Morning 22 April 2004 | FINAL EXAMINATION

PAPER NO.: 616 |

DEPARTMENT & COURSE NO: Mathematics - 136.382 | Time: 3 hours

EXAMINATION: 136.382 - Introduction to Mathematical Modelling | EXAMINER: Dr. T. G. Berry

VALUES

Instructions:

This is an OPEN-BOOK examination.
Any literature may be consulted.
Electronic calculators are permitted.
Attempt any combination of problems.
The total number of marks available is 135.
However, a score of 100 (or more) will be regarded as "full marks".

[10] 1. Assume that a given set of data

$$\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$$

may be approximated by a function of the form

$$y = Ce^{Dx^2},$$

where C and D are unknown constants to be determined.

Introduce a transformation of variables which will allow you to rewrite this function in the form of a polynomial, and thus obtain a *linear system of equations* which may be solved to provide *least-squares estimates* for the constants C and D appearing in the assumed approximating function.

It is **not** necessary to determine the numerical values for either the coefficients of the linear system or the solutions of this system.

However, the resulting system **must** be expressed in terms of the original variables and parameters.

VALUES

[10] 2. Suppose that the data in the following table illustrates an (unknown) relationship between an independent variable x and a dependent variable y :

x	y
0	1
1	3
2	17
3	50
4	105
5	191
6	314
7	477
8	689
9	955

Assume that y may be approximated by a polynomial (of unknown degree) in x . Indicate, <u>showing all supporting calculations and explanations for your conclusions</u>, how you would determine an appropriate value for the degree of the desired unknown polynomial.

[NOTE: It is <u>not</u> necessary to determine least-squares estimates for the coefficients of this polynomial.]

VALUES

[15] 3. We have studied in detail the logistic model for single-species population growth in a limited environment, namely

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right), \qquad (k > 0\,, C > 0),$$

and found its solution to be of the form

$$N(t) = \frac{C}{1 + Fe^{-kt}}.$$

Modification of this model, which incorporates a "harvesting term" in which the "harvesting rate" is directly proportional to the instantaneous population, is given by

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{C}\right) - hN, \qquad (k > h > 0\,, C > 0)\,.$$

(a) Show that this model is simply a logistic law with "instantaneous initial per capita growth rate"

$$k^* = k - h\ ,$$

and "carrying capacity"

$$C^* = \frac{(k-h)}{k}C\ .$$

(b) Write down an expression for the instantaneous population $N = N^*(t)$ of this model as a function of time. [HINT: It is <u>not</u> necessary to solve the differential equation in this case.]

(c) On a single diagram, sketch anticipated graphs of typical solutions $N = N(t)$ and $N = N^*(t)$ of these two models, both starting at a common fixed initial value $N_0 = N(0) = N^*(0)$ with $N_0 < \frac{C^*}{2}$. Explain the differences in these two graphs, indicating clearly the reasons for these differences.

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[20] 4. As a ***modification of the logistic law*** for single-species population growth, consider the model based on the first order differential equation

$$\frac{dN}{dt} = kN^2\left(1 - \frac{N}{C}\right)^2$$

in which k and C are positive constants and $k << C$.

(a) Sketch a graph of $\dfrac{dN}{dt}$ versus N , and identify all equilibrium points of this model.

(b) Show that $\dfrac{d^2N}{dt^2} = 2k^2N^3\left(1 - \dfrac{N}{C}\right)^3\left(1 - \dfrac{2N}{C}\right)$.

(c) Sketch a graph of $\dfrac{d^2N}{dt^2}$ versus N , and thus determine conditions under which a solution $N = N(t)$ of this model will have a point of inflection.

(d) Use the above information in order to sketch anticipated graphs of typical solutions $N = N(t)$ of this model in each of the following cases:

(i) $0 < N(0) < \dfrac{C}{2}$

(ii) $\dfrac{C}{2} < N(0) < C$

(iii) $C < N(0)$.

(e) In the case when $0 < N(0) < \dfrac{C}{2}$, compare the growth rate of the population as predicted by the above model with the corresponding growth rate as predicted by the logistic model, and explain your findings in terms of the graphs of the solutions of these two models.

VALUES

[25] 5. A ***modified version of the predator-prey model***, in which a fixed number x^* of the prey [known as the prey "reserve"] are excluded from interacting with the predator species, is given by the system

$$\frac{dx}{dt} = \ell x - n\left(x - x^*\right)y$$
$$\frac{dy}{dt} = m\left(x - x^*\right)y - ky$$

in which k, ℓ, m and n are **positive** constants, and $x(t)$ and $y(t)$ respectively denote the instantaneous prey and predator population sizes.

(a) Identify, and sketch on a phase-plane diagram, the nullclines of this model. **Be especially careful** when sketching the graph of the nullcline for the prey.

(b) Determine the equilibrium points of this model.

(c) In each of the regions into which the phase-plane is divided by the nullclines of part (a), indicate the direction to be followed by a trajectory of this model.

(d) Sketch graphs of anticipated trajectories of this model.

(e) Based on the above information, predict whether each of the equilibrium points of part (b) is **stable** or **unstable.**

VALUES

[25] 6. Consider the following ***special case*** of the ***logistic competitive-hunters model***

$$\frac{dx}{dt} = x(\ell - ny - px)$$
$$\frac{dy}{dt} = a\,y(\ell - ny - px)$$

with a , ℓ , n and p positive constants.

(a) Identify the equilibrium point(s) of the model.

(b) On a phase-plane diagram sketch anticipated trajectories of this model.

(c) Does this model support or violate the "principle of competitive exclusion"?

(d) Find the equation of the trajectory which passes through the "initial" point (x_0 , y_0) .

(e) If $a = \frac{1}{2}$, $\ell = 60$, $p = \frac{3}{1000}$ and $n = \frac{1}{500}$, and the trajectory begins at the initial point (40000 , 30000) , determine the "ultimate outcome" of the competition between these two competing species.

[15] 7. The probabilistic model for a pure-birth process, for a population whose initial size is $N(0) = N_0$, is given by the initial-value problem

$$\frac{dP_N(t)}{dt} + bNP_N(t) = b(N-1)P_{N-1}(t) \quad (\text{for } N \geq N_0, \quad t \geq 0),$$

with

$$P_N(0) = \begin{cases} 1 & \text{for } N = N_0 \\ 0 & \textit{otherwise} \end{cases} .$$

We have shown that, when $N = N_0$, this model has solution

$$P_{N_0}(t) = e^{-bN_0 t},$$

while for $N = N_0 + 1$ and $N = N_0 + 2$, this model has solutions

$$P_{N_0+1}(t) = N_0 e^{-bN_0 t}\left(1 - e^{-bt}\right)$$

and

$$P_{N_0+2}(t) = \frac{N_0(N_0+1)}{2} e^{-bN_0 t}\left(1 - e^{-bt}\right)^2,$$

respectively.

(a) Show that, when $N = N_0 + 3$, the model has solution

$$P_{N_0+3}(t) = \frac{N_0(N_0+1)(N_0+2)}{2(3)} e^{-bN_0 t}\left(1 - e^{-bt}\right)^3 .$$

(b) Use the result of (a) in order to find $\dfrac{dP_{N_0+3}(t)}{dt}$, and hence determine the time at which $P_{N_0+3}(t)$ attains its maximum value.

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[15] 8. Consider a stable population of size N (constant). As discussed in class, an epidemic spreads through this population, in such a way that, at any instant in time, each member belongs to precisely one of the following "compartments":

"susceptibles" (size $S(t)$),

"latent infecteds" (size $L(t)$),

"infecteds" (size $I(t)$),

or "removeds" (size $R(t)$).

Through the use of a number of assumptions which we made, we showed that this epidemic is modelled by the equations

$$\frac{dS}{dt} = -\beta SI$$

$$L(t) = 0$$

$$\frac{dR}{dt} = rI$$

$$S + R + I = N$$

in which β , r and N are positive constants. Throughout the remainder of this problem, assume that these constants have the values

$$r = 0.9 , \quad b = 0.0002 , \quad N = 10000 .$$

If $S(0) = 9990$ and $I(0) = 10$, find the maximum value of $I(t)$.

THE END