

# MATH 4370/7370 – Linear Algebra and Matrix Analysis

Factorisations, canonical forms and decompositions

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# Outline

Unitary matrices and QR factorisation

Schur's Form

Consequences of Schur's triangularisation theorem

Normal Matrices

Jordan Canonical Form

## Unitary matrices and QR factorisation

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## Definition 4.1

Let  $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{C}^n$ . We say that  $\mathbf{x}_1, \dots, \mathbf{x}_k$  is an **orthogonal list** if  $\mathbf{x}_i^* \mathbf{x}_j = 0$  for all  $i \neq j$ . If, in addition, we have that  $\mathbf{x}_i^* \mathbf{x}_i = 1$ , then we say that the list is **orthonormal**

## Theorem 4.2

*Every orthonormal list of vectors in  $\mathbb{C}^n$  is linearly independent*

## Remark 4.3

*In Theorem 4.2, if we have “only” orthogonal vectors, we need to replace “list of vectors” by “list of non-zero vectors” in the statement*

## Definition 4.4

Let  $U \in \mathcal{M}_n$ , we say that  $U$  is an **unitary matrix** if  $U^*U = \mathbb{I}$ . Furthermore, we say that  $U \in \mathcal{M}_n(\mathbb{R})$  is a **(real) orthogonal matrix** if  $U^T U = \mathbb{I}$

## Theorem 4.5

Let  $U \in \mathcal{M}_n$ . TFAE:

1.  $U$  is unitary
2.  $U$  is non-singular and  $U^* = U^{-1}$
3.  $UU^* = \mathbb{I}$
4.  $U^*$  is unitary
5. the columns of  $U$  are orthonormal
6. the rows of  $U$  are orthonormal
7. for all  $\mathbf{x} \in \mathbb{C}^n$  we have  $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$

## Definition 4.6

A **linear transformation**  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a **Euclidean isometry** if  $\|\mathbf{x}\|_2 = \|T\mathbf{x}\|_2$  for all  $\mathbf{x} \in \mathbb{C}^n$

## Corollary 4.7

Let  $U \in \mathcal{M}_n$ .  $U$  is a Euclidean isometry if and only if  $U$  is unitary

### Remark 4.8

Let  $U, V \in \mathcal{M}_n$  are unitary matrices (respectively real orthogonal), then  $UV$  is unitary (respectively real orthogonal).

Indeed,  $U, V$  unitary  $\Leftrightarrow U^{-1}, V^{-1}$  exist and  $U^{-1} = U^*, V^{-1} = V^*$ . Then

$$\begin{aligned} UV \text{ unitary} &\Leftrightarrow (UV)^* UV = \mathbb{I} \\ &\Leftrightarrow V^* U^* UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I} \end{aligned}$$

**Notation:**  $\mathrm{GL}(n, \mathbb{F})$  is the general linear group, where the elements are non-singular matrices in  $\mathcal{M}_n(\mathbb{F})$

### Theorem 4.9

*The set of unitary (respectively real orthogonal) matrices in  $\mathcal{M}_n$  forms a group, the  $n \times n$  unitary (respectively real orthogonal) subgroup of  $\mathrm{GL}(n, \mathbb{C})$  (respectively  $\mathrm{GL}(n, \mathbb{R})$ )*

### Theorem 4.10 (Selection Principle)

*Suppose that we have a sequence of unitary matrices  $U_1, U_2 \dots \in \mathcal{M}_n$ . Then there exists a subsequence  $U_{k_1}, U_{k_2} \dots$  such that the entries of  $U_{k_i}$  converge to entries of a unitary matrix as  $i \rightarrow \infty$*

### Lemma 4.11

Let  $U \in \mathcal{M}_n$  be a unitary matrix partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with  $U_{ii} \in \mathcal{M}_k$ . Then  $\text{rank } U_{12} = \text{rank } U_{21}$  and  $\text{rank } U_{22} = \text{rank } U_{11} + n - 2k$ . If, furthermore,  $U_{21} = 0$  and  $U_{12} = 0$ , then  $U_{11}$  and  $U_{22}$  are unitary

## Theorem 4.12 (QR factorisation)

Let  $A \in \mathcal{M}_{nm}$

1. If  $n \geq m$ , there is a  $Q \in \mathcal{M}_{nm}$  with orthonormal columns and upper triangular  $R \in \mathcal{M}_m$  with non-negative main diagonal entries such that  $A = QR$
2. If  $\text{rank } A = m$  then the factors  $Q$  and  $R$  in (1) are uniquely determined and the main diagonal entries of  $R$  are all positive
3. If  $n = m$ , Then the factor  $Q$  in (1) is unitary
4. There is a unitary  $Q \in \mathcal{M}_n$  and an upper triangular  $R \in \mathcal{M}_{nm}$  with nonnegative diagonal entries such that  $A = QR$
5. If  $A$  is real, then  $Q$  and  $R$  are in (1), (2), (3), and (4) may be taken to be real

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For a unitary matrix  $U$ ,  $U^* = U$ , so the transformation  $A \mapsto U^*AU$  is a **similarity transformation**, provided that  $U$  is unitary. This is a **unitary similarity**

### Definition 4.13 (Unitarily similar matrices)

Let  $A, B \in \mathcal{M}_n$ . We say that  $A$  is **unitarily similar** to  $B$  if there exists  $U \in \mathcal{M}_n$  unitary such that

$$A = U^*BU$$

If  $U$  can be taken real (i.e., if  $U$  is real orthogonal) than  $A$  is real orthogonal similar to  $B$  (if  $A = U^TBU$ )

## Remark 4.14

1. Unitary similarity is an equivalence relation
2. Unitary similarity implies similarity. However, the converse is not true
3. Similarity is a change of bases. Unitary similarity is a change of orthonormal bases

## Definition 4.15 (Householder matrix)

Let  $0 \neq \omega \in \mathbb{C}^n$ . The Householder matrix  $U_\omega \in \mathcal{M}_n$  is

$$U_\omega = \mathbb{I} - 2(\omega^* \omega)^{-1} \omega \omega^*$$

## Remark 4.16

1. If  $\|\omega\| = 1$  then  $U_\omega = \mathbb{I} - 2\omega\omega^*$
2. Householder matrix are unitary and Hermitian, thus  $U_\omega^{-1} = U_\omega$ .
3. The eigenvalues of a Householder matrix are  $-1, 1, \dots, 1$  and  $|U_\omega| = 1$

## Theorem 4.17

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$  and assume that  $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 > 0$

- ▶ If  $\mathbf{y} = e^{i\theta}\mathbf{x}$  for some  $\theta \in \mathbb{R}$  [ $\mathbf{x}, \mathbf{y}$  are linearly dependent], define  $U(\mathbf{y}, \mathbf{x}) = e^{i\theta}\mathbb{I}$
- ▶ Otherwise, let  $\phi \in [0, 2\pi)$  be such that  $\mathbf{x}^*\mathbf{y} = e^{i\phi}|\mathbf{x}^*\mathbf{y}|$  (taking  $\phi = 0$  if  $\mathbf{x}^*\mathbf{y} = 0$ ). Let  $\omega = e^{i\phi}\mathbf{x} - \mathbf{y}$  and define

$$U(\mathbf{y}, \mathbf{x}) = e^{i\phi} U_\omega$$

where  $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$  is Householder

1.  $U(\mathbf{y}, \mathbf{x})$  unitary and essentially Hermitian
2.  $U(\mathbf{y}, \mathbf{x})\mathbf{x} = \mathbf{y}$
3.  $U(\mathbf{y}, \mathbf{x})\mathbf{z} \perp \mathbf{y}$ , when  $\mathbf{z} \perp \mathbf{y}$
4. If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then  $U(\mathbf{y}, \mathbf{x})$  is real and  $U(\mathbf{y}, \mathbf{x}) = \mathbb{I}$  if  $\mathbf{y} = \mathbf{x}$  and  $U(\mathbf{y}, \mathbf{x}) = U_{\mathbf{x}-\mathbf{y}} \in \mathcal{M}_n(\mathbb{R})$  otherwise

## Remark 4.18

For all  $A \in \mathcal{M}_n$ ,  $U(y, x)^*AU(y, x) = U_\omega^*AU_\omega$ . This is called a Householder transformation.

## Theorem 4.19 (Schur's Form)

Let  $A \in \mathcal{M}_n$  with eigenvalues  $\lambda_1, \dots, \lambda_n$  in any prescribed order (including multiplicities). Let  $x \in \mathbb{C}^n$ ,  $\|x\| = 1$ , be such that  $Ax = \lambda_1 x$

1. There exists  $U = [x \ u_2 \dots u_n] \in \mathcal{M}_n$  unitary such that  $U^*AU = T$ , where  $T$  is upper triangular such that  $t_{ii} = \lambda_i$ ,  $i = 1, \dots, n$ .
2. If  $A \in \mathcal{M}_n(\mathbb{R})$  and has real eigenvalues, then  $x$  can be chosen to be real and there exists

$$Q = [x \ q_2 \dots q_n] \in \mathcal{M}_n(\mathbb{R})$$

real orthogonal and such that  $Q^T A Q = T$ , with  $T$  upper triangular with  $t_{ii} = \lambda_1$   $i = 1, \dots, n$ .

## Theorem 4.20 (Schur version 2)

Let  $A \in \mathcal{M}_n$  with eigenvalues  $\lambda_1, \dots, \lambda_n$  (including mutiplicities). Then there esists  $U \in \mathcal{M}_n$  such that

$$U^*AU = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

## Remark 4.21

*The decomposition is not unique*

## Theorem 4.22

Let  $U \in \mathcal{M}_n$ ,  $A, B \in \mathcal{M}_n$ . Suppose  $A$  is unitarily similar to  $B$ , then

$$\sum_{i,j=1}^n |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

## Corollary 4.23

Let  $A \in \mathcal{M}_n$  have eigenvalues  $\lambda_1, \dots, \lambda_n$ ,  $T = UAU^*$  upper triangular. Then

$$\sum_{i=1}^n |\lambda_i|^2 = \sum_{i,j=1}^n |a_{ij}|^2 - \sum_{i < j} |t_{ij}|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2 = \text{tr}AA^*$$

with equality if  $T$  is diagonal.

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## Theorem 4.24 (Cayley-Hamilton)

Let  $A \in \mathcal{M}_n$  and  $p_A(t)$  is the characteristic polynomial of  $A$ , then  $p_A(A) = 0$ .

## Theorem 4.25 (Sylvester's theorem – pole placement)

Assume  $A \in \mathcal{M}_n$  has eigenvalues  $\lambda_1, \dots, \lambda_n$  with multiplicities  $n_1, \dots, n_d$  ( $\sum_{i=1}^d n_i = n$ ).

Then  $A$  is unitary similar to a  $d \times d$  block upper triangular matrix  $T$ , where  $T_{i,j} \in M_{n_i m_j}$ ,  $T_{ij} = 0$  if  $i > j$ ,  $T_{ii}$  upper triangular with diagonal  $\lambda_i$ ,  $T_{ii} = \lambda_i \mathbb{I} + R_i$ ,  $R_i$  strictly upper triangular, and  $A$  is similar to a matrix to  $\bigoplus_{i=1}^d T_{ii}$  [standard similarity, not unitary]

## Theorem 4.26

(Every square matrix is almost diagonalisable) Let  $A \in \mathcal{M}_n$  for all  $\varepsilon > 0$ , there exists  $A(\varepsilon)[a_{ij}(\varepsilon)] \in \mathcal{M}$  with distinct eigenvalues such that

$$\sum_{i,j} |a_{ij} - a_{ij}(\varepsilon)|^2 < \varepsilon$$

## Theorem 4.27

If  $A \in \mathcal{M}_n$  for all  $\varepsilon > 0$  there exists  $S(\varepsilon) \in \mathcal{M}_n$  non-singular such that

$$S^{-1}(\varepsilon)AS(\varepsilon) = T(\varepsilon),$$

where  $T(\varepsilon)$  is upper triangular and  $|t_{ij}(\varepsilon)| < \varepsilon$  for all  $i, j$ , with  $i < j$ .

## Lemma 4.28

Let  $(A_k)_{k \in \mathbb{N}}$  a sequence of matrices such that  $\lim_{k \rightarrow \infty} A_k = A$  (entry-wise). Then there exists  $k_1 < k_2 < \dots$  and  $U_{k_i} \in \mathcal{M}$  such that

1.  $T_i = U_{k_i}^* A_{k_i} U_{k_i}$  upper triangular
2.  $U + \lim_{i \rightarrow \infty} U_{k_i}$  exists and is unitary
3.  $T = U^* A U$  upper triangular
4.  $\lim_{i \rightarrow \infty} T_i = T$

## Theorem 4.29

Let  $(A_k)_{k \in \mathbb{N}}$  a sequence of matrices such that  $\lim_{k \rightarrow \infty} A_k = A$  (entry-wise). Then let

$$\lambda(A) = [\lambda_1(A) \quad \dots \quad \lambda_n(A)]^T$$

and

$$\lambda(A_k) = [\lambda_1(A_k) \quad \dots \quad \lambda_n(A_k)]^T$$

be presentations of the eigenvalues of  $A$  and  $A_k$ . Define

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1, \dots, n\}\}.$$

Then for all  $\varepsilon > 0$  there exists  $N(\varepsilon) \in \mathbb{N} \setminus \{0\}$  such that

$$\min_{\pi \in S_n} \max_{i=1, \dots} \{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \leq \varepsilon \quad \forall k \geq N(\varepsilon)$$

Recall that if  $\mathbf{x}, \mathbf{y}$  are two (column) vectors in  $\mathbb{F}^n$ , then  $\mathbf{x}\mathbf{y}^*$  is a rank 1 matrix in  $\mathcal{M}_n(\mathbb{F})$ . (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly

### Theorem 4.30 (Brauer)

Suppose  $A \in \mathcal{M}_n$  has eigenvalues  $\lambda, \lambda_2, \dots, \lambda_n$ . Let  $\mathbf{x}$  be an eigenvector associated to  $\lambda$ . Then for every vector  $\mathbf{v} \in \mathbb{C}^n$ , the eigenvalues of  $A + \mathbf{x}^*\mathbf{v}$  are  $\lambda + \mathbf{v}^*\mathbf{x}, \lambda_2, \dots, \lambda_n$ .

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### Definition 4.31 (Normal matrix)

A matrix  $A \in \mathcal{M}_n$  is **normal** if  $AA^* = A^*A$

All unitary, Hermitian or skew-Hermitian and diagonal matrices are normal

## Theorem 4.32

Let  $A \in \mathcal{M}_n$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ . TFAE:

1.  $A$  is normal
2.  $A$  is unitary diagonalisable
3.  $\sum_{i,j} |a_{i,j}|^2 = \sum_i |\lambda_i|^2$
4.  $A$  has  $n$  orthogonal eigenvectors

### Theorem 4.33

Let  $A \in \mathcal{M}_n$  be a hermitian matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$ . Let

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

Then

1.  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
2.  $A$  is unitary diagonalisable
3. there exists  $U \in \mathcal{M}_n$  such that  $A = U\Lambda U^*$

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### Definition 4.34

A **Jordan block**  $J_k(\lambda)$  is a  $k \times k$  upper triangular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

### Theorem 4.35

Let  $A \in \mathcal{M}_n$  then there exists  $S \in \mathcal{M}_n$  non-singular such that

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^k J_{n_i}(\lambda_i) S^{-1}$$

### Theorem 4.36

Let  $A \in \mathcal{M}_n$  with real eigenvalues. Then there exists a basis of generalised eigenvectors for  $\mathbb{R}^n$ , and if  $\{v_1, \dots, v_n\}$  is a basis of generalised eigenvectors of  $\mathbb{R}^n$ , then

$P = [v_1 \ \dots \ v_n]$  is non-singular and  $A = D + N$  where  $P^{-1}DP = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $N = A - D$  is nilpotent<sup>1</sup> of order  $k \leq n$ , and  $D$  and  $N$  commute.

## References I