

MATH 4370/7370 – Linear Algebra and Matrix Analysis

Norms and Matrix Norms

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Fall 2023



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of Manitoba**

Outline

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Definition 1 (Norm)

Let V be a vector space over a field \mathbb{F} . A function $\|\cdot\| : V \rightarrow \mathbb{R}_+$ is a **norm** if for all $\mathbf{x}, \mathbf{y} \in V$ and for all $c \in \mathbb{F}$

1. $\|\mathbf{x}\| \geq 0$ [Nonnegativity]
2. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$ [Positivity]
3. $\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$ [Homogeneity]
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ [Triangle Inequality]

Remark 2

*If we have 1, 3, and 4 but not 2, then we have a **seminorm***

Definition 3 (Inner product)

Let V be a vector space over \mathbb{F} . A function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ is an **inner product** if for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and all $c \in \mathbb{F}$

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
3. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
4. $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

Theorem 4 (Cauchy-Schwartz)

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V over \mathbb{F} , then

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$$

Corollary 5

If $\langle \cdot, \cdot \rangle$ is an inner product on a real or complex vector space V , then $\| \cdot \| : V \rightarrow \mathbb{R}_+$ defined by $\| \mathbf{x} \| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is a norm on V

Remark 6

If $\langle \cdot, \cdot \rangle$ is a semi-inner product, then the resulting $\| \mathbf{x} \| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is a seminorm

Theorem 7

Consider the norm $\|\cdot\|$. Then $\|\cdot\|$ is derived from an inner product if and only if it satisfies the parallelogram identity

$$\frac{1}{2} (\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Theorem 8

If $\|\cdot\|$ is a norm on \mathbb{C}^n and a matrix $T \in \mathcal{M}_n$ which is non-singular. Then

$$\|\mathbf{x}\|_T = \|T\mathbf{x}\|$$

is also a norm on \mathbb{C}^n

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Definition 9

Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Take a norm $\|\cdot\|$ on V . The sequence $\{\mathbf{x}^{(k)}\}$ of vectors in V converges to $\mathbf{x} \in V$ with respect to the norm $\|\cdot\|$ if and only if $\|\mathbf{x}^{(k)} - \mathbf{x}\| \rightarrow 0$ as $k \rightarrow \infty$

We write $\lim_{k \rightarrow \infty} \mathbf{x}^{(k)} = \mathbf{x}$ with respect to $\|\cdot\|$ or

$$\mathbf{x}^{(k)} \xrightarrow{\|\cdot\|} \mathbf{x}$$

Theorem 10

Every (vector) norm in \mathbb{C}^n is uniformly continuous

Corollary 11

Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two norms on a finite-dimensional vector space V . Then there exist $C_m, C_r > 0$ such that

$$C_m \|\mathbf{x}\|_\alpha \leq \|\mathbf{x}\|_\beta \leq C_r \|\mathbf{x}\|_\alpha, \forall \mathbf{x} \in V$$

Corollary 12

Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ norms on a finite-dimensional vector space V over \mathbb{R} or \mathbb{C} , $\{\mathbf{x}^{(k)}\}$ a given sequence in V , then

$$\mathbf{x}^{(k)} \xrightarrow{\|\cdot\|_\alpha} \mathbf{x} \iff \mathbf{x}^{(k)} \xrightarrow{\|\cdot\|_\beta} \mathbf{x}$$

Definition 13 (Equivalent norms)

Two norms are **equivalent** if whenever a sequence $\{\mathbf{x}^{(k)}\}$ converges to \mathbf{x} with respect to one of the norm, it converges to \mathbf{x} in the other norm

Theorem 14

In finite-dimensional vector spaces, all norm are equivalent

Definition 15 (Dual norm)

Let f be a pre-norm on $V = \mathbb{R}^n$ or \mathbb{C}^n . The function

$$f_d = (\mathbf{y}) \max_{f(\mathbf{x})=1} \operatorname{Re} \mathbf{y}^* \mathbf{x}$$

is the **dual norm** of f

Remark 16

The dual norm is well defined. $\operatorname{Re} \mathbf{y}^ \mathbf{x}$ is a continuous function for all $\mathbf{y} \in V$ fixed. The set $\{f(\mathbf{x}) = 1\}$ is compact*

Equivalent definition for dual norm: $f^D(\mathbf{y}) = \max_{f(\mathbf{x})=1} |\mathbf{y}^* \mathbf{x}|$

Lemma 17 (Extension of Cauchy-Schwartz)

Let f be a prenorm on $V = \mathbb{R}^n$ or \mathbb{C}^n for all $\mathbf{x}, \mathbf{y} \in V$. Then

$$|\mathbf{y}^* \mathbf{x}| \leq f(\mathbf{x}) f^D(\mathbf{y})$$

$$|\mathbf{y}^* \mathbf{x}| \leq f^D(\mathbf{x}) f(\mathbf{x})$$

Remark 18

- ▶ The dual norm of a pre-norm is a norm
- ▶ The only norm that equals its dual norm is the Euclidean norm

Theorem 19

Let $\|\cdot\|$ be a norm on \mathbb{C}^n or \mathbb{R}^n , and $\|\cdot\|^D$ its dual, $c > 0$ given. Then for all $\mathbf{x} \in V$, $\|\mathbf{x}\| = c\|\mathbf{x}\|^d \iff \|\cdot\| = \sqrt{c}\|\cdot\|^d$. In particular, $\|\cdot\| = \|\cdot\|^2 \iff \|\cdot\| = \|\cdot\|_2$

Definition 20

Let $x \in \mathbb{F}^n$. Denote $|x| = [|x_i|]$ ($|\cdot|$ entry-wise), and write that $|x| \leq |y|$ if $|x_i| \leq |y_i|$ for all $i = 1, \dots, n$. Assume $\|\cdot\|$ is

1. monotone if $|x| \leq |y| \implies \|\mathbf{x}\| \leq \|\mathbf{y}\|$ for all \mathbf{x}, \mathbf{y}
2. absolute if $\|\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in V$

Theorem 21

Let $\|\cdot\|$ be a norm on \mathbb{F}^n . Then

1. If $\|\cdot\|$ is absolute, then

$$\|\mathbf{y}\|^D = \max_{\mathbf{x} \neq 0} \frac{|\mathbf{y}|^T \mathbf{x}|}{\|\mathbf{x}\|}$$

for all $\mathbf{y} \in V$

2. If $\|\cdot\|$ is absolute, then $\|\cdot\|^D$ is absolute and monotone
3. $\|\cdot\|$ is absolute if and only if $\|\cdot\|$

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Definition 22 (Matrix norm)

Let $\|\cdot\|$ be a function from $\mathcal{M}_n \rightarrow \mathbb{R}$. $\|\cdot\|$ is a **matrix norm** if for all $A, B \in \mathcal{M}_n$ and $c \in \mathbb{C}$, it satisfies the following

1. $\|A\| \geq 0$ [nonnegativity]
2. $\|A\| = 0 \iff A = 0$ [positivity]
3. $\|cA\| = |c| \|A\|$ [homogeneity]
4. $\|A + B\| \leq \|A\| + \|B\|$ [triangle inequality]
5. $\|AB\| \leq \|A\| \|B\|$ [submultiplicativity]

Remark 23

As with vector norms, if property 2 does not hold, $\|\cdot\|$ is a **matrix semi-norm**

Remark 24

$\|A^2\| = \|AA\| \leq \|A\|^2$ [for any matrix norm].

If $A^2 = A$, then

$$\|A^2\| = \|A\| \leq \|A\|^2 \implies \|A\| \geq 1.$$

In particular, $\|I\| \geq 1$ for any matrix norm.

Assume that A is invertible, then $AA^{-1} = I$, thus

$$\|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| \tag{1}$$

$$\|A^{-1}\| \geq \frac{\|I\|}{\|A\|} \tag{2}$$

Definition 25 (Induced matrix norm)

Let $\|\cdot\|$ be a norm on \mathbb{C}^n . Define $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{C})$ by

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Then $\|\cdot\|$ is the **matrix norm induced** by $\|\cdot\|$

Theorem 26

The function $\|\cdot\|$ defined in Definition 25 has the following properties

1. $\|\mathbb{I}\| = 1$
2. $\|Ay\| \leq \|A\| \|y\|$ for all $A \in \mathcal{M}_n(\mathbb{C})$ and all $y \in \mathbb{C}^n$
3. $\|\cdot\|$ is a matrix norm on $\mathcal{M}_n(\mathbb{C})$.
4. $\|A\| = \max_{\|x\|=\|y\|^D} |y^* Ax|$

Definition 27 (Induced norm/Operator norm)

$\|\cdot\|$ defined from $\|\cdot\|$ by any of the previous methods is the matrix norm induced by $\|\cdot\|$. It is also called the **operator norm**

Definition 28 (Unital norm)

A norm such that $\|\mathbb{I}\| = 1$ is **unital**

Remark 29

Every induced matrix norm is unital. Every induced norm is a matrix norm

Proposition 30

For all U, V unitary matrices, we have $\|UAV\|_2 = \|A\|_2$

Theorem 31

Let $\|\cdot\|$ be a matrix norm in \mathcal{M}_n and let $S \in \mathcal{M}_n$ be nonsingular. Then for all $A \in \mathcal{M}_n$, $\|A\|_S = \|SAS^{-1}\|$ is a matrix norm. Furthermore, if $\|\cdot\|$ on \mathbb{C}^n , then $\|\mathbf{x}\|_S = \|S\mathbf{x}\|$ induces $\|\cdot\|_S$ on \mathcal{M}_n

Theorem 32

Let $\|\cdot\|$ be a matrix norm on \mathcal{M}_n , $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. Then

1. $|\lambda| \leq \rho(A) \leq \|A\|$
2. *If A is nonsingular, then*

$$\rho(A) \geq |\lambda| \geq \frac{1}{\|A^{-1}\|}$$

Lemma 33

Let $A \in \mathcal{M}_n$. If there exists a norm $\|\cdot\|$ on \mathcal{M}_n such that $\|A\| < 1$, then $\lim_{k \rightarrow \infty} A^k = 0$ entry-wise

Remark 34

When $\|A\| < 1$ for some norm, we say that A is **convergent**

Theorem 35

Let $A \in \mathcal{M}_n$, then

$$\lim_{k \rightarrow \infty} A^k = 0 \iff \rho(A) < 1$$

Theorem 36 (Gelfand Formula)

Let $\|\cdot\|$ be a matrix norm on \mathcal{M}_n , let $A \in \mathcal{M}_n$. Then

$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$$

Theorem 37

Let R be the radius of convergence of the (scalar) power series $\sum_{k=0}^{\infty} a_k z^k$ and $A \in \mathcal{M}_n$.

Then the matrix power series $\sum_{k=1}^{\infty} a_k A^k$ converges if $\rho(A) < R$

Remark 38

The convergence condition for the matrix power series is “there exists a matrix norm $\|\cdot\|$ such that $\|A\| < R$ ”

Corollary 39

Let $A \in \mathcal{M}_n$ be nonsingular, if there $\|\cdot\|$ matrix norm such that $\|\mathbb{I} - A\| \leq 1$

Corollary 40

Let $A \in \mathcal{M}_n$ is such that $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all $i = 1, \dots, n$. Then A is invertible

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Let $V = \mathcal{M}_{mn}(\mathbb{C})$ with Frobenius inner product

$$\langle A, B \rangle_F = \text{tr}(B^* A)$$

The norm derived from the Frobenius inner product is

$$\|A\|_2 = (\text{tr}(A^* A))^{1/2}$$

is the ℓ -2 norm (or Frobenius norm)

The spectral norm $\|\cdot\|$ defined on \mathcal{M}_n by

$$\|A\|_2 = \sigma_1(A),$$

where $\sigma_1(A)$ is the largest singular value of A is induced by the ℓ_2 norm on \mathbb{C}^n .
Indeed, from the singular value decomposition theorem, let

$$A = V\Sigma W^*$$

be a singular value decomposition of A , where V, W unitary, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ and $\sigma_1 \geq \dots \geq \sigma_n \geq 0$ are the non-increasingly ordered singular values of A

From unitary invariance and monotonicity of the Euclidean norm, we say that

$$\begin{aligned}\max_{\|x\|_1} \|Ax\|_1 &= \max_{\|x\|_1} \|V\Sigma W^*\|_2 \\ &= \max_{\|x\|_2} \|\Sigma W^*x\|_2 \\ &= \max_{\|Wy\|_2=1} \|\Sigma y\|_2 \\ &= \max_{\|y\|_2} \|\Sigma y\|_2 \\ &\leq \max_{\|y\|_2} \|\sigma_1 y\|_2 \\ &= \sigma_1 \max_{\|y\|_2} \|y\|_2 \\ &= \sigma_1\end{aligned}$$

Since $\|\Sigma y\|_2 = \sigma_1$ for $y = e_1$,

$$\max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$$

We could have used

$$\begin{aligned}\max_{\|x\|_2=1} \|Ax\|_2^2 &= \max_{\|x\|_2=1} x^* A^* A x \\ &= \lambda_{\max}(A^* A) \\ &= \sigma_1(A)\end{aligned}$$

Remark 41

For all U, V unitary \mathcal{M}_n matrices, for all $A \in \mathcal{M}_n$, $\|UAV\|_2 = \|A\|_2$