University of Manitoba
Math 3820 – Winter 2011

Midterm

Thursday, March 10, 2011

Instructions

This test is 2 hours; it has 2 questions on 2 pages. Notes are allowed. In marking, attention will be paid to the overall legibility of solutions; so detail and structure your answers and make good use of scrap paper.

1. The following equation

$$N_{t+1} = N_t e^{r(1-N_t/K)} \tag{1}$$

is called Ricker's equation and was introduced in the context of a model of fisheries. The parameter $r \geq 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity. We consider equation (1) together with a nonnegative initial condition N_0.

1.a. Give an interpretation of the form of the equation.

1.b. Discuss the existence of solutions to (1). Do solutions always remain nonnegative? bounded?

1.c. Study the existence of fixed points for (1) and its/their stability as a function of parameter values.

2. We consider the following system of ordinary differential equations

$$\frac{d}{dt}S = D(S^0 - S) - \frac{\mu_1}{Y_1}SP \tag{2a}$$

$$\frac{d}{dt}P = \mu_1 SP - DP - \frac{mu_2}{Y_2}PZ \tag{2b}$$

$$\frac{d}{dt}Z = \mu_2 PZ - DZ \tag{2c}$$

together with the nonnegative initial condition $(S(0), P(0), Z(0)) = (N_0, P_0, Z_0)$. With this system, we aim to describe the interactions between a substrate S, phytoplankton P and zooplankton Z in a chemostat. All parameters are assumed to be positive.

2.a. Give an interpretation of the form of (2). The parameters Y_1, Y_2 are called *yield coefficients*. Can you see why? Also, beside the addition of a zooplankton equation, what is the difference with the chemostat model we studied? Discuss this difference and its implication in terms of the assumptions made on the nutrient uptake/growth mechanism.

2.b. Do solutions to (2) exist? are they unique? do they remain nonnegative? are they bounded?

2.c. We propose to *nondimensionalize* the system to transform it into a system easier to study yet possessing the same properties. For this, we introduce the following change of variables

$$x = \frac{S}{S^0}, \quad y = \frac{P}{Y_1 S^0}, \quad z = \frac{Z}{Y_1 Y_2 S^0}$$

and of time

$$t = DT.$$

(Note that $d/dT = D\, d/dt$.) Show that, using these, (2) can be transformed into the following system

$$\frac{d}{dt}x = 1 - x - Axy \tag{3a}$$

$$\frac{d}{dt}y = Axy - y - Byz \tag{3b}$$

$$\frac{d}{dt}z = Byz - z, \tag{3c}$$

where

$$A = \frac{\mu_1 S^0}{D}, \quad \beta = \frac{\mu_2 S^0 Y_1}{D}.$$

2.d. Verify that the properties established in **2.b** are still true for (3).

2.e. Show that an (asymptotic) mass conservation property holds for (3), so that, for all nonnegative initial conditions,

$$\lim_{t\to\infty} x(t) + y(t) + z(t) = 1.$$

2.f. Use the result of **2.e** to write (3) as a 2-dimensional system involving the variables y and z. This new system has 2 equilibria. Find them and study their local stability properties as a function of the parameters they involve. Summarize the situation as a function of the parameters.

2.g. Express the results of **2.f** for system (3). Can you interpret these results? Can you relate these results to the original system (2)? Discuss the pros and cons of the original system versus the nondimensionalized system.