

## Graphs – Introduction (theory)

# Graphs versus networks

Mostly a terminology difference:

- ▶ graphs in the mathematical world
- ▶ networks elsewhere

I will mostly say *graphs* (this is a math course) but might oscillate

Beware: language is not consistent, so make sure you read the definitions at the start of whatever source you are using

# Graphs vs digraphs vs multigraphs vs multidigraphs vs ...

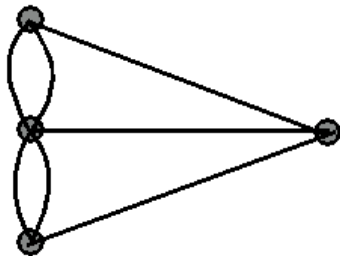
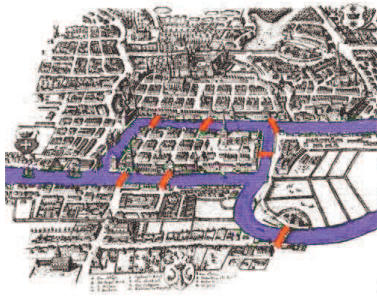
Name-wise and notation-wise, this domain is a bit of a mess. We see definitions later, but as far as possible, in these notes:

- ▶ The vertex set  $V$  is essentially the only constant in what follows
- ▶ An undirected graph is denoted  $G = (V, E)$ , where  $E$  are the *edges*
- ▶ An undirected multigraph is denoted  $G_M = (V, E)$ . We will not be using these much
- ▶ A *directed graph* (or *digraph*) is denoted  $G = (V, A)$ , where  $A$  are the *arcs*
- ▶ A *directed multigraph* (or *multidigraph*) is denoted  $G_M = (V, A)$
- ▶ Any of the above is called a *graph* and is denoted  $G = (V, X)$ , when we seek generality

And just to confuse the whole thing more: we often say *graph* for *unoriented graph*.

# The bridges of Königsberg

A “real life” problem from Euler: is it possible to cross the seven bridges in a continuous walk without recrossing any of them?



## Mathematical problem

Is it possible to find a trail containing all edges of the graph?

# Finding a cycle with all vertices

A salesperson must visit a couple of cities for their job. Is it possible for them to plan a round trip using highways enabling them to visit each specified city exactly once?



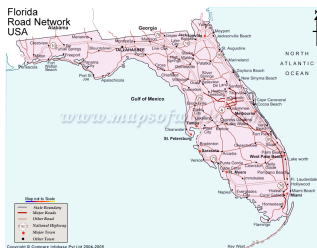
- ▶ vertices correspond to cities
- ▶ two vertices are connected iff a highway connects the corresponding cities and does not pass through any other city.

## Mathematical problem

Is it possible to find a cycle containing all graph vertices?

# How far is it to drive through $n$ cities?

What is the minimal length of driving needed to drive through  $n$  cities?



- ▶ vertices correspond to the cities
- ▶ all cities are connected; each edge has a value assigned to it

## Mathematical problem

What is the minimal spanning tree associated to the graph?

Binary relations

Undirected graphs

Directed graphs

Trees

Matrices associated to a graph/digraph

# Binary relation

## Definition 1 (Binary relation)

- ▶ A **binary relation** is an arbitrary association of elements of one set with elements of another (maybe the same) set
- ▶ A binary relation over the sets  $X$  and  $Y$  is defined as a subset of the Cartesian product  $X \times Y = \{(x, y) | x \in X, y \in Y\}$
- ▶  $(x, y) \in R$  is read “ $x$  is  $R$ -related to  $y$ ” and is denoted  $xRy$
- ▶ If  $(x, y) \notin R$ , we write not  $xRy$



## Definition 2 (Properties of binary relations)

A binary relation  $R$  over a set  $X$  is

- ▶ **Reflexive** if  $\forall x \in X, xRx$
- ▶ **Irreflexive** if there does not exist  $x \in X$  such that  $xRx$
- ▶ **Symmetric** if  $xRy \Rightarrow yRx$
- ▶ **Asymmetric** if  $xRy \Rightarrow \text{not } yRx$
- ▶ **Antisymmetric** if  $xRy$  and  $yRx \Rightarrow x = y$
- ▶ **Transitive** if  $xRy$  and  $yRz \Rightarrow xRz$
- ▶ **Total** (or **complete**) if  $\forall x, y \in X, xRy$  or  $yRx$

### Definition 3 (Equivalence relation)

A relation which is reflexive, symmetric and transitive is called an **equivalence relation**

### Definition 4 (Partial order)

A relation which is reflexive, antisymmetric and transitive is called a **partial order**

### Definition 5 (Total order)

A partial order which is total is called a **total order**

## Binary relations

### Undirected graphs

- Undirected graph

- Degree of a vertex

- Complete, bipartite and other notable graphs

- Isomorphic graphs

- Subgraphs, unions of graphs

- Walks, trails, paths

- Connectedness

- Planar graphs

### Directed graphs

### Trees

### Matrices associated to a graph/digraph

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# Graph

Intuitively: a graph is a set of points, and a set of relations between the points

The points are called the *vertices* of the graph and the relations are the *edges* of the graph

We can also think of the relations as being one directional, in which case the relations are the *arcs* of the digraph (a contraction of “directed graph”)

# Graph, vertex and edge

## Definition 6 (Graph)

An **undirected graph** is a pair  $G = (V, E)$  of sets such that

- ▶  $V$  is a set of points:  $V = \{v_1, v_2, v_3, \dots, v_p\}$
- ▶  $E$  is a set of 2-element subsets of  $V$ :  
 $E = \{\{v_i, v_j\}, \{v_i, v_k\}, \dots, \{v_n, v_p\}\}$ , also noted  
 $E = \{v_i v_j, v_i v_k, \dots, v_n v_p\}$

## Definition 7 (Vertex)

The elements of  $V$  are the **vertices** (or nodes, or points) of the graph  $G$ .  $V$  (or  $V(G)$ ) is the vertex set of the graph  $G$

## Definition 8 (Edge)

The elements of  $E$  are the **edges** (or lines) of the graph  $G$ .  $E$  (or  $E(G)$ ) is the edge set of the graph  $G$

# Order and Size

## Definition 9 (Order of a graph)

The number of vertices in  $G$  is the **order** of  $G$ . Using the notation  $|V(G)|$  for the *cardinality* of  $V(G)$ ,

$$|V(G)| = \text{order of } G$$

## Definition 10 (Size of a graph)

The number of edges in  $G$  is the **size** of  $G$ ,

$$|E(G)| = \text{size of } G$$

- ▶ A graph having order  $p$  and size  $q$  is called a  $(p, q)$ -graph
- ▶ A graph is finite if  $|V(G)| < \infty$

# Adjacent - Incident

## Definition 11 (Incident)

- ▶ A vertex  $v$  is **incident** with an edge  $e$  if  $v \in e$ ; then  $e$  is an edge at  $v$
- ▶ If  $e = uv \in E(G)$ , then  $u$  and  $v$  are each incident with  $e$
- ▶ The two vertices incident with an edge are its ends
- ▶ An edge  $e = uv$  is incident with both vertices  $u$  and  $v$

## Definition 12 (Adjacent)

- ▶ Two vertices  $u$  and  $v$  are **adjacent** in a graph  $G$  if  $uv \in E(G)$
- ▶ If  $uv$  and  $uw$  are distinct edges (i.e.  $v \neq w$ ) of a graph  $G$ , then  $uv$  and  $uw$  are adjacent edges



### Definition 13 (Multiple edge)

**Multiple edges** are two or more edges connecting the same two vertices within a multigraph

### Definition 14 (Loop)

A **loop** is an edge with both the same ends; e.g.  $\{u, u\}$  is a loop

### Definition 15 (Simple graph)

A **simple graph** is a graph which contains no loops or multiple edges

### Definition 16 (Multigraph)

A **multigraph** is a graph which can contain multiple edges or loops

## Graph and binary relations

A (simple) graph  $G$  can be defined in term of a vertex set  $V$  and an irreflexive and symmetric binary relation over  $V$

$R$  is symmetric if  $(u, v) \in R \Rightarrow (v, u) \in R$  (or, in other words,  $uRv \Rightarrow vRu$ )

Hence,  $\{(u, v), (v, u)\} \in E(G)$  ( $\{(u, v), (v, u)\}$  is an edge)

The set of edges  $E(G)$  is the set of symmetric pairs in  $R$

## Undirected graphs

Undirected graph

Degree of a vertex

Complete, bipartite and other notable graphs

Isomorphic graphs

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Planar graphs

## Definition 17 (Degree of a vertex)

Let  $v$  be a vertex of  $G = (V, E)$ .

- ▶ The number of edges of  $G$  incident with  $v$  is the **degree** of  $v$  in  $G$
- ▶ The number of edges of  $G$  at  $v$  is the **degree** of  $v$  in  $G$
- ▶ The degree of  $v$  in  $G$  is noted  $d_G(v)$  or  $\deg_G(v)$

## Theorem 18

Let  $G$  be a  $(p, q)$ -graph with vertices  $v_1, \dots, v_p$ , then

$$\sum_{i=1}^p d_G(v_i) = 2q$$

### Definition 19 (Odd vertex)

A vertex is an **odd vertex** if its degree is odd.

### Definition 20 (Even vertex)

A vertex is called **even vertex** if its degree is even.

### Theorem 21

*Every graph contains an even number of odd vertices.*

# Regular graph

## Definition 22 (Regular graph)

If all the vertices of  $G$  have the same degree  $k$ , then the graph  $G$  is  $k$ -regular

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### Definition 23 (Complete graph)

A graph is complete if every two of its vertices are adjacent

### Definition 24 ( $n$ -clique)

A simple, complete graph on  $n$  vertices is called an  $n$ -**clique** and is often denoted  $K_n$

Note that a complete graph of order  $p$  is  $(p - 1)$ -regular



# Bipartite graph

## Definition 25 (Bipartite graph)

A graph  $G = (V, E)$  is called **bipartite** if it is possible to partition the vertex set of  $G$  into two subsets, say  $V_1$  and  $V_2$  so that every edge of  $G$  joins a vertex of  $V_1$  with a vertex of  $V_2$  and no vertex joins another vertex of its own set

## Definition 26 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets  $X_1$  and  $X_2$ , s.t. no two vertices in the same set are adjacent. This graph may be written  $G = (X_1, X_2, U)$

# Complete Bipartite graph

## Definition 27 (Complete bipartite graph)

A bipartite graph in which every two vertices from the 2 different partitions are adjacent is called a complete bipartite graph

## Definition 28 (Complete bipartite graph)

If  $\forall x_1 \in X_1, \forall x_2 \in X_2$ , we have  $m_G(x_1, x_2) \geq 1$ , then  $G = (X_1, X_2, U)$  is a **complete bipartite** graph

A simple, complete bipartite graph with  $|X_1| = p$  and  $|X_2| = q$  is often denoted  $K_{p,q}$

## Some specific graphs

### Definition 29 (Tree)

Any connected graph that has no cycles is a **tree**

### Definition 30 (Cycle $C_n$ )

For  $n \geq 3$ , the **cycle**  $C_n$  is a connected graph of order  $n$  that is a cycle on  $n$  vertices

### Definition 31 (Path $P_n$ )

The **path**  $P_n$  is a connected graph that consists of  $n \geq 2$  vertices and  $n - 1$  edges. Two vertices of  $P_n$  have degree 1 and the rest are of degree 2

### Definition 32 (Star $S_n$ )

The **star** of order  $n$  is the complete bipartite graph  $K_{1,n-1}$  (1 vertex of degree  $n - 1$  and  $n - 1$  vertices of degree 1)

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# Isomorphic graphs

## Definition 33 (Isomorphic graphs)

Let  $G_1 = (V(G_1), E(G_1))$  and  $G_2 = (V(G_2), E(G_2))$  be two graphs.  $G_1$  and  $G_2$  are **isomorphic** if there exists an isomorphism  $\phi$  from  $G_1$  to  $G_2$ , that is defined as an injective mapping  $\phi : V(G_1) \rightarrow V(G_2)$  such that two vertices  $u_1$  and  $v_1$  are adjacent in  $G_1$  if and only if the vertices  $\phi(u_1)$  and  $\phi(v_1)$  are adjacent in  $G_2$

## Definition 34 (Another formulation of isomorphic graphs)

Let  $G_1 = (V(G_1), E(G_1))$  and  $G_2 = (V(G_2), E(G_2))$  be two graphs;  $G_1$  and  $G_2$  are isomorphic if there exists an isomorphism  $\phi$  from  $G_1$  to  $G_2$ , defined as an injective mapping  $\phi : V(G_1) \rightarrow V(G_2)$  such that  $\{u_1, v_1\} \in E(G_1)$  if and only if  $\{\phi(u_1), \phi(v_1)\} \in E(G_2)$  for all  $u_1, v_1 \in V(G_1)$

If  $\phi$  is an isomorphism from  $G_1$  to  $G_2$ , then the inverse mapping  $\phi^{-1}$  from  $V(G_2)$  to  $V(G_1)$  also satisfies the definition of an isomorphism. As a consequence, if  $G_1$  and  $G_2$  are isomorphic graphs, then

- ▶  $G_1$  is isomorphic to  $G_2$
- ▶  $G_2$  is isomorphic to  $G_1$

### Theorem 35

*The relation “is isomorphic to” is an equivalence relation on the set of all graphs*

### Theorem 36

*If  $G_1$  and  $G_2$  are isomorphic graphs, then the degrees of vertices of  $G_1$  are exactly the degrees of vertices of  $G_2$*

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# Subgraph

## Definition 37 (Subgraph)

Let  $G = (V(G), E(G))$  be a graph. A graph  $H = (V(H), E(H))$  is a **subgraph** of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$

## Definition 38 (Another definition of subgraph)

If a graph  $F$  is isomorphic to a subgraph  $H$  of  $G$ , then  $F$  is also called a subgraph of  $G$



Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs.

Definition 39 (Union of  $G_1$  and  $G_2$ )

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

Definition 40 (Intersection of  $G_1$  and  $G_2$ )

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

Definition 41 (Disjoint graphs)

If  $G_1 \cap G_2 = (\emptyset, \emptyset) = \emptyset$  (empty graph) then  $G_1$  and  $G_2$  are disjoint.

Definition 42 (Complement of  $G_1$ )

The complement  $\bar{G}_1$  of  $G_1$  is the graph on  $V_1$ , with the edge set  $E(\bar{G}_1) = [V_1]^2 \setminus E_1$  ( $e \in E(\bar{G}_1)$  iff  $e \notin E_1$ )

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# Walk

## Definition 43 (Walk)

A **walk** in a graph  $G = (V, E)$  is a non-empty alternating sequence  $v_0 e_0 v_1 e_1 v_2 \dots e_{k-1} v_k$  of vertices and edges in  $G$  such that  $e_i = \{v_i, v_{i+1}\}$  for all  $i < k$ . This walk begins with  $v_0$  and ends with  $v_k$

## Definition 44 (Length of a walk)

The **length** of a walk is equal to the number of edges in the walk

## Definition 45 (Closed walk)

If  $v_0 = v_k$ , the walk is **closed**

# Trail and path

## Definition 46 (Trail)

If the edges in the walk are all distinct, it defines a **trail** in  $G = (V, E)$

## Definition 47 (Path)

If the vertices in the walk are all distinct, it defines a **path** in  $G$

The sets of vertices and edges determined by a trail is a subgraph

## Distance between two vertices

### Definition 48 (Distance between two vertices)

The distance  $d(u, v)$  in  $G = (V, E)$  between two vertices  $u$  and  $v$  is the length of the shortest path linking  $u$  and  $v$  in  $G$ . If no such path exists, we assume  $d(u, v) = \infty$

# Circuit and cycle

## Definition 49 (Circuit)

A trail linking  $u$  to  $v$ , containing at least 3 edges and in which  $u = v$ , is a **circuit**

## Definition 50 (Cycle)

A circuit which does not repeat any vertices (except the first and the last) is a **cycle** (or **simple circuit**)

## Definition 51 (Length of a cycle)

The **length of a cycle** is its number of edges

### Definition 52 (Eulerian trail)

A trail containing all the vertices and edges of a multigraph  $M$  is a **Eulerian trail** of  $M$

### Definition 53 (Traversable graph)

If a graph  $G$  has a Eulerian trail, then  $G$  is a **traversable graph**

### Definition 54 (Eulerian circuit)

A circuit containing all the vertices and edges of a multigraph  $M$  is a **Eulerian circuit** of  $M$

### Definition 55 (Eulerian graph)

A graph (resp. multigraph) containing an Eulerian circuit is a **Eulerian graph** (resp. **Eulerian multigraph**)

### Theorem 56

*A multigraph  $M$  is traversable if and only if  $M$  is connected and has exactly two odd vertices. Furthermore, any Eulerian trail of  $M$  begins at one of the odd vertices and ends at the other odd vertex*

### Theorem 57

*A multigraph  $M$  is Eulerian  $\iff M$  is connected and every vertex of  $M$  is even*



# Fleury's algorithm to find a Eulerian circuit

(for a connected graph with no odd vertices)

- ▶ Pick any vertex as a starting point.
- ▶ Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to.
- ▶ Continue until you return to your starting point.

RESULT: a Eulerian circuit

# Fleury's algorithm to find a Eulerian trail

(for a connected graph with exactly 2 odd vertices)

- ▶ Start at one of the odd vertices.
- ▶ Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to.
- ▶ Continue until every edge has been traveled.

RESULT: a Eulerian trail

### Definition 58 (Hamiltonian path)

A path containing all vertices of a graph  $G$  is a **Hamiltonian path** of  $G$

### Definition 59 (Traceable graph)

If a graph  $G$  has an Hamiltonian path, then  $G$  is a **traceable graph**

### Definition 60 (Hamiltonian cycle)

A cycle containing all vertices of a graph  $G$  is a **Hamiltonian cycle** of  $G$

### Definition 61 (Hamiltonian graph)

A graph containing a Hamiltonian cycle is a **Hamiltonian graph**

### Theorem 62 (Dirac's theorem)

*If  $G$  is a graph of order  $p \geq 3$  such that  $\deg(v) \geq p/2$  for every vertex  $v$  of  $G$ , then  $G$  is Hamiltonian*

### Theorem 63 (Ore's theorem)

*If  $G$  is a graph of order  $p \geq 3$  such that for all distinct nonadjacent vertices  $u$  and  $v$  of  $G$ ,*

$$\deg(u) + \deg(v) \geq p,$$

*then  $G$  is Hamiltonian*

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Degree of a vertex

Complete, bipartite and other notable graphs

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**Connectedness**

Planar graphs

# Connected vertices and graph, components

## Definition 64 (Connected vertices)

Two vertices  $u$  and  $v$  in a graph  $G$  are **connected** if  $u = v$ , or if  $u \neq v$  and there exists a path in  $G$  that links  $u$  and  $v$

## Definition 65 (Connected graph)

A graph is **connected** if every two vertices of  $G$  are connected; otherwise,  $G$  is **disconnected**

## A necessary condition for connectedness

### Theorem 66

*A connected graph on  $p$  vertices has at least  $p - 1$  edges*

In other words, a connected graph  $G$  of order  $p$  has  $\text{size}(G) \geq p - 1$

# Connectedness is an equivalence relation

Denote  $x \equiv y$  the relation “ $x = y$ , or  $x \neq y$  and there exists a path in  $G$  connecting  $x$  and  $y$ ”.  $\equiv$  is an equivalence relation since

1.  $x \equiv y$  [reflexivity]
2.  $x \equiv y \implies y \equiv x$  [symmetry]
3.  $x \equiv y, y \equiv z \implies x \equiv z$  [transitivity]

## Definition 67 (Connected component of a graph)

The classes of the equivalence relation  $\equiv$  partition  $X$  into connected sub-graphs of  $G$  called **connected components** (or **components** for short) of  $G$

A connected subgraph  $H$  of a graph  $G$  is a component of  $G$  if  $H$  is not contained in any connected subgraph of  $G$  having more vertices or edges than  $H$



## Vertex deletion & cut vertices

### Definition 68 (Vertex deletion)

If  $v$  is a vertex of  $G$ , the graph  $G - v$  is the graph formed from  $G$  by removing  $v$  and all edges incident with  $v$

### Definition 69 (Cut-vertices)

If  $v$  is a cut-vertex of a connected graph  $G$ , then  $G - v$  is disconnected

# Edge deletion & bridges

## Definition 70 (Edge deletion)

If  $e$  is an edge of  $G$ , the graph  $G - e$  is the graph formed from  $G$  by removing  $e$  from  $G$

## Definition 71 (Bridge)

An edge  $e$  in a connected graph  $G$  is called a bridge if  $G - e$  is disconnected

## Theorem 72

*Let  $G$  be a connected graph. An edge  $e$  of  $G$  is a bridge of  $G$  if and only if  $e$  does not lie on any cycle of  $G$*

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# Planar graph

## Definition 73 (Planar graph)

A graph is **planar** if it can be drawn in the plane with no crossing edges. Otherwise it is nonplanar

## Definition 74 (Plane graph)

A **plane graph** is a graph that is drawn in the plane with no crossing edges. (This is only possible if the graph is planar)

Let  $G$  be a plane graph

- ▶ the connected parts of the plane are called **regions**
- ▶ vertices and edges that are incident with a region  $R$  make up a **boundary** of  $R$

### Theorem 75 (Euler's formula)

*Let  $G$  be a connected plane graph with  $p$  vertices,  $q$  edges, and  $r$  regions, then*

$$p - q + r = 2$$

### Corollary 76

*Let  $G$  be a plane graph with  $p$  vertices,  $q$  edges,  $r$  regions, and  $k$  connected components, then*

$$p - q + r = k + 1$$

## Theorem 77

*Let  $G$  be a connected planar graph with  $p$  vertices and  $q$  edges, where  $p \geq 3$ , then*

$$q \leq 3p - 6.$$

*(a maximal connected planar graph with  $p$  vertices has  $q = 3p - 6$  edges)*

## Corollary 78

*If  $G$  is a planar graph, then  $\delta(G) \leq 5$ , where  $\delta(G)$  is the minimal degree of  $G$ . (every planar graph contains a vertex of degree less than 6)*

## Two well-known non-planar graphs

$K_{3,3}$  is nonplanar

$K_5$  is nonplanar

As a matter of fact, a graph having a subgraph that is  $K_{3,3}$  or  $K_5$  is a sufficient condition for non-planarity, as we see later.

### Definition 79 (Subdivision of $G$ )

Given a graph  $G$ , a subdivision of  $G$  is a graph that can be obtained by inserting any number of vertices of degree 2 along a edges of  $G$

### Theorem 80 (Kuratowski Theorem)

*A graph  $G$  is planar if and only if it contains no subgraph isomorphic to  $K_5$  or  $K_{3,3}$  or any subdivision of  $K_5$  or  $K_{3,3}$*

**Note:** If a graph  $G$  is nonplanar and  $G$  is a subgraph of  $G'$ , then  $G'$  is also nonplanar



### Definition 81 (Colouring of a graph $G$ )

A **colouring** of a graph  $G$  is an assignment of colours to the vertices of  $G$  such that adjacent vertices have different colours

### Definition 82 ( $n$ -colouring of $G$ )

A  **$n$ -colouring** is a colouring of  $G$  using  $n$  colours

### Definition 83 ( $n$ -colourable)

$G$  is  **$n$ -colourable** if there exists a colouring of  $G$  that uses  $n$  colours

## Definition 84 (Chromatic number)

The **chromatic number**  $\chi(G)$  of a graph  $G$  is the minimal value  $n$  for which an  $n$ -colouring of  $G$  exists

## Theorem 85 (Some properties)

- ▶  $\chi(G) = 1$  if and only if  $G$  have no edges
- ▶ If  $G = K_{n,m}$ , then  $\chi(G) = 2$
- ▶ If  $G = K_n$ , then  $\chi(G) = n$
- ▶ For any graph  $G$ ,

$$\chi(G) \leq 1 + \Delta(G)$$

where  $\Delta(G)$  is the maximum degree of  $G$

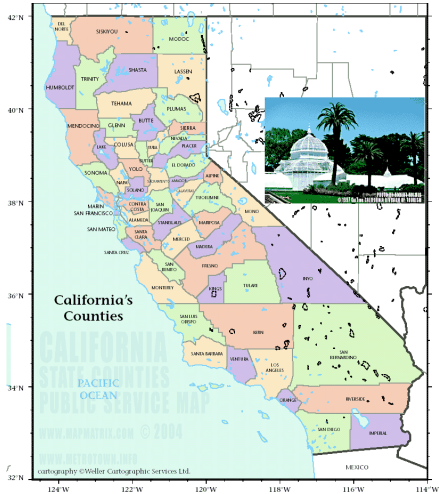
- ▶ If  $G$  is a planar graph, then  $\chi(G) \leq 4$
- ▶ If  $G$  is a planar graph, then  $\chi(G) \leq 5$

# Welch-Powell algorithm for colouring a graph $G$

1. Order the vertices of  $G$  in decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree)
2. Use one colour to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this colour
3. Start again at the top of the list and repeat the process painting previously unpainted vertices using a second colour
4. Continue repeating with additional colours until all the vertices have been painted

## “Real life” problem

What is the minimal number of colors to color all counties in the map? (two adjacent counties must have different colors)



## “Real life” problem

What is the minimal number of colors to color all counties in the map? (two adjacent counties must have different colors)

Mathematical representation:

- ▶ vertices correspond to the counties
- ▶ two vertices are linked iff the two counties are adjacent.

### Mathematical problem

What is the chromatic number of the graph associated to the map?

Binary relations

Undirected graphs

Directed graphs

- Directed graph

- Degrees in digraphs

- Walks, paths, etc.

- Connectivity in digraphs

- Orientable graphs

Trees

Matrices associated to a graph/digraph

## Directed graphs

Directed graph

Degrees in digraphs

Walks, paths, etc.

Connectivity in digraphs

Orientable graphs

# Definitions

## Definition 86 (Digraph)

A directed graph (or **digraph**) is a pair  $G = (V, A)$  of sets such that

- ▶  $V$  is a set of points:  $V = \{v_1, v_2, v_3, \dots, v_p\}$
- ▶  $A$  is a set of ordered pairs of  $V$ :  
 $A = \{(v_i, v_j), (v_i, v_k), \dots, (v_n, v_p)\}$ , also noted  
 $A = \{v_i v_j, v_i v_k, \dots, v_n v_p\}$ .

## Definition 87 (Vertex)

The elements of  $V$  are the vertices of the digraph  $G$ .  $V$  is the vertex set of the digraph  $G$ , also noted  $V(G)$ .

## Definition 88 (Arc)

The elements of  $A$  are the **arcs** (directed edges) of the digraph  $G$ .  $A$  is the arc set of the digraph  $G$ , also noted  $A(G)$ .



## Digraph and binary relation

A digraph  $D$  can be defined in term of a vertex set  $V$  and an irreflexive relation  $R$  over  $V$

The defining relation  $R$  of the digraph  $G$  need not be symmetric

# Directed network

## Definition 89 (Directed network)

A directed network is a digraph together with a function  $f$ ,

$$f : A \rightarrow \mathbb{R},$$

which maps the arc set  $A$  into the set of real number. The value of the arc  $uv \in A$  is  $f(uv)$

# Loops & Multiple arcs

## Definition 90 (Loop)

A **loop** is an arc with both the same ends; e.g.  $(u, u)$  is a loop.

## Definition 91 (Multiple arcs)

**Multiple arcs** (or multi-arcs) are two or more arcs connecting the same two vertices.

# Multidigraph/Digraph

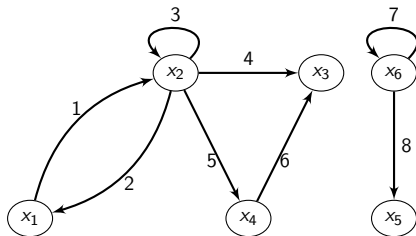
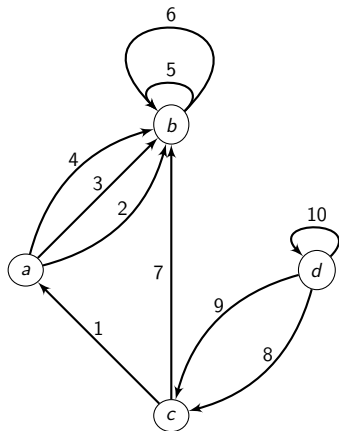
## Definition 92 (Multidigraph)

A **multidigraph** is a digraph which allows repetition of arcs or loops.

## Definition 93 (Digraph)

In a digraph, no more than one arc can join any pair of vertices.

# Examples



Let  $G = (V, A)$  be a digraph.

### Definition 94 (Arc endpoints)

For an arc  $u = (x, y)$ , vertex  $x$  is the **initial endpoint**, and vertex  $y$  is the **terminal endpoint**

### Definition 95 (Predecessor - Successor)

If  $(u, v) \in A(G)$  is an arc of  $G$ , then

- ▶  $u$  is a **predecessor** of  $v$ ,
- ▶  $v$  is a **successor** of  $u$ .

### Definition 96 (Neighbours of a vertex)

Let  $x \in V$  be a vertex. The **neighbours** of  $x$  is the set  $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$ , where  $\Gamma_G^+(x)$  and  $\Gamma_G^-(x)$  are, respectively, the set of successors and predecessors of  $x$ .

# Sources and sinks

## Definition 97 (Directed away - Directed towards)

If  $a = (u, v) \in A(G)$  is an arc of  $G$ , then

- ▶ the arc  $a$  is said to be **directed away** from  $u$
- ▶ the arc  $a$  is said to be **directed towards**  $v$

## Definition 98 (Source - Sink)

- ▶ Any vertex which has no arcs directed towards it is a **source**
- ▶ Any vertex which has no arcs directed away from it is a **sink**

# Adjacent arcs

## Definition 99 (Adjacent arcs)

Two arcs are **adjacent** if they have at least one endpoint in common.



## Arc incident out of a subset of arcs

### Definition 100 (Arc incident out of $A \subset X$ )

If the initial endpoint of an arc  $u$  belongs to  $A$ , and if the terminal endpoint of arc  $u$  does not belong to  $A$ , then  $u$  is said to be incident out of  $A$ , and we write  $u \in \omega^+(A)$ . Similarly, we define an arc incident into  $A$ , and the set  $\omega^-(A)$ . Finally, the set of arcs incident to  $A$  is denoted

$$\omega(A) = \omega^+(A) \cup \omega^-(A).$$

### Definition 101 (Symmetric graph)

If  $m_G^+(x, y) = m_G^-(x, y)$  for all  $x, y \in X$ , the graph  $G$  is **symmetric**. A 1-graph  $G = (X, U)$  is symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \in U$$

### Definition 102 (Anti-symmetric graph)

If for each pair  $(x, y) \in X \times X$ ,

$$m_G^+(x, y) + m_G^-(x, y) \leq 1$$

then the graph  $G$  is **anti-symmetric**. A 1-graph  $G = (X, U)$  is anti-symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \notin U$$

An anti-symmetric 1-graph without its direction is a simple graph

### Definition 103 (Subgraph of $G$ generated by $A \subset X$ )

The **subgraph** of  $G$  generated by  $A$  is the graph with  $A$  as its vertex set and with all the arcs in  $G$  that have both their endpoints in  $A$ . If  $G = (X, \Gamma)$  is a 1-graph, then the subgraph generated by  $A$  is the 1-graph  $G_A = (A, \Gamma_A)$  where

$$\Gamma_A(x) = \Gamma(x) \cap A \quad (x \in A)$$

### Definition 104 (Partial graph of $G$ generated by $V \subset U$ )

The graph  $(X, V)$  whose vertex set is  $X$  and whose arc set is  $V$ . In other words, it is graph  $G$  without the arcs  $U - V$

### Definition 105 (Partial subgraph of $G$ )

A partial subgraph of  $G$  is the subgraph of a partial graph of  $G$

## Directed graphs

Directed graph

**Degrees in digraphs**

Walks, paths, etc.

Connectivity in digraphs

Orientable graphs

# Degree

Let  $v$  be a vertex of a digraph  $G = (V, A)$ .

## Definition 106 (Outdegree of a vertex)

The number of arcs directed away from a vertex  $v$ , in a digraph is called the **outdegree** of  $v$  and is written  $d^+(v)$  or  $outdeg(v)$ .

## Definition 107 (Indegree of a vertex)

The number of arcs directed towards a vertex  $v$ , in a digraph is called the **indegree** of  $v$  and is written  $d^-(v)$  or  $indeg(v)$ .

## Definition 108 (Degree)

For any vertex  $v$  in a digraph, the **degree** of  $v$  is defined as  $d(v) = d^+(v) + d^-(v)$ .

### Theorem 109

*For any (di)graph, the sum of the degrees of the vertices equals twice the number of edges (arcs).*

### Corollary 110

*In any (di)graph, the sum of the degrees of the vertices is a nonnegative even integer.*

### Theorem 111

*If  $G$  is a digraph with a vertex set  $V(G) = \{v_1, \dots, v_p\}$  and having  $q$  arcs then*

$$\sum_{i=1}^p d^+(v_i) = \sum_{i=1}^p d^-(v_i) = q.$$

### Definition 112 (Regular digraph)

A digraph  $G$  is  $r$ -regular if  $\text{indeg}(v) = \text{outdeg}(v) = r$  for each vertex  $v$  of  $G$ .

## Directed graphs

Directed graph

Degrees in digraphs

**Walks, paths, etc.**

Connectivity in digraphs

Orientable graphs



# Walks

Let  $G = (V, A)$  be a digraph.

## Definition 113 (Directed walk)

A **directed walk** in a digraph  $G$  is a non-empty alternating sequence  $v_0 a_0 v_1 a_1 v_2 \dots a_{k-1} v_k$  of vertices and arcs in  $G$  such that  $a_i = (v_i, v_{i+1})$  for all  $i < k$ . This walk begins with  $v_0$  and ends with  $v_k$ .

## Definition 114 (Length of a directed walk)

The length of a directed walk is equal to the number of arcs in the directed walk.

## Definition 115 (Closed walk)

If  $v_0 = v_k$ , the walk is closed.

# Trails

Let  $G = (V, A)$  be a digraph.

## Definition 116 (Directed trail)

A directed walk in  $G$  in which all arcs are distinct is a **directed trail** in  $G$ .

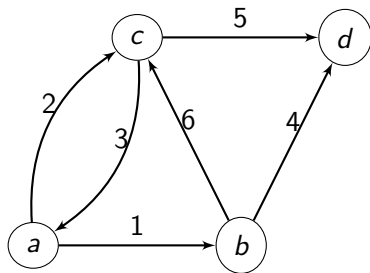
## Definition 117 (Directed path)

A directed walk in  $G$  in which all vertices are distinct is a **directed path** in  $G$ .

## Definition 118 (Directed cycle)

A closed walk is a **directed cycle** if it contains at least three vertices and all its vertices are distinct except for  $v_0 = v_k$ .

## Examples of directed cycles



Cycles:

►  $\mu^1 = (1, 6, 2) = [abca]$

►  $\mu^2 = (1, 6, 3) = [abca]$

►  $\mu^3 = (2, 3) = [aca]$

►  $\mu^4 = (1, 4, 5, 2) = [abdca]$

►  $\mu^5 = (6, 5, 4) = [acdb]$

►  $\mu^6 = (1, 4, 5, 3) = [abdca]$

Given a cycle  $\mu$ , denote  $\mu^+$  the set of all arcs in  $\mu$  that are in the direction that the cycle is traversed and  $\mu^-$  the set of all the other arcs in  $\mu$

Number the arcs in  $G$  as  $1, 2, \dots, m$ , then the cycle  $\mu$  is the vector

$$\mu = (\mu_1, \dots, \mu_m)$$

where

$$\mu_i = \begin{cases} 0 & \text{if } i \notin \mu^+ \cup \mu^- \\ +1 & \text{if } i \in \mu^+ \\ -1 & \text{if } i \in \mu^- \end{cases}$$

# Cocycles

Let  $A \subset X$  be nonempty and denote  $\omega^+(A)$  the set of arcs that have only their initial endpoint in  $A$  and  $\omega^-(A)$  the set of arcs that have only their terminal endpoint in  $A$ . Let

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

A **cocycle** is a nonempty set of arcs of the form  $\omega(A)$ , partitioned into two sets  $\omega^+(A)$  and  $\omega^-(A)$

An **elementary cocycle** is the set of arcs joining two connected subgraphs  $A_1$  and  $A_2$  s.t.

- ▶  $A_1, A_2 \neq \emptyset$
- ▶  $A_1 \cap A_2 = \emptyset$
- ▶  $A_1 \cup A_2 = C$ , with  $C$  a connected component of the graph

# A colouring lemma

## Lemma 119 (Arc colouring Lemma)

*Consider  $G$  with arcs  $1, \dots, m$ . Colour arc 1 black and arbitrarily colour the remaining arcs red, black or green. Then exactly one of the following holds true:*

- 1. there is an elementary cycle containing arc 1 and only red and black arcs with the property that all black arcs in the cycle have the same direction*
- 2. there is an elementary cocycle containing arc 1 and only green and black arcs, with the property that all black arcs in the cocycle have the same direction*

# Independent cycles and cycle bases

Consider cycles  $\mu^1, \mu^2, \dots, \mu^k$ . The cycles are **independent** if

$$c_1\mu^1 + c_2\mu^2 + \dots + c_k\mu^k = \mathbf{0}$$

$$\iff c_1 = c_2 = \dots = c_k = 0$$

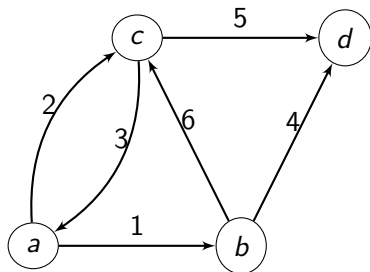
A **cycle basis** is an independent set  $\{\mu^1, \mu^2, \dots, \mu^k\}$  of cycles such that any cycle  $\mu$  can be written as

$$\mu = c_1\mu^1 + c_2\mu^2 + \dots + c_k\mu^k$$

for  $c_1, \dots, c_k \in \mathbb{R}$

The constant  $k$  is the **cyclomatic number** of  $G$ , denoted  $\nu(G)$

## Example



Elementary cycles:

- ▶  $\mu^1 = (1, 6, 2) = [abca]$
- ▶  $\mu^2 = (1, 6, 3) = [abca]$
- ▶  $\mu^3 = (2, 3) = [aca]$
- ▶  $\mu^4 = (1, 4, 5, 2) = [abdca]$
- ▶  $\mu^5 = (6, 5, 4) = [acdb]$
- ▶  $\mu^6 = (1, 4, 5, 3) = [abdca]$

We have  $\mu^1 - \mu^2 + \mu^3 = \mathbf{0}$



# An important result

## Theorem 120

*Let  $G$  be a graph with  $n$  vertices,  $m$  arcs and  $p$  connected components. Then the cyclomatic number of  $G$  is*

$$\nu(G) = m - n + p$$

## Directed graphs

Directed graph

Degrees in digraphs

Walks, paths, etc.

Connectivity in digraphs

Orientable graphs

# Definitions

## Definition 121 (Underlying graph)

Given a digraph, the undirected graph with each arc replaced by an edge is called the **underlying graph**.

## Definition 122 (Weakly connected digraph)

If the underlying graph is a connected graph, then the digraph is **weakly connected**.

## Definition 123 (Strongly connected digraph)

A digraph  $G$  is **strongly connected** if for every two distinct vertices  $u$  and  $v$  of  $G$ , there exists a directed path from  $u$  to  $v$ .

## Definition 124 (Disconnected digraph)

A digraph is said to be **disconnected** if it is not weakly connected.

## Strong connectedness is an equivalence relation

Denote  $x \equiv y$  the relation “ $x = y$ , or  $x \neq y$  and there exists a directed path in  $G$  from  $x$  to  $y$ ”.  $\equiv$  is an equivalence relation since

1.  $x \equiv x$  [reflexivity]
2.  $x \equiv y \implies y \equiv x$  [symmetry]
3.  $x \equiv y, y \equiv z \implies x \equiv z$  [transitivity]

### Definition 125 (Connected component of a graph)

Sets of the form

$$A(x_0) = \{x : x \in X, x \equiv x_0\}$$

are equivalence classes. They partition  $X$  into strongly connected sub-digraphs of  $G$  called **strongly connected components** (or **strong components**) of  $G$ .

A strong component in  $G$  is a maximal strongly connected subdigraph of  $G$ .

Let  $G = (V, A)$  be a digraph.

## Theorem 126

### *Properties*

- ▶ *If a digraph is strongly connected, it has only one strongly connected component.*
- ▶ *The strongly connected components partition the vertices in the digraph, with every vertex in exactly one strongly connected component.*

## Algorithm for determining strongly connected components in $G = (V, A)$

- ▶ Determine the strongly connected component  $C(v)$  containing the vertex  $v$ ; if  $V - C(v)$  is non-empty, re-do the same operation on the sub-digraph  $G' = (V - C(v), A')$ .
- ▶ To determine  $C(v)$ , the strongly connected component containing  $v$ : let  $v$  be a vertex of a digraph, which is not already in any strongly connected component.
  1. Mark the vertex  $v$  with  $\pm$
  2. Mark with  $+$  all successors (not already marked with  $+$ ) of a vertex marked with  $+$
  3. Mark with  $-$  all predecessors (not already marked with  $-$ ) of a vertex marked with  $-$
  4. Repeat until no more possible marking with  $+$  or  $-$

All vertices marked with  $\pm$  belong to the same strongly connected component  $C(v)$  containing the vertex  $v$ .

# Condensation of a digraph

## Definition 127 (Condensation of a digraph)

The condensation  $G^*$  of a digraph  $G$  is a digraph having as vertices the strongly connected components (SCC) of  $G$  and such that there exists an arc in  $G^*$  from a SCC  $C_i$  to another SCC  $C_j$  if there is an arc in  $G$  from some vertex of  $S_i$  to a vertex of  $S_j$ .

### Definition 128 (Articulation set)

For a connected graph, a set  $A$  of vertices is called an **articulation set** (or a **cutset**) if the subgraph of  $G$  generated by  $X - A$  is not connected

### Definition 129 (Stable set)

A set  $S$  of vertices is called a stable set if no arc joins two distinct vertices in  $S$



## Directed graphs

Directed graph

Degrees in digraphs

Walks, paths, etc.

Connectivity in digraphs

**Orientable graphs**

# Orientation

## Definition 130 (Orienting a graph)

Given a connected *graph*, we describe the act of assigning a direction to each edge (edge  $\rightarrow$  arc) as **orienting the graph**.

## Definition 131 (Strong orientation)

If the resulting digraph is strongly connected the orientation is called a **strong orientation**.

# Orientable graph

## Definition 132 (Orientable graph)

A connected graph  $G$  is called **orientable** if it is possible to assign a direction to each edge of  $G$  to produce a strongly connected digraph  $D$ . (If there exists a strong orientation of a connected graph, then the graph is orientable.)

## Theorem 133

*A connected graph  $G$  is orientable (has a strong orientation) if and only if  $G$  contains no bridges; that is every edge is contained in a cycle.*

Binary relations

Undirected graphs

Directed graphs

Trees

Matrices associated to a graph/digraph

# Trees

## Definition 134 (Forest, trees and branches)

- ▶ A connected graph with no cycle is a **tree**.
- ▶ A tree is a connected acyclic graph, its edges are called **branches**.
- ▶ A graph (connected or not) without any cycle is a **forest**. Each component is a tree. (A forest is a graph whose connected components are trees)

## Theorem 135 (Properties)

- ▶ *Every edge of a tree is a bridge (the deletion of any edge of a tree disconnects it)*
- ▶ *Given two vertices  $u$  and  $v$  of a tree, there is an unique path linking  $u$  to  $v$ .*
- ▶ *A tree with  $p$  vertices and  $q$  edges satisfies  $q = p - 1$ . Thus, a tree is minimally connected.*

# Spanning tree

## Definition 136 (Spanning tree)

A **spanning tree** of a connected graph  $G$  is a subgraph of  $G$  that contains all the vertices of  $G$  and is a tree.

A graph may have many spanning trees.

# Minimal spanning tree

## Definition 137 (Value of a spanning tree)

The **value of a spanning tree**  $T$  of order  $p$  is

$$\sum_{i=1}^{p-1} f(e_i)$$

where  $f$  is the function that maps the edge set into the set of real number.

## Definition 138 (Minimal spanning tree)

Let  $G$  be an undirected network, and let  $T$  be a **minimal spanning tree** of  $G$ . Then  $T$  is a spanning tree whose the value is minimum.

## Algorithm to find a minimal spanning tree

Let  $G = (V(G), E(G))$  be an undirected network, and let  $T$  be a minimal spanning tree.

1. sort the edges of  $G$  in increasing order by value
2.  $T = (V(G), \emptyset)$
3. for each edge  $e$  in sorted order if the endpoints of  $e$  are disconnected in  $T$  add  $e$  to  $T$



# Minimal connector problem

- ▶ Model: a graph  $G$  such that edges represent all possible connections, and each edge has a positive value which represents its cost; an undirected network  $G$
- ▶ Solution: a minimal spanning tree  $T$  of  $G$ 
  - ▶ a spanning tree of  $G$  is a subgraph of  $G$  that contains all the vertices of  $G$  and is a tree.
  - ▶ the cost of the spanning tree is the sum of values of the edges of  $T$
  - ▶ a spanning tree such that no other spanning tree has a smaller cost is a minimal spanning tree.

## Theorem 139 (Characterisation of trees)

*$H = (X, U)$  a graph of order  $|X| = n > 2$ . The following are equivalent and all characterise a tree :*

- 1.  $H$  connected and has no cycles*
- 2.  $H$  has  $n - 1$  arcs and no cycles*
- 3.  $H$  connected and has exactly  $n - 1$  arcs*
- 4.  $H$  has no cycles, and if an arc is added to  $H$ , exactly one cycle is created*
- 5.  $H$  connected, and if any arc is removed, the remaining graph is not connected*
- 6. Every pair of vertices of  $H$  is connected by one and only one chain*

## Proof of Theorem 139

We make abundant use of Theorem 120

[1  $\implies$  2] Let  $p$  be the number of connected components,  $m$  the number of arcs and  $\nu(H)$  the cyclomatic number. Since  $H$  connected,  $p = 1$ . Since  $H$  has no cycles,  $\nu(H) = m - n + p = 0 \implies m = n - p = n - 1$

[2  $\implies$  3] Assume  $H$  has no cycles ( $\nu(H) = 0$ ) and has  $n - 1$  arcs ( $m = n - 1$ ). Then, since

$$\nu(H) = m - n + p$$

$p = \nu(H) - m + n = 0 - (n - 1) - n = 1$ , i.e.,  $H$  is connected

## Proof of Theorem 139 (cont.)

[3  $\implies$  4] Assume  $H$  connected ( $p = 1$ ) and contains exactly  $n - 1$  arcs ( $m = n - 1$ ). Then

$$\nu(H) = m - n + p = (n - 1) - n + 1 = 0$$

and  $H$  has no cycles

Now add an arc, i.e., suppose  $m = n$ . Then

$\nu(H) = m - n + p = n - n + 1 = 1$  and there is one cycle in the new graph

## Proof of Theorem 139 (cont.)

[4  $\implies$  5] Assume  $H$  has no cycles ( $\nu(H) = 0$ ) and that addition of an arc to  $H$  creates exactly one cycle

Suppose  $H$  not connected. Then there are two vertices, say  $a$  and  $b$ , that are not connected and adding the arc  $(a, b)$  does not create a cycle, a contradiction with “addition of an arc to  $H$  creates exactly one cycle”

$\implies p = 1$ . Since  $\nu(H) = 0$ , this implies that  $m = n - 1$

Now suppose we remove an arc. We obtain graph  $H'$  with

$$m' = n' - 2 \quad \text{and} \quad \nu(H') = 0$$

So

$$p' = \nu(H') - m' + n' = 2$$

$\implies H'$  not connected

## Proof of Theorem 139 (cont.)

[5  $\implies$  6] Assume  $H$  connected and if any arc is removed, the remaining graph is not connected

Any vertices  $a, b \in X$  are connected by a chain (since  $H$  connected). That chain is unique: suppose there is a second chain connecting  $a$  to  $b$ ; then removing an arc from that chain does not disconnect the graph, since there is still the original chain connecting  $a$  and  $b$

[6  $\implies$  1] Assume every pair of vertices of  $H$  is connected by one and only one chain

Now assume  $H$  has a cycle. Then at least one pair of vertices would be connected by two distinct chains, a contradiction  $\square$

## Definition 140 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

## Theorem 141

*A tree of order  $n \geq 2$  has at least two pendant vertices*

**Proof:** Suppose  $H$  tree of order  $n \geq 2$  with 0 or 1 pendant vertices

Consider a traveller traversing the graph edges starting from a pendant vertex (if there is one) or anywhere if there is no pendant vertex

If he does not allow himself to use same edge twice, he cannot go to the same vertex twice ( $H$  has no cycle)

If he arrives at vertex  $x$ , he can depart  $x$  using a new edge ( $x$  is not a pendant vertex as there are 0 or 1 pendant vertex and if there is 1, that's where he started)

## Theorem 142

*A graph  $G = (X, U)$  has a partial graph that is a tree  $\iff G$  connected*

Recall that a partial graph is a graph generated by a subset of the arcs (Definition 104 slide 63)

**Proof:** Suppose  $G$  not connected. Then no partial graph of  $G$  is connected  $\implies$  there is no partial graph of  $G$  that is a tree [we want to show  $P \iff Q$ , we start with  $\wedge Q \implies \wedge P$  (which  $\iff P \implies Q$ )]

Suppose  $G$  connected. Look for an arc whose removal does not disconnect  $G$

- ▶ if there is none,  $G$  is a tree by Theorem 139(5)
- ▶ if there is, remove it and look for another one, etc. When no more arcs can be removed, the remaining graph is a tree with vertex set  $X$   $\square$



# Spanning tree

The procedure in the proof of Theorem 142 gives a **spanning tree**

Can also build a spanning tree as follows:

- ▶ Consider any arc  $u_0$
- ▶ Find arc  $u_1$  that does not form a cycle with  $u_0$
- ▶ Find arc  $u_2$  that does not form a cycle with  $\{u_0, u_1\}$
- ▶ Continue
- ▶ When you cannot continue anymore, you have a spanning tree

## Theorem 143

*G connected graph with  $\geq 1$  arc. TFAE*

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

**Proof:** [1  $\implies$  2]  $(x, y)$  an arc of  $G$ ; there is a path from  $y$  to  $x$  ( $G$  strongly connected), so arc  $(x, y)$  is contained in a circuit of  $G$   
[2  $\implies$  3] Suppose  $G$  has a cocircuit containing arc  $(x, y)$ ; then  $G$  cannot have a circuit containing this arc by the Arc Colouring Lemma (Lemma 119 slide 72) with all arcs coloured black. This contradicts (2)

[3  $\implies$  1] Assume  $G$  connected graph without cocircuits, but  $G$  not strongly connected. Since  $G$  not strongly connected, it has  $> 1$  strongly connected component. Since  $G$  connected, there exist two distinct strongly connected components that are joined by an arc  $(a, b)$ . Arc  $(a, b)$  is not contained in any circuit because otherwise  $a$  and  $b$  would be in the same strongly connected component. By Lemma 119, arc  $(a, b)$  is contained in some co-

### Theorem 144

*$G$  graph with  $\geq 1$  arc. TFAE*

- 1.  $G$  is a graph without circuits*
- 2. Each arc is contained in a cocircuit*

### Theorem 145

*If  $G$  is a strongly connected graph of order  $n$ , then  $G$  has a cycle basis of  $\nu(G)$  circuits*

### Definition 146 (Node, anti-node, branch)

$G = (X, U)$  strongly connected without loops and  $> 1$  vertex. For each  $x \in X$ , there is a path from it and a path to it so  $x$  has at least 2 incident arcs. Specifically,

- ▶  $x \in X$  with  $> 2$  incident arcs is a **node**
- ▶  $x \in X$  with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

### Definition 147 (Minimally connected graph)

$G$  is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

### Definition 148 (Contraction)

$G = (X, U)$ . The **contraction** of the set  $A \subset X$  of vertices consists in replacing  $A$  by a single vertex  $a$  and replacing each arc into (resp. out of)  $A$  by an arc with same index into (resp. out of)  $a$

## Theorem 149

*$G$  minimally connected,  $A \subset X$  generating a strongly connected subgraph of  $G$ . Then the contraction of  $A$  gives a minimally connected graph*

**Proof:** First, show that contraction of  $A$  yields a 1-graph. If this were not the case, there would exist  $x \notin A$  and  $a, a' \in A$  s.t.  $(x, a), (x, a') \in U$  (or, with  $(a, x), (a', x) \in U$  but this would not change the proof). If one of these arcs is removed, the graph remains strongly connected. Thus,  $G$  is not minimally connected, a contradiction

Now show that the contraction of  $A$  yields a graph  $G'$  that is minimally connected. Clearly,  $G'$  strongly connected. If an arc  $u$  is removed, the remaining graph is not strongly connected, since the graph  $(X, U - \{u\})$  not strongly connected □

### Theorem 150

*$G$  a minimally connected graph,  $G'$  be the minimally connected graph obtained by the contraction of an elementary circuit of  $G$ . Then*

$$\nu(G) = \nu(G') + 1$$

### Theorem 151

*$G$  minimally connected of order  $n \geq 2 \implies G$  has  $\geq 2$  anti-nodes*

### Theorem 152

*$G = (X, U)$ . Then the graph  $C'$  obtained by contracting each strongly connected component of  $G$  contains no circuits*

# Arborescences

## Definition 153 (Root)

Vertex  $a \in X$  in  $G = (X, U)$  is a **root** if all vertices of  $G$  can be reached by paths *starting* from  $a$

Not all graphs have roots

## Definition 154 (Quasi-strong connectedness)

$G$  is **quasi-strongly connected** if  $\forall x, y \in X$ , exists  $z \in X$  (denoted  $z(x, y)$  to emphasize dependence on  $x, y$ ) from which there is a path to  $x$  and a path to  $y$

Strongly connected  $\implies$  quasi-strongly connected (take  $z(x, y) = x$ ); converse not true

Quasi-strongly connected  $\implies$  connected

# Arborescence

## Definition 155 (Arborescence)

An **arborescence** is a tree that has a root

## Lemma 156

$G = (X, U)$  has a root  $\iff G$  quasi-strongly connected



## Theorem 157

*H graph of order  $n > 1$ . TFAE (and all characterise an arborescence)*

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with  $n - 1$  arcs*
3. *H tree having a root  $a$*
4.  *$\exists a \in X$  s.t. all other vertices are connected with  $a$  by 1 and only 1 path from  $a$*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and  $\exists a \in X$  s.t.*

$$d_H^-(a) = 0$$

$$d_H^-(x) = 1 \quad \forall x \neq a$$

7. *H has no cycles and  $\exists a \in X$  s.t.*

$$d_H^-(a) = 0$$

### Theorem 158

*$G$  has a partial graph that is an arborescence  $\iff G$  quasi-strongly connected*

### Theorem 159

*$G = (X, E)$  simple connected graph and  $x_1 \in X$ . It is possible to direct all edges of  $E$  so that the resulting graph  $G_0 = (X, U)$  has a spanning tree  $H$  s.t.*

- 1.  $H$  is an arborescence with root  $x_1$*
- 2. The cycles associated with  $H$  are circuits*
- 3. The only elementary circuits of  $G_0$  are the cycles associated with  $H$*

# Counting trees

## Proposition 160

*$X$  a set with  $n$  distinct objects,  $n_1, \dots, n_p$  nonnegative integers s.t.  $n_1 + \dots + n_p = n$ . The number of ways to place the  $n$  objects into  $p$  boxes  $X_1, \dots, X_p$  containing  $n_1, \dots, n_p$  objects respectively is*

$$\binom{n}{n_1, \dots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

## Proposition 161 (Multinomial formula)

*Let  $a_1, \dots, a_p \in \mathbb{R}$  be  $p$  real numbers, then*

$$(a_1 + \dots + a_p)^n = \sum_{n_1, \dots, n_p \geq 0} \binom{n}{n_1, \dots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

## Theorem 162

Denote  $T(n; d_1, \dots, d_n)$  the number of distinct trees  $H$  with vertices  $x_1, \dots, x_n$  and with degrees  $d_H(x_1) = d_1, \dots, d_H(x_n) = d_n$ . Then

$$T(n; d_1, \dots, d_n) = \binom{n-2}{d_1-1, \dots, d_n-1}$$

## Theorem 163

The number of different trees with vertices  $x_1, \dots, x_n$  is  $n^{n-2}$

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..

Binary relations

Undirected graphs

Directed graphs

Trees

Matrices associated to a graph/digraph

- Adjacency matrices

- Other matrices associated to a graph/digraph

- Linking graphs and linear algebra

# Matrices associated to a graph/digraph

There are multiple matrices associated to a graph/digraph.

The branch of graph theory that studies the properties of matrices derived from graphs and uses of these matrices in determining graph properties is *spectral graph theory*.

Graphs greatly simplify some problems in linear algebra and vice versa.

## Matrices associated to a graph/digraph

Adjacency matrices

Other matrices associated to a graph/digraph

Linking graphs and linear algebra

## Adjacency matrix (undirected case)

Let  $G = (V, E)$  be a graph of order  $p$  and size  $q$ , with vertices  $v_1, v_2, \dots, v_p$  and edges  $e_1, e_2, \dots, e_q$ .

### Definition 164 (Adjacency matrix)

The adjacency matrix is

$$M_A = M_A(G) = [m_{ij}]$$

is a  $p \times p$  matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$



### Theorem 165 (Adjacency matrix and degree)

*The sum of the entries in row  $i$  of the adjacency matrix is the degree of  $v_i$  in the graph.*

## Definition 166 (Multiplicity of a pair)

The **multiplicity** of a pair  $x, y$  is the number  $m_G^+(x, y)$  of arcs with initial endpoint  $x$  and terminal endpoint  $y$ . Let

$$m_G^-(x, y) = m_G^+(y, x)$$

$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y)$$

If  $x \neq y$ , then  $m_G(x, y)$  is number of arcs with both  $x$  and  $y$  as endpoints. If  $x = y$ , then  $m_G(x, y)$  equals twice the number of loops attached to vertex  $x$ . If  $A, B \subset X$ ,  $A \neq B$ , let

$$m_G^+(A, B) = \{u : u \in U, u = (x, y), x \in A, y \in B\}$$

$$m_G(A, B) = m_G^+(A, B) + m_G^+(B, A)$$

# Adjacency matrix of a multigraph

## Definition 167 (Matrix associated with $G$ )

If  $G$  has vertices  $x_1, x_2, \dots, x_n$ , then the **matrix associated** with  $G$  is

$$a_{ij} = m_G^+(x_i, x_j)$$

## Definition 168 (Adjacency matrix)

The matrix  $a_{ij} + a_{ji}$  is the **adjacency matrix** associated with  $G$

# Adjacency matrix

Let  $D = (V, A)$  be a digraph of order  $p$  with vertices denoted by  $v_1, v_2, \dots, v_p$ .

## Definition 169 (Adjacency matrix)

The adjacency matrix  $M = M(D) = [m_{ij}]$  is a  $p \times p$  matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if arc } v_i v_j \in A \\ 0 & \text{otherwise} \end{cases}$$

## Theorem 170 (Properties)

- ▶  $M$  is not necessarily symmetric.
- ▶ The sum of any column of  $M$  is equal to the number of arcs directed towards  $v_j$
- ▶ The sum of the entries in row  $i$  is equal to the number of arcs directed away from vertex  $v_i$ .
- ▶ The  $(i, j)$ -entry of  $M^n$  is equal to the number of walks of length  $n$  from vertex  $v_i$  to  $v_j$ .

# Incidence matrix

Let  $D = (V, A)$  be a digraph of order  $p$ , and size  $q$ , with vertices denoted by  $v_1, v_2, \dots, v_p$ , and arcs denoted  $a_1, a_2, \dots, a_q$ .

## Definition 171

Definition The incidence matrix  $B = B(D) = [b_{ij}]$  is a  $p \times q$  matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if arc } a_j \text{ is directed away from a vertex } v_i \\ -1 & \text{if arc } a_j \text{ is directed towards a vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

# Adjacency matrix

We have already seen adjacency matrices, let us recall the definition here

## Definition 172 (Adjacency matrix)

$G$  a 1-graph, then the **adjacency matrix**  $A = [a_{ij}]$  is defined as follows

$$a_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \in U \\ 0 & \text{otherwise} \end{cases}$$

We often write  $A(G)$  and, reciprocally, if  $A$  is an adjacency matrix,  $G(A)$  the corresponding graph

$G$  undirected  $\implies A(G)$  symmetric

$A(G)$  has nonzero diagonal entries if  $G$  is not simple

## Adjacency matrix (multigraph case)

### Definition 173 (Adjacency matrix of a multigraph)

$G$  an  $\ell$ -graph, then the adjacency matrix  $M_A = [m_{ij}]$  is defined as follows

$$m_{ij} = \begin{cases} k & \text{if arc there are } k \text{ arcs } (i, j) \in U \\ 0 & \text{otherwise} \end{cases}$$

with  $k \leq \ell$

$G$  undirected  $\implies M_A(G)$  symmetric

$M_A(G)$  has nonzero diagonal entries if  $G$  is not simple.

# Weighted adjacency matrices

Sometimes, adjacency matrices (typically for 1-graphs) have real entries, usually positive

This means that the arcs/edges have been given a weight



## Theorem 174 (Number of walks of length $n$ )

*Let  $A$  be the adjacency matrix of a graph  $G = (V(G), E(G))$ , where  $V(G) = \{v_1, v_2, \dots, v_p\}$ . Then the  $(i, j)$ -entry of  $A^n$ ,  $n \geq 1$ , is the number of different walks linking  $v_i$  to  $v_j$  of length  $n$  in  $G$ .*

(two walks of the same length are equal if their edges occur in exactly the same order)

Example: let  $A$  be the adjacency matrix of a graph  $G = (V(G), E(G))$ .

- ▶ the  $(i, i)$ -entry of  $A^2$  is equal to the degree of  $v_i$ .
- ▶ the  $(i, i)$ -entry of  $A^3$  is equal to twice the number of  $C_3$  containing  $v_i$ .

## Matrices associated to a graph/digraph

Adjacency matrices

Other matrices associated to a graph/digraph

Linking graphs and linear algebra

## Incidence matrix

Let  $G = (V, E)$  be a graph of order  $p$ , and size  $q$ , with vertices denoted by  $v_1, v_2, \dots, v_p$ , and edges denoted  $e_1, e_2, \dots, e_q$ .

### Definition 175 (Incidence matrix)

The incidence matrix is

$$B = B(G) = [b_{ij}]$$

is that  $p \times q$  matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

### Theorem 176 (Incidence matrix and degrees)

*The sum of the entries in row  $i$  of the incidence matrix is the degree of  $v_i$  in the graph.*

# Incidence matrix

## Definition 177 (Incidence matrix – Undirected case)

$G = (X, E)$  an undirected graph of order  $n$  with  $p$  edges. The **incidence** matrix of  $G$  is an  $n \times p$  matrix with vertices as rows and edges as columns and where  $B = [b_{ij}]$  satisfies

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incident to vertex } i \\ 0 & \text{otherwise} \end{cases}$$

## Definition 178 (Incidence matrix – Directed case)

$G = (X, U)$  a directed graph of order  $n$  with  $p$  arcs. The **incidence** matrix of  $G$  is an  $n \times p$  matrix with vertices as rows and edges as columns and where  $B = [b_{ij}]$  satisfies

$$b_{ij} = \begin{cases} 1 & \text{if arc } j \text{ "enters" vertex } i \\ -1 & \text{if arc } j \text{ "leaves" vertex } i \\ 0 & \text{otherwise} \end{cases}$$

# Spectrum of a graph

We will come back to this later, but for now..

## Definition 179 (Spectrum of a graph)

The **spectrum** of a graph  $G$  is the spectrum (set of eigenvalues) of its associated adjacency matrix  $A(G)$

This is regardless of the type of adjacency matrix or graph

# Degree matrix

## Definition 180 (Degree matrix)

The **degree** matrix  $D = [d_{ij}]$  for  $G$  is a  $n \times n$  diagonal matrix defined as

$$d_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In an undirected graph, this means that each loop increases the degree of a vertex by two

In a directed graph, the term “degree” may refer either to indegree (the number of incoming edges at each vertex) or outdegree (the number of outgoing edges at each vertex)

# Laplacian matrix

## Definition 181

$G = (X, U)$  a simple graph with  $n$  vertices. The **Laplacian** matrix is

$$L = D(G) - A(G)$$

where  $D(G)$  is the degree matrix and  $A(G)$  is the adjacency matrix

$G$  simple graph  $\implies A(G)$  only contains 1 or 0 and its diagonal elements are all 0

For directed graphs, either the indegree or outdegree is used, depending on the application

Elements of  $L$  are given by

$$\ell_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

# Distance matrix

Let  $G$  be a graph of order  $p$  with vertices denoted by  $v_1, v_2, \dots, v_p$ .

## Definition 182 (Distance matrix)

The distance matrix  $DIST = DIST(D) = [d_{ij}]$  is a  $p \times p$  matrix in which

$$d_{ij} = \text{dist}(v_i, v_j).$$

Note  $d_{ii} = 0$  for  $i = 1, \dots, p$ .

## Definition 183

### Properties

- ▶  $M$  is not necessarily symmetric.
- ▶ The sum of any column of  $M$  is equal to the number of arcs directed towards  $v_j$
- ▶ The sum of the entries in row  $i$  is equal to the number of arcs directed away from vertex  $v_i$ .
- ▶ The  $(i, j)$ -entry of  $M^n$  is equal to the number of walks of length  $n$  from vertex  $v_i$  to  $v_j$ .



## Matrices associated to a graph/digraph

Adjacency matrices

Other matrices associated to a graph/digraph

Linking graphs and linear algebra

## Counting paths

To count paths between vertices  $x$  and  $y$  in a graph, we use the adjacency matrix

### Theorem 184

*$G$  a graph and  $A(G)$  its adjacency matrix. Denote  $P = [p_{ij}]$  the matrix  $P = A^k$ . Then  $p_{ij}$  is the number of distinct paths of length  $k$  from  $i$  to  $j$  in  $G$*

This provides an interesting connection with linear algebra

### Definition 185 (Irreducible matrix)

A matrix  $A \in \mathcal{M}_n$  is **reducible** if  $\exists P \in \mathcal{M}_n$ , permutation matrix, s.t.  $P^T A P$  can be written in block triangular form. If no such  $P$  exists,  $A$  is **irreducible**

### Theorem 186

$A$  irreducible  $\iff G(A)$  strongly connected

### Theorem 187

*Let  $A$  be the adjacency matrix of a graph  $G$  on  $p$  vertices. A graph  $G$  on  $p$  vertices is connected if and only if*

$$I + A + A^2 + \cdots + A^{p-1} = C$$

*has no zero entries.*

### Theorem 188

*Let  $M$  be the adjacency matrix of a digraph  $D$  on  $p$  vertices. A digraph  $D$  on  $p$  vertices is strongly connected if and only if*

$$I + M + M^2 + \cdots + M^{p-1} = C$$

*has no zero entries.*

# Perron-Frobenius theorem

$A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$  **nonnegative** if  $a_{ij} \geq 0 \ \forall i, j = 1, \dots, n$ ;  $\mathbf{v} \in \mathbb{R}^n$  nonnegative if  $v_i \geq 0 \ \forall i = 1, \dots, n$ . **Spectral radius** of  $A$

$$\rho(A) = \max_{\lambda \in \text{Sp}(A)} \{|\lambda|\}$$

$\text{Sp}(A)$  the **spectrum** of  $A$

## Theorem 189 (PF – Nonnegative case)

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$ . Then  $\exists \mathbf{v} \geq \mathbf{0}$  s.t.

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

## Theorem 190 (PF – Irreducible case)

Let  $0 \leq A \in \mathcal{M}_n(\mathbb{R})$  irreducible. Then  $\exists \mathbf{v} > \mathbf{0}$  s.t.

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

$\rho(A) > 0$  and with algebraic multiplicity 1. No nonnegative

# Primitive matrices

## Definition 191

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$  **primitive** (with **primitivity index**  $k \in \mathbb{N}_+^*$ ) if  $\exists k \in \mathbb{N}_+^*$  s.t.

$$A^k > 0,$$

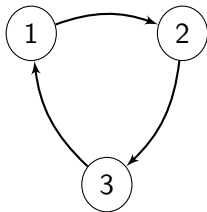
with  $k$  the smallest integer for which this is true.  $A$  **imprimitive** if it is not primitive

$A$  primitive  $\implies A$  irreducible; converse false

## Theorem 192

$A \in \mathcal{M}_n(\mathbb{R})$  irreducible and  $\exists i = 1, \dots, n$  s.t.  $a_{ii} > 0 \implies A$  primitive

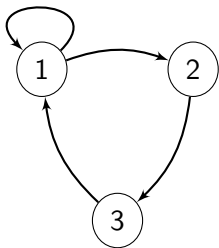
Here  $d$  is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as  $\lambda_p = \rho(A)$ ). If  $d = 1$ , then  $A$  is primitive. We have that  $d = \gcd$  of all the lengths of closed walks in  $G(A)$



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walks in  $G(A)$  (lengths):  $1 \rightarrow 1$  (3),  $2 \rightarrow 2$  (3),  $2 \rightarrow 2$  (3)  
 $\implies \gcd = 3 \implies d = 3$  (here, all eigenvalues have modulus 1)



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walk  $1 \rightarrow 1$  has length 1  $\implies \gcd$  of lengths of closed walks is 1  $\implies A$  primitive

### Theorem 193

$0 \leq A \in \mathcal{M}_n$ .  $A$  primitive  $\implies A^k > 0$  for some  $0 < k \leq (n-1)n^n$

### Theorem 194

$A \geq 0$  primitive. Suppose the shortest simple directed cycle in  $G(A)$  has length  $s$ , then primitivity index is  $\leq n + s(n-1)$

### Theorem 195

$0 \leq A \in \mathcal{M}_n$  primitive  $\iff A^{n^2-2n+2} > 0$

### Theorem 196

$0 \leq A \in \mathcal{M}_n$  irreducible.  $A$  has  $d$  positive entries on the diagonal  $\implies$  primitivity index  $\leq 2n - d - 1$

## Theorem 197

$0 \leq A \in \mathcal{M}_n$ ,  $\lambda_P = \rho(A)$  the Perron root of  $A$ ,  $\mathbf{w}_P$  and  $\mathbf{v}_P$  the corresponding right and left Perron vectors of  $A$ , respectively,  $d$  the index of imprimitivity of  $A$  (with  $d = 1$  when  $A$  is primitive) and  $\lambda_j \in \sigma(A)$  the spectrum of  $A$ , with  $j = 2, \dots, n$  unless otherwise specified (assuming  $\lambda_1 = \lambda_P$ )

