Morning 18 April 2005	FINAL EXAMINATION
PAPER NO.: 268	PAGE NO.: 1 of 7
DEPARTMENT & COURSE NO: Mathematics - 136.382	Time: 3 hours
EXAMINATION: 136.382 - Introduction to Mathematical Modelling	EXAMINER: Dr. T. G. Berry

VALUES

Instructions:

This is an OPEN-BOOK examination.
Any literature may be consulted.
Electronic calculators are permitted.
Attempt any combination of problems.
The total number of marks available is 120.
However, a score of 90 (or more) will be regarded as "full marks".

10 1. The monomolecular law for single-species population growth, namely

$$\frac{dN}{dt} = kN\frac{be^{-kt}}{1-be^{-kt}} \quad (k>0\,, \quad 1>b>0)$$

has solution

$$N(t) = C\left(1 - be^{-kt}\right) \ . \qquad \text{(I)}$$

Since $N \to C$ as $t \to \infty$, the parameter C is interpreted as the "carrying capacity" for the model.

Assume that a given set of data $\left\{ (t_i, N_i) \mid i = 1,2,...,n \right\}$ can be approximated by the monomolecular function (I) with known carrying capacity $C = 100000$.

Introduce a transformation of variables which will allow you to rewrite (I) in the form of a polynomial in t, and thus obtain a *linear system of equations* which can be solved to provide *least-squares estimates* for the parameters k and b appearing in (I) .

VALUES

10 2. You are given a set of data $\{(x_i, y_i) \mid i = 0,1,2,\ldots,n\}$ with $x_i = x_0 + i\Delta x$ and $\Delta x > 0$.
Suppose that

$$Y_i = \ell n(y_i)$$

and that

$$\Delta^2 Y_i = k \ \text{(a constant)} .$$

(a) Find the **form** of the explicit dependence of y_i on x_i .
Note: It is only necessary to find the form of this dependence. It is not necessary to find the values of the parameters in this function.

(b) Show that **recursively** one may find y_i in terms of y_{i-1} and y_{i-2} by the relation

$$y_i = K\frac{y_{i-1}^2}{y_{i-2}}$$

with $K = e^k$.

10 3. The differential equation

$$\frac{dN}{dt} = 1 - e^{-k\left(1-\frac{N}{C}\right)}, \quad (k>0, C>0)$$

is sometimes used as an ***alternative*** to the ***logistic law*** for single-species population dynamics with "carrying capacity" C.

Find $\dfrac{d^2N}{dt^2}$, and show that (unlike the case of the logistic model) a solution $N = N(t)$ of this alternative model never possesses a point of inflection.

VALUES

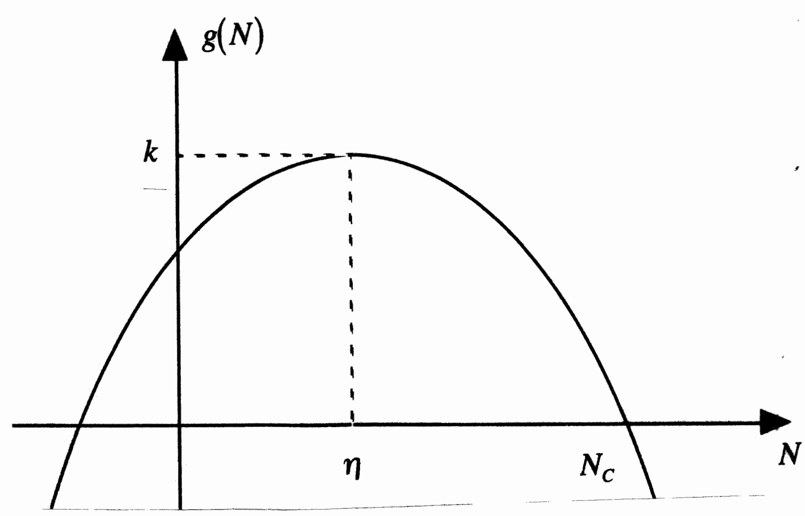
15 4. One motivation that we discussed for the logistic law for population growth involved the introduction of a *variable* relative growth rate $g(N)$ into the Malthusian model to yield

$$\frac{dN}{dt} = Ng(N) \tag{II}$$

with $g(N)$ being chosen to be $g(N) = k\left(1 - \frac{N}{C}\right)$, in which $k > 0$ is interpreted as the initial relative growth rate, and C is interpreted as the logistic carrying capacity. The above choice of $g(N)$ is made to guarantee that $g(N) \to 0$ as N increases (from its initial value $N_0 = N(0)$), so that $N = C$ becomes a stable equilibrium point of the logistic model.

The above development has been criticized in that it does not recognize the so-called ***Allee effect*** which requires that "*the relative growth rate is small when the population is small, reaches a maximum value at some intermediate population size* η *, and then decreases toward zero as* N *continues to increase*".

In an effort to incorporate the Allee effect into a single-species population model, let us adopt equation (II) as the basis of the model, and furthermore suppose that the graph of $g(N)$ is shown below:



NOTE: for the above diagram the following facts should be noted:

(i) $g(\eta) = k$ is the ***maximum value*** of $g(N)$,

(ii) $N = N_C$ is the ***only positive zero*** of $g(N)$.

continued . . .

problem 4 continued . . .

(a) Assume that $g(N)$ is a quadratic function of N of the form

$$g(N) = k - \alpha(N-\eta)^2 ,$$

and show that for this function to be consistent with the above diagram

$$\alpha < \frac{k}{\eta^2} \quad \text{and} \quad N_C = \eta + \sqrt{\frac{k}{\alpha}} \; .$$

(b) Construct a phase diagram for the resulting model, and use this information to sketch anticipated graphs of solutions of this model.

(c) Compare and contrast these anticipated solutions with those of the standard logistic model , under the assumption that the parameter k has the same value in these two models and that $C = N_C$.
[HINT: You might find it useful to compare the graphs of the functions $g(N)$ for these two models.]

25 5. Consider the following ***special case*** of the ***logistic competitive-hunters model***

$$\frac{dx}{dt} = x(\ell - ny - px)$$
$$\frac{dy}{dt} = a\,y(\ell - ny - px)$$

with a , ℓ , n and p positive constants.

(a) Identify the equilibrium point(s) of the model.

(b) On a phase-plane diagram sketch anticipated trajectories of this model.

(c) Does this model support or violate the "principle of competitive exclusion"?

(d) Find the equation of the trajectory which passes through the "initial" point (x_0 , y_0) .

(e) If $a = \frac{1}{2}$, $\ell = 60$, $p = \frac{3}{1000}$ and $n = \frac{1}{500}$, and the trajectory begins at the initial point (40000 , 30000), determine the "ultimate outcome" of the competition between these two competing species.

15 6. As ***an alternative to the standard Lotka-Volterra predator-prey model***, the following model has been proposed:

$$\frac{dx}{dt} = x(\ell - ny)$$
$$\frac{dy}{dt} = y\left(k - \lambda \frac{y}{x}\right)$$

with k , ℓ , n and positive constants . In this model $x = x(t)$ and $y = y(t)$ denote the instantaneous sizes of the prey and predator populations respectively.

Clearly the evolutionary equation for the prey species is identical to that of the Lotka-Volterra model, and thus it may be interpreted in exactly the same manner as done in lectures.

(a) Consider the evolutionary equation for the predator species, namely

$$\frac{dy}{dt} = y\left(k - \lambda \frac{y}{x}\right) .$$

What does this equation indicate about the growth rate of the predator population in each of the two cases:

(i) $y << x$,

(ii) $x << y$?

Explain the significance of these observations.

(b) Identify, and sketch on a phase plane diagram, the nullclines of this model.

(c) Determine the equilibrium point(s) of this model.

(d) In each of the regions into which the phase plane is divided by the nullclines, indicate the direction to be followed by a trajectory of this model.

(e) Use the above information to sketch anticipated graphs of trajectories for this model.

VALUES

20 7. Consider the following ***special case*** of ***the "mutual grievance" version of the Richardson's Arms Race model*** for the total expenditures $x = x(t)$ and $y = y(t)$ for countries X and Y:

$$\frac{dx}{dt} = ky - mx + r$$
$$\frac{dy}{dt} = kx - my + s$$

with k, m, r and s positive constants.

Note: ***In this version of the model the "escalation coefficients" are identical for the two countries, as are the "braking coefficients".***

(a) If $z = z(t)$ denotes the total expenditure of the two countries

i.e., $z(t) = x(t) + y(t)$,

show that it must satisfy the differential equation

$$\frac{dz}{dt} = (k - m)z + (r + s).$$

(b) If $z_0 = z(0)$ denotes the initial value of $z(t)$ at time $t = 0$, show that

$$z(t) = \begin{cases} \left(\frac{r+s}{m-k}\right) + \left(z_0 - \frac{r+s}{m-k}\right)e^{(k-m)t}, & \text{for } k \neq m \\ (r+s)t + z_0, & \text{for } k = m. \end{cases}$$

(c) Evaluate the limit of $z(t)$ as $t \to \infty$.

(d) Explain the significance of the results of parts (b) and (c), making reference to the phase-plane diagrams discussed in lectures.
To help you understand this, it might be useful to show the appropriate phase-plane diagrams for this modified model, which may easily be obtained from the corresponding diagrams discussed in lectures by making the changes required to obtain the modified model.

VALUES

15 8. For the probabilistic single-species population dynamics model we discussed in lectures, the probability $P_N(t)$ that the population is of size $N \geq N_0$ at time $t \geq 0$ is given by

$$P_N(t) = \binom{N-1}{N_0-1} e^{-bN_0 t}\left[1-e^{-bt}\right]^{(N-N_0)} .$$

For each $N \geq N_0$, $P_N(t)$ attains a single relative maximum value $(P_N)_{MAX}$ at time $(t_M)_N = \frac{1}{b}\ell n\left(\frac{N}{N_0}\right)$.

(a) Show that, as a function of N and N_0, $(P_N)_{MAX}$ is given by

$$(P_N)_{MAX} = \frac{(N-1)!}{(N_0-1)!(N-N_0)!} \frac{N_0^{N_0}(N-N_0)^{N-N_0}}{N^N} \quad \text{for } N \geq N_0 .$$

(b) Consider the sequence $\left\{(P_N)_{MAX}\right\}_{N=N_0}^{\infty}$ of maximum probabilities. Verify the claim that this sequence is monotone decreasing. Show all your work and explain fully why you may draw this conclusion.

(c) Explain why one should intuitively expect the result of part (b).

THE END

HAVE A GREAT SUMMER!!