University of Manitoba
Math 38200 – Winter 2008

Assignment 1

Solutions

Remarks

The maximum mark was 55. Please look carefully at the method in exercises 4 and 5, for the existence. You had all the tools, just not the experience.. But from now on, you will be expected to be able to tackle such a problem (of reasonable difficulty).

The Ricker model of growth of a single population takes the form

$$N_{t+1} = N_t e^{r\left(1-\frac{N_t}{K}\right)}, \tag{1}$$

with $r, K > 0$ and initial condition $N_0 > 0$.

1. (5 points) Setting $x_t = N_t/K$ is similar to saying that $N_t = Kx_t$. Substituting this value into (1), we get

$$\begin{aligned} Kx_{t+1} &= Kx_t e^{r\left(1-\frac{Kx_t}{K}\right)} \\ &= Kx_t e^{r(1-x_t)}, \end{aligned}$$

and therefore,

$$x_{t+1} = x_t e^{r(1-x_t)}. \tag{2}$$

Note that N has units *number of individuals*, K has units *number of individuals*, and therefore N/K is dimensionless.

2. (5 points) Suppose that $x_0 > 0$. Then $x_1 > 0$, since $e^{r(1-x_0)} > 0$ and $x_0 > 0$. Now suppose that $x_k > 0$. Then $x_{k+1} > 0$, since $e^{r(1-x_k)} > 0$ and $x_k > 0$. Therefore, by induction, $x_t > 0$ for all t.

3. (15 points) We set

$$f(x) = xe^{r(1-x)}$$

and seek p such that $p = f(p)$. We have

$$\begin{aligned} p = f(p) &\Leftrightarrow p = pe^{r(1-p)} \\ &\Leftrightarrow p = 0 \text{ or } 1 = e^{r(1-p)} \\ &\Leftrightarrow p = 0 \text{ or } 0 = r(1-p) \qquad \text{(taking the ln of both sides)} \\ &\Leftrightarrow p = 0 \text{ or } p = 1. \end{aligned}$$

So there are two fixed points, 0 and 1. To study the stability, we compute

$$f'(x) = (1 - rx)e^{r(1-x)}.$$

Then

$$|f'(0)| = |e^r| = e^r,$$

so that 0 is attracting if $r > 0$ and repelling if $r < 0$. Since it is assumed that $r > 0$, 0 is always attracting. Also,

$$|f'(1)| = |1 - r|.$$

So

$$\begin{aligned} |f'(1)| < 1 &\Leftrightarrow |1 - r| < 1 \\ &\Leftrightarrow -1 < 1 - r < 1 \\ &\Leftrightarrow -2 < -r < 0 \\ &\Leftrightarrow 0 < r < 2. \end{aligned}$$

Therefore, if $0 < r < 2$, 1 is attracting, and if $r > 2$, 1 is repelling.

4. (10 points) We have

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f(x)e^{r(1-f(x))} \\ &= xe^{r(1-x)}e^{r(1-xe^{r(1-x)})} \end{aligned}$$

We have a period 2 point if $f^2(p) = p$ for some p, that is,

$$p = pe^{r(1-p)}e^{r(1-pe^{r(1-p)})}.$$

The first conclusion is that $p = 0$ is a fixed point (but since 0 is also a fixed point, it is not a point with *least* period 2). Supposing now that $p \neq 0$, we thus must find p such that

$$1 = e^{r(1-p)}e^{r(1-pe^{r(1-p)})}.$$

Taking the ln of both sides of the equation, we have

$$0 = r(1-p) + r\left(1 - pe^{r(1-p)}\right).$$

Note that it is clear from this equation that $p = 1$ is a fixed point of f^2 but, as for $p = 0$, it is not a point with *least* period 2, since $p = 1$ is also a fixed point. Therefore, we must solve (for p) the equation

$$r(1-p) + r(1 - pe^{r(1-p)}) = 0,$$

that is,

$$r\left\{p\left(e^{r(1-p)}+1\right) - 2\right\} = 0.$$

Since $r > 0$, solving this is equivalent to finding the roots of the equation

$$\Gamma(p) = p\left(e^{r(1-p)}+1\right) - 2. \tag{3}$$

We have $\Gamma(0) = -2$, and $\lim_{p\to\infty} \Gamma(p) = \infty$, therefore, since Γ is continuous, it must have at least one root. Clearly, $p = 1$ is such a root, but are there others? We have

$$\Gamma'(p) = 1 + (1 - rp)e^{r(1-p)} \tag{4}$$

and

$$\Gamma''(p) = -r(2 - rp)e^{r(1-p)}$$

At 0, $\Gamma''(0) = -2re^r < 0$, that is, Γ is concave down. It then changes concavity when $rp = 2$, i.e., for $p = 2/r$, and then remains concave up for all values of $p > r/2$. Now note that since $0 < r < 2$, it follows that $1 < 2/r < \infty$, so that the inflection point occurs after $p = 1$.

On the other hand, $\Gamma'(0) = 1 + e^r > 0$, so Γ is initially increasing, from the value -2. Putting all this together, we have that Γ has the shape shown in Figures 1 and 2.

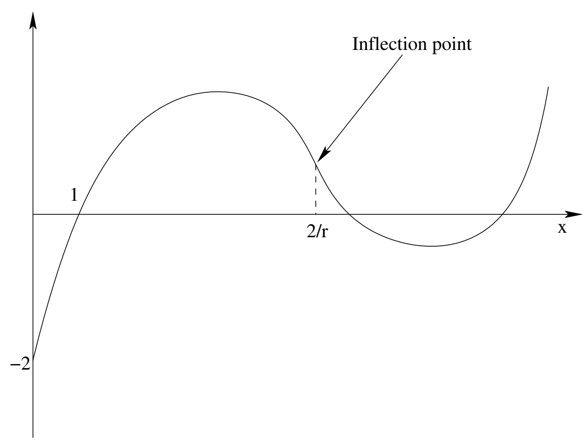

Figure 1: One possibility for Γ that would lead to two positive period 2 points for (2).

To be able to exclude the possibility of the case of Figure 1 happening, we reason as follows. The main difference between Figures 1 and 2 is that when the inflection point is reached, Γ is increasing in Figure 2. So a sufficient condition for period 2 point is that

$$\Gamma'\left(\frac{2}{r}\right) > 0.$$

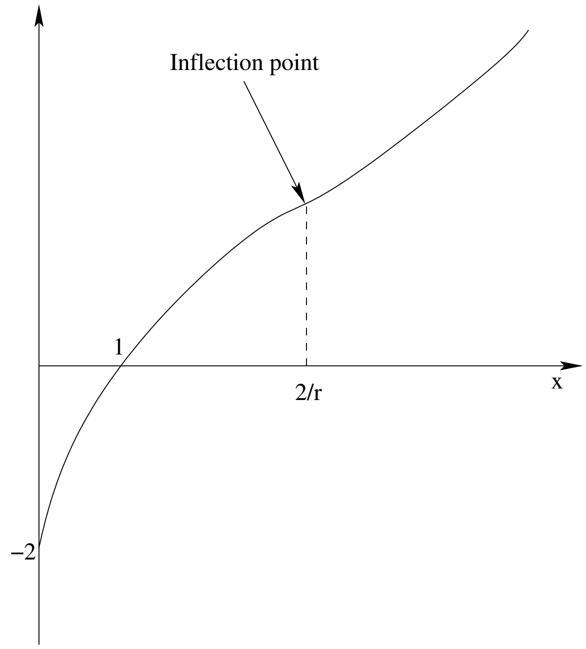

Figure 2: One possibility for Γ that would lead to $x = 1$ as a positive period 2 point for (2).

(Note that this is only sufficient: in Figure 1, we see that if Γ is decreasing at the inflexion point, it is still possible for the graph of Γ not to intersect the p-axis after the inflexion point.) We have

$$\Gamma'\left(\frac{2}{r}\right) = 1 - e^{r\left(1-\frac{2}{r}\right)}$$

Recall that since $r < 2$, we have $2/r > 1$ and thus $1 - 2/r < 0$. Therefore, since $r > 0$,

$$e^{r\left(1-\frac{2}{r}\right)} < 1$$

and thus it is always true that $\Gamma'(2/r) > 0$ when $r < 2$. This means that the sufficient condition we derived for there not to exist a point of least period 2 is always satisfied, when $0 < r < 2$, i.e., when $0 < r < 2$, it is the situation depicted in Figure 2 that prevails.

Therefore, when $0 < r < 2$, there are no points of least period 2.

5. (10 points) We use the same reasoning as in question 4, but here, we assume that $r > 2$. The first conclusion is that the inflection point $p = 2/r < 1$. At $p = 2/r$,

$$\Gamma\left(\frac{2}{r}\right) = \frac{2}{r}\left(e^{r-2} + 1 - r\right).$$

This is clearly positive for $r > 2$, since the function $\Theta(r) = e^{r-2} + 1 - r$ is such that $\Theta(2) = 0$ and $\Theta'(r) = e^{r-2} - 1 > 0$ for all $r > 2$. It follows that $\Gamma(2/r) > 0$, that is, $\Gamma > 0$ at the inflection point $2/r$. As a consequence, there is a period 2 point p_1^* in $(0, 2/r)$.

Since Γ is increasing from 0 to $2/r$ and that $2/r < 1$, this implies that there exists a fixed point in the interval $(0, 1)$.

Figure 3: One possibility for Γ that leads to the existence of positive period 2 points for (2). (Case of Exercise 5, the inflexion point $2/r$ of Γ lies to the left of $p = 1$.)

The curve Γ then decreases and hits the p-axis for $p = 1$. (Recall that it was shown in Exercise 4 that $\Gamma(1) = 0$.) Now, at the point $p = 1$, we have

$$\Gamma'(1) = 2 - r,$$

which is negative since $r > 2$. It follows, by continuity of Γ, that there exists $\varepsilon > 2$ such that $\Gamma(\varepsilon) < 0$. There is no other change of concavity of Γ after $2/r$, Γ is thus concave up for $p > 2/r$. As $\Gamma(\varepsilon) < 0$ and $\lim_{p\to\infty} \Gamma(p) = \infty$, this implies that there exists a unique $p_2^* > 1$ such that $\Gamma(p_2^*) = 0$.

Therefore, for $r > 2$, there exists $p_1^* \in (0, 2/r)$ and $p_2^* > 1$, period 2 points of f. To study the stability of these period two points, we need to compute the derivative of

$$f^2(x) = xe^{r(1-x)}e^{r(1-xe^{r(1-x)})},$$

which is given by

$$\frac{d}{dx}f^2(x) = (1 - rp)e^{r(1-p)}\left(1 - rpe^{r(1-p)}\right)e^{r\left(1-pe^{r(1-p)}\right)}$$

6. (10 points) Using numerical software, draw a bifurcation diagram for (2), for r varying in $(0, 5]$. What do you observe?