MATH 4370/7370 – Linear Algebra and Matrix Analysis

Factorisations, canonical forms and decompositions

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Fall 2023

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Outline

Unitary matrices and QR factorisation

Schur's Form

Consequences of Schur's triangularisation theorem

Normal Matrices

Jordan Canonical Form

Singular values and the Singular value decomposition

Properties of Singular Values

Definition 4.1

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{C}^n$. We say that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is an **orthogonal list** if $\mathbf{x}_i^* \mathbf{x}_j = 0$ for all $i \neq j$. If, in addition, we have that $\mathbf{x}_i^* \mathbf{x}_i = 1$, then we say that the list is **orthonormal**

Theorem 4.2

Every orthonormal list of vectors in $\mathbb{C}^n$ is linearly independent

Remark 4.3

In Theorem 4.2, if we have "only" orthogonal vectors, we need to replace "list of vectors" by "list of non-zero vectors" in the statement

Definition 4.4

Let $U \in \mathcal{M}_n$, we say that U is an **unitary matrix** if $U^*U = \mathbb{I}$. Furthermore, we say that $U \in \mathcal{M}_n(\mathbb{R})$ is a **(real) orthogonal matrix** if $U^TU = \mathbb{I}$

Theorem 4.5

Let $U \in \mathcal{M}_n$. TFAE:

1. *U is unitary*
2. *U is non-singular and $U^* = U^{-1}$*
3. *$UU^* = \mathbb{I}$*
4. *U^* is unitary*
5. *the columns of U are orthonormal*
6. *the rows of U are orthonormal*
7. *for all $\mathbf{x} \in \mathbb{C}^n$ we have $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$*

Definition 4.6

A **linear transformation** $T : \mathbb{C}^n \to \mathbb{C}^n$ is a **Euclidean isometry** if $\|\mathbf{x}\|_2 = \|T\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$

Corollary 4.7

Let $U \in \mathcal{M}_n$. U is a Euclidean isometry if and only if U is unitary

Remark 4.8

Let U, $V \in \mathcal{M}_n$ *are unitary matrices (respectively real orthogonal), then* UV *is unitary (respectively real orthogonal).*
Indeed, U, V *unitary* $\Leftrightarrow U^{-1}, V^{-1}$ *exist and* $U^{-1} = U^*, V^{-1} = V^*$. *Then*

$$\begin{aligned} UV \ \ unitary \ &\Leftrightarrow (UV)^* UV = \mathbb{I} \\ &\Leftrightarrow V^* U^* UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I} \end{aligned}$$

Notation: $\mathrm{GL}(n, \mathbb{F})$ is the general linear group, where the elements are non-singular matrices in $\mathcal{M}_n(\mathbb{F})$

Theorem 4.9

The set of unitary (respectively real orhogonal) matrices in $\mathcal{M}_n$ forms a group, the $n \times n$ unitary (respectively real orthogonal) subgroup of $\mathrm{GL}(n, \mathbb{C})$ (respectively $\mathrm{GL}(n, \mathbb{R})$)

Theorem 4.10 (Selection Principle)

Suppose that we have a sequence of unitary matrices $U_1, U_2 \ldots, \in \mathcal{M}_n$. Then there exists a subsequence $U_{k_1}, U_{k_2} \ldots$ such that the entries of U_{k_i} converge to entries of a unitary matrix as $i \to \infty$

Lemma 4.11

Let $U \in \mathcal{M}_n$ be a unitary matrix partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with $U_{ii} \in \mathcal{M}_k$. Then $\operatorname{rank} U_{12} = \operatorname{rank} U_{21}$ and $\operatorname{rank} U_{22} = \operatorname{rank} U_{11} + n - 2k$. If, furthermore, $U_{21} = 0$ and $U_{12} = 0$, then U_{11} and U_{22} are unitary

Theorem 4.12 (QR factorisation)

Let $A \in \mathcal{M}_{nm}$

1. *If $n \geq m$, there is a $Q \in \mathcal{M}_{nm}$ with orthogonormal columns and upper triangular $R \in \mathcal{M}_m$ with non-negative main diagonal entries such that $A = QR$*
2. *If $\operatorname{rank} A = m$ then the factors Q and R in (1) are uniquely determined and the main diagonal entries of R are all positive*
3. *If $n = m$, Then the factor Q in (1) is unitary*
4. *There is a unitary $Q \in \mathcal{M}_n$ and an upper triangular $R \in \mathcal{M}_{nm}$ with nonnegative diagonal entries such that $A = QR$*
5. *If A is real, then Q and R are in (1), (2), (3), and (4) may be taken to be real*

For a unitary matrix U, $U^* = U$, so the transformation $A \mapsto U^*AU$ is a **similarity transformation**, provided that U is unitary. This is a **unitary similarity**

Definition 4.13 (Unitarily similar matrices)

Let $A, B \in \mathcal{M}_n$. We say that A is **unitarily similar** to B if there exists $U \in \mathcal{M}_n$ unitary such that

$$A = U^*BU$$

If U can be taken real (i.e., if U is real orthogonal) than A is real orthogonal similar to B (if $A = U^TBU$)

Remark 4.14

1. *Unitary similarity is an equivalence relation*
2. *Unitary similarity implies similarity. However, the converse is not true*
3. *Similarity is a change of bases. Unitary similarity is a change of orthonormal bases*

Definition 4.15 (Householder matrix)

Let $0 \neq \omega \in \mathbb{C}^n$. The Householder matrix $U_\omega \in \mathcal{M}_n$ is

$$U_\omega = \mathbb{I} - 2(\omega^* \omega)^{-1} \omega \omega^*$$

Remark 4.16

1. *If* $\|\omega\| = 1$ *then* $U_\omega = \mathbb{I} - 2\omega\omega^*$
2. *Householder matrix are unitary and Hermitian, thus* $U_\omega^{-1} = U_\omega$.
3. *The eigenvalues of a Householder matrix are* $-1, 1, \ldots, 1$ *and* $|U_\omega| = 1$

Theorem 4.17

Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *and assume that* $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 > 0$

- *If* $\mathbf{y} = e^{i\theta}\mathbf{x}$ *for some* $\theta \in \mathbb{R}$ *[*$\mathbf{x}, \mathbf{y}$ *are linearly dependent], define* $U(\mathbf{y}, \mathbf{x}) = e^{i\theta}\mathbb{I}$
- *Otherwise, let* $\phi \in [0, 2\pi)$ *be such that* $\mathbf{x}^*\mathbf{y} = e^{i\phi}|\mathbf{x}^*\mathbf{y}|$ *(taking* $\phi = 0$ *if* $\mathbf{x}^*\mathbf{y} = 0$*).* *Let* $\omega = e^{i\phi}\mathbf{x} - \mathbf{y}$ *and define*

$$U(\mathbf{y}, \mathbf{x}) = e^{i\phi} U_\omega$$

where $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$ *is Householder*

1. $U(\mathbf{y}, \mathbf{x})$ *unitary and essentially Hermitian*
2. $U(\mathbf{y}, \mathbf{x})\mathbf{x} = \mathbf{y}$
3. $U(\mathbf{y}, \mathbf{x})\mathbf{z} \perp \mathbf{y}$, *when* $\mathbf{z} \perp \mathbf{y}$
4. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $U(\mathbf{y}, \mathbf{x})$ *is real and* $U(\mathbf{y}, \mathbf{x}) = \mathbb{I}$ *if* $\mathbf{y} = \mathbf{x}$ *and* $U(\mathbf{y}, \mathbf{x}) = U_{\mathbf{x}-\mathbf{y}} \in \mathcal{M}_n(\mathbb{R})$ *otherwise*

Remark 4.18

For all $A \in \mathcal{M}_n$, $U(y,x)^* A U(y,x) = U_\omega^* A U_\omega$. *This is called a Householder transformation.*

Theorem 4.19 (Schur's Form)

Let $A \in \mathcal{M}_n$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *in any prescribed order (including multiplicities). Let* $x \in \mathbb{C}^n$, $\|x\| = 1$, *be such that* $Ax = \lambda_1 x$

1. *There exists* $U = [x\, u_2 \ldots u_n] \in \mathcal{M}_n$ *unitary such that* $U^* A U = T$, *where* T *is upper triangular such that* $t_i i = \lambda_i$, $i = 1, \ldots, n$.
2. *If* $A \in \mathcal{M}_n(\mathbb{R})$ *and has real eigenvalues, then* x *can be chosen to be real and there exists*

$$Q = [x\, q_2 \ldots q_n] \in \mathcal{M}_n(\mathbb{R})$$

real orthogonal and such that $Q^T A Q = T$, *with* T *upper triangular with* $t_{ii} = \lambda_1$ $i = 1, \ldots, n$.

Theorem 4.20 (Schur version 2)

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ (including mutiplicities). Then there esists $U \in \mathcal{M}_n$ such that

$$U^*AU = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

Remark 4.21

The decomposition is not unique

Theorem 4.22

Let $U \in \mathcal{M}_n$, A, $B \in \mathcal{M}_n$. Suppose A is unitarily similar to B, then

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

Corollary 4.23

Let $A \in \mathcal{M}_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, $T = UAU^*$ upper triangular. Then

$$\sum_{i=1}^{n} |\lambda_1|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \leq \sum_{i,j=1} |a_{ij}|^2 = \mathrm{tr} AA^*$$

with equality if T is diagonal.

Theorem 4.24 (Cayley-Hamilton)

Let $A \in \mathcal{M}_n$ and $p_A(t)$ is the characteristic polynomial of A, then $p_A(A) = 0$.

Theorem 4.25 (Sylvester's theorem – pole placement)

Assume $A \in \mathcal{M}_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicities $n_1, \ldots, n_d$ $(\sum_{i=1}^{d} n_i = n)$. Then A is unitary similar to a $d \times d$ block upper triangular matrix T, where $T_{i,j} \in M_{n_i m_j}$, $T_{ij} = 0$ if $i > i$, T_{ii} upper triangular with diagonal λ_i, $T_{ii} = \lambda \mathbb{I} + R_i$, R_i strictly upper triangular, and A is similar to a matrix to $\bigoplus_{i=1}^{d} T_{ii}$ [standard similarity, not unitary]

Theorem 4.26

(Every square matrix is almost diagonalisble) Let $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$, *there exists* $A(\varepsilon)[a_{ij}(\varepsilon)] \in \mathcal{M}$ *with distinct eigenvalues such that*

$$\sum_{i,j} |a_{ij} - a_{ij}(\varepsilon)|^2 < \varepsilon$$

Theorem 4.27

If $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$ *there exists* $S(\varepsilon) \in \mathcal{M}_n$ *non-singular such that*

$$S^{-1}(\varepsilon)AS(\varepsilon) = T(\varepsilon),$$

where $T(\varepsilon)$ *is upper triangular and* $|t_{ij}(\varepsilon)| < \varepsilon$ *for all* i, j, *with* $i < j$.

Lemma 4.28

Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim\limits_{k\to\infty} A_k = A$ (entry-wise). Then there exists $k_1 < k_2 < \ldots$ and $U_{k_i} \in \mathcal{M}$ such that

1. $T_i = U_{k_i}^* A_{k_i} U_{k_i}$ *upper triangular*
2. $U + \lim\limits_{i\to\infty} U_{k_i}$ *exists and is unitary*
3. $T = U^* A U$ *upper triangular*
4. $\lim\limits_{i\to\infty} T_i = T$

Theorem 4.29

Let $(A_k)_{k\in\mathbb{N}}$ *a sequence of matrices such that* $\lim_{k\to\infty} A_k = A$ *(entry-wise). Then let*

$$\lambda(A) = \begin{bmatrix}\lambda_1(A) & \dots & \lambda_n(A)\end{bmatrix}^T$$

and

$$\lambda(A_k) = \begin{bmatrix}\lambda_1(A_k) & \dots & \lambda_n(A_k)\end{bmatrix}^T$$

be presentations of the eigenvalues of A *and* A_k. *Define*

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1,\dots,n\}\}.$$

Then for all $\varepsilon > 0$ *there exists* $N(\varepsilon) \in \mathbb{N} \setminus \{0\}$ *such that*

$$\min_{\pi\in S_n} \max_{i=1,\dots} \{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \le \varepsilon \qquad \forall k \ge N(\varepsilon)$$

Recall that if $\mathbf{x}, \mathbf{y}$ are two (column) vectors in $\mathbb{F}^n$, then $\mathbf{x}\mathbf{y}^*$ is a rank 1 matrix in $\mathcal{M}_n(\mathbb{F})$. (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly

Theorem 4.30 (Brauer)

Suppose $A \in \mathcal{M}_n$ has eigenvalues $\lambda, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}$ be an eigenvector associated to λ. Then for every vector $\mathbf{v} \in \mathbb{C}^n$, the eigenvalues of $A + \mathbf{x}^*\mathbf{v}$ are $\lambda + \mathbf{v}^*\mathbf{x}, \lambda_2, \ldots, \lambda_n$.

Definition 4.31 (Normal matrix)

A matrix $A \in \mathcal{M}_n$ is **normal** if $AA^* = A^*A$

All unitary, Hermitian or skew-Hermitian and diagonal matrices are normal

Theorem 4.32

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. TFAE:

1. *A is normal*
2. *A is unitary diagonalisable*
3. $\sum\limits_{i,j} |a_{i,j}|^2 = \sum\limits_{i} |\lambda_i|^2$
4. *A has n orthogonal eigenvectors*

Theorem 4.33

Let $A \in \mathcal{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

Then

1. $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
2. *A is unitary diagonalisable*
3. *there exists $U \in \mathcal{M}_n$ such that $A = U \Lambda U^*$*

Definition 4.34

A **Jordan block** $J_k(\lambda)$ is a $k \times k$ upper triagular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

Theorem 4.35

Let $A \in \mathcal{M}_n$ then there exists $S \in \mathcal{M}_n$ non-singular such that

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) S^{-1}$$

Theorem 4.36

Let $A \in \mathcal{M}_n$ with real eigenvalues. Then there exists a basis of generalised eigenvectors for $\mathbb{R}^n$, and if $\{v_1, \dots, v_n\}$ is a basis of generalised eigenvectors of $\mathbb{R}^n$, then $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ is non-singular and $A = D + N$ where $P^{-1}DP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $N = A - D$ is nilpotent[1] *of order $k \leq n$, and D and N commute.*

Definition 4.37

Let A be a Hermitian matrix in $\mathcal{M}_n$. We say that A is **positive definite** if for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* A \mathbf{x} > 0$. We say that A is **positive semidefinite** if for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^* A \mathbf{x} \geq 0$

Theorem 4.38

Let $A \in \mathcal{M}_n$ be a Hermitian matrix. Then

1. *for all $\mathbf{x} \in \mathbb{C}^*$, $\mathbf{x}^* A \mathbf{x} \in \mathbb{R}$*
2. $\sigma(A) \subset \mathbb{R}$
3. *$S^* A S$ is Hermitian for any $S \in \mathcal{M}_n$*

Theorem 4.39

Each eigenvalue of a positive definite matrix (respectively positive semidefinite matrix) is positive (respectively nonnegative)

Proposition 4.40

Let A be a positive semidefinite (respectively positive definite) matrix. Then $\operatorname{tr}(A)$, $\det(A)$, *the principal minors of A are all nonnegative (respectively positive). Also,* $\operatorname{tr}(A) = 0$ *if and only if* $A = 0$

Theorem 4.41

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and $\mathbf{x} \in \mathbb{C}^n$. Then

$$\mathbf{x}^* A\mathbf{x} = 0 \iff A\mathbf{x} = \mathbf{0}$$

Corollary 4.42

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix. Then A is positive definite if and only if A is nonsingular

Theorem 4.43 (Somewhat unrelated)

Let $B \in \mathcal{M}_n$ be a Hermitian matrix, $\mathbf{y} \in \mathbb{C}^n$, and $a \in \mathbb{R}$. Let

$$A = \begin{pmatrix} B & \mathbf{y} \\ \mathbf{y}^* & a \end{pmatrix} \in \mathcal{M}_{n+1}$$

Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

Definition 4.44

The singular values of a matrix A are the (nonnegative) square roots of the eigenvalues of A^*A

Remark 4.45

A^*A is positive semidefinite

Theorem 4.46 (Zhang)

Let $A \in \mathcal{M}_{mn}$ with nonzero singular values $\sigma_1, \ldots, \sigma_r$. Then there exists $U \in \mathcal{M}_n$ and $V \in \mathcal{M}_n$ unitary such that

$$A = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V,$$

where $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{mn}$ and $D_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$

Theorem 4.47 (H & J)

Let $A \in \mathcal{M}_{nm}$ *,* $q = \min\{m, n\}$*. Assume that the rank of A is n. Then*

1. $\exists V \in M_n$ *and* $W \in \mathcal{M}_m$ *unitary matrices and* $\Sigma_q \in \mathcal{M}_q = \mathrm{diag}(\sigma_1, \ldots, \sigma_q)$ *s.t.*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$$

   and

$$A\Sigma W$$

   where

$$\Sigma = \begin{cases} \Sigma_1, & m = n \\ \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{nm}, & m > n \\ \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{nm}, & n > m \end{cases}$$

2. *The parameters* $\sigma_1, \ldots, \sigma_r$ *are the positive square roots of the decreasingly ordered eigenvalues of* $A^* A$

Remark 4.48

Let $A \in \mathcal{M}_{mn}$. Then $A, \overline{A}, A^T$, and A^* have the same singular values

Remark 4.49

Let $A \in \mathcal{M}_n$ with singular values $\sigma_1, \ldots, \sigma_n$, then

$$\sigma_1 \ldots \sigma_n = \det(A)$$

and

$$\sigma_1^2 + \ldots + \sigma_n^2 = \mathrm{tr}(A^* A)$$

Theorem 4.50

Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, and $\sigma_1 \geq \cdots \geq \sigma_q$ nonincresingly ordered singular values of A. Define

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

to be a Hermitian matrix. Then the ordered eigenvalues of $\mathcal{A}$ are

$$-\sigma_1 \leq \cdots \leq -\sigma_q \leq \underbrace{0 = \cdots = 0}_{} |n - m| \leq \sigma_q \leq \cdots \leq \sigma_1$$

Theorem 4.51 (An interlacing result)

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\hat{A}$ be the matrix obtained from A by deleting one row and one column. Let $\sigma_1 \geq \cdots \geq \sigma_q$ and $\hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_q$ be the nonsingular ordered singular values of A and $\hat{A}$, respectively, where $\hat{\sigma}_q = 0$ if $n \geq m$ and a column is deleted or if $n \geq m$ and a row is deleted. Then

$$\sigma_1 \geq \hat{\sigma_1} \geq \sigma_2 \geq \hat{\sigma_2} \geq \ldots \sigma_q \geq \hat{\sigma_q}.$$

Theorem 4.52 (von Neumann)

Let A, $B \in \mathcal{M}_{mn}$, $q = \min\{m, n\}$, $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ and $\sigma_1(B) \geq \cdots \geq \sigma_q(B)$ the non-increasingly singular values of A and B, respectively. Then

$$\operatorname{Re}\operatorname{tr}(AB^*) \leq \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B).$$

Theorem 4.53

Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, and $\sigma_1 \geq \cdots \geq \sigma_q$ nonincreasingly ordered singular values of A, and $\alpha = \{1, \ldots, q\}$. Then

$$\operatorname{Re}\operatorname{tr}(A) \leq \sum_{i=1}^{q} \sigma_i$$

with equality if and only if $A[\alpha]$ (principal leading submatrix of A) is positive semidefinite and A has no nonzero entries outside $A[\alpha]$.

- Let $A \in \mathcal{M}_2$

$$\sigma_1, \sigma_2 = \frac{1}{2}\left((\mathrm{tr} A^* A) \mp \sqrt{(\mathrm{tr} A^* A)^2 - 4|\mathrm{det} A|^2} \right)$$

- The nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & & \\ & \ddots & & \\ & & & a_{n-1,m} \\ & & & 0 \end{pmatrix}$$

has singular values $0, |a_{12}|, \ldots, |a_{n-1,n}|$.

Theorem 4.54

Let $A_1, A_2, \dots \in \mathcal{M}_{nm}$ *given (infinite) sequence with* $\lim_{k\to\infty} A_k = A$ *(entrywise). Let* $q = \min(m, n)$. *Let* $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ *and* $\sigma_1(A_k) \geq \cdots \geq \sigma_q(A_k)$ *be the non-increasinly ordered singular values of* A *and* A_k*, respectively (for all* k*). Then*

$$\lim_{k\to\infty} \sigma_i(A_k) = \sigma_i(A)$$

Theorem 4.55

Let $A \in \mathcal{M}_n$ *where* $n = \text{rank } A$

1. $A = A^T$ *if and only if there exists* $U \in \mathcal{M}_n$ *unitary and a nonnegative diagonal matrix* Σ *such that* $A = U\Sigma U^T$. *Then the diagonal entries of* Σ *are the singular values of* A
2. *If* $A = -A^T$, *then* n *is even and there exists* $U \in \mathcal{M}_n$ *unitary and positive real scalars* $s_1, \ldots, s_{r/2}$ *such that*

$$U\left(\begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_{r/2} \\ -s_{r/2} & 0 \end{pmatrix}\right) U^T$$

The non-zero singular values of A *are* $s_1, s_1, \ldots, s_{r/2}, s_{r/2}$. *Conversely, any matrix of the above form is skew-symetric*

References I