y(t)$ denote the instantaneous sizes of the prey and predator populations respectively.

Clearly the evolutionary equation for the prey species is identical to that of the Lotka-Volterra model, and thus it may be interpreted in exactly the same manner as done in lectures.

(a) Consider the evolutionary equation for the predator species, namely

$$\frac{dy}{dt} = y\left(k - \lambda\frac{y}{x}\right).$$

What does this equation indicate about the growth rate of the predator population in each of the two cases:
(i) $y << x$,
(ii) $x << y$?
Explain the physical significance of these observations.

(b) Identify, and sketch on a phase plane diagram, the nullclines of this model.

(c) Determine the equilibrium point(s) of this model.

(d) In each of the regions into which the phase plane is divided by the nullclines, indicate the direction to be followed by a trajectory of this model.

(e) Based on the above information, predict whether each of the equilibrium points of this model is stable or unstable.

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15 6. Consider the following special case of the "mutual grievance" version of the Richardson's Arms Race model for the total expenditures $x = x(t)$ and $y = y(t)$ for countries X and Y:

$$\left.\begin{array}{l} \dfrac{dx}{dt} = ky - mx + r \\ \dfrac{dy}{dt} = kx - my + s \end{array}\right\} \text{ with } r > 0\ ,\ s > 0\ .$$

Note: In this version of the model the "escalation coefficients" are identical for the two countries, as are the "braking coefficients",

i.e., $\ell = k$ and $n = m$.

(a) If $z = z(t)$ denotes the *total expenditure of the two countries* , show that it must satisfy the differential equation

$$\frac{dz}{dt} = (k - m)z + (r + s)\ .$$

(b) If $z_0 = z(0)$ denotes the initial value of $z(t)$ at time $t = 0$, show that

$$z(t) = \begin{cases} \left(\dfrac{r+s}{m-k}\right) + \left(z_0 - \dfrac{r+s}{m-k}\right) e^{(k-m)t}\ , & for\ \ k \neq m \\ (r+s)t + z_0\ , & for\ \ k = m\ . \end{cases}$$

(c) Discuss the limit of $z(t)$ as $t \to \infty$. Explain the physical significance of this result.

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20 7. To develop a probabilistic model for the spread of an epidemic through a population several assumptions are made:

(i) the duration of the epidemic is short enough that we may assume that the population size N remains constant,

(ii) the population may be divided into simply two "classes", namely, the "susceptibles" and the "infecteds", the sizes of which are denoted by $S(t)$ and $I(t)$ respectively,

(iii) to become infected a susceptible must come into contact with an infected,

(iv) once infected, an individual remains infected,

(v) the number of encounters between members of these two groups at any time is directly proportional to the product of the sizes of the two groups,

(vi) the probability p that any susceptible will become infected during a short time interval of length Δt is directly proportional to the product of the length of the time interval and the number of encounters between members of the two groups, with factor of proportionality β , so that

$$p = \beta I(t)[N - I(t)]\Delta t \ .$$

In the limit as $\Delta t \to 0$, these assumptions give rise to the following system of differential-difference equations for the probability $P_I(t)$ that at time t there are I infected individuals:

$$\frac{dP_I(t)}{dt} + \beta I[N - I]P_I(t) = \beta(I-1)[N-I+1] \ P_{I-1}(t) \quad .$$

(a) Under the assumption that at time $t = 0$ there is precisely one infected individual in the population, find

$$P_1(t), \quad P_2(t) \quad for \quad t \geq 0 .$$

(b) Sketch graphs of $P_1(t)$, $P_2(t)$ for $t \geq 0$.

THE END