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MODELING THE RELATIVISTIC ROCKET

This paper models a rocket in Special Relativity. It is shown that the Photon Rocket has relativistic mass limit that prevents it from using more than 50% of its fuel. This limit implies that the Photon rocket can reach the speed of light.

The Problem

n · zero

It has been know for some time that the velocity of an object with non zero rest mass can never reach the speed of light C. ?

The object with non rest mass Mo requires energy to accelerate which increase its velocity. As the object approaches c, relativistic effects occur. An increase in the objects mass is one effect that occurs and is called the Relativistic mass increase. This means the mass of the object is increasing and the object requires more and more energy to accelerate the object. The object of non zero rest mass Mo would require an infinite amount of energy to accelerate its velocity to C because the object's relativistic mass is increasing to infinity.

The rocket seems to be the only thing capable of attaining C because its rest mass decreases as its relativistic mass increases. explain

Introduction to the Non Relativistic Rocket

$\underline{V(M) = V_P \ln(M_O/M)}$ is the equation for a non relativistic rocket in the absence of a gravitational field, where V_P = the effective velocity of propellant through the rocket nozzle exit plane. M_O is the initial mass of the rocket. M is the rocket mass at any time. The above equation may be used to find the burnout velocity of a vehicle when it has used all of its propellant. ?

We will assume in our model that the initial mass of the propellant is very much larger than the mass of the rockets engines and its other structures. This will allow the variable M to approach 0 in our calculations. This is a good approximation because most of a rocket's mass is propellant.

We will assume the rocket starts off in space and gravity fields are 0. Gravity is more important for a rocket starting from the surface of planet.

The term of <u>propellent exhaust velocity</u> Vp or Bp = Vp/C will be used throughout out this paper to describe the concept of <u>propellant economy</u>.

When describing rockets the terms of propellent velocity Bp, and specific impulse I_S, are used to describe propellant economy. Bp and I_S are crucial in determining upper limits on a rocket's velocity.

I_s = lb force seconds / lb mass = N S/Kg, where 1 lb F S / lb M = 9.8067 N S/Kg

Rocket thrust describes engine effectiveness in terms of the amount of force the engine exerts and how quickly the engine can accelerate the rocket. This is not very important in our model. Normally thrust would be the most important feature in a model of a rocket.

Propulsion System Performance

System	Thrust (lbf)	I_s (lbf.s/lbm)	Thrust/weight	Propellants
Chemical Liquid Solid	 2×10^6 5×10^4	 410 260	 100 200	reaction products of C, H, O, N, F
Nuclear Heat transfer Gaseous	 10^6 10^6	 1200 3000	 30 10	 H, He, NH_3 U_{235} fission, H, He
Electro Thermal Magnetic Static	 10 1 1	 2500 15,000 25,000	 0.01 0.001 0.0001	 H, He, NH_3 H, Li, Na, K, Cs H, Li, Na, K, Cs

From this table, we can see a difference in each type of propulsion system.

The Electrostatic rocket is one rocket we will consider in our model because its propellent can have any Vp value except C. The Electrostatic rocket uses plasma ions as propellent and accelerates these ions to high speeds. In theory, any speed can be reached except C.

The Photon rocket is the other rocket we will consider in our model because its propellent has a Vp value of C. The Photon rocket does not appear in the table above. Its Vp and I_S are very much greater than the conventional rockets.

Equation for the Velocity of a Rocket in Special Relativity

$V(M) = Vp \ln(Mo/M)$, is the equation for non relativistic rocket velocity (in the absence of a gravitation field) and is explained in Appendix A. 7
M variable is the mass of rocket from Mo to 0
Mo constant is the initial rocket mass
Vp parameter is the propellant velocity

$B(M) = Bp \ln(Mo/M)$, is same equation setting $B = V/C$, where B goes from 0 to C.

We want to model the rocket equation in Special Relativity such that it is consistent with the two postulates of Special Relativity:

1) All inertial frames of reference should be equivalent.
2) The speed of light should be constant in all inertial frames of reference.

Imagine the rocket is firing a bit of propellant out the back of the rocket each second and the rocket is slowly gaining velocity in the series $Vo + V1 + V2 + V3 + \dots Vn$. This suggests we should look at the equation in Special Relativity for the transformations of velocities along the direction of motion.

$V(V1,V2) = T(V1+V2) = (V1 + V2)/(1 + (V1/C)(V2/C))$, or setting $B = V/C$

$B(B1,B2) = T(B1 + B2) = (B1 + B2)/(1 + B1\,B2))$, where $0 <= B <= 1$

This is the law for addition of hyperbolic tangents which is,

$\mathrm{Tanh}(A1 + A2) = (\mathrm{Tanh}(A1) + \mathrm{Tanh}(A2))/(1 + \mathrm{Tanh}(A1)\,\mathrm{Tanh}(A2))$

We can equate:

$B(B1,B2) = \mathrm{Tanh}(A1 + A2) = \mathrm{Tanh}(Bp \ln(Mo/M1) + Bp \ln(Mo/M2))$

$B(M) = \mathrm{Tanh}(Bp \ln(Mo/M))$

$= (1 - m\,\hat{}\,2Bp)/(1 + m\,\hat{}\,2Bp)$, becomes the equation for the velocity of a rocket in Special Relativity.

Equation for the Velocity of a Relativistic Rocket

Starting with $B(M) = \text{Tanh}\left[B_p \ln\left(\frac{M_0}{M}\right)\right]$,

we will set $M = m M_0$, or $m = \frac{M_0}{M}$,

where <u>m is the fraction of the intial rocket mass M_0 remaining.</u>

$$\beta(m) = \text{Tanh}\left(\beta_p \ln \frac{1}{m}\right)$$

, $m \in [0,1]$, $\beta_p \in [0,1]$
β_p is the velocity of the propellent used, $\beta_p = V_p/c$.

$$= \text{Tanh}\left[\ln\left(m^{-\beta_p}\right)\right]$$

$$= \frac{\text{Sinh}\left[\ln\left(m^{-\beta_p}\right)\right]}{\text{Cosh}\left[\ln\left(m^{-\beta_p}\right)\right]}$$

$$= \frac{e^{\ln u} - e^{-\ln u}}{e^{\ln u} + e^{-\ln u}}$$

, where $u = m^{-\beta_p}$

$$= \frac{u - u^{-1}}{u + u^{-1}}$$

$$= \frac{1 - u^{-2}}{1 + u^{-2}}$$

$$\beta(m) = \frac{1 - m^{2\beta_p}}{1 + m^{2\beta_p}}$$

, is the velocity of a relativistic rocket as measured by an intial observer
$0 \le \beta \le 1$, and $\beta = V/c$

Checking the limits of $\beta(m)$
to compare with the physical model

$$\beta(m, \beta_p) = \frac{1 - m^{2\beta_p}}{1 + m^{2\beta_p}}$$

$$\lim_{m \to 0} \beta(m, \beta_p) = \frac{1 - 0^{2\beta_p}}{1 + 0^{2\beta_p}} = 1,$$ the upper limit of the rocket is the speed of light.

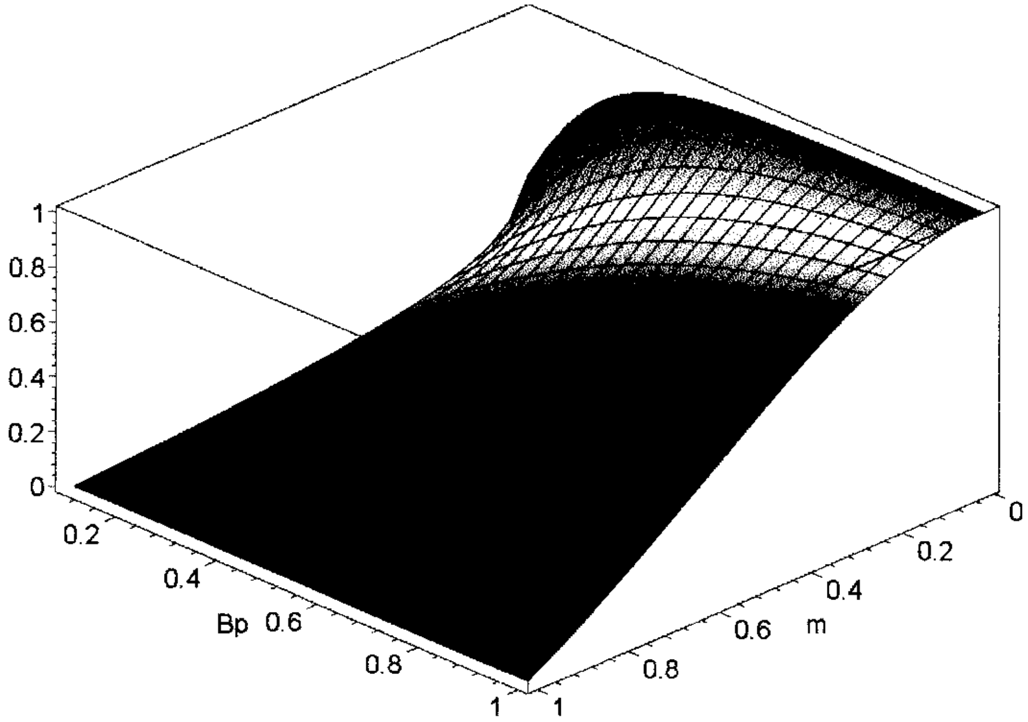
For $m = 1$, $$\beta(1, \beta_p) = \frac{1 - 1^{2\beta_p}}{1 + 1^{2\beta_p}} = 0,$$ velocity is 0 before the rocket loses any mass.

Conclusion: $\beta(m)$ behaves as one would expect.

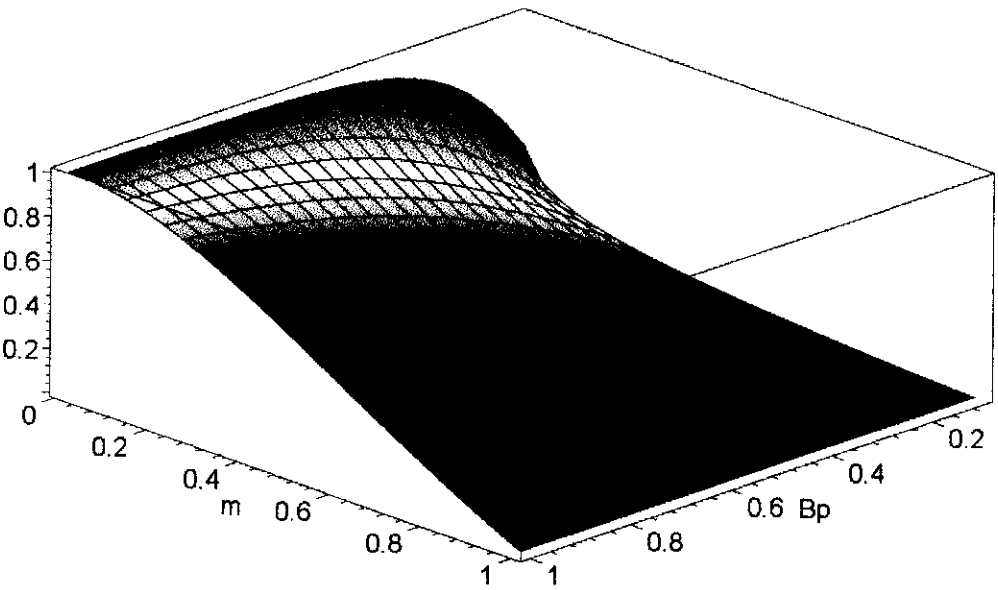
$\beta(m)$ is now a function of mass instead of a function of velocity.

$\beta(m)$ can be used to rewrite the Lorentz equations and any of the equations related to the Lorentz equations.

Speed V/C of a Rocket



Speed V/C of a Rocket



The equation for $\gamma(m)$, to use in the relativistic mass.

$$\gamma(m) = \frac{1}{\sqrt{1-\beta}}$$, β is the velocity of a relativistic rocket.

$$= \frac{1}{\sqrt{1-\left(\frac{1-u^2}{1+u^2}\right)}}$$, setting $u = m^{\beta_p}$

$$= \frac{1+u^2}{\sqrt{(1+u^2)^2\left(1-\left(\frac{1-u^2}{1+u^2}\right)\right)}}$$

$$= \frac{1+u^2}{\sqrt{(1+u^2)^2-(1-u^2)^2}}$$

$$= \frac{1+u^2}{\sqrt{(1+2u^2+u^4)-(1-2u^2+u^4)}}$$

$$= \frac{1+u^2}{\sqrt{4u^2}}$$

$$\gamma(m) = \frac{1+m^{2\beta_p}}{2m^{\beta_p}}$$

Checking the limits for $\gamma(m)$,
and comparing with the expectations from the physical model.

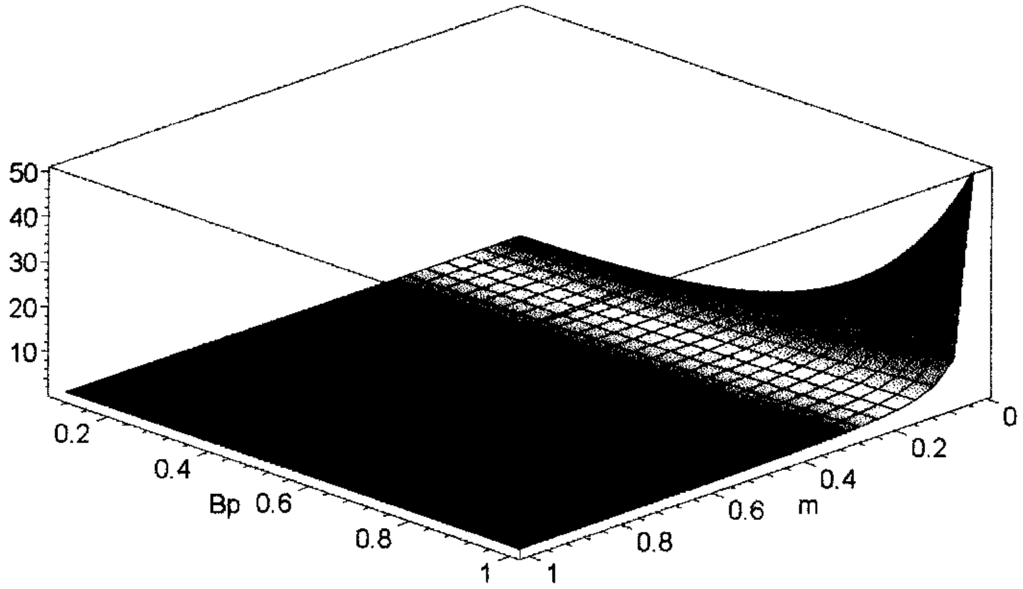
$$\gamma(m, \beta_p) = \frac{1 + m^{2\beta_p}}{2m^{\beta_p}}$$

$$\lim_{m \to 0} \gamma(m, \beta_p) = \frac{1 + 0^{2\beta_p}}{2(0^{\beta_p})} = \infty$$, γ becomes unbounded as $m \to 0$ and $\beta \to 1$

$$\gamma(1, \beta_p) = \frac{1 + 1^{2\beta_p}}{2(1^{\beta_p})} = 1$$, γ is non relativistic as $\beta = 0$ for the start

Conclusion: $\gamma(m)$ behaves as one would expect.

Relativistic Gamma for a Rocket



Equation for the Relativistic Mass of a Rocket

$m \in [0,1]$ is the fraction of mass remaining in the rocket.

$$\bar{m}\,(m, \beta_p) = \gamma(m)\; m$$, $m \in [0,1]$ is the fraction of initial mass M_0 remaining.

$$= \left[\frac{1 + m^{2\beta_p}}{2\, m^{\beta_p}} \right] m$$, the relativistic mass of a rocket measured by an initial observer.

For the case of the Photon Rocket with $\beta_p = 1$,

$$\bar{m}\,(m, 1) = \frac{1 + m^2}{2}$$, the relativistic mass of a photon rocket as measured by an initial observer.

$\bar{M} = \bar{m} M_0$ is the mass remaining in the rocket from an inertial frame of reference. (in kg)

$M = m M_0$ is the mass remaining in the rocket from the rocket's frame of reference. (in kg)

Checking the limits for the relativistic mass

$$\overline{m}(m,1) = \frac{1+m^2}{2}$$, for the case of the Photon Rocket with $B_p = 1$.

$$\lim_{m\to 0} \overline{m}(m,1) = \frac{1+0}{2} = \frac{1}{2}$$, $\overline{M} = \frac{1}{2} M_0$ is the limit in Kg.

Conclusion: The Photon Rocket can use $\frac{1}{2}$ of its intial mass M_0.
A lack of energy does not prevent the Photon Rocket from attaining C.

$$\lim_{m\to 0} \overline{m}(m, B_p<1) = \left[\frac{1+m^{2B_p}}{2m^{B_p}}\right] m$$

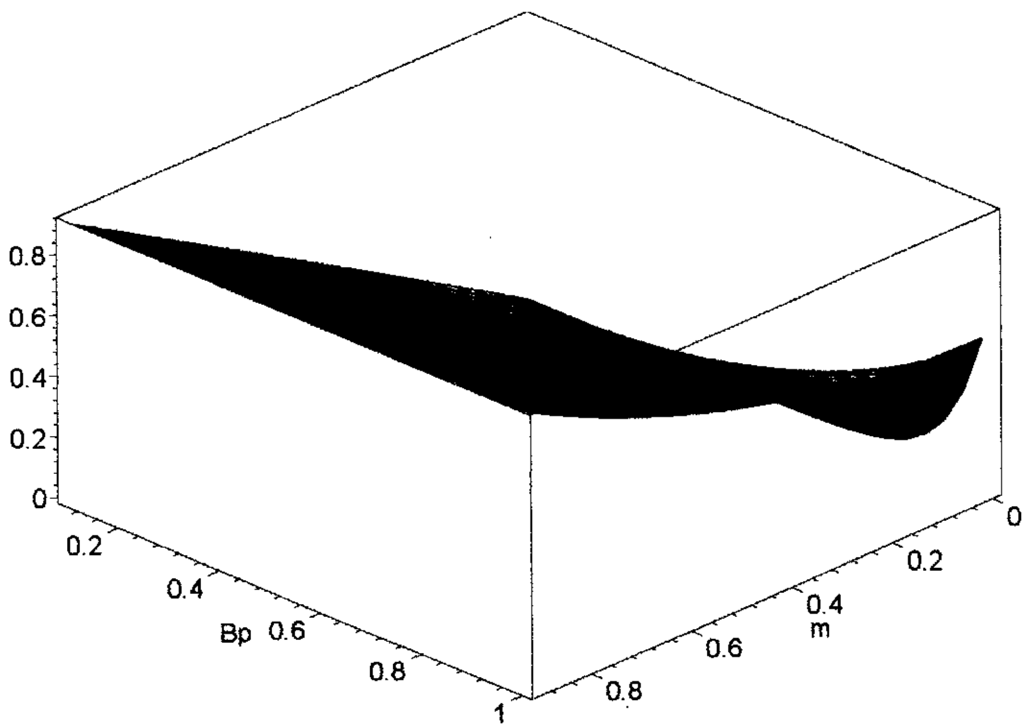
$$= (1+m^{2B_p})\frac{1}{2}(m^{1-B_p})$$

$$\lim_{m\to 0} \overline{m}(m, B_p<1) = (1+0^{2B_p})\frac{1}{2}(0^{1-B_p})$$

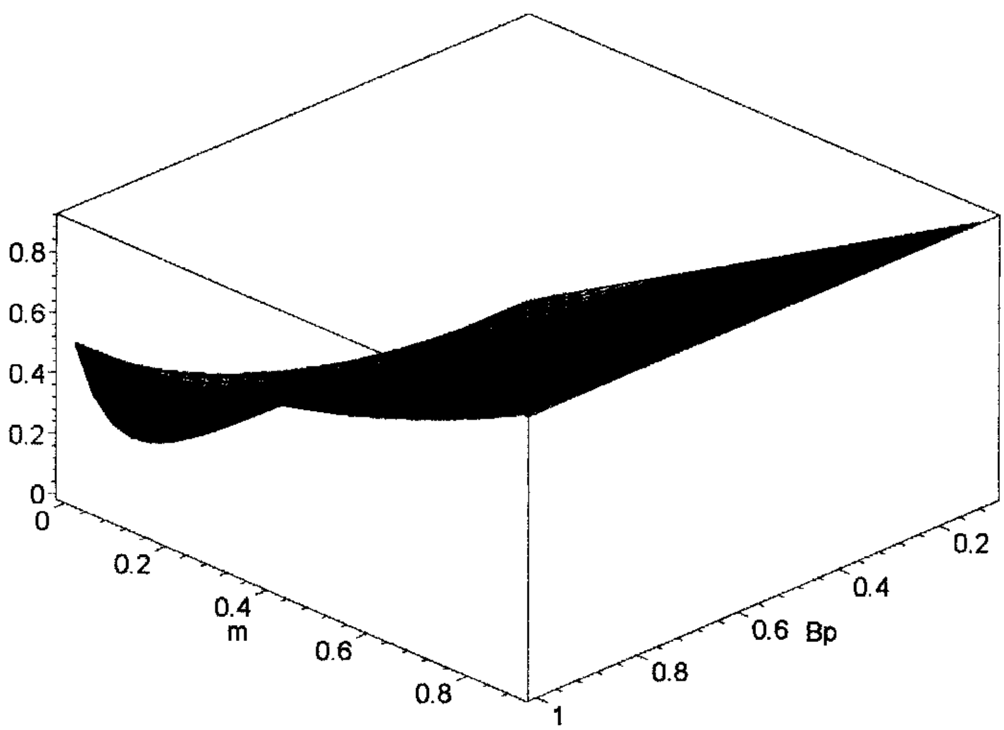
$$= 0 \quad \text{for } B_p < 1$$

Conclusion: Non-Photon Rockets will use all of its M_0
Non-Photon Rockets can not attain C.

Relativistic Mass of the Rocket



Relativistic Mass of the Rocket



Equation for the Relativistic Density of a Rocket.

$$\bar{\rho} = \bar{m} / \overline{Volumn}$$

$$= \frac{\bar{m}}{\bar{x}\, y_0 z_0}$$, x_0, y_0, z_0 are the dimensions of the rocket in its inertial rest frame

$$= \frac{\gamma m \; M_0}{\left(\frac{1}{\gamma} x_0\right) y_0 z_0}$$, $\bar{x} = \frac{1}{\gamma} x$ is the dimension of the rocket measured by an inertial observer.

$$= \frac{\gamma^2 m \left(\frac{M_0}{x_0 y_0 z_0}\right)}{}$$, $m \in [0,1]$ is the fraction of intial mass M_0 remaini[ng]
ρ_0 is the intial density

$$\bar{\rho} = \gamma^2 m \rho_0$$, is the relativistic density of a rocket as measured by an intial observer.

Checking the limits for the relativistic density

$$\bar{\rho}(m, \beta_p) = \gamma^2 m \rho_0$$

$$= \left[\frac{1 + m^{2\beta_p}}{2 m^{\beta_p}} \right]^2 m \rho_0$$

$$= (1 + m^{2\beta_p})^2 \frac{\rho_0}{2} (m^{1-2\beta_p})$$

$$= (1 + 2m^{2\beta_p} + m^{4\beta_p}) \frac{\rho_0}{2} (m^{1-2\beta_p})$$

$$\lim_{m \to 0} \bar{\rho}(m, \beta_p) = (1 + 0 + 0) \frac{\rho_0}{2} \lim_{m \to 0} (m^{1-2\beta_p})$$

$$= \frac{\rho_0}{2} \lim_{m \to 0} (m^{1-2\beta_p})$$

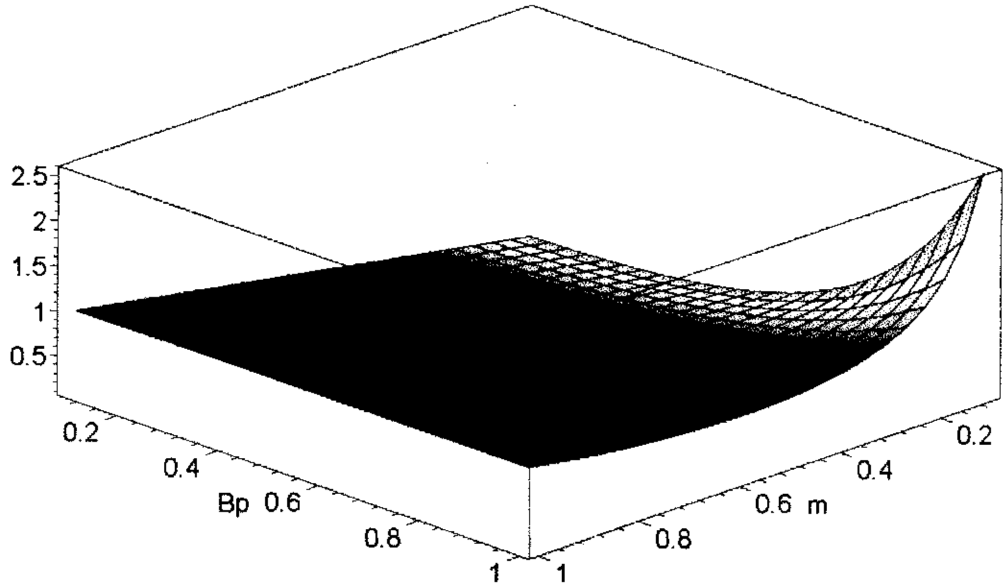
$$\lim_{m \to 0} \bar{\rho}(m, \beta_p > 1/2) = \infty$$

$$\lim_{m \to 0} \bar{\rho}(m, \beta_p = 1/2) = \rho_0/2$$

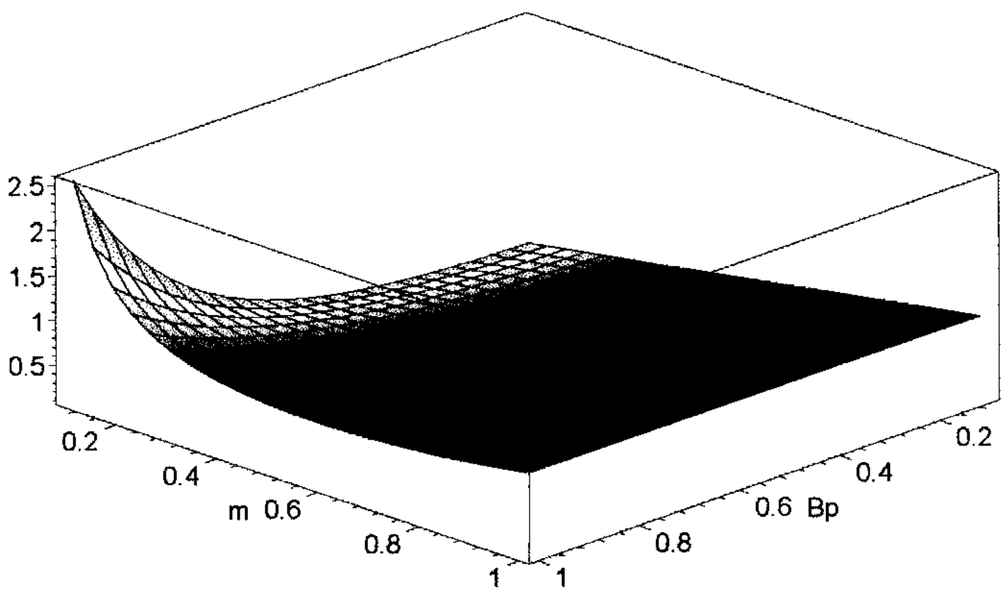
$$\lim_{m \to 0} \bar{\rho}(m, \beta_p < 1/2) = 0$$

Conclusion: The rockets with $\beta_p = 1/2$ attain a relativistic density limit, Rockets with $\beta_p > 1/2$ have their relativistic densities increase unbounded.

Relativistic Density of the Rocket



Relativistic Density of a Rocket



Conclusions and Suggestions for Improvements

We want to use the Relativistic Rocket as a tool to model other systems.

The Relativistic Rocket can be used to model nuclear fission and nuclear collisions.

The propellant that leaves the rocket is effectively equivalent to particles emitted from nuclear fission. A model can have a variable Bp to encompass the different particles emitted by a nucleon. The Photon Rocket can model reactions that emit gamma rays and photons of other energy levels.

A Photon Rocket with very little rest mass traveling close to the speed of light would make a good model for the Neutrino particle.

Nuclear collisions can be modeled by looking at the Relativistic Rocket backwards in time and seeing the propellant colliding with the rocket in a series of inelastic collisions.

The Photon Rocket can be used to model a casual event horizon where information can not be exchanged between regions of space.

A person viewing the rocket from an inertial frame of reference would see the photon propellent red shifted more and more as the rocket approaches the speed of light. There is a point were the photons are red shifted so greatly that their wavelength is not distinguishable because their energies are too small to be detected.

The red shifting is a physical reason for the Relativistic Mass limit. Each photon propellent is getting smaller and smaller in energy and its mass equivalence because of this red shifting. As the rocket approaches the speed of light, the mass of its emitted photons approaches 0.

The Relativistic Density of a rocket should also be looked at in greater detail as the density in some cases increases unbounded.

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Taylor, Edwin E. and Wheeler, John Archibald (1966). Spacetime Physics. W. H. Freeman and Company. San Francisco.

```
> restart;
> mParameter:= M/Mo;
```

$$mParameter := \frac{M}{Mo}$$

```
> beta:= (1- m^(2*Bp)) /(1+m^(2*Bp));
```

$$\beta := \frac{1 - m^{(2\,Bp)}}{1 + m^{(2\,Bp)}}$$

```
> Gamma:= (1-beta^2)^(-0.5);
```

$$\Gamma := \frac{1}{\left(1 - \frac{\left(1 - m^{(2\,Bp)}\right)^2}{\left(1 + m^{(2\,Bp)}\right)^2}\right)^{.5}}$$

```
> mRel:= Gamma * m;
```

$$mRel := \frac{m}{\left(1 - \frac{\left(1 - m^{(2\,Bp)}\right)^2}{\left(1 + m^{(2\,Bp)}\right)^2}\right)^{.5}}$$

```
> densityRel:= Gamma^2 * m;
```

$$densityRel := \frac{m}{\left(1 - \frac{\left(1 - m^{(2\,Bp)}\right)^2}{\left(1 + m^{(2\,Bp)}\right)^2}\right)^{1.0}}$$

```
> plot3d(beta,m=.01..1,Bp=0.1..1,color=m,title=`Speed V/C of a
  Rocket` ,axes=BOXED);
> plot3d(beta,Bp=0.1..1,m=.01..1,color=m,title=`Speed V/C of a
  Rocket`,axes=BOXED);
>
> plot3d(Gamma,m=0.01..1,Bp=0.1..1,color=m,title=`Relativistic Gamma
  for a Rocket`,axes=BOXED);
>
> plot3d(mRel,Bp=0.1..0.9,m=.000001..1,color=m,title=`Relativistic
  Mass of a Rocket`,axes=BOXED);
> plot3d(mRel,m=.000001..0.9,Bp=0.1..1,color=m,title=`Relativistic
  Mass of a Rocket`,axes=BOXED);
>
> plot3d(densityRel,Bp=0.1..1,m=.1..1,color=m,title=`Relativistic
  Density of a Rocket`,axes=BOXED);
> plot3d(densityRel,m=.1..1,Bp=0.1..1,color=m,title=`Relativistic
  Density of a Rocket`,axes=BOXED);
```