



University  
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## Network models MATH 8xyz – Lecture 27

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.

We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

# Outline



## Why network models?

- ▶ Classical models allow some flexibility (see e.g. the section in Course 04 on incidence functions or Course 11 on group models), but this remains limited and an approximation.
- ▶ Like agent-based models (Course 18), network models are considered to make the description of contact processes in pathogen transmission more realistic.

# Human life is organized in networks

- ▶ Family
- ▶ Circles of friends
- ▶ Professional network
- ▶ ...
- ▶ Social network theory has existed and been used for years, e.g., in professional settings (how to improve interactions in a company)

# Social networks

- ▶ Before considering epidemics on networks, a few notions of social networks, because they are generally useful for understanding networks
- ▶ Social network methods introduce measures to evaluate certain properties of graphs that are useful to know
- ▶ A network is a (mathematical) graph, directed or not, in which edges represent connections (of any kind) between individuals, who are the nodes of the graph

## Context

- ▶  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  an undirected graph
- ▶  $\mathcal{D}(\mathcal{V}, \mathcal{A})$  a digraph (directed graph)
- ▶  $\mathcal{V}$  the set of nodes (vertices)
- ▶  $\mathcal{E}$  the set of edges in the undirected case
- ▶  $\mathcal{A}$  the set of arcs in the directed graph

## Example: Air transport network

- ▶ We illustrate with data from the global air transport network
- ▶ Data is quite good (sometimes very good), and a clear advantage:
- ▶ When a plane leaves somewhere and arrives elsewhere, it's quite ... deterministic

## Graph density

An (un)directed graph is **complete** if every pair of nodes is connected (by an edge in each direction for digraphs).

If there are  $n = |\mathcal{V}|$  nodes, then there are  $n(n-1)/2$  (resp.  $n(n-1)$ ) edges in the complete graph (resp. digraph).

(We do not count self-loops.)

Density of  $\mathcal{G}$  (undirected):

$$\text{dens}_{\mathcal{G}} = \frac{2 |\mathcal{E}|}{n(n-1)}$$

Density of  $\mathcal{D}$  (directed):

$$\text{dens}_{\mathcal{D}} = \frac{|\mathcal{A}|}{n(n-1)}$$



## Density of considered digraphs

Digraph	# nodes	# arcs	density
Manitoba	24	64	0.1159
Canada	222	804	0.0164
North America	934	7,814	0.009
Global	3403	32,576	0.0028

# Degree

- ▶ **Degree**  $d_{\mathcal{G}}(v)$  of node  $v \in \mathcal{V}$  in  $\mathcal{G}$ : number of edges incident to  $v$
- ▶ **In-degree**  $d_{\mathcal{D}}^{-}(v)$  in  $\mathcal{D}$ : number of arcs with head  $v$
- ▶ **Out-degree**  $d_{\mathcal{D}}^{+}(v)$  in  $\mathcal{D}$ : number of arcs with tail  $v$
- ▶ **Degree**  $d_{\mathcal{D}}(v)$  in  $\mathcal{D}(\mathcal{V}, \mathcal{A})$ : number of arcs incident to  $v$  in the underlying undirected graph  $\mathcal{G}$  (where every arc is considered bidirectional)

## Global in-degree of the air transport network

City	Country	In-degree	Rank
London	UK	365	1
Paris	France	294	2
Frankfurt	Germany	287	3
Atlanta	USA	249	4
New York	USA	241	5
Moscow	Russia	225	6
Amsterdam	Netherlands	204	7
Chicago	USA	203	8
Munich	Germany	200	9
Milan	Italy	181	10

## Shortest path

Let  $\mathcal{D}$  be a digraph. The (or the) shortest path(s) from  $i$  to  $j$  in  $\mathcal{V}$ :

$$d_{\mathcal{D}}(i, j) = \min_{p \in \mathcal{P}(i, j)} f(p)$$

where  $\mathcal{P}(i, j)$  is the set of paths from  $i$  to  $j$  and  $f(p)$  is a valuation of the arcs in path  $p$ . We define  $d_{\mathcal{D}}(i, j) = \infty$  if there is no path from  $i$  to  $j$ .

$f(p)$  can be:

- ▶ the number of arcs in  $p$  from  $i$  to  $j$  (**geodesic distance**)
- ▶ great circle distance of the arcs in  $p$
- ▶ flight duration of the arcs in  $p$

## Eccentricity

**Eccentricity** (or **König number**) of node  $v$  in  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ :

$$e(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v, v')$$

**In-eccentricity** in  $\mathcal{D}(\mathcal{V}, \mathcal{A})$ :

$$e^{-}(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v', v)$$

**Out-eccentricity** in  $\mathcal{D}(\mathcal{V}, \mathcal{A})$ :

$$e^{+}(v) = \max_{v' \in \mathcal{V}} d_{\mathcal{D}}(v, v')$$

# Radius

**Radius** of  $\mathcal{G}$

$$\rho_{\mathcal{G}} = \min_{v \in \mathcal{V}} e(v)$$

**In-radius** of  $\mathcal{D}$

$$\rho_{\mathcal{D}}^{-} = \min_{v \in \mathcal{V}} e^{-}(v)$$

**Out-radius** of  $\mathcal{D}$

$$\rho_{\mathcal{D}}^{+} = \min_{v \in \mathcal{V}} e^{+}(v)$$

Radius =  $\min(\max(\cdot)) \rightarrow$  directionality

## Radius of the air transport network

Graph	$\rho^-$	$\rho^+$
Manitoba	2	3
Canada	6	6
North America	6	7
Global	7	7

## Center of a graph

**Center** of  $\mathcal{D}$ :

$$\mathcal{C}_{\mathcal{D}} = \{v \in \mathcal{V} : e(v) = \rho_{\mathcal{D}}\}$$



## Center of the air transport network

Graph	$\mathcal{C}^-$	$\ \mathcal{C}^-\ $	$\mathcal{C}^+$	$\ \mathcal{C}^+\ $
Manitoba	2	1 (YWG)	3	7
Canada	6	1 (YTO)	6	1 (YTO)
North America	6	1 (YTO)	7	18
Global	7	131	7	20

{YYC, YEA, Halifax, Kelowna, Moncton, YMQ, YOW, Quebec, St John's, YTO, YVR, Victoria, Y

$\mathcal{C}^-$

{Toronto, Vancouver}  $\subset \mathcal{C}^+$

# Diameter

Diameter of  $\mathcal{D}$

$$\text{diam}_{\mathcal{D}} = \max_{v \in \mathcal{V}} e(v)$$

diameter =  $\max(\max(.)) \rightarrow$  no directionality

## Diameter of the air transport network

Graph	Diameter
Manitoba	5
Canada	12
North America	13
Global	13

## Periphery of a graph

**Periphery of  $\mathcal{D}$**

$$\mathcal{P}_{\mathcal{D}} = \{v \in \mathcal{V} : e(v) = \text{diam}_{\mathcal{D}}\}$$

## Periphery of the air transport network

Graph	In-periphery	Out-periphery
Manitoba	Lynn Lake	Cross Lake, Red Sucker Lake
Canada	Peawanuck	Peawanuck, Port Hope Simpson
North America	Stony River	Peawanuck, Port Hope Simpson
Global	Stony River, Hooker Creek, Peawanuck	Hooker Creek, Beni, Peawanuck,

## Many other measures

- ▶ betweenness
- ▶ closeness
- ▶  $k$ -cores
- ▶ ...

# General framework for network models

See for example:

- ▶ Newman. Spread of epidemic disease on networks, 2002
- ▶ Keeling & Eames. Networks and epidemic models, 2005
- ▶ Meyers et al. Network theory and SARS: predicting outbreak diversity, 2005
- ▶ Meyers, Newman & Pourbohloul. Predicting epidemics on directed contact networks, 2006
- ▶ Bansal et al. The dynamic nature of contact networks in infectious disease epidemiology, 2010

## Typical network model

Typically, we consider a graph (or digraph) in which:

- ▶ each node is an individual
- ▶ the existence of an arc from  $i$  to  $j$  indicates that  $i$  is in contact with  $j$  and can transmit the infection
- ▶ in the undirected case, the existence of an arc from  $i$  to  $j$  implies the same arc from  $j$  to  $i$  and establishes that the two individuals are connected
- ▶ the connection is not permanent, but rather describes the possibility of a connection:  $i$  and  $j$  come into contact regularly



## Adjacency matrix

We often use the **adjacency matrix**  $A = [a_{ij}]$ , where  $a_{ij} = 1$  if node  $i$  has a link to node  $j$  and  $a_{ij} = 0$  otherwise.

Sometimes we write  $A(\mathcal{D})$  to indicate that  $A$  is the adjacency matrix of digraph  $\mathcal{D}$ , and conversely,  $\mathcal{D}(A)$  to indicate that the graph is built using the adjacency matrix.

If the graph is undirected, then  $A$  is symmetric.

## Nature of the network

- ▶ Sometimes we have precise data on links between individuals (surveys, etc.)
- ▶ Often we idealize networks, choosing networks with given properties

## Degree distribution of the (di)graph

The **transmissibility**  $T$  of a disease in a graph is the average probability that an infectious individual transmits the disease to a susceptible individual with whom they are in contact.

In an uncorrelated network,

$$T_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

where  $\langle k \rangle$  and  $\langle k^2 \rangle$  are the mean degree and the mean squared degree.

It is necessary that  $T > T_c$  for an outbreak to become a major epidemic.

# The EpiModel library

Jenness SM, Goodreau SM and Morris M. EpiModel: An R Package for Mathematical Modeling of Infectious Disease over Networks. Journal of Statistical Software. 2018; 84(8): 1-47

# EpiModel

- ▶ R package providing tools to simulate and analyze network-based epidemiological models
- ▶ Provides two types of approaches:
  - ▶ Simulation of compartmental ODE models (not very interesting)
  - ▶ Simulation of network models
- ▶ Their website contains some useful tutorials
- ▶ Part of the meta-package statnet

# Bibliography I