SIAM REVIEW
Vol. 15, No. 2, April 1973

MATHEMATICS AND POPULATION*

NATHAN KEYFITZ†

Abstract. The larger questions of population and the environment cannot be formulated with present knowledge. We will have to know much more before we can even state the problems clearly, let alone apply mathematics to their solution. But some smaller problems can be stated and solved. Examplcs are listed, in the hope that the methods can be extended from these minor issues to the larger environmental problems.

Population dilemmas. We are told that our rapid population growth threatens the future of the human race: we are growing so fast that we will soon crowd ourselves to death. Our technology, which is man's ability to control the environment in his own interest, has become so effective that the environment is being destroyed. The economy is booming, but many of the services we took for granted yesterday when we were poorer—good schools, good hospitals, clean cities and safe streets—have now become too expensive for most of us to afford. Each of these three propositions, by no means original with me but widely heard in one form or another, seems self-contradictory, and we may well ask whether current writers who discuss population and its related problems have abandoned logic in favor of some other form of discourse.

The same statements can be made in a less surrealistic manner. The more there are of us the more likely we are to destroy the support base of human life on the planet, and this could make life impossible even for a small population. The threat of self-destruction through excessive numbers is used by those who would have us treat the planet like permanent farmers rather than like slash-and-burn primitives. But the mechanisms by which increased human population would provoke a crash like that which affects a lynx population are by no means understood, and the analogy is often presented in an incoherent way.

The illogic is not confined to rich countries. Many of the poor countries of the world, especially those that have empty spaces, are told by their governments that they need more people; they must fill those empty spaces just as the United States filled its spaces in the 19th century. But no one is moving towards the empty spaces of Mexico, and not many to the outer islands of Indonesia. Far from setting out to clear their jungles, peasants are leaving well-settled and relatively comfortable countrysides and moving into cities. Brazilians dream of settling their interior as did the heroes of the American west, but most do not imitate the action; rather they strive to find an apartment in Sao Paulo or Rio de Janeiro, where they can watch western stories on their television sets.

One apparent paradox that needs explaining is that a poor country ambitious for large population ought to start using birth control as soon as possible. If it

* Received by invitation of the editors December 7, 1972.

† Andelot Professor of Demography and Sociology, Harvard University, Cambridge, Massachusetts 02138. The publishing of this article was supported in part by grants from the Alfred P. Sloan Foundation and the Research Corporation.

continues to increase rapidly it will be burdened with children whose upbringing and schooling will drain away resources and so hinder the accumulation of industrial capital. If births are initially fewer, saving can increase and thus increase industry, and industrial power can be applied to improve agriculture and otherwise build the means for supporting population. At that point births can rise again, though people may not want more children; we can nonetheless assert that a poor country can have a larger ultimate population if it begins by reducing its births.

The suspicion that logic has been abandoned is confirmed when we see some of the mechanisms that are proposed to analyze environmental constraints. Air pollution can raise the death rate; when the death rate gets high enough and remains high enough for a long enough period, population density will decrease; with fewer people there will be fewer automobiles and other sources of pollution; the diminution of the sources of pollution will lower the death rate and allow population to rise again; ultimately an equilibrium level of population will be reached, determined by the amount of usable air. This unattractive picture is presented in the good cause of persuading us to do something about population and pollution, but no one so far knows enough about the effect of pollution on mortality to establish such a mechanism.

To self-contradictory statements of the population problem and mechanisms that go far beyond known empirical relations are added some unilluminating forms of statistical description of the economy. Accounting systems for individual businesses are scrupulous in allowing for the depreciation of the firm's assets; an airline that failed to depreciate its planes before declaring a dividend would find itself bankrupt the moment the planes had to be replaced. Even a corner grocery store keeps books that financially protect those assets that will be physically wasted.

But no such elementary precautions are taken for the human enterprise as a whole. No world accounting system reckons the oil in the ground as an asset that will be needed for lubrication for the next million years. This invaluable resource is burned on the highways in an inefficient means of transportation; it pollutes the air; it kills 55,000 persons every year in the United States alone; it makes quiet and pleasant streets barbarously noisy and dangerous. Extracting the oil from the ground and burning it is added into the wrong column in the national accounts; it is taken as income rather than as loss. The principles of accounting that ensure a degree of permanence in the operation of an airline are not applied to ensure the permanence of the human habitat. The global affairs of mankind are not run with the caution and foresight applied by a grocery store.

The large problem. Underlying all the dilemmas and paradoxes of population are difficulties of housekeeping on the planet Earth, the only home man is ever likely to have. After being one species competing with many others in approximate equilibrium for most of a million years, man has taken over; his present four billion compares with possibly as few as four million only 10,000 years ago. He changes the course of evolution of plants, extracts fossil fuels, flies through the stratosphere. His technical triumphs are so recent that there has not yet been time to adjust them to ensure the long-term stability of the environment. The question is how much time he has to bring his techniques into balance with the environment.

This phrasing of the problem applies especially to the rich populations. They are increasing at about one-half percent per year and may not take long to drop to stationarity. For the poor countries rates of increase are as high as three and one-half percent per year, and their crisis is not due to improving technology and rising incomes, but to more people with distressingly low incomes. They both crowd their environments and hinder capital accumulation. Their question is how long it will be until they have only our problems to cope with. Like the question of the preceding paragraph this is not an easy one to answer.

The few things that can be said about the big problems of mankind and its environment without gross violation of the rules of logic are negative. Human population growth cannot continue indefinitely in a finite world. On the average of the long-term future, the number of births cannot be much larger than the number of deaths. With present low death rates, an average of three children per couple leads quickly to population explosion; two children per couple to population extinction.

One of the large problems, that crucially affects what line of action is appropriate, is whether population or income is responsible for environmental troubles. Is it our numbers or the way we live that makes the air unbreathable in an increasing number of places and threatens the supplies of essential materials? The answer must be both: more people living the way we live would make things worse, and expansion along the lines of our present incomes with the same number of people would likewise make things worse. Just as production is the joint outcome of the action of capital and labor, so pollution and exhaustion of materials are the joint outcome of the action of population and affluence, possibly representable by a function like $P^a\bar{Y}^b$, where P is population and $\bar{Y}$ average income. The task is to find the values of a and b and to lower them by suitably oriented changes in technology and in the pattern of life. Among other changes, our throwaway economy may give place to one in which houses, automobiles and clothing are meant to last. We know some of the technical means for attaining durability, but they are not valued sufficiently in the market, and the economic means to implement them in a free society are elusive.

These examples are enough to show that the major problems of population cannot in the present state of our knowledge be clearly formulated, let alone solved. Certainly I am in no position to state the population question in a sharp enough way that the techniques of mathematics can be brought to bear on it. This lack of incisiveness, if not actual self-contradiction, with which we tend to confront the major issues effectively prevents for the moment a scientific attack on them.

Formulation of smaller problems. But I can show that the case is not hopeless by dropping down to some smaller problems and showing how these can be sharply formulated and precisely solved. Their solution ought to encourage us to try for the larger problems, and may even provide some hints as to the methods to use.

The most fully developed model of formal demography deals with one species (man) and one sex only, and is deterministic in supposing that the probability of an event occurring to an individual gives also the fraction of individuals in the population to whom the event will occur. Usually ages are recognized, and at each age the fraction dying and the fraction bearing a child are obtained from data on a real population; the fractions are called age-specific rates, and they are taken as fixed and given. This stable population theory can hardly be called realistic, but it nonetheless gives useful answers to many questions: what is the effect on age distribution of a fall in births as against a fall in deaths? what is the effect on the rate of increase of a population of a stream of emigration at specified ages? how are age distribution and rate of increase affected by a drop in the death rate at one given age? Sensitivity analysis (for example, finding the effect on certain output parameters of varying a birth or death input) is carried out with any complete mathematical model and is indeed a main object of constructing such models; it is especially convenient with the stable model.

Population theory has gone beyond the stable model in a number of directions. Present-day writings in the mathematics of population often recognize two sexes and two or more species, as well as age-specific rates of birth and death that vary through time. The theory is often stochastic in allowing to each individual member of the population his own separate risk, rather than deterministic in supposing that whatever probability applies to each individual is also the fraction of the population that succumbs to the risk.

Applied mathematical demography. What may be called the pure mathematics of population, that is, the formal demography described above, is population analysis in separation from other disciplines. Applied mathematical demography overlaps with substantive fields of natural and social science: biology, sociology, economics, actuarial science, and many others. A few examples of the nature of the overlap can be given here.

A demographic-sociological model may be constructed for social mobility to show in what degree the increase of a population accelerates the upward movement of its individual members, for example, their promotion within an office or factory. Population increase favors mobility more than does mortality (even for those who survive). This is true whether the population consists of a firm, a government agency, a nation, or other organization. For the stable model, the relation of promotion to rate of increase can be expressed as a simple equation, and a set of age specific rates of birth and death suffices to calculate average or expected rank as a function of population increase. An extension can be worked out for an irregularly increasing population, in which the amount of advantage for a person born in a valley of births, who through his life stays in a valley of the age distribution, is calculated, as well as the disadvantage for a person born at the peak of a baby boom. The point can be discussed in terms of the size of the cohorts, which is to say groups of individuals born in given years, each such cohort being followed through its work-career and childbearing.

But the relatively advantaged cohort, as Easterlin (1961) has suggested, is then likely to have more children, and the disadvantaged fewer. This phenomenon may be shown to produce waves in births of length twice the generation. As the large number of births proceeds through the life of a cohort, it will continue to be larger than that for the cohorts born just before or just after, who will be accordingly disadvantaged in promotion and have fewer children. We have here a chain of causation that proceeds from the demographic to the social-economic and then feeds back into the demographic again.

The relation of population to insurance, and in particular to social insurance, has been studied by actuaries, but they have not fully explored the overlap with demography. For a fixed rate of increase of the population, consider the premium for old-age pensions in a no-reserve system, i.e., a system where current pensions are paid out of current premiums. Contrast this with the premium in a reserve system with given rate of interest. It can be shown that the two are identical if the rate of increase of the population in the no-reserve system is the same as the rate of interest in the reserve system. In the reserve system each cohort looks after itself, in the no-reserve system each cross section of time looks after itself, but no difference arises except at the point where the scheme is initiated or terminated.

If the population growth rate changes, then gross inequities arise in the period or no-reserve system. Those belonging to small cohorts will have higher premiums if the pension is fixed, and higher pensions than they have paid for if the premium is fixed. The amount of these inequities can be established by a demographic-actuarial analysis.

The educational system is especially sensitive to demographic changes. Until a few years ago it seemed that college enrollments were steadily growing, existing graduate schools were increasing in size, and new schools were being initiated; indefinite expansion at all levels was projected. We now know that such extrapolations give nonsensical results—they show that ultimately the entire population goes to graduate school and finds jobs as college teachers. A demographic-educational model in which two or more levels of schooling are provided for avoids such absurdities. It shows how fluctuations in the population input to the educational system result in greatly amplified fluctuations in jobs for the output of graduate schools.

The area common to anthropology and demography centers on the study of kinship. We know that expected numbers of all kin are implied by a given set of age-specific rates of birth and death, and explicit formulas for these have been established. Formulas for the variances of numbers of kin are not yet discovered, and work requiring them has had to be carried out by simulation. Further research is required for the inverse problem: given the various mean numbers of kin, taken from a genealogical table, to find age-specific rates of birth and death. Some results on this are now available; it is hoped they can be made more precise. The study is important both for historical demography and for the present-day populations in which birth and death registrations are not to be had; it can be shown that any two observations of kin—average number of aunts along with the fraction of grandmothers alive, for instance—will in principle give the regime of mortality and fertility, if these regimes can each be represented as a one-dimensional set of model tables. Different combinations of kin will select the appropriate model table with different degrees of precision.

Demography overlaps with economics on an especially wide front. Since the work of Coale and Hoover we have known that economic and social development will take place more rapidly with slower population increase, a fact shown by alternative demographic-economic projections. One further task is to extend this by setting up a variety of economic-demographic models, each containing at least a production function and a savings function, and see in what degree the drawback of increasing population is a consequence of the functions assumed.

Within biology demographers are interested in genetics, and human population genetics is a well developed field. Demography also overlaps with reproductive biology. Given the probability of conception per month for individual couples, what is the birth rate in the population as a whole? How does frequency of intercourse affect the birth rate? How does modification of the probability of conception—which is the immediate object of contraception—lower the number of births, which is its ultimate object? We know that abortion reduces the births, but does it do so in a number equal to the number of abortions? Such questions are answerable by looking at conception and birth as a stochastic process.

Many more examples of this detailed reasoning are to be found in the writings of Ansley J. Coale, Louis Henry, Robert G. Potter, Norman B. Ryder, Mindel C. Sheps, and other workers in the mathematics of population. It is no depreciation of their results to say that they have not tackled the larger problems of population. In contrasting the precise work that they have done with the self-contradictory statements on the larger issues, I express the hope that precise methods that have worked well in the small will ultimately prove applicable in the large.

FURTHER READING

ANSLEY J. COALE, *The Growth and Structure of Human Populations: A Mathematical Investigation*, Princeton University Press, Princeton, 1972.

ANSLEY J. COALE AND EDGAR M. HOOVER, *Population Growth and Economic Development in Low-Income Countries*, Princeton University Press, Princeton, 1958.

R. A. EASTERLIN, *The baby boom in perspective*, American Economic Review, 51 (1961), pp. 869–911.

W. FELLER, *On the integral equation of renewal theory*, Ann. Math. Statist., 12 (1941), pp. 243–267.

NATHAN KEYFITZ, *Introduction to the Mathematics of Population*, Addison-Wesley, Reading, Mass., 1968.

———, *On the momentum of population growth*, Demography, 8 (1971), pp. 71–80.

A. J. LOTKA, *Théorie analytique des associations biologiques, Part II. Analyse démographique avec application particulière à l'espèce humaine*, Actualités Scientifiques et Industrielles, No. 780, Hermann & Cie, Paris, 1939.