

# MATH 4370/7370 – Linear Algebra and Matrix Analysis

## Norms and Matrix Norms

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# Outline

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## Definition 1 (Norm)

Let  $V$  be a vector space over a field  $\mathbb{F}$ . A function  $\|\cdot\| : V \rightarrow \mathbb{R}_+$  is a **norm** if for all  $\mathbf{x}, \mathbf{y} \in V$  and for all  $c \in \mathbb{F}$

1.  $\|\mathbf{x}\| \geq 0$  [Nonnegativity]
2.  $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$  [Positivity]
3.  $\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$  [Homogeneity]
4.  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$  [Triangle Inequality]

## Remark 2

*If we have 1, 3, and 4 but not 2, then we have a **seminorm***

### Definition 3 (Inner product)

Let  $V$  be a vector space over  $\mathbb{F}$ . A function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$  is an **inner product** if for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and all  $c \in \mathbb{F}$

1.  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
2.  $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
3.  $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
4.  $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$
5.  $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

### Theorem 4 (Cauchy-Schwartz)

Let  $\langle \cdot, \cdot \rangle$  be an inner product on a vector space  $V$  over  $\mathbb{F}$ , then

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$$

### Corollary 5

*If  $\langle \cdot, \cdot \rangle$  is an inner product on a real or complex vector space  $V$ , then  $\| \cdot \| : V \rightarrow \mathbb{R}_+$  defined by  $\| \mathbf{x} \| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$  is a norm on  $V$*

### Remark 6

*If  $\langle \cdot, \cdot \rangle$  is a semi-inner product, then the resulting  $\| \mathbf{x} \| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$  is a seminorm*

### Theorem 7

*Consider the norm  $\| \cdot \|$ . Then  $\| \cdot \|$  is derived from an inner product if and only if it satisfies the parallelogram identity*

$$\frac{1}{2} (\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

### Theorem 8

*If  $\| \cdot \|$  is a norm on  $\mathbb{C}^n$  and a matrix  $T \in \mathcal{M}_n$  which is non-singular. Then*

$$\|\mathbf{x}\|_T = \|T\mathbf{x}\|$$

*is also a norm on  $\mathbb{C}^n$*

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## Definition 9

Let  $V$  be a vector space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . Take a norm  $\|\cdot\|$  on  $V$ . The sequence  $\{\mathbf{x}^{(k)}\}$  of vectors in  $V$  converges to  $\mathbf{x} \in V$  with respect to the norm  $\|\cdot\|$  if and only if  $\|\mathbf{x}^{(k)} - \mathbf{x}\| \rightarrow 0$  as  $k \rightarrow \infty$

We write  $\lim_{k \rightarrow \infty} \mathbf{x}^{(k)} = \mathbf{x}$  with respect to  $\|\cdot\|$  or

$$\mathbf{x}^{(k)} \xrightarrow{\|\cdot\|} \mathbf{x}$$

## Theorem 10

*Every (vector) norm in  $\mathbb{C}^n$  is uniformly continuous*

## Corollary 11

*Let  $\|\cdot\|_\alpha$  and  $\|\cdot\|_\beta$  be any two norms on a finite-dimensional vector space  $V$ . Then there exist  $C_m, C_r > 0$  such that*

$$C_m \|\mathbf{x}\|_\alpha \leq \|\mathbf{x}\|_\beta \leq C_r \|\mathbf{x}\|_\alpha, \forall \mathbf{x} \in V$$

## Corollary 12

*Let  $\|\cdot\|_\alpha$  and  $\|\cdot\|_\beta$  norms on a finite-dimensional vector space  $V$  over  $\mathbb{R}$  or  $\mathbb{C}$ ,  $\{\mathbf{x}^{(k)}\}$  a given sequence in  $V$ , then*

$$\mathbf{x}^{(k)} \xrightarrow{\|\cdot\|_\alpha} \mathbf{x} \iff \mathbf{x}^{(k)} \xrightarrow{\|\cdot\|_\beta} \mathbf{x}$$

### Definition 13 (Equivalent norms)

Two norms are **equivalent** if whenever a sequence  $\{\mathbf{x}^{(k)}\}$  converges to  $\mathbf{x}$  with respect to one of the norm, it converges to  $\mathbf{x}$  in the other norm

### Theorem 14

*In finite-dimensional vector spaces, all norm are equivalent*

### Definition 15 (Dual norm)

Let  $f$  be a pre-norm on  $V = \mathbb{R}^n$  or  $\mathbb{C}^n$ . The function

$$f_d = (\mathbf{y}) \max_{f(\mathbf{x})=1} \operatorname{Re} \mathbf{y}^* \mathbf{x}$$

is the **dual norm** of  $f$

### Remark 16

*The dual norm is well defined.  $\operatorname{Re} \mathbf{y}^* \mathbf{x}$  is a continuous function for all  $\mathbf{y} \in V$  fixed. The set  $\{f(\mathbf{x}) = 1\}$  is compact*

Equivalent definition for dual norm:  $f^D(\mathbf{y}) = \max_{f(\mathbf{x})=1} |\mathbf{y}^* \mathbf{x}|$

### Lemma 17 (Extension of Cauchy-Schwartz)

Let  $f$  be a prenorm on  $V = \mathbb{R}^n$  or  $\mathbb{C}^n$  for all  $\mathbf{x}, \mathbf{y} \in V$ . Then

$$|\mathbf{y}^* \mathbf{x}| \leq f(\mathbf{x}) f^D(\mathbf{y})$$

$$|\mathbf{y}^* \mathbf{x}| \leq f^D(\mathbf{x}) f(\mathbf{x})$$

### Remark 18

- ▶ The dual norm of a pre-norm is a norm
- ▶ The only norm that equals its dual norm is the Euclidean norm

## Theorem 19

Let  $\|\cdot\|$  be a norm on  $\mathbb{C}^n$  or  $\mathbb{R}^n$ , and  $\|\cdot\|^D$  its dual,  $c > 0$  given. Then for all  $\mathbf{x} \in V$ ,  $\|\mathbf{x}\| = c\|\mathbf{x}\|^d \iff \|\cdot\| = \sqrt{c}\|\cdot\|^d$ . In particular,  $\|\cdot\| = \|\cdot\|^2 \iff \|\cdot\| = \|\cdot\|_2$

## Definition 20

Let  $x \in \mathbb{F}^n$ . Denote  $|x| = [|x_i|]$  ( $|\cdot|$  entry-wise), and write that  $|x| \leq |y|$  if  $|x_i| \leq |y_i|$  for all  $i = 1, \dots, n$ . Assume  $\|\cdot\|$  is

1. monotone if  $|x| \leq |y| \implies \|\mathbf{x}\| \leq \|\mathbf{y}\|$  for all  $\mathbf{x}, \mathbf{y}$
2. absolute if  $\|\mathbf{x}\| = \|\mathbf{x}\|$  for all  $\mathbf{x} \in V$

## Theorem 21

Let  $\|\cdot\|$  be a norm on  $\mathbb{F}^n$ . Then

1. If  $\|\cdot\|$  is absolute, then

$$\|\mathbf{y}\|^D = \max_{\mathbf{x} \neq 0} \frac{|\mathbf{y}|^T \mathbf{x}|}{\|\mathbf{x}\|}$$

for all  $\mathbf{y} \in V$

2. If  $\|\cdot\|$  is absolute, then  $\|\cdot\|^D$  is absolute and monotone
3.  $\|\cdot\|$  is absolute if and only if  $\|\cdot\|$



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## Definition 22 (Matrix norm)

Let  $\|\cdot\|$  be a function from  $\mathcal{M}_n \rightarrow \mathbb{R}$ .  $\|\cdot\|$  is a **matrix norm** if for all  $A, B \in \mathcal{M}_n$  and  $c \in \mathbb{C}$ , it satisfies the following

1.  $\|A\| \geq 0$  [nonnegativity]
2.  $\|A\| = 0 \iff A = 0$  [positivity]
3.  $\|cA\| = |c| \|A\|$  [homogeneity]
4.  $\|A + B\| \leq \|A\| + \|B\|$  [triangle inequality]
5.  $\|AB\| \leq \|A\| \|B\|$  [submultiplicativity]

## Remark 23

As with vector norms, if property 2 does not hold,  $\|\cdot\|$  is a **matrix semi-norm**

## Remark 24

$\|A^2\| = \|AA\| \leq \|A\|^2$  [for any matrix norm].

If  $A^2 = A$ , then

$$\|A^2\| = \|A\| \leq \|A\|^2 \implies \|A\| \geq 1.$$

In particular,  $\|I\| \geq 1$  for any matrix norm.

Assume that  $A$  is invertible, then  $AA^{-1} = I$ , thus

$$\|I\| = \|AA^{-1}\| \leq \|A\|\|A^{-1}\| \tag{1}$$

$$\|A^{-1}\| \geq \frac{\|I\|}{\|A\|} \tag{2}$$

## Definition 25 (Induced matrix norm)

Let  $\|\cdot\|$  be a norm on  $\mathbb{C}^n$ . Define  $\|\cdot\|$  on  $\mathcal{M}_n(\mathbb{C})$  by

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Then  $\|\cdot\|$  is the **matrix norm induced** by  $\|\cdot\|$

## Theorem 26

*The function  $\|\cdot\|$  defined in Definition 25 has the following properties*

1.  $\|\mathbb{I}\| = 1$
2.  $\|Ay\| \leq \|A\| \|y\|$  for all  $A \in \mathcal{M}_n(\mathbb{C})$  and all  $y \in \mathbb{C}^n$
3.  $\|\cdot\|$  is a matrix norm on  $\mathcal{M}_n(\mathbb{C})$ .
4.  $\|A\| = \max_{\|x\|=\|y\|^D} |y^* Ax|$

### Definition 27 (Induced norm/Operator norm)

$\|\cdot\|$  defined from  $\|\cdot\|$  by any of the previous methods is the matrix norm induced by  $\|\cdot\|$ . It is also called the **operator norm**

### Definition 28 (Unital norm)

A norm such that  $\|\mathbb{I}\| = 1$  is **unital**

### Remark 29

*Every induced matrix norm is unital. Every induced norm is a matrix norm*

### Proposition 30

*For all  $U, V$  unitary matrices, we have  $\|UAV\|_2 = \|A\|_2$*

### Theorem 31

*Let  $\|\cdot\|$  be a matrix norm in  $\mathcal{M}_n$  and let  $S \in \mathcal{M}_n$  be nonsingular. Then for all  $A \in \mathcal{M}_n$ ,  $\|A\|_S = \|SAS^{-1}\|$  is a matrix norm. Furthermore, if  $\|\cdot\|$  on  $\mathbb{C}^n$ , then  $\|\mathbf{x}\|_S = \|S\mathbf{x}\|$  induces  $\|\cdot\|_S$  on  $\mathcal{M}_n$*

### Theorem 32

*Let  $\|\cdot\|$  be a matrix norm on  $\mathcal{M}_n$ ,  $A \in \mathcal{M}_n$  and  $\lambda \in \sigma(A)$ . Then*

1.  $|\lambda| \leq \rho(A) \leq \|A\|$
2. *If  $A$  is nonsingular, then*

$$\rho(A) \geq |\lambda| \geq \frac{1}{\|A^{-1}\|}$$

### Lemma 33

Let  $A \in \mathcal{M}_n$ . If there exists a norm  $\|\cdot\|$  on  $\mathcal{M}_n$  such that  $\|A\| < 1$ , then  $\lim_{k \rightarrow \infty} A^k = 0$  entry-wise

### Remark 34

When  $\|A\| < 1$  for some norm, we say that  $A$  is **convergent**

### Theorem 35

Let  $A \in \mathcal{M}_n$ , then

$$\lim_{k \rightarrow \infty} A^k = 0 \iff \rho(A) < 1$$

### Theorem 36 (Gelfand Formula)

Let  $\|\cdot\|$  be a matrix norm on  $\mathcal{M}_n$ , let  $A \in \mathcal{M}_n$ . Then

$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$$

### Theorem 37

Let  $R$  be the radius of convergence of the (scalar) power series  $\sum_{k=0}^{\infty} a_k z^k$  and  $A \in \mathcal{M}_n$ .

Then the matrix power series  $\sum_{k=1}^{\infty} a_k A^k$  converges if  $\rho(A) < R$



### Remark 38

*The convergence condition for the matrix power series is “there exists a matrix norm  $\|\cdot\|$  such that  $\|A\| < R$ ”*

### Corollary 39

*Let  $A \in \mathcal{M}_n$  be nonsingular, if there  $\|\cdot\|$  matrix norm such that  $\|\mathbb{I} - A\| \leq 1$*

### Corollary 40

*Let  $A \in \mathcal{M}_n$  is such that  $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$  for all  $i = 1, \dots, n$ . Then  $A$  is invertible*

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Let  $V = \mathcal{M}_{mn}(\mathbb{C})$  with Frobenius inner product

$$\langle A, B \rangle_F = \text{tr}(B^* A)$$

The norm derived from the Frobenius inner product is

$$\|A\|_2 = (\text{tr}(A^* A))^{1/2}$$

is the  $\ell$ -2 norm (or Frobenius norm)

The spectral norm  $\|\cdot\|_2$  defined on  $\mathcal{M}_n$  by

$$\|A\|_2 = \sigma_1(A),$$

where  $\sigma_1(A)$  is the largest singular value of  $A$  is induced by the  $\ell_2$  norm on  $\mathbb{C}^n$ .  
Indeed, from the singular value decomposition theorem, let

$$A = V\Sigma W^*$$

be a singular value decomposition of  $A$ , where  $V, W$  unitary,  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$  and  $\sigma_1 \geq \dots \geq \sigma_n \geq 0$  are the non-increasingly ordered singular values of  $A$

From unitary invariance and monotonicity of the Euclidean norm, we say that

$$\begin{aligned}\max_{\|x\|_1} \|Ax\|_1 &= \max_{\|x\|_1} \|V\Sigma W^*\|_2 \\ &= \max_{\|x\|_2} \|\Sigma W^*x\|_2 \\ &= \max_{\|Wy\|_2=1} \|\Sigma y\|_2 \\ &= \max_{\|y\|_2} \|\Sigma y\|_2 \\ &\leq \max_{\|y\|_2} \|\sigma_1 y\|_2 \\ &= \sigma_1 \max_{\|y\|_2} \|y\|_2 \\ &= \sigma_1\end{aligned}$$

Since  $\|\Sigma y\|_2 = \sigma_1$  for  $y = e_1$ ,

$$\max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$$

We could have used

$$\begin{aligned}\max_{\|x\|_2=1} \|Ax\|_2^2 &= \max_{\|x\|_2=1} x^* A^* A x \\ &= \lambda_{\max}(A^* A) \\ &= \sigma_1(A)\end{aligned}$$

#### Remark 41

*For all  $U, V$  unitary  $\mathcal{M}_n$  matrices, for all  $A \in \mathcal{M}_n$ ,  $\|UAV\|_2 = \|A\|_2$*

# References I