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MATHEMATICAL MODEL OF THE COCHLEA. I: FORMULATION AND SOLUTION*

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Abstract. A mathematical model of the cochlea of the inner ear is formulated. It consists of two communicating rectangular chambers (the scalae) filled with a viscous incompressible fluid and separated by a thin uniform Euler–Bernoulli beam representing the basilar membrane. The system is driven at one end by the piston-like movement of the stapes. The solution of the resulting initial and boundary value problem is obtained in closed form. The representation of acoustical information in terms of the motion of the basilar membrane and some conclusions relating cochlear function to its structure are presented in a companion paper.

1. Introduction. The ear is conveniently subdivided into three parts—outer, middle, and inner ear (Fig. 1). The outer ear consists of the external flap and the

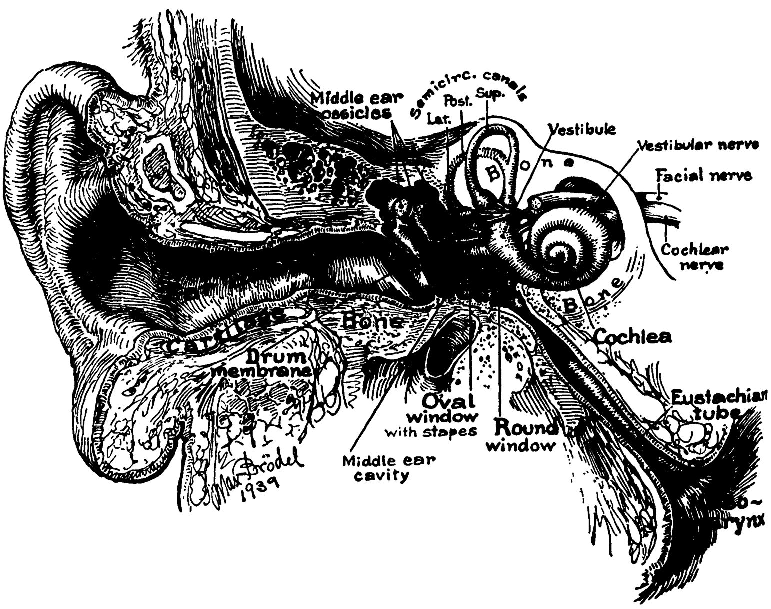

FIG. 1. *Schematic of the ear. The cochlea is rotated somewhat from the normal orientation to show its coils more clearly. From "3 Unpublished Drawings of the Anatomy of the Human Ear", by Max Brodel, 1946. © 1946, W. B. Saunders Co., Philadelphia. Used by permission.*

ear canal leading to the drum (tympanic)[1] membrane of the middle ear. Attached to this membrane is a chain of three small bones called the middle ear ossicles. The innermost ossicle, called stapes, has its footplate implanted at the oval window of the inner ear.

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[1] Various anatomical terms have a number of common synonyms which are indicated here in parentheses.

The auditory portion of the inner ear is a snail-like structure called the cochlea. With the exception of an initial bulge at the basal end, where the stapes is located, the cochlea narrows gradually towards the apex. In man, it winds about $2\frac{3}{4}$ turns, and its uncoiled length is about 35 mm. Internally (Fig. 2), the cochlea is composed

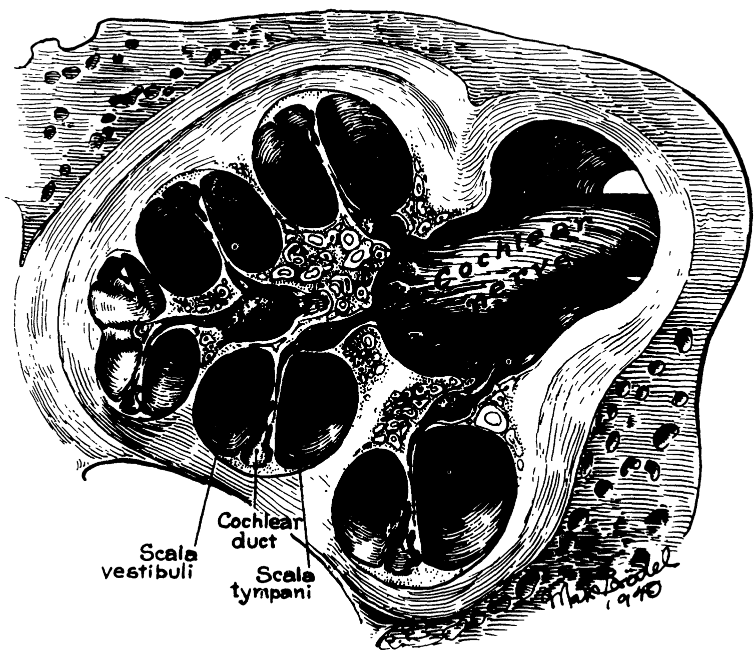

FIG. 2. *Cochlear cross section. From "Year Book of Eye, Ear, Nose and Throat", L. Bothman and S. J. Crowe, eds.,* 1940. © 1940 *Year Book Medical Publisher, Inc., Chicago. Used by permission.*

of three fluid-filled chambers (the scalae). The sense organ proper, that is, the organ of Corti with its accessory structures, is contained (Fig. 3) in the cochlear duct (scala media). The organ of Corti is supported by the fibrous basilar membrane. The thin and pliant Reissner's membrane separates the cochlear duct from the scala vestibuli which, at the basal end of the cochlea, communicates with the middle ear through the oval window. The scala tympani ends at the round window, an opening on the cochlear wall which is covered by a small flexible membrane. The cochlear duct, including both the basilar and Reissner's membranes, ends a little short of the cochlear apex leaving a small opening called the helicotrema. While the scale media ends "blindly", the scala tympani and scala vestibuli join through the helicotrema (see also Fig. 5). The sensory surface of the cochlea is then a narrow spiral ribbon mounted on the basilar membrane between two fluid-filled chambers.

Hearing is the result of two kinds of processes, one mechanical and the other electrochemical. Conduction of the sound signal, via the outer and middle ear, to the cochlear fluids and the resulting oscillations of the basilar membrane are

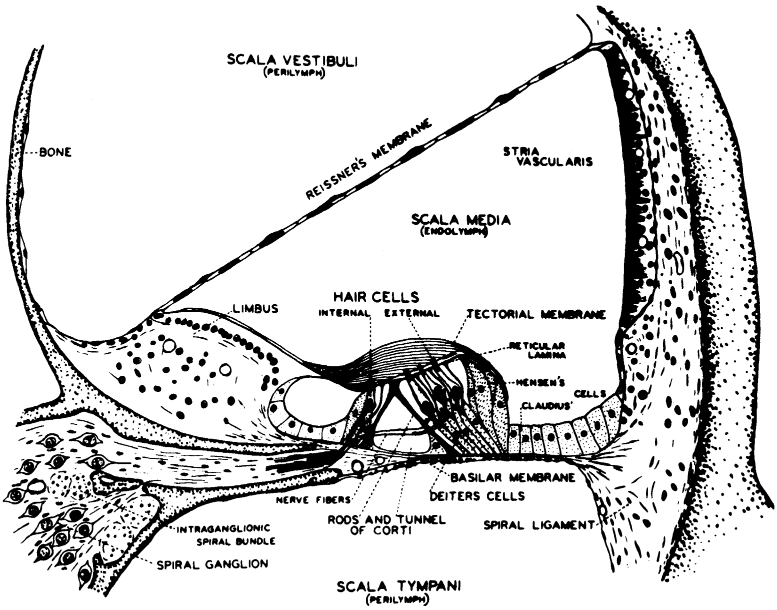

FIG. 3. *Cross section of the cochlear duct. From "Acoustic trauma in the guinea pig", H. Davis and Associates, J. Acoust. Soc. of Amer., 1953. Used by permission.*

the mechanical events in the auditory process (Fig. 4). While the conductive mechanism is relatively well understood, the representation of the acoustical information in terms of the motion of the basilar membrane is not. This is due to the scarcity of direct experimental measurements on cochlear function, which in turn is due to the inaccessible position of the cochlea and the delicate nature of the basilar membrane.

As early as the 1850's, Helmholtz [1] realized that mathematical models of the cochlea can overcome many of the experimental limitations and provide

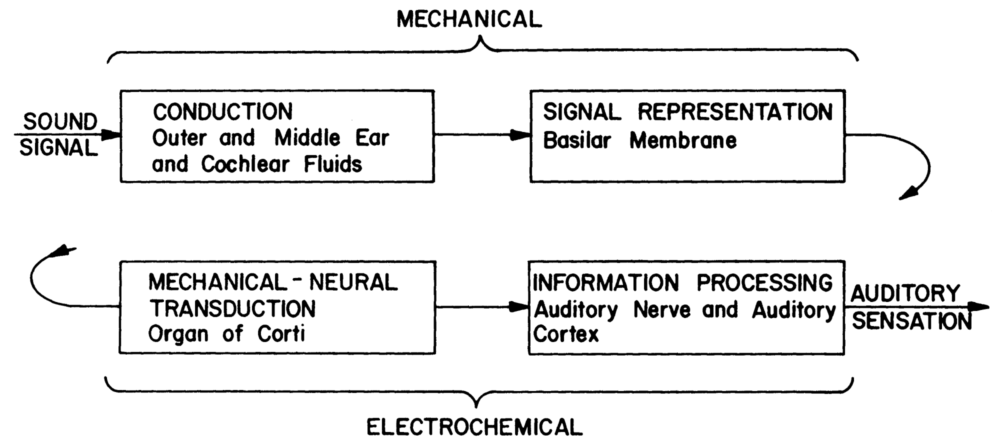

FIG. 4. *Sequence of auditory events*

crucial information towards a viable theory of hearing. What is generally meant by "theory of hearing" is an explanation of the process of pitch discrimination in the auditory system. At present, this process is incompletely understood. To proceed further, it is essential that cochlear dynamics be well understood since the output of the cochlea is the input to the higher mechanical to neural transduction process.

A brief review of the current status of mathematical cochlear models will clarify the scope of the model proposed here. For a theoretician studying cochlear mechanics, the most important observations of von Bekesy [2] are listed below:

(a) The basilar membrane has neither longitudinal nor transverse tension in the resting state (see also [3]).

(b) The stiffness of the basilar membrane decreases by about two orders of magnitude from the stapes to the helicotrema; the taper of the membrane can account for this. The basilar membrane does not exhibit any static anisotropic elastic properties except very close to the stapes.

(c) The damping of the basilar membrane due to the fluid, as calculated from the logarithmic decrement, is essentially constant at all points on the membrane except near the helicotrema, where it increases.

(d) Traveling waves exist in the motion of the basilar membrane for excitation frequencies above 25 cps. The velocity of propagation decreases from the stapes towards the helicotrema, the waves being spatially damped on the apical side of the position of maximum displacement.

(e) There is a Place principle; that is, a one-to-one correspondence exists between excitation frequencies and positions of the maximum membrane displacement. Low frequencies result in maxima close to the helicotrema and high frequencies near the stapes.

The most elegant mathematical model of the cochlea to appear was the first one [1], long before von Bekesy's data was available. Helmholtz considered the basilar membrane as tapered and supported by longitudinal and transverse tension. Under the action of a sinusoidally time-dependent pressure difference, the model showed a Place principle as in (e). Had Helmholtz known of (a) and (b), he would most probably have considered an unstressed tapered elastically isotropic plate vibrating within a viscous and incompressible fluid, which is closer to reality.

The simplest cochlear model is [4], where the basilar membrane is considered as a uniform homogeneous elastic beam vibrating within a viscous medium. Its oscillations consist of temporally damped standing waves as well as steady state traveling waves. Separate standing and traveling wave theories have been held at various times in physiological acoustics, the later being the most prominent. This model's behavior suggests that the theories can be unified. Though the membrane is untapered, a Place principle is shown with the activity concentrated on the stapes side. With taper, the activity is spread, and the sensitivity of pitch discrimination is increased. The model also shows auditory thresholds and provides the dependence of the low and high frequency thresholds on the physical properties of the system.

In all the remaining models, the basilar membrane is treated as a second order system with no elastic coupling between adjacent segments. The most recent of these is the model of Lesser and Berkley [5]. The authors concentrated on the two-dimensional aspects of the velocity field in the cochlear canals and clarified some of the unsolved details in [6]. A Place principle was shown by using an empirical damping function. Incidentally, that function does not agree with von Bekesy's observations (c). A justification for using two-dimensional theory is given in order to describe von Bekesy's eddies (in von Bekesy's model studies, eddies formed near the place of maximum membrane excitation) and for introducing a new length scale necessitated by the rapidly changing elastic properties of the basilar membrane. The reasoning is based on the assumption that the behavior of the membrane can be described by a second order differential equation. By the way, it is still unclear if the eddies occur under physiological sound pressures.

In the wake of von Bekesy's experiments came the work of Peterson and Bogert [7], Fletcher [8], Zwislocki [9], Ranke [6], Flanagan [10] and Hause [11], all of which modeled the basilar membrane as a second order system. It is not practical to discuss here the merits and shortcomings of these papers. All these models reproduced in varying degrees von Bekesy's observations. In general, those with the closer agreement are also the more empirical. These empirical assumptions often are either contrary to what is known or else defy a physical interpretation. Also several authors discuss complicated phenomena in terms of electrical analogies which, it seems to us, obscure the physics of the problem.

Rather than strive for close agreement with inconclusive experimental data, where such agreement is based partly on "matching functions" whose credibility and physical interpretation are in doubt, we opt in favor of a simple yet realistic model having a mathematical description that is tractable and derived from first principles. Such a model will provide a viable qualitative understanding of the relationship between the structure and the function of the cochlea. This is what is really needed at this time.

The model proposed here is based on [4]. It is extended to include the dynamics of an unsteady, viscous incompressible fluid in an idealized cochlea, coupled with the motion of the basilar membrane. Structurally the membrane is modeled as a uniform Euler–Bernoulli beam. When the uniform case is well understood, the nonuniformities may be introduced, and their effect can then be pinpointed.

2. Formulation. An idealized cochlea is shown in Fig. 5. The line segment $y = 0, 0 < x < L$, is the initial position of a thin uniform Euler–Bernoulli beam which represents the basilar membrane. The beam divides the cochlea into two chambers which are filled with a viscous incompressible fluid called the perilymph. The upper chamber corresponds to the scalae vestibuli and media, and the lower to the scala tympani. At $0 \leqq y \leqq h$ and $-h \leqq y \leqq 0$ for $x = 0$, the boundaries are deformable and depict the oval and round windows, respectively. The other boundaries are the rigid cochlear walls. At the apical end ($x = L$) of the basilar membrane, there is a small opening called the helicotrema which allows the fluid to communicate. While the geometry of this passage is not explicitly considered, its major effect of equalizing the pressure in the two chambers is used as a boundary condition.

A detailed analysis of the complex hydrodynamic and elastic interactions between the contents of the scala media and the basilar membrane is not given

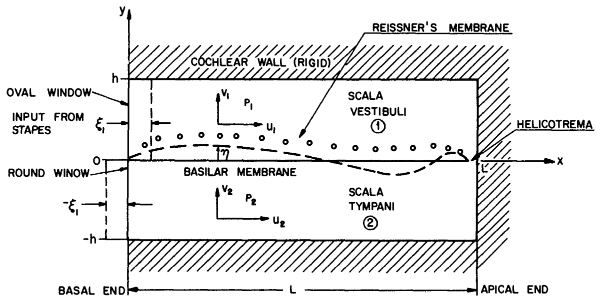

FIG. 5. *Idealized two-dimensional cochlea*

here. These effects are important in the local mechanical stimulation of the sensory hair cells of the organ of Corti, but presumably result in higher order effects on the motion of the basilar membrane. Reissner's membrane, being only two microns thick, can transmit normal pressure, but the Stokes' shear waves which form on its upper surface no doubt will be attenuated by the contents of the scala media before reaching the upper surface of the basilar membrane. To model these effects, and yet maintain mathematical simplicity, the entire scala media and Reissner's membrane will be represented by an idealized surface capable of transmitting normal pressure but not shear from above, and which is contiguous to the upper surface of the basilar membrane. It is further assumed that no relative motion exists between the basilar membrane and this idealized surface. The entire system is driven by the displacement of the stapes of the middle ear acting on the oval window. The determination of the subsequent motion of the perilymphatic fluid and the basilar membrane is the objective of this paper.

The equations of motion (1)–(3), of the fluid in both chambers are the continuity equation and the Navier–Stokes equations for the x- and y-components of momentum. The horizontal and vertical fluid velocities, pressure, density, and kinematic viscosity, are denoted by u, v, p, ρ and ν, respectively:

$$u_x + v_y = 0, \tag{1}$$

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = \nu(u_{xx} + u_{yy}), \tag{2}$$

$$v_t + uv_x + vv_y + \frac{1}{\rho}p_y = \nu(v_{xx} + v_{yy}). \tag{3}$$

These equations are considerably simplified with the following assumptions:

$$\frac{\eta_{\max}}{h} \ll 1, \quad \frac{h}{\lambda} \ll 1. \tag{4}$$

The first condition is a small amplitude approximation. The maximum beam displacement $\eta_{\max}$ is typically $\sim 10^{-7}$ m at normal sound pressures, while the chamber height, $h \sim 10^{-3}$ m [2]. The second condition in (4) is a long wavelength approximation. The ratio $h/L \sim 3 \times 10^{-2}$ in the human cochlea. The characteristic horizontal length scale of motion λ (a typical wavelength of the beam) is of the order of L for the case of uniform beam excited at audio frequencies.

The small amplitude approximation implies that the nonlinear convective terms in (2) and (3) can be neglected, when compared to the unsteady terms. The following gross estimates clearly show this. If T is the oscillatory period of the beam, then

$$\frac{vv_y}{v_t} \sim \frac{(\eta_{\max}/T)(\eta_{\max}/Th)}{\eta_{\max}/T^2} \sim \frac{\eta_{\max}}{h} \ll 1,$$

$$\frac{uu_x}{u_t} \sim \frac{uv_y}{u_t} \sim \frac{u\eta_{\max}/Th}{u/T} \sim \frac{\eta_{\max}}{h} \ll 1.$$

The remaining nonlinear terms can be treated in the same manner.

Actually, all viscous terms except νu_{yy} can also be neglected. To show this, from (4)

$$\frac{v_{xx}}{v_{yy}} \sim \left(\frac{h}{\lambda}\right)^2 \ll 1.$$

Furthermore, it is well known [12, p. 354] that an unsteady boundary layer with thickness $\delta = O(\nu T)^{1/2}$ is formed on the walls. Hence the layer grows thicker as the oscillatory frequency decreases. Taking $\nu \sim 10^{-6}\,\mathrm{m}^2/\mathrm{sec}$ for the cochlear fluid [4], $h \sim 10^{-3}$ m and $T \sim 10^{-2}$ sec (a very low frequency audio signal), then $\delta/h \sim 10^{-1}$. Therefore,

$$\frac{\delta}{h} \ll 1. \tag{5}$$

The additional estimates obtained from (5) show that νu_{yy} is the only viscous term which must be retained. To wit

$$\frac{\nu v_{yy}}{v_t} \sim \frac{\nu T}{h^2} \sim \left(\frac{\delta}{h}\right)^2 \ll 1,$$

$$\frac{\nu u_{yy}}{u_t} \sim \frac{\nu T}{\delta^2} = O(1), \qquad \frac{u_{xx}}{u_{yy}} \sim \left(\frac{\delta}{\lambda}\right)^2 < \left(\frac{\delta}{h}\right)^2 \ll 1.$$

At this point, it is convenient to average (1) and (2) across the chamber height. Let the average velocity and pressure (see Note added in proof) be

$$\left\{\begin{array}{c} \bar{u}_i(x,t) \\ \bar{p}_i(x,t) \end{array}\right\} = \frac{(-1)^{i+1}}{h} \int^{(-1)^{i+1}h} \left\{\begin{array}{c} u_i(x,y,t) \\ p_i(x,y,t) \end{array}\right\} dy, \qquad i = 1,2. \tag{6}$$

where the subscripts 1 and 2 refer to the upper and lower chambers, respectively.

By applying (6) to (1), the averaged continuity equations are obtained

$$\bar{u}_{ix} = \frac{(-1)^{i+1}}{h}\eta_t, \qquad i = 1,2, \tag{7}$$

where the following boundary conditions

$$\begin{aligned} v_i(x,0,t) &= \eta_t, \\ v_i(x,(-1)^{i+1}h,t) &= 0, \end{aligned} \qquad i = 1,2, \tag{8}$$

have been used.

Equation (9)

$$\bar{u}_{it} + \frac{1}{\rho}\bar{p}_{ix} = \frac{\nu}{h}u_{iy}\Big|_0^{(-1)^{i+1}h}(-1)^{i+1}, \qquad i = 1,2, \tag{9}$$

is obtained by averaging (2).

If the driving amplitude of the oval window is small enough and such that

$$\frac{\eta_{\max}}{\delta} \ll 1, \tag{10}$$

the right-hand side of (9) can be simplified. Condition (10) is amply justified by the data of Johnston and Boyle [13], where for a driving frequency, ω, of 1.8 khz and a driving pressure of 90 db re 2×10^{-10} atm, $\eta_{\max} \sim 10^{-7}$ m. For the same ω, the thickness of the boundary layer is approximately 10^{-5} m. Evidently, (10) is satisfied over the entire audio range.

For simplicity, the deflection of the oval window is assumed to be flat so that the magnitude of the velocity in the inviscid core is just $\bar{u}(x,t)$, as in Fig. 6.[2] From (10), the shear stress acting on the beam may be calculated by assuming the beam to be at rest.[3] In such a case, the problem of evaluating the wall shear stresses in

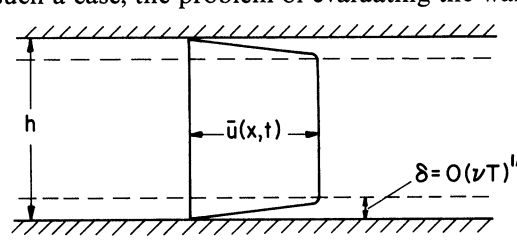

FIG. 6. *Velocity profile in scala*

[2] The referee has pointed out that the assumption of a single moded flat velocity profile is good for the human range of 20 Hz–20 kHz. The frequency range corresponds to the low end of what Scarton and Rouleau [16] call the intermediate frequency range defined by $F_{L-I} \leqq F \leqq F_{I-H}$, where $F_{L-I} = 11.14\pi D$, $F_{I-H} = \pi$, $F = \omega h/c_0$, $D = \nu/hc_0$ and c_0 is the speed of sound in the fluid. Taking $h \sim 10^{-3}$ m, $\nu \sim 10^{-6}$ m²/sec, $c_0 \sim 1.5 \times 10^3$ m/sec gives $\omega_{L-I} \sim 5$ Hz, which corresponds to the transition region between parabolic and flat velocity profile, and $\omega_{I-H} \sim 750$ kHz which is the region where compressibility effects and higher mode velocity profiles exist.

[3] This is plausible in view of Lyne's [15] results. He showed for a viscous oscillating flow past a *rigid* wavy wall, (10) implies that the wall may be considered flat in the first approximation. Whether this applies also for a nonrigid wavy wall is a matter for further investigation and not within the scope of this work. Also, the transient contribution to the shear stress is not considered here.

(9) is reduced to the well-known Stokes' problem of calculating the unsteady boundary layer on a flat plate due to an oscillatory outer flow.

The steady state average velocity can be represented in the form of a traveling wave

$$\bar{u}(x, t) = \bar{U}_0(x) \cos(\omega t + \psi(x)),$$

where $\bar{U}_0(x)$ and $\psi(x)$ are an amplitude and phase function, respectively. The corresponding solution for the wall shear stress is given by [12]:

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \left(\frac{\omega}{\nu}\right)^{1/2} \bar{U}_0(x) \cos\left(\omega t + \psi(x) + \frac{\pi}{4}\right)$$
$$= \left(\frac{\omega}{2\nu}\right)^{1/2} \left[\bar{u}(x, t) + \frac{1}{\omega}\bar{u}_t(x, t)\right].$$

Hence, the wall shear stress has a phase lead of $\pi/4$ over the velocity fluctuations. It follows that

$$(11) \qquad (-1)^{i+1} u_{iy}\Big|_0^{(-1)^{i+1}h} = -\left(\frac{2\omega}{\nu}\right)^{1/2} \left[\bar{u}_i(x, t) + \frac{1}{\omega}\bar{u}_{it}(x, t)\right], \qquad i = 1, 2,$$

and (9) becomes

$$(12) \qquad \left[1 + \left(\frac{2\nu}{\omega h^2}\right)^{1/2}\right]\bar{u}_{it} + \frac{1}{\rho}\bar{p}_{ix} + r\bar{u}_i = 0,$$

where

$$r = \left(\frac{2\omega\nu}{h^2}\right)^{1/2}.$$

But

$$\left(\frac{2\nu}{\omega h^2}\right)^{1/2} \sim \frac{\delta}{h} \ll 1$$

from (5). It is interesting that the phase lead of the shear stress gives rise to an apparent increase of fluid inertia, which can be neglected by the thin boundary-layer approximation.

It is advantageous to define the new variables as in [7]:

$$(13) \qquad \begin{aligned} u^+ &= \tfrac{1}{2}(\bar{u}_1 + \bar{u}_2), \\ u^- &= \tfrac{1}{2}(\bar{u}_1 - \bar{u}_2), \\ p^+ &= \tfrac{1}{2}(\bar{p}_1 + \bar{p}_2), \\ p^- &= \tfrac{1}{2}(\bar{p}_1 - \bar{p}_2). \end{aligned}$$

Substitution of (13) into (7) and (12) (neglecting the apparent increase in fluid inertia) gives the following equations describing the motion of the fluid in both chambers:

$$(14) \qquad u_t^+ + \frac{1}{\rho}p_x^+ + ru^+ = 0,$$

$$u_t^- + \frac{1}{\rho} p_x^- + r u^- = 0, \tag{15}$$

$$u_x^+ = 0, \tag{16}$$

$$u_x^- = \frac{1}{h} \eta_t \tag{17}$$

The five unknown quantities u^+, u^-, p^+, p^- and η are contained in (14)–(17) and the beam equation which is yet to be derived. However, (14) and (16) are uncoupled from the beam equation and they can be trivially solved.

$$u_x^+(x, t) = 0 \Rightarrow u^+(x, t) = f(t),$$
$$u^+(0, t) = \tfrac{1}{2}[\bar{u}_1(0, t) + \bar{u}_2(0, t)] = \tfrac{1}{2}[\xi_1 + \xi_2]_t = 0 = u^+(x, t).$$

Here ξ_1 and ξ_2 are the displacements of the oval and round windows, which must be equal and opposite in sign due to the incompressibility of the fluid. The result

$$u^+(x, t) = 0 \tag{18}$$

implies that the average fluid velocities are equal but in opposite directions on each side of the beam.

From $u^+ = u_t^+ = 0$ in (14) it follows that

$$p_x^+(x, t) = 0 \Rightarrow p^+(x, t) = g(t).$$

This implies that the average pressure in the scalae is constant along the length of the cochlea at a given instant of time. The function $g(t)$ can be determined from a force balance on the oval and round windows. By assuming that the windows have both negligible damping and mass but identical elastic properties, it follows that

$$g(t) = p^+(0, t) = p^+(x, t) = \tfrac{1}{2} p_0' \sin \omega t, \tag{19}$$

where $p_0' \sin \omega t$ is the effective pressure acting on the oval window from the stapes. The same force balance gives another condition which will be needed later, that[4]

$$p^-(0, t) = \left(\tfrac{1}{2} p_0' - \frac{k}{A} \xi_0\right) \sin \omega t = P_0 \sin \omega t. \tag{20}$$

Here k represents the equivalent elastic constant of the windows, A is the area of each window, and the displacement magnitude of the oval window ξ_0 is defined by $\xi_1 = \xi_0 \sin \omega t$.

It remains to develop the equation describing the motion of the beam. The forces and moments acting on an element of the beam of unit depth, thickness d and density ρ_b, are shown in Fig. 7. Equilibrium of forces and moments gives the

[4] The referee has questioned (20) by saying that the effective inertia, elasticity and damping of the tympanic membrane and middle ear ossicles effect the magnitude of $p^-(0, t)$. Indeed, this is true, and, in fact, the coupled middle ear-cochlea system must be solved in order to determine p_0' and ξ_0. What is important here is that even with these complications, the force transmitted by the stapes to the oval window is a simple harmonic function of time. In addition, any possible phase difference between the motions of the tympanic membrane and stapes can be accommodated by a time translation.

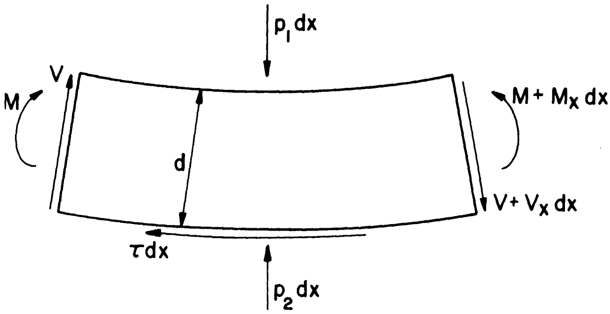

FIG. 7. *Forces on element of basilar membrane*

equation (see Note added in proof)

(21) $$EI\eta_{xxxx} + \rho_b d\eta_{tt} + 2p^- + \tfrac{1}{2}d\tau_x = 0,$$

where τ is the shear stress acting on the beam as a result of the viscosity of the fluid and EI is the flexural rigidity of the beam. However,

(22) $$\tau = \rho\nu u_{2y}(x,0,t) = -\left(\frac{\nu\omega\rho^2}{2}\right)^{1/2}\left[\bar{u}_2 + \frac{1}{\omega}\bar{u}_t\right],$$
$$\tau_x = \left(\frac{\nu\omega\rho^2}{2h^2}\right)^{1/2}\left[\eta_t + \frac{1}{\omega}\eta_{tt}\right],$$

where use has been made of (11) and (7). It can be seen from (22) that the viscous shear force acting on the beam produces a frequency-dependent damping force proportional to the velocity of the beam, as well as an apparent increase in the inertia of the beam. The shear force acting on the lower surface of the beam also contributes the term $-S\eta_{xx}$ on the left-hand side of (21), where S is the local variable tension in the beam and is related to the shear stress by $\tau = -S_x$. This term can be neglected by comparison to the bending term, even at high frequency, as can be seen from the estimate

$$\frac{S\eta_{xx}}{EI\eta_{xxxx}} \sim \frac{\rho\nu(\omega/\nu)^{1/2}\omega\xi_0\lambda^3}{EI} \sim \frac{\rho\nu^{1/2}\omega^{3/2}\xi_0}{E}\left(\frac{\lambda}{d}\right)^3.$$

Taking $\rho \sim 10^3$ kgm/m^3, $\nu \sim 10^{-6}$ m^2/sec, $\lambda \sim 10^{-2}$ m, $d \sim 10^{-4}$ m, $\xi_0 \sim 10^{-6}$ m, $E \sim 10^9$ N/m^2 and $\omega \sim 10^5$ sec^{-1}, the ratio of the terms is $\sim 10^{-2}$. On the other hand, the ratio of the shear induced damping term to the tension term tends to be large for high frequencies (when the damping term is comparable to the bending term):

$$\frac{d\tau_x}{S\eta_{xx}} \sim \frac{dS_{xx}}{S\eta_{xx}} \sim \frac{d}{\eta_{\max}} \gg 1.$$

The inertia term can be written

$$\rho_b d\left[1 + \left(\frac{\rho}{\rho_b}\right)\frac{(2\nu/\omega)^{1/2}}{4h}\right]\eta_{tt},$$

but

$$\left(\frac{\rho}{\rho_b}\right)\frac{(2\nu/\omega)^{1/2}}{4h} \sim \frac{\delta}{h} \ll 1$$

and can be neglected.

From (21) and (22), the final equation of the beam is

$$\eta_{xxxx} + \alpha^2\eta_{tt} + \gamma\eta_t + \beta p^- = 0, \tag{23}$$

where the coefficients are given by

$$\alpha^2 = \rho_b d/EI,$$
$$\beta = 2/EI,$$
$$\gamma = \frac{d\rho}{4EIh}(2\nu\omega)^{1/2}.$$

The solution of (15), (17) and (23) with the appropriate initial and boundary conditions will complete the description of the dynamical behavior of the cochlea.

3. Solution of the initial and boundary value problem. The equations of motion for the model are

$$\eta_{xxxx} + \alpha^2\eta_{tt} + \gamma\eta_t = -\beta p, \tag{23'}$$

$$u_x = \frac{1}{h}\eta_t, \tag{17'}$$

$$u_t = -\frac{1}{\rho}p_x - ru \tag{15'}$$

for $0 < x < L$ and $t > 0$.

Notice that, for the sake of simplicity, the superscript $-$ has been dropped. Unless otherwise specified, u and p stand for the averaged quantities, with negative superscript defined by (13).

The membrane and fluid are initially at rest; that is,

$$\eta(x,0) = \eta_t(x,0) = 0, \tag{24}$$

$$u(x,0) = 0 \tag{25}$$

for $0 \leqq x \leqq L$.

The basilar membrane is assumed to be hinged at both ends resulting in the boundary conditions[5]

$$\begin{aligned} \eta(0,t) &= \eta(L,t) = 0, \\ \eta_{xx}(0,t) &= \eta_{xx}(L,t) = 0. \end{aligned} \tag{26}$$

The pressure equalization by the helicotrema at $x = L$ yields[6]

$$p(L,t) = 0. \tag{27}$$

[5] These end conditions result in static deflections (low frequency behavior) which compare more favorably with the uniform basilar membrane supported along its lengthwise edges, than, say, a beam cantilevered at the stapes and free at the helicotrema.

[6] The authors are grateful to the reviewer for pointing out that a pressure drop may exist at the helicotrema. This is carefully considered in a following paper.

The whole system is driven by

$$p(0, t) = P_0 \sin \omega t. \tag{20'}$$

Conditions (26), (27) and (20′) are valid for $t \geqq 0$.

Three independent 6th order partial differential equations, one for each one of the dependent variables, can be obtained from (23′), (17′) and (15′). The resulting equation for η involves only even derivatives with respect to x. This together with the boundary conditions (26) shows that it is natural to assume a sine series expansion in x for η. Let then

$$\eta(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L}. \tag{28}$$

Notice that (28) satisfies (26).

From (28), (17′) and (15′) employing the initial conditions together with the boundary conditions on p, the expansions

$$u(x, t) = \frac{1}{h} \sum_{n=1}^{\infty} \left(-\frac{L}{n}\right) \dot{f}_n \cos \frac{n\pi x}{L} + \frac{P_0}{\rho L(\omega^2 + r^2)} [\omega e^{-rt} + r \sin \omega t - \omega \cos \omega t] \tag{29}$$

and

$$p(x, t) = \frac{\rho}{h} \sum_{n=1}^{\infty} \frac{L^2}{n^2\pi^2} (\ddot{f}_n + r\dot{f}_n) \sin \frac{n\pi x}{L} + P_0 \left(-\frac{x}{L} + 1\right) \sin \omega t \tag{30}$$

are easily obtained. The dot superscript indicates, of course, differentiation with respect to t.

There remains the determination of the functions $f_n(t)$. If (28) and (30) are substituted into (23′), the left-hand side of (23′), since it involves only terms in η, yields a sine series. The right-hand side, however, yields in addition a term involving $\sin \omega t$ stemming from the corresponding term in (30). It does not seem possible then to match coefficients in this manner. The difficulty is only apparent. There is a subtlety involved due to the fact that (23′) is valid for $0 < x < L$ and $t > 0$ and not necessarily at the endpoints. Equation (30), however, is valid for $0 \leqq x \leqq L$ and $t \geqq 0$ with the "troublesome" $\sin \omega t$ term being due to the boundary condition (20′). The problem is resolved by expanding the $\sin \omega t$ term in (30) into a sine series in x resulting in

$$p^*(x, t) = \frac{\rho}{h} \sum_{n=1}^{\infty} \left\{ \frac{L^2}{n^2\pi^2} (\ddot{f}_n + r\dot{f}_n) + \frac{2P_0 h}{n\pi\rho} \sin \omega t \right\} \sin \frac{n\pi x}{L}, \tag{31}$$

Note that $p^*(0, t) \neq p(0, t)$, but $p^*(x, t) = p(x, t)$, $\forall\, x \in (0, L]$ and $t \geqq 0$.

From (23′), (28) and (31) via some obvious algebraic manipulation, the f_n are determined by:

(32)

$$f_n(t) = e^{-\lambda_n t}(c_{1n} \sin E_n t + c_{2n} \cos E_n t) + d_{1n} \sin \omega t + d_{2n} \cos \omega t, \quad n = 1, 2, \cdots,$$

where

$$
\begin{aligned}
d_{1n} &= (-\omega^2 A_n + C_n)D_n/\Delta_n, \qquad d_{2n} = -\omega B_n D_n/\Delta_n,\\
\Delta_n &= (-\omega^2 A_n + C_n)^2 + \omega^2 B_n^2,\\
A_n &= \frac{L^2\rho}{n^2\pi^2 h} + \frac{\alpha^2}{\beta}, \qquad B_n = \frac{L^2 r\rho}{n^2\pi^2 h} + \frac{\gamma}{\beta},\\
C_n &= \frac{n^4\pi^4}{L^4\beta}, \qquad D_n = -2P_0/n\pi,\\
E_n &= \sqrt{B_n^2 - 4A_nC_n}/2A_n, \qquad \lambda_n = B_n/2A_n,
\end{aligned} \tag{33}
$$

and

$$
c_{1n} = -(\lambda_n d_{2n} + \omega d_{1n})/E_n, \qquad c_{2n} = -d_{2n}. \tag{34}
$$

The constants c_{1n} and c_{2n} were obtained from the conditions

$$
f_n(0) = \dot{f}_n(0) = 0
$$

resulting from (24).

With the f_n as prescribed, (28), (29) and (30) provide a closed form solution for the complete *initial* and boundary problem, a fact which can also be verified by direct substitution. All previous models, with the exception of [4], gave approximate solutions for the boundary but *not* the initial value problem.

An easy application of the Weirstrass M-test shows that the convergence of the series of the solution functions is uniform for $0 \leqq x \leqq L$ and $t \geqq 0$. In fact, the convergence is very rapid (d_{1n} as $1/n^5$ and d_{2n} as $1/n^9$).

The behavior of the solutions as well as the plausible implications to physiological acoustics are presented in a companion paper [14].

Note added in proof. In writing (21), it was assumed that the difference of the *averaged* pressures in the two chambers drives the beam. In more recent work [18], it was shown that using (3), corrections can be calculated to account for the pressure gradients normal to the beam.

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