6.337 - INTRODUCTION TO MATHEMATICAL MODELLING
MID-TERM TEST - 6 March 2001

Instructions:

This is an open-book examination.
Any literature may be consulted.
Electronic calculators are permitted.
Attempt any combination of problems.
The total number of marks available is 70.
However, a score of 50 (or more) will be regarded as "full marks".

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10 1. Assume that a given set of data

$$\{ (x_i , y_i) \mid i = 0,1,2, \ldots ,n \}$$

may be approximated by a function of the form

$$y = ae^{bx^2} . \qquad (*)$$

Find a set of linear equations which may be solved to provide least-squares estimates for the constants a and b appearing in the assumed approximating function (*).

10 2. You are given a set of data $\{ (x_i , y_i) \mid i = 0,1,2, \ldots ,n \}$ with $x_i = x_0 + i\Delta x$ for $\Delta x > 0$.
Suppose that

$$Y_i = \ell n(y_i)$$

and that

$$\Delta^2 Y_i = k \ \text{(a constant)} .$$

(a) Find the **form** of the explicit dependence of y_i on x_i .
Note: It is only necesary to find the form of this dependence. It is not necessary to attempt to find the values of the parameters in this function.

(b) Show that **recursively** one may find y_i in terms of y_{i-1} and y_{i-2} by the relation

$$y_i = K \frac{y_{i-1}^2}{y_{i-2}}$$

with $K = e^k$.

15 3. An experimental laboratory population with **known** (constant) relative growth rate $k > 0$ is established at time $t = t_0$ with exactly N_0 individuals.

(a) If it is assumed that the population growth is governed by the **Malthusian law**

$$\frac{dN}{dt} = kN \ ,$$

show that the population size is given by

$$N(t) = N_0 e^{k(t-t_0)} \ .$$

(b) Similarly, show that, under the assumption that the population growth is governed by the **logistic law**

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right)$$

with **specified** (constant) carrying capacity C, the population size at time t is given by

$$N(t) = \frac{C}{1 + \left(\frac{C}{N_0} - 1\right) e^{-k(t-t_0)}} \ .$$

(c) The "doubling time" for a population is defined to be the length T of the time interval, measured from the initial time t_0, for the population to double its initial value [i.e., $N(t_0 + T) = 2N_0$].

In each of the two cases discussed in parts (a) and (b), find a formula for the "doubling time" for the given population, it being assumed in the case of the logistic law of part (b) that the initial population size N_0 is less than $\frac{C}{2}$.

(d) Show that if N_0 is **very much smaller** than $\frac{C}{2}$, then the "doubling time" for the logistic law is approximately the same as the "doubling time" for the Malthusian law.

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4. The model

$$\frac{dN}{dt} = k\left(1 - \frac{N}{C}\right)(N - M) \text{ with initial condition } N(0) = N_0$$

has been proposed to describe the ***evolution of a single-species population*** having "initial relative growth rate" k, "carrying capacity" C and "minimum viable population" M, where it is assumed that $k > 0$ and $C > M > 0$.

(a) Sketch a graph of $\frac{dN}{dt}$ vs N, and hence identify the equilibrium populations for this model and the sign of $\frac{dN}{dt}$ for various ranges of values of N.

(b) Use the information of part (a) in order to sketch anticipated graphs of solutions $N = N(t)$ of this model for various appropriate choices of the initial population N_0.

(c) In the preamble to this problem, the parameters C and M have been given physical interpretations as the "carrying capacity" and "minimum viable population" respectively. Based on the results of parts (a) and (b), ***explain*** why these names are appropriate.

(d) By writing the differential equation in the "separated" form

$$\frac{C}{(C-N)(N-M)}\, dN = k\, dt\ ,$$

show that the solution of the initial-value problem

$$\frac{dN}{dt} = k\left(1 - \frac{N}{C}\right)(N - M)\ ;\ N(0) = N_0$$

is given implicitly by the relation

$$\ell n\left|\left(\frac{N-M}{C-N}\right)\left(\frac{C-N_0}{N_0-M}\right)\right| = k\left(\frac{C-M}{C}\right)t\ .$$

(e) Verify that, when $N_0 < M$, extinction of the population must occur in a finite time, as suggested by the graph of anticipated solutions of part (b). In particular, find the time at which this extinction must occur.

15 5. As an ***alternative*** to the standard ***Lotka-Volterra predator-prey model***, the following model has been proposed:

$$\left.\begin{aligned} \frac{dx}{dt} &= x(\ell - ny) \\ \frac{dy}{dt} &= y\left(k - \lambda\frac{y}{x}\right) \end{aligned}\right\} \text{ for } k, \ell, n, \lambda > 0 ,$$

in which $x = x(t)$ and $y = y(t)$ denote the instantaneous sizes of the prey and predator populations respectively.
Clearly the evolutionary equation for the prey species is identical to that of the Lotka-Volterra model, and thus it may be interpreted in exactly the same manner as done in lectures.

(a) Consider the evolutionary equation for the predator species, namely

$$\frac{dy}{dt} = y\left(k - \lambda\frac{y}{x}\right).$$

What does this equation indicate about the growth rate of the predator population in each of the two cases:

(i) $y << x$,

(ii) $x << y$?

Explain the physical significance of these observations.

(b) Identify, and sketch on a phase-plane diagram, the nullclines of this model.

(c) Determine the equilibrium point(s) of this model.

(d) In each of the regions into which the phase-plane is divided by the nullclines, indicate the direction to be followed by trajectories of this model.

(e) Based on the above information, predict whether each of the equilibrium points of this model is stable or unstable. [It is not necessary to confirm your predictions.]