6.337 Mathematical Modelling

Term Project:

Population and Technology:

A Professor - Student Model for

The Growth and Progress of Knowledge

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1.0 Introduction:

1.1 Background:

Throughout the Modern Period, the growth of knowledge, prosperity, and technology has been unprecedented. The level of technology, as we today see it, is higher than ever before. Developments are occurring at an ever faster rate. Those living twenty years ago could make exactly the same statement about the growth of technology. Again, the same statement applies to those living one hundred years ago. All through the past five hundred years, what we call the Modern Period, people commented on the growth of knowledge and technology as never seen before.

Also through the Modern Period, the population of all parts of the world has experienced unprecedented growth. History does not record any other time where there were so many people on this planet. While there physically has to be some limitation on the world's population, nobody knows what it is or what population level is ideal. The growth of population and its implications have been a source of much speculation for at least two hundred years.

Among the most famous discussions of population growth, overpopulation and their effects on civilization comes from Thomas Malthus, for whom the Malthusian Model is named. Population, he argues, grows exponentially, while the means of supporting them only grows linearly. In times where population growth outstrips economic growth, there is increasing crime, war, misery, starvation, and a general decay of civilization.

History shows that tough times due to overpopulation did not outlive Malthus for very long, since he lived on the very eve of the Industrial Revolution. A large number of people is also a large number of minds to solve problems and innovate. It is very difficult to foresee how a larger number of people can live in this world since we can see only what technology exists today. Again, today, people speculate on the influence of overpopulation on civilization.

1.2 Objective:

The objective of this study is to explore ~~the possibility that~~ how population growth. affects and drives the growth of knowledge and technology. This requires a mathematical model which links population and knowledge. With a model linking the growth of knowledge to population, it is then possible to show what happens when populations increase, stabilize, ~~and~~ or decay.

1.3 Modelling of Relative Quantities:

The great challenge in developing a model to link the growth of knowledge to population is that the level of knowledge and technology is a relative quantity. While it is possible to physically measure a population by counting individuals, there is no absolute measure to quantify knowledge. The challenge is to develop a mathematical model, a system of equations relating absolute quantities, that meaningfully follows the relative quantity of knowledge.

Discussion of relative quantities tends to be argumentative rather than analytical in nature. This allows much liberty in making assumptions to base a mathematical model. While the assumptions are quite necessary in simplifying a model to meaningful terms, it is also quite possible to make a model imply anything to fit an argument. There is no means of analytically testing the validity of a mathematical model based on relative quantities. This is the principal weakness of mathematical models based on relative quantities.

1.4 Analogy to Economics:

A closely related field in this respect is economics, since wealth is a relative quantity. An economist studies and compares the wealth of individuals, families, businesses, and nations during various time periods and under various conditions. Again, the challenge in developing an economic model is in using absolute quantities to model relative quantities.

Economists attempt to circumvent this problem through the use of indicators. An indicator is an absolute, measurable quantity that bears a general relationship to the relative quantity that one is really trying to measure. Ideally, there is a direct relationship between the indicator and the relative quantity to support the validity of the indicator.

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In the field of economics, there are many indicators to measure the wealth of individuals, families, businesses, and nations. For individuals and families, these include real income and net worth. For businesses, revenues and profits indicate wealth. And for nations, the common economic indicators include productivity, auto and steel production, and GNP among many others. These indicators attempt to measure wealth, which is again a relative term comparing the living conditions of people from one time to another.

With the aid of indicators, and a general relationship between the indicator and the relative quantity, it is then possible to compare the wealth of people living under different conditions. Economists can use the indicators do develop models to follow the fluctuations of these indicators. These models then allow conclusions about the overall health of the economy and the wealth of its people. Naturally, the validity of an economic model and its implications depends on the validity of the indicators themselves.

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1.5 Population, Wealth, and Technology:

The first step in modelling a relative quantity, such as the growth of knowledge and technology, is to find a simple, reliable, and appropriate indicator. Perhaps the simplest indicator of technology growth is population change. The number of people working in a field of science reflects the amount of knowledge accumulated and the corresponding level of technology.

The greater the number of people working in a field, the more possible it is for them to specialize and become more effective at what they do. This is, as Adam Smith [1] puts it, an example of the division of labour. When a number of people work together to perform a task, it is possible to subdivide the task into a multitude of separate tasks. The people working at the subtasks develop a specialized knowledge of their individual subtask and become more effective at it. In the classic pin manufacture example, [1] a manufacturer employing ten people could produce 12 pins a day when a man working by himself could scarcely produce a single pin. As well, by subdividing a task, it is also possible to develop a device to replace the need of a person to perform a subtask. Thus, through specialization and the division of labour, it is possible for a group of people to improve their wealth and knowledge.

A large population of available workers makes specialization possible, and it supports a high level of technology and wealth. A highly specialized population is expected to be wealthy and technologically advanced, and will improve as it becomes more specialized. A population of unspecialized people tends to be poor and technologically backwards. As a population increases, a higher degree of specialization is possible, and with it, corresponding improvements in wealth and technology.

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1.6 Examples of Population and Technology in Civilizations:

In order to establish some basis for population change as an indicator of technological growth, it is necessary to go over a few examples. These examples discuss the effects of rising, stable, and decaying population within a civilization and their effects on technological growth. On the basis of these examples, it is then possible to derive a relationship between population and technology and knowledge.

The most familiar example of a civilization where population levels were rising rapidly is our own western civilization. Braudel [2] comments on population as a socio-economic indicator. He states that, "In principle and in fact, therefore, an increase in population has always helped the growth of civilization." His primary example of this case is Europe through the modern period, especially the thirteenth, sixteenth, eighteenth, nineteenth, and twentieth centuries. In all of these examples, improvements of technology and living standards followed the increase in population. It follows from the example of western civilization that it is indeed possible that population, technology, and living conditions can improve together.

Between the years AD. and 1500AD, the world population was fairly stable, varying between 300 and 500 million. During this period, technology has changed very little, especially when compared to the modern period. One example of a civilization which thrived and reached its peak during this time period is the Islamic civilization.

The Arab civilization grew as a result of massive conquests which peaked during the eighth century AD. They were the dominant force in the old world for the next few centuries, while having a population of 30 to 50 million people at most. [2] While Islamic people at the time actively pursued science and astronomy, they produced little that was original. "While originating little, the Arabs transmitted both instruments and data and translated Greek manuscripts." [3] The best the Islamic world could do with their limited manpower was maintain the current knowledge of the world.

It is rather challenging to find a civilization in a region where the population is in decline and technologies are forgotten, as these civilizations themselves tend to be lost to time. History records quite a variety of civilizations which rose to a peak of wealth and innovation, and then declined and fell. In most cases, these civilizations were then replaced with new, rising civilizations. If there was no civilization to replace a defunct old civilization, then the local world fell into a dark age.

One example which stands out is the Mayan civilization of Central America, where the population, literacy, prosperity and technology decayed to near extinction. It is not known to what extent the Mayan people expanded their knowledge. It is not even known what caused their decline. Both of these are the source of much speculation. Gilbert and Cottered [4] speculate that it was a simple decline in the birth rate which caused a decay in Mayan population levels and a decline of their civilization. If this is true, then it is quite possible that a decay in population levels causes a corresponding decay in knowledge and technology since the population no longer supports the previous level of technology.

These examples show the possible link between population and technology. When the population increases, technology improves. While the population is stable, technology remains relatively constant, with little original innovation. When the population decays, old knowledge and technologies become forgotten as there are not enough people to support them. The next step is to develop a mathematical model linking technology growth to population growth that accounts for these considerations.

2.0 Mathematical Modelling of Population and Technology:

2.1 Problem Statement:

Upon establishing population ~~as~~ size an indicator of technological growth and innovation, the next task is to develop a model that makes logical sense in the physical world. This model for technological innovation must reasonably follow the change of population. When a population is increasing, the model should reflect an increase in knowledge. When the population decreases, the knowledge should stabilize and then decay. Finally, the model should have a physical meaning in the real world.

2.2 Assumptions:

1. It is assumed that a population of people working within a field can be divided into two interacting groups. One group represents the students, whether they are university students, or apprentice and journeyman tradesmen. The second group represents the teachers, whether they are university professors or trades masters. The reason for this assumption is to account for the fact that it is far easier to acquire knowledge when there is somebody there to teach it.

2. It is assumed that the number of teachers is directly proportional to the number of students. This could be seen as a faculty department admitting a number of students proportional to the number of professors, or hiring a number of professors proportional to the number of students. The assumption could also be visualized by imagining that a certain percentage is selected among

the most knowledgable to be teachers. The reason for this assumption is to simplify the mathematical model and make a quasi phase-plane analysis possible.

3. It is assumed that over a short period of time, that the population of students can be described using the Malthusian Model. The number of students entering the system is proportional to the total number of students. As well, the number of students leaving the system is proportional to the overall number of students. It is necessary to separate the number of students entering and leaving a system, since we must keep track of their effect on the overall knowledge.

4a. It is assumed that the growth of student knowledge is proportional to the number of students. This assumption accounts for the students performing research.

4b. It is assumed that student knowledge also grows when there are professors available to teach them. This is proportional to the number of students multiplied by the number of professors and the amount of professor knowledge per student. In other words, the amount of knowledge gained through teaching is the number of professors multiplied by their knowledge. This assumption accounts for professor to student interaction.

4c. It is assumed that a student removes a constant amount of knowledge when the student retires from the system. That is, an amount of knowledge proportional to the number of students is removed from the system as students retire. This assumption keeps the equations simple.

5a. It is assumed that the growth of professor knowledge due to research is proportional to the number of professors.

5b. It is assumed that the growth of professor knowledge also increases due to interaction with students. This amount is proportional to the number of professors multiplied by the number of students and the amount of student knowledge per professor. This, in more simple terms, is the number of students multiplied by their knowledge. This assumption accounts for professor to student interaction improving the professor's knowledge as teaching techniques improve.

5c. It is assumed that ~~the~~ a professor removes a constant amount of knowledge when the professor retires from the system. That is, an amount of knowledge proportional to the number of professors is removed from the system as professors retire. This assumption, along with 4c, places a limit on how much knowledge an individual can acquire, but it is valid and physically meaningful only if the average knowledge of the retiree is greater average amount of knowledge.

2.3 Functions, Variables and Parameters:

With the assumptions established, the next stage is to establish functions, variables, and coefficients based on these assumptions. It is the basis of these parameters that the mathematical model can be built. The variables, functions and coefficients are listed below.

Variables and Functions:

t - Time is represented by the variable, t. For the purposes of this project, it is most practical to think of time as being measured in years.

x(t) The function, x(t) represents the amount of professor knowledge. This represents what it is that the professors know that the students have yet to learn. average amount?

y(t) The function, y(t) represents the amount of student knowledge. average amount?

Nx(t) The function Nx(t) represents the number of professors at a given time, t.

Ny(t) The function Ny(t) represents the number of students at a given time, t.

Constants and Coefficients:

k - The coefficient, k, is a Malthusian growth factor accounting for students and professors entering the academic system.

l - The coefficient, l, is a Malthusian decay factor accounting for students and professors leaving the acedemic system through retirement.

m - The coefficient, m, represents the professor to student ratio. (Constant?)

p - The coefficient, p, accounts for students learning when there are professors available to teach them.

q- The coefficient, q, accounts for students learning by working on their own and peforming research.

r - The coefficient, r, accounts for professors learning when they have students available to teach.

s - The coefficient, s, accounts for professors learning from their own research.

v - The coefficient, v, represents the average knowledge of a student retiring from the academic system. This coefficient accounts for knowledge lost to retiring students.

w - The coefficient, w, represents the average knowledge of a professor retiring from the academic system. This coefficient accounts for knowledge lost to retiring professors.

2.4 Equations:

The first set equations (1 through 4) in this mathematical model determine how student and professor populations changes with time. To simplify the analysis, since any model which shows expanding, stable and decaying populations is adequate, the Malthusian model for population growth was chosen. It is also possible to modify the assumptions and equations dealing with population growth if this is desired. The equations below show the Malthusian model for population dynamics.

$$dNy(t)/dt = (k-l)Ny(t) \tag{1}$$

Solving equation (1) yields the exponential:

$$Ny(t) = EXP[(k-l)t] \tag{2}$$

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$$Nx(t) = mNy(t) = mEXP[(k-l)t] \tag{3}$$

Differentiating equation 3 yields:

$$dNx(t)/dt = mdNy(t)/dt = m(k-l)Ny(t) \tag{4}$$

The second set of equations (5 to 6) relates the population of students and professors to their knowledge. By referring to the definitions of the coefficients in the previous section, the equations essentially become a mathematical restatement of the assumptions on which the model is based. The first term of the equation represents the amount of learning due to teaching. The second term is knowledge gained due to research. The third term represents knowledge lost to students or professors retiring from the academic system. The differential equations relating to the functions x(t) and y(t) are as follows:

$$dy/dt = pxNx(t) + qNy(t) - lvNy(t) \tag{5}$$

$$dx/dt = ryNy(t) + sNx(t) - mlwNx(t) \tag{6}$$

It is possible to simplify Equations 5 and 6. Since the student and professor populations, Nx(t) and Ny(t), are related, it is possible to make substitutions. After making the substitutions, it is then possible to isolate the terms of population in terms of time. The simplified equations are listed below.

$$dy/dt = (mpx+q-lv)Ny(t) \tag{7}$$

$$dx/dt = (ry + sm - lwm^2)Ny(t) \tag{8}$$

With these simplified equations, it is then possible to eliminate the term Ny(t) by dividing one equation by the other. On the good side, the system of equations becomes semi autonomous. On the negative side, this eliminates the time and population factors from the differential equations. It is no longer possible to completely analyse what happens when populations increase, stabilise, and decrease according to this model. However, it is possible to perform a quasi phase plane analysis and obtain some insight into how the system of equations works.

3.0 Analysis and Discussion:

3.1 Quasi Phase Plane Analysis:

By inspecting Equations 7 and 8, it is possible to eliminate the term, Ny(t) and perform some kind of phase plane analysis. This first step is to determine where the nullclines are located. A nullcline is a point where the derivative of one function is equal to zero. Finding the nullclines requires setting Equations 7 and 8 equal to zero. This yields the following:

$$x = (lv-q)/(mp) \qquad (9)$$

$$y = (lwm^2 - ms)/(r) \qquad (10)$$

By referring to the list of variables and coefficients, it is then possible to understand the literal meaning of Equations 9 and 10. Quite literally, Equation 9 states that when the knowledge lost to retiring students exceeds the knowledge gained through research, there is a level of professor knowledge at which the level of student knowledge remains constant. In other words, if there is a deficit in the knowledge gained by students through their own research, then this deficit has to be made up through teaching or the student knowledge will decay. Similarly, Equation 10 states that if the knowledge lost to retiring professors exceeds the knowledge gained, through their research, then there is a certain level of student knowledge at which the level of professor knowledge remains constant. The interpretations of these equations then make real physical sense.

Also, the intersection point of these nullclines determines the equilibrium point of professor and student knowledge. If the equilibrium point is in the first quadrant, (both nullclines are positive constants) then there is a level of student knowledge and professor knowledge at which neither changes with time. For the purposes of this discussion, in order that we do not consider negative quantities of knowledge, it is assumed that there is an equilibrium point in the first quadrant.

If the student and professor knowledge are at their respective equilibrium ~~points~~ values, then the knowledge of each group remains the same. In this case, the professors are able to teach just enough to maintain student knowledge as well as their own knowledge. If the student knowledge is greater than its equilibrium value, and the professor knowledge is greater than its equilibrium value, then the knowledge of students and professors both increase. If the student knowledge is less than its equilibrium value, and the professor knowledge is less than its equilibrium value, then the knowledge of each group decays. It is apparent that the equilibrium point is unstable.

If the professor knowledge is below its equilibrium point, but the student knowledge is greater than its equilibrium point, then student knowledge decays while professor knowledge increases. Three things can happen. Student knowledge could decay to the point where it is also below its equilibrium point and professor knowledge also begins to decay. Professor knowledge could increase above its equilibrium point and then allow student knowledge to increase. It is also possible that the student knowledge and professor knowledge reach their equilibrium points simultaneously, and equilibrium point is reached. A similar argument exists for the case where professor knowledge is above its equilibrium its equilibrium point, but student knowledge is below its equilibrium point.

3.2 Quasi Phase Plane Diagram:

The quasi phase plane analysis shows what happens to professor and student knowledge when the professor and student knowledge begin at certain points. The next step is to find student knowledge as a function of professor knowledge and sketch the trajectories onto the quasi phase plane diagram. Dividing Equation 7 by Equation 8 yields a differential equation which yields y as a function of x.

$$dy/dx = (mpx+q-lv)/(ry+sm-lwm^2) \qquad (11)$$

This equation is separable and the solution to this equation is quite trivial, since most of the equation consists of constant coefficients. Solving requires separating the variables x and y and integrating. This yields the hyperbolic equation below, with C as a constant of integration and determined from the initial point.

$$0.5ry^2 + (sm-lwm^2)y = 0.5mpx^2 + (q-lv)x + C \qquad (12)$$

To further simplify the parameters, it is possible to make substitutions. The modified hyperbolic equation is as follows:

$$Ay^2 + By = Dx^2 + Ex + C \qquad (13)$$

$$A = 0.5r \qquad (14)$$

$$B = sm-lwm^2 \qquad (15)$$

$$D = 0.5mp \qquad (16)$$

$$E = q-lv \qquad (17)$$

Finding y as a function of x to determine the trajectories requires a solution to the quadratic formula. The values of A,B,D, and E determine the trajectories. The value of C is the constant of integration which is determined by the starting point. The quadratic formula yields two possibilities for student knowledge as a function of professor knowledge.

$$y(x) = -(B/2A)+SQRT[(B/2A)^2+(Dx^2+Ex+C)/A] \quad (18)$$

and

$$y(x) = -(B/2A)-SQRT[(B/2Z)^2+(Dx^2+Ex+C)/A] \quad (19)$$

Interestingly enough, by replacing the -(B/2A) terms with their coefficients yields the nullcline for student knowledge at which professor knowledge remains constant. That is, when the student knowledge reaches its nullcline value, the second term of the equation becomes equal to zero.

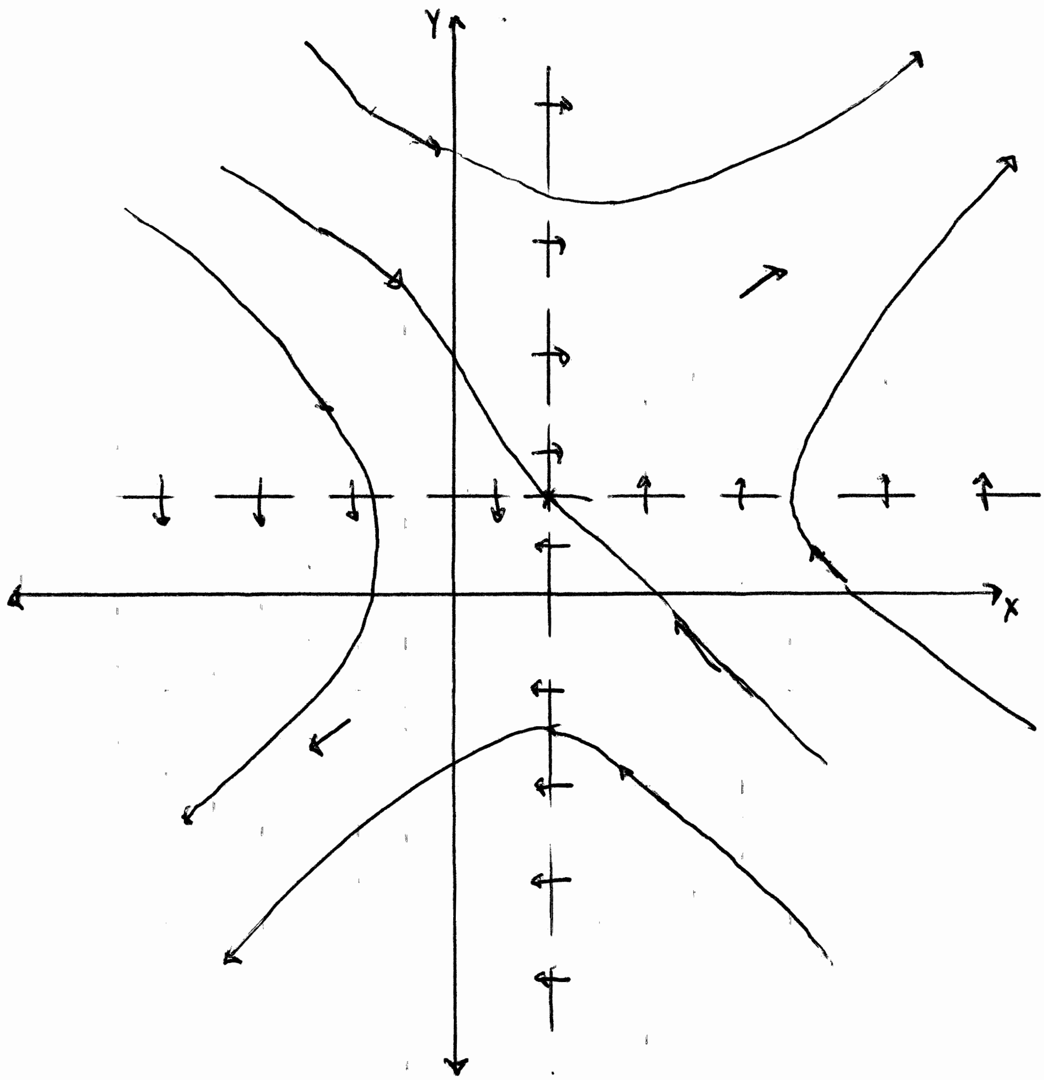
By making setting arbitrary values for A,B,C,D,and E, then it is possible to plot its trajectory. A sample quasi phase plane diagram and trajectory plot appear on the next page. A simple example occurs by setting A=C=D=1 and B=E=-1. This plot yields an equilibrium point in the first quadrant (at 1,1) and has equations as follows:

$$y(x) = 1 + SQRT[x^2-x+2] \quad (20)$$

and

$$y(x) = 1 - SQRT[x^2-x+2] \quad (21)$$

Phase - Plane Diagram for the growth and decay of Professor and student Knowledge



4.0 Conclusions:

The objective of this project is to derive and analyse a mathematical model which relates population growth to technological growth. The primary challenge in achieving this objective is in determining indicators for technological growth, since it is a relative quantity that the equations are trying to model. The There is no means of determining a number to attach to the level of technology. From attempting to derive and analyse the mathematical model, several conclusions can be drawn.

1. By simplifying the model so that population and time no longer contribute to the equations, then the knowledge of students becomes an independent function of the knowledge of professors. A project objective is to show what happens when population grows, stabilises, and decays. When student knowledge becomes an independent function of professor knowledge, then population no longer has a role in determining the levels of knowledge of professors and students.

2. It was found by analysing the equations 7 and 8 that a nullcline occurs when the level of professor knowledge is sufficient to make up for any deficit resulting from the loss of student knowledge due to students leaving the academic system. The level of professor knowledge determines whether the level of student knowledge rises, stabilises, or decays. Also, the level of student knowledge determines whether the level of professor knowledge increases, decreases, or remains the same. These observations are encouraging to the hypothesis that population is an indicator of knowledge, since the number of academics retiring and the knowledge they take with them also determines whether the knowledge of their group rises or decays.

3. A weakness in one of the assumptions of the mathematical model is the fact that a retiring academic takes a constant amount of knowledge away from the system. It is quite possible that this could vary. The assumption was made to allow for a simple analysis of the system. Again, this goes back to the basic problem that knowledge is a relative quantity that physically cannot be measured by an absolute quantity.

4. Since this mathematical model deals with relative quantities, it is not practical to make an analytic example of the knowledge of an actual civilisation. A quasi phase plane diagram does allow some insight into how knowledge increases and decreases with time, but it does not reveal the rate at which this occurs. Since Equations 7 and 8 show that the growth of knowledge is also proportional to the population as a function of time, it is easy to speculate that population change according to this model speeds or slows the growth of knowledge.

4.1 Suggestions for Improvement:

The mathematical model developed in this project is far from perfect. There are many means of improvements. The first suggestion for improvement is in the strength of the assumptions. The assumptions simplify the mathematical model by removing the effect of population from the equations. Yet the objective is to determine the influence of population on the level of technology. Another mathematical model which more directly includes the effect of population would be helpful to this study.

5.0 References:

[1] Smith, Adam; The Wealth of Nations, Volume I, Oxford University Press, 1904

The example of pin production is to be found on page 6.

[2] Braudel, Fernand, A History of Civilizations, Translated by Richard Mayne, Penguin Books, 1993

[3] Trefil, James S., Space Time Infinity: The Smithsonian Views the Universe, Smithsonian Institution, 1985

[4] Gilbert, Adrian G; Cotterell, Maurice M, The Mayan Prophecies, Element Books, 1995