# MATH 4370/7370 – Linear Algebra and Matrix Analysis

## Factorisations, canonical forms and decompositions

Julien Arino

Fall 2025

University of Manitoba

# Outline

## Definition 4.1

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{C}^n$. We say that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is an **orthogonal list** if $\mathbf{x}_i^* \mathbf{x}_j = 0$ for all $i \neq j$. If, in addition, we have that $\mathbf{x}_i^* \mathbf{x}_i = 1$, then we say that the list is **orthonormal**

## Theorem 4.2

*Every orthonormal list of vectors in $\mathbb{C}^n$ is linearly independent*

## Remark 4.3

*In Theorem 4.2, if we have "only" orthogonal vectors, we need to replace "list of vectors" by "list of non-zero vectors" in the statement*

## Definition 4.4

Let $U \in \mathcal{M}_n$, we say that $U$ is an **unitary matrix** if $U^* U = \mathbb{I}$. Furthermore, we say that $U \in \mathcal{M}_n(\mathbb{R})$ is a **(real) orthogonal matrix** if $U^T U = \mathbb{I}$

## Theorem 4.5

*Let $U \in \mathcal{M}_n$. TFAE:*

1. *$U$ is unitary*
2. *$U$ is non-singular and $U^* = U^{-1}$*
3. *$UU^* = \mathbb{I}$*
4. *$U^*$ is unitary*
5. *the columns of $U$ are orthonormal*
6. *the rows of $U$ are orthonormal*
7. *for all $\mathbf{x} \in \mathbb{C}^n$ we have $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$*

### Definition 4.6

A **linear transformation** $T : \mathbb{C}^n \to \mathbb{C}^n$ is a **Euclidean isometry** if $\|\mathbf{x}\|_2 = \|T\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$

### Corollary 4.7

*Let $U \in \mathcal{M}_n$. $U$ is a Euclidean isometry if and only if $U$ is unitary*

## Remark 4.8

*Let $U$, $V \in \mathcal{M}_n$ are unitary matrices (respectively real orthogonal), then $UV$ is unitary (respectively real orthogonal).*
*Indeed, $U, V$ unitary $\Leftrightarrow U^{-1}, V^{-1}$ exist and $U^{-1} = U^*, V^{-1} = V^*$. Then*

$$\begin{aligned} UV \ \mathit{unitary} \ &\Leftrightarrow (UV)^* UV = \mathbb{I} \\ &\Leftrightarrow V^* U^* UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I} \end{aligned}$$

**Notation**: $\mathrm{GL}(n, \mathbb{F})$ is the general linear group, where the elements are non-singular matrices in $\mathcal{M}_n(\mathbb{F})$

### Theorem 4.9

*The set of unitary (respectively real orhogonal) matrices in $\mathcal{M}_n$ forms a group, the $n \times n$ unitary (respectively real orthogonal) subgroup of $\mathrm{GL}(n, \mathbb{C})$ (respectively $\mathrm{GL}(n, \mathbb{R})$)*

### Theorem 4.10 (Selection Principle)

*Suppose that we have a sequence of unitary matrices $U_1, U_2 \ldots, \in \mathcal{M}_n$. Then there exists a subsequence $U_{k_1}, U_{k_2} \ldots$ such that the entries of $U_{k_i}$ converge to entries of a unitary matrix as $i \to \infty$*

## Lemma 4.11

*Let $U \in \mathcal{M}_n$ be a unitary matrix partitioned as*

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

*with $U_{ii} \in \mathcal{M}_k$. Then* $\operatorname{rank} U_{12} = \operatorname{rank} U_{21}$ *and* $\operatorname{rank} U_{22} = \operatorname{rank} U_{11} + n - 2k$*. If, furthermore, $U_{21} = 0$ and $U_{12} = 0$, then $U_{11}$ and $U_{22}$ are unitary*

## Theorem 4.12 (QR factorisation)

*Let* $A \in \mathcal{M}_{nm}$

1. *If* $n \geq m$, *there is a* $Q \in \mathcal{M}_{nm}$ *with orthogonormal columns and upper triangular* $R \in \mathcal{M}_m$ *with non-negative main diagonal entries such that* $A = QR$
2. *If* $\operatorname{rank} A = m$ *then the factors* $Q$ *and* $R$ *in (1) are uniquely determined and the main diagonal entries of* $R$ *are all positive*
3. *If* $n = m$, *Then the factor* $Q$ *in (1) is unitary*
4. *There is a unitary* $Q \in \mathcal{M}_n$ *and an upper triangular* $R \in \mathcal{M}_{nm}$ *with nonnegative diagonal entries such that* $A = QR$
5. *If* $A$ *is real, then* $Q$ *and* $R$ *are in (1), (2), (3), and (4) may be taken to be real*

For a unitary matrix $U$, $U^* = U$, so the transformation $A \mapsto U^*AU$ is a **similarity transformation**, provided that $U$ is unitary. This is a **unitary similarity**

### Definition 4.13 (Unitarily similar matrices)

Let $A, B \in \mathcal{M}_n$. We say that $A$ is **unitarily similar** to $B$ if there exists $U \in \mathcal{M}_n$ unitary such that

$$A = U^*BU$$

If $U$ can be taken real (i.e., if $U$ is real orthogonal) than $A$ is real orthogonal similar to $B$ (if $A = U^TBU$)

## Remark 4.14

1. *Unitary similarity is an equivalence relation*
2. *Unitary similarity implies similarity. However, the converse is not true*
3. *Similarity is a change of bases. Unitary similarity is a change of orthonormal bases*

## Definition 4.15 (Householder matrix)

Let $0 \neq \omega \in \mathbb{C}^n$. The Householder matrix $U_\omega \in \mathcal{M}_n$ is

$$U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$$

## Remark 4.16

1. *If* $\|\omega\| = 1$ *then* $U_\omega = \mathbb{I} - 2\omega\omega^*$
2. *Householder matrix are unitary and Hermitian, thus* $U_\omega^{-1} = U_\omega$.
3. *The eigenvalues of a Householder matrix are* $-1, 1, \ldots, 1$ *and* $|U_\omega| = 1$

## Theorem 4.17

*Let* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *and assume that* $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 > 0$

- *If* $\mathbf{y} = e^{i\theta}\mathbf{x}$ *for some* $\theta \in \mathbb{R}$ *[*$\mathbf{x}, \mathbf{y}$ *are linearly dependent], define* $U(\mathbf{y}, \mathbf{x}) = e^{i\theta}\mathbb{I}$
- *Otherwise, let* $\phi \in [0, 2\pi)$ *be such that* $\mathbf{x}^*\mathbf{y} = e^{i\phi}|\mathbf{x}^*\mathbf{y}|$ *(taking* $\phi = 0$ *if* $\mathbf{x}^*\mathbf{y} = 0$*). Let* $\omega = e^{i\phi}\mathbf{x} - \mathbf{y}$ *and define*

$$U(\mathbf{y}, \mathbf{x}) = e^{i\phi} U_\omega$$

*where* $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$ *is Householder*

1. $U(\mathbf{y}, \mathbf{x})$ *unitary and essentially Hermitian*
2. $U(\mathbf{y}, \mathbf{x})\mathbf{x} = \mathbf{y}$
3. $U(\mathbf{y}, \mathbf{x})\mathbf{z} \perp \mathbf{y}$*, when* $\mathbf{z} \perp \mathbf{y}$
4. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*, then* $U(\mathbf{y}, \mathbf{x})$ *is real and* $U(\mathbf{y}, \mathbf{x}) = \mathbb{I}$ *if* $\mathbf{y} = \mathbf{x}$ *and* $U(\mathbf{y}, \mathbf{x}) = U_{\mathbf{x}-\mathbf{y}} \in \mathcal{M}_n(\mathbb{R})$ *otherwise*

## Remark 4.18

*For all* $A \in \mathcal{M}_n$, $U(y,x)^* A U(y,x) = U_\omega^* A U_\omega$. *This is called a Householder transformation.*

## Theorem 4.19 (Schur's Form)

*Let* $A \in \mathcal{M}_n$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *in any prescribed order (including multiplicities). Let* $x \in \mathbb{C}^n$, $\|x\| = 1$, *be such that* $Ax = \lambda_1 x$

1. *There exists* $U = [x\, u_2 \ldots u_n] \in \mathcal{M}_n$ *unitary such that* $U^* A U = T$, *where* $T$ *is upper triangular such that* $t_i i = \lambda_i$, $i = 1, \ldots, n$.
2. *If* $A \in \mathcal{M}_n(\mathbb{R})$ *and has real eigenvalues, then* $x$ *can be chosen to be real and there exists*

$$Q = [x\, q_2 \ldots q_n] \in \mathcal{M}_n(\mathbb{R})$$

*real orthogonal and such that* $Q^T A Q = T$, *with* $T$ *upper triangular with* $t_{ii} = \lambda_1$ $i = 1, \ldots, n$.

## Theorem 4.20 (Schur version 2)

*Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ (including mutiplicities). Then there esists $U \in \mathcal{M}_n$ such that*

$$U^* A U = \begin{pmatrix} \lambda_1 & * & \ldots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

## Remark 4.21

*The decomposition is not unique*

## Theorem 4.22

*Let $U \in \mathcal{M}_n, A,\ B \in \mathcal{M}_n$. Suppose $A$ is unitarily similar to $B$, then*

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

## Corollary 4.23

*Let $A \in \mathcal{M}_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, $T = UAU^*$ upper triangular. Then*

$$\sum_{i=1}^{n} |\lambda_1|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \leq \sum_{i,j=1} |a_{ij}|^2 = \mathrm{tr} AA^*$$

*with equality if $T$ is diagonal.*

## Theorem 4.24 (Cayley-Hamilton)

*Let $A \in \mathcal{M}_n$ and $p_A(t)$ is the characteristic polynomial of $A$, then $p_A(A) = 0$.*

## Theorem 4.25 (Sylvester's theorem – pole placement)

*Assume $A \in \mathcal{M}_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicities $n_1, \ldots, n_d$ $(\sum_{i=1}^{d} n_i = n)$. Then $A$ is unitary similar to a $d \times d$ block upper triangular matrix $T$, where $T_{i,j} \in M_{n_i m_j}$, $T_{ij} = 0$ if $i > i$, $T_{ii}$ upper triangular with diagonal $\lambda_i$, $T_{ii} = \lambda \mathbb{I} + R_i$, $R_i$ strictly upper triangular, and $A$ is similar to a matrix to $\bigoplus_{i=1}^{d} T_{ii}$ [standard similarity, not unitary]*

### Theorem 4.26

*(Every square matrix is almost diagonalisble) Let* $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$, *there exists* $A(\varepsilon)[a_{ij}(\varepsilon)] \in \mathcal{M}$ *with distinct eigenvalues such that*

$$\sum_{i,j} |a_{ij} - a_{ij}(\varepsilon)|^2 < \varepsilon$$

### Theorem 4.27

*If* $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$ *there exists* $S(\varepsilon) \in \mathcal{M}_n$ *non-singular such that*

$$S^{-1}(\varepsilon)AS(\varepsilon) = T(\varepsilon),$$

*where* $T(\varepsilon)$ *is upper triangular and* $|t_{ij}(\varepsilon)| < \varepsilon$ *for all* $i, j$, *with* $i < j$.

## Lemma 4.28

*Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim\limits_{k\to\infty} A_k = A$ (entry-wise). Then there exists $k_1 < k_2 < \ldots$ and $U_{k_i} \in \mathcal{M}$ such that*

1. $T_i = U_{k_i}^* A_{k_i} U_{k_i}$ *upper triangular*
2. $U + \lim\limits_{i\to\infty} U_{k_i}$ *exists and is unitary*
3. $T = U^* A U$ *upper triangular*
4. $\lim\limits_{i\to\infty} T_i = T$

## Theorem 4.29

*Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim\limits_{k\to\infty} A_k = A$ (entry-wise). Then let*

$$\lambda(A) = \begin{bmatrix}\lambda_1(A) & \dots & \lambda_n(A)\end{bmatrix}^T$$

*and*

$$\lambda(A_k) = \begin{bmatrix}\lambda_1(A_k) & \dots & \lambda_n(A_k)\end{bmatrix}^T$$

*be presentations of the eigenvalues of $A$ and $A_k$. Define*

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1,\dots,n\}\}.$$

*Then for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}\setminus\{0\}$ such that*

$$\min_{\pi\in S_n}\max_{i=1,\dots}\{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \leq \varepsilon \qquad \forall k \geq N(\varepsilon)$$

Recall that if $\mathbf{x}, \mathbf{y}$ are two (column) vectors in $\mathbb{F}^n$, then $\mathbf{x}\mathbf{y}^*$ is a rank 1 matrix in $\mathcal{M}_n(\mathbb{F})$. (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly

### Theorem 4.30 (Brauer)

*Suppose $A \in \mathcal{M}_n$ has eigenvalues $\lambda, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}$ be an eigenvector associated to $\lambda$. Then for every vector $\mathbf{v} \in \mathbb{C}^n$, the eigenvalues of $A + \mathbf{x}^*\mathbf{v}$ are $\lambda + \mathbf{v}^*\mathbf{x}, \lambda_2, \ldots, \lambda_n$.*

**Definition 4.31 (Normal matrix)**

A matrix $A \in \mathcal{M}_n$ is **normal** if $AA^* = A^*A$

All unitary, Hermitian or skew-Hermitian and diagonal matrices are normal

## Theorem 4.32

*Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. TFAE:*

1. *$A$ is normal*
2. *$A$ is unitary diagonalisable*
3. $\sum\limits_{i,j} |a_{i,j}|^2 = \sum\limits_{i} |\lambda_i|^2$
4. *$A$ has $n$ orthogonal eigenvectors*

## Theorem 4.33

*Let $A \in \mathcal{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let*

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

*Then*

1. $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$
2. *$A$ is unitary diagonalisable*
3. *there exists $U \in \mathcal{M}_n$ such that $A = U \Lambda U^*$*

## Definition 4.34

A **Jordan block** $J_k(\lambda)$ is a $k \times k$ upper triagular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

## Theorem 4.35

*Let $A \in \mathcal{M}_n$ then there exists $S \in \mathcal{M}_n$ non-singular such that*

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) S^{-1}$$

## Theorem 4.36

*Let $A \in \mathcal{M}_n$ with real eigenvalues. Then there exists a basis of generalised eigenvectors for $\mathbb{R}^n$, and if $\{v_1, \dots, v_n\}$ is a basis of generalised eigenvectors of $\mathbb{R}^n$, then $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ is non-singular and $A = D + N$ where $P^{-1}DP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $N = A - D$ is nilpotent*[1] *of order $k \leq n$, and $D$ and $N$ commute.*

## Definition 4.37

Let $A$ be a Hermitian matrix in $\mathcal{M}_n$. We say that $A$ is **positive definite** if for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* A \mathbf{x} > 0$. We say that $A$ is **positive semidefinite** if for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^* A \mathbf{x} \geq 0$

## Theorem 4.38

*Let $A \in \mathcal{M}_n$ be a Hermitian matrix. Then*

1. *for all $\mathbf{x} \in \mathbb{C}^*$, $\mathbf{x}^* A \mathbf{x} \in \mathbb{R}$*
2. $\sigma(A) \subset \mathbb{R}$
3. *$S^* A S$ is Hermitian for any $S \in \mathcal{M}_n$*

## Theorem 4.39

*Each eigenvalue of a positive definite matrix (respectively positive semidefinite matrix) is positive (respectively nonnegative)*

## Proposition 4.40

*Let $A$ be a positive semidefinite (respectively positive definite) matrix. Then* $\mathrm{tr}(A)$, $\det(A)$, *the principal minors of $A$ are all nonnegative (respectively positive). Also,* $\mathrm{tr}(A) = 0$ *if and only if* $A = 0$

## Theorem 4.41

*Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and $\mathbf{x} \in \mathbb{C}^n$. Then*

$$\mathbf{x}^* A\mathbf{x} = 0 \iff A\mathbf{x} = \mathbf{0}$$

## Corollary 4.42

*Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix. Then $A$ is positive definite if and only if $A$ is nonsingular*

## Theorem 4.43 (Somewhat unrelated)

*Let $B \in \mathcal{M}_n$ be a Hermitian matrix, $\mathbf{y} \in \mathbb{C}^n$, and $a \in \mathbb{R}$. Let*

$$A = \begin{pmatrix} B & \mathbf{y} \\ \mathbf{y}^* & a \end{pmatrix} \in \mathcal{M}_{n+1}$$

*Then*

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

### Definition 4.44

The singular values of a matrix $A$ are the (nonnegative) square roots of the eigenvalues of $A^*A$

### Remark 4.45

*$A^*A$ is positive semidefinite*

## Theorem 4.46 (Zhang)

*Let* $A \in \mathcal{M}_{mn}$ *with nonzero singular values* $\sigma_1, \ldots, \sigma_r$*. Then there exists* $U \in \mathcal{M}_n$ *and* $V \in \mathcal{M}_n$ *unitary such that*

$$A = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V,$$

*where* $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{mn}$ *and* $D_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$

## Theorem 4.47 (H & J)

*Let* $A \in \mathcal{M}_{nm}$ *,* $q = \min\{m, n\}$*. Assume that the rank of* $A$ *is* $n$*. Then*

1. $\exists V \in M_n$ *and* $W \in \mathcal{M}_m$ *unitary matrices and* $\Sigma_q \in \mathcal{M}_q = \mathrm{diag}(\sigma_1, \ldots, \sigma_q)$ *s.t.*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$$

   *and*

$$A \Sigma W$$

   *where*

$$\Sigma = \begin{cases} \Sigma_1, & m = n \\ \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{nm}, & m > n \\ \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{nm}, & n > m \end{cases}$$

2. *The parameters* $\sigma_1, \ldots, \sigma_r$ *are the positive square roots of the decreasingly ordered eigenvalues of* $A^* A$

## Remark 4.48

*Let $A \in \mathcal{M}_{mn}$. Then $A, \overline{A}, A^T$, and $A^*$ have the same singular values*

## Remark 4.49

*Let $A \in \mathcal{M}_n$ with singular values $\sigma_1, \ldots, \sigma_n$, then*

$$\sigma_1 \ldots \sigma_n = \det(A)$$

*and*

$$\sigma_1^2 + \ldots + \sigma_n^2 = \operatorname{tr}(A^* A)$$

## Theorem 4.50

*Let* $A \in \mathcal{M}_{nm}$, $q = \min m, n$, *and* $\sigma_1 \geq \cdots \geq \sigma_q$ *nonincresingly ordered singular values of* $A$. *Define*

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

*to be a Hermitian matrix. Then the ordered eigenvalues of* $\mathcal{A}$ *are*

$$-\sigma_1 \leq \cdots \leq -\sigma_q \leq \underbrace{0 = \cdots = 0}_{|n - m|} \leq \sigma_q \leq \cdots \leq \sigma_1$$

## Theorem 4.51 (An interlacing result)

*Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\hat{A}$ be the matrix obtained from $A$ by deleting one row and one column. Let $\sigma_1 \geq \cdots \geq \sigma_q$ and $\hat{\sigma}_1 \geq \cdots \geq \hat{\sigma}_q$ be the nonsingular ordered singular values of $A$ and $\hat{A}$, respectively, where $\hat{\sigma}_q = 0$ if $n \geq m$ and a column is deleted or if $n \geq m$ and a row is deleted. Then*

$$\sigma_1 \geq \hat{\sigma_1} \geq \sigma_2 \geq \hat{\sigma_2} \geq \ldots \sigma_q \geq \hat{\sigma_q}.$$

## Theorem 4.52 (von Neumann)

*Let $A$, $B \in \mathcal{M}_{mn}$, $q = \min\{m, n\}$, $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ and $\sigma_1(B) \geq \cdots \geq \sigma_q(B)$ the non-increasingly singular values of $A$ and $B$, respectively. Then*

$$\operatorname{Re}\operatorname{tr}(AB^*) \leq \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B).$$

## Theorem 4.53

*Let* $A \in \mathcal{M}_{nm}$*,* $q = \min m, n$*, and* $\sigma_1 \geq \cdots \geq \sigma_q$ *nonincreasingly ordered singular values of* $A$*, and* $\alpha = \{1, \ldots, q\}$*. Then*

$$\operatorname{Re}\operatorname{tr}(A) \leq \sum_{i=1}^{q} \sigma_i$$

*with equality if and only if* $A[\alpha]$ *(principal leading submatrix of* $A$*) is positive semidefinite and* $A$ *has no nonzero entries outside* $A[\alpha]$*.*

- Let $A \in \mathcal{M}_2$

$$\sigma_1, \sigma_2 = \frac{1}{2}\left((\mathrm{tr} A^* A) \mp \sqrt{(\mathrm{tr} A^* A)^2 - 4|\det A|^2}\right)$$

- The nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & & \\ & \ddots & & \\ & & & a_{n-1,m} \\ & & & 0 \end{pmatrix}$$

has singular values $0, |a_{12}|, \ldots, |a_{n-1,n}|$.

## Theorem 4.54

*Let* $A_1, A_2, \cdots \in \mathcal{M}_{nm}$ *given (infinite) sequence with* $\lim_{k\to\infty} A_k = A$ *(entrywise). Let* $q = \min(m, n)$*. Let* $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ *and* $\sigma_1(A_k) \geq \cdots \geq \sigma_q(A_k)$ *be the non-increasinly ordered singular values of* $A$ *and* $A_k$*, respectively (for all* $k$*). Then*

$$\lim_{k\to\infty} \sigma_i(A_k) = \sigma_i(A)$$

## Theorem 4.55

*Let* $A \in \mathcal{M}_n$ *where* $n = \text{rank } A$

1. $A = A^T$ *if and only if there exists* $U \in \mathcal{M}_n$ *unitary and a nonnegative diagonal matrix* $\Sigma$ *such that* $A = U\Sigma U^T$. *Then the diagonal entries of* $\Sigma$ *are the singular values of* $A$
2. *If* $A = -A^T$, *then* $n$ *is even and there exists* $U \in \mathcal{M}_n$ *unitary and positive real scalars* $s_1, \ldots, s_{r/2}$ *such that*

$$U\left(\begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_{r/2} \\ -s_{r/2} & 0 \end{pmatrix}\right) U^T$$

*The non-zero singular values of* $A$ *are* $s_1, s_1, \ldots, s_{r/2}, s_{r/2}$. *Conversely, any matrix of the above form is skew-symetric*

# References I