

MATH 4370/7370 – Linear Algebra and Matrix Analysis

The Singular Value Decomposition

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Outline

Singular values and the Singular value decomposition

Properties of Singular Values

Matrix norms and Singular values

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Properties of Singular Values

Matrix norms and Singular values

Definition 5.1

Let A be a Hermitian matrix in \mathcal{M}_n . We say that A is **positive definite** if for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* A \mathbf{x} > 0$. We say that A is **positive semidefinite** if for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^* A \mathbf{x} \geq 0$

Theorem 5.2

Let $A \in \mathcal{M}_n$ be a Hermitian matrix. Then

1. for all $\mathbf{x} \in \mathbb{C}^*$, $\mathbf{x}^* A \mathbf{x} \in \mathbb{R}$
2. $\sigma(A) \subset \mathbb{R}$
3. $S^* A S$ is Hermitian for any $S \in \mathcal{M}_n$

Theorem 5.3

Every eigenvalue of a positive definite matrix (respectively positive semidefinite matrix) is positive (respectively nonnegative)

Proposition 5.4

Let A be a positive semidefinite (respectively positive definite) matrix. Then $\text{tr}(A)$, $\det(A)$ and the principal minors of A are all nonnegative (respectively positive). Also, $\text{tr}(A) = 0$ if and only if $A = 0$

Theorem 5.5

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and $\mathbf{x} \in \mathbb{C}^n$. Then

$$\mathbf{x}^* A \mathbf{x} = 0 \iff A \mathbf{x} = \mathbf{0}$$

Corollary 5.6

Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix. Then

$$A \text{ positive definite} \iff A \text{ nonsingular}$$

Theorem 5.7 (An eigenvalue interlacing theorem)

Let $B \in \mathcal{M}_n$ be a Hermitian matrix, $\mathbf{y} \in \mathbb{C}^n$ and $a \in \mathbb{R}$. Let

$$A = \begin{pmatrix} B & \mathbf{y} \\ \mathbf{y}^* & a \end{pmatrix} \in \mathcal{M}_{n+1}$$

Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$$

Definition 5.8 (Singular values)

The **singular values** of a matrix $A \in \mathcal{M}_{mn}$ are the (nonnegative) square roots of the eigenvalues of A^*A

Remark 5.9

Let $A \in \mathcal{M}_{mn}$. Then A , \bar{A} , A^T , and A^* have the same nonzero singular values

Remark 5.10

A^*A is positive semidefinite

Theorem 5.11 (Zhang)

Let $A \in M_{mn}$ with nonzero singular values $\sigma_1, \dots, \sigma_r$. Then there exists $U \in M_n$ and $V \in M_n$ unitary such that

$$A = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V,$$

where $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{mn}$ and $D_r = \text{diag}(\sigma_1, \dots, \sigma_r)$

Remark about SVs/SVD in H & J

H & J like to work with matrices in \mathcal{M}_n

If $A \in \mathcal{M}_{nm}$, then $A^*A \in \mathcal{M}_n$

So all of their SV/SVD results are with matrices in \mathcal{M}_{nm}

Pfff...

Theorem 5.12 (H & J)

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$. Assume that the rank of A is r . Then

1. $\exists V \in M_n$ and $W \in M_m$ unitary matrices and $\Sigma_q \in \mathcal{M}_q = \text{diag}(\sigma_1, \dots, \sigma_q)$ s.t.

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0 = \sigma_{r+1} = \dots = \sigma_q$$

and

$$A = V\Sigma W^*$$

where

$$\Sigma = \begin{cases} \Sigma_q, & m = n \\ \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{nm}, & m > n \\ \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{nm}, & n > m \end{cases}$$

2. The parameters $\sigma_1, \dots, \sigma_r$ are the positive square roots of the decreasingly ordered eigenvalues of A^*A or AA^*

Remark 5.13

Let $A \in \mathcal{M}_n$ with singular values $\sigma_1, \dots, \sigma_n$, then

$$\sigma_1 \cdots \sigma_n = \det(A)$$

and

$$\sigma_1^2 + \cdots + \sigma_n^2 = \text{tr}(A^*A)$$

Theorem 5.14

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\sigma_1 \geq \dots \geq \sigma_q$ nonincreasingly ordered singular values of A . Define the Hermitian matrix

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

Then the ordered eigenvalues of \mathcal{A} are

$$-\sigma_1 \leq \dots \leq -\sigma_q \leq \underbrace{0 = \dots = 0}_{|n-m|} \leq \sigma_q \leq \dots \leq \sigma_1$$

Theorem 5.15 (An interlacing result)

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and \hat{A} be the matrix obtained from A by deleting one row or one column of A

Let $\sigma_1 \geq \dots \geq \sigma_q$ and $\hat{\sigma}_1 \geq \dots \geq \hat{\sigma}_q$ be the nonincreasingly ordered singular values of A and \hat{A} , respectively, where $\hat{\sigma}_q = 0$ if $n \geq m$ and a column is deleted or if $n \geq m$ and a row is deleted

Then

$$\sigma_1 \geq \hat{\sigma}_1 \geq \sigma_2 \geq \hat{\sigma}_2 \geq \dots \geq \sigma_q \geq \hat{\sigma}_q$$

Theorem 5.16 (von Neumann)

Let $A, B \in \mathcal{M}_{mn}$, $q = \min\{m, n\}$, $\sigma_1(A) \geq \dots \geq \sigma_q(A)$ and $\sigma_1(B) \geq \dots \geq \sigma_q(B)$ the nonincreasingly ordered singular values of A and B , respectively. Then

$$\operatorname{Re} \operatorname{tr}(AB^*) \leq \sum_{i=1}^q \sigma_i(A)\sigma_i(B)$$

Theorem 5.17

Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\sigma_1 \geq \dots \geq \sigma_q$ nonincreasingly ordered singular values of A , and $\alpha = \{1, \dots, q\}$. Then

$$\operatorname{Re} \operatorname{tr}(A) \leq \sum_{i=1}^q \sigma_i$$

with equality if and only if $A[\alpha]$ (principal leading submatrix of A) is positive semidefinite and A has no nonzero entries outside $A[\alpha]$

Singular values and the Singular value decomposition

Properties of Singular Values

Matrix norms and Singular values

- ▶ Let $A \in \mathcal{M}_2$

$$\sigma_1, \sigma_2 = \frac{1}{2} \left(\text{tr}(A^*A) \mp \sqrt{(\text{tr}(A^*A))^2 - 4|\det A|^2} \right)$$

- ▶ The nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & & \\ & \ddots & & \\ & & a_{n-1,m} & \\ & & & 0 \end{pmatrix}$$

has singular values $0, |a_{12}|, \dots, |a_{n-1,n}|$

Theorem 5.18

Let $A_1, A_2, \dots \in \mathcal{M}_{nm}$ be a sequence with $\lim_{k \rightarrow \infty} A_k = A$ (entrywise)

Let $q = \min\{m, n\}$. Let $\sigma_1(A) \geq \dots \geq \sigma_q(A)$ and $\sigma_1(A_k) \geq \dots \geq \sigma_q(A_k)$ be the nonincreasingly ordered singular values of A and A_k , respectively

Then

$$\lim_{k \rightarrow \infty} \sigma_i(A_k) = \sigma_i(A)$$

Theorem 5.19

Let $A \in \mathcal{M}_n$ where $r = \text{rank } A$

1. $A = A^T$ if and only if there exists $U \in \mathcal{M}_n$ unitary and a nonnegative diagonal matrix Σ such that $A = U\Sigma U^T$. The diagonal entries of Σ are the singular values of A
2. If $A = -A^T$, then n is even and there exists $U \in \mathcal{M}_n$ unitary and positive real scalars $s_1, \dots, s_{r/2}$ such that

$$U \left(\begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_{r/2} \\ -s_{r/2} & 0 \end{pmatrix} \right) U^T$$

The non-zero singular values of A are $s_1, s_1, \dots, s_{r/2}, s_{r/2}$. Conversely, any matrix of the above form is skew-symmetric

Singular values and the Singular value decomposition

Properties of Singular Values

Matrix norms and Singular values

Let $V = \mathcal{M}_{mn}(\mathbb{C})$ with Frobenius inner product

$$\langle A, B \rangle_F = \text{tr}(B^*A)$$

The norm derived from the Frobenius inner product is

$$\|A\|_2 = (\text{tr}(A^*A))^{1/2}$$

is the ℓ -2 norm (or Frobenius norm)

The spectral norm $\|\cdot\|$ defined on \mathcal{M}_n by

$$\|A\|_2 = \sigma_1(A),$$

where $\sigma_1(A)$ is the largest singular value of A is induced by the ℓ_2 norm on \mathbb{C}^n .
Indeed, from the singular value decomposition theorem, let

$$A = V\Sigma W^*$$

be a singular value decomposition of A , where V, W unitary, $\Sigma = \sigma(\sigma_1, \dots, \sigma_n)$ and $\sigma_1 \geq \dots \geq \sigma_n \geq 0$ are the non-increasingly ordered singular values of A

From unitary invariance and monotonicity of the Euclidean norm, we have

$$\begin{aligned}\max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 &= \max_{\|\mathbf{x}\|_2=1} \|V\Sigma W^*\mathbf{x}\|_2 \\&= \max_{\|\mathbf{x}\|_2=1} \|\Sigma W^*\mathbf{x}\|_2 \\&= \max_{\|W\mathbf{y}\|_2=1} \|\Sigma\mathbf{y}\|_2 \\&= \max_{\|\mathbf{y}\|_2=1} \|\Sigma\mathbf{y}\|_2 \\&\leq \max_{\|\mathbf{y}\|_2=1} \|\sigma_1 \mathbf{y}\|_2 \\&= \sigma_1 \max_{\|\mathbf{y}\|_2=1} \|\mathbf{y}\|_2 \\&= \sigma_1\end{aligned}$$

Since $\|\Sigma\mathbf{y}\|_2 = \sigma_1$ for $\mathbf{y} = \mathbf{e}_1$,

$$\max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 = \sigma_1(A)$$

We could have used

$$\begin{aligned}\max_{\|x\|_2=1} &= \|Ax\|_2^2 = \max_{\|x\|_2=1} x^* A^* A X \\ &= \lambda_{\max}(A^* A) \\ &= \sigma_1(A)\end{aligned}$$

Proposition 5.20

For all $U, V \in \mathcal{M}_n$ unitary matrices, we have $\|UAV\|_2 = \|A\|_2$

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