6.337 PROJECT

Introduction to Mathematical Modelling

TWO FORCE ARMS-RACE:

A MODIFIED LANCHESTER COMBAT MODEL

Submitted to:
Dr. T.G. Berry

Submitted By:
Trevor Oltsher
6604835

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The problem discussed in the following, deals with the possible outcomes of two opposing forces in combat against one another. For this comparison, the representation of the forces will be by X and Y. An example of such a scenario, would be a tank force (X) vs. an anti-tank weapon force (Y). This problem will be examined using an extension of a Lanchester-type combat model named after F.W. Lanchester, who modelled such real-life battles.[1]

The strength of the forces X and Y at time t {x(t) & y(t)}, will be measured by the number of components in operation. Other factors such as the craftsmanship of the weapons, the strategic decisions by each side, & the level of technology used in the creation of the weapons, all play a role in the real-life strength of a force. For the purpose of this discussion, these will be disregarded, as only the count of each weapon will be considered for the measure of strength. good

These strength levels will change over the course of a battle, based on the initial strengths of each force & the rates by which weapons are destroyed & created. The loss of a component of a force, will only be the result of the presence of the other force. ? Thus, such factors as mechanical or operator failure will not be considered. OK Before the start of the battle, each force has a carrying capacity that limits the number of components that make up its force at any time t. This capacity could be determined by the amount of storage space or financial restrictions. good The rate of change of one force is based on its growth rate minus the casualty rate ~~created~~ caused by the amount of the other force present. Thus, the model is comprised of both a logistic and malthusian model. The rate of change in x(t) and y(t) is shown in (1) and (2).

$$\frac{dx}{dt}=k*x*(1-\frac{x}{R})-a*y \qquad \textbf{(1)}$$

$$\frac{dy}{dt}=l*y*(1-\frac{y}{S})-b*x \quad \textbf{(2)}$$

l

where a, b, k, l, R, S are constants > 0.

The parameter a is the anti-tank weapon kill rate, which reflects the degree to which a single anti-tank weapon can destroy tanks. The growth rate of the number of tanks is denoted by k, while the upper limit for the number of tanks is R. The parameter b is the rate by which a tank can destroy an anti-tank weapon. The parameter l l reflects the growth rate of the number of anti-tank weapons, while S denotes the upper limit for the number of anti-tank weapons.

To determine the general pattern of combat, the nullclines of the model must be determined. By setting each of (1) and (2) to equal zero, the equations of the nullclines are:

$$Nx: y=\frac{k}{a}*x-\frac{k}{R*a}*x^2 \quad \textbf{(3)}$$

$$Ny: x=\frac{l}{b}*y-\frac{l}{S*b}*y^2 \quad \textbf{(4)}$$

opening down

opening right

These nullclines are both parabolas, with Nx inverted in the vertical direction and Ny inverted in the horizontal direction. Both pass through the origin, with Nx passing through (R,0) and Ny passing through (0,S). Only the positive values of X & Y are being considered to reflect real-life situations (ie. Cannot have a negative amount of weapons).

x, y

?

surface

Figure 1 shows the general nullcline Nx, with the maximum y-value relative to the unknown parameters.

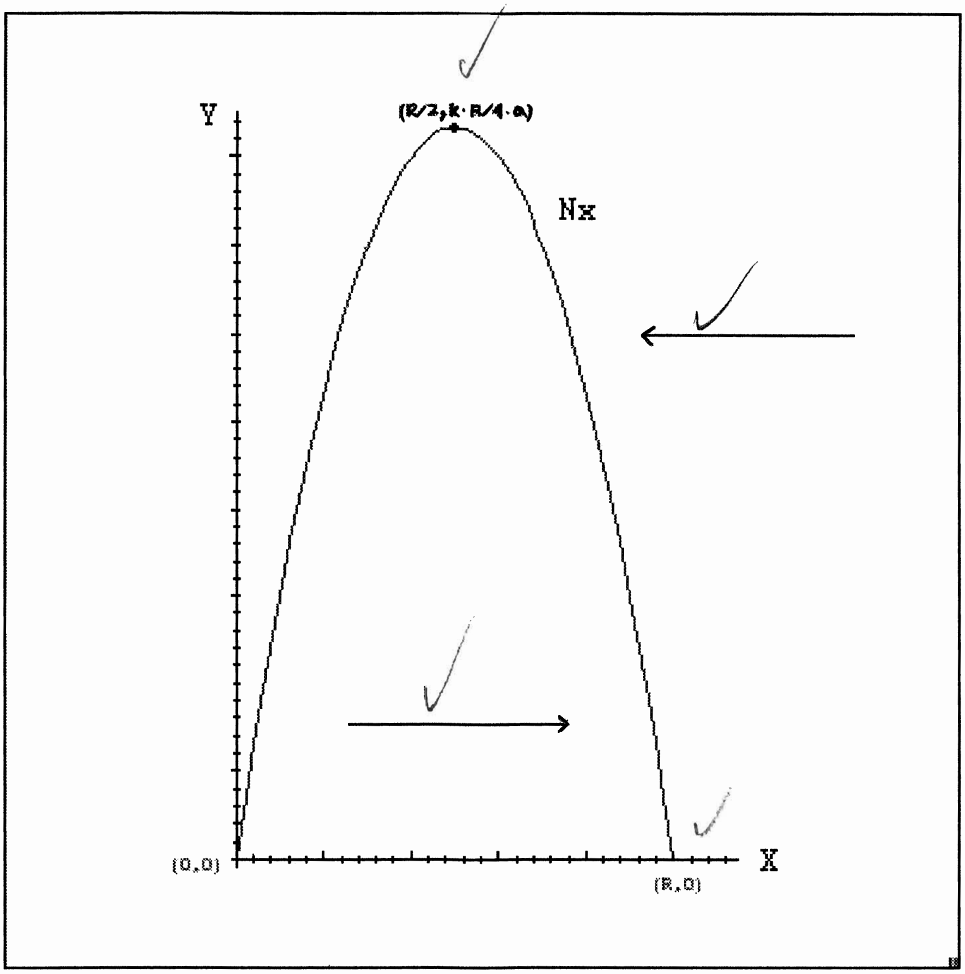


Figure 1

Below

The arrows represent the sign of dx/dt. Inside the nullcline Nx, dx/dt is > 0 (increasing), while outside, it is < 0 (increasing).

Above it

de

Figure 2 shows the general nullcline Ny, with the maximum x-value relative to the unknown parameters.

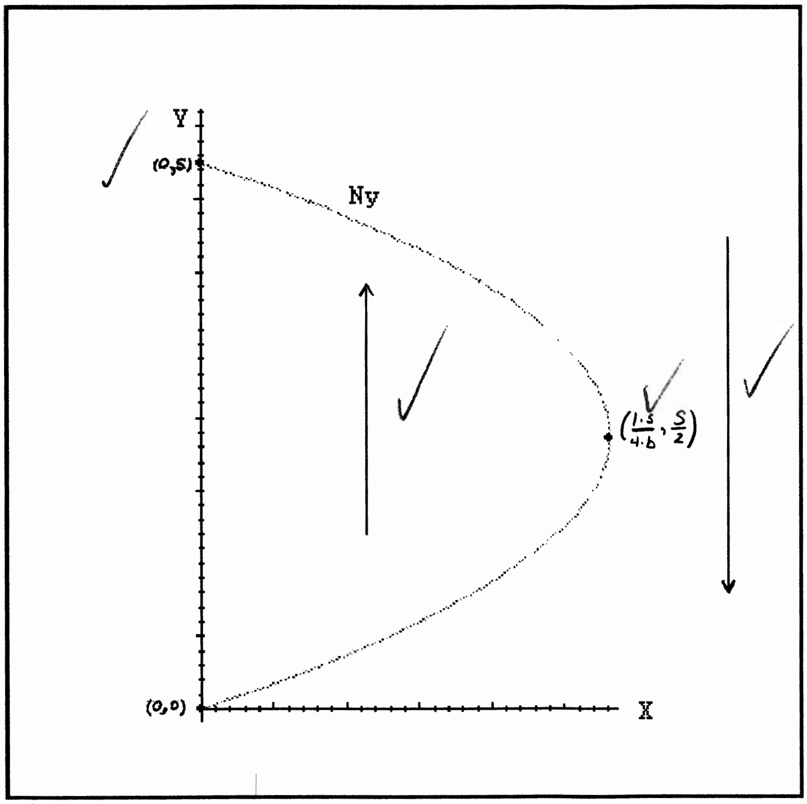


Figure 2

To left of

The arrows represent the sign of dy/dt. Inside the nullcline Ny, dy/dt is > 0 (increasing), while outside, it is < 0 (decreasing).

to right of

The intersection of these nullclines, are the points of most interest. By solving equations (3) and (4) simultaneously, these equilibrium points can be determined. The resulting equation is shown in (5). Information can only be extracted from the points that are both real and positive. good

$$\frac{l*k^2}{R^2*S*a^2*b}*x^4-\frac{2*k^2*l}{R*S*a^2*b}*x^3+\frac{l*k}{R*a*b}+\frac{l*k^2}{S*a^2*b}*x^2-\left(\frac{l*k}{a*b}-1\right)*x=0 \qquad (5)$$

The corresponding y-values are obtained by inserting each x-value into equation (3). These are points at which the model stabilizes to a draw between the two forces. OK!

only if y=0

One can see that x=0 is always a solution to equation to (5). This value will always correspond to a y=0 value, giving an equilibrium point of (0,0) to all models. If a given model moves to this point, total disarmament occurs, as both supplies are depleted to zero.

Depending on the original parameters, equation (5) may yield 1, 2, 3 or 4 positive, real values for x. This will yield many different scenarios, that may produce different outcomes in the battle. agreed - show details!

no! Because system is autonomous equations of trajectories can be found!

A solution to the model can be useful to obtain possible trajectories that may lead to the result of the battle. Theoretically, by determining dy/dx and integrating, this gives the solution trajectories for a given model. Solving for dy/dx gives:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{l*y-\frac{l*y^2}{S}-b*x}{k*x-\frac{k*x^2}{R}-a*y} \qquad (6)$$

OK because the system is autonomous

The complexity of the model and its nullclines, make the integration of equation (6) quite difficult. ~~However,~~ by determining the direction of dx/dt and dy/dt in certain areas of the nullclines, these can be estimated with ~~reasonable accuracy~~.

The following three cases each refer to a specific figure outlining the general scenarios that could occur. The direction arrows depict the general direction that the outcome of the battle will take. These directions are derived from the earlier findings regarding the sign of dx/dt and dy/dt inside and outside their respective nullclines. A win for force Y, where graphically the battle ends on the y-axis, is depicted with blue arrows. A win for force X, where the battle ends on the x-axis, is depicted with black arrows. A draw, where the battle finishes at an equilibrium point, is shown by red arrows.

CASE I

The first case considered, has only the one equilibrium point at (0,0). Two of the other points were imaginary, with the other ~~lying~~ outside the positive x-y quadrant. Such a situation occurs when Nx lies entirely below and Ny lies entirely above the line y=x. Thus, the magnitude of the slope at any point on Nx is less than one. The magnitude of the slope at any point on Ny is greater than 1, except at maximum x, where the slope is undefined. These shallow ~~nulliclines~~ could be the result of both forces having ~~low~~ upper limits, low growth rates or high kill rates. Depending on the initial strengths of X and Y at time t=0, the outcome of the battle may be different. Figure 3 shows a general sketch of a few possible scenarios of case I. One possible situation would involve the starting point being located above the y=x line. Thus, force Y begins with more strength than force X. Any start point in this region would lead to a victory by Y, with X having lost all assets.

If true then there will be no other point of intersection

depends on whether the two parabolae cross at x=b or are tgt at x=0

The strength of the victor at finish time, varies from being greater than zero to less than S, the upper limit. If the starting strength of both forces is identical, total disarmament occurs. Here, both forces fight to a draw, as the battle ends up at the lone equilibrium point of (0,0). Predictably, if force X has a larger stock than force Y at start time, X will be the winner, with a maximum of R weapons remaining. Case I produces the simplest results of those discussed here.

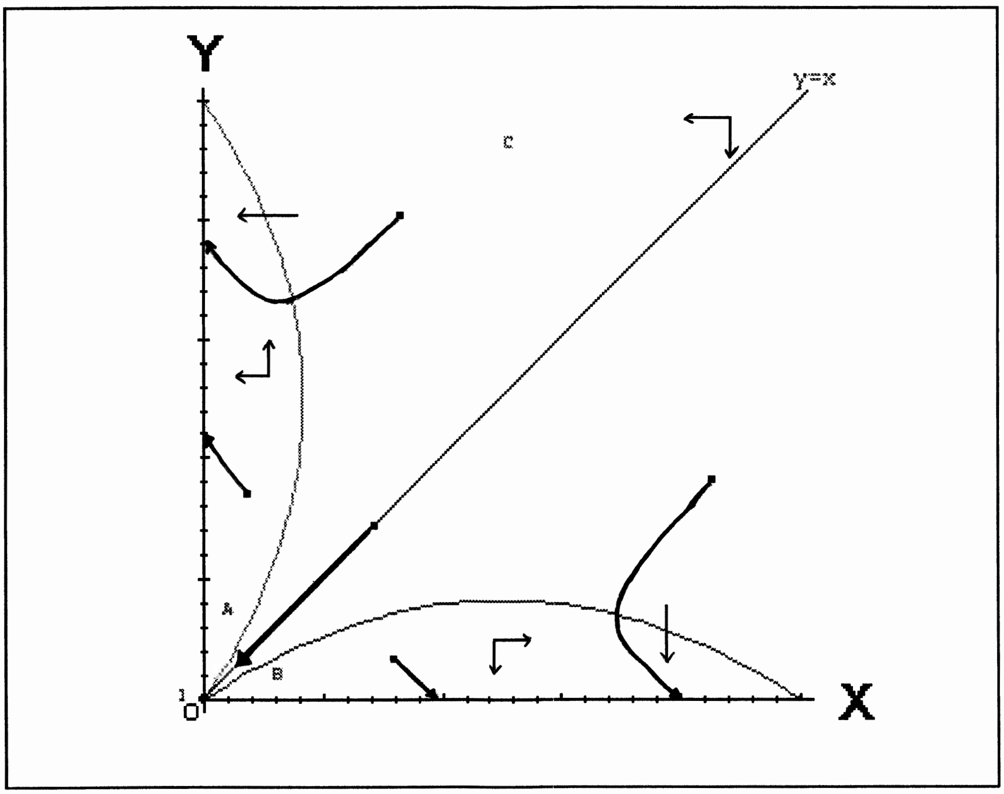
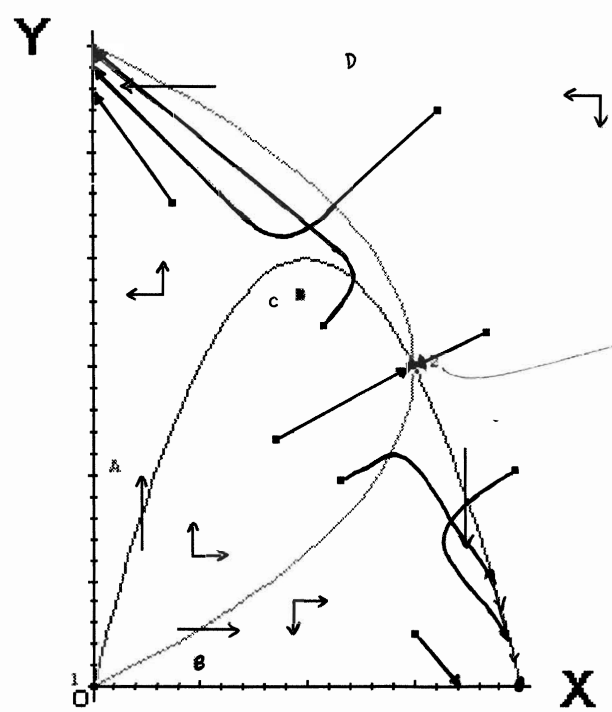


Figure 3

Case II

Case II deals with the intersection of the nullclines producing 2 equilibrium points. As before, one is at (0,0) and the other one is dependent on the parameters. The only real, positive solution to (5) and its corresponding y-coordinate, is this point. This creates four different regions, as depicted in Figure 4. Once again, the arrows show the general path the battle would take based on initial strengths. This situation is commonly created by having either the maximum x-value of Ny less than R, or by having the maximum y-value of Nx less than S. A starting point in region A, would guarantee a win for force Y. Similarly, if the starting point is located in region B, force X would win over force Y. However, if region C or region D encloses the starting point, many outcomes are possible. X winning, Y winning or a draw are all possible. For a win to occur, the path of the battle starts in region C or D, then crosses into region A or B producing a win similar to the above. A draw will occur if the equilibrium point 2 is reached. Here, the battle is at a permanent stand-still, with neither side able to continue. In Case II, the parameters can play a larger role in determining the outcome than the actual starting strength of the forces can. A scenario where force X begins with more strength than force Y, does not automatically mean that X will be the winner.



at extreme value of Ny? NOT nec.

more possibilities should be considered.

Figure 4

Case III

Case III involves the scenario where all four roots of equation (5) are positive and real. Thus, the intersection of Nx and Ny produce 4 equilibrium points, one of which is at (0,0). Six regions are created from this intersection, as shown in figure 5. As in the previous two cases, an initial point in section A, produces a win for Y and an initial point in section B, produces a win for X. In section C, the most likely scenario would have the battle move up to and stop at equilibrium point 3. There is a slight possibility that the battle could cross over into region A or B, creating a victory for either side or the equilibrium points 2 and 4 could be reached. But, because of the angles created by the intersection, along with the direction of the arrows, these are not usually attained. In both regions D and E, an initial point will go to equilibrium point 3, resulting in a draw between the two forces. In D, each of the initial strengths increase until the eq. point 3 is reached. In E, each of the initial strengths decrease until the eq. point 3 is reached. If the initial point is located outside the nullclines in region F, the strengths of each force become reduced until equilibrium point 2, 3 or 4 is reached or one side wins out. Because of the four intersection points, both nullclines are required to be similar in size. Thus, region C is likely be quite large. This, combined with the possible results obtainable in regions C, D and E, gives a Case III scenario the high probability of draw occurring. The results discovered here, are very similar to those cases where there are 3 equilibrium points. Thus, detailed examination of this case would not be necessary.

only slight

Yes!

2 & 4 prob not stable

because of direction fields

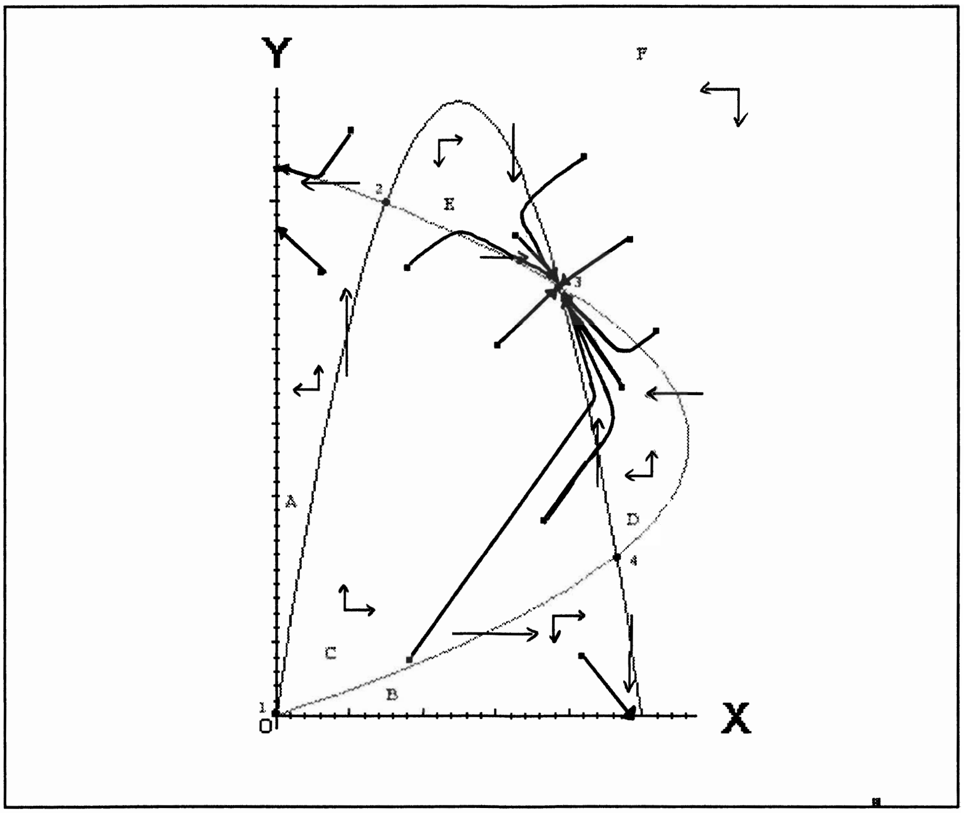


Figure 5

CONCLUSION

Many factors contribute to the accuracy of the portrayal of such a two force battle. As mentioned earlier, the strength of each force depends only on the number of components that make up its fleet. Few real-life battles operate in this manner, producing a major flaw of this model. Such factors such as strategic moves or accuracy of the weapons, would require additions to the original model. Another possible extension would be to include more forces, either as independents or as allies. These factors may require introducing probability or be difficult to mathematically measure, increasing the difficulty of the model. However, the simple model discussed here does give an accurate view of such possible battles. Because of the carrying capacity restraint, there is no possibility of a run-away arms race. Thus, every situation has a definite ending point. This model allows for a variety of outcomes based on the value of the parameters regarding kill and growth rates, as well as the carrying capacities of each force. As mentioned above, the force with the greater initial strength is not always the winner. Large values of k and b, combined with small values for a and l, could easily turn an initial advantage for Y into a win for X. Thus, this model handles a variety of circumstances and determines a reasonable solution to a battle between the two forces. Yes

REFERENCES

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3. Corless, Robert M. (1995). ESSENTIAL MAPLE, Springer - Verlag, New York, New York.