

MATH 4370/7370 – Linear Algebra and Matrix Analysis

Factorisations, canonical forms and decompositions

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Outline

Unitary matrices and QR factorisation

Schur's Form

Consequences of Schur's triangularisation theorem

Normal Matrices

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Definition 4.1

Let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{C}^n$. We say that $\mathbf{x}_1, \dots, \mathbf{x}_k$ is an **orthogonal list** if $\mathbf{x}_i^* \mathbf{x}_j = 0$ for all $i \neq j$. If, in addition, we have that $\mathbf{x}_i^* \mathbf{x}_i = 1$, then we say that the list is **orthonormal**

Theorem 4.2

Every orthonormal list of vectors in \mathbb{C}^n is linearly independent

Remark 4.3

In Theorem 4.2, if we have “only” orthogonal vectors, we need to replace “list of vectors” by “list of non-zero vectors” in the statement

Definition 4.4

Let $U \in \mathcal{M}_n$, we say that U is an **unitary matrix** if $U^*U = \mathbb{I}$. Furthermore, we say that $U \in \mathcal{M}_n(\mathbb{R})$ is a **(real) orthogonal matrix** if $U^T U = \mathbb{I}$

Theorem 4.5

Let $U \in \mathcal{M}_n$. TFAE:

1. U is unitary
2. U is non-singular and $U^* = U^{-1}$
3. $UU^* = \mathbb{I}$
4. U^* is unitary
5. the columns of U are orthonormal
6. the rows of U are orthonormal
7. for all $\mathbf{x} \in \mathbb{C}^n$ we have $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$

Definition 4.6

A **linear transformation** $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a **Euclidean isometry** if $\|\mathbf{x}\|_2 = \|T\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$

Corollary 4.7

Let $U \in \mathcal{M}_n$. U is a Euclidean isometry if and only if U is unitary

Remark 4.8

Let $U, V \in \mathcal{M}_n$ be unitary matrices (respectively real orthogonal), then UV is unitary (respectively real orthogonal).

Indeed, U, V unitary $\Leftrightarrow U^{-1}, V^{-1}$ exist and $U^{-1} = U^*, V^{-1} = V^*$. Then

$$\begin{aligned} UV \text{ unitary} &\Leftrightarrow (UV)^* UV = \mathbb{I} \\ &\Leftrightarrow V^* U^* UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I} \end{aligned}$$

Notation: $\text{GL}(n, \mathbb{F})$ is the general linear group, where the elements are non-singular matrices in $\mathcal{M}_n(\mathbb{F})$

Theorem 4.9

The set of unitary (respectively real orthogonal) matrices in \mathcal{M}_n forms a group, the $n \times n$ unitary (respectively real orthogonal) subgroup of $GL(n, \mathbb{C})$ (respectively $GL(n, \mathbb{R})$)

Theorem 4.10 (Selection Principle)

Suppose that we have a sequence of unitary matrices $U_1, U_2, \dots \in \mathcal{M}_n$. Then there exists a subsequence U_{k_1}, U_{k_2}, \dots such that the entries of U_{k_i} converge to entries of a unitary matrix as $i \rightarrow \infty$

Lemma 4.11

Let $U \in \mathcal{M}_n$ be a unitary matrix partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with $U_{ii} \in \mathcal{M}_k$. Then $\text{rank} U_{12} = \text{rank} U_{21}$ and $\text{rank} U_{22} = \text{rank} U_{11} + n - 2k$. If, furthermore, $U_{21} = 0$ and $U_{12} = 0$, then U_{11} and U_{22} are unitary

Theorem 4.12 (QR factorisation)

Let $A \in \mathcal{M}_{nm}$

1. If $n \geq m$, there is a $Q \in \mathcal{M}_{nm}$ with orthonormal columns and upper triangular $R \in \mathcal{M}_m$ with non-negative main diagonal entries such that $A = QR$
2. If $\text{rank} A = m$ then the factors Q and R in (1) are uniquely determined and the main diagonal entries of R are all positive
3. If $n = m$, Then the factor Q in (1) is unitary
4. There is a unitary $Q \in \mathcal{M}_n$ and an upper triangular $R \in \mathcal{M}_{nm}$ with nonnegative diagonal entries such that $A = QR$
5. If A is real, then Q and R are in (1), (2), (3), and (4) may be taken to be real

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For a unitary matrix U , $U^* = U^{-1}$, so the transformation $A \mapsto U^*AU$ is a **similarity transformation**, provided that U is unitary. This is a **unitary similarity**

Definition 4.13 (Unitarily similar matrices)

Let $A, B \in \mathcal{M}_n$. We say that A is **unitarily similar** to B if there exists $U \in \mathcal{M}_n$ unitary such that

$$A = U^*BU$$

If U can be taken real (i.e., if U is real orthogonal) then A is real orthogonal similar to B (if $A = U^TBU$)

Remark 4.14

1. *Unitary similarity is an equivalence relation*
2. *Unitary similarity implies similarity. However, the converse is not true*
3. *Similarity is a change of bases. Unitary similarity is a change of orthonormal bases*

Definition 4.15 (Householder matrix)

Let $0 \neq \omega \in \mathbb{C}^n$. The Householder matrix $U_\omega \in \mathcal{M}_n$ is

$$U_\omega = \mathbb{I} - 2(\omega^* \omega)^{-1} \omega \omega^*$$

Remark 4.16

1. *If $\|\omega\| = 1$ then $U_\omega = \mathbb{I} - 2\omega\omega^*$*
2. *Householder matrix are unitary and Hermitian, thus $U_\omega^{-1} = U_\omega$.*
3. *The eigenvalues of a Householder matrix are $-1, 1, \dots, 1$ and $|U_\omega| = 1$*

Theorem 4.17

Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and assume that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 > 0$

- ▶ If $\mathbf{y} = e^{i\theta}\mathbf{x}$ for some $\theta \in \mathbb{R}$ [\mathbf{x}, \mathbf{y} are linearly dependent], define $U(\mathbf{y}, \mathbf{x}) = e^{i\theta}\mathbb{I}$
- ▶ Otherwise, let $\phi \in [0, 2\pi)$ be such that $\mathbf{x}^*\mathbf{y} = e^{i\phi}|\mathbf{x}^*\mathbf{y}|$ (taking $\phi = 0$ if $\mathbf{x}^*\mathbf{y} = 0$). Let $\omega = e^{i\phi}\mathbf{x} - \mathbf{y}$ and define

$$U(\mathbf{y}, \mathbf{x}) = e^{i\phi} U_\omega$$

where $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$ is Householder

1. $U(\mathbf{y}, \mathbf{x})$ unitary and essentially Hermitian
2. $U(\mathbf{y}, \mathbf{x})\mathbf{x} = \mathbf{y}$
3. $U(\mathbf{y}, \mathbf{x})\mathbf{z} \perp \mathbf{y}$, when $\mathbf{z} \perp \mathbf{y}$
4. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $U(\mathbf{y}, \mathbf{x})$ is real and $U(\mathbf{y}, \mathbf{x}) = \mathbb{I}$ if $\mathbf{y} = \mathbf{x}$ and $U(\mathbf{y}, \mathbf{x}) = U_{\mathbf{x}-\mathbf{y}} \in \mathcal{M}_n(\mathbb{R})$ otherwise

Remark 4.18

*For all $A \in \mathcal{M}_n$, $U(y, x)^*AU(y, x) = U_\omega^*AU_\omega$. This is called a Householder transformation.*

Theorem 4.19 (Schur's Form)

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ in any prescribed order (including multiplicities). Let $x \in \mathbb{C}^n$, $\|x\| = 1$, be such that $Ax = \lambda_1 x$

- 1. There exists $U = [x \ u_2 \ \dots \ u_n] \in \mathcal{M}_n$ unitary such that $U^*AU = T$, where T is upper triangular such that $t_{ii} = \lambda_i$, $i = 1, \dots, n$.*
- 2. If $A \in \mathcal{M}_n(\mathbb{R})$ and has real eigenvalues, then x can be chosen to be real and there exists*

$$Q = [x \ q_2 \ \dots \ q_n] \in \mathcal{M}_n(\mathbb{R})$$

real orthogonal and such that $Q^T A Q = T$, with T upper triangular with $t_{ii} = \lambda_i$ $i = 1, \dots, n$.

Theorem 4.20 (Schur version 2)

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ (including multiplicities). Then there exists $U \in \mathcal{M}_n$ such that

$$U^*AU = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

Remark 4.21

The decomposition is not unique

Theorem 4.22

Let $U \in \mathcal{M}_n$, $A, B \in \mathcal{M}_n$. Suppose A is unitarily similar to B , then

$$\sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

Corollary 4.23

Let $A \in \mathcal{M}_n$ have eigenvalues $\lambda_1, \dots, \lambda_n$, $T = UAU^*$ upper triangular. Then

$$\sum_{i=1}^n |\lambda_1|^2 = \sum_{i,j=1}^n |a_{ij}|^2 - \sum_{i < j} |t_{ij}|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2 = \operatorname{tr} AA^*$$

with equality if T is diagonal.

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Theorem 4.24 (Cayley-Hamilton)

Let $A \in \mathcal{M}_n$ and $p_A(t)$ is the characteristic polynomial of A , then $p_A(A) = 0$.

Theorem 4.25 (Sylvester's theorem – pole placement)

Assume $A \in \mathcal{M}_n$ has eigenvalues $\lambda_1, \dots, \lambda_n$ with multiplicities n_1, \dots, n_d ($\sum_{i=1}^d n_i = n$).

Then A is unitary similar to a $d \times d$ block upper triangular matrix T , where $T_{i,j} \in M_{n_i m_j}$, $T_{ij} = 0$ if $i > j$, T_{ii} upper triangular with diagonal λ_i , $T_{ii} = \lambda_i \mathbb{I} + R_i$, R_i strictly upper triangular, and A is similar to a matrix to $\bigoplus_{i=1}^d T_{ii}$ [standard similarity, not unitary]

Theorem 4.26

(Every square matrix is almost diagonalisable) Let $A \in \mathcal{M}_n$ for all $\varepsilon > 0$, there exists $A(\varepsilon)[a_{ij}(\varepsilon)] \in \mathcal{M}$ with distinct eigenvalues such that

$$\sum_{i,j} |a_{ij} - a_{ij}(\varepsilon)|^2 < \varepsilon$$

Theorem 4.27

If $A \in \mathcal{M}_n$ for all $\varepsilon > 0$ there exists $S(\varepsilon) \in \mathcal{M}_n$ non-singular such that

$$S^{-1}(\varepsilon)AS(\varepsilon) = T(\varepsilon),$$

where $T(\varepsilon)$ is upper triangular and $|t_{ij}(\varepsilon)| < \varepsilon$ for all i, j , with $i < j$.

Lemma 4.28

Let $(A_k)_{k \in \mathbb{N}}$ a sequence of matrices such that $\lim_{k \rightarrow \infty} A_k = A$ (entry-wise). Then there exists $k_1 < k_2 < \dots$ and $U_{k_i} \in \mathcal{M}$ such that

1. $T_i = U_{k_i}^* A_{k_i} U_{k_i}$ upper triangular
2. $U + \lim_{i \rightarrow \infty} U_{k_i}$ exists and is unitary
3. $T = U^* A U$ upper triangular
4. $\lim_{i \rightarrow \infty} T_i = T$

Theorem 4.29

Let $(A_k)_{k \in \mathbb{N}}$ a sequence of matrices such that $\lim_{k \rightarrow \infty} A_k = A$ (entry-wise). Then let

$$\lambda(A) = [\lambda_1(A) \quad \dots \quad \lambda_n(A)]^T$$

and

$$\lambda(A_k) = [\lambda_1(A_k) \quad \dots \quad \lambda_n(A_k)]^T$$

be presentations of the eigenvalues of A and A_k . Define

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1, \dots, n\}\}.$$

Then for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N} \setminus \{0\}$ such that

$$\min_{\pi \in S_n} \max_{i=1, \dots} \{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \leq \varepsilon \quad \forall k \geq N(\varepsilon)$$

Recall that if \mathbf{x}, \mathbf{y} are two (column) vectors in \mathbb{F}^n , then $\mathbf{x}\mathbf{y}^*$ is a rank 1 matrix in $\mathcal{M}_n(\mathbb{F})$. (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly

Theorem 4.30 (Brauer)

Suppose $A \in \mathcal{M}_n$ has eigenvalues $\lambda, \lambda_2, \dots, \lambda_n$. Let \mathbf{x} be an eigenvector associated to λ . Then for every vector $\mathbf{v} \in \mathbb{C}^n$, the eigenvalues of $A + \mathbf{x}^\mathbf{v}$ are $\lambda + \mathbf{v}^*\mathbf{x}, \lambda_2, \dots, \lambda_n$.*

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Definition 4.31 (Normal matrix)

A matrix $A \in \mathcal{M}_n$ is **normal** if $AA^* = A^*A$

All unitary, Hermitian or skew-Hermitian and diagonal matrices are normal

Theorem 4.32

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. TFAE:

1. A is normal
2. A is unitary diagonalisable
3. $\sum_{i,j} |a_{i,j}|^2 = \sum_i |\lambda_i|^2$
4. A has n orthogonal eigenvectors

Theorem 4.33

Let $A \in \mathcal{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

Then

1. $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
2. A is unitary diagonalisable
3. there exists $U \in \mathcal{M}_n$ such that $A = U\Lambda U^*$

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Definition 4.34

A **Jordan block** $J_k(\lambda)$ is a $k \times k$ upper triangular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

Theorem 4.35

Let $A \in \mathcal{M}_n$ then there exists $S \in \mathcal{M}_n$ non-singular such that

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^k J_{n_i}(\lambda_i) S^{-1}$$

Theorem 4.36

Let $A \in \mathcal{M}_n$ with real eigenvalues. Then there exists a basis of generalised eigenvectors for \mathbb{R}^n , and if $\{v_1, \dots, v_n\}$ is a basis of generalised eigenvectors of \mathbb{R}^n , then $P = [v_1 \ \dots \ v_n]$ is non-singular and $A = D + N$ where $P^{-1}DP = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $N = A - D$ is nilpotent¹ of order $k \leq n$, and D and N commute.

References I