Graphs – Introduction (theory) – 2

MATH 2740 – Mathematics of Data Science – Lecture 16

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Undirected graphs

Directed graphs

Undirected graphs

Directed graphs

Undirected graphs

Connectedness

Walks, trails, paths

Complete, bipartite and other notable graphs

Planar graphs

Graph colouring

Connected vertices and graph, components

Definition 31 (Connected vertices)

Two vertices u and v in a graph G are **connected** if $u = v$, or if $u \neq v$ and there exists a path in G that links u and v

(For *path*, see Definition 44 later)

Definition 32 (Connected graph)

A graph is **connected** if every two vertices of G are connected; otherwise, G is **disconnected**

A necessary condition for connectedness

Theorem 33

A connected graph on p vertices has at least $p - 1$ edges

In other words, a connected graph G of order p has $\text{size}(G) \geq p - 1$

Connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a path in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 34 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition V into connected sub-graphs of G called **connected components** (or **components** for short) of G

A connected subgraph H of a graph G is a component of G if H is not contained in any connected subgraph of G having more vertices or edges than H

Vertex deletion & cut vertices

Definition 35 (Vertex deletion)

If $v \in V(G)$ is a vertex of G, the graph $G - v$ is the graph formed from G by removing v and all edges incident with v

Definition 36 (Cut-vertices)

Let G be a connected graph. Then v is a **cut-vertex** G if $G - v$ is disconnected

Edge deletion & bridges

Definition 37 (Edge deletion)

If e is an edge of G, the graph $G - e$ is the graph formed from G by removing e from G

Definition 38 (Bridge)

An edge e in a connected graph G is a **bridge** if $G - e$ is disconnected

Theorem 39

Let G be a connected graph. An edge e of G is a bridge of G $\iff$ e does not lie on any cycle of G

(For *cycle*, see Definition 47 later)

Undirected graphs

- Connectedness
- **Walks, trails, paths**
- Complete, bipartite and other notable graphs
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Walk

Definition 40 (Walk)

A **walk** in a graph $G = (V, E)$ is a non-empty alternating sequence $v_0 e_0 v_1 e_1 v_2 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. This walk begins with v_0 and ends with v_k

Definition 41 (Length of a walk)

The **length** of a walk is equal to the number of edges in the walk

Definition 42 (Closed walk)

If $v_0 = v_k$, the walk is **closed**

Trail and path

Definition 43 (Trail)

If the edges in the walk are all distinct, it defines a **trail** in $G = (V, E)$

Definition 44 (Path)

If the vertices in the walk are all distinct, it defines a **path** in G

The sets of vertices and edges determined by a trail is a subgraph

Distance between two vertices

Definition 45 (Distance between two vertices)

The **distance** $d(u, v)$ in $G = (V, E)$ between two vertices u and v is the length of the shortest path linking u and v in G

If no such path exists, we assume $d(u, v) = \infty$

Circuit and cycle

Definition 46 (Circuit)

A trail linking u to v, containing at least 3 edges and in which $u = v$, is a **circuit**

Definition 47 (Cycle)

A circuit which does not repeat any vertices (except the first and the last) is a **cycle** (or **simple circuit**)

Definition 48 (Length of a cycle)

The **length of a cycle** is its number of edges

Definition 49 (Eulerian trail)

A walk in an undirected multigraph M that uses each edge **exactly once** is a **Eulerian trail** of M

Definition 50 (Traversable graph)

If a graph G has a Eulerian trail, then G is a **traversable graph**

Definition 51 (Eulerian circuit)

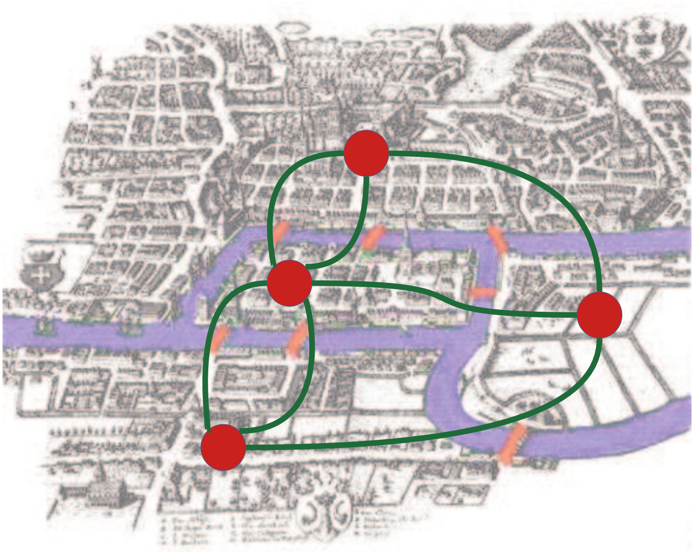
A circuit containing all the vertices and edges of a multigraph M is a **Eulerian circuit** of M

Definition 52 (Eulerian graph)

A graph (resp. multigraph) containing an Eulerian circuit is a **Eulerian graph** (resp. **Eulerian multigraph**)

Remember Euler's bridges of Königsberg?

Cross the 7 bridges in a single walk without recrossing any of them?



Theorem 53

A multigraph M is traversable $\iff$ M is connected and has exactly two odd vertices

Furthermore, any Eulerian trail of M begins at one of the odd vertices and ends at the other odd vertex

Theorem 54

A multigraph M is Eulerian $\iff$ M is connected and every vertex of M is even

Fleury's algorithm to find a Eulerian *trail*

For a connected graph with exactly 2 odd vertices

- Start at one of the odd vertices
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to
- Continue until every edge has been traveled

RESULT: a Eulerian trail

Fleury's algorithm to find a Eulerian *circuit*

For a connected graph with no odd vertices

- Pick any vertex as a starting point
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to
- Continue until you return to your starting point

RESULT: a Eulerian circuit

Definition 55 (Hamiltonian path)

A path containing all vertices of a graph G is a **Hamiltonian path** of G

Definition 56 (Traceable graph)

If a graph G has an Hamiltonian path, then G is a **traceable graph**

Definition 57 (Hamiltonian cycle)

A cycle containing all vertices of a graph G is a **Hamiltonian cycle** of G

Definition 58 (Hamiltonian graph)

A graph containing a Hamiltonian cycle is a **Hamiltonian graph**

Theorem 59 (Dirac's theorem)

If G is a graph of order $p \geq 3$ such that $deg(v) \geq p/2$ for every vertex v of G, then G is Hamiltonian

Theorem 60 (Ore's theorem)

If G is a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v of G,

$$deg(u) + deg(v) \geq p$$

then G is Hamiltonian

Undirected graphs

- Connectedness
- Walks, trails, paths
- **Complete, bipartite and other notable graphs**
- Planar graphs
- Graph colouring

Definition 61 (Complete graph)

A graph is complete if every two of its vertices are adjacent

Definition 62 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Note that a complete graph of order p is $(p-1)$-regular

Bipartite graph

Definition 63 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets V_1 and V_2, such that no two vertices in the same set are adjacent. This graph may be written $G = (V_1, V_2, E)$

Definition 64 (Complete bipartite graph)

A bipartite graph in which every two vertices from the 2 different partitions are adjacent is called a **complete bipartite graph**

We often denote $K_{p,q}$ a simple, complete bipartite graph with $|V_1| = p$ and $|V_2| = q$

Some specific graphs

Definition 65 (Tree)

Any connected graph that has no cycles is a **tree**

Definition 66 (Cycle C_n)

For $n \geq 3$, the **cycle** C_n is a connected graph of order n that is a cycle on n vertices

Definition 67 (Path P_n)

The **path** P_n is a connected graph that consists of $n \geq 2$ vertices and $n - 1$ edges. Two vertices of P_n have degree 1 and the rest are of degree 2

Definition 68 (Star S_n)

The **star** of order n is the complete bipartite graph $K_{1,n-1}$ (1 vertex of degree $n - 1$ and $n - 1$ vertices of degree 1)

Undirected graphs

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Planar graph

Definition 69 (Planar graph)

A graph is **planar** if it *can be* drawn in the plane with no crossing edges (except at the vertices). Otherwise, it is **nonplanar**

Definition 70 (Plane graph)

A **plane graph** is a graph *that is drawn* in the plane with no crossing edges. (This is only possible if the graph is planar)

(To see the difference, have you ever played this game?)

Let G be a plane graph

- the connected parts of the plane are called **regions**
- vertices and edges that are incident with a region R make up a **boundary** of R

Theorem 71 (Euler's formula)

Let G be a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2$$

Corollary 72

Let G be a plane graph with p vertices, q edges, r regions, and k connected components, then

$$p - q + r = k + 1$$

Theorem 73

Let G be a connected planar graph with p vertices and q edges, where $p \geq 3$, then

$$q \leq 3p - 6.$$

(a maximal connected planar graph with p vertices has $q = 3p - 6$ edges)

Corollary 74

If G is a planar graph, then $\delta(G) \leq 5$, where $\delta(G)$ is the minimal degree of G.
(every planar graph contains a vertex of degree less than 6)

Two well-known non-planar graphs

$K_{3,3}$ and K_5 are nonplanar

Theorem 75 (Kuratowski Theorem)

A graph G is planar $\iff$ it contains no subgraph isomorphic to K_5 or $K_{3,3}$ or any subdivision of K_5 or $K_{3,3}$

Note: If a graph G is nonplanar and G is a subgraph of G', then G' is also nonplanar

Undirected graphs

- Connectedness
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Definition 76 (Colouring of a graph G)

A **colouring** of a graph G is an assignment of colours to the vertices of G such that adjacent vertices have different colours

Definition 77 (n-colouring of G)

A **n-colouring** is a colouring of G using n colours

Definition 78 (n-colourable)

G is **n-colourable** if there exists a colouring of G that uses n colours

Definition 79 (Chromatic number)

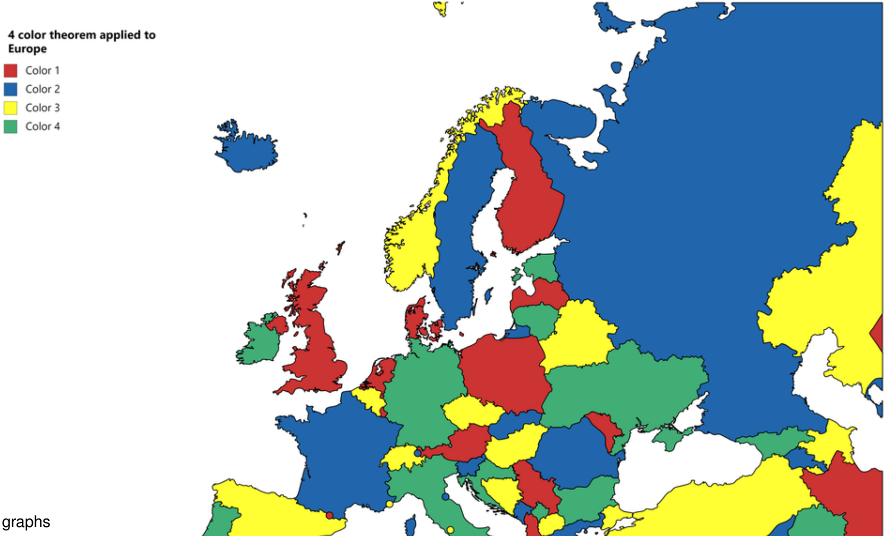
The **chromatic number** $\chi(G)$ of a graph G is the minimal value n for which an n-colouring of G exists

Property 80

- $\chi(G) = 1 \iff$ *G have no edges*
- *If $G = K_{n,m}$, then $\chi(G) = 2$*
- *If $G = K_n$, then $\chi(G) = n$*
- *For any graph G,*
  $$\chi(G) \leq 1 + \Delta(G)$$
  where $\Delta(G)$ is the maximum degree of G
- *If G is a planar graph, then $\chi(G) \leq 4$*

“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?



“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?

Mathematical representation:

- vertices correspond to the states
- vertices are adjacent $\iff$ the two states are adjacent (sharing an isolated point such as the “Four Corners” does not count)

Mathematical problem

What is the chromatic number of the graph associated to the map?

Welch-Powell algorithm for colouring a graph G

1. Order the vertices of G by decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree)

2. Use one colour to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this colour

3. Start again at the top of the list and repeat the process, painting previously unpainted vertices using a second colour

4. Repeat with additional colours until all vertices have been painted

- Undirected graphs
- **Directed graphs**

Definitions

Definition 81 (Digraph)

A directed graph (or **digraph**) is a pair $G = (V, A)$ of sets such that

- V is a set of points: $V = \{v_1, v_2, v_3, .., v_p\}$
- A is a set of ordered pairs of V: $A = \{(v_i, v_j), (v_i, v_k), \dots, (v_n, v_p)\}$ or $A = \{v_i v_j, v_i v_k, \dots, v_n v_p\}$

Definition 82 (Vertex)

The elements of V are the vertices of the digraph G. V or $V(G)$ is the vertex set of the digraph G

Definition 83 (Arc)

The elements of A are the **arcs** (directed edges) of the digraph G. A or $A(G)$ is the arc set of the digraph G

Digraph and binary relation

A (simple) digraph D can be defined in term of a vertex set V and an irreflexive relation R over V

The defining relation R of the digraph G need not be symmetric

Directed network

Definition 84 (Directed network)

A directed network is a digraph together with a function f,

$$f : A \to \mathbb{R},$$

which maps the arc set A into the set of real number. The value of the arc $uv \in A$ is $f(uv)$

Loops & Multiple arcs

Definition 85 (Loop)

A **loop** is an arc with both the same ends; *e.g.* (u, u) is a loop

Definition 86 (Multiple arcs)

Multiple arcs (or multi-arcs) are two or more arcs connecting the same two vertices

Multidigraph/Digraph

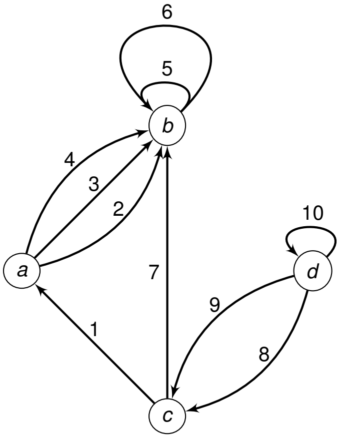
Definition 87 (Multidigraph)

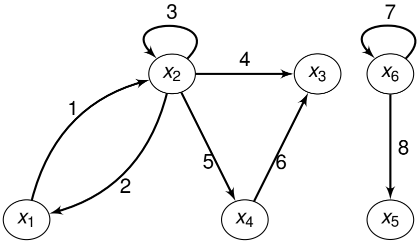
A **multidigraph** is a digraph which allows repetition of arcs or loops

Definition 88 (Digraph)

In a digraph, no more than one arc can join any pair of vertices

Examples





Let $G = (V, A)$ be a digraph

Definition 89 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the **initial endpoint**, and vertex y is the **terminal endpoint**

Definition 90 (Predecessor - Successor)

If $(u, v) \in A(G)$ is an arc of G, then

- u is a **predecessor** of v
- v is a **successor** of u

Definition 91 (Neighbours of a vertex)

Let $x \in V$ be a vertex. The **neighbours** of x is the set $\Gamma(x) = \Gamma^+_G(x) \cup \Gamma^-_G(x)$, where $\Gamma^+_G(x)$ and $\Gamma^-_G(x)$ are, respectively, the set of successors and predecessors of v

Sources and sinks

Definition 92 (Directed away - Directed towards)

If $a = (u, v) \in A(G)$ is an arc of G, then

- the arc a is said to be **directed away** from u
- the arc a is said to be **directed towards** v

Definition 93 (Source - Sink)

- Any vertex which has no arcs directed towards it is a **source**
- Any vertex which has no arcs directed away from it is a **sink**

Adjacent arcs

Definition 94 (Adjacent arcs)

Two arcs are **adjacent** if they have at least one endpoint in common

Arcs incident to a subset of arcs

Definition 95 (Arc incident out of $X \subset A(G)$)

If the initial endpoint of an arc u belongs to $X \subset A(G)$ and if the terminal endpoint of arc u does not belong to X, then u is said to be **incident out of** X; we write $u \in \omega^+(X)$

Similarly, we define an **arc incident into** X and the set $\omega^-(X)$

Finally, the set of arcs **incident to** X is denoted

$$\omega(X) = \omega^+(X) \cup \omega^-(X)$$

Definition 96 (Subgraph of G generated by $A \subset V$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (V, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 97 (Partial graph of G generated by $V \subset U$)

The graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 98 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G