Green Lights & Traffic Humps

Matt Greenberg

In this paper we attempt to construct a model describing traffic flow on a long, single lane road. Our treatment of this problem is macroscopic in that we do not concern ourselves with individual cars, but instead treat traffic as a "density wave" moving at a certain velocity. We will attempt to use our model to predict what happens when a traffic light at an intersection turns green, allowing traffic to proceed. We will also look at our model's (disappointing) reaction to traffic humps. Throughout our treatment, x will denote the position on the road, and $t > 0$ will denote time, both measured in some convenient units.

Our model is described in terms of three fundamental quantities: The *flow rate*, $q(x,t)$ is the number of cars passing through x per unit time at a time t. Thus if $t_1 < t_2$,

$$\int_{t_1}^{t_2} q(x,t)\,dt = \text{the number of cars passing through position } x \text{ during the time interval } [t_1, t_2].$$

The *car density* $k(x,t)$ is the number of cars per unit road at position x at time t. If $x_1 < x_2$, then

$$\int_{x_1}^{x_2} k(x,t)\,dt = \text{the number of cars on the stretch of road between } x_1 \text{ and } x_2 \text{ at time } t.$$

We will let $u(x,t)$ denote the velocities of the cars located at position x at a time t.

In order to describe the traffic flow quantitatively, we need to derive mathematical relationships between $q(x,t)$, $k(x,t)$, and $u(x,t)$. We begin by developing a simple conservation law.

Let $t_1 < t_2$ and $x_1 < x_2$. Then

$$\begin{pmatrix}\text{\# of cars in } [x_1,x_2] \\ \text{at time } t_2\end{pmatrix} = \begin{pmatrix}\text{\# of cars in } [x_1,x_2] \\ \text{at time } t_1\end{pmatrix} \quad \ldots\ldots (1)$$

$$+ \begin{pmatrix}\text{\# of cars entering} \\ [x_1,x_2] \text{ in } [t_1,t_2]\end{pmatrix} - \begin{pmatrix}\text{\# of cars leaving} \\ [x_1,x_2] \text{ in } [t_1,t_2]\end{pmatrix}.$$

ASSUMPTION. All cars are moving in the same direction, say in the direction of increasing x.

Two consequences of this assumption are

$$\begin{pmatrix}\text{\# of cars entering} \\ [x_1,x_2] \text{ in } [t_1,t_2]\end{pmatrix} = \begin{matrix}\text{flux through } x_1 \\ \text{in } [t_1,t_2]\end{matrix} = \int_{t_1}^{t_2} q(x_1,t)\,dt,$$

$$\begin{pmatrix}\text{\# of cars leaving} \\ [x_1,x_2] \text{ in } [t_1,t_2]\end{pmatrix} = \begin{matrix}\text{flux through } x_2 \\ \text{in } [t_1,t_2]\end{matrix} = \int_{t_1}^{t_2} q(x_2,t)\,dt.$$

Thus we can reformulate (1) more quantitatively:

$$\int_{x_1}^{x_2} k(x,t_2)\,dx - \int_{x_1}^{x_2} k(x,t_1)\,dx = \int_{t_1}^{t_2} q(x_1,t)\,dt - \int_{t_1}^{t_2} q(x_2,t)\,dt. \quad (2)$$

or $\displaystyle\int_{x_1}^{x_2}\left[k(x,t_2)-k(x,t_1)\right]dx=\int_{t_1}^{t_2}\left[q(x_1,t)-q(x_2,t)\right]dt.$

If we assume suitable differentiability conditions on k and q, we can write this as

$$\int_{x_1}^{x_2}\left(\int_{t_1}^{t_2}k_t(x,t)\,dt\right)dx=-\int_{t_1}^{t_2}\left(\int_{x_1}^{x_2}q_x(x,t)\,dx\right)dt$$

or changing order of integration,

$$\iint_{[x_1,x_2]\times[t_1,t_2]}\left(k_t(x,t)+q_x(x,t)\right)dA=0 \quad\ldots (3)$$

for <u>every</u> choice of $x_1<x_2$ and $t_1<t_2$.
Since k_t+q_x can be assumed to be continuous, and (3) holds for every choice of $x_1<x_2$ and $t_1<t_2$, we must have

$$k_t(x,t)+q_x(x,t)=0 \quad\ldots (4).$$

Even if we know the initial traffic density $k_0(x)=k(x,0)$, we cannot solve for $k(x,t)$ with (4) unless we have some explicit relationship between k and q. As a first step in this direction it is easy to see that $q(x,t)=k(x,t)\cdot u(x,t)$. It is reasonable to assume that car velocity is dependent upon local vehicle density; if traffic is heavy, cars move slowly. Suppose we let K_j denote "traffic jam density," i.e. if $k(x,t)=K_j$, then $u(x,t)=0$. Let's abuse the simplicity principle and assume

$$u(x,t)=\mu\left(K_j-k(x,t)\right) \quad\ldots (5)$$

This is the simplest possible functional form that will give us $u=0$ when $k=K_j$ and $u>0$ when $k<K_j$. Thus we have postulated that

$$q(x,t) = k(x,t)\,u(x,t) = \mu k(x,t)\left[K_j - k(x,t)\right],$$

or less messily, $q = \mu k(K_j - k) \quad \cdots \quad (6)$

Using the chain rule, (4) becomes

$$k_t + c(k)\,k_x = 0 \quad \text{where}$$

$$c(k) = \frac{dq}{dk} = \mu(K_j - 2k) \quad \cdots\cdots \quad (7).$$

This is a quasi-linear first order PDE and is best attacked using the method of characteristics.

To use the method of characteristics to solve this PDE, we must investigate the sets of points in the x-t plane on which k is constant. The curves along which k is constant are known as characteristics (or characteristic curves) of the PDE. Take a curve $x(t)$. Along this curve, $k(x,t) = k(x(t),t)$ and

$$\frac{dk}{dt} = \frac{dx}{dt}\,k_x + k_t$$

by the chain rule.

The curve $x(t)$ is a characteristic iff $k(x(t),t)$ is constant, i.e. $dk/dt = 0$. But since k is assumed to satisfy (7), we have that $x(t)$ is a characteristic iff

$$\frac{dx(t)}{dt} = c(k(x,t)). \quad \cdots \quad (8)$$

Suppose we know the initial traffic density function $k_0(x) = k(x,t)$. Let $x(0) = \eta$. Then since k is constant on $x(t)$,

$$c(k(x(t),t)) = c(k(\eta,0)) = c(k_0(\eta)).$$

Integrating (8), we have

$$x = c(k_0(\eta))t + \eta. \cdots \quad (9)$$

Equation (9) defines η implicitly as a function of x and t wherever the hypotheses of the implicit function theorem are satisfied. Thus if we can solve for $\eta = \eta(x,t)$, then

$$k(x,t) = k_0(\eta(x,t)) \quad \cdots \quad (10)$$

and we have solved the system (7).

We are now prepared to test our model...

THE GREEN LIGHT PROBLEM

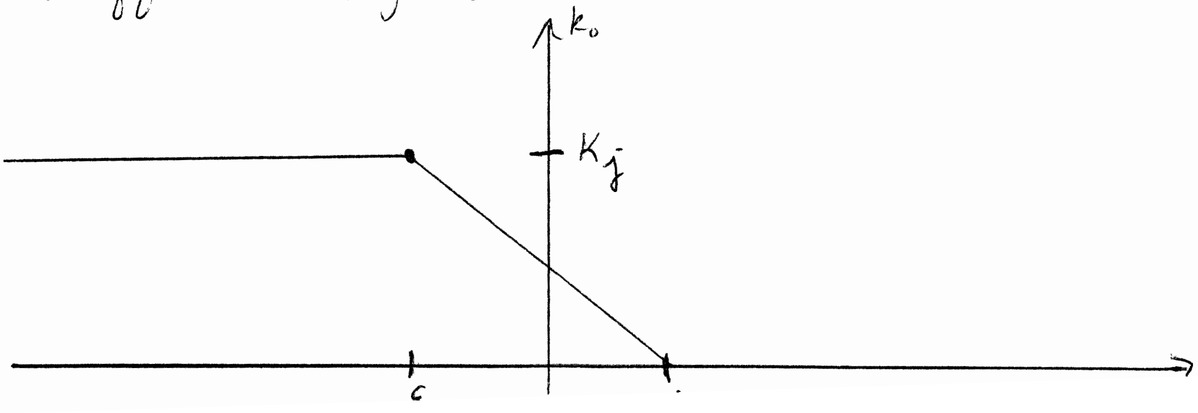
Suppose a traffic light is located at $x=0$ and it turns green at $t=0$. Cars should begin to move and traffic should "thin out." To study the green light problem, it seems natural to define an initial traffic density by

$$k_0(x) = \begin{cases} K_j & \text{if } x \le 0 \\ 0 & \text{if } x > 0. \end{cases}$$

This would cause difficulties, though, due to the discontinuity at $x=0$. We will take a slightly modified approach; we define the initial traffic density k_0 by

$$k_0(x) = \begin{cases} K_j & \text{if } x < -\varepsilon \\ \frac{K_j}{2}\left(1-\frac{x}{\varepsilon}\right) & \text{if } |x| \le \varepsilon \\ 0 & \text{if } x > \varepsilon \end{cases}$$

where ε is an arbitrary positive number. We will be interested, of course, in the limiting case where $\varepsilon \to 0^+$. The initial traffic density looks like this:



As described earlier, to solve for $k(x,t)$, we must first solve the equation

$$x = c(k_0(\eta))t + \eta$$

for η. Since k_0 is defined piecewise, we consider three cases:

CASE 1. $\eta < -\varepsilon$.

If $\eta < -\varepsilon$, then $k_0(\eta) = K_j$. Therefore,

$$x = c(K_j)t + \eta = -\mu K_j t + \eta$$

$$\Longleftrightarrow \quad \eta = x + \mu K_j t$$

So $x < -\mu K_j t - \varepsilon \Rightarrow k(x,t) = k_0(\eta(x,t))$
$= K_j$

CASE 2. $-\varepsilon \le \eta \le \varepsilon$.

If $-\varepsilon \le \eta \le \varepsilon$, then $k_0(\eta) = \frac{K_j}{2}\left(1 - \frac{\eta}{\varepsilon}\right)$. Therefore,

$$x = c\left(\frac{K_j}{2}\left(1 - \frac{\eta}{\varepsilon}\right)\right)t + \eta$$

$$= \frac{\eta}{\varepsilon}(\mu K_j t + \varepsilon).$$

$$\Longleftrightarrow \quad \eta = \frac{\varepsilon x}{\mu K_j t + \varepsilon}.$$

So $|x| \leq \mu K_j t + \varepsilon \Rightarrow$

$$k(x,t) = k_0(\eta(x,t)) = \frac{K_j}{2}\left(1 - \frac{x}{\mu K_j t + \varepsilon}\right)$$

CASE 3. $\eta > \varepsilon$

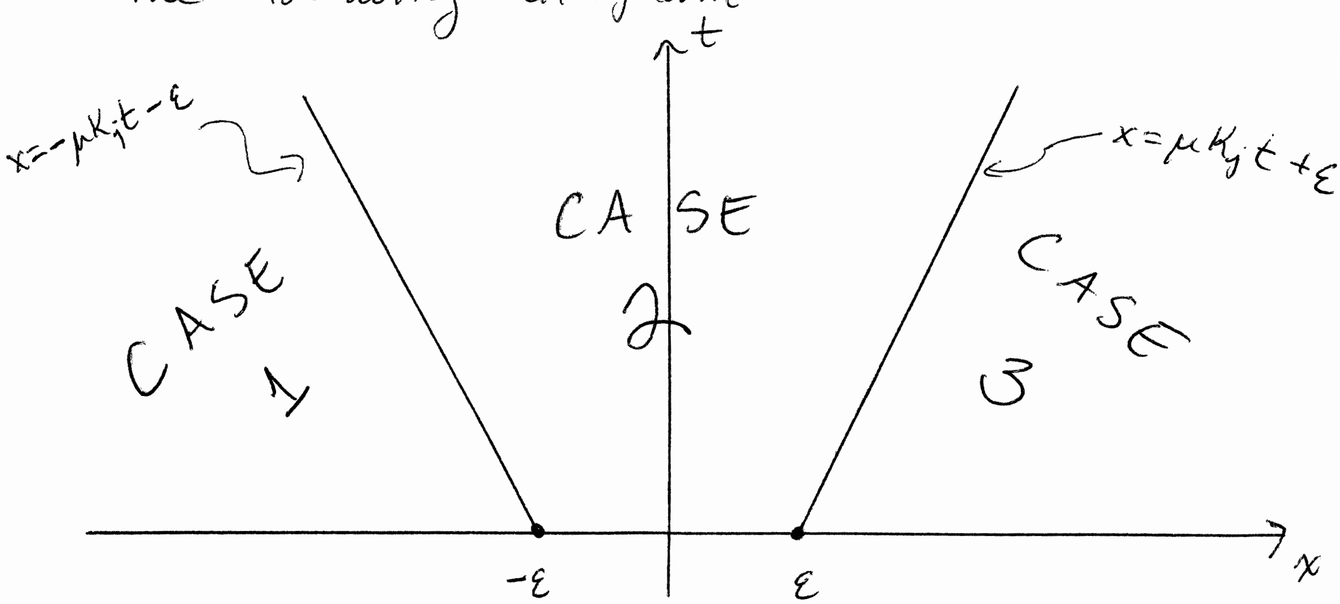
If $\eta > \varepsilon$, then $k_0(\eta) = 0$. Therefore,

$$x = c(0)t + \eta = \mu K_j t + \eta.$$

$$\Leftrightarrow \quad \eta = x - \mu K_j t.$$

So $x > \mu K_j t + \varepsilon \Rightarrow k(x,t) = k_0(\eta(x,t)) = 0$.

Cases 1, 2, and 3 define $k(x,t)$ for all x, and for all $t > 0$ according to the following diagram:



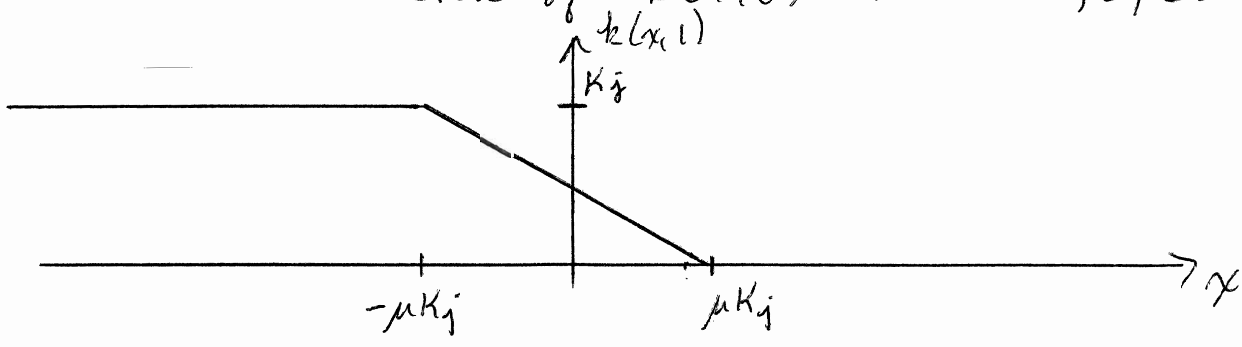
In summary, our solution $k(x,t)$ is given by

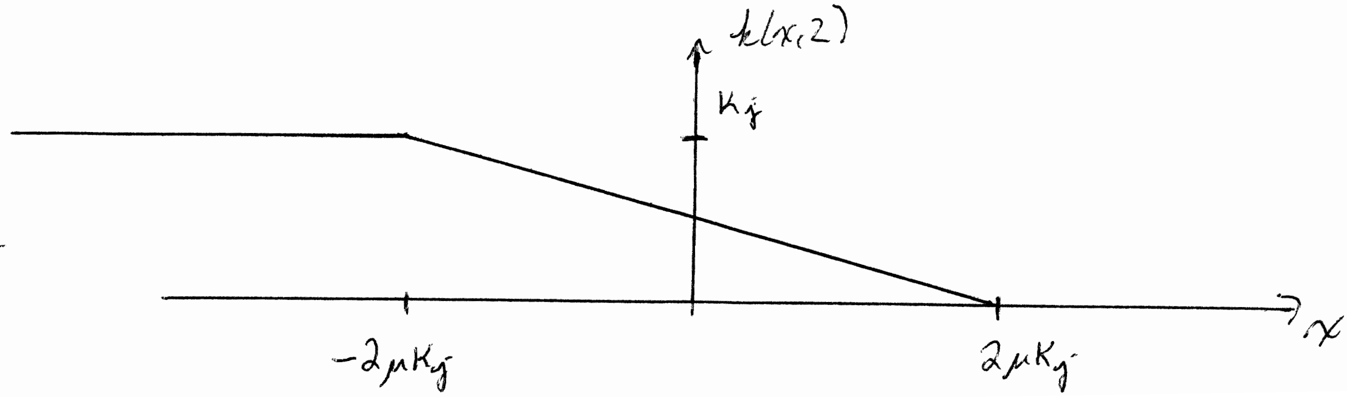
$$k(x,t) = \begin{cases} K_j & \text{if } x < -\mu K_j t - \varepsilon \\ \frac{K_j}{2}\left(1 - \frac{x}{\mu K_j t + \varepsilon}\right) & \text{if } |x| \leq \mu K_j t + \varepsilon \\ 0 & \text{if } x > \mu K_j t + \varepsilon \end{cases}$$

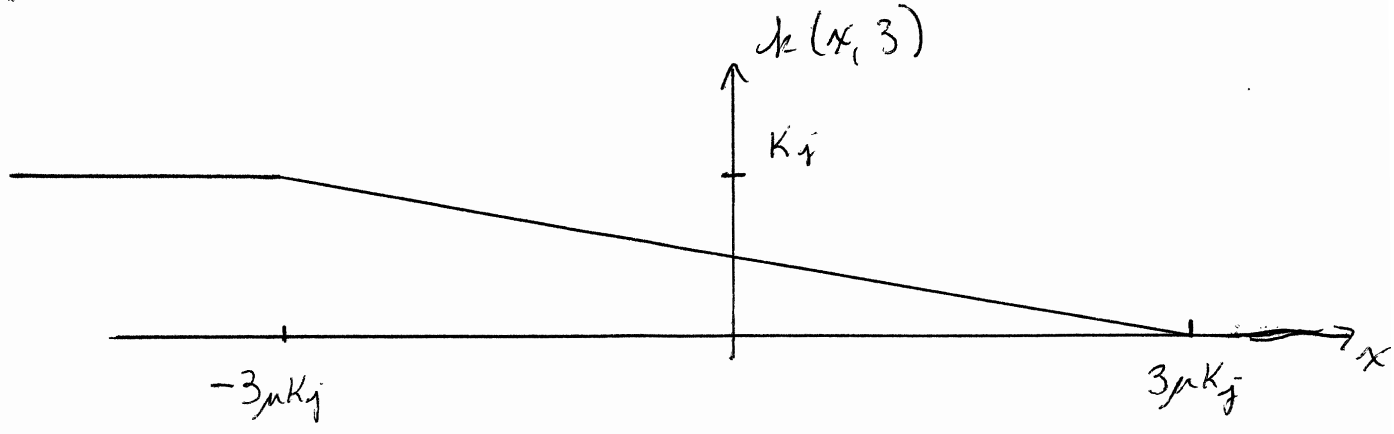
Letting $\varepsilon \to 0$, we get

$$k(x,t) = \begin{cases} K_j & \text{if } x < -\mu K_j t \\ \frac{K_j}{2}\left(1 - \frac{x}{\mu K_j t}\right) & \text{if } |x| \leq \mu K_j t \\ 0 & \text{if } x > \mu K_j t \end{cases}$$

Here is a sketch of $k(x,t)$ for $t=1,2,3$.







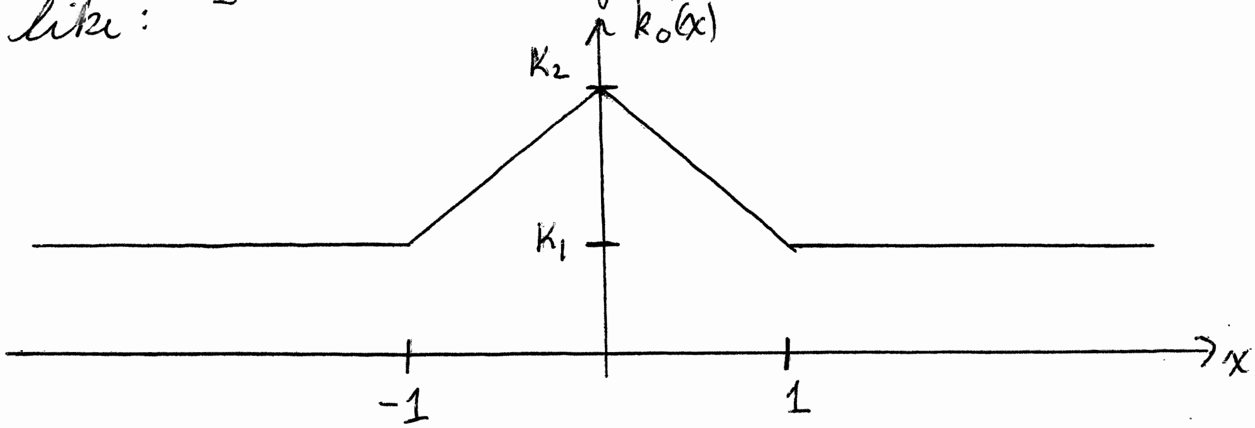
COMMENTS. Our solution does capture many of the qualitative features we were looking for in our solution; when the light turns green, the lead car begins to move and the cars behind follow causing the traffic density to thin out. There are several flaws inherent in this solution, though. Curiously, $k(0,t) = K_j/2$ for all $t > 0$. This fact is difficult to interpret physically and is certainly the result of the simplistic nature of our model. A more alarming deficiency in our model, though, is that it allows for instaneous accelleration. Since for every $\varepsilon > 0$, $k_0(\varepsilon) = 0$. Therefore, the lead car accellerates to a speed of μK_j instantly, since in our model, velocity is a function of local density alone. A refined model should not allow for such instantaneous accellerations. Further deficiencies in this model will become apparent as we study traffic humps.

TRAFFIC HUMPS.

In this section, we investigate the diffusion of traffic from traffic humps. As usual, we will let $k(x,t)$ denote the traffic density at a point x on the road at a time $t \geq 0$. Suppose at $t=0$, there is a hump in the traffic centred at $x=0$. Mathematically, we can describe this as follows. We define the initial traffic density $k_0(x) = k(x,0)$ by

$$k_0(x) = \begin{cases} K_1 & \text{if } |x| \geq 1 \\ K_2 - (K_2 - K_1)|x| & \text{if } |x| \leq 1 \end{cases}$$

where $K_1 < K_2$. The graph of $k_0(x)$ looks like:



For convenience, we let $\Delta K = K_2 - K_1$. Physically, ΔK represents the "height" of the traffic hump. There is no loss of generality in assuming that the traffic hump extends only from $x=-1$ to $x=+1$ since we could always simply rescale the x-axis in order to make this occur. We make one more assumption in order to simplify our analysis. We assume that initially for $|x| \geq 1$, the traffic is "light"; or to give this a precise representation, that $2K_1 < K_j$, where K_j, we recall, is traffic jam density.

To solve for $k(x,t)$ subject to $k(x,0) = k_0(x)$, we use the solution

$$k(x,y) = k_0(\eta(x,t))$$

where $\eta(x,t)$ is given implicitly in terms of x and t by

$$x = c(k_0(\eta))t + \eta.$$

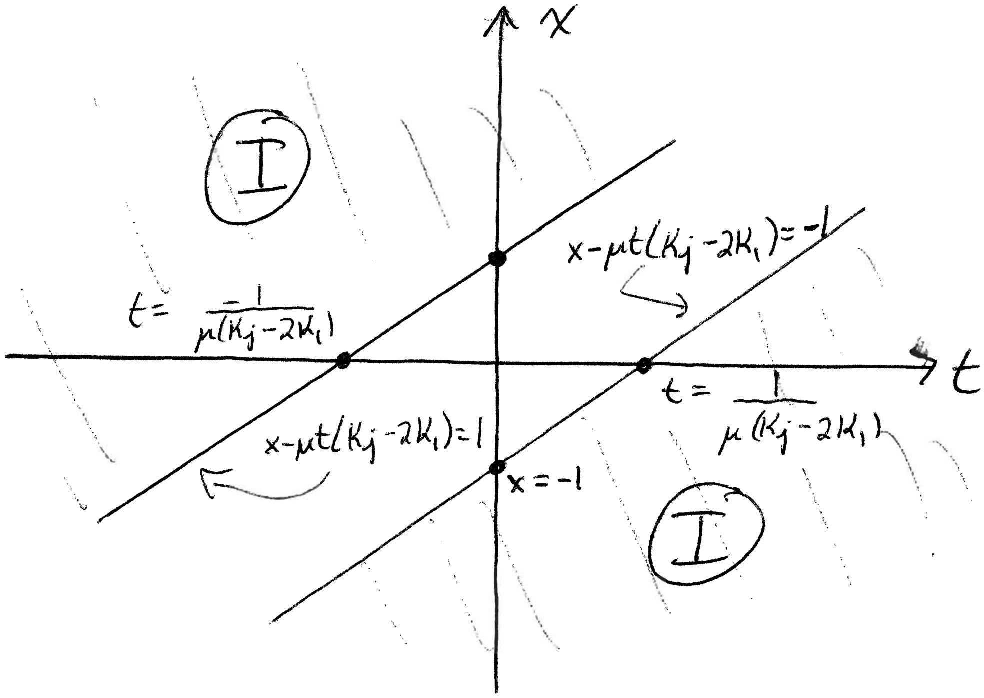
Recall that $c(k) = \mu(K_j - 2k)$.

If $|\eta| \geq 1$, then $k_0(\eta) = K_1$, so $\eta(x,y)$ is given by

$$x = c(K_1)t + \eta = \mu t(K_j - 2K_1) + \eta, \text{ or}$$

$$\eta = x - \mu t(K_j - 2K_1).$$

$|\eta| \geq 1$ implies $x - \mu t(K_j - 2K_1) \geq 1$ or $x - \mu t(K_j - 2K_1) \leq -1$. Let region I in the xt-plane be the region described by these two inequalities. We can draw region 1 like this.



Note that the slopes of the boundary lines $x-\mu t(K_j-2K_1)=\pm 1$ are positive because we assume $2K_1 < K_j$. For $\eta = x-\mu t(K_j-2K_1)$, $(x,t)\in I$, $k_0(\eta(x,y)) = K_1$.

Now suppose $-1 \le \eta \le 0$. Then $|\eta| = -\eta$, so

$$x = \mu t\left(K_j - 2(K_2+\eta\Delta K)\right)+\eta$$
$$= \eta(1-2\mu t\Delta K) + \mu t(K_j-2K_2).$$

Assuming $t \ne 1/2\mu\Delta K$, we have

$$\eta = \frac{x-\mu t(K_j-2K_2)}{1-2\mu t\Delta K}.$$

$$-1\le \eta \le 0 \Rightarrow -1 \le \frac{x-\mu t(K_j-2K_2)}{1-2\mu t\Delta K} \le 0$$

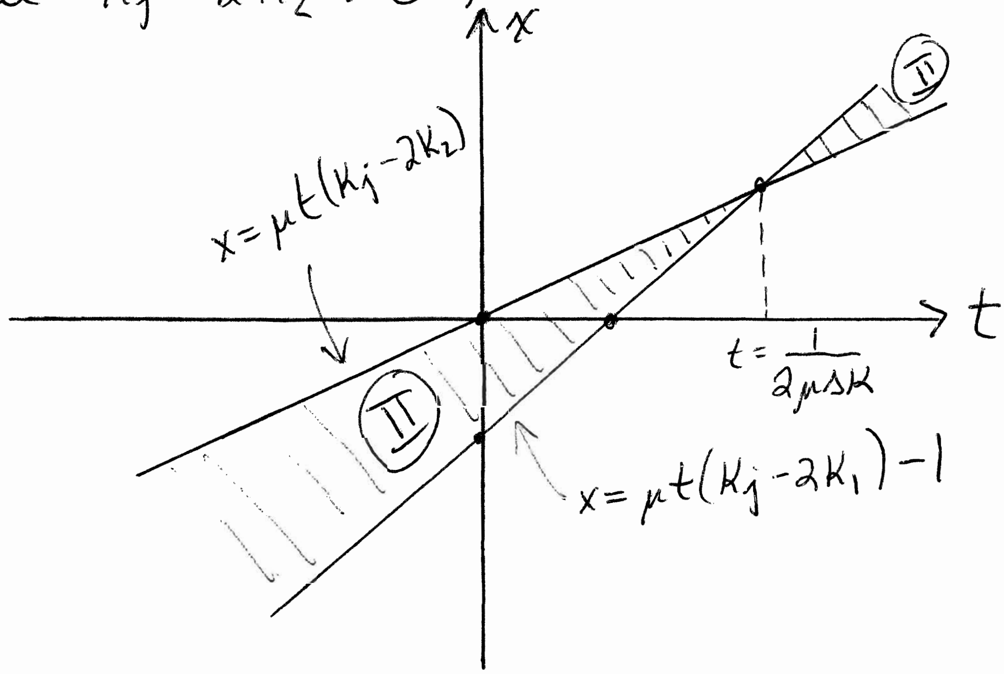
If $t < 1/2\mu\Delta K$, then this becomes (after a little algebraic manipulation)

$$\mu t(K_j-2K_1)-1 \le x \le \mu t(K_j-2K_2).$$

If $t > 1/2\mu\Delta K$, the inequalities are reversed, i.e.

$$\mu t(K_j - 2K_2) \le x \le \mu t(K_j - 2K_1) - 1$$

Let region II in the xt-plane be described by these inequalities. We draw region II as follows (here we only draw the case where $K_j - 2K_2 > 0$):



If $(x,t) \in \mathrm{II}$, then we have a solution
$\eta(x,t) = \dfrac{x - \mu t(K_j - 2K_2)}{1 - 2\mu t\Delta K}$ and

$$k(x,t) = k_0(\eta(x,t)) = K_2 + (\overbrace{K_2 - K_1}^{\Delta K})\left[\frac{x - \mu t(K_j - 2K_2)}{1 - 2\mu t \Delta K}\right].$$

Finally, let us assume that $0 \le \eta \le 1$. Then $|\eta| = \eta$ and $k_0(\eta) = K_2 - \Delta K \eta$.

Therefore, $x = c(K_2 - \eta \Delta K)t + \eta$
$= \eta(1 + 2\mu t \Delta K) + \mu t(K_j - 2K_2)$.

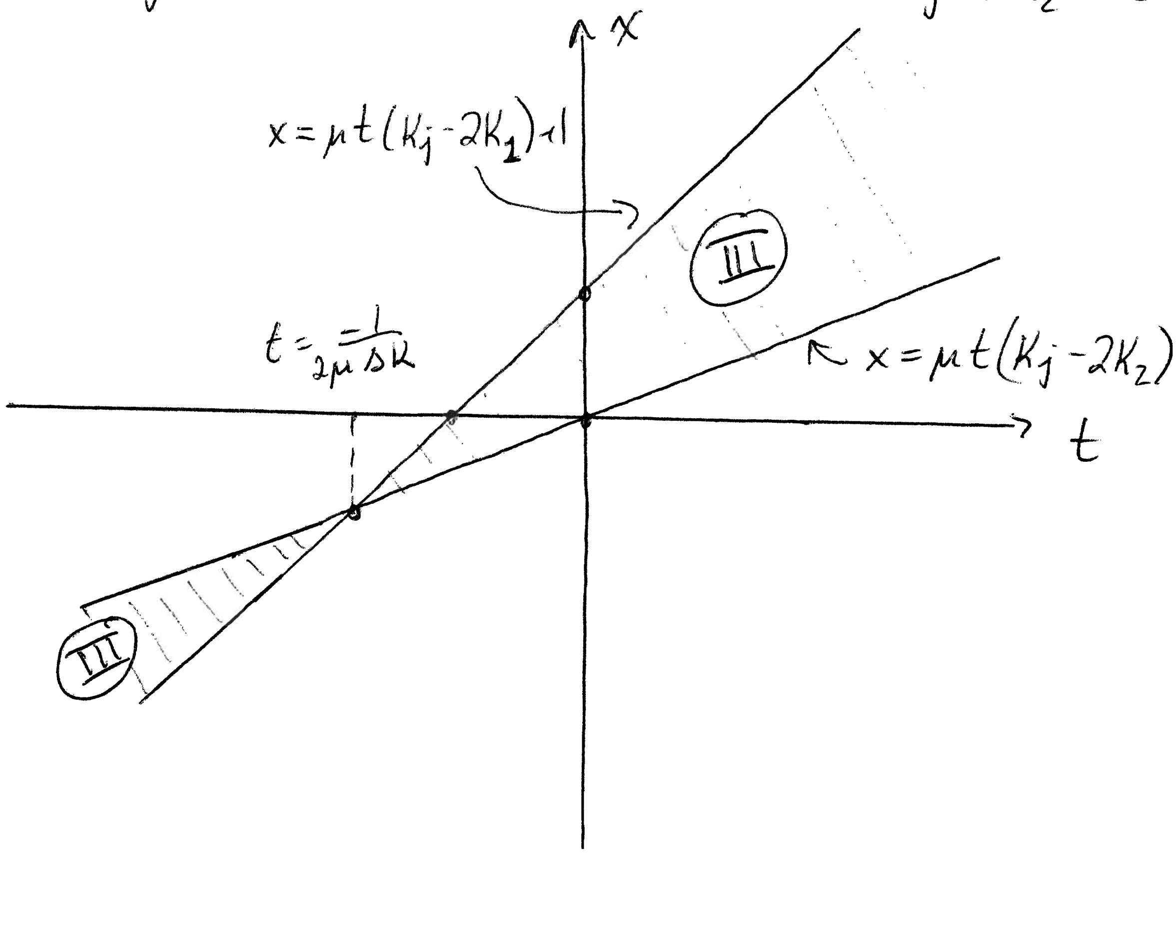
Solving, we have $\eta = \dfrac{x - \mu t(K_j - 2K_2)}{1 + 2\mu t \Delta K}$.

$0 \le \eta \le 1 \Longrightarrow \quad \mu t(K_j - 2K_2) \le x \le 1 + \mu t(K_j - 2K_1)$

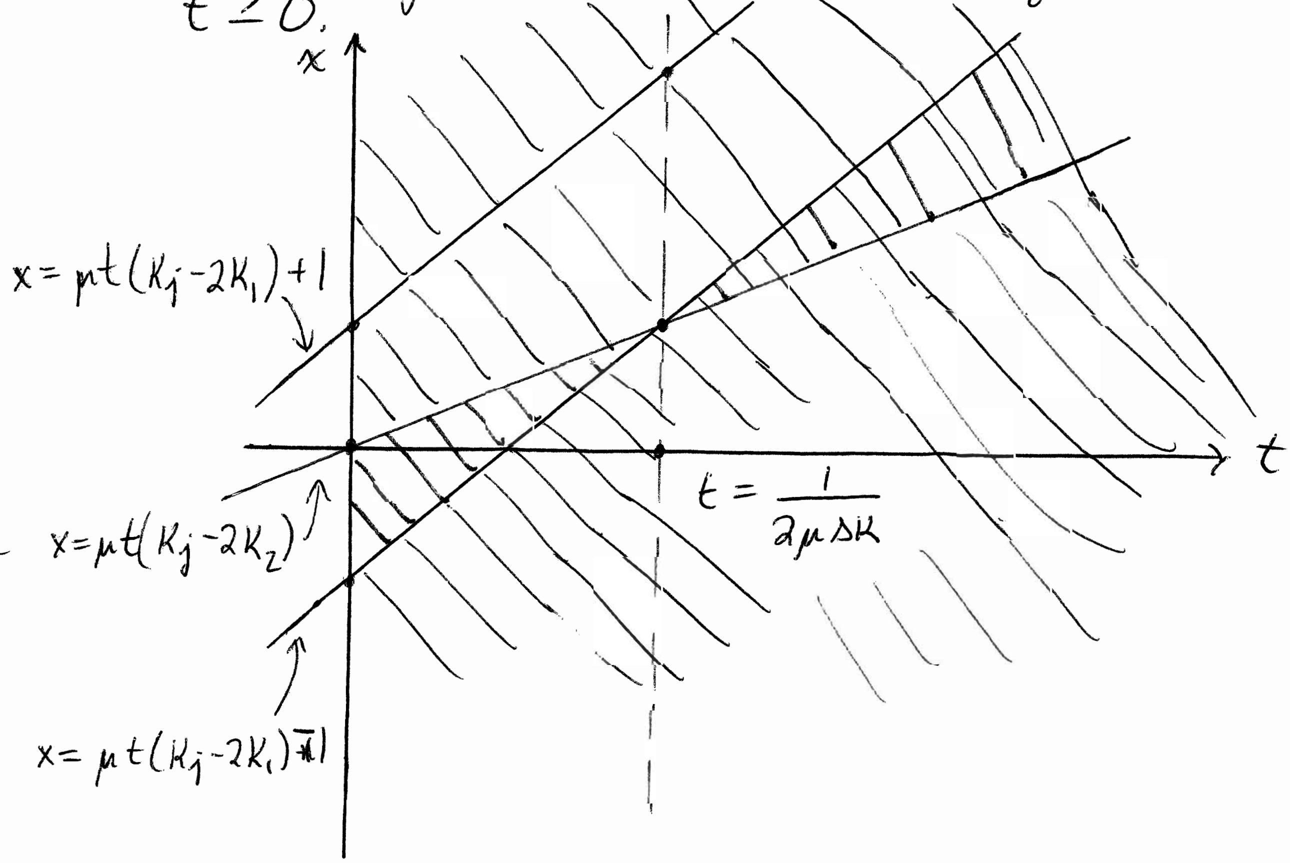
if $t > -1/2\mu\Delta K$ and

$$1 + \mu t(K_j - 2K_1) \le x \le \mu t(K_j - 2K_2)$$

if $t < -1/2\mu\Delta K$. Let region III in the xt-plane be described by these inequalities. We draw region III as follows. Again, we only draw the case where $K_j - 2K_2 > 0$.



Now that we have considered the relevant cases individually, let us plot regions I, II, and III on the same plane in order to reconcile our findings. We only plot our domain of interest, $t \geq 0$.



Red thatches: Region I
Green thatches: Region II
Blue thatches: Region III.

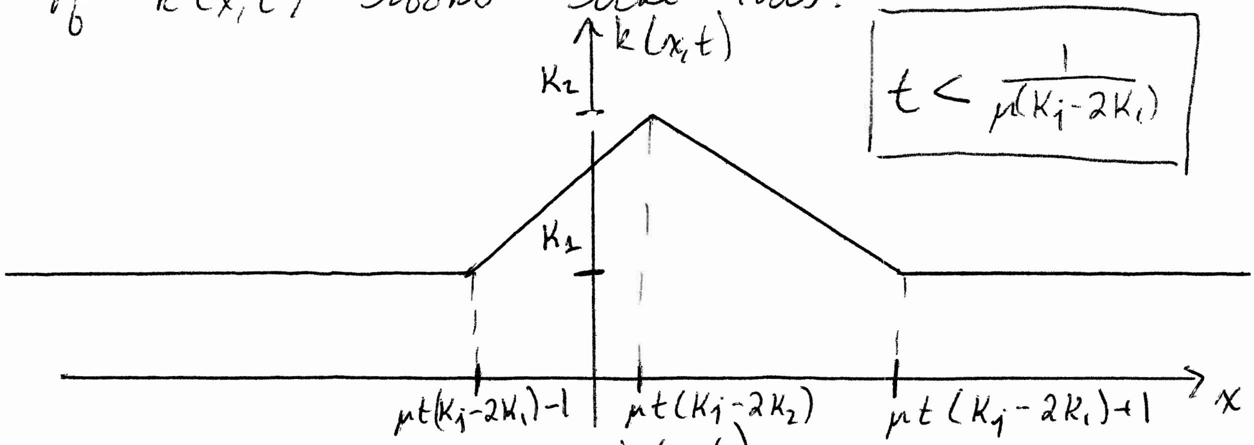
This diagram shows that if $(x,t) \in \mathbb{R} \times [0, \frac{1}{2\mu \Delta k}]$ then the equation

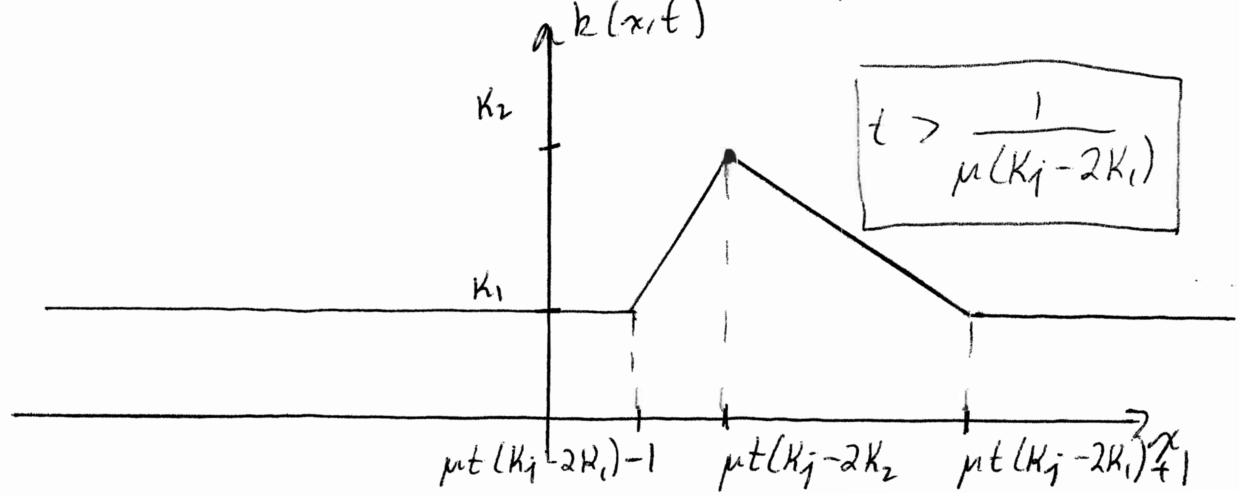
$$x = c(k(\eta))t + \eta$$

does define η implicitly as a function of x and t. We can use the expressions for η derived in the last few pages to produce an explicit solution to our traffic hump problem valid for $0 < t < 1/2\mu\Delta K$.

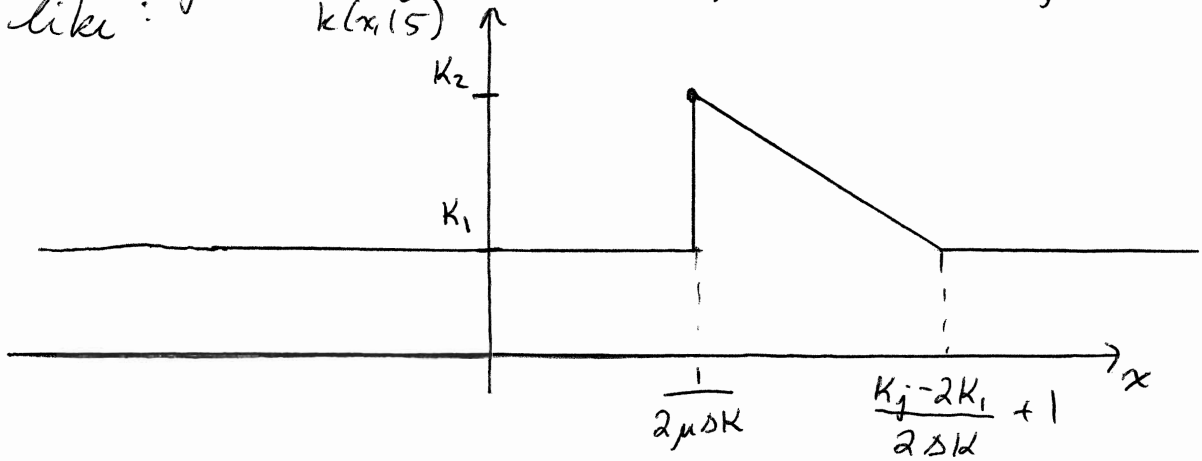
$$k(x,t) = \begin{cases} K_1 & \text{if } x \le \mu t(K_j - 2K_1) - 1 \text{ or } x \ge \mu t(K_j - 2K_1) + 1 \\ K_2 + \Delta K\left[\dfrac{x - \mu t(K_j - 2K_2)}{1 - 2\mu t \Delta K}\right] & \text{if } \mu t(K_j - 2K_1) - 1 \le x \le \mu t(K_j - 2K_2) \\ K_2 - \Delta K\left[\dfrac{x - \mu t(K_j - 2K_2)}{1 + 2\mu t \Delta K}\right] & \text{if } \mu t(K_j - 2K_2) \le x \le \mu t(K_j - 2K_1) + 1 \end{cases}$$

For a *fixed* $t \in [0, 1/2\mu\Delta K)$, the graph of $k(x,t)$ looks like this:





Taking the limit as $t \to 1/2\mu\Delta K^-$, we get the graph of $k(x, 1/2\mu\Delta K)$ looking like:



This finding is somewhat disturbing; it says that no matter what the height of the traffic hump, even if it is very small, the traffic density profile will <u>always</u> have a jump discontinuity at some finite time ($t = 1/2\mu\Delta K$).

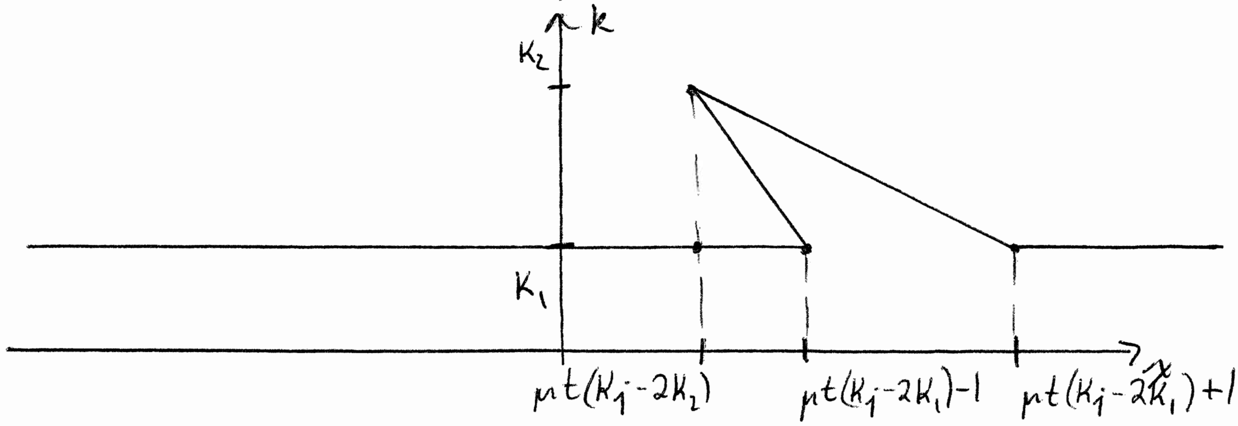
Also disturbing/interesting is that if $t > 1/2\mu\Delta K$, $\mu t(K_j - 2K_2) \le x \le \mu t(K_j - 2K_1) - 1$, we can no longer solve

$$x = c(k_0(\eta))\,t + \eta$$

uniquely for η. Since regions I, II, and <u>III</u> overlap there, we have three solutions for η. Consequently, $k(x,t)$ is <u>multivalued</u> in the region

$$t > 1/2\mu\Delta K,$$
$$\mu t(K_j - 2K_2) \le x \le \mu t(K_j - 2K_1) - 1.$$

A cross section of the solution surface at some $t > 1/2\mu s k_2$ looks like:



Having a multivalued solution is clearly nonphysical since at any time t the traffic density at x is uniquely determined. Thus our solution for $t > 1/2\mu s k$ should be ignored.

Our analysis shows that our model does not faithfully describe the traffic hump phenomenon. Experience leads us to believe that if the traffic hump is not too high, then eventually the traffic density should diffuse from the hump, eventually becoming uniform. Our analysis shows that this would never happen; we always get a shock appearing in the traffic density profile at $t = 1/2\mu s k_2$. But what is the deficiency in our model that prevents it from being able to faithfully describe the traffic hump phenomenon? In reality, drivers look ahead and if they see slower moving traffic, they will slow down in advance to prevent collisions.

good.

In our model, though, car velocity u is a function of $k(x,t)$, the car density at a specific point on the road at a given instant. Thus our model does not allow for looking ahead. Therefore, the cars do not notice the traffic hump until they are on top of it. This is why our model does not properly describe drivers' reactions to traffic humps. In order to allow for looking ahead, it would be reasonable to assume that car velocity $u(x,t)$ depends on $k_x(x,t)$. One modification is to say

$$q(k,k_x) = \mu k(K_j - k) - \frac{\nu}{2} k_x.$$

With this modification, our PDE becomes

$$k_t + c(k) k_x = \frac{\nu}{2} k_{xx}$$

which resembles the heat/diffusion equation. Due to this resemblance, this refined model would likely describe the traffic diffusion phenomena we have looked at better than the models we have been working with; but that is a project for another time...

References.

1. F. Wan, *Mathematical Models and their Analysis*, Harper and Row, New York. 1989.

2. P. O'Neil, *Beginning Partial Differential Equations*, Wiley-Interscience, New York, 1999.