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A Dynamical Model for Human Population

(sex ratio/monogamy/birth rates/death rates/fertility)

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ABSTRACT We analyze a simple, deterministic model
of the dynamics of population changes in a bisexual,
reproductive system based on marriage. Our model is one
of a general class, special cases of which have been pre-
viously discussed within the framework of population
biology by D. G. Kendall, L. A. Goodman, J. H. Pollard,
and others. Here, we extend and complete previous
analyses of systems characterized by first-degree homo-
geneous, unbounded marriage functions, allowing for
arbitrary birth and death rates.

The dynamics of the model is determined by three
coupled first-order, nonlinear differential equations,
similar to those used in the description of chemical re-
actions and of radioactive decay chains. Solutions of the
differential equation system are classified according to the
associated patterns of birth and death rates of the two
sexes, and growth and stability properties are discussed.

This preliminary report gives conditions sufficient to
insure the existence of a unique, exponential mode of
population growth or decay, with a finite ratio of the
sexes. We also exhibit other conditions which, in contrast
to the standard, linear demographic analysis of Lotka,
guarantee that the sex ratio asymptotically becomes
infinite.

The model manifests a delicate balance between the
vital parameters that alerts one to the possibility of self-
aggravating distortions of the sex ratio, once a monog-
amous society's fertility falls below the replacement
value.

This is a preliminary report of the results of an investigation
into the stability properties of reproductive systems whose
dynamics are independent of overall scale, and in which age-
specific fertility and mortality rates are (provisionally) as-
sumed to be non-age-specific, and constant through time.
Special cases of this general class have previously been par-
tially analyzed by D. G. Kendall (1), L. A. Goodman (2), J. H.
Pollard (3), and others.

Scale-independent model

Our population model is defined by the following assumptions:
1. There are two sexes with total numbers $T_1(t)$ and $T_2(t)$.
2. Monogamy prevails, with $T_3(t)$ marriages existing at time t.
   The number of single individuals of each sex is therefore

$$N_1(t) = T_1(t) - T_3(t); \quad N_2(t) = T_2(t) - T_3(t). \quad [1]$$

3. The population is replenished only through births to mar-
   ried couples. The fertility rates, per marriage, for female
   and male births are f_1 and f_2 respectively, and are inde-
   pendent of parental ages.
4. The (instantaneous) rate at which marriages occur de-
   pends nonlinearly on the then-existing stocks of single in-

dividuals $N_1(t)$ and $N_2(t)$, vanishing when either N_i van-
ishes, and increasing without limit as either N_i increases.
We will write this (nonnegative) rate as $M[N_1(t),N_2(t)]$.
5. The mortality rates (λ_1,λ_2) of single individuals are con-
   stants, independent of age, as is the mortality rate of
   couples, λ_3, due to death (or divorce, or permanent steriliza-
   tion, etc.).
6. The dynamics of the system is independent of its overall
   scale, and of the common scale of its components.

Dynamical equations

Given the assumptions above, the behavior of the system is
determined by the following set of differential equations:

$$dN_1/dt = -\lambda_1 N_1 + f_1 T_3 - M[N_1,N_2] \quad [2a]$$

$$dN_2/dt = -\lambda_2 N_2 + f_2 T_3 - M[N_1,N_2] \quad [2b]$$

$$dT_3/dt = -\lambda_3 T_3 + M[N_1,N_2]. \quad [2c]$$

The interpretation of [2] is straightforward. For example,
in [2a] the rate of change of the number of single women is
negatively proportional to the fraction of single women dying
at time t, $-\lambda_1 N_1(t)$, and to the marriage rate $-M$. The rate of
female births is proportional to the number of existing mar-
riages, and makes a positive contribution of $f_1 T_3(t)$ to $dN_1(t)/$
dt. Similarly, for the growth conditions on the number of
single men $N_2(t)$.

Though we take mortality and fertility rates to be fixed
quantities independent of population growth, a more realistic
model would include, for example, the nonlinearity which
arises from the relations between agricultural production, the
supply of arable land, and population density.

It is often convenient to work with the totals (T_1,T_2), in
which case the set [2] translates as:

$$dT_1/dt = -\lambda_1 T_1 + (f_1 - \lambda_3 + \lambda_1)T_3 \equiv -\lambda_1 T_1 + \beta_1 T_3 \quad [3a]$$

$$dT_2/dt = -\lambda_2 T_2 + (f_2 - \lambda_3 + \lambda_2)T_3 \equiv -\lambda_2 T_2 + \beta_2 T_3 \quad [3b]$$

$$dT_3/dt = -\lambda_3 T_3 + M[T_1 - T_3, T_2 - T_3] \quad [3c]$$

which defines the quantities β_i. Note that for population
growth, both the β's must be positive. However, as we shall
see below, positivity of the β's, while necessary, is not suf-
ficient to insure a growing population.

In [2] and [3], λ_3 is to be interpreted as a disappearance rate
of marriages, due to the death of either partner, or to divorce.
In this simple treatment, divorce or the death of one marriage
partner can be considered to result in an accretion to the

stocks of single individuals available for later fecund marriages; such an accretion is tacitly included in our fertility rates f_i.

In the discusssion to follow, we allow the fundamental parameters (f_i, λ_i) to range over all positive values, and we then qualitatively analyze the associated dynamical behavior of the system. Our procedure differs from that of previous writers, who imposed certain further restrictions on the parameters in order to explicitly take the accretions mentioned above into account. An example of such additional restrictions may be found in refs (1) and (3), where it is assumed that $\lambda_3 = \lambda_1 + \lambda_2, f_1 > \lambda_2, f_2 > \lambda_1$, and therefore that the resulting values of the β_i—i.e., $\beta_1 = f_1 - \lambda_2; \beta_2 = f_2 - \lambda_1$—are positive.

Our analysis is not restricted to these special cases, since in the age-specific models needed for more realistic description there will often be a higher average age for married persons, and for surviving partners of terminating marriages, than for single persons of the same sex—with implied effects on average (f_i, λ_i) parameters not consistent with any such simple restrictions. We do, however, note the special implications of any such postulated restrictions on the parameters.

Restrictions on the marriage function $M[N_1, N_2]$

To insure scale-independence, we ask that the (positive) marriage function M be first-degree homogeneous in its arguments, so that

$$M[qN_1, qN_2] = qM[N_1, N_2] \qquad [4]$$

for all positive q. We do not insist that $M[x, y]$ be a symmetric function of x and y.

We further suppose that no marriages take place if either stock of single individuals vanishes, so that for positive N_i

$$M[0, N_2] = 0 = M[N_1, 0]. \qquad [5]$$

Furthermore, we require that the marriage rate increase when the number of single individuals of either sex rises, so that

$$\partial M[N_1, N_2]/\partial N_i > 0 \qquad [i = 1, 2], \qquad [6]$$

again for positive (N_1, N_2).

The general class of models we consider is characterized by the set [3]— or [2]— and by Eqs. [4]–[6]. Particular cases within this class are distinguished by further detailed assumptions on the behavior of the function M. In this preliminary note, we consider the class of M functions for which

$$\lim_{N_i/N_j \to \infty} M[N_1, N_2]/N_i = 0 \qquad (i, j = 1, 2; \ i \neq j) \quad [7a]$$

and

$$\lim_{N_i/N_j \to \infty} M[N_1, N_2]/N_j = \infty \qquad (i, j = 1, 2; \ i \neq j). \quad [7b]$$

Equation [7a], which places restrictions on the behavior of $M[1, x]/x$ and $M[x, 1]/x$ for large x, can be interpreted as expressing the assumption that the more single individuals there are of one sex alone, the lower is the ultimate marriage rate, per unit of that sex.

Equation [7b], which translates as unbounded behavior of $M[1, x]$ and $M[x, 1]$, can be interpreted as the assumption that single females can, for a fixed positive supply of single males, achieve any preset rate of marriages per unit time. (That does not imply, however, that there are ever more weddings than bachelors available to be grooms!)

We note parenthetically that there exists an alternative possibility of bounded M, with

$$\lim_{N_i/N_j \to \infty} M[N_1, N_2]/N_j = C_j < \infty, \qquad (i, j = 1, 2; \ i \neq j) \quad [8]$$

in contrast to [7b]. This asymptotic requirement is favored by many demographers, and is manifested by the harmonic mean $M[x, y] = xy/(x + y)$, and also by $M[x, y] = \text{Min}(x, y)$. [See e.g., Pollard (3), Chap. 7].

It turns out that the bounded behavior [8] increases the chances of pathological sex-ratio distortion. In this preliminary report we therefore concentrate primarily on the *a priori* more favorable case [7b], and show that unstable, infinitely unbalanced, sex-ratios can still occur.

Two-dimensional intensive system

Since our marriage functions are first-order homogeneous, we can choose to write the system in terms of the ratios

$$x_1 = N_1/T_3, \qquad x_2 = N_2/T_3 \qquad [9]$$

of single individuals to married couples. This reduces [2] and [3] to the two-dimensional system

$$dx_i/dt = (\lambda_3 - \lambda_i)x_i + f_i$$
$$- (1 + x_i)M[x_1, x_2], \qquad (i = 1, 2). \quad [10]$$

The question of stability of the sex ratio reduces to a discussion of the possible existence of stable nodes in the (finite) positive quadrant of the (x_1, x_2) plane. As we shall see below, the number and distribution of critical points depend on the detailed behavior of the marriage function, $M[N_1, N_2]$.

Qualitative dynamics

For any particular marriage function, our system is defined in a five-dimensional space of positive parameters $(\lambda_1, \lambda_2, \lambda_3, f_1, f_2)$. Our basic question then is: given a particular pattern of relative sizes of these parameters, is there positive population growth, immediately or ultimately, and does the sex ratio

$$R(t) = T_2(t)/T_1(t) \qquad [11]$$

approach a constant, positive, finite value which is stable under perturbation of the initial conditions?

In particular, we are interested in determining whether there exist trajectories $T_i(t)$ which represent "interior rays" of balanced-growth, exponential change, in which

$$\mathbf{T}(t) = \mathbf{k}\, e^{rt} \qquad [12]$$

where the constant vector $\mathbf{k}$ has components $k_i > 0$, and where $k_1/k_3 > 1$, $k_2/k_3 > 1$, and r is real. (The exponential rate r is positive for population growth, negative for decay, and zero for stationary replacement.) Furthermore, if such an interior ray (or rays) does exist, we would like to know if it is stable under small and/or arbitrary perturbations of the initial conditions, and whether its stability and growth properties are altered under small changes in the fundamental parameters (f_i, λ_i).

We know that there always do exist two modes of one-sex exponential decay, which we call "exterior rays." For example, if the system begins with females only, they die out at an exponential rate λ_1, and the whole system moves along a trajectory

$$\mathbf{T}(t) = [T_1(0), 0, 0]\exp(-\lambda_1 t). \qquad [13]$$

Likewise there is also a T_2 exterior ray $[0,T_2(0),0]\exp(-\lambda_2 t)$. Note that there is no T_3 exterior ray, since any initial stock of married couples $T_3(0)$ immediately produces births at a rate

$$dN_i(0)/dt = f_i T_3(0), \qquad (i = 1,2). \qquad [14]$$

A major part of our problem is to distinguish between situations in which there exist interior rays that asymptotically determine the behavior of the system, and those in which $R(t)$ tends to 0 or ∞, so that the system eventually reaches an exterior ray on the T_1 or T_2 axis.

We can easily derive necessary conditions for the existence of interior rays by substituting [12] into [2]. This leads to the three conditions

$$k_i/k_3 = (f_i - r - \lambda_3)/(r + \lambda_i) > 0, \qquad (i = 1,2) \quad [15a]$$

$$r + \lambda_3 = M[k_1/k_3, k_2/k_3] > 0. \qquad [15b]$$

Any generalized "eigenvalue," or exponential rate r, must satisfy, along with the associated "eigenvector" $\mathbf{k}$, the basic nonlinear relations [15], in order that there can exist an internal ray. Equations [15] lead to a rather complicated classification of trajectories of [3] with respect to the general topology of our parameter space, and we therefore first discuss the simple, symmetric situation in which the birth and death rates of the two sexes are equal.

Symmetric case

Let us suppose that the birth and death rates of the two sexes satisfy the symmetry conditions

$$\lambda_1 = \lambda_2 = \lambda; \quad f_1 = f_2 = f. \qquad [16]$$

Then for symmetric initial values $N_1(t) = N_2(t) = N(t)$, [1] reduces to a *linear* system in $[N(t),N_3(t)]$ and [15] to a *linear* system in $[k,k_3]$. This case has been previously discussed in part by other writers, who suggested that it is always associated with one interior ray along which there is balanced exponential change. However, complete analysis reveals that: (a) for symmetric growth rates $r^* \leq -\lambda$—a range of values resulting from negative β's in [3]—such an internal ray need not be asymptotically reattained if the initial sex ratio $R(0)$ is perturbed; (b) even the slightest perturbation of $\lambda_1 - \lambda_2$ and/or $f_1 - f_2$ away from zero can quite possibly lead to the nonexistence of any interior ray whatsoever; (c) in the particular symmetric case with $f + \lambda = \lambda_3$, a degeneracy occurs in which there are an infinity of critical points in the (x_1,x_2) plane, and a corresponding infinity of interior rays in $[T_i]$ space.

The degenerate case is a key indicator of the topology of our parameter space, and we briefly discuss its properties. To achieve degeneracy, it is necessary to impose, in place of the symmetry conditions [16], the stronger conditions

$$\lambda_3 - \lambda = \lambda_3 - \lambda_i = f_i = f; \quad \beta_i = 0. \qquad [17]$$

The relations [17] eliminate the T_3 terms from [3a] and [3b], so that the male and female populations decay at the common exponential rate λ, and the sex ratio $R = T_2(0)/T_1(0)$ is permanently fixed.

Equation [17] implies there are an infinity of critical points of [10] in the finite (x_1,x_2) plane, and an associated infinity of interior rays in the $[T_i]$ space. To derive that result, substitute [17] in [10], and note that dx_1/dt and dx_2/dt then vanish simultaneously everywhere on the curve

$$f = M[x_1,x_2]. \qquad [18]$$

Clearly, if we break the degeneracy [17], even by an infinitesmal amount, the curves $dx_i/dt = 0$ will no longer be coincident, and, generally, there will be, at most, only a finite number of critical points in the (x_1,x_2) plane.

For illustrative purposes, let us examine a particular simple perturbation from degeneracy. We will still treat both sexes symmetrically, choosing

$$f - \lambda_3 + \lambda = f_i - \lambda_3 + \lambda_i \equiv \beta, \qquad [19]$$

but keeping $f_1 = f_2$ and $\lambda_1 = \lambda_2$.

It is easily verified that negative β in [19] corresponds to a declining population, while positive β may be associated with either growth or decline. Explicitly, from the now-linear case of [15], the general symmetric growth rate is found to be

$$r^* = -\lambda + \frac{-f - m + \beta + \sqrt{(f - \beta - m)^2 + 4fm}}{2}, \qquad [20]$$

where $r^* \gtreqless -\lambda$ depending on whether $\beta \gtreqless 0$, and where $m = M[1,1]$. The replacement criterion for the watershed between growth and decay is $r^* \gtreqless 0$ depending on whether $mf \gtreqless \lambda_3(\lambda + m)$. For $\beta < 0$, the symmetric critical point is a saddle; for $\beta > 0$ it is a stable node.

We will see in the following section that the incompatibility of unbalanced sex-ratio pathology with population increase, found in this special symmetric case, carries over to more general considerations in which there is an arbitrary pattern of birth and death rates (λ_i, f_i).

Our summary of the general case will further show that infinitesmal variations of (λ_i, f_i) around the degeneracy point [17] or around the symmetry axes of [16] can cause the system to lose all its finite critical points, so that its asymptotic behavior in the extensive space $[T_i]$ takes it to an exterior ray with a sex ratio of zero or infinity.

Summary of results

To display our general results, it is helpful to subdivide the parameter space (λ_i, f_i) into regions defined by certain inequality relations. For that purpose, we introduce the quantities

$$\Delta_i = \lambda_3 - \lambda_i. \qquad (i = 1,2) \qquad [21]$$

which incapsulate the relevant roles of the λ's in [10]. Each distinct sector of parameter space will then be categorized, for example as

$$\Delta_1 < f_1 < f_2 < \Delta_2, \qquad [22]$$

there being essentially 4! such distinct cases to analyze. The detailed behavior of trajectories $\mathbf{T}(t)$ associated with each such sector is summarized in Table 1, where we have avoided redundancy by listing only the $4!/2$ sectors with $f_2 > f_1$.

As shown in table 1, there turn out to be five topologically distinct possibilities:

(A) One stable critical point in the finite (x_1,x_2) plane. This occurs for sectors 1 and 2 in Table 1. Here, as in Lotka's linear analysis, all trajectories asymptotically result in a balanced sex ratio; overall, the population may grow or shrink. Case (A) is the only one which can be compatible with the growth of all components.

(A') K+1 stable nodes, K saddle points in the finite x_i plane. This applies to sector 3 in Table 1. In this case, and also for (B) and (C) below, it can be shown that in general a chain of critical points exists, with saddle points and stable nodes

TABLE 1. *Dynamics of scale- and age-independent reproductive systems, with unbounded, homogeneous marriage functions of order one*

| Sector | Condition | Behavior in (x_1,x_2) plane | | | Behavior in $[T_i]$ space | |
| | | Critical points[a] | | | | |
		Stable	Saddle	Case[d]	Range of growth rates r, for interior rays[e]	Population growth or decay
1	$\Delta_2 < \Delta_1 < f_1 < f_2$	1	0	A[f]	$[-\lambda_1, f_1 - \lambda_3]$	Either
2	$\Delta_1 < \Delta_2 < f_1 < f_2$	1	0	A[f]	$[-\lambda_2, f_1 - \lambda_3]$	Either
3	$\Delta_2 < f_1 < \Delta_1 < f_2$	$1 + K$	K	A'	$[f_1 - \lambda_3, -\lambda_1]$	Decay
4	$f_1 < f_2 < \Delta_2 < \Delta_1$	K	$1 + K$	B	$[f_2 - \lambda_3, -\lambda_2]$	Decay
5	$f_1 < f_2 < \Delta_1 < \Delta_2$	K	$1 + K$	B	$[f_2 - \lambda_3, -\lambda_1]$	Decay
6	$f_1 < \Delta_2 < f_2 < \Delta_1$	K	$1 + K$	B	$[-\lambda_2, f_2 - \lambda_3]$	Decay
7	$\Delta_2 < f_1 < f_2 < \Delta_1$	K[b]	K[b]	C	$[f_1 - \lambda_3, f_2 - \lambda_3]$	Decay
8	$f_1 < \Delta_2 < \Delta_1 < f_2$	K[c]	K[c]	C	$[-\lambda_2, -\lambda_1]$	Decay
9	$\Delta_1 < f_1 < \Delta_2 < f_2$	No critical points		D[f]	Decaying exterior ray	
10	$\Delta_1 < f_1 < f_2 < \Delta_2$	No critical points		D	Decaying exterior ray	
11	$f_1 < \Delta_1 < f_2 < \Delta_2$	No critical points		D	Decaying exterior ray	
12	$f_1 < \Delta_1 < \Delta_2 < f_2$	No critical points		D	Decaying exterior ray	
—	$f = f_1 = \Delta_1 = f_2 = \Delta_2$	∞	0	Degenerate[d]	$-\lambda$	Decay
—	$\Delta_1 = \Delta_2 < f_1 = f_2$	1	0	Symmetric[d]	$[-\lambda, f - \lambda_3]$	Either
—	$f_1 = f_2 < \Delta_1 = \Delta_2$	K	$1 + K$	Symmetric[d]	$[f - \lambda_3, -\lambda]$	Decay

Shown here is the qualitative behavior of trajectories of the differential equation system [3], and of the associated two-dimensional system [10].

The parameter space $(\Delta_1 = \lambda_3 - \lambda_1; \Delta_2 = \lambda_3 - \lambda_2; f_1; f_2)$ has been divided into four sectors defined by the inequality relations listed in column two. Behavior in the twelve regions with $f_2 > f_1$ is shown; behavior in the remaining twelve regions is easily abstracted from the table by consistently interchanging subscripts 1 and 2. Also shown is trajectory behavior for two symmetric cases discussed in the *text*.

[a] Our listing of critical points assumes, as is almost always the case, no points of tangency between the curves $dx_i/dt = 0$ of equation [10]. Note also that by suitably choosing the marriage rate scale $M[1,1]$, the integer K characterizing the number of critical points in sectors 3 through 8 can be made to vanish.

[b] These critical points exist only for sufficiently large $M[1,1]$.

[c] These critical points exist only for sufficiently small $M[1,1]$.

[d] See discussion in *text*.

[e] Range of possible exponential rates r, in parameter space, for interior ray solutions of the form $\mathbf{k}e^{rt}$, with least values listed first. As the marriage rate scale $M[1,1]$ runs from 0 to ∞, r takes all values in the given range. Note that the range of r in each case corresponds to the central interval in column two. When there are multiple possible r's, as in sectors 3–8, all fall in the specified range.

[f] Sectors 1, 2, and 9 have two positive β's, as required by the Kendall-Pollard reasoning discussed in the *text*.

distributed roughly along a hyperbolic curve whose asymptotes lie along the axes $x_1 = 0$ and $x_2 = 0$. Saddle points and stable nodes alternate along this hyperbola. For case (A'), the two nodes at the ends of the chain are stable; all trajectories therefore result in balanced sex ratios, just as in case (A). For case (A'), the population shrinks.

Remark: In cases (A'), (B) and (C) certain singular values of the parameters result in points of tangency between the two curves $dx_i/dt = 0$ of [10], and there can even be an uncountable infinity of points in common. Furthermore, it can be shown that for particularly simple choices of $M[x_1,x_2]$, e.g. $M = x_1^\alpha x_2^{1-\alpha}$, $0 < \alpha < 1$, there is at most a pair of critical points in the finite plane. We note also that all our critical points are associated with real exponents. There are no complex foci, and there is no limit cycle behavior in the x_i plane.

(B) $K+1$ saddle points, K stable nodes in the finite x_i plane.

This occurs for sectors 4–6 in Table 1, in which at least one of the β_i is negative. In general, we again have a chain of critical points, but here, unlike (A'), the two ends of the chain are saddle points. Therefore, trajectories lying beyond the two separatrices dividing the saddle points at the ends of the chain from their stable-node neighbors will always result in unbalanced sex ratios. In case (B) the population shrinks.

Cases (A) and (B) occur when

$$U = (f_1 - \Delta_2)(f_2 - \Delta_1) = (f_1 - \lambda_3 + \lambda_2)(f_2 - \lambda_3 + \lambda_1) > 0$$
$$[23]$$

It can be shown from [10] that the relation [23] insures that the curves $dx_1/dt = 0$ and $dx_2/dt = 0$ cross at least once in the positive quadrant of the (x_1,x_2) plane. Cases (C) and (D), to be discussed below, are associated with negative U.

(C) K stable nodes, K saddle points in x_i plane.

This occurs for sectors 7 and 8 in Table 1. The chain of critical points now has a saddle point on one end, and a stable node on the other. There is now a watershed defined by the separatrix dividing the last saddle point from its neighboring stable node. Trajectories on the saddle point side of this separatrix always result in unbalanced sex ratios.

Cases (A'), (B), and (C) have the property that K can be reduced to zero, independently of the shape of $M[x_1,x_2]$, by changing the relative scale of the marriage function. In particular, if we write $M[N_1,N_2] = M[1,1] \left\{ M[N_1,N_2]/M[1,1] \right\}$ and permanently fix the expression in curly brackets, then $M[1,1]$ becomes a parameter characterizing the "strength"—or scale—of the marriage function, and this strength plays a role in determining K in sectors 3 through 8.

In case (C)—as in Cases (A') and (B)—the population shrinks.

(D) No critical points in the finite (x_1,x_2) plane. In these

cases, sectors 9–12 in Table 1, one sex has both a higher birth rate and a lower death rate than the other, with the result that the sex ratio asymptotically approaches 0 or ∞. Specifically, a sufficient condition for the nonexistence of a balanced-growth path is

$$(\Delta_1 - \Delta_2)(f_1 - f_2)(f_2 - \Delta_1)(f_1 - \Delta_2) < 0. \qquad [24]$$

The Kendall-Pollard restriction to positive β's rules out the saddles of sections 4–6, and also the exterior ray sectors 10–12, while permitting sector 9. Sector 9's disastrous blowup of the sex ratio may seem intuitively obvious from the fact that one of the sexes there has a higher birth rate and a lower death rate than the other. But such intuitive explanations are dangerous for at least two reasons: (1) $\lambda_1 > \lambda_2$ and $f_1 < f_2$ causes *no* infinite distortion of the asymptotic sex ratio in sector 2. In sector 2 birth rates will be sufficiently high, and the sex ratio N_2/N_1 asymptotically approaches a positive limit, greater than unity but finite. (2) The following brief discussion of a non-marriage, bisexual deterministic model shows one can always avoid infinite sex ratios, even with $\lambda_1 \gg \lambda_2$ and $f_1 \ll f_2$, if monogamy does not prevail.

A bisexual, non-marriage model

Consider, along with the earlier cited writers, a system with no marriage institution, and write, instead of [2] and [3]:

$$dN_1(t)/dt = -\lambda_1 N_1(t) + f_1 H[N_1(t),N_2(t)] \qquad [25a]$$

$$dN_2(t)/dt = -\lambda_2 N_2(t) + f_2 H[N_1(t),N_2(t)] \qquad [25b]$$

where $H(x,y)$ is again an unbounded, positive function, homogeneous of order 1, vanishing on the boundaries of the positive quadrant, and possessing a positive gradient, just as did M.

The system [25] always moves asymptotically to a finite sex-ratio, as one easily sees by transforming to a one-dimensional system in $Z = N_2/N_1$,

$$dZ/dt = (\lambda_1 - \lambda_2)Z + H(1,Z)(f_2 - f_1 Z) \qquad [26]$$

and using the properties of H to show that dZ/dt changes sign from positive to negative once along the positive Z axis, independent of the signs or magnitudes of $\lambda_2 - \lambda_1$, f_2 and f_1.

Population decline and sex-ratio pathology

Finally, we note that a striking feature of the results of Table 1 is the absence of pathological behavior of the sex ratio in the presence of overall population growth. Growth is always associated with a unique, stable interior ray. There is a simple geometric explanation of why this is so. Visualize $[T_i]$ space, with a vertical T_3 axis. Then note, from [3], that above (below) their respective planes

$$-\lambda_i T_i + (\lambda_i + f_i - \lambda_3)\ T_3 = 0 \qquad (i = 1,2), \qquad [27]$$

dT_1/dt and dT_2/dt are positive (negative). Similarly, below (above) the hyperbolic surface

$$-\lambda_3 T_3 + M[T_1,T_2] = 0, \qquad [28]$$

dT_3/dt is positive (negative). For sufficiently large f_1 and f_2, the planes [27] intersect the surface [28] in such a way as to create an irregularly shaped cone, with its apex at $T_i = 0$, within which all three dT_i/dt are positive. With the exception of the apex, the surface of this cone lies completely within the positive octant, and all trajectories entering it can never escape, but instead go off to infinity. Again, the cone is bounded above by the surface [28], and below by the planes [27].

This trapping phenomenon for growing population is a feature of systems with marriage functions even more general than those we consider here. The essential properties of M which insure trapping are that it vanish for vanishing stocks of single individuals, and that it grow at least linearly as its arguments increase together.

Conclusion

In conclusion, it is important to emphasize that models such as ours lack two features which play crucial roles in the population dynamics of real societies: (1) the effects of biomedical and cultural change on mortality and fertility; (2) the feedback caused by resource limitations on output. Such effects might be admitted by allowing our fundamental parameters to be explicit functions of time, and of population density itself.

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