Matrix methods – Absorbing Markov chains

MATH 2740 – Mathematics of Data Science – Lecture 14

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Absorbing Markov chains

Analysing absorbing Markov chains

Numerics

Absorbing Markov chains

Analysing absorbing Markov chains

Numerics

Modifying the breeding experiment

We modify the setup in Lecture 13 by now repeatedly mating a female offspring with an orange male ($X^O Y$).

The state of the chain is the genotype of the female offspring from the previous generation

There are 3 possible states for the female offspring:

- S_1: $X^O X^O$ (orange female)
- S_2: $X^o X^o$ (black female)
- S_3: $X^O X^o$ (tortoiseshell female)

State 1 – orange female ($X^O X^O$)

Current state is S_1. Mate this $X^O X^O$ female with the $X^O Y$ male

Mother		X^O	Y
	X^O	$X^O X^O$ (S_1)	Orange male
	X^O	$X^O X^O$ (S_1)	Orange male

The female offspring are:

- 100% $X^O X^O$ (state S_1)

State 2 – black female ($X^o X^o$)

Current state is S_2. Mate this $X^o X^o$ female with an $X^O Y$ male

		X^O	Y
Mother	X^o	$X^O X^o$ (S_1)	Black male
	X^o	$X^O X^o$ (S_3)	Black male

The female offspring are:

- 100% $X^O X^o$ (State S_3) $\implies \mathbb{P}(S_2 \to S_3) = 1$

State 3 – tortoiseshell female ($X^O X^o$)

Current state is S_3. Mate this $X^O X^o$ female with the $X^O Y$ male

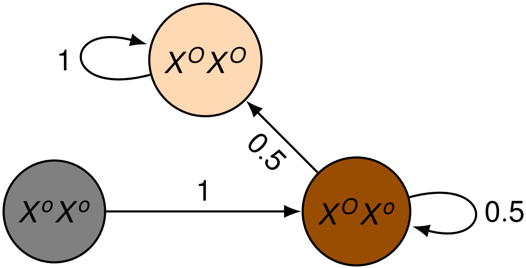
Mother	X^O	Y
X^O	$X^O X^O$ (S_1)	Orange male
X^o	$X^O X^o$ (S_3)	Black male

The female offspring have a 50/50 chance:

- 50% $X^O X^O$ (State S_1) $\implies \mathbb{P}(S_3 \to S_1) = 1/2$
- 50% $X^O X^o$ (State S_3) $\implies \mathbb{P}(S_3 \to S_3) = 1/2$

Summary of the chain

$$P = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$



Absorbing Markov chains

Definition 1 (Absorbing state)

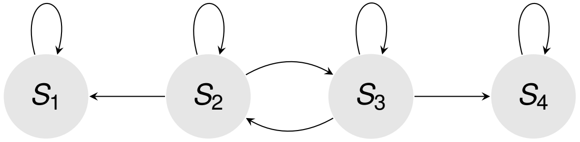
A state S_i in a Markov chain is **absorbing** if whenever it occurs on the t^{th} generation of the experiment, it then occurs on every subsequent step. In other words, S_i is absorbing if $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$

Definition 2 (Absorbing chain)

A Markov chain is **absorbing** if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state. In an absorbing Markov chain, a state that is not absorbing is called **transient**

Questions about absorbing chains

Suppose we have a chain like the following



1. Does the process eventually reach an absorbing state?
2. What is the average number of steps spent in a transient state, if starting in a transient state?
3. What is the average number of steps before entering an absorbing state?
4. What is the probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?

- Absorbing Markov chains
- **Analysing absorbing Markov chains**
- Numerics

The answer to the first question (“Does the process eventually reach an absorbing state?”) is given by the following result

Theorem 3

In an absorbing Markov chain, the probability of reaching an absorbing state is 1

To answer the other questions, write the transition matrix in **standard** form

For an absorbing chain with k absorbing states and $r-k$ transient states, write transition matrix as

$$P = \begin{pmatrix} \mathbb{I}_k & R \\ \mathbf{0} & Q \end{pmatrix}$$

with following meaning

	Absorbing states	Transient states
Absorbing states	$\mathbb{I}_k$	R
Transient states	$\mathbf{0}$	Q

with $\mathbb{I}_k$ the $k \times k$ identity matrix, $\mathbf{0}$ an $(r-k) \times k$ matrix of zeros, R an $k \times (r-k)$ matrix and Q an $(r-k) \times (r-k)$ matrix. The matrix $\mathbb{I}_{r-k} - Q$ is invertible. Let

- $N = (\mathbb{I}_{r-k} - Q)^{-1}$ the **fundamental matrix** of the MC
- T_i sum of the entries on column i of N
- $B = RN$

Answers to our remaining questions:

2. N_{ij} average number of times the process is in the *i*th transient state if it starts in the *j*th transient state

3. T_i average number of steps before the process enters an absorbing state if it starts in the *i*th transient state

4. B_{ij} probability of eventually entering the *i*th absorbing state if the process starts in the *j*th transient state

Back to the genetic example

The matrix is already in standard form

$$P = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 1/2 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_1 & R \\ \mathbf{0} & Q \end{pmatrix}$$

with $\mathbb{I}_1 = 1$, $\mathbf{0} = (0 \;\; 0)^T$ and

$$R = \begin{pmatrix} 0 & 1/2 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix}$$

We have

$$\mathbb{I}_2 - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1/2 \end{pmatrix}$$

so the fundamental matrix is

$$N = (\mathbb{I}_2 - Q)^{-1} = 2 \begin{pmatrix} 1/2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Summary about absorption

Fundamental matrix, whose (i, j) entry is the average number of visits to the ith transient state if it starts in the jth transient state

$$N = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Average number of steps before absorption (i.e., only orange females) if we start in a transient state is

$$T = \mathbb{1}^T N = (3, 2)$$

Probability of eventually entering the ith absorbing state if we start in the jth transient state

$$B = RN = \begin{pmatrix} 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Absorbing Markov chains

Analysing absorbing Markov chains

Numerics

Numerics

Regular MC example

Absorbing MC example

```r
# Library
library(markovchain)
# Make the transition matrix
P_reg = matrix(c(0.5, 0, 0.25,
                 0, 0.5, 0.25,
                 0.5, 0.5, 0.5),
               nr = 3, byrow = TRUE)
# Library: markovchain
mc_reg <- new("markovchain",
              states = sprintf("S_%d", 1:3),
              transitionMatrix = t(P_reg),
              name = "orange_reg")
```

```r
summary(mc_reg)
```

```
## orange_reg  Markov chain that is composed by:
## Closed classes:
## S_1 S_2 S_3
## Recurrent classes:
## {S_1,S_2,S_3}
## Transient classes:
## NONE
## The Markov chain is irreducible
## The absorbing states are: NONE
```

The equilibrium distribution

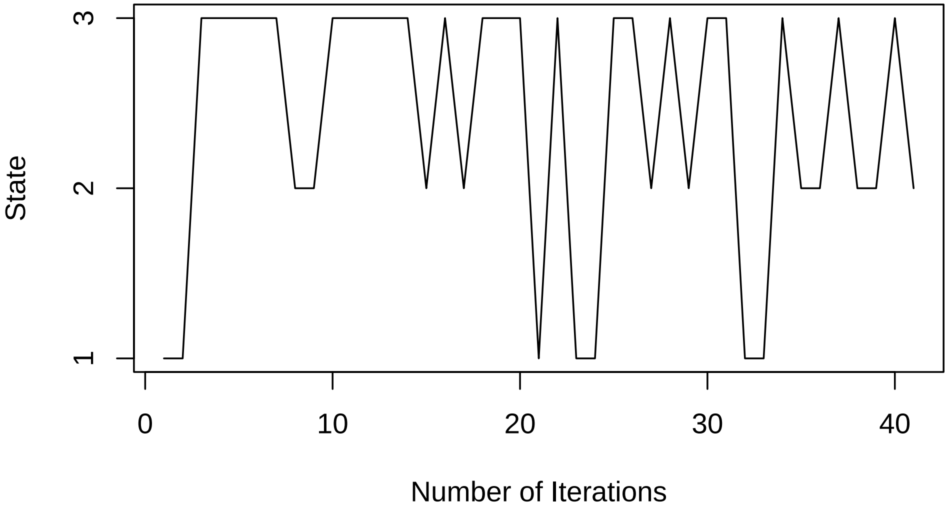
```r
steadyStates(mc_reg)

##       S_1  S_2 S_3
## [1,] 0.25 0.25 0.5
```

Showing a realisation

```
library(DTMCPack)
IC = rep(0, dim(P_reg)[1])
IC[1] = 1
sol = DTMC(t(P_reg), IC, 41, trace=TRUE)

## NULL
```


State
3
2
1
0
10
20
30
40
Number of Iterations

Numerics

- Regular MC example
- **Absorbing MC example**

The absorbing version

```r
P_abs = matrix(c(1, 0, 0.5,
                 0, 0, 0,
                 0, 1, 0.5),
               nr = 3, byrow = TRUE)
mc_abs <- new("markovchain",
              states = sprintf("S_%d", 1:3),
              transitionMatrix = t(P_abs),
              name = "orange_abs")
```

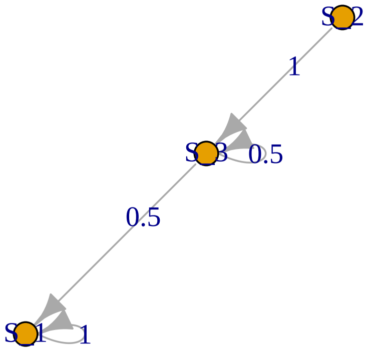
Show some info about the chain

```
summary(mc_abs)

## orange_abs  Markov chain that is composed by:
## Closed classes:
## S_1
## Recurrent classes:
## {S_1}
## Transient classes:
## {S_2},{S_3}
## The Markov chain is not irreducible
## The absorbing states are: S_1
```

Plotting the chain

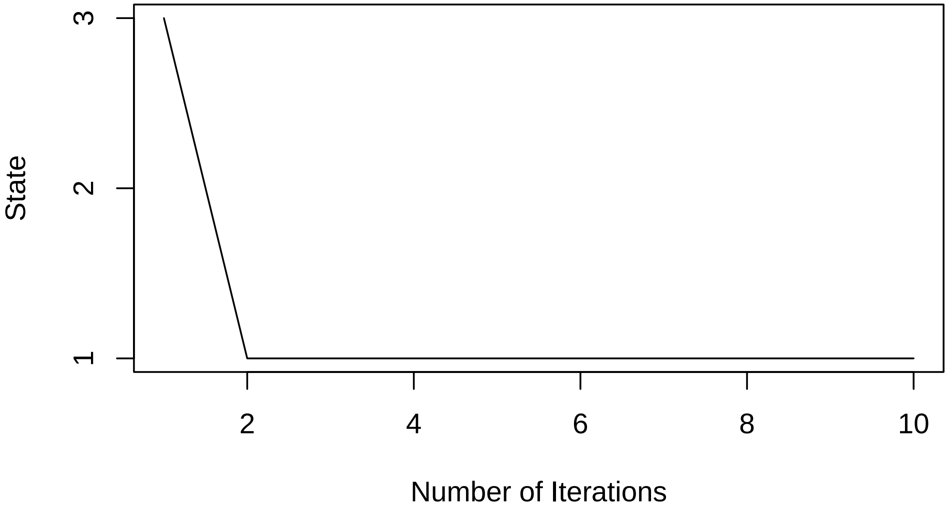
```
plot(mc_abs)
```


1
0.5
0.5
1

Showing a realisation

```
IC = rep(0, dim(P_abs)[1])
IC[3] = 1
sol = DTMC(t(P_abs), IC, 10, trace=TRUE)

## NULL
```


State
3
2
1
2
4
6
8
10
Number of Iterations

Additional information about absorbing chains

```
canonicForm(mc_abs)
```

```
meanAbsorptionTime(mc_abs)
```

```
absorptionProbabilities(mc_abs)
```

```
hittingProbabilities(mc_abs)
```

Canonical form

```
canonicForm(mc_abs)

## orange_abs
##  A  3 - dimensional discrete Markov Chain defined by the following states
##  S_1, S_2, S_3
##  The transition matrix  (by rows)  is defined as follows:
##     S_1 S_2 S_3
## S_1 1.0   0 0.0
## S_2 0.0   0 1.0
## S_3 0.5   0 0.5
```

Mean absorption time

```r
meanAbsorptionTime(mc_abs)

## S_2 S_3
##   3   2
```

Absorption probabilities

```
absorptionProbabilities(mc_abs)

##       S_1
## S_2     1
## S_3     1
```

Hitting probabilities

```r
hittingProbabilities(mc_abs)
```

```
##     S_1 S_2 S_3
## S_1   1   0 0.0
## S_2   1   0 1.0
## S_3   1   0 0.5
```