MATH 4370/7370 – Linear Algebra and Matrix Analysis

Quick review of 2nd year linear algebra

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Outline of these slides

Source of the material

The material in these slides is mostly derived from [AxI15]

Some notation and basic stuff

Sets and elements

Logic

Sets and elements

Logic

Sets and elements

Definition 2.1 (Set)

A **set** X is a collection of **elements**.

We write $x \in X$ or $x \notin X$ to indicate that the element x belongs to the set X or does not belong to the set X, respectively.

Definition 2.2 (Subset)

Let X be a set. The set S is a **subset** of X, which is denoted $S \subset X$ or $S \subseteq X$, if all its elements belong to X. S is a **proper subset** of X if it is a subset of X and not equal to X; we then write $S \subsetneq X$.

Smith reserves $\subset$ for $\subsetneq$. I learned $\subset$ for not specified (proper or not) and $\subsetneq$ for proper. So beware!

Quantifiers

- A shorthand notation for "for all elements x belonging to X" is $\forall x \in X$. For example, if $X = \mathbb{R}$, the field of real numbers, then $\forall x \in \mathbb{R}$ means "for all real numbers x".
- A shorthand notation for "there exists an element x in the set X" is $\exists x \in X$.
- Sometimes we write $\exists! x \in X$ for "there exists a **unique** x in X".
- $\forall$ and $\exists$ are **quantifiers**.

Intersection and union of sets

Let X and Y be two sets.

Definition 2.3 (Intersection)

The intersection of X and Y, $X \cap Y$, is the set of elements that belong to X **and** to Y

$$X \cap Y = \{x : x \in X \textbf{ and } x \in Y\}$$

Definition 2.4 (Union)

The union of X and Y, $X \cup Y$, is the set of elements that belong to X **or** to Y

$$X \cup Y = \{x : x \in X \textbf{ or } x \in Y\}$$

Use of the expression "and/or" is *strictly* forbidden in this course! "Or but not and" (a.k.a. **xor**, exclusive or) is $(X \cup Y) \setminus (X \cap Y)$.

Sets and elements

Logic

A few notions of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. "The sky is blue" is also a proposition.
Let A be a proposition. We generally write

$$A$$

to mean that A is true, and

$$\textbf{not } A$$

to mean that A is false. We also write $\neg A$. **not** A is the **negation** of A.

A few notions of logic (cont.)

Let A, B be propositions. Then

- $A \Rightarrow B$ (read A implies B) means that whenever A is true, then so is B.
- $A \Leftrightarrow B$, also denoted A if and only if B (A iff B for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$. We also say that A and B are equivalent.

Let A and B be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\textbf{not } B \Rightarrow \textbf{not } A)$$

This is useful for proving some results.

Necessary and/or sufficient conditions

Suppose we want to establish whether a given statement P is true, depending on the truth value of a statement H. Then we say that

- H is a **necessary condition** if $P \Rightarrow H$.
  (It is necessary that H be true for P to be true; so whenever P is true, so is H).
- H is a **sufficient condition** if $H \Rightarrow P$.
  (It suffices for H to be true for P to also be true).
- H is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., H and P are equivalent.

Playing with quantifiers

For the quantifiers $\forall$ (for all) and $\exists$ (there exists),

$$\exists \text{ is the negation of } \forall$$

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

This is also regularly used in proofs.

Vector spaces

Operations

Definition 2.5 (Operations – Addition and multiplication)

An **operation** on a set V is a mapping that associates an element of the set V to *every* pair of its elements

- The result of the **addition** of a and b is the *sum* $a + b$ of a and b
- The result of the **multiplication** of a and b is the *product* ab (or $a \cdot b$) of a and b

Field

Definition 2.6 (Field)

A **field** is a set $\mathbb{F}$ together with two (binary) operations, *addition* and *multiplication*, which are required to satisfy the following *field axioms*, where $a, b, c \in \mathbb{F}$:

- **Associativity** of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$
- **Commutativity** of addition and multiplication: $a + b = b + a$ and $ab = ba$
- **Additive** and **multiplicative identity**: $\exists 0, 1 \in \mathbb{F}$, $0 \neq 1$, s.t. $a + 0 = a$ and $a1 = a$
- **Additive inverses**: $\forall a \in \mathbb{F}$, $\exists -a \in \mathbb{F}$ s.t. $a + (-a) = 0$
- **Multiplicative inverses**: $\forall a \neq 0 \in \mathbb{F}$, $\exists a^{-1} \in \mathbb{F}$ s.t. $aa^{-1} = 1$
- **Distributivity** (of multiplication over addition): $a(b + c) = (ab) + (ac)$

Notation

- Both $\mathbb{R}$ and $\mathbb{C}$ are fields.
- From now on, $\mathbb{F}$ refers to $\mathbb{R}$ or $\mathbb{C}$.
- Some results are specific to $\mathbb{R}$ xor $\mathbb{C}$, in which case we specify the relevant field.
- If we use $\mathbb{F}$, we mean the result applies to both $\mathbb{R}$ and $\mathbb{C}$.

Addition and Scalar multiplication

Definition 2.7 (Addition and scalar multiplication on a set)

- An **addition** on a set V is a function that assigns an element $\mathbf{u} + \mathbf{v} \in V$ to each pair of elements $\mathbf{u}, \mathbf{v} \in V$
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda\mathbf{v}$ to each $\lambda \in \mathbb{F}$ and each $\mathbf{v} \in V$

Vector space

Definition 2.8 (Vector space)

A **vector space** (over $\mathbb{F}$) is a set V along with an addition on V and a scalar multiplication on V such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [**commutativity**]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall a, b \in \mathbb{F}$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and $(ab)\mathbf{v} = a(b\mathbf{v})$ [**associativity**]
3. $\exists \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V$, $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [**additive identity**]
4. $\forall \mathbf{v} \in V$, $\exists \mathbf{w} \in V$ s.t. $\mathbf{v} + \mathbf{w} = \mathbf{0}_V$ [**additive inverse**]
5. $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$ [**multiplicative identity**]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [**distributivity**]

Results

Theorem 2.9 (Uniqueness of the additive identity)

A vector space V has a unique additive identity $\mathbf{0}_V \in V$

Theorem 2.10 (Existence and uniqueness of additive inverse)

Let V be a vector space. Then each $\mathbf{v} \in V$ has a unique additive inverse, denoted $-\mathbf{v}$

We also define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$.

Theorem 2.11

- $\forall \mathbf{v} \in V$, $0_{\mathbb{F}}\mathbf{v} = \mathbf{0}_V$.
- $\forall a \in \mathbb{F}$, $a\mathbf{0}_V = \mathbf{0}_V$.
- $\forall \mathbf{v} \in V$, $(-1)\mathbf{v} = -\mathbf{v}$.

Vector space

Definition 2.12 (Vector space)

A **vector space** (over $\mathbb{F}$) is a set V along with an addition on V and a scalar multiplication on V such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [**commutativity of $+$**]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ [**associativity of $+$**]
3. $\exists! \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V$, $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [**additive identity**]
4. $\forall \mathbf{v} \in V$, $\exists! - \mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_V$ [**additive inverse**]
5. $\forall a \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ [**distributivity of $\cdot$ over $+$**]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u} \in V$, $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [**distributivity of $+$ over $\cdot$**]
7. $\forall a, b \in \mathbb{F}$, $(ab)\mathbf{v} = a(b\mathbf{v})$ [**associativity of $\cdot$**]
8. $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$ [**multiplicative identity**]

$\mathbb{F}^n$ is a vector space

Typically called *Euclidean space* when $\mathbb{F} = \mathbb{R}$.

Definition 2.13

Let $0 \neq n \in \mathbb{N}$. An n-**tuple** is an ordered collection of n elements,

$$(x_1, \ldots, x_n)$$

Definition 2.14

Let $0 \neq n \in \mathbb{N}$. $\mathbb{F}^n$ is the set of all n-tuples of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

- Often write $x = (x_1, \ldots, x_n)$ for short.
- For a given $j \in \{1, \ldots, n\}$, x_j is the jth **coordinate** of x.
- Think of $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ that you saw in whatever flavour of Linear Algebra 1 you took.

Addition in $\mathbb{F}^n$

Definition 2.15 (Addition in $\mathbb{F}^n$)

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}^n$. Then

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Property 2.16 (Commutativity of addition in $\mathbb{F}^n$)

Let $x, y \in \mathbb{F}^n$, then

$$x + y = y + x$$

0 and additive inverse in $\mathbb{F}^n$

Definition 2.17 (0)

0 denotes the *n*-tuple whose coordinates are all 0,

$$0 = (0, \ldots, 0)$$

If any ambiguity arises, will write $0_{\mathbb{F}^n}$

Definition 2.18 (Additive inverse)

Let $x \in \mathbb{F}^n$. The **additive inverse** of x is $-x \in \mathbb{F}^n$ s.t.

$$x + (-x) = 0$$

If $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$

Scalar multiplication in $\mathbb{F}^n$

Definition 2.19 (Scalar multiplication)

The **product** of $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$ is

$$\lambda x = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Complex numbers

Definition 2.20 (Complex numbers)

A **complex number** is an ordered pair (a, b), where $a, b \in \mathbb{R}$. Usually written $a + ib$ or $a + bi$, where $i^2 = -1$
The set of all complex numbers is denoted $\mathbb{C}$,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

Definition 2.21 (Addition and multiplication on $\mathbb{C}$)

Letting $a + ib$ and $c + id \in \mathbb{C}$, addition on $\mathbb{C}$ is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on $\mathbb{C}$ is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter equality easy to obtain using regular multiplication and $i^2 = -1$

Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C}$,

- $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ [**commutativity**]
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ [**associativity**]
- $\gamma + 0 = \gamma$ and $\gamma 1 = \gamma$ [**identities**]
- $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha + \beta = 0$ [**additive inverse**]
- $\forall \alpha \neq 0 \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha\beta = 1$ [**multiplicative inverse**]
- $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ [**distributivity**]

Thus $\mathbb{C}$ is a field.

Additive & multiplicative inverse, subtraction, division

Definition 2.22

Let $\alpha, \beta \in \mathbb{C}$

- $-\alpha$ is the **additive inverse** of α, i.e., the unique number in $\mathbb{C}$ s.t. $\alpha + (-\alpha) = 0$
- **Subtraction** on $\mathbb{C}$:
$$\beta - \alpha = \beta + (-\alpha)$$
- For $\alpha \neq 0$, $1/\alpha$ is the **multiplicative inverse** of α, i.e., the unique number in $\mathbb{C}$ s.t.
$$\alpha(1/\alpha) = 1$$
- **Division** on $\mathbb{C}$:
$$\beta/\alpha = \beta(1/\alpha)$$

Definition 2.23 (Real and imaginary parts)

Let $z = a + ib$. Then $\mathrm{Re}\, z = a$ is **real part** and $\mathrm{Im}\, z = b$ is **imaginary part** of z

If ambiguous, write $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$

Definition 2.24 (Conjugate and Modulus)

Let $z = a + ib \in \mathbb{C}$. Then

- **Complex conjugate** of z is

$$\bar{z} = \mathrm{Re}\, z - i(\mathrm{Im}\, z) = a - ib$$

- **Modulus** (or **absolute value**) of z is

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \sqrt{a^2 + b^2} \geq 0$$

Properties of complex numbers

Let $w, z \in \mathbb{C}$, then

- $z + \bar{z} = 2\mathrm{Re}\ z$
- $z - \bar{z} = 2i\mathrm{Im}\ z$
- $z\bar{z} = |z|^2$
- $\overline{w+z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$
- $\bar{\bar{z}} = z$
- $|\mathrm{Re}\ z| \leq |z|$ and $|\mathrm{Im}\ z| \leq |z|$
- $|\bar{z}| = |z|$
- $|wz| = |w|\ |z|$
- $|w + z| \leq |w| + |z|$ [**triangle inequality**]

Subspace

Definition 2.25 (Subspace)

Let V be a vector space over $\mathbb{F}$. Let $U \subseteq V$ be a subset of V. Then U is a **subspace** of V if U is a vector space over $\mathbb{F}$ for the same operations of addition and scalar multiplication as V

Theorem 2.26 (Conditions for a subspace)

$U \subseteq V$ is a subspace of V $\iff$ U satisfies the following three conditions:

- $\mathbf{0}_V \in U$ *[additive identity]*
- $\forall \mathbf{u}, \mathbf{v} \in U$, $\mathbf{u} + \mathbf{v} \in U$ *[closed under addition]*
- $\forall \mathbf{u} \in U$, $\forall a \in \mathbb{F}$, $a\mathbf{u} \in U$ *[closed under scalar multiplication]*

The smallest possible subspace of V is $\{\mathbf{0}_V\}$, the largest is V.

Sums of subspaces

Definition 2.27 (Sum of subsets)

Let V be a vector space and $U_1, \ldots, U_m$ be *subsets* of V. The **sum** of $U_1, \ldots, U_m$ is

$$U_1 + \cdots + U_m = \{\mathbf{u}_1 + \cdots + \mathbf{u}_m : \mathbf{u}_1 \in U_1, \ldots, \mathbf{u}_m \in U_m\}$$

Theorem 2.28

Let V be a vector space and $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$

Direct sums

Definition 2.29 (Direct sum)

Suppose $U_1, \ldots, U_m$ are subspaces of a vector space V. The sum $U_1 + \cdots + U_m$ is a **direct sum** and is then written $U_1 \oplus \cdots \oplus U_m$ if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $\mathbf{u}_1 + \cdots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$

Theorem 2.30 (Condition for a direct sum)

Suppose $U_1, \ldots, U_m$ are subspaces of a vector space V. Then $U_1 + \cdots + U_m$ is a direct sum $\iff$ the only way to write $\mathbf{0}$ as a sum $\mathbf{u}_1 + \cdots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$, is by taking each $\mathbf{u}_j$ equal to $\mathbf{0}_V$

Theorem 2.31 (Direct sum of two subspaces)

Let U, W be subspaces of a vector space V. Then $U + W$ is a direct sum $\iff$ $U \cap W = \{\mathbf{0}_V\}$

Finite-dimensional vector spaces

Span and Linear independence

Bases

Dimension

Span and Linear independence

Bases

Dimension

Definition 2.32 (Linear combination)

A **linear combination** of a list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in V is a vector

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$

Definition 2.33 (Span)

The set of all linear combinations of a list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_m$,

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \{a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m : a_1, \ldots, a_m \in \mathbb{F}\}$$

The span of the empty list ( ) is $\{\mathbf{0}_V\}$

Finite/infinite-dimensional vector spaces

Theorem 2.34

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list

Definition 2.35 (List of vectors spanning a space)

If $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = V$, we say $\mathbf{v}_1, \ldots, \mathbf{v}_m$ **spans** V

Definition 2.36 (Finite-dimensional vector space)

A vector space V is **finite-dimensional** if some list of vectors in it spans V

Definition 2.37 (Infinite-dimensional vector space)

A vector space V is **infinite-dimensional** if it is not finite-dimensional

Linear (in)dependence

Definition 2.38 (Linear independence/Linear dependence)

A list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in a vector space V is **linearly independent** if

$$(a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = 0) \Leftrightarrow (a_1 = \cdots = a_m = 0),$$

where $a_1, \ldots, a_m \in \mathbb{F}$. A list of vectors is **linearly dependent** if it is not linearly independent.

The empty list ( ) is assumed to be linearly independent

Lemma 2.39 (Linear dependence)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *be a linearly dependent list in a vector space* V*. Then there exists* $j \in \{1, 2, \ldots, m\}$ *s.t.*

1. $\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$
2. *if the* j*th term is removed from* $\mathbf{v}_1, \ldots, \mathbf{v}_m$*, the span of the remaining list equals* $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$

Theorem 2.40

Let V *be a finite-dimensional vector space. Then the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors*

Theorem 2.41 (Subspace of a finite-dimensional vector space)

Every subspace of a finite-dimensional vector space is finite-dimensional

Basis

Definition 2.42 (Basis)

Let V be a vector space. A **basis** of V is a list of vectors in V that is both linearly independent and spanning

Theorem 2.43 (Criterion for a basis)

A list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in a vector space V is a basis of V iff $\forall \mathbf{v} \in V$, v can be written uniquely in the form

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$

Theorem 2.44 (All spanning lists contain a basis)

Every spanning list in a vector space can be reduced to a basis of the vector space

Theorem 2.45 (Basis of finite-dimensional vector space)

Every finite-dimensional vector space has a basis

Theorem 2.46 (Extension to a basis)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space

Theorem 2.47

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V. Then $\exists W \subset V$ subspace of V s.t. $V = U \oplus W$

Span and Linear independence

Bases

Dimension

Theorem 2.48 (Bases of a finite-dim. space have equal length)

Any two bases of a finite-dimensional vector space have the same length

Definition 2.49 (Dimension)

The **dimension** $\dim V$ of a finite-dimensional vector space V is the length of any basis of the vector space

Theorem 2.50 (Dimension of a subspace)

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V. Then $\dim U \leq \dim V$

Theorem 2.51

Let V be a finite-dimensional vector space. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V

Theorem 2.52

Let V be a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V

Theorem 2.53 (Dimension of a sum of subspaces)

Let U_1, U_2 be subspaces of a finite-dimensional vector space V. Then

$$\dim (U_1 + U_2) = \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2)$$

Linear maps

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

Definition 2.54 (Linear map/transformation)

Let V, W be vector spaces. A **linear map** (or **linear transformation**) from V to W is a function $T : V \to W$ that has the following properties:

1. **Additivity** $\forall \mathbf{u}, \mathbf{v} \in V,\ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. **Homogeneity** $\forall \lambda \in \mathbb{F}, \forall \mathbf{v} \in V,\ T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$.

Often, parentheses are omitted, $T(\mathbf{u})$ is written $T\mathbf{u}$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$

Theorem 2.55 (Linear maps and basis of domain)

Let V, W be two vector spaces and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of V. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n \in W$. Then there exists a unique linear map $T : V \to W$ s.t.

$$\forall j = 1, \ldots, n, \qquad T\mathbf{v}_j = \mathbf{w}_j$$

Definition 2.56 (Addition & Scalar multiplication)

Let V, W be vector spaces, $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$ and **product** λT are the linear maps from V to W defined, $\forall \mathbf{v} \in V$, by

$$(S+T)(\mathbf{v}) = S\mathbf{v} + T\mathbf{v} \text{ and } (\lambda T)(\mathbf{v}) = \lambda(T\mathbf{v}).$$

Theorem 2.57 (Linear maps are vector spaces)

Let V, W be vector spaces. Equipped with addition and scalar multiplication as just defined, $\mathcal{L}(V, W)$ is a vector space.

Product of linear maps

Definition 2.58 (Product of linear maps)

Let U, V, W be vector spaces, $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$. The **product** $ST \in \mathcal{L}(U, W)$ is defined for $\mathbf{u} \in U$ by

$$(ST)(\mathbf{u}) = S(T\mathbf{u}).$$

This means that the product of linear maps is the composition $S \circ T$, although because of the linearity, we often omit the $\circ$ composition sign.

Properties of products of linear maps

Theorem 2.59

1. **Associativity** *If* V, V_2, V_3, W *vector spaces,* $T_1 \in \mathcal{L}(V, V_2)$, $T_2 \in \mathcal{L}(V_2, V_3)$, $T_3 \in \mathcal{L}(V_3, W)$, *then*
$$(T_1 T_2)T_3 = T_1(T_2 T3)$$
2. **Identity** V, W *vector spaces. Then for* $T \in \mathcal{L}(V, W)$,
$$TI_V = I_W T = T$$
3. **Distributivity** U, V, W *vector spaces,* $T, T_1, T_2 \in \mathcal{L}(U, V)$, $S, S_1, S_2 \in \mathcal{L}(V, W)$, *then*
$$(S_1 + S_2)T = S_1 T + S_2 T \text{ and } S(T_1 + T_2) = ST_1 + ST_2$$

Theorem 2.60 (Linear maps take 0 to 0)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T(\mathbf{0}_V) = \mathbf{0}_W.$$

Definition 2.61 (Null space)

Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. The **null space** $\operatorname{null} T$ (or **kernel** $\ker T$) of T is the subet of V consisting of those vectors that T maps to $\mathbf{0}_W$:

$$\operatorname{null} T = \{\mathbf{v} \in V; T\mathbf{v} = \mathbf{0}_W\}.$$

Theorem 2.62 (Null space is a subspace)

Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then $\operatorname{null} T$ is a subspace of V

Definition 2.63 (Injectivity)

A function $T : V \to W$ is **injective** (or **one-to-one**) if

$$T\mathbf{u} = T\mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}.$$

We can also use the contrapositive: T injective if $\mathbf{u} \neq \mathbf{v} \Rightarrow T\mathbf{u} \neq T\mathbf{v}$.

Theorem 2.64 (Linking injectivity and null space)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \textit{ injective} \Leftrightarrow \operatorname{null} T = \{\mathbf{0}_V\}$$

Definition 2.65 (Range)

Let V, W be finite-dimensional vector spaces, $T : V \to W$ a function. The **range** (or **image**) of T is the subset of W defined by

$$\text{range } T = \{T\mathbf{v}; \mathbf{v} \in V\}.$$

When talking about the image, we write $\text{Im } T$.

Theorem 2.66 (Range is a subspace)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then range$T$ *is a subspace of W.*

Definition 2.67 (Surjectivity)

A function $T : V \to W$ is **surjective** (or **onto**) if

$$\operatorname{range} T = W$$

Theorem 2.68 (Fundamental theorem of linear maps)

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T < \infty$ *and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Theorem 2.69 (Linear map onto a smaller space is not injective)

Let V, W *be finite-dimensional vector spaces such that* $\dim V > \dim W$. *Then* $\nexists T \in \mathcal{L}(V, W)$ *that is injective*

Theorem 2.70 (Linear map onto a larger space is not surjective)

Let V, W *be finite-dimensional vector spaces such that* $\dim V < \dim W$. *Then* $\nexists T \in \mathcal{L}(V, W)$ *that is surjective*

Do as exercises..

Theorem 2.71

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Theorem 2.72

A nonhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms

Definition 2.73 (Matrix)

An m-by-n or $m \times n$ matrix is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Remember that we always list indices as "row,column"
We denote $\mathcal{M}_{mn}(\mathbb{F})$ the set of $m \times n$ matrices with entries in $\mathbb{F}$

Definition 2.74 (Matrix of a linear map)

Let V, W be finite-dimensional vector spaces, $v_1, \dots, v_n$ a basis of V and $w_1, \dots, w_m$ a basis of W. The **matrix of** T with respect to these bases is the matrix $M(T) \in \mathcal{M}_{mn}$ with entries a_{jk} defined by

$$Tv_k = a_{1k}w_1 + \cdots + a_{mk}w_m$$

for $1 \leq l \leq n$. If the bases are not clear from the context, then write

$$M(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$$

I will often write M_T rather than $M(T)$.

Most definitions are assumed known

Theorem 2.75 (Matrix of sums of linear maps)

Suppose $S, T \in \mathcal{L}(V, W)$. *Then* $M(S+T) = M(S) + M(T)$

Theorem 2.76 (Matrix of a scalar times a linear map)

Suppose $T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$. *Then* $M(\lambda T) = \lambda M(T)$

Theorem 2.77 (Dimension of $\mathcal{M}_{mn}$)

$\dim \mathbb{F}^{mn} = mn$

Theorem 2.78 (Matrix of products of linear maps)

Suppose $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$. *Then* $M(ST) = M(S)M(T)$

Theorem 2.79

Let $A \in \mathcal{M}_{mn}$, $C \in \mathcal{M}_{np}$. *Then*

$$(AC)_{jk} = A_{j\bullet} C_{\bullet k}, \qquad 1 \leq j \leq m, 1 \leq k \leq p$$

and

$$(AC)_{\bullet k} = AC_{\bullet k}, \qquad 1 \leq k \leq p$$

Theorem 2.80

Let $A \in \mathcal{M}_{mn}$, $c = (c_1, \ldots, c_n)^T \in \mathcal{M}_{n1}$. *Then*

$$Ac = c_1 A_{\bullet 1} + \cdots + c_n A_{\bullet n}$$

Change of basis

Definition 2.81 (Change of basis matrix)

$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ bases of vector space V The **change of basis matrix** $P_{\mathcal{C}\leftarrow\mathcal{B}} \in \mathcal{M}_n$,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

has columns the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ w.r.t. $\mathcal{C}$

Theorem 2.82

$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ bases of vector space V and $P_{\mathcal{C}\leftarrow\mathcal{B}}$ a change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$

1. $\forall \mathbf{x} \in V$, $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$
2. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *s.t.* $\forall \mathbf{x} \in V$, $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ *is* **unique**
3. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *invertible and* $P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$

Row-reduction method for changing bases

Theorem 2.83

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ bases of vector space V. Let $\mathcal{E}$ be any basis for V,

$$B = [[\mathbf{u}_1]_{\mathcal{E}}, \dots, [\mathbf{u}_n]_{\mathcal{E}}] \text{ and } C = [[\mathbf{v}_1]_{\mathcal{E}}, \dots, [\mathbf{v}_n]_{\mathcal{E}}]$$

and let $[C|B]$ be the augmented matrix constructed using C and B. Then

$$RREF([C|B]) = [\mathbb{I}|P_{\mathcal{C}\leftarrow\mathcal{B}}]$$

If working in $\mathbb{R}^n$, this is quite useful with $\mathcal{E}$ the standard basis of $\mathbb{R}^n$ (it does not matter if $\mathcal{B} = \mathcal{E}$)

More on changing bases

Theorem 2.84 (NSC for two matrices representing the same linear map)

Let $A, B \in \mathcal{M}_{mn}$, V and W be n and m dimensional vector spaces, respectively. Then A and B represent the same linear transformation $T \in \mathcal{L}(V, W)$ relative to perhaps different bases of V and W $\iff$ $\exists P \in \mathcal{M}_m$, $Q \in \mathcal{M}_n$ nonsingular and such that

$$A = PBQ^{-1}$$

Definition 2.85 (Inverse/Invertibility)

$T \in \mathcal{L}(V, W)$ is **invertible** if $\exists S \in \mathcal{L}(W, V)$ s.t. $ST = I_V$ and $TS = I_W$. Such a map is the **inverse** of T

Theorem 2.86 (Uniqueness of inverse)

An invertible linear map $T \in \mathcal{L}(V, W)$ has a unique inverse denoted T^{-1}

Theorem 2.87 (NSC for invertibility)

$T \in \mathcal{L}(V, W)$ *invertible* $\Leftrightarrow$ *(T injective and surjective)*

Definition 2.88 (Isomorphism/Isomorphic spaces)

$T \in \mathcal{L}(V, W)$ is an **isomorphism** if it invertible. Two vector spaces are **isomorphic** if there exists an isomorphism from one to the other

Theorem 2.89 (NSC for isomorphicity)

Let V, W be finite-dimensional vector spaces over $\mathbb{F}$. Then

$$V \textit{ and } W \textit{ are isomorphic} \Leftrightarrow \dim V = \dim W$$

Theorem 2.90

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W. Then M is an isomorphism between $\mathcal{M}_{mn}$ and $\mathcal{L}(V, W)$

Theorem 2.91 (Dimension of $\mathcal{L}(V, W)$)

Let V, W be finite-dimensional vector spaces. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = \dim V \ \dim W$$

Definition 2.92 (Matrix of a vector)

Let V be a finite-dimensional vector space, $v \in V$ and $v_1, \ldots, v_n$ a basis of V. The **matrix** of v with respect to the basis $v_1, \ldots, v_n$ is the $n \times 1$ matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \ldots, c_n \in \mathbb{F}$ are s.t.

$$v = c_1 v_1 + \cdots + c_n v_n$$

Theorem 2.93

Let V, W be finite-dimensional vector spaces, $v_1, \ldots, v_n$ a basis of V, $w_1, \ldots, w_m$ a basis of W and $T \in \mathcal{L}(V, W)$. For $k \in \{1, \ldots, n\}$, $M(T)_{\bullet k} = M(Tv_k)$

Theorem 2.94 (Linear maps act like matrix multiplication)

Let V, W be finite-dimensional vector spaces, $v_1, \ldots, v_n$ a basis of V, $w_1, \ldots, w_m$ a basis of W, $T \in \mathcal{L}(V, W)$ and $v \in V$. Then

$$M(Tv) = M(T)M(v)$$

Operator/Endomorphism

Definition 2.95 (Operator/Endomorphism)

Let V be a vector space. A linear map $\mathcal{L}(V, V)$ is an **operator** (or an **endomorphism**). $\mathcal{L}(V) = \mathcal{L}(V, V)$ denotes the set of all operators on V

Theorem 2.96 (Injectivity equiv. to surjectivity in finite-dim.)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. TFAE:

1. *T invertible*
2. *T injective*
3. *T surjective*

Rank of an operator/endomorphism

Proposition 2.97 (Rank)

Let $T \in \mathcal{L}(V)$ with V finite-dimensional. Then there exists bases $\mathcal{B}_U = \{u_1, \ldots, u_n\}$ and $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ for V such that the matrix M_T of T can be written as the block matrix

$$M_T = \begin{pmatrix} \mathrm{diag}(1, \ldots, 1) & \mathbf{0}_{k,n-k} \\ \mathbf{0}_{n-k,k} & \mathbf{0}_{n-k,n-k} \end{pmatrix}$$

for some $k \in \mathbb{N}$ called the **rank** *of T, with $k = \mathrm{rank}(T) = \dim(\mathrm{range}\ T)$.*

Definition 2.98 (Row and column rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$ be a matrix

- The **row rank** of A is the dimension of the span of the rows of A in $\mathcal{M}_{1n}(\mathbb{F})$
- The **column rank** of A is the dimension of the span of the columns of A in $\mathcal{M}_{m1}(\mathbb{F})$

Row and column ranks are the dimensions of the row and column spaces of Definition 2.102.

Theorem 2.99 ($\dim \operatorname{range} T$ equals column rank of $M(T)$)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $M(T)$

Theorem 2.100 (Row rank equals column rank)

Let $A \in \mathcal{M}_{mn}$. Then the row rank of A equals the column rank of A

Definition 2.101 (Rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$. The **rank** of A is the column (or row, by Theorem 2.100) rank of A

Row space and column space of a matrix

Definition 2.102 (Row and column spaces)

Let $A \in \mathcal{M}_{mn}$. The subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ spanned by the row and column vectors of A are the **row space** and **column space** of A, respectively.

Definition 2.103 (Null space/kernel)

Let $A \in \mathcal{M}_{mn}$. The null space (or kernel) of A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

This makes explicit the already seen definition in the special case of a matrix. As previously seen, the null space is a subspace of $\mathbb{R}^n$.

Definition 2.104 (Nullity)

The dimension of the null space of $A \in \mathcal{M}_{mn}$ is called the **nullity** of A.

Theorem 2.105

Let $A \in \mathcal{M}_{mn}$. Then

1. $\text{rank}(A) = \text{rank}(A^T)$
2. $\text{rank}(A) + \text{nullity}(A) = n$
3. $\text{rank}(A) \leq \min(m, n)$

Theorem 2.106 (Consistency)

Consider the linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathcal{M}_{mn}$. TFAE:

- $A\mathbf{x} = \mathbf{b}$ *is consistent*
- $\mathbf{b} \in$ *column space of* A
- A *and* $[A|\mathbf{b}]$ *have the same rank*

Proposition 2.107

Let $A \in \mathcal{M}_{mn}$ be in row-echelon form. Then

- *The row vectors ($\in \mathbb{R}^n$) with leading ones form a basis for the row space of A.*
- *The column vectors ($\in \mathbb{R}^m$) with leading ones form a basis for the column space of A.*

Definition 2.108 (Product of vector spaces)

Let $V_1, \ldots, V_m$ be vector spaces over $\mathbb{F}$. The **product** $V_1 \times \cdots \times V_m$ is

$$V_1 \times \cdots \times V_m = \{(\mathbf{v}_1, \ldots, \mathbf{v}_m); \mathbf{v}_1 \in V_1, \ldots, \mathbf{v}_m \in \ V_m\}$$

Theorem 2.109 (Products of vector spaces are vector spaces)

Let $V_1, \ldots, V_m$ be vector spaces over $\mathbb{F}$. Define

- *addition on $V_1 \times \cdots \times V_m$ by*

$$(\mathbf{u}_1, \ldots, \mathbf{u}_m) + (\mathbf{v}_1, \ldots, \mathbf{v}_m) = (\mathbf{u}_1 + \mathbf{v}_1, \ldots, \mathbf{u}_m + \mathbf{v}_m)$$

- *scalar multiplication on $V_1 \times \cdots \times V_m$ by*

$$\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_m) = (\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_m)$$

With these operations, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$

Theorem 2.110 (Dimension of product space)

Let $V_1, \ldots, V_m$ be finite-dimensional vector spaces. Then

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m < \infty$$

Theorem 2.111 (Product spaces and direct sums)

Let $U_1, \ldots, U_m \subset V$ be subspaces of V. Let

$$\begin{aligned} \Gamma : \quad U_1 \times \cdots \times U_m &\to U_1 + \cdots + U_m \\ (\mathbf{u}_1, \ldots, \mathbf{u}_m) &\mapsto \mathbf{u}_1 + \cdots + \mathbf{u}_m \end{aligned}$$

Then

$$U_1 + \cdots + U_m \text{ direct sum} \Leftrightarrow \Gamma \text{ injective}$$

Theorem 2.112 (NSC for direct sum)

Let V be a finite-dimensional vector space, $U_1, \ldots, U_m$ subspaces of V. Then

$$U_1 \oplus \cdots \oplus U_m \Leftrightarrow \dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

Definition 2.113 ($\mathbf{v} + U$)

Let V be a vector space, U a subspace of V and $\mathbf{v} \in V$. Then $\mathbf{v} + U$ is the subset of V defined by

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u}; \mathbf{u} \in U\}$$

Definition 2.114 (Affine subset/Parallel affine subset)

Let V be a vector space

- An **affine subset** of V is a subset of V of the form $\mathbf{v} + U$ for some $\mathbf{v} \in V$ and some subspace U of V
- For $\mathbf{v} \in V$ and U subspace of V, the affine subset $\mathbf{v} + U$ is **parallel** to U

Definition 2.115 (Quotient space)

Let V be a vector space, U a subspace of V. The **quotient space** V/U is the set of all affine subsets of V parallel to U, i.e.,

$$V/U = \{\mathbf{v} + U; \mathbf{v} \in V\}$$

Theorem 2.116 (2 affine subsets // to U are equal or disjoint)

Let V be a vector space, U subspace of V and $v, w \in V$. TFAE

1. $\mathbf{v} - \mathbf{w} \in U$
2. $\mathbf{v} + U = \mathbf{w} + U$
3. $(\mathbf{v} + U) \cap (\mathbf{w} + U) \neq \emptyset$

Definition 2.117 (Addition and scalar multiplication on V/U)

Let V be a vector space, U subspace of V. Then **addition** and **scalar multiplication** on V/U are defined for $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$ by

$$(\mathbf{v} + U) + (\mathbf{w} + U) = (\mathbf{v} + \mathbf{w}) + U$$

and

$$\lambda(\mathbf{v} + U) = (\lambda \mathbf{v}) + U$$

Theorem 2.118 (Quotient space is a vector space)

Let V be a vector space and U subspace of V. Equipped with addition and scalar multiplication as above, V/U is a vector space

Definition 2.119 (Quotient map)

Let V be a vector space, U subspace of V. The **quotient map** π is the linear map $\pi \in \mathcal{L}(V, V/U)$ defined by

$$\pi(\mathbf{v}) = \mathbf{v} + U$$

for $\mathbf{v} \in V$

Theorem 2.120 (Dimension of quotient space)

Let V be a finite-dimensional vector space and U subspace of V. Then

$$\dim V/U = \dim V - \dim U$$

Definition 2.121 ($\tilde{T}$)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Define $\tilde{T}$ by

$$\begin{aligned} \tilde{T} : \quad V/(\operatorname{null} T) &\to W \\ \tilde{T}(\mathbf{v} + \operatorname{null} T) &= T\mathbf{v} \end{aligned}$$

Theorem 2.122 (Null space and range of $\tilde{T}$)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$
2. $\tilde{T}$ *injective*
3. $\operatorname{range} \tilde{T} = \operatorname{range} T$
4. $V/\operatorname{null} T$ *isomorphic to* $\operatorname{range} T$

Definition 2.123 (Linear functional/form)

A **linear functional** (or **linear form**) on a vector space V is a linear map in $\mathcal{L}(V, \mathbb{F})$

Definition 2.124 (Dual space)

The **dual space** $V^\star$ of V is the vector space $V^\star = \mathcal{L}(V, \mathbb{F})$ of linear functionals on V

Theorem 2.125 ($\dim V^\star = \dim V$)

Suppose V is a finite-dimensional vector space. Then $\dim V^\star = \dim V < \infty$

Definition 2.126 (Dual basis)

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of the vector space V, then the **dual basis** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of $V^\star$, where for $j = 1, \ldots, n$, φ_j is the linear functional on V s.t.

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Theorem 2.127 (Dual basis is a basis of the dual space)

Suppose V is a finite-dimensional vector space. Then the dual basis of a basis of V is a basis of $V^\star$

Definition 2.128 (Dual map)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. The **dual map** of T is the linear map $T^\star \in \mathcal{L}(W^\star, V^\star)$ defined by $T^\star(\varphi) = \varphi \circ T$ for $\varphi \in W^\star$

Property 2.129 (Algebraic properties of dual maps)

Let U, V, W be vector spaces

- $(S + T)^\star = S^\star + T^\star$ *for all* $S, T \in \mathcal{L}(V, W)$
- $(\lambda T)^\star = \lambda T^\star$ *for all* $\lambda \in \mathbb{F}$ *and all* $T \in \mathcal{L}(V, W)$
- $(ST)^\star = T^\star S^\star$ *for all* $T \in \mathcal{L}(U, V)$ *and all* $S \in \mathcal{L}(V, W)$

Definition 2.130 (Annihilator)

Let V be a vector space, $U \subseteq V$. The **annihilator** U^0 of U is defined by

$$U^0 = \{\varphi \in V^\star : \forall \mathbf{u} \in U, \quad \varphi(\mathbf{u}) = 0_{\mathbb{F}}\}$$

Theorem 2.131 (The annihilator is a subspace)

Let V be a vector space and $U \subseteq V$. Then the annihilator U^0 is a subspace of $V^\star$

Theorem 2.132 (Dimension of the annihilator)

Let V be a finite-dimensional vector space, $U \subseteq V$ a subspace of V. Then

$$\dim U + \dim U^0 = \dim V$$

Theorem 2.133 (Null space of $T^\star$)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\operatorname{null} T^\star = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T^\star = \dim \operatorname{null} T + \dim W - \dim V$

Theorem 2.134 (T surjective $\Leftrightarrow$ $T^\star$ injective)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \textit{ surjective} \Leftrightarrow T^\star \textit{ injective}$$

Theorem 2.135 (Range of $T^\star$)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\dim \operatorname{range} T^\star = \dim \operatorname{range} T$
2. $\operatorname{range} T^\star = (\operatorname{null} T)^0$

Theorem 2.136 (T injective $\Leftrightarrow$ $T^\star$ surjective)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \textit{ injective} \Leftrightarrow T^\star \textit{ surjective}$$

Theorem 2.137 (Matrix of $T^\star$ is transpose of matrix of T)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then $M(T^\star) = M(T)^T$, where ^{T} denotes the transpose

Eigenvalues, eigenvectors and invariant subspaces

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Definition 2.138 (Invariant subspace)

Let V be a vector space, $T \in \mathcal{L}(V)$. A subspace U of V is **invariant** under T if

$$\mathbf{u} \in U \Rightarrow T\mathbf{u} \in U$$

In other words, U invariant under T if $T|_U \in \mathcal{L}(U)$ [see Definition 2.144]

Definition 2.139 (Eigenvalue)

Let V be a vector space, $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of T if

$$\exists \mathbf{v} \in V, v \neq \mathbf{0}_V, \text{ s.t. } T(\mathbf{v}) = \lambda \mathbf{v}.$$

I use the notation $T(\mathbf{v})$ instead of $T\mathbf{v}$ to emphasise that $T \in \mathcal{L}(V)$.

Theorem 2.140 (Conditions equivalent to being an eigenvalue)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Denote $I_{\mathcal{L}(V)}$ the identity operator, $I_{\mathcal{L}(V)} \in \mathcal{L}(V)$ s.t. $\forall \mathbf{v} \in V$, $I_{\mathcal{L}(V)}\mathbf{v} = \mathbf{v}$. TFAE:

1. *λ eigenvalue of T*
2. *$T - \lambda I_{\mathcal{L}(V)}$ not injective*
3. *$T - \lambda I_{\mathcal{L}(V)}$ not surjective*
4. *$T - \lambda I_{\mathcal{L}(V)}$ not invertible*

Definition 2.141 (Eigenvector)

Let V be a vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ be an eigenvalue of T. A vector $\mathbf{v} \in V$ is an **eigenvector** of T corresponding to λ if $\mathbf{v} \neq 0$ and $T(\mathbf{v}) = \lambda\mathbf{v}$

Theorem 2.142 (Linearly independent eigenvectors)

Let V be a vector space, $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ linearly independent

Theorem 2.143 (Number of eigenvalues)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Then T has at most $\dim V$ distinct eigenvalues

Definition 2.144 (Restriction and quotient operators)

Let V be a vector space, $T \in \mathcal{L}(V)$ and U a subspace of V invariant under T (Def. 2.138)

- The **restriction operator** $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U = T\mathbf{u}, \quad \mathbf{u} \in U$$

- The **quotient operator** $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(\mathbf{v} + U) = T\mathbf{v} + U, \quad \mathbf{v} \in V$$

For the quotient space $\mathcal{L}(V/U)$, see Definition 2.138 and the results that follow

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Definition 2.145

Let V be a vector space, $T \in \mathcal{L}(V)$, $m \in \mathbb{N} \setminus \{0\}$

- $T^m = \underbrace{T \cdots T}_{m \text{ times}}$
- $T^0 = I$, the identity operator on V
- If T invertible with inverse T^{-1}, then $T^{-m} = (T^{-1})^m$

Definition 2.146

Let V be a vector space, $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ be the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad z \in \mathbb{F}$$

Then $p(T)$ is the operator on $\mathcal{L}(V)$ defined by

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

where I is the identity operator

Definition 2.147 (Product of polynomials)

Let $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial

$$(pq)(z) = p(z)q(z), \quad z \in \mathbb{F}$$

Theorem 2.148 (Multiplicative properties)

Let $p, q \in \mathcal{P}(\mathbb{F})$, V a vector space and $T \in \mathcal{L}(V)$. Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

Theorem 2.149 (Operators on complex v.s. have an eigenvalue)

Let V *be a vector space over* $\mathbb{C}$ *with* $\dim V = n < \infty$. *Assume* $T \in \mathcal{L}(V)$. *Then* V *has an eigenvalue*

Definition 2.150 (Matrix of an operator)

Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space, let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of V. The **matrix** of T with respect to the basis is the $n \times n$ matrix

$$M(T) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

with entries a_{jk} defined by

$$Tv_k = a_{1k}\mathbf{v}_1 + \cdots + a_{nk}\mathbf{v}_n$$

If basis is not clear from the context, write $M(T, (\mathbf{v}_1, \ldots, \mathbf{v}_n))$

Definition 2.151 (Diagonal of a matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The **diagonal** of A consists of the entries a_{ii}, $i = 1, \ldots, n$

Definition 2.152 (Upper-triangular matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The matrix A is **upper-triangular** if all entries below the diagonal are 0, i.e.,

$$a_{ij} = 0, \quad \forall i, j \text{ such that } i > j$$

Theorem 2.153 (Conditions for an upper-triangular matrix)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a basis of V. TFAE:

1. *$M(T)$ with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is upper-triangular*
2. $Tv_j \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j)$, $\forall j = 1, \ldots, n$
3. $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j)$ *invariant under T*, $\forall j = 1, \ldots, n$

Theorem 2.154 (Every operator over $\mathbb{C}$ has an UT matrix)

Let V be a finite-dimensional vector space over $\mathbb{C}$, $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V

Theorem 2.155 (Determination of invertibility from UT matrix)

Let V be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then

$$T \textit{ invertible } \Leftrightarrow \forall i = 1, \ldots, n, \quad a_{ii} \neq 0$$

Theorem 2.156 (Determination of eigenvalues from UT matrix)

Let V be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then

$$\lambda \textit{ eigenvalue of } T \Leftrightarrow \lambda \in \{a_{ii}, \quad i = 1, \ldots, n\}$$

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Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Definition 2.157 (Diagonal matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. A is a **diagonal** matrix if all entries of A are zero except possibly on the diagonal, i.e.,

$$\forall i, j,\ i \neq j, \quad a_{ij} = 0.$$

Definition 2.158 (Eigenspace)

Let V be a vector space, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$. The **eigenspace** $E(\lambda, T)$ of T corresponding to λ is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I).$$

Thus λ eigenvalue of $T \Leftrightarrow E(\lambda, T) \neq \{\mathbf{0}_V\}$.

Theorem 2.159 (Sum of eigenspaces is a direct sum)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Assume $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum and

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

Definition 2.160 (Diagonalisable operator)

Let V be a vector space, $T \in \mathcal{L}(V)$. T is **diagonalisable** if T has a diagonal matrix with respect to some basis of V.

Theorem 2.161 (Conditions equivalent to diagonalisability)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of T. TFAE:

1. *T diagonalisable*
2. *V has a basis consisting of eigenvectors of T*
3. *$\exists U_1, \ldots, U_n$ 1-dimensional subspaces of V invariant under T s.t.*

$$V = U_1 \oplus \cdots \oplus U_n$$

4. $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$
5. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

Theorem 2.162 (Sufficient condition for diagonalisability)

Let V be a vector space, $T \in \mathcal{L}(V)$. If T has $\dim V$ distinct eigenvalues, then T diagonalisable

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Definition 2.163 (Inner product)

Let V be a vector space over $\mathbb{F}$. An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ having the following properties, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \lambda \in \mathbb{F}$,

- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ **[positivity]**
- $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$ **[definiteness]**
- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[additivity in first slot]**
- $\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$ **[homogeneity in first slot]**
- $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ **[conjugate symmetry]**

Definition 2.164 (Inner product space)

An **inner product space** is a vector space V along with an inner product on V

Theorem 2.165 (Basic properties of inner product)

Let V be an inner product space over $\mathbb{F}$. Then

1. *For each fixed $\mathbf{u} \in V$, the function $\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{u} \rangle$ is a linear map from V to $\mathbb{F}$*
2. $\forall \mathbf{u} \in V$, $\langle \mathbf{0}_V, \mathbf{u} \rangle = 0$
3. $\forall \mathbf{u} \in V$, $\langle \mathbf{u}, \mathbf{0}_V \rangle = 0$
4. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
5. $\forall \mathbf{u}, \mathbf{v} \in V$ *and* $\forall \lambda \in \mathbb{F}$, $\langle \mathbf{u}, \lambda \mathbf{v} \rangle = \overline{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$

Definition 2.166 (Norm)

Let V be an inner product space over $\mathbb{F}$. For $\mathbf{v} \in V$, the **norm** of v is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Theorem 2.167 (Basic properties of the norm)

Let V be an inner product space, $\mathbf{v} \in V$. Then

1. $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = 0$
2. $\forall \lambda \in \mathbb{F}$, $\|\lambda \mathbf{v}\| = |\lambda| \, \|\mathbf{v}\|$

Definition 2.168 (Orthogonality)

Let V be an inner product space over $\mathbb{F}$. Two vectors $\mathbf{u}, \mathbf{v} \in V$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. We sometimes denote $\mathbf{u} \perp \mathbf{v}$

Theorem 2.169 ($\mathbf{0}$ and orthogonality)

Let V be an inner product space over $\mathbb{F}$. Then

1. $\mathbf{0}_V$ *is orthogonal to every vector in* V
2. $\mathbf{0}_V$ *is the only vector in* V *that is orthogonal to itself*

Theorem 2.170 (Pythagorean theorem)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ s.t. $\mathbf{u} \perp \mathbf{v}$. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Theorem 2.171 (An orthogonal decomposition)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq 0$. Let

$$c = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \ (\in \mathbb{F}) \ \textit{and} \ \mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} \ (\in V).$$

Then

$$\langle \mathbf{w}, \mathbf{v} \rangle = 0 \ \textit{and} \ \mathbf{u} = c\mathbf{v} + \mathbf{w}.$$

Theorem 2.172 (Cauchy-Schwarz inequality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \ \|\mathbf{v}\|$$

with $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \ \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ for some $0 \neq k \in \mathbb{F}$.

Theorem 2.173 (Triangle inequality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

with $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ for some $0 \leq k \in \mathbb{R}$.

Theorem 2.174 (Parallelogram equality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right).$$

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Definition 2.175 (Orthonormal list)

A list of vectors is **orthonormal** if each vector in the list has norm 1 and is orthogonal to all other vectors in the list, i.e., the list $\mathbf{e}_1, \ldots, \mathbf{e}_m$ of vectors in the inner product space V is orthonormal if

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Theorem 2.176 (Norm of an orthonormal linear combination)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space V. Then

$$\|a_1\mathbf{e}_1 + \cdots + a_m\mathbf{e}_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

Theorem 2.177 (Orthonormal lists are LI)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *be an orthonormal list of vectors in an inner product space* V*. Then* $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *is linearly independent*

Definition 2.178 (Orthonormal basis)

An **orthonormal basis** of an inner product space V is an orthonormal list of vectors in V that is also a basis of V

Theorem 2.179 (Orthonormal list & orthonormal basis)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *be an orthonormal list of vectors in an inner product space* V*. If* $\dim V = m$*, then* $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *orthonormal basis of* V*.*

Theorem 2.180 (Vector as LC of orthonormal basis)

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of the inner product space V, $\mathbf{v} \in V$. Then

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n$$

and

$$\|\mathbf{v}\|^2 = |\langle \mathbf{v}, \mathbf{e}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{e}_n \rangle|^2$$

Theorem 2.181 (Gram-Schmidt procedure)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *be a linearly independent list of vectors in an inner product space* V*. Let*

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

For $j = 2, \ldots, m$*, define* $\mathbf{e}_j$ *inductively by*

$$\mathbf{e}_j = \frac{\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}}{\|\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}\|}$$

Then $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *is an orthonormal list of vectors in* V *such that*

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j) = \operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_j), \quad j = 1, \ldots, m$$

Theorem 2.182 (Existence of orthonormal basis)

Let V be a finite-dimensional inner product space. Then V has an orthonormal basis

Theorem 2.183 (Extending orthonormal list to basis)

Let V be a finite-dimensional inner product space. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V

Theorem 2.184 (UT matrix wrt orthonormal basis)

Let V be a finite-dimensional inner product space, $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some orthonormal basis of V

Theorem 2.185 (Schur's Theorem)

Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V

Theorem 2.186 (Riesz representation Theorem)

Let V be a finite-dimensional inner product space, $\varphi \in \mathcal{L}(V, \mathbb{F})$ a linear functional on V. Then $\exists \mathbf{u} \in V$ unique s.t.

$$\forall \mathbf{v} \in V, \quad \varphi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle.$$

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Definition 2.187 (Orthogonal complement)

Let V be an inner product space, $U \subset V$. The **orthogonal complement** $U^\perp$ of U is the set

$$U^\perp = \{\mathbf{v} \in V : \forall \mathbf{u} \in U, \quad \langle \mathbf{v}, \mathbf{u} \rangle = 0\}$$

Property 2.188 (Basic properties of orthogonal complement)

1. *If $U \subset V$, then $U^\perp$ subspace of V*
2. $\{\mathbf{0}_V\}^\perp = V$
3. $V^\perp = \{\mathbf{0}_V\}$
4. *If $U \subset V$, then $U \cap U^\perp \subset \{0\}$*
5. *If $U \subset W \subset V$, then $W^\perp \subset U^\perp$*

Theorem 2.189 (Direct sum U and $U^\perp$)

Let U be a finite-dimensional subspace of V, inner product space. Then

$$V = U \oplus U^\perp$$

Theorem 2.190 (Dimension of $U^\perp$)

Let V be a finite-dimensional inner product space, U subspace of V. Then

$$\dim U^\perp = \dim V - \dim U$$

Theorem 2.191 (Orth. complement of orth. complement)

Let U be a finite-dimensional subspace of the inner product space V. Then

$$\left(U^\perp\right)^\perp = U$$

Definition 2.192 (Orthogonal projection P_U)

Let V be an inner product space, U a finite-dimensional subspace of V. The **orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined by

$$P_U \mathbf{v} = \mathbf{u},$$

where $\mathbf{v} \in V$ is written $\mathbf{v} = \mathbf{u} + \mathbf{w}$, with $\mathbf{u} \in U$ and $\mathbf{w} \in U^{\perp}$

Property 2.193 (Properties of the orthogonal projection P_U)

Let V be an inner product space, U a finite-dimensional subspace of V, $v \in V$. Then

1. $P_U \in \mathcal{L}(V)$
2. $\forall \mathbf{u} \in U$, $P_U \mathbf{u} = \mathbf{u}$
3. $\forall \mathbf{w} \in U^\perp$, $P_U \mathbf{w} = \mathbf{0}_V$
4. $\text{range} P_U = U$
5. $\text{null} P_U = U^\perp$
6. $\mathbf{v} - P_U \mathbf{v} \in U^\perp$
7. $P_U^2 = P_U$
8. $\|P_U \mathbf{v}\| \leq \|\mathbf{v}\|$
9. *for every orthonormal basis* $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *of* U,

$$P_U \mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_m \rangle \mathbf{e}_m.$$

Theorem 2.194 (Minimising distance to a subspace)

Let V be an inner product space, U a finite-dimensional subspace of V, $\mathbf{v} \in V$, $u \in U$. Then

$$\|\mathbf{v} - P_U\mathbf{v}\| \leq \|\mathbf{v} - \mathbf{u}\|$$

with equality if and only if $\mathbf{u} = P_U\mathbf{v}$

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Definition 2.195 (Adjoint)

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^\star : W \to V$ such that

$$\forall \mathbf{v} \in V, \forall \mathbf{w} \in W, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^\star \mathbf{w} \rangle$$

Theorem 2.196 (Adjoint is a linear map)

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Then

$$T^\star \in \mathcal{L}(W, V)$$

Property 2.197 (Properties of the adjoint)

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$. Then

1. $\forall S, T \in \mathcal{L}(V, W)$, $(S + T)^\star = S^\star + T^\star$
2. $\forall T \in \mathcal{L}(V, W)$, $\forall \lambda \in \mathbb{F}$, $(\lambda T)^\star = \overline{\lambda} T^\star$
3. $\forall T \in \mathcal{L}(V, W)$, $(T^\star)^\star = T$
4. $I^\star = I$ *if* I *is the identity operator on* V
5. *Let* U *be an inner product space over* $\mathbb{F}$, *then* $\forall T \in \mathcal{L}(V, W)$ *and* $\forall S \in \mathcal{L}(W, U)$, $(ST)^\star = T^\star S^\star$

Theorem 2.198 (Null space and range of $T^\star$)

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Then

1. $\operatorname{null} T^\star = (\operatorname{range} T)^\perp$
2. $\operatorname{range} T^\star = (\operatorname{null} T)^\perp$
3. $\operatorname{null} T = (\operatorname{range} T^\star)^\perp$
4. $rangeT = (\operatorname{null} T^\star)^\perp$

Definition 2.199 (Conjugate transpose)

Let $M \in \mathcal{M}_{mn}(\mathbb{F})$, $M = [m_{ij}]$. The **conjugate transpose** of M, often denoted $M^\star$, is the matrix

$$M^\star = [\overline{m_{ji}}] \in \mathcal{M}_{nm}$$

i.e, the matrix obtained by transposing M then taking the (complex) conjugate of each entry

Theorem 2.200 (Matrix of $T^\star$)

Let V, W be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be orthonormal bases of V and W, respectively. Then

$$M(T^\star, (\mathbf{f}_1, \ldots, \mathbf{f}_m), (\mathbf{e}_1, \ldots, \mathbf{e}_n))$$

is the conjugate transpose of

$$M(T, (\mathbf{e}_1, \ldots, \mathbf{e}_n), (\mathbf{f}_1, \ldots, \mathbf{f}_m))$$

Definition 2.201 (Self-adjoint operator)

Let V be an inner product space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. T is **self-adjoint** (or **Hermitian**) if

$$T = T^\star$$

In other words, $T \in \mathcal{L}(V)$ self-adjoint $\iff$

$$\forall \mathbf{v}, \mathbf{w} \in V, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T\mathbf{w} \rangle$$

Theorem 2.202 (Eigenvalues of self-adjoint operators are real)

Let V be an inner product space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Then all eigenvalues of T are real

Theorem 2.203

Let V be a complex inner product space, $T \in \mathcal{L}(V)$. Then

$$(\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = \mathbf{0}$$

Theorem 2.204

Let V be a complex inner product space, $T \in \mathcal{L}(V)$. Then

$$(T \ \textit{self-adjoint}) \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v} \rangle \in \mathbb{R})$$

Theorem 2.205

Let V be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint. Then

$$(\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = 0$$

Definition 2.206 (Normal operator)

Let V be an inner product space, $T \in \mathcal{L}(V)$. T is **normal** if

$$TT^\star = T^\star T$$

In words, T is normal if it commutes with its adjoint

Theorem 2.207 (T normal $\Leftrightarrow \|T\mathbf{v}\| = \|T^\star\mathbf{v}\|$)

Let V be an inner product space, $T \in \mathcal{L}(V)$. Then

$$T \ normal \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V, \ \|T\mathbf{v}\| = \|T^\star\mathbf{v}\|)$$

Theorem 2.208 (T normal and $T^\star$ have same eigenvectors)

Let V be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. Then

$$(\lambda, \mathbf{v}) \text{ eigenpair of } T \quad \Leftrightarrow \quad (\overline{\lambda}, \mathbf{v}) \text{ eigenpair of } T^\star$$

Theorem 2.209 (Orthogonal eigenvectors for normal operators)

Let V be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. If $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ eigenpairs of T with $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1 \perp \mathbf{v}_2$.

Theorem 2.210

Let V be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint and $b, c \in \mathbb{R}$ s.t. $b^2 < 4c$. Then $T^2 + bT + cI$ invertible

Theorem 2.211 (Self-adjoint operators have eigenvalues)

Let $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ be self-adjoint. Then T has an eigenvalue

Theorem 2.212 (Self-adjoint operators & invariant subspaces)

Let V be an inner product space, $T \in \mathcal{L}(V)$ be self-adjoint and U be a subspace of V invariant under T. Then

1. *$U^\perp$ invariant under T*
2. *$T|_U \in \mathcal{L}(U)$ self-adjoint*
3. *$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ self-adjoint*

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Theorem 2.213 (Complex spectral theorem)

Let V be an inner product space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. TFAE:

1. *T normal*
2. *V has an orthonormal basis consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some orthonormal basis of V*

Theorem 2.214 (Real spectral theorem)

Let V be an inner product space over $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$. TFAE:

1. *T self-adjoint*
2. *V has an orthonormal basis consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some orthonormal basis of V*

Definition 2.215 (Positive (semidefinite) operator)

Let V be an inner product space. An operator $T \in \mathcal{L}(V)$ is **positive** (or **positive semidefinite**) if T is self-adjoint and

$$\forall \mathbf{v} \in V, \quad \langle T\mathbf{v}, \mathbf{v} \rangle \geq 0$$

Definition 2.216 (Square root operator)

Let V be an inner product space. An operator $R \in \mathcal{L}(V)$ is a **square root** of an operator $T \in \mathcal{L}(V)$ if

$$R^2 = T$$

Theorem 2.217 (Characterisation of positive operators)

Let $T \in \mathcal{L}(V)$, where V is an inner product space. TFAE:

1. *T positive semidefinite*
2. *T self-adjoint and all eigenvalues of T are nonnegative*
3. *T has a positive semidefinite square root*
4. *T has a self-adjoint square root*
5. *$\exists R \in \mathcal{L}(V)$ s.t. $T = R^\star R$*

Theorem 2.218 (Uniqueness of positive semidefinite square root)

Let $T \in \mathcal{L}(V)$ be a positive semidefinite operator on an inner product space V. Then T has a unique positive semidefinite square root

Definition 2.219 (Isometry)

Let V be an inner product space. $S \in \mathcal{L}(V)$ is an **isometry** if

$$\forall \mathbf{v} \in V, \quad \|S\mathbf{v}\| = \|\mathbf{v}\|$$

Theorem 2.220 (Characterisation of isometries)

Let V be an inner product space, $S \in \mathcal{L}(V)$. TFAE:

1. S *isometry*
2. $\forall \mathbf{u}, \mathbf{v} \in V$, $\langle S\mathbf{u}, S\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$
3. $\forall \mathbf{e}_1, \ldots, \mathbf{e}_n \in V$ *orthonormal list,* $S\mathbf{e}_1, \ldots, S\mathbf{e}_n$ *orthonormal*
4. $\exists \mathbf{e}_1, \ldots, \mathbf{e}_n$ *orthonormal basis of* V *s.t.* $S\mathbf{e}_1, \ldots, S\mathbf{e}_n$ *orthonormal*
5. $S^\star S = I$
6. $SS^\star = I$
7. $S^\star$ *isometry*
8. S *invertible and* $S^{-1} = S^\star$

Theorem 2.221 (Isometries when $\mathbb{F} = \mathbb{C}$)

Let V be a complex inner product space, $S \in \mathcal{L}(V)$. TFAE:

1. *S isometry*
2. *$\exists$ orthonormal basis of V consisting of eigenvectors of S with corresponding eigenvalues all having modulus 1*

Let T be a positive semidefinite operator, then denote $\sqrt{T}$ the unique positive semidefinite square root of T

Theorem 2.222 (Polar decomposition)

Let V be an inner product space, $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ s.t.

$$T = S\sqrt{T^\star T}$$

Definition 2.223 (Singular values)

Let V be an inner product space, $T \in \mathcal{L}(V)$. The **singular values** of T are the eigenvalues of $\sqrt{T^\star T}$, with each eigenvalue λ repeated $\dim E(\lambda, \sqrt{T^\star T})$ times. All are nonnegative

Theorem 2.224 (Singular value decomposition – SVD)

Let V be an inner product space. Assume $T \in \mathcal{L}(V)$ has singular values $s_1, \dots, s_n$. Then $\exists \mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{f}_1, \dots, \mathbf{f}_n$ orthonormal bases of V s.t.

$$\forall \mathbf{v} \in V, \quad T\mathbf{v} = s_1\langle v, \mathbf{e}_1\rangle \mathbf{f}_1 + \cdots + s_n\langle \mathbf{v}, \mathbf{e}_n\rangle \mathbf{f}_n$$

Theorem 2.225 (SV without square root)

Let V be an inner product space. The singular values of T are the nonnegative square roots of the eigenvalues of $T^\star T$, with each eigenvalue λ repeated $\dim E(\lambda, T^\star T)$ times

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Theorem 2.226 (Sequence of increasing null spaces)

Let V be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Then

$$\{0\} = \operatorname{null} T^0 \subset \operatorname{null} T^1 \subset \cdots \subset \operatorname{null} T^k \subset \operatorname{null} T^{k+1} \subset \cdots$$

Theorem 2.227 (Equality in sequence of null spaces)

Let V be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Assume $m \in \mathbb{N} \setminus \{0\}$ is s.t.

$$\operatorname{null} T^m = \operatorname{null} T^{m+1}$$

Then

$$\forall k \in \mathbb{N}, \quad \operatorname{null} T^{m+k} = \operatorname{null} T^m$$

Theorem 2.228 (Null spaces stop growing)

Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim V = n$, $T \in \mathcal{L}(V)$. Then

$$\forall k \in \mathbb{N}, \quad \operatorname{null} T^{n+k} = \operatorname{null} T^n$$

Theorem 2.229 ($V = \operatorname{null} T^{\dim V} \oplus \operatorname{range} T^{\dim V}$)

Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim V = n$, $T \in \mathcal{L}(V)$. Then

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n$$

Definition 2.230 (Generalised eigenvector)

Let V be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ an eigenvalue of T. $\mathbf{v} \in V$ is a **generalised eigenvector** of T corresponding to λ if $\mathbf{v} \neq 0$ and

$$\exists j \in \mathbb{N} \setminus \{0\}, \quad (T - \lambda I)^j \mathbf{v} = \mathbf{0}_V$$

Definition 2.231 (Generalised eigenspace)

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalised eigenspace** $G(\lambda, T)$ of T corresponding to λ is the set of all generalised eigenvectors of T corresponding to λ together with the $\mathbf{0}_V$ vector

Theorem 2.232 (Description of generalised eigenspaces)

Let $T \in \mathcal{L}(V)$ *and* $\lambda \in \mathbb{F}$. *Then*

$$G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}$$

Theorem 2.233 (LI generalised eigenvectors)

Let $T \in \mathcal{L}(V)$. *Assume* $\lambda_1, \ldots, \lambda_m$ *are distinct eigenvalues of* T, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *corresponding generalised eigenvectors. Then* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *linearly independent.*

Definition 2.234 (Nilpotent operator)

An operator is **nilpotent** if $\exists k \in \mathbb{N}$ s.t. $T^k = 0$

Theorem 2.235 (A loose upper bound on power required)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then

$$N^{\dim V} = 0$$

Theorem 2.236 (Matrix of a nilpotent operator)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then there exists a basis of V with respect to which $M(N)$ is **strictly upper triangular**, *i.e.,*

$$M(N) = [m_{ij}] \text{ is s.t. } m_{ij} = 0 \text{ if } i \geq j$$

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Theorem 2.237 (& range of $p(T)$ invariant under T)

Let $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then $p(T)$ and range *$p(T)$ invariant under T*

Theorem 2.238 (Description of operators when $\mathbb{F} = \mathbb{C}$)

Suppose V complex vector space, $T \in \mathcal{L}(V)$. Assume $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Then

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
2. *each $G(\lambda_j, T)$ invariant under T*
3. $\forall j = 1, \dots, m$, $(T - \lambda_j I)|_{G(\lambda_j, T)}$ *nilpotent*

Theorem 2.239 (Basis of generalised eigenvectors)

Let V be a complex vector space and $T \in \mathcal{L}(V)$. Then there exists a basis of V consisting of generalised eigenvectors of T

Definition 2.240 (Multiplicity of an eigenvalue)

Let $T \in \mathcal{L}(V)$. The (**algebraic**) **multiplicity** of an eigenvalue λ of T is

- $\dim G(\lambda, T)$
- $\dim (T - \lambda I))^{\dim V}$

Theorem 2.241 ($\sum$ multiplicities $= \dim V$)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be distinct eigenvalues of T with multiplicities $d_1, \ldots, d_n$. Then

$$\sum_{k=1}^{n} d_k = \dim V$$

Definition 2.242 (Block diagonal matrix)

Let $A_1, \ldots, A_m$ be square matrices (not necessarily of the same size). A **block matrix** is a matrix of the form

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

We also write

$$A = \operatorname{diag}(A_1, \ldots, A_m)$$

You will also see (not in this book)

$$A = A_1 \oplus \cdots \oplus A_m$$

Theorem 2.243 (Block diagonal matrix with UT blocks)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, \ldots, d_m$. Then there exists a basis of V s.t. T has a block diagonal matrix

$$\operatorname{diag}(A_1, \ldots, A_m)$$

with each A_j a $d_j \times d_j$ upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

Theorem 2.244 (Identity plus nilpotent has square root)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then $I + N$ has a square root

Theorem 2.245 (T invertible has square root when $\mathbb{F} = \mathbb{C}$)

Let V be a complex vector space, $T \in \mathcal{L}(V)$ invertible. Then T has a square root

Definition 2.246 (Characteristic polynomial)

Let V be a complex vector space, $T \in \mathcal{L}(V)$, $\lambda_1, \dots, \lambda_m$ the distinct eigenvalues of T with multiplicities $d_1, \dots, d_m$. The **characteristic polynomial** of T is

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

Theorem 2.247 (Degree and zeros of char. polyn.)

V a complex vector space, $T \in \mathcal{L}(V)$. Then

1. *the characteristic polynomial of T has degree* $\dim V$
2. *zeros of the characteristic polynomial of T are the eigenvalues of T*

Theorem 2.248 (Cayley-Hamilton)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let q be the characteristic polynomial of T. Then $q(T) = 0$

Definition 2.249 (Monic polynomial)

A **monic polynomial** is a polynomial with highest degree coefficient equal to 1

Theorem 2.250 (Minimal polynomial)

Let $T \in \mathcal{L}(V)$. Then there exists a unique monic polynomial p of smallest degree s.t. $p(T) = 0$

Definition 2.251 (Minimal polynomial)

Let $T \in \mathcal{L}(V)$. The **minimal polynomial** of T is the unique monic polynomial p of smallest degree s.t. $p(T) = 0$

Theorem 2.252

Let $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0 \Leftrightarrow q$ polynomial multiple of the minimal polynomial of T

Theorem 2.253 (Char. polyn. is multiple of min. polyn.)

Assume V vector space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T

Theorem 2.254 (Eigenvalues are zeros of min. polyn.)

Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T

Theorem 2.255 (Basis corresponding to nilpotent operator)

Let $N \in \mathcal{L}(V)$ *be nilpotent. Then* $\exists \mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ *and* $m_1, \ldots, m_n \in \mathbb{N}$ *s.t.*

1. $N^{m_1}\mathbf{v}_1, \ldots, N\mathbf{v}_1, \mathbf{v}_1, N^{m_n}\mathbf{v}_n, \ldots, N\mathbf{v}_n, \mathbf{v}_n$ *is a basis of* V
2. $N^{m_1+1}\mathbf{v}_1 = \cdots = N^{m_n+1}\mathbf{v}_n = 0$

Definition 2.256 (Jordan basis)

Let $T \in \mathcal{L}(V)$. A **Jordan basis** for T is a basis of V s.t. with respect to this basis, T has a block diagonal matrix

$$\operatorname{diag}(A_1, \ldots, A_p)$$

where each A_j is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Theorem 2.257 (Jordan form)

Let V be a complex vector space. If $T \in \mathcal{L}(V)$, then $\exists$ a Jordan basis for T

An algorithm for finding the Jordan form

An algorithm to compute the Jordan canonical form of an $n \times n$ matrix A [MM82].

1. Compute the eigenvalues of A. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A with multiplicities $n_1, \ldots, n_m$, respectively.
2. Compute n_1 linearly independent generalized eigenvectors of A associated with λ_1 as follows. Compute
$$(A - \lambda_1 E_n)^i$$
for $i = 1, 2, \ldots$ until the rank of $(A - \lambda_1 E_n)^k$ is equal to the rank of $(A - \lambda_1 E_n)^{k+1}$. Find a generalized eigenvector of rank k, say u. Define $u_i = (A - \lambda_1 E_n)^{k-1} u$, for $i = 1, \ldots, k$. If $k = n_1$, proceed to step 3. If $k < n_1$, find another linearly independent generalized eigenvector with rank k. If this is not possible, try $k - 1$, and so forth, until n_1 linearly independent generalized eigenvectors are determined. Note that if $\rho(A - \lambda_1 E_n) = r$, then there are totally $(n - r)$ chains of generalized eigenvectors associated with λ_1.
3. Repeat step 2 for $\lambda_2, \ldots, \lambda_m$.

1. Let $u_1, \ldots, u_k, \ldots$ be the new basis. Observe that Thus in the new basis, A has the desired representation
2. The similarity transformation which yields $J = Q^{-1}AQ$ is given by $Q = [u_1, \ldots, u_k, \ldots]$.

References I

- Sheldon Axler, *Linear algebra done right*, Springer, 2015.
- R.K Miller and A.N. Michel, *Ordinary differential equations*, Academic Press, 1982.