MATH 4370/7370 – Linear Algebra and Matrix Analysis

The Singular Value Decomposition

Julien Arino

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University of Manitoba

Outline

Singular values and the Singular value decomposition

Properties of Singular Values

Matrix norms and Singular values

Definition 5.1

Let A be a Hermitian matrix in $\mathcal{M}_n$. We say that A is **positive definite** if for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* A \mathbf{x} > 0$. We say that A is **positive semidefinite** if for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^* A \mathbf{x} \geq 0$

Theorem 5.2

Let $A \in \mathcal{M}_n$ be a Hermitian matrix. Then

1. *for all $\mathbf{x} \in \mathbb{C}^*$, $\mathbf{x}^* A \mathbf{x} \in \mathbb{R}$*
2. $\sigma(A) \subset \mathbb{R}$
3. *$S^* AS$ is Hermitian for any $S \in \mathcal{M}_n$*

Theorem 5.3

Each eigenvalue of a positive definite matrix (respectively positive semidefinite matrix) is positive (respectively nonnegative)

Proposition 5.4

Let A be a positive semidefinite (respectively positive definite) matrix. Then $\mathrm{tr}(A)$, $\det(A)$, *the principal minors of A are all nonnegative (respectively positive). Also,* $\mathrm{tr}(A) = 0$ *if and only if* $A = 0$

Theorem 5.5

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and $\mathbf{x} \in \mathbb{C}^n$. Then

$$\mathbf{x}^* A\mathbf{x} = 0 \iff A\mathbf{x} = \mathbf{0}$$

Corollary 5.6

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix. Then A is positive definite if and only if A is nonsingular

Theorem 5.7 (Somewhat unrelated)

Let $B \in \mathcal{M}_n$ be a Hermitian matrix, $\mathbf{y} \in \mathbb{C}^n$, and $a \in \mathbb{R}$. Let

$$A = \begin{pmatrix} B & \mathbf{y} \\ \mathbf{y}^* & a \end{pmatrix} \in \mathcal{M}_{n+1}$$

Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

Definition 5.8

The singular values of a matrix A are the (nonnegative) square roots of the eigenvalues of A^*A

Remark 5.9

A^*A is positive semidefinite

Theorem 5.10 (Zhang)

Let $A \in \mathcal{M}_{mn}$ with nonzero singular values $\sigma_1, \ldots, \sigma_r$. Then there exists $U \in \mathcal{M}_n$ and $V \in \mathcal{M}_n$ unitary such that

$$A = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V,$$

where $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{mn}$ *and* $D_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$

Theorem 5.11 (H & J)

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$*. Assume that the rank of* A *is* n*. Then*

1. $\exists V \in M_n$ *and* $W \in \mathcal{M}_m$ *unitary matrices and* $\Sigma_q \in \mathcal{M}_q = \mathrm{diag}(\sigma_1, \ldots, \sigma_q)$ *s.t.*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$$

   and

$$A \Sigma W$$

   where

$$\Sigma = \begin{cases} \Sigma_1, & m = n \\ \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{nm}, & m > n \\ \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{nm}, & n > m \end{cases}$$

2. *The parameters* $\sigma_1, \ldots, \sigma_r$ *are the positive square roots of the decreasingly ordered eigenvalues of* $A^* A$

Remark 5.12

Let $A \in \mathcal{M}_{mn}$. Then $A, \overline{A}, A^T$, and A^* have the same singular values

Remark 5.13

Let $A \in \mathcal{M}_n$ with singular values $\sigma_1, \ldots, \sigma_n$, then

$$\sigma_1 \ldots \sigma_n = \det(A)$$

and

$$\sigma_1^2 + \ldots + \sigma_n^2 = \mathrm{tr}(A^*A)$$

Theorem 5.14

Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, *and* $\sigma_1 \geq \cdots \geq \sigma_q$ *nonincresingly ordered singular values of* A. *Define*

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

to be a Hermitian matrix. Then the ordered eigenvalues of $\mathcal{A}$ *are*

$$-\sigma_1 \leq \cdots \leq -\sigma_q \leq \underbrace{0 = \cdots = 0}_{|n-m|} \leq \sigma_q \leq \cdots \leq \sigma_1$$

Theorem 5.15 (An interlacing result)

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\hat{A}$ be the matrix obtained from A by deleting one row and one column. Let $\sigma_1 \geq \cdots \geq \sigma_q$ and $\hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_q$ be the nonsingular ordered singular values of A and $\hat{A}$, respectively, where $\hat{\sigma}_q = 0$ if $n \geq m$ and a column is deleted or if $n \geq m$ and a row is deleted. Then

$$\sigma_1 \geq \hat{\sigma_1} \geq \sigma_2 \geq \hat{\sigma_2} \geq \ldots \sigma_q \geq \hat{\sigma_q}.$$

Theorem 5.16 (von Neumann)

Let A, $B \in \mathcal{M}_{mn}$, $q = \min\{m, n\}$, $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ and $\sigma_1(B) \geq \cdots \geq \sigma_q(B)$ the non-increasingly singular values of A and B, respectively. Then

$$\operatorname{Re}\operatorname{tr}(AB^*) \leq \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B).$$

Theorem 5.17

Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, and $\sigma_1 \geq \cdots \geq \sigma_q$ nonincreasingly ordered singular values of A, and $\alpha = \{1, \ldots, q\}$. Then

$$\operatorname{Re}\operatorname{tr}(A) \leq \sum_{i=1}^{q} \sigma_i$$

with equality if and only if $A[\alpha]$ (principal leading submatrix of A) is positive semidefinite and A has no nonzero entries outside $A[\alpha]$.

- Let $A \in \mathcal{M}_2$

$$\sigma_1, \sigma_2 = \frac{1}{2}\left((\mathrm{tr}A^*A) \mp \sqrt{(\mathrm{tr}A^*A)^2 - 4|\mathrm{det}A|^2}\right)$$

- The nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & & \\ & \ddots & & \\ & & & a_{n-1,m} \\ & & & 0 \end{pmatrix}$$

has singular values $0, |a_{12}|, \ldots, |a_{n-1,n}|$.

Theorem 5.18

Let $A_1, A_2, \cdots \in \mathcal{M}_{nm}$ *given (infinite) sequence with* $\lim_{k\to\infty} A_k = A$ *(entrywise). Let* $q = \min(m, n)$*. Let* $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ *and* $\sigma_1(A_k) \geq \cdots \geq \sigma_q(A_k)$ *be the non-increasinly ordered singular values of* A *and* A_k*, respectively (for all* k*). Then*

$$\lim_{k\to\infty} \sigma_i(A_k) = \sigma_i(A)$$

Theorem 5.19

Let $A \in \mathcal{M}_n$ *where* $n = \text{rank } A$

1. $A = A^T$ *if and only if there exists* $U \in \mathcal{M}_n$ *unitary and a nonnegative diagonal matrix* Σ *such that* $A = U\Sigma U^T$. *Then the diagonal entries of* Σ *are the singular values of* A
2. *If* $A = -A^T$, *then* n *is even and there exists* $U \in \mathcal{M}_n$ *unitary and positive real scalars* $s_1, \dots, s_{r/2}$ *such that*

$$U\left(\begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_{r/2} \\ -s_{r/2} & 0 \end{pmatrix}\right) U^T$$

The non-zero singular values of A *are* $s_1, s_1, \dots, s_{r/2}, s_{r/2}$. *Conversely, any matrix of the above form is skew-symetric*

Let $V = \mathcal{M}_{mn}(\mathbb{C})$ with Frobenius inner product

$$\langle A, B \rangle_F = \mathrm{tr}(B^* A)$$

The norm derived from the Frobenius inner product is

$$\|A\|_2 = (\mathrm{tr}(A^* A))^{1/2}$$

is the ℓ-2 norm (or Frobenius norm)

The spectral norm $\|\cdot\|$ defined on $\mathcal{M}_n$ by

$$\|A\|_2 = \sigma_1(A),$$

where $\sigma_1(A)$ is the largest singular value of A is induced by the ℓ-2 norm on $\mathbb{C}^n$. Inded, from the singular value decomposition theorem, let

$$A = V\Sigma W^*$$

be a singular value decomposition of A, where V, W unitary, $\Sigma = \sigma(\sigma_1, \ldots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are the non-increasingly ordered singular values of A

From unitary invariance and monotonicity of the Euclidean norm, we say that

$$
\begin{aligned}
\max \|Ax\|_1 &= \max_{\|x\|_1} \|V\Sigma W^*\|_2 \\
&= \max_{\|x\|_2} \|\Sigma W^* x\|_2 \\
&= \max_{\|Wy\|_2=1} \|\Sigma y\|_2 \\
&= \max_{\|y\|_2} \|\Sigma y\|_2 \\
&\leq \max_{\|y\|_2} \|\sigma_1 y\|_2 \\
&= \sigma_1 \max_{\|y\|_2} \|y\|_2 \\
&= \sigma_1
\end{aligned}
$$

Since $\|\Sigma y\|_2 = \sigma_1$ for $y = e_1$,

$$
\max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)
$$

We could have used

$$\max_{\|x\|_2=1} = \|Ax\|_2^2 = \max_{\|x\|_2=1} x^*A^*AX$$
$$= \lambda_{max}(A^*A)$$
$$= \sigma_1(A)$$

Proposition 5.20

For all $U, V \in \mathcal{M}_n$ unitary matrices, we have $\|UAV\|_2 = \|A\|_2$

References I