# MATH 3610 – 06
# Traffic flow – Linear cascades & Linear systems
# Delay differential equations & Laplace transform

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.
We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

# Outline

Traffic flow – ODE model

Linear systems of ODE – Brief theory

Linear systems – Our case

Traffic flow – DDE model

The Laplace transform

Laplace transform of our DDE traffic flow model

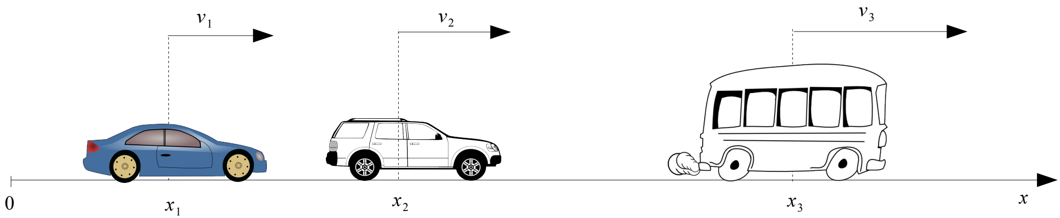
# Problem formulation

Want to model

- $N$ cars
- on a straight road
- no overtaking
- adjustment of speed on driver in front

# Hypotheses

- $N$ cars in total.
- Road is the $x$-axis.
- $x_n(t)$ position of the $n$th car at time $t$.
- $v_n(t) \overset{\Delta}{=} x_n'(t)$ velocity of the $n$th car at time $t$.




- All cars start with the same initial speed $v_0$ before time $t = 0$.

# Moving frame coordinates

To make computations easier, express velocity of cars in a reference frame moving at speed $u_0$.

Remark that here, speed=velocity, since movement is 1-dimensional.

Let

$$u_n(t) = v_n(t) - u_0.$$

Then $u_n(t) = 0$ for $t \leq 0$, and $u_n$ is the speed of the $n$th car in the moving frame coordinates.

# Modeling driver behavior

Assume that

- Driver adjusts his/her speed according to relative speed between his/her car and the car in front.
- This adjustment is a linear term, equal to $\lambda$ for all drivers.

- First car: evolution of speed remains to be determined.
- Second car:
$$u_2' = \lambda(u_1 - u_2).$$
- Third car:
$$u_3' = \lambda(u_2 - u_3)$$
- Thus, for $n = 1, \ldots, N - 1$,
$$u_{n+1}' = \lambda(u_n - u_{n+1}). \tag{1}$$

This can be solved using *linear cascades*: if $u_1(t)$ is known, then

$$u_2' = \lambda(u_1(t) - u_2)$$

is a linear first-order nonhomogeneous equation. Solution (integrating factors, or variation of constants) is

$$u_2(t) = \lambda e^{-\lambda t} \int_0^t u_1(s) e^{\lambda s} ds$$

Then use this function $u_2(t)$ in $u_3'$ to get $u_3(t)$,

$$u_3(t) = \lambda e^{-\lambda t} \int_0^t u_2(s) e^{\lambda s} ds$$

Thus

$$
\begin{aligned}
u_3(t) &= \lambda e^{-\lambda t} \int_0^t u_2(s) e^{\lambda s} ds \\
&= \lambda e^{-\lambda t} \int_0^t \left( \lambda e^{-\lambda s} \int_0^s u_1(q) e^{\lambda q} dq \right) ds \\
&= \lambda^3 e^{-\lambda t} \int_0^t e^{-\lambda s} \int_0^s u_1(q) e^{\lambda q} dq ds
\end{aligned}
$$

Continue the process to get the solution.

# Example

Suppose driver of car 1 follows this function

$$u_1(t) = \alpha \sin(\omega t)$$

that is, $\omega$-periodic, 0 at $t = 0$ (we want all cars to start with speed relative to the moving reference equal to 0), and with amplitude $\alpha$.

Then

$$\begin{aligned}
u_2(t) &= \lambda\alpha e^{-\lambda t} \int_0^t \sin(\omega s) e^{\lambda s} ds \\
&= \lambda\alpha e^{-\lambda t} \left( \frac{\omega - \omega e^{\lambda t} \cos(\omega t) + \lambda e^{\lambda t} \sin(\omega t)}{\lambda^2 + \omega^2} \right) \\
&= \frac{\lambda\alpha}{\lambda^2 + \omega^2} \left( \omega e^{-\lambda t} + \lambda \sin(\omega t) - \omega \cos(\omega t) \right).
\end{aligned}$$

When $t \to \infty$, first term goes to 0, we are left with a $\omega$-periodic term.

Continuing the process,

$$u_3(t) = \frac{\lambda^2\alpha}{\lambda^2+\omega^2} e^{-\lambda t} \times$$
$$\int_0^t \left(\omega e^{-\lambda s} + \lambda \sin(\omega s) - \omega \cos(\omega s)\right) e^{\lambda s} ds$$

that is,

$$\begin{aligned}
u_3(t) &= \frac{\lambda^2\alpha}{\lambda^2+\omega^2} e^{-\lambda t} \left(\omega t + \int_0^t \left(\lambda \sin(\omega s) - \omega \cos(\omega s)\right) e^{\lambda s} ds\right) \\
&= \frac{\lambda^2\alpha}{\lambda^2+\omega^2} \left(\omega t + \frac{2\lambda\omega}{\lambda^2+\omega^2}\right) e^{-\lambda t} \\
&\quad - \frac{\lambda^2\alpha}{(\lambda^2+\omega^2)^2} \left(2\lambda\omega \cos(\omega t) - \lambda^2 \sin(\omega t) + \omega^2 \sin(\omega t)\right)
\end{aligned}$$

Once again, the terms in $e^{-\lambda t}$ vanishes for large $t$, and we are left with 3 $\omega$-periodic terms.

# Linear ODEs

## Definition 1 (Linear ODE)

A *linear* ODE is a differential equation taking the form

$$\frac{d}{dt}x = A(t)x + B(t), \tag{LNH}$$

where $A(t) \in \mathcal{M}_n(\mathbb{R})$ with continuous entries, $B(t) \in \mathbb{R}^n$ with real valued, continuous coefficients, and $x \in \mathbb{R}^n$. The associated IVP takes the form

$$\begin{aligned} \frac{d}{dt}x &= A(t)x + B(t) \\ x(t_0) &= x_0. \end{aligned} \tag{2}$$

# Types of systems

- $x' = A(t)x + B(t)$ is linear nonautonomous ($A(t)$ depends on $t$) nonhomogeneous (also called *affine* system).
- $x' = A(t)x$ is linear nonautonomous homogeneous.
- $x' = Ax + B$, that is, $A(t) \equiv A$ and $B(t) \equiv B$, is linear autonomous nonhomogeneous (or affine autonomous).
- $x' = Ax$ is linear autonomous homogeneous.

- If $A(t + T) = A(t)$ for some $T > 0$ and all $t$, then linear periodic.

# Existence and uniqueness of solutions

**Theorem 2 (Existence and Uniqueness)**

*Solutions to (2) exist and are unique on the whole interval over which $A$ and $B$ are continuous.*
*In particular, if $A, B$ are constant, then solutions exist on $\mathbb{R}$.*

# Autonomous linear systems

Consider the autonomous affine system

$$\frac{d}{dt}x = Ax + B, \tag{A}$$

and the associated homogeneous autonomous system

$$\frac{d}{dt}x = Ax. \tag{L}$$

# Exponential of a matrix

**Definition 3 (Matrix exponential)**

Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of $A$, denoted $e^{At}$, is a matrix in $\mathcal{M}_n(\mathbb{K})$, defined by

$$e^{At} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k,$$

where $\mathbb{I}$ is the identity matrix in $\mathcal{M}_n(\mathbb{K})$.

# Properties of the matrix exponential

- $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \in \mathbb{R}$. 1
- $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.
- $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.
- The unique solution $\phi$ of (L) with $\phi(t_0) = x_0$ is given by

$$\phi(t) = e^{A(t-t_0)}x_0.$$

# Computing the matrix exponential

Let $P$ be a nonsingular matrix in $\mathcal{M}_n(\mathbb{R})$. We transform the IVP

$$\begin{aligned} \frac{d}{dt}x &= Ax \\ x(t_0) &= x_0 \end{aligned} \tag{L_IVP}$$

using the transformation $x = Py$ or $y = P^{-1}x$.

The dynamics of $y$ is

$$\begin{aligned} y' &= (P^{-1}x)' \\ &= P^{-1}x' \\ &= P^{-1}Ax \\ &= P^{-1}APy \end{aligned}$$

The initial condition is $y_0 = P^{-1}x_0$.

We have thus transformed IVP (L_IVP) into

$$
\begin{aligned}
\frac{d}{dt}y &= P^{-1}APy \\
y(t_0) &= P^{-1}x_0
\end{aligned}
\tag{L\_IVP\_y}
$$

From the earlier result, we then know that the solution of (L_IVP_y) is given by

$$\psi(t) = e^{P^{-1}AP(t-t_0)}P^{-1}x_0,$$

and since $x = Py$, the solution to (L_IVP) is given by

$$\phi(t) = Pe^{P^{-1}AP(t-t_0)}P^{-1}x_0.$$

So everything depends on $P^{-1}AP$.

# Diagonalizable case

Assume $P$ nonsingular in $\mathcal{M}_n(\mathbb{R})$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with all eigenvalues $\lambda_1, \ldots, \lambda_n$ different.

We have

$$e^{P^{-1}AP} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k$$

For a (block) diagonal matrix $M$ of the form

$$M = \begin{pmatrix} m_{11} & & 0 \\ & \ddots & \\ 0 & & m_{nn} \end{pmatrix}$$

there holds

$$M^k = \begin{pmatrix} m_{11}^k & & 0 \\ & \ddots & \\ 0 & & m_{nn}^k \end{pmatrix}$$

Therefore,

$$
\begin{aligned}
e^{P^{-1}AP} &= \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}
\end{aligned}
$$

And so the solution to (L_IVP) is given by

$$\phi(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0.$$

# Nondiagonalizable case

The Jordan canonical form is

$$P^{-1}AP = \begin{pmatrix} J_0 & & 0 \\ & \ddots & \\ 0 & & J_s \end{pmatrix}$$

so we use the same property as before (but with block matrices now), and

$$e^{P^{-1}APt} = \begin{pmatrix} e^{J_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_s t} \end{pmatrix}$$

The first block in the Jordan canonical form takes the form

$$J_0 = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

and thus, as before,

$$e^{J_0 t} = \begin{pmatrix} e^{\lambda_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}$$

Other blocks $J_i$ are written as

$$J_i = \lambda_{k+i}\mathbb{I} + N_i$$

with $\mathbb{I}$ the $n_i \times n_i$ identity and $N_i$ the $n_i \times n_i$ nilpotent matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$\lambda_{k+i}\mathbb{I}$ and $N_i$ commute, and thus

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}$$

Since $N_i$ is nilpotent, $N_i^k = 0$ for all $k \geq n_i$, and the series $e^{N_i t}$ terminates, and

$$e^{J_i t} = e^{\lambda_{k+i} t} \begin{pmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & & \\ 0 & & & 1 \end{pmatrix}$$

### Theorem 4

*For all* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$*, there is a unique solution* $x(t)$ *to* (L_IVP) *defined for all* $t \in \mathbb{R}$*. Each coordinate function of* $x(t)$ *is a linear combination of functions of the form*

$$t^k e^{\alpha t} \cos(\beta t) \quad and \quad t^k e^{\alpha t} \sin(\beta t)$$

*where* $\alpha + i\beta$ *is an eigenvalue of* $A$ *and* $k$ *is less than the algebraic multiplicity of the eigenvalue.*

# Generalized eigenvectors

**Definition 5 (Generalized eigenvectors)**

Let $A \in \mathcal{M}_r(\mathbb{R})$. Suppose $\lambda$ is an eigenvalue of $A$ with multiplicity $m \leq n$. Then, for $k = 1, \ldots, m$, any nonzero solution $v$ of

$$(A - \lambda \mathbb{I})^k v = 0$$

is called a *generalized eigenvector* of $A$.

# Nilpotent matrix

**Definition 6 (Nilpotent matrix)**

Let $A \in \mathcal{M}_n(\mathbb{R})$. $A$ is *nilpotent* (of order $k$) if $A^j \neq 0$ for $j = 1, \ldots, k-1$, and $A^k = 0$.

# Jordan normal form

## Theorem 7 (Jordan normal form)

*Let $A \in \mathcal{M}_n(\mathbb{R})$ have eigenvalues $\lambda_1, \dots, \lambda_n$, repeated according to their multiplicities.*

- *Then there exists a basis of generalized eigenvectors for $\mathbb{R}^n$.*
- *And if $\{v_1, \dots, v_n\}$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, then the matrix $P = [v_1 \cdots v_n]$ is invertible, and $A$ can be written as*

$$A = S + N,$$

*where*

$$P^{-1}SP = \mathrm{diag}(\lambda_j),$$

*the matrix $N = A - S$ is nilpotent of order $k \leq n$, and $S$ and $N$ commute, i.e., $SN = NS$.*

## Theorem 8

*Under conditions of the Jordan normal form Theorem, the linear system $x' = Ax$ with initial condition $x(0) = x_0$, has solution*

$$x(t) = P\mathrm{diag}\left(e^{\lambda_j t}\right) P^{-1} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

The result is particularly easy to apply in the following case.

## Theorem 9 (Case of an eigenvalue of multiplicity $n$)

*Suppose that $\lambda$ is an eigenvalue of multiplicity $n$ of $A \in \mathcal{M}_n(\mathbb{R})$. Then $S = \mathrm{diag}(\lambda)$, and the solution of $x' = Ax$ with initial value $x_0$ is given by*

$$x(t) = e^{\lambda t} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

In the simplified case, we do not need the matrix $P$ (the basis of generalized eigenvectors)

# A variation of constants formula

**Theorem 10 (Variation of constants formula)**

*Consider the IVP*

$$x' = Ax + B(t) \tag{3a}$$

$$x(t_0) = x_0, \tag{3b}$$

*where* $B : \mathbb{R} \to \mathbb{R}^n$ *a smooth function on* $\mathbb{R}$*, and let* $e^{A(t-t_0)}$ *be matrix exponential associated to the homogeneous system* $x' = Ax$*. Then the solution* $\phi$ *of* (3) *is given by*

$$\phi(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)ds. \tag{4}$$

# Computation in our case

Consider the case of 3 cars. Let

$$X = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

Then the system can be written as

$$X' = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} U + \begin{pmatrix} \lambda u_1(t) \\ 0 \end{pmatrix}$$

which we write for short as $X' = AX + B(t)$.

The matrix $A$ has the eigenvalue $-\lambda$ with multiplicity 2. Its Jordan form is obtained by using the maple function `JordanForm`:

```
> with(LinearAlgebra)
> A := <<-lambda, lambda> | <0, -lambda>>:
> J := JordanForm(A)
```

giving

$$J = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

To get the matrix of change of basis,

```
> P := JordanForm(A,output='Q')
```

giving

$$P = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

which is such that $P^{-1}AP = J$.

Because $-\lambda$ is an eigenvalue with multiplicity 2 (same as the size of the matrix), we can use the simplified theorem, and only need $N$.

We have

$$\begin{aligned} N &= A - S \\ &= \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \end{aligned}$$

Clearly, $N^2 = 0$, so, by the theorem in the simplified case,

$$x(t) = e^{-\lambda t} \left( \mathbb{I} + Nt \right) x_0$$

But we know that solutions are unique, and that the solution to the differential equation is given by $x(t) = e^{At} x_0$. This means that

$$\begin{aligned}
e^{At} &= e^{-\lambda t} \left( \mathbb{I} + Nt \right) \\
&= e^{-\lambda t} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda t & 0 \end{pmatrix} \right) \\
&= e^{-\lambda t} \begin{pmatrix} 1 & 0 \\ \lambda t & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-\lambda t} & 0 \\ \lambda t e^{-\lambda t} & e^{-\lambda t} \end{pmatrix}
\end{aligned}$$

Now notice that the solution to

$$X' = AX$$

is trivially established here, since

$$X(0) = \begin{pmatrix} u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus

$$X(t) = e^{At}0 = 0.$$

$e^{At}$ does however play a role in the solution (fortunately), since it is involved in the variation of constants formula:

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B(s)ds$$

Thus we need to compute $e^{A(t-s)}B(s)$, and then the integral.

$$
\begin{aligned}
e^{A(t-s)}B(s) &= \begin{pmatrix} e^{-\lambda(t-s)} & 0 \\ \lambda(t-s)e^{-\lambda(t-s)} & e^{-\lambda(t-s)} \end{pmatrix} \begin{pmatrix} \lambda u_1(s) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda e^{-\lambda(t-s)} u_1(s) \\ \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s) \end{pmatrix}
\end{aligned}
$$

and thus

$$
\begin{aligned}
\int_0^t e^{A(t-s)}B(s)ds &= \int_0^t \begin{pmatrix} \lambda e^{-\lambda(t-s)}u_1(s) \\ \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s) \end{pmatrix} ds \\
&= \begin{pmatrix} \int_0^t \lambda e^{-\lambda(t-s)}u_1(s)ds \\ \int_0^t \lambda^2 e^{-\lambda(t-s)}(t-s)u_1(s)ds \end{pmatrix} \\
&= \begin{pmatrix} \lambda e^{-\lambda t}\int_0^t e^{\lambda s}u_1(s)ds \\ \lambda^2 e^{-\lambda t}\int_0^t e^{\lambda s}(t-s)u_1(s)ds \end{pmatrix} \\
&= \begin{pmatrix} \lambda e^{-\lambda t}\int_0^t e^{\lambda s}u_1(s)ds \\ \lambda^2 e^{-\lambda t}\left(t\int_0^t e^{\lambda s}u_1(s)ds - \int_0^t s e^{\lambda s}u_1(s)ds\right) \end{pmatrix}
\end{aligned}
$$

Let

$$\Psi(t) = \int_0^t e^{\lambda s} u_1(s) ds$$

and

$$\Phi(t) = \int_0^t s e^{\lambda s} u_1(s) ds$$

These can be computed when we choose a function $u_1(t)$. Then, finally, we have

$$\begin{aligned} X(t) &= \int_0^t e^{A(t-s)} B(s) ds \\ &= \begin{pmatrix} \lambda e^{-\lambda t} \Psi(t) \\ \lambda^2 e^{-\lambda t} \left( t\Psi(t) - \Phi(t) \right) \end{pmatrix} \end{aligned}$$

# Case of the $\alpha \sin(\omega t)$ driver

We set

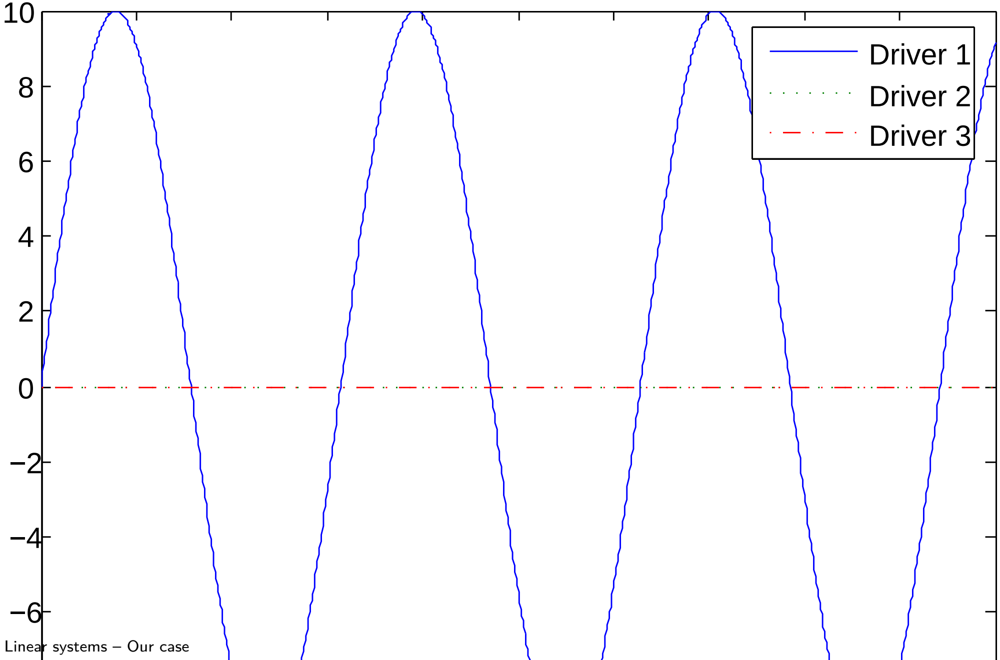
$$u_1(t) = \alpha \sin(\omega t).$$

Then

$$\Psi(t) = \frac{\alpha(\omega - \omega e^{\lambda t}\cos(\omega t) + \lambda e^{\lambda t}\sin(\omega t))}{\lambda^2 + \omega^2}$$

and

$$\Phi(t) = \frac{\alpha(\lambda^3 t + \lambda t\omega^2 - \lambda^2 + \omega^2)\sin(\omega t)e^{\lambda t}}{(\lambda^2+\omega^2)^2} - \frac{\alpha\omega\cos(\omega t)(t\lambda^2 + t\omega^2 - 2\lambda)e^{\lambda t} + 2\alpha\lambda\omega}{(\lambda^2+\omega^2)^2}$$



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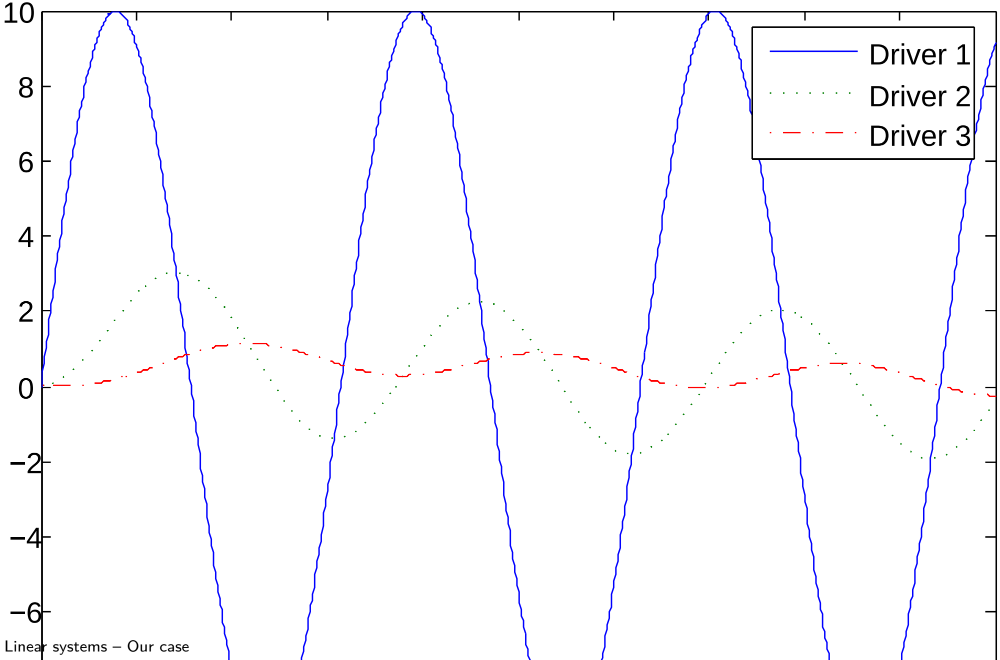

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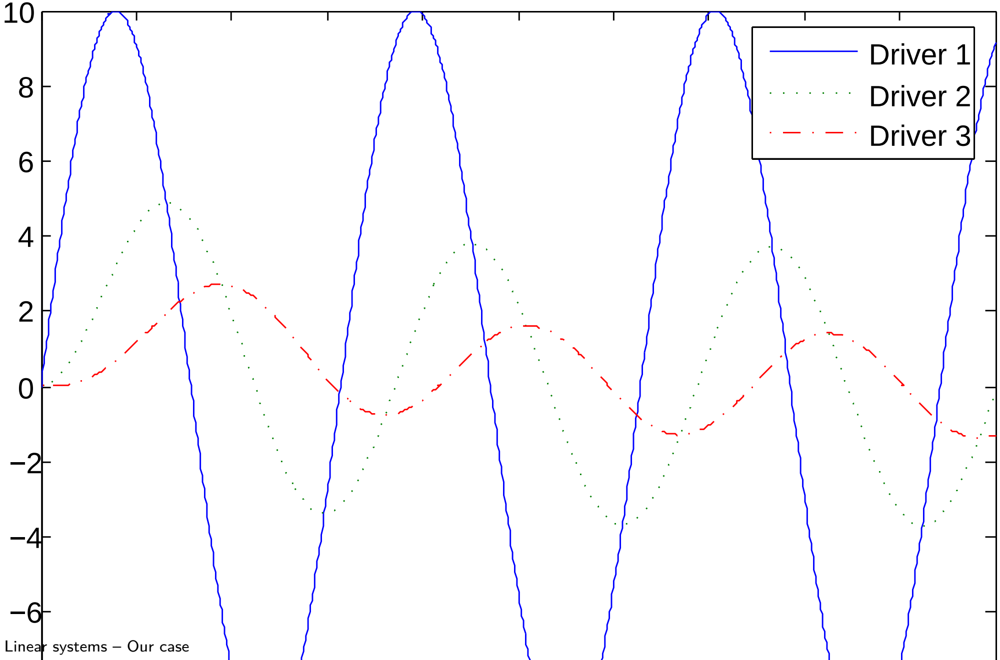
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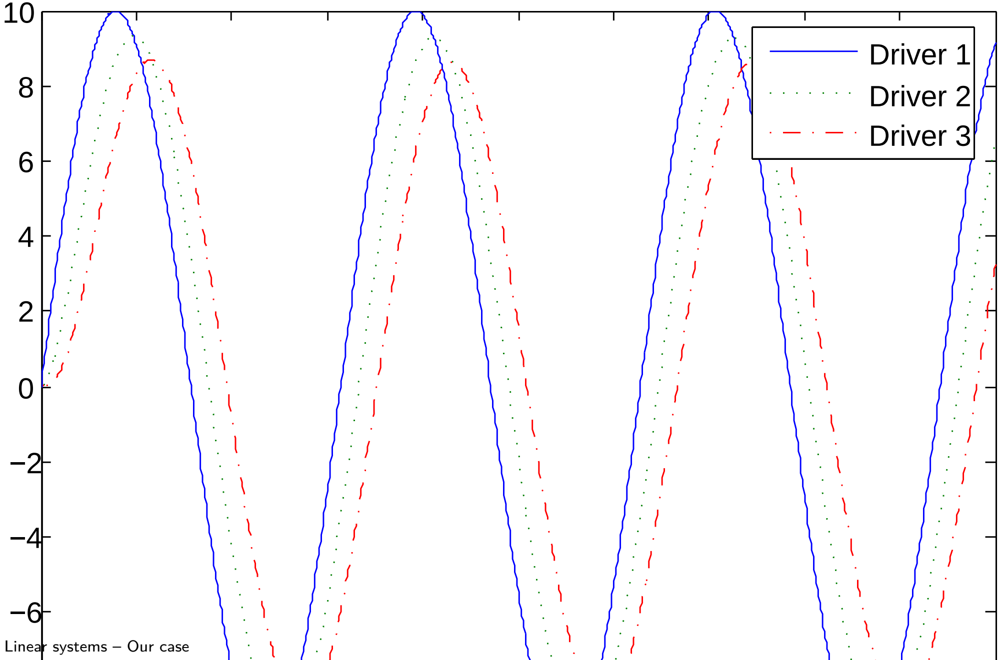

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# A delay differential equations model

- In the previous model, reaction time is instantaneous.
- In practice, this is known to be incorrect: reflexes and psychology play a role.
- It takes at least a few instants to acknowledge a change of speed in the car in front.
- If the change of speed is not threatening, then you may not want to react right away.
- When you press the accelerator or the brake, there is a delay between the action and the reaction..

# A delayed model of traffic flow

We consider the same setting as previously, except that now, for $t > 0$,

$$u'_{n+1}(t) = \lambda(u_n(t-\tau) - u_{n+1}(t-\tau)), \tag{5}$$

for $n = 1, \ldots, N-1$. Here, $\tau \geq 0$ is called the *time delay* (or *time lag*), or for short, *delay* (or *lag*).

If $\tau = 0$, we are back to the previous model.

# Initial data

For a delay equation such as (5), the initial conditions become *initial data*. This initial data must be specified on an interval of length $\tau$, left of zero.

This is easy to see by looking at the terms: $u(t-\tau)$ involves, at time $t$, the state of $u$ at time $t-\tau$. So if $t<\tau$, we need to know what happened for $t \in [-\tau, 0]$.

So, normally, we specify initial data as

$$u_n(t) = \phi(t) \text{ for } t \in [-\tau, 0],$$

where $\phi$ is some function, that we assume to be continuous. We assume $u_1(t)$ is known.

Here, we assume, for $n = 1, \ldots, N$,

$$u_n(t) = 0, \qquad t \leq (n-1)\tau$$

# Important remark

Although (5) looks very similar to (1), you must keep in mind that it is in fact much more complicated.

- A solution to (1) is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (or to $\mathbb{R}^n$ if we consider the system).
- A solution to (5) is a continuous function in the space of continuous functions.
- The space $\mathbb{R}^n$ has dimension $n$. The space of continuous functions has dimension $\infty$.

We can use the Laplace transform to get some understanding of the nature of the solutions.

# The Laplace transform

**Definition 11 (Laplace transform)**

Let $f(t)$ be a function defined for $t \geq 0$. The *Laplace transform* of $f$ is the function $F(s)$ defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The Laplace transform is a linear operator:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

# Rules of transformation

| $t$-domain | $s$-domain |
|---|---|
| $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| $tf(t)$ | $-F'(s)$ |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ |
| $f'$ | $sF(s) - f(0)$ |
| $f''$ | $s^2 F(s) - sf(0) - f'(0)$ |
| $f^{(n)}$ | $s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$ |
| $\frac{f(t)}{t}$ | $\int_s^\infty F(u)du$ |
| $\int_0^t f(u)du = u(t) * f(t)$ | $\frac{1}{s} F(s)$ |
| $f(at)$ | $\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$ |
| $e^{at} f(t)$ | $F(s-a)$ |
| $f(t-a)u(t-a)$ | $e^{-as} F(s)$ |
| $f(t) * g(t)$ | $F(s)G(s)$ |

Here $f^{(n)}$ represents the $n$th derivative, not the $n$th iterate. $*$ is the convolution product.

# Dirac delta – Heaviside function

In the table on the following slide,

- $\delta(t)$ is the Dirac delta,

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0. \end{cases}$$

- $H(t)$ is the Heaviside function,

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Note that $H(t) = \int_{-\infty}^{t} \delta(s) ds$.

# Transforms of common functions

| $t$-domain | $s$-domain |
|---|---|
| $\delta(t)$ | $1$ |
| $\delta(t-\tau)$ | $e^{-\tau s}$ |
| $H(t)$ | $\frac{1}{s}$ |
| $H(t-\tau)$ | $\frac{e^{-\tau s}}{s}$ |
| $\frac{t^n}{n!}H(t)$ | $\frac{1}{s^{n+1}}$ |
| $e^{-\alpha t}H(t)$ | $\frac{1}{s+\alpha}$ |
| $\sin(\omega t)H(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)H(t)$ | $\frac{s}{s^2+\omega^2}$ |

# Inverse Laplace transform

## Definition 12

Given a function $F(s)$, if there exists $f(t)$, continuous on $[0, \infty)$ and such that

$$\mathcal{L}\{f\} = F,$$

then $f(t)$ is the *inverse Laplace transform* of $F(s)$, and is denoted $f = \mathcal{L}^{-1}\{F\}$.

## Theorem 13

*The inverse Laplace transform is a linear operator. Assume that* $\mathcal{L}^{-1}\{F_1\}$ *and* $\mathcal{L}^{-1}\{F_2\}$ *exist, then*

$$\mathcal{L}^{-1}\{aF_1 + bF_2\} = a\mathcal{L}^{-1}\{F_1\} + b\mathcal{L}^{-1}\{F_2\}.$$

# Solving differential equations using the Laplace transform

1. Take the Laplace transform of both sides of the equation.
2. Using the initial conditions, deduce an algebraic system of equations in $s$-space.
3. Solve the algebraic system in $s$-space.
4. Take the inverse Laplace transform of the solution in $s$-space, to obtain the solution of the differential equation in $t$-space.

Let

$$U_{k+1}(s) = \mathcal{L}\{u_{k+1}(t)\} = \int_0^\infty e^{-st} u_{k+1}(t) dt.$$

Since we have assumed initial data of the form

$$u_n(t) = 0 \qquad \text{for } t \leq (n-1)\tau,$$

we have

$$U_{k+1}(s) = \int_{k\tau}^\infty e^{-st} u_{k+1}(t) ds.$$

Since $u_{n+1}(t) = 0$ for $t \leq n\tau$,

$$\int_0^\infty e^{-st} u'_{n+1}(t) dt = \left[u_{k+1}(t) e^{-st}\right]_{k\tau}^\infty + s \int_{k\tau}^\infty e^{-st} u_{k+1}(t) dt$$
$$= sU_{k+1}(s)$$

and

$$\int_0^\infty e^{-st} u_{k+1}(t-\tau) dt = \int_{(k-1)\tau}^\infty e^{-st} u_{k+1}(t-\tau) dt$$
$$= \int_{(k-2)\tau}^\infty e^{-s(t+\tau)} u_k(\tau) d\tau$$
$$= e^{-s\tau} U_k(s),$$

since $e^{-st} u_{k+1}(t) \to 0$ for the improper integral to exist.
Note that we could have obtained this directly using the properties of the Laplace transform.

Multiply

$$u'_{n+1}(t) = \lambda(u_n(t-\tau) - u_{n+1}(t-\tau))$$

by $e^{-st}$,

$$e^{-st}u'_{n+1}(t) = \lambda e^{-st}(u_n(t-\tau) - u_{n+1}(t-\tau))$$

integrate over $(0, \infty)$ (using the expressions found above),

$$sU_{n+1}(s) = \lambda(e^{-s\tau}U_n(s) - e^{-s\tau}U_{n+1}(s))$$

which is equivalent to

$$U_{n+1}(s) = \frac{\lambda U_n(s)}{\lambda + se^{s\tau}}$$

Thus, when $U_1(s)$ is known, we can deduce the values for all $U_n$.

Suppose

$$u_1(t) = \alpha \sin(\omega t)$$

From the table of Laplace transforms, it follows that

$$U_1(s) = \alpha \frac{\omega}{s^2 + \omega^2}$$

Therefore,

$$U_2 = \frac{\lambda U_1(s)}{\lambda + se^{st}} = \alpha \frac{\lambda}{\lambda + se^{st}} \, \frac{\omega}{s^2 + \omega^2}$$

and we can continue..

However, even though we know the solution in $s$-space, it is difficult to get the behavior in $t$-space, by hand, and maple does not help us either.