

# MATH 4370/7370 – Linear Algebra and Matrix Analysis

Quick review of 2nd year linear algebra

Julien Arino

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**University  
of Manitoba**

# OUTLINE OF THESE SLIDES

- Part 1: Some notation and basic stuff
- Part 2: Vector spaces
- Part 3: Finite-dimensional vector spaces
- Part 4: Linear maps
- Part 5: Eigenvalues, eigenvectors and invariant subspaces
- Part 6: Inner product spaces
- Part 7: Operators on inner product spaces
- Part 8: Operators on complex vector spaces

## Source of the material

The material in these slides is mostly derived from [Axl15]

# Some notation and basic stuff

Sets and elements

Logic

Sets and elements

Logic

# Sets and elements

## Definition 1 (Set)

A **set**  $X$  is a collection of **elements**.

We write  $x \in X$  or  $x \notin X$  to indicate that the element  $x$  belongs to the set  $X$  or does not belong to the set  $X$ , respectively.

## Definition 2 (Subset)

Let  $X$  be a set. The set  $S$  is a **subset** of  $X$ , which is denoted  $S \subset X$  or  $S \subseteq X$ , if all its elements belong to  $X$ .  $S$  is a **proper subset** of  $X$  if it is a subset of  $X$  and not equal to  $X$ ; we then write  $S \subsetneq X$ .

Smith reserves  $\subset$  for  $\subsetneq$ . I learned  $\subset$  for not specified (proper or not) and  $\subsetneq$  for proper. So beware!

# Quantifiers

- ▶ A shorthand notation for “for all elements  $x$  belonging to  $X$ ” is  $\forall x \in X$ . For example, if  $X = \mathbb{R}$ , the field of real numbers, then  $\forall x \in \mathbb{R}$  means “for all real numbers  $x$ ”.
- ▶ A shorthand notation for “there exists an element  $x$  in the set  $X$ ” is  $\exists x \in X$ .
- ▶ Sometimes we write  $\exists! x \in X$  for “there exists a **unique**  $x$  in  $X$ ”.
- ▶  $\forall$  and  $\exists$  are **quantifiers**.

# Intersection and union of sets

Let  $X$  and  $Y$  be two sets.

## Definition 3 (Intersection)

The intersection of  $X$  and  $Y$ ,  $X \cap Y$ , is the set of elements that belong to  $X$  **and** to  $Y$

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

## Definition 4 (Union)

The union of  $X$  and  $Y$ ,  $X \cup Y$ , is the set of elements that belong to  $X$  **or** to  $Y$

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}$$

Use of the expression “and/or” is *strictly* forbidden in this course! “Or but not and” (a.k.a. **xor**, exclusive or) is  $(X \cup Y) \setminus (X \cap Y)$ .



Sets and elements

Logic

## A few notions of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. “The sky is blue” is also a proposition.

Let  $A$  be a proposition. We generally write

$A$

to mean that  $A$  is true, and

**not**  $A$

to mean that  $A$  is false. We also write  $\neg A$ . **not**  $A$  is the **negation** of  $A$ .

## A few notions of logic (cont.)

Let  $A, B$  be propositions. Then

- ▶  $A \Rightarrow B$  (read  $A$  implies  $B$ ) means that whenever  $A$  is true, then so is  $B$ .
- ▶  $A \Leftrightarrow B$ , also denoted  $A$  if and only if  $B$  ( $A$  iff  $B$  for short), means that  $A \Rightarrow B$  **and**  $B \Rightarrow A$ . We also say that  $A$  and  $B$  are equivalent.

Let  $A$  and  $B$  be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\text{not } B \Rightarrow \text{not } A)$$

This is useful for proving some results.

## Necessary and/or sufficient conditions

Suppose we want to establish whether a given statement  $P$  is true, depending on the truth value of a statement  $H$ . Then we say that

- ▶  $H$  is a **necessary condition** if  $P \Rightarrow H$ .  
(It is necessary that  $H$  be true for  $P$  to be true; so whenever  $P$  is true, so is  $H$ ).
- ▶  $H$  is a **sufficient condition** if  $H \Rightarrow P$ .  
(It suffices for  $H$  to be true for  $P$  to also be true).
- ▶  $H$  is a **necessary and sufficient condition** if  $H \Leftrightarrow P$ , i.e.,  $H$  and  $P$  are equivalent.

## Playing with quantifiers

For the quantifiers  $\forall$  (for all) and  $\exists$  (there exists),

$\exists$  is the negation of  $\forall$

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

This is also regularly used in proofs.

# Vector spaces

Fields

Definition of vector spaces

Example – Space  $\mathbb{F}^n$

Example – Complex numbers

Subspaces

## Fields

Definition of vector spaces

Example – Space  $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Operations

## Definition 5 (Operations – Addition and multiplication)

An **operation** on a set  $V$  is a mapping that associates an element of the set  $V$  to every pair of its elements

- ▶ The result of the **addition** of  $a$  and  $b$  is the *sum*  $a + b$  of  $a$  and  $b$
- ▶ The result of the **multiplication** of  $a$  and  $b$  is the *product*  $ab$  (or  $a \cdot b$ ) of  $a$  and  $b$



## Definition 6 (Field)

A **field** is a set  $\mathbb{F}$  together with two (binary) operations, *addition* and *multiplication*, which are required to satisfy the following *field axioms*, where  $a, b, c \in \mathbb{F}$ :

- ▶ **Associativity** of addition and multiplication:  $a + (b + c) = (a + b) + c$  and  $a(bc) = (ab)c$
- ▶ **Commutativity** of addition and multiplication:  $a + b = b + a$  and  $ab = ba$
- ▶ **Additive and multiplicative identity**:  $\exists 0, 1 \in \mathbb{F}$ ,  $0 \neq 1$ , s.t.  $a + 0 = a$  and  $a1 = a$
- ▶ **Additive inverses**:  $\forall a \in \mathbb{F}$ ,  $\exists -a \in \mathbb{F}$  s.t.  $a + (-a) = 0$
- ▶ **Multiplicative inverses**:  $\forall a \neq 0 \in \mathbb{F}$ ,  $\exists a^{-1} \in \mathbb{F}$  s.t.  $aa^{-1} = 1$
- ▶ **Distributivity** (of multiplication over addition):  $a(b + c) = (ab) + (ac)$

# Notation

- ▶ Both  $\mathbb{R}$  and  $\mathbb{C}$  are fields.
- ▶ From now on,  $\mathbb{F}$  refers to  $\mathbb{R}$  or  $\mathbb{C}$ .
- ▶ Some results are specific to  $\mathbb{R}$  xor  $\mathbb{C}$ , in which case we specify the relevant field.
- ▶ If we use  $\mathbb{F}$ , we mean the result applies to both  $\mathbb{R}$  and  $\mathbb{C}$ .

Fields

Definition of vector spaces

Example – Space  $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Addition and Scalar multiplication

## Definition 7 (Addition and scalar multiplication on a set)

- ▶ An **addition** on a set  $V$  is a function that assigns an element  $\mathbf{u} + \mathbf{v} \in V$  to each pair of elements  $\mathbf{u}, \mathbf{v} \in V$
- ▶ A **scalar multiplication** on a set  $V$  is a function that assigns an element  $\lambda \mathbf{v}$  to each  $\lambda \in \mathbb{F}$  and each  $\mathbf{v} \in V$

# Vector space

## Definition 8 (Vector space)

A **vector space** (over  $\mathbb{F}$ ) is a set  $V$  along with an addition on  $V$  and a scalar multiplication on  $V$  such that the following properties (*axioms*) hold

1.  $\forall \mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  [commutativity]
2.  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and  $\forall a, b \in \mathbb{F}, (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  and  $(ab)\mathbf{v} = a(b\mathbf{v})$  [associativity]
3.  $\exists \mathbf{0}_V \in V$  s.t.  $\forall \mathbf{v} \in V, \mathbf{v} + \mathbf{0}_V = \mathbf{v}$  [additive identity]
4.  $\forall \mathbf{v} \in V, \exists \mathbf{w} \in V$  s.t.  $\mathbf{v} + \mathbf{w} = \mathbf{0}_V$  [additive inverse]
5.  $\forall \mathbf{v} \in V, 1\mathbf{v} = \mathbf{v}$  [multiplicative identity]
6.  $\forall a, b \in \mathbb{F}$  and  $\forall \mathbf{u}, \mathbf{v} \in V, a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$  and  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$  [distributivity]

# Results

## Theorem 9 (Uniqueness of the additive identity)

*A vector space  $V$  has a unique additive identity  $\mathbf{0}_V \in V$*

## Theorem 10 (Existence and uniqueness of additive inverse)

*Let  $V$  be a vector space. Then each  $\mathbf{v} \in V$  has a unique additive inverse, denoted  $-\mathbf{v}$*

We also define  $\mathbf{v} - \mathbf{w}$  as  $\mathbf{v} + (-\mathbf{w})$ .

## Theorem 11

- ▶  $\forall \mathbf{v} \in V, 0_{\mathbb{F}}\mathbf{v} = \mathbf{0}_V.$
- ▶  $\forall a \in \mathbb{F}, a\mathbf{0}_V = \mathbf{0}_V.$
- ▶  $\forall \mathbf{v} \in V, (-1)\mathbf{v} = -\mathbf{v}.$

# Vector space

## Definition 12 (Vector space)

A **vector space** (over  $\mathbb{F}$ ) is a set  $V$  along with an addition on  $V$  and a scalar multiplication on  $V$  such that the following properties (*axioms*) hold

1.  $\forall \mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  [commutativity of  $+$ ]
2.  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  [associativity of  $+$ ]
3.  $\exists! \mathbf{0}_V \in V$  s.t.  $\forall \mathbf{v} \in V, \mathbf{v} + \mathbf{0}_V = \mathbf{v}$  [additive identity]
4.  $\forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V$  s.t.  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_V$  [additive inverse]
5.  $\forall a \in \mathbb{F}$  and  $\forall \mathbf{u}, \mathbf{v} \in V, a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$  [distributivity of  $\cdot$  over  $+$ ]
6.  $\forall a, b \in \mathbb{F}$  and  $\forall \mathbf{u} \in V, (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$  [distributivity of  $+$  over  $\cdot$ ]
7.  $\forall a, b \in \mathbb{F}, (ab)\mathbf{u} = a(b\mathbf{u})$  [associativity of  $\cdot$ ]
8.  $\forall \mathbf{u} \in V, 1\mathbf{u} = \mathbf{u}$  [multiplicative identity]

Fields

Definition of vector spaces

Example – Space  $\mathbb{F}^n$

Example – Complex numbers

Subspaces



$\mathbb{F}^n$  is a vector space

Typically called *Euclidean space* when  $\mathbb{F} = \mathbb{R}$ .

### Definition 13

Let  $0 \neq n \in \mathbb{N}$ . An  $n$ -**tuple** is an ordered collection of  $n$  elements,

$$(x_1, \dots, x_n)$$

### Definition 14

Let  $0 \neq n \in \mathbb{N}$ .  $\mathbb{F}^n$  is the set of all  $n$ -tuples of elements of  $\mathbb{F}$ :

$$\mathbb{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n\}$$

- ▶ Often write  $x = (x_1, \dots, x_n)$  for short.
- ▶ For a given  $j \in \{1, \dots, n\}$ ,  $x_j$  is the  $j$ th **coordinate** of  $x$ .
- ▶ Think of  $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$  that you saw in whatever flavour of Linear Algebra 1 you took.

## Addition in $\mathbb{F}^n$

### Definition 15 (Addition in $\mathbb{F}^n$ )

Let  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{F}^n$ . Then

$$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

### Property 16 (Commutativity of addition in $\mathbb{F}^n$ )

*Let  $x, y \in \mathbb{F}^n$ , then*

$$x + y = y + x$$

## 0 and additive inverse in $\mathbb{F}^n$

### Definition 17 (0)

0 denotes the  $n$ -tuple whose coordinates are all 0,

$$0 = (0, \dots, 0)$$

If any ambiguity arises, will write  $0_{\mathbb{F}^n}$

### Definition 18 (Additive inverse)

Let  $x \in \mathbb{F}^n$ . The **additive inverse** of  $x$  is  $-x \in \mathbb{F}^n$  s.t.

$$x + (-x) = 0$$

If  $x = (x_1, \dots, x_n)$ , then  $-x = (-x_1, \dots, -x_n)$

## Scalar multiplication in $\mathbb{F}^n$

### Definition 19 (Scalar multiplication)

The **product** of  $\lambda \in \mathbb{F}$  and  $x \in \mathbb{F}^n$  is

$$\lambda x = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Fields

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Subspaces

# Complex numbers

## Definition 20 (Complex numbers)

A **complex number** is an ordered pair  $(a, b)$ , where  $a, b \in \mathbb{R}$ . Usually written  $a + ib$  or  $a + bi$ , where  $i^2 = -1$

The set of all complex numbers is denoted  $\mathbb{C}$ ,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

### Definition 21 (Addition and multiplication on $\mathbb{C}$ )

Letting  $a + ib$  and  $c + id \in \mathbb{C}$ , addition on  $\mathbb{C}$  is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on  $\mathbb{C}$  is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter equality easy to obtain using regular multiplication and  $i^2 = -1$



# Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C},$

▶  $\alpha + \beta = \beta + \alpha$  and  $\alpha\beta = \beta\alpha$

[commutativity]

▶  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  and  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

[associativity]

▶  $\gamma + 0 = \gamma$  and  $\gamma 1 = \gamma$

[identities]

▶  $\forall \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C}$  unique s.t.  $\alpha + \beta = 0$

[additive inverse]

▶  $\forall \alpha \neq 0 \in \mathbb{C}, \exists \beta \in \mathbb{C}$  unique s.t.  $\alpha\beta = 1$

[multiplicative inverse]

▶  $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$

[distributivity]

Thus  $\mathbb{C}$  is a field.

# Additive & multiplicative inverse, subtraction, division

## Definition 22

Let  $\alpha, \beta \in \mathbb{C}$

- ▶  $-\alpha$  is the **additive inverse** of  $\alpha$ , i.e., the unique number in  $\mathbb{C}$  s.t.  $\alpha + (-\alpha) = 0$
- ▶ **Subtraction** on  $\mathbb{C}$ :

$$\beta - \alpha = \beta + (-\alpha)$$

- ▶ For  $\alpha \neq 0$ ,  $1/\alpha$  is the **multiplicative inverse** of  $\alpha$ , i.e., the unique number in  $\mathbb{C}$  s.t.

$$\alpha(1/\alpha) = 1$$

- ▶ **Division** on  $\mathbb{C}$ :

$$\beta/\alpha = \beta(1/\alpha)$$

### Definition 23 (Real and imaginary parts)

Let  $z = a + ib$ . Then  $\operatorname{Re} z = a$  is **real part** and  $\operatorname{Im} z = b$  is **imaginary part** of  $z$

If ambiguous, write  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$

### Definition 24 (Conjugate and Modulus)

Let  $z = a + ib \in \mathbb{C}$ . Then

► **Complex conjugate** of  $z$  is

$$\bar{z} = \operatorname{Re} z - i(\operatorname{Im} z) = a - ib$$

► **Modulus** (or **absolute value**) of  $z$  is

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{a^2 + b^2} \geq 0$$

# Properties of complex numbers

Let  $w, z \in \mathbb{C}$ , then

▶  $z + \bar{z} = 2\operatorname{Re} z$

▶  $z - \bar{z} = 2i\operatorname{Im} z$

▶  $z\bar{z} = |z|^2$

▶  $\overline{w + z} = \bar{w} + \bar{z}$  and  $\overline{wz} = \bar{w}\bar{z}$

▶  $\bar{\bar{z}} = z$

▶  $|\operatorname{Re} z| \leq |z|$  and  $|\operatorname{Im} z| \leq |z|$

▶  $|\bar{z}| = |z|$

▶  $|wz| = |w| |z|$

▶  $|w + z| \leq |w| + |z|$

[triangle inequality]

Fields

Definition of vector spaces

Example – Space  $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Subspace

## Definition 25 (Subspace)

Let  $V$  be a vector space over  $\mathbb{F}$ . Let  $U \subseteq V$  be a subset of  $V$ . Then  $U$  is a **subspace** of  $V$  if  $U$  is a vector space over  $\mathbb{F}$  for the same operations of addition and scalar multiplication as  $V$

## Theorem 26 (Conditions for a subspace)

$U \subseteq V$  is a subspace of  $V \iff U$  satisfies the following three conditions:

- ▶  $\mathbf{0}_V \in U$  [additive identity]
- ▶  $\forall \mathbf{u}, \mathbf{v} \in U, \mathbf{u} + \mathbf{v} \in U$  [closed under addition]
- ▶  $\forall \mathbf{u} \in U, \forall a \in \mathbb{F}, a\mathbf{u} \in U$  [closed under scalar multiplication]

The smallest possible subspace of  $V$  is  $\{\mathbf{0}_V\}$ , the largest is  $V$ .

# Sums of subspaces

## Definition 27 (Sum of subsets)

Let  $V$  be a vector space and  $U_1, \dots, U_m$  be *subsets* of  $V$ . The **sum** of  $U_1, \dots, U_m$  is

$$U_1 + \cdots + U_m = \{\mathbf{u}_1 + \cdots + \mathbf{u}_m : \mathbf{u}_1 \in U_1, \dots, \mathbf{u}_m \in U_m\}$$

## Theorem 28

*Let  $V$  be a vector space and  $U_1, \dots, U_m$  be subspaces of  $V$ . Then  $U_1 + \cdots + U_m$  is the smallest subspace of  $V$  containing  $U_1, \dots, U_m$*

## Direct sums

### Definition 29 (Direct sum)

Suppose  $U_1, \dots, U_m$  are subspaces of a vector space  $V$ . The sum  $U_1 + \dots + U_m$  is a **direct sum** and is then written  $U_1 \oplus \dots \oplus U_m$  if each element of  $U_1 + \dots + U_m$  can be written in only one way as a sum  $\mathbf{u}_1 + \dots + \mathbf{u}_m$ , where each  $\mathbf{u}_j \in U_j$

### Theorem 30 (Condition for a direct sum)

*Suppose  $U_1, \dots, U_m$  are subspaces of a vector space  $V$ . Then  $U_1 + \dots + U_m$  is a direct sum  $\iff$  the only way to write  $\mathbf{0}$  as a sum  $\mathbf{u}_1 + \dots + \mathbf{u}_m$ , where each  $\mathbf{u}_j \in U_j$ , is by taking each  $\mathbf{u}_j$  equal to  $\mathbf{0}_V$*

### Theorem 31 (Direct sum of two subspaces)

*Let  $U, W$  be subspaces of a vector space  $V$ . Then  $U + W$  is a direct sum  $\iff U \cap W = \{\mathbf{0}_V\}$*



# Finite-dimensional vector spaces

Span and Linear independence

Bases

Dimension

## Span and Linear independence

Bases

Dimension

### Definition 32 (Linear combination)

A **linear combination** of a list  $\mathbf{v}_1, \dots, \mathbf{v}_m$  of vectors in  $V$  is a vector

$$a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m,$$

where  $a_1, \dots, a_m \in \mathbb{F}$

### Definition 33 (Span)

The set of all linear combinations of a list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$  is the **span** of  $\mathbf{v}_1, \dots, \mathbf{v}_m$ ,

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = \{a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m : a_1, \dots, a_m \in \mathbb{F}\}$$

The span of the empty list  $()$  is  $\{\mathbf{0}_V\}$

# Finite/infinite-dimensional vector spaces

## Theorem 34

*The span of a list of vectors in  $V$  is the smallest subspace of  $V$  containing all the vectors in the list*

## Definition 35 (List of vectors spanning a space)

If  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = V$ , we say  $\mathbf{v}_1, \dots, \mathbf{v}_m$  **spans**  $V$

## Definition 36 (Finite-dimensional vector space)

A vector space  $V$  is **finite-dimensional** if some list of vectors in it spans  $V$

## Definition 37 (Infinite-dimensional vector space)

A vector space  $V$  is **infinite-dimensional** if it is not finite-dimensional

## Linear (in)dependence

### Definition 38 (Linear independence/Linear dependence)

A list  $\mathbf{v}_1, \dots, \mathbf{v}_m$  of vectors in a vector space  $V$  is **linearly independent** if

$$(a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m = 0) \Leftrightarrow (a_1 = \dots = a_m = 0),$$

where  $a_1, \dots, a_m \in \mathbb{F}$ . A list of vectors is **linearly dependent** if it is not linearly independent.

The empty list  $()$  is assumed to be linearly independent

### Lemma 39 (Linear dependence)

*Let  $\mathbf{v}_1, \dots, \mathbf{v}_m$  be a linearly dependent list in a vector space  $V$ . Then there exists  $j \in \{1, 2, \dots, m\}$  s.t.*

- 1.  $\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$*
- 2. if the  $j$ th term is removed from  $\mathbf{v}_1, \dots, \mathbf{v}_m$ , the span of the remaining list equals  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$*

### Theorem 40

*Let  $V$  be a finite-dimensional vector space. Then the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors*

### Theorem 41 (Subspace of a finite-dimensional vector space)

*Every subspace of a finite-dimensional vector space is finite-dimensional*

Span and Linear independence

Bases

Dimension

# Basis

## Definition 42 (Basis)

Let  $V$  be a vector space. A **basis** of  $V$  is a list of vectors in  $V$  that is both linearly independent and spanning

## Theorem 43 (Criterion for a basis)

*A list  $\mathbf{v}_1, \dots, \mathbf{v}_m$  of vectors in a vector space  $V$  is a basis of  $V$  iff  $\forall \mathbf{v} \in V$ ,  $\mathbf{v}$  can be written uniquely in the form*

$$\mathbf{v} = a_1 \mathbf{v}_1 + \dots + a_m \mathbf{v}_m,$$

*where  $a_1, \dots, a_m \in \mathbb{F}$*



#### Theorem 44 (All spanning lists contain a basis)

*Every spanning list in a vector space can be reduced to a basis of the vector space*

#### Theorem 45 (Basis of finite-dimensional vector space)

*Every finite-dimensional vector space has a basis*

#### Theorem 46 (Extension to a basis)

*Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space*

#### Theorem 47

*Let  $V$  be a finite-dimensional vector space and  $U \subset V$  be a subspace of  $V$ . Then  $\exists W \subset V$  subspace of  $V$  s.t.  $V = U \oplus W$*

Span and Linear independence

Bases

Dimension

### Theorem 48 (Bases of a finite-dim. space have equal length)

*Any two bases of a finite-dimensional vector space have the same length*

### Definition 49 (Dimension)

The **dimension**  $\dim V$  of a finite-dimensional vector space  $V$  is the length of any basis of the vector space

### Theorem 50 (Dimension of a subspace)

*Let  $V$  be a finite-dimensional vector space and  $U \subset V$  be a subspace of  $V$ . Then  $\dim U \leq \dim V$*

### Theorem 51

*Let  $V$  be a finite-dimensional vector space. Then every linearly independent list of vectors in  $V$  with length  $\dim V$  is a basis of  $V$*

### Theorem 52

*Let  $V$  be a finite-dimensional vector space. Then every spanning list of vectors in  $V$  with length  $\dim V$  is a basis of  $V$*

### Theorem 53 (Dimension of a sum of subspaces)

*Let  $U_1, U_2$  be subspaces of a finite-dimensional vector space  $V$ . Then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

# Linear maps

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### Definition 54 (Linear map/transformation)

Let  $V, W$  be vector spaces. A **linear map** (or **linear transformation**) from  $V$  to  $W$  is a function  $T : V \rightarrow W$  that has the following properties:

1. **Additivity**  $\forall \mathbf{u}, \mathbf{v} \in V, T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ .
2. **Homogeneity**  $\forall \lambda \in \mathbb{F}, \forall \mathbf{v} \in V, T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ .

Often, parentheses are omitted,  $T(\mathbf{u})$  is written  $T\mathbf{u}$

The set of all linear maps from  $V$  to  $W$  is denoted  $\mathcal{L}(V, W)$

### Theorem 55 (Linear maps and basis of domain)

*Let  $V, W$  be two vector spaces and  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be a basis of  $V$ . Let  $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$ . Then there exists a unique linear map  $T : V \rightarrow W$  s.t.*

$$\forall j = 1, \dots, n, \quad T\mathbf{v}_j = \mathbf{w}_j$$



### Definition 56 (Addition & Scalar multiplication)

Let  $V, W$  be vector spaces,  $S, T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbb{F}$ . The **sum**  $S + T$  and **product**  $\lambda T$  are the linear maps from  $V$  to  $W$  defined,  $\forall \mathbf{v} \in V$ , by

$$(S + T)(\mathbf{v}) = S\mathbf{v} + T\mathbf{v} \text{ and } (\lambda T)(\mathbf{v}) = \lambda(T\mathbf{v}).$$

### Theorem 57 (Linear maps are vector spaces)

*Let  $V, W$  be vector spaces. Equipped with addition and scalar multiplication as just defined,  $\mathcal{L}(V, W)$  is a vector space.*

## Product of linear maps

### Definition 58 (Product of linear maps)

Let  $U, V, W$  be vector spaces,  $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$ . The **product**  $ST \in \mathcal{L}(U, W)$  is defined for  $\mathbf{u} \in U$  by

$$(ST)(\mathbf{u}) = S(T\mathbf{u}).$$

This means that the product of linear maps is the composition  $S \circ T$ , although because of the linearity, we often omit the  $\circ$  composition sign.

# Properties of products of linear maps

## Theorem 59

1. **Associativity** If  $V, V_2, V_3, W$  vector spaces,  
 $T_1 \in \mathcal{L}(V, V_2), T_2 \in \mathcal{L}(V_2, V_3), T_3 \in \mathcal{L}(V_3, W)$ , then

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$

2. **Identity**  $V, W$  vector spaces. Then for  $T \in \mathcal{L}(V, W)$ ,

$$T I_V = I_W T = T$$

3. **Distributivity**  $U, V, W$  vector spaces,  $T, T_1, T_2 \in \mathcal{L}(U, V), S, S_1, S_2 \in \mathcal{L}(V, W)$ ,  
then

$$(S_1 + S_2) T = S_1 T + S_2 T \text{ and } S(T_1 + T_2) = S T_1 + S T_2$$

### Theorem 60 (Linear maps take 0 to 0)

*Let  $V, W$  be vector spaces,  $T \in \mathcal{L}(V, W)$ . Then*

$$T(\mathbf{0}_V) = \mathbf{0}_W.$$

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### Definition 61 (Null space)

Let  $V, W$  be finite-dimensional vector spaces and  $T \in \mathcal{L}(V, W)$ . The **null space**  $\text{null } T$  (or **kernel**  $\ker T$ ) of  $T$  is the subset of  $V$  consisting of those vectors that  $T$  maps to  $\mathbf{0}_W$ :

$$\text{null } T = \{\mathbf{v} \in V; T\mathbf{v} = \mathbf{0}_W\}.$$

### Theorem 62 (Null space is a subspace)

*Let  $V, W$  be finite-dimensional vector spaces and  $T \in \mathcal{L}(V, W)$ . Then  $\text{null } T$  is a subspace of  $V$*

### Definition 63 (Injectivity)

A function  $T : V \rightarrow W$  is **injective** (or **one-to-one**) if

$$T\mathbf{u} = T\mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}.$$

We can also use the contrapositive:  $T$  injective if  $\mathbf{u} \neq \mathbf{v} \Rightarrow T\mathbf{u} \neq T\mathbf{v}$ .

### Theorem 64 (Linking injectivity and null space)

*Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then*

$$T \text{ injective} \Leftrightarrow \text{null } T = \{\mathbf{0}_V\}$$

### Definition 65 (Range)

Let  $V, W$  be finite-dimensional vector spaces,  $T : V \rightarrow W$  a function. The **range** (or **image**) of  $T$  is the subset of  $W$  defined by

$$\text{range } T = \{T\mathbf{v}; \mathbf{v} \in V\}.$$

When talking about the image, we write  $\text{Im } T$ .

### Theorem 66 (Range is a subspace)

*Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then  $\text{range } T$  is a subspace of  $W$ .*



### Definition 67 (Surjectivity)

A function  $T : V \rightarrow W$  is **surjective** (or **onto**) if

$$\text{range } T = W$$

### Theorem 68 (Fundamental theorem of linear maps)

*Let  $V$  be a finite-dimensional vector space and  $T \in \mathcal{L}(V, W)$ . Then  $\dim \text{range } T < \infty$  and*

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

### Theorem 69 (Linear map onto a smaller space is not injective)

*Let  $V, W$  be finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then  $\nexists T \in \mathcal{L}(V, W)$  that is injective*

### Theorem 70 (Linear map onto a larger space is not surjective)

*Let  $V, W$  be finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then  $\nexists T \in \mathcal{L}(V, W)$  that is surjective*

Do as exercises..

### Theorem 71

*A homogeneous system of linear equations with more variables than equations has nonzero solutions.*

### Theorem 72

*A nonhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms*

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### Definition 73 (Matrix)

An  $m$ -by- $n$  or  $m \times n$  matrix is a rectangular array of elements of  $\mathbb{F}$  with  $m$  rows and  $n$  columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Remember that we always list indices as “row,column”

We denote  $\mathcal{M}_{mn}(\mathbb{F})$  the set of  $m \times n$  matrices with entries in  $\mathbb{F}$

### Definition 74 (Matrix of a linear map)

Let  $V, W$  be finite-dimensional vector spaces,  $v_1, \dots, v_n$  a basis of  $V$  and  $w_1, \dots, w_m$  a basis of  $W$ . The **matrix of**  $T$  with respect to these bases is the matrix  $M(T) \in \mathcal{M}_{mn}$  with entries  $a_{jk}$  defined by

$$Tv_k = a_{1k}w_1 + \cdots + a_{mk}w_m$$

for  $1 \leq k \leq n$ . If the bases are not clear from the context, then write

$$M(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$$

I will often write  $M_T$  rather than  $M(T)$ .

Most definitions are assumed known

### Theorem 75 (Matrix of sums of linear maps)

*Suppose  $S, T \in \mathcal{L}(V, W)$ . Then  $M(S + T) = M(S) + M(T)$*

### Theorem 76 (Matrix of a scalar times a linear map)

*Suppose  $T \in \mathcal{L}(V, W)$ ,  $\lambda \in \mathbb{F}$ . Then  $M(\lambda T) = \lambda M(T)$*

### Theorem 77 (Dimension of $\mathcal{M}_{mn}$ )

$$\dim \mathbb{F}^{mn} = mn$$

### Theorem 78 (Matrix of products of linear maps)

*Suppose  $T \in \mathcal{L}(U, V)$ ,  $S \in \mathcal{L}(V, W)$ . Then  $M(ST) = M(S)M(T)$*

### Theorem 79

Let  $A \in \mathcal{M}_{mn}$ ,  $C \in \mathcal{M}_{np}$ . Then

$$(AC)_{jk} = A_{j\bullet} C_{\bullet k}, \quad 1 \leq j \leq m, 1 \leq k \leq p$$

and

$$(AC)_{\bullet k} = AC_{\bullet k}, \quad 1 \leq k \leq p$$

### Theorem 80

Let  $A \in \mathcal{M}_{mn}$ ,  $c = (c_1, \dots, c_n)^T \in \mathcal{M}_{n1}$ . Then

$$Ac = c_1 A_{\bullet 1} + \dots + c_n A_{\bullet n}$$



## Change of basis

### Definition 81 (Change of basis matrix)

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  bases of vector space  $V$  The **change of basis matrix**  $P_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathcal{M}_n$ ,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

has columns the coordinate vectors  $[\mathbf{u}_1]_{\mathcal{C}}, \dots, [\mathbf{u}_n]_{\mathcal{C}}$  of the vectors in  $\mathcal{B}$  w.r.t.  $\mathcal{C}$

### Theorem 82

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  bases of vector space  $V$  and  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  a change of basis matrix from  $\mathcal{B}$  to  $\mathcal{C}$

1.  $\forall \mathbf{x} \in V, P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$
2.  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  s.t.  $\forall \mathbf{x} \in V, P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$  is **unique**
3.  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  invertible and  $P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$

## Row-reduction method for changing bases

### Theorem 83

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  bases of vector space  $V$ . Let  $\mathcal{E}$  be any basis for  $V$ ,

$$B = [[\mathbf{u}_1]_{\mathcal{E}}, \dots, [\mathbf{u}_n]_{\mathcal{E}}] \text{ and } C = [[\mathbf{v}_1]_{\mathcal{E}}, \dots, [\mathbf{v}_n]_{\mathcal{E}}]$$

and let  $[C|B]$  be the augmented matrix constructed using  $C$  and  $B$ . Then

$$\text{RREF}([C|B]) = [\mathbb{I} | P_{C \leftarrow B}]$$

If working in  $\mathbb{R}^n$ , this is quite useful with  $\mathcal{E}$  the standard basis of  $\mathbb{R}^n$  (it does not matter if  $\mathcal{B} = \mathcal{E}$ )

## More on changing bases

### Theorem 84 (NSC for two matrices representing the same linear map)

*Let  $A, B \in \mathcal{M}_{mn}$ ,  $V$  and  $W$  be  $n$  and  $m$  dimensional vector spaces, respectively. Then  $A$  and  $B$  represent the same linear transformation  $T \in \mathcal{L}(V, W)$  relative to perhaps different bases of  $V$  and  $W \iff \exists P \in \mathcal{M}_m, Q \in \mathcal{M}_n$  nonsingular and such that*

$$A = PBQ^{-1}$$

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### Definition 85 (Inverse/Invertibility)

$T \in \mathcal{L}(V, W)$  is **invertible** if  $\exists S \in \mathcal{L}(W, V)$  s.t.  $ST = I_V$  and  $TS = I_W$ . Such a map is the **inverse** of  $T$

### Theorem 86 (Uniqueness of inverse)

*An invertible linear map  $T \in \mathcal{L}(V, W)$  has a unique inverse denoted  $T^{-1}$*

### Theorem 87 (NSC for invertibility)

$T \in \mathcal{L}(V, W)$  invertible  $\Leftrightarrow (T \text{ injective and surjective})$

### Definition 88 (Isomorphism/Isomorphic spaces)

$T \in \mathcal{L}(V, W)$  is an **isomorphism** if it is invertible. Two vector spaces are **isomorphic** if there exists an isomorphism from one to the other

### Theorem 89 (NSC for isomorphicity)

*Let  $V, W$  be finite-dimensional vector spaces over  $\mathbb{F}$ . Then*

$$V \text{ and } W \text{ are isomorphic} \Leftrightarrow \dim V = \dim W$$

### Theorem 90

*Let  $v_1, \dots, v_n$  be a basis of  $V$  and  $w_1, \dots, w_m$  be a basis of  $W$ . Then  $M$  is an isomorphism between  $\mathcal{M}_{mn}$  and  $\mathcal{L}(V, W)$*

### Theorem 91 (Dimension of $\mathcal{L}(V, W)$ )

*Let  $V, W$  be finite-dimensional vector spaces. Then  $\mathcal{L}(V, W)$  is finite-dimensional and*

$$\dim \mathcal{L}(V, W) = \dim V \dim W$$

## Definition 92 (Matrix of a vector)

Let  $V$  be a finite-dimensional vector space,  $v \in V$  and  $v_1, \dots, v_n$  a basis of  $V$ . The **matrix** of  $v$  with respect to the basis  $v_1, \dots, v_n$  is the  $n \times 1$  matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where  $c_1, \dots, c_n \in \mathbb{F}$  are s.t.

$$v = c_1 v_1 + \dots + c_n v_n$$

## Theorem 93

*Let  $V, W$  be finite-dimensional vector spaces,  $v_1, \dots, v_n$  a basis of  $V$ ,  $w_1, \dots, w_m$  a basis of  $W$  and  $T \in \mathcal{L}(V, W)$ . For  $k \in \{1, \dots, n\}$ ,  $M(T)_{\bullet k} = M(Tv_k)$*



### Theorem 94 (Linear maps act like matrix multiplication)

*Let  $V, W$  be finite-dimensional vector spaces,  $v_1, \dots, v_n$  a basis of  $V$ ,  $w_1, \dots, w_m$  a basis of  $W$ ,  $T \in \mathcal{L}(V, W)$  and  $v \in V$ . Then*

$$M(Tv) = M(T)M(v)$$

# Operator/Endomorphism

## Definition 95 (Operator/Endomorphism)

Let  $V$  be a vector space. A linear map  $\mathcal{L}(V, V)$  is an **operator** (or an **endomorphism**).  $\mathcal{L}(V) = \mathcal{L}(V, V)$  denotes the set of all operators on  $V$

## Theorem 96 (Injectivity equiv. to surjectivity in finite-dim.)

Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$ . TFAE:

1.  $T$  invertible
2.  $T$  injective
3.  $T$  surjective

# Rank of an operator/endomorphism

## Proposition 97 (Rank)

Let  $T \in \mathcal{L}(V)$  with  $V$  finite-dimensional. Then there exists bases  $\mathcal{B}_U = \{u_1, \dots, u_n\}$  and  $\mathcal{B}_V = \{v_1, \dots, v_n\}$  for  $V$  such that the matrix  $M_T$  of  $T$  can be written as the block matrix

$$M_T = \begin{pmatrix} \text{diag}(1, \dots, 1) & \mathbf{0}_{k, n-k} \\ \mathbf{0}_{n-k, k} & \mathbf{0}_{n-k, n-k} \end{pmatrix}$$

for some  $k \in \mathbb{N}$  called the **rank** of  $T$ , with  $k = \text{rank}(T) = \dim(\text{range } T)$ .

### Definition 98 (Row and column rank)

Let  $A \in \mathcal{M}_{mn}(\mathbb{F})$  be a matrix

- ▶ The **row rank** of  $A$  is the dimension of the span of the rows of  $A$  in  $\mathcal{M}_{1n}(\mathbb{F})$
- ▶ The **column rank** of  $A$  is the dimension of the span of the columns of  $A$  in  $\mathcal{M}_{m1}(\mathbb{F})$

Row and column ranks are the dimensions of the row and column spaces of  
Definition 102.

### Theorem 99 ( $\dim \text{range } T$ equals column rank of $M(T)$ )

*Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then  $\dim \text{range } T$  equals the column rank of  $M(T)$*

### Theorem 100 (Row rank equals column rank)

*Let  $A \in \mathcal{M}_{mn}$ . Then the row rank of  $A$  equals the column rank of  $A$*

### Definition 101 (Rank)

Let  $A \in \mathcal{M}_{mn}(\mathbb{F})$ . The **rank** of  $A$  is the column (or row, by Theorem 100) rank of  $A$

## Row space and column space of a matrix

### Definition 102 (Row and column spaces)

Let  $A \in \mathcal{M}_{mn}$ . The subspaces of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  spanned by the row and column vectors of  $A$  are the **row space** and **column space** of  $A$ , respectively.

### Definition 103 (Null space/kernel)

Let  $A \in \mathcal{M}_{mn}$ . The null space (or kernel) of  $A$  is the solution space of the homogeneous system  $A\mathbf{x} = \mathbf{0}$ .

This makes explicit the already seen definition in the special case of a matrix. As previously seen, the null space is a subspace of  $\mathbb{R}^n$ .

### Definition 104 (Nullity)

The dimension of the null space of  $A \in \mathcal{M}_{mn}$  is called the **nullity** of  $A$ .

## Theorem 105

Let  $A \in \mathcal{M}_{mn}$ . Then

1.  $\text{rank}(A) = \text{rank}(A^T)$
2.  $\text{rank}(A) + \text{nullity}(A) = n$
3.  $\text{rank}(A) \leq \min(m, n)$

## Theorem 106 (Consistency)

Consider the linear system  $A\mathbf{x} = \mathbf{b}$ , with  $A \in \mathcal{M}_{mn}$ . TFAE:

- ▶  $A\mathbf{x} = \mathbf{b}$  is consistent
- ▶  $\mathbf{b} \in \text{column space of } A$
- ▶  $A$  and  $[A|\mathbf{b}]$  have the same rank

## Proposition 107

*Let  $A \in \mathcal{M}_{mn}$  be in row-echelon form. Then*

- ▶ *The row vectors ( $\in \mathbb{R}^n$ ) with leading ones form a basis for the row space of  $A$ .*
- ▶ *The column vectors ( $\in \mathbb{R}^m$ ) with leading ones form a basis for the column space of  $A$ .*



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## Definition 108 (Product of vector spaces)

Let  $V_1, \dots, V_m$  be vector spaces over  $\mathbb{F}$ . The **product**  $V_1 \times \dots \times V_m$  is

$$V_1 \times \dots \times V_m = \{(\mathbf{v}_1, \dots, \mathbf{v}_m); \mathbf{v}_1 \in V_1, \dots, \mathbf{v}_m \in V_m\}$$

## Theorem 109 (Products of vector spaces are vector spaces)

Let  $V_1, \dots, V_m$  be vector spaces over  $\mathbb{F}$ . Define

► addition on  $V_1 \times \dots \times V_m$  by

$$(\mathbf{u}_1, \dots, \mathbf{u}_m) + (\mathbf{v}_1, \dots, \mathbf{v}_m) = (\mathbf{u}_1 + \mathbf{v}_1, \dots, \mathbf{u}_m + \mathbf{v}_m)$$

► scalar multiplication on  $V_1 \times \dots \times V_m$  by

$$\lambda(\mathbf{v}_1, \dots, \mathbf{v}_m) = (\lambda\mathbf{v}_1, \dots, \lambda\mathbf{v}_m)$$

With these operations,  $V_1 \times \dots \times V_m$  is a vector space over  $\mathbb{F}$

### Theorem 110 (Dimension of product space)

*Let  $V_1, \dots, V_m$  be finite-dimensional vector spaces. Then*

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m < \infty$$

## Theorem 111 (Product spaces and direct sums)

Let  $U_1, \dots, U_m \subset V$  be subspaces of  $V$ . Let

$$\begin{aligned}\Gamma : U_1 \times \dots \times U_m &\rightarrow U_1 + \dots + U_m \\ (\mathbf{u}_1, \dots, \mathbf{u}_m) &\mapsto \mathbf{u}_1 + \dots + \mathbf{u}_m\end{aligned}$$

Then

$$U_1 + \dots + U_m \text{ direct sum} \Leftrightarrow \Gamma \text{ injective}$$

## Theorem 112 (NSC for direct sum)

Let  $V$  be a finite-dimensional vector space,  $U_1, \dots, U_m$  subspaces of  $V$ . Then

$$U_1 \oplus \dots \oplus U_m \Leftrightarrow \dim(U_1 + \dots + U_m) = \dim U_1 + \dots + \dim U_m$$

### Definition 113 ( $\mathbf{v} + U$ )

Let  $V$  be a vector space,  $U$  a subspace of  $V$  and  $\mathbf{v} \in V$ . Then  $\mathbf{v} + U$  is the subset of  $V$  defined by

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u}; \mathbf{u} \in U\}$$

### Definition 114 (Affine subset/Parallel affine subset)

Let  $V$  be a vector space

- ▶ An **affine subset** of  $V$  is a subset of  $V$  of the form  $\mathbf{v} + U$  for some  $\mathbf{v} \in V$  and some subspace  $U$  of  $V$
- ▶ For  $\mathbf{v} \in V$  and  $U$  subspace of  $V$ , the affine subset  $\mathbf{v} + U$  is **parallel** to  $U$

### Definition 115 (Quotient space)

Let  $V$  be a vector space,  $U$  a subspace of  $V$ . The **quotient space**  $V/U$  is the set of all affine subsets of  $V$  parallel to  $U$ , i.e.,

$$V/U = \{\mathbf{v} + U; \mathbf{v} \in V\}$$

### Theorem 116 (2 affine subsets // to $U$ are equal or disjoint)

Let  $V$  be a vector space,  $U$  subspace of  $V$  and  $v, w \in V$ . TFAE

1.  $\mathbf{v} - \mathbf{w} \in U$
2.  $\mathbf{v} + U = \mathbf{w} + U$
3.  $(\mathbf{v} + U) \cap (\mathbf{w} + U) \neq \emptyset$

### Definition 117 (Addition and scalar multiplication on $V/U$ )

Let  $V$  be a vector space,  $U$  subspace of  $V$ . Then **addition** and **scalar multiplication** on  $V/U$  are defined for  $\mathbf{v}, \mathbf{w} \in V$  and  $\lambda \in \mathbb{F}$  by

$$(\mathbf{v} + U) + (\mathbf{w} + U) = (\mathbf{v} + \mathbf{w}) + U$$

and

$$\lambda(\mathbf{v} + U) = (\lambda\mathbf{v}) + U$$

### Theorem 118 (Quotient space is a vector space)

*Let  $V$  be a vector space and  $U$  subspace of  $V$ . Equipped with addition and scalar multiplication as above,  $V/U$  is a vector space*

### Definition 119 (Quotient map)

Let  $V$  be a vector space,  $U$  subspace of  $V$ . The **quotient map**  $\pi$  is the linear map  $\pi \in \mathcal{L}(V, V/U)$  defined by

$$\pi(\mathbf{v}) = \mathbf{v} + U$$

for  $\mathbf{v} \in V$

### Theorem 120 (Dimension of quotient space)

*Let  $V$  be a finite-dimensional vector space and  $U$  subspace of  $V$ . Then*

$$\dim V/U = \dim V - \dim U$$



### Definition 121 ( $\tilde{T}$ )

Let  $V, W$  be vector spaces,  $T \in \mathcal{L}(V, W)$ . Define  $\tilde{T}$  by

$$\begin{aligned}\tilde{T} : \quad V/(\text{null } T) &\rightarrow W \\ \tilde{T}(\mathbf{v} + \text{null } T) &= T\mathbf{v}\end{aligned}$$

### Theorem 122 (Null space and range of $\tilde{T}$ )

Let  $V, W$  be vector spaces,  $T \in \mathcal{L}(V, W)$ . Then

1.  $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$
2.  $\tilde{T}$  injective
3.  $\text{range } \tilde{T} = \text{range } T$
4.  $V/\text{null } T$  isomorphic to  $\text{range } T$

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### Definition 123 (Linear functional/form)

A **linear functional** (or **linear form**) on a vector space  $V$  is a linear map in  $\mathcal{L}(V, \mathbb{F})$

### Definition 124 (Dual space)

The **dual space**  $V^*$  of  $V$  is the vector space  $V^* = \mathcal{L}(V, \mathbb{F})$  of linear functionals on  $V$

### Theorem 125 ( $\dim V^* = \dim V$ )

*Suppose  $V$  is a finite-dimensional vector space. Then  $\dim V^* = \dim V < \infty$*

### Definition 126 (Dual basis)

If  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a basis of the vector space  $V$ , then the **dual basis** of  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is the list  $\varphi_1, \dots, \varphi_n$  of elements of  $V^*$ , where for  $j = 1, \dots, n$ ,  $\varphi_j$  is the linear functional on  $V$  s.t.

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

### Theorem 127 (Dual basis is a basis of the dual space)

*Suppose  $V$  is a finite-dimensional vector space. Then the dual basis of a basis of  $V$  is a basis of  $V^*$*

### Definition 128 (Dual map)

Let  $V, W$  be vector spaces,  $T \in \mathcal{L}(V, W)$ . The **dual map** of  $T$  is the linear map  $T^* \in \mathcal{L}(W^*, V^*)$  defined by  $T^*(\varphi) = \varphi \circ T$  for  $\varphi \in W^*$

### Property 129 (Algebraic properties of dual maps)

*Let  $U, V, W$  be vector spaces*

- ▶  $(S + T)^* = S^* + T^*$  for all  $S, T \in \mathcal{L}(V, W)$
- ▶  $(\lambda T)^* = \lambda T^*$  for all  $\lambda \in \mathbb{F}$  and all  $T \in \mathcal{L}(V, W)$
- ▶  $(ST)^* = T^*S^*$  for all  $T \in \mathcal{L}(U, V)$  and all  $S \in \mathcal{L}(V, W)$

### Definition 130 (Annihilator)

Let  $V$  be a vector space,  $U \subseteq V$ . The **annihilator**  $U^0$  of  $U$  is defined by

$$U^0 = \{\varphi \in V^* : \forall \mathbf{u} \in U, \quad \varphi(\mathbf{u}) = 0_{\mathbb{F}}\}$$

### Theorem 131 (The annihilator is a subspace)

*Let  $V$  be a vector space and  $U \subseteq V$ . Then the annihilator  $U^0$  is a subspace of  $V^*$*

### Theorem 132 (Dimension of the annihilator)

*Let  $V$  be a finite-dimensional vector space,  $U \subseteq V$  a subspace of  $V$ . Then*

$$\dim U + \dim U^0 = \dim V$$

### Theorem 133 (Null space of $T^*$ )

Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then

1.  $\text{null } T^* = (\text{range } T)^0$
2.  $\dim \text{null } T^* = \dim \text{null } T + \dim W - \dim V$

### Theorem 134 ( $T$ surjective $\Leftrightarrow T^*$ injective)

Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ surjective} \Leftrightarrow T^* \text{ injective}$$

### Theorem 135 (Range of $T^*$ )

Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then

1.  $\dim \operatorname{range} T^* = \dim \operatorname{range} T$
2.  $\operatorname{range} T^* = (\operatorname{null} T)^0$

### Theorem 136 ( $T$ injective $\Leftrightarrow T^*$ surjective)

Let  $V, W$  be finite-dimensional vector spaces,  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ injective} \Leftrightarrow T^* \text{ surjective}$$



**Theorem 137 (Matrix of  $T^*$  is transpose of matrix of  $T$ )**

*Let  $V, W$  be vector spaces,  $T \in \mathcal{L}(V, W)$ . Then  $M(T^*) = M(T)^T$ , where  $T$  denotes the transpose*

# Eigenvalues, eigenvectors and invariant subspaces

Invariant subspaces

Eigenvectors and upper-triangular matrices

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### Definition 138 (Invariant subspace)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ . A subspace  $U$  of  $V$  is **invariant** under  $T$  if

$$\mathbf{u} \in U \Rightarrow T\mathbf{u} \in U$$

In other words,  $U$  invariant under  $T$  if  $T|_U \in \mathcal{L}(U)$  [see Definition 144]

### Definition 139 (Eigenvalue)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ . A scalar  $\lambda \in \mathbb{F}$  is an **eigenvalue** of  $T$  if

$$\exists \mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V, \text{ s.t. } T(\mathbf{v}) = \lambda \mathbf{v}.$$

I use the notation  $T(\mathbf{v})$  instead of  $T\mathbf{v}$  to emphasise that  $T \in \mathcal{L}(V)$ .

## Theorem 140 (Conditions equivalent to being an eigenvalue)

Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . Denote  $I_{\mathcal{L}(V)}$  the identity operator,  $I_{\mathcal{L}(V)} \in \mathcal{L}(V)$  s.t.  $\forall \mathbf{v} \in V, I_{\mathcal{L}(V)}\mathbf{v} = \mathbf{v}$ . TFAE:

1.  $\lambda$  eigenvalue of  $T$
2.  $T - \lambda I_{\mathcal{L}(V)}$  not injective
3.  $T - \lambda I_{\mathcal{L}(V)}$  not surjective
4.  $T - \lambda I_{\mathcal{L}(V)}$  not invertible

## Definition 141 (Eigenvector)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$  be an eigenvalue of  $T$ . A vector  $\mathbf{v} \in V$  is an **eigenvector** of  $T$  corresponding to  $\lambda$  if  $\mathbf{v} \neq 0$  and  $T(\mathbf{v}) = \lambda\mathbf{v}$

### Theorem 142 (Linearly independent eigenvectors)

*Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ . Suppose  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$  with corresponding eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$ . Then  $\mathbf{v}_1, \dots, \mathbf{v}_m$  linearly independent*

### Theorem 143 (Number of eigenvalues)

*Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$ . Then  $T$  has at most  $\dim V$  distinct eigenvalues*

### Definition 144 (Restriction and quotient operators)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$  and  $U$  a subspace of  $V$  invariant under  $T$  (Def. 138)

- ▶ The **restriction operator**  $T|_U \in \mathcal{L}(U)$  is defined by

$$T|_U = T\mathbf{u}, \quad \mathbf{u} \in U$$

- ▶ The **quotient operator**  $T/U \in \mathcal{L}(V/U)$  is defined by

$$(T/U)(\mathbf{v} + U) = T\mathbf{v} + U, \quad \mathbf{v} \in V$$

For the quotient space  $\mathcal{L}(V/U)$ , see Definition 138 and the results that follow

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### Definition 145

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ ,  $m \in \mathbb{N} \setminus \{0\}$

►  $T^m = \underbrace{T \cdots T}_{m \text{ times}}$

►  $T^0 = I$ , the identity operator on  $V$

► If  $T$  invertible with inverse  $T^{-1}$ , then  $T^{-m} = (T^{-1})^m$

### Definition 146

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(\mathbb{F})$  be the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad z \in \mathbb{F}$$

Then  $p(T)$  is the operator on  $\mathcal{L}(V)$  defined by

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

where  $I$  is the identity operator

### Definition 147 (Product of polynomials)

Let  $p, q \in \mathcal{P}(\mathbb{F})$ , then  $pq \in \mathcal{P}(\mathbb{F})$  is the polynomial

$$(pq)(z) = p(z)q(z), \quad z \in \mathbb{F}$$

### Theorem 148 (Multiplicative properties)

Let  $p, q \in \mathcal{P}(\mathbb{F})$ ,  $V$  a vector space and  $T \in \mathcal{L}(V)$ . Then

1.  $(pq)(T) = p(T)q(T)$
2.  $p(T)q(T) = q(T)p(T)$

### Theorem 149 (Operators on complex v.s. have an eigenvalue)

*Let  $V$  be a vector space over  $\mathbb{C}$  with  $\dim V = n < \infty$ . Assume  $T \in \mathcal{L}(V)$ . Then  $V$  has an eigenvalue*

### Definition 150 (Matrix of an operator)

Let  $T \in \mathcal{L}(V)$ , where  $V$  is a finite-dimensional vector space, let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be a basis of  $V$ . The **matrix** of  $T$  with respect to the basis is the  $n \times n$  matrix

$$M(T) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

with entries  $a_{jk}$  defined by

$$T\mathbf{v}_k = a_{1k}\mathbf{v}_1 + \cdots + a_{nk}\mathbf{v}_n$$

If basis is not clear from the context, write  $M(T, (\mathbf{v}_1, \dots, \mathbf{v}_n))$

### Definition 151 (Diagonal of a matrix)

Let  $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$  be a square matrix. The **diagonal** of  $A$  consists of the entries  $a_{ii}$ ,  $i = 1, \dots, n$

### Definition 152 (Upper-triangular matrix)

Let  $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$  be a square matrix. The matrix  $A$  is **upper-triangular** if all entries below the diagonal are 0, i.e.,

$$a_{ij} = 0, \quad \forall i, j \text{ such that } i > j$$

### Theorem 153 (Conditions for an upper-triangular matrix)

Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$  and  $\mathbf{v}_1, \dots, \mathbf{v}_n$  a basis of  $V$ .  
TFAE:

1.  $M(T)$  with respect to  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is upper-triangular
2.  $T\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$ ,  $\forall j = 1, \dots, n$
3.  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$  invariant under  $T$ ,  $\forall j = 1, \dots, n$

### Theorem 154 (Every operator over $\mathbb{C}$ has an UT matrix)

Let  $V$  be a finite-dimensional vector space over  $\mathbb{C}$ ,  $T \in \mathcal{L}(V)$ . Then  $T$  has an upper-triangular matrix with respect to some basis of  $V$

### Theorem 155 (Determination of invertibility from UT matrix)

*Let  $V$  be finite-dimensional vector space. Assume that  $T \in \mathcal{L}(V)$  has an upper-triangular matrix with respect to some basis of  $V$ . Then*

$$T \text{ invertible} \Leftrightarrow \forall i = 1, \dots, n, \quad a_{ii} \neq 0$$

### Theorem 156 (Determination of eigenvalues from UT matrix)

*Let  $V$  be finite-dimensional vector space. Assume that  $T \in \mathcal{L}(V)$  has an upper-triangular matrix with respect to some basis of  $V$ . Then*

$$\lambda \text{ eigenvalue of } T \Leftrightarrow \lambda \in \{a_{ii}, \quad i = 1, \dots, n\}$$

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### Definition 157 (Diagonal matrix)

Let  $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$  be a square matrix.  $A$  is a **diagonal** matrix if all entries of  $A$  are zero except possibly on the diagonal, i.e.,

$$\forall i, j, \ i \neq j, \quad a_{ij} = 0.$$

### Definition 158 (Eigenspace)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ ,  $\lambda \in \mathbb{F}$ . The **eigenspace**  $E(\lambda, T)$  of  $T$  corresponding to  $\lambda$  is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I).$$

Thus  $\lambda$  eigenvalue of  $T \Leftrightarrow E(\lambda, T) \neq \{\mathbf{0}_V\}$ .

### Theorem 159 (Sum of eigenspaces is a direct sum)

*Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$ . Assume  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$ . Then*

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

*is a direct sum and*

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

## Definition 160 (Diagonalisable operator)

Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ .  $T$  is **diagonalisable** if  $T$  has a diagonal matrix with respect to some basis of  $V$ .

## Theorem 161 (Conditions equivalent to diagonalisability)

*Let  $V$  be a finite-dimensional vector space,  $T \in \mathcal{L}(V)$ . Let  $\lambda_1, \dots, \lambda_m$  be distinct eigenvalues of  $T$ . TFAE:*

1.  $T$  diagonalisable
2.  $V$  has a basis consisting of eigenvectors of  $T$
3.  $\exists U_1, \dots, U_n$  1-dimensional subspaces of  $V$  invariant under  $T$  s.t.

$$V = U_1 \oplus \dots \oplus U_n$$

4.  $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$
5.  $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$

### Theorem 162 (Sufficient condition for diagonalisability)

*Let  $V$  be a vector space,  $T \in \mathcal{L}(V)$ . If  $T$  has  $\dim V$  distinct eigenvalues, then  $T$  diagonalisable*

# Inner product spaces

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### Definition 163 (Inner product)

Let  $V$  be a vector space over  $\mathbb{F}$ . An **inner product** on  $V$  is a function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$  having the following properties,  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and  $\forall \lambda \in \mathbb{F}$ ,

- ▶  $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$  [positivity]
- ▶  $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$  [definiteness]
- ▶  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$  [additivity in first slot]
- ▶  $\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$  [homogeneity in first slot]
- ▶  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$  [conjugate symmetry]

### Definition 164 (Inner product space)

An **inner product space** is a vector space  $V$  along with an inner product on  $V$

## Theorem 165 (Basic properties of inner product)

*Let  $V$  be an inner product space over  $\mathbb{F}$ . Then*

1. *For each fixed  $\mathbf{u} \in V$ , the function  $\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{u} \rangle$  is a linear map from  $V$  to  $\mathbb{F}$*
2.  $\forall \mathbf{u} \in V, \langle \mathbf{0}_V, \mathbf{u} \rangle = 0$
3.  $\forall \mathbf{u} \in V, \langle \mathbf{u}, \mathbf{0}_V \rangle = 0$
4.  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
5.  $\forall \mathbf{u}, \mathbf{v} \in V$  and  $\forall \lambda \in \mathbb{F}, \langle \mathbf{u}, \lambda \mathbf{v} \rangle = \bar{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$



### Definition 166 (Norm)

Let  $V$  be an inner product space over  $\mathbb{F}$ . For  $\mathbf{v} \in V$ , the **norm** of  $\mathbf{v}$  is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

### Theorem 167 (Basic properties of the norm)

*Let  $V$  be an inner product space,  $\mathbf{v} \in V$ . Then*

1.  $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = 0$
2.  $\forall \lambda \in \mathbb{F}, \|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|$

### Definition 168 (Orthogonality)

Let  $V$  be an inner product space over  $\mathbb{F}$ . Two vectors  $\mathbf{u}, \mathbf{v} \in V$  are **orthogonal** if  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ . We sometimes denote  $\mathbf{u} \perp \mathbf{v}$

### Theorem 169 ( $\mathbf{0}$ and orthogonality)

*Let  $V$  be an inner product space over  $\mathbb{F}$ . Then*

1.  $\mathbf{0}_V$  is orthogonal to every vector in  $V$
2.  $\mathbf{0}_V$  is the only vector in  $V$  that is orthogonal to itself

### Theorem 170 (Pythagorean theorem)

*Let  $V$  be an inner product space,  $\mathbf{u}, \mathbf{v} \in V$  s.t.  $\mathbf{u} \perp \mathbf{v}$ . Then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

### Theorem 171 (An orthogonal decomposition)

Let  $V$  be an inner product space,  $\mathbf{u}, \mathbf{v} \in V$  with  $\mathbf{v} \neq 0$ . Let

$$c = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \ (\in \mathbb{F}) \text{ and } \mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \ (\in V).$$

Then

$$\langle \mathbf{w}, \mathbf{v} \rangle = 0 \text{ and } \mathbf{u} = c\mathbf{v} + \mathbf{w}.$$

### Theorem 172 (Cauchy-Schwarz inequality)

Let  $V$  be an inner product space,  $\mathbf{u}, \mathbf{v} \in V$ . Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

with  $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$  for some  $0 \neq k \in \mathbb{F}$ .

### Theorem 173 (Triangle inequality)

Let  $V$  be an inner product space,  $\mathbf{u}, \mathbf{v} \in V$ . Then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

with  $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$  for some  $0 \leq k \in \mathbb{R}$ .

### Theorem 174 (Parallelogram equality)

Let  $V$  be an inner product space,  $\mathbf{u}, \mathbf{v} \in V$ . Then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

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### Definition 175 (Orthonormal list)

A list of vectors is **orthonormal** if each vector in the list has norm 1 and is orthogonal to all other vectors in the list, i.e., the list  $\mathbf{e}_1, \dots, \mathbf{e}_m$  of vectors in the inner product space  $V$  is orthonormal if

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

### Theorem 176 (Norm of an orthonormal linear combination)

*Let  $\mathbf{e}_1, \dots, \mathbf{e}_m$  be an orthonormal list of vectors in an inner product space  $V$ . Then*

$$\|a_1\mathbf{e}_1 + \dots + a_m\mathbf{e}_m\|^2 = |a_1|^2 + \dots + |a_m|^2$$

*for all  $a_1, \dots, a_m \in \mathbb{F}$ .*

### Theorem 177 (Orthonormal lists are LI)

*Let  $\mathbf{e}_1, \dots, \mathbf{e}_m$  be an orthonormal list of vectors in an inner product space  $V$ . Then  $\mathbf{e}_1, \dots, \mathbf{e}_m$  is linearly independent*

### Definition 178 (Orthonormal basis)

An **orthonormal basis** of an inner product space  $V$  is an orthonormal list of vectors in  $V$  that is also a basis of  $V$

### Theorem 179 (Orthonormal list & orthonormal basis)

*Let  $\mathbf{e}_1, \dots, \mathbf{e}_m$  be an orthonormal list of vectors in an inner product space  $V$ . If  $\dim V = m$ , then  $\mathbf{e}_1, \dots, \mathbf{e}_m$  orthonormal basis of  $V$ .*

### Theorem 180 (Vector as LC of orthonormal basis)

*Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an orthonormal basis of the inner product space  $V$ ,  $\mathbf{v} \in V$ . Then*

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n$$

*and*

$$\|\mathbf{v}\|^2 = |\langle \mathbf{v}, \mathbf{e}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{e}_n \rangle|^2$$



### Theorem 181 (Gram-Schmidt procedure)

Let  $\mathbf{v}_1, \dots, \mathbf{v}_m$  be a linearly independent list of vectors in an inner product space  $V$ . Let

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

For  $j = 2, \dots, m$ , define  $\mathbf{e}_j$  inductively by

$$\mathbf{e}_j = \frac{\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \dots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}}{\|\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \dots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}\|}$$

Then  $\mathbf{e}_1, \dots, \mathbf{e}_m$  is an orthonormal list of vectors in  $V$  such that

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_j), \quad j = 1, \dots, m$$

### Theorem 182 (Existence of orthonormal basis)

*Let  $V$  be a finite-dimensional inner product space. Then  $V$  has an orthonormal basis*

### Theorem 183 (Extending orthonormal list to basis)

*Let  $V$  be a finite-dimensional inner product space. Then every orthonormal list of vectors in  $V$  can be extended to an orthonormal basis of  $V$*

### Theorem 184 (UT matrix wrt orthonormal basis)

*Let  $V$  be a finite-dimensional inner product space,  $T \in \mathcal{L}(V)$ . If  $T$  has an upper-triangular matrix with respect to some basis of  $V$ , then  $T$  has an upper-triangular matrix with respect to some orthonormal basis of  $V$*

### Theorem 185 (Schur's Theorem)

*Suppose  $V$  is a finite-dimensional complex vector space,  $T \in \mathcal{L}(V)$ . Then  $T$  has an upper-triangular matrix with respect to some orthonormal basis of  $V$*

### Theorem 186 (Riesz representation Theorem)

*Let  $V$  be a finite-dimensional inner product space,  $\varphi \in \mathcal{L}(V, \mathbb{F})$  a linear functional on  $V$ . Then  $\exists \mathbf{u} \in V$  unique s.t.*

$$\forall \mathbf{v} \in V, \quad \varphi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle.$$

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### Definition 187 (Orthogonal complement)

Let  $V$  be an inner product space,  $U \subset V$ . The **orthogonal complement**  $U^\perp$  of  $U$  is the set

$$U^\perp = \{\mathbf{v} \in V : \forall \mathbf{u} \in U, \quad \langle \mathbf{v}, \mathbf{u} \rangle = 0\}$$

### Property 188 (Basic properties of orthogonal complement)

1. If  $U \subset V$ , then  $U^\perp$  subspace of  $V$
2.  $\{\mathbf{0}_V\}^\perp = V$
3.  $V^\perp = \{\mathbf{0}_V\}$
4. If  $U \subset V$ , then  $U \cap U^\perp \subset \{0\}$
5. If  $U \subset W \subset V$ , then  $W^\perp \subset U^\perp$

### Theorem 189 (Direct sum $U$ and $U^\perp$ )

*Let  $U$  be a finite-dimensional subspace of  $V$ , inner product space. Then*

$$V = U \oplus U^\perp$$

### Theorem 190 (Dimension of $U^\perp$ )

*Let  $V$  be a finite-dimensional inner product space,  $U$  subspace of  $V$ . Then*

$$\dim U^\perp = \dim V - \dim U$$

### Theorem 191 (Orth. complement of orth. complement)

*Let  $U$  be a finite-dimensional subspace of the inner product space  $V$ . Then*

$$(U^\perp)^\perp = U$$

### Definition 192 (Orthogonal projection $P_U$ )

Let  $V$  be an inner product space,  $U$  a finite-dimensional subspace of  $V$ . The **orthogonal projection** of  $V$  onto  $U$  is the operator  $P_U \in \mathcal{L}(V)$  defined by

$$P_U \mathbf{v} = \mathbf{u},$$

where  $\mathbf{v} \in V$  is written  $\mathbf{v} = \mathbf{u} + \mathbf{w}$ , with  $\mathbf{u} \in U$  and  $\mathbf{w} \in U^\perp$



## Property 193 (Properties of the orthogonal projection $P_U$ )

Let  $V$  be an inner product space,  $U$  a finite-dimensional subspace of  $V$ ,  $v \in V$ . Then

1.  $P_U \in \mathcal{L}(V)$
2.  $\forall \mathbf{u} \in U, P_U \mathbf{u} = \mathbf{u}$
3.  $\forall \mathbf{w} \in U^\perp, P_U \mathbf{w} = \mathbf{0}_V$
4.  $\text{range } P_U = U$
5.  $\text{null } P_U = U^\perp$
6.  $\mathbf{v} - P_U \mathbf{v} \in U^\perp$
7.  $P_U^2 = P_U$
8.  $\|P_U \mathbf{v}\| \leq \|\mathbf{v}\|$
9. for every orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_m$  of  $U$ ,

$$P_U \mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \dots + \langle \mathbf{v}, \mathbf{e}_m \rangle \mathbf{e}_m.$$

### Theorem 194 (Minimising distance to a subspace)

*Let  $V$  be an inner product space,  $U$  a finite-dimensional subspace of  $V$ ,  $\mathbf{v} \in V$ ,  $\mathbf{u} \in U$ . Then*

$$\|\mathbf{v} - P_U \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{u}\|$$

*with equality if and only if  $\mathbf{u} = P_U \mathbf{v}$*

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### Definition 195 (Adjoint)

Let  $V, W$  be finite-dimensional inner product spaces over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V, W)$ . The **adjoint** of  $T$  is the function  $T^* : W \rightarrow V$  such that

$$\forall \mathbf{v} \in V, \forall \mathbf{w} \in W, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^*\mathbf{w} \rangle$$

### Theorem 196 (Adjoint is a linear map)

*Let  $V, W$  be finite-dimensional inner product spaces over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V, W)$ . Then*

$$T^* \in \mathcal{L}(W, V)$$

## Property 197 (Properties of the adjoint)

*Let  $V, W$  be finite-dimensional inner product spaces over  $\mathbb{F}$ . Then*

1.  $\forall S, T \in \mathcal{L}(V, W), (S + T)^* = S^* + T^*$
2.  $\forall T \in \mathcal{L}(V, W), \forall \lambda \in \mathbb{F}, (\lambda T)^* = \overline{\lambda} T^*$
3.  $\forall T \in \mathcal{L}(V, W), (T^*)^* = T$
4.  $I^* = I$  if  $I$  is the identity operator on  $V$
5. *Let  $U$  be an inner product space over  $\mathbb{F}$ , then  $\forall T \in \mathcal{L}(V, W)$  and  $\forall S \in \mathcal{L}(W, U)$ ,  $(ST)^* = T^* S^*$*

### Theorem 198 (Null space and range of $T^*$ )

Let  $V, W$  be finite-dimensional inner product spaces over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V, W)$ . Then

1.  $\text{null } T^* = (\text{range } T)^\perp$
2.  $\text{range } T^* = (\text{null } T)^\perp$
3.  $\text{null } T = (\text{range } T^*)^\perp$
4.  $\text{range } T = (\text{null } T^*)^\perp$

### Definition 199 (Conjugate transpose)

Let  $M \in \mathcal{M}_{mn}(\mathbb{F})$ ,  $M = [m_{ij}]$ . The **conjugate transpose** of  $M$ , often denoted  $M^*$ , is the matrix

$$M^* = [\overline{m_{ji}}] \in \mathcal{M}_{nm}$$

i.e, the matrix obtained by transposing  $M$  then taking the (complex) conjugate of each entry

### Theorem 200 (Matrix of $T^*$ )

*Let  $V, W$  be finite-dimensional inner product spaces over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V, W)$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{f}_1, \dots, \mathbf{f}_m$  be orthonormal bases of  $V$  and  $W$ , respectively. Then*

$$M(T^*, (\mathbf{f}_1, \dots, \mathbf{f}_m), (\mathbf{e}_1, \dots, \mathbf{e}_n))$$

*is the conjugate transpose of*

$$M(T, (\mathbf{e}_1, \dots, \mathbf{e}_n), (\mathbf{f}_1, \dots, \mathbf{f}_m))$$



### Definition 201 (Self-adjoint operator)

Let  $V$  be an inner product space over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V)$ .  $T$  is **self-adjoint** (or **Hermitian**) if

$$T = T^*$$

In other words,  $T \in \mathcal{L}(V)$  self-adjoint  $\iff$

$$\forall \mathbf{v}, \mathbf{w} \in V, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T\mathbf{w} \rangle$$

### Theorem 202 (Eigenvalues of self-adjoint operators are real)

*Let  $V$  be an inner product space over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V)$ . Then all eigenvalues of  $T$  are real*

### Theorem 203

*Let  $V$  be a complex inner product space,  $T \in \mathcal{L}(V)$ . Then*

$$(\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = \mathbf{0}$$

### Theorem 204

*Let  $V$  be a complex inner product space,  $T \in \mathcal{L}(V)$ . Then*

$$(T \text{ self-adjoint}) \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle \in \mathbb{R})$$

### Theorem 205

*Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$  self-adjoint. Then*

$$(\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = 0$$

### Definition 206 (Normal operator)

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$ .  $T$  is **normal** if

$$TT^* = T^*T$$

In words,  $T$  is normal if it commutes with its adjoint

Theorem 207 ( $T$  normal  $\Leftrightarrow \|T\mathbf{v}\| = \|T^*\mathbf{v}\|$ )

*Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$ . Then*

$$T \text{ normal} \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V, \|T\mathbf{v}\| = \|T^*\mathbf{v}\|)$$

### Theorem 208 ( $T$ normal and $T^*$ have same eigenvectors)

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$  a normal operator. Then

$$(\lambda, \mathbf{v}) \text{ eigenpair of } T \quad \Leftrightarrow \quad (\bar{\lambda}, \mathbf{v}) \text{ eigenpair of } T^*$$

### Theorem 209 (Orthogonal eigenvectors for normal operators)

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$  a normal operator. If  $(\lambda_1, \mathbf{v}_1)$  and  $(\lambda_2, \mathbf{v}_2)$  eigenpairs of  $T$  with  $\lambda_1 \neq \lambda_2$ , then  $\mathbf{v}_1 \perp \mathbf{v}_2$ .

### Theorem 210

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$  self-adjoint and  $b, c \in \mathbb{R}$  s.t.  $b^2 < 4c$ .  
Then  $T^2 + bT + cI$  invertible

### Theorem 211 (Self-adjoint operators have eigenvalues)

Let  $V \neq \{0\}$  and  $T \in \mathcal{L}(V)$  be self-adjoint. Then  $T$  has an eigenvalue

### Theorem 212 (Self-adjoint operators & invariant subspaces)

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$  be self-adjoint and  $U$  be a subspace of  $V$  invariant under  $T$ . Then

1.  $U^\perp$  invariant under  $T$
2.  $T|_U \in \mathcal{L}(U)$  self-adjoint
3.  $T|_{U^\perp} \in \mathcal{L}(U^\perp)$  self-adjoint

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### Theorem 213 (Complex spectral theorem)

Let  $V$  be an inner product space over  $\mathbb{F} = \mathbb{C}$ ,  $T \in \mathcal{L}(V)$ . TFAE:

1.  $T$  normal
2.  $V$  has an orthonormal basis consisting of eigenvectors of  $T$
3.  $T$  has a diagonal matrix with respect to some orthonormal basis of  $V$

### Theorem 214 (Real spectral theorem)

Let  $V$  be an inner product space over  $\mathbb{F} = \mathbb{R}$ ,  $T \in \mathcal{L}(V)$ . TFAE:

1.  $T$  self-adjoint
2.  $V$  has an orthonormal basis consisting of eigenvectors of  $T$
3.  $T$  has a diagonal matrix with respect to some orthonormal basis of  $V$



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### Definition 215 (Positive (semidefinite) operator)

Let  $V$  be an inner product space. An operator  $T \in \mathcal{L}(V)$  is **positive** (or **positive semidefinite**) if  $T$  is self-adjoint and

$$\forall \mathbf{v} \in V, \quad \langle T\mathbf{v}, \mathbf{v} \rangle \geq 0$$

### Definition 216 (Square root operator)

Let  $V$  be an inner product space. An operator  $R \in \mathcal{L}(V)$  is a **square root** of an operator  $T \in \mathcal{L}(V)$  if

$$R^2 = T$$

### Theorem 217 (Characterisation of positive operators)

Let  $T \in \mathcal{L}(V)$ , where  $V$  is an inner product space. TFAE:

1.  $T$  positive semidefinite
2.  $T$  self-adjoint and all eigenvalues of  $T$  are nonnegative
3.  $T$  has a positive semidefinite square root
4.  $T$  has a self-adjoint square root
5.  $\exists R \in \mathcal{L}(V)$  s.t.  $T = R^*R$

### Theorem 218 (Uniqueness of positive semidefinite square root)

Let  $T \in \mathcal{L}(V)$  be a positive semidefinite operator on an inner product space  $V$ . Then  $T$  has a unique positive semidefinite square root

### Definition 219 (Isometry)

Let  $V$  be an inner product space.  $S \in \mathcal{L}(V)$  is an **isometry** if

$$\forall \mathbf{v} \in V, \quad \|S\mathbf{v}\| = \|\mathbf{v}\|$$

## Theorem 220 (Characterisation of isometries)

Let  $V$  be an inner product space,  $S \in \mathcal{L}(V)$ . TFAE:

1.  $S$  isometry
2.  $\forall \mathbf{u}, \mathbf{v} \in V, \langle S\mathbf{u}, S\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$
3.  $\forall \mathbf{e}_1, \dots, \mathbf{e}_n \in V$  orthonormal list,  $S\mathbf{e}_1, \dots, S\mathbf{e}_n$  orthonormal
4.  $\exists \mathbf{e}_1, \dots, \mathbf{e}_n$  orthonormal basis of  $V$  s.t.  $S\mathbf{e}_1, \dots, S\mathbf{e}_n$  orthonormal
5.  $S^*S = I$
6.  $SS^* = I$
7.  $S^*$  isometry
8.  $S$  invertible and  $S^{-1} = S^*$

## Theorem 221 (Isometries when $\mathbb{F} = \mathbb{C}$ )

*Let  $V$  be a complex inner product space,  $S \in \mathcal{L}(V)$ . TFAE:*

- 1.  $S$  isometry*
- 2.  $\exists$  orthonormal basis of  $V$  consisting of eigenvectors of  $S$  with corresponding eigenvalues all having modulus 1*

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Let  $T$  be a positive semidefinite operator, then denote  $\sqrt{T}$  the unique positive semidefinite square root of  $T$

### Theorem 222 (Polar decomposition)

*Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$ . Then there exists an isometry  $S \in \mathcal{L}(V)$  s.t.*

$$T = S\sqrt{T^*T}$$



## Definition 223 (Singular values)

Let  $V$  be an inner product space,  $T \in \mathcal{L}(V)$ . The **singular values** of  $T$  are the eigenvalues of  $\sqrt{T^*T}$ , with each eigenvalue  $\lambda$  repeated  $\dim E(\lambda, \sqrt{T^*T})$  times. All are nonnegative

## Theorem 224 (Singular value decomposition – SVD)

*Let  $V$  be an inner product space. Assume  $T \in \mathcal{L}(V)$  has singular values  $s_1, \dots, s_n$ . Then  $\exists \mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{f}_1, \dots, \mathbf{f}_n$  orthonormal bases of  $V$  s.t.*

$$\forall \mathbf{v} \in V, \quad T\mathbf{v} = s_1 \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{f}_1 + \dots + s_n \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{f}_n$$

## Theorem 225 (SV without square root)

*Let  $V$  be an inner product space. The singular values of  $T$  are the nonnegative square roots of the eigenvalues of  $T^*T$ , with each eigenvalue  $\lambda$  repeated  $\dim E(\lambda, T^*T)$  times*

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### Theorem 226 (Sequence of increasing null spaces)

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V)$ . Then

$$\{0\} = \text{null } T^0 \subset \text{null } T^1 \subset \cdots \subset \text{null } T^k \subset \text{null } T^{k+1} \subset \cdots$$

### Theorem 227 (Equality in sequence of null spaces)

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V)$ . Assume  $m \in \mathbb{N} \setminus \{0\}$  is s.t.

$$\text{null } T^m = \text{null } T^{m+1}$$

Then

$$\forall k \in \mathbb{N}, \quad \text{null } T^{m+k} = \text{null } T^m$$

### Theorem 228 (Null spaces stop growing)

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$  with  $\dim V = n$ ,  $T \in \mathcal{L}(V)$ . Then

$$\forall k \in \mathbb{N}, \quad \text{null } T^{n+k} = \text{null } T^n$$

### Theorem 229 ( $V = \text{null } T^{\dim V} \oplus \text{range } T^{\dim V}$ )

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$  with  $\dim V = n$ ,  $T \in \mathcal{L}(V)$ . Then

$$V = \text{null } T^n \oplus \text{range } T^n$$

### Definition 230 (Generalised eigenvector)

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ ,  $T \in \mathcal{L}(V)$ ,  $\lambda \in \mathbb{F}$  an eigenvalue of  $T$ .  $\mathbf{v} \in V$  is a **generalised eigenvector** of  $T$  corresponding to  $\lambda$  if  $\mathbf{v} \neq \mathbf{0}$  and

$$\exists j \in \mathbb{N} \setminus \{0\}, \quad (T - \lambda I)^j \mathbf{v} = \mathbf{0}_V$$

### Definition 231 (Generalised eigenspace)

Let  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . The **generalised eigenspace**  $G(\lambda, T)$  of  $T$  corresponding to  $\lambda$  is the set of all generalised eigenvectors of  $T$  corresponding to  $\lambda$  together with the  $\mathbf{0}_V$  vector

### Theorem 232 (Description of generalised eigenspaces)

*Let  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . Then*

$$G(\lambda, T) = \text{null}(T - \lambda I)^{\dim V}$$

### Theorem 233 (LI generalised eigenvectors)

*Let  $T \in \mathcal{L}(V)$ . Assume  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$ ,  $\mathbf{v}_1, \dots, \mathbf{v}_m$  corresponding generalised eigenvectors. Then  $\mathbf{v}_1, \dots, \mathbf{v}_m$  linearly independent.*

### Definition 234 (Nilpotent operator)

An operator is **nilpotent** if  $\exists k \in \mathbb{N}$  s.t.  $T^k = 0$

### Theorem 235 (A loose upper bound on power required)

*Let  $N \in \mathcal{L}(V)$  be nilpotent. Then*

$$N^{\dim V} = 0$$

### Theorem 236 (Matrix of a nilpotent operator)

*Let  $N \in \mathcal{L}(V)$  be nilpotent. Then there exists a basis of  $V$  with respect to which  $M(N)$  is **strictly upper triangular**, i.e.,*

$$M(N) = [m_{ij}] \text{ is s.t. } m_{ij} = 0 \text{ if } i \geq j$$



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### Theorem 237 ( & range of $p(T)$ invariant under $T$ )

*Let  $T \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(\mathbb{F})$ . Then  $p(T)$  and  $\text{range } p(T)$  invariant under  $T$*

### Theorem 238 (Description of operators when $\mathbb{F} = \mathbb{C}$ )

*Suppose  $V$  complex vector space,  $T \in \mathcal{L}(V)$ . Assume  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $T$ . Then*

1.  $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$
2. each  $G(\lambda_j, T)$  invariant under  $T$
3.  $\forall j = 1, \dots, m, (T - \lambda_j I)|_{G(\lambda_j, T)}$  nilpotent

### Theorem 239 (Basis of generalised eigenvectors)

*Let  $V$  be a complex vector space and  $T \in \mathcal{L}(V)$ . Then there exists a basis of  $V$  consisting of generalised eigenvectors of  $T$*

### Definition 240 (Multiplicity of an eigenvalue)

Let  $T \in \mathcal{L}(V)$ . The (**algebraic**) **multiplicity** of an eigenvalue  $\lambda$  of  $T$  is

- ▶  $\dim G(\lambda, T)$
- ▶  $\dim (T - \lambda I)^{\dim V}$

### Theorem 241 ( $\sum$ multiplicities = $\dim V$ )

*Let  $V$  be a complex vector space,  $T \in \mathcal{L}(V)$ . Let  $\lambda_1, \dots, \lambda_n$  be distinct eigenvalues of  $T$  with multiplicities  $d_1, \dots, d_n$ . Then*

$$\sum_{k=1}^n d_k = \dim V$$

## Definition 242 (Block diagonal matrix)

Let  $A_1, \dots, A_m$  be square matrices (not necessarily of the same size). A **block matrix** is a matrix of the form

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

We also write

$$A = \text{diag}(A_1, \dots, A_m)$$

You will also see (not in this book)

$$A = A_1 \oplus \dots \oplus A_m$$

### Theorem 243 (Block diagonal matrix with UT blocks)

*Let  $V$  be a complex vector space,  $T \in \mathcal{L}(V)$ . Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $T$  with multiplicities  $d_1, \dots, d_m$ . Then there exists a basis of  $V$  s.t.  $T$  has a block diagonal matrix*

$$\text{diag}(A_1, \dots, A_m)$$

*with each  $A_j$  a  $d_j \times d_j$  upper-triangular matrix of the form*

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

### Theorem 244 (Identity plus nilpotent has square root)

*Let  $N \in \mathcal{L}(V)$  be nilpotent. Then  $I + N$  has a square root*

### Theorem 245 ( $T$ invertible has square root when $\mathbb{F} = \mathbb{C}$ )

*Let  $V$  be a complex vector space,  $T \in \mathcal{L}(V)$  invertible. Then  $T$  has a square root*

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### Definition 246 (Characteristic polynomial)

Let  $V$  be a complex vector space,  $T \in \mathcal{L}(V)$ ,  $\lambda_1, \dots, \lambda_m$  the distinct eigenvalues of  $T$  with multiplicities  $d_1, \dots, d_m$ . The **characteristic polynomial** of  $T$  is

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

### Theorem 247 (Degree and zeros of char. polyn.)

*$V$  a complex vector space,  $T \in \mathcal{L}(V)$ . Then*

- 1. the characteristic polynomial of  $T$  has degree  $\dim V$*
- 2. zeros of the characteristic polynomial of  $T$  are the eigenvalues of  $T$*

### Theorem 248 (Cayley-Hamilton)

*Let  $V$  be a complex vector space,  $T \in \mathcal{L}(V)$ . Let  $q$  be the characteristic polynomial of  $T$ . Then  $q(T) = 0$*



### Definition 249 (Monic polynomial)

A **monic polynomial** is a polynomial with highest degree coefficient equal to 1

### Theorem 250 (Minimal polynomial)

*Let  $T \in \mathcal{L}(V)$ . Then there exists a unique monic polynomial  $p$  of smallest degree s.t.  $p(T) = 0$*

### Definition 251 (Minimal polynomial)

Let  $T \in \mathcal{L}(V)$ . The **minimal polynomial** of  $T$  is the unique monic polynomial  $p$  of smallest degree s.t.  $p(T) = 0$

### Theorem 252

*Let  $T \in \mathcal{L}(V)$  and  $q \in \mathcal{P}(\mathbb{F})$ . Then  $q(T) = 0 \Leftrightarrow q$  polynomial multiple of the minimal polynomial of  $T$*

### Theorem 253 (Char. polyn. is multiple of min. polyn.)

*Assume  $V$  vector space over  $\mathbb{F} = \mathbb{C}$ ,  $T \in \mathcal{L}(V)$ . Then the characteristic polynomial of  $T$  is a polynomial multiple of the minimal polynomial of  $T$*

### Theorem 254 (Eigenvalues are zeros of min. polyn.)

*Let  $T \in \mathcal{L}(V)$ . Then the zeros of the minimal polynomial of  $T$  are precisely the eigenvalues of  $T$*

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### Theorem 255 (Basis corresponding to nilpotent operator)

Let  $N \in \mathcal{L}(V)$  be nilpotent. Then  $\exists \mathbf{v}_1, \dots, \mathbf{v}_n \in V$  and  $m_1, \dots, m_n \in \mathbb{N}$  s.t.

1.  $N^{m_1}\mathbf{v}_1, \dots, N\mathbf{v}_1, \mathbf{v}_1, N^{m_n}\mathbf{v}_n, \dots, N\mathbf{v}_n, \mathbf{v}_n$  is a basis of  $V$
2.  $N^{m_1+1}\mathbf{v}_1 = \dots = N^{m_n+1}\mathbf{v}_n = 0$

### Definition 256 (Jordan basis)

Let  $T \in \mathcal{L}(V)$ . A **Jordan basis** for  $T$  is a basis of  $V$  s.t. with respect to this basis,  $T$  has a block diagonal matrix

$$\text{diag}(A_1, \dots, A_p)$$

where each  $A_j$  is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

### Theorem 257 (Jordan form)

*Let  $V$  be a complex vector space. If  $T \in \mathcal{L}(V)$ , then  $\exists$  a Jordan basis for  $T$*

## An algorithm for finding the Jordan form

An algorithm to compute the Jordan canonical form of an  $n \times n$  matrix  $A$  [MM82].

1. Compute the eigenvalues of  $A$ . Let  $\lambda_1, \dots, \lambda_m$  be the distinct eigenvalues of  $A$  with multiplicities  $n_1, \dots, n_m$ , respectively.
2. Compute  $n_1$  linearly independent generalized eigenvectors of  $A$  associated with  $\lambda_1$  as follows. Compute



$$(A - \lambda_1 E_n)^i$$

for  $i = 1, 2, \dots$  until the rank of  $(A - \lambda_1 E_n)^k$  is equal to the rank of  $(A - \lambda_1 E_n)^{k+1}$ . Find a generalized eigenvector of rank  $k$ , say  $u$ . Define  $u_i = (A - \lambda_1 E_n)^{k-1}u$ , for  $i = 1, \dots, k$ . If  $k = n_1$ , proceed to step 3. If  $k < n_1$ , find another linearly independent generalized eigenvector with rank  $k$ . If this is not possible, try  $k - 1$ , and so forth, until  $n_1$  linearly independent generalized eigenvectors are determined. Note that if  $\rho(A - \lambda_1 E_n) = r$ , then there are totally  $(n - r)$  chains of generalized eigenvectors associated with  $\lambda_1$ .

3. Repeat step 2 for  $\lambda_2, \dots, \lambda_m$ .

1. Let  $u_1, \dots, u_k, \dots$  be the new basis. Observe that Thus in the new basis,  $A$  has the desired representation
2. The similarity transformation which yields  $J = Q^{-1}AQ$  is given by  $Q = [u_1, \dots, u_k, \dots]$ .

# References I

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-  R.K Miller and A.N. Michel, *Ordinary differential equations*, Academic Press, 1982.