

## Review of first-year linear algebra

In MATH 2740, we rely on notions you acquired in MATH 1210/1220/1300

So let us (briefly) go over material in these courses

I also add (for some of you) a few things that will be handy and establish some terminology that we use throughout the course

# OUTLINE

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

Matrix arithmetic

Diagonalisation

Linear independence/Bases/Dimension

Linear algebra in a nutshell

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# Sets and elements

## Definition 1 (Set)

A **set**  $X$  is a collection of **elements**

We write  $x \in X$  or  $x \notin X$  to indicate that the element  $x$  belongs to the set  $X$  or does not belong to the set  $X$ , respectively

## Definition 2 (Subset)

Let  $X$  be a set. The set  $S$  is a **subset** of  $X$ , which is denoted  $S \subset X$ , if all its elements belong to  $X$

Not used here but worth noting: we say  $S$  is a **proper subset** of  $X$  and write  $S \subsetneq X$ , if it is a subset of  $X$  and not equal to  $X$

## Quantifiers

A shorthand notation for “for all elements  $x$  belonging to  $X$ ” is  $\forall x \in X$

For example, if  $X = \mathbb{R}$ , the *field* of real numbers, then  $\forall x \in \mathbb{R}$  means “for all real numbers  $x$ ”

A shorthand notation for “there exists an element  $x$  in the set  $X$ ” is  $\exists x \in X$

$\forall$  and  $\exists$  are **quantifiers**

## Intersection and union of sets

Let  $X$  and  $Y$  be two sets

### Definition 3 (Intersection)

The intersection of  $X$  and  $Y$ ,  $X \cap Y$ , is the set of elements that belong to  $X$  **and** to  $Y$ ,

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

### Definition 4 (Union)

The union of  $X$  and  $Y$ ,  $X \cup Y$ , is the set of elements that belong to  $X$  **or** to  $Y$ ,

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}$$

In mathematics, or=and/or in common parlance. We also have an **exclusive or** (xor)

## A teeny bit of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. “The sky is blue” is also a proposition

Let  $A$  be a proposition. We generally write

$A$

to mean that  $A$  is true, and

**not**  $A$

to mean that  $A$  is false. **not**  $A$  is the **contraposition** of  $A$  (or **not**  $A$  is the contrapositive of  $A$ )

## A teeny bit of logic (cont.)

Let  $A, B$  be propositions. Then

- ▶  $A \Rightarrow B$  (read  $A$  implies  $B$ ) means that whenever  $A$  is true, then so is  $B$
- ▶  $A \Leftrightarrow B$ , also denoted  $A$  if and only if  $B$  ( $A$  iff  $B$  for short), means that  $A \Rightarrow B$  **and**  $B \Rightarrow A$

We also say that  $A$  and  $B$  are **equivalent**

Let  $A$  and  $B$  be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\text{not } B \Rightarrow \text{not } A)$$

## Necessary or sufficient conditions

Suppose we want to establish whether a given statement  $P$  is true, depending on the truth value of a statement  $H$ . Then we say that

- ▶  $H$  is a **necessary condition** if  $P \Rightarrow H$   
(It is necessary that  $H$  be true for  $P$  to be true; so whenever  $P$  is true, so is  $H$ )
- ▶  $H$  is a **sufficient condition** if  $H \Rightarrow P$   
(It suffices for  $H$  to be true for  $P$  to also be true)
- ▶  $H$  is a **necessary and sufficient condition** if  $H \Leftrightarrow P$ , i.e.,  $H$  and  $P$  are equivalent

## Playing with quantifiers

For the quantifiers  $\forall$  (for all) and  $\exists$  (there exists),

$\exists$  is the contrapositive of  $\forall$

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

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Linear independence/Bases/Dimension

Linear algebra in a nutshell

## Complex numbers

### Definition 5 (Complex numbers)

A **complex number** is an ordered pair  $(a, b)$ , where  $a, b \in \mathbb{R}$ . Usually written  $a + ib$  or  $a + bi$ , where  $i^2 = -1$  (i.e.,  $i = \sqrt{-1}$ )

The set of all complex numbers is denoted  $\mathbb{C}$ ,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

## Definition 6 (Addition and multiplication on $\mathbb{C}$ )

Letting  $a + ib$  and  $c + id \in \mathbb{C}$ , addition on  $\mathbb{C}$  is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on  $\mathbb{C}$  is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter is easy to obtain using regular multiplication and  $i^2 = -1$

## Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C}$ ,

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \quad [\text{commutativity}]$$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \text{ and } (\alpha\beta)\gamma = \alpha(\beta\gamma) \quad [\text{associativity}]$$

$$\gamma + 0 = \gamma \text{ and } \gamma 1 = \gamma \quad [\text{identities}]$$

$$\forall \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C} \text{ unique s.t. } \alpha + \beta = 0 \quad [\text{additive inverse}]$$

$$\forall \alpha \neq 0 \in \mathbb{C}, \exists \beta \in \mathbb{C} \text{ unique s.t. } \alpha\beta = 1 \quad [\text{multiplicative inverse}]$$

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta \quad [\text{distributivity}]$$

## Additive & multiplicative inverse, subtraction, division

### Definition 7

Let  $\alpha, \beta \in \mathbb{C}$

- ▶  $-\alpha$  is the **additive inverse** of  $\alpha$ , i.e., the unique number in  $\mathbb{C}$  s.t.  $\alpha + (-\alpha) = 0$
- ▶ **Subtraction** on  $\mathbb{C}$ :

$$\beta - \alpha = \beta + (-\alpha)$$

- ▶ For  $\alpha \neq 0$ ,  $1/\alpha$  is the **multiplicative inverse** of  $\alpha$ , i.e., the unique number in  $\mathbb{C}$  s.t.

$$\alpha(1/\alpha) = 1$$

- ▶ **Division** on  $\mathbb{C}$ :

$$\beta/\alpha = \beta(1/\alpha)$$

## Definition 8 (Real and imaginary parts)

Let  $z = a + ib$ . Then  $\operatorname{Re} z = a$  is **real part** and  $\operatorname{Im} z = b$  is **imaginary part** of  $z$

If ambiguous, write  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$

## Definition 9 (Conjugate and Modulus)

Let  $z = a + ib \in \mathbb{C}$ . Then

- ▶ **Complex conjugate** of  $z$  is

$$\bar{z} = a - ib$$

- ▶ **Modulus (or absolute value)** of  $z$  is

$$|z| = \sqrt{a^2 + b^2} \geq 0$$

## Properties of complex numbers

Let  $w, z \in \mathbb{C}$ , then

- ▶  $z + \bar{z} = 2\operatorname{Re} z$
- ▶  $z - \bar{z} = 2i\operatorname{Im} z$
- ▶  $z\bar{z} = |z|^2$
- ▶  $\overline{w+z} = \bar{w} + \bar{z}$  and  $\overline{wz} = \bar{w}\bar{z}$
- ▶  $\overline{\bar{z}} = z$
- ▶  $|\operatorname{Re} z| \leq |z|$  and  $|\operatorname{Im} z| \leq |z|$
- ▶  $|\bar{z}| = |z|$
- ▶  $|wz| = |w| |z|$
- ▶  $|w+z| \leq |w| + |z|$  [triangle inequality]

## Solving quadratic equations

Consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where  $x, a_0, a_1, a_2 \in \mathbb{R}$ . Letting

$$\Delta = a_1^2 - 4a_0a_2$$

you know that if  $\Delta > 0$ , then

$$P(x) = 0$$

has two distinct *real* solutions,

$$x_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2} \quad \text{and} \quad x_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_2}$$

if  $\Delta = 0$ , then there is a (multiplicity 2) unique *real* solution

$$x_1 = \frac{-a_1}{2a_2}$$

while if  $\Delta < 0$ , there is no solution

## Solving quadratic equations with complex numbers

Consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where  $x, a_0, a_1, a_2 \in \mathbb{R}$ . If instead of seeking  $x \in \mathbb{R}$ , we seek  $x \in \mathbb{C}$ , then the situation is the same, except when  $\Delta < 0$

In the latter case, note that

$$\sqrt{\Delta} = \sqrt{(-1)(-\Delta)} = \sqrt{-1}\sqrt{-\Delta} = i\sqrt{-\Delta}$$

Since  $\Delta < 0$ ,  $-\Delta > 0$  and the square root is the usual one

## Solving quadratic equations with complex numbers

To summarize, consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where  $x, a_0, a_1, a_2 \in \mathbb{R}$ . Letting

$$\Delta = a_1^2 - 4a_0a_2$$

Then

$$P(x) = 0$$

has two solutions,

$$x_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2a_2}$$

where, if  $\Delta < 0$ ,  $x_1, x_2 \in \mathbb{C}$  and take the form

$$x_{1,2} = \frac{-a_1 \pm i\sqrt{-\Delta}}{2a_2}$$

## Why this matters

Recall (we will come back to this later) that to find the *eigenvalues* of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we seek  $\lambda$  solutions to  $\det(A - \lambda\mathbb{I}) = 0$ , i.e.,  $\lambda$  solutions to

$$|A - \lambda\mathbb{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

i.e.,  $\lambda$  solutions to

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

## Why this matters (cont.)

Let

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

From previous discussion, letting

$$\begin{aligned}\Delta &= (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \\ &= a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4a_{11}a_{22} + 4a_{12}a_{21} \\ &= a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} \\ &= (a_{11} - a_{22})^2 + 4a_{12}a_{21}\end{aligned}$$

we have two (potentially equal) solutions to  $P(\lambda) = 0$

$$x_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{\Delta}}{2}$$

that are complex if  $\Delta < 0$

Example:  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

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Linear algebra in a nutshell

## Vectors

A **vector**  $\mathbf{v}$  is an ordered  $n$ -tuple of real or complex numbers

Denote  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  (real or complex numbers). For  $v_1, \dots, v_n \in \mathbb{F}$ ,

$$\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{F}^n$$

is a vector.  $v_1, \dots, v_n$  are the **components** of  $\mathbf{v}$

If unambiguous, we write  $v$ . Otherwise,  $\mathbf{v}$  or  $\vec{v}$

# Vector space

## Definition 10 (Vector space)

A **vector space** over  $\mathbb{F}$  is a set  $V$  together with two binary operations, **vector addition**, denoted  $+$ , and **scalar multiplication**, that satisfy the relations:

1.  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2.  $\forall \mathbf{v}, \mathbf{w} \in V, \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
3.  $\exists \mathbf{0} \in V$ , the zero vector, such that  $\mathbf{v} + \mathbf{0} = \mathbf{v}$  for all  $\mathbf{v} \in V$
4.  $\forall \mathbf{v} \in V$ , there exists an element  $\mathbf{w} \in V$ , the additive inverse of  $\mathbf{v}$ , such that  $\mathbf{v} + \mathbf{w} = \mathbf{0}$
5.  $\forall \alpha \in \mathbb{R}$  and  $\forall \mathbf{v}, \mathbf{w} \in V, \alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
6.  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall \mathbf{v} \in V, (\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7.  $\forall \alpha, \beta \in \mathbb{R}$  and  $\forall \mathbf{v} \in V, \alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
8.  $\forall \mathbf{v} \in V, 1\mathbf{v} = \mathbf{v}$

## Definition 11 (Norm)

Let  $V$  be a vector space over  $\mathbb{F}$ , and  $\mathbf{v} \in V$  be a vector. The **norm** of  $\mathbf{v}$ , denoted  $\|\mathbf{v}\|$ , is a function from  $V$  to  $\mathbb{R}_+$  that has the following properties:

1. For all  $\mathbf{v} \in V$ ,  $\|\mathbf{v}\| \geq 0$  with  $\|\mathbf{v}\| = 0$  iff  $\mathbf{v} = \mathbf{0}$
2. For all  $\alpha \in \mathbb{F}$  and all  $\mathbf{v} \in V$ ,  $\|\alpha\mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
3. For all  $\mathbf{u}, \mathbf{v} \in V$ ,  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Let  $V$  be a vector space (for example,  $\mathbb{R}^2$  or  $\mathbb{R}^3$ )

The **zero element** (or **zero vector**) is the vector  $\mathbf{0} = (0, \dots, 0)$

The **additive inverse** of  $\mathbf{v} = (v_1, \dots, v_n)$  is  $-\mathbf{v} = (-v_1, \dots, -v_n)$

For  $\mathbf{v} = (v_1, \dots, v_n) \in V$ , the length (or Euclidean norm) of  $\mathbf{v}$  is the **scalar**

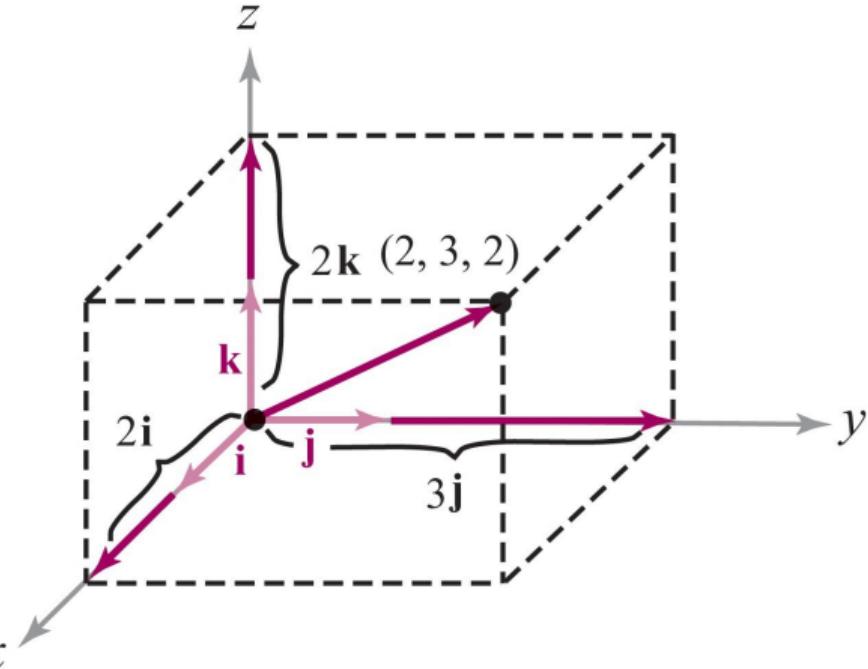
$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

To **normalize** the vector  $\mathbf{v}$  consists in considering  $\tilde{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$ , i.e., the vector in the same direction as  $\mathbf{v}$  that has unit length

## Standard basis vectors

Vectors  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$  and  $\mathbf{k} = (0, 0, 1)$  are the **standard basis vectors** of  $\mathbb{R}^3$ . A vector  $\mathbf{v} = (v_1, v_2, v_3)$  can then be written

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$



For  $V(\mathbb{R}^n)$ , the standard basis vectors are usually denoted  $\mathbf{e}_1, \dots, \mathbf{e}_n$ , with

$$\mathbf{e}_k = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k+1})$$

## Dot product

### Definition 12 (Dot product)

Let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ ,  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ . The **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the scalar

$$\mathbf{a} \bullet \mathbf{b} = \sum_{i=1}^n a_i b_i = a_1 b_1 + \cdots + a_n b_n$$

The dot product is a special case of **inner product**

# Properties of the dot product

## Theorem 13

For  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ ,

- ▶  $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$  (so  $\mathbf{a} \bullet \mathbf{a} \geq 0$ , with  $\mathbf{a} \bullet \mathbf{a} = 0$  iff  $\mathbf{a} = \mathbf{0}$ )
- ▶  $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$  ( $\bullet$  is commutative)
- ▶  $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$  ( $\bullet$  distributive over  $+$ )
- ▶  $(\alpha \mathbf{a}) \bullet \mathbf{b} = \alpha(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (\alpha \mathbf{b})$
- ▶  $\mathbf{0} \bullet \mathbf{a} = 0$

## Some results stemming from the dot product

### Theorem 14

If  $\theta$  is the angle between the vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

### Corollary 15 (Cauchy-Schwarz inequality)

For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$ , we have

$$|\mathbf{a} \bullet \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

with equality if and only if  $\mathbf{a}$  is a scalar multiple of  $\mathbf{b}$ , or one of them is  $\mathbf{0}$ .

### Theorem 16

$\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \bullet \mathbf{b} = 0$ .

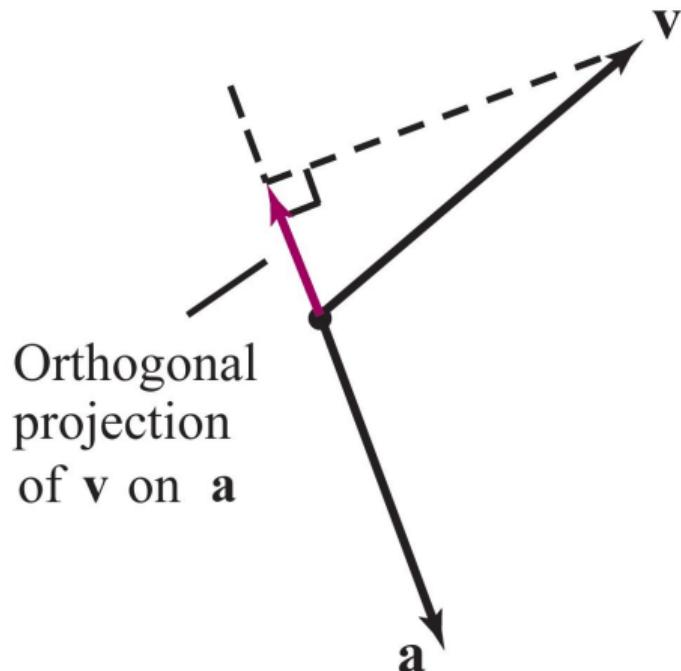
## Scalar and vector projections

Scalar projection of  $\mathbf{v}$  onto  $\mathbf{a}$  (or component of  $\mathbf{v}$  along  $\mathbf{a}$ ):

$$\text{comp}_{\mathbf{a}} \mathbf{v} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|}$$

Vector (or orthogonal) projection of  $\mathbf{v}$  onto  $\mathbf{a}$ :

$$\text{proj}_{\mathbf{a}} \mathbf{v} = \left( \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|^2} \mathbf{a}$$



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Linear algebra in a nutshell

## Linear systems

### Definition 17 (Linear system)

A **linear system** of  $m$  equations in  $n$  unknowns takes the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned} \tag{1}$$

The  $a_{ij}$ ,  $x_j$  and  $b_j$  could be in  $\mathbb{R}$  or  $\mathbb{C}$ , although here we typically assume they are in  $\mathbb{R}$

The aim is to find  $x_1, x_2, \dots, x_n$  that satisfy all equations simultaneously

## Theorem 18 (Nature of solutions to a linear system)

A *linear system* can have

- ▶ *no solution*
- ▶ *a unique solution*
- ▶ *infinitely many solutions*

## Operations on linear systems

You learned to manipulate linear systems using

- ▶ Gaussian elimination
- ▶ Gauss-Jordan elimination

with the aim to put the system in **row echelon form** (REF) or **reduced row echelon form** (RREF)

# Matrices and linear systems

Writing

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where  $A$  is an  $m \times n$  **matrix**,  $\mathbf{x}$  and  $\mathbf{b}$  are  $n$  (column) **vectors** (or  $n \times 1$  matrices), then the linear system in the previous slide takes the form

$$A\mathbf{x} = \mathbf{b}$$

## Notation for vectors

We usually assume vectors are column vectors and thus write, e.g.,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \dots, x_n)^T$$

Here,  $T$  is the **transpose operator** (more on this soon)

Consider the system

$$Ax = \mathbf{b}$$

If  $\mathbf{b} = \mathbf{0}$ , the system is **homogeneous** and always has the solution  $\mathbf{x} = \mathbf{0}$  and so the “no solution” option in Theorem 18 goes away

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

**Matrix arithmetic**

Diagonalisation

Linear independence/Bases/Dimension

Linear algebra in a nutshell

## Definition 19 (Matrix)

An  $m$ -by- $n$  or  $m \times n$  matrix is a rectangular array of elements of  $\mathbb{R}$  or  $\mathbb{C}$  with  $m$  rows and  $n$  columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We always list indices as “row,column”

We denote  $\mathcal{M}_{mn}(\mathbb{F})$  or  $\mathbb{F}^{mn}$  the set of  $m \times n$  matrices with entries in  $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$ . Often, we omit  $\mathbb{F}$  in  $\mathcal{M}_{mn}$  if the nature of  $\mathbb{F}$  is not important

When  $m = n$ , we usually write  $\mathcal{M}_n$

## Basic matrix arithmetic

Let  $A \in \mathcal{M}_{mn}$ ,  $B \in \mathcal{M}_{mn}$  be matrices (of the same size) and  $c \in \mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$  be a scalar

- ▶ **Scalar multiplication**

$$cA = [ca_{ij}]$$

- ▶ **Addition**

$$A + B = [a_{ij} + b_{ij}]$$

- ▶ **Subtraction** (addition of  $-B = (-1)B$  to  $A$ )

$$A - B = A + (-1)B = [a_{ij} + (-1)b_{ij}] = [a_{ij} - b_{ij}]$$

- ▶ **Transposition** of  $A$  gives a matrix  $A^T = \mathcal{M}_{nm}$  with

$$A^T = [a_{ji}], \quad j = 1, \dots, n, \quad i = 1, \dots, m$$

## Matrix multiplication

The (matrix) **product** of  $A$  and  $B$ ,  $AB$ , requires the “inner dimensions” to match, i.e., the number of columns in  $A$  must equal the number of rows in  $B$

Suppose that is the case, i.e., let  $A \in \mathcal{M}_{mn}$ ,  $B \in \mathcal{M}_{np}$ . Then the  $i,j$  entry in  $C := AB$  takes the form

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Recall that the matrix product is not commutative, i.e., in general,  $AB \neq BA$  (when both those products are defined, i.e., when  $A, B \in \mathcal{M}_n$ )

## Special matrices

### Definition 20 (Zero and identity matrices)

The **zero** matrix is the matrix  $0_{mn}$  whose entries are all zero. The **identity** matrix is a square  $n \times n$  matrix  $\mathbb{I}_n$  with all entries on the main diagonal equal to one and all off diagonal entries equal to zero

### Definition 21 (Symmetric matrix)

A square matrix  $A \in \mathcal{M}_n$  is **symmetric** if  $\forall i, j = 1, \dots, n$ ,  $a_{ij} = a_{ji}$ . In other words,  $A \in \mathcal{M}_n$  is symmetric if  $A = A^T$

## Properties of symmetric matrices

### Theorem 22

1. If  $A \in \mathcal{M}_n$ , then  $A + A^T$  is symmetric
2. If  $A \in \mathcal{M}_{mn}$ , then  $AA^T \in \mathcal{M}_m$  and  $A^TA \in \mathcal{M}_n$  are symmetric

$X$  symmetric  $\iff X = X^T$ , so use  $X$  = the matrix whose symmetric property you want to check

1. True if  $A + A^T = (A + A^T)^T$ . We have

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

2.  $AA^T$  symmetric if  $AA^T = (AA^T)^T$ . We have

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$A^TA$  works similarly

## Determinants

### Definition 23 (Determinant)

Let  $A \in \mathcal{M}_n$  with  $n \geq 2$ . The **determinant** of  $A$  is the scalar

$$\det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij}$$

where  $C_{ij} = (-1)^{i+j} \det(A_{ij})$  is the  $(i,j)$ -**cofactor** of  $A$  and  $A_{ij}$  is the submatrix of  $A$  from which the  $i$ th row and  $j$ th column have been removed

This is a cofactor expansion along the  $i$ th row

This is a recursive formula: it gives result in terms of  $n \mathcal{M}_{n-1}$  matrices, to which it must in turn be applied, all the way down to

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

## Two special matrices and their determinants

### Definition 24

$A \in \mathcal{M}_n$  is **upper triangular** if  $a_{ij} = 0$  when  $i > j$ , **lower triangular** if  $a_{ij} = 0$  when  $j > i$ , **triangular** if it is either upper or lower triangular and **diagonal** if it is both upper and lower triangular

When  $A$  diagonal, we often write  $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$

### Theorem 25

Let  $A \in \mathcal{M}_n$  be triangular or diagonal. Then

$$\det(A) = \prod_{i=1}^n a_{ii} = a_{11}a_{22} \cdots a_{nn}$$

# Inversion/Singularity

## Definition 26 (Matrix inverse)

$A \in \mathcal{M}_n$  is **invertible** (or **nonsingular**) if  $\exists A^{-1} \in \mathcal{M}_n$  s.t.

$$AA^{-1} = A^{-1}A = \mathbb{I}$$

$A^{-1}$  is the **inverse** of  $A$ . If  $A^{-1}$  does not exist,  $A$  is **singular**

## Theorem 27

Let  $A \in \mathcal{M}_n$ ,  $\mathbf{x}, \mathbf{b} \in \mathbb{F}^n$ . Then

- ▶  $A$  invertible  $\iff \det(A) \neq 0$
- ▶ If  $A$  invertible,  $A^{-1}$  is unique
- ▶ If  $A$  invertible, then  $A\mathbf{x} = \mathbf{b}$  has the unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$

## Revisiting matrix arithmetic

With addition, subtraction, scalar multiplication, multiplication, transposition and inversion, you can perform arithmetic on matrices essentially as on scalar, if you bear in mind a few rules

- ▶ The sizes have to be compatible
- ▶ The order is important since matrix multiplication is not commutative
- ▶ Transposition and inversion change the order of products:

$$(AB)^T = B^T A^T \text{ and } (AB)^{-1} = B^{-1}A^{-1}$$

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

Matrix arithmetic

Diagonalisation

Linear independence/Bases/Dimension

Linear algebra in a nutshell

# Eigenvalues / Eigenvectors / Eigenpairs

## Definition 28

Let  $A \in \mathcal{M}_n$ . A vector  $\mathbf{x} \in \mathbb{F}^n$  such that  $\mathbf{x} \neq \mathbf{0}$  is an **eigenvector** of  $A$  if  $\exists \lambda \in \mathbb{F}$  called an **eigenvalue**, s.t.

$$A\mathbf{x} = \lambda\mathbf{x}$$

A couple  $(\lambda, \mathbf{x})$  with  $\mathbf{x} \neq \mathbf{0}$  s.t.  $A\mathbf{x} = \lambda\mathbf{x}$  is an **eigenpair**

If  $(\lambda, \mathbf{x})$  eigenpair, then for  $c \neq 0$ ,  $(\lambda, c\mathbf{x})$  also eigenpair since  $A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x}$  and dividing both sides by  $c$ .

# Similarity

## Definition 29 (Similarity)

$A, B \in \mathcal{M}_n$  are **similar** ( $A \sim B$ ) if  $\exists P \in \mathcal{M}_n$  invertible s.t.

$$P^{-1}AP = B$$

## Theorem 30 ( $\sim$ is an equivalence relation)

$A, B, C \in \mathcal{M}_n$ , then

- ▶  $A \sim A$  ( $\sim$  **reflexive**)
- ▶  $A \sim B \implies B \sim A$  ( $\sim$  **symmetric**)
- ▶  $A \sim B$  and  $B \sim C \implies A \sim C$  ( $\sim$  **transitive**)

## Similarity (cont.)

### Theorem 31

$A, B \in \mathcal{M}_n$  with  $A \sim B$ . Then

- ▶  $\det A = \det B$
- ▶  $A$  invertible  $\iff B$  invertible
- ▶  $A$  and  $B$  have the same eigenvalues

## Diagonalisation

Definition 32 (Diagonalisability)

$A \in \mathcal{M}_n$  is **diagonalisable** if  $\exists D \in \mathcal{M}_n$  diagonal s.t.  $A \sim D$

In other words,  $A \in \mathcal{M}_n$  is diagonalisable if there exists a diagonal matrix  $D \in \mathcal{M}_n$  and a nonsingular matrix  $P \in \mathcal{M}_n$  s.t.  $P^{-1}AP = D$

Could of course write  $PAP^{-1} = D$  since  $P$  invertible, but  $P^{-1}AP$  makes more sense for computations

### Theorem 33

$A \in M_n$  diagonalisable  $\iff A$  has  $n$  linearly independent eigenvectors

### Corollary 34 (Sufficient condition for diagonalisability)

$A \in M_n$  has all its eigenvalues distinct  $\implies A$  diagonalisable

For  $P^{-1}AP = D$ : in  $P$ , put the linearly independent eigenvectors as columns and in  $D$ , the corresponding eigenvalues

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

Matrix arithmetic

Diagonalisation

Linear independence/Bases/Dimension

Linear algebra in a nutshell

## Linear combination and span

### Definition 35 (Linear combination)

Let  $V$  be a vector space. A **linear combination** of a set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  of vectors in  $V$  is a *vector*

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

where  $c_1, \dots, c_k \in \mathbb{F}$

### Definition 36 (Span)

The set of all linear combinations of a set of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is the **span** of  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ ,

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \dots, c_k \in \mathbb{F}\}$$

# Finite/infinite-dimensional vector spaces

## Theorem 37

*The span of a set of vectors in  $V$  is the smallest subspace of  $V$  containing all the vectors in the set*

## Definition 38 (Set of vectors spanning a space)

If  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = V$ , we say  $\mathbf{v}_1, \dots, \mathbf{v}_k$  **spans**  $V$

## Definition 39 (Dimension of a vector space)

A vector space  $V$  is **finite-dimensional** if some set of vectors in it spans  $V$ . A vector space  $V$  is **infinite-dimensional** if it is not finite-dimensional

## Linear (in)dependence

Definition 40 (Linear independence/Linear dependence)

A set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  of vectors in a vector space  $V$  is **linearly independent** if

$$(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0) \Leftrightarrow (c_1 = \cdots = c_k = 0),$$

where  $c_1, \dots, c_k \in \mathbb{F}$ . A set of vectors is **linearly dependent** if it is not linearly independent.

If linearly dependent, assume w.l.o.g. that  $c_1 \neq 0$ , then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \cdots - \frac{c_k}{c_1}\mathbf{v}_k$$

i.e.,  $\mathbf{v}_1$  is a linear combination of the other vectors in the set

## Theorem 41

*Let  $V$  be a finite-dimensional vector space. Then the **cardinal** (number of elements) of every linearly independent set of vectors is less than or equal to the number of elements in every spanning set of vectors*

E.g., in  $\mathbb{R}^3$ , a set with 4 or more vectors is automatically linearly dependent

## Basis

### Definition 42 (Basis)

Let  $V$  be a vector space. A **basis** of  $V$  is a set of vectors in  $V$  that is both linearly independent and spanning

### Theorem 43 (Criterion for a basis)

A set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  of vectors in a vector space  $V$  is a basis of  $V$   $\iff \forall \mathbf{v} \in V, \mathbf{v}$  can be written uniquely in the form

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k,$$

where  $c_1, \dots, c_k \in \mathbb{F}$

# Plus/Minus Theorem

## Theorem 44 (Plus/Minus Theorem)

$S$  a nonempty set of vectors in vector space  $V$

- ▶ If  $S$  is linearly independent and  $V \ni \mathbf{v} \notin \text{span}(S)$ , then  $S \cup \{\mathbf{v}\}$  is linearly independent
- ▶ If  $\mathbf{v} \in S$  is linear combination of other vectors in  $S$ , then  $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$

## More on bases

Theorem 45 (Basis of finite-dimensional vector space)

*Every finite-dimensional vector space has a basis*

Theorem 46

*Any two bases of a finite-dimensional vector space have the same number of vectors*

Definition 47 (Dimension)

The **dimension**  $\dim V$  of a finite-dimensional vector space  $V$  is the number of vectors in any basis of the vector space

Theorem 48 (Dimension of a subspace)

*Let  $V$  be a finite-dimensional vector space and  $U \subset V$  be a subspace of  $V$ . Then  $\dim U \leq \dim V$*

## Constructing bases

### Theorem 49

*Let  $V$  be a finite-dimensional vector space. Then every linearly independent set of vectors in  $V$  with  $\dim V$  elements is a basis of  $V$*

### Theorem 50

*Let  $V$  be a finite-dimensional vector space. Then every spanning set of vectors in  $V$  with  $\dim V$  elements is a basis of  $V$*

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

Matrix arithmetic

Diagonalisation

Linear independence/Bases/Dimension

Linear algebra in a nutshell

## To finish: the “famous” “growing result”

### Theorem 51

Let  $A \in \mathcal{M}_n$ . The following statements are equivalent (TFAE)

1. The matrix  $A$  is invertible
2.  $\forall \mathbf{b} \in \mathbb{F}^n$ ,  $A\mathbf{x} = \mathbf{b}$  has a unique solution ( $\mathbf{x} = A^{-1}\mathbf{b}$ )
3. The only solution to  $A\mathbf{x} = \mathbf{0}$  is the trivial solution  $\mathbf{x} = \mathbf{0}$
4.  $RREF(A) = \mathbb{I}_n$
5. The matrix  $A$  is equal to a product of elementary matrices
6.  $\forall \mathbf{b} \in \mathbb{F}^n$ ,  $A\mathbf{x} = \mathbf{b}$  has a solution
7. There is a matrix  $B \in \mathcal{M}_n$  such that  $AB = \mathbb{I}_n$
8. There is an invertible matrix  $B \in \mathcal{M}_n$  such that  $AB = \mathbb{I}_n$
9.  $\det(A) \neq 0$
10. 0 is not an eigenvalue of  $A$