
This is a 180 minutes exam, with 6 questions for a total of 115 marks. Lecture Notes are allowed. PLEASE SHOW YOUR WORK CLEARLY. A correct answer without explanation will not get full marks.

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1. (20 points) **(An arms race model)** Two countries are involved in an arms race. Let $x_i(t)$ be the defense expenditure of country i at time t. For a given country $i = 1, 2$, in the absence of the other country, the expenditure tends to decrease at a rate $-a_{ii}$ proportional to the current expenditure. Each country also spends a fixed amount p_i on military equipment. When other countries are present, country i buys military equipment at a rate a_{ij} proportional to the expenditure of country j.
   (a) Explain briefly why the model
   $$\begin{aligned} x_1' &= -a_{11}x_1 + a_{12}x_2 + p_1 \\ x_2' &= a_{21}x_1 - a_{22}x_2 + p_2 \end{aligned} \tag{1}$$
   with $a_{ij} > 0$, is an appropriate model for the arms race described above.
   (b) Write (1) in vector form.
   (c) Discuss the well-posedness of (1) (existence and uniqueness of solutions, nonnegativity, boundedness..).
   (d) Suppose first that $p_i = 0$ for $i = 1, 2$. Determine the equilibrium of (1).
   (e) From now on, assume that $p_i > 0$. Determine the equilibrium of (1).
   (f) The matrix of the system in vector form is diagonalizable. Find this diagonal form, and express the general solution to (1).

2. (20 points) The probability that a machine functions without failure today is:
   i) 0.7 if the machine functioned without failure yesterday and the day before yesterday,
   ii) 0.5 if the machine functioned without failure yesterday, but not the day before yesterday,
   iii) 0.4 if the machine functioned without failure the day before yesterday, but not yesterday,
   iv) 0.2 if the machine had failures yesterday and the day before yesterday.

   Answer the following questions.

(a) Explain why the chain with two states, "machine functions without failure at day n" and "machine not functions without failure at day n" is not a Markov chain.

(b) What states should you consider to have a Markov chain? From now on, consider the Markov chain with those states.

(c) Find the transition matrix associated to the Markov chain.

(d) Draw the transition graph associated to the Markov chain.

(e) Is the chain regular?

(f) Calculate the probability that the machine will function correctly tomorrow, given that it functioned correctly yesterday and the day before yesterday.

3. (15 points) Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ \alpha & 1-\alpha & 0 \\ 0 & \alpha & 1-\alpha \end{pmatrix},$$

where $0 \leq \alpha \leq 1$.

(a) For what values of α is the Markov chain regular?

(b) Calculate the equilibrium probability distribution for the values of α found in (a).

4. (10 points) Show that the two-dimensional system

$$\begin{aligned} x_{t+1} &= x_t(1 + x_t + y_t)/3 \\ y_{t+1} &= y_t(1 - x_t + y_2)/2 \end{aligned}$$

has four fixed points, but only one that is locally stable.

5. (20 points) Consider the following competition model for two species with population sizes x and y:

$$\begin{aligned} x' &= 2x(1 - 3x - y) \\ y' &= 6y(1 - 2x - 4y). \end{aligned} \tag{2}$$

(a) Discuss the well-posedness of (2).

(b) Sketch the phase plane, including the nullclines and equilibria. Show the general direction of flow at different parts of the phase plane.

(c) Study the stability of the equilibria.

(d) If $x(0), y(0) > 0$, find $\lim_{t\to\infty}(x(t), y(t))$, if this limit exists.

6. (30 points) **(Spreading rumors)** Construct a model for spreading rumors making use of the following assumptions. The total population size N is constant with no one entering or leaving the population. The population is split into three subgroups: the uncool (who have not heard the rumor), the rumor spreaders, and the cool (who have heard the rumor, but are no longer spreading it). Let the sizes of these groups be given by U, R, and C respectively.

   As soon as someone in the uncool group hears the rumor, they enter the group of rumor spreaders. Each rumor spreader tells the rumor at rate k. Thus, the rate at which the rumor is being told is k times the number of rumor spreaders. So, the rate at which uncool people hear the rumor is k times the number of rumor spreaders times the *fraction* of the population consisting of uncool people.

   When a rumor spreader tells the rumor to someone who has already heard it (i.e. someone in the rumor spreading group or the cool group), they stop spreading the rumor and enter the cool group. Thus, the rate at which rumor spreaders enter the cool group is k times the number of rumor spreaders times the *fraction* of the population consisting of rumor spreaders and cool people.

   (a) Draw the transfer diagram.

   (b) Write down the differential equations.

   (c) Discuss the well-posedness of your system.

   (d) Rewrite the equations for U' and R' without using the variable C by making the substitution $C = N - U - R$.

   (e) Find the equilibria for the two dimensional system given in part (d).

   (f) Suppose that when the rumor starts, it is initially known by a very small fraction (much less than half) of the population and that this group of people are all rumor spreaders. Suppose that everyone else in the population is in the uncool group.

      (a) Initially, this should give $R' > 0$ and R should increase. Check that your equations in part (d) support this.

      (b) Then U will decrease (from nearly N) until the rumor dies out. Check that your equations in part (d) support this.

      (c) At what value of U does R stop increasing?

      (d) Eventually the rumor will die out when R goes to zero. When the rumor dies out, are there more people in the uncool group or in the cool group?

   (e) For bonus marks, study the stability of the equilibria of the system found in part (d).

   (f) Consider the state variables u, r and c representing the *proportion* of U, R and C in the population. Write your system in terms of these variables.