Graphs – Introduction (theory) – 2

MATH 2740 – Mathematics of Data Science – Lecture 16

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Undirected graphs

Directed graphs

Undirected graphs

Directed graphs

Undirected graphs

- **Connectedness**
- Walks, trails, paths
- Complete, bipartite and other notable graphs
- Planar graphs
- Graph colouring

Connected vertices and graph, components

Definition 31 (Connected vertices)

Two vertices u and v in a graph G are **connected** if $u = v$, or if $u \neq v$ and there exists a path in G that links u and v

(For *path*, see Definition 44 later)

Definition 32 (Connected graph)

A graph is **connected** if every two vertices of G are connected; otherwise, G is **disconnected**

A necessary condition for connectedness

Theorem 33

A connected graph on p vertices has at least $p - 1$ edges

In other words, a connected graph G of order p has $\text{size}(G) \geq p - 1$

Connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a path in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 34 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition V into connected sub-graphs of G called **connected components** (or **components** for short) of G

A connected subgraph H of a graph G is a component of G if H is not contained in any connected subgraph of G having more vertices or edges than H

Vertex deletion & cut vertices

Definition 35 (Vertex deletion)

If $v \in V(G)$ is a vertex of G, the graph $G - v$ is the graph formed from G by removing v and all edges incident with v

Definition 36 (Cut-vertices)

Let G be a connected graph. Then v is a **cut-vertex** G if $G - v$ is disconnected

Edge deletion & bridges

Definition 37 (Edge deletion)

If e is an edge of G, the graph $G - e$ is the graph formed from G by removing e from G

Definition 38 (Bridge)

An edge e in a connected graph G is a **bridge** if $G - e$ is disconnected

Theorem 39

Let G be a connected graph. An edge e of G is a bridge of G $\iff$ e does not lie on any cycle of G

(For *cycle*, see Definition 47 later)

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Walk

Definition 40 (Walk)

A **walk** in a graph $G = (V, E)$ is a non-empty alternating sequence $v_0 e_0 v_1 e_1 v_2 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. This walk begins with v_0 and ends with v_k

Definition 41 (Length of a walk)

The **length** of a walk is equal to the number of edges in the walk

Definition 42 (Closed walk)

If $v_0 = v_k$, the walk is **closed**

Trail and path

Definition 43 (Trail)

If the edges in the walk are all distinct, it defines a **trail** in $G = (V, E)$

Definition 44 (Path)

If the vertices in the walk are all distinct, it defines a **path** in G

The sets of vertices and edges determined by a trail is a subgraph

Distance between two vertices

Definition 45 (Distance between two vertices)

The (**geodesic**) **distance** $d(u, v)$ in $G = (V, E)$ between two vertices u and v is the length of the shortest path linking u and v in G

If no such path exists, we assume $d(u, v) = \infty$

Circuit and cycle

Definition 46 (Circuit)

A trail linking u to v, containing at least 3 edges and in which $u = v$, is a **circuit**

Definition 47 (Cycle)

A circuit which does not repeat any vertices (except the first and the last) is a **cycle** (or **simple circuit**)

Definition 48 (Length of a cycle)

The **length of a cycle** is its number of edges

Eulerian trails and circuits

Definition 49 (Eulerian trail)

A walk in an undirected multigraph M that uses each edge **exactly once** is a **Eulerian trail** of M

Definition 50 (Traversable graph)

If a graph G has a Eulerian trail, then G is a **traversable graph**

Definition 51 (Eulerian circuit)

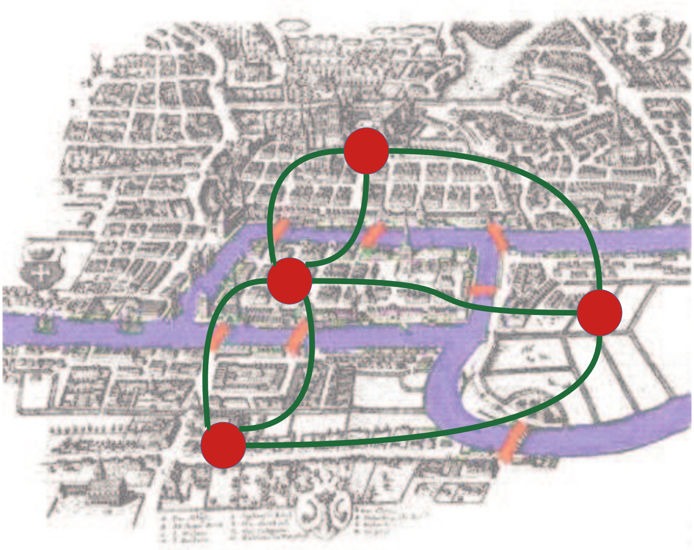
A circuit containing all the vertices and edges of a multigraph M is a **Eulerian circuit** of M

Definition 52 (Eulerian graph)

A graph (resp. multigraph) containing an Eulerian circuit is a **Eulerian graph** (resp. **Eulerian multigraph**)

Remember Euler's bridges of Königsberg?

Cross the 7 bridges in a single walk without recrossing any of them?



Theorem 53

A multigraph M is traversable $\iff$ M is connected and has exactly two odd vertices

Furthermore, any Eulerian trail of M begins at one of the odd vertices and ends at the other odd vertex

Theorem 54

A multigraph M is Eulerian $\iff$ M is connected and every vertex of M is even

Definition 55 (Hamiltonian path)

A path containing all vertices of a graph G is a **Hamiltonian path** of G

Definition 56 (Traceable graph)

If a graph G has an Hamiltonian path, then G is a **traceable graph**

Definition 57 (Hamiltonian cycle)

A cycle containing all vertices of a graph G is a **Hamiltonian cycle** of G

Definition 58 (Hamiltonian graph)

A graph containing a Hamiltonian cycle is a **Hamiltonian graph**

Theorem 59 (Dirac's theorem)

If G is a graph of order $p \geq 3$ such that $deg(v) \geq p/2$ for every vertex v of G, then G is Hamiltonian

Eulerian and Hamiltonian trails and circuits

Eulerian	Hamiltonian
A walk in an undirected multigraph M that uses each edge **exactly once** is a **Eulerian trail** of M	A path containing all vertices of a graph G is a **Hamiltonian path** of G
If a graph G has a Eulerian trail, then G is a **traversable graph**	If a graph G has an Hamiltonian path, then G is a **traceable graph**
A circuit containing all the vertices and edges of a multigraph M is a **Eulerian circuit** of M	A cycle containing all vertices of a graph G is a **Hamiltonian cycle** of G
A graph (resp. multigraph) containing an Eulerian circuit is a **Eulerian graph** (resp. **Eulerian multigraph**)	A graph containing a Hamiltonian cycle is a **Hamiltonian graph**

Undirected graphs

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Definition 60 (Complete graph)

A graph is complete if every two of its vertices are adjacent

Definition 61 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Note that a complete graph of order p is $(p-1)$-regular

Bipartite graph

Definition 62 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets V_1 and V_2, such that no two vertices in the same set are adjacent. This graph may be written $G = (V_1, V_2, E)$

Definition 63 (Complete bipartite graph)

A bipartite graph in which every two vertices from the 2 different partitions are adjacent is called a **complete bipartite graph**

We often denote $K_{p,q}$ a simple, complete bipartite graph with $|V_1| = p$ and $|V_2| = q$

Some specific graphs

Definition 64 (Tree)

Any connected graph that has no cycles is a **tree**

Definition 65 (Cycle C_n)

For $n \geq 3$, the **cycle** C_n is a connected graph of order n that is a cycle on n vertices

Definition 66 (Path P_n)

The **path** P_n is a connected graph that consists of $n \geq 2$ vertices and $n - 1$ edges. Two vertices of P_n have degree 1 and the rest are of degree 2

Definition 67 (Star S_n)

The **star** of order n is the complete bipartite graph $K_{1,n-1}$ (1 vertex of degree $n - 1$ and $n - 1$ vertices of degree 1)

Undirected graphs

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Planar graph

Definition 68 (Planar graph)

A graph is **planar** if it *can be* drawn in the plane with no crossing edges (except at the vertices). Otherwise, it is **nonplanar**

Definition 69 (Plane graph)

A **plane graph** is a graph *that is drawn* in the plane with no crossing edges. (This is only possible if the graph is planar)

(To see the difference, have you ever played this game?)

Let G be a plane graph

- the connected parts of the plane are called **regions**
- vertices and edges that are incident with a region R make up a **boundary** of R

Theorem 70 (Euler's formula)

Let G be a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2$$

Corollary 71

Let G be a plane graph with p vertices, q edges, r regions, and k connected components, then

$$p - q + r = k + 1$$

Two well-known non-planar graphs

$K_{3,3}$ and K_5 are nonplanar

Theorem 72 (Kuratowski Theorem)

A graph G is planar $\iff$ it contains no subgraph isomorphic to K_5 or $K_{3,3}$ or any subdivision of K_5 or $K_{3,3}$

Note: If a graph G is nonplanar and G is a subgraph of G', then G' is also nonplanar

Undirected graphs

- Connectedness
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- **Graph colouring**

Definition 73 (Colouring of a graph G)

A **colouring** of a graph G is an assignment of colours to the vertices of G such that adjacent vertices have different colours

Definition 74 (n-colouring of G)

A **n-colouring** is a colouring of G using n colours

Definition 75 (n-colourable)

G is **n-colourable** if there exists a colouring of G that uses n colours

Definition 76 (Chromatic number)

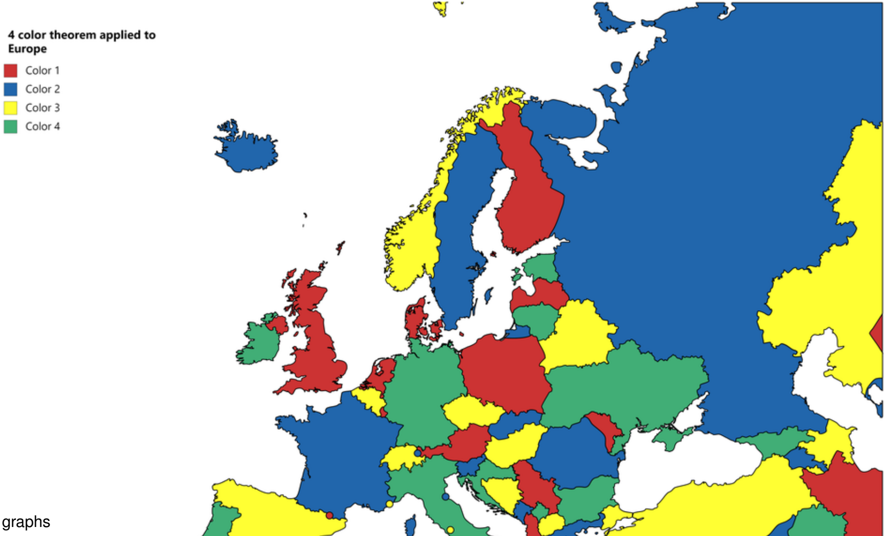
The **chromatic number** $\chi(G)$ of a graph G is the minimal value n for which an n-colouring of G exists

Property 77

- $\chi(G) = 1 \iff$ *G have no edges*
- *If* $G = K_{n,m}$, *then* $\chi(G) = 2$
- *If* $G = K_n$, *then* $\chi(G) = n$
- *For any graph G,*
$$\chi(G) \leq 1 + \Delta(G)$$
where $\Delta(G)$ *is the maximum degree of G*
- *If G is a planar graph, then* $\chi(G) \leq 4$

“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?



“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?

Mathematical representation:

- vertices correspond to the states
- vertices are adjacent $\iff$ the two states are adjacent (sharing an isolated point such as the “Four Corners” does not count)

Mathematical problem

What is the chromatic number of the graph associated to the map?

Welch-Powell algorithm for colouring a graph G

1. Order the vertices of G by decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree)

2. Use one colour to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this colour

3. Start again at the top of the list and repeat the process, painting previously unpainted vertices using a second colour

4. Repeat with additional colours until all vertices have been painted

Undirected graphs

Directed graphs

Definitions

Definition 78 (Digraph)

A directed graph (or **digraph**) is a pair $G = (V, A)$ of sets such that

- V is a set of points: $V = \{v_1, v_2, v_3, .., v_p\}$
- A is a set of ordered pairs of V: $A = \{(v_i, v_j), (v_i, v_k), \dots, (v_n, v_p)\}$ or $A = \{v_i v_j, v_i v_k, \dots, v_n v_p\}$

Definition 79 (Vertex)

The elements of V are the vertices of the digraph G. V or $V(G)$ is the vertex set of the digraph G

Definition 80 (Arc)

The elements of A are the **arcs** (directed edges) of the digraph G. A or $A(G)$ is the arc set of the digraph G

Digraph and binary relation

A (simple) digraph D can be defined in term of a vertex set V and an irreflexive relation R over V

The defining relation R of the digraph G need not be symmetric

Directed network/weighted (di)graph

Definition 81 (Directed network)

A directed network is a digraph together with a function f,

$$f : A \to \mathbb{R},$$

which maps the arc set A into the set of real number. The value of the arc $uv \in A$ is $f(uv)$

Another name is **weighted** (di)graph

Loops & Multiple arcs

Definition 82 (Loop)

A **loop** is an arc with both the same ends; *e.g.* (u, u) is a loop

Definition 83 (Multiple arcs)

Multiple arcs (or multi-arcs) are two or more arcs connecting the same two vertices

Multidigraph/Digraph

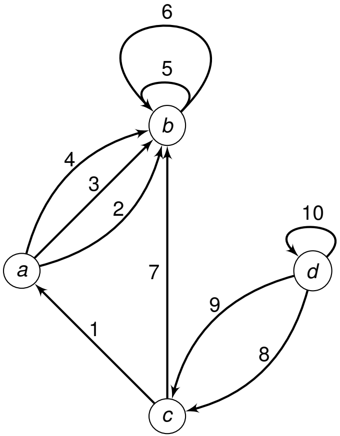
Definition 84 (Multidigraph)

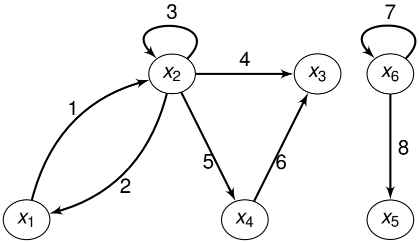
A **multidigraph** is a digraph which allows repetition of arcs or loops

Definition 85 (Digraph)

In a digraph, no more than one arc can join any pair of vertices

Examples





Let $G = (V, A)$ be a digraph

Definition 86 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the **initial endpoint**, and vertex y is the **terminal endpoint**

Definition 87 (Predecessor - Successor)

If $(u, v) \in A(G)$ is an arc of G, then

- u is a **predecessor** of v
- v is a **successor** of u

Definition 88 (Neighbours of a vertex)

Let $x \in V$ be a vertex. The **neighbours** of x is the set $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$, where $\Gamma_G^+(x)$ and $\Gamma_G^-(x)$ are, respectively, the set of successors and predecessors of v

Sources and sinks

Definition 89 (Directed away - Directed towards)

If $a = (u, v) \in A(G)$ is an arc of G, then

- *the arc a is said to be **directed away** from u*
- *the arc a is said to be **directed towards** v*

Definition 90 (Source - Sink)

- *Any vertex which has no arcs directed towards it is a **source***
- *Any vertex which has no arcs directed away from it is a **sink***

Adjacent arcs

Definition 91 (Adjacent arcs)

Two arcs are **adjacent** if they have at least one endpoint in common

Arcs incident to a subset of arcs

Definition 92 (Arc incident out of $X \subset A(G)$)

If the initial endpoint of an arc u belongs to $X \subset A(G)$ and if the terminal endpoint of arc u does not belong to X, then u is said to be **incident out of** X; we write $u \in \omega^+(X)$

Similarly, we define an **arc incident into** X and the set $\omega^-(X)$

Finally, the set of arcs **incident to** X is denoted

$$\omega(X) = \omega^+(X) \cup \omega^-(X)$$

Definition 93 (Subgraph of G generated by $A \subset V$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (V, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 94 (Partial graph of G generated by $V \subset U$)

The graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 95 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G