Matrix methods – QR factorisation (2)

MATH 2740 – Mathematics of Data Science – Lecture 08

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Fall 202X

The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

The QR factorisation

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Theorem 1

Let $A \in \mathcal{M}_{mn}$ with LI columns. Then A can be factored as

$$A = QR$$

where $Q \in \mathcal{M}_{mn}$ has orthonormal columns and $R \in \mathcal{M}_n$ is nonsingular upper triangular

Back to least squares

So what was the point of all that..?

Theorem 2 (Least squares with QR factorisation)

$A \in \mathcal{M}_{mn}$ with LI columns, $\boldsymbol{b} \in \mathbb{R}^m$. If $A = QR$ is a QR factorisation of A, then the unique least squares solution $\tilde{\boldsymbol{x}}$ of $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\tilde{\boldsymbol{x}} = R^{-1} Q^T \boldsymbol{b}$$

Proof of Theorem 2

A has LI columns so

- least squares $A\boldsymbol{x} = \boldsymbol{b}$ has unique solution $\tilde{\boldsymbol{x}} = (A^T A)^{-1} A^T \boldsymbol{b}$
- by Theorem 1, A can be written as $A = QR$ with $Q \in \mathcal{M}_{mn}$ with orthonormal columns and $R \in \mathcal{M}_n$ nonsingular and upper triangular

So

$$\begin{aligned}
A^T A\tilde{\boldsymbol{x}} = A^T \boldsymbol{b} &\implies (QR)^T QR\tilde{\boldsymbol{x}} = (QR)^T \boldsymbol{b} \\
&\implies R^T Q^T QR\tilde{\boldsymbol{x}} = R^T Q^T \boldsymbol{b} \\
&\implies R^T \mathbb{I}_n R\tilde{\boldsymbol{x}} = R^T Q^T \boldsymbol{b} \\
&\implies R^T R\tilde{\boldsymbol{x}} = R^T Q^T \boldsymbol{b} \\
&\implies (R^T)^{-1} R\tilde{\boldsymbol{x}} = (R^T)^{-1} R^T Q^T \boldsymbol{b} \\
&\implies R\tilde{\boldsymbol{x}} = Q^T \boldsymbol{b} \\
&\implies \tilde{\boldsymbol{x}} = R^{-1} Q^T \boldsymbol{b} \quad \square
\end{aligned}$$