Graphs – Introduction (theory) – 3

MATH 2740 – Mathematics of Data Science – Lecture 16

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Directed graphs

Matrices associated to a graph/digraph

FIGS-transitions/Gemini_Generated_Image_duutxqduutxqduut.jpeg

Trees

Directed graphs

Matrices associated to a graph/digraph

Trees

FIGS-transitions/Gemini_Generated_Image_fto8nofto8nofto8.jpeg

Directed graphs

- **Degrees in digraphs**
- Walks, paths, etc.
- Connectivity in digraphs
- Orientable graphs

Degree

Let v be a vertex of a digraph $G = (V, A)$

Definition 31 (Outdegree of a vertex)

The number of arcs directed away from a vertex v, in a digraph is called the **outdegree** of v and is written $d_G^+(v)$

Definition 32 (Indegree of a vertex)

The number of arcs directed towards a vertex v, in a digraph is called the **indegree** of v and is written $d_G^-(v)$

Definition 33 (Degree)

For any vertex v in a digraph, the **degree** of v is defined as

$$d_G(v) = d_G^+(v) + d_G^-(v)$$

Theorem 34

For any (di)graph, the sum of the degrees of the vertices equals twice the number of edges (arcs)

Corollary 35

In any (di)graph, the sum of the degrees of the vertices is a nonnegative even integer

Theorem 36

If G is a digraph with vertex set $V(G) = \{v_1, \ldots, v_p\}$ and q arcs, then

$$\sum_{i=1}^{p} d_G^+(v_i) = \sum_{i=1}^{p} d_G^-(v_i) = q$$

Definition 37 (Regular digraph)

A digraph G is r-regular if $d^+_G(v) = d^-_G(v) = r$ for all $v \in V(G)$

Symmetric/antisymmetric digraphs

Definition 38 (Symmetric digraph)

Let $G = (V, A)$ be a digraph with associated binary relation R. If R is *symmetric*, the digraph is symmetric

Definition 39 (Anti-symmetric digraph)

Let $G = (V, A)$ be a digraph with associated binary relation R. The digraph G is **anti-symmetric** if

$$xRy \implies y \not{R} x$$

Definition 40 (Symmetric multidigraph)

Let $G = (V, A)$ be a multidigraph. G is symmetric if $\forall x, y \in V(G)$, the number of arcs from x to y equals the number of arcs from y to x

Directed graphs

- Degrees in digraphs
- **Walks, paths, etc.**
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Walks

Let $G = (V, A)$ be a digraph.

Definition 41 (Directed walk)

A **directed walk** in a digraph G is a non-empty alternating sequence $v_0 a_0 v_1 a_1 v_2 \ldots a_{k-1} v_k$ of vertices and arcs in G such that $a_i = (v_i, v_{i+1})$ for all $i < k$. This walk begins with v_0 and ends with v_k

Definition 42 (Length of a directed walk)

The length of a directed walk is equal to the number of arcs in the directed walk

Definition 43 (Closed walk)

If $v_0 = v_k$, the walk is closed

Trails

Let $G = (V, A)$ be a digraph.

Definition 44 (Directed trail)

A directed walk in G in which all arcs are distinct is a **directed trail** in G

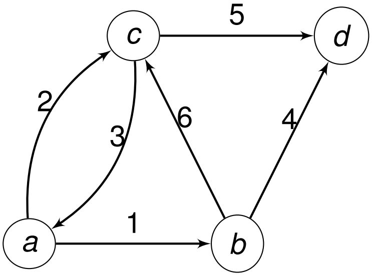
Definition 45 (Directed path)

A directed walk in G in which all vertices are distinct is a **directed path** in G

Definition 46 (Directed cycle)

A closed walk is a **directed cycle** if it contains at least three vertices and all its vertices are distinct except for $v_0 = v_k$

Examples of directed cycles



Cycles:

- $\mu^1 = (1, 6, 2) = [abca]$
- $\mu^2 = (1, 6, 3) = [abca]$
- $\mu^3 = (2, 3) = [aca]$
- $\mu^4 = (1, 4, 5, 2) = [abdca]$
- $\mu^5 = (6, 5, 4) = [acdb]$
- $\mu^6 = (1, 4, 5, 3) = [abdca]$

Directed graphs

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- **Connectivity in digraphs**
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Definitions

Definition 47 (Underlying graph)

Given a digraph, the undirected graph with each arc replaced by an edge is called the **underlying graph**

Definition 48 (Weakly connected digraph)

If the underlying graph is a connected graph, then the digraph is **weakly connected**

Definition 49 (Strongly connected digraph)

A digraph G is **strongly connected** if for every two distinct vertices u and v of G, there exists a directed path from u to v

Definition 50 (Disconnected digraph)

A digraph is said to be **disconnected** if it is not weakly connected

Strong connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a directed path in G from x to y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 51 (Connected component of a graph)

Sets of the form

$$A(x_0) = \{x : x \in V, x \equiv x_0\}$$

are equivalence classes. They partition V into strongly connected sub-digraphs of G called **strongly connected components** (or **strong components**) of G

A strong component in G is a maximal strongly connected subdigraph of G

Theorem 52 (Properties)

Let $G = (V, A)$ be a digraph

- *If G is strongly connected, it has only one strongly connected component*
- *The strongly connected components partition the vertices $V(G)$, with every vertex in exactly one strongly connected component*

Algorithm for determining strongly connected components in $G = (V, A)$

- Determine the strongly connected component $C(v)$ containing the vertex v; if $V - C(v)$ is non-empty, re-do the same operation on the sub-digraph $G' = (V - C(v), A')$
- To determine $C(v)$, the strongly connected component containing v: let v be a vertex of a digraph , which is not already in any strongly connected component
  1. Mark the vertex v with $\pm$
  2. Mark with $+$ all successors (not already marked with $+$) of a vertex marked with $+$
  3. Mark with $-$ all predecessors (not already marked with $-$) of a vertex marked with $-$
  4. Repeat until no more possible marking with $+$ or $-$

  All vertices marked with $\pm$ belong to the same strongly connected component $C(v)$ containing the vertex v

Condensation of a digraph

Definition 53 (Condensation of a digraph)

The condensation G^* of a digraph G is a digraph having as vertices the strongly connected components (SCC) of G and such that there exists an arc in G^* from a SCC C_i to another SCC C_j if there is an arc in G from some vertex of S_i to a vertex of S_j

Definition 54 (Articulation set)

For a connected graph, a set X of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $V - X$ is not connected

Definition 55 (Stable set)

A set S of vertices is called a **stable set** if no arc joins two distinct vertices in S

Directed graphs

- Degrees in digraphs
- Walks, paths, etc.
- Connectivity in digraphs
- **Orientable graphs**

Orientation

Definition 56 (Orienting a graph)

Given a connected *graph*, we describe the act of assigning a direction to each edge (edge $\rightarrow$ arc) as **orienting the graph**

Definition 57 (Strong orientation)

If the digraph resulting from orienting a graph is strongly connected, the orientation is a **strong orientation**

Orientable graph

Definition 58 (Orientable graph)

A connected graph G is **orientable** if it admits a strong orientation

Theorem 59

A connected graph $G = (V, E)$ is orientable $\iff$ G contains no bridges

(in other words, iff every edge is contained in a cycle)

Matrices associated to a graph/digraph

There are multiple matrices associated to a graph/digraph

The branch of graph theory that studies the properties of matrices derived from graphs and uses of these matrices in determining graph properties is *spectral graph theory*

Graphs greatly simplify some problems in linear algebra and vice versa

Matrices associated to a graph/digraph

- **Adjacency matrices**
- Other matrices associated to a graph/digraph
- Linking graphs and linear algebra

Adjacency matrix (undirected case)

Let $G = (V, E)$ be a graph of order p and size q, with vertices $v_1, \dots, v_p$ and edges $e_1, \dots, e_q$

Definition 60 (Adjacency matrix)

The **adjacency matrix** is

$$M_A = M_A(G) = [m_{ij}]$$

is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 61 (Adjacency matrix and degree)

The sum of the entries in row i of the adjacency matrix is the degree of v_i in the graph

We often write $A(G)$ and, reciprocally, if A is an adjacency matrix, $G(A)$ the corresponding graph

G undirected $\implies A(G)$ symmetric

$A(G)$ has nonzero diagonal entries if G is not simple

Adjacency matrix (directed case)

Let $G = (V, A)$ be a digraph of order p with vertices $v_1, \ldots, v_p$

Definition 62 (Adjacency matrix)

The **adjacency matrix** $M = M(G) = [m_{ij}]$ is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if arc } v_i v_j \in A \\ 0 & \text{otherwise} \end{cases}$$

Theorem 63 (Properties)

- *M is not necessarily symmetric*
- *The sum of any column of M is equal to the number of arcs directed towards v_j*
- *The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i*
- *The $(i, j)-$entry of M^n is equal to the number of walks of length n from vertex v_i to v_j*

Definition 64 (Multiplicity of a pair)

The **multiplicity** of a pair x, y is the number $m_G^+(x, y)$ of arcs with initial endpoint x and terminal endpoint y. Let

$$m_G^-(x, y) = m_G^+(y, x)$$
$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y)$$

If $x \neq y$, then $m_G(x, y)$ is number of arcs with both x and y as endpoints. If $x = y$, then $m_G(x, y)$ equals twice the number of loops attached to vertex x. If $A, B \subset V$, $A \neq B$, let

$$m_G^+(A, B) = \{u : u \in U, u = (x, y), x \in A, y \in B\}$$
$$m_G(A, B) = m_G^+(A, B) + m_G^+(A, B)$$

Adjacency matrix of a multigraph

Definition 65 (Matrix associated with G)

If G has vertices $x_1, x_2, \ldots, x_n$, then the **matrix associated** with G is

$$a_{ij} = m_G^+(x_i, x_j)$$

Definition 66 (Adjacency matrix)

The matrix $a_{ij} + a_{ji}$ is the **adjacency matrix** associated with G

Adjacency matrix (multigraph case)

Definition 67 (Adjacency matrix of a multigraph)

G an ℓ-graph, then the adjacency matrix $M_A = [m_{ij}]$ is defined as follows

$$m_{ij} = \begin{cases} k & \text{if arc there are } k \text{ arcs } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

with $k \leq \ell$

G undirected $\implies M_A(G)$ symmetric

$M_A(G)$ has nonzero diagonal entries if G is not simple.

Weighted adjacency matrices

Sometimes, adjacency matrices (typically for 1-graphs) have real entries, usually positive

This means that the arcs/edges have been given a weight

Theorem 68 (Number of walks of length n)

Let A be the adjacency matrix of a graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then the $(i, j)-$entry of A^n, $n \geq 1$, is the number of different walks linking v_i to v_j of length n in G.

(two walks of the same length are equal if their edges occur in exactly the same order)

Example: let A be the adjacency matrix of a graph $G = (V(G), E(G))$.

- the $(i, i)-$entry of A^2 is equal to the degree of v_i.
- the $(i, i)-$entry of A^3 is equal to twice the number of C_3 containing v_i.

Matrices associated to a graph/digraph

- Adjacency matrices
- **Other matrices associated to a graph/digraph**
- Linking graphs and linear algebra

Incidence matrix (undirected case)

Let $G = (V, E)$ be a graph of order p, and size q, with vertices $v_1, \dots, v_p$, and edges $e_1, \dots, e_q$

Definition 69 (Incidence matrix)

The incidence matrix is

$$B = B(G) = [b_{ij}]$$

is that $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 70 (Incidence matrix and degrees)

The sum of the entries in row i of the incidence matrix is the degree of v_i in the graph

Incidence matrix (directed case)

Let $G = (V, A)$ be a digraph of order p and size q, with vertices $v_1, \ldots, v_p$ and arcs $a_1, \ldots, a_q$

Definition 71 (Incidence matrix)

The **incidence matrix** $B = B(G) = [b_{ij}]$ is a $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if arc } a_j \text{ is directed away from a vertex } v_i \\ -1 & \text{if arc } a_j \text{ is directed towards a vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of a graph

We will come back to this later, but for now..

Definition 72 (Spectrum of a graph)

The **spectrum** of a graph G is the spectrum (set of eigenvalues) of its associated adjacency matrix $M(G)$

This is regardless of the type of adjacency matrix or graph

Degree matrix

Definition 73 (Degree matrix)

The **degree** matrix $D = [d_{ij}]$ for G is a $n \times n$ diagonal matrix defined as

$$d_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In an undirected graph, this means that each loop increases the degree of a vertex by two

In a directed graph, the term "degree" may refer either to indegree (the number of incoming edges at each vertex) or outdegree (the number of outgoing edges at each vertex)

Laplacian matrix

Definition 74 (Laplacian matrix)

$G = (V, A)$ a simple graph with n vertices. The **Laplacian** matrix is

$$L = D(G) - M(G)$$

where $D(G)$ is the degree matrix and $M(G)$ is the adjacency matrix

Laplacian matrix (continued)

G simple graph $\implies M(G)$ only contains 1 or 0 and its diagonal elements are all 0

For directed graphs, either the indegree or outdegree is used, depending on the application

Elements of L are given by

$$\ell_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Distance matrix

Let G be a graph of order p with vertices $v_1, \ldots, v_p$

Definition 75 (Distance matrix)

The distance matrix $\Delta(G) = [d_{ij}]$ is a $p \times p$ matrix in which

$$\delta_{ij} = d_G(v_i, v_j)$$

Note $\delta_{ii} = 0$ for $i = 1, \ldots, p$

Property 76

- *M is not necessarily symmetric*
- *The sum of any column of M is equal to the number of arcs directed towards v_j*
- *The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i*
- *The $(i, j)-$entry of M^n is equal to the number of walks of length n from vertex v_i to v_j*

Matrices associated to a graph/digraph

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Counting paths

Theorem 77

G a digraph and $M_A(G)$ its adjacency matrix. Denote $P = [p_{ij}]$ the matrix $P = M_A^k$. Then p_{ij} is the number of distinct paths of length k from i to j in G

Definition 78 (Irreducible matrix)

A matrix $A \in \mathcal{M}_n$ is **reducible** if $\exists P \in \mathcal{M}_n$, permutation matrix, s.t. $P^T AP$ can be written in block triangular form. If no such P exists, A is **irreducible**

Theorem 79

A irreducible $\iff$ $G(A)$ strongly connected

Theorem 80

Let A be the adjacency matrix of a graph G on p vertices. A graph G on p vertices is connected $\iff$

$$I + A + A^2 + \cdots + A^{p-1} = C$$

has no zero entries

Theorem 81

Let M be the adjacency matrix of a digraph D on p vertices. A digraph D on p vertices is strongly connected $\iff$

$$I + M + M^2 + \cdots + M^{p-1} = C$$

has no zero entries

Nonnegative matrix

$A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ **nonnegative** if $a_{ij} \geq 0$ $\forall i, j = 1, \ldots, n$; $\boldsymbol{v} \in \mathbb{R}^n$ nonnegative if $v_i \geq 0$ $\forall i = 1, \ldots, n$. **Spectral radius** of A

$$\rho(A) = \max_{\lambda \in \mathrm{Sp}(A)} \{|\lambda|\}$$

$\mathrm{Sp}(A)$ the **spectrum** of A

Perron-Frobenius (PF) theorem

Theorem 82 (PF – Nonnegative case)

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$. Then $\exists \mathbf{v} \geq \mathbf{0}$ s.t.

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

Theorem 83 (PF – Irreducible case)

Let $0 \leq A \in \mathcal{M}_n(\mathbb{R})$ irreducible. Then $\exists \mathbf{v} > \mathbf{0}$ s.t.

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

$\rho(A) > 0$ and with algebraic multiplicity 1. No nonnegative eigenvector is associated to any other eigenvalue of A

Primitive matrices

Definition 84

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$ **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if $\exists k \in \mathbb{N}_+^*$ s.t.

$$A^k > 0,$$

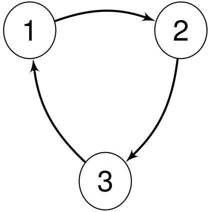
with k the smallest integer for which this is true. A **imprimitive** if it is not primitive

A primitive $\implies$ A irreducible; the converse is false

Theorem 85

$A \in \mathcal{M}_n(\mathbb{R})$ *irreducible and* $\exists i = 1, \ldots, n$ *s.t.* $a_{ii} > 0 \implies$ *A primitive*

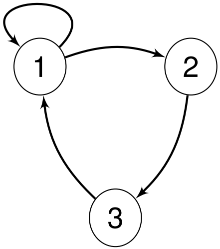
Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d = \gcd$ of all the lengths of closed walks in $G(A)$



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walks in $G(A)$ (lengths): $1 \to 1$ (3), $2 \to 2$ (3), $2 \to 2$ (3) $\implies \mathrm{gcd} = 3 \implies d = 3$ (here, all eigenvalues have modulus 1)



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walk $1 \to 1$ has length 1 $\implies$ gcd of lengths of closed walks is 1 $\implies$ A primitive

Let $\mathbf{0} \leq A \in \mathcal{M}_n$

Theorem 86

A primitive $\implies \exists\, 0 < k \leq (n-1)n^n$ such that $A^k > 0$

Theorem 87

If A is primtive and the shortest simple directed cycle in $G(A)$ has length s, then the primitivity index is $\leq n + s(n-1)$

Theorem 88

A primitive $\iff A^{n^2-2n+2} > \mathbf{0}$

Theorem 89

If A is irreducible and has d positive entries on the diagonal, then the primitivity index $\leq 2n - d - 1$

Theorem 90

$\mathbf{0} \leq A \in \mathcal{M}_n$, $\lambda_P = \rho(A)$ *the Perron root of* A, $\boldsymbol{v}_P$ *and* $\boldsymbol{w}_P$ *the corresponding right and left Perron vectors of* A, *respectively,* d *the index of imprimitivity of* A *(with* $d = 1$ *when* A *is primitive) and* $\lambda_j \in \sigma(A)$ *the spectrum of* A, *with* $j = 2, \ldots, n$ *unless otherwise specified (assuming* $\lambda_1 = \lambda_P$*)*

NONNEGATIVE

REDUCIBLE

- $\lambda_P \geq 0$
- $\boldsymbol{w}_P \geq 0$
- $\boldsymbol{v}_P \geq 0$
- $\lambda_P \geq |\lambda_j|$

IRREDUCIBLE

IMPRIMITIVE

- $\lambda_P > 0$
- $\boldsymbol{w}_P > 0$
- $\boldsymbol{v}_P > 0$
- $\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
- $\lambda_P > |\lambda_j|$, $j > d$

PRIMITIVE

- $\lambda_P > 0$
- $\boldsymbol{w}_P > 0$
- $\boldsymbol{v}_P > 0$
- $\lambda_P > |\lambda_j|$, $j \neq P$

Directed graphs

Matrices associated to a graph/digraph

Trees

Trees

Definition 91 (Forest, trees and branches)

- A connected graph with no cycle is a **tree**
- A tree is a connected acyclic graph, its edges are called **branches**
- A graph (connected or not) without any cycle is a **forest**. Each component is a tree

(A forest is a graph whose connected components are trees)

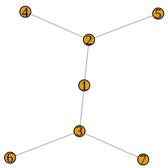
Is the “Karate graph” a tree?

```
is_acyclic(G)

## Error:  object 'G' not found

is_tree(G)

## Error:  object 'G' not found
```

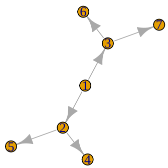
So we need friend to play with!

```
G_tu <- make_tree(7, 2, mode = "undirected")
G_td <- make_tree(7, 2)
```

An undirected tree



A (out) directed tree



Property 92

- *Every edge of a tree is a bridge*
- *Given two vertices u and v of a tree, there is an unique path linking u to v*
- *A tree with p vertices and q edges satisfies $q = p - 1$. Thus, a tree is minimally connected*

(First property: the deletion of any edge of a tree diconnects it)

Every edge of a tree is a bridge

```r
E(G_tu)

## + 6/6 edges from f6e1019:
## [1] 1--2 1--3 2--4 2--5 3--6 3--7

bridges(G_tu)

## + 6/6 edges from f6e1019:
## [1] 2--4 2--5 1--2 3--6 3--7 1--3

all(sort(E(G_tu)) == sort(bridges(G_tu)))

## [1] TRUE
```

Spanning tree

Definition 93 (Spanning tree)

A **spanning tree** of a connected graph G is a subgraph of G that contains all the vertices of G and is a tree.

A graph may have many spanning trees

Minimal spanning tree

Definition 94 (Value of a spanning tree)

The **value of a spanning tree** T of order p is

$$\sum_{i=1}^{p-1} f(e_i)$$

where f is the function that maps the edge set into $\mathbb{R}$

Definition 95 (Minimal spanning tree)

Let G be an undirected network, and let T be a **minimal spanning tree** of G.
Then T is a spanning tree whose the value is minimum

Algorithm to find a minimal spanning tree

Let $G = (V(G), E(G))$ be an undirected network and T be a minimal spanning tree

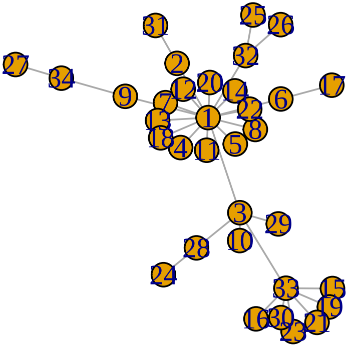
1. Sort the edges of G in increasing order by value
2. $T = (V(G), \emptyset)$
3. For each edge e in sorted order if the endpoints of e are disconnected in T add e to T

Finding a minimal spanning tree of the Karate graph

The function `mst` finds minimal spanning trees, using distances if no edge weights are provided

```
G_mst = mst(G)

## Error:  object 'G' not found
## Error in h(simpleError(msg, call)):  error in evaluating the
argument 'x' in selecting a method for function 'plot':  object
'G_mst' not found
```

A minimal spanning tree of the Karate graph



Minimal connector problem

- Model: a graph G such that edges represent all possible connections, and each edge has a positive value which represents its cost; an undirected network G
- Solution: a minimal spanning tree T of G
  - a spanning tree of G is a subgraph of G that contains all the vertices of G and is a tree.
  - the cost of the spanning tree is the sum of values of the edges of T
  - a spanning tree such that no other spanning tree has a smaller cost is a minimmal spanning tree.

Theorem 96 (Characterisation of trees)

$H = (V, U)$ a graph of order $|V| = n > 2$. The following are equivalent and all characterise a tree :

1. *H connected and has no cycles*
2. *H has $n - 1$ arcs and no cycles*
3. *H connected and has exactly $n - 1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is removed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Definition 97 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

Theorem 98

A tree of order $n \geq 2$ has at least two pendant vertices

Theorem 99

A graph $G = (V, U)$ *has a partial graph that is a tree* $\iff$ G *connected*

Recall that a partial graph is a graph generated by a subset of the arcs (Definition **??** slide **??**)

Spanning tree

The procedure in the proof of Theorem 99 gives a **spanning tree**

Can also build a spanning tree as follows:

- Consider any arc u_0
- Find arc u_1 that does not form a cycle with u_0
- Find arc u_2 that does not form a cycle with $\{u_0, u_1\}$
- Continue
- When you cannot continue anymore, you have a spanning tree

Theorem 100

G connected graph with ≥ 1 arc. TFAE

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

Theorem 101

G graph with ≥ 1 *arc. TFAE*

1. *G is a graph without circuits*
2. *Each arc is contained in a cocircuit*

Theorem 102

If G is a strongly connected graph of order n, then G has a cycle basis of $\nu(G)$ *circuits*

Definition 103 (Node, anti-node, branch)

$G = (V, U)$ strongly connected without loops and > 1 vertex. For each $x \in V$, there is a path from it and a path to it so x has at least 2 incident arcs. Specifically,

- $x \in V$ with > 2 incident arcs is a **node**
- $x \in V$ with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

Definition 104 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

Definition 105 (Contraction)

$G = (V, U)$. The **contraction** of the set $A \subset V$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Theorem 106

G minimally connected, $A \subset V$ generating a strongly connected subgraph of G. Then the contraction of A gives a minimally connected graph

Theorem 107

G a minimally connected graph, G' be the minimally connected graph obtained by the contraction of an elementary circuit of G. Then

$$\nu(G) = \nu(G') + 1$$

Theorem 108

G minimally connected of order $n \geq 2 \implies G$ has ≥ 2 anti-nodes

Theorem 109

$G = (V, U)$. Then the graph C' obtained by contracting each strongly connected component of G contains no circuits

Arborescences

Definition 110 (Root)

Vertex $a \in V$ in $G = (V, U)$ is a **root** if all vertices of G can be reached by paths *starting* from a

Not all graphs have roots

Definition 111 (Quasi-strong connectedness)

G is **quasi-strongly connected** if $\forall x, y \in V$, exists $z \in V$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected $\implies$ quasi-strongly connected (take $z(x, y) = x$); converse not true

Quasi-strongly connected $\implies$ connected

Arborescence

Definition 112 (Arborescence)

An **arborescence** is a tree that has a root

Lemma 113

$G = (V, U)$ has a root $\iff$ G quasi-strongly connected

Theorem 114

H graph of order $n > 1$. TFAE (and all characterise an arborescence)

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with $n-1$ arcs*
3. *H tree having a root a*
4. *$\exists a \in V$ s.t. all other vertices are connected with a by 1 and only 1 path from a*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

7. *H has no cycles and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

Theorem 115

G has a partial graph that is an arborescence $\iff$ G quasi-strongly connected

Theorem 116

$G = (V, E)$ simple connected graph and $x_1 \in V$. It is possible to direct all edges of E so that the resulting graph $G_0 = (V, U)$ has a spanning tree H s.t.

1. *H is an arborescence with root x_1*
2. *The cycles associated with H are circuits*
3. *The only elementary circuits of G_0 are the cycles associated with H*

Counting trees

Proposition 117

X a set with n distinct objects, $n_1, \ldots, n_p$ nonnegative integers s.t. $n_1 + \cdots + n_p = n$. The number of ways to place the n objects into p boxes $X_1, \ldots, X_p$ containing $n_1, \ldots, n_p$ objects respectively is

$$\binom{n}{n_1, \ldots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

Proposition 118 (Multinomial formula)

Let $a_1, \ldots, a_p \in \mathbb{R}$ be p real numbers, then

$$(a_1 + \cdots + a_p)^n = \sum_{n_1, \ldots, n_p \geq 0} \binom{n}{n_1, \ldots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

Theorem 119

Denote $T(n; d_1, \ldots, d_n)$ the number of distinct trees H with vertices $x_1, \ldots, x_n$ and with degrees $d_H(x_1) = d_1, \ldots, d_H(x_n) = d_n$. Then

$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_n - 1}$$

Theorem 120

The number of different trees with vertices $x_1, \ldots, x_n$ is n^{n-2}

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..