I/ **Inhibited enzymatic reaction.** *Write down the differential equations describing the following enzymatic reaction, where enzyme E is inhibited by inhibitor I:*

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B_1 \overset{k_2}{\rightarrow} E + Q,$$

$$E + I \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B_2.$$

II/ **A feedback mechanism for oscillatory reactions.** *Write down a differential equation model for the following pathway:*

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} A.$$

III/ **Enzymatic reaction with two intermediate steps.** *Write down the equations describing the following reaction:*

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C_2 \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} E + P.$$

IV/ **Linear systems.** *We study* $2 \times 2$ *systems of linear ODEs:*

$$y' = Ay, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Classify the origin* $\binom{0}{0}$ *as a stable/unstable spiral, node, or saddle, and plot (or sketch) the phase portrait for each of the following cases:*

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

V/ Determine whether the origin is stable or unstable, a node, spiral, saddle, or center for the system $dX/dt = AX$.

(a) $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} -2 & 4 \\ -1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} -1 & a \\ 0 & a \end{pmatrix}, a \neq 0$

VI/ Apply Gershgorin's Theorem to determine sufficient conditions on the parameters $a$, $b$, and $c$ such that the eigenvalues of $A$ are negative or have negative real part.

(a) $A = \begin{pmatrix} a & -1 & 0 \\ -1 & b & 1 \\ 0 & -2 & c \end{pmatrix}$

(b) $A = \begin{pmatrix} a & b \\ c & -2 \end{pmatrix}$

VII/ For the linear differential systems $dX/dt = AX$, the matrices $A$ are given below. For each system, find the eigenvalues and eigenvectors of $A$. Then find the general solution to each differential system.

(a) $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

VIII For the following nonlinear differential equations, find the stable equilibria as a function of the bifurcation parameter $r$. Then draw the bifurcation diagram.

(a) $dx/dt = rx - x^3$
(b) $dx/dt = rx + x^2$

IX For the following nonlinear differential equations, find the stable equilibria as a function of the bifurcation parameter $r$. Then draw the bifurcation diagram.
(a) $dx/dt = rx + x^3$ (pitchfork)
(b) $dx/dt = r - x^2$ (saddle node)
(c) $dx/dt = r + 2x + x^2$ (saddle node)
(d) $dx/dt = rx - x^2$ (transcritical)
(e) $dx/dt = x(r - e^x)$ (transcritical)

X Show that the following system has a saddle node bifurcation at $r = 0$:

$$\frac{dx}{dt} = r + x^2,$$
$$\frac{dy}{dt} = -y.$$

XI The following system represents the concentrations of two chemicals, $x$ and $y$ (Hale and Koçak, 1991, pages 360–361):

$$\frac{dx}{dt} = 1 - (r + 1)x + x^2 y,$$
$$\frac{dy}{dt} = x(r - xy),$$

where $r > 0$.
(a) Show that there exists a unique positive equilibrium, $(\bar{x}, \bar{y}) = (1, r)$.
(b) Determine the local asymptotic stability of the positive equilibrium.
(c) Show that there is a Hopf bifurcation at $r = 2$.

XII A chemical compound undergoes the transformation

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C.$$

This process is described by the following kinetic equations:

$$\frac{dA}{dt} = -k_1 A,$$
$$\frac{dB}{dt} = k_1 A - k_2 B,$$
$$\frac{dC}{dt} = k_2 B,$$

where $k_1 \neq k_2$ and $k_1, k_2 > 0$. The variables $A$, $B$, and $C$ represent the concentration of each of the chemicals (Kaplan and Glass, 1995). The initial conditions are $A(0) = N$, $B(0) = 0$, and $C(0) = 0$.
(a) Find the solution for $A(t)$.
(b) Use (a) to find the solution for $B(t)$.
(c) Use (b) to find the solution for $C(t)$.
(d) If $k_1 = 2k_2$, at what time is $B$ a maximum?
(e) Find the following limits, if they exist, $\lim_{t\to\infty} A(t)$, $\lim_{t\to\infty} B(t)$, and $\lim_{t\to\infty} C(t)$.

XIII

**Self-intoxicating population.** *Some populations produce waste products, which in high concentrations are toxic to the population itself. For example, algae or bacteria show the structure in Figure 3.25.*

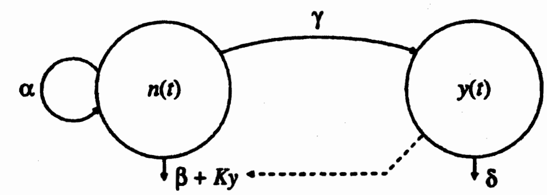


**Figure 3.25.** *Arrow diagram for a self-intoxicating population.*

*Let the population density be denoted by $n(t)$ and the toxin concentration by $y(t)$. Then*

$$\dot{n} = (\alpha - \beta - Ky)n,$$
$$\dot{y} = \gamma n - \delta y,$$

*with $\alpha, \beta, \gamma, \delta, K \geq 0$.*

(a) *Explain each term of the above system.*

(b) *Find the nullclines, the steady states, and sketch a phase portrait.*

(c) *Sketch the vector field.*

(d) *Linearize the system and characterize each of the steady states (stable/unstable, saddle, node, spiral, center, etc.). Find the regions in parameter space such that the nontrivial (coexistence) equilibrium is either a node or a spiral.*

(e) *Sketch some trajectories for the case of $\delta < 4(\alpha - \beta)$, and explain what you see in terms of the biology.*

(f) *Consider the case of higher dilution: $\delta \gg 1$, $\gamma/\delta < \infty$.*

XIV

**Fish populations in a pond.** *Imagine a small pond that is mature enough to support wildlife. We desire to stock the pond with game fish, say trout and bass. Let $T(t)$ denote the population of the trout at any time $t$, and let $B(t)$ denote the bass population.*

(a) *Initially, assume that the pond environment can support an unlimited number of trout in isolation (i.e., growth of the trout population is exponential). Write down an equation that describes the evolution of the trout population in the absence of competition.*

(b) *Modify the equation to account for competition of the trout with the bass population for living space and a common food supply. You may assume that the growth rate of the trout population depends linearly on the bass population.*

(c) *Repeat* (a) *and* (b) *for the bass population.*

(d) *Explain the meaning of the parameters you introduced into the model.*

(e) *What are the steady states of the system? Determine the stability of the steady states using linearization.*

(f) *Perform a graphical analysis of the model. That is, find the nullclines, and sketch the phase portrait, taking into account the information obtained in* (e).

(g) *Is coexistence of the two species in the pond possible? If so, how sensitive is the final solution of the population levels to the initial stocked levels and external perturbations? Explain.*

(h) *Replace the exponential growth term in each equation with a logistic growth term. Use $r_t$ and $r_b$ to denote the intrinsic growth rate of the trout and bass, respectively, and $K_t$ and $K_b$ to denote the respective carrying capacities. Analyze the following specific case: $K_t > r_b/I_b$ and $K_b > r_t/I_t$, where $I_t$ ($I_b$) represents the strength of the effect of the bass (trout) population on the rate of change of the trout (bass) population. How does the final outcome differ from before? Explain.*

(i) *The second model is a lot more realistic than the first model, as it no longer assumes unlimited growth in the absence of competition. Think of at least one further improvement to the model. How would the equations be affected? You should write down the*

Consider an epidemic model, where $S + I + R = N =$ constant, so that $R = N - S - I$. The original model with three differential equations can be simplified to one with just two differential equations,

$$\frac{dS}{dt} = -\frac{\beta}{N}SI + \nu(N - S - I),$$
$$\frac{dI}{dt} = I\left(\frac{\beta}{N}S - \gamma\right).$$

The parameters are all positive: $\beta, N, \gamma, \nu > 0$. The parameter $\beta$ represents the contact rate, $N$ the total population size, $\gamma$ the recovery rate, and $\nu$ the rate of loss of immunity. Define the basic reproduction number

$$\mathcal{R}_0 = \frac{\beta}{\gamma},$$

the number of secondary infections caused by one infectious individual.

(a) If $\mathcal{R}_0 < 1$, find all of the nonnegative equilibria and determine the local asymptotic stability for each of the equilibria. Do a phase plane analysis for initial conditions satisfying

$$S(0) > 0, \quad I(0) > 0, \quad \text{and} \quad S(0) + I(0) \leq N.$$

(b) Do the same as in part (a) but assume $\mathcal{R}_0 > 1$.