University of Manitoba
Math 3820 – Winter 2008

Midterm

Tuesday, March 18, 2008

Instructions

This test is 1 hour and 20 minutes. It comprises 3 questions on 3 pages. Notes are allowed; calculators and computers are **not** allowed. In marking, attention will be paid to the overall legibility of solutions; so detail and structure your answers.

1. We have the following setup, illustrated in Figure 1: a tank contains initially, at time $t = 0$,

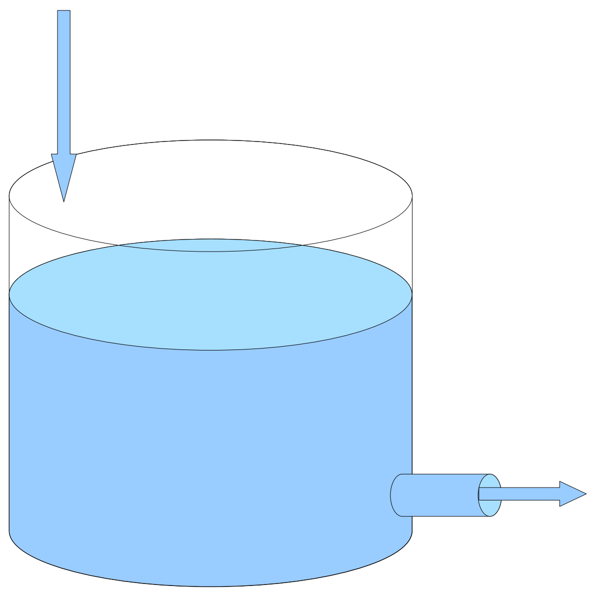

Figure 1: Situation modelled in exercise **1**.

100 litres of water. There is an inflow, at the rate of r_{in} litres per minute, and an outflow at the rate of r_{out} litres per minute. Before the experiment starts, the tank contains a concentration of salt of C_0. At the start of the experiment, liquid starts flowing into the tank, and the contents of the tank start flowing out.

1.a. Write a differential equation for the variation of the volume $V(t)$ of liquid in the tank at time t.

1.b. Find the expression of $V(t)$, the volume of liquid in the tank at time t.
1.c. Assume that $r_{in} = 10$ litres per minute. What is the value of r_{out}, if after 1 hour, there remains exactly 50 litres of liquid in the tank.
1.d. Assume now that $r_{in} = r_{out}$, so that the amount of liquid remains constant in the tank. Denote $r(t)$ this rate of in/out-flow, which we assume can vary with time. The inflow contains a concentration S_0 of salt. Assuming that the tank is well stirred, so that the concentration of salt is uniform in the tank, write a differential equation for the concentration $C(t)$ of salt in the tank at time t.
1.e. The general solution to the linear equation $x' + p(t)x = q(t)$ is given by

$$x(t) = e^{-\int p(t)dt}\left(\int e^{\int p(t)dt} q(t)dt + K\right), \quad K \in \mathbb{R}.$$

Using this, solve the differential equation you found in **1.d**, with the initial condition $C(0) = C_0$.

2. Consider the difference equation

$$x_{t+1} = \frac{a x_t}{b + x_t}, \tag{1}$$

with $a, b > 0$ and $x_0 \geq 0$.
2.a. Show that for $x_0 \geq 0$, $x_t \geq 0$ for all t.
2.b. Find the fixed points of (1).
2.c. Study the relevance (nonnegativity) and stability of these fixed points, as a function of the (relative) values of a and b.
2.d. Summarize your findings of **2.c** in a diagram having a on the x-axis (assuming a fixed value of b), and the value of the fixed points on the y-axis, as shown in Figure 2. Indicate an attracting equilibrium by a thick line, a repelling one by a dashed line.

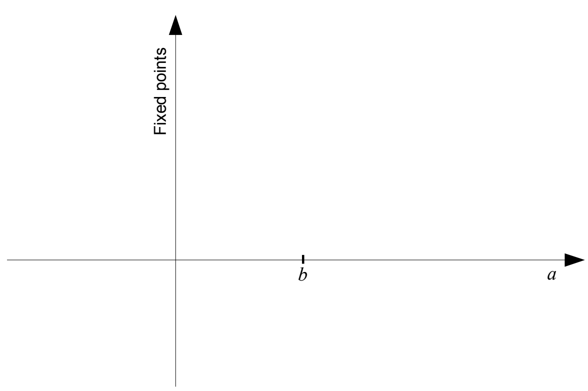

Figure 2: Setup of the bifurcation diagram for exercise **2.d**.

2.e. Can (1) have period 2 points other than its fixed points?

3. Consider the following model for hares and fox, where $H(t)$ and $F(t)$ are the numbers of hares and fox at time t, respectively.

$$\begin{aligned} H' &= (b_H - d_H)H - \pi HF \\ F' &= \sigma\pi HF - d_F F. \end{aligned} \tag{2}$$

The parameters are b_H, birth rate of hares, d_H, death rate of hares, d_F, death rate of fox, π, the predation rate, and σ, the conversion coefficient. We assume that $b_H, d_H, d_F > 0$, while $\pi \geq 0$ and $0 \leq \sigma \leq 1$. System (2) is considered together with initial conditions $(H(0), F(0)) = (H_0, F_0)$, with $H_0, F_0 > 0$.

3.a. Suppose that there is no predation, i.e., $\pi = 0$. Solve system (2), and discuss the behavior of its solutions as a function of the relative values of b_H and d_H.

3.b. Suppose now that $\pi > 0$ and $\sigma > 0$. Draw the nullclines of (2); show the direction field in each region of $\mathbb{R}_+^2$ hence delimited; identify the equilibria.

3.c. Discuss the relevance (nonnegativity) and local stability of the equilibria, as a function of the relative values of b_H, d_H and d_F, in the $\pi, \sigma > 0$ case.