

# The Attractiveness of the Droop Equations

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## ABSTRACT

The Droop equations are a system of three coupled, nonlinear ordinary differential equations describing the growth of a microorganism in a chemostat. The growth rate of the organism is limited by the availability of a single nutrient. In contrast to the better known Monod equations, the nutrient is divided into external and internal cellular pools. Only the internal pool can catalyze growth. This paper proves that the Droop equations are globally stable. Based on a single combination of parameters, either the chemostat organism goes extinct or it tends to a fixed, positive concentration.

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## INTRODUCTION

When M. R. Droop [4] studied the growth-limiting effect of vitamin B<sub>12</sub> deficiency on the alga *Monochrysis lutheri* under culture in a chemostat, he was perplexed that the usual Monod [15] model gave a poor fit to his data. This led him to postulate the cell quota model for nutrient limitation. What distinguishes this model from the earlier Monod [15] model is the notion of an internal nutrient pool. The rate-limiting substrate is subdivided into the ambient substrate outside the cells and the substrate internal to individual cells. Only the internal substrate is immediately available for cell growth and metabolism. Passage of nutrient from outside to inside the cells introduces inevitable time delays. These time delays are absent in the Monod model and account for the inaccuracies of the Monod equations in nonequilibrium situations.

For instance, if there is a step increase in the external nutrient concentration, the Monod equations imply a step increase in the specific growth rate. The Droop equations, in contrast, imply a gradual rise in the specific growth rate. Even in nonchemostat settings the differ-

ence can be seen. Droop [9] remarks:

A case in point in the algal field was the luxury uptake of phosphorus, first observed by Ketchum [13] and later studied in detail with the marine diatom *Phaeodactylum tricornutum* by Kuenzler and Ketchum [14]. When cells of this alga were placed in fresh medium virtually all the phosphorus was observed to be taken up by the cells from the new medium before even the first cell division had taken place. Thereafter it was apportioned among the progeny of the subsequent cell divisions, which continued until cell phosphorus had dropped sufficiently to arrest cell division. It is apparent that cell division during the life of the culture must have occurred without direct reference to the external concentration of the substrate.

In the present paper, we explore the global stability of the Droop equations. As developed by Droop [5, 6] and Burmaster [1], these three coupled, nonlinear ordinary differential equations relate the ambient concentration  $S$  of the limiting nutrient, the total biomass concentration  $X$  of the organism in the chemostat, and the cell quota  $Q$ , which is defined as the concentration of the limiting nutrient in the internal pool.  $S$ ,  $X$ , and  $Q$  have units of mass/volume, biomass/volume, and mass/biomass, respectively. If the average mass of a cell is fairly constant over the course of an experiment, then cell number can be substituted for biomass in these definitions [8].

In contrast to the Monod equations, which have been thoroughly investigated mathematically [16], the Droop equations have very little visibility in the biomathematics literature. This is unfortunate because of the empirical superiority of the Droop equations to the Monod equations [9]. We show here that the Droop equations tend to a single global equilibrium. Depending on a particular combination of the underlying parameters, this global equilibrium represents either extinction for the chemostat organism or long-term survival at a definite fixed level. There can be no limit cycles or chaotic behavior. Approach to the global equilibrium should also entail no systematic oscillations, as the eigenvalues of the variational matrix at the equilibrium are all negative and real.

## REVIEW OF THE DROOP EQUATIONS

Let us begin by simply listing the Droop equations:

$$S'(T) = D[S_0 - S(T)] - \rho_m \frac{S(T)X(T)}{K_\rho + S(T)}, \quad (1a)$$

$$X'(T) = \mu_m \left(1 - \frac{K_Q}{Q(T)}\right) X(T) - DX(T), \quad (1b)$$

and

$$Q'(T) = \rho_m \frac{S(T)}{K_p + S(T)} - \mu_m [Q(T) - K_Q]. \quad (1c)$$

In these equations  $T$  represents time. (We will pass to lowercase letters in a moment when we introduce nondimensional versions of the equations.) Also, all parameters and the initial values of  $S(0)$ ,  $X(0)$ , and  $Q(0)$  are taken positive.

In (1a), the equation for  $S'(T)$ , the first term on the right is the difference between the rate of ambient nutrient concentration entering the chemostat chamber and the rate of ambient nutrient concentration leaving the chamber. The parameter  $S_0$  is the concentration of the influent nutrient, and  $D$  is the dilution rate; these have units of mass/volume and  $\text{time}^{-1}$ , respectively. The second term gives the rate of loss of  $S$  due to uptake by the organism. Here typical hyperbolic kinetics are assumed. The parameter  $\rho_m$  is the maximum uptake rate, and  $K_p$  is the associated half-saturation constant. These two parameters have units of mass/(biomass  $\times$  time) and mass/volume, respectively.

In (1b), the equation for  $X'(T)$ , the factor  $\mu_m \left(1 - \frac{K_Q}{Q(T)}\right)$  gives the growth rate per unit biomass concentration of the organism. This growth rate is multiplied by  $X(T)$  to give the growth rate of the total biomass concentration. The parameter  $\mu_m$  is the maximum growth rate, and  $K_Q$  is the minimum cell quota. When  $Q$  drops below  $K_Q$ , there is insufficient internal nutrient to sustain the organism. The units of  $\mu_m$  and  $K_Q$  are  $\text{time}^{-1}$  and mass/biomass, respectively. The second term on the right of (1b) accounts for the washout of the organism in the effluent of the chemostat.

In Equation (1c), the first term on the right represents recruitment of nutrient into the internal pool. The factor  $X(T)$  in the corresponding term of the equation for  $S'(T)$  is missing here because  $Q$  is measured as a concentration relative to  $X$ . The second term on the right of Equation (1c) represents the reduction of the concentration of  $Q$  due to growth in biomass. Up to sign, this is just the growth rate of Equation (1b) multiplied by  $Q(T)$ . As biomass expands, the cell quota  $Q$  must fall unless balanced by the influx of nutrient from the external pool.

It is worth noting some of the limitations of these equations. First, growth and cell division are not distinguished in the model. Second, the lumped variable  $Q$  is forced to describe possibly many complex biochemical pathways for the nutrient in the cell. Third, the model makes no provision for excretion or sequestration of the limiting nutrient by a cell [7]. Finally, the growth and uptake functions are somewhat arbitrary. Fortunately, in algae the empirical data supporting these functional forms are impressive [2, 9].

It is convenient to change to nondimensional variables in Equations (1a)–(1c). Perhaps the simplest approach is to define the variables

$$t = DT; \quad s = \frac{S}{S_0}; \quad x = \frac{\rho_m X}{DS_0}; \quad q = \frac{Q}{K_Q}$$

and the new combinations of parameters

$$\alpha_1 = \frac{K_\rho}{S_0}; \quad \alpha_2 = \frac{\mu_m}{D}; \quad \alpha_3 = \frac{\rho_m}{DK_Q}.$$

With these conventions, the nondimensional Droop equations become

$$s'(t) = 1 - s(t) - \frac{s(t)x(t)}{\alpha_1 + s(t)}, \quad (2a)$$

$$x'(t) = \alpha_2 \left[ 1 - \frac{1}{q(t)} \right] x(t) - x(t), \quad (2b)$$

and

$$q'(t) = \frac{\alpha_3 s(t)}{\alpha_1 + s(t)} - \alpha_2 [q(t) - 1]. \quad (2c)$$

Burmaster [1] identified a first integral of the system (2a)–(2c) based on the total concentration of limiting nutrient in the chemostat chamber. The nondimensional form of this first integral can be derived by multiplying Equation (2b) by  $\alpha_3^{-1}q(t)$ , multiplying Equation (2c) by  $\alpha_3^{-1}x(t)$ , and adding the results to Equation (2a). The outcome of these manipulations is the linear differential equation

$$\frac{d}{dt} [\alpha_3^{-1}q(t)x(t) + s(t) - 1] = -[\alpha_3^{-1}q(t)x(t) + s(t) - 1],$$

with the obvious solution

$$\alpha_3^{-1}q(t)x(t) + s(t) - 1 = [\alpha_3^{-1}q(0)x(0) + s(0) - 1]e^{-t}. \quad (3)$$

This first integral shows that the Droop system (2a)–(2c) converges to the locus of the equation  $\alpha_3^{-1}xq + s - 1 = 0$  as  $t \rightarrow \infty$ .

Equation (2b) yields immediately

$$x(t) = x(0) \exp \left[ (\alpha_2 - 1)t - \alpha_2 \int_0^t \frac{1}{q(s)} ds \right]. \quad (4)$$

If  $\alpha_2 \leq 1$ , then it is intuitively clear that  $\lim_{t \rightarrow \infty} x(t) = 0$  because the dilution rate exceeds the maximum growth rate. This extinction property follows directly from Equation (4) if  $q(t)$  is bounded above and below by positive constants. Parts (a) and (b) of Lemma 2 (proved later) affirm these facts, and consequently we assume  $\alpha_2 > 1$  in the sequel.

### LOCAL STABILITY ANALYSIS

The Droop system (2a)–(2c) possesses either one or two critical points in the first orthant  $\{(s, x, q): s \geq 0, x \geq 0, q \geq 0\}$ . Setting  $x' = 0$  in Equation (2b) gives either  $x = 0$  or  $\alpha_2(1 - 1/q) - 1 = 0$ . In the former case, setting  $s' = 0$  in Equation (2a) then implies  $s = 1$ , and setting  $q' = 0$  further implies  $q = (\alpha_2 + \alpha_3 + \alpha_1 \alpha_2) / [\alpha_2(\alpha_1 + 1)]$ . We will refer to the point

$$\left( \begin{array}{c} 1 \\ 0 \\ \frac{\alpha_2 + \alpha_3 + \alpha_1 \alpha_2}{\alpha_2(\alpha_1 + 1)} \end{array} \right) \quad (5)$$

as the boundary critical point of the Droop system.

If  $x = 0$  is not assumed, then

$$q = \frac{\alpha_2}{\alpha_2 - 1}$$

solves  $x' = 0$  in (2b). Substitution of  $q$  in  $q' = 0$  now produces

$$s = \frac{\alpha_1 \alpha_2}{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3},$$

which in turn yields

$$x = \frac{(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2) \alpha_3 (\alpha_2 - 1)}{\alpha_2 (\alpha_2 \alpha_3 - \alpha_2 - \alpha_3)}$$

when  $s$  is substituted in  $s' = 0$ . If this critical point is to lie in the first orthant, then the condition  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 \geq 0$  must hold.

When this last inequality is strict, we will refer to the point

$$\left( \begin{array}{c} \frac{\alpha_1 \alpha_2}{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3} \\ \frac{(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2) \alpha_3 (\alpha_2 - 1)}{\alpha_2 (\alpha_2 \alpha_3 - \alpha_2 - \alpha_3)} \\ \frac{\alpha_2}{\alpha_2 - 1} \end{array} \right) \quad (6)$$

as the *interior critical point* of the Droop system. The two critical points (5) and (6) coincide when  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 = 0$ .

The local stability of either critical point depends on the eigenvalues of the variational matrix

$$A = \begin{pmatrix} -1 - \frac{\alpha_1 x}{(\alpha_1 + s)^2} & -\frac{s}{\alpha_1 + s} & 0 \\ 0 & \alpha_2 \left(1 - \frac{1}{q}\right) - 1 & \frac{\alpha_2 x}{q^2} \\ \frac{\alpha_1 \alpha_3}{(\alpha_1 + s)^2} & 0 & -\alpha_2 \end{pmatrix}. \quad (7)$$

At the boundary critical point, (7) reduces to

$$A = \begin{pmatrix} -1 & -\frac{1}{\alpha_1 + 1} & 0 \\ 0 & \frac{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2}{\alpha_2 + \alpha_3 + \alpha_1 \alpha_2} & 0 \\ \frac{\alpha_1 \alpha_3}{(\alpha_1 + 1)^2} & 0 & -\alpha_2 \end{pmatrix},$$

and the characteristic polynomial of  $A$  is

$$\det(\lambda I - A) = (\lambda + 1) \left( \lambda - \frac{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2}{\alpha_2 + \alpha_3 + \alpha_1 \alpha_2} \right) (\lambda + \alpha_2).$$

The eigenvalues of  $A$ ,  $\lambda = -1$ ,  $\frac{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2}{\alpha_2 + \alpha_3 + \alpha_1 \alpha_2}$ , and  $-\alpha_2$ , are all negative when  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 < 0$ . In this case, the boundary critical point is asymptotically stable, and the interior critical point does not exist.

When  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 = 0$ , one eigenvalue of the variational matrix  $A$  is 0 at the boundary critical point, and the other two eigenvalues are negative. In this instance, the eigenvectors associated with the eigenvalues  $-1$ ,  $0$ , and  $-\alpha_2$  are, from left to right, respectively, the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -(\alpha_1 + 1) & 0 \\ \frac{\alpha_1 \alpha_3}{(\alpha_2 + 1)^2 (\alpha_2 - 1)} & \frac{\alpha_1 \alpha_3}{(\alpha_1 + 1)^2 \alpha_2} & 1 \end{pmatrix}.$$

Despite the presence of 0 as an eigenvalue, we will demonstrate that the boundary critical point is still asymptotically stable.

When  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 > 0$ , the boundary critical point is a saddle point. The interior critical point is asymptotically stable. In general, the characteristic polynomial of  $A$  is

$$\begin{aligned} \det(\lambda I - A) &= \left( \lambda + 1 + \frac{\alpha_1 x}{(\alpha_1 + s)^2} \right) \left[ \lambda - \alpha_2 \left( 1 - \frac{1}{q} \right) + 1 \right] (\lambda + \alpha_2) \\ &\quad + \left( \frac{\alpha_1 \alpha_3}{(\alpha_1 + s)^2} \right) \left( \frac{s}{\alpha_1 + s} \right) \left( \frac{\alpha_2 x}{q^2} \right). \end{aligned}$$

At the interior critical point this becomes

$$\begin{aligned} \det(\lambda I - A) &= (\lambda + 1 + c) \lambda (\lambda + \alpha_2) + c(\alpha_2 - 1) \\ &= (\lambda + 1) [\lambda^2 + (c + \alpha_2) \lambda + c(\alpha_2 - 1)], \end{aligned} \quad (8)$$

where the positive constant  $c$  is

$$c = \frac{(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2)(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3)}{\alpha_1 \alpha_2 \alpha_3 (\alpha_2 - 1)}.$$

In verifying (8) it is helpful to note the relations

$$\begin{aligned} s/(\alpha_1 + s) &= \alpha_2 / [\alpha_3 (\alpha_2 - 1)], \\ \alpha_1 x / (\alpha_1 + s)^2 &= c \end{aligned}$$

at the interior critical point. In any event, the eigenvalues of  $A$  are now

$\lambda = -1$  and

$$\begin{aligned}\lambda &= \frac{-(c + \alpha_2) \pm \left[ (c + \alpha_2)^2 - 4c(\alpha_2 - 1) \right]^{1/2}}{2} \\ &= \frac{-(c + \alpha_2) \pm \left[ (c - \alpha_2)^2 + 4c \right]^{1/2}}{2}.\end{aligned}$$

These are all real and negative.

A simple bifurcation diagram can clarify the local solution structure. Suppose, for instance, that all parameters except the concentration  $S_0 = K_p / \alpha_1$  of the influent nutrient are fixed. The equilibrium biomass in nondimensional units is either  $x(\infty) = 0$  or

$$\begin{aligned}x(\infty) &= \frac{(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2) \alpha_3 (\alpha_2 - 1)}{\alpha_2 (\alpha_2 \alpha_3 - \alpha_2 - \alpha_3)} \\ &= c_1 - c_2 / S_0,\end{aligned}$$

with the constants  $c_1$  and  $c_2$  defined as

$$c_1 = \alpha_3 (\alpha_2 - 1) / \alpha_2$$

and

$$c_2 = K_p \alpha_3 (\alpha_2 - 1) / (\alpha_2 \alpha_3 - \alpha_2 - \alpha_3).$$

A transcritical bifurcation with an exchange of stability occurs when

$$\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 = 0.$$

At this point the trivial and nontrivial branches of solutions cross. Figure 1 illustrates the situation.

## GLOBAL STABILITY ANALYSIS

Our proof of global stability depends strongly on the following well-known comparison result [3]:

### LEMMA 1

*Suppose  $y(t)$  and  $z(t)$  satisfy the ordinary differential equations*

$$\begin{aligned}y'(t) &= F(t, y(t)), \\ z'(t) &= G(t, z(t))\end{aligned}$$



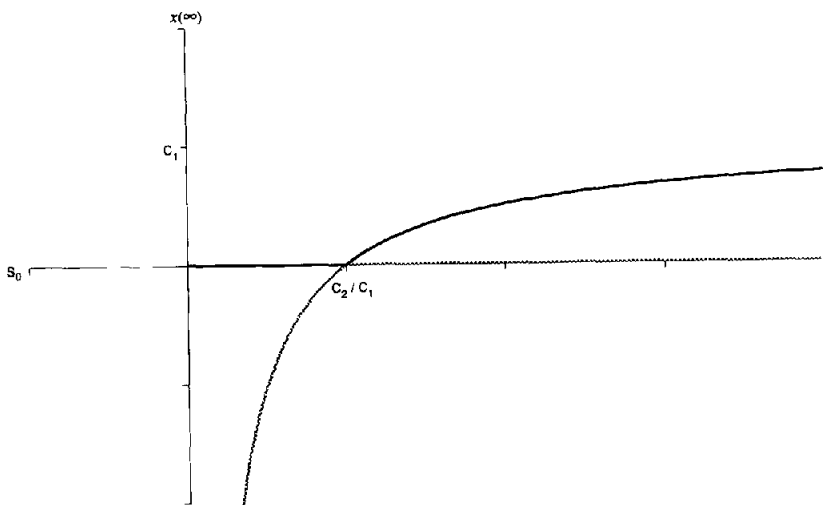


FIG. 1. Bifurcation diagram for the equilibrium concentration  $x(\infty)$  of the microorganism versus the influent concentration  $S_0$  of the limiting nutrient.

on  $t \in [a, b]$ . If, in addition,  $y(a) = z(a)$  and  $F(t, u) \leq G(t, u)$  for all relevant  $t$  and  $u$ , then  $y(t) \leq z(t)$  for all  $t \in [a, b]$ .

The most important special case of the lemma, and the only one we use, is the linear case  $G(t, u) = -cu + d$ . If  $c \neq 0$ , then

$$y(t) \leq y(a)e^{-c(t-a)} + \frac{d}{c}[1 - e^{-c(t-a)}],$$

and if  $c = 0$ , then

$$y(t) \leq y(a) + d(t - a).$$

Obviously, the reverse inequality holds when  $F(t, u) \geq G(t, u)$ .

Our first global results follow directly from Lemma 1.

#### LEMMA 2

Suppose  $s(t)$ ,  $x(t)$  and  $q(t)$  are all positive for  $t \in [0, a]$ . Then on  $[0, a]$ ,

(a)

$$q(t) \geq q(0)e^{-\alpha_2 t} + 1 - e^{-\alpha_2 t} \geq \min[q(0), 1],$$

(b)

$$q(t) \leq q(0)e^{-\alpha_2 t} + \left(1 - \frac{\alpha_3}{\alpha_2}\right)(1 - e^{-\alpha_2 t}) \leq \max\left[q(0), 1 + \frac{\alpha_3}{\alpha_2}\right],$$

(c)

$$s(t) \leq s(0)e^{-t} + 1 - e^{-t} \leq \max[s(0), 1],$$

(d)

$$x(t) \geq x(0) \exp\left[\left(\alpha_2 - 1 - \frac{\alpha_2}{\min[q(0), 1]}\right)t\right],$$

(e)

$$\begin{aligned} x(t) &\leq \frac{(\alpha_3^{-1}q(0)x(0) + s(0) - 1)e^{-t} + 1}{\alpha_3^{-1}\min[q(0), 1]} \\ &\leq \frac{\max[\alpha_3^{-1}q(0)x(0) + s(0), 1]}{\alpha_3^{-1}\min[q(0), 1]} = x_{\max}, \end{aligned}$$

(f) With  $x_{\max}$  as defined in (e) and  $\gamma = 1 + x_{\max}/\alpha_1$ ,

$$\begin{aligned} s(t) &\geq s(0)e^{-\gamma t} + \frac{1}{\gamma}(1 - e^{-\gamma t}) \\ &\geq \min\left[s(0), \frac{1}{\gamma}\right]. \end{aligned}$$

*Proof.* For parts (a) and (b), use

$$-\alpha_2[q(t) - 1] \leq q'(t) \leq \alpha_3 - \alpha_2[q(t) - 1]$$

and the linear version of Lemma 1. Similarly, (c) follows from  $s'(t) \leq 1 - s(t)$  and Lemma 1. For (d), insert the bound (a) in the representation (4). Part (e) follows from the first integral (3) and (a). Finally, (f) is a consequence of

$$\begin{aligned} s'(t) &\geq 1 - s(t) - \frac{x_{\max}s(t)}{\alpha_1 + s(t)} \\ &\geq 1 - s(t) - \frac{x_{\max}s(t)}{\alpha_1} \end{aligned}$$

and Lemma 1. ■

Lemma 2 shows the solution of the Droop system (2a)–(2c) exists for all  $t \geq 0$  and is confined to the positive first orthant  $\{(s, x, q): s > 0, x > 0, q > 0\}$ . In addition, all solutions are bounded. It is possible for  $\lim_{t \rightarrow \infty} x(t) = 0$ , but  $s(t)$  and  $q(t)$  are both bounded below by positive constants. More precise asymptotic estimates for  $q(t)$  can be found by pushing the comparison method farther.

LEMMA 3

(a) The nondimensional cell quota  $q(t)$  of the Droop system satisfies

$$\limsup_{t \rightarrow \infty} q(t) \leq 1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)}.$$

(b) If  $\lim_{t \rightarrow \infty} s(t) = 1$ , then

$$\liminf_{t \rightarrow \infty} q(t) \geq 1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)},$$

and hence

$$\lim_{t \rightarrow \infty} q(t) = 1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)}.$$

*Proof.* By (c) of Lemma 2,  $\limsup_{t \rightarrow \infty} s(t) \leq 1$ . Also  $s/(\alpha_1 + s)$  is an increasing function of  $s$ . Thus given  $\epsilon > 0$ , there exists a  $t_0$  such that

$$\frac{\alpha_3 s(t)}{\alpha_1 + s(t)} \leq \frac{\alpha_3}{\alpha_1 + 1} + \epsilon$$

for  $t \geq t_0$ . Applying Lemma 1 to

$$q'(t) \leq \frac{\alpha_3}{\alpha_1 + 1} + \epsilon - \alpha_2[q(t) - 1]$$

shows that

$$\begin{aligned} q(t) &\leq q(t_0) e^{-\alpha_2(t-t_0)} \\ &\quad + \left(1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)} + \frac{\epsilon}{\alpha_2}\right) (1 - e^{-\alpha_2(t-t_0)}) \end{aligned}$$

for  $t \geq t_0$ . Consequently,

$$\limsup_{t \rightarrow \infty} q(t) \leq 1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)} + \frac{\epsilon}{\alpha_2},$$

and since  $\epsilon$  is arbitrary, this proves (a).

If  $\lim_{t \rightarrow \infty} s(t) = 1$ , then given  $\epsilon > 0$  there exists some  $t_0$  such that

$$\frac{\alpha_3 s(t)}{\alpha_1 + s(t)} \geq \frac{\alpha_3}{\alpha_1 + 1} - \epsilon$$

for  $t \geq t_0$ . To prove (b), now argue as in the proof of part (a), reversing inequality signs and substituting  $\liminf$  for  $\limsup$  throughout. ■

#### LEMMA 4

When  $t$  is large,  $|1 - s(t)|$  is small if and only if  $x(t)$  is small. Thus, the two conditions  $\lim_{t \rightarrow \infty} s(t) = 1$  and  $\lim_{t \rightarrow \infty} x(t) = 0$  are equivalent.

*Proof.* This is evident from the first integral (3) because  $q(t)$  is bounded above and below by positive constants. ■

These preliminaries provide a basis for our first global stability result.

#### THEOREM 1

Suppose  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 \leq 0$ . Then

$$\begin{aligned} \lim_{t \rightarrow \infty} s(t) &= 1, & \lim_{t \rightarrow \infty} x(t) &= 0, \\ \lim_{t \rightarrow \infty} q(t) &= 1 + \frac{\alpha_3}{\alpha_2(\alpha_1 + 1)}. \end{aligned}$$

In other words, the boundary critical point is approached, and the chemostat organism goes extinct as  $t \rightarrow \infty$ .

*Proof.* Suppose  $s(t) \geq 1$  for all  $t$ . Since  $\limsup_{t \rightarrow \infty} s(t) \leq 1$ , it follows that  $\lim_{t \rightarrow \infty} s(t) = 1$ . In this situation, application of Lemma 4 and (b) of Lemma 3 completes the proof of the theorem. Hence, we can assume that  $s(t) < 1$  for at least one  $t$ . Once  $s(t) < 1$  is achieved, it continues to hold for all subsequent  $t$  by (c) of Lemma 2. Without loss of generality, we now take  $s(0) < 1$ .

Our strategy will be to show that the function

$$L(t) = (\alpha_2 - 1) \ln[1 - s(t)] + \ln[x(t)]$$

satisfies  $\lim_{t \rightarrow \infty} L(t) = -\infty$ . Lemma 4 then implies that  $\lim_{t \rightarrow \infty} s(t) = 1$  and  $\lim_{t \rightarrow \infty} x(t) = 0$ , which in view of (b) of Lemma 3 finishes the proof.

The function  $L(t)$  is designed to have Lyapunov-like behavior. Consider the derivative

$$\begin{aligned} L'(t) &= -\frac{\alpha_2 - 1}{1 - s(t)} \left( 1 - s(t) - \frac{s(t)x(t)}{\alpha_1 + s(t)} \right) + \alpha_2 - 1 - \frac{\alpha_2}{q(t)} \\ &= \left( \frac{\alpha_2 x(t)}{\alpha_3 [1 - s(t)]} - \frac{\alpha_2}{q(t)} \right) \\ &\quad + \left( \frac{(\alpha_2 - 1)s(t)x(t)}{[1 - s(t)][\alpha_1 + s(t)]} - \frac{\alpha_2 x(t)}{\alpha_3 [1 - s(t)]} \right) \\ &= \frac{\alpha_2}{[1 - s(t)]q(t)} \left( \frac{q(t)x(t)}{\alpha_3} + s(t) - 1 \right) \\ &\quad + \frac{x(t)}{1 - s(t)} \left( \frac{(\alpha_2 - 1)s(t)}{\alpha_1 + s(t)} - \frac{\alpha_2}{\alpha_3} \right). \end{aligned}$$

Since

$$\begin{aligned} \frac{(\alpha_2 - 1)s(t)}{\alpha_1 + s(t)} - \frac{\alpha_2}{\alpha_3} &= \frac{(\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2)s(t) - \alpha_1 \alpha_2 [1 - s(t)]}{\alpha_3 [\alpha_1 + s(t)]} \\ &\leq -\frac{\alpha_1 \alpha_2 [1 - s(t)]}{\alpha_3 [\alpha_1 + s(t)]}, \end{aligned}$$

it follows that

$$L'(t) \leq \frac{\alpha_2}{[1 - s(t)]q(t)} \left( \frac{q(t)x(t)}{\alpha_3} + s(t) - 1 \right) - \frac{\alpha_1 \alpha_2 x(t)}{\alpha_3 [\alpha_1 + s(t)]}. \quad (9)$$

Given  $\delta > 0$  we need to prove that  $L(t) \leq -\delta$  for all  $t$  large enough. To achieve this, consider the auxiliary function  $H(t)$  defined by

$$\begin{aligned} H(t) &= L(t) + \gamma G(t), \\ G(t) &= \frac{q(t)x(t)}{\alpha_3} + s(t) - 1, \end{aligned}$$

where  $\gamma$  is some nonnegative contrast to be specified. If we can demonstrate that  $H(t) \leq -2\delta$  for all  $t$  large enough, then because  $\lim_{t \rightarrow \infty} G(t) = 0$ , it follows that  $L(t) \leq -\delta$  for all  $t$  large enough.

Suppose that we choose  $\gamma \geq 0$ ,  $\epsilon > 0$ , and  $t_0$  such that  $H'(t) \leq -\epsilon$  whenever  $H(t) \geq -2\delta$  and  $t \geq t_0$ . These choices, if possible, imply the existence of a point  $t_1 \geq t_0$  such that  $H(t_1) \leq -2\delta$ . Indeed, the contrary assumption cannot be true because Lemma 1 then provides the contradictory inequality

$$H(t) \leq H(t_0) - \epsilon(t - t_0)$$

for all  $t \geq t_0$ . Given the existence of such a point  $t_1$ , it is then a simple consequence of the mean value theorem that  $H(t) \leq -2\delta$  for all  $t \geq t_1$ . This result is also graphically obvious.

Now we choose  $\gamma$ ,  $\epsilon$ , and  $t_0$ . If  $G(0) < 0$ , take  $\gamma = 0$ ; otherwise, take

$$\gamma = \frac{\alpha_2}{d \min[q(0), 1]},$$

where the constant  $d$  is defined by

$$d = \exp\left[-\frac{3\delta + \ln(x_{\max})}{\alpha_2 - 1}\right],$$

with  $x_{\max}$  coming from (e) of Lemma 2. (The reason for this odd choice of the constant  $d$  will be evident in a moment.) Because  $G(t)$  has constant sign, and  $q(t)$  has lower bound  $\min[q(0), 1]$ ,  $\gamma$  is chosen to satisfy

$$\left(\frac{\alpha_2}{[1 - s(t)]q(t)} - \gamma\right)G(t) \leq 0$$

whenever  $1 - s(t) \geq d$ .

Given this  $\gamma$ , take  $t_0$  so that  $\gamma|G(t_0)| \leq \delta$  for all  $t \geq t_0$ . If  $H(t) \geq -2\delta$  for some  $t \geq t_0$ , it then follows that  $L(t) \geq -3\delta$ . This in turn implies

$$\begin{aligned} (\alpha_2 - 1)\ln[1 - s(t)] &\geq -3\delta - \ln[x(t)] \\ &\geq -3\delta - \ln(x_{\max}), \end{aligned}$$

and so  $1 - s(t) \geq d$ . In similar fashion,  $x(t) \geq e^{-3\delta}$  whenever  $H(t) \geq -2\delta$  and  $t \geq t_0$ .

These considerations determine the choice of  $\epsilon$ . For  $t \geq t_0$  and  $H(t) \geq -2\delta$ , we deduce from (9) that

$$\begin{aligned}
H'(t) &= L'(t) + \gamma G'(t) \\
&\leq \left( \frac{\alpha_2}{[1-s(t)]q(t)} - \gamma \right) G(t) - \frac{\alpha_1 \alpha_2 x(t)}{\alpha_3 [\alpha_1 + s(t)]} \\
&\leq -\alpha_1 \alpha_2 x(t) / (\alpha_3 [\alpha_1 + s(t)]) \\
&\leq -\alpha_1 \alpha_2 e^{-3\delta} / (\alpha_3 [\alpha_1 + 1]).
\end{aligned}$$

Thus, when we choose

$$\epsilon = \alpha_1 \alpha_2 e^{-3\delta} / \alpha_3 [\alpha_1 + 1],$$

the required inequality  $H'(t) \leq -\epsilon$  holds. ■

The converse of Theorem 1 is also true.

#### LEMMA 5

Suppose  $\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2 > 0$ . Under our assumption that  $s(0)$ ,  $x(0)$ , and  $q(0)$  are all positive, the boundary critical point cannot be the limit of the Droop system as  $t \rightarrow \infty$ . Both of the weaker assertions  $\lim_{t \rightarrow \infty} s(t) = 1$  and  $\lim_{t \rightarrow \infty} x(t) = 0$  are also false.

*Proof.* Assume that  $\lim_{t \rightarrow \infty} s(t) = 1$ . Then by (b) of Lemma 3,

$$\lim_{t \rightarrow \infty} \frac{1}{q(t)} = \frac{1}{1 + \alpha_3 / [\alpha_2 (\alpha_1 + 1)]}.$$

Thus, given  $\epsilon > 0$ , there is a  $t_0$  such that

$$\frac{1}{q(t)} \leq \frac{1}{1 + \alpha_3 / [\alpha_2 (\alpha_1 + 1)]} + \epsilon$$

for  $t \geq t_0$ . But then the representation (4) with initial time  $t_0$  gives

$$\begin{aligned}
x(t) &\geq x(t_0) \exp \left[ \alpha_2 \left( 1 - \frac{1}{\alpha_2} - \frac{1}{1 + \alpha_3 / [\alpha_2 (\alpha_1 + 1)]} - \epsilon \right) (t - t_0) \right] \\
&= x(t_0) \exp \left[ \left( \frac{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2}{\alpha_2 (\alpha_1 + 1) + \alpha_3} - \alpha_2 \epsilon \right) (t - t_0) \right]
\end{aligned}$$

for  $t \geq t_0$ . If  $\epsilon > 0$  is sufficiently small, then

$$\frac{\alpha_2 \alpha_3 - \alpha_2 - \alpha_3 - \alpha_1 \alpha_2}{\alpha_2 (\alpha_1 + 1) + \alpha_3} - \alpha_2 \epsilon > 0,$$

and  $x(t)$  grows exponentially fast. This contradicts the boundedness of  $x(t)$  stated in (e) of Lemma 2. ■

Since the boundary critical point cannot be the limit of the Droop system when  $\alpha_2\alpha_3 - \alpha_2 - \alpha_3 - \alpha_1\alpha_2 > 0$ , it is reasonable to conjecture that the interior critical point plays this role. Our strategy for proving this conjecture depends on using the first integral (3) to reduce the three-dimensional Droop system to a limiting two-dimensional system. On the set  $\alpha_3^{-1}xq + s - 1 = 0$ , substitution of  $\alpha_3[1 - s(t)]/x(t)$  for  $q(t)$  in the Droop Equation (2b) yields

$$x'(t) = \alpha_2 \left( 1 - \frac{x(t)}{\alpha_3[1 - s(t)]} \right) x(t) - x(t). \quad (10)$$

The Droop Equation (2a) and Equation (10) constitute what we will term the two-dimensional Droop system. Note that  $1 - s(t) > 0$  must hold for (10) to make sense. As in the proof of Theorem 1, either  $\lim_{t \rightarrow \infty} s(t) = 1$  or  $s(t) < 1$  for some  $t$  and hence for all subsequent  $t$ . Lemma 5 excludes the first possibility, and again without loss of generality we assume that  $s(0) < 1$ .

These considerations enable us to prove global stability with the concentration of the chemostat organism tending to a positive constant.

#### THEOREM 2

*Suppose  $\alpha_2\alpha_3 - \alpha_2 - \alpha_3 - \alpha_1\alpha_2 > 0$ . Then*

$$\lim_{t \rightarrow \infty} s(t) = \frac{\alpha_1\alpha_2}{\alpha_2\alpha_3 - \alpha_2 - \alpha_3},$$

$$\lim_{t \rightarrow \infty} x(t) = \frac{(\alpha_2\alpha_3 - \alpha_2 - \alpha_3 - \alpha_1\alpha_2)\alpha_3(\alpha_2 - 1)}{\alpha_2(\alpha_2\alpha_3 - \alpha_2 - \alpha_3)},$$

and

$$\lim_{t \rightarrow \infty} q(t) = \frac{\alpha_2}{\alpha_2 - 1}.$$

*In other words, the interior critical point is approached at  $t \rightarrow \infty$ .*

*Proof.* The omega limit set of any initial point  $(s(0), x(0), q(0))$  for the three-dimensional Droop system is nonempty, invariant, compact, and contained in the locus of the equation  $\alpha_3^{-1}xq + s - 1 = 0$  [11]. By Lemma 5 this omega limit set must contain some point  $(s^\infty, x^\infty, q^\infty)$  other than the boundary critical point. If the omega limit set of  $(s^\infty, x^\infty, q^\infty)$  contains the interior critical point, then the omega limit set of  $(s(0), x(0), q(0))$  does also. Because the interior critical point is locally



attractive, it then necessarily is the only limit point of the original point  $(s(0), x(0), q(0))$ .

It now clearly suffices to prove that the two-dimensional omega limit set of  $(s^\infty, x^\infty)$  contains the two-dimensional interior critical point. Outside  $U = \{(s, x): 1 > s > 0, x > 0\}$ , the only possible limit point of the two-dimensional Droop system is the boundary critical point  $s = 1, x = 0$ . As noted above, this cannot be the only limit point. Hence, the intersection of  $U$  and the omega limit set of  $(s^\infty, x^\infty)$  is nonempty. According to the Poincaré-Bendixson theorem [10, 11], the omega limit set of  $(s^\infty, x^\infty)$  either contains the interior critical point or reduces to a closed orbit. We can rule out the second contingency by Dulac's criterion [12]. This entails showing that

$$\frac{\partial}{\partial s} [\beta(s, x)f(s, x)] + \frac{\partial}{\partial x} [\beta(s, x)g(s, x)]$$

never changes sign on  $U$ , where  $\beta(s, x)$  is some arbitrary smooth function and

$$\begin{aligned} s'(t) &= f(s(t), x(t)), \\ x'(t) &= g(s(t), x(t)) \end{aligned}$$

is shorthand for the two-dimensional Droop system. If we take  $\beta(s, x) = 1/x$ , then an easy computation demonstrates that

$$\begin{aligned} &\frac{\partial}{\partial s} [\beta(s, x)f(s, x)] + \frac{\partial}{\partial x} [\beta(s, x)g(s, x)] \\ &= -\frac{1}{x} - \frac{\alpha_1}{(\alpha_1 + s)^2} - \frac{\alpha_2}{\alpha_3(1-s)} \\ &< 0. \end{aligned}$$

This completes the proof except for a subtle detail. The Poincaré-Bendixson theorem is usually proved under the hypothesis that the domain  $U$  is compact. This condition obviously fails in the current problem, but it can be checked that the Poincaré-Bendixson theorem [10, 11] continues to hold provided every omega limit set for an orbit in  $U$  contains a point in  $U$ . Lemma 5 fortunately gives us this assurance. ■

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