Sojourn time in compartments

MATH 8xyz – Lecture 06

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Outline

Distributions of times to events

Two "extreme" distributions and a nicer one

A simple cohort model with death

Possible fixes to the exponential distribution issue

Sojourn times in an SIS disease transmission model

Distributions of times to events

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Sojourn times in an SIS disease transmission model

See some of the work of Horst Thieme on the subject

If one considers time of sojourn in compartments from a more detailed perspective, one obtains integro-differential models

We use here continuous random variables. See chapters 12 and 13 in [**?**] (link) for arbitrary distributions

Time to events

Suppose that a system can be in two states *A* and *B*

- At time $t = 0$, the system is in state *A*

- An event happens at some time $t = \tau$, which triggers the switch from state *A* to state *B*

Let *T* be the random variable "time spent in state *A* before switching into state *B*"

The states can be anything:

- ▶ *A*: working, *B*: broken
- ▶ *A*: infected, *B*: recovered
- ▶ *A*: alive, *B*: dead
- ▶ . . .

We take a collection of objects or individuals that are in state *A* and want some law for the **distribution** of the times spent in *A*, i.e., a law for *T*

For example, we make light bulbs and would like to tell our customers that on average, our light bulbs last 200 years...

We conduct an **infinite** number of experiments, and observe the time that it takes, in every experiment, to switch from *A* to *B*
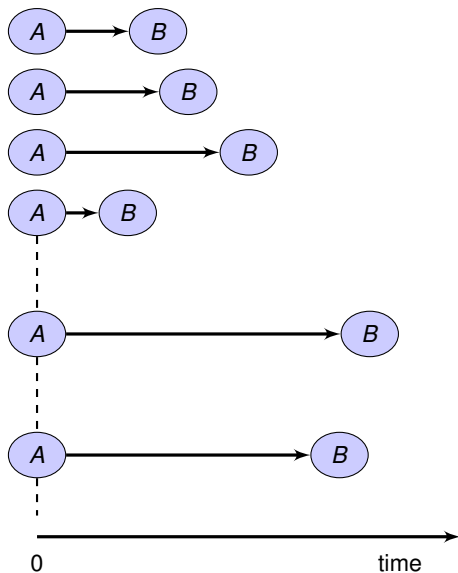
A distribution of probability is a model

From the sequence of experiments, we deduce a model, which in this context is called a **probability distribution**
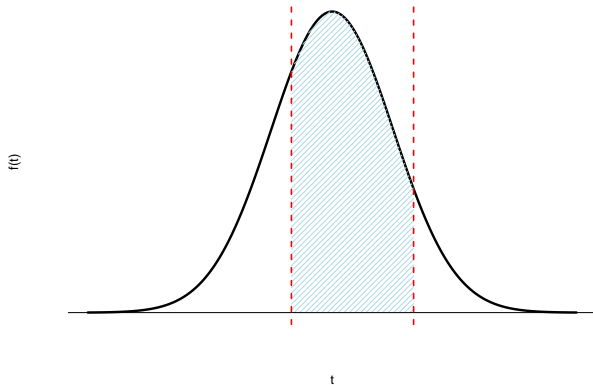
We assume that T is a **continuous** random variable

Probability density function (p.d.f.)

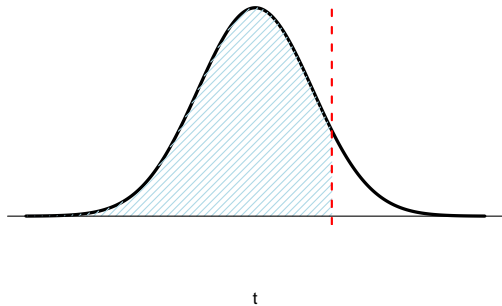Since *T* is continuous, it has a continuous **probability density function** *f*

- $f \geq 0$
- $\int_{-\infty}^{+\infty} f(s)ds = 1$
- $\mathbb{P}(a \leq T \leq b) = \int_{a}^{b} f(t)dt$

Cumulative distribution function (c.d.f.)

The **cumulative distribution function** is a function $F(t)$ that characterizes the distribution of T, and defined by

$$F(s) = \mathbb{P}(T \le s) = \int_{-\infty}^{s} f(x)dx$$

Survival function

Another characterization of the distribution of the random variable *T* is through the **survival** (or **sojourn**) function

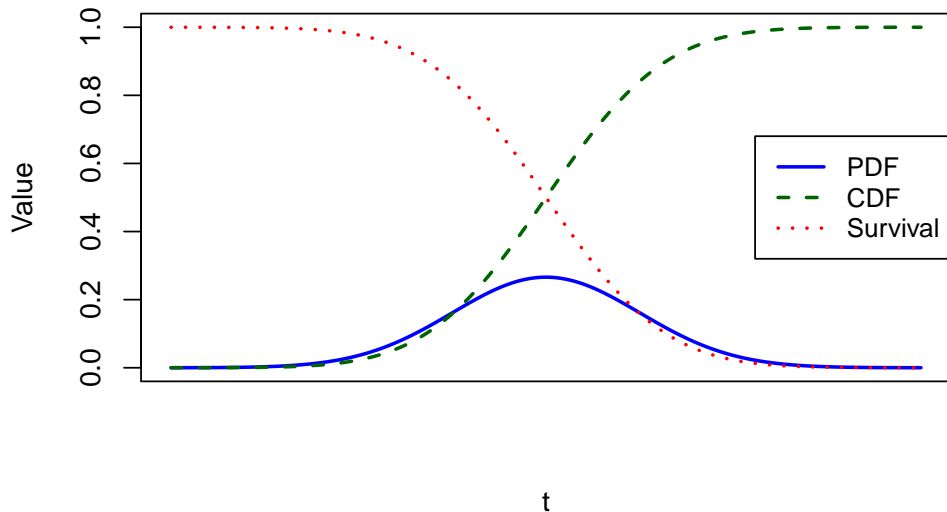The survival function of state *A* is given by

$$\mathcal{S}(t) = 1 - F(t) = \mathbb{P}(T > t) \tag{1}$$

This gives a description of the **sojourn time** of a system in a particular state (the time spent in the state)

$\mathcal{S}$ is a nonincreasing function (since $\mathcal{S} = 1 - F$ with F a c.d.f.), and $\mathcal{S}(0) = 1$ (since *T* is a nonnegative random variable)

PD, CD and Survival functions

The **average sojourn time** τ in state *A* is given by

$$\tau = E(T) = \int_0^\infty t f(t) dt$$

Since $\lim_{t \to \infty} t\mathcal{S}(t) = 0$, it follows that

$$\tau = \int_0^\infty \mathcal{S}(t) dt$$

Expected future lifetime:

$$\frac{1}{\mathcal{S}(t_0)} \int_0^\infty t f(t + t_0) dt$$

$$
\begin{aligned}
\mathcal{S}(t) - \mathcal{S}(a) &= \mathbb{P}\{\text{survive during } (a, t) \text{ having survived until } a\} \\
&= \exp\left(-\int_a^t h(u) du\right)
\end{aligned}
$$

Hazard rate

The **hazard rate** (or **failure rate**) is

$$h(t) = \lim_{\Delta t \to 0} \frac{\mathcal{S}(t) - \mathcal{S}(t + \Delta t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\mathbb{P}(T < t + \Delta t | T \geq t)}{\Delta t}$$
$$= \frac{f(t)}{\mathcal{S}(t)}$$

It gives probability of failure between t and Δt, given survival to t.

We have

$$h(t) = -\frac{d}{dt} \ln \mathcal{S}(t)$$

PD, CD and Survival functions & Hazard rate
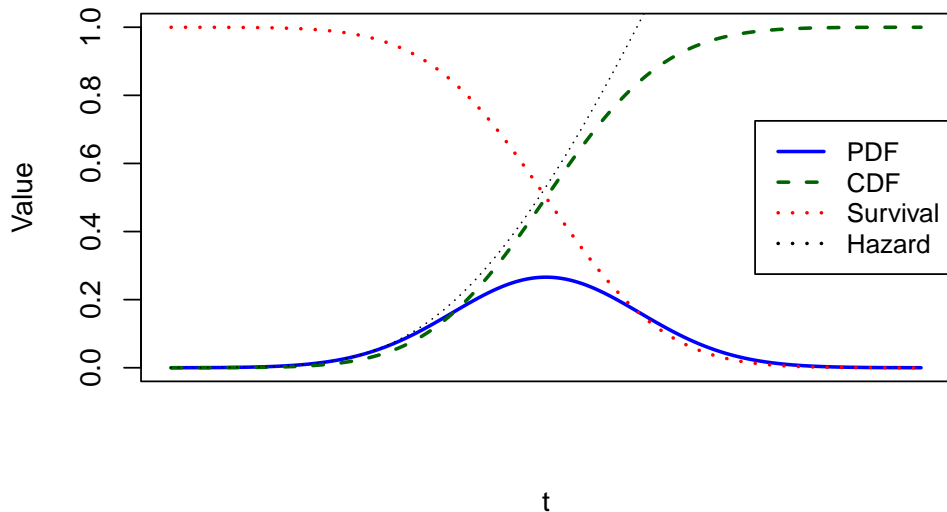
Competing risks

Suppose now that the system starts in state *A* at time $t = 0$ and that depending on which of the two events $\mathcal{E}_1$ or $\mathcal{E}_2$ takes place first, it switches to state B_1 or B_2, respectively

Consider the random variables T_A, *time spent* in state *A* (or sojourn time in *A*), T_{AB_1}, *time before switch to* B_1 and T_{AB_2}, *time before switch to* B_2

If we consider state *A*, we cannot observe the variables T_{AB_1} or T_{AB_2}. What is observable is the sojourn time in *A*

$$T_A^* = \min\left(T_{AB_1}, T_{AB_2}\right)$$

(where * indicates that a quantity is observable)

Failure rate by type of event

We have two (or more) types of events whose individual failure rates have to be accounted for

$$h_j(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}(T < t + \Delta t, S = S_j | T \geq t)}{\Delta t}$$

where $\mathbb{P}(T < t + \Delta t, S = S_j | T \geq t)$ is the probability of failure due to cause S_j ($j = 1, 2$ ici), i.e., S is a discrete r.v. representing the event that is taking place

By the law of total probability, since only one of the event can take place, if there are *n* risks, then

$$h(t) = \sum_{i=1}^{n} h_j(t)$$

or, identically,

$$S(t) = \exp\left(-\int_0^t \sum_{j=1}^n h_j(s)\, ds\right)$$

As a consequence, suppose a process is subject to two competing exponential risks with respective distributions with parameters θ_1 and θ_2

Then the mean sojourn time in the initial state before being affected by one of the two risks is

$$\frac{1}{\theta_1 + \theta_2}$$

The exponential distribution

The random variable *T* has an **exponential** distribution if its probability density function takes the form

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \theta e^{-\theta t} & \text{if } t \geq 0, \end{cases} \tag{2}$$

with $\theta > 0$. Then the survival function for state *A* is of the form $\mathcal{S}(t) = e^{-\theta t}$, for $t \geq 0$, and the average sojourn time in state *A* is

$$\tau = \int_0^\infty e^{-\theta t} dt = \frac{1}{\theta}$$

Particularities of the exponential distribution

The standard deviation of an exponential distribution is also $1/\theta$. When estimating θ, it is impossible to distinguish the mean and the standard deviation
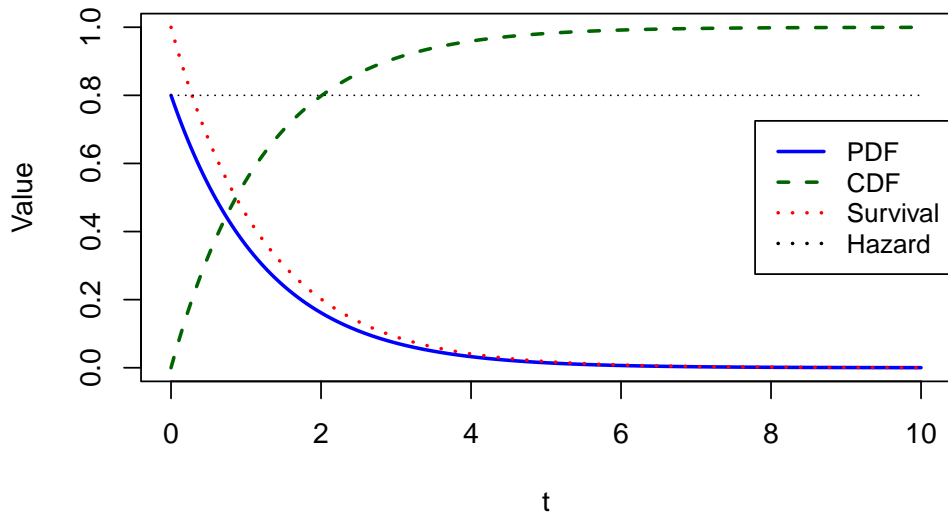
The exponential distribution is **memoryless**: its conditional probability obeys

$$P(T > s + t \mid T > s) = P(T > t), \quad \forall s, t \geq 0$$

The exponential and geometric distributions are the only memoryless probability distributions

The exponential distribution has a constant hazard function $h(t) \equiv \theta$

PD, CD and Surv. functions & Hazard rate of exponential

The Dirac delta distribution

If for some constant $\omega > 0$,

$$\mathcal{S}(t) = \left\{ \begin{array}{ll} 1, & 0 \leq t \leq \omega \\ 0, & \omega < t \end{array} \right.$$

meaning that T has a Dirac delta distribution $\delta_\omega(t)$, then the average sojourn time is

$$\tau = \int_0^\omega dt = \omega$$

with standard deviation $\sigma = 0$

The Gamma distribution

R.v. *X* is **Gamma** distributed ($X \sim \Gamma(k, \theta)$) with **shape parameter** *k* and **scale parameter** θ (or **rate** $\beta = 1/\theta$) (all positive) if its probability density function takes the form

$$f(x; k, \theta) = \frac{x^{k-1} \mathrm{e}^{-\frac{x}{\theta}}}{\Gamma(k) \theta^k} \tag{3}$$

where $x > 0$ and Γ is the Euler Gamma function, defined for all $z \in \mathbb{C}$ s.t. $\mathrm{Re}\,(z) > 0$ by

$$\Gamma : z \mapsto \int_0^{+\infty} t^{z-1}\, \mathrm{e}^{-t}\, \mathrm{d}t$$

Properties of the Gamma distribution

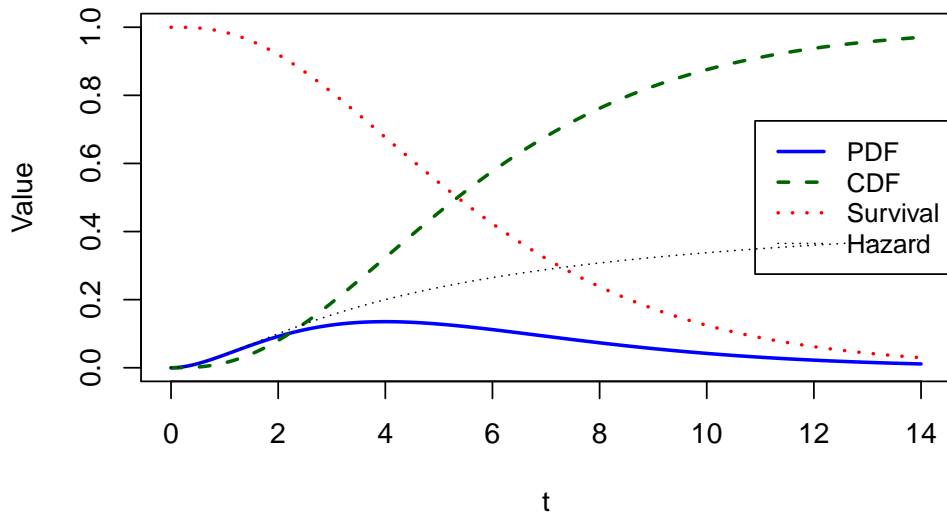Mean $k\theta$, variance $k\theta^2$

Survival function

$$\mathcal{S}(t) = 1 - \frac{1}{\Gamma(k)}\gamma\left(k, \frac{t}{\theta}\right) = 1 - \frac{1}{\Gamma(k)}\gamma\left(k, \beta t\right)$$

where

$$\gamma(a, x) = \int_0^x t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$$

is an incomplete Gamma function

PDF, CDF, Survival & Hazard of Gamma Distribution

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A model for a cohort with one cause of death

Consider a **cohort** of individuals born at the same time, e.g., the same year

- At time $t = 0$, there are initially $N_0 > 0$ individuals
- All causes of death are compounded together
- The time until death, for a given individual, is a random variable T, with continuous probability density distribution $f(t)$ and survival function $\mathcal{S}(t)$

$N(t)$ the cohort population at time $t \geq 0$

$$N(t) = N_0 \mathcal{S}(t) \tag{4}$$

$\mathcal{S}(t)$ proportion of initial population still alive at time t, so $N_0 \mathcal{S}(t)$ number in the cohort still alive at time t

Case where *T* is exponentially distributed

Suppose that *T* has an exponential distribution with mean $1/d$ (or parameter d), $f(t) = de^{-dt}$. Then the survival function is $\mathcal{S}(t) = e^{-dt}$, and (4) takes the form

$$N(t) = N_0 e^{-dt} \tag{5}$$
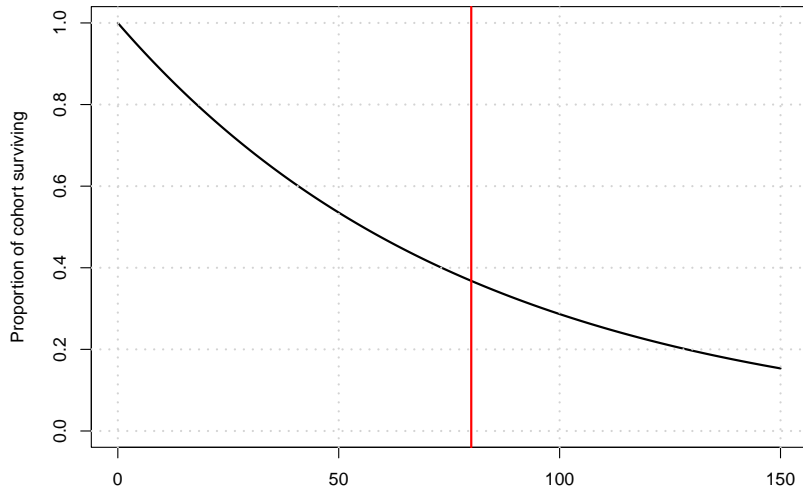
Now note that

$$\begin{aligned}
\frac{d}{dt} N(t) &= -dN_0 e^{-dt} \\
&= -dN(t)
\end{aligned}$$

with $N(0) = N_0$.

$\Rightarrow$ The ODE $N' = -dN$ makes the assumption that the life expectancy at birth is exponentially distributed

Survival function, $\mathcal{S}(t) = \mathbb{P}(T > t)$, for an exponential distribution with mean 80 years

Case where *T* has a Dirac delta distribution

Suppose that *T* has a Dirac delta distribution at $t = \omega$, giving the survival function

$$\mathcal{S}(t) = \begin{cases} 1, & 0 \leq t \leq \omega \\ 0, & t > \omega \end{cases}$$

Then (4) takes the form
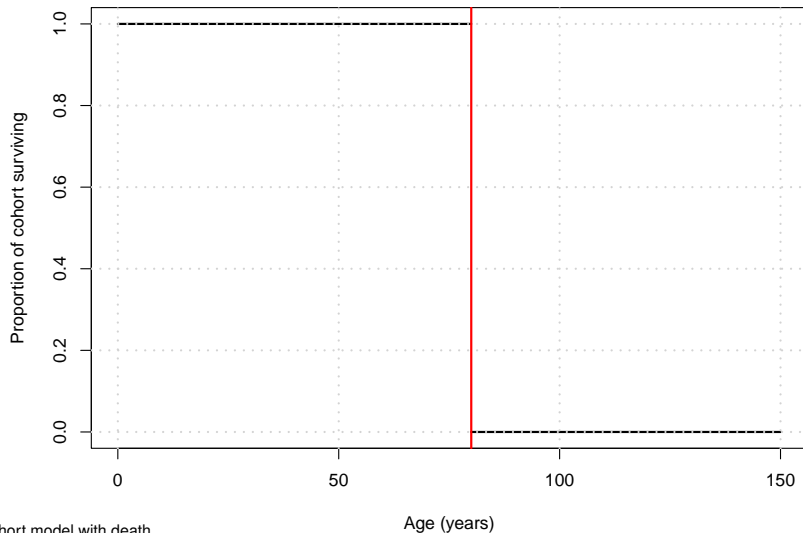
$$N(t) = \begin{cases} N_0, & 0 \leq t \leq \omega \\ 0, & t > \omega \end{cases} \tag{6}$$

All individuals survive until time ω, then they all die at time ω

Here, $N' = 0$ everywhere except at $t = \omega$, where it is undefined

Survival function, $\mathcal{S}(t) = \mathbb{P}(T > t)$, for a Dirac distribution with mean 80 years

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Possible fixes to the exponential distribution issue

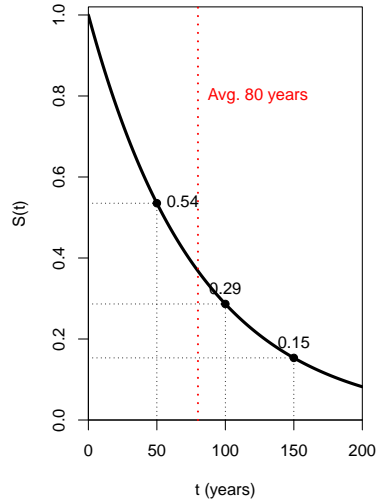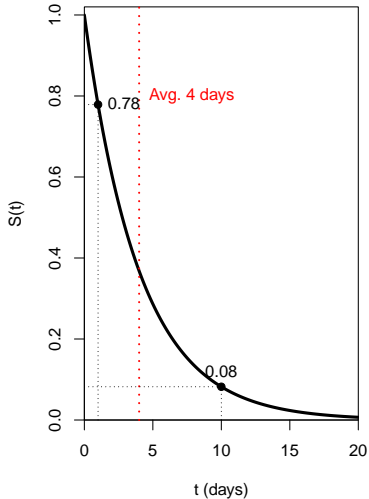**The issue with exponential distributions**

Fix 1 – Use info on the distribution as well

Fix 2 – Use an Erlang distribution

Finding the right Erlang

Example – A COVID-19 model

Survival for the exponential distribution

Issues with the exponential distribution

▶ Survival drops quickly

▶ Survival continues way beyond the mean

Acceptable if what matters is the average duration of sojourn in a compartment (e.g., long term dynamics)

More iffy if one is interested in short-term dynamics

▶ Exponential distribution with parameter θ has same mean and standard deviation $1/\theta$, i.e., a single parameter controls mean and dispersion about the mean

Possible fixes to the exponential distribution issue

Possible fixes to the exponential distribution issue

The issue with exponential distributions
Fix 1 – Use info on the distribution as well
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Side note – What is a convolution?

▶ The *convolution product* is used to find the probability distribution of the *sum of two independent random variables*

▶ If X and Y are two continuous random variables, and $Z = X + Y$, use the convolution to find the p.d.f. $f_Z(z)$ of Z

The setting

▶ Consider two independent continuous random variables X and Y with p.d.f. $f_X(x)$ and $f_Y(y)$, respectively

▶ Want to find the p.d.f. of their sum $Z = X + Y$

▶ Simply adding the p.d.f. or multiplying them does not yield the correct distribution for the sum Z

▶ Convolution accounts for all possible combinations of X and Y that sum to a specific value z

Deriving the convolution formula

Start with the c.d.f. of Z:

- $F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z)$
- Since X and Y are independent, their joint p.d.f. is $f_{X,Y}(x, y) = f_X(x)f_Y(y)$
- $\mathbb{P}(X + Y \leq z)$ found by integrating the joint p.d.f. over the region where $x + y \leq z$

$$F_Z(z) = \iint_{x+y \leq z} f_X(x)f_Y(y) \, dx \, dy$$

- Change the order of integration. For a fixed x, y must be less than or equal to $z - x$

$$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_Y(y) \, dy \right) f_X(x) \, dx$$

- Inner integral is c.d.f. of Y evaluated at $z - x$, i.e., $F_Y(z - x)$

$$F_Z(z) = \int_{-\infty}^{\infty} F_Y(z - x)f_X(x) \, dx$$

Deriving the convolution formula (2)

To obtain the PDF $f_Z(z)$, we differentiate the CDF $F_Z(z)$ with respect to z

► $f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} F_Y(z-x)f_X(x)\,dx$

► Using Leibniz integral rule (differentiating under the integral sign)

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z}F_Y(z-x)f_X(x)\,dx$$

► Since $\frac{d}{du}F_Y(u) = f_Y(u)$, and here $u = z - x$, we have $\frac{\partial}{\partial z}F_Y(z-x) = f_Y(z-x)$

The convolution pProduct formula

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

▶ This is the **convolution product** of f_X and f_Y, often denoted as $(f_X * f_Y)(z)$

▶ Alternatively, by symmetry

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y)\, dy$$

▶ **Key condition –** Valid only if X and Y are *independent* r.v.

Exponential distributions are "bad" but also cool

X_1 and X_2 2 i.i.d. (independent and identically distributed) exponential r.v. with parametres θ_1 and θ_2. Then the probability density function of the r.v. $Z = X_1 + X_2$ is given by the convolution

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z - x_1) \, dx_1 \\
&= \int_0^z \theta_1 e^{-\theta_1 x_1} \theta_2 e^{-\theta_2 (z - x_1)} \, dx_1 \\
&= \theta_1 \theta_2 e^{-\theta_2 z} \int_0^z e^{(\theta_2 - \theta_1) x_1} \, dx_1 \\
&= \begin{cases} \dfrac{\theta_1 \theta_2}{\theta_2 - \theta_1} \left(e^{-\theta_1 z} - e^{-\theta_2 z} \right) & \text{if } \theta_1 \neq \theta_2 \\ \theta^2 z e^{-\theta z} & \text{if } \theta_1 = \theta_2 =: \theta \end{cases}
\end{aligned} \tag{7}
$$

The tool we use

Theorem 1

Let X_i be independent exponentially distributed random variables with parameter ξ and $Y = \sum_{i=1}^{n} X_i$

Then the random variable $Y \rightsquigarrow E(n, \xi)$, an Erlang distribution with shape parameter n and scale parameter ξ

(Erlang distribution: Gamma distribution with integer shape parameter)

Consequences for compartmental models

If n compartments are traversed successively by individuals, with each compartment having an outflow rate of $1/\xi$ (or a mean sojourn time of ξ), then the time of sojourn from entry into the first compartment to exit from the last is Erlang distributed with mean $E(Y) = n\xi$ and variance $\mathrm{Var}(Y) = n\xi^2$

Average sojourn time $1/\mu$

Average sojourn time N/ε
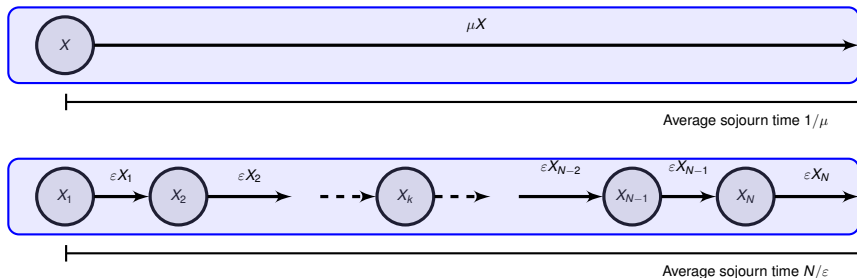
I have a Shiny app for this :)

Possible fixes to the exponential distribution issue

The issue with exponential distributions

Fix 1 – Use info on the distribution as well

Fix 2 – Use an Erlang distribution

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Example – A COVID-19 model

Example: EVD incubation periods

During the 2014 Ebola Virus Disease (EVD) crisis in Western Africa, the WHO Ebola Response Team estimated incubation periods in a 2015 paper

Table S2 in the Supplementary Information in that paper gives the best fit for the distribution of incubation periods for EVD as a Gamma distribution with mean 10.3 days and standard deviation 8.2, i.e., $n\varepsilon = 10.3$ and $\varepsilon\sqrt{n} = 8.2$

From this, $\varepsilon = 8.2^2/10.3 \simeq 6.53$ and $n = 10.3^2/8.2^2 \simeq 1.57$. However, that is a Gamma distribution

Switching to a compartmental model approach

To use multiple compartments to better fit residence times, we need to find the closest possible Erlang distribution to this Gamma distribution

$\implies$ compute RSS errors between data points generated from the given Gamma distribution and an Erlang

```r
error_Gamma <- function(theta,shape,t,data) {
  test_points <- dgamma(t, shape = shape, scale = theta)
  ls_error <- sum((data-test_points)^2)
  return(ls_error)
}
```

```r
optimize_gamma <- function(t,d) {
  max_shape <- 10
  error_vector <- mat.or.vec(max_shape,1)
  scale_vector <- mat.or.vec(max_shape,1)
  for (i in 1:max_shape) {
    result_optim <- try(optim(par = 3,
                              fn = error_Gamma,
                              lower = 0,
                              method = "L-BFGS-B",
                              shape = i,
                              t = t,
                              data = d),
                        TRUE)
    if (!inherits(result_optim,"try-error")) {
      error_vector[i] <- result_optim$value
      scale_vector[i] <- result_optim$par
    } else {
      error_vector[i] <- NaN
```

```r
time_points <- seq(0,60)
data_points <- dgamma(time_points, shape = 1.57,
                      scale = 6.53)
# Run the minimization
optim_fits <- optimize_gamma(time_points,data_points)
# Which is the best Erlang to fit the data
idx_best <- which.min(optim_fits$error)
```

We find the best fit below, which is obtained using 2 compartments
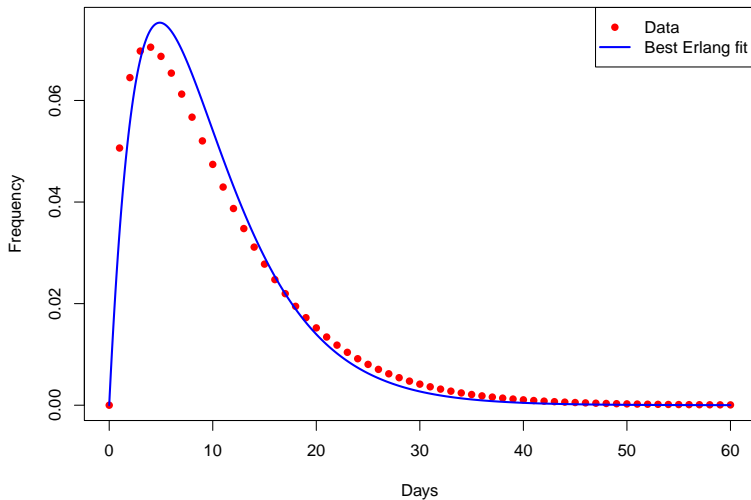
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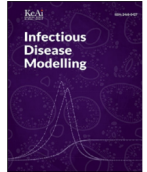
A simple model for COVID-19

Julien Arino [a,b,c,*], Stéphanie Portet [a]

[a] Department of Mathematics & Data Science NEXUS, University of Manitoba, Canada
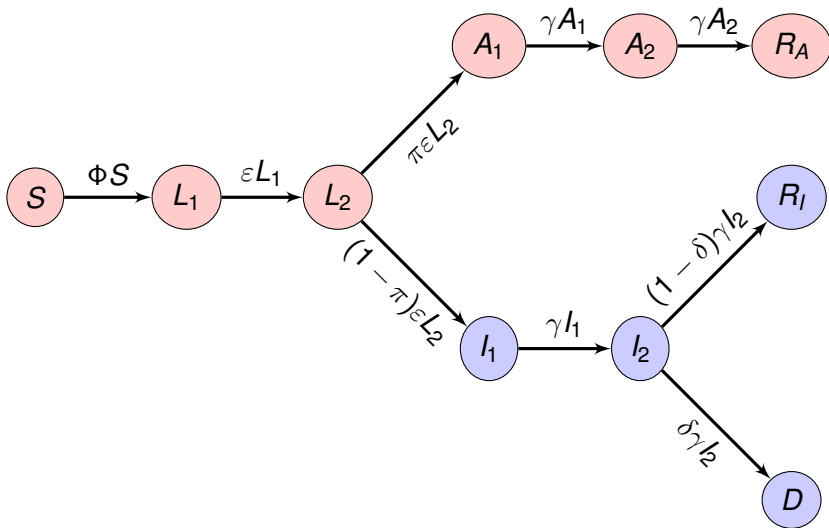[b] Centre for Disease Modelling, Canada
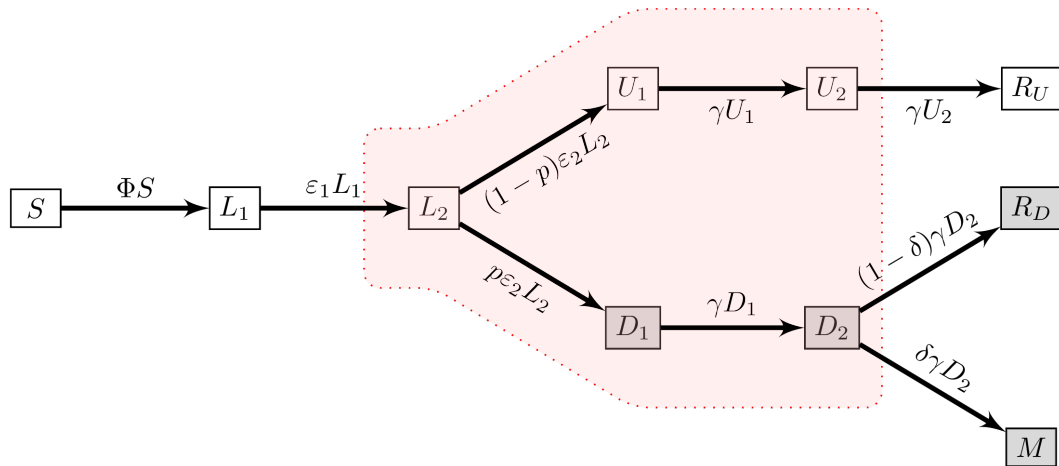[c] Canadian COVID-19 Mathematical Modelling Task Force, Canada

Extends the SLIAR model to take into account non-exponentially distributed stage durations (see lecture 05)

The original model (well, almost the first one)

Reinterpreting terms

Here *D* stands for *detected*, *U* is *undetected*

Working out when the first COVID-19 case occurred

▶ Details of emergence and precise timeline before amplification started unknown

▶ Amplification in Wuhan
  ▶ Cluster of pneumonia cases mostly related to the Huanan Seafood Market
  ▶ 27 December 2019: first report to local government
  ▶ 31 December 2019: publication
  ▶ 8 January 2020: identification of SARS-CoV-2 as causative agent
  ▶ $\sim$ 23 January 2020: lockdown Wuhan and Hubei province + face mask mandates

▶ By 2020-01-29, virus in all provinces of mainland CHN

Evidence of earlier spread

► Report to Wuhan authorities on 27 December 2019

► First export detections in Thailand and Japan on 13 and 16 January 2020 (with actual importations on 8 and 6 January)

$\implies$ amplification must have been occuring for a while longer

► France: sample taken from 42-year-old male (last foreign travel to Algeria in August 2019) who presented to ICU on 27 December 2019

► Retrospective studies in United Kingdom and Italy also showed undetected COVID-19 cases in prepandemic period

Untangling the first case issue

▶ Robert, Rossman & Jaric. Dating first cases of COVID-19. *PLoS Pathogens* (2021)
Find likely timing of first case of COVID-19 in China as November 17 (95% CI October 4)

▶ Pekar, Worobey, Moshiri, Scheffler & Wertheim. Timing the SARS-CoV-2 index case in Hubei province. *Science* (2021)
Period between mid-October and mid-November 2019 is plausible interval when the first case of SARS-CoV-2 emerged in Hubei province

Important when trying to understand global spread, so let me illustrate with the model I used, taking into account model evolution since

Back-calculating the start of spread (example of China)

Cumulative confirmed case counts in China as reported to WHO was $c = 547$ cases on $t_c = 2020\text{-}01\text{-}22$

Let u be a point in parameter space. Solve ODE numerically over $[0, t]$, with $S(0)$ the population of China, $L_1(0) = 1$ and other state variables 0. This gives a solution $x(t, t_0 = 0, u)$

Extracting $L_2(t, t_0 = 0, u)$ from this solution, obtain cumulative number of new detections as

$$C(t) = \int_{t_0=0}^{t} p\varepsilon_2 L_2(s, t_0, u) \, ds$$

Let $t^\star$ be s.t. $C(t^\star) = 547$; then $t_i = 2020\text{-}01\text{-}22 - t^\star$
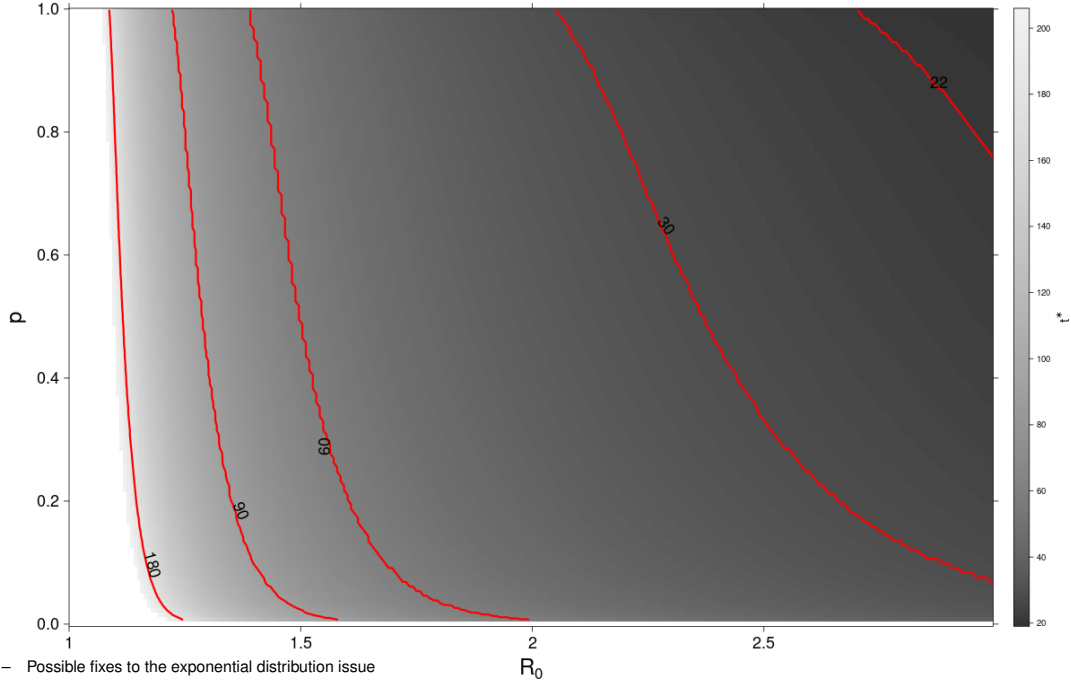
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An SIS model

Hypotheses

▶ Individuals typically recover from the disease

▶ The disease does not confer immunity

▶ There is no birth or death (from the disease or natural)
  $\Rightarrow$ Constant total population $N \equiv N(t) = S(t) + I(t)$

▶ Infection is of **standard incidence** type

Recovery

- ► Traditional models suppose that recovery occurs with rate constant γ

- ► Here, of the individuals that become infective at time t_0, a fraction $\mathcal{S}(t - t_0)$ remain infective at time $t \geq t_0$

- ► $\Rightarrow$ For $t \geq 0$, $\mathcal{S}(t)$ is a survival function. As such, it verifies $\mathcal{S}(0) = 1$ and $\mathcal{S}$ is nonnegative and nonincreasing

Model for infectious individuals

Since N is constant, $S(t) = N - I(t)$ and we need only consider the following equation (where S is used for clarity)

$$I(t) = I_0(t) + \int_0^t \beta \frac{S(u)I(u)}{N} \mathcal{S}(t - u) du \tag{8}$$

▶ $I_0(t)$ number of individuals who were infective at time $t = 0$ and still are at time t

  ▶ $I_0(t)$ is nonnegative, nonincreasing, and such that $\lim_{t \to \infty} I_0(t) = 0$

▶ $\mathcal{S}(t - u)$ proportion of individuals who became infective at time u and who still are at time t

Expression under the integral

Integral equation for the number of infective individuals:

$$I(t) = I_0(t) + \int_0^t \beta \frac{(N - I(u))I(u)}{N} \mathcal{S}(t - u) du \qquad (8)$$

The term

$$\beta \frac{(N - I(u))I(u)}{N} \mathcal{S}(t - u)$$

▶ $\beta(N - I(u))I(u)/N$ is the rate at which new infectives are created, at time u
▶ multiplying by $\mathcal{S}(t - u)$ gives the proportion of those who became infectives at time u and who still are at time t

Summing over $[0, t]$ gives the number of infective individuals at time t

Case of an exponentially distributed time to recovery

Suppose $\mathcal{S}(t)$ such that sojourn time in the infective state has exponential distribution with mean $1/\gamma$, *i.e.*, $\mathcal{S}(t) = e^{-\gamma t}$

Initial condition function $I_0(t)$ takes the form

$$I_0(t) = I_0(0)e^{-\gamma t}$$

with $I_0(0)$ the number of infective individuals at time $t = 0$. Obtained by considering the cohort of initially infectious individuals, giving a model such as (4)

Equation (8) becomes

$$I(t) = I_0(0)e^{-\gamma t} + \int_0^t \beta \frac{(N - I(u))I(u)}{N} e^{-\gamma(t-u)} du \qquad (9)$$

Taking the time derivative of (9) yields

$$I'(t) = -\gamma I_0(0)e^{-\gamma t} - \gamma \int_0^t \beta \frac{(N-I(u))I(u)}{N} e^{-\gamma(t-u)} du$$
$$+ \beta \frac{(N-I(t))I(t)}{N}$$
$$= -\gamma \left(I_0(0)e^{-\gamma t} + \int_0^t \beta \frac{(N-I(u))I(u)}{N} e^{-\gamma(t-u)} du \right)$$
$$+ \beta \frac{(N-I(t))I(t)}{N}$$
$$= \beta \frac{(N-I(t))I(t)}{N} - \gamma I(t)$$

This is the classical logistic type ordinary differential equation (ODE) for I in an SIS model without vital dynamics (no birth or death)

Case of a step function survival function

Consider case where the time spent infected has survival function

$$\mathcal{S}(t) = \begin{cases} 1, & 0 \le t \le \omega, \\ 0, & t > \omega. \end{cases}$$

i.e., the sojourn time in the infective state is a constant $\omega > 0$

In this case (8) becomes

$$I(t) = I_0(t) + \int_{t-\omega}^{t} \beta \frac{(N - I(u))I(u)}{N} du. \tag{10}$$

Here, it is more difficult to obtain an expression for $I_0(t)$. It is however assumed that $I_0(t)$ vanishes for $t > \omega$

When differentiated, (10) gives, for $t \geq \omega$,

$$I'(t) = I_0'(t) + \beta \frac{(N - I(t))I(t)}{N} - \beta \frac{(N - I(t-\omega))\,I(t-\omega)}{N}.$$

Since $I_0(t)$ vanishes for $t > \omega$, this gives the delay differential equation (DDE)

$$I'(t) = \beta \frac{(N - I(t))I(t)}{N} - \beta \frac{(N - I(t-\omega))I(t-\omega)}{N}.$$

Bibliography I