# MATH 4370/7370 – Linear Algebra and Matrix Analysis

## Quick review of 2nd year linear algebra

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# Outline of these slides


Part 1: Some notation and basic stuff
Part 2: Vector spaces
Part 3: Finite-dimensional vector spaces
Part 4: Linear maps
Part 5: Eigenvalues, eigenvectors and invariant subspaces
Part 6: Inner product spaces
Part 7: Operators on inner product spaces
Part 8: Operators on complex vector spaces

# Source of the material

The material in these slides is mostly derived from [**?**]

# Some notation and basic stuff

Sets and elements

Logic

Sets and elements

Logic

# Sets and elements

**Definition 2.1 (Set)**

A **set** $X$ is a collection of **elements**.

We write $x \in X$ or $x \notin X$ to indicate that the element $x$ belongs to the set $X$ or does not belong to the set $X$, respectively.

**Definition 2.2 (Subset)**

Let $X$ be a set. The set $S$ is a **subset** of $X$, which is denoted $S \subset X$ or $S \subseteq X$, if all its elements belong to $X$. $S$ is a **proper subset** of $X$ if it is a subset of $X$ and not equal to $X$; we then write $S \subsetneq X$.

Smith reserves $\subset$ for $\subsetneq$. I learned $\subset$ for not specified (proper or not) and $\subsetneq$ for proper. So beware!

# Quantifiers

- A shorthand notation for "for all elements $x$ belonging to $X$" is $\forall x \in X$. For example, if $X = \mathbb{R}$, the field of real numbers, then $\forall x \in \mathbb{R}$ means "for all real numbers $x$".
- A shorthand notation for "there exists an element $x$ in the set $X$" is $\exists x \in X$.
- Sometimes we write $\exists! x \in X$ for "there exists a **unique** $x$ in $X$".
- $\forall$ and $\exists$ are **quantifiers**.

# Intersection and union of sets

Let $X$ and $Y$ be two sets.

**Definition 2.3 (Intersection)**

The intersection of $X$ and $Y$, $X \cap Y$, is the set of elements that belong to $X$ **and** to $Y$

$$X \cap Y = \{x : x \in X \textbf{ and } x \in Y\}$$

**Definition 2.4 (Union)**

The union of $X$ and $Y$, $X \cup Y$, is the set of elements that belong to $X$ **or** to $Y$

$$X \cup Y = \{x : x \in X \textbf{ or } x \in Y\}$$

Use of the expression "and/or" is *strictly* forbidden in this course! "Or but not and" (a.k.a. **xor**, exclusive or) is $(X \cup Y) \setminus (X \cap Y)$.

Sets and elements

Logic

# A few notions of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. “The sky is blue” is also a proposition.
Let $A$ be a proposition. We generally write

$$A$$

to mean that $A$ is true, and

$$\textbf{not } A$$

to mean that $A$ is false. We also write $\neg A$. **not** $A$ is the **negation** of $A$.

# A few notions of logic (cont.)

Let $A, B$ be propositions. Then

- $A \Rightarrow B$ (read $A$ implies $B$) means that whenever $A$ is true, then so is $B$.
- $A \Leftrightarrow B$, also denoted $A$ if and only if $B$ ($A$ iff $B$ for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$. We also say that $A$ and $B$ are equivalent.

Let $A$ and $B$ be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\textbf{not } B \Rightarrow \textbf{not } A)$$

This is useful for proving some results.

# Necessary and/or sufficient conditions

Suppose we want to establish whether a given statement $P$ is true, depending on the truth value of a statement $H$. Then we say that

- $H$ is a **necessary condition** if $P \Rightarrow H$.
  (It is necessary that $H$ be true for $P$ to be true; so whenever $P$ is true, so is $H$).
- $H$ is a **sufficient condition** if $H \Rightarrow P$.
  (It suffices for $H$ to be true for $P$ to also be true).
- $H$ is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., $H$ and $P$ are equivalent.

# Playing with quantifiers

For the quantifiers $\forall$ (for all) and $\exists$ (there exists),

$$\exists \text{ is the negation of } \forall$$

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

This is also regularly used in proofs.

# Vector spaces


Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Operations

**Definition 2.5 (Operations – Addition and multiplication)**

An **operation** on a set $V$ is a mapping that associates an element of the set $V$ to *every* pair of its elements

- The result of the **addition** of $a$ and $b$ is the *sum* $a + b$ of $a$ and $b$
- The result of the **multiplication** of $a$ and $b$ is the *product* $ab$ (or $a \cdot b$) of $a$ and $b$

# Field

### Definition 2.6 (Field)

A **field** is a set $\mathbb{F}$ together with two (binary) operations, *addition* and *multiplication*, which are required to satisfy the following *field axioms*, where $a, b, c \in \mathbb{F}$:

- **Associativity** of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$
- **Commutativity** of addition and multiplication: $a + b = b + a$ and $ab = ba$
- **Additive** and **multiplicative identity**: $\exists 0, 1 \in \mathbb{F}$, $0 \neq 1$, s.t. $a + 0 = a$ and $a1 = a$
- **Additive inverses**: $\forall a \in \mathbb{F}$, $\exists - a \in \mathbb{F}$ s.t. $a + (-a) = 0$
- **Multiplicative inverses**: $\forall a \neq 0 \in \mathbb{F}$, $\exists a^{-1} \in \mathbb{F}$ s.t. $aa^{-1} = 1$
- **Distributivity** (of multiplication over addition): $a(b + c) = (ab) + (ac)$

# Notation

- Both $\mathbb{R}$ and $\mathbb{C}$ are fields.
- From now on, $\mathbb{F}$ refers to $\mathbb{R}$ or $\mathbb{C}$.
- Some results are specific to $\mathbb{R}$ xor $\mathbb{C}$, in which case we specify the relevant field.
- If we use $\mathbb{F}$, we mean the result applies to both $\mathbb{R}$ and $\mathbb{C}$.

Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Addition and Scalar multiplication

**Definition 2.7 (Addition and scalar multiplication on a set)**

- An **addition** on a set $V$ is a function that assigns an element $\mathbf{u} + \mathbf{v} \in V$ to each pair of elements $\mathbf{u}, \mathbf{v} \in V$
- A **scalar multiplication** on a set $V$ is a function that assigns an element $\lambda \mathbf{v}$ to each $\lambda \in \mathbb{F}$ and each $\mathbf{v} \in V$

# Vector space

## Definition 2.8 (Vector space)

A **vector space** (over $\mathbb{F}$) is a set $V$ along with an addition on $V$ and a scalar multiplication on $V$ such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [**commutativity**]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall a, b \in \mathbb{F}$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and $(ab)\mathbf{v} = a(b\mathbf{v})$ [**associativity**]
3. $\exists \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V$, $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [**additive identity**]
4. $\forall \mathbf{v} \in V$, $\exists \mathbf{w} \in V$ s.t. $\mathbf{v} + \mathbf{w} = \mathbf{0}_V$ [**additive inverse**]
5. $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$ [**multiplicative identity**]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [**distributivity**]

# Results

**Theorem 2.9 (Uniqueness of the additive identity)**

*A vector space $V$ has a unique additive identity $\mathbf{0}_V \in V$*

**Theorem 2.10 (Existence and uniqueness of additive inverse)**

*Let $V$ be a vector space. Then each $\mathbf{v} \in V$ has a unique additive inverse, denoted $-\mathbf{v}$*

We also define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$.

**Theorem 2.11**

- $\forall \mathbf{v} \in V$, $0_{\mathbb{F}}\mathbf{v} = \mathbf{0}_V$.
- $\forall a \in \mathbb{F}$, $a\mathbf{0}_V = \mathbf{0}_V$.
- $\forall \mathbf{v} \in V$, $(-1)\mathbf{v} = -\mathbf{v}$.

# Vector space

**Definition 2.12 (Vector space)**

A **vector space** (over $\mathbb{F}$) is a set $V$ along with an addition on $V$ and a scalar multiplication on $V$ such that the following properties (*axioms*) hold

1. $\forall \mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ [commutativity of $+$]
2. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ [associativity of $+$]
3. $\exists! \mathbf{0}_V \in V$ s.t. $\forall \mathbf{v} \in V$, $\mathbf{v} + \mathbf{0}_V = \mathbf{v}$ [additive identity]
4. $\forall \mathbf{v} \in V$, $\exists! - \mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_V$ [additive inverse]
5. $\forall a \in \mathbb{F}$ and $\forall \mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ [distributivity of $\cdot$ over $+$]
6. $\forall a, b \in \mathbb{F}$ and $\forall \mathbf{u} \in V$, $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ [distributivity of $+$ over $\cdot$]
7. $\forall a, b \in \mathbb{F}$, $(ab)\mathbf{v} = a(b\mathbf{v})$ [associativity of $\cdot$]
8. $\forall \mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$ [multiplicative identity]

Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# $\mathbb{F}^n$ is a vector space

Typically called *Euclidean space* when $\mathbb{F} = \mathbb{R}$.

## Definition 2.13

Let $0 \neq n \in \mathbb{N}$. An $n$-**tuple** is an ordered collection of $n$ elements,

$$(x_1, \ldots, x_n)$$

## Definition 2.14

Let $0 \neq n \in \mathbb{N}$. $\mathbb{F}^n$ is the set of all $n$-tuples of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

- Often write $x = (x_1, \ldots, x_n)$ for short.
- For a given $j \in \{1, \ldots, n\}$, $x_j$ is the $j$th **coordinate** of $x$.
- Think of $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ that you saw in whatever flavour of Linear Algebra 1 you took.

# Addition in $\mathbb{F}^n$

**Definition 2.15 (Addition in $\mathbb{F}^n$)**

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}^n$. Then

$$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

**Property 2.16 (Commutativity of addition in $\mathbb{F}^n$)**

*Let $x, y \in \mathbb{F}^n$, then*

$$x + y = y + x$$

# 0 and additive inverse in $\mathbb{F}^n$

**Definition 2.17 (0)**

0 denotes the *n*-tuple whose coordinates are all 0,

$$0 = (0, \ldots, 0)$$

If any ambiguity arises, will write $0_{\mathbb{F}^n}$

**Definition 2.18 (Additive inverse)**

Let $x \in \mathbb{F}^n$. The **additive inverse** of $x$ is $-x \in \mathbb{F}^n$ s.t.

$$x + (-x) = 0$$

If $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$

# Scalar multiplication in $\mathbb{F}^n$

**Definition 2.19 (Scalar multiplication)**

The **product** of $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$ is

$$\lambda x = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Complex numbers

**Definition 2.20 (Complex numbers)**

A **complex number** is an ordered pair $(a, b)$, where $a, b \in \mathbb{R}$. Usually written $a + ib$ or $a + bi$, where $i^2 = -1$
The set of all complex numbers is denoted $\mathbb{C}$,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

## Definition 2.21 (Addition and multiplication on $\mathbb{C}$)

Letting $a + ib$ and $c + id \in \mathbb{C}$, addition on $\mathbb{C}$ is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on $\mathbb{C}$ is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter equality easy to obtain using regular multiplication and $i^2 = -1$

# Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C}$,

- $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ [**commutativity**]
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ [**associativity**]
- $\gamma + 0 = \gamma$ and $\gamma 1 = \gamma$ [**identities**]
- $\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha + \beta = 0$ [**additive inverse**]
- $\forall \alpha \neq 0 \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha\beta = 1$ [**multiplicative inverse**]
- $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ [**distributivity**]

Thus $\mathbb{C}$ is a field.

# Additive & multiplicative inverse, subtraction, division

**Definition 2.22**

Let $\alpha, \beta \in \mathbb{C}$

- $-\alpha$ is the **additive inverse** of $\alpha$, i.e., the unique number in $\mathbb{C}$ s.t. $\alpha + (-\alpha) = 0$
- **Subtraction** on $\mathbb{C}$:
$$\beta - \alpha = \beta + (-\alpha)$$
- For $\alpha \neq 0$, $1/\alpha$ is the **multiplicative inverse** of $\alpha$, i.e., the unique number in $\mathbb{C}$ s.t.
$$\alpha(1/\alpha) = 1$$
- **Division** on $\mathbb{C}$:
$$\beta/\alpha = \beta(1/\alpha)$$

## Definition 2.23 (Real and imaginary parts)

Let $z = a + ib$. Then $\mathrm{Re}\, z = a$ is **real part** and $\mathrm{Im}\, z = b$ is **imaginary part** of $z$

If ambiguous, write $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$

## Definition 2.24 (Conjugate and Modulus)

Let $z = a + ib \in \mathbb{C}$. Then

- **Complex conjugate** of $z$ is

$$\bar{z} = \mathrm{Re}\, z - i(\mathrm{Im}\, z) = a - ib$$

- **Modulus** (or **absolute value**) of $z$ is

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \sqrt{a^2 + b^2} \geq 0$$

# Properties of complex numbers

Let $w, z \in \mathbb{C}$, then

- $z + \bar{z} = 2\mathrm{Re}\ z$
- $z - \bar{z} = 2i\mathrm{Im}\ z$
- $z\bar{z} = |z|^2$
- $\overline{w+z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$
- $\bar{\bar{z}} = z$
- $|\mathrm{Re}\ z| \leq |z|$ and $|\mathrm{Im}\ z| \leq |z|$
- $|\bar{z}| = |z|$
- $|wz| = |w|\ |z|$
- $|w + z| \leq |w| + |z|$ [**triangle inequality**]

Fields

Definition of vector spaces

Example – Space $\mathbb{F}^n$

Example – Complex numbers

Subspaces

# Subspace

**Definition 2.25 (Subspace)**

Let $V$ be a vector space over $\mathbb{F}$. Let $U \subseteq V$ be a subset of $V$. Then $U$ is a **subspace** of $V$ if $U$ is a vector space over $\mathbb{F}$ for the same operations of addition and scalar multiplication as $V$

**Theorem 2.26 (Conditions for a subspace)**

*$U \subseteq V$ is a subspace of $V \iff U$ satisfies the following three conditions:*

- $\mathbf{0}_V \in U$ *[**additive identity**]*
- $\forall \mathbf{u}, \mathbf{v} \in U$, $\mathbf{u} + \mathbf{v} \in U$ *[**closed under addition**]*
- $\forall \mathbf{u} \in U$, $\forall a \in \mathbb{F}$, $a\mathbf{u} \in U$ *[**closed under scalar multiplication**]*

The smallest possible subspace of $V$ is $\{\mathbf{0}_V\}$, the largest is $V$.

# Sums of subspaces

**Definition 2.27 (Sum of subsets)**

Let $V$ be a vector space and $U_1, \dots, U_m$ be *subsets* of $V$. The **sum** of $U_1, \dots, U_m$ is

$$U_1 + \cdots + U_m = \{\mathbf{u}_1 + \cdots + \mathbf{u}_m : \mathbf{u}_1 \in U_1, \dots, \mathbf{u}_m \in U_m\}$$

**Theorem 2.28**

*Let $V$ be a vector space and $U_1, \dots, U_m$ be subspaces of $V$. Then $U_1 + \cdots + U_m$ is the smallest subspace of $V$ containing $U_1, \dots, U_m$*

# Direct sums

**Definition 2.29 (Direct sum)**

Suppose $U_1, \ldots, U_m$ are subspaces of a vector space $V$. The sum $U_1 + \cdots + U_m$ is a **direct sum** and is then written $U_1 \oplus \cdots \oplus U_m$ if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $\mathbf{u}_1 + \cdots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$

**Theorem 2.30 (Condition for a direct sum)**

*Suppose $U_1, \ldots, U_m$ are subspaces of a vector space $V$. Then $U_1 + \cdots + U_m$ is a direct sum $\iff$ the only way to write $\mathbf{0}$ as a sum $\mathbf{u}_1 + \cdots + \mathbf{u}_m$, where each $\mathbf{u}_j \in U_j$, is by taking each $\mathbf{u}_j$ equal to $\mathbf{0}_V$*

**Theorem 2.31 (Direct sum of two subspaces)**

*Let $U, W$ be subspaces of a vector space $V$. Then $U + W$ is a direct sum $\iff$ $U \cap W = \{\mathbf{0}_V\}$*

# Finite-dimensional vector spaces

Span and Linear independence

Bases

Dimension

Span and Linear independence

Bases

Dimension

## Definition 2.32 (Linear combination)

A **linear combination** of a list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in $V$ is a vector

$$a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$

## Definition 2.33 (Span)

The set of all linear combinations of a list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_m$,

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \{a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m : a_1, \ldots, a_m \in \mathbb{F}\}$$

The span of the empty list ( ) is $\{\mathbf{0}_V\}$

# Finite/infinite-dimensional vector spaces

### Theorem 2.34

*The span of a list of vectors in $V$ is the smallest subspace of $V$ containing all the vectors in the list*

### Definition 2.35 (List of vectors spanning a space)

If $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = V$, we say $\mathbf{v}_1, \ldots, \mathbf{v}_m$ **spans** $V$

### Definition 2.36 (Finite-dimensional vector space)

A vector space $V$ is **finite-dimensional** if some list of vectors in it spans $V$

### Definition 2.37 (Infinite-dimensional vector space)

A vector space $V$ is **infinite-dimensional** if it is not finite-dimensional

# Linear (in)dependence

**Definition 2.38 (Linear independence/Linear dependence)**

A list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in a vector space $V$ is **linearly independent** if

$$(a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m = 0) \Leftrightarrow (a_1 = \cdots = a_m = 0),$$

where $a_1, \ldots, a_m \in \mathbb{F}$. A list of vectors is **linearly dependent** if it is not linearly independent.

The empty list ( ) is assumed to be linearly independent

## Lemma 2.39 (Linear dependence)

*Let* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *be a linearly dependent list in a vector space* $V$*. Then there exists* $j \in \{1, 2, \ldots, m\}$ *s.t.*

1. $\mathbf{v}_j \in \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{j-1})$
2. *if the* $j$*th term is removed from* $\mathbf{v}_1, \ldots, \mathbf{v}_m$*, the span of the remaining list equals* $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$

## Theorem 2.40

*Let* $V$ *be a finite-dimensional vector space. Then the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors*

## Theorem 2.41 (Subspace of a finite-dimensional vector space)

*Every subspace of a finite-dimensional vector space is finite-dimensional*

Span and Linear independence

Bases

Dimension

# Basis

**Definition 2.42 (Basis)**

Let $V$ be a vector space. A **basis** of $V$ is a list of vectors in $V$ that is both linearly independent and spanning

**Theorem 2.43 (Criterion for a basis)**

*A list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of vectors in a vector space $V$ is a basis of $V$ iff $\forall \mathbf{v} \in V$, $v$ can be written uniquely in the form*

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m,$$

*where $a_1, \ldots, a_m \in \mathbb{F}$*

### Theorem 2.44 (All spanning lists contain a basis)

*Every spanning list in a vector space can be reduced to a basis of the vector space*

### Theorem 2.45 (Basis of finite-dimensional vector space)

*Every finite-dimensional vector space has a basis*

### Theorem 2.46 (Extension to a basis)

*Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space*

### Theorem 2.47

*Let $V$ be a finite-dimensional vector space and $U \subset V$ be a subspace of $V$. Then $\exists W \subset V$ subspace of $V$ s.t. $V = U \oplus W$*

Span and Linear independence

Bases

Dimension

**Theorem 2.48 (Bases of a finite-dim. space have equal length)**

*Any two bases of a finite-dimensional vector space have the same length*

**Definition 2.49 (Dimension)**

The **dimension** $\dim V$ of a finite-dimensional vector space $V$ is the length of any basis of the vector space

**Theorem 2.50 (Dimension of a subspace)**

*Let $V$ be a finite-dimensional vector space and $U \subset V$ be a subspace of $V$. Then* $\dim U \leq \dim V$

### Theorem 2.51

*Let $V$ be a finite-dimensional vector space. Then every linearly independent list of vectors in $V$ with length* $\dim V$ *is a basis of $V$*

### Theorem 2.52

*Let $V$ be a finite-dimensional vector space. Then every spanning list of vectors in $V$ with length* $\dim V$ *is a basis of $V$*

### Theorem 2.53 (Dimension of a sum of subspaces)

*Let $U_1, U_2$ be subspaces of a finite-dimensional vector space $V$. Then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

# Linear maps

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

## Definition 2.54 (Linear map/transformation)

Let $V, W$ be vector spaces. A **linear map** (or **linear transformation**) from $V$ to $W$ is a function $T : V \to W$ that has the following properties:

1. **Additivity** $\forall \mathbf{u}, \mathbf{v} \in V,\ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. **Homogeneity** $\forall \lambda \in \mathbb{F}, \forall \mathbf{v} \in V,\ T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$.

Often, parentheses are omitted, $T(\mathbf{u})$ is written $T\mathbf{u}$

The set of all linear maps from $V$ to $W$ is denoted $\mathcal{L}(V, W)$

### Theorem 2.55 (Linear maps and basis of domain)

*Let $V, W$ be two vector spaces and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $V$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n \in W$. Then there exists a unique linear map $T : V \to W$ s.t.*

$$\forall j = 1, \ldots, n, \qquad T\mathbf{v}_j = \mathbf{w}_j$$

## Definition 2.56 (Addition & Scalar multiplication)

Let $V, W$ be vector spaces, $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$ and **product** $\lambda T$ are the linear maps from $V$ to $W$ defined, $\forall \mathbf{v} \in V$, by

$$(S + T)(\mathbf{v}) = S\mathbf{v} + T\mathbf{v} \text{ and } (\lambda T)(\mathbf{v}) = \lambda(T\mathbf{v}).$$

## Theorem 2.57 (Linear maps are vector spaces)

*Let $V, W$ be vector spaces. Equipped with addition and scalar multiplication as just defined, $\mathcal{L}(V, W)$ is a vector space.*

# Product of linear maps

**Definition 2.58 (Product of linear maps)**

Let $U, V, W$ be vector spaces, $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$. The **product** $ST \in \mathcal{L}(U, W)$ is defined for $\mathbf{u} \in U$ by

$$(ST)(\mathbf{u}) = S(T\mathbf{u}).$$

This means that the product of linear maps is the composition $S \circ T$, although because of the linearity, we often omit the $\circ$ composition sign.

# Properties of products of linear maps

## Theorem 2.59

1. **Associativity** *If* $V, V_2, V_3, W$ *vector spaces,* $T_1 \in \mathcal{L}(V, V_2)$, $T_2 \in \mathcal{L}(V_2, V_3)$, $T_3 \in \mathcal{L}(V_3, W)$, *then*

$$(T_1 T_2) T_3 = T_1 (T_2 T3)$$

2. **Identity** $V, W$ *vector spaces. Then for* $T \in \mathcal{L}(V, W)$,

$$T I_V = I_W T = T$$

3. **Distributivity** $U, V, W$ *vector spaces,* $T, T_1, T_2 \in \mathcal{L}(U, V), S, S_1, S_2 \in \mathcal{L}(V, W)$, *then*

$$(S_1 + S_2) T = S_1 T + S_2 T \text{ and } S(T_1 + T_2) = ST_1 + ST_2$$

## Theorem 2.60 (Linear maps take 0 to 0)

*Let $V, W$ be vector spaces, $T \in \mathcal{L}(V, W)$. Then*

$$T(\mathbf{0}_V) = \mathbf{0}_W.$$

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

**Definition 2.61 (Null space)**

Let $V, W$ be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. The **null space** $\operatorname{null} T$ (or **kernel** $\ker T$) of $T$ is the subet of $V$ consisting of those vectors that $T$ maps to $\mathbf{0}_W$:

$$\operatorname{null} T = \{\mathbf{v} \in V; T\mathbf{v} = \mathbf{0}_W\}.$$

**Theorem 2.62 (Null space is a subspace)**

*Let $V, W$ be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then $\operatorname{null} T$ is a subspace of $V$*

**Definition 2.63 (Injectivity)**

A function $T : V \to W$ is **injective** (or **one-to-one**) if

$$T\mathbf{u} = T\mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}.$$

We can also use the contrapositive: $T$ injective if $\mathbf{u} \neq \mathbf{v} \Rightarrow T\mathbf{u} \neq T\mathbf{v}$.

**Theorem 2.64 (Linking injectivity and null space)**

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ injective} \Leftrightarrow \operatorname{null} T = \{\mathbf{0}_V\}$$

### Definition 2.65 (Range)

Let $V, W$ be finite-dimensional vector spaces, $T : V \to W$ a function. The **range** (or **image**) of $T$ is the subset of $W$ defined by

$$\text{range } T = \{T\mathbf{v}; \mathbf{v} \in V\}.$$

When talking about the image, we write $\text{Im } T$.

### Theorem 2.66 (Range is a subspace)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then* $\text{range} T$ *is a subspace of $W$.*

## Definition 2.67 (Surjectivity)

A function $T : V \to W$ is **surjective** (or **onto**) if

$$\operatorname{range} T = W$$

## Theorem 2.68 (Fundamental theorem of linear maps)

*Let $V$ be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then* $\dim \operatorname{range} T < \infty$ *and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

## Theorem 2.69 (Linear map onto a smaller space is not injective)

*Let* $V, W$ *be finite-dimensional vector spaces such that* $\dim V > \dim W$*. Then* $\nexists T \in \mathcal{L}(V, W)$ *that is injective*

## Theorem 2.70 (Linear map onto a larger space is not surjective)

*Let* $V, W$ *be finite-dimensional vector spaces such that* $\dim V < \dim W$*. Then* $\nexists T \in \mathcal{L}(V, W)$ *that is surjective*

# Do as exercises..

**Theorem 2.71**

*A homogeneous system of linear equations with more variables than equations has nonzero solutions.*

**Theorem 2.72**

*A nonhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms*

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

## Definition 2.73 (Matrix)

An $m$-by-$n$ or $m \times n$ matrix is a rectangular array of elements of $\mathbb{F}$ with $m$ rows and $n$ columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Remember that we always list indices as “row,column”

We denote $\mathcal{M}_{mn}(\mathbb{F})$ the set of $m \times n$ matrices with entries in $\mathbb{F}$

## Definition 2.74 (Matrix of a linear map)

Let $V, W$ be finite-dimensional vector spaces, $v_1, \ldots, v_n$ a basis of $V$ and $w_1, \ldots, w_m$ a basis of $W$. The **matrix of** $T$ with respect to these bases is the matrix $M(T) \in \mathcal{M}_{mn}$ with entries $a_{jk}$ defined by

$$Tv_k = a_{1k}w_1 + \cdots + a_{mk}w_m$$

for $1 \leq l \leq n$. If the bases are not clear from the context, then write

$$M(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$$

I will often write $M_T$ rather than $M(T)$.

Most definitions are assumed known

### Theorem 2.75 (Matrix of sums of linear maps)

*Suppose* $S, T \in \mathcal{L}(V, W)$. *Then* $M(S + T) = M(S) + M(T)$

### Theorem 2.76 (Matrix of a scalar times a linear map)

*Suppose* $T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$. *Then* $M(\lambda T) = \lambda M(T)$

### Theorem 2.77 (Dimension of $\mathcal{M}_{mn}$)

$\dim \mathbb{F}^{mn} = mn$

### Theorem 2.78 (Matrix of products of linear maps)

*Suppose* $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$. *Then* $M(ST) = M(S)M(T)$

## Theorem 2.79

*Let* $A \in \mathcal{M}_{mn}$, $C \in \mathcal{M}_{np}$. *Then*

$$(AC)_{jk} = A_{j\bullet}C_{\bullet k}, \qquad 1 \leq j \leq m, 1 \leq k \leq p$$

*and*

$$(AC)_{\bullet k} = AC_{\bullet k}, \qquad 1 \leq k \leq p$$

## Theorem 2.80

*Let* $A \in \mathcal{M}_{mn}$, $c = (c_1, \ldots, c_n)^T \in \mathcal{M}_{n1}$. *Then*

$$Ac = c_1 A_{\bullet 1} + \cdots + c_n A_{\bullet n}$$

# Change of basis

**Definition 2.81 (Change of basis matrix)**

$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ bases of vector space $V$ The **change of basis matrix** $P_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathcal{M}_n$,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{u}_1]_{\mathcal{C}} \cdots [\mathbf{u}_n]_{\mathcal{C}}]$$

has columns the coordinate vectors $[\mathbf{u}_1]_{\mathcal{C}}, \ldots, [\mathbf{u}_n]_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ w.r.t. $\mathcal{C}$

**Theorem 2.82**

*$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ bases of vector space $V$ and $P_{\mathcal{C} \leftarrow \mathcal{B}}$ a change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$*

1. $\forall \mathbf{x} \in V$, $P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$
2. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ *s.t.* $\forall \mathbf{x} \in V$, $P_{\mathcal{C} \leftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ *is* **unique**
3. $P_{\mathcal{C} \leftarrow \mathcal{B}}$ *invertible and* $P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$

# Row-reduction method for changing bases

**Theorem 2.83**

*$\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ bases of vector space $V$. Let $\mathcal{E}$ be any basis for $V$,*

$$B = [[\mathbf{u}_1]_{\mathcal{E}}, \ldots, [\mathbf{u}_n]_{\mathcal{E}}] \text{ and } C = [[\mathbf{v}_1]_{\mathcal{E}}, \ldots, [\mathbf{v}_n]_{\mathcal{E}}]$$

*and let $[C|B]$ be the augmented matrix constructed using $C$ and $B$. Then*

$$RREF([C|B]) = [\mathbb{I}|P_{\mathcal{C}\leftarrow\mathcal{B}}]$$

If working in $\mathbb{R}^n$, this is quite useful with $\mathcal{E}$ the standard basis of $\mathbb{R}^n$ (it does not matter if $\mathcal{B} = \mathcal{E}$)

# More on changing bases

**Theorem 2.84 (NSC for two matrices representing the same linear map)**

*Let $A, B \in \mathcal{M}_{mn}$, $V$ and $W$ be $n$ and $m$ dimensional vector spaces, respectively. Then $A$ and $B$ represent the same linear transformation $T \in \mathcal{L}(V, W)$ relative to perhaps different bases of $V$ and $W$ $\iff$ $\exists P \in \mathcal{M}_m$, $Q \in \mathcal{M}_n$ nonsingular and such that*

$$A = PBQ^{-1}$$

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

### Definition 2.85 (Inverse/Invertibility)

$T \in \mathcal{L}(V, W)$ is **invertible** if $\exists S \in \mathcal{L}(W, V)$ s.t. $ST = I_V$ and $TS = I_W$. Such a map is the **inverse** of $T$

### Theorem 2.86 (Uniqueness of inverse)

*An invertible linear map $T \in \mathcal{L}(V, W)$ has a unique inverse denoted $T^{-1}$*

### Theorem 2.87 (NSC for invertibility)

*$T \in \mathcal{L}(V, W)$ invertible $\Leftrightarrow$ ($T$ injective and surjective)*

## Definition 2.88 (Isomorphism/Isomorphic spaces)

$T \in \mathcal{L}(V, W)$ is an **isomorphism** if it invertible. Two vector spaces are **isomorphic** if there exists an isomorphism from one to the other

## Theorem 2.89 (NSC for isomorphicity)

*Let $V, W$ be finite-dimensional vector spaces over $\mathbb{F}$. Then*

$$V \textit{ and } W \textit{ are isomorphic} \Leftrightarrow \dim V = \dim W$$

## Theorem 2.90

*Let $v_1, \ldots, v_n$ be a basis of $V$ and $w_1, \ldots, w_m$ be a basis of $W$. Then $M$ is an isomorphism between $\mathcal{M}_{mn}$ and $\mathcal{L}(V, W)$*

### Theorem 2.91 (Dimension of $\mathcal{L}(V, W)$)

*Let $V, W$ be finite-dimensional vector spaces. Then $\mathcal{L}(V, W)$ is finite-dimensional and*

$$\dim \mathcal{L}(V, W) = \dim V \ \dim W$$

## Definition 2.92 (Matrix of a vector)

Let $V$ be a finite-dimensional vector space, $v \in V$ and $v_1, \ldots, v_n$ a basis of $V$. The **matrix** of $v$ with respect to the basis $v_1, \ldots, v_n$ is the $n \times 1$ matrix

$$M(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \ldots, c_n \in \mathbb{F}$ are s.t.

$$v = c_1 v_1 + \cdots + c_n v_n$$

## Theorem 2.93

*Let $V, W$ be finite-dimensional vector spaces, $v_1, \ldots, v_n$ a basis of $V$, $w_1, \ldots, w_m$ a basis of $W$ and $T \in \mathcal{L}(V, W)$. For $k \in \{1, \ldots, n\}$, $M(T)_{\bullet k} = M(Tv_k)$*

### Theorem 2.94 (Linear maps act like matrix multiplication)

*Let $V, W$ be finite-dimensional vector spaces, $v_1, \ldots, v_n$ a basis of $V$, $w_1, \ldots, w_m$ a basis of $W$, $T \in \mathcal{L}(V, W)$ and $v \in V$. Then*

$$M(Tv) = M(T)M(v)$$

# Operator/Endomorphism

**Definition 2.95 (Operator/Endomorphism)**

Let $V$ be a vector space. A linear map $\mathcal{L}(V, V)$ is an **operator** (or an **endomorphism**). $\mathcal{L}(V) = \mathcal{L}(V, V)$ denotes the set of all operators on $V$

**Theorem 2.96 (Injectivity equiv. to surjectivity in finite-dim.)**

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. TFAE:*

1. *$T$ invertible*
2. *$T$ injective*
3. *$T$ surjective*

# Rank of an operator/endomorphism

### Proposition 2.97 (Rank)

*Let $T \in \mathcal{L}(V)$ with $V$ finite-dimensional. Then there exists bases $\mathcal{B}_U = \{u_1, \ldots, u_n\}$ and $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ for $V$ such that the matrix $M_T$ of $T$ can be written as the block matrix*

$$M_T = \begin{pmatrix} \mathrm{diag}(1, \ldots, 1) & \mathbf{0}_{k,n-k} \\ \mathbf{0}_{n-k,k} & \mathbf{0}_{n-k,n-k} \end{pmatrix}$$

*for some $k \in \mathbb{N}$ called the* **rank** *of $T$, with $k = \mathrm{rank}(T) = \dim(\mathrm{range}\ T)$.*

## Definition 2.98 (Row and column rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$ be a matrix

- The **row rank** of $A$ is the dimension of the span of the rows of $A$ in $\mathcal{M}_{1n}(\mathbb{F})$
- The **column rank** of $A$ is the dimension of the span of the columns of $A$ in $\mathcal{M}_{m1}(\mathbb{F})$

Row and column ranks are the dimensions of the row and column spaces of Definition 2.102.

### Theorem 2.99 ($\dim \operatorname{range} T$ equals column rank of $M(T)$)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $M(T)$*

### Theorem 2.100 (Row rank equals column rank)

*Let $A \in \mathcal{M}_{mn}$. Then the row rank of $A$ equals the column rank of $A$*

### Definition 2.101 (Rank)

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$. The **rank** of $A$ is the column (or row, by Theorem 2.100) rank of $A$

# Row space and column space of a matrix

### Definition 2.102 (Row and column spaces)

Let $A \in \mathcal{M}_{mn}$. The subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ spanned by the row and column vectors of $A$ are the **row space** and **column space** of $A$, respectively.

### Definition 2.103 (Null space/kernel)

Let $A \in \mathcal{M}_{mn}$. The null space (or kernel) of $A$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

This makes explicit the already seen definition in the special case of a matrix. As previously seen, the null space is a subspace of $\mathbb{R}^n$.

### Definition 2.104 (Nullity)

The dimension of the null space of $A \in \mathcal{M}_{mn}$ is called the **nullity** of $A$.

## Theorem 2.105

*Let $A \in \mathcal{M}_{mn}$. Then*

1. $\text{rank}(A) = \text{rank}(A^T)$
2. $\text{rank}(A) + \text{nullity}(A) = n$
3. $\text{rank}(A) \leq \min(m, n)$

## Theorem 2.106 (Consistency)

*Consider the linear system $A\mathbf{x} = \mathbf{b}$, with $A \in \mathcal{M}_{mn}$. TFAE:*

- $A\mathbf{x} = \mathbf{b}$ *is consistent*
- $\mathbf{b} \in$ *column space of* $A$
- $A$ *and* $[A|\mathbf{b}]$ *have the same rank*

## Proposition 2.107

*Let $A \in \mathcal{M}_{mn}$ be in row-echelon form. Then*

- *The row vectors ($\in \mathbb{R}^n$) with leading ones form a basis for the row space of $A$.*
- *The column vectors ($\in \mathbb{R}^m$) with leading ones form a basis for the column space of $A$.*

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

## Definition 2.108 (Product of vector spaces)

Let $V_1, \ldots, V_m$ be vector spaces over $\mathbb{F}$. The **product** $V_1 \times \cdots \times V_m$ is

$$V_1 \times \cdots \times V_m = \{(\mathbf{v}_1, \ldots, \mathbf{v}_m); \mathbf{v}_1 \in V_1, \ldots, \mathbf{v}_m \in \ V_m\}$$

## Theorem 2.109 (Products of vector spaces are vector spaces)

*Let $V_1, \ldots, V_m$ be vector spaces over $\mathbb{F}$. Define*

- *addition on $V_1 \times \cdots \times V_m$ by*

$$(\mathbf{u}_1, \ldots, \mathbf{u}_m) + (\mathbf{v}_1, \ldots, \mathbf{v}_m) = (\mathbf{u}_1 + \mathbf{v}_1, \ldots, \mathbf{u}_m + \mathbf{v}_m)$$

- *scalar multiplication on $V_1 \times \cdots \times V_m$ by*

$$\lambda(\mathbf{v}_1, \ldots, \mathbf{v}_m) = (\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_m)$$

*With these operations, $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$*

### Theorem 2.110 (Dimension of product space)

*Let $V_1, \ldots, V_m$ be finite-dimensional vector spaces. Then*

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m < \infty$$

## Theorem 2.111 (Product spaces and direct sums)

*Let $U_1, \ldots, U_m \subset V$ be subspaces of $V$. Let*

$$\begin{aligned} \Gamma : \quad U_1 \times \cdots \times U_m &\to U_1 + \cdots + U_m \\ (\mathbf{u}_1, \ldots, \mathbf{u}_m) &\mapsto \mathbf{u}_1 + \cdots + \mathbf{u}_m \end{aligned}$$

*Then*

$$U_1 + \cdots + U_m \textit{ direct sum} \Leftrightarrow \Gamma \textit{ injective}$$

## Theorem 2.112 (NSC for direct sum)

*Let $V$ be a finite-dimensional vector space, $U_1, \ldots, U_m$ subspaces of $V$. Then*

$$U_1 \oplus \cdots \oplus U_m \Leftrightarrow \dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

## Definition 2.113 ($\mathbf{v} + U$)

Let $V$ be a vector space, $U$ a subspace of $V$ and $\mathbf{v} \in V$. Then $\mathbf{v} + U$ is the subset of $V$ defined by

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u}; \mathbf{u} \in U\}$$

## Definition 2.114 (Affine subset/Parallel affine subset)

Let $V$ be a vector space

- An **affine subset** of $V$ is a subset of $V$ of the form $\mathbf{v} + U$ for some $\mathbf{v} \in V$ and some subspace $U$ of $V$
- For $\mathbf{v} \in V$ and $U$ subspace of $V$, the affine subset $\mathbf{v} + U$ is **parallel** to $U$

## Definition 2.115 (Quotient space)

Let $V$ be a vector space, $U$ a subspace of $V$. The **quotient space** $V/U$ is the set of all affine subsets of $V$ parallel to $U$, i.e.,

$$V/U = \{\mathbf{v} + U; \mathbf{v} \in V\}$$

## Theorem 2.116 (2 affine subsets // to $U$ are equal or disjoint)

*Let $V$ be a vector space, $U$ subspace of $V$ and $v, w \in V$. TFAE*

1. $\mathbf{v} - \mathbf{w} \in U$
2. $\mathbf{v} + U = \mathbf{w} + U$
3. $(\mathbf{v} + U) \cap (\mathbf{w} + U) \neq \emptyset$

## Definition 2.117 (Addition and scalar multiplication on $V/U$)

Let $V$ be a vector space, $U$ subspace of $V$. Then **addition** and **scalar multiplication** on $V/U$ are defined for $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in \mathbb{F}$ by

$$(\mathbf{v} + U) + (\mathbf{w} + U) = (\mathbf{v} + \mathbf{w}) + U$$

and

$$\lambda(\mathbf{v} + U) = (\lambda \mathbf{v}) + U$$

## Theorem 2.118 (Quotient space is a vector space)

*Let $V$ be a vector space and $U$ subspace of $V$. Equipped with addition and scalar multiplication as above, $V/U$ is a vector space*

### Definition 2.119 (Quotient map)

Let $V$ be a vector space, $U$ subspace of $V$. The **quotient map** $\pi$ is the linear map $\pi \in \mathcal{L}(V, V/U)$ defined by

$$\pi(\mathbf{v}) = \mathbf{v} + U$$

for $\mathbf{v} \in V$

### Theorem 2.120 (Dimension of quotient space)

*Let $V$ be a finite-dimensional vector space and $U$ subspace of $V$. Then*

$$\dim V/U = \dim V - \dim U$$

## Definition 2.121 ($\tilde{T}$)

Let $V, W$ be vector spaces, $T \in \mathcal{L}(V, W)$. Define $\tilde{T}$ by

$$\begin{aligned} \tilde{T} : \quad V/(\operatorname{null} T) &\to W \\ \tilde{T}(\mathbf{v} + \operatorname{null} T) &= T\mathbf{v} \end{aligned}$$

## Theorem 2.122 (Null space and range of $\tilde{T}$)

*Let $V, W$ be vector spaces, $T \in \mathcal{L}(V, W)$. Then*

1. $\tilde{T} \in \mathcal{L}(V/\operatorname{null} T, W)$
2. $\tilde{T}$ *injective*
3. $\operatorname{range} \tilde{T} = \operatorname{range} T$
4. $V/\operatorname{null} T$ *isomorphic to* $\operatorname{range} T$

Vector space of linear maps

Null spaces and Ranges

Matrices

Invertibility and Isomorphic vector spaces

Products and quotients of vector spaces

Duality

### Definition 2.123 (Linear functional/form)

A **linear functional** (or **linear form**) on a vector space $V$ is a linear map in $\mathcal{L}(V, \mathbb{F})$

### Definition 2.124 (Dual space)

The **dual space** $V^\star$ of $V$ is the vector space $V^\star = \mathcal{L}(V, \mathbb{F})$ of linear functionals on $V$

### Theorem 2.125 ($\dim V^\star = \dim V$)

*Suppose $V$ is a finite-dimensional vector space. Then* $\dim V^\star = \dim V < \infty$

## Definition 2.126 (Dual basis)

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of the vector space $V$, then the **dual basis** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of $V^\star$, where for $j = 1, \ldots, n$, $\varphi_j$ is the linear functional on $V$ s.t.

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

## Theorem 2.127 (Dual basis is a basis of the dual space)

*Suppose $V$ is a finite-dimensional vector space. Then the dual basis of a basis of $V$ is a basis of $V^\star$*

## Definition 2.128 (Dual map)

Let $V, W$ be vector spaces, $T \in \mathcal{L}(V, W)$. The **dual map** of $T$ is the linear map $T^\star \in \mathcal{L}(W^\star, V^\star)$ defined by $T^\star(\varphi) = \varphi \circ T$ for $\varphi \in W^\star$

## Property 2.129 (Algebraic properties of dual maps)

*Let $U, V, W$ be vector spaces*

- $(S + T)^\star = S^\star + T^\star$ *for all* $S, T \in \mathcal{L}(V, W)$
- $(\lambda T)^\star = \lambda T^\star$ *for all* $\lambda \in \mathbb{F}$ *and all* $T \in \mathcal{L}(V, W)$
- $(ST)^\star = T^\star S^\star$ *for all* $T \in \mathcal{L}(U, V)$ *and all* $S \in \mathcal{L}(V, W)$

### Definition 2.130 (Annihilator)

Let $V$ be a vector space, $U \subseteq V$. The **annihilator** $U^0$ of $U$ is defined by

$$U^0 = \{\varphi \in V^\star : \forall \mathbf{u} \in U, \quad \varphi(\mathbf{u}) = 0_{\mathbb{F}}\}$$

### Theorem 2.131 (The annihilator is a subspace)

*Let $V$ be a vector space and $U \subseteq V$. Then the annihilator $U^0$ is a subspace of $V^\star$*

### Theorem 2.132 (Dimension of the annihilator)

*Let $V$ be a finite-dimensional vector space, $U \subseteq V$ a subspace of $V$. Then*

$$\dim U + \dim U^0 = \dim V$$

### Theorem 2.133 (Null space of $T^\star$)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then*

1. $\operatorname{null} T^\star = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T^\star = \dim \operatorname{null} T + \dim W - \dim V$

### Theorem 2.134 ($T$ surjective $\Leftrightarrow$ $T^\star$ injective)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then*

$$T \textit{ surjective} \Leftrightarrow T^\star \textit{ injective}$$

## Theorem 2.135 (Range of $T^\star$)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then*

1. $\dim \operatorname{range} T^\star = \dim \operatorname{range} T$
2. $\operatorname{range} T^\star = (\operatorname{null} T)^0$

## Theorem 2.136 ($T$ injective $\Leftrightarrow$ $T^\star$ surjective)

*Let $V, W$ be finite-dimensional vector spaces, $T \in \mathcal{L}(V, W)$. Then*

$$T \ \textit{injective} \Leftrightarrow T^\star \ \textit{surjective}$$

### Theorem 2.137 (Matrix of $T^\star$ is transpose of matrix of $T$)

*Let $V, W$ be vector spaces, $T \in \mathcal{L}(V, W)$. Then $M(T^\star) = M(T)^T$, where $^T$ denotes the transpose*

# Eigenvalues, eigenvectors and invariant subspaces

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

## Definition 2.138 (Invariant subspace)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$. A subspace $U$ of $V$ is **invariant** under $T$ if

$$\mathbf{u} \in U \Rightarrow T\mathbf{u} \in U$$

In other words, $U$ invariant under $T$ if $T|_U \in \mathcal{L}(U)$ [see Definition 2.144]

## Definition 2.139 (Eigenvalue)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of $T$ if

$$\exists \mathbf{v} \in V, v \neq \mathbf{0}_V, \text{ s.t. } T(\mathbf{v}) = \lambda \mathbf{v}.$$

I use the notation $T(\mathbf{v})$ instead of $T\mathbf{v}$ to emphasise that $T \in \mathcal{L}(V)$.

## Theorem 2.140 (Conditions equivalent to being an eigenvalue)

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Denote $I_{\mathcal{L}(V)}$ the identity operator, $I_{\mathcal{L}(V)} \in \mathcal{L}(V)$ s.t. $\forall \mathbf{v} \in V$, $I_{\mathcal{L}(V)}\mathbf{v} = \mathbf{v}$. TFAE:*

1. *$\lambda$ eigenvalue of $T$*
2. *$T - \lambda I_{\mathcal{L}(V)}$ not injective*
3. *$T - \lambda I_{\mathcal{L}(V)}$ not surjective*
4. *$T - \lambda I_{\mathcal{L}(V)}$ not invertible*

## Definition 2.141 (Eigenvector)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ be an eigenvalue of $T$. A vector $\mathbf{v} \in V$ is an **eigenvector** of $T$ corresponding to $\lambda$ if $\mathbf{v} \neq 0$ and $T(\mathbf{v}) = \lambda\mathbf{v}$

### Theorem 2.142 (Linearly independent eigenvectors)

*Let $V$ be a vector space, $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ linearly independent*

### Theorem 2.143 (Number of eigenvalues)

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Then $T$ has at most $\dim V$ distinct eigenvalues*

## Definition 2.144 (Restriction and quotient operators)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$ and $U$ a subspace of $V$ invariant under $T$ (Def. 2.138)

- The **restriction operator** $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U = T\mathbf{u}, \quad \mathbf{u} \in U$$

- The **quotient operator** $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(\mathbf{v} + U) = T\mathbf{v} + U, \quad \mathbf{v} \in V$$

For the quotient space $\mathcal{L}(V/U)$, see Definition 2.138 and the results that follow

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

## Definition 2.145

Let $V$ be a vector space, $T \in \mathcal{L}(V)$, $m \in \mathbb{N} \setminus \{0\}$

- $T^m = \underbrace{T \cdots T}_{m \text{ times}}$
- $T^0 = I$, the identity operator on $V$
- If $T$ invertible with inverse $T^{-1}$, then $T^{-m} = (T^{-1})^m$

## Definition 2.146

Let $V$ be a vector space, $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ be the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad z \in \mathbb{F}$$

Then $p(T)$ is the operator on $\mathcal{L}(V)$ defined by

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

where $I$ is the identity operator

## Definition 2.147 (Product of polynomials)

Let $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial

$$(pq)(z) = p(z)q(z), \quad z \in \mathbb{F}$$

## Theorem 2.148 (Multiplicative properties)

*Let $p, q \in \mathcal{P}(\mathbb{F})$, $V$ a vector space and $T \in \mathcal{L}(V)$. Then*

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

**Theorem 2.149 (Operators on complex v.s. have an eigenvalue)**

*Let $V$ be a vector space over $\mathbb{C}$ with $\dim V = n < \infty$. Assume $T \in \mathcal{L}(V)$. Then $V$ has an eigenvalue*

## Definition 2.150 (Matrix of an operator)

Let $T \in \mathcal{L}(V)$, where $V$ is a finite-dimensional vector space, let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $V$. The **matrix** of $T$ with respect to the basis is the $n \times n$ matrix

$$M(T) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

with entries $a_{jk}$ defined by

$$Tv_k = a_{1k}\mathbf{v}_1 + \cdots + a_{nk}\mathbf{v}_n$$

If basis is not clear from the context, write $M(T, (\mathbf{v}_1, \ldots, \mathbf{v}_n))$

## Definition 2.151 (Diagonal of a matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The **diagonal** of $A$ consists of the entries $a_{ii}$, $i = 1, \dots, n$

## Definition 2.152 (Upper-triangular matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. The matrix $A$ is **upper-triangular** if all entries below the diagonal are 0, i.e.,

$$a_{ij} = 0, \quad \forall i, j \text{ such that } i > j$$

### Theorem 2.153 (Conditions for an upper-triangular matrix)

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a basis of $V$. TFAE:*

1. $M(T)$ *with respect to* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is upper-triangular*
2. $Tv_j \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j)$, $\forall j = 1, \ldots, n$
3. $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j)$ *invariant under* $T$, $\forall j = 1, \ldots, n$

### Theorem 2.154 (Every operator over $\mathbb{C}$ has an UT matrix)

*Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, $T \in \mathcal{L}(V)$. Then $T$ has an upper-triangular matrix with respect to some basis of $V$*

### Theorem 2.155 (Determination of invertibility from UT matrix)

*Let $V$ be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of $V$. Then*

$$T \textit{ invertible } \Leftrightarrow \forall i = 1, \ldots, n, \quad a_{ii} \neq 0$$

### Theorem 2.156 (Determination of eigenvalues from UT matrix)

*Let $V$ be finite-dimensional vector space. Assume that $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of $V$. Then*

$$\lambda \textit{ eigenvalue of } T \Leftrightarrow \lambda \in \{a_{ii}, \quad i = 1, \ldots, n\}$$

Invariant subspaces

Eigenvectors and upper-triangular matrices

Eigenspaces and diagonal matrices

## Definition 2.157 (Diagonal matrix)

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. $A$ is a **diagonal** matrix if all entries of $A$ are zero except possibly on the diagonal, i.e.,

$$\forall i, j,\ i \neq j, \quad a_{ij} = 0.$$

## Definition 2.158 (Eigenspace)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$. The **eigenspace** $E(\lambda, T)$ of $T$ corresponding to $\lambda$ is defined by

$$E(\lambda, T) = \mathrm{null}(T - \lambda I).$$

Thus $\lambda$ eigenvalue of $T \Leftrightarrow E(\lambda, T) \neq \{\mathbf{0}_V\}$.

## Theorem 2.159 (Sum of eigenspaces is a direct sum)

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Assume $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T$. Then*

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

*is a direct sum and*

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V$$

### Definition 2.160 (Diagonalisable operator)

Let $V$ be a vector space, $T \in \mathcal{L}(V)$. $T$ is **diagonalisable** if $T$ has a diagonal matrix with respect to some basis of $V$.

### Theorem 2.161 (Conditions equivalent to diagonalisability)

*Let $V$ be a finite-dimensional vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of $T$. TFAE:*

1. *$T$ diagonalisable*
2. *$V$ has a basis consisting of eigenvectors of $T$*
3. *$\exists U_1, \ldots, U_n$ 1-dimensional subspaces of $V$ invariant under $T$ s.t.*

$$V = U_1 \oplus \cdots \oplus U_n$$

4. $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$
5. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

**Theorem 2.162 (Sufficient condition for diagonalisability)**

*Let $V$ be a vector space, $T \in \mathcal{L}(V)$. If $T$ has $\dim V$ distinct eigenvalues, then $T$ diagonalisable*

# Inner product spaces

Inner products and norms

Orthonormal bases

Orthogonal complements & Minimisation problems

Inner products and norms

Orthonormal bases

Orthogonal complements & Minimisation problems

## Definition 2.163 (Inner product)

Let $V$ be a vector space over $\mathbb{F}$. An **inner product** on $V$ is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ having the following properties, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \lambda \in \mathbb{F}$,

- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ **[positivity]**
- $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$ **[definiteness]**
- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[additivity in first slot]**
- $\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle$ **[homogeneity in first slot]**
- $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ **[conjugate symmetry]**

## Definition 2.164 (Inner product space)

An **inner product space** is a vector space $V$ along with an inner product on $V$

## Theorem 2.165 (Basic properties of inner product)

*Let $V$ be an inner product space over $\mathbb{F}$. Then*

1. *For each fixed $\mathbf{u} \in V$, the function $\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{u} \rangle$ is a linear map from $V$ to $\mathbb{F}$*
2. $\forall \mathbf{u} \in V$, $\langle \mathbf{0}_V, \mathbf{u} \rangle = 0$
3. $\forall \mathbf{u} \in V$, $\langle \mathbf{u}, \mathbf{0}_V \rangle = 0$
4. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
5. $\forall \mathbf{u}, \mathbf{v} \in V$ *and* $\forall \lambda \in \mathbb{F}$, $\langle \mathbf{u}, \lambda \mathbf{v} \rangle = \overline{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$

## Definition 2.166 (Norm)

Let $V$ be an inner product space over $\mathbb{F}$. For $\mathbf{v} \in V$, the **norm** of $v$ is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

## Theorem 2.167 (Basic properties of the norm)

*Let $V$ be an inner product space, $\mathbf{v} \in V$. Then*

1. $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = 0$
2. $\forall \lambda \in \mathbb{F}$, $\|\lambda \mathbf{v}\| = |\lambda| \, \|\mathbf{v}\|$

### Definition 2.168 (Orthogonality)

Let $V$ be an inner product space over $\mathbb{F}$. Two vectors $\mathbf{u}, \mathbf{v} \in V$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. We sometimes denote $\mathbf{u} \perp \mathbf{v}$

### Theorem 2.169 ($\mathbf{0}$ and orthogonality)

*Let $V$ be an inner product space over $\mathbb{F}$. Then*

1. $\mathbf{0}_V$ *is orthogonal to every vector in* $V$
2. $\mathbf{0}_V$ *is the only vector in* $V$ *that is orthogonal to itself*

### Theorem 2.170 (Pythagorean theorem)

*Let $V$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ s.t. $\mathbf{u} \perp \mathbf{v}$. Then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

### Theorem 2.171 (An orthogonal decomposition)

*Let $V$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{v} \neq 0$. Let*

$$c = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \ (\in \mathbb{F}) \ \textit{and} \ \mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} \ (\in V).$$

*Then*

$$\langle \mathbf{w}, \mathbf{v} \rangle = 0 \ \textit{and} \ \mathbf{u} = c\mathbf{v} + \mathbf{w}.$$

### Theorem 2.172 (Cauchy-Schwarz inequality)

*Let $V$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

*with $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \, \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ for some $0 \neq k \in \mathbb{F}$.*

## Theorem 2.173 (Triangle inequality)

*Let $V$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then*

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

*with* $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\| \Leftrightarrow \mathbf{u} = k\mathbf{v}$ *for some* $0 \leq k \in \mathbb{R}$.

## Theorem 2.174 (Parallelogram equality)

*Let $V$ be an inner product space, $\mathbf{u}, \mathbf{v} \in V$. Then*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right).$$

Inner products and norms

Orthonormal bases

Orthogonal complements & Minimisation problems

## Definition 2.175 (Orthonormal list)

A list of vectors is **orthonormal** if each vector in the list has norm 1 and is orthogonal to all other vectors in the list, i.e., the list $\mathbf{e}_1, \ldots, \mathbf{e}_m$ of vectors in the inner product space $V$ is orthonormal if

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

## Theorem 2.176 (Norm of an orthonormal linear combination)

*Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space $V$. Then*

$$\|a_1\mathbf{e}_1 + \cdots + a_m\mathbf{e}_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

*for all $a_1, \ldots, a_m \in \mathbb{F}$.*

**Theorem 2.177 (Orthonormal lists are LI)**

*Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space $V$. Then $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is linearly independent*

**Definition 2.178 (Orthonormal basis)**

An **orthonormal basis** of an inner product space $V$ is an orthonormal list of vectors in $V$ that is also a basis of $V$

**Theorem 2.179 (Orthonormal list & orthonormal basis)**

*Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal list of vectors in an inner product space $V$. If* $\dim V = m$, *then $\mathbf{e}_1, \ldots, \mathbf{e}_m$ orthonormal basis of $V$.*

## Theorem 2.180 (Vector as LC of orthonormal basis)

*Let* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *be an orthonormal basis of the inner product space* $V$*,* $\mathbf{v} \in V$*. Then*

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n$$

*and*

$$\|\mathbf{v}\|^2 = |\langle \mathbf{v}, \mathbf{e}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{e}_n \rangle|^2$$

## Theorem 2.181 (Gram-Schmidt procedure)

*Let* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *be a linearly independent list of vectors in an inner product space* $V$. *Let*

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

*For* $j = 2, \ldots, m$, *define* $\mathbf{e}_j$ *inductively by*

$$\mathbf{e}_j = \frac{\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}}{\|\mathbf{v}_j - \langle \mathbf{v}_j, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_j, \mathbf{e}_{j-1} \rangle \mathbf{e}_{j-1}\|}$$

*Then* $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *is an orthonormal list of vectors in* $V$ *such that*

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_j) = \operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_j), \quad j = 1, \ldots, m$$

## Theorem 2.182 (Existence of orthonormal basis)

*Let $V$ be a finite-dimensional inner product space. Then $V$ has an orthonormal basis*

## Theorem 2.183 (Extending orthonormal list to basis)

*Let $V$ be a finite-dimensional inner product space. Then every orthonormal list of vectors in $V$ can be extended to an orthonormal basis of $V$*

### Theorem 2.184 (UT matrix wrt orthonormal basis)

*Let $V$ be a finite-dimensional inner product space, $T \in \mathcal{L}(V)$. If $T$ has an upper-triangular matrix with respect to some basis of $V$, then $T$ has an upper-triangular matrix with respect to some orthonormal basis of $V$*

### Theorem 2.185 (Schur's Theorem)

*Suppose $V$ is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$. Then $T$ has an upper-triangular matrix with respect to some orthonormal basis of $V$*

## Theorem 2.186 (Riesz representation Theorem)

*Let $V$ be a finite-dimensional inner product space, $\varphi \in \mathcal{L}(V, \mathbb{F})$ a linear functional on $V$. Then $\exists \mathbf{u} \in V$ unique s.t.*

$$\forall \mathbf{v} \in V, \quad \varphi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle.$$

Inner products and norms

Orthonormal bases

Orthogonal complements & Minimisation problems

## Definition 2.187 (Orthogonal complement)

Let $V$ be an inner product space, $U \subset V$. The **orthogonal complement** $U^\perp$ of $U$ is the set

$$U^\perp = \{\mathbf{v} \in V : \forall \mathbf{u} \in U, \quad \langle \mathbf{v}, \mathbf{u} \rangle = 0\}$$

## Property 2.188 (Basic properties of orthogonal complement)

1. *If $U \subset V$, then $U^\perp$ subspace of $V$*
2. $\{\mathbf{0}_V\}^\perp = V$
3. $V^\perp = \{\mathbf{0}_V\}$
4. *If $U \subset V$, then $U \cap U^\perp \subset \{0\}$*
5. *If $U \subset W \subset V$, then $W^\perp \subset U^\perp$*

## Theorem 2.189 (Direct sum $U$ and $U^\perp$)

*Let $U$ be a finite-dimensional subspace of $V$, inner product space. Then*

$$V = U \oplus U^\perp$$

## Theorem 2.190 (Dimension of $U^\perp$)

*Let $V$ be a finite-dimensional inner product space, $U$ subspace of $V$. Then*

$$\dim U^\perp = \dim V - \dim U$$

## Theorem 2.191 (Orth. complement of orth. complement)

*Let $U$ be a finite-dimensional subspace of the inner product space $V$. Then*

$$\left(U^\perp\right)^\perp = U$$

## Definition 2.192 (Orthogonal projection $P_U$)

Let $V$ be an inner product space, $U$ a finite-dimensional subspace of $V$. The **orthogonal projection** of $V$ onto $U$ is the operator $P_U \in \mathcal{L}(V)$ defined by

$$P_U \mathbf{v} = \mathbf{u},$$

where $\mathbf{v} \in V$ is written $\mathbf{v} = \mathbf{u} + \mathbf{w}$, with $\mathbf{u} \in U$ and $\mathbf{w} \in U^{\perp}$

## Property 2.193 (Properties of the orthogonal projection $P_U$)

*Let $V$ be an inner product space, $U$ a finite-dimensional subspace of $V$, $v \in V$. Then*

1. $P_U \in \mathcal{L}(V)$
2. $\forall \mathbf{u} \in U$, $P_U\mathbf{u} = \mathbf{u}$
3. $\forall \mathbf{w} \in U^\perp$, $P_U\mathbf{w} = \mathbf{0}_V$
4. $\mathrm{range} P_U = U$
5. $\mathrm{null} P_U = U^\perp$
6. $\mathbf{v} - P_U\mathbf{v} \in U^\perp$
7. $P_U^2 = P_U$
8. $\|P_U\mathbf{v}\| \leq \|\mathbf{v}\|$
9. *for every orthonormal basis* $\mathbf{e}_1, \ldots, \mathbf{e}_m$ *of* $U$,

$$P_U\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1\rangle\mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_m\rangle\mathbf{e}_m.$$

## Theorem 2.194 (Minimising distance to a subspace)

*Let $V$ be an inner product space, $U$ a finite-dimensional subspace of $V$, $\mathbf{v} \in V$, $u \in U$. Then*

$$\|\mathbf{v} - P_U\mathbf{v}\| \leq \|\mathbf{v} - \mathbf{u}\|$$

*with equality if and only if* $\mathbf{u} = P_U\mathbf{v}$

# Operators on inner product spaces

Self-adjoint and normal operators

Spectral theorems

Positive (semidefinite) operators & Isometries

Polar and Singular value decompositions

Self-adjoint and normal operators

Spectral theorems

Positive (semidefinite) operators & Isometries

Polar and Singular value decompositions

## Definition 2.195 (Adjoint)

Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. The **adjoint** of $T$ is the function $T^\star : W \to V$ such that

$$\forall \mathbf{v} \in V, \forall \mathbf{w} \in W, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T^\star \mathbf{w} \rangle$$

## Theorem 2.196 (Adjoint is a linear map)

*Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Then*

$$T^\star \in \mathcal{L}(W, V)$$

## Property 2.197 (Properties of the adjoint)

*Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{F}$. Then*

1. $\forall S, T \in \mathcal{L}(V, W)$, $(S + T)^\star = S^\star + T^\star$
2. $\forall T \in \mathcal{L}(V, W)$, $\forall \lambda \in \mathbb{F}$, $(\lambda T)^\star = \overline{\lambda} T^\star$
3. $\forall T \in \mathcal{L}(V, W)$, $(T^\star)^\star = T$
4. $I^\star = I$ *if* $I$ *is the identity operator on* $V$
5. *Let* $U$ *be an inner product space over* $\mathbb{F}$, *then* $\forall T \in \mathcal{L}(V, W)$ *and* $\forall S \in \mathcal{L}(W, U)$, $(ST)^\star = T^\star S^\star$

## Theorem 2.198 (Null space and range of $T^\star$)

*Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Then*

1. $\operatorname{null} T^\star = (\operatorname{range} T)^\perp$
2. $\operatorname{range} T^\star = (\operatorname{null} T)^\perp$
3. $\operatorname{null} T = (\operatorname{range} T^\star)^\perp$
4. $rangeT = (\operatorname{null} T^\star)^\perp$

## Definition 2.199 (Conjugate transpose)

Let $M \in \mathcal{M}_{mn}(\mathbb{F})$, $M = [m_{ij}]$. The **conjugate transpose** of $M$, often denoted $M^\star$, is the matrix

$$M^\star = [\overline{m_{ji}}] \in \mathcal{M}_{nm}$$

i.e, the matrix obtained by transposing $M$ then taking the (complex) conjugate of each entry

## Theorem 2.200 (Matrix of $T^\star$)

*Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{F}$, $T \in \mathcal{L}(V, W)$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be orthonormal bases of $V$ and $W$, respectively. Then*

$$M(T^\star, (\mathbf{f}_1, \ldots, \mathbf{f}_m), (\mathbf{e}_1, \ldots, \mathbf{e}_n))$$

*is the conjugate transpose of*

$$M(T, (\mathbf{e}_1, \ldots, \mathbf{e}_n), (\mathbf{f}_1, \ldots, \mathbf{f}_m))$$

## Definition 2.201 (Self-adjoint operator)

Let $V$ be an inner product space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. $T$ is **self-adjoint** (or **Hermitian**) if

$$T = T^\star$$

In other words, $T \in \mathcal{L}(V)$ self-adjoint $\iff$

$$\forall \mathbf{v}, \mathbf{w} \in V, \quad \langle T\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, T\mathbf{w} \rangle$$

## Theorem 2.202 (Eigenvalues of self-adjoint operators are real)

*Let $V$ be an inner product space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Then all eigenvalues of $T$ are real*

## Theorem 2.203

*Let $V$ be a complex inner product space, $T \in \mathcal{L}(V)$. Then*

$$(\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v}\rangle = 0) \quad \Rightarrow \quad T = \mathbf{0}$$

## Theorem 2.204

*Let $V$ be a complex inner product space, $T \in \mathcal{L}(V)$. Then*

$$(T \ \textit{self-adjoint}) \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v}\rangle \in \mathbb{R})$$

## Theorem 2.205

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint. Then*

$$(\forall \mathbf{v} \in V,\ \langle T\mathbf{v}, \mathbf{v} \rangle = 0) \quad \Rightarrow \quad T = 0$$

## Definition 2.206 (Normal operator)

Let $V$ be an inner product space, $T \in \mathcal{L}(V)$. $T$ is **normal** if

$$TT^\star = T^\star T$$

In words, $T$ is normal if it commutes with its adjoint

## Theorem 2.207 ($T$ normal $\Leftrightarrow \|T\mathbf{v}\| = \|T^\star\mathbf{v}\|$)

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$. Then*

$$T \text{ normal} \quad \Leftrightarrow \quad (\forall \mathbf{v} \in V,\ \|T\mathbf{v}\| = \|T^\star\mathbf{v}\|)$$

### Theorem 2.208 ($T$ normal and $T^\star$ have same eigenvectors)

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. Then*

$$(\lambda, \mathbf{v}) \text{ eigenpair of } T \quad \Leftrightarrow \quad (\bar{\lambda}, \mathbf{v}) \text{ eigenpair of } T^\star$$

### Theorem 2.209 (Orthogonal eigenvectors for normal operators)

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$ a normal operator. If $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ eigenpairs of $T$ with $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1 \perp \mathbf{v}_2$.*

### Theorem 2.210

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$ self-adjoint and $b, c \in \mathbb{R}$ s.t. $b^2 < 4c$. Then $T^2 + bT + cI$ invertible*

### Theorem 2.211 (Self-adjoint operators have eigenvalues)

*Let $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ be self-adjoint. Then $T$ has an eigenvalue*

### Theorem 2.212 (Self-adjoint operators & invariant subspaces)

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$ be self-adjoint and $U$ be a subspace of $V$ invariant under $T$. Then*

1. $U^\perp$ *invariant under* $T$
2. $T|_U \in \mathcal{L}(U)$ *self-adjoint*
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ *self-adjoint*

Self-adjoint and normal operators

Spectral theorems

Positive (semidefinite) operators & Isometries

Polar and Singular value decompositions

### Theorem 2.213 (Complex spectral theorem)

*Let $V$ be an inner product space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. TFAE:*

1. *$T$ normal*
2. *$V$ has an orthonormal basis consisting of eigenvectors of $T$*
3. *$T$ has a diagonal matrix with respect to some orthonormal basis of $V$*

### Theorem 2.214 (Real spectral theorem)

*Let $V$ be an inner product space over $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$. TFAE:*

1. *$T$ self-adjoint*
2. *$V$ has an orthonormal basis consisting of eigenvectors of $T$*
3. *$T$ has a diagonal matrix with respect to some orthonormal basis of $V$*

Self-adjoint and normal operators

Spectral theorems

Positive (semidefinite) operators & Isometries

Polar and Singular value decompositions

## Definition 2.215 (Positive (semidefinite) operator)

Let $V$ be an inner product space. An operator $T \in \mathcal{L}(V)$ is **positive** (or **positive semidefinite**) if $T$ is self-adjoint and

$$\forall \mathbf{v} \in V, \quad \langle T\mathbf{v}, \mathbf{v} \rangle \geq 0$$

## Definition 2.216 (Square root operator)

Let $V$ be an inner product space. An operator $R \in \mathcal{L}(V)$ is a **square root** of an operator $T \in \mathcal{L}(V)$ if

$$R^2 = T$$

## Theorem 2.217 (Characterisation of positive operators)

*Let $T \in \mathcal{L}(V)$, where $V$ is an inner product space. TFAE:*

1. *$T$ positive semidefinite*
2. *$T$ self-adjoint and all eigenvalues of $T$ are nonnegative*
3. *$T$ has a positive semidefinite square root*
4. *$T$ has a self-adjoint square root*
5. *$\exists R \in \mathcal{L}(V)$ s.t. $T = R^\star R$*

## Theorem 2.218 (Uniqueness of positive semidefinite square root)

*Let $T \in \mathcal{L}(V)$ be a positive semidefinite operator on an inner product space $V$. Then $T$ has a unique positive semidefinite square root*

## Definition 2.219 (Isometry)

Let $V$ be an inner product space. $S \in \mathcal{L}(V)$ is an **isometry** if

$$\forall \mathbf{v} \in V, \quad \|S\mathbf{v}\| = \|\mathbf{v}\|$$

## Theorem 2.220 (Characterisation of isometries)

*Let $V$ be an inner product space, $S \in \mathcal{L}(V)$. TFAE:*

1. $S$ *isometry*
2. $\forall \mathbf{u}, \mathbf{v} \in V$, $\langle S\mathbf{u}, S\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$
3. $\forall \mathbf{e}_1, \ldots, \mathbf{e}_n \in V$ *orthonormal list*, $S\mathbf{e}_1, \ldots, S\mathbf{e}_n$ *orthonormal*
4. $\exists \mathbf{e}_1, \ldots, \mathbf{e}_n$ *orthonormal basis of* $V$ *s.t.* $S\mathbf{e}_1, \ldots, S\mathbf{e}_n$ *orthonormal*
5. $S^\star S = I$
6. $SS^\star = I$
7. $S^\star$ *isometry*
8. $S$ *invertible and* $S^{-1} = S^\star$

### Theorem 2.221 (Isometries when $\mathbb{F} = \mathbb{C}$)

*Let $V$ be a complex inner product space, $S \in \mathcal{L}(V)$. TFAE:*

1. *$S$ isometry*
2. *$\exists$ orthonormal basis of $V$ consisting of eigenvectors of $S$ with corresponding eigenvalues all having modulus 1*

Self-adjoint and normal operators

Spectral theorems

Positive (semidefinite) operators & Isometries

Polar and Singular value decompositions

Let $T$ be a positive semidefinite operator, then denote $\sqrt{T}$ the unique positive semidefinite square root of $T$

## Theorem 2.222 (Polar decomposition)

*Let $V$ be an inner product space, $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ s.t.*

$$T = S\sqrt{T^\star T}$$

## Definition 2.223 (Singular values)

Let $V$ be an inner product space, $T \in \mathcal{L}(V)$. The **singular values** of $T$ are the eigenvalues of $\sqrt{T^\star T}$, with each eigenvalue $\lambda$ repeated $\dim E(\lambda, \sqrt{T^\star T})$ times. All are nonnegative

## Theorem 2.224 (Singular value decomposition – SVD)

*Let $V$ be an inner product space. Assume $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then $\exists \mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ orthonormal bases of $V$ s.t.*

$$\forall \mathbf{v} \in V, \quad T\mathbf{v} = s_1 \langle v, \mathbf{e}_1 \rangle \mathbf{f}_1 + \cdots + s_n \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{f}_n$$

## Theorem 2.225 (SV without square root)

*Let $V$ be an inner product space. The singular values of $T$ are the nonnegative square roots of the eigenvalues of $T^\star T$, with each eigenvalue $\lambda$ repeated $\dim E(\lambda, T^\star T)$ times*

# Operators on complex vector spaces

Generalised eigenvectors & Nilpotent operators

Decomposition of an operator

Characteristic and minimal polynomials

Jordan form

Generalised eigenvectors & Nilpotent operators

Decomposition of an operator

Characteristic and minimal polynomials

Jordan form

## Theorem 2.226 (Sequence of increasing null spaces)

*Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Then*

$$\{0\} = \operatorname{null} T^0 \subset \operatorname{null} T^1 \subset \cdots \subset \operatorname{null} T^k \subset \operatorname{null} T^{k+1} \subset \cdots$$

## Theorem 2.227 (Equality in sequence of null spaces)

*Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$. Assume $m \in \mathbb{N} \setminus \{0\}$ is s.t.*

$$\operatorname{null} T^m = \operatorname{null} T^{m+1}$$

*Then*

$$\forall k \in \mathbb{N}, \quad \operatorname{null} T^{m+k} = \operatorname{null} T^m$$

## Theorem 2.228 (Null spaces stop growing)

*Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ with* $\dim V = n$, $T \in \mathcal{L}(V)$. *Then*

$$\forall k \in \mathbb{N}, \quad \operatorname{null} T^{n+k} = \operatorname{null} T^n$$

## Theorem 2.229 ($V = \operatorname{null} T^{\dim V} \oplus \operatorname{range} T^{\dim V}$)

*Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ with* $\dim V = n$, $T \in \mathcal{L}(V)$. *Then*

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n$$

## Definition 2.230 (Generalised eigenvector)

Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$ an eigenvalue of $T$. $\mathbf{v} \in V$ is a **generalised eigenvector** of $T$ corresponding to $\lambda$ if $\mathbf{v} \neq 0$ and

$$\exists j \in \mathbb{N} \setminus \{0\}, \quad (T - \lambda I)^j \mathbf{v} = \mathbf{0}_V$$

## Definition 2.231 (Generalised eigenspace)

Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalised eigenspace** $G(\lambda, T)$ of $T$ corresponding to $\lambda$ is the set of all generalised eigenvectors of $T$ corresponding to $\lambda$ together with the $\mathbf{0}_V$ vector

## Theorem 2.232 (Description of generalised eigenspaces)

*Let* $T \in \mathcal{L}(V)$ *and* $\lambda \in \mathbb{F}$. *Then*

$$G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}$$

## Theorem 2.233 (LI generalised eigenvectors)

*Let* $T \in \mathcal{L}(V)$. *Assume* $\lambda_1, \ldots, \lambda_m$ *are distinct eigenvalues of* $T$, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *corresponding generalised eigenvectors. Then* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *linearly independent.*

### Definition 2.234 (Nilpotent operator)

An operator is **nilpotent** if $\exists k \in \mathbb{N}$ s.t. $T^k = 0$

### Theorem 2.235 (A loose upper bound on power required)

*Let $N \in \mathcal{L}(V)$ be nilpotent. Then*

$$N^{\dim V} = 0$$

### Theorem 2.236 (Matrix of a nilpotent operator)

*Let $N \in \mathcal{L}(V)$ be nilpotent. Then there exists a basis of $V$ with respect to which $M(N)$ is* **strictly upper triangular***, i.e.,*

$$M(N) = [m_{ij}] \ \mathit{is\ s.t.}\ m_{ij} = 0 \ \mathit{if}\ i \geq j$$

Generalised eigenvectors & Nilpotent operators

Decomposition of an operator

Characteristic and minimal polynomials

Jordan form

## Theorem 2.237 ( & range of $p(T)$ invariant under $T$)

*Let* $T \in \mathcal{L}(V)$ *and* $p \in \mathcal{P}(\mathbb{F})$*. Then* $p(T)$ *and* range $p(T)$ *invariant under* $T$

## Theorem 2.238 (Description of operators when $\mathbb{F} = \mathbb{C}$)

*Suppose* $V$ *complex vector space,* $T \in \mathcal{L}(V)$*. Assume* $\lambda_1, \ldots, \lambda_m$ *be the distinct eigenvalues of* $T$*. Then*

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
2. *each* $G(\lambda_j, T)$ *invariant under* $T$
3. $\forall j = 1, \ldots, m,$ $(T - \lambda_j I)|_{G(\lambda_j, T)}$ *nilpotent*

## Theorem 2.239 (Basis of generalised eigenvectors)

*Let* $V$ *be a complex vector space and* $T \in \mathcal{L}(V)$*. Then there exists a basis of* $V$ *consisting of generalised eigenvectors of* $T$

## Definition 2.240 (Multiplicity of an eigenvalue)

Let $T \in \mathcal{L}(V)$. The (**algebraic**) **multiplicity** of an eigenvalue $\lambda$ of $T$ is

- $\dim G(\lambda, T)$
- $\dim (T - \lambda I))^{\dim V}$

## Theorem 2.241 ($\sum$ multiplicities $= \dim V$)

*Let $V$ be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be distinct eigenvalues of $T$ with multiplicities $d_1, \ldots, d_n$. Then*

$$\sum_{k=1}^{n} d_k = \dim V$$

## Definition 2.242 (Block diagonal matrix)

Let $A_1, \dots, A_m$ be square matrices (not necessarily of the same size). A **block matrix** is a matrix of the form

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

We also write

$$A = \operatorname{diag}(A_1, \dots, A_m)$$

You will also see (not in this book)

$$A = A_1 \oplus \cdots \oplus A_m$$

## Theorem 2.243 (Block diagonal matrix with UT blocks)

*Let $V$ be a complex vector space, $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$ with multiplicities $d_1, \ldots, d_m$. Then there exists a basis of $V$ s.t. $T$ has a block diagonal matrix*

$$\operatorname{diag}(A_1, \ldots, A_m)$$

*with each $A_j$ a $d_j \times d_j$ upper-triangular matrix of the form*

$$A_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$

### Theorem 2.244 (Identity plus nilpotent has square root)

*Let $N \in \mathcal{L}(V)$ be nilpotent. Then $I + N$ has a square root*

### Theorem 2.245 ($T$ invertible has square root when $\mathbb{F} = \mathbb{C}$)

*Let $V$ be a complex vector space, $T \in \mathcal{L}(V)$ invertible. Then $T$ has a square root*

Generalised eigenvectors & Nilpotent operators

Decomposition of an operator

Characteristic and minimal polynomials

Jordan form

### Definition 2.246 (Characteristic polynomial)

Let $V$ be a complex vector space, $T \in \mathcal{L}(V)$, $\lambda_1, \ldots, \lambda_m$ the distinct eigenvalues of $T$ with multiplicities $d_1, \ldots, d_m$. The **characteristic polynomial** of $T$ is

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

### Theorem 2.247 (Degree and zeros of char. polyn.)

*$V$ a complex vector space, $T \in \mathcal{L}(V)$. Then*

1. *the characteristic polynomial of $T$ has degree* $\dim V$
2. *zeros of the characteristic polynomial of $T$ are the eigenvalues of $T$*

### Theorem 2.248 (Cayley-Hamilton)

*Let $V$ be a complex vector space, $T \in \mathcal{L}(V)$. Let $q$ be the characteristic polynomial of $T$. Then $q(T) = 0$*

### Definition 2.249 (Monic polynomial)

A **monic polynomial** is a polynomial with highest degree coefficient equal to 1

### Theorem 2.250 (Minimal polynomial)

*Let $T \in \mathcal{L}(V)$. Then there exists a unique monic polynomial $p$ of smallest degree s.t. $p(T) = 0$*

### Definition 2.251 (Minimal polynomial)

Let $T \in \mathcal{L}(V)$. The **minimal polynomial** of $T$ is the unique monic polynomial $p$ of smallest degree s.t. $p(T) = 0$

### Theorem 2.252

*Let $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0 \Leftrightarrow q$ polynomial multiple of the minimal polynomial of $T$*

### Theorem 2.253 (Char. polyn. is multiple of min. polyn.)

*Assume $V$ vector space over $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$. Then the characteristic polynomial of $T$ is a polynomial multiple of the minimal polynomial of $T$*

### Theorem 2.254 (Eigenvalues are zeros of min. polyn.)

*Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of $T$ are precisely the eigenvalues of $T$*

Generalised eigenvectors & Nilpotent operators

Decomposition of an operator

Characteristic and minimal polynomials

Jordan form

### Theorem 2.255 (Basis corresponding to nilpotent operator)

*Let* $N \in \mathcal{L}(V)$ *be nilpotent. Then* $\exists \mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ *and* $m_1, \ldots, m_n \in \mathbb{N}$ *s.t.*

1. $N^{m_1}\mathbf{v}_1, \ldots, N\mathbf{v}_1, \mathbf{v}_1, N^{m_n}\mathbf{v}_n, \ldots, N\mathbf{v}_n, \mathbf{v}_n$ *is a basis of* $V$
2. $N^{m_1+1}\mathbf{v}_1 = \cdots = N^{m_n+1}\mathbf{v}_n = 0$

### Definition 2.256 (Jordan basis)

Let $T \in \mathcal{L}(V)$. A **Jordan basis** for $T$ is a basis of $V$ s.t. with respect to this basis, $T$ has a block diagonal matrix

$$\operatorname{diag}(A_1, \ldots, A_p)$$

where each $A_j$ is an upper-triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

### Theorem 2.257 (Jordan form)

*Let $V$ be a complex vector space. If $T \in \mathcal{L}(V)$, then $\exists$ a Jordan basis for $T$*

# An algorithm for finding the Jordan form

An algorithm to compute the Jordan canonical form of an $n \times n$ matrix $A$ [MM82].

1. Compute the eigenvalues of $A$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $A$ with multiplicities $n_1, \ldots, n_m$, respectively.
2. Compute $n_1$ linearly independent generalized eigenvectors of $A$ associated with $\lambda_1$ as follows. Compute
$$(A - \lambda_1 E_n)^i$$
for $i = 1, 2, \ldots$ until the rank of $(A - \lambda_1 E_n)^k$ is equal to the rank of $(A - \lambda_1 E_n)^{k+1}$. Find a generalized eigenvector of rank $k$, say $u$. Define $u_i = (A - \lambda_1 E_n)^{k-1} u$, for $i = 1, \ldots, k$. If $k = n_1$, proceed to step 3. If $k < n_1$, find another linearly independent generalized eigenvector with rank $k$. If this is not possible, try $k - 1$, and so forth, until $n_1$ linearly independent generalized eigenvectors are determined. Note that if $\rho(A - \lambda_1 E_n) = r$, then there are totally $(n - r)$ chains of generalized eigenvectors associated with $\lambda_1$.
3. Repeat step 2 for $\lambda_2, \ldots, \lambda_m$.

1. Let $u_1, \ldots, u_k, \ldots$ be the new basis. Observe that Thus in the new basis, $A$ has the desired representation
2. The similarity transformation which yields $J = Q^{-1}AQ$ is given by $Q = [u_1, \ldots, u_k, \ldots]$.

# References I


R.K Miller and A.N. Michel, *Ordinary differential equations*, Academic Press, 1982.