Absorbing Markov chains

MATH 2740 – Mathematics of Data Science – Lecture 25

Julien Arino

julien.arino@umanitoba.ca

Department of Mathematics @ University of Manitoba

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Absorbing Markov chains

Analysing absorbing Markov chains

A few examples

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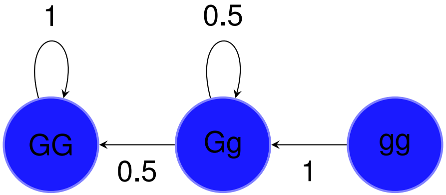
Changing the setting of the genetic experiment

Suppose now the same type of experiment, but mate each new generation with a GG individual instead of a Gg individual



↙	GG	Gg	gg
GG	1	0.5	0
Gg	0	0.5	1
gg	0	0	0

$$P = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- leave gg after 1 iteration and can never return
- when we leave Gg, we can never return
- we can never leave GG when we get there

Absorbing Markov chains

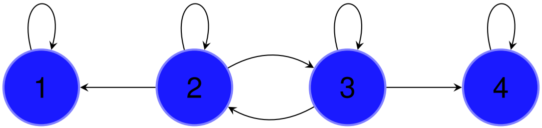
Definition 1 (Absorbing state)

A state S_i in a Markov chain is **absorbing** if whenever it occurs on the t^{th} generation of the experiment, it then occurs on every subsequent step. In other words, S_i is absorbing if $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$

Definition 2 (Absorbing chain)

A Markov chain is **absorbing** if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state. In an absorbing Markov chain, a state that is not absorbing is called **transient**

Suppose we have a chain like the following



1. Does the process eventually reach an absorbing state?
2. What is the average number of steps spent in a transient state, if starting in a transient state?
3. What is the average number of steps before entering an absorbing state?
4. What is the probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?

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A few examples

The answer to the first question ("Does the process eventually reach an absorbing state?") is given by the following result

Theorem 3

In an absorbing Markov chain, the probability of reaching an absorbing state is 1

To answer the other questions, write the transition matrix in **standard** form

For an absorbing chain with k absorbing states and $r - k$ transient states, write transition matrix as

$$P = \begin{pmatrix} \mathbb{I}_k & R \\ \mathbf{0} & Q \end{pmatrix}$$

with following meaning

	Absorbing states	Transient states
Absorbing states	$\mathbb{I}_k$	R
Transient states	$\mathbf{0}$	Q

with $\mathbb{I}_k$ the $k \times k$ identity matrix, $\mathbf{0}$ an $(r-k) \times k$ matrix of zeros, R an $k \times (r-k)$ matrix and Q an $(r-k) \times (r-k)$ matrix. The matrix $\mathbb{I}_{r-k} - Q$ is invertible. Let

- $N = (\mathbb{I}_{r-k} - Q)^{-1}$ the **fundamental matrix** of the MC
- T_i sum of the entries on row i of N
- $B = RN$

Answers to our remaining questions:

2. N_{ij} average number of times the process is in the jth transient state if it starts in the ith transient state

3. T_i average number of steps before the process enters an absorbing state if it starts in the ith transient state

4. B_{ij} probability of eventually entering the ith absorbing state if the process starts in the jth transient state

Back to the genetic example

The matrix is already in standard form

$$P = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_1 & R \\ \mathbf{0} & Q \end{pmatrix}$$

with $\mathbb{I}_1 = 1$, $\mathbf{0} = (0 \;\; 0)^T$ and

$$R = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix}$$

We have

$$\mathbb{I}_2 - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix}$$

so

$$N = (\mathbb{I}_2 - Q)^{-1} = 2 \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

We have

$$N = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

So

$$T = N\mathbb{1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$B = RN = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

2. N_{ij} average number of times the process is in the jth transient state if it starts in the ith transient state

$$N = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

3. T_i average number of steps before the process enters an absorbing state if it starts in the ith transient state

$$T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

4. B_{ij} probability of eventually entering the ith absorbing state if the process starts in the jth transient state

$$B = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Make an absorbing random walk

```r
# Total population
nb_states = 7
proba_left = 0.5  # Probability of moving left
proba_right = 0.5 # Probability of moving right
proba_stay = 1-(proba_left+proba_right)
# Make the transition matrix
T = mat.or.vec(nr = nb_states, nc = nb_states)
for (row in 2:(nb_states-1)) {
  T[row,(row-1)] = proba_left
  T[row,(row+1)] = proba_right
  T[row, row] = proba_stay
}
T[1,1] = 1 # First state is absorbing
T[nb_states, nb_states] = 1 # Last too
```

Create MC object

```r
mcRW <- new("markovchain",
            states = sprintf("S_%d", 1:nb_states),
            transitionMatrix = T,
            name = "RW_abs")
```

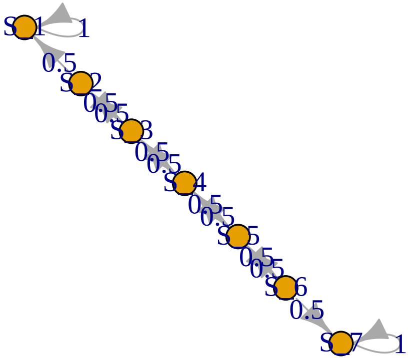
Show some info about the chain

```r
summary(mcRW)

## RW_abs  Markov chain that is composed by:
## Closed classes:
## S_1
## S_7
## Recurrent classes:
## {S_1},{S_7}
## Transient classes:
## {S_2,S_3,S_4,S_5,S_6}
## The Markov chain is not irreducible
## The absorbing states are: S_1 S_7
```

Plotting the chain

```r
plot(mcRW)
```

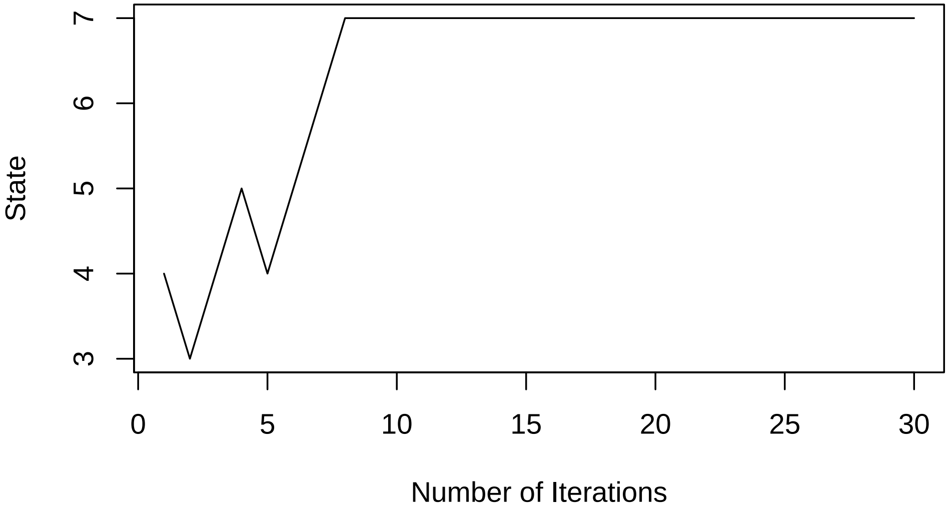


S1
1
0.5
S2
0.5
0.5
S3
0.5
0.5
S4
0.5
0.5
S5
0.5
0.5
S6
0.5
S7
1

Showing a realisation

```
# Library: DTMCPack
IC = rep(0, nb_states)
IC[4] = 1
sol = DTMC(T, IC, 30, trace=TRUE)

## NULL
```



State
3
4
5
6
7
0
5
10
15
20
25
30
Number of Iterations

Additional information about absorbing chains

```
canonicForm(mcRW)
```

```
meanAbsorptionTime(mcRW)
```

```
absorptionProbabilities(mcRW)
```

```
hittingProbabilities(mcRW)
```

Canonical form

```
canonicForm(mcRW)

## RW_abs
##  A  7 - dimensional discrete Markov Chain defined by the following states
##  S_1, S_7, S_2, S_3, S_4, S_5, S_6
##  The transition matrix  (by rows)  is defined as follows:
##     S_1 S_7 S_2 S_3 S_4 S_5 S_6
## S_1 1.0 0.0 0.0 0.0 0.0 0.0 0.0
## S_7 0.0 1.0 0.0 0.0 0.0 0.0 0.0
## S_2 0.5 0.0 0.0 0.5 0.0 0.0 0.0
## S_3 0.0 0.0 0.5 0.0 0.5 0.0 0.0
## S_4 0.0 0.0 0.0 0.5 0.0 0.5 0.0
## S_5 0.0 0.0 0.0 0.0 0.5 0.0 0.5
## S_6 0.0 0.5 0.0 0.0 0.0 0.5 0.0
```

Mean absorption time

```
meanAbsorptionTime(mcRW)

## S_2 S_3 S_4 S_5 S_6
##   5   8   9   8   5
```

Absorption probabilities

```
absorptionProbabilities(mcRW)

##            S_1       S_7
## S_2 0.8333333 0.1666667
## S_3 0.6666667 0.3333333
## S_4 0.5000000 0.5000000
## S_5 0.3333333 0.6666667
## S_6 0.1666667 0.8333333
```

Hitting probabilities

```r
hittingProbabilities(mcRW)
```

```
##           S_1 S_2   S_3       S_4   S_5 S_6       S_7
## S_1 1.0000000 0.0 0.000 0.0000000 0.000 0.0 0.0000000
## S_2 0.8333333 0.4 0.500 0.3333333 0.250 0.2 0.1666667
## S_3 0.6666667 0.8 0.625 0.6666667 0.500 0.4 0.3333333
## S_4 0.5000000 0.6 0.750 0.6666667 0.750 0.6 0.5000000
## S_5 0.3333333 0.4 0.500 0.6666667 0.625 0.8 0.6666667
## S_6 0.1666667 0.2 0.250 0.3333333 0.500 0.4 0.8333333
## S_7 0.0000000 0.0 0.000 0.0000000 0.000 0.0 1.0000000
```

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