MATH 4370/7370 – Linear Algebra and Matrix Analysis

Essentially nonnegative matrices and M-matrices

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Outline

Essentially nonnegative matrices

Z-matrices

Class K_0

M-matrices

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The Perron-Frobenius can be applied not only to nonnegative matrices, but also to matrices that are *essentially nonnegative*, in the sense that they are nonnegative except perhaps along the main diagonal

Definition 1

A matrix $A \in \mathcal{M}_n$ is **essentially nonegative** (or **quasi-positive**) if there exist $\alpha \in \mathbb{R}$ such that $A + \alpha\mathbb{I} \geq 0$

Remark 2

An essentially nonnegative matrix A has non-negative off-diagonal entries. The sign of the diagonal entries is not relevant

Remark 3

Irreducibility of a matrix is not affected by the nature of its diagonal entries. Indeed, consider an essentially nonnegative matrix A. The existence of a directed path in $G(A)$ does not depend on the existence of "self-loops". The same is not true of primitive matrices, where the presence of negative entries on the main diagonal has an influence on the values of A^k and thus ultimately, on the capacity to find k such that $A^k > 0$

So we can apply the "weak" versions of the Perron-Frobenius Theorem (the imprimitive cases in Theorem **??**) to $A + \alpha\mathbb{I}$, which is a nonnegative matrix (potentially irreducible). One important ingredient is a result that was proved as Theorem **??**. Namely, that perturbations of the entire diagonal by the same scalar lead to a shift of the spectrum; this is summarised as

$$\sigma(A + \alpha\mathbb{I}) = \{\lambda_1 + \alpha, \ldots, \lambda_n + \alpha, \quad \lambda_i \in \sigma(A)\}$$

Definition 4 (Spectral abscissa)

Let $A \in \mathcal{M}_n$. The **spectral abscissa** of A, $s(A)$, is

$$s(A) = \max\{\mathrm{Re}(\lambda), \lambda \in \sigma(A)\}$$

Theorem 5

Let $A \in \mathcal{M}_n(\mathbb{R})$ be essentially nonnegative. Then $s(A)$ is an eigenvalue of A and is associated to a nonnegative eigenvector. If, additionally, A is irreducible, then $s(A)$ is simple and is associated to a positive eigenvector

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Definition 6

A matrix is of class Z_n if it is in $\mathcal{M}_n(\mathbb{R})$ and such that $a_{i,j} \leq 0$, $i \neq j$, $i,j = 1,\ldots,n$

$$Z_n = \{A \in \mathcal{M}_n : a_{i,j} \leq 0, i \neq j\}$$

We also say that $A \in Z_n$ has the **Z-sign pattern**

Theorem 7 ([?])

Let $A \in Z_n$. TFAE

1. *There is a nonnegative vector x such that $Ax > 0$*
2. *There is a positive vector x such that $Ax > 0$*
3. *There is a diagonal matrix $\operatorname{diag}(D) > 0$ such that the entries in $AD = [w_{ik}]$ are such that*
$$w_{ii} > \sum_{k \neq i} |w_{ik}| \forall i$$
4. *For any $B \in Z_n$ such that $A \geq A$, then B is nonsingular*
5. *Every real eigenvalue of any principal submatrix of A is positive.*
6. *All principal minors of A are positive*

Theorem 7 (Continued)

7. *For all* $k = 1, \dots, n$*, the sum of all principal minors is positive*
8. *Every real eigenvalue of* A *is positive*
9. *There exists a matrix* $C \geq 0$ *and a number* $k > \rho(A)$ *such that* $A = k\mathbb{I} - C$
10. *There exists a splitting* $A = P - Q$ *of the matrix* A *such that* $P^{-1} \geq 0$*,* $Q \geq 0$*, and* $\rho(P^{-1}Q < 1)$
11. A *is nonsingular and* $A^{-1} \geq 0$
12. ...

18 *The real part of any eigenvalue of* A *is positive*

Notation: $A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class K (or a nonsingular M-matrix).

Theorem 8

Let $A \in Z = \bigcap_{i=1,\dots} Z_n$ *be symmetric. Then* $A \in K$ *if and only if* A *is positive define.*

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Theorem 9

Let $A \in Z_n$. TFAE

1. $A + \varepsilon\mathbb{I} \in K$ *for all* $\varepsilon > 0$
2. *Every real eigenvalue of a principal submatrix of A is nonnegative*
3. *All principal minors of A are nonnegative*
4. *The sum of all principal minors of order $k = 1, \ldots, n$ is nonnegative*
5. *Every real eigenvaue of A is nonegative*
6. *There exists $C \geq 0$ and $k \geq \rho(C)$ such that $A = k\mathbb{I} - C$*
7. *Every eigenvalue of A has nonnegative real part*

$A \in Z_n$ such that any of these properties holds is a matrix of class K_0

Theorem 10

Let $A \in Z_n$. Assume $A \in K_0$. Then $A \in K \iff A$ nonsingular

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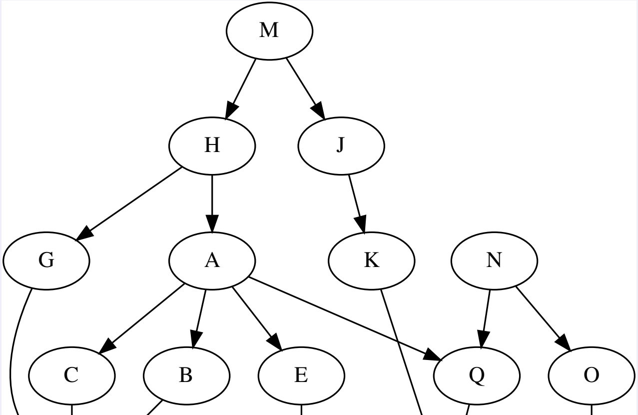
The following theorem can be found in [**?**], pages 134–138. It is famous for .. well, you will understand soon.

Theorem 11 ([?])

Let $A \in \mathcal{M}_n$. Then for each fixed letter $\mathcal{C}$ representing one of the following conditions, conditions $\mathcal{C}_i$ are equivalent for each i. Moreover, letting $\mathcal{C}$ then represent any of the equivalent conditions $\mathcal{C}_i$, the following implication tree holds:



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