

# MATH 4370/7370 – Linear Algebra and Matrix Analysis

Eigenvalues, eigenvectors, similarity and Geršgorin disks

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# Outline

Eigenpairs

Characteristic equation and algebraic multiplicity

Similarity

Left and right eigenvectors, geometric multiplicity

The Geršgorin Theorem

Extensions of Geršgorin disks using graph theory

## Eigenpairs

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## Definition 1

Let  $A \in \mathcal{M}_n(\mathbb{F})$ . If  $\lambda \in \mathbb{C}$  and  $\mathbf{v} \neq \mathbf{0} \in \mathbb{F}^n$  are such that  $A\mathbf{v} = \lambda\mathbf{v}$ , then  $\lambda$  is an **eigenvalue** of  $A$  associated to the **eigenvector**  $\mathbf{v}$ . We also say that  $(\lambda, \mathbf{v})$  form an **eigenpair**.

The eigenpair equation takes the form  $A\mathbf{v} = \lambda\mathbf{v}$ , for  $\mathbf{v} \neq \mathbf{0}$ . Rewriting this,

$$A\mathbf{v} = \lambda\mathbf{v} \iff A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \iff A\mathbf{v} - \lambda\mathbb{I}\mathbf{v} = \mathbf{0} \iff (A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$$

(We could also have obtained  $(\lambda\mathbb{I} - A)\mathbf{v} = \mathbf{0}$ )

Hence, since we seek  $\mathbf{v} \neq \mathbf{0}$ , the homogeneous system  $(A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$  must have non-trivial solutions; this implies that  $A - \lambda\mathbb{I}$  must be *singular*. So, if  $\lambda$  is an eigenvalue, there must hold that  $\det(A - \lambda\mathbb{I}) = 0$

## Remark 2

*It is essential to remember that one seeks a nonzero vector  $\mathbf{v}$ . Clearly, if  $\mathbf{v} = \mathbf{0}$ , then  $A\mathbf{v} = \lambda\mathbf{v}$  for any  $\lambda$ , since this just means that  $\mathbf{0} = \mathbf{0}$*

Often, we use normalised eigenvectors,  $\tilde{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$ , so that  $\|\tilde{\mathbf{v}}\| = 1$

Also, for eigenvectors  $\mathbf{v}$  that have all their components nonpositive, we typically use  $-\mathbf{v}$ , so that all components are nonnegative.

### Definition 3 (Spectrum of a matrix)

The **spectrum** of  $A \in \mathcal{M}_n$  is the set of all its eigenvalues and its denoted  $\sigma(A)$

### Theorem 4

$0 \in \sigma(A) \iff A \text{ is singular}$

### Theorem 5

$A \in \mathcal{M}_n(\mathbb{F})$ ,  $\lambda, \mu \in \mathbb{C}$  given. Then  $\lambda \in \sigma(A)$  if and only if  $\lambda + \mu \in \sigma(A + \mu\mathbb{I})$

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## Definition 6 (Characteristic polynomial/equation)

The **characteristic polynomial** of  $A \in \mathcal{M}_n$  is

$$p_A(z) = \det(A - z\mathbb{I}).$$

The **characteristic equation** of  $A$  is  $p_A(z) = 0$

By the Fundamental Theorem of Algebra, if  $p_A(z)$  has degree  $n$ , then  $p_A(z)$  has  $n$  complex roots including multiplicity (or at most  $n$  roots if not counting multiplicity)

These roots are the eigenvalues of  $A$  and thus  $\sigma(A)$  has at most  $n$  elements in  $\mathbb{C}$

## Theorem 7

*Let  $A \in \mathcal{M}_n$ . Then*

$$\operatorname{tr}(A) = \sum_{i=1}^n \lambda_i \quad \text{and} \quad \det(A) = \prod_{i=1}^n \lambda_i$$

## Theorem 8

*Let  $p(T)$  be a  $k$ -degree polynomial. If  $(\lambda, \mathbf{v})$  eigenpair of  $A \in \mathcal{M}_n$ , then  $(p(\lambda), \mathbf{v})$  is an eigenpair for  $p(A)$*

## Definition 9 (Algebraic multiplicity of an eigenvalue)

Let  $A \in \mathcal{M}_n$ . The **(algebraic) multiplicity** of  $\lambda \in \sigma(A)$  is its multiplicity as a zero of the characteristic polynomial  $p_A(\lambda)$

### Definition 10 (Spectral radius of a matrix)

The **spectral radius** of  $A \in \mathcal{M}_n$  is

$$\rho(A) = \max\{|\lambda|, \mid \lambda \in \sigma(A)\}$$

### Proposition 11

*For all  $\lambda \in \sigma(A)$ ,  $A \in \mathcal{M}_n$ ,  $\lambda$  lies in the closed bounded disk in  $\mathbb{C}$ ,*

$$\{z \in \mathbb{C} : |z| \leq \rho(A)\}$$

### Theorem 12 (Every square matrix is close to nonsingular matrices)

*Let  $A \in \mathcal{M}_n$ , then there exists  $\delta > 0$  such that  $A + \varepsilon \mathbb{I}$  is non-singular for  $0 < |\varepsilon| < \delta$*

### Theorem 13

*Let  $A \in \mathcal{M}_n$ . Suppose that  $\lambda \in \sigma(A)$  has algebraic multiplicity  $k$ . Then*

$$\text{rank}(A - \lambda\mathbb{I}) \geq n - k$$

*with equality when  $k = 1$*

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### Definition 14 (Similarity/permutation similarity)

Let  $A, B \in \mathcal{M}_n$ . We say that  $B$  is **similar** to  $A$  if there exists a nonsingular  $S \in \mathcal{M}_n$  such that

$$B = S^{-1}AS$$

The transformation  $A \mapsto S^{-1}AS$  is a **similarity transformation** with similarity matrix  $S$ . If  $S = P$  with  $P$  a **permutation matrix** and that  $B = P^TAP$ ,  $A$  and  $B$  are **permutation similar**. In both cases, we denote “ $A$  similar to  $B$ ” as  $A \sim B$

### Theorem 15

*Similarity is an equivalence relation, i.e., it is reflexive, symmetric, and transitive*

## Theorem 16

*Let  $A, B \in \mathcal{M}_n$ . If  $A$  is similar to  $B$ , then they have the same characteristic polynomial, i.e.,*

$$p_A(t) = p_B(t)$$

## Corollary 17

*Let  $A, B \in \mathcal{M}_n$ . If  $A \sim B$ , then*

- 1.  $A$  and  $B$  have the same eigenvalues*
- 2. If  $B$  is a diagonal matrix, then the main diagonal entries are the eigenvalues of  $A$*
- 3.  $B = 0 \iff A = 0$*
- 4.  $B = \mathbb{I} \iff A = \mathbb{I}$*



### Definition 18

If  $A \in \mathcal{M}_n$ . If  $A$  is similar to a diagonal matrix, then  $A$  is **diagonalisable**

## Theorem 19

Let  $A \in \mathcal{M}_n$ .

1.

$$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix} \quad (1)$$

with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$ ,  $D \in \mathcal{M}_{n-k}$ ,  $1 \leq k \leq n \iff k$  linear independent vectors in  $\mathbb{C}^n$ , each of which is an eigenvector of  $A$

2.  $A$  diagonalisable  $\iff$  there are  $n$  linearly independent eigenvectors of  $A$

## Theorem 19 (continued)

3. If  $x^{(1)}, \dots, x^{(n)}$  are linear independent eigenvectors of  $A$ , define

$$S = [x^{(1)} \dots x^{(n)}].$$

Then  $S^{-1}AS$  is diagonal.

4. If

$$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix},$$

then the diagonal entries of  $\Lambda$  are eigenvalues of  $A$ , if  $A \sim \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ , then  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$

### Lemma 20

*Let  $\lambda_1, \dots, \lambda_k$ ,  $k \geq 2$  be  $k$  distinct eigenvalue of  $A$ . Let  $x^{(i)}$  be an eigenvector associated to  $\lambda_i$ ,  $i = 1, \dots, k$ . Then  $x^{(1)}, \dots, x^{(k)}$  are linear independent*

### Theorem 21

*If  $A \in \mathcal{M}_n$  has  $n$  distinct eigenvalues, then it is diagonalisable*

### Lemma 22

*Let  $B = \bigoplus_{i=1}^d B_{ii}$ . Then  $B$  is diagonalisable if and only if each of the  $B_{ii}$  is diagonalisable*

### Definition 23

Two matrices  $A$  and  $B$  in  $\mathcal{M}_n$  are **simultaneously diagonalisable** if there exists a matrix  $S \in \mathcal{M}_n$  non-singular such that  $S^{-1}AS$  and  $S^{-1}BS$  are diagonal

### Theorem 24

*Let  $A, B \in \mathcal{M}_n$  be diagonalisable. Then  $A$  and  $B$  commute if and only if  $A$  and  $B$  are simultaneously diagonalisable*

### Remark 25

*See Definition 1.3.16 and following for commuting families and simultaneously diagonalisable families*

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## Theorem 26

Let  $A \in \mathcal{M}_n$ , then

1.  $\sigma(A) = \sigma(A^T)$
2.  $\sigma(A^*) = \overline{\sigma(A)}$

## Definition 27

Take  $A \in \mathcal{M}_n$ , for a given  $\lambda \in \sigma(A)$ , the set of  $\mathbf{x} \in \mathbb{C}^n$  such that  $A\mathbf{x} = \lambda\mathbf{x}$  is the eigenspace associated to  $\lambda$ . Every non-zero vector in the eigenspace associated to  $\lambda \in \sigma(A)$  is an eigenvector of  $A$  associated to  $\lambda$

## Definition 28

Let  $A \in \mathcal{M}_n$  and  $\lambda \in \sigma(A)$ . The dimension of the eigenspace associated to  $\lambda$  is the **geometric multiplicity** of  $\lambda$ . We say that  $\lambda$  is **simple** if its algebraic multiplicity is one, it is **semisimple** if its algebraic and geometric multiplicities are equal

## Proposition 29

*Let  $\lambda$  be an eigenvalue of  $A$ . We have that the algebraic multiplicity is greater or equal to the geometric multiplicity. Furthermore, if the algebraic multiplicity is one then the geometric multiplicity is one as well*



## Definition 30

Let  $A \in \mathcal{M}_n$ . We say that  $A$  is

- ▶ **defective** if the geometric multiplicity is less than the algebraic multiplicity for *some* eigenvalue
- ▶ **non-defective** if *for all* eigenvalues, the geometric multiplicity equals the algebraic multiplicity
- ▶ **non-derogatory** if *for all* eigenvalues, the geometric multiplicity is one
- ▶ **derogatory** otherwise

## Theorem 31

Let  $A \in \mathcal{M}_n$

1.  $A$  is diagonalisable if and only if it is nondefective
2.  $A$  has distinct eigenvalues if and only if  $A$  is nonderogatory and non-defective

### Remark 32

$\sigma(A) = \sigma(A^T)$ , however they might have different spaces associated to each eigenvalue

### Definition 33 (Left wigenvector)

Let  $\mathbf{0} \neq \mathbf{y} \in \mathbb{C}^n$ , then we say that  $\mathbf{y}$  is a **left eigenvector** of  $A \in \mathcal{M}_n$  associated to  $\lambda \in \sigma(A)$  if  $\mathbf{y}^* A = \lambda \mathbf{y}^*$

### Theorem 34

Let  $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$ ,  $A \in \mathcal{M}_n$ . Assume that  $A\mathbf{x} = \lambda\mathbf{x}$  for some  $\lambda$ . If  $\mathbf{x}^* A = \mu\mathbf{x}^*$ , then  $\lambda = \mu$

### Remark 35

*$\mathbf{y}$  is a left eigenvector associated to  $\lambda$  is also a right eigenvector of  $A^*$  associated to  $\bar{\lambda}$ .  
 $\bar{\mathbf{y}}$  eigenvector of  $A^T$  associated to  $\lambda$*

Let  $A \in \mathcal{M}_n$  diagonalisable,  $S$  non-singular matrix,  $S^{-1}AS = \Lambda$ . Partition  $S = [\mathbf{x}_1, \dots, \mathbf{x}_n]$  and  $S^{-*} = [\mathbf{y}_1, \dots, \mathbf{y}_n]$ , where  $\mathbf{x}_i$  and  $\mathbf{y}_i$  are the right and left eigenvectors associated to  $\lambda_i$ , respectively.

## Theorem 36

Let  $A \in \mathcal{M}_n$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ ,  $\lambda, \mu \in \mathbb{C}$ . Assume  $A\mathbf{x} = \lambda\mathbf{x}$  and  $\mathbf{y}^*A = \mu\mathbf{y}^*$

1. If  $\lambda \neq \mu$ , then  $\mathbf{y}^*\mathbf{x} = 0$ , then  $\mathbf{x} \perp \mathbf{y}$
2. If  $\lambda = \mu$  and  $\mathbf{y}^*\mathbf{x} \neq 0$ , then there exists  $S$  non-singular of the form  $S = [\mathbf{x}S_1]$  such that  $S^{-*} = [\mathbf{y}/(\mathbf{x}^*\mathbf{y})Z_1]$  and  $A = S \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} S^{-1}$

Conversely, if  $A$  is similar to a block matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, B \in \mathcal{M}_{n-1}$$

then it has a non-orthogonal pair of left and right eigenvectors associated to  $\lambda$

### Theorem 37

*Let  $A, B \in \mathcal{M}_n$ , with  $A \sim B$  with similarity matrix  $S$ . If  $(\lambda, \mathbf{x})$  is an eigenpair of  $B$ , then  $(\lambda, S\mathbf{x})$  is an eigenpair of  $A$ . If  $(\lambda, \mathbf{y})$  is a left eigenpair of  $B$ , then  $(\lambda, S^{-*}\mathbf{y})$  is a left eigenpair of  $A$*

### Theorem 38

*Let  $A \in \mathcal{M}_n$ ,  $\lambda \in \mathbb{C}$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$  non-zero. Suppose that  $\lambda \in \sigma(A)$  and  $A\mathbf{x} = \lambda\mathbf{x}$  and  $\mathbf{y}^*A = \lambda\mathbf{y}^*$*

- 1. If  $\lambda$  has algebraic multiplicity 1, then  $\mathbf{y}^*\mathbf{x} \neq 0$*
- 2. If  $\lambda$  has geometric multiplicity 1, then it has algebraic multiplicity 1 if and only if  $\mathbf{y}^*\mathbf{x} \neq 0$ .*

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This section is based mostly on Varga's book *Geršgorin and His Circles* [Var10], which is highly recommended reading if you enjoy matrix theory.

Let  $A \in \mathcal{M}_n(\mathbb{C})$ . Denote  $N = \{1, \dots, n\}$ . For  $i \in N$ , define

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|$$

to be the  $i$ th deleted row sums of  $A$ . Assume that  $r_i(A) = 0$  if  $n = 1$ . Let

$$\Gamma_i(A) = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq r_i(A)\} \quad i \in N$$

be the  $i$ th **Gershgorin disk** of  $A$  and

$$\Gamma(A) = \bigcup_{i \in N} \Gamma_i(A)$$

be the **Gershgorin set** of  $A$ .  $\Gamma_i$  and  $\Gamma$  are closed and bounded in  $\mathbb{C}$ .  $\Gamma_i(A)$  is a disk centred at  $a_{ii}$  and with radius  $r_i(A)$ ,  $i \in N$ .

### Theorem 39 (Gershgorin, 1931)

*For all  $A \in \mathcal{M}_n(\mathbb{C})$  and for all  $\lambda \in \sigma(A)$ , there exists  $k \in \mathbb{N}$  such that*

$$|\lambda - a_{kk}| \leq r_k(A)$$

*i.e.,  $\lambda \in \Gamma_k(A)$  and thus  $\lambda \in \Gamma(A)$ . Since this is true for all  $\lambda$ , we have*

$$\sigma(A) \subseteq \Gamma(A)$$

### Remark 40

*This also works with deleted column sums; indeed, just consider  $A^T$  in this case. However, this typically gives different disks*



## Corollary 41

Let  $A \in \mathcal{M}_n(\mathbb{C})$ , then

$$\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\} \leq \max_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}|$$

Definition 42 (Strictly diagonally dominant matrix)

$A \in \mathcal{M}_n(\mathbb{C})$  is **strictly diagonally dominant** (SDD) if

$$\forall i \in N, |a_{ii}| > r_i(A)$$

## Theorem 43

Let  $A \in \mathcal{M}_n(\mathbb{C})$ . If  $A$  SDD then  $A$  is nonsingular

Let  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{x} > \mathbf{0}$ , i.e.,  $\mathbf{x} = (x_1, \dots, x_n)$  is such that  $x_i > 0$  for all  $i$ . Let  $X = \text{diag}(\mathbf{x}) = \text{diag}(x_1, \dots, x_n)$  such that  $X$  is invertible. Let  $A \in \mathcal{M}_n(\mathbb{C})$ , then  $X^{-1}AX = \left[ \frac{a_{ij}x_j}{x_i} \right]_{i,j \in N}$ . Also  $X^{-1}AX$  similar to  $A$ , so  $\sigma(X^{-1}AX) = \sigma(A)$ .b

Let  $r_i^{x_i}(A) = r_i(X^{-1}AX) = \sum_{j \in N \setminus \{i\}} \frac{|a_{ij}|x_j}{x_i}$  be the  $i$ th weighted rows sums of  $A$ . Let

$$\Gamma_i^{r^x} = \{z \in \mathbb{C}, |z - a_{ii}| \leq r_i^x(A)\}$$

and

$$\Gamma^{r^x} = \bigcup_{i \in N} \Gamma_i^{r^x}$$

be the  $i$ th **weighted Gershgorin disk** and the **weighted Gershgorin set** of  $A$ , respectively

## Corollary 44

For any matrix  $A \in \mathcal{M}_n(\mathbb{C})$  and  $x \in \mathbb{R}^n$ ,  $x > 0$ ,

$$\sigma(A) \subset \Gamma^{r^x}(A)$$

**Question:** How many eigenvalues are contained in each “component”?

Assume  $n \geq 2$ . Let  $S$  be a proper subset of  $N$ , i.e.,  $\emptyset \neq S \subsetneq N$ , with  $|S|$  its cardinality.

Let  $A \in \mathcal{M}_n(\mathbb{C})$ ,  $\mathbf{x} > \mathbf{0}$  in  $\mathbb{R}^n$  and

$$F_S^{\mathbf{x}} = \bigcup_{i \in S} \Gamma_i^{\mathbf{x}}(A)$$

Then

$$\Gamma_S^{\mathbf{x}}(A) \cap \Gamma_{N \setminus S}^{\mathbf{x}}(A) = \emptyset$$

### Theorem 45

For all  $A \in \mathcal{M}_m(\mathbb{C})$ , for all  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{x} > \mathbf{0}$  for which

$$\Gamma_S^{\mathbf{x}}(A) \cap \Gamma_{N \setminus S}^{\mathbf{x}}(A) = \emptyset$$

for some proper subset  $S$  of  $N$ , then  $\Gamma_S^{\mathbf{x}}(A)$  contains exactly  $|S|$  eigenvalues of  $A$

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We have seen that a matrix  $A$  that is SDD is nonsingular. Can we weaken this? What if diagonal dominance is not strict, i.e.,  $|a_{ii}| = r_i(A)$  for some  $i \in N$ ,  $|a_{ii}| \geq r_i(A)$  for all  $i \in N$ . This is not sufficient for nonsingularity. If we take the matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

is DD and singular, however,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is DD and singular.

### Definition 46 (Reducible/irreducible matrices)

$A \in \mathcal{M}_n(\mathbb{C})$  is **reducible** if there exists a permutation matrix  $P \in \mathcal{M}_n(\mathbb{R})$  and  $r \in N = \{1, \dots, n\}$  such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where  $A_{11} \in \mathcal{M}_r$ ,  $A_{22} \in \mathcal{M}_{n-r}$ . If there is no such  $P$ , then we say that  $A$  is **irreducible**

### Remark 47

If  $A \in \mathcal{M}_1$ , then  $A$  irreducible if  $a_{11} \neq 0$

In the reducible case, we can continue the process and find a matrix  $P$  (permutation) such that

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ & & \ddots & \\ 0 & & \dots & R_{nm} \end{pmatrix}$$

with the diagonal block  $R_{ii}$  irreducible. This is the **normal reduced form** of  $A$



### Remark 48

*Establishing irreducibility this way is hard. If no obvious permutation of rows and columns gives rise to a matrix in reduced form, then deciding on irreducibility requires to exhaust all possible permutation matrices to assert none exists. There are  $n!$  permutation matrices of size  $n \times n$ ...*

Let  $A \in \mathcal{M}_n(\mathbb{C})$ . Let  $\{v_1, \dots, v_n\}$  be  $n$  distinct points called **vertices**

For any  $(i, j)$ ,  $i, j \in N$ , for which  $a_{ij} \neq 0$ , connect  $v_i$  to  $v_j$  using a directed arc  $\overrightarrow{v_i v_j}$

If  $a_{ii} \neq 0$ , there is a loop from  $v_i$  to  $v_i$

The collection of all the directed arcs (and loops) obtained thusly is called the **directed graph** (or **digraph**) associated to  $A$  and is denoted  $\mathcal{G}(A)$

A **directed path** in  $\mathcal{G}(A)$  is a collection of directed arcs from  $v_i$  to  $v_j$ , i.e.,

$$\overrightarrow{v_{i_1} v_{i_2}}, \dots, \overrightarrow{v_{i_{n-1}} v_{i_n}}$$

Along a directed path

$$\prod_{k=1}^{n-1} a_{i_k} a_{i_{k+1}} \neq 0$$

#### Remark 49

Given a graph  $\mathcal{G}$ , the matrix  $A$  such that  $\mathcal{G}(A) = \mathcal{G}$  is the **adjacency matrix** of  $\mathcal{G}$

## Definition 50

Let  $\mathcal{G}$  be a digraph with vertex set  $\{v_1, \dots, v_n\}$ .  $\mathcal{G}$  is **strongly connected** if for all ordered pairs  $(v_i, v_j)$  of vertices, there is a directed path from  $v_i$  to  $v_j$  in  $\mathcal{G}$

## Remark 51

*If  $\mathcal{G}(A)$  is strongly connected, then  $A$  cannot have a row with only zero off-diagonal entries. Indeed, suppose  $\mathcal{G}(A)$  is strongly connected. Without loss of generality, assume row 1 in  $A$  has only zero off-diagonal entries. Then because of the way  $\mathcal{G}(A)$  is constructed, this means there are no directed arcs terminating in  $v_1$  and as a consequence, there is no directed path terminating in  $v_1$ , contradicting strong connectedness of  $\mathcal{G}(A)$ .*

### Remark 52

*$\mathcal{G}(A)$  is strongly connected if and only if for any permutation matrix  $P$ , we have that  $\mathcal{G}(P^T A P)$  is strongly connected. [Because permutation is a relabelling of vertices.]*

### Theorem 53

*Let  $A \in \mathcal{M}_n(\mathbb{C})$ . Then  $A$  is irreducible if and only  $\mathcal{G}(A)$  is strongly connected*

### Definition 54 (Irreducibly diagonally dominant matrix)

$A \in \mathcal{M}_n(\mathbb{C})$  is **irreducibly diagonally dominant** (IDD) if  $A$  is irreducible, diagonally dominant, i.e.,

$$\forall i \in N, \quad |a_{ii}| \geq r_i(A)$$

and there exists  $i \in N$  for which diagonal dominance is strict, i.e., there exists  $i$  such that  $|a_{ii}| = r_i(A)$ .

### Theorem 55 (Taussky 1949 [Tau49])

*For any  $A \in \mathcal{M}_n(\mathbb{C})$ ,  $A$  IDD  $\Rightarrow A$  non-singular*

## Another result of Taussky



### Theorem 56

*Let  $A \in \mathcal{M}_n(\mathbb{C})$  be irreducible. Suppose  $\lambda \in \sigma(A)$  be such that  $\forall i \in N, \lambda \notin \text{Int } \Gamma_i(A)$   
Then*

$$\forall i \in N, \quad |\lambda - a_{ii}| = r_i(A) \quad (2)$$

*In particular, if  $\lambda \in \partial\Gamma(A)$  [the boundary of  $\Gamma(A)$ ] for some  $\lambda \in \sigma(A)$ , then (2) holds for  $\lambda$*

# References I

-  Olga Taussky, *A recurring theorem on determinants*, The American Mathematical Monthly **56** (1949), no. 10P1, 672–676.
-  Richard S Varga, *Geršgorin and his circles*, vol. 36, Springer Science & Business Media, 2010.