ASSUMPTIONS, INCLUDING REASONS AND DESCRIPTION	10	[illegible]	7.
MATHEMATICAL ANALYSIS	20	a few math errors - very detailed calculations - mathematically complex!	15
CONCLUSIONS AND SUGGESTIONS FOR IMPROVEMENT	5		3
EXAMPLE(S)	5	yes!	4
REFERENCES	5		5
ADDITIONAL COMMENTS		TOTAL (OUT OF 60)	44.

Confusing # system.
does not understand "autonomous"

EPIDEMICS

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EPIDEMICS

Infectious diseases are responsible for many nonfatal or fatal cases each year, and are a large concern in todays society. The prevention and control of these numerous diseases is an important, pratical health problem. An infectious disease may become so common within a population that scientists classify it as an epidemic. An epidemic is described as a disease that is, "prevalent and spreading rapidly among many people in a community at the same time", (Websters New Word Dictionary,college edition,p.488). Hence, models descibing infectious diseases are studied by epidemologists to help control, predict and understand epidemics.The most basic model used for desribing highly infectious diseases is the SI model. Even more interesting and challenging is a modified version , called the Kermack-McKendrick model or the SIR model.

(fig. 1) $\boxed{S \xrightarrow{\beta} \cancel{R}\, I}$

The SI model is used for a disease that is highly infectious but not serious enough for cases to be withdrawn by death or permanent immunity . Any individual may transmit the disease almost essentially as soon as he/she is infected. Given a community of people, classify this group into two types ,those individuals who are susceptible to the disease and those who are infected. Denote this relationship as N=S(t)+I(t). At time t=0, assume that only one individual is infected with the disease, $I(t_0) = I(0) = I_0 = 1$ and $S(t_0) = N - I_0 = N - 1$. Thus the SI model, is written as

motivation?

$$1.\ \frac{dS}{dt} = -\beta S(t)I(t) \qquad\qquad 2.\ \frac{dI}{dt} = \beta S(t)I(t)$$

where β is the infection rate. Observing these two equations, $-\frac{dS}{dt} = \frac{dI}{dt} = \beta S(t)I(t)$. Writing (1) in terms of I only, using S=N-I, (2) becomes, $\frac{dI}{dt} = \beta(N-I)I$. (3). Solve (3) explicitly for I by the method of separation of variables, and integrate. $\int \frac{dI}{(N-I)I} = \int \beta dt$, (4) .In order to integrate

essentially logistic.

the left hand side of (4) implement the method of partial fractions to produce

$\int\left(\frac{A}{(N-I)}+\frac{B}{I}\right)dI=\int \beta dt$ where A and B are $\frac{1}{N}$.

Thus, $\int\left(\frac{1/N}{(N-I)}+\frac{1/N}{I}\right)dI=\int \beta dt$

$$\frac{1}{N}\int\left(\frac{1}{(N-I)}+\frac{1}{N}\right)dI=\int \beta dt$$

$$\int\left(\frac{1}{(N-I)}+\frac{1}{N}\right)dI=\int \beta N dt$$

$$-\ln|N-I|+\ln|I|=\beta Nt+C$$

$\ln\left|\frac{I}{N-I}\right|=\beta Nt+C$, with N>>0 and I>0

$\left|\frac{I}{N-I}\right|=e^{\beta Nt+C}$, let $K=e^{C}>0$ so that $\left|\frac{I}{N-I}\right|=Ke^{\beta Nt}$

$\frac{I}{N-I}=De^{\beta Nt}$ where $D=\pm K\neq 0$, (5)

Solve (5) for D by putting t=0, $D=\frac{I}{N-I}$ 1 1. Now applying the initial condition $I_0=1$,

$D=\frac{1}{N-1}$.

Substitute the value for D back into (5) and solve explicitly for I,

$$\frac{I}{N-I}=\frac{1}{N-1}e^{\beta Nt}$$

$$I=\frac{N-I}{N-1}e^{\beta Nt}=\frac{N}{N-1}e^{\beta Nt}-\frac{I}{N-1}e^{\beta Nt}$$

$$I\left(1+\frac{e^{\beta Nt}}{N-1}\right)=\frac{N}{N-1}e^{\beta Nt}$$

$$I\left(\frac{N-1+e^{\beta Nt}}{N-1}\right)=\frac{N}{N-1}e^{\beta Nt}$$

$$I = \left(\frac{N-1}{N-1+e^{\beta Nt}}\right)\left(\frac{N}{N-1}e^{\beta Nt}\right) = \frac{Ne^{\beta Nt}}{N-1+e^{\beta Nt}} = \frac{N}{(N-1)e^{-\beta Nt}+1}, \quad (6)$$

Taking the limit, of (6), $\lim_{t\to\infty} I = \lim_{t\to\infty}\frac{N}{(N-1)e^{-\beta Nt}+1} = \frac{N}{1} = N$ (7). Since evaluating the denominator, $e^{-\beta Nt}$ approaches zero as time goes to infinity. (i.e.,.. $\lim_{t\to\infty} e^{-\beta Nt} = 0$). This means as time proceeds everyone will contact the disease.

The number of susceptible individuals S(t), is found by ,

$$S(t) = N - I(t) = N - \frac{N}{(N-1)e^{-\beta Nt}+1} = \frac{N(N-1)e^{-\beta Nt}+N-N}{(N-1)e^{-\beta Nt}+1}$$

$S(t) = \frac{N(N-1)e^{-\beta Nt}}{(N-1)e^{-\beta Nt}+1} = \frac{N(N-1)}{(N-1)+e^{\beta Bt}}$, thus $S(t) = \frac{N}{1+\left(e^{\beta Nt}\right)/(N-1)}$ (8). Equations (6) and (8) give explicit equations for the number of infected and susceptible individuals in a population. It is of more interest to find the rate of which a disease may spread.

The number of cases arising with respect to time is found by taking the first derivative of (6).

$\frac{dI}{dt} = \frac{0\left((N-1)e^{-\beta Nt}+1\right) - N\left((N-1)(-\beta N)e^{-\beta Nt}\right)}{\left((N-1)e^{-\beta Nt}+1\right)^2} = \frac{N^2\beta(N-1)e^{-\beta Nt}}{\left((N-1)e^{-\beta Nt}+1\right)^2}$,(9). Equation (9) describes the epidemic curve. Determining as to where the epidemic starts, put t=0 into (9),

?

$$I'(0) = \frac{N^2\beta(N-1)}{\left((N-1)+1\right)^2} = \frac{N^2\beta(N-1)}{N^2} = \beta(N-1), (10).$$

OK

To determine when the epidemic curve (9) reaches it's maximum (when the disease is spreading most rapidly) take the second derivative of (10),

$$I''(t) = \frac{\left[(N-1)e^{-\beta Nt}+1\right]^2\left(N^2(N-1)\beta(-\beta N)e^{-\beta Nt}\right) - \left(N^2(N-1)\beta e^{-\beta Nt}\right)(2)\left[(N-1)e^{-\beta Nt}+1\right]\left((N-1)(-\beta N)e\right.}{\left[(N-1)e^{-\beta Nt}+1\right]^4}$$

(11). After considerable simplification and factoring out common terms in the numerator of (11),

$I''(t) = \dfrac{\left((N-1)\beta^2 N^3 e^{-\beta Nt}\right)\left((N-1)e^{-\beta Nt}-1\right)}{\left((N-1)e^{-\beta Nt}+1\right)^3}$, (12). In analyzing (12), note that $(N-1)\beta^2 N^3 e^{-\beta Nt}$ is always >0 , (since $N>0, \beta^2>0, N^3>0, e^{-\beta Nt}>0$). Therefore if $\dfrac{d^2I}{dt^2}$ were to change signs this would occur in $\left((N-1)e^{-\beta Nt}-1\right)$, $\lim_{t\to\infty}(N-1)e^{-\beta Nt}-1 = 0-1 = -1$ and the epidemic curve reaches it's maximum height when $\dfrac{N-1}{e^{\beta Nt}}-1=0$ or $\dfrac{N-1}{e^{\beta Nt}}=1$, $N-1=e^{\beta Nt}$, (13) .

Solving explicitly for t in (13) $\ln(N-1)=\beta Nt$, the epidemic curve reaches its maximum at $t_{max}=\dfrac{\ln(N-1)}{\beta N}$, (14). To find the rate of infection at t_{max} take equation (9) and substitute (14) for t. $\dfrac{dI}{dt_{max}} = \dfrac{N^2(N-1)\beta e^{-\beta N\left(\log\frac{N-1}{\beta N}\right)}}{\left((N-1)e^{-\beta N\left(\log\frac{N-1}{\beta N}\right)}+1\right)^2} = \dfrac{N^2(N-1)\beta(N-1)^{-1}}{\left((N-1)(N-1)^{-1}+1\right)^2} = \dfrac{N^2\beta}{4}$, (15). The maximum number of infected individuals when the epidemic is at it's peak is (take equation (6) and substitute (14) for t),

$$I(t_{max}) = \frac{N}{(N-1)e^{-\beta N\left(\log\frac{N-1}{\beta N}\right)}+1} = \frac{N}{(N-1)(N-1)^{-1}+1} = \frac{N}{2}, (15).$$

As well, the maximum number of susceptibles when the epidemic is at it's maximum is,

$$S(t_{max}) = \frac{N(N-1)}{(N-1)e^{-\beta N\left(\log\frac{N-1}{\beta N}\right)}+1} = \frac{N}{(N-1)(N-1)^{-1}+1} = \frac{N}{2}, (16)$$

The epidemic curve is symmetrical about it's maximum coordinate. Is this clear? - not a

To illustrate the SI model, a numerical example is necessary. Suppose a community of Morris, 2500 individuals, one individual having the common winter cold. Given a infectious rate of 0.0009 we can determine how many people will catch the cold in weekly periods. Defining our

EXAMPLE

functions (with the aid of Maple) a tables of values of the following is shown for t, S(t), I(t) and the infection rate I'(t).(See Appendix 1 for tables and graphs)

Observing graph (1), it is obvious that as the weeks increase the number of susceptible individuals decrease. Graph (2) the opposite occurs, as the weeks tend to increase the number of people becoming infected with a cold increases. Graph (3) illustrates the epidemic curve. Notice that the graph is symmetrical about it's maximum rate of infection. To locate precisely when the epidemic will be at it's peak the following calculations are performed.

$$t\max = \ln\frac{(N-1)}{\beta N} = \ln\frac{(2500-1)}{(.0009)2500} = 3.477175969$$

$$I'(t_{\max}) = \frac{N^2\beta(N-1)e^{-\beta N t_{\max}}}{((N-1)e^{-\beta N t_{\max}}+1)^2} = \frac{(2500)^2.0009(2500-1)e^{-.0009(2500)3.477175969}}{((2500-1)e^{-.0009(2500)3.477175969}+1)^2} = 1406.250000$$

In conclusion, from our numerical example it would take approximately 7 weeks for everyone in the community of Morris to contract the cold. The maximum rate in which the cold would spread throughout Morris is 1406 cases per week. This occurs in the middle of the third week. In addition, if one was to change the infection rate β this would greatly affect the epidemic. The smaller the infection rate the longer it would take for the disease to affect the population, and the longer it takes for the epidemic curve to reach it's peak.

Good

The SI model is a very simple model, however the prediction that everyone will contract the disease is somewhat unrealistic, (as shown in the cold example). In reality, there are individuals who escape the disease. A more appropriate model describing epidemics is the SIR model, otherwise known as the Kermack-McKendrick model. The SIR model is appropriate for describing an outbreak in a short period of time, it is more used also for more sever cases as well it is a bit more difficult to comprehend in comparison to the SI model.

In order to completely comprehend the SIR model, some basic assumptions about the disease and the population must be stated;

i) The population of the community has a constant size N. N is fixed.(no births or deaths are are introduced throughout the epidemic)

ii) In the SIR model, we are only dealing with the case of a single homogeneously mixing group.

iii) Every individual (whether a child or a senior) is susceptible to the disease. No one is immune to the illness. initially

iv) Once an individual has gone into the "removal" stage they become immune to it. These individuals that pass away from the disease are also considered a part of the "removal "stage.

v) The epidemic starts at initial time, $t_0 = 0$.

figure 2: $\boxed{S \to I \to R}$

Similar to the SI model by separating the population, N into three categories, (instead of two):

not a sentence

1. Let S represent the susceptible class. The total number of individuals who aren't infected with the disease however they may contact the disease and can become infected. — not a sentence

2. Let I represent the infective class. These individuals already have the disease and are able to transmit the disease to other individuals.

3.Let R represent those individuals in the removal class. These individuals have had the disease and have either passed away, recovered and are permanently immune to the disease, or are isolated until recovery and permanent immunity occur.

One must also assume that the total population N is composed of all three types of categories. The total population is equal to the sum of the susceptible class, the infected class and the removal class. That is, $N = S + I + R$. These different classes of the model, include a time factor. Each class may be written as a function of time t, since one treats the population as a continuum. $N = S(t) + I(t) + R(t)$.

The SIR model starts at time $t_0 = 0$, and assume that there are no individuals who are part of the removal class. Since the disease hasn't had a chance to grow throughout the population, (N) , therefore no one has yet to recover or pass away from the disease. Hence, this can be expressed as the following condition, $R(t_0) = R(0) = 0$.

At initial time $t_0 = 0$, we don't know exactly how many infectives there are, but since the epidemic is at a very early stage, one can assume the number of infectives is relatively small in comparison to the total population. We will denote this number as I_0. Hence, the following condition can be stated as $I(t_0) = I(0) = I_0$.

The number of susceptible individuals at initial time $t_0 = 0$, is equal to the total population, N excluding those who already have the disease.The number of individuals who have the disease is relatively small, implying the number of susceptible individuals is approximately equal to the total population. $S(t_0) = S(0) = N - I_0 = S_0$. Where S_0 denotes the initial number of susceptible individuals at time t=0. It is also important to note that when dealing with the SIR model it only makes sense to talk about positive number of individuals.

$R(t) \geq 0, S(t) \geq 0, I(t) \geq 0$

Recapturing our initial conditions i) $R(0) = 0$, ii) $I(0) = I_0$, iii) $S(0) = S_0$, looking at figure 1, there are two paths that may occur in the SIR model. The first one is when the susceptible individuals (S), become infected with the disease. These individuals are now considered part of the infected class, (I). This process is denoted by the infectious rate β. (as mentioned in the SI model) The second path in the SIR model is when the infectious individuals recover from the disease (either from passing away or becoming immune), and then become part of the removed class (R). This process is denoted by the removal rate γ.

characterize

(figure 3) $\boxed{S \xrightarrow{\beta} I \xrightarrow{\gamma} R}$

There are two ways a disease may spread rapidly. Firstly, if the number of infected individuals increases, then there are more cases of disease in a certain area, resulting in the probability of an individual catching the disease to increase. Secondly, a disease may spread more rapidly if the number of susceptible individuals increases. In this case, there will be more individuals who can carry the disease. From both of these cases, the rate of change of the number of susceptible people is proportional to both the number of susceptibles and to the number of infectives. We may denote the rate of change in the susceptible population with

S S

respect to time as $\frac{ds}{dt}$ and we can write $\frac{ds}{dt} = -\beta S(t)I(t)$, where the negative sign indicates that the number of susceptible individuals is decreasing. This equation is known as the law of mass action.

The rate at which individuals who pass from the infectious stage to the recovery stage is proportional to the size of the infective class. Let $\frac{dR}{dt}$ denote the rate of change of recovery of individuals with respect to time, and $\frac{dR}{dt} = \gamma I(t)$.

The rate of change in the number of infectives over time t is equal to the product of the number of susceptibles, S(t), and the number of infectives, I(t) minus the removal rate of the infectives, $\frac{dI}{dt} = \beta S(t)I(t) - \gamma I(t)$.

Summarizing, SIR model is the following set of differential equations:

$$\boxed{\begin{array}{l} 1.\ \frac{dS}{dt} = -\beta S(t)I(t) \\[1ex] 2.\ \frac{dI}{dt} = \beta S(t)I(t) - \gamma I(t) \\[1ex] 3.\ \frac{dR}{dt} = \gamma I(t) \end{array}}$$

with initial conditions, $S(0) = S_0, I(0) = I_0, R(0) = 0$ where $S_0, I_0 \geq 0$.

The number of susceptibles, infectives and removals are always between 0 and N, (the total population). As time goes on, the number of people who enter the removal class reaches a limit. The function $R(t)$ has an upper bound and $\lim_{t\to\infty} R(t)$ exists. Denote this limit as R^*. Mathematically, $\lim_{t\to\infty} R(t) = R^* \leq N$. Similarly, the function $S(t)$ must be greater than or equal to zero. (Zero is a lower bound) and usually decreases as time goes on. ($S(t)$ is nonincreasing function). Hence, a limit for $S(t)$ exists as time increases. Mathematically, $\lim_{t\to\infty} S(t) = S^* \geq 0$. Using our previous assumption, $I(t) = N - S(t) - R(t)$ the

$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}(N - S(t) - R(t))$ implies $\lim_{t\to\infty} N$ - $\lim_{t\to\infty} S(t)$-

$\lim_{t\to\infty} R(t) = N - S^* - R^* = I^*$, where I^* represents the limit of $I(t)$.

The relationship between the number of susceptibles and number of removals, during an epidemic is determined by an approximate solution of equations (1) and (3) from the SIR model. Divide equation 1 by 3 to create an autonomous system of differential equations (eliminate dt).

the original system was autonomous!

$$\frac{dS/dt}{dR/dt} = \frac{-\beta S(t)I(t)}{\gamma I(t)}$$
$$\frac{dS}{dt}\left(\frac{dt}{dR}\right) = \frac{-\beta S(t)I(t)}{\gamma I(t)} \quad (4)$$

$\frac{dS}{dR} = \frac{-\beta S(t)}{\gamma}$ (5), implement the method of separation of variables in (5) $\frac{dS}{S(t)} = \frac{-\beta}{\gamma} dR$, (6) . Integrate

(6) $\int \frac{dS}{S(t)} = \int \frac{-\beta}{\gamma} dR$. (7) After integration, $\ln S(t) = \frac{-\beta}{\gamma} R(t) + C$, (8) where C is the constant of integration. Put t=0 and using the initial condition $R(0) = 0$ and substituting this into (8)

$$\ln S(0) = \frac{-\beta}{\gamma} R(0) + C,$$

$$\ln S(0) = \frac{-\beta}{\gamma}(0) + C$$

$\ln S(0) = C$. Substituting $C = \ln(S(0))$ into (8) produces $\ln S(t) = \frac{-\beta}{\gamma} R(t) + \ln S(0)$, (9). Taking $\ln S(0)$ to the left hand side produces $\ln S(t) - \ln S(0) = \frac{-\beta}{\gamma} R(t)$. Combine the natural logrithmic functions together, $\ln\left[\frac{S(t)}{S(0)}\right] = -\frac{\beta}{\gamma} R(t)$,(10) and exponentiating both sides of (10),

gives $\left[\frac{S(t)}{S(0)}\right] = e^{-\frac{\beta}{\gamma}R(t)}$. Solving explicitly for $S(t)$ by multiplying the left and right hand sides by $S(0)$,

$S(t) = S(0)e^{-\frac{\beta}{\gamma}R(t)}$, (11). Applying the initial condition $S(0) = 0$ our solution is $S(t) = S_0 e^{-\frac{\beta}{\gamma}R(t)}$.

$$\boxed{S(t) = S_0 e^{-\frac{\beta}{\gamma}R(t)} \qquad (12)}$$

Equation (12) explicitly defines the number of individuals who are susceptible to the disease and any given time . Since $R(t) \leq R^* \leq N$ for all t, $e^{\frac{-\beta}{\gamma}R(t)} \geq e^{\frac{-\beta}{\gamma}N}$ or $-\frac{\beta}{\gamma}R(t) \geq -\frac{\beta}{\gamma}N$ implies $S(t) \geq S_0 e^{\frac{-\beta}{\gamma}N}$. Knowing already S(t) is a positive number and $\lim_{t \to \infty} S(t) = S^* > 0$ there will always be some individuals who will not become infected by the disease.

Having solved for $S(t)$, one must determine if our epidemic is under control and whether or not an epidemic would occur. To evaluate if an epidemic is rising, observe the rate of infectious individuals over time, (ie., $\frac{dI}{dt}$).If $\frac{dI}{dt}$ is increasing or rather large, then the disease is spreading the contrary, if $\frac{dI}{dt} \leq 0$, the infection rate is decreasing then the epidemic is under control. In otherwords, the number of people being infected with the disease is diminishing, (perhaps due to a medical intervention). An extreme example of this occurs is during some time t^* in the epidemic, there is no one left within the population that is susceptible to the disease, is S(t^*)=0. Eventually, since all the individuals are infected with the disease individuals will either recover or die. The number of infected individuals will decrease $\frac{dI}{dt} \leq 0$. Implying the epidemic is under control when $\frac{dI}{dt} \leq 0$ for all $t^* > t_0$.

Mathematically, $I(t) = N - S(t) - R(t)$,(13) differentiating $I(t)$ with respect to t , (13) becomes $\frac{dI}{dt} = -\frac{dS}{dt} - \frac{dR}{dt}$. Substitute (1) and (3) into (13) to obtain, $\frac{dI}{dt} = \beta S(t)I(t) - \gamma I(t)$. Factoring out $I(t)$ produces $\frac{dI}{dt} = I(t)[\beta S(t) - \gamma]$,(14). As stated previously an epidemic is controlled when $\frac{dI}{dt} \leq 0$, or ?

Hence, $\frac{dI}{dt}$ is negative when $\beta S(t) - \gamma \leq 0$, or $S(t) \leq \frac{\gamma}{\beta}$. The term $\frac{\gamma}{\beta}$ is known as the relative removal rate. If The number of susceptible people is less than the removal rate $\frac{\gamma}{\beta}$ then the epidemic is under control. This statement is known as the threshold phenomenon , (critical level). In other words, an epidemic will be controlled in the start (time t=0) if the number of susceptible individuals is less than the removal rate, $S(0) \le \frac{\gamma}{\beta}$. An epidemic will not start if the initial number of susceptibles is less then the relative removal rate.

To develop a better understanding of how the SIR model travels throughout a population, a phase plane analysis of the S-I plane is necessary, Taking equations (1) and (2) from the SIR model,we can find the trajectories by dividing equation (2) by (1) and eliminating dt.

$$\frac{dI/dt}{dS/dt} = \frac{\beta S(t)I(t) - \gamma I(t)}{-\beta S(t)I(t)}$$

$$\frac{dI}{dS} = \frac{\beta S(t)I(t)}{-\beta S(t)I(t)} + \frac{\gamma I(t)}{\beta S(t)I(t)}$$

$$\frac{dI}{dS} = -1 + \frac{\gamma}{\beta S(t)}, (15)$$

Solve for I by implementing the method of separation of variables and integration.

$$dI = \left(-1 + \frac{\gamma}{\beta S(t)}\right) dS$$

$$\int dI = \int \left(-1 + \frac{\gamma}{\beta S(t)}\right) dS$$

$$I(t) = -S(t) + \frac{\gamma}{\beta} \ln S(t) + C, (16)$$

where C is the constant of integration. To determine C put t=0 into (16).

$$I(0) = -S(0) + \frac{\gamma}{\beta} \ln(S(0)) + C$$

$$C = I(0) + S(0) - \frac{\gamma}{\beta} \ln(S(0))$$

Applying the initial condition, $I(0) = I_0$, $C = I_0 + S(0) - \frac{\gamma}{\beta}\ln(S(0))$. Substitute C back into (16). $I(t) = -S(t) + \frac{\gamma}{\beta}\ln(S(t)) + I_0 + S(0) - \frac{\gamma}{\beta}\ln(S(0))$ and replace $S(t) = N - I_0 = S_0$.

$$I(t) = -S(t) + S(0) + I_0 + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{S(0)}\right) = I_0 - S(t) + S_0 + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{S(0)}\right) = I_0 - S(t) + (N - I_0) + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{S(0)}\right)$$

$$I(t) = N - S(t) + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{S(0)}\right),(17)$$

The trajectories in the phase plane analysis are described by the curve $I(t) = N - S(t) + \frac{\gamma}{\beta}\ln\left(\frac{S(t)}{S(0)}\right)$. To determine the critical points, equations (1), (2),(3) all add up to zero (ie,.. $-\beta S(t)I(t) + \beta S(t)I(t) - \gamma I(t) + \gamma I(t) = 0$) implies linear dependence. Taking equations (1) and (2) and settting $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$, produces I(t)=0 and S(t)=0 or S(t)= $\frac{\gamma}{\beta}$. . (note:I(t)>0 and S(t)>0) Therefore the critical points are (S,I)=(0,0) and (S,I)=($\frac{\gamma}{\beta}$,0). A phase plane analysis, determing the direction of a trajectoris is included in Appendix A.

Once an epidemic occurs, the number of susceptible individuals diminishes,(S(t) decreases). Hence the trajectories move from right to left. Since the critical points lie only on I=0, the trajectories approach $S(\infty)$ as time goes to infinity, so $I(\infty) = 0$. The maximum number of infectives occur at S= $\frac{\gamma}{\beta}$ (when $\frac{dI}{dS}$ =0). This is shown mathematically by setting the first derivative of equation (17) equal to zero. $I' = -S' + \frac{\gamma}{\beta}\left(\frac{1}{S/S_0}\left(\frac{S'}{S_0}\right)\right) = -S' + \frac{\gamma}{\beta}\left(\frac{S'}{S}\right) = S'\left(\frac{\gamma}{\beta S} - 1\right)$. Since the maximum occurs when $I'(0)$=0, or $\frac{\gamma}{\beta S} - 1$=0 implies S= $\frac{\gamma}{\beta}$. Substitute S= $\frac{\gamma}{\beta}$ into (17), the maximum number of infectives is, $\mathrm{I}max = N + \frac{\gamma}{\beta}\left(\ln\frac{\gamma}{\beta S_0} - 1\right)$.In otherwords the threshold or maximum value for the rate of new cases occurs at S= $\frac{\gamma}{\beta}$. Depicting the direction of trajectories in the S-I plane is included in Appendix

B. If (S_0, I_0) fall below S=$\frac{\gamma}{\beta}$. then no epidemic will occur,(I(t) will go to 0). If $S_0 > \frac{\gamma}{\beta}$ the number of infected cases increases until the threshold of $\frac{\gamma}{\beta}$ is reached. Once S has reached this threshold the epidemic starts to decrease its number of infectives and the epidemic becomes under control. I(t) falls towards zero.

The trajectory crosses the line at some positive value of S less than S_0 and appraoches S^*.To see if a point of inflection exists, from equation (15), taking the second derivative with respect to S $\frac{d^2 I}{dS^2} = \frac{-\gamma}{\beta S^2}$. This equation is always negative (there is never a sign change) and a point of inflection doesn't exist. The graph of the trajectory is always concave down.

An approximate solution as to how many individuals will recover from the illness may be found by starting with equation (3) from the SIR model. Since we know $I = N - S(t) - R(t)$ substituting this into (3) gives $\frac{dR}{dt} = \gamma(N - S(t) - R(t))$,(18),where γ >0. Substitute equation (12) into this expression, $\frac{dR}{dt} = \gamma\{N - (S_0)e^{\frac{-\beta}{\gamma}R(t)} - R(t)\}$, (19). Since an approximate solution is what we are trying to find, one may expand the function $e^{\frac{-\beta}{\gamma}R(t)}$ as a maclaurin series to the $R^2(t)$ term. The maclaurin expansion of $e^{\frac{-\beta}{\gamma}R(t)}$ is, $e^{\frac{-\beta}{\gamma}R(t)} = 1 - \frac{\beta}{\gamma}R(t) + \frac{1}{2!}\left(\frac{-\beta}{\gamma}\right)^2 R^2(t) = 1 - \frac{\beta}{\gamma}R(t) + \frac{\beta^2}{2\gamma^2}R^2(t)$. (20) Replacing (20) for $e^{\frac{-\beta}{\gamma}R(t)}$ in (19) the differential equation becomes $\frac{dR}{dt} = \gamma\{N - S_0(1 - \frac{\beta}{\gamma}R(t) + \frac{\beta^2}{2\gamma^2}R^2(t)) - R(t)\}$,(21), which results in a solution of $R(t) = \frac{\gamma^2}{\beta^2 S_0}\left(\frac{\beta S_0}{\gamma} - 1\right) + \frac{\gamma^2}{\beta^2 S_0}\left(\sqrt{\left(\frac{\beta S_0}{\gamma} - 1\right)^2 + \frac{2\beta^2 S_0 I_0}{\gamma^2}}\right)\tanh\left(\frac{1}{2}\sqrt{\left(\frac{\beta S_0}{\gamma} - 1\right)^2 + \frac{2\beta^2 S_0 I_0}{\gamma^2}}\gamma t + C_2\right)$(22) The rather tedious details for solution to (22) are included in Appendix A. The derivative of (22) with respect to t is,

$$\frac{dR}{dt}=\frac{\gamma^2\left(\left(\frac{\beta S_0}{\gamma}-1\right)^2+\frac{2\beta^2 S_0 I_0}{\gamma^2}\right)\gamma}{2\beta^2 S_0}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\left(\frac{\beta S_0}{\gamma}-1\right)^2+\frac{2\beta^2 S_0 I_0}{\gamma^2}}\gamma t\right)$$,(23). Equation (23), is the equation of the bell shaped curve, usually known as the "epidemic curve", and is also symmetrical about it's peak.

The centre of the epidemic curve is determined by differentiating equation, (19).

$\frac{d}{dt}\left(\frac{dR}{dt}\right)=\gamma\left(\frac{-dR}{dt}-S_0 e^{\frac{-\beta}{\gamma}R(t)}\left(\frac{-dR}{dt}\left(\frac{-\gamma}{\beta}\right)\right)\right)$, (24). Substitute equations (3) and (12) into (24) to obtain

$\frac{d}{dt}\left(\frac{dR}{dt}\right)=\gamma\left(\gamma I(t)-\frac{\gamma}{\beta}S(t)I(t)\right)$. After some simplification the final equation for the centre of the epidemic curve is, $\frac{d}{dt}\left(\frac{dR}{dt}\right)=\gamma^2 I(t)\left(1-\frac{\gamma}{\beta}S(t)\right)$. (25)

Nota perfecta

Whether a fatal disease such as ebola or pnemonnia it is always of interest to consider the total size o the epidemic after a long period of time. An approximation of the extent of the epidemic, denoted by E will indicate how many individuals were infected. Let $t\to\infty$, as mentioned previously when this occurs, $\lim_{t\to\infty}S(t)=S^*,\lim_{t\to\infty}I(t)=0$ then equation (17) becomes $S^*-N=\frac{\gamma}{\beta}\ln\left(\frac{S^*}{S_0}\right)$. Let $E=N-S^*$ denote t extent of the epidemic then $-E=\frac{\gamma}{\beta}\ln\left(\frac{S^*}{S_0}\right)$, (26). Multiply both sides by $\frac{\beta}{\gamma}$ then after some rearranging (2 becomes, $N-W-S_0 e^{\frac{\beta}{\gamma}W}=0$, (27). It is impossible to solve for W explicilty, however a numerical examp is necessary to see it's implications.

The university of Manitoba's population for the year 1990 is 25,000. In January of 1991, a sudder outbreak of menegitis occurs. The initial number of cases reported is 5000. The infection rate among stude and staff is β=.00001 and the removal rate γ =0.1.Our time intervals will be in months.

First, one must determine if an epidemic will ensue or not,

$S_0=N-I_0=25000-5000=20000$

$$\frac{\gamma}{\beta} = \frac{0.1}{.00001} = 10000$$

An epidemic will occur at the University since $S_0 > \frac{\gamma}{\beta}$. To determine as to how severe the outbreak cases define the extent of the epidemic as the function $f(E) = N - S_0 e^{\frac{-\beta}{\gamma}E} - E = 0$. By using newton's method an estimation of the epidemic is possible.

Newton's method: $E_{n+1} = E_n - \dfrac{f(E)}{f'(E)}$ where $f'(E) = \dfrac{\beta}{\gamma} S_0 e^{\frac{-\beta}{\gamma}E} - 1$

$E_{n+1} = E_n - \dfrac{N - S_0 e^{\frac{-\beta}{\gamma}E} - E}{\frac{\beta}{\gamma} S_0 e^{\frac{-\beta}{\gamma}E} - 1}$, and assume a starting value of $E_0 = 8000$ then the following formula is used to approximate the extent of the epidemic. $E_{n+1} = E_n - \left(\dfrac{25000 - 20000 e^{\frac{-E_n}{10000}} - E_n}{\frac{20000}{10000} e^{\frac{-E_n}{10000}} - 1} \right)$, (28).

Formula (28) produces the table of values:

N	Wn	Wn+1
0	8000	87072.9908917
1	87072.9908917	24976.1540702
2	24976.1540702	23034.9287614
3	23034.9287614	22993.5456252
4	22993.5456252	22993.5241907
5	22993.5241907	22993.52419

After 5 iterations of Newton's Method $W_5 \cong 22993.52419$. A verification of this approximation can be done by substituting W_5 into $f(E) = 25000 - 20000 e^{-.0001E} - E = 0$. Therefore, by the time the epidemic has run it's course out of 25000 students and staff 22994 have been infected,($E \approx 22994$) .

Determining the maximum number of infectives (per month)

$$I_{max} = N + \frac{\gamma}{\beta}\left(\ln\left(\frac{\gamma}{\beta S_0}\right) - 1\right) = 25000 + \frac{.1}{.00001}\left(\ln\left(\frac{.1}{.00001(20000)} - 1\right)\right) = 8068.528194$$

Demonstrating this graphically, the trajectory I=25000-S+10000ln(S/20000) moves from right to left. It's maximum occurs at I=8068.528194 where S=10000. The graph I vs S is concave down.

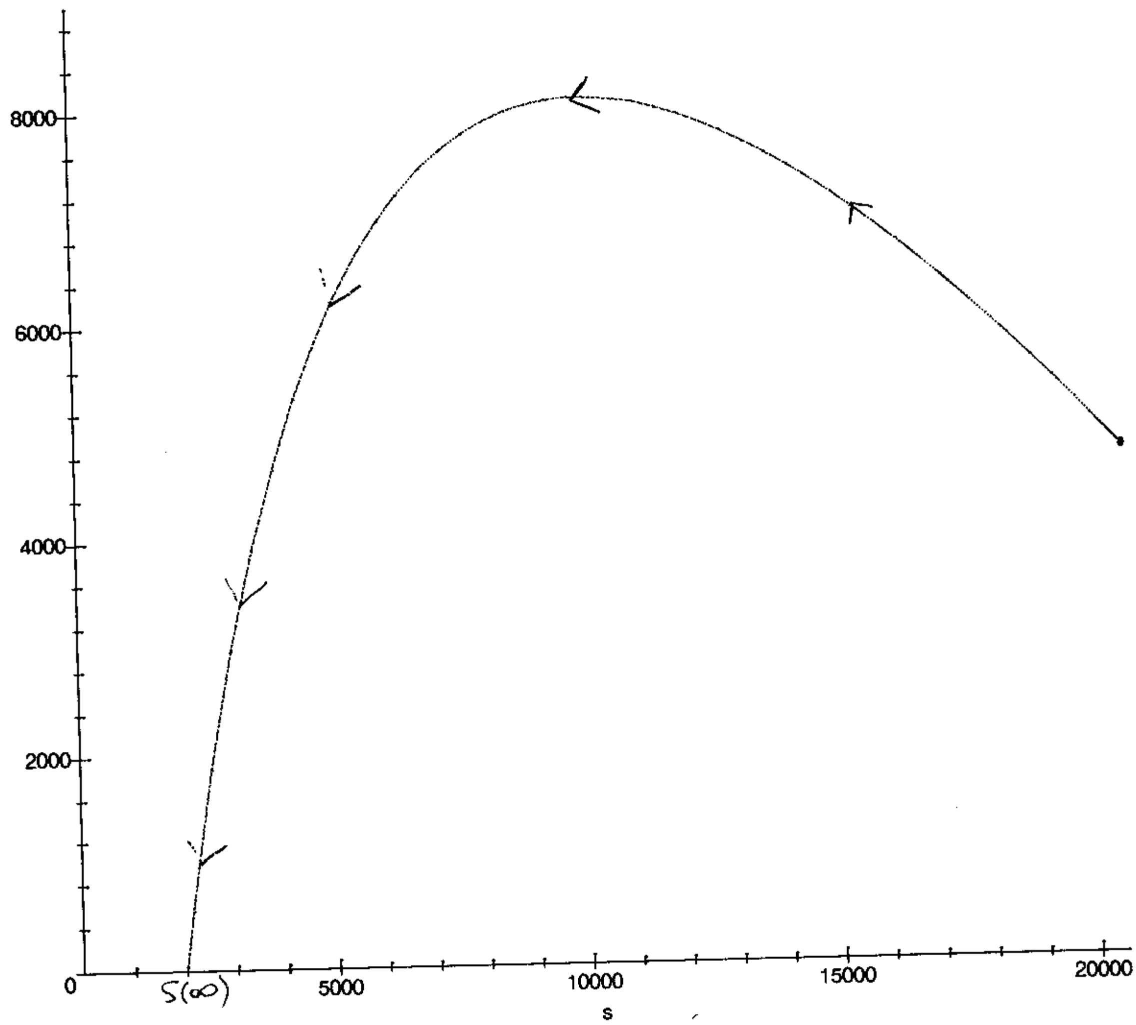
Both the SI and SIR model, individuals are capable of passing on the disease as soon as he/she succumbed to it. This may be the case in very few situations. In reality there is usually an incubation period where individuals cannot transmit the disease to others. Other limitations which might affect an epidemic are weather conditions, vaccinations, medical interventions and living conditions. These factors need to be taken into affect if a more precise study is to be taken.

```
> N:=25000:
> so:=20000:
> g:=0.1:
> b:=0.00001:
> i:=s->N-s+(g/b)*ln(s/so);
```

$$i := s \to N - s + \frac{g \ln\left(\frac{s}{so}\right)}{b}$$

```
> plot(i(s),s=0..20500,0..9000);
```

<u>I vs S</u>



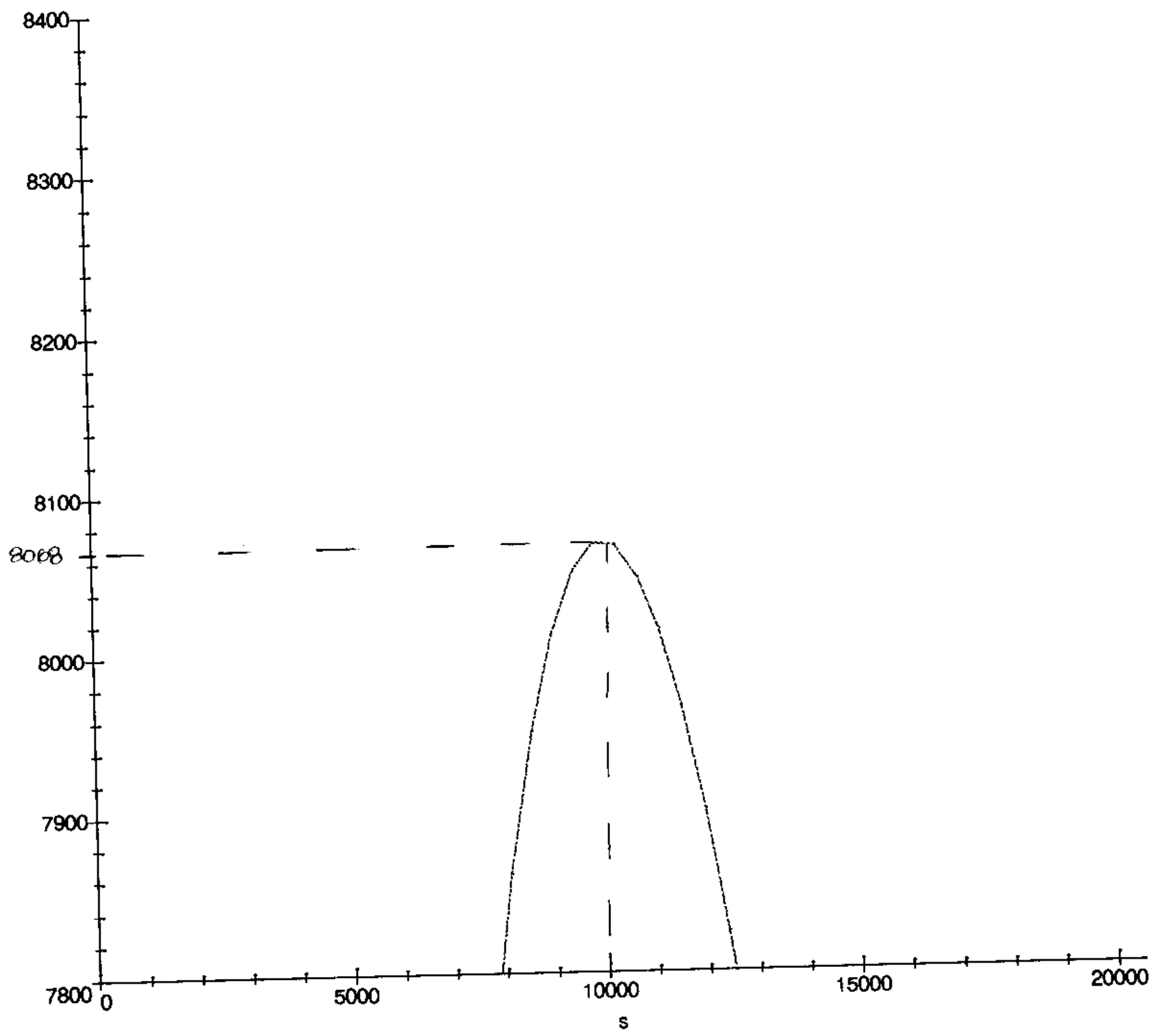

```
>
```

```
> N:=25000:
> so:=20000:
> g:=0.1:
> b:=0.00001:
> i:=s->N-s+(g/b)*ln(s/so);
```

$$i := s \to N - s + \frac{g \ln\left(\frac{s}{so}\right)}{b}$$

```
> plot(i(s),s=0..20500,7800..8400);
```

I vs S (close up)




```
>
```

APPENDIX 1

```
> s:=t->N/(1+exp(B*N*t)/(N-1));
```

$$s := t \to \frac{N}{1 + \frac{e^{B\,N\,t}}{N-1}}$$

```
> i:=t->N/(1+(N-1)*exp(-B*N*t));
```

$$i := t \to \frac{N}{1 + (N-1)\,e^{-B\,N\,t}}$$

```
> epidmc:=t->(N^2*B*(N-1)*exp(-B*N*t))/((N-1)*exp(-B*N*t)+1)^2;
```

$$epidmc := t \to \frac{N^2\,B\,(N-1)\,e^{-B\,N\,t}}{\left(1 + (N-1)\,e^{-B\,N\,t}\right)^2}$$

```
> N:=2500:
```

```
> B:=.0009:
```

```
> m:='m':
```

```
> for m from 1 to 10 do
evalf(m),evalf(s(m)),evalf(i(m)),evalf(epidmc(m));
od;
```

1., 2490.544367, 9.455633066, 21.19470629

2., 2413.077891, 86.92210860, 188.7748366

3., 1863.224131, 636.7758686, 1067.810548

4., 589.2709436, 1910.729057, 1013.343404

5., 78.70477476, 2421.295225, 171.5107458

6., 8.535822936, 2491.464176, 19.14002733

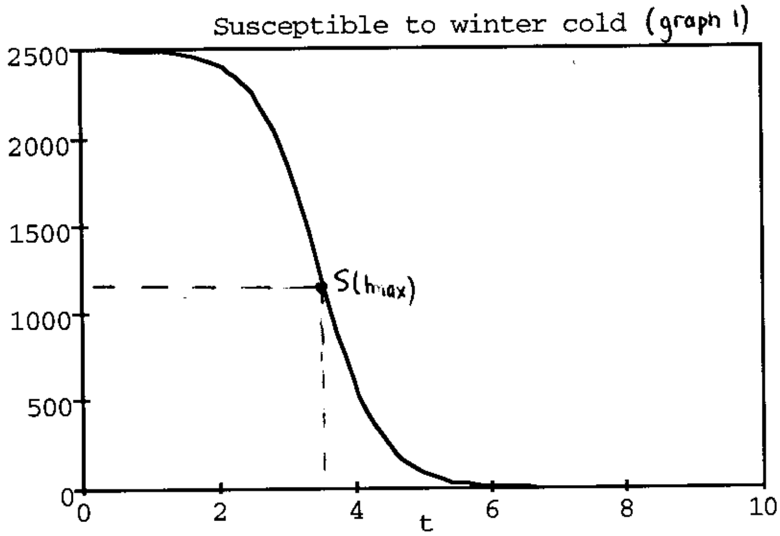
7., .9024255425, 2499.097573, 2.029724535

8., .09514567725, 2499.904854, .2140696263

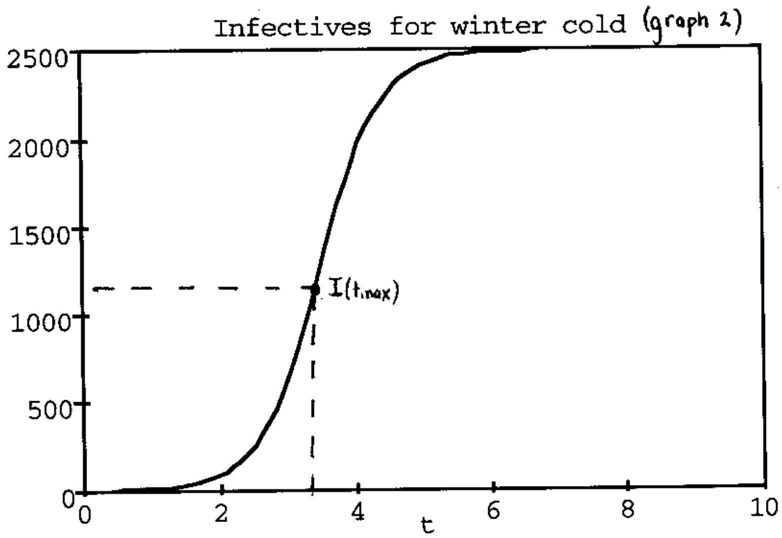
9., .01002862204, 2499.989973, .02256430911

10., .001057012780, 2499.998943, .002378277750

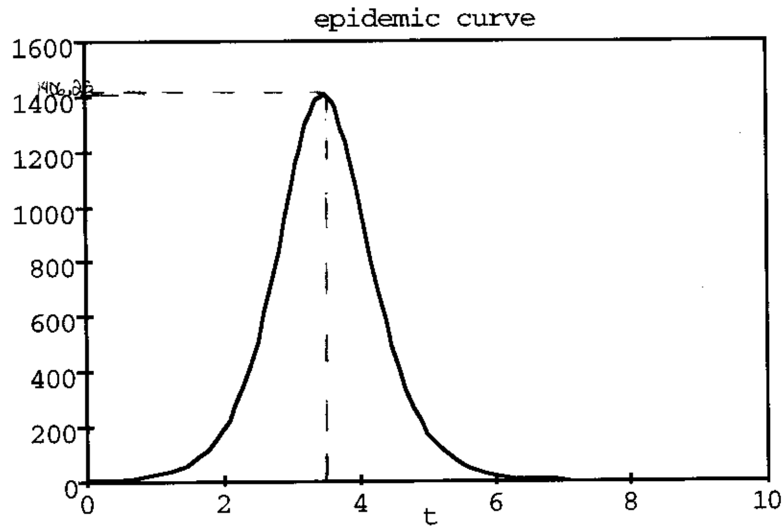
```
> plot(s(t),t=0..10,0..2500);
```




```
> plot(i(t),t=0..10,0..2500);
```

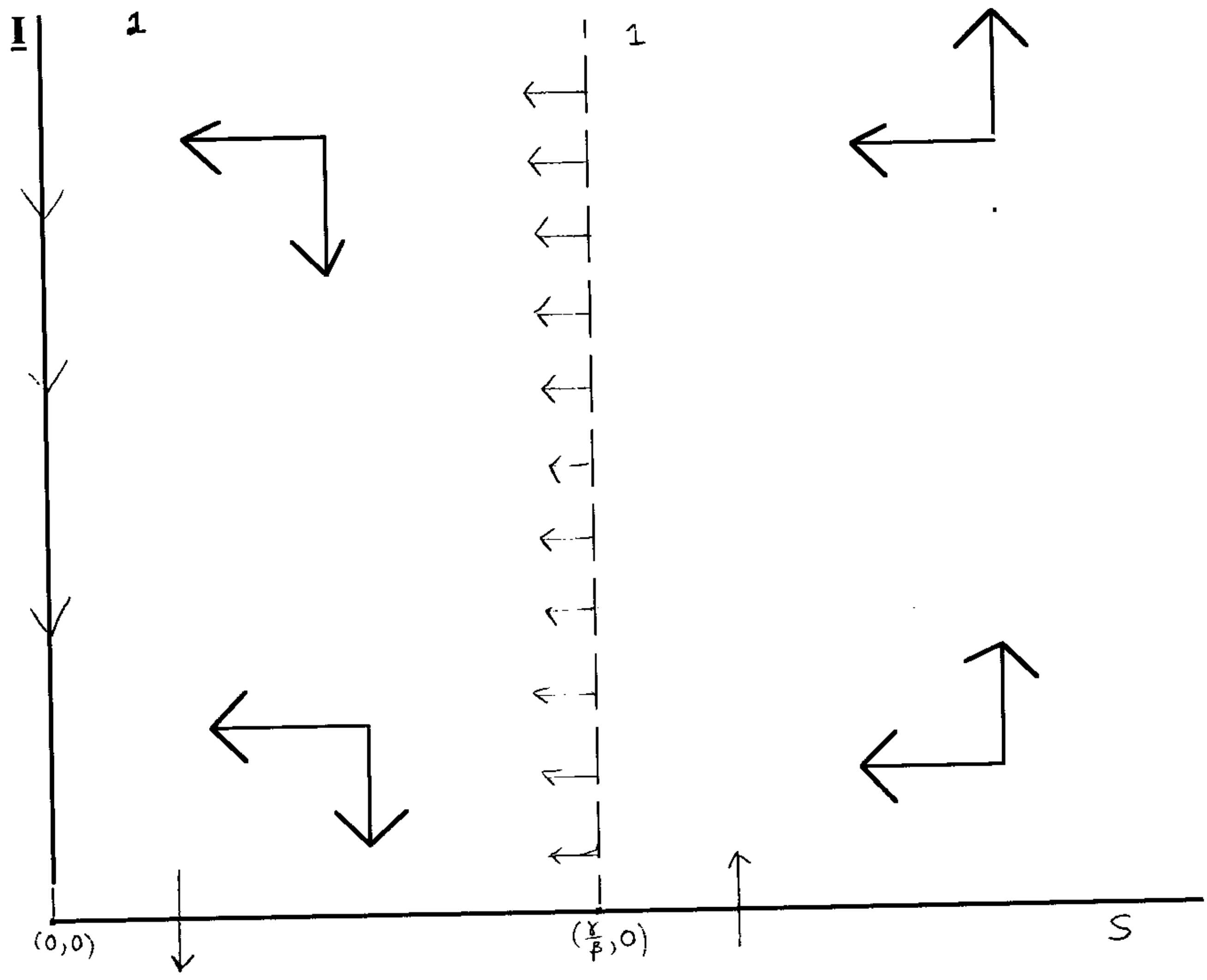



```
> plot(epidmc(t),t=0..10,0..1600);
```


epidemic curve
1600
1400
1200
1000
800
600
400
200
0
0
2
4
6
8
10
t

APPENDIX A:
PHASE PLANE ANALYSIS IN THE S-I PLANE

Diagram: S-I plane



The direction of the trajectories depends upon the following two equations:

1. $\frac{dS}{dt} = -\beta SI$ 2. $\frac{dI}{dt} = \beta SI - \gamma I = I[\beta S - \gamma]$

Along line S=0, from (2) $\frac{dI}{dt} = -\gamma I < 0$

Along line S=$\frac{\gamma}{\beta}$, from (1) $\frac{dS}{dt} = -\gamma I < 0$

<u>In quadrant 1:</u> the region where S>$\frac{\gamma}{\beta}$ from (2) $\frac{dI}{dt}$>0 (negative direction)

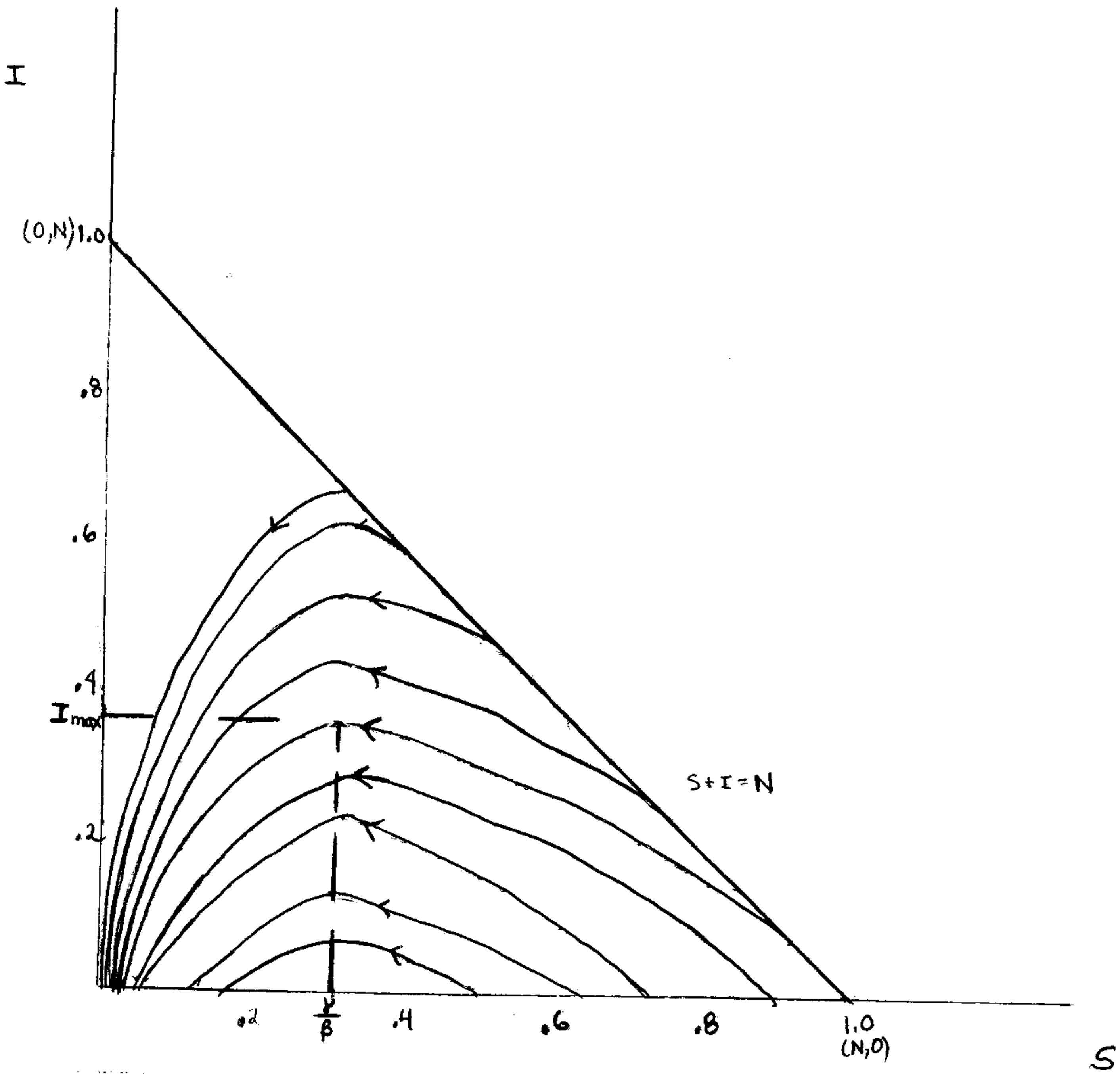
<u>In quadrant 2:</u> the region where S<$\frac{\gamma}{\beta}$, from (2) $\frac{dI}{dt}$<0 (negative direction)

The trajectory moves from right to left and concave down towards the line I=0. The critical point ($\frac{\gamma}{\beta}$,0) is unstable since the trajectory moves away from this point. The critical point (0,0) is stable since the trajectory approches (0,0) but not necessarily hitting it direcly.

(note: It would not make sense for the trajectory to directly hit (0,0). This would indicate all the population has either passed away or have recovered from the disease. In actual reality, there are always some individuals who never contract the disease.)

APPENDIX B

DIRECTION OF TRAJECTORIES :



APPENDIX C

$$\frac{dR}{dt}=\gamma\{N-S_0(1-\frac{\beta}{\gamma}R(t)+\frac{\beta^2}{2\gamma^2}R^2(t))-R(t)\}(1)$$

rearrange the terms as coefficients of $R(t)$ to construct a quadratic in $R^2(t)$ on the right hand side of (1)

$$\frac{dR}{dt}=\gamma\{(\frac{-\beta^2}{2\gamma^2}S_0)R^2(t)+(\frac{\beta}{\gamma}S_0-1)R(t)+(N-S_0)\}(2)$$

for simplicity, let the coefficient of a= $(\frac{-\beta^2}{2\gamma^2}S_0)$

let the coefficient of b= $(\frac{\beta}{\gamma}S_0-1)$

let the coefficient of c= $(N-S_0)$

Evaluating a,b and c, assume that N>>>0, (N is realtively large) and S_0>0 then c= $(N-S_0)$>0. Since γ >0 and β>0 then a= $(\frac{-\beta^2}{2\gamma^2}S_0)$<0 and b= $(\frac{\beta}{\gamma}S_0-1)$>0.

Furthermore (2) can be written as,

$$\frac{dR}{dt}=\gamma(aR^2+bR+c)\ (3)$$

Using the method of separation of variables (3) becomes, $\frac{dR}{(aR^2+bR+c)}=\gamma dt$ (4)

In order to integrate, complete the square for the quadratic in $R^2(t)$.

$$(aR^2+bR+c)=a\left(R^2+\frac{b}{a}R\right)+c=a\left(R^2+\frac{b}{a}R+\left(\frac{b}{a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c$$

$$=a\left(R+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

It has been already concluded that a<0, b>0 and c>0, thus $\frac{b^2}{4a}<0$, or $\frac{-b^2}{4a}>0$, and the term $c-\frac{b^2}{4a}$>0. (Note: $c-\frac{b^2}{4a}$ may be written as $\frac{4ac-b^2}{4a}$.)

$$(aR^2+bR+c)=a\left(R+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a}\right)\quad(5)$$

Factor out $\left(\frac{4ac-b^2}{4a}\right)$ from (5),

$$(aR^2+bR+c)=\left(\frac{4ac-b^2}{4a}\right)\left(1+\frac{a}{\frac{4ac-b^2}{4a}}\left(R+\frac{b}{2a}\right)^2\right)$$

$$(aR^2+bR+c)=\left(\frac{4ac-b^2}{4a}\right)\left(1+\frac{4a^2}{4ac-b^2}\left(R+\frac{b}{2a}\right)^2\right)\ (6)$$

The term $\frac{4a^2}{4ac-b^2}$ <0 since $4a^2$>0 and $4ac$<0, $-b^2$<0 thus $4ac-b^2$<0.

Factoring out a negative from $\frac{4a^2}{4ac-b^2}$, (6) can be rewritten as,

$$(aR^2+bR+c)=\left(\frac{4ac-b^2}{4a}\right)\left(1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2\right),\ (7).$$

$\frac{4a^2}{b^2-4ac}$ is a positive term, and $\left(\frac{4ac-b^2}{4a}\right)$ is a positive term.Substituting (7) into (4),

$$\int\frac{dR}{\left(\frac{4ac-b^2}{4a}\right)\left(1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2\right)}=\int\gamma dt\ ,(8)\ .$$

Solve the integral by using a trigonometric substitution.

From the trigonometric identity, $\sin^2\theta+\cos^2\theta=1$ or $\cos^2\theta=1-\sin^2\theta$, let

$\sin^2\theta=\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2$ (9). Then taking the square root of (9),

$$\sin\theta=\left(\frac{2|a|}{\sqrt{b^2-4ac}}\right)\left(R+\frac{b}{2a}\right)\text{ or }\sin\theta=\left(\frac{-2a}{\sqrt{b^2-4ac}}\right)\left(R+\frac{b}{2a}\right),(10).$$

After some minor Rearranging of (10) $R=\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\sin\theta-\frac{b}{2a}$, (11) and

$dR=\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\cos\theta d\theta$, (12). Substitute (9) and (12) into (8),

$$\int\frac{\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\cos\theta d\theta}{\left(\frac{4ac-b^2}{4a}\right)\left(1-\sin^2\theta\right)}=\int\gamma dt\quad(13)$$

or $$\int\frac{\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\cos\theta d\theta}{\left(\frac{4ac-b^2}{4a}\right)\cos^2\theta}=\int\gamma dt$$

$$\int\frac{\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)d\theta}{\left(\frac{4ac-b^2}{4a}\right)\cos\theta}=\int\gamma dt,$$

$$\int\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\left(\frac{4a}{4ac-b^2}\right)\frac{1}{\cos\theta}d\theta=\int\gamma dt$$

$$\int\left(\frac{\sqrt{b^2-4ac}}{-2a}\right)\left(\frac{-4a}{b^2-4ac}\right)\sec\theta d\theta=\int\gamma dt$$

Therefore (13) reduces to, $\int\frac{2}{\sqrt{b^2-4ac}}\sec\theta d\theta=\int\gamma dt$, (14) Integrating both the left and right hand sides of (14), $\frac{2}{\sqrt{b^2-4ac}}\ln\left|\sec\theta+\tan\theta\right|=\gamma t+C_1$ (15).Where C_1 is the constant of integration.

Write $\sec\theta$ and $\tan\theta$ interms of the coefficients of the quadratic.

$$\sec\theta=\frac{1}{\cos\theta}=\frac{1}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}},(16)$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\frac{-2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}},(17)$$

Substitute (16) and (17) into (15)

$$\frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{1}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}+\frac{\frac{-2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}\right|=\gamma t+C_1$$

$$\ln\left|\frac{1}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}+\frac{\frac{-2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}\right|=\left(\frac{\sqrt{b^2-4ac}}{2}\right)\gamma t+C_2,\ (18)$$

where $C_2=\frac{\sqrt{b^2-4ac}}{2}C_1$

To eliminate the ln, exponentiate both the left and right hand sides of (24),

$$\left|\frac{1}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}+\frac{\frac{-2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}\right|=e^{\left(\frac{\sqrt{b^2-4ac}}{2}\right)\gamma t+C2},\ (19)$$

For simplicity, let $X=\frac{\sqrt{b^2-4ac}}{2}(\gamma t)+C_2$ and combine the two terms on the left hand side of (19), (eliminating the absolute values)

$$\left[\frac{1-\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)}{\sqrt{1-\left(\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2}}\right]=e^X,(20)$$

Multiply both sides by the denominator of the left hand side and square both sides.

$$\left[1-\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)\right]^2=\left[e^X\sqrt{1-\frac{4a^2}{b^2-4ac}\left(R+\frac{b}{2a}\right)^2}\right]^2,(21)$$

After completing the operations of both sides of (21), the following is produced

$$1-\frac{4a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)+\frac{4a^2}{b^2-4ac}\left(R+\frac{b}{2a}\right)^2=e^{2X}\left(1-\frac{4a^2}{b^2-4ac}\left(R+\frac{b}{2a}\right)^2\right),\ (22)$$

From (22), moving everything over to the left hand side of the equation,

$$1-\frac{4a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)+\frac{4a^2}{b^2-4ac}\left(R+\frac{b}{2a}\right)^2-e^{2X}\left(1-\frac{4a^2}{b^2-4ac}\left(R+\frac{b}{2a}\right)^2\right)=0,\ (23)$$

Rearrange (23), to a quadratic in $\left(R+\frac{b}{2a}\right)$ terms,

$$\left(1-e^{2X}\right)+\frac{-4a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)+\left(\frac{4a^2}{b^2-4ac}+e^{2X}\frac{4a^2}{b^2-4ac}\right)\left(R+\frac{b}{2a}\right)^2=0,\ (24)$$

or, rewrite (24) as

$$\left\{\left(1+e^{2X}\right)\left(\frac{4a^2}{b^2-4ac}\right)\right\}\left(R+\frac{b}{2a}\right)^2-\left\{\frac{4a}{\sqrt{b^2-4ac}}\right\}\left(R+\frac{b}{2a}\right)+\left(1-e^{2X}\right)=0,\ (25)$$

Using the quadratic formula. solve (25) for $\left(R+\frac{b}{2a}\right)$.

For simplicity let $Q=\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)$ in (25). Then (25) is rewritten as,

$\left(1+e^{2X}\right)Q^2-2Q+\left(1-e^{2X}\right)=0$, (26). Applying the quadratic formula to (26),

$$Q=\frac{2\pm\sqrt{(-2)^2-4\left(1+e^{2X}\right)\left(1-e^{2X}\right)}}{2\left(1+e^{2x}\right)},\ (27).$$

After considerable simplification of (27), Q turns out to be, $Q=\frac{1\pm e^{2X}}{1+e^{2X}}$, (ie,...

$Q_1=\frac{1-e^{2X}}{1+e^{2X}}, Q_2=1$) (28).

Multiplying both the numerator and denominator of Q_1 in (28) by e^{-X}, Q_1 becomes, $Q_1=\frac{e^{-X}-e^{X}}{e^{-X}+e^{X}}=-\tanh X$, (29).

Having solved the quadratic for Q, using (29) and our variable substitution of Q (25), $Q=\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)$, (26) is transformed to $\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)=-\tanh X$,(30).

Substituting back in $X=\frac{\sqrt{b^2-4ac}}{2}(\gamma t)+C_2$ in (30), (26) may be further written as,

$$\frac{2a}{\sqrt{b^2-4ac}}\left(R+\frac{b}{2a}\right)=-\tanh\left(\frac{\sqrt{b^2-4ac}}{2}(\gamma t)+C_2\right),\ (31).$$

Solving (31) for R, $R=\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{\sqrt{b^2-4ac}}{2}(\gamma t)+C_2\right)$, (32).

Recall, from the beginning of Appendix C, replace

$a=\left(\frac{-\beta^2}{2\gamma^2}S_0\right), b=\left(\frac{\beta}{\gamma}S_0-1\right), c=(N-S_0)=I_0$, variables in (32).After substitution, and simplification,

$$\frac{-b}{2a}=\frac{\gamma^2}{\beta^2 S_0}\left(\frac{\beta S_0}{\gamma}-1\right),(33)$$

$$\frac{\sqrt{b^2-4ac}}{2a}=\frac{-\gamma^2}{\beta^2 S_0}\left(\sqrt{\left(\frac{\beta S_0}{\gamma}-1\right)^2+\frac{2\beta^2 S_0 I_0}{\gamma^2}}\right),\ (34)$$

Substituting, (33), (34) into (32) and simplyfying , the equation for the number of individuals in the removal class as a function of time is,

$$R(t)=\frac{\gamma^2}{\beta^2 S_0}\left(\frac{\beta S_0}{\gamma}-1\right)+\frac{\gamma^2}{\beta^2 S_0}\left(\sqrt{\left(\frac{\beta S_0}{\gamma}-1\right)^2+\frac{2\beta^2 S_0 I_0}{\gamma^2}}\right)\tanh\left(\frac{1}{2}\sqrt{\left(\frac{\beta S_0}{\gamma}-1\right)^2+\frac{2\beta^2 S_0 I_0}{\gamma^2}}\gamma t+C_2\right)$$

BIBLIOGRAPHY

[1] Bailey, N.T.J. (1957), "The Mathematical Theory of Epidemics", Charles Griffin and Company Limited.

[2]Busenberg, S., and Cooke, K. (1981), "Differential Equations and Applications in Ecology,Epidemics, and Population Problems', Academic Press.

[3]Cooke, K and Ludwig, D. (1974), "Epidemiology", Society for Industrial and Applied Mathematics.

[4] Derrick and Grossman, (1978), " Introduction to Differential Equations", West Publishing Company.

[5]Edelstein-Keshet, L. (1988), "Mathematical Models In Biology", Random House.

[6] Jones, D.S. and Sleeman, B.D. (1989), " Differential Equations and Mathematical Biology", George Allen and Unwin.

[7] Olinick, M. (1978), " An Introduction to Mathematical Models in the Social and Life Sciences", Addison-Wesley.

[8] Waltman, P. (1974), " Deterministic Threshold Models in the Theory of Epidemics. Lecture Notes in Biomathematics, vol. 1.", New York: Springer-Verlag.