Newton's Law of Cooling: A Mathematical Model of Heat Exchange in a Building.

Isaac Newton 1643 - 1727

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Problem:

To formulate a mathematical model that describes: *i.)* The temperature inside a building as a function of the outside temperature, the heat generated inside the building, and the furnace heating or air conditioner cooling. *ii.)* The temperature of a house which consists of an attic and a living area as a function of time.

Since there are changing temperatures all around us, we would like to determine how to mathematically describe these changes in temperature. For instance, a cold glass of water put on the table in a warm room will eventually get warm. A furnace will warm a house while an open window during the winter will cool a house. What is the law that governs the rate of temperature change? This fundamental law of physics is known as **Newton's Law of Cooling**.

Newton's law of cooling states that the rate of change dT/dt, in the temperature T of a body placed in a medium of temperature M is proportional to the difference between the temperature of the body and the temperature of the medium.[1] Intuitively, we can see that the temperature of the object will *fall* if its' temperature is *greater* than that of the surrounding medium, and similarly, the temperature of an object will *rise* if its' temperature is *less* than that of the surrounding medium.

Newton's law of cooling can be written mathematically as:

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[1] Farlow, S., *An Introduction to Differential Equations and Their Applications*, McGraw-Hill Inc., 1994.

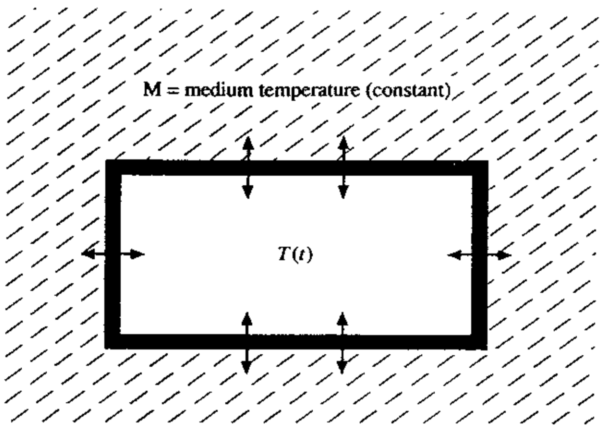
$$dT/dt = -k(T - M)$$

where:

M = temperature of the medium (constant value)

T = temperature of an object (body)

k > 0 is a constant of proportionality.



Assume that the temperature in each of the two bodies (i.e. surrounding, and object) are spatially uniform, which implies that the coefficient of thermal conductivity of each body is large enough that we do not have to worry about variations in temperature throughout these bodies.[2] Assume also that the temperature of a surrounding/ medium is a constant value M. If an object is placed in a surrounding at time t=0, where the object has an initial temperature $T(0) = T_o$, then the temperature of the object can be determined as a function of time, T(t), by solving:

$$dT/dt = -k(T - M), \quad T(0) = T_o.$$

This equation is separable when M is constant, so we separate variables and integrate:

[2] Berry, T.G., *Mathematical Modeling Lecture Notes*, February 13, 1997.

$$dT/(T - M) = -kdt$$

$$\int dT/(T - M) = -k \int dt$$

$$\ln | T - M | = -kt + c, \quad \text{where c is a constant of integration.}$$

Taking the exponential of both sides yields:

$$| T - M | = e^{-kt + c}$$

$$| T - M | = De^{-kt}, \quad \text{where } D = e^{c}$$

$$T - M = Fe^{-kt}, \quad \text{where } F = \pm D$$

$$T(t) = M + Fe^{-kt}. \qquad \text{(equation 1)}$$

Using the initial condition $T(0) = T_0$, we find that

$$T(0) = T_0 = M + F.$$

Solving for F yields

$$F = -M + T_0,$$

which we can substitute into (equation 1):

$$T(t) = M + (-M + T_0)e^{-kt}$$

$$T(t) = T_0 \, e^{-kt} + M(1 - e^{-kt}). \qquad \text{(equation 2)}$$

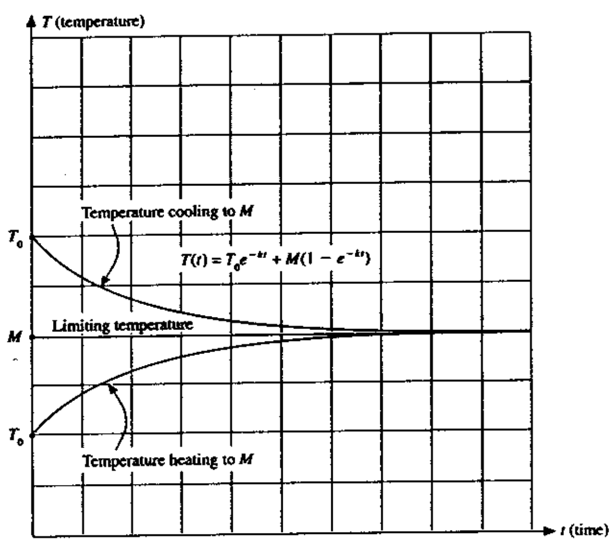


Figure 1

We can sketch the solutions given in (equation 2) where M, which is assumed to be constant, is the temperature the object will approach. The temperature T_0 either increases or decreases exponentially towards M depending on whether T_0 was greater than or less than M initially (see Figure 1).

Here is an example of Newton's law of cooling:

Example 1——————————

The temperature in Machray Hall was 21°C at 8:00 p.m. when the custodian turned the heat off. At 10:00 p.m. the temperature inside had fallen to 9°C. If the temperature outside was constant at -15°C, what will the temperature be at 12:00 midnight?

Solution:——————————

Using the equation for Newton's law of cooling and the fact that the temperature outside can be considered the surrounding medium to the object, namely Machray Hall, we can write:

$$dT/dt = -k(T - (-15)), \qquad T(0) = 21^{o}C \text{ (corresponding to 8:00pm)}$$
$$T(2) = 9^{o}C \text{ (corresponding to 10:00pm).}$$

Solving, we have:

$$\int dT/(T + 15) = -k\int dt$$

$$\ln | T + 15 | = -kt + c, \text{ where c is a constant of integration}$$

$$T + 15 = Fe^{-kt}, \qquad F = \pm e^{c}$$

$$T(t) = -15 + Fe^{-kt}.$$

Using the initial condition $T(0) = 21 = -15 + F$

$$F = 36,$$

and $T(2) = 9 = -15 + 36e^{-kt}$ and solving for k yields:

$$24/36 = e^{-2k}$$

$$\ln (24/36) = -2k$$

$$k = 0.2027.$$

Putting this together gives:

$$T(t) = -15 + 36e^{-0.2027t}.$$

Therefore, at midnight (4 hours after initial temperature), the temperature in Machray Hall is:

$$T(4) = -15 + 36e^{(-0.2027*4)}$$

$$T(4) = 1^{o}C.$$

One might ask whether the constant k has a physical interpretation and indeed it does. The reciprocal of k has units of time and is called the "time constant" of the equation.[3] Its value indicates the rate at which heat is transferred between the object and surrounding medium. The higher the value of ~~1/k,~~ k the faster the rate of heat transfer. In buildings, it is obvious that a low value of k is preferable since this suggests that heat is lost more slowly from the building.

Since 1/k has units of time, let's substitute the value t=1/k into (equation 2) to see if we can glean some information about k.

$$T(1/k) = T_o\, e^{-1} + M(1 - e^{-1})$$

$$T(1/k) = M + (T_o - M)/e \qquad \text{(equation 3)}$$

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[3] Farlow, S., *An Introduction to Differential Equations and Their Applications*, McGraw-Hill Inc., 1994.

This shows that 1/k is the time required for the temperature to drop approximately (1-1/e) $\cong$ 0.632 or $\simeq$ 63% of the total difference in temperature (T_O - M). good

Example 1 gave a rather simplistic determination of the temperature change in a building. There are many factors that we have to look at to more accurately predict the temperature at a time t. We also have to take into consideration the constant changes in temperature outside. Some of the factors that we have to look at that affect the temperature in a building are: the heat given off from people and lights, the heat from the furnace and cold from outside in the winter, the cold from an air-conditioner and heat from outside in the summer. As well the effect of sunlight streaming in through the windows, open windows, type of materials used to build the house, cracks in the wall, drafts under a door and still other things contribute to a change in the temperature in a building.

For simplicity, we shall consider these as three main factors; the result is: a model of the temperature in a building described as a function of the outside temperature, the heat generated inside the building (by people, lights etc.), and the heating or cooling given off by a furnace or air-conditioner.

Let T(t) be the temperature inside a building at time t. The rate of change in the temperature is given by the difference between the rate that the temperature is increasing at due to factors inside, heat from outside (in the summer) or the furnace and the rate that the temperature is decreasing at due to cold air, and air conditioners.

NB

H(t) is the rate of increase of temperature caused by factors inside a building, namely people, lights, machinery and appliances. H(t) is a non-negative term.

NB

Let U(t) be the rate of increase in temperature caused by the furnace heating or the rate of decrease in temperature caused by the air conditioner cooling. This term is positive for furnace heating and negative for air-conditioner cooling.

M(t) is the outside temperature which we know from (example 1) can be expressed using Newton's law of cooling. There is a rate of change in temperature T(t) proportional to [M(t) - T(t)] and hence, the rate of change in the temperature in the building due to M(t) is given by:

$$K[\, M(t) - T(t)\,].$$

K is a constant greater than zero that is independent of M, T and t. K depends on the construction of the building, the insulation used and the effects of drafts coming from doors and windows.

Combining the factors affecting the temperature in the building, we have the model:

$$dT/dt = K[\, M(t) - T(t)\,] + H(t) + U(t). \qquad \text{(equation 4)}$$

In order to solve this differential equation, we can substitute

$$Q(t) := KM(t) + H(t) + U(t) \qquad \text{(equation 5)}$$

$$P(t) = K$$

and write the linear equation in standard form:

$$dT(t)/dt + P(t)T(t) = Q(t).$$

An integrating factor is $e^{\int P(t)dt} = e^{\int Kdt} = e^{Kt}$, so we multiply through by e^{Kt} and obtain:

$$e^{Kt}dT(t)/dt + e^{Kt}KT(t) = e^{Kt}Q(t)$$

$$d(Te^{Kt})/dt = e^{Kt}Q(t).$$

We now take integrals of both sides with respect to t:

$$Te^{Kt} = \int e^{Kt}Q(t)dt + c, \text{ where c is a constant of integration.}$$

$$T(t) = e^{-Kt}[\int e^{Kt}Q(t)dt + c], \text{ substitute Q(t) from (equation 5)}$$

$$T(t) = e^{-Kt}\{\int e^{Kt}[\ KM(t) + H(t) + U(t)\ ]dt + c\}. \qquad \text{(equation 6)}$$

This is the general solution to the differential equation given in (equation 4). Note that when H(t)=0, U(t)=0, $T(0)=T_0$ and $M=M_0$ is a constant, (equation 6) simply reduces to the basic Newton's law of cooling:

$$T(t) = e^{-Kt}\{\int e^{Kt}KM_0 dt + c\}$$

$$T(t) = e^{-Kt}[M_0 e^{Kt} + c]$$

$$T(t) = M_0 + ce^{-Kt}.$$

Using the initial condition and simplifying yields:

$$T(t) = T_0 e^{-Kt} + M_0(1 - e^{-Kt})$$

the same as

and this is ~~equal to~~ (equation 2).

There are certain pieces of information that we would like to obtain from our model, namely:

- How long does it take for the temperature in the building to change substantially?

- How does the building temperature vary when the furnace and air conditioner are off (in the spring or fall)?
- How does the building temperature vary in the summer when there is air conditioning, or in the winter when there is furnace heating?[4]

We would like to establish how long it takes for the temperature to change by a considerable amount. This can be found using the time constant 1/k. We have already determined the physical interpretation of k in (equation 3) and it is obviously the same in this case (not taking into account heating or air conditioning). This gives us information as to how long it takes for the temperature in the building to change noticeably: the temperature in the building increases or decreases exponentially with a time constant 1/k. Most buildings have a time constant of 2 to 4 hours.[5] good

Previously, we assumed that U(t)=0 and H(t)=0. The value U(t)=0 is only feasible in the spring and fall when there is no air conditioning or heating. H(t) would equal zero only in a building that is not inhabited by any living beings, and has no lights or appliances. In other words, for the most part, H(t) should be given a value greater than zero. We would also like to improve on assuming that the temperature outside, M, is constant by taking into account the fact that the temperature outside is usually cooler at night and warmer in the daytime. good

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[4] Nagle, R. & Saff, E., *Fundamentals of Differential Equations*, Addison-Wesley, 3rd Edition, 1993.

[5] Nagle, R. & Saff, E., *Fundamentals of Differential Equations*, Addison-Wesley, 3rd Edition, 1993.

$u(t) = 0$

Example 2 ————————————

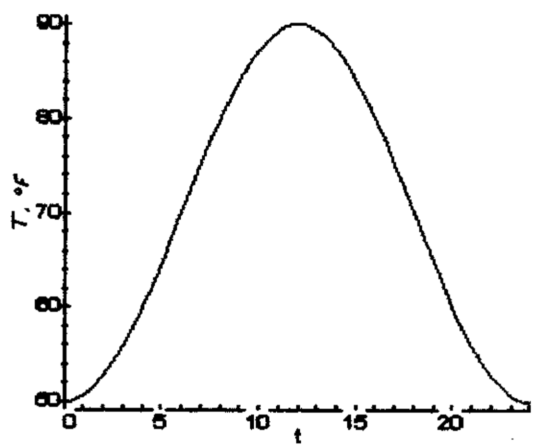
Suppose that, during the spring or fall when there is no heating or air conditioning, the temperature outside varies as a sine wave over a 24-hour period. Let the maximum temperature occur at t=12(noon) and the minimum temperature occur at t=0(midnight). Then the temperature outside is given by:

$$M(t) = M_O - B\cos\omega t$$

where B is a positive constant, M_O is the average outside temperature, and $\omega = 2\pi/24 = \pi/12$.

Solution:————————————

The graph of the temperature outside as a function of t is given by:



Since there is no heating or cooling, U(t) = 0, and we will let $H(0) = H_O$ be a constant. Then from (equation 4) we have:

$$dT/dt = K[\,(M_O - B\cos\omega t) - T(t)\,] + H_O$$

$$dT(t)/dt = (-K)T(t) + [\,KM_O - KB\cos\omega t + H_O\,]$$

$$dT(t)/dt + KT(t) = Q(t) \qquad \text{(equation 7)}$$

where $Q(t) := KM_O - KB\cos\omega t + H_O$

and $Q(0) = KM_o + H_o$.

Equation 7 is a linear equation in standard form so we find an integrating factor, $e^{\int K dt} = e^{Kt}$. Multiplying through by the integrating factor gives:

$$e^{Kt}\, dT/dt + e^{Kt}\, KT(t) = e^{Kt}\, Q(t)$$

$$d(Te^{Kt})/dt = e^{Kt}\, Q(t)$$

Taking the integral of both sides and substituting Q(t) yields:

$$Te^{Kt} = \int e^{Kt}\, [KM_o - KB\cos\omega t + H_o] + c,$$

where c is a constant of integration. Dividing both sides by e^{Kt}:

$$T(t) = e^{-Kt}\{\int e^{Kt}\, [\, KM_o - KB\cos\omega t + H_o\,] + c\}$$

$$T(t) = e^{-Kt}\, e^{Kt}\, (KM_o)/K + e^{-Kt}\, e^{Kt}\, (H_o)/K - KBe^{-Kt} \int e^{Kt}\cos\omega t\; dt.$$

Simplification and integration by parts gives:

$$T(t) = M_o + H_o/K + ce^{-Kt} - KBe^{-Kt}\, [\, \cos\omega t\, (e^{Kt})/K + \omega/K \int \sin \omega t\; e^{Kt}\, dt\,].$$

Using integration by parts a second time,

$$T(t) = M_o + H_o/K + ce^{-Kt} - KBe^{-Kt}\, \{\, \cos\omega t\, (e^{Kt})/K + \omega/K\, [\sin\omega t\, (e^{Kt})/K - (\omega/K)\int \cos\omega t\; e^{Kt}\, dt\,]\, \}.$$

Solving for the integral $\int \cos\omega t\; e^{Kt}\, dt$ and simplifying, we have:

$$T(t) = M_o + H_o/K + ce^{-Kt} - B\, [\, (\cos\omega t + \omega/K\, \sin\omega t\,) \,/\, (1 + \omega^2/K^2\,)\,],$$

(equation 8)

where $M_o + H_o/K$ represents the daily average temperature inside the building. Using the initial condition that at midnight $T(0) = T_o$,

$$T_o = M_o + H_o/K + c - B / (1 + \omega^2/K^2),$$

and solving for c gives:

$$c = B / (1 + \omega^2/K^2) - M_o - H_o/K + T_o.$$

Substituting c into (equation 8) gives us the temperature as a function of time.

Example 3

Using the above information and let $H_o = 0.5$, $K = 0.25$ and $T(0) = 60$, while the temperature outside varies as a sine wave with the minimum temperature, $M(0) = 50$ (corresponding to midnight) and maximum temperature, $M(12) = 90$ (corresponding to noon). Find the temperature as a function of time.

Solution:

Since $M(t) = M_o - B\cos\omega t$ and $\omega = \pi/12$, we have:

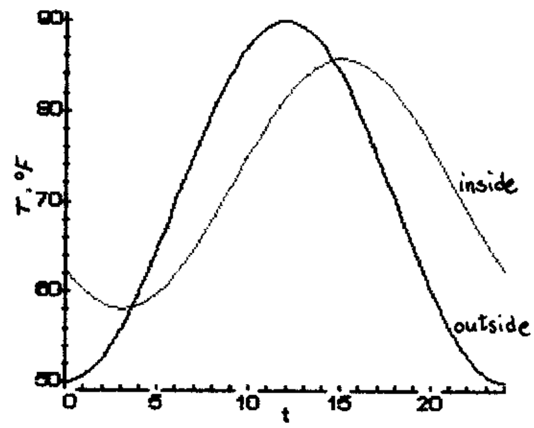
$$M(0) = 50 = M_o + B \text{, and}$$

$$M(12) = 90 = M_o - B.$$

By adding these 2 equations, we can solve for M_o: $M_o = 70$, and by subtracting the 2 equations we find that $B = 20$. When we substitute these values into (equation 8), we find the temperature:

$$T(t) = 70 + 0.5/0.25 + \{ 20 / [1 + (\pi/12)^2/0.25^2] - 70 - 0.5/0.25 + 60 \} *e^{-0.25t} - 20[\cos(\pi/12\ t) + \pi/3 \sin(\pi/12\ t)] / (1 + (\pi/3)^2),$$

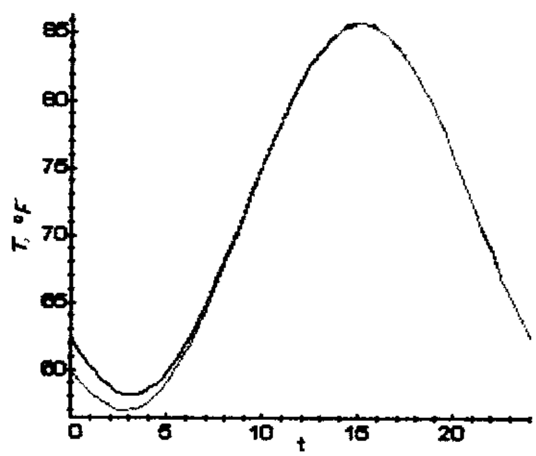
$$T(t) = 72 - 2.460849302e^{-0.25t} - 9.539150698[\cos(\pi/12\ t) + \pi/3 *\sin(\pi/12\ t)]. \quad \text{(equation 9)}$$



Directly from the graph (above) of this function, we can see that the lag between the inside and outside temperature is somewhere between 2 and 3 hours. This means that the temperature in the building will be affected by a change in temperature outside only after that time. The exponential term in the above function has little effect on the temperature since it tends to zero as time increases, so we have plotted the temperature indoors without the exponential term. To verify this, we have plotted (equation 9) below with the exponential term (green curve) and without (red curve).

good

good



Example 2 gave a model of the temperature as a function of time in the spring or fall. In the summer we can use an air conditioner for cooling and in the winter, a furnace for heating. This causes the U(t) term to be non-zero.

Suppose we installed a temperature unit in a home to regulate the temperature where the furnace would run if it became too cold and the air conditioner would run if it became too hot. T(t) is the actual temperature in the home and we will let Tr be the comfortable room temperature desired. We now have:

$$U(t) = C[\, Tr - T(t)\,], \text{ where } C \text{ is a positive constant.}$$

When: $T(t) > Tr$, the air conditioner runs,

$T(t) < Tr$, the furnace runs,

$T(t) = Tr$, neither the air conditioner nor the furnace runs.

we would not normally allow both to run during a given season.

see page 17

Substituting the value of U(t) into (equation 8), we have:

$$dT/dt = K\,[\, M(t) - T(t)\,] + H(t) + C[\, Tr - T(t)\,].$$

Example 4————————————

If we let H(t) = Ho, a constant, and let M vary as a sine wave (as in example 3) we can write:

$$dT/dt + P(t)T(t) = Q(t)$$

where: $Q(t) := K[\, M_o - B\cos\omega t\,] + H_o + CTr$

$P(t) := C + K.$

Determine the temperature as a function of time.

Solution:——————————————

We can find an integrating factor for the linear equation, $e^{\int P(t)dt} = e^{\int (C+K)dt} = e^{(C+K)t}$.

Let (C+K) = K', and multiply through by $e^{K't}$:

$$e^{K't}\, dT/dt + TK'e^{K't} = e^{K't}\,[\, K(M_o - B\cos\omega t) + H_o + CTr\,]$$

$$d(Te^{K't})/dt = e^{K't}\,\{\, K[M_o - B\cos\omega t\,] + H_o + CTr\,\}.$$

Taking the integral of each side yields and dividing by $e^{K't}$ yields:

$$T(t) = e^{-K't}\ \{\textstyle\int e^{K't}\,[\, K(M_o - B\cos\omega t\,) + H_o + CTr] + c\,\}$$

where c is a constant of integration. Integration and integration by parts gives:

$$T(t) = KM_o/K' + (Ho + CTr)/K' - KBe^{-K't}\,\{\,(e^{K't}/K')\cos\omega t + \omega/K'\textstyle\int \sin\omega t\, e^{K't}\, dt\,\} + ce^{-K't}.$$

Taking integration by parts again on the $\int \sin\omega t\, e^{K't}\, dt$ term, we have :

$$\int \sin\omega t\, e^{K't}\, dt = \{\, \sin\omega t\, e^{K't} / K' - \omega/K'\textstyle\int \cos\omega t\, e^{K't}\, dt\,\}$$

Solving for the integral $\int \cos\omega t\, e^{K't}\, dt$ and simplifying, we find:

$$T(t) = KM_o/K' + (H_o + CTr)/K' - KB/K'\,[\,\cos\omega t + \omega/K' \sin\omega t\,] / (1+ \omega^2/K'^2) + ce^{-K't}. \qquad \text{(equation 10)}$$

Let $(CTr + KM_o + H_o) / K' = N$, a constant then

$$T(t) = N - KB/K'\,[\,\cos\omega t + \omega/K' \sin\omega t\,] / (1+ \omega^2/K'^2) + ce^{-K't}.$$

The initial condition $T(0) = T_o$ corresponding to midnight gives:

$$T(0) = T_o = N - KB/K'\,[\, 1 / (\, 1 + \omega^2/K'^2\,)\,] + c$$

Solving for $c = To - N + KB/K' [1 / (1 + \omega^2/K'^2)]$.

Combining c with (equation 10) gives the temperature as a function of time for a temperature controlled home.

The time constant in this example is $1/K' = 1/(C+K)$ which is not the same as that in the previous example; in this case, the constant takes into consideration the effect of heating and air conditioning. Using the same value for K from (example 3), K=0.25 and the fact that a typical value for heating and cooling systems is near C=2, we find that the time constant with heating and air conditioning is approximately $1/K' = 1/(C+K) = 1/(2 + 0.25) \cong 0.44$ hour $\simeq$ 26 minutes.

good

What this value means is that when the heating or cooling is turned on, it takes close to 30 minutes for the exponential term to decrease to a point where its' value is essentially negligible. Since this term dies off in about 30 minutes and since (K+C)>>K in general, and H_0 is small, we are essentially left with:

$$T(t) \simeq CTr/K' - KB/K' [\cos\omega t + \omega/K' \sin\omega t] / (1+ \omega^2/K'^2)$$

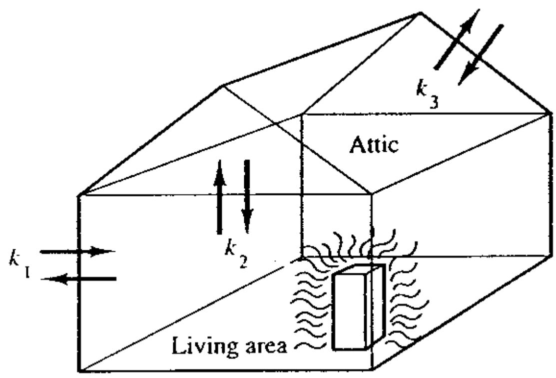
where $KB/K' [\cos\omega t + w/K' \sin\omega t] / (1+ \omega^2/K'^2)$ only contributes a slight sinusoidal variation, and $CTr/K' \simeq Tr$. Therefore, we see that after about 30 minutes, the temperature in the room is essentially Tr with only a slight sinusoidal variation.

Although this example takes into consideration the varying temperature outside and incorporates a heating and cooling system, this model is still unrealistic. For example, the assumption that $U(t) = C [Tr - T(t)]$ implies that any variation in the constant room temperature Tr will result in either furnace heating or air conditioner cooling. A 'comfort zone' would be more appropriate and realistic, where Tr can vary by several degrees and the heating or cooling would start only when the temperature was above a certain maximum, or below a certain minimum temperature.

good

One should also realize that the temperature does not vary in a perfect sine wave, in fact the temperature outside can be unpredictable with large variations from a simple sine wave, but this is still an improvement over assuming that the temperature is a constant fixed value.

Since most buildings and houses consist of more than one large room, ideally our model should also incorporate several rooms. For our purposes, we shall look at a simple case where a house has a heated (or cooled) living area and an attic (which is not heated or cooled). This model shall deal with heat exchange through the walls, ceiling and roof, and incorporates a heating/cooling device. First we will deal with the model in the winter when furnace heating is necessary.



Let x(t) be the temperature of the living area and y(t) that of the attic. The temperature is affected by loss of heat to the attic and outside and by the addition of heat from the furnace. The rate that the furnace heats the living area at is given by the heat capacity of the house multiplied by the number of BTU's (British Thermal Units) per hour that the furnace generates. Assume that, in this particular house the heat capacity is given by 0.2°F per thousand BTU's and the furnace generates 75,000 BTU/hr. This means that the furnace can provide 75(0.2) = 15°F per hour to the living area. Suppose the initial temperature outside and in the house is only 35°F. Then the heat lost through the outside walls contributes a change in temperature of $k_1(35 - x)$ and the heat lost to the attic is $k_2(y - x)$. The heat lost to the roof is given by $k_3(35 - y)$.

should be rates of loss or gain

The constants of proportionality, k_1, k_2 and k_3 are all positive. K_1 depends on the materials, insulation, and thickness of the outside walls, and k_2 depends on that of the ceiling. Since heat rises, ceilings are generally more insulated and thicker than walls, so we would assume that $k_1 > k_2$ since the heat exchange will be faster through the walls than through the ceiling. K_3 represents the roofing materials, and since there is generally little insulation in the roof, we will let this constant be the largest (i.e. the fastest rate of heat exchange in comparison to k_1 and k_2). Therefore, we have logically assumed that $k_3 > k_1 > k_2$. We will arbitrarily choose values for these constants using this criterion: $k_3 = 1/2$, $k_1 = 1/3$, and $k_2 = 1/4$.

ok

Combining the above assumptions and the information we have, we end up with two equations:

$$dx/dt = 15 + k_2(35 - x) + k_2(y - x) \qquad \text{(equation 11)}$$
$$= 15 + 1/3\,(35 - x) + 1/4\,(y - x)$$
$$dy/dt = k_2(x - y) + k_3(35 - y) \qquad \text{(equation 12)}$$
$$= 1/4\,(x - y) + 1/2\,(35 - y)$$

where dx/dt is the rate of change in temperature in the living area, and dy/dt the rate of change in temperature in the attic.

In order to solve these differential equations, we will use the method of operators:

$(D + 7/12)x - 1/4\ y = 80/3$ $\qquad$ $(D + 3/4)y - 1/4\ x = 35/2.$

Multiplying the first equation by $(D + 3/4)$ and the second by 1/4, then adding the equations causes the y term to disappear, and we are left with:

$$(D^2 + 4/3\ D + 7/16 - 1/16)x = 195/8$$

$$(D^2 + 4/3\ D + 3/8)x = 195/8. \qquad \text{(equation 13)}$$

The auxiliary equation of (equation 13) is: $m^2 + 4/3\ m + 3/8 = 0$. We can solve for m using the quadratic formula: $m = [-4/3 \pm \sqrt{16/9 - 12/8}\,] / 2$. Simplifying, we find the roots are: $m = -2/3 \pm 1/6\sqrt{5/2}$, but for simplicity we shall label these as

$$\alpha = -2/3 + 1/6\sqrt{5/2} \quad \text{and} \quad \beta = -2/3 - 1/6\sqrt{5/2}.$$

A general solution to the associated homogeneous equation is:

$x_h(t) = C_1e^{\alpha t} + C_2e^{\beta t}$, where C_1 and C_2 are constants.

A particular solution can be calculated using the method of undetermined coefficients: $x_p = A195/8$

$x_p' = x_p'' = 0.$

When we substitute these values into (equation 13),

$$(D^2 + 4/3\ D + 3/8)x_p = 195/8$$

$$(D^2 + 4/3\ D + 3/8)A195/8 = 195/8$$

$$(3/8)\ A195/8 = 195/8$$

$$x_p = A195/8 = 65.$$

Therefore, the particular solution to (equation 13) is x_p = 65, and combining the homogeneous and particular solutions, we find a general solution for x which gives the temperature in the living area as a function of time:

$$x(t) = C_1e^{\alpha t} + C_2e^{\beta t} + 65. \qquad \text{(equation 14)}$$

We can find y(t) by substituting (equation 14) into (equation 12):

$$dy/dt + 3/4\ y = 1/4(C_1e^{\alpha t} + C_2e^{\beta t} + 65\ ) + 35/2.$$

This is a linear equation so we multiply through by its integrating factor: $e^{3/4t}$

$$d(\ ye^{3/4t})/dt = (C_1/4)e^{3/4t}e^{\alpha t} + (C_2/4)e^{3/4t}e^{\beta t} + (135/4)\ e^{3/4t}.$$

Integrating and dividing by $e^{3/4\ t}$ yields:

$$y(t) = C_1e^{\alpha t}/[4(\ \alpha + 3/4)] + C_2e^{\beta t}/[4(\ \beta + 3/4)] + 45$$

which is the temperature in the attic as a function of time.

In order to solve for C_1 and C_2, we shall use the initial conditions x(0) = 35 and y(0) = 35.

$$x(0) = 35 = C_1 + C_2 + 65 \text{ which implies } -C_2 = C_1 + 30,$$

$$y(0) = 35 = 0.72075922C_1 - 1.387425 89C_2 + 45$$

$$0 = 0.72075922C_1 + 1.38742589(C_1 + 30) + 10.$$

We find C_1 = -24.486833 and C_2 = -5.513167 and so the temperature in the living area and attic are given by:

$$x(t) = -24.486833e^{-.403153 52t} - 5.513167e^{-.930189t} + 65, \qquad \text{(equation 15)}$$

$$y(t) = -17.64911065e^{-.40314352t} + 7.6491106e^{-.930189t} + 45. \qquad \text{(equation 16)}$$

Since the exponential terms in the above 2 equations have negative powers, these terms will decrease to zero and so the limiting value of the living area is 65^oF and that of the attic, 45^oF. From these values, we see that if a house were indeed built with materials that resulted in the constants k_1, k_2 and k_3 used above then the house would not be very warm. We could improve on this by decreasing the values of these constants and hence slowing the rate of heat loss.

Good

Example 5:

When will the temperature of the living area be 58^oF?

Solution:

We must use Newton's or the bisection method to solve equations 15 and 16 for time, t when $T(t) = 58^oF$. A computer program using the bisection method with the initial approximation of t between a = 3 and b = 4 hours gave the following results:

a	b	time
3.000000	4.00000	3.5000000
3.000000	3.50000	3.2500000
3.000000	3.25000	3.1250000
3.125000	3.25000	3.1875000
3.187500	3.25000	3.2187500
3.187500	3.21875	3.2031250
3.203125	3.21875	3.2109375
3.203125	3.2109375	3.20703125
3.2070313	3.2109375	3.20898438
3.2070313	3.20898438	3.20800781

We have found that it takes approximately 3.21 hours for the living area to heat to 58^oF. Since this is quite a long time, we have even more evidence that we made non-ideal choices for k_1, k_2, and k_3.

Suppose now that it is $90^{o}F$ in the summer and we would like to cool our house with a 50,000 BTU air conditioner. Recall that the heat capacity of the house is $0.2^{o}F$ per thousand BTU's , so the air conditioner can remove $50(0.2) = 10^{o}F$ per hour from the living area. The constants of proportionality will remain the same.

Example 6:——————————

What will the temperature be in the air conditioned living area described above after 2 hours?

Solution:——————————

Since we have only changed the constant value from 15 to -10 in equation 11 and equation 12 remains the same, we see that the auxiliary equation is the same and we have only to determine the particular solution.

After combining the 2 differential equations and eliminating y, we are left with:

$$(D^2 + 4/3\ D + 3/8)x = 105/4$$

$$x_p = A105/4$$

$$x_p' = x_p'' = 0.$$

Plugging in $x_p = A105/4$

$$(D^2 + 4/3\ D + 3/8)x_p = 105/4$$

$$(D^2 + 4/3\ D + 3/8)A105/4 = 105/4$$

$$x_p = 70.$$

The temperature in the living area in the summer is given by:

$$x(t) = C_1e^{\alpha t} + C_2e^{\beta t} + 70$$

where $\alpha = -2/3 + 1/6\sqrt{5/2}$ and $\beta = -2/3 - 1/6\sqrt{5/2}$.

Similarly, we can determine the temperature in the attic:

$$y(t) = C_1e^{\alpha t}/[4(\alpha + 3/4)] + C_2e^{\beta t}/[4(\beta + 3/4)] + 250/3.$$

Calculating coefficients we find:

$$x(t) = 16.324555e^{-.40314325t} + 3.675444674e^{-.9301898t} + 70$$

and after 2 hours the temperature in the living area is,

$$x(2) = 77.86.$$

Once again, in the above example the exponential terms have negative powers, so the temperature in the living area and attic will approach 70°F and 250/3 ≈ 83°F respectively.

good

Since heat rises, we would like to have a ceiling with a great deal of insulation in order to slow the rate of heat exchange between the attic and ceiling. However, it is not practical to have an excessive amount of insulation, but what if we doubled the amount in the ceiling of the house in question?

Suppose the thickness of insulation is inversely proportional to the proportionality constant k_2 (i.e. when we double the thickness, we halve the value of k_2). Let us use this new value of k_2 on the two previous examples in order to see what effect this has on the heating/cooling of the house. The value was previously 1/4, but now we will let $k_2 = 1/8$. Incorporating this change into (equation 11) and (equation 12), we obtain:

Heating:

$$dx/dt = 15 + 1/3\,(35 - x) + 1/8\,(y - x)$$

$$dy/dt = 1/8\,(x - y) + 1/2\,(35 - y)$$

The above equations can be solved using the same method as before, so the details are omitted. The solutions to the modified equations are:

$$x(t) = C_1 e^{\alpha t} + C_2 e^{\beta t} + 905/13$$
$$y(t) = C_1 e^{\alpha t}/[8(\alpha + 5/8)] + C_2 e^{\beta t}/[8(\beta + 5/8)] + 545/13$$

where $\alpha = -13/24 + \sqrt{13/24}$ and $\beta = -13/24 - \sqrt{13/24}$.

Now, the limiting values when heating the house when it is 35^oF outside is $\approx 70^oF$ for the living area and $\approx 42^oF$ for the attic, compared to 65^oF and 45^oF. This is an improvement because now the living area can attain a higher temperature.

When we determine how long it takes to heat the house to 58^oF (as we did in example 5) we find that takes only 0.56 of an hour or about 34 minutes, which is a drastic improvement over the 3.21 hours necessary to heat the house previously, using $k_2 = 1/4$.

Cooling:

The modified system for cooling is:

$$dx/dt = -10 + 1/3\,(35 - x) + 1/8\,(y - x)$$
$$dy/dt = 1/8\,(x - y) + 1/2\,(35 - y)$$

whose solutions are given by:

$$x(t) = C_1 e^{\alpha t} + C_2 e^{\beta t} + 870/13$$
$$y(t) = C_1 e^{\alpha t}/[8(\alpha + 5/8)] + C_2 e^{\beta t}/[8(\beta + 5/8)] + 1110/13$$

where $\alpha = -13/24 + \sqrt{13/24}$ and $\beta = -13/24 - \sqrt{13/24}$.

The temperature that the living area and attic can attain when it is $90^{o}F$ outside are: $\approx 67^{o}F$ and $\approx 85^{o}F$ respectively compared to $70^{o}F$ and $83^{o}F$. After 2 hours, the temperature in the living area will be $\approx 77^{o}F$ which is about $2^{o}F$ lower than what we obtained when we used $k_2 = 1/4$.

Thus, we have determined that doubling the thickness of the insulation in the ceiling has a very positive effect on the heating and cooling of a house. While the improvement is only marginal in the cooling situation, we notice a *very* significant reduction in the time it takes to heat a house. Moreover, once the house is heated, heat will escape much slower in this case and hence will take longer for the house to cool.

Hence, we have considered two models; one which determines the temperature as a function of the outside temperature, the heat generated inside the building, and the effect of heating / cooling; the other which gives the temperature of a two-compartment house as a function of time. Although both were relatively simplistic, perhaps incorporating the factors affecting the first model into a several-compartment model (like that of our second model) would more accurately predict the temperature in a house. This way, we are taking into consideration the effects of all the factors, like people, drafty windows, etc., and applying these results to a house where there are several rooms. We could also look at a situation where there are more that one heating / cooling devices in the house, and when the temperature outside is not constant.

A problem that arises in these models is: How do we determine the values of the parameters, for instance, H(t) which is the rate of increase of temperature caused by lights, people and appliances? Values such as this are very hard to approximate to any reasonable degree of accuracy since they can vary widely and it is essentially impossible to isolate these factors in order to measure them. Even with these imperfections, we have still come up with a feasible means of determining the temperature in a building as a function of time.

Good

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