

MATH 4370/7370 – Linear Algebra and Matrix Analysis

Quick review of 2nd year linear algebra

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OUTLINE OF THESE SLIDES

- Part 1: Some notation and basic stuff
- Part 2: Vector spaces
- Part 3: Finite-dimensional vector spaces
- Part 4: Linear maps
- Part 5: Eigenvalues, eigenvectors and invariant subspaces
- Part 6: Inner product spaces
- Part 7: Operators on inner product spaces
- Part 8: Operators on complex vector spaces

Source of the material

The material in these slides is mostly derived from [Axl15]

Some notation and basic stuff

Sets and elements

Logic

Sets and elements

Logic

Sets and elements

Definition 1 (Set)

A **set** X is a collection of **elements**.

We write $x \in X$ or $x \notin X$ to indicate that the element x belongs to the set X or does not belong to the set X , respectively.

Definition 2 (Subset)

Let X be a set. The set S is a **subset** of X , which is denoted $S \subset X$ or $S \subseteq X$, if all its elements belong to X . S is a **proper subset** of X if it is a subset of X and not equal to X ; we then write $S \subsetneq X$.

Smith reserves \subset for \subsetneq . I learned \subset for not specified (proper or not) and \subsetneq for proper. So beware!

Quantifiers

- ▶ A shorthand notation for “for all elements x belonging to X ” is $\forall x \in X$. For example, if $X = \mathbb{R}$, the field of real numbers, then $\forall x \in \mathbb{R}$ means “for all real numbers x ”.
- ▶ A shorthand notation for “there exists an element x in the set X ” is $\exists x \in X$.
- ▶ Sometimes we write $\exists! x \in X$ for “there exists a **unique** x in X ”.
- ▶ \forall and \exists are **quantifiers**.

Intersection and union of sets

Let X and Y be two sets.

Definition 3 (Intersection)

The intersection of X and Y , $X \cap Y$, is the set of elements that belong to X **and** to Y

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

Definition 4 (Union)

The union of X and Y , $X \cup Y$, is the set of elements that belong to X **or** to Y

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}$$

Use of the expression “and/or” is *strictly* forbidden in this course! “Or but not and” (a.k.a. **xor**, exclusive or) is $(X \cup Y) \setminus (X \cap Y)$.

Sets and elements

Logic

A few notions of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. “The sky is blue” is also a proposition.

Let A be a proposition. We generally write

A

to mean that A is true, and

not A

to mean that A is false. We also write $\neg A$. **not** A is the **negation** of A .

A few notions of logic (cont.)

Let A, B be propositions. Then

- ▶ $A \Rightarrow B$ (read A implies B) means that whenever A is true, then so is B .
- ▶ $A \Leftrightarrow B$, also denoted A if and only if B (A iff B for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$. We also say that A and B are equivalent.

Let A and B be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\text{not } B \Rightarrow \text{not } A)$$

This is useful for proving some results.

Necessary and/or sufficient conditions

Suppose we want to establish whether a given statement P is true, depending on the truth value of a statement H . Then we say that

- ▶ H is a **necessary condition** if $P \Rightarrow H$.
(It is necessary that H be true for P to be true; so whenever P is true, so is H).
- ▶ H is a **sufficient condition** if $H \Rightarrow P$.
(It suffices for H to be true for P to also be true).
- ▶ H is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., H and P are equivalent.

Playing with quantifiers

For the quantifiers \forall (for all) and \exists (there exists),

\exists is the negation of \forall

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

This is also regularly used in proofs.

Vector spaces

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

Operations

Definition 5 (Operations – Addition and multiplication)

An **operation** on a set V is a mapping that associates an element of the set V to every pair of its elements

- ▶ The result of the **addition** of a and b is the *sum* $a + b$ of a and b
- ▶ The result of the **multiplication** of a and b is the *product* ab (or $a \cdot b$) of a and b

Definition 6 (Field)

A **field** is a set \mathbb{F} together with two (binary) operations, *addition* and *multiplication*, which are required to satisfy the following *field axioms*, where $a, b, c \in \mathbb{F}$:

- ▶ **Associativity** of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$
- ▶ **Commutativity** of addition and multiplication: $a + b = b + a$ and $ab = ba$
- ▶ **Additive and multiplicative identity**: $\exists 0, 1 \in \mathbb{F}$, $0 \neq 1$, s.t. $a + 0 = a$ and $a1 = a$
- ▶ **Additive inverses**: $\forall a \in \mathbb{F}$, $\exists -a \in \mathbb{F}$ s.t. $a + (-a) = 0$
- ▶ **Multiplicative inverses**: $\forall a \neq 0 \in \mathbb{F}$, $\exists a^{-1} \in \mathbb{F}$ s.t. $aa^{-1} = 1$
- ▶ **Distributivity** (of multiplication over addition): $a(b + c) = (ab) + (ac)$

Notation

- ▶ Both \mathbb{R} and \mathbb{C} are fields.
- ▶ From now on, \mathbb{F} refers to \mathbb{R} or \mathbb{C} .
- ▶ Some results are specific to \mathbb{R} xor \mathbb{C} , in which case we specify the relevant field.
- ▶ If we use \mathbb{F} , we mean the result applies to both \mathbb{R} and \mathbb{C} .

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

Addition and Scalar multiplication

Definition 7 (Addition and scalar multiplication on a set)

- ▶ An **addition** on a set V is a function that assigns an element $\mathbf{u} + \mathbf{v} \in V$ to each pair of elements $\mathbf{u}, \mathbf{v} \in V$
- ▶ A **scalar multiplication** on a set V is a function that assigns an element $\lambda \mathbf{v}$ to each $\lambda \in \mathbb{F}$ and each $\mathbf{v} \in V$

Vector space

Definition 8 (Vector space)

A **vector space** (over \mathbb{F}) is a set V along with an addition on V and a scalar multiplication on V such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [commutativity]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall a, b \in \mathbb{F}, (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and $(ab)\mathbf{v} = a(b\mathbf{v})$ [associativity]
3. $\exists \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V, \mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [additive identity]
4. $\forall \mathbf{v} \in V, \exists \mathbf{w} \in V$ s.t. $\mathbf{v} + \mathbf{w} = \mathbf{0}_V$ [additive inverse]
5. $\forall \mathbf{v} \in V, 1\mathbf{v} = \mathbf{v}$ [multiplicative identity]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V, a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [distributivity]

Results

Theorem 9 (Uniqueness of the additive identity)

A vector space V has a unique additive identity $\mathbf{0}_V \in V$

Theorem 10 (Existence and uniqueness of additive inverse)

Let V be a vector space. Then each $\mathbf{v} \in V$ has a unique additive inverse, denoted $-\mathbf{v}$

We also define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$.

Theorem 11

- ▶ $\forall \mathbf{v} \in V, 0_{\mathbb{F}}\mathbf{v} = \mathbf{0}_V.$
- ▶ $\forall a \in \mathbb{F}, a\mathbf{0}_V = \mathbf{0}_V.$
- ▶ $\forall \mathbf{v} \in V, (-1)\mathbf{v} = -\mathbf{v}.$

Vector space

Definition 12 (Vector space)

A **vector space** (over \mathbb{F}) is a set V along with an addition on V and a scalar multiplication on V such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [commutativity of $+$]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ [associativity of $+$]
3. $\exists! \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V, \mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [additive identity]
4. $\forall \mathbf{v} \in V, \exists! -\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_V$ [additive inverse]
5. $\forall a \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V, a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ [distributivity of \cdot over $+$]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u} \in V, (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ [distributivity of $+$ over \cdot]
7. $\forall a, b \in \mathbb{F}, (ab)\mathbf{u} = a(b\mathbf{u})$ [associativity of \cdot]
8. $\forall \mathbf{u} \in V, 1\mathbf{u} = \mathbf{u}$ [multiplicative identity]

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

\mathbb{F}^n is a vector space

Typically called *Euclidean space* when $\mathbb{F} = \mathbb{R}$.

Definition 13

Let $0 \neq n \in \mathbb{N}$. An n -**tuple** is an ordered collection of n elements,

$$(x_1, \dots, x_n)$$

Definition 14

Let $0 \neq n \in \mathbb{N}$. \mathbb{F}^n is the set of all n -tuples of elements of \mathbb{F} :

$$\mathbb{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n\}$$

- ▶ Often write $x = (x_1, \dots, x_n)$ for short.
- ▶ For a given $j \in \{1, \dots, n\}$, x_j is the j th **coordinate** of x .
- ▶ Think of $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ that you saw in whatever flavour of Linear Algebra 1 you took.

Addition in \mathbb{F}^n

Definition 15 (Addition in \mathbb{F}^n)

Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{F}^n$. Then

$$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

Property 16 (Commutativity of addition in \mathbb{F}^n)

Let $x, y \in \mathbb{F}^n$, then

$$x + y = y + x$$

0 and additive inverse in \mathbb{F}^n

Definition 17 (0)

0 denotes the n -tuple whose coordinates are all 0,

$$0 = (0, \dots, 0)$$

If any ambiguity arises, will write $0_{\mathbb{F}^n}$

Definition 18 (Additive inverse)

Let $x \in \mathbb{F}^n$. The **additive inverse** of x is $-x \in \mathbb{F}^n$ s.t.

$$x + (-x) = 0$$

If $x = (x_1, \dots, x_n)$, then $-x = (-x_1, \dots, -x_n)$

Scalar multiplication in \mathbb{F}^n

Definition 19 (Scalar multiplication)

The **product** of $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$ is

$$\lambda x = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

Complex numbers

Definition 20 (Complex numbers)

A **complex number** is an ordered pair (a, b) , where $a, b \in \mathbb{R}$. Usually written $a + ib$ or $a + bi$, where $i^2 = -1$

The set of all complex numbers is denoted \mathbb{C} ,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

Definition 21 (Addition and multiplication on \mathbb{C})

Letting $a + ib$ and $c + id \in \mathbb{C}$, addition on \mathbb{C} is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on \mathbb{C} is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter equality easy to obtain using regular multiplication and $i^2 = -1$

Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C},$

▶ $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$

[commutativity]

▶ $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

[associativity]

▶ $\gamma + 0 = \gamma$ and $\gamma 1 = \gamma$

[identities]

▶ $\forall \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C}$ unique s.t. $\alpha + \beta = 0$

[additive inverse]

▶ $\forall \alpha \neq 0 \in \mathbb{C}, \exists \beta \in \mathbb{C}$ unique s.t. $\alpha\beta = 1$

[multiplicative inverse]

▶ $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$

[distributivity]

Thus \mathbb{C} is a field.

Additive & multiplicative inverse, subtraction, division

Definition 22

Let $\alpha, \beta \in \mathbb{C}$

- ▶ $-\alpha$ is the **additive inverse** of α , i.e., the unique number in \mathbb{C} s.t. $\alpha + (-\alpha) = 0$
- ▶ **Subtraction** on \mathbb{C} :

$$\beta - \alpha = \beta + (-\alpha)$$

- ▶ For $\alpha \neq 0$, $1/\alpha$ is the **multiplicative inverse** of α , i.e., the unique number in \mathbb{C} s.t.

$$\alpha(1/\alpha) = 1$$

- ▶ **Division** on \mathbb{C} :

$$\beta/\alpha = \beta(1/\alpha)$$

Definition 23 (Real and imaginary parts)

Let $z = a + ib$. Then $\operatorname{Re} z = a$ is **real part** and $\operatorname{Im} z = b$ is **imaginary part** of z

If ambiguous, write $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$

Definition 24 (Conjugate and Modulus)

Let $z = a + ib \in \mathbb{C}$. Then

► **Complex conjugate** of z is

$$\bar{z} = \operatorname{Re} z - i(\operatorname{Im} z) = a - ib$$

► **Modulus** (or **absolute value**) of z is

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{a^2 + b^2} \geq 0$$

Properties of complex numbers

Let $w, z \in \mathbb{C}$, then

▶ $z + \bar{z} = 2\operatorname{Re} z$

▶ $z - \bar{z} = 2i\operatorname{Im} z$

▶ $z\bar{z} = |z|^2$

▶ $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$

▶ $\bar{\bar{z}} = z$

▶ $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$

▶ $|\bar{z}| = |z|$

▶ $|wz| = |w| |z|$

▶ $|w + z| \leq |w| + |z|$

[triangle inequality]

Fields

Definition of vector spaces

Example – Space \mathbb{F}^n

Example – Complex numbers

Subspaces

Subspace

Definition 25 (Subspace)

Let V be a vector space over \mathbb{F} . Let $U \subseteq V$ be a subset of V . Then U is a **subspace** of V if U is a vector space over \mathbb{F} for the same operations of addition and scalar multiplication as V

Theorem 26 (Conditions for a subspace)

$U \subseteq V$ is a subspace of $V \iff U$ satisfies the following three conditions:

- ▶ $\mathbf{0}_V \in U$ [additive identity]
- ▶ $\forall \mathbf{u}, \mathbf{v} \in U, \mathbf{u} + \mathbf{v} \in U$ [closed under addition]
- ▶ $\forall \mathbf{u} \in U, \forall a \in \mathbb{F}, a\mathbf{u} \in U$ [closed under scalar multiplication]

The smallest possible subspace of V is $\{\mathbf{0}_V\}$, the largest is V .

Sums of subspaces

Definition 27 (Sum of subsets)

Let V be a vector space and U_1, \dots, U_m be *subsets* of V . The **sum** of U_1, \dots, U_m is

$$U_1 + \dots + U_m = \{\mathbf{u}_1 + \dots + \mathbf{u}_m : \mathbf{u}_1 \in U_1, \dots, \mathbf{u}_m \in U_m\}$$

Theorem 28

Let V be a vector space and U_1, \dots, U_m be subspaces of V . Then $U_1 + \dots + U_m$ is the smallest subspace of V containing U_1, \dots, U_m

Direct sums

Definition 29 (Direct sum)

Suppose U_1, \dots, U_m are subspaces of a vector space V . The sum $U_1 + \dots + U_m$ is a **direct sum** and is then written $U_1 \oplus \dots \oplus U_m$ if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $\mathbf{u}_1 + \dots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$

Theorem 30 (Condition for a direct sum)

Suppose U_1, \dots, U_m are subspaces of a vector space V . Then $U_1 + \dots + U_m$ is a direct sum \iff the only way to write $\mathbf{0}$ as a sum $\mathbf{u}_1 + \dots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$, is by taking each \mathbf{u}_j equal to $\mathbf{0}_V$

Theorem 31 (Direct sum of two subspaces)

Let U, W be subspaces of a vector space V . Then $U + W$ is a direct sum $\iff U \cap W = \{\mathbf{0}_V\}$

Finite-dimensional vector spaces

Span and Linear independence

Bases

Dimension

Span and Linear independence

Bases

Dimension

Definition 32 (Linear combination)

A **linear combination** of a list $\mathbf{v}_1, \dots, \mathbf{v}_m$ of vectors in V is a vector

$$a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$

Definition 33 (Span)

The set of all linear combinations of a list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ is the **span** of $\mathbf{v}_1, \dots, \mathbf{v}_m$,

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = \{a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m : a_1, \dots, a_m \in \mathbb{F}\}$$

The span of the empty list $()$ is $\{\mathbf{0}_V\}$

Finite/infinite-dimensional vector spaces

Theorem 34

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list

Definition 35 (List of vectors spanning a space)

If $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = V$, we say $\mathbf{v}_1, \dots, \mathbf{v}_m$ **spans** V

Definition 36 (Finite-dimensional vector space)

A vector space V is **finite-dimensional** if some list of vectors in it spans V

Definition 37 (Infinite-dimensional vector space)

A vector space V is **infinite-dimensional** if it is not finite-dimensional

Linear (in)dependence

Definition 38 (Linear independence/Linear dependence)

A list $\mathbf{v}_1, \dots, \mathbf{v}_m$ of vectors in a vector space V is **linearly independent** if

$$(a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m = 0) \Leftrightarrow (a_1 = \dots = a_m = 0),$$

where $a_1, \dots, a_m \in \mathbb{F}$. A list of vectors is **linearly dependent** if it is not linearly independent.

The empty list $()$ is assumed to be linearly independent

Lemma 39 (Linear dependence)

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be a linearly dependent list in a vector space V . Then there exists $j \in \{1, 2, \dots, m\}$ s.t.

- 1. $\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_{j-1})$*
- 2. if the j th term is removed from $\mathbf{v}_1, \dots, \mathbf{v}_m$, the span of the remaining list equals $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$*

Theorem 40

Let V be a finite-dimensional vector space. Then the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors

Theorem 41 (Subspace of a finite-dimensional vector space)

Every subspace of a finite-dimensional vector space is finite-dimensional

Span and Linear independence

Bases

Dimension

Basis

Definition 42 (Basis)

Let V be a vector space. A **basis** of V is a list of vectors in V that is both linearly independent and spanning

Theorem 43 (Criterion for a basis)

A list $\mathbf{v}_1, \dots, \mathbf{v}_m$ of vectors in a vector space V is a basis of V iff $\forall \mathbf{v} \in V$, \mathbf{v} can be written uniquely in the form

$$\mathbf{v} = a_1 \mathbf{v}_1 + \dots + a_m \mathbf{v}_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$

Theorem 44 (All spanning lists contain a basis)

Every spanning list in a vector space can be reduced to a basis of the vector space

Theorem 45 (Basis of finite-dimensional vector space)

Every finite-dimensional vector space has a basis

Theorem 46 (Extension to a basis)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space

Theorem 47

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V . Then $\exists W \subset V$ subspace of V s.t. $V = U \oplus W$

Span and Linear independence

Bases

Dimension

Theorem 48 (Bases of a finite-dim. space have equal length)

Any two bases of a finite-dimensional vector space have the same length

Definition 49 (Dimension)

The **dimension** $\dim V$ of a finite-dimensional vector space V is the length of any basis of the vector space

Theorem 50 (Dimension of a subspace)

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V . Then $\dim U \leq \dim V$

Theorem 51

Let V be a finite-dimensional vector space. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V

Theorem 52

Let V be a finite-dimensional vector space. Then every spanning list of vectors in V with length $\dim V$ is a basis of V

Theorem 53 (Dimension of a sum of subspaces)

Let U_1, U_2 be subspaces of a finite-dimensional vector space V . Then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Linear maps

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

Definition 54 (Linear map/transformation)

Let V, W be vector spaces. A **linear map** (or **linear transformation**) from V to W is a function $T : V \rightarrow W$ that has the following properties:

1. **Additivity** $\forall \mathbf{u}, \mathbf{v} \in V, T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. **Homogeneity** $\forall \lambda \in \mathbb{F}, \forall \mathbf{v} \in V, T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$.

Often, parentheses are omitted, $T(\mathbf{u})$ is written $T\mathbf{u}$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$

Theorem 55 (Linear maps and basis of domain)

Let V, W be two vector spaces and $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V . Let $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$. Then there exists a unique linear map $T : V \rightarrow W$ s.t.

$$\forall j = 1, \dots, n, \quad T\mathbf{v}_j = \mathbf{w}_j$$

Definition 56 (Addition & Scalar multiplication)

Let V, W be vector spaces, $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$ and **product** λT are the linear maps from V to W defined, $\forall \mathbf{v} \in V$, by

$$(S + T)(\mathbf{v}) = S\mathbf{v} + T\mathbf{v} \text{ and } (\lambda T)(\mathbf{v}) = \lambda(T\mathbf{v}).$$

Theorem 57 (Linear maps are vector spaces)

Let V, W be vector spaces. Equipped with addition and scalar multiplication as just defined, $\mathcal{L}(V, W)$ is a vector space.

Product of linear maps

Definition 58 (Product of linear maps)

Let U, V, W be vector spaces, $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$. The **product** $ST \in \mathcal{L}(U, W)$ is defined for $\mathbf{u} \in U$ by

$$(ST)(\mathbf{u}) = S(T\mathbf{u}).$$

This means that the product of linear maps is the composition $S \circ T$, although because of the linearity, we often omit the \circ composition sign.

Properties of products of linear maps

Theorem 59

1. **Associativity** If V, V_2, V_3, W vector spaces,
 $T_1 \in \mathcal{L}(V, V_2), T_2 \in \mathcal{L}(V_2, V_3), T_3 \in \mathcal{L}(V_3, W)$, then

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$

2. **Identity** V, W vector spaces. Then for $T \in \mathcal{L}(V, W)$,

$$T I_V = I_W T = T$$

3. **Distributivity** U, V, W vector spaces, $T, T_1, T_2 \in \mathcal{L}(U, V), S, S_1, S_2 \in \mathcal{L}(V, W)$,
then

$$(S_1 + S_2) T = S_1 T + S_2 T \text{ and } S(T_1 + T_2) = S T_1 + S T_2$$

Theorem 60 (Linear maps take 0 to 0)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T(\mathbf{0}_V) = \mathbf{0}_W.$$

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

Definition 61 (Null space)

Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. The **null space** $\text{null } T$ (or **kernel** $\ker T$) of T is the subset of V consisting of those vectors that T maps to $\mathbf{0}_W$:

$$\text{null } T = \{\mathbf{v} \in V; T\mathbf{v} = \mathbf{0}_W\}.$$

Theorem 62 (Null space is a subspace)

Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then $\text{null } T$ is a subspace of V

Definition 63 (Injectivity)

A function $T : V \rightarrow W$ is **injective** (or **one-to-one**) if

$$T\mathbf{u} = T\mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}.$$

We can also use the contrapositive: T injective if $\mathbf{u} \neq \mathbf{v} \Rightarrow T\mathbf{u} \neq T\mathbf{v}$.

Theorem 64 (Linking injectivity and null space)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \text{ injective} \Leftrightarrow \text{null } T = \{\mathbf{0}_V\}$$

Definition 65 (Range)

Let V, W be finite-dimensional vector spaces, $T : V \rightarrow W$ a function. The **range** (or **image**) of T is the subset of W defined by

$$\text{range } T = \{T\mathbf{v}; \mathbf{v} \in V\}.$$

When talking about the image, we write $\text{Im } T$.

Theorem 66 (Range is a subspace)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then $\text{range } T$ is a subspace of W .

Definition 67 (Surjectivity)

A function $T : V \rightarrow W$ is **surjective** (or **onto**) if

$$\text{range } T = W$$

Theorem 68 (Fundamental theorem of linear maps)

Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then $\dim \text{range } T < \infty$ and

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

Theorem 69 (Linear map onto a smaller space is not injective)

Let V, W be finite-dimensional vector spaces such that $\dim V > \dim W$. Then $\nexists T \in \mathcal{L}(V, W)$ that is injective

Theorem 70 (Linear map onto a larger space is not surjective)

Let V, W be finite-dimensional vector spaces such that $\dim V < \dim W$. Then $\nexists T \in \mathcal{L}(V, W)$ that is surjective

Do as exercises..

Theorem 71

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Theorem 72

A nonhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms

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Definition 73 (Matrix)

An m -by- n or $m \times n$ matrix is a rectangular array of elements of \mathbb{F} with m rows and n columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Remember that we always list indices as “row,column”

We denote $\mathcal{M}_{mn}(\mathbb{F})$ the set of $m \times n$ matrices with entries in \mathbb{F}

Definition 74 (Matrix of a linear map)

Let V, W be finite-dimensional vector spaces, v_1, \dots, v_n a basis of V and w_1, \dots, w_m a basis of W . The **matrix of** T with respect to these bases is the matrix $M(T) \in \mathcal{M}_{mn}$ with entries a_{jk} defined by

$$Tv_k = a_{1k}w_1 + \cdots + a_{mk}w_m$$

for $1 \leq k \leq n$. If the bases are not clear from the context, then write

$$M(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$$

I will often write M_T rather than $M(T)$.

Most definitions are assumed known

Theorem 75 (Matrix of sums of linear maps)

Suppose $S, T \in \mathcal{L}(V, W)$. Then $M(S + T) = M(S) + M(T)$

Theorem 76 (Matrix of a scalar times a linear map)

Suppose $T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$. Then $M(\lambda T) = \lambda M(T)$

Theorem 77 (Dimension of \mathcal{M}_{mn})

$\dim \mathbb{F}^{mn} = mn$

Theorem 78 (Matrix of products of linear maps)

Suppose $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$. Then $M(ST) = M(S)M(T)$

Theorem 79

Let $A \in \mathcal{M}_{mn}$, $C \in \mathcal{M}_{np}$. Then

$$(AC)_{jk} = A_{j\bullet} C_{\bullet k}, \quad 1 \leq j \leq m, 1 \leq k \leq p$$

and

$$(AC)_{\bullet k} = AC_{\bullet k}, \quad 1 \leq k \leq p$$

Theorem 80

Let $A \in \mathcal{M}_{mn}$, $c = (c_1, \dots, c_n)^T \in \mathcal{M}_{n1}$. Then

$$Ac = c_1 A_{\bullet 1} + \dots + c_n A_{\bullet n}$$

Change of basis

Definition 81 (Change of basis matrix)

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ bases of vector space V The **change of basis matrix** $P_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathcal{M}_n$,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

has columns the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \dots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in \mathcal{B} w.r.t. \mathcal{C}

Theorem 82

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ bases of vector space V and $P_{\mathcal{C} \leftarrow \mathcal{B}}$ a change of basis matrix from \mathcal{B} to \mathcal{C}

1. $\forall \mathbf{x} \in V, P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$
2. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t. $\forall \mathbf{x} \in V, P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ is **unique**
3. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ invertible and $P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$

Row-reduction method for changing bases

Theorem 83

$\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ bases of vector space V . Let \mathcal{E} be any basis for V ,

$$B = [[\mathbf{u}_1]_{\mathcal{E}}, \dots, [\mathbf{u}_n]_{\mathcal{E}}] \text{ and } C = [[\mathbf{v}_1]_{\mathcal{E}}, \dots, [\mathbf{v}_n]_{\mathcal{E}}]$$

and let $[C|B]$ be the augmented matrix constructed using C and B . Then

$$\text{RREF}([C|B]) = [\mathbb{I} | P_{C \leftarrow B}]$$

If working in \mathbb{R}^n , this is quite useful with \mathcal{E} the standard basis of \mathbb{R}^n (it does not matter if $\mathcal{B} = \mathcal{E}$)

More on changing bases

Theorem 84 (NSC for two matrices representing the same linear map)

Let $A, B \in \mathcal{M}_{mn}$, V and W be n and m dimensional vector spaces, respectively. Then A and B represent the same linear transformation $T \in \mathcal{L}(V, W)$ relative to perhaps different bases of V and $W \iff \exists P \in \mathcal{M}_m, Q \in \mathcal{M}_n$ nonsingular and such that

$$A = PBQ^{-1}$$

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Definition 85 (Inverse/Invertibility)

$T \in \mathcal{L}(V, W)$ is **invertible** if $\exists S \in \mathcal{L}(W, V)$ s.t. $ST = I_V$ and $TS = I_W$. Such a map is the **inverse** of T

Theorem 86 (Uniqueness of inverse)

An invertible linear map $T \in \mathcal{L}(V, W)$ has a unique inverse denoted T^{-1}

Theorem 87 (NSC for invertibility)

$T \in \mathcal{L}(V, W)$ invertible $\Leftrightarrow (T \text{ injective and surjective})$

Definition 88 (Isomorphism/Isomorphic spaces)

$T \in \mathcal{L}(V, W)$ is an **isomorphism** if it is invertible. Two vector spaces are **isomorphic** if there exists an isomorphism from one to the other

Theorem 89 (NSC for isomorphicity)

Let V, W be finite-dimensional vector spaces over \mathbb{F} . Then

$$V \text{ and } W \text{ are isomorphic} \Leftrightarrow \dim V = \dim W$$

Theorem 90

Let v_1, \dots, v_n be a basis of V and w_1, \dots, w_m be a basis of W . Then M is an isomorphism between \mathcal{M}_{mn} and $\mathcal{L}(V, W)$

Theorem 91 (Dimension of $\mathcal{L}(V, W)$)

Let V, W be finite-dimensional vector spaces. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = \dim V \dim W$$

Definition 92 (Matrix of a vector)

Let V be a finite-dimensional vector space, $v \in V$ and v_1, \dots, v_n a basis of V . The **matrix** of v with respect to the basis v_1, \dots, v_n is the $n \times 1$ matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \dots, c_n \in \mathbb{F}$ are s.t.

$$v = c_1 v_1 + \dots + c_n v_n$$

Theorem 93

Let V, W be finite-dimensional vector spaces, v_1, \dots, v_n a basis of V , w_1, \dots, w_m a basis of W and $T \in \mathcal{L}(V, W)$. For $k \in \{1, \dots, n\}$, $M(T)_{\bullet k} = M(Tv_k)$

Theorem 94 (Linear maps act like matrix multiplication)

Let V, W be finite-dimensional vector spaces, v_1, \dots, v_n a basis of V , w_1, \dots, w_m a basis of W , $T \in \mathcal{L}(V, W)$ and $v \in V$. Then

$$M(Tv) = M(T)M(v)$$

Operator/Endomorphism

Definition 95 (Operator/Endomorphism)

Let V be a vector space. A linear map $\mathcal{L}(V, V)$ is an **operator** (or an **endomorphism**). $\mathcal{L}(V) = \mathcal{L}(V, V)$ denotes the set of all operators on V

Theorem 96 (Injectivity equiv. to surjectivity in finite-dim.)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. TFAE:

1. T invertible
2. T injective
3. T surjective

Rank of an operator/endomorphism

Proposition 97 (Rank)

Let $T \in \mathcal{L}(V)$ with V finite-dimensional. Then there exists bases $\mathcal{B}_U = \{u_1, \dots, u_n\}$ and $\mathcal{B}_V = \{v_1, \dots, v_n\}$ for V such that the matrix M_T of T can be written as the block matrix

$$M_T = \begin{pmatrix} \text{diag}(1, \dots, 1) & \mathbf{0}_{k, n-k} \\ \mathbf{0}_{n-k, k} & \mathbf{0}_{n-k, n-k} \end{pmatrix}$$

for some $k \in \mathbb{N}$ called the **rank** of T , with $k = \text{rank}(T) = \dim(\text{range } T)$.

Definition 98 (Row and column rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$ be a matrix

- ▶ The **row rank** of A is the dimension of the span of the rows of A in $\mathcal{M}_{1n}(\mathbb{F})$
- ▶ The **column rank** of A is the dimension of the span of the columns of A in $\mathcal{M}_{m1}(\mathbb{F})$

Row and column ranks are the dimensions of the row and column spaces of
Definition 102.

Theorem 99 ($\dim \text{range } T$ equals column rank of $M(T)$)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then $\dim \text{range } T$ equals the column rank of $M(T)$

Theorem 100 (Row rank equals column rank)

Let $A \in \mathcal{M}_{mn}$. Then the row rank of A equals the column rank of A

Definition 101 (Rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$. The **rank** of A is the column (or row, by Theorem 100) rank of A

Row space and column space of a matrix

Definition 102 (Row and column spaces)

Let $A \in \mathcal{M}_{mn}$. The subspaces of \mathbb{R}^n and \mathbb{R}^m spanned by the row and column vectors of A are the **row space** and **column space** of A , respectively.

Definition 103 (Null space/kernel)

Let $A \in \mathcal{M}_{mn}$. The null space (or kernel) of A is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

This makes explicit the already seen definition in the special case of a matrix. As previously seen, the null space is a subspace of \mathbb{R}^n .

Definition 104 (Nullity)

The dimension of the null space of $A \in \mathcal{M}_{mn}$ is called the **nullity** of A .

Theorem 105

Let $A \in \mathcal{M}_{mn}$. Then

1. $\text{rank}(A) = \text{rank}(A^T)$
2. $\text{rank}(A) + \text{nullity}(A) = n$
3. $\text{rank}(A) \leq \min(m, n)$

Theorem 106 (Consistency)

Consider the linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathcal{M}_{mn}$. TFAE:

- ▶ $A\mathbf{x} = \mathbf{b}$ is consistent
- ▶ $\mathbf{b} \in \text{column space of } A$
- ▶ A and $[A|\mathbf{b}]$ have the same rank

Proposition 107

Let $A \in \mathcal{M}_{mn}$ be in row-echelon form. Then

- ▶ *The row vectors ($\in \mathbb{R}^n$) with leading ones form a basis for the row space of A .*
- ▶ *The column vectors ($\in \mathbb{R}^m$) with leading ones form a basis for the column space of A .*

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Definition 108 (Product of vector spaces)

Let V_1, \dots, V_m be vector spaces over \mathbb{F} . The **product** $V_1 \times \dots \times V_m$ is

$$V_1 \times \dots \times V_m = \{(\mathbf{v}_1, \dots, \mathbf{v}_m); \mathbf{v}_1 \in V_1, \dots, \mathbf{v}_m \in V_m\}$$

Theorem 109 (Products of vector spaces are vector spaces)

Let V_1, \dots, V_m be vector spaces over \mathbb{F} . Define

► addition on $V_1 \times \dots \times V_m$ by

$$(\mathbf{u}_1, \dots, \mathbf{u}_m) + (\mathbf{v}_1, \dots, \mathbf{v}_m) = (\mathbf{u}_1 + \mathbf{v}_1, \dots, \mathbf{u}_m + \mathbf{v}_m)$$

► scalar multiplication on $V_1 \times \dots \times V_m$ by

$$\lambda(\mathbf{v}_1, \dots, \mathbf{v}_m) = (\lambda\mathbf{v}_1, \dots, \lambda\mathbf{v}_m)$$

With these operations, $V_1 \times \dots \times V_m$ is a vector space over \mathbb{F}

Theorem 110 (Dimension of product space)

Let V_1, \dots, V_m be finite-dimensional vector spaces. Then

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m < \infty$$

Theorem 111 (Product spaces and direct sums)

Let $U_1, \dots, U_m \subset V$ be subspaces of V . Let

$$\begin{aligned}\Gamma : U_1 \times \cdots \times U_m &\rightarrow U_1 + \cdots + U_m \\ (\mathbf{u}_1, \dots, \mathbf{u}_m) &\mapsto \mathbf{u}_1 + \cdots + \mathbf{u}_m\end{aligned}$$

Then

$$U_1 + \cdots + U_m \text{ direct sum} \Leftrightarrow \Gamma \text{ injective}$$

Theorem 112 (NSC for direct sum)

Let V be a finite-dimensional vector space, U_1, \dots, U_m subspaces of V . Then

$$U_1 \oplus \cdots \oplus U_m \Leftrightarrow \dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

Definition 113 ($\mathbf{v} + U$)

Let V be a vector space, U a subspace of V and $\mathbf{v} \in V$. Then $\mathbf{v} + U$ is the subset of V defined by

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u}; \mathbf{u} \in U\}$$

Definition 114 (Affine subset/Parallel affine subset)

Let V be a vector space

- ▶ An **affine subset** of V is a subset of V of the form $\mathbf{v} + U$ for some $\mathbf{v} \in V$ and some subspace U of V
- ▶ For $\mathbf{v} \in V$ and U subspace of V , the affine subset $\mathbf{v} + U$ is **parallel** to U

Definition 115 (Quotient space)

Let V be a vector space, U a subspace of V . The **quotient space** V/U is the set of all affine subsets of V parallel to U , i.e.,

$$V/U = \{\mathbf{v} + U; \mathbf{v} \in V\}$$

Theorem 116 (2 affine subsets // to U are equal or disjoint)

Let V be a vector space, U subspace of V and $v, w \in V$. TFAE

1. $\mathbf{v} - \mathbf{w} \in U$
2. $\mathbf{v} + U = \mathbf{w} + U$
3. $(\mathbf{v} + U) \cap (\mathbf{w} + U) \neq \emptyset$

Definition 117 (Addition and scalar multiplication on V/U)

Let V be a vector space, U subspace of V . Then **addition** and **scalar multiplication** on V/U are defined for $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$ by

$$(\mathbf{v} + U) + (\mathbf{w} + U) = (\mathbf{v} + \mathbf{w}) + U$$

and

$$\lambda(\mathbf{v} + U) = (\lambda\mathbf{v}) + U$$

Theorem 118 (Quotient space is a vector space)

Let V be a vector space and U subspace of V . Equipped with addition and scalar multiplication as above, V/U is a vector space

Definition 119 (Quotient map)

Let V be a vector space, U subspace of V . The **quotient map** π is the linear map $\pi \in \mathcal{L}(V, V/U)$ defined by

$$\pi(\mathbf{v}) = \mathbf{v} + U$$

for $\mathbf{v} \in V$

Theorem 120 (Dimension of quotient space)

Let V be a finite-dimensional vector space and U subspace of V . Then

$$\dim V/U = \dim V - \dim U$$

Definition 121 (\tilde{T})

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Define \tilde{T} by

$$\begin{aligned}\tilde{T} : \quad V/(\text{null } T) &\rightarrow W \\ \tilde{T}(\mathbf{v} + \text{null } T) &= T\mathbf{v}\end{aligned}$$

Theorem 122 (Null space and range of \tilde{T})

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\tilde{T} \in \mathcal{L}(V/\text{null } T, W)$
2. \tilde{T} injective
3. $\text{range } \tilde{T} = \text{range } T$
4. $V/\text{null } T$ isomorphic to $\text{range } T$

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Definition 123 (Linear functional/form)

A **linear functional** (or **linear form**) on a vector space V is a linear map in $\mathcal{L}(V, \mathbb{F})$

Definition 124 (Dual space)

The **dual space** V^* of V is the vector space $V^* = \mathcal{L}(V, \mathbb{F})$ of linear functionals on V

Theorem 125 ($\dim V^* = \dim V$)

Suppose V is a finite-dimensional vector space. Then $\dim V^ = \dim V < \infty$*

Definition 126 (Dual basis)

If $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis of the vector space V , then the **dual basis** of $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V^* , where for $j = 1, \dots, n$, φ_j is the linear functional on V s.t.

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Theorem 127 (Dual basis is a basis of the dual space)

*Suppose V is a finite-dimensional vector space. Then the dual basis of a basis of V is a basis of V^**

Definition 128 (Dual map)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. The **dual map** of T is the linear map $T^* \in \mathcal{L}(W^*, V^*)$ defined by $T^*(\varphi) = \varphi \circ T$ for $\varphi \in W^*$

Property 129 (Algebraic properties of dual maps)

Let U, V, W be vector spaces

- ▶ $(S + T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(V, W)$
- ▶ $(\lambda T)^* = \lambda T^*$ for all $\lambda \in \mathbb{F}$ and all $T \in \mathcal{L}(V, W)$
- ▶ $(ST)^* = T^*S^*$ for all $T \in \mathcal{L}(U, V)$ and all $S \in \mathcal{L}(V, W)$

Definition 130 (Annihilator)

Let V be a vector space, $U \subseteq V$. The **annihilator** U^0 of U is defined by

$$U^0 = \{\varphi \in V^* : \forall \mathbf{u} \in U, \quad \varphi(\mathbf{u}) = 0_{\mathbb{F}}\}$$

Theorem 131 (The annihilator is a subspace)

*Let V be a vector space and $U \subseteq V$. Then the annihilator U^0 is a subspace of V^**

Theorem 132 (Dimension of the annihilator)

Let V be a finite-dimensional vector space, $U \subseteq V$ a subspace of V . Then

$$\dim U + \dim U^0 = \dim V$$

Theorem 133 (Null space of T^*)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\text{null } T^* = (\text{range } T)^0$
2. $\dim \text{null } T^* = \dim \text{null } T + \dim W - \dim V$

Theorem 134 (T surjective $\Leftrightarrow T^*$ injective)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \text{ surjective} \Leftrightarrow T^* \text{ injective}$$

Theorem 135 (Range of T^*)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

1. $\dim \operatorname{range} T^* = \dim \operatorname{range} T$
2. $\operatorname{range} T^* = (\operatorname{null} T)^0$

Theorem 136 (T injective $\Leftrightarrow T^*$ surjective)

Let V, W be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then

$$T \text{ injective} \Leftrightarrow T^* \text{ surjective}$$

Theorem 137 (Matrix of T^* is transpose of matrix of T)

Let V, W be vector spaces, $T \in \mathcal{L}(V, W)$. Then $M(T^) = M(T)^T$, where T denotes the transpose*

Eigenvalues, eigenvectors and invariant subspaces

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Definition 138 (Invariant subspace)

Let V be a vector space, $T \in \mathcal{L}(V)$. A subspace U of V is **invariant** under T if

$$\mathbf{u} \in U \Rightarrow T\mathbf{u} \in U$$

In other words, U invariant under T if $T|_U \in \mathcal{L}(U)$ [see Definition 144]

Definition 139 (Eigenvalue)

Let V be a vector space, $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of T if

$$\exists \mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V, \text{ s.t. } T(\mathbf{v}) = \lambda \mathbf{v}.$$

I use the notation $T(\mathbf{v})$ instead of $T\mathbf{v}$ to emphasise that $T \in \mathcal{L}(V)$.

Theorem 140 (Conditions equivalent to being an eigenvalue)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Denote $I_{\mathcal{L}(V)}$ the identity operator, $I_{\mathcal{L}(V)} \in \mathcal{L}(V)$ s.t. $\forall \mathbf{v} \in V, I_{\mathcal{L}(V)}\mathbf{v} = \mathbf{v}$. TFAE:

1. λ eigenvalue of T
2. $T - \lambda I_{\mathcal{L}(V)}$ not injective
3. $T - \lambda I_{\mathcal{L}(V)}$ not surjective
4. $T - \lambda I_{\mathcal{L}(V)}$ not invertible

Definition 141 (Eigenvector)

Let V be a vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ be an eigenvalue of T . A vector $\mathbf{v} \in V$ is an **eigenvector** of T corresponding to λ if $\mathbf{v} \neq 0$ and $T(\mathbf{v}) = \lambda\mathbf{v}$

Theorem 142 (Linearly independent eigenvectors)

Let V be a vector space, $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_m$. Then $\mathbf{v}_1, \dots, \mathbf{v}_m$ linearly independent

Theorem 143 (Number of eigenvalues)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Then T has at most $\dim V$ distinct eigenvalues

Definition 144 (Restriction and quotient operators)

Let V be a vector space, $T \in \mathcal{L}(V)$ and U a subspace of V invariant under T (Def. 138)

- ▶ The **restriction operator** $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U = T\mathbf{u}, \quad \mathbf{u} \in U$$

- ▶ The **quotient operator** $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(\mathbf{v} + U) = T\mathbf{v} + U, \quad \mathbf{v} \in V$$

For the quotient space $\mathcal{L}(V/U)$, see Definition 138 and the results that follow

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Eigenspaces and diagonal matrices

Definition 145

Let V be a vector space, $T \in \mathcal{L}(V)$, $m \in \mathbb{N} \setminus \{0\}$

► $T^m = \underbrace{T \cdots T}_{m \text{ times}}$

► $T^0 = I$, the identity operator on V

► If T invertible with inverse T^{-1} , then $T^{-m} = (T^{-1})^m$

Definition 146

Let V be a vector space, $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ be the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad z \in \mathbb{F}$$

Then $p(T)$ is the operator on $\mathcal{L}(V)$ defined by

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

where I is the identity operator

Definition 147 (Product of polynomials)

Let $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial

$$(pq)(z) = p(z)q(z), \quad z \in \mathbb{F}$$

Theorem 148 (Multiplicative properties)

Let $p, q \in \mathcal{P}(\mathbb{F})$, V a vector space and $T \in \mathcal{L}(V)$. Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

Theorem 149 (Operators on complex v.s. have an eigenvalue)

Let V be a vector space over \mathbb{C} with $\dim V = n < \infty$. Assume $T \in \mathcal{L}(V)$. Then V has an eigenvalue

Definition 150 (Matrix of an operator)

Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space, let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V . The **matrix** of T with respect to the basis is the $n \times n$ matrix

$$M(T) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

with entries a_{jk} defined by

$$T\mathbf{v}_k = a_{1k}\mathbf{v}_1 + \cdots + a_{nk}\mathbf{v}_n$$

If basis is not clear from the context, write $M(T, (\mathbf{v}_1, \dots, \mathbf{v}_n))$

Definition 151 (Diagonal of a matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The **diagonal** of A consists of the entries a_{ii} , $i = 1, \dots, n$

Definition 152 (Upper-triangular matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The matrix A is **upper-triangular** if all entries below the diagonal are 0, i.e.,

$$a_{ij} = 0, \quad \forall i, j \text{ such that } i > j$$

Theorem 153 (Conditions for an upper-triangular matrix)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ a basis of V .
TFAE:

1. $M(T)$ with respect to $\mathbf{v}_1, \dots, \mathbf{v}_n$ is upper-triangular
2. $T\mathbf{v}_j \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$, $\forall j = 1, \dots, n$
3. $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j)$ invariant under T , $\forall j = 1, \dots, n$

Theorem 154 (Every operator over \mathbb{C} has an UT matrix)

Let V be a finite-dimensional vector space over \mathbb{C} , $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V

Theorem 155 (Determination of invertibility from UT matrix)

Let V be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V . Then

$$T \text{ invertible} \Leftrightarrow \forall i = 1, \dots, n, \quad a_{ii} \neq 0$$

Theorem 156 (Determination of eigenvalues from UT matrix)

Let V be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V . Then

$$\lambda \text{ eigenvalue of } T \Leftrightarrow \lambda \in \{a_{ii}, \quad i = 1, \dots, n\}$$

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Definition 157 (Diagonal matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. A is a **diagonal** matrix if all entries of A are zero except possibly on the diagonal, i.e.,

$$\forall i, j, \quad i \neq j, \quad a_{ij} = 0.$$

Definition 158 (Eigenspace)

Let V be a vector space, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$. The **eigenspace** $E(\lambda, T)$ of T corresponding to λ is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I).$$

Thus λ eigenvalue of $T \Leftrightarrow E(\lambda, T) \neq \{\mathbf{0}_V\}$.

Theorem 159 (Sum of eigenspaces is a direct sum)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Assume $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T . Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum and

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

Definition 160 (Diagonalisable operator)

Let V be a vector space, $T \in \mathcal{L}(V)$. T is **diagonalisable** if T has a diagonal matrix with respect to some basis of V .

Theorem 161 (Conditions equivalent to diagonalisability)

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T . TFAE:

1. T diagonalisable
2. V has a basis consisting of eigenvectors of T
3. $\exists U_1, \dots, U_n$ 1-dimensional subspaces of V invariant under T s.t.

$$V = U_1 \oplus \dots \oplus U_n$$

4. $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$
5. $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$

Theorem 162 (Sufficient condition for diagonalisability)

Let V be a vector space, $T \in \mathcal{L}(V)$. If T has $\dim V$ distinct eigenvalues, then T is diagonalisable

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Definition 163 (Inner product)

Let V be a vector space over \mathbb{F} . An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ having the following properties, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \lambda \in \mathbb{F}$,

- ▶ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ [positivity]
- ▶ $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$ [definiteness]
- ▶ $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ [additivity in first slot]
- ▶ $\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$ [homogeneity in first slot]
- ▶ $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ [conjugate symmetry]

Definition 164 (Inner product space)

An **inner product space** is a vector space V along with an inner product on V

Theorem 165 (Basic properties of inner product)

Let V be an inner product space over \mathbb{F} . Then

1. *For each fixed $\mathbf{u} \in V$, the function $\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{u} \rangle$ is a linear map from V to \mathbb{F}*
2. $\forall \mathbf{u} \in V, \langle \mathbf{0}_V, \mathbf{u} \rangle = 0$
3. $\forall \mathbf{u} \in V, \langle \mathbf{u}, \mathbf{0}_V \rangle = 0$
4. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
5. $\forall \mathbf{u}, \mathbf{v} \in V$ and $\forall \lambda \in \mathbb{F}, \langle \mathbf{u}, \lambda \mathbf{v} \rangle = \bar{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$

Definition 166 (Norm)

Let V be an inner product space over \mathbb{F} . For $\mathbf{v} \in V$, the **norm** of \mathbf{v} is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Theorem 167 (Basic properties of the norm)

Let V be an inner product space, $\mathbf{v} \in V$. Then

1. $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = 0$
2. $\forall \lambda \in \mathbb{F}, \|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|$

Definition 168 (Orthogonality)

Let V be an inner product space over \mathbb{F} . Two vectors $\mathbf{u}, \mathbf{v} \in V$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. We sometimes denote $\mathbf{u} \perp \mathbf{v}$

Theorem 169 ($\mathbf{0}$ and orthogonality)

Let V be an inner product space over \mathbb{F} . Then

1. $\mathbf{0}_V$ is orthogonal to every vector in V
2. $\mathbf{0}_V$ is the only vector in V that is orthogonal to itself

Theorem 170 (Pythagorean theorem)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ s.t. $\mathbf{u} \perp \mathbf{v}$. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Theorem 171 (An orthogonal decomposition)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq 0$. Let

$$c = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \ (\in \mathbb{F}) \text{ and } \mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \ (\in V).$$

Then

$$\langle \mathbf{w}, \mathbf{v} \rangle = 0 \text{ and } \mathbf{u} = c\mathbf{v} + \mathbf{w}.$$

Theorem 172 (Cauchy-Schwarz inequality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

with $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ for some $0 \neq k \in \mathbb{F}$.

Theorem 173 (Triangle inequality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

with $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ for some $0 \leq k \in \mathbb{R}$.

Theorem 174 (Parallelogram equality)

Let V be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2).$$

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Definition 175 (Orthonormal list)

A list of vectors is **orthonormal** if each vector in the list has norm 1 and is orthogonal to all other vectors in the list, i.e., the list $\mathbf{e}_1, \dots, \mathbf{e}_m$ of vectors in the inner product space V is orthonormal if

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Theorem 176 (Norm of an orthonormal linear combination)

Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space V . Then

$$\|a_1\mathbf{e}_1 + \dots + a_m\mathbf{e}_m\|^2 = |a_1|^2 + \dots + |a_m|^2$$

for all $a_1, \dots, a_m \in \mathbb{F}$.

Theorem 177 (Orthonormal lists are LI)

Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space V . Then $\mathbf{e}_1, \dots, \mathbf{e}_m$ is linearly independent

Definition 178 (Orthonormal basis)

An **orthonormal basis** of an inner product space V is an orthonormal list of vectors in V that is also a basis of V

Theorem 179 (Orthonormal list & orthonormal basis)

Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space V . If $\dim V = m$, then $\mathbf{e}_1, \dots, \mathbf{e}_m$ orthonormal basis of V .

Theorem 180 (Vector as LC of orthonormal basis)

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an orthonormal basis of the inner product space V , $\mathbf{v} \in V$. Then

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n$$

and

$$\|\mathbf{v}\|^2 = |\langle \mathbf{v}, \mathbf{e}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{e}_n \rangle|^2$$

Theorem 181 (Gram-Schmidt procedure)

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be a linearly independent list of vectors in an inner product space V . Let

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

For $j = 2, \dots, m$, define \mathbf{e}_j inductively by

$$\mathbf{e}_j = \frac{\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \dots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}}{\|\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \dots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}\|}$$

Then $\mathbf{e}_1, \dots, \mathbf{e}_m$ is an orthonormal list of vectors in V such that

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_j), \quad j = 1, \dots, m$$

Theorem 182 (Existence of orthonormal basis)

Let V be a finite-dimensional inner product space. Then V has an orthonormal basis

Theorem 183 (Extending orthonormal list to basis)

Let V be a finite-dimensional inner product space. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V

Theorem 184 (UT matrix wrt orthonormal basis)

Let V be a finite-dimensional inner product space, $T \in \mathcal{L}(V)$. If T has an upper-triangular matrix with respect to some basis of V , then T has an upper-triangular matrix with respect to some orthonormal basis of V

Theorem 185 (Schur's Theorem)

Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V

Theorem 186 (Riesz representation Theorem)

Let V be a finite-dimensional inner product space, $\varphi \in \mathcal{L}(V, \mathbb{F})$ a linear functional on V . Then $\exists \mathbf{u} \in V$ unique s.t.

$$\forall \mathbf{v} \in V, \quad \varphi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle.$$

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Definition 187 (Orthogonal complement)

Let V be an inner product space, $U \subset V$. The **orthogonal complement** U^\perp of U is the set

$$U^\perp = \{\mathbf{v} \in V : \forall \mathbf{u} \in U, \quad \langle \mathbf{v}, \mathbf{u} \rangle = 0\}$$

Property 188 (Basic properties of orthogonal complement)

1. If $U \subset V$, then U^\perp subspace of V
2. $\{\mathbf{0}_V\}^\perp = V$
3. $V^\perp = \{\mathbf{0}_V\}$
4. If $U \subset V$, then $U \cap U^\perp \subset \{0\}$
5. If $U \subset W \subset V$, then $W^\perp \subset U^\perp$

Theorem 189 (Direct sum U and U^\perp)

Let U be a finite-dimensional subspace of V , inner product space. Then

$$V = U \oplus U^\perp$$

Theorem 190 (Dimension of U^\perp)

Let V be a finite-dimensional inner product space, U subspace of V . Then

$$\dim U^\perp = \dim V - \dim U$$

Theorem 191 (Orth. complement of orth. complement)

Let U be a finite-dimensional subspace of the inner product space V . Then

$$(U^\perp)^\perp = U$$

Definition 192 (Orthogonal projection P_U)

Let V be an inner product space, U a finite-dimensional subspace of V . The **orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined by

$$P_U \mathbf{v} = \mathbf{u},$$

where $\mathbf{v} \in V$ is written $\mathbf{v} = \mathbf{u} + \mathbf{w}$, with $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$

Property 193 (Properties of the orthogonal projection P_U)

Let V be an inner product space, U a finite-dimensional subspace of V , $v \in V$. Then

1. $P_U \in \mathcal{L}(V)$
2. $\forall \mathbf{u} \in U, P_U \mathbf{u} = \mathbf{u}$
3. $\forall \mathbf{w} \in U^\perp, P_U \mathbf{w} = \mathbf{0}_V$
4. $\text{range } P_U = U$
5. $\text{null } P_U = U^\perp$
6. $\mathbf{v} - P_U \mathbf{v} \in U^\perp$
7. $P_U^2 = P_U$
8. $\|P_U \mathbf{v}\| \leq \|\mathbf{v}\|$
9. for every orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_m$ of U ,

$$P_U \mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \dots + \langle \mathbf{v}, \mathbf{e}_m \rangle \mathbf{e}_m.$$

Theorem 194 (Minimising distance to a subspace)

Let V be an inner product space, U a finite-dimensional subspace of V , $\mathbf{v} \in V$, $\mathbf{u} \in U$. Then

$$\|\mathbf{v} - P_U \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{u}\|$$

with equality if and only if $\mathbf{u} = P_U \mathbf{v}$

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Definition 195 (Adjoint)

Let V, W be finite-dimensional inner product spaces over \mathbb{F} , $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \rightarrow V$ such that

$$\forall \mathbf{v} \in V, \forall \mathbf{w} \in W, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^*\mathbf{w} \rangle$$

Theorem 196 (Adjoint is a linear map)

Let V, W be finite-dimensional inner product spaces over \mathbb{F} , $T \in \mathcal{L}(V, W)$. Then

$$T^* \in \mathcal{L}(W, V)$$

Property 197 (Properties of the adjoint)

Let V, W be finite-dimensional inner product spaces over \mathbb{F} . Then

1. $\forall S, T \in \mathcal{L}(V, W), (S + T)^* = S^* + T^*$
2. $\forall T \in \mathcal{L}(V, W), \forall \lambda \in \mathbb{F}, (\lambda T)^* = \overline{\lambda} T^*$
3. $\forall T \in \mathcal{L}(V, W), (T^*)^* = T$
4. $I^* = I$ if I is the identity operator on V
5. *Let U be an inner product space over \mathbb{F} , then $\forall T \in \mathcal{L}(V, W)$ and $\forall S \in \mathcal{L}(W, U)$,*
 $(ST)^* = T^* S^*$

Theorem 198 (Null space and range of T^*)

Let V, W be finite-dimensional inner product spaces over \mathbb{F} , $T \in \mathcal{L}(V, W)$. Then

1. $\text{null } T^* = (\text{range } T)^\perp$
2. $\text{range } T^* = (\text{null } T)^\perp$
3. $\text{null } T = (\text{range } T^*)^\perp$
4. $\text{range } T = (\text{null } T^*)^\perp$

Definition 199 (Conjugate transpose)

Let $M \in \mathcal{M}_{mn}(\mathbb{F})$, $M = [m_{ij}]$. The **conjugate transpose** of M , often denoted M^* , is the matrix

$$M^* = [\overline{m_{ji}}] \in \mathcal{M}_{nm}$$

i.e, the matrix obtained by transposing M then taking the (complex) conjugate of each entry

Theorem 200 (Matrix of T^*)

Let V, W be finite-dimensional inner product spaces over \mathbb{F} , $T \in \mathcal{L}(V, W)$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{f}_1, \dots, \mathbf{f}_m$ be orthonormal bases of V and W , respectively. Then

$$M(T^*, (\mathbf{f}_1, \dots, \mathbf{f}_m), (\mathbf{e}_1, \dots, \mathbf{e}_n))$$

is the conjugate transpose of

$$M(T, (\mathbf{e}_1, \dots, \mathbf{e}_n), (\mathbf{f}_1, \dots, \mathbf{f}_m))$$

Definition 201 (Self-adjoint operator)

Let V be an inner product space over \mathbb{F} , $T \in \mathcal{L}(V)$. T is **self-adjoint** (or **Hermitian**) if

$$T = T^*$$

In other words, $T \in \mathcal{L}(V)$ self-adjoint \iff

$$\forall \mathbf{v}, \mathbf{w} \in V, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T\mathbf{w} \rangle$$

Theorem 202 (Eigenvalues of self-adjoint operators are real)

Let V be an inner product space over \mathbb{F} , $T \in \mathcal{L}(V)$. Then all eigenvalues of T are real

Theorem 203

Let V be a complex inner product space, $T \in \mathcal{L}(V)$. Then

$$(\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = \mathbf{0}$$

Theorem 204

Let V be a complex inner product space, $T \in \mathcal{L}(V)$. Then

$$(T \text{ self-adjoint}) \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle \in \mathbb{R})$$

Theorem 205

Let V be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint. Then

$$(\forall \mathbf{v} \in V, \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = 0$$

Definition 206 (Normal operator)

Let V be an inner product space, $T \in \mathcal{L}(V)$. T is **normal** if

$$TT^* = T^*T$$

In words, T is normal if it commutes with its adjoint

Theorem 207 (T normal $\Leftrightarrow \|T\mathbf{v}\| = \|T^*\mathbf{v}\|$)

Let V be an inner product space, $T \in \mathcal{L}(V)$. Then

$$T \text{ normal} \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V, \|T\mathbf{v}\| = \|T^*\mathbf{v}\|)$$

Theorem 208 (T normal and T^* have same eigenvectors)

Let V be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. Then

$$(\lambda, \mathbf{v}) \text{ eigenpair of } T \quad \Leftrightarrow \quad (\bar{\lambda}, \mathbf{v}) \text{ eigenpair of } T^*$$

Theorem 209 (Orthogonal eigenvectors for normal operators)

Let V be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. If $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ eigenpairs of T with $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1 \perp \mathbf{v}_2$.

Theorem 210

Let V be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint and $b, c \in \mathbb{R}$ s.t. $b^2 < 4c$.
Then $T^2 + bT + cI$ invertible

Theorem 211 (Self-adjoint operators have eigenvalues)

Let $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ be self-adjoint. Then T has an eigenvalue

Theorem 212 (Self-adjoint operators & invariant subspaces)

Let V be an inner product space, $T \in \mathcal{L}(V)$ be self-adjoint and U be a subspace of V invariant under T . Then

1. U^\perp invariant under T
2. $T|_U \in \mathcal{L}(U)$ self-adjoint
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ self-adjoint

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Theorem 213 (Complex spectral theorem)

Let V be an inner product space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. TFAE:

1. T normal
2. V has an orthonormal basis consisting of eigenvectors of T
3. T has a diagonal matrix with respect to some orthonormal basis of V

Theorem 214 (Real spectral theorem)

Let V be an inner product space over $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$. TFAE:

1. T self-adjoint
2. V has an orthonormal basis consisting of eigenvectors of T
3. T has a diagonal matrix with respect to some orthonormal basis of V

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Definition 215 (Positive (semidefinite) operator)

Let V be an inner product space. An operator $T \in \mathcal{L}(V)$ is **positive** (or **positive semidefinite**) if T is self-adjoint and

$$\forall \mathbf{v} \in V, \quad \langle T\mathbf{v}, \mathbf{v} \rangle \geq 0$$

Definition 216 (Square root operator)

Let V be an inner product space. An operator $R \in \mathcal{L}(V)$ is a **square root** of an operator $T \in \mathcal{L}(V)$ if

$$R^2 = T$$

Theorem 217 (Characterisation of positive operators)

Let $T \in \mathcal{L}(V)$, where V is an inner product space. TFAE:

1. T positive semidefinite
2. T self-adjoint and all eigenvalues of T are nonnegative
3. T has a positive semidefinite square root
4. T has a self-adjoint square root
5. $\exists R \in \mathcal{L}(V)$ s.t. $T = R^*R$

Theorem 218 (Uniqueness of positive semidefinite square root)

Let $T \in \mathcal{L}(V)$ be a positive semidefinite operator on an inner product space V . Then T has a unique positive semidefinite square root

Definition 219 (Isometry)

Let V be an inner product space. $S \in \mathcal{L}(V)$ is an **isometry** if

$$\forall \mathbf{v} \in V, \quad \|S\mathbf{v}\| = \|\mathbf{v}\|$$

Theorem 220 (Characterisation of isometries)

Let V be an inner product space, $S \in \mathcal{L}(V)$. TFAE:

1. S isometry
2. $\forall \mathbf{u}, \mathbf{v} \in V, \langle S\mathbf{u}, S\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$
3. $\forall \mathbf{e}_1, \dots, \mathbf{e}_n \in V$ orthonormal list, $S\mathbf{e}_1, \dots, S\mathbf{e}_n$ orthonormal
4. $\exists \mathbf{e}_1, \dots, \mathbf{e}_n$ orthonormal basis of V s.t. $S\mathbf{e}_1, \dots, S\mathbf{e}_n$ orthonormal
5. $S^*S = I$
6. $SS^* = I$
7. S^* isometry
8. S invertible and $S^{-1} = S^*$

Theorem 221 (Isometries when $\mathbb{F} = \mathbb{C}$)

Let V be a complex inner product space, $S \in \mathcal{L}(V)$. TFAE:

- 1. S isometry*
- 2. \exists orthonormal basis of V consisting of eigenvectors of S with corresponding eigenvalues all having modulus 1*

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Let T be a positive semidefinite operator, then denote \sqrt{T} the unique positive semidefinite square root of T

Theorem 222 (Polar decomposition)

Let V be an inner product space, $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ s.t.

$$T = S\sqrt{T^*T}$$

Definition 223 (Singular values)

Let V be an inner product space, $T \in \mathcal{L}(V)$. The **singular values** of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ repeated $\dim E(\lambda, \sqrt{T^*T})$ times. All are nonnegative

Theorem 224 (Singular value decomposition – SVD)

Let V be an inner product space. Assume $T \in \mathcal{L}(V)$ has singular values s_1, \dots, s_n . Then $\exists \mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{f}_1, \dots, \mathbf{f}_n$ orthonormal bases of V s.t.

$$\forall \mathbf{v} \in V, \quad T\mathbf{v} = s_1 \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{f}_1 + \dots + s_n \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{f}_n$$

Theorem 225 (SV without square root)

*Let V be an inner product space. The singular values of T are the nonnegative square roots of the eigenvalues of T^*T , with each eigenvalue λ repeated $\dim E(\lambda, T^*T)$ times*

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Theorem 226 (Sequence of increasing null spaces)

Let V be a finite-dimensional vector space over \mathbb{F} , $T \in \mathcal{L}(V)$. Then

$$\{0\} = \text{null } T^0 \subset \text{null } T^1 \subset \cdots \subset \text{null } T^k \subset \text{null } T^{k+1} \subset \cdots$$

Theorem 227 (Equality in sequence of null spaces)

Let V be a finite-dimensional vector space over \mathbb{F} , $T \in \mathcal{L}(V)$. Assume $m \in \mathbb{N} \setminus \{0\}$ is s.t.

$$\text{null } T^m = \text{null } T^{m+1}$$

Then

$$\forall k \in \mathbb{N}, \quad \text{null } T^{m+k} = \text{null } T^m$$

Theorem 228 (Null spaces stop growing)

Let V be a finite-dimensional vector space over \mathbb{F} with $\dim V = n$, $T \in \mathcal{L}(V)$. Then

$$\forall k \in \mathbb{N}, \quad \text{null } T^{n+k} = \text{null } T^n$$

Theorem 229 ($V = \text{null } T^{\dim V} \oplus \text{range } T^{\dim V}$)

Let V be a finite-dimensional vector space over \mathbb{F} with $\dim V = n$, $T \in \mathcal{L}(V)$. Then

$$V = \text{null } T^n \oplus \text{range } T^n$$

Definition 230 (Generalised eigenvector)

Let V be a finite-dimensional vector space over \mathbb{F} , $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ an eigenvalue of T . $\mathbf{v} \in V$ is a **generalised eigenvector** of T corresponding to λ if $\mathbf{v} \neq \mathbf{0}$ and

$$\exists j \in \mathbb{N} \setminus \{0\}, \quad (T - \lambda I)^j \mathbf{v} = \mathbf{0}_V$$

Definition 231 (Generalised eigenspace)

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalised eigenspace** $G(\lambda, T)$ of T corresponding to λ is the set of all generalised eigenvectors of T corresponding to λ together with the $\mathbf{0}_V$ vector

Theorem 232 (Description of generalised eigenspaces)

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then

$$G(\lambda, T) = \text{null}(T - \lambda I)^{\dim V}$$

Theorem 233 (LI generalised eigenvectors)

Let $T \in \mathcal{L}(V)$. Assume $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T , $\mathbf{v}_1, \dots, \mathbf{v}_m$ corresponding generalised eigenvectors. Then $\mathbf{v}_1, \dots, \mathbf{v}_m$ linearly independent.

Definition 234 (Nilpotent operator)

An operator is **nilpotent** if $\exists k \in \mathbb{N}$ s.t. $T^k = 0$

Theorem 235 (A loose upper bound on power required)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then

$$N^{\dim V} = 0$$

Theorem 236 (Matrix of a nilpotent operator)

*Let $N \in \mathcal{L}(V)$ be nilpotent. Then there exists a basis of V with respect to which $M(N)$ is **strictly upper triangular**, i.e.,*

$$M(N) = [m_{ij}] \text{ is s.t. } m_{ij} = 0 \text{ if } i \geq j$$

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Theorem 237 (& range of $p(T)$ invariant under T)

Let $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then $p(T)$ and $\text{range } p(T)$ invariant under T

Theorem 238 (Description of operators when $\mathbb{F} = \mathbb{C}$)

Suppose V complex vector space, $T \in \mathcal{L}(V)$. Assume $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Then

1. $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$
2. each $G(\lambda_j, T)$ invariant under T
3. $\forall j = 1, \dots, m, (T - \lambda_j I)|_{G(\lambda_j, T)}$ nilpotent

Theorem 239 (Basis of generalised eigenvectors)

Let V be a complex vector space and $T \in \mathcal{L}(V)$. Then there exists a basis of V consisting of generalised eigenvectors of T

Definition 240 (Multiplicity of an eigenvalue)

Let $T \in \mathcal{L}(V)$. The (**algebraic**) **multiplicity** of an eigenvalue λ of T is

- ▶ $\dim G(\lambda, T)$
- ▶ $\dim (T - \lambda I)^{\dim V}$

Theorem 241 (\sum multiplicities = $\dim V$)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues of T with multiplicities d_1, \dots, d_n . Then

$$\sum_{k=1}^n d_k = \dim V$$

Definition 242 (Block diagonal matrix)

Let A_1, \dots, A_m be square matrices (not necessarily of the same size). A **block matrix** is a matrix of the form

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

We also write

$$A = \text{diag}(A_1, \dots, A_m)$$

You will also see (not in this book)

$$A = A_1 \oplus \dots \oplus A_m$$

Theorem 243 (Block diagonal matrix with UT blocks)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T with multiplicities d_1, \dots, d_m . Then there exists a basis of V s.t. T has a block diagonal matrix

$$\text{diag}(A_1, \dots, A_m)$$

with each A_j a $d_j \times d_j$ upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

Theorem 244 (Identity plus nilpotent has square root)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then $I + N$ has a square root

Theorem 245 (T invertible has square root when $\mathbb{F} = \mathbb{C}$)

Let V be a complex vector space, $T \in \mathcal{L}(V)$ invertible. Then T has a square root

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Definition 246 (Characteristic polynomial)

Let V be a complex vector space, $T \in \mathcal{L}(V)$, $\lambda_1, \dots, \lambda_m$ the distinct eigenvalues of T with multiplicities d_1, \dots, d_m . The **characteristic polynomial** of T is

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

Theorem 247 (Degree and zeros of char. polyn.)

V a complex vector space, $T \in \mathcal{L}(V)$. Then

- 1. the characteristic polynomial of T has degree $\dim V$*
- 2. zeros of the characteristic polynomial of T are the eigenvalues of T*

Theorem 248 (Cayley-Hamilton)

Let V be a complex vector space, $T \in \mathcal{L}(V)$. Let q be the characteristic polynomial of T . Then $q(T) = 0$

Definition 249 (Monic polynomial)

A **monic polynomial** is a polynomial with highest degree coefficient equal to 1

Theorem 250 (Minimal polynomial)

Let $T \in \mathcal{L}(V)$. Then there exists a unique monic polynomial p of smallest degree s.t. $p(T) = 0$

Definition 251 (Minimal polynomial)

Let $T \in \mathcal{L}(V)$. The **minimal polynomial** of T is the unique monic polynomial p of smallest degree s.t. $p(T) = 0$

Theorem 252

Let $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0 \Leftrightarrow q$ polynomial multiple of the minimal polynomial of T

Theorem 253 (Char. polyn. is multiple of min. polyn.)

Assume V vector space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T

Theorem 254 (Eigenvalues are zeros of min. polyn.)

Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T

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Theorem 255 (Basis corresponding to nilpotent operator)

Let $N \in \mathcal{L}(V)$ be nilpotent. Then $\exists \mathbf{v}_1, \dots, \mathbf{v}_n \in V$ and $m_1, \dots, m_n \in \mathbb{N}$ s.t.

1. $N^{m_1}\mathbf{v}_1, \dots, N\mathbf{v}_1, \mathbf{v}_1, N^{m_n}\mathbf{v}_n, \dots, N\mathbf{v}_n, \mathbf{v}_n$ is a basis of V
2. $N^{m_1+1}\mathbf{v}_1 = \dots = N^{m_n+1}\mathbf{v}_n = 0$

Definition 256 (Jordan basis)

Let $T \in \mathcal{L}(V)$. A **Jordan basis** for T is a basis of V s.t. with respect to this basis, T has a block diagonal matrix

$$\text{diag}(A_1, \dots, A_p)$$

where each A_j is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Theorem 257 (Jordan form)

Let V be a complex vector space. If $T \in \mathcal{L}(V)$, then \exists a Jordan basis for T

An algorithm for finding the Jordan form

An algorithm to compute the Jordan canonical form of an $n \times n$ matrix A [MM82].

1. Compute the eigenvalues of A . Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of A with multiplicities n_1, \dots, n_m , respectively.
2. Compute n_1 linearly independent generalized eigenvectors of A associated with λ_1 as follows. Compute



$$(A - \lambda_1 E_n)^i$$

for $i = 1, 2, \dots$ until the rank of $(A - \lambda_1 E_n)^k$ is equal to the rank of $(A - \lambda_1 E_n)^{k+1}$. Find a generalized eigenvector of rank k , say u . Define $u_i = (A - \lambda_1 E_n)^{k-1}u$, for $i = 1, \dots, k$. If $k = n_1$, proceed to step 3. If $k < n_1$, find another linearly independent generalized eigenvector with rank k . If this is not possible, try $k - 1$, and so forth, until n_1 linearly independent generalized eigenvectors are determined. Note that if $\rho(A - \lambda_1 E_n) = r$, then there are totally $(n - r)$ chains of generalized eigenvectors associated with λ_1 .

3. Repeat step 2 for $\lambda_2, \dots, \lambda_m$.

1. Let u_1, \dots, u_k, \dots be the new basis. Observe that Thus in the new basis, A has the desired representation
2. The similarity transformation which yields $J = Q^{-1}AQ$ is given by $Q = [u_1, \dots, u_k, \dots]$.

References I

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-  R.K Miller and A.N. Michel, *Ordinary differential equations*, Academic Press, 1982.