Network models

MATH 8xyz – Lecture 27

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Outline

Why network models?

- Classical models allow some flexibility (see e.g. the section in Course 04 on incidence functions or Course 11 on group models), but this remains limited and an approximation.
- Like agent-based models (Course 18), network models are considered to make the description of contact processes in pathogen transmission more realistic.

Human life is organized in networks

- Family
- Circles of friends
- Professional network
- ...
- Social network theory has existed and been used for years, e.g., in professional settings (how to improve interactions in a company)

Social networks

- Before considering epidemics on networks, a few notions of social networks, because they are generally useful for understanding networks
- Social network methods introduce measures to evaluate certain properties of graphs that are useful to know
- A network is a (mathematical) graph, directed or not, in which edges represent connections (of any kind) between individuals, who are the nodes of the graph

Context

- $\mathcal{G}(\mathcal{V}, \mathcal{E})$ an undirected graph
- $\mathcal{D}(\mathcal{V}, \mathcal{A})$ a digraph (directed graph)
- $\mathcal{V}$ the set of nodes (vertices)
- $\mathcal{E}$ the set of edges in the undirected case
- $\mathcal{A}$ the set of arcs in the directed graph

Example: Air transport network

- We illustrate with data from the global air transport network
- Data is quite good (sometimes very good), and a clear advantage:
- When a plane leaves somewhere and arrives elsewhere, it's quite ... deterministic

Graph density

An (un)directed graph is **complete** if every pair of nodes is connected (by an edge in each direction for digraphs).
If there are $n = |\mathcal{V}|$ nodes, then there are $n(n-1)/2$ (resp. $n(n-1)$) edges in the complete graph (resp. digraph).
(We do not count self-loops.)
Density of $\mathcal{G}$ (undirected):

$$\text{dens}_{\mathcal{G}} = \frac{2\,|\mathcal{E}|}{n(n-1)}$$

Density of $\mathcal{D}$ (directed):

$$\text{dens}_{\mathcal{D}} = \frac{|\mathcal{A}|}{n(n-1)}$$

Density of considered digraphs

Digraph	# nodes	# arcs	density
Manitoba	24	64	0.1159
Canada	222	804	0.0164
North America	934	7,814	0.009
Global	3403	32,576	0.0028

Degree

- **Degree** $d_{\mathcal{G}}(v)$ of node $v \in \mathcal{V}$ in $\mathcal{G}$: number of edges incident to v
- **In-degree** $d^{-}_{\mathcal{D}}(v)$ in $\mathcal{D}$: number of arcs with head v
- **Out-degree** $d^{+}_{\mathcal{D}}(v)$ in $\mathcal{D}$: number of arcs with tail v
- **Degree** $d_{\mathcal{D}}(v)$ in $\mathcal{D}(\mathcal{V}, \mathcal{A})$: number of arcs incident to v in the underlying undirected graph $\mathcal{G}$ (where every arc is considered bidirectional)

Global in-degree of the air transport network

City	Country	In-degree	Rank
London	UK	365	1
Paris	France	294	2
Frankfurt	Germany	287	3
Atlanta	USA	249	4
New York	USA	241	5
Moscow	Russia	225	6
Amsterdam	Netherlands	204	7
Chicago	USA	203	8
Munich	Germany	200	9
Milan	Italy	181	10

Shortest path

Let $\mathcal{D}$ be a digraph. The (or the) shortest path(s) from i to j in $\mathcal{V}$:

$$d_{\mathcal{D}}(i,j) = \min_{p \in \mathcal{P}(i,j)} f(p)$$

where $\mathcal{P}(i,j)$ is the set of paths from i to j and $f(p)$ is a valuation of the arcs in path p. We define $d_{\mathcal{D}}(i,j) = \infty$ if there is no path from i to j.
$f(p)$ can be:

- the number of arcs in p from i to j (**geodesic distance**)
- great circle distance of the arcs in p
- flight duration of the arcs in p

Eccentricity

Eccentricity (or **König number**) of node v in $\mathcal{G}(\mathcal{V}, \mathcal{E})$:

$$e(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v, v')$$

In-eccentricity in $\mathcal{D}(\mathcal{V}, \mathcal{A})$:

$$e^{-}(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v', v)$$

Out-eccentricity in $\mathcal{D}(\mathcal{V}, \mathcal{A})$:

$$e^{+}(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v, v')$$

Radius

Radius of $\mathcal{G}$

$$\rho_{\mathcal{G}} = \min_{v \in \mathcal{V}} e(v)$$

In-radius of $\mathcal{D}$

$$\rho^{-}_{\mathcal{D}} = \min_{v \in \mathcal{V}} e^{-}(v)$$

Out-radius of $\mathcal{D}$

$$\rho^{+}_{\mathcal{D}} = \min_{v \in \mathcal{V}} e^{+}(v)$$

Radius $=$ min(max($\cdot$)) $\rightarrow$ directionality

Radius of the air transport network

Graph	ρ^-	ρ^+
Manitoba	2	3
Canada	6	6
North America	6	7
Global	7	7

Center of a graph

Center of $\mathcal{D}$:

$$\mathcal{C}_{\mathcal{D}} = \{v \in \mathcal{V} : e(v) = \rho_{\mathcal{D}}\}$$

Center of the air transport network

Graph	$\mathcal{C}^-$	$\|\mathcal{C}^-\|$	$\mathcal{C}^+$	$\|\mathcal{C}^+\|$
Manitoba	2	1 (YWG)	3	7
Canada	6	1 (YTO)	6	1 (YTO)
North America	6	1 (YTO)	7	18
Global	7	131	7	20

{YYC, YEA, Halifax, Kelowna, Moncton, YMQ, YOW, Quebec, St John's, YTO, YVR, Victoria, Y

$\mathcal{C}^-$

{Toronto, Vancouver} $\subset \mathcal{C}^+$

Diameter

Diameter of $\mathcal{D}$

$$\mathrm{diam}_{\mathcal{D}} = \max_{v \in \mathcal{V}} e(v)$$

diameter $=$ max(max(.)) $\rightarrow$ no directionality

Diameter of the air transport network

Graph	Diameter
Manitoba	5
Canada	12
North America	13
Global	13

Periphery of a graph

Periphery of $\mathcal{D}$

$$\mathcal{P}_{\mathcal{D}} = \{v \in \mathcal{V} : e(v) = \mathrm{diam}_{\mathcal{D}}\}$$

Periphery of the air transport network

Graph	In-periphery	Out-periphery
Manitoba	Lynn Lake	Cross Lake, Red Sucker La
Canada	Peawanuck	Peawanuck, Port Hope Simpson |
North America	Stony River	Peawanuck, Port Hope
Global	Stony River, Hooker Creek, Peawanuck	Hooker Creek, Beni, Peawanuck,

Many other measures

- betweenness
- closeness
- k-cores
- ...

General framework for network models

See for example:

- Newman. Spread of epidemic disease on networks, 2002
- Keeling & Eames. Networks and epidemic models, 2005
- Meyers et al. Network theory and SARS: predicting outbreak diversity, 2005
- Meyers, Newman & Pourbohloul. Predicting epidemics on directed contact networks, 2006
- Bansal et al. The dynamic nature of contact networks in infectious disease epidemiology, 2010

Typical network model

Typically, we consider a graph (or digraph) in which:

- each node is an individual
- the existence of an arc from i to j indicates that i is in contact with j and can transmit the infection
- in the undirected case, the existence of an arc from i to j implies the same arc from j to i and establishes that the two individuals are connected
- the connection is not permanent, but rather describes the possibility of a connection: i and j come into contact regularly

Adjacency matrix

We often use the **adjacency matrix** $A = [a_{ij}]$, where $a_{ij} = 1$ if node i has a link to node j and $a_{ij} = 0$ otherwise.
Sometimes we write $A(\mathcal{D})$ to indicate that A is the adjacency matrix of digraph $\mathcal{D}$, and conversely, $\mathcal{D}(A)$ to indicate that the graph is built using the adjacency matrix.
If the graph is undirected, then A is symmetric.

Nature of the network

- Sometimes we have precise data on links between individuals (surveys, etc.)
- Often we idealize networks, choosing networks with given properties

Degree distribution of the (di)graph

The **transmissibility** T of a disease in a graph is the average probability that an infectious individual transmits the disease to a susceptible individual with whom they are in contact.

In an uncorrelated network,

$$T_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

where $\langle k \rangle$ and $\langle k^2 \rangle$ are the mean degree and the mean squared degree.

It is necessary that $T > T_c$ for an outbreak to become a major epidemic.

The EpiModel library

Jenness SM, Goodreau SM and Morris M. EpiModel: An R Package for Mathematical Modeling of Infectious Disease over Networks. Journal of Statistical Software. 2018; 84(8): 1-47

EpiModel

- R package providing tools to simulate and analyze network-based epidemiological models
- Provides two types of approaches:
  - Simulation of compartmental ODE models (not very interesting)
  - Simulation of network models
- Their website contains some useful tutorials
- Part of the meta-package statnet

Bibliography I