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# Essentially nonnegative matrices and M-matrices

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# Outline

Essentially nonnegative matrices

Z-matrices

Class $K_0$

M-matrices

The Perron-Frobenius can be applied not only to nonnegative matrices, but also to matrices that are *essentially nonnegative*, in the sense that they are nonnegative except perhaps along the main diagonal

### Definition 7.1

A matrix $A \in \mathcal{M}_n$ is **essentially nonnegative** (or **quasi-positive**) if there exist $\alpha \in \mathbb{R}$ such that $A + \alpha\mathbb{I} \geq 0$

### Remark 7.2

*An essentially nonnegative matrix A has non-negative off-diagonal entries. The sign of the diagonal entries is not relevant*

### Remark 7.3

*Irreducibility of a matrix is not affected by the nature of its diagonal entries. Indeed, consider an essentially nonnegative matrix $A$. The existence of a directed path in $G(A)$ does not depend on the existence of "self-loops". The same is not true of primitive matrices, where the presence of negative entries on the main diagonal has an influence on the values of $A^k$ and thus ultimately, on the capacity to find $k$ such that $A^k > 0$*

So we can apply the "weak" versions of the Perron-Frobenius Theorem (the imprimitive cases in Theorem **??**) to $A + \alpha\mathbb{I}$, which is a nonnegative matrix (potentially irreducible). One important ingredient is a result that was proved as Theorem **??**. Namely, that perturbations of the entire diagonal by the same scalar lead to a shift of the spectrum; this is summarised as

$$\sigma(A + \alpha\mathbb{I}) = \{\lambda_1 + \alpha, \ldots, \lambda_n + \alpha, \quad \lambda_i \in \sigma(A)\}$$

### Definition 7.4 (Spectral abscissa)

Let $A \in \mathcal{M}_n$. The **spectral abscissa** of $A$, $s(A)$, is

$$s(A) = \max\{\mathrm{Re}(\lambda), \lambda \in \sigma(A)\}$$

### Theorem 7.5

*Let $A \in \mathcal{M}_n(\mathbb{R})$ be essentially nonnegative. Then $s(A)$ is an eigenvalue of $A$ and is associated to a nonnegative eigenvector. If, additionally, $A$ is irreducible, then $s(A)$ is simple and is associated to a positive eigenvector*

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## Definition 7.6

A matrix is of class $Z_n$ if it is in $\mathcal{M}_n(\mathbb{R})$ and such that $a_{i,j} \leq 0$, $i \neq j$, $i,j = 1, \ldots, n$

$$Z_n = \{A \in \mathcal{M}_n : a_{i,j} \leq 0, i \neq j\}$$

We also say that $A \in Z_n$ has the **$Z$-sign pattern**

## Theorem 7.7 ([?])

*Let $A \in Z_n$. TFAE*

1. *There is a nonnegative vector $x$ such that $Ax > 0$*
2. *There is a positive vector $x$ such that $Ax > 0$*
3. *There is a diagonal matrix* $\mathrm{diag}(D) > 0$ *such that the entries in $AD = [w_{ik}]$ are such that*
$$w_{ii} > \sum_{k \neq i} |w_{ik}| \forall i$$
4. *For any $B \in Z_n$ such that $A \geq A$, then $B$ is nonsingular*
5. *Every real eigenvalue of any principal submatrix of $A$ is positive.*
6. *All principal minors of $A$ are positive*

## Theorem 7.7 (Continued)

7. *For all* $k = 1, \ldots, n$*, the sum of all principal minors is positive*
8. *Every real eigenvalue of* $A$ *is positive*
9. *There exists a matrix* $C \geq 0$ *and a number* $k > \rho(A)$ *such that* $A = k\mathbb{I} - C$
10. *There exists a splitting* $A = P - Q$ *of the matrix* $A$ *such that* $P^{-1} \geq 0$*,* $Q \geq 0$*, and* $\rho(P^{-1}Q < 1)$
11. $A$ *is nonsingular and* $A^{-1} \geq 0$
12. ...

18 *The real part of any eigenvalue of* $A$ *is positive*

**Notation**: $A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class $K$ (or a nonsingular $M$-matrix).

### Theorem 7.8

*Let $A \in Z = \bigcap_{i=1,\ldots} Z_n$ be symmetric. Then $A \in K$ if and only if $A$ is positive define.*

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## Theorem 7.9

*Let $A \in Z_n$. TFAE*

1. $A + \varepsilon\mathbb{I} \in K$ *for all* $\varepsilon > 0$
2. *Every real eigenvalue of a principal submatrix of A is nonnegative*
3. *All principal minors of A are nonnegative*
4. *The sum of all principal minors of order* $k = 1, \ldots, n$ *is nonnegative*
5. *Every real eigenvaue of A is nonegative*
6. *There exists* $C \geq 0$ *and* $k \geq \rho(C)$ *such that* $A = k\mathbb{I} - C$
7. *Every eigenvalue of A has nonnegative real part*

$A \in Z_n$ *such that any of these properties holds is a matrix of class* $K_0$

## Theorem 7.10

*Let* $A \in Z_n$. *Assume* $A \in K_0$. *Then* $A \in K \iff A$ *nonsingular*

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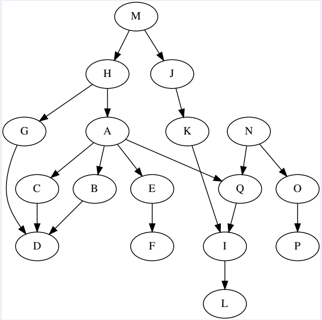
### Definition 7.11 (Signature matrix)

A **signature matrix** is is a diagonal matrix $S$ with diagonal entries $\pm 1$

## Theorem 7.12 ([?])

*Let $A \in \mathcal{M}_n$. Then for each fixed letter $\mathcal{C}$ representing one of the following conditions, conditions $\mathcal{C}_i$ are equivalent for each $i$. Moreover, letting $\mathcal{C}$ then represent any of the equivalent conditions $\mathcal{C}_i$, the following implication tree holds:*




*If $A \in Z_n$, each of the following conditions is equivalent to the statement "A is a nonsingular M-matrix"*

## Theorem 7.12 (Continued)

($A_1$) *All the principal minors of $A$ are positive*

($A_2$) *Every real eigenvalue of each principal submatrix of $A$ is positive*

($A_3$) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a positive diagonal matrix $D$ such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

($A_4$) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a nonnegative diagonal matrix $D$ such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

($A_5$) *$A$ does not reverse the sign of any vector; that is, if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then for some subscript $i$, $x_i y_i > 0$*

($A_6$) *For each signature matrix $S$, there exists an $\mathbf{x} \gg \mathbf{0}$ such that*

$$SAS\mathbf{x} \gg \mathbf{0}$$

## Theorem 7.12 (Continued)

($B_7$) *The sum of all the* $k \times k$ *principal minors of* $A$ *is positive for* $k = 1, \ldots, n$

($C_8$) $A$ *is nonsingular and all the principal minors of* $A$ *are nonnegative*

($C_9$) $A$ *is nonsingular and every real eigenvalue of each principal submatrix of* $A$ *is nonnegative*

($C_{10}$) $A$ *is nonsingular and* $A + D$ *is nonsingular for each positive diagonal matrix* $D$

($C_{11}$) $A + D$ *is nonsingular for each nonnegative diagonal matrix* $D$

($C_{12}$) $A$ *is nonsingular and for each* $\mathbf{x} \neq \mathbf{0}$ *there exists a nonnegative diagonal matrix* $D$ *such that*

$$\mathbf{x}^T D\mathbf{x} \neq 0 \quad \textit{and} \quad \mathbf{x}^T AD\mathbf{x} > 0$$

($C_{13}$) $A$ *is nonsingular and if* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{y} = A\mathbf{x}$*, then for some subscript* $i$*,* $x_i \neq 0$ *and* $x_i y_i \geq 0$*.*

($C_{14}$) $A$ *is nonsingular and for each signature matrix* $S$ *there exists a vector* $\mathbf{x} > \mathbf{0}$ *such that*

$$SAS\mathbf{x} \geq \mathbf{0}$$

## Theorem 7.12 (Continued)

($D_{15}$) $A + \alpha\mathbb{I}$ *is nonsingular for each* $\alpha \geq 0$

($D_{16}$) *Every real eigenvalue of* $A$ *is positive*

($E_{17}$) *All the leading principal minors of* $A$ *are positive*

($E_{18}$) *There exists lower and upper triangular matrices* $L$ *and* $U$*, respectively, with positive diagonals such that*

$$A = LU$$

($F_{19}$) *There exists a permutation matrix* $P$ *such that* $PAP^T$ *satisfies* ($E_{17}$) *or* ($E_{18}$)

## Theorem 7.12 (Continued)

($G_{20}$) *A is* **positive stable***; that is, the real part of each eigenvalue of A is positive*

($G_{21}$) *There exists a symmetric positive definite matrix W such that*

$$AW + WA^T$$

*is positive definite.*

($G_{22}$) $A + \mathbb{I}$ *is nonsingular and*

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

*is convergent*

## Theorem 7.12 (Continued)

($G_{23}$) *$A + \mathbb{I}$ is nonsingular and for*

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

*there exists a positive definite matrix $W$ such that*

$$W - G^T WG$$

*is positive definite*

## Theorem 7.12 (Continued)

($H_{24}$) *There exists a positive diagonal matrix $D$ such that*

$$AD + DA^T$$

*is positive definite*

($H_{25}$) *The exists a positive diagonal matrix $E$ such that for $B = E^{-1}AE$, the matrix*

$$(B + B^T)/2$$

*is positive definite*

($H_{26}$) *For each positive semidefinite matrix $Q$, the matrix $QA$ has a positive diagonal element*

## Theorem 7.12 (Continued)

($I_{27}$) *A is* **semipositive**; *that is, there exists* $\mathbf{x} \gg \mathbf{0}$ *with* $A\mathbf{x} \gg \mathbf{0}$

($I_{28}$) *There exists* $\mathbf{x} > \mathbf{0}$ *with* $A\mathbf{x} \gg \mathbf{0}$

($I_{29}$) *There exists a positive diagonal matrix* $D$ *such that* $AD$ *has all positive row sums*

($J_{30}$) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $A\mathbf{x} > \mathbf{0}$ *and*

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \quad i = 1, \ldots, n$$

($K_{31}$) *There exists a permutation matrix* $P$ *such that* $PAP^T$ *satisfies* ($J_{30}$)

## Theorem 7.12 (Continued)

($L_{32}$) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ *such that if* $y_{i_0} = 0$, *then there exists a sequence of indices* $i_1, \ldots, i_r$ *with* $a_{i_{j-1}i_j} \neq 0$, $j = 1, \ldots, r$ *and with* $y_{i_r} \neq 0$

($L_{33}$) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ *such that the matrix* $\hat{A} = [\hat{a}_{ij}]$ *defined by*

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \text{ or } y_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

*is irreducible*

## Theorem 7.12 (Continued)

$(M_{34})$ *There exists* $\mathbf{x} \gg \mathbf{0}$ *such that for each signature matrix* $S$

$$SAS\mathbf{x} \gg \mathbf{0}$$

$(M_{35})$ *$A$ has all positive diagonal elements and there exists a positive diagonal matrix $D$ such that $AD$ is* **strictly diagonally dominant**; *that is*

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}d_j|, \qquad i = 1, \dots, n$$

$(M_{36})$ *$A$ has all positive diagonal elements and there exists a positive diagonal matrix $E$ such that $E^{-1}AE$ is strictly diagonally dominant*

## Theorem 7.12 (Continued)

$(M_{37})$ *A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is* **lower semistrictly diagonally dominant**; *that is,*

$$a_{ii}d_i \geq \sum_{j \neq i} |a_{ij}d_j|, \qquad i = 1, \ldots, n$$

*and*

$$a_{ii}d_i > \sum_{j=1}^{i-1} |a_{ij}d_j|, \qquad i = 2, \ldots, n.$$

## Theorem 7.12 (Continued)

($N_{38}$) *A is* **inverse-positive**; *that is,* $A^{-1}$ *exists and*

$$A^{-1} \geq 0$$

($N_{39}$) *A is* **monotone**; *that is,*

$$Ax \geq 0 \Rightarrow x \geq 0 \quad \textit{for all } x \in \mathbb{R}^n$$

($N_{40}$) *There exists inverse-positive matrices* $B_1$ *and* $B_2$ *such that*

$$B_1 \leq A \leq B_2$$

($N_{41}$) *There exists an inverse-positive matrix* $B \geq A$ *such that* $I - B^{-1}A$ *is convergent*

($N_{42}$) *There exists an inverse-positive matrix* $B \geq A$ *and A satisfies* ($I_{27}$), ($I_{28}$) *and* ($I_{29}$)

## Theorem 7.12 (Continued)

($N_{43}$) *There exists an inverse-positive matrix $B \geq A$ and a nonsingular M-matrix $C$ such that*

$$A = BC$$

($N_{44}$) *There exists an inverse-positive matrix $B$ and a nonsingular M-matrix $C$ such that*

$$A = BC$$

($N_{45}$) *$A$ has a* **convergent regular splitting**; *that is, $A$ has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad N \geq 0$$

*where $M^{-1}N$ is convergent*

## Theorem 7.12 (Continued)

($N_{46}$) *A has a* **convergent weak regular splitting***; that is, A has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad M^{-1}N \geq 0$$

*where $M^{-1}N$ is convergent*

($O_{47}$) *Each weak regular splitting of A is convergent*

($P_{48}$) *Every regular splitting of A is convergent*

($Q_{49}$) *For each $\mathbf{y} \geq \mathbf{0}$ the set*

$$S_{\mathbf{y}} = \{\mathbf{x} \geq \mathbf{0} : A^T\mathbf{x} \leq \mathbf{y}\}$$

*is bounded and A is nonsingular*

($Q_{50}$) $S_{\mathbf{0}} = \{\mathbf{0}\}$*; that is, the inequalities $A^b x \leq \mathbf{0}$ and $\mathbf{x} \geq \mathbf{0}$ have only the trivial solution $\mathbf{x} = \mathbf{0}$ and A is nonsingular*

# References I