Math 3820: Introduction to Mathematical Modelling
Prof. **J. Arino**

Assignment No.2:

Explanation of the model

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Position of the problem

In the paper "Nonlinear standing waves in an acoustical resonator," by Ilinski et al, we find an interesting problem being tackled. Can we, mathematically, correctly describe standing sound waves (waves that are in a constant position) that develop in a prescribed geometry? Moreover, can we correctly characterize the non-linearity of the sound waves, given that we are using high amplitude (and therefore the non-linearity is accentuated)?

The model that is presented is a one-dimensional model, based on the restriction of using axially symmetric resonators. The benefits of having a model like this is that, once derived, it would be able to predict waveforms in geometrically different resonators. Then, resonators could be designed in such a way to optimally produce resonators suitable for specific applications. The use of dimensionless parameters allows us to generalize the results and makes this a useful tool for analysis and for industrial research applications.

Description of the Physical System

The simplest way to describe the system is that it is a resonator (solid cavity of a prescribed geometry, in this case axisymmetric) in which there is external harmonic force driving the oscillations in the ideal gas within the resonator. Once the waves have comingled and formed standing waves (by the combination of the travelling waves), or reached a steady state, we are interested to know what shape the resulting waveform has taken. There are many possible ways in which the resulting waveform can result.

The authors have developed the following terminology to describe whether a resonator has higher resonances within its upper natural harmonics. If a resonator has higher resonances at frequencies which are integer multiples (these are called natural harmonics) of the fundamental sound/vibration, they are called consonant resonators. It they instead have a non-equidistant spectrum, that is, high-resonance frequencies that fall outside of the natural harmonics, they are called dissonant resonators. The different resonator geometries, given the same driving force, give rise to various different standing waveforms. It is to be able to predict these forms, as well as to shape these forms that the analyses in this paper are useful for.

Model Hypotheses

There are a few governing hypotheses that are intrinsic to this analysis. Some of the assumptions that are made have to do with the behaviour of the gas inside the chamber (modelled as an ideal gas….that is, that gas particles act in non-interactive ways that obey the laws of conservation of energy), as well as a restriction on the geometry of the resonator. These assumptions are necessary, because otherwise the model would become exponentially complicated (and it is already a very complicated model). If we could not assume axial symmetry, especially, it would be very hard indeed to analytically account for the various spatial relationships of the waveform. This 1-D standing wave model, therefore, saves a lot of headache. We will now expand on the assumptions a little more:

1) Ideal Gas. This assumption is not a bad approximation of gas behaviour at standard pressure and temperature, but fails to account for physical phenomena at low temperatures or high pressures (when individual atoms are closer to each and can interact more directly). Towards the end of the analytical derivation, we see the authors making the assumption of no energy loss (no energy dissipation in the waves).

2) Axial Symmetry. This is more of a restriction on the scope of the analysis than an assumption. As it is, many fabricated resonators are of this unique geometry, since it allows for more control over the shape of the produced waveform.

3) No acceleration. Also towards the end of the derivation, the authors make an assumption of $a(t) = 0$. That is, the resonator body is not accelerating with respect to its frame of reference. This serves to further reduce the dynamic equations and therefore the scope of the problem.

Model Derivation

Assuming that we have an ideal gas, and that we have an axially symmetric geometry, we can start deriving the equations that will allow us to model the waveforms. For sake of simplicity, I will be summarizing the derivations done in Ilinsky et al, since it is a very long derivation (and they have put it quite succinctly themselves.

There are three main equations that need to be established to describe the motion of the viscous gas: mass conservation, momentum equation, and the state equation. We will treat each one individually at first.

1) <u>Conservation of mass</u>. The general form of this equation is: $\frac{\partial M}{\partial t}+\frac{\partial F}{\partial x}=0.$

Defining M as mass per unit length, the radius r as a function of x (useful for describing axisymmetric geometry), and the gas flow F through the cross sectional area of radius r, we get: $M=\rho\pi r^2$ and $F=\rho u\pi r^2$ .Combining these equations, then, we obtain as our mass conservation equation:

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial x}(r^2\rho u)=0$$

2) <u>Momentum equation</u>. The 1-D momentum equation, with an in-built dissipative term is that shown pictured on the right. Here 'p' means pressure, a(t) is the acceleration of the resonator, and the dissipative term includes coefficients of the gas viscosity. This dissipative term is derived from the general three-dimensional dissipative term for the *i* velocity component, in its potential form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-a(t)+\frac{(\zeta+4\eta/3)}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{r^2}\frac{\partial}{\partial x}(r^2u)\right)$$

$$u_i=\frac{\partial\varphi}{\partial x_i}$$

3) <u>State equation</u>. The form of the state equation for an ideal gas, is: $p=p_0\left(\frac{\rho}{\rho_0}\right)^\gamma$

Substituting this expression, along with our definition of the velocity component potential into our main expression derived from the momentum equation, and integrating the equation over x, we obtain the following:

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2=-\frac{\gamma p_0}{(\gamma-1)\rho_0^\gamma}\rho^{(\gamma-1)}-a(t)x+\frac{\delta}{r^2}\frac{\partial}{\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right)+\phi(t)$$

Here, $\phi(t)$ refers to some function of t, which is the integration constant resulting from integrating the momentum equation, and delta is used to abbreviate the equation: $\delta=\frac{(\zeta+4\eta/3)}{\rho_0}$

The replacement of ρ by ρ_o in the equation means that we are ignoring the nonlinear terms, including absorption, in the gas dynamics. This therefore means that we assume that energy dissipation is quite small. We use the following expression for the wave propagation speed (c). $c^2 = \frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0^\gamma} \rho^{(\gamma-1)}$ Also, choosing the time function of integration to be constant, for the linear case of the limit as ρ approaches ρ_o, we have that: $\phi(t) = \text{const} = \frac{c_0^2}{\gamma - 1}$

Then, substituting this back into the main equation, we have the following equation:

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 = -\frac{c^2 - c_0^2}{\gamma - 1} - a(t)x + \frac{\delta}{r^2}\frac{\partial}{\partial x}\left(r^2 \frac{\partial\varphi}{\partial x}\right)$$

Finally, after differentiating with respect to *t*, and replacing $\partial\rho/\partial t$, we get:

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial x}\right)^2 = -\left[\frac{\partial^2\varphi}{\partial x\,\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)^2 + a(t)\right]\frac{\partial\varphi}{\partial x} + \frac{c^2}{r^2}\frac{\partial}{\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right) - \frac{da}{dt}x + \frac{\delta}{r^2}\frac{\partial^2}{\partial t\,\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right)$$

Then, we can also substitute the c^2 with an expression using c_o^2. Then, for the case that a(t) = 0 and δ=0 (lossless ideal gas) we will obtain:

$$\frac{c_0^2}{r^2}\frac{\partial}{\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right) - \frac{\partial^2\varphi}{\partial t^2} = 2\frac{\partial^2\varphi}{\partial x\,\partial t}\frac{\partial\varphi}{\partial x} + \frac{\gamma-1}{r^2}\frac{\partial\varphi}{\partial t}\frac{\partial}{\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)^3 + \frac{\gamma-1}{2r^2}\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial}{\partial x}\left(r^2\frac{\partial\varphi}{\partial x}\right)$$

We can then apply any arbitrary geometry for a resonator that is symmetric around its axis and derive the final specific equations.

Numerical Analysis

Integral to the numerical analytical approach is its preparation by the transformation of variables into dimensionless parameters. The authors also try to derive a final equation in which only first derivatives with respect to x are involved. For the ensuing dimensionless transformations, please refer to Ilinsky et al, as it would be hard to summarize it further. However, in brief, after deriving the dimensionless form of the equation, the authors transform it into a equation in the frequency domain (to better characterize the sound waves, and predict the higher resonance frequencies). Finally, they develop a numerical algorithm for solving the final (4N+1) first order ODE’s equation using a Runge-Kutta method, subject to the boundary conditions at either end of the resonator. Here N refers to the number of complex amplitudes of the harmonic components.

Synthesis

Our model consists of (4N +1) coupled first order ordinary differential equations:

$$\frac{dV_l}{dX} = S_l(V_k, \Phi_k, X) \qquad \frac{d\Phi_k}{dX} = \frac{V_k}{R^2}$$

Here, S_l refers to functions from the solutions of:

$$D_{kl}\frac{dV_l}{dX} = F_k$$

where: $$D_{kl} = \left(\frac{1}{\pi^2} + \frac{ikG\Omega}{\pi^3}\right)\delta_{kl} + D'_{k-l}$$ and

$$F_k = -k^2\Omega^2R^2\Phi_k + ik\Omega R^2XA_k + \frac{ik\Omega}{R^2}[V^2]_k + \sum_{l=-N+k}^{N}\{A_{k-l}V_l\} - \frac{2}{R^5}\frac{dR}{dX}\sum_{l=-N}^{N}\{[V^2]_{k-l}V_l\}$$

For further explanation of all the variables involved, see Ilinsky et al. Our boundary conditions, at the resonator axial boundaries are:

$$V_k = 0, \quad V_{dc} = 0, \quad \text{at} \quad X = 0,$$

$$V_k = 0, \quad V_{dc} = 0, \quad \text{at} \quad X = 1.$$

We have seen the derivation and assumptions that are the basis for the paper “Nonlinear standing waves in an acoustical resonator,” by Ilinski et al. We have briefly discussed the reason behind some of the transformations and the final form of the dynamic equations for a lossless perfect gas in a non-accelerating framework in a axisymmetric resonator. The authors show their method functioning for three specific geometries (cylinder, conical, and bulbous) and found that their model had good agreement with experimental values.

References

Yurii A. Ilinskii, Bart Lipkens, Timothy S. Lucas, Thomas W. Van Doren, and Evgenia A. Zabolotskaya, “Nonlinear standing waves in an acoustical resonator,” J. Acoust. Soc. Am. **104**, 2664–2674, 1998.