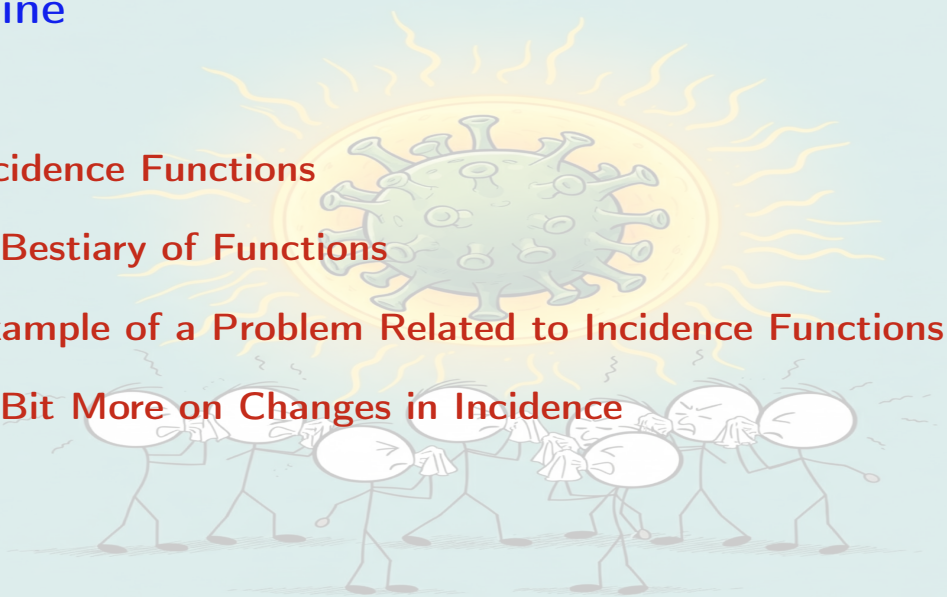
Incidence functions

MATH 8xyz – Lecture 12

Julien Arino
Department of Mathematics @ University of Manitoba
Maud Menten Institute @ PIMS
julien.arino@umanitoba.ca

Winter 20XX

Outline



Incidence Functions

A Bestiary of Functions

Example of a Problem Related to Incidence Functions

A Bit More on Changes in Incidence

Introduction to Incidence Functions

- Before continuing, let's discuss **incidence functions**, which describe how contacts between individuals occur and how they result in disease transmission

- See in particular McCallum, Barlow & Hone, How should pathogen transmission be modelled?, Trends in Ecology & Evolution **16** (2001)

Incidence Function versus Force of Infection

- Two different forms of the function representing the rate at which individuals move from compartment S to infected compartments:

- $S' = -f(S, I, N)$ is an **incidence function**

- $S' = -\lambda(S, I, N)S$ is a **force of infection**

- The two are equivalent; the context usually determines which form is used. For example, PDE models structured by age of infection must integrate $I(t, a)$ over age and therefore often use the force of infection.

Interactions - Infection

- The rate at which new cases appear is the **incidence function**

$$f(S, I, N) \tag{1}$$

- Depends on the number S of susceptibles, I of infectious individuals, and sometimes the total population N

- An incidence function includes two components:
  - a count of the number of contacts occurring
  - a description of the probability that such a contact, when it occurs, leads to pathogen transmission

Difficulty of the Choice

- Choosing a good function is difficult and is probably one of the most "unstable" parts of modelling the spread of infectious diseases

The Two Most Common Incidence Functions

- The two most commonly used incidence functions are the **mass action incidence**

$$f(S, I) = \beta SI \tag{2}$$

and the **standard incidence** (or **proportional incidence**)

$$f(S, I) = \beta \frac{SI}{S + I} \tag{3}$$

- In both cases, β is the **disease transmission coefficient**, although its exact interpretation varies

Units of β

- If $X(t)$ is the population of compartment X at time t, then X' has units number/time

- In a differential equation, the terms on the left and right sides of the "=" sign must have the same units

- The incidence function therefore has units number/time

- (And if a force of infection is used, the units are 1/time)

Units – Mass Action Incidence

- Mass action incidence

$$\beta SI \propto \beta \times \text{number} \times \text{number}$$

has units number/time if β has units $1/(\text{number} \times \text{time})$

- Standard incidence

$$\beta SI/N \propto \beta \times \text{number} \times \text{number/number} \propto \beta \times \text{number}$$

has units number/time if β has units $1/\text{time}$

Mass Action Incidence

$$f(S, I) = \beta SI \tag{2}$$

- The mixing of susceptibles and infectious individuals is homogeneous
- This is a strong assumption: the number of contacts is the product of the number of susceptibles and the number of infectious individuals, so each susceptible individual can potentially meet each infectious individual
- (hence the name, by analogy with gas dynamics in chemistry/physics)
- When the population is large, this assumption becomes unrealistic

Standard Incidence (Proportional)

- Another widely used form of the incidence function

$$f(S, I, N) = \beta \frac{SI}{N} \tag{3}$$

- Each susceptible meets a fraction of the infectious individuals

- Or vice-versa! See, e.g., Hethcote, Qualitative analyses of communicable disease models, *Mathematical Biosciences* (1976)

- Case of a larger population

Constant Population $\implies$ Equivalence of Incidences

- When the total population is constant, many incidence functions are qualitatively equivalent

- Suppose that $N(t) \equiv N_0$, then

$$\beta SI = \tilde{\beta}\frac{SI}{N} \iff \tilde{\beta} = N_0\beta$$

with this $\tilde{\beta}$, (2) and (3) are identical

- Remember that the units differ, however

General Incidence

$$f(S, I, N) = \beta S^q I^p \tag{4}$$

- These functions were introduced with the aim of fitting data: for fitting, this adds two parameters p, q. We will see, however, that much theoretical work uses this incidence

Refuge Incidence

- Refuge effect; a proportion $0 < q < 1$ of the population is truly susceptible, for example due to spatial heterogeneities

$$f(S, I, N) = \begin{cases} \beta I \left(N - \dfrac{I}{q} \right), & \text{if } I < qN \\ 0, & \text{if } I \geq qN \end{cases} \tag{5}$$

Negative Binomial Incidence

$$f(S, I, N) = kS \ln\left(1 + \beta \frac{I}{k}\right) \tag{6}$$

- For small values of k, this describes a highly concentrated infection process, while as $k \to \infty$, we tend towards a mass action incidence

Asymptotic Contact

$$f(S, I, N) = \frac{N}{1 - \varepsilon + \varepsilon N} \frac{F(S, I)}{N} \tag{7}$$

where F is one of the functions already described

- When $\varepsilon = 0$, contacts are proportional to N, while when $\varepsilon = 1$, contacts are independent of N

Asymptomatic Transmission

$$f(S, I, N) = \beta \frac{SI}{c + S + I} \tag{8}$$

where c is a constant. For example,

$$\frac{C(N)}{N} F(S, I)$$

with $C(N) = N/(1 - \varepsilon + \varepsilon N)$ the function describing the contact rate and $F(S, I)$ the function describing disease spread, which is assumed here to be a negative binomial incidence

Changing Incidence

$$F(S, I, N) = \begin{cases} \beta SI & \text{if } N \leq \hat{N} \\ \beta \dfrac{SI}{N} & \text{if } N > \hat{N} \end{cases} \tag{9}$$

- Arino & McCluskey, Effect of a sharp change of the incidence function on the dynamics of a simple disease, *Journal of Biological Dynamics* (2010)

Change Shape or Saturate?

- C. Kribs-Zaleta. To switch or taper off: the dynamics of saturation, *Mathematical Biosciences* (2004)

Two Different Incidence Functions

- We will consider the role of two different functions: a continuous and differentiable function

$$\beta_{sm}(N) = \beta_0 \frac{N}{N + A} \tag{10}$$

and a continuous but with a *switch* (abrupt transition?)

$$\beta_{sw}(N) = \begin{cases} \beta_0 \dfrac{N}{A}, & N < A \\ \beta_0, & N \geq A \end{cases} \tag{11}$$

- These functions represent saturation in different ways

The Model

- SIS model in non-constant population

$$I' = \beta(N)(N - I)\frac{I}{N} - (d + \gamma + \delta)I \tag{1}$$

$$N' = bN\left(1 - \frac{N}{K}\right) - dN - \delta I \tag{2}$$

Non-dimensionalization of the Model

- Let $\bar{N} = (1 - \frac{d}{b})K$, $i = I/N$ and $n = N/\bar{N}$. Then

$$i' = (\beta(\bar{N}n) - \delta)i(1 - i) - (b + \gamma)i + (b - d)in \tag{12a}$$

$$n' = (b - d)n(1 - n) - \delta in \tag{12b}$$

The Basic Reproduction Number

- Whatever incidence function is used

$$\mathcal{R}_0 = \frac{\beta(\bar{N})}{d + \gamma + \delta}$$

Analysis with $\beta = \beta_{sw}$

- With $\beta = \beta_{sw}$, there are 2 sub-models
- **Case I** $\beta(N) = \beta_0 N/A$; 3 equilibria $(i^\star, n^\star)$ in $[0,1]^2$:
  - the origin, always unstable
  - the ESMP $(0,1)$, GAS when $\mathcal{R}_0 < 1$
  - a unique EE that exists and is LAS when $\mathcal{R}_0 > 1$
- **Case II** $\beta(N) = \beta_0$. We have 4 equilibria $(i^\star, n^\star)$:
  - the origin, always unstable
  - the ESMP, GAS when $\mathcal{R}_0 < 1$
  - a unique EE that exists and is LAS when $\mathcal{R}_0 > 1$
  - an extinction PE that exists when $\beta \geq b + \gamma + \delta$ and is LAS iff, i.e., when $\mathcal{R}_0 > 1$ but the EE does not exist

Detailed Analysis

- Let $a = A/\bar{N}$
- If $a > 1$, all equilibria are below the switching point and the model reduces to Case I (model without saturation)
- If $a < 1$, there is a single ESMP (as well as the unstable trivial equilibrium) and there is a unique EE which is the EE of Case I iff

$$a > 1 - \frac{1}{k}\left(1 - \frac{d + \gamma + \delta}{\beta_0}\right)$$

and the EE of Case II otherwise

- The model with switch does not have the extinction PE of Case II

$\mathcal{R}_0$ in the Model with Switch

- In the model with switch

$$\mathcal{R}_0 = \frac{\beta_0}{d+\gamma+\delta}\,\frac{1}{\max(a,1)}$$

$\mathcal{R}_0$ in the Model with Differentiable Saturation

- In the model with differentiable saturation (10)

$$\mathcal{R}_0 = \frac{\beta_0}{d+\gamma+\delta}\,\frac{1}{a+1}$$

- We have the trivial PE (unstable), the ESMP (GAS when $\mathcal{R}_0 < 1$) and a unique EE that exists and is LAS when $\mathcal{R}_0 > 1$

Main Differences Between the Models

- The main difference between the models is quantitative: the point where $\mathcal{R}_0 < 1$ changes; this reflects that $\beta_{sm} < \beta_{sw}$
- Both models predict the eradication of the disease for a large region of the parameter space and exclude the possibility of extinction that the classical model with standard incidence allows
- Also, in both cases, the EE is such that

$$\lim_{\mathcal{R}_0 \to \infty} i^\star = \min(k, 1)$$

i.e., for $k < 1$ (i.e., $r < d + \delta$), the prevalence at the EE never approaches 100% because the reproductive resilience of the population is so low that the population becomes too small for the disease to spread to all, regardless of the value of $\mathcal{R}_0$

A Bit More on Changes in Incidence

The Basic Model

$$S' = bN - dS - F(S, I) + \gamma I \tag{13a}$$
$$I' = F(S, I) - (d + \delta + \gamma)I \tag{13b}$$

- So the dynamics of the total population are

$$N' = (b - d)N - \delta I$$

and the system has EEs iff $d < b < d + \delta$

System in Proportions

- We switch to the system in proportions $s = S/N$ and $i = I/N$. The total population is not constant and we therefore consider

$$i' = f(i, N) - (b + \delta + \gamma)i + \delta i^2 \tag{14a}$$
$$N' = (b - d - \delta i)N \tag{14b}$$

with

$$f(i, N) = \frac{F(S, I)}{N}$$

- We define

$$f(i, 0) = \lim_{N \to 0} f(i, N)$$

and we assume that this limit exists (we exclude, for example, that $F(\alpha S, \alpha I)$ is sub-linear in α near 0)

Existence of Solutions

Theorem 1

The solutions of (14a)–(14b) *in the positively invariant band*

$$\mathcal{D} = \{(i, N): \quad 0 \leq i \leq 1,\ N \geq 0\}$$

exist for all $t > 0$

A Bit More on Changes in Incidence

Density-Dependent Incidence

- The first case we consider is one with an incidence of the form we have already seen

$$F(S, I, N) = \begin{cases} \beta SI & \text{if } N \leq \hat{N} \\ \beta \dfrac{SI}{N} & \text{if } N > \hat{N} \end{cases} \tag{9}$$

where we normalize so that $\hat{N} = 1$

- We denote by $\mathcal{D}_L$ the part of $\mathcal{D}$ where $N \leq 1$ and by $\mathcal{D}_H$ where $N \geq 1$

Explicit Solution in $\mathcal{D}_H$

Theorem 2

Suppose that at time $t = \tau \geq 0$, we have $(i_\tau, N_\tau) := (i(\tau), N(\tau)) \in \mathcal{D}_H$. Then there exists a potentially infinite interval $\mathcal{I}$ with left endpoint the point $t = \tau$, such that for all $t \in \mathcal{I}$, we have $(i(t), N(t)) \in \mathcal{D}_H$ with

$$i(t) = \frac{K i_\tau}{\Psi(t-\tau)} \tag{3}$$

$$N(t) = N_\tau e^{(b-d)(t-\tau)} \exp\left(-\delta K i_\tau \int_\tau^t \frac{du}{\Psi(u-\tau)}\right) \tag{4}$$

where $K = \beta - (d + \gamma + \delta)$ and

$$\Psi(u) = i_\tau(\beta - \delta)(1 - e^{-Ku}) + K e^{-Ku}$$

Equilibria of the System

Theorem 3

Let

$$N_\Delta = \frac{\delta(d+\delta+\gamma)}{\beta(d+\delta-b)}$$

For all parameter values, the ESMP is

$$e_0 := (i_0, N_0) = (0, 0)$$

- *If $b < d$, there is no other PE and e_0 is GAS*
- *If $d < b$, e_0 is unstable*
- *If $d < b < d + \delta$, then the presence of EE $e_\star = (i_\star, N_\star)$, $i_\star = (b-d)/\delta$ depends on*
  - *If $N_\Delta < 1$, $e_\star = (i_\star, N_\Delta)$ LAS*
  - *If $N_\Delta = 1$, $e_\star = (i_\star, N)$ for all $N \geq 1$*
  - *If $N_\Delta > 1$, $e_\star$ does not exist*

A Bit More on Changes in Incidence

Incidence with Exceeding Capacities

- Suppose

$$F(S, I, N) = \begin{cases} \beta_1 \dfrac{SI}{N} & \text{if } I \leq \hat{I} \\ \beta_1 \dfrac{SI}{N} + \beta_2 S(I - \hat{I}) & \text{if } I > \hat{I} \end{cases} \tag{15}$$

or, in proportions, $f(i, N)$ is of the form

$$\begin{cases} \beta_1(1 - i)i & \text{if } iN \leq \hat{I} \\ \beta_1(1 - i)i + \beta_2(1 - i)(iN - \hat{I}) & \text{if } iN > \hat{I} \end{cases} \tag{16}$$

The Regions in the Plane

- Here, the change does not occur for a value of N but along the hyperbola $N = \hat{I}/i$, for $i \in]0, 1]$, and we adapt the regions $\mathcal{D}_L$ and $\mathcal{D}_H$

Equilibria

- We potentially have 3 PEs

$$e_0 = (0, 0) \quad \bar{e} = (i_{LS}, 0) \quad e_\star = (i_N, N_{HS}(i_N))$$

with

$$i_{LS} = 1 - \frac{b + \gamma}{\beta_1 - \delta}, \tag{5}$$

$$i_N = \frac{b - d}{\delta} \tag{6}$$

and

$$N_{HS}(i) = \frac{\hat{I}}{i} + \frac{b + \gamma}{\beta_2(1 - i)} - \frac{\beta_1 - \delta}{\beta_2}$$

Stability of the Equilibria

Theorem 4

The system (14a)–(14b) *with incidence* (16) *has, potentially, 3 PEs, whose stability is given by the following table, in which we use*

$$\mathcal{E} = \frac{\beta_1}{b + \delta + \gamma}$$

and

$$\mathcal{S}_N = \frac{(b+\gamma)(b-d)}{d+\delta-b} + \frac{\beta_2 \hat{I}(d+\delta-b)}{d-b}$$

Stability Table

	e_0	$\bar{e}$	$e_\star$
$b < d, \mathcal{E} < 1$	GAS	Does not exist	Does not exist
$b < d, \mathcal{E} > 1$	Unstable	GAS	Does not exist
$d < b < d + \delta$, $i_{LS} < i_N$, $\mathcal{S}_N > 0$	Unstable	Unstable	LAS
$d < b < d + \delta$, $i_{LS} < i_N$, $\mathcal{S}_N < 0$	Unstable	Unstable	Unstable
$d < b < d + \delta$, $i_{LS} > i_N$	Unstable	GAS	Does not exist
$d + \delta < b$	Unstable	Unstable	Does not exist

- The SAG is obtained using an extension of Dulac's Theorem taking into account the existence, in a C^1 field, of a curve on which the field is C^0

Periodic Solutions

- This system admits periodic solutions
- We consider the case

$d < b < d + \delta$, $i_{LS} < i_N$, $\mathcal{S}_N > 0$	Unstable	Unstable	LAS
$d < b < d + \delta$, $i_{LS} < i_N$, $\mathcal{S}_N < 0$	Unstable	Unstable	Unstable

Theorem 5

For all $\varepsilon > 0$ and all $\mathcal{S}_N^0 > 0$, there exists $\bar{\mathcal{S}}_N \in]-\mathcal{S}_N^0, 0[$ such that the system (14a)–(14b) *with incidence* (16) *has a non-trivial periodic orbit in $\mathcal{B}_\varepsilon(e_\star)$ (open ball centered at $e_\star$ and radius ε) for $\mathcal{S}_N = \bar{\mathcal{S}}_N$*

Bibliography I


Arino, J. and McCluskey, C. C. (2010). Effect of a sharp change of the incidence function on the dynamics of a simple disease. *Journal of Biological Dynamics*, 4(5):490–505.


McCallum, H., Barlow, N., and Hone, J. (2001). How should pathogen transmission be modelled? *Trends in Ecology & Evolution*, 16(6):295–300.