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Some Distributions of Time to Failure for Reliability Applications

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Knowledge gained in the study of the physics of failure has lead to more meaningful distributions of time to failure for components subject to failure. This work expands the application of this knowledge by developing mathematical models for failure based upon physical failure models. The form of the distribution of time to failure may then be derived under certain conditions. This model applies to components for which failure can be associated with some extreme random phenomenon such as the largest flaw or impurity, etc. The method is illustrated for two cases which are likely to be found in applications. In their simplest form these two cases reduce to distributions found empirically to be useful in reliability.

Key Words

Failure Model
Extreme Value Distribution

Reliability ($R(t)$) is a measure of the ability of a device to perform without failure for a given period of time t. In terms of the distribution of time to failure random variable (W) for the device $R(t)$ is defined as

$$R(t) = 1 - F_W(t) = P(W > t),$$

where $F_W(t)$ is the distribution function of time to failure. Several distributions have, through experience, been found useful in reliability work [Gnedenko, 1969, page 91]. Because there is no general theory to guide a practitioner in the choice of a distribution, some distributions such as the Weibull have become overworked.

A relatively new body of knowledge has been assembled which has led to the development of high reliability components. The knowledge has been gained through the study of the physics of failures. These studies have also led to more meaningful distributions of time to failure [Shooman, 1968].

The purpose of this paper is to further investigate the application of failure models to distribution theory. The contribution of this work lies in the definition of a mathematical structure for deriving density functions from a failure model based on extreme values. Two special cases are considered which conform to much that is known or postulated concerning the physics of failure. This work is a generalization of a model for failures given by Gumbel [1958, p. 249]. This work is also similar to that of Cohen [1974] in that assumptions governing the failure mechanisms are similar and based upon extreme values. A special case of theorem 3.1 of this work is proved in Cohen [1974].

It is assumed throughout that a component has potentially many sources of failure. This assumption is the basis of a model developed by Shooman [1968]. In his work he assumes that a component is made up of many elementary building blocks. The strength of the part is dependent on the strength and configuration of these blocks, the strength of each block may be visualized as an observed random variable. In this paper it is assumed that there may be associated with each block or potential failure within a component a random variable "time to failure." The actual time to failure is the minimum value of all those random variables describing the device. This association with extreme value theory leads very naturally to extensions of the classical theory.

The first section of this paper reviews the portions of extreme value theory and background material basic to the development which follows. A failure model is described in the second section. In the third section distributions of time to failure are derived which correspond to two typical applications. A method of estimation is given in section four.

1. Review

Two distribution functions are said to be of the same type if one may be obtained from the other by a linear transformation of the random variable associated with it.

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Let $X_1, X_2, \cdots$ be a sequence of independent random variables with common d.f. $F(x)$. Let $Z_n = \min(X_1, X_2, \cdots, X_n)$, the d.f. of Z_n is given by $P(Z_n \leq x) = 1 - P(X_1 > x, X_2 > x, \cdots, X_n > x) = 1 - [1 - F(x)]^n$. It has been shown by Gnedenko [1943] that if there exists a d.f. $\Phi(x)$ and norming sequences $\{a_n\}$, $a_n > 0$ and $\{b_n\}$ such that

$$\lim_{n\to\infty} P[a_n^{-1}(Z_n - b_n) \leq x] = \lim_{n\to\infty} \{1 - [1 - F(a_n x + b_n)]^n\} = \Phi(x)$$

at each continuity point of $\Phi(x)$, then $\Phi(x)$ belongs to one of three types denoted $\Phi_i(x) i = 1, 2, 3$. Of particular interest in reliability is the so called "limited" type which includes the Weibull distribution. The limiting form is of the type

$$\Phi(x) = \begin{cases} 1 - \exp[-x^\alpha] & x \geq 0, \quad \alpha > 0 \\ 0 & x < 0 \end{cases} \tag{1.1}$$

The elements of the sequence b_n are constant. The great majority of distributions found useful in reliability have this extreme value form. However, another type may be possible for distributions which are limited [Gnedenko, 1943].

Gnedenko has also shown that there exists sequences $\{a_n\}$, $a_n > 0$ and $\{b_n\}$ such that

$$\lim_{n\to\infty} [1 - [1 - F(a_n x + b_n)]^n] = 1 - \Phi(x)$$

if and only if

$$\lim_{n\to\infty} nF(a_n x + b_n) = -\ln(1 - \Phi(x)]. \tag{1.2}$$

The following generalization of Polya's Theorem [1920] is needed. The proof follows exactly that of Polya's Theorem.

Theorem 1.1 Let $G_n(x)$ be a sequence of non-decreasing functions which converge to a continuous distribution function $F(x)$ at each point x. The convergence is uniform.

2. Model of Device Failure

In this section a model of device failure is given which motivates an extension of the classical extreme value theory.

An analysis of the failures of many different populations indicates that the failures for some populations of components may be classified into one or more different modes or causes (Wright, 1968). For example, three primary modes of failure have been classified as being typical of electromechanical devices: 1) failures due to manufacturing defects, 2) random failures for which no direct cause can be determined, and 3) wear-out failures [Krohn, 1969].

The existance of several modes of failure motivates the following failure model for components.

A component is construed to be made up of many "elements." These "elements" are such that: 1) each is capable of failure; 2) the failures of the elements are independent chance events; 3) the total number of elements within a component (N) is a chance event; 4) of the N elements there are at most K distinct types, each having its own probability distribution of time to failure.

The actual time to component failure for a component which satisfies this model is the minimum value of the random variables, time to failure, describing each of the elements.

Since the "elements" of the forgoing definition are the sources of failure they will be referred to as potential failures. We shall be interested in the distribution of time to component failure as the expected number of potential failures diverges to infinity.

The microscopic study of defects in semiconductor materials illustrates the suitability of this model. For example it is reported that defects originating within germanium epitaxial films show up in two forms. The difference may be only the presence or absence of stress while the defect is forming [Schroeder, Ed., 1961, p. 149]. Thus the hypothesis of a multinomial distribution of type of defect seems reasonable while the total number of defects is independent of the conditional distribution of type. When postulating the use of the Poisson distribution in a particular application, it is seldom possible to verify mathematically the verasity of the Poisson assumptions. This verification is usually left to intuition. Here it seems reasonable that two defects cannot occupy the same position, for small areas the expected number of defects is proportional to the size of the area being observed, and that the distribution of defects is plane invariant. Thus the generality of the Poisson assumptions leads to a reasonable distribution for total number of defects in a germanium film of carefully controlled dimension. The hypothesis that a large number of potential failures exists is verified in this case as Schroeder, Ed. [1961, p. 149] reports concentrations as high as 3×10^8 cm^{-2}. A Poisson distribution of number of potential failures will be referred to as Model I.

Schroeder, Ed. [1961, p. 168], reports that the number of dislocation faults in evaporated germanium films depends strongly on the cleanliness of the substrate surface prior to evaporation. It seems reasonable to suppose that the number of faults is again Poisson distributed. However, the environmental conditions which affect surface contamination may vary from day to day. This leads to consideration of the negative binomial distribution in lue of the Poisson as discussed by Williamsen and Bretherton [1963, p. 9]. The negative binomial distribution of the number of potential failures will be referred to as Model II.

These cases serve as examples only. The development which follows is carried out in sufficient generality to permit a wide variety of applications.

3. Mathematical Models for Component Reliability

In this section the model given in Section II is developed mathematically to provide reliability functions which are consistent with the failure concepts.

Suppose that a component has K modes of failure (e.g., $K = 3$, manufacturing defects, random, and wear out). Associated with these modes is the set of random variables

$$R_K = \{X_1, \cdots, X_K\}$$

where R_K contains exactly K random variables and where X_k is the time to failure for potential failures of mode k, $k = 1, 2, \cdots, K$.

Let N_k be the number of potential failures of type k, $k = 1, 2, \cdots, K$. Let $N = \sum_{k=1} N_k$ be the number of potential failures in a component. It is assumed that the conditional distribution of the N_k given the total N is multinominial. This does not seem to be overly restrictive since an N_k which is independent of the rest can be treated separately. Also for the important case of the Poisson distribution for the number of faults of type k if the N_k are independent, the conditional distribution of types of defects is multinomial given the total.

Theorem 3.1 Let $R_K = \{X_1, \cdots, X_K\}$ be a set of K random variables with distribution functions $F_k(x)$, $k = 1, 2, \cdots, K$. Let N_k, $k = 1, 2, \cdots K$ be any set of K integer-valued random variables such that the conditional distribution of N_k given $N = \sum_{k=1}^{K} N_k$ is multinomial, i.e.,

$$P(N_1 = n_1, \cdots, N_K = n_K \mid N = n) = \frac{n!}{n_1! \cdots n_K!} p_1^{n_1} \cdots p_K^{n_K}.$$

where $p_k > 0$, $k = 1, 2, \cdots, K$ and $\sum_{k=1}^{K} p_k = 1$ Let N be a random variable with probability generating function $\gamma(s)$. Let the following sequence of random length N

$$Y_1, Y_2, \cdots Y_N$$

be any collection of N random independent variables in R_K such that N_k are of the type X_k with distribution function $F_k(x)$, $k = 1, 2, \cdots, K$. Define

$$Z_N = \begin{cases} & \text{if } N = 0 \\ \sum\limits_{min}^{\infty} \{Y_1 . \cdots, Y_N\} & N > 0. \end{cases}$$

Then

$$P(Z_N > t) = \gamma\left(1 - \sum_{k=1}^{K} p_k F_k(t)\right)$$

Proof.

$$P(Z_N > t) = \sum_{n=0}^{\infty} P(Z_N > t \mid N = n) p(N = n). \quad (3.1)$$

But

$$P(Z_N > t \mid N = n) = \sum{}^{*} P(Z_N > t \mid N_1 = n_1, \cdots N_K = n_K) P(N_1 = n_1, \cdots N_K = n_K)$$

where $\sum^{*}$ is over all sets of $n_1, \cdots, n_K$ such that $\sum_{k=1}^{K} n_k = n$. Thus

$$\begin{aligned} P(Z_N > t \mid N = n) &= \sum^{*} \left[\prod_{k=1}^{K} (1 - F_k(t))^{n_k}\right]\left[\frac{n!}{n_1 \cdots n_k!} \prod_{k=1}^{K} p_k^{n_k}\right] \\ &= \sum^{*} \frac{n!}{n_1! \cdots n_k!} \prod_{k=1}^{K} [p_k(1 - F_k(t))]^{n_k} \\ &= \left[\sum_{k=1}^{K} p_k(1 - F_k(t))\right]^n \\ &= \left[1 - \sum_{k=1}^{K} p_k F_k(t)\right]^n. \end{aligned} \quad (3.2)$$

Combining (3.1) and (3.2)

$$\begin{aligned} P(Z_N > t) &= \sum_{n=0}^{\infty} \left[1 - \sum_{k=1}^{K} p_k F_k(t)\right]^n p(N = n) \\ &= \gamma\left(1 - \sum_{k=1}^{K} p_k F_k(t)\right) \end{aligned}$$

Model 1.

Theorem 3.2 Let Z_N, R_K, $F_k(x)$ and N_k, $k = 1, 2, \cdots, K$ be defined as in the Theorem 3.1. Let N have a Poisson distribution with parameter λ. Suppose each $F_k(x)$ is in the domain of attraction of one of the classical limits $\Phi_{i_k}(x)$ ($i_k = 1, 2$, or 3). Then given $\epsilon > 0$ there exists a positive number Λ such that for integer values of $\lambda \geq \Lambda$

$$\left|P(Z_N > t) - \prod_{k=1}^{K} \left\{1 - \Phi_{i_k}\left(\frac{t - b_\lambda(k)}{a_\lambda(k)}\right)^{p_k}\right\}\right| < \epsilon$$

for all t.

Proof. The probability generating function of a Poisson random variable with mean λ is $\gamma(s) = e^{-\lambda(1-s)}$. By Theorem 3.1,

$$P(Z_N > t) = \exp\left\{-\lambda \sum_{k=1}^{K} p_k F_k(t)\right\}.$$

By Gnedenko [1943], and (1.2),

$$\lim_{\lambda \to \infty} \{\exp - \lambda F_k(a_\lambda(k) x_k + b_\lambda(k))\} = 1 - \Phi_{i_k}(x_k).$$

By theorem 1.1 the convergence above is uniform. Therefore

$$\lim_{\lambda\to\infty} \exp\left\{-\lambda \sum_{k=1}^{K} p_k F_k(a_\lambda(k)x_k + b_\lambda(k))\right\} = \prod_{k=1}^{K} [1 - \Phi_{i_k}(x_k)]^{p_k}$$

uniformly in x_k, $k = 1, \cdots, K$. Thus given $\epsilon > 0$ there exists $\Lambda > 0$ such that for all integer values of $\lambda \geq \Lambda$

$$\left|\prod_{k=1}^{K} \exp\left[-p_k\lambda F_k(a_\lambda(k)x_k - b_\lambda(k))\right] - \prod_{k=1}^{K} [1 - \Phi_{i_k}(x_k)]^{p_k}\right| < \epsilon. \quad (3.3)$$

For fixed $\lambda > \Lambda$ choose x_k such that

$$a_\lambda(k)x_k + b_\lambda(k) = t \text{ for all } k.$$

Then (3.3) becomes

$$\left|\prod_{k=1}^{K} \exp\{-p_k\lambda F_k(t)\} - \prod_{k=1}^{K}\left[1 - \Phi_{i_k}\left(\frac{t - b_\lambda(k)}{a_\lambda(k)}\right)\right]^{p_k}\right| = \left|P(Z_N > t) - \prod_{k=1}^{K}\left[1 - \Phi_{i_k}\left(\frac{t - b_\lambda(k)}{a_\lambda(k)}\right)\right]^{p_k}\right| < \epsilon$$

Model 2. Negative binomial N

Theorem 3.3 Let R_K, $F_k(x)$ and N_k, $k = 1, \cdots K$ be defined as in Theorem 3.1. Let N have the negative binomial distribution with parameters p and m and $\mu = p/(1 - p)$. Suppose each $F_k(t)$ is in the domain of attraction of one of the classical limits $\Phi_{i_k}(t)$ ($i_k =$ 1, 2, or 3). Then given $\epsilon > 0$ there exists a positive number Λ such that for all integer values of $\mu > \Lambda$

$$\left|P(Z_N > t) - \left[1 - \sum_{k=1}^{K} \ln\left(1 - \Phi_{i_k}\left(\frac{t - b_\mu(k)}{a_\lambda(k)}\right)\right)^{p_k}\right]^{-m}\right| < \epsilon.$$

Proof. The probability generating function for the negative binomial distribution is

$$\gamma_N(s) = \left(\frac{1 - p}{1 - sp}\right)^m.$$

By Theorem 3.1

$$P(Z_N > t) = \gamma_N\left(1 - \sum_{k-1}^{K} p_k F_k(t)\right) = \left[\frac{1 - p}{1 - p\left(1 - \sum_{k=1}^{K} p_k F_k(t)\right)}\right]^m$$

$$= \left[\frac{1 - p}{1 - p + \sum_{k=1}^{K} p_k p F_k(t)}\right]^m = \left[1 + \sum_{k=1}^{K} p_k \mu F_k(t)\right]^{-m}$$

where $\mu = p/(1 - p)$.

As in Theorem 3.2 for integral values of μ

$$\lim_{\mu\to\infty}\left[1 + \sum_{k=1}^{K} p_k\mu F(a_\mu(k)x_k + b_\mu(k)\right]^{-m} = \left[1 - \sum_{k=1}^{K} p_k \ln(1 - \Phi_{i_k}(x_k))\right]^{-m}$$

uniformly in the x_k. Let

$$x_k = \frac{t - b_\mu(k)}{a_\mu(k)}.$$

Then given $\epsilon > 0$ for sufficiently large μ

$$\left|P(Z_N > t) - \left[1 - \sum_{k=1}^{K} p_k \ln\left(1 - \Phi_{i_k}\left(\frac{t - b_\mu(k)}{a_\mu(k)}\right)\right)\right]^{-m}\right| = \left|P(Z_N > t) - \left[\prod_{k=1}^{K} \ln\left(1 - \Phi_{i_k} \cdot\left(\frac{t - b_\tau(k)}{a_\tau(k)}\right)\right)^{p_k}\right]^{-m}\right| < \epsilon.$$

Assuming extreme value limits of the type (1.1) reliability formulas corresponding to Theorem 3.2 and 3.3 may be summarized as follows

Model 1:

$$R(t) \simeq \exp\left\{-\sum_{k=1}^{K} (t/\theta_k)^{\alpha_k}\right\}$$

where $\theta_k = a_\lambda(k)/p_k^{1/\alpha_k}$.

Model 2:

$$R(t) \simeq \left[1 + \sum_{k=1}^{K} (t/\theta_k)^{\alpha_k}\right]^{-m}$$

where $\theta_k = a_\mu(k)/p_k^{1/\alpha_k}$.

It is assumed here that time to failure is limited below by 0.

Complex hazard functions can be formed by proper choice of the parameters θ_k, α_k. For example if $K = 3$, the bathtub hazard function results if $\alpha_1 < 1$, $\alpha_2 = 1$ and $\alpha_3 > 1$.

It is interesting to view these reliability functions for $K = 1$, i.e., one mode of failure. Model 1 results in the Weibull formula and Model 2 results in the power distribution [Gnedenko, 1969, p. 93]. These distributions have been found useful in reliability theory.

4. Estimation

The usefullness of this model depends upon the availability of estimation techniques. Since the reliability models derived here apply to complicated failure structures, several parameters are necessary in the distributions. Accordingly estimation is difficult in the general case. In certain cases of special interest to reliability, numerical solutions to the estimation equations are manageable. These cases are considered after the general case is presented. Since numerical solutions will certainly involve computers, the estimation equations are presented in matrix form.

A least squares technique reported by Bain and Antle (1967) for the Weibull distribution is particularly suited to the models given here. Consider first the problem of estimation of the parameters of Model 1.

The problem of estimation is: Given the random sample $X_1, X_2, \cdots, X_n$ from a member of the family of distributions

$$F(x) = 1 - \exp\left[-\sum_{k=1}^{K} \theta_k x^{\alpha_k}\right],$$

estimate θ_k and α_k, $k = 1 \cdots K$. For convenience the parameterization of the model has been changed.

Let $X_{(i)}$ be the ith order statistic. Then

$$E[1 - F(X_{(i)})] = \frac{n+1-i}{n+1}.$$

The estimates of the parameters are taken to be those values which minimize the expression

$$\begin{aligned}\Psi &= \sum_{i=1}^{n} \{\ln E[1 - F(X_{(i)})] - \ln(1 - F(X_{(i)}))\}^2 \\ &= \sum_{i=1}^{n} \left\{\ln\left[\frac{n+1-i}{n+1}\right] + \sum_{k=1}^{K} \theta_k X_{(i)}{}^{\alpha_k}\right\}^2 \qquad (4.1)\end{aligned}$$

Let

$$\Theta' = (\theta_1, \theta_2, \cdots, \theta_K)$$

$$L' = -\left(\ln\frac{n}{n+1}, \ln\frac{n-1}{n+1}, \ln\frac{n-2}{n+1}, \cdots, \ln\frac{1}{n+1}\right)$$

$$D = \operatorname{diag}(\ln X_{(1)}, \ln X_{(2)}, \cdots, \ln X_{(n)})$$

and A a $K \times n$ matrix such that

$$a_{ij} = X_{(j)}{}^{\alpha_i}.$$

By evaluating the appropriate derivatives, the values of the parameters which minimize (4.1) are those which satisfy the following two equations.

$$AA'\Theta = AL \qquad (4.2)$$

$$ADA'\Theta = ADL \qquad (4.3)$$

If the $X_{(j)}$ and the α_i are unique A will be full rank so that

$$\Theta = (AA')^{-1}AL. \qquad (4.4)$$

Substituting (4.4) into (4.3) leads to

$$ADA'(AA')^{-1}AL = ADL \qquad (4.5)$$

These equations involve the α_k's only. Unfortunately an analytical solution is not available. If there are several α_k's a numerical solution is difficult to say the least. However, consider the following case.

As was noted in Section 2, it is common to have three modes of failure, i.e., manufacturing defects, random and wearout failures—$K = 3$; and further the random mode is described as having a constant hazard function. For this mode the $\alpha_2 = 1$. Thus the reliability may be written

$$R(t) = \exp[-\theta_1 t^{\alpha_1} - \theta_2 t - \theta_3 t^{\alpha_3}] \qquad (4.6)$$

For this case (4.5) may be simplified to

$$BDB'(BB')^{-1}BL = BDL \qquad (4.7)$$

where the B matrix is formed by deleting the second row of the A matrix. There are only 2 equations in (4.7) for which a numerical solution for α_1 and α_3 is manageable. The solution for Θ remains (4.4).

It is typical of electronic components to exhibit an extremely long life so that in general it is not possible to conduct life tests until all items on test fail. Thus in general no data is available on the wearout mode of failure if it exists at all. The parameters Θ_3 and α_3 may then be dropped from (4.6) and estimation simplifies accordingly. Note that the method of estimation permits truncation of data, that is, all items on test need not fail for the method to be used.

A similar method of estimation may be used for Model 2. Estimates for the parameters may be found by minimizing

$$\psi = \sum_{i=1}^{n} \{E[(1 - F(X_{(i)})]^{-1/m} - (1 - F(X_{(i)})^{-1/m}\}^2$$

with respect to the θ_k, α_k, $k = 1, \cdots K$ and m. The simplifications described for Model 1 also apply here.

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