

# Nonnegative matrices

Julien Arino

Fall 2025

# Outline

Definitions and some preliminary results

The Perron-Frobenius theorem

Stochastic matrices

Definitions and some preliminary results

Zero-nonzero structure of a matrix

The Perron-Frobenius theorem

Stochastic matrices

## Definition 6.1 (Nonnegative/positive matrix)

A matrix  $A \in \mathcal{M}_{mn}(\mathbb{R})$  is a **nonnegative matrix** if  $a_{ij} \geq 0$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . We write  $A \geq 0$ .  $A$  is a **positive matrix** if  $a_{ij} > 0$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . We write  $A > 0$

## Remark 6.2

*In other references, you will see*

- ▶  $A \geq 0 \iff a_{ij} \geq 0$
- ▶  $A > 0 \iff A \geq 0 \text{ and there exists } (i,j), a_{ij} > 0$  [positive]
- ▶  $A \gg 0 \iff a_{ij} > 0 \text{ for all } i,j$  [strongly positive]

*I tend to favour the latter notation over the one used in these notes, but since the former is more common in matrix theory, I use the notation of Definition 6.1 here*

## Notation

Let  $A, B \in \mathcal{M}_{mn}(\mathbb{R})$ . Nonnegativity and positivity are used to define partial orders on  $\mathcal{M}_{mn}(\mathbb{R})$

- ▶  $A \geq B \iff A - B \geq 0$
- ▶  $A > B \iff A - B > 0$

The same is used for vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ :  $\mathbf{x} \geq \mathbf{y}$  and  $\mathbf{x} > \mathbf{y}$  if, respectively,  $\mathbf{x} - \mathbf{y} \geq \mathbf{0}$  and  $\mathbf{x} - \mathbf{y} > \mathbf{0}$ . Note that the order is only partial: if  $A \geq 0$  and  $B \geq 0$ , for instance, it is not necessarily possible to decide on the ordering of  $A$  and  $B$  with respect to one another

### Theorem 6.3

Let  $A$  and  $B$  be nonnegative matrices of appropriate sizes. Then  $A + B$  and  $AB$  are nonnegative. If  $A > 0$  and  $B \geq 0$ ,  $B \neq 0$ , then  $AB \geq 0$  and  $AB \neq 0$

### Corollary 6.4

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  be such that  $\mathbf{x} \geq \mathbf{y}$  and  $A \in \mathcal{M}_{mn}$  be nonnegative. Then  $A\mathbf{x} \geq A\mathbf{y}$ . Assume additionally that  $\mathbf{x} \geq \mathbf{y}$ ,  $\mathbf{x} \neq \mathbf{y}$  and  $A > 0$ . Then  $A\mathbf{x} > A\mathbf{y}$

Definitions and some preliminary results

Zero-nonzero structure of a matrix

The Perron-Frobenius theorem

Stochastic matrices

## Definition 6.5

Let  $P, Q \in \mathcal{M}_{nm}(\mathbb{F})$ .  $P$  and  $Q$  have the same **zero-nonzero structure** if for all  $i, j$ ,  
 $p_{ij} \neq 0 \iff q_{ij} \neq 0$

Zero-nonzero structure defines an equivalence relation. Therefore, as with all equivalence relations, one only needs one representative from the equivalence class. One typical representative is defined using Boolean matrices

## Definition 6.6

A **Boolean matrix** is a matrix whose entries are Boolean  $\{0, 1\}$  and use Boolean arithmetics:

- ▶  $0 + 0 = 0$
- ▶  $1 + 0 = 0 + 1 = 1$
- ▶  $1 + 1 = 1$
- ▶  $0 \cdot 1 = 1$  and  $1 = 0 = 0 \cdot 0$
- ▶  $1 \cdot 1 = 1$

## Definition 6.7

Let  $A \in \mathcal{M}_{nm}(\mathbb{F})$ . Then  $A_B$  denotes the **Boolean representation** of  $A$ , defined as follows. If  $A = [a_{ij}]$ , then  $A_B = [\alpha_{ij}]$  with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

## Stochastic matrices

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

## Stochastic matrices

## Theorem 6.15 (Perron-Frobenius)

Let  $A \geq 0 \in \mathcal{M}_n$  be irreducible. Then the spectral radius  $\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\}$  is an eigenvalue of  $A$ . It is simple (has algebraic multiplicity 1), positive and is associated with a positive eigenvector. Furthermore, there is no nonnegative eigenvector associated to any other eigenvalue of  $A$

## Remark 6.16

We often say that  $\rho(A)$  is the **Perron root** of  $A$ ; the corresponding eigenvector is the **Perron vector** of  $A$

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

## Stochastic matrices

### Lemma 6.17 (Perron)

Let  $\mathcal{M}_n \ni A > 0$ . Then  $\rho(A)$  is a positive eigenvalue of  $A$  and there is only one linearly independent eigenvector associated to  $\rho(A)$ , which can be taken to be positive

### Lemma 6.18

Let  $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+^*$  and  $v_1, \dots, v_n \in \mathbb{C}$ . Then

$$\left| \sum_{i=1}^n \alpha_i v_i \right| \leq \sum_{i=1}^n \alpha |v_i| \quad (1)$$

with equality if and only if there exists  $\eta \in \mathbb{C}$ ,  $|\eta| = 1$ , such that  $\eta v_i \geq 0$  for all  $i = 1, \dots, n$

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

## Stochastic matrices

## Theorem 6.19

Let  $A \in \mathcal{M}_n$  and  $f(x)$  a polynomial. Then

$$\sigma(f(A)) = \{f(\lambda_1), \dots, f(\lambda_n), \lambda_i \in \sigma(A)\}$$

If we have  $g(\lambda_i) \neq 0$  for  $\lambda_i \in \sigma(A)$ , for some polynomial  $g$ , then the matrix  $g(A)$  is non-singular and

$$\sigma(f(A)g(A)^{-1}) = \left\{ \frac{f(\lambda_1)}{g(\lambda_1)}, \dots, \frac{f(\lambda_n)}{g(\lambda_n)}, \lambda_i \in \sigma(A) \right\}$$

If  $x \neq 0$  eigenvector of  $A$  associated to  $\lambda \in \sigma(A)$ , then  $x$  is also an eigenvector of  $f(A)$  and  $f(A)g(A)^{-1}$  associated to eigenvalue  $f(\lambda)$  and  $f(\lambda)/g(\lambda)$ , respectively

## Lemma 6.20 (Schur's lemma)

Let  $A \in \mathcal{M}_n$  and  $\lambda \in \sigma(A)$ . Then  $\lambda$  is simple if and only if both the following conditions are satisfied:

1. There exists only one linear independent eigenvector of  $A$  associated to  $\lambda$ , say  $\mathbf{u}$ , and thus only one linear independent eigenvector of  $A^T$  associated to  $\lambda$ , say  $\mathbf{v}$
2. Vectors  $\mathbf{u}$  and  $\mathbf{v}$  in (1) satisfy  $\mathbf{v}^T \mathbf{u} \neq 0$

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

#### Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

### Stochastic matrices

### Definition 6.21

Let  $\mathcal{M}_n(\mathbb{R}) \ni A \geq 0$ . We say that  $A$  is **primitive** (with **primitivity index**  $k \in \mathbb{N}_+^*$ ) if there exists  $k \in \mathbb{N}_+^*$  such that

$$A^k > 0$$

with  $k$  the smallest integer for which this is true. We say that a matrix is **imprimitive** if it is not primitive

### Remark 6.22

*Primitivity implies irreducibility. The converse is not true*

### Theorem 6.23

*A sufficient condition for primitivity is irreducibility with at least one positive diagonal entry*

Here  $d$  is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as  $\lambda_p = \rho(A)$ ). If  $d = 1$ , then  $A$  is primitive. We have that  $d = \gcd$  of all the lengths of closed walks in  $G(A)$

### Theorem 6.24

*Let  $A \in \mathcal{M}_n$  be a non-negative matrix. If  $A$  is primitive, then  $A^k > 0$  for some  $0 < k \leq (n - 1)n^n$*

### Theorem 6.25

Let  $A \geq 0$  primitive. Suppose the shortest simple directed cycle in  $G(A)$  has length  $s$ , then primitivity index is  $\leq n + s(n - 1)$

### Theorem 6.26

Let  $A \in \mathcal{M}_n$  be a nonnegative matrix.  $A$  is primitive if and only if  $A^{n^2-2n+2} > 0$

### Theorem 6.27

Let  $A \in \mathcal{M}_n$  be a nonnegative irreducible matrix . Suppose that  $A$  has  $d$  positive entries on the diagonal. Then the primitivity index is  $\leq 2n - d - 1$

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

### Stochastic matrices

### Theorem 6.28

*Let  $A \geq 0$  in  $\mathcal{M}_n$ . Then there exists  $0 \neq v \geq 0$  such that  $Av = \rho(A)v$*

## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

**The Perron-Frobenius Theorem (revamped)**

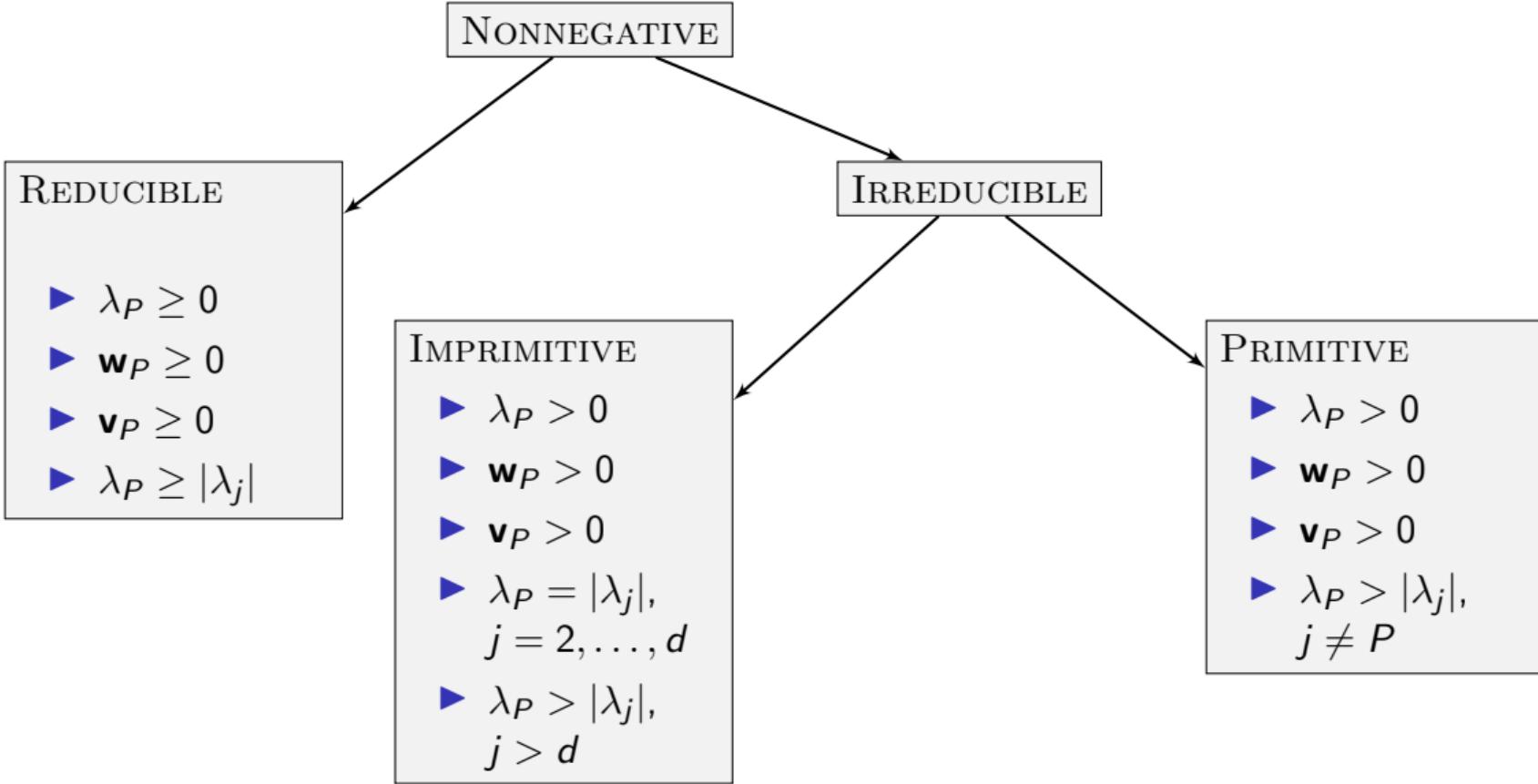
Application of the Perron-Frobenius Theorem

### Stochastic matrices

Let us restate the Perron-Frobenius theorem, taking into account the different cases. The classification in the following result is inspired by the presentation in [?].

### Theorem 6.29

Let  $M_n \ni A \geq 0$ . Denote  $\lambda_P$  the Perron root of  $A$ , i.e.,  $\lambda_P = \rho(A)$ ,  $\mathbf{v}_P$  and  $\mathbf{w}_P$  the corresponding right and left Perron vectors of  $A$ , respectively. Denote  $d$  the index of imprimitivity of  $A$  (with  $d = 1$  when  $A$  is primitive) and  $\lambda_j \in \sigma(A)$  the spectrum of  $A$ , with  $j = 2, \dots, n$  unless otherwise specified (assuming  $\lambda_1 = \lambda_P$ ). Then conclusions of the Perron-Frobenius Theorem can be summarised as follows.



## Definitions and some preliminary results

### The Perron-Frobenius theorem

The Perron-Frobenius Theorem for irreducible matrices

Proof of a result of Perron for positive matrices

Proof of the Perron-Frobenius theorem for irreducible matrices

Primitive matrices

The Perron-Frobenius Theorem for nonnegative matrices

The Perron-Frobenius Theorem (revamped)

Application of the Perron-Frobenius Theorem

## Stochastic matrices

### Theorem 6.30

Let  $A \in \mathcal{M}_n$  be a nonnegative irreducible matrix and  $\lambda \in \mathbb{N}_+$ . Then the following are equivalent:

1. there exists exactly  $h$  distinct eigenvalues such that  $|\lambda| = \rho(A)$ .
2. there exists  $P$  a permutation matrix such that

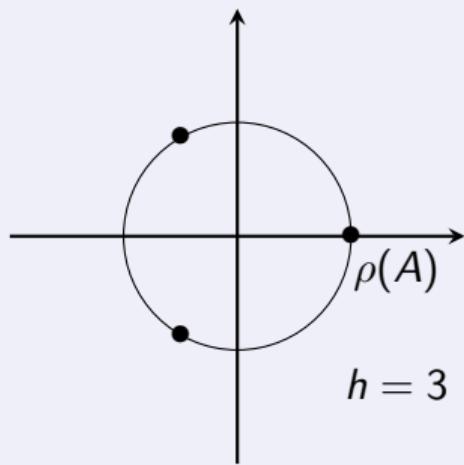
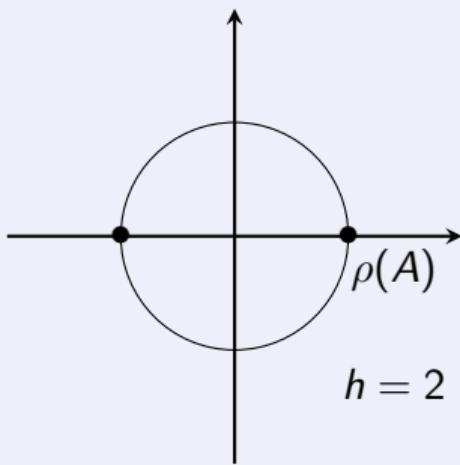
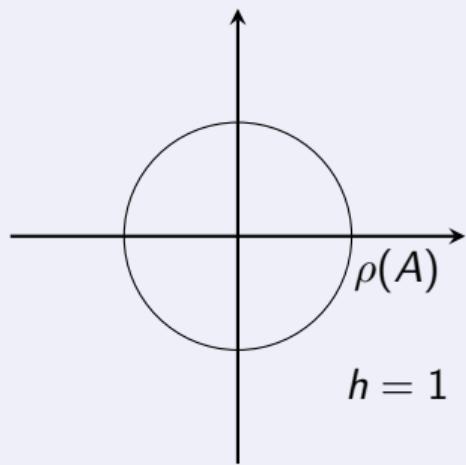
$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ \vdots & & A_{23} & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 \end{pmatrix}$$

where the diagonal blocks are square, and there does not exist other permutation matrix giving less than  $h$  horizontal blocks.

3. the greatest common divisor of the lengths of all cycles in  $G(A)$  is  $h$ .
4.  $h$  is the maximal positive integer  $k$  such that

### Corollary 6.31

Let  $A \in \mathcal{M}_n$ ,  $A \geq 0$  irreducible with exactly  $h$  distinct eigenvalues of modulus  $\rho(A)$ . Then, we can consider this eigenvalues as points in the complex plan, the eigenvalues are the vertices of a regular polygon of  $h$  sides with centre at the origin and are of the vertices being  $\rho(A)$



### Remark 6.32

*For Fiedler, a primitive matrix is defined as an irreducible nonnegative matrix such that  $h = 1$*

### Theorem 6.33

Let  $A \geq 0$  in  $\mathcal{M}_n$ ,  $n \geq 2$ . TFAE

1.  $A^n = 0$
2. there exists  $\mathbb{N} \ni k > 0$  such that  $A^k = 0$
3.  $G(A)$  acyclic
4.  $\exists P$ , permutation matrix, .t.  $PAP^T$  is upper-triangular with zeros on main diagonal
5.  $\rho(A) = 0$

### Theorem 6.34

Let  $A \geq 0$  be a nonnegative matrix in  $\mathcal{M}_n$ . Assume that  $A$  has a positive eigenvector. Then that eigenvector is the Perron vector and is associated to  $\rho(A)$

Definitions and some preliminary results

The Perron-Frobenius theorem

Stochastic matrices

Row- and column-stochastic matrices

Doubly stochastic matrices

Definitions and some preliminary results

The Perron-Frobenius theorem

Stochastic matrices

Row- and column-stochastic matrices

Doubly stochastic matrices

### Definition 6.35 (Stochastic matrix)

The matrix  $A \in \mathcal{M}_n$  is **stochastic** if

- $A \geq 0$  [The matrix is nonnegative]
- $A\mathbb{1} = \mathbb{1}$ ,  $\mathbb{1} = (1, \dots, 1)^T$  [All rows sum to 1]

Equivalently, the matrix is stochastic if its column sums all equal 1

### Definition 6.36

The matrix is **row-stochastic** or **column-stochastic**, respectively, if the rows or columns sum to 1. The terms **right stochastic** and **left stochastic** are also used. If both rows and columns sum to 1, then the matrix is **doubly stochastic**

### Theorem 6.37

Let  $A \in \mathcal{M}_n$  be stochastic. Then  $\rho(A) = 1$

### Theorem 6.38

Let  $P \in \mathcal{M}_n$ ,  $P \geq 0$ . Assume that  $P$  has a positive eigenvector  $u$  and that  $\rho(P) > 0$ . Then there exists  $D$ , diagonal matrix with  $\text{diag}(D) > 0$ , and  $k > 0$ ,  $k \in \mathbb{R}$  such that

$$A = kDPD^{-1}$$

is stochastic, with  $k = \rho(P)^{-1}$

### Theorem 6.39

Let  $A, B \in \mathcal{M}_n$  be stochastic. Then  $AB$  is stochastic

### Theorem 6.40

Let  $A$  be a primitive stochastic. Then  $A^k \rightarrow \mathbf{1}\mathbf{v}^T$ ,  $k \rightarrow \infty$ , where  $\mathbf{1}\mathbf{v}^T$  has rank 1 and  $\mathbf{v}$  is the (left) eigenvector of  $A^T$  associated to  $\rho(A) = 1$  and normalised so that  $\mathbf{v}^T \mathbf{1} = 1$

### Remark 6.41

This is a result that is used to compute the limit of a regular Markov chain

Definitions and some preliminary results

The Perron-Frobenius theorem

Stochastic matrices

Row- and column-stochastic matrices

Doubly stochastic matrices

### Definition 6.42

The matrix  $A \in \mathcal{M}_n$ ,  $A \geq 0$  is **doubly stochastic** if  $A\mathbb{1} = \mathbb{1}$  and  $\mathbb{1}^T A = \mathbb{1}^T$

### Remark 6.43

Here  $\rho(A) = 1$  is associated to  $\mathbb{1}$  for  $A$  and for  $A^T$

Consider  $E$  the Euclidean space. A set  $K$  of points in  $E$  is **convex** if  $A_1, A_2$  points in  $K$ ,  $\lambda_1, \lambda_2 \in \mathbb{R}_+$  such that  $\lambda_1 + \lambda_2 = 1$ , then

$$\lambda_1 A_1 + \lambda_2 A_2 \in K.$$

A **convex polyhedron**  $K$  is the set of all points of the form

$$\sum_{i=1}^N \lambda_i A_i$$

where  $A_i$  are points in  $E$  and  $\lambda_1 \in \mathbb{R}_+$

Let  $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ . Consider this matrix as a point in  $E$  with coordinates  $[a_{11}, a_{12}, \dots, a_{nn}]$  ( $\dim E = n^2$ )

### Theorem 6.44

Let  $A \in \mathcal{M}_n$ ,  $A = [a_{ij}]$ , if  $A$  is doubly stochastic, then this forms an  $(n - 1)^2$  dimensional subspace of  $\tilde{E} = \mathbb{R}^{n^2}$

### Theorem 6.45 (Birkhoff)

In the space  $\tilde{E} = \mathbb{R}^{n^2}$ , the set of doubly stochastic matrices of order  $n$  is a convex polyhedron in  $E$  (the subspace of stochastic matrices). The vertices of the polyhedron are the points corresponding to all the permutation matrices

## References I