MATH 4370/7370 – Linear Algebra and Matrix Analysis

Nonnegative matrices

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Outline

Definition 1 (Nonnegative/positive matrix)

A matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$ is a **nonnegative matrix** if $a_{ij} \geq 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. We write $A \geq 0$. A is a **positive matrix** if $a_{ij} > 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. We write $A > 0$

Remark 2

In other references, you will see

- $A \geq 0 \iff a_{ij} \geq 0$
- $A > 0 \iff A \geq 0$ *and there exists* (i, j), $a_{ij} > 0$ *[positive]*
- $A \gg 0 \iff a_{ij} > 0$ *for all* i, j *[strongly positive]*

I tend to favour the latter notation over the one used in these notes, but since the former is more common in matrix theory, I use the notation of Definition 1 here

Notation

Let $A, B \in \mathcal{M}_{mn}(\mathbb{R})$. Nonnegativity and positivity are used to define partial orders on $\mathcal{M}_{mn}(\mathbb{R})$

- $A \geq B \iff A - B \geq 0$
- $A > B \iff A - B > 0$

The same is used for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$: $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} > \mathbf{y}$ if, respectively, $\mathbf{x} - \mathbf{y} \geq \mathbf{0}$ and $\mathbf{x} - \mathbf{y} > \mathbf{0}$. Note that the order is only partial: if $A \geq 0$ and $B \geq 0$, for instance, it is not necessarily possible to decide on the ordering of A and B with respect to one another

Theorem 3

Let A and B be nonnegative matrices of appropriate sizes. Then $A + B$ and AB are nonnegative. If $A > 0$ and $B \geq 0$, $B \neq 0$, then $AB \geq 0$ and $AB \neq 0$

Corollary 4

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be such that $\mathbf{x} \geq \mathbf{y}$ and $A \in \mathcal{M}_{mn}$ be nonnegative. Then $A\mathbf{x} \geq A\mathbf{y}$. Assume additionally that $\mathbf{x} \geq \mathbf{y}$, $\mathbf{x} \neq \mathbf{y}$ and $A > 0$. Then $A\mathbf{x} > A\mathbf{y}$

Definition 5

Let $P, Q \in \mathcal{M}_{nm}(\mathbb{F})$. P and Q have the same **zero-nonzero structure** if for all i, j, $p_{ij} \neq 0 \iff q_{ij} \neq 0$

Zero-nonzero structure defines an equivalence relation. Therefore, as with all equivalence relations, one only needs one representative from the equivalence class. One typical representative is defined using Boolean matrices

Definition 6

A **Boolean matrix** is a matrix whose entries are Boolean $\{0, 1\}$ and use Boolean arithmetics:

- $0 + 0 = 0$
- $1 + 0 = 0 + 1 = 1$
- $1 + 1 = 1$
- $0 \cdot 1 = 1$ and $1 = 0 = 0 \cdot 0$
- $1 \cdot 1 = 1$

Definition 7

Let $A \in \mathcal{M}_{nm}(\mathbb{F})$. Then A_B denotes the **Boolean representation** of A, defined as follows. If $A = [a_{ij}]$, then $A_B = [\alpha_{ij}]$ with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

Theorem 15 (Perron-Frobenius)

Let $A \geq 0 \in \mathcal{M}_n$ be irreducible. Then the spectral radius $\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\}$ is an eigenvalue of A. It is simple (has algebraic multiplicity 1), positive and is associated with a positive eigenvector. Furthermore, there is no nonnegative eigenvector associated to any other eigenvalue of A

Remark 16

We often say that $\rho(A)$ is the **Perron root** *of A; the corresponding eigenvector is the* **Perron vector** *of A*

Lemma 17 (Perron)

Let $\mathcal{M}_n \ni A > 0$. Then $\rho(A)$ is a positive eigenvalue of A and there is only one linearly independent eigenvector associated to $\rho(A)$, which can be taken to be positive

Lemma 18

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+^*$ and $v_1, \ldots, v_n \in \mathbb{C}$. Then

$$\left| \sum_{i=1}^{n} \alpha_i v_i \right| \leq \sum_{i=1}^{n} \alpha |v_i| \tag{1}$$

with equality if and only if there exists $\eta \in \mathbb{C}$, $|\eta| = 1$, such that $\eta v_i \geq 0$ for all $i = 1, \ldots, n$

Theorem 19

Let $A \in \mathcal{M}_n$ and $f(x)$ a polynomial. Then

$$\sigma(f(A)) = \{f(\lambda_1), \ldots, f(\lambda_n), \lambda_i \in \sigma(A)\}$$

If we have $g(\lambda_i) \neq 0$ for $\lambda_i \in \sigma(A)$, for some polynomial g, then the matrix $g(A)$ is non-singular and

$$\sigma\left(f(A)g(A)^{-1}\right) = \left\{\frac{f(\lambda_1)}{g(\lambda_1)}, \ldots, \frac{f(\lambda_n)}{g(\lambda_n)}, \lambda_i \in \sigma(A)\right\}$$

If $x \neq 0$ eigenvector of A associated to $\lambda \in \sigma(A)$, then x is also an eigenvector of $f(A)$ and $f(A)g(A)^{-1}$ associated to eigenvalue $f(\lambda)$ and $f(\lambda)/g(\lambda)$, respectively

Lemma 20 (Schur's lemma)

Let $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. Then λ is simple if and only if both the following conditions are statisfied:

1. *There exists only one linear independent eigenvector of A associated to λ, say $\mathbf{u}$, and thus only one linear independent eigenvector of A^T associated to λ, say $\mathbf{v}$*
2. *Vectors $\mathbf{u}$ and $\mathbf{v}$ in (1) satisfy $\mathbf{v}^T\mathbf{u} \neq 0$*

Definition 21

Let $\mathcal{M}_n(\mathbb{R}) \ni A \geq 0$. We say that A is **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if there exists $k \in \mathbb{N}_+^*$ such that

$$A^k > 0$$

with k the smallest integer for which this is true. We say that a matrix is **imprimitive** if it is not primitive

Remark 22

Primitivity implies irreducibility. The converse is not true

Theorem 23

A sufficient condition for primitivity is irreducibility with at least one positive diagonal entry

Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d = \gcd$ of all the lengths of closed walks in $G(A)$

Theorem 24

Let $A \in \mathcal{M}_n$ be a non-negative matrix. If A is primitive, then $A^k > 0$ for some $0 < k \leq (n-1)n^n$

Theorem 25

Let $A \geq 0$ primtive. Suppose the shortest simple directed cycle in $G(A)$ has length s, then primitivity index is $\leq n + s(n-1)$

Theorem 26

Let $A \in \mathcal{M}_n$ be a nonnegative matrix. A is primitive if and only if $A^{n^2-2n+2} > 0$

Theorem 27

Let $A \in \mathcal{M}_n$ be a nonnegative irreducible matrix . Suppose that A has d positive entries on the diagonal. Then the primitivity index is $\leq 2n - d - 1$

Theorem 28

Let $A \geq 0$ *in* $\mathcal{M}_n$. *Then there exists* $0 \neq v \geq 0$ *such that* $A\mathbf{v} = \rho(A)\mathbf{v}$

Let us restate the Perron-Frobenius theorem, taking into account the different cases. The classification in the following result is inspired by the presentation in [**?**].

Theorem 29

Let $\mathcal{M}_n \ni A \geq 0$. Denote λ_P the Perron root of A, i.e., $\lambda_P = \rho(A)$, $\mathbf{v}_P$ and $\mathbf{w}_P$ the corresponding right and left Perron vectors of A, respectively. Denote d the index of imprimitivity of A (with $d = 1$ when A is primitive) and $\lambda_j \in \sigma(A)$ the spectrum of A, with $j = 2, \ldots, n$ unless otherwise specified (assuming $\lambda_1 = \lambda_P$). Then conclusions of the Perron-Frobenius Theorem can be summarised as follows.

Nonnegative

Reducible

- $\lambda_P \geq 0$
- $\mathbf{w}_P \geq 0$
- $\mathbf{v}_P \geq 0$
- $\lambda_P \geq |\lambda_j|$

Irreducible

Imprimitive

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
- $\lambda_P > |\lambda_j|$, $j > d$

Primitive

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P > |\lambda_j|$, $j \neq P$

Theorem 30

Let $A \in \mathcal{M}_n$ be a nonnegative irreducible matrix and $\in \mathbb{N}_+$. Then the following ar eequivalent:

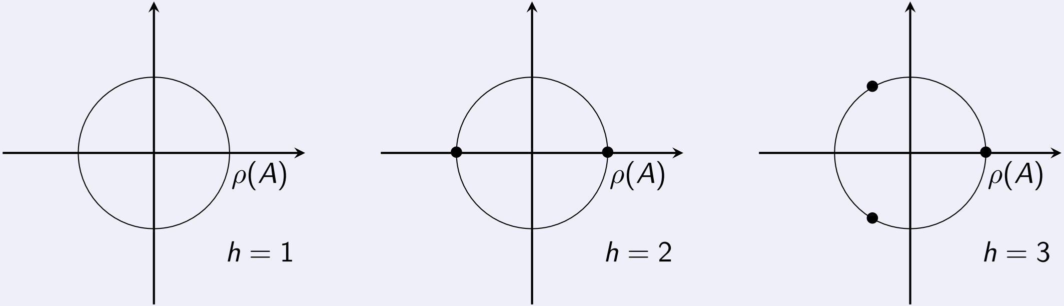
1. *there exists exactly h distinct eigenvalues such that $|\lambda| = \rho(A)$.*
2. *there exists P a permutation matrix such that*

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ \vdots & & A_{23} & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 \end{pmatrix}$$

   where the diagonal blocks are square, and there does not exists other permutation matrix giving less than h horizontal blocks.
3. *the greatest common divisor of the lengths of all cycles in $G(A)$ is h.*
4. *h is the maximal positive integer k such that*

Corollary 31

Let $A \in \mathcal{M}_n$, $A \geq 0$ irreducible with exactly h distinct eigenvalues of modulus $\rho(A)$. Then, we can consider this eigenvalues as points in the complex plan, the eigenvalues are the vertices of a regular polygon of h sides with centre at the origin and are of the vertices being $\rho(A)$



Remark 32

For Fiedler, a primitive matrix is defined as an irreducible nonnegative matrix such that $h = 1$

Theorem 33

Let $A \geq 0$ in $\mathcal{M}_n$, $n \geq 2$. TFAE

1. $A^n = 0$
2. *there exists* $\mathbb{N} \ni k > 0$ *such that* $A^k = 0$
3. $G(A)$ *acyclic*
4. $\exists P$, *permutation matrix, .t.* PAP^T *is upper-triangular with zeros on main diagonal*
5. $\rho(A) = 0$

Theorem 34

Let $A \geq 0$ be a nonnegative matrix in $\mathcal{M}_n$. Assume that A has a positive eigenvector. Then that eigenvector is the Perron vector and is associated to $\rho(A)$

Definition 35 (Stochastic matrix)

The matrix $A \in \mathcal{M}_n$ is **stochastic** if

- $A \geq 0$ [The matrix is nonnegative]
- $A\mathbb{1} = \mathbb{1}$, $\mathbb{1} = (1, \ldots, 1)^T$ [All rows sum to 1]

Equivalently, the matrix is stochastic if its column sums all equal 1

Definition 36

The matrix is **row-stochastic** or **column-stochastic**, respectively, if the rows or columns sum to 1. The terms **right stochastic** and **left stochastic** are also used. If both rows and columns sum to 1, then the matrix is **doubly stochastic**

Theorem 37

Let $A \in \mathcal{M}_n$ be stochastic. Then $\rho(A) = 1$

Theorem 38

Let $P \in \mathcal{M}_n$, $P \geq 0$. Assume that P has a positive eigenvector u and that $\rho(P) > 0$. Then there exists D, diagonal matrix with $\operatorname{diag}(D) > 0$, and $k > 0$, $k \in \mathbb{R}$ such that

$$A = kDPD^{-1}$$

is stochastic, with $k = \rho(P)^{-1}$

Theorem 39

Let A, $B \in \mathcal{M}_n$ be stochastic. Then AB is stochastic

Theorem 40

Let A be a primitive stochastic. Then $A^k \to \mathbb{1}\mathbf{v}^T$, $k \to \infty$, where $\mathbb{1}\mathbf{v}^T$ has rank 1 and $\mathbf{v}$ is the (left) eigenvector of A^T associated to $\rho(A) = 1$ and normalised so that $\mathbf{v}^T\mathbb{1} = 1$

Remark 41

This is a result that is used to compute the limit of a regular Markov chain

Definition 42

The matrix $A \in \mathcal{M}_n$, $A \geq 0$ is **doubly stochastic** if $A\mathbb{1} = \mathbb{1}$ and $\mathbb{1}^T A = \mathbb{1}^T$

Remark 43

Here $\rho(A) = 1$ is associated to $\mathbb{1}$ for A and for A^T

Consider E the Euclidean space. A set K of points in E is **convex** if A_1, A_2 points in K, $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $\lambda_1 + \lambda_2 = 1$, then

$$\lambda_1 A_1 + \lambda_2 A_2 \in K.$$

A **convex polyhedron** K is the set of all points of the form

$$\sum_{i=1}^{N} \lambda_i A_i$$

where A_i are points in E and $\lambda_1 \in \mathbb{R}_+$

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$. Consider this matrix as a point in E with coordinates $[a_{11}, a_{12}, \ldots, a_{nn}]$ ($\dim E = n^2$)

Theorem 44

Let $A \in \mathcal{M}_n$, $A = [a_{ij}]$, if A is doubly stochastic, then this forms an $(n-1)^2$ dimensional subspace of $\tilde{E} = \mathbb{R}^{n^2}$

Theorem 45 (Birkhoff)

In the space $\tilde{E} = R^{n^2}$, the set of doubly stochastic matrices of order n is a convex polyhedron in E (the subspace of stochastic matrices). The vertices of the polyhedron are the points corresponding to all the permutation matrices

References I