Graphs – Introduction (theory) – 4

MATH 2740 – Mathematics of Data Science – Lecture 18

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Trees

Trees

Trees

Definition 96 (Forest, trees and branches)

- A connected graph with no cycle is a **tree**
- A tree is a connected acyclic graph, its edges are called **branches**
- A graph (connected or not) without any cycle is a **forest**. Each component is a tree

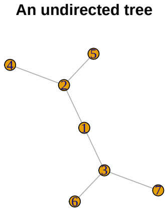
(A forest is a graph whose connected components are trees)

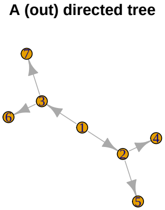
Is the “Karate graph” a tree?

```
is_acyclic(G_Z)

## [1] FALSE

is_tree(G_Z)

## [1] FALSE
```

So we need friend to play with!

```
G_tu <- make_tree(7, 2, mode = "undirected")
G_td <- make_tree(7, 2)
```


An undirected tree
4
5
2
1
3
6
7


A (out) directed tree
7
3
6
1
2
4
5

Property 97

- *Every edge of a tree is a bridge*
- *Given two vertices u and v of a tree, there is an unique path linking u to v*
- *A tree with p vertices and q edges satisfies $q = p - 1$. Thus, a tree is minimally connected*

(First property: the deletion of any edge of a tree diconnects it)

Every edge of a tree is a bridge

```r
E(G_tu)

## + 6/6 edges from a572df3:
## [1] 1--2 1--3 2--4 2--5 3--6 3--7

bridges(G_tu)

## + 6/6 edges from a572df3:
## [1] 2--4 2--5 1--2 3--6 3--7 1--3

all(sort(E(G_tu)) == sort(bridges(G_tu)))

## [1] TRUE
```

Spanning tree

Definition 98 (Spanning tree)

A **spanning tree** of a connected graph G is a subgraph of G that contains all the vertices of G and is a tree.

A graph may have many spanning trees

Minimal spanning tree

Definition 99 (Value of a spanning tree)

The **value of a spanning tree** T of order p is

$$\sum_{i=1}^{p-1} f(e_i)$$

where f is the function that maps the edge set into $\mathbb{R}$

Definition 100 (Minimal spanning tree)

Let G be an undirected network, and let T be a **minimal spanning tree** of G. Then T is a spanning tree whose the value is minimum

Algorithm to find a minimal spanning tree

Let $G = (V(G), E(G))$ be an undirected network and T be a minimal spanning tree

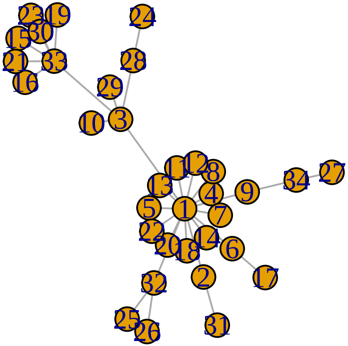
1. Sort the edges of G in increasing order by value
2. $T = (V(G), \emptyset)$
3. For each edge e in sorted order if the endpoints of e are disconnected in T add e to T

Finding a minimal spanning tree of the Karate graph

The function `mst` finds minimal spanning trees, using distances if no edge weights are provided

```
G_mst = mst(G_Z)
```

A minimal spanning tree of the Karate graph



Minimal connector problem

- Model: a graph G such that edges represent all possible connections, and each edge has a positive value which represents its cost; an undirected network G
- Solution: a minimal spanning tree T of G
  - a spanning tree of G is a subgraph of G that contains all the vertices of G and is a tree.
  - the cost of the spanning tree is the sum of values of the edges of T
  - a spanning tree such that no other spanning tree has a smaller cost is a minimmal spanning tree.

Theorem 101 (Characterisation of trees)

$H = (V, U)$ a graph of order $|V| = n > 2$. The following are equivalent and all characterise a tree :

1. *H connected and has no cycles*
2. *H has $n - 1$ arcs and no cycles*
3. *H connected and has exactly $n - 1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is removed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Definition 102 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

Theorem 103

A tree of order $n \geq 2$ has at least two pendant vertices

Theorem 104

A graph $G = (V, U)$ has a partial graph that is a tree $\iff$ G connected

Recall that a partial graph is a graph generated by a subset of the arcs (Definition **??** slide **??**)

Spanning tree

The procedure in the proof of Theorem 104 gives a **spanning tree**

Can also build a spanning tree as follows:

- Consider any arc u_0
- Find arc u_1 that does not form a cycle with u_0
- Find arc u_2 that does not form a cycle with $\{u_0, u_1\}$
- Continue
- When you cannot continue anymore, you have a spanning tree

Theorem 105

G connected graph with ≥ 1 arc. TFAE

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

Theorem 106

G graph with ≥ 1 arc. TFAE

1. *G is a graph without circuits*
2. *Each arc is contained in a cocircuit*

Theorem 107

If G is a strongly connected graph of order n, then G has a cycle basis of $\nu(G)$ circuits

Definition 108 (Node, anti-node, branch)

$G = (V, U)$ strongly connected without loops and > 1 vertex. For each $x \in V$, there is a path from it and a path to it so x has at least 2 incident arcs. Specifically,

- $x \in V$ with > 2 incident arcs is a **node**
- $x \in V$ with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

Definition 109 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

Definition 110 (Contraction)

$G = (V, U)$. The **contraction** of the set $A \subset V$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Theorem 111

G minimally connected, $A \subset V$ generating a strongly connected subgraph of G. Then the contraction of A gives a minimally connected graph

Theorem 112

G a minimally connected graph, G' be the minimally connected graph obtained by the contraction of an elementary circuit of G. Then

$$\nu(G) = \nu(G') + 1$$

Theorem 113

G minimally connected of order $n \geq 2 \implies G$ has ≥ 2 anti-nodes

Theorem 114

$G = (V, U)$. Then the graph C' obtained by contracting each strongly connected component of G contains no circuits

Arborescences

Definition 115 (Root)

Vertex $a \in V$ in $G = (V, U)$ is a **root** if all vertices of G can be reached by paths *starting* from a

Not all graphs have roots

Definition 116 (Quasi-strong connectedness)

G is **quasi-strongly connected** if $\forall x, y \in V$, exists $z \in V$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected $\implies$ quasi-strongly connected (take $z(x, y) = x$); converse not true

Quasi-strongly connected $\implies$ connected

Arborescence

Definition 117 (Arborescence)

An **arborescence** is a tree that has a root

Lemma 118

$G = (V, U)$ has a root $\iff$ G quasi-strongly connected

Theorem 119

H graph of order $n > 1$. TFAE (and all characterise an arborescence)

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with $n-1$ arcs*
3. *H tree having a root a*
4. *$\exists a \in V$ s.t. all other vertices are connected with a by 1 and only 1 path from a*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

7. *H has no cycles and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

Theorem 120

G has a partial graph that is an arborescence $\iff$ G quasi-strongly connected

Theorem 121

$G = (V, E)$ simple connected graph and $x_1 \in V$. It is possible to direct all edges of E so that the resulting graph $G_0 = (V, U)$ has a spanning tree H s.t.

1. *H is an arborescence with root x_1*
2. *The cycles associated with H are circuits*
3. *The only elementary circuits of G_0 are the cycles associated with H*

Counting trees

Proposition 122

X a set with n distinct objects, $n_1, \ldots, n_p$ nonnegative integers s.t. $n_1 + \cdots + n_p = n$. The number of ways to place the n objects into p boxes $X_1, \ldots, X_p$ containing $n_1, \ldots, n_p$ objects respectively is

$$\binom{n}{n_1, \ldots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

Proposition 123 (Multinomial formula)

Let $a_1, \ldots, a_p \in \mathbb{R}$ be p real numbers, then

$$(a_1 + \cdots + a_p)^n = \sum_{n_1, \ldots, n_p \geq 0} \binom{n}{n_1, \ldots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

Theorem 124

Denote $T(n; d_1, \dots, d_n)$ the number of distinct trees H with vertices $x_1, \dots, x_n$ and with degrees $d_H(x_1) = d_1, \dots, d_H(x_n) = d_n$. Then

$$T(n; d_1, \dots, d_n) = \binom{n-2}{d_1 - 1, \dots, d_n - 1}$$

Theorem 125

The number of different trees with vertices $x_1, \dots, x_n$ is n^{n-2}

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..