



University  
of Manitoba

## Incidence functions MATH 8xyz – Lecture 12

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.

We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

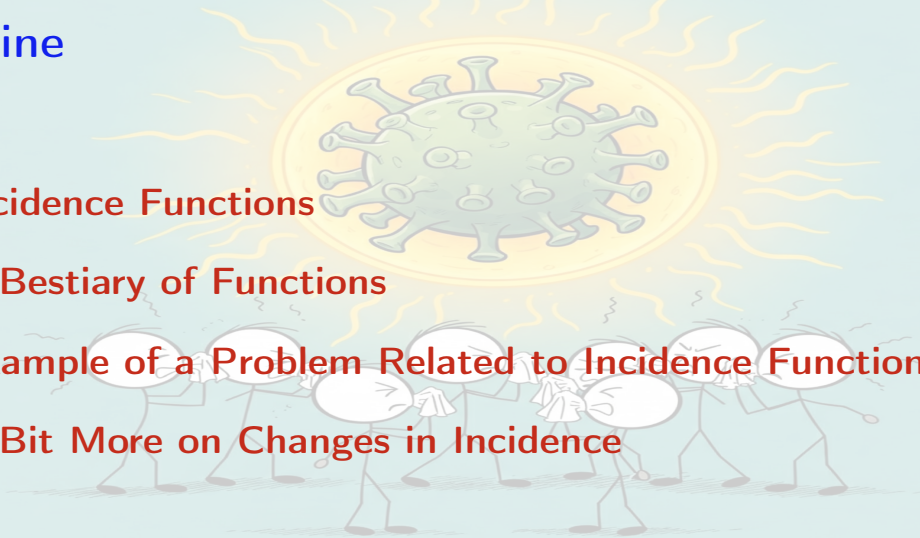
# Outline

Incidence Functions

A Bestiary of Functions

Example of a Problem Related to Incidence Functions

A Bit More on Changes in Incidence





**Incidence Functions**

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# Introduction to Incidence Functions

- ▶ Before continuing, let's discuss **incidence functions**, which describe how contacts between individuals occur and how they result in disease transmission
- ▶ See in particular McCallum, Barlow & Hone, How should pathogen transmission be modelled?, Trends in Ecology & Evolution **16** (2001)

# Incidence Function versus Force of Infection

- ▶ Two different forms of the function representing the rate at which individuals move from compartment  $S$  to infected compartments:
- ▶  $S' = -f(S, I, N)$  is an **incidence function**
- ▶  $S' = -\lambda(S, I, N)S$  is a **force of infection**
- ▶ The two are equivalent; the context usually determines which form is used. For example, PDE models structured by age of infection must integrate  $I(t, a)$  over age and therefore often use the force of infection.

## Interactions - Infection

- ▶ The rate at which new cases appear is the **incidence function**

$$f(S, I, N) \tag{1}$$

- ▶ Depends on the number  $S$  of susceptibles,  $I$  of infectious individuals, and sometimes the total population  $N$
- ▶ An incidence function includes two components:
  - ▶ a count of the number of contacts occurring
  - ▶ a description of the probability that such a contact, when it occurs, leads to pathogen transmission

## Difficulty of the Choice

- Choosing a good function is difficult and is probably one of the most "unstable" parts of modelling the spread of infectious diseases

## The Two Most Common Incidence Functions

- The two most commonly used incidence functions are the **mass action incidence**

$$f(S, I) = \beta SI \quad (2)$$

and the **standard incidence** (or **proportional incidence**)

$$f(S, I) = \beta \frac{SI}{S + I} \quad (3)$$

- In both cases,  $\beta$  is the **disease transmission coefficient**, although its exact interpretation varies



## Units of $\beta$

- ▶ If  $X(t)$  is the population of compartment  $X$  at time  $t$ , then  $X'$  has units number/time
- ▶ In a differential equation, the terms on the left and right sides of the “=” sign must have the same units
- ▶ The incidence function therefore has units number/time
- ▶ (And if a force of infection is used, the units are 1/time)

## Units – Mass Action Incidence

- ▶ Mass action incidence

$$\beta SI \propto \beta \times \text{number} \times \text{number}$$

has units number/time if  $\beta$  has units  $1/(\text{number} \times \text{time})$

- ▶ Standard incidence

$$\beta SI/N \propto \beta \times \text{number} \times \text{number}/\text{number} \propto \beta \times \text{number}$$

has units number/time if  $\beta$  has units  $1/\text{time}$

## Mass Action Incidence

$$f(S, I) = \beta SI \quad (2)$$

- ▶ The mixing of susceptibles and infectious individuals is homogeneous
- ▶ This is a strong assumption: the number of contacts is the product of the number of susceptibles and the number of infectious individuals, so each susceptible individual can potentially meet each infectious individual
- ▶ (hence the name, by analogy with gas dynamics in chemistry/physics)
- ▶ When the population is large, this assumption becomes unrealistic

## Standard Incidence (Proportional)

- ▶ Another widely used form of the incidence function

$$f(S, I, N) = \beta \frac{SI}{N} \quad (3)$$

- ▶ Each susceptible meets a fraction of the infectious individuals
- ▶ Or vice-versa! See, e.g., Hethcote, Qualitative analyses of communicable disease models, *Mathematical Biosciences* (1976)
- ▶ Case of a larger population

## Constant Population $\implies$ Equivalence of Incidences

- ▶ When the total population is constant, many incidence functions are qualitatively equivalent
- ▶ Suppose that  $N(t) \equiv N_0$ , then

$$\beta SI = \tilde{\beta} \frac{SI}{N} \iff \tilde{\beta} = N_0 \beta$$

with this  $\tilde{\beta}$ , (2) and (3) are identical

- ▶ Remember that the units differ, however

The background of the slide features a stylized, semi-transparent globe showing the continents of North and South America. Floating above the globe are several 3D models of virus particles, each with a green spherical core and red, spike-like protrusions. The overall color palette is muted, with greys, blues, and greens.

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## General Incidence

$$f(S, I, N) = \beta S^q I^p \quad (4)$$

► These functions were introduced with the aim of fitting data: for fitting, this adds two parameters  $p, q$ . We will see, however, that much theoretical work uses this incidence

## Refuge Incidence

- Refuge effect; a proportion  $0 < q < 1$  of the population is truly susceptible, for example due to spatial heterogeneities

$$f(S, I, N) = \begin{cases} \beta I \left( N - \frac{I}{q} \right), & \text{if } I < qN \\ 0, & \text{if } I \geq qN \end{cases} \quad (5)$$



## Negative Binomial Incidence

$$f(S, I, N) = kS \ln \left( 1 + \beta \frac{I}{k} \right) \quad (6)$$

- For small values of  $k$ , this describes a highly concentrated infection process, while as  $k \rightarrow \infty$ , we tend towards a mass action incidence

## Asymptotic Contact

$$f(S, I, N) = \frac{N}{1 - \varepsilon + \varepsilon N} \frac{F(S, I)}{N} \quad (7)$$

where  $F$  is one of the functions already described

► When  $\varepsilon = 0$ , contacts are proportional to  $N$ , while when  $\varepsilon = 1$ , contacts are independent of  $N$

## Asymptomatic Transmission

$$f(S, I, N) = \beta \frac{SI}{c + S + I} \quad (8)$$

where  $c$  is a constant. For example,

$$\frac{C(N)}{N} F(S, I)$$

with  $C(N) = N/(1 - \varepsilon + \varepsilon N)$  the function describing the contact rate and  $F(S, I)$  the function describing disease spread, which is assumed here to be a negative binomial incidence

## Changing Incidence

$$F(S, I, N) = \begin{cases} \beta SI & \text{if } N \leq \hat{N} \\ \beta \frac{SI}{N} & \text{if } N > \hat{N} \end{cases} \quad (9)$$

- Arino & McCluskey, Effect of a sharp change of the incidence function on the dynamics of a simple disease, *Journal of Biological Dynamics* (2010)



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## Change Shape or Saturate?

- C. Kribs-Zaleta. To switch or taper off: the dynamics of saturation, *Mathematical Biosciences* (2004)

## Two Different Incidence Functions

- We will consider the role of two different functions: a continuous and differentiable function

$$\beta_{sm}(N) = \beta_0 \frac{N}{N + A} \quad (10)$$

and a continuous but with a *switch* (abrupt transition?)

$$\beta_{sw}(N) = \begin{cases} \beta_0 \frac{N}{A}, & N < A \\ \beta_0, & N \geq A \end{cases} \quad (11)$$

- These functions represent saturation in different ways

# The Model

- SIS model in non-constant population

$$I' = \beta(N)(N - I)\frac{I}{N} - (d + \gamma + \delta)I \quad (1)$$

$$N' = bN \left(1 - \frac{N}{K}\right) - dN - \delta I \quad (2)$$



## Non-dimensionalization of the Model

► Let  $\bar{N} = (1 - \frac{d}{b})K$ ,  $i = I/N$  and  $n = N/\bar{N}$ . Then

$$i' = (\beta(\bar{N}n) - \delta)i(1 - i) - (b + \gamma)i + (b - d)in \quad (12a)$$

$$n' = (b - d)n(1 - n) - \delta in \quad (12b)$$

# The Basic Reproduction Number

- Whatever incidence function is used

$$\mathcal{R}_0 = \frac{\beta(\bar{N})}{d + \gamma + \delta}$$

## Analysis with $\beta = \beta_{sw}$

- ▶ With  $\beta = \beta_{sw}$ , there are 2 sub-models
- ▶ **Case I**  $\beta(N) = \beta_0 N/A$ ; 3 equilibria  $(i^*, n^*)$  in  $[0, 1]^2$ :
  - ▶ the origin, always unstable
  - ▶ the ESMP  $(0, 1)$ , GAS when  $\mathcal{R}_0 < 1$
  - ▶ a unique EE that exists and is LAS when  $\mathcal{R}_0 > 1$
- ▶ **Case II**  $\beta(N) = \beta_0$ . We have 4 equilibria  $(i^*, n^*)$ :
  - ▶ the origin, always unstable
  - ▶ the ESMP, GAS when  $\mathcal{R}_0 < 1$
  - ▶ a unique EE that exists and is LAS when  $\mathcal{R}_0 > 1$
  - ▶ an extinction PE that exists when  $\beta \geq b + \gamma + \delta$  and is LAS iff, i.e., when  $\mathcal{R}_0 > 1$  but the EE does not exist

## Detailed Analysis

- ▶ Let  $a = A/\bar{N}$
- ▶ If  $a > 1$ , all equilibria are below the switching point and the model reduces to Case I (model without saturation)
- ▶ If  $a < 1$ , there is a single ESMP (as well as the unstable trivial equilibrium) and there is a unique EE which is the EE of Case I iff

$$a > 1 - \frac{1}{k} \left( 1 - \frac{d + \gamma + \delta}{\beta_0} \right)$$

and the EE of Case II otherwise

- ▶ The model with switch does not have the extinction PE of Case II

## $\mathcal{R}_0$ in the Model with Switch

- In the model with switch

$$\mathcal{R}_0 = \frac{\beta_0}{d + \gamma + \delta} \frac{1}{\max(a, 1)}$$

## $\mathcal{R}_0$ in the Model with Differentiable Saturation

- In the model with differentiable saturation (10)

$$\mathcal{R}_0 = \frac{\beta_0}{d + \gamma + \delta} \frac{1}{a + 1}$$

- We have the trivial PE (unstable), the ESMP (GAS when  $\mathcal{R}_0 < 1$ ) and a unique EE that exists and is LAS when  $\mathcal{R}_0 > 1$

## Main Differences Between the Models

- ▶ The main difference between the models is quantitative: the point where  $\mathcal{R}_0 < 1$  changes; this reflects that  $\beta_{sm} < \beta_{sw}$
- ▶ Both models predict the eradication of the disease for a large region of the parameter space and exclude the possibility of extinction that the classical model with standard incidence allows
- ▶ Also, in both cases, the EE is such that

$$\lim_{\mathcal{R}_0 \rightarrow \infty} i^* = \min(k, 1)$$

i.e., for  $k < 1$  (i.e.,  $r < d + \delta$ ), the prevalence at the EE never approaches 100% because the reproductive resilience of the population is so low that the population becomes too small for the disease to spread to all, regardless of the value of  $\mathcal{R}_0$



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# A Bit More on Changes in Incidence

The Basic Model

Density-Dependent Incidence

Cases Where Infection Exceeds Treatment Capacities

## The Basic Model

$$S' = bN - dS - F(S, I) + \gamma I \quad (13a)$$

$$I' = F(S, I) - (d + \delta + \gamma)I \quad (13b)$$

► So the dynamics of the total population are

$$N' = (b - d)N - \delta I$$

and the system has EEs iff  $d < b < d + \delta$

## System in Proportions

► We switch to the system in proportions  $s = S/N$  and  $i = I/N$ . The total population is not constant and we therefore consider

$$i' = f(i, N) - (b + \delta + \gamma)i + \delta i^2 \quad (14a)$$

$$N' = (b - d - \delta i)N \quad (14b)$$

with

$$f(i, N) = \frac{F(S, I)}{N}$$

► We define

$$f(i, 0) = \lim_{N \rightarrow 0} f(i, N)$$

and we assume that this limit exists (we exclude, for example, that  $F(\alpha S, \alpha I)$  is sub-linear in  $\alpha$  near 0)

# Existence of Solutions

## Theorem 1

*The solutions of (14a)–(14b) in the positively invariant band*

$$\mathcal{D} = \{(i, N) : 0 \leq i \leq 1, N \geq 0\}$$

*exist for all  $t > 0$*

The background of the slide features a stylized world map with landmasses in a light tan color and oceans in a light blue color. Scattered across the map and the surrounding grey background are several 3D models of virus particles. These particles are spherical with a greenish-blue base and numerous red, spike-like protrusions extending from their surfaces.

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## Density-Dependent Incidence

- The first case we consider is one with an incidence of the form we have already seen

$$F(S, I, N) = \begin{cases} \beta SI & \text{if } N \leq \hat{N} \\ \beta \frac{SI}{N} & \text{if } N > \hat{N} \end{cases} \quad (9)$$

where we normalize so that  $\hat{N} = 1$

- We denote by  $\mathcal{D}_L$  the part of  $\mathcal{D}$  where  $N \leq 1$  and by  $\mathcal{D}_H$  where  $N \geq 1$

## Explicit Solution in $\mathcal{D}_H$

### Theorem 2

*Suppose that at time  $t = \tau \geq 0$ , we have  $(i_\tau, N_\tau) := (i(\tau), N(\tau)) \in \mathcal{D}_H$ . Then there exists a potentially infinite interval  $\mathcal{I}$  with left endpoint the point  $t = \tau$ , such that for all  $t \in \mathcal{I}$ , we have  $(i(t), N(t)) \in \mathcal{D}_H$  with*

$$i(t) = \frac{Ki_\tau}{\Psi(t - \tau)} \quad (3)$$

$$N(t) = N_\tau e^{(b-d)(t-\tau)} \exp \left( -\delta Ki_\tau \int_\tau^t \frac{du}{\Psi(u - \tau)} \right) \quad (4)$$

*where  $K = \beta - (d + \gamma + \delta)$  and*

$$\Psi(u) = i_\tau(\beta - \delta)(1 - e^{-Ku}) + Ke^{-Ku}$$

# Equilibria of the System

## Theorem 3

Let

$$N_{\Delta} = \frac{\delta(d + \delta + \gamma)}{\beta(d + \delta - b)}$$

For all parameter values, the ESMP is

$$e_0 := (i_0, N_0) = (0, 0)$$

- ▶ If  $b < d$ , there is no other PE and  $e_0$  is GAS
- ▶ If  $d < b$ ,  $e_0$  is unstable
- ▶ If  $d < b < d + \delta$ , then the presence of EE  $e_{\star} = (i_{\star}, N_{\star})$ ,  $i_{\star} = (b - d)/\delta$  depends on
  - ▶ If  $N_{\Delta} < 1$ ,  $e_{\star} = (i_{\star}, N_{\Delta})$  LAS
  - ▶ If  $N_{\Delta} = 1$ ,  $e_{\star} = (i_{\star}, N)$  for all  $N \geq 1$
  - ▶ If  $N_{\Delta} > 1$ ,  $e_{\star}$  does not exist





# A Bit More on Changes in Incidence

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## Incidence with Exceeding Capacities

► Suppose

$$F(S, I, N) = \begin{cases} \beta_1 \frac{SI}{N} & \text{if } I \leq \hat{I} \\ \beta_1 \frac{SI}{N} + \beta_2 S(I - \hat{I}) & \text{if } I > \hat{I} \end{cases} \quad (15)$$

or, in proportions,  $f(i, N)$  is of the form

$$\begin{cases} \beta_1(1-i)i & \text{if } iN \leq \hat{I} \\ \beta_1(1-i)i + \beta_2(1-i)(iN - \hat{I}) & \text{if } iN > \hat{I} \end{cases} \quad (16)$$

## The Regions in the Plane

► Here, the change does not occur for a value of  $N$  but along the hyperbola  $N = \hat{I}/i$ , for  $i \in ]0, 1]$ , and we adapt the regions  $\mathcal{D}_L$  and  $\mathcal{D}_H$

## Equilibria

- We potentially have 3 PEs

$$e_0 = (0, 0) \quad \bar{e} = (i_{LS}, 0) \quad e_\star = (i_N, N_{HS}(i_N))$$

with

$$i_{LS} = 1 - \frac{b + \gamma}{\beta_1 - \delta}, \quad (5)$$

$$i_N = \frac{b - d}{\delta} \quad (6)$$

and

$$N_{HS}(i) = \frac{\hat{I}}{i} + \frac{b + \gamma}{\beta_2(1 - i)} - \frac{\beta_1 - \delta}{\beta_2}$$

# Stability of the Equilibria

## Theorem 4

*The system (14a)–(14b) with incidence (16) has, potentially, 3 PEs, whose stability is given by the following table, in which we use*

$$\mathcal{E} = \frac{\beta_1}{b + \delta + \gamma}$$

*and*

$$\mathcal{S}_N = \frac{(b + \gamma)(b - d)}{d + \delta - b} + \frac{\beta_2 \hat{I}(d + \delta - b)}{d - b}$$

## Stability Table

	$e_0$	$\bar{e}$	$e_*$
$b < d, \mathcal{E} < 1$	GAS	Does not exist	Does not exist
$b < d, \mathcal{E} > 1$	Unstable	GAS	Does not exist
$d < b < d + \delta, i_{LS} < i_N, \mathcal{S}_N > 0$	Unstable	Unstable	LAS
$d < b < d + \delta, i_{LS} < i_N, \mathcal{S}_N < 0$	Unstable	Unstable	Unstable
$d < b < d + \delta, i_{LS} > i_N$	Unstable	GAS	Does not exist
$d + \delta < b$	Unstable	Unstable	Does not exist

► The SAG is obtained using an extension of Dulac's Theorem taking into account the existence, in a  $C^1$  field, of a curve on which the field is  $C^0$

# Periodic Solutions

- ▶ This system admits periodic solutions
- ▶ We consider the case

$d < b < d + \delta, i_{LS} < i_N, S_N > 0$	Unstable	Unstable	LAS
$d < b < d + \delta, i_{LS} < i_N, S_N < 0$	Unstable	Unstable	Unstable

## Theorem 5

*For all  $\varepsilon > 0$  and all  $S_N^0 > 0$ , there exists  $\bar{S}_N \in ] - S_N^0, 0[$  such that the system (14a)–(14b) with incidence (16) has a non-trivial periodic orbit in  $\mathcal{B}_\varepsilon(e_\star)$  (open ball centered at  $e_\star$  and radius  $\varepsilon$ ) for  $S_N = \bar{S}_N$*

# Bibliography I