



University  
of Manitoba

## The endemic SLIRS model

MATH 8xyz – Lecture 07

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.

We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

# Outline



**SLIRS model with constant population**



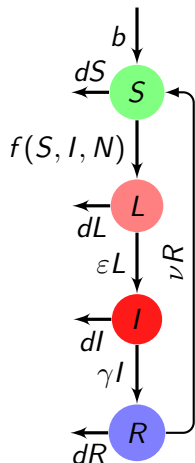
## Incubation periods

- ▶ SIS and SIR: progression from S to I is instantaneous
- ▶ Several incubation periods:

| Disease                       | Incubation period |
|-------------------------------|-------------------|
| Yersinia Pestis               | 2-6 days          |
| Ebola haemorrhagic fever (HF) | 2-21 days         |
| Marburg HF                    | 5-10 days         |
| Lassa fever                   | 1-3 weeks         |
| Tse-tse                       | weeks–months      |
| HIV/AIDS                      | months–years      |

# Hypotheses

- ▶ There is demography
- ▶ New individuals are born at a constant rate  $b$
- ▶ There is no vertical transmission: all “newborns” are susceptible
- ▶ The disease is non lethal, it causes no additional mortality
- ▶ New infections occur at the rate  $f(S, I, N)$
- ▶ There is a period of incubation for the disease
- ▶ There is a period of time after recovery during which the disease confers immunity to reinfection (immune period)



The model is as follows:

$$S' = b + \nu R - dS - f(S, I, N) \quad (1a)$$

$$L' = f(S, I, N) - (d + \varepsilon)L \quad (1b)$$

$$I' = \varepsilon L - (d + \gamma)I \quad (1c)$$

$$R' = \gamma I - (d + \nu)R \quad (1d)$$

Meaning of the parameters:

- ▶  $1/\varepsilon$  average duration of the incubation period
- ▶  $1/\gamma$  average duration of infectious period
- ▶  $1/\nu$  average duration of immune period

## Example of the SLIRS model (1)

Variation of the infected variables in (1) are described by

$$L' = f(S, I, N) - (\varepsilon + d)L$$

$$I' = \varepsilon L - (d + \gamma)I$$

Write

$$\mathcal{I}' = \begin{pmatrix} L \\ I \end{pmatrix}' = \begin{pmatrix} f(S, I, N) \\ 0 \end{pmatrix} - \begin{pmatrix} (\varepsilon + d)L \\ (d + \gamma)I - \varepsilon L \end{pmatrix} =: \mathcal{F} - \mathcal{V} \quad (2)$$

Denote

$$f_L^* := \left. \frac{\partial}{\partial L} f \right|_{(S,I,R)=\mathbf{E}_0} \quad f_I^* := \left. \frac{\partial}{\partial I} f \right|_{(S,I,R)=\mathbf{E}_0}$$

the values of the partials of the incidence function at the DFE  $\mathbf{E}_0$

Compute the Jacobian matrices of vectors  $\mathcal{F}$  and  $\mathcal{V}$  at the DFE  $\mathbf{E}_0$

$$F = \begin{pmatrix} f_L^* & f_I^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \varepsilon + d & 0 \\ -\varepsilon & d + \gamma \end{pmatrix} \quad (3)$$



Thus

$$V^{-1} = \frac{1}{(d + \varepsilon)(d + \gamma)} \begin{pmatrix} d + \gamma & 0 \\ \varepsilon & d + \varepsilon \end{pmatrix}$$

Also, in the case  $N$  is constant,  $\partial f / \partial L = 0$  and thus

$$FV^{-1} = \frac{f_I^*}{(d + \varepsilon)(d + \gamma)} \begin{pmatrix} \varepsilon & d + \varepsilon \\ 0 & 0 \end{pmatrix}$$

As a consequence,

$$\mathcal{R}_0 = \varepsilon \frac{f_I^*}{(d + \varepsilon)(d + \gamma)}$$

## Theorem 1

Let

$$\mathcal{R}_0 = \frac{\varepsilon f_I^*}{(d + \varepsilon)(d + \gamma)} \quad (4)$$

Then

- ▶ if  $\mathcal{R}_0 < 1$ , the DFE is LAS
- ▶ if  $\mathcal{R}_0 > 1$ , the DFE is unstable

It is important here to stress that the result we obtain concerns the **local** asymptotic stability. We see later that even when  $\mathcal{R}_0 < 1$ , there can be several locally asymptotically stable equilibria

## Application

The DFE is

$$(\bar{S}, \bar{L}, \bar{I}, \bar{R}) = (N, 0, 0, 0)$$

- Mass action incidence (frequency-dependent contacts):

$$f_I^* = \beta \bar{S} \Rightarrow \mathcal{R}_0 = \frac{\epsilon \beta N}{(\epsilon + d)(\gamma + d)}$$

- Standard incidence (proportion-dependent contacts):

$$f_I^* = \frac{\beta \bar{S}}{N} \Rightarrow \mathcal{R}_0 = \frac{\epsilon \beta}{(\epsilon + d)(\gamma + d)}$$

## Links between SLIRS-type models

$$S' = b + \nu R - dS - f(S, I, N)$$

$$L' = f(S, I, N) - (d + \varepsilon)L$$

$$I' = \varepsilon L - (d + \gamma)I$$

$$R' = \gamma I - (d + \nu)R$$

|      |  |
|------|--|
| SLIR | SLIRS where $\nu = 0$                                |
| SLIS | Limit of SLIRS when $\nu \rightarrow \infty$         |
| SLI  | SLIR where $\gamma = 0$                              |
| SIRS | Limit of SLIRS when $\varepsilon \rightarrow \infty$ |
| SIR  | SIRS where $\nu = 0$                                 |
| SIS  | Limit of SIRS when $\nu \rightarrow \infty$          |
|      | Limit SLIS when $\varepsilon \rightarrow \infty$     |
| SI   | SIS where $\nu = 0$                                  |

## Values of $\mathcal{R}_0$

$(\bar{S}, \bar{I}, \bar{N})$  values of  $S, I$  and  $N$  at DFE. Denote  $\bar{f}_I = \partial f / \partial I(\bar{S}, \bar{I}, \bar{N})$ .

|       |   |
|-------|---|
| SLIRS | $\frac{\varepsilon \bar{f}_I}{(d+\varepsilon)(d+\gamma)}$ |
| SLIR  | $\frac{\varepsilon \bar{f}_I}{(d+\varepsilon)(d+\gamma)}$ |
| SLIS  | $\frac{\varepsilon \bar{f}_I}{(d+\varepsilon)(d+\gamma)}$ |
| SLI   | $\frac{\varepsilon \bar{f}_I}{(d+\varepsilon)(d+\gamma)}$ |
| SIRS  | $\frac{\varepsilon \bar{f}_I}{d+\gamma}$                  |
| SIR   | $\frac{\bar{f}_I}{d+\gamma}$                              |
| SIS   | $\frac{\bar{f}_I}{d+\gamma}$                              |
| SI    | $\frac{\bar{f}_I}{d+\gamma}$                              |

## Can I have this wrapped up to go?

To finish, we use the command `purl` to generate an R file (course-01-introduction-math-epi.R) in the CODE directory with all the code chunks in this Rnw file

```
# From https://stackoverflow.com/questions/36868287/purl-within-knit-duplic
rmd_chunks_to_r_temp <- function(file){
  callr::r(function(file, temp){
    out_file = sprintf("../CODE/%s", gsub(".Rnw", ".R", file))
    knitr::purl(file, output = out_file, documentation = 1)
  }, args = list(file))
}
rmd_chunks_to_r_temp("L09-endemic-SLIRS.Rnw")

## [1] "../CODE/L09-endemic-SLIRS.R"
```

# Bibliography I