

Review of first-year linear algebra

In MATH 2740, we rely on notions you acquired in MATH
1210/1220/1300

So let us (briefly) go over material in these courses

I also add (for some of you) a few things that will be handy and establish some terminology that we use throughout the course

OUTLINE

Sets and elements

Definition 1 (Set)

A **set** X is a collection of **elements**

We write $x \in X$ or $x \notin X$ to indicate that the element x belongs to the set X or does not belong to the set X , respectively

Definition 2 (Subset)

Let X be a set. The set S is a **subset** of X , which is denoted $S \subset X$, if all its elements belong to X

Not used here but worth noting: we say S is a **proper subset** of X and write $S \subsetneq X$, if it is a subset of X and not equal to X

Quantifiers

A shorthand notation for “for all elements x belonging to X ” is

$$\forall x \in X$$

For example, if $X = \mathbb{R}$, the *field* of real numbers, then $\forall x \in \mathbb{R}$ means “for all real numbers x ”

A shorthand notation for “there exists an element x in the set X ” is $\exists x \in X$

\forall and \exists are **quantifiers**

Intersection and union of sets

Let X and Y be two sets

Definition 3 (Intersection)

The intersection of X and Y , $X \cap Y$, is the set of elements that belong to X **and** to Y ,

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

Definition 4 (Union)

The union of X and Y , $X \cup Y$, is the set of elements that belong to X **or** to Y ,

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}$$

In mathematics, **or**=**and/or** in common parlance. We also have an **exclusive or** (**xor**)

A teeny bit of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. “The sky is blue” is also a proposition

Let A be a proposition. We generally write

A

to mean that A is true, and

not A

to mean that A is false. **not** A is the **contraposition** of A (or **not** A is the contrapositive of A)

A teeny bit of logic (cont.)

Let A, B be propositions. Then

- ▶ $A \Rightarrow B$ (read A implies B) means that whenever A is true, then so is B
- ▶ $A \Leftrightarrow B$, also denoted A if and only if B (A iff B for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$
We also say that A and B are **equivalent**

Let A and B be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\text{not } B \Rightarrow \text{not } A)$$

Necessary or sufficient conditions

Suppose we want to establish whether a given statement P is true, depending on the truth value of a statement H . Then we say that

- ▶ H is a **necessary condition** if $P \Rightarrow H$
(It is necessary that H be true for P to be true; so whenever P is true, so is H)
- ▶ H is a **sufficient condition** if $H \Rightarrow P$
(It suffices for H to be true for P to also be true)
- ▶ H is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., H and P are equivalent

Playing with quantifiers

For the quantifiers \forall (for all) and \exists (there exists),

\exists is the contrapositive of \forall

Therefore, for example, the contrapositive of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

Complex numbers

Definition 5 (Complex numbers)

A **complex number** is an ordered pair (a, b) , where $a, b \in \mathbb{R}$.
Usually written $a + ib$ or $a + bi$, where $i^2 = -1$ (i.e., $i = \sqrt{-1}$)
The set of all complex numbers is denoted \mathbb{C} ,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

Definition 6 (Addition and multiplication on \mathbb{C})

Letting $a + ib$ and $c + id \in \mathbb{C}$, addition on \mathbb{C} is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on \mathbb{C} is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter is easy to obtain using regular multiplication and $i^2 = -1$

Properties

$$\forall \alpha, \beta, \gamma \in \mathbb{C},$$

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha \quad [\mathbf{commutativity}]$$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \text{ and } (\alpha\beta)\gamma = \alpha(\beta\gamma) \quad [\mathbf{associativity}]$$

$$\gamma + 0 = \gamma \text{ and } \gamma 1 = \gamma \quad [\mathbf{identities}]$$

$$\forall \alpha \in \mathbb{C}, \exists \beta \in \mathbb{C} \text{ unique s.t. } \alpha + \beta = 0 \quad [\mathbf{additive\ inverse}]$$

$$\forall \alpha \neq 0 \in \mathbb{C}, \exists \beta \in \mathbb{C} \text{ unique s.t. } \alpha\beta = 1 \quad [\mathbf{multiplicative\ inverse}]$$

$$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta \quad [\mathbf{distributivity}]$$

Additive & multiplicative inverse, subtraction, division

Definition 7

Let $\alpha, \beta \in \mathbb{C}$

- ▶ $-\alpha$ is the **additive inverse** of α , i.e., the unique number in \mathbb{C} s.t. $\alpha + (-\alpha) = 0$
- ▶ **Subtraction** on \mathbb{C} :

$$\beta - \alpha = \beta + (-\alpha)$$

- ▶ For $\alpha \neq 0$, $1/\alpha$ is the **multiplicative inverse** of α , i.e., the unique number in \mathbb{C} s.t.

$$\alpha(1/\alpha) = 1$$

- ▶ **Division** on \mathbb{C} :

$$\beta/\alpha = \beta(1/\alpha)$$

Definition 8 (Real and imaginary parts)

Let $z = a + ib$. Then $\operatorname{Re} z = a$ is **real part** and $\operatorname{Im} z = b$ is **imaginary part** of z

If ambiguous, write $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$

Definition 9 (Conjugate and Modulus)

Let $z = a + ib \in \mathbb{C}$. Then

- ▶ **Complex conjugate** of z is

$$\bar{z} = a - ib$$

- ▶ **Modulus (or absolute value)** of z is

$$|z| = \sqrt{a^2 + b^2} \geq 0$$

Properties of complex numbers

Let $w, z \in \mathbb{C}$, then

- ▶ $z + \bar{z} = 2\operatorname{Re} z$
- ▶ $z - \bar{z} = 2i\operatorname{Im} z$
- ▶ $z\bar{z} = |z|^2$
- ▶ $\overline{w+z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$
- ▶ $\overline{\bar{z}} = z$
- ▶ $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$
- ▶ $|\bar{z}| = |z|$
- ▶ $|wz| = |w| |z|$
- ▶ $|w+z| \leq |w| + |z|$ **[triangle inequality]**

Solving quadratic equations

Consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. Letting

$$\Delta = a_1^2 - 4a_0a_2$$

you know that if $\Delta > 0$, then

$$P(x) = 0$$

has two distinct *real* solutions,

$$x_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2} \quad \text{and} \quad x_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_2}$$

if $\Delta = 0$, then there is a (multiplicity 2) unique *real* solution

$$x_1 = \frac{-a_1}{2a_2}$$

while if $\Delta < 0$, there is no solution

Solving quadratic equations with complex numbers

Consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. If instead of seeking $x \in \mathbb{R}$, we seek $x \in \mathbb{C}$, then the situation is the same, except when $\Delta < 0$

In the latter case, note that

$$\sqrt{\Delta} = \sqrt{(-1)(-\Delta)} = \sqrt{-1}\sqrt{-\Delta} = i\sqrt{-\Delta}$$

Since $\Delta < 0$, $-\Delta > 0$ and the square root is the usual one

Solving quadratic equations with complex numbers

To summarize, consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. Letting

$$\Delta = a_1^2 - 4a_0a_2$$

Then

$$P(x) = 0$$

has two solutions,

$$x_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2a_2}$$

where, if $\Delta < 0$, $x_1, x_2 \in \mathbb{C}$ and take the form

$$x_{1,2} = \frac{-a_1 \pm i\sqrt{-\Delta}}{2a_2}$$

Why this matters

Recall (we will come back to this later) that to find the *eigenvalues* of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we seek λ solutions to $\det(A - \lambda\mathbb{I}) = 0$, i.e., λ solutions to

$$|A - \lambda\mathbb{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

i.e., λ solutions to

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

Why this matters (cont.)

Let

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

From previous discussion, letting

$$\begin{aligned}\Delta &= (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \\ &= a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4a_{11}a_{22} + 4a_{12}a_{21} \\ &= a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} \\ &= (a_{11} - a_{22})^2 + 4a_{12}a_{21}\end{aligned}$$

we have two (potentially equal) solutions to $P(\lambda) = 0$

$$x_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{\Delta}}{2}$$

that are complex if $\Delta < 0$

Example: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Vectors

A **vector** \mathbf{v} is an ordered n -tuple of real or complex numbers

Denote $\mathbb{F} = \mathbb{R}$ or \mathbb{C} (real or complex numbers). For $v_1, \dots, v_n \in \mathbb{F}$,

$$\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{F}^n$$

is a vector. v_1, \dots, v_n are the **components** of \mathbf{v}

If unambiguous, we write v . Otherwise, \mathbf{v} or \vec{v}

Vector space

Definition 10 (Vector space)

A **vector space** over \mathbb{F} is a set V together with two binary operations, **vector addition**, denoted $+$, and **scalar multiplication**, that satisfy the relations:

1. $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2. $\forall \mathbf{v}, \mathbf{w} \in V, \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
3. $\exists \mathbf{0} \in V$, the zero vector, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
4. $\forall \mathbf{v} \in V$, there exists an element $\mathbf{w} \in V$, the additive inverse of \mathbf{v} , such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$
5. $\forall \alpha \in \mathbb{R}$ and $\forall \mathbf{v}, \mathbf{w} \in V, \alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
6. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \mathbf{v} \in V, (\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \mathbf{v} \in V, \alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
8. $\forall \mathbf{v} \in V, 1\mathbf{v} = \mathbf{v}$

Norms

Definition 11 (Norm)

Let V be a vector space over \mathbb{F} , and $\mathbf{v} \in V$ be a vector. The **norm** of \mathbf{v} , denoted $\|\mathbf{v}\|$, is a function from V to \mathbb{R}_+ that has the following properties:

1. For all $\mathbf{v} \in V$, $\|\mathbf{v}\| \geq 0$ with $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$
2. For all $\alpha \in \mathbb{F}$ and all $\mathbf{v} \in V$, $\|\alpha\mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
3. For all $\mathbf{u}, \mathbf{v} \in V$, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Let V be a vector space (for example, \mathbb{R}^2 or \mathbb{R}^3)

The **zero element** (or **zero vector**) is the vector $\mathbf{0} = (0, \dots, 0)$

The **additive inverse** of $\mathbf{v} = (v_1, \dots, v_n)$ is $-\mathbf{v} = (-v_1, \dots, -v_n)$

For $\mathbf{v} = (v_1, \dots, v_n) \in V$, the length (or Euclidean norm) of \mathbf{v} is the **scalar**

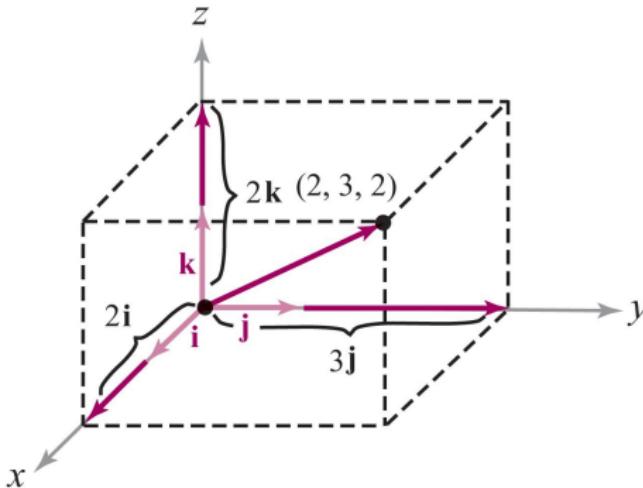
$$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

To **normalize** the vector \mathbf{v} consists in considering $\tilde{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$, i.e., the vector in the same direction as \mathbf{v} that has unit length

Standard basis vectors

Vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ are the **standard basis vectors** of \mathbb{R}^3 . A vector $\mathbf{v} = (v_1, v_2, v_3)$ can then be written

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$



For $V(\mathbb{R}^n)$, the standard basis vectors are usually denoted $\mathbf{e}_1, \dots, \mathbf{e}_n$, with

$$\mathbf{e}_k = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k+1})$$

Dot product

Definition 12 (Dot product)

Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$. The **dot product** of \mathbf{a} and \mathbf{b} is the **scalar**

$$\mathbf{a} \bullet \mathbf{b} = \sum_{i=1}^n a_i b_i = a_1 b_1 + \cdots + a_n b_n$$

The dot product is a special case of **inner product**

Properties of the dot product

Theorem 13

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

- ▶ $\mathbf{a} \bullet \mathbf{a} = \|\mathbf{a}\|^2$ (so $\mathbf{a} \bullet \mathbf{a} \geq 0$, with $\mathbf{a} \bullet \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$)
- ▶ $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ (\bullet is commutative)
- ▶ $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ (\bullet distributive over $+$)
- ▶ $(\alpha \mathbf{a}) \bullet \mathbf{b} = \alpha(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (\alpha \mathbf{b})$
- ▶ $\mathbf{0} \bullet \mathbf{a} = 0$

Some results stemming from the dot product

Theorem 14

If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

Corollary 15 (Cauchy-Schwarz inequality)

For any two vectors \mathbf{a} and \mathbf{b} , we have

$$|\mathbf{a} \bullet \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

with equality if and only if \mathbf{a} is a scalar multiple of \mathbf{b} , or one of them is $\mathbf{0}$.

Theorem 16

\mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \bullet \mathbf{b} = 0$.

Scalar and vector projections

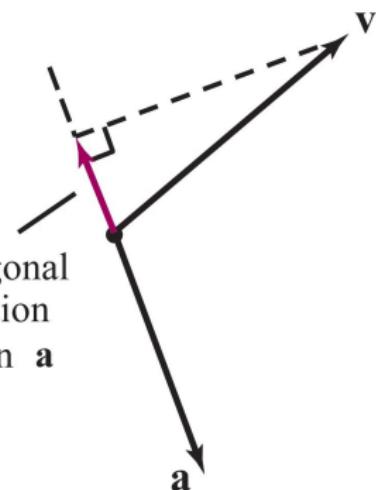
Scalar projection of \mathbf{v} onto \mathbf{a} (or component of \mathbf{v} along \mathbf{a}):

$$\text{comp}_{\mathbf{a}} \mathbf{v} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|}$$

Vector (or orthogonal) projection of \mathbf{v} onto \mathbf{a} :

$$\text{proj}_{\mathbf{a}} \mathbf{v} = \left(\frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \bullet \mathbf{v}}{\|\mathbf{a}\|^2} \mathbf{a}$$

Orthogonal
projection
of \mathbf{v} on \mathbf{a}



Linear systems

Definition 17 (Linear system)

A **linear system** of m equations in n unknowns takes the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned} \tag{1}$$

The a_{ij} , x_j and b_j could be in \mathbb{R} or \mathbb{C} , although here we typically assume they are in \mathbb{R}

The aim is to find x_1, x_2, \dots, x_n that satisfy all equations simultaneously

Theorem 18 (Nature of solutions to a linear system)

A linear system can have

- ▶ *no solution*
- ▶ *a unique solution*
- ▶ *infinitely many solutions*

Operations on linear systems

You learned to manipulate linear systems using

- ▶ Gaussian elimination
- ▶ Gauss-Jordan elimination

with the aim to put the system in **row echelon form** (REF) or
reduced row echelon form (RREF)

Matrices and linear systems

Writing

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where A is an $m \times n$ **matrix**, \mathbf{x} and \mathbf{b} are n (column) **vectors** (or $n \times 1$ matrices), then the linear system in the previous slide takes the form

$$A\mathbf{x} = \mathbf{b}$$

Notation for vectors

We usually assume vectors are column vectors and thus write, e.g.,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \dots, x_n)^T$$

Here, T is the **transpose operator** (more on this soon)

Consider the system

$$Ax = \mathbf{b}$$

If $\mathbf{b} = \mathbf{0}$, the system is **homogeneous** and always has the solution $\mathbf{x} = \mathbf{0}$ and so the “no solution” option in Theorem ?? goes away

Definition 19 (Matrix)

An m -by- n or $m \times n$ matrix is a rectangular array of elements of \mathbb{R} or \mathbb{C} with m rows and n columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We always list indices as “row,column”

We denote $\mathcal{M}_{mn}(\mathbb{F})$ or \mathbb{F}^{mn} the set of $m \times n$ matrices with entries in $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Often, we omit \mathbb{F} in \mathcal{M}_{mn} if the nature of \mathbb{F} is not important

When $m = n$, we usually write \mathcal{M}_n

Basic matrix arithmetic

Let $A \in \mathcal{M}_{mn}$, $B \in \mathcal{M}_{mn}$ be matrices (of the same size) and $c \in \mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$ be a scalar

- ▶ **Scalar multiplication**

$$cA = [ca_{ij}]$$

- ▶ **Addition**

$$A + B = [a_{ij} + b_{ij}]$$

- ▶ **Subtraction** (addition of $-B = (-1)B$ to A)

$$A - B = A + (-1)B = [a_{ij} + (-1)b_{ij}] = [a_{ij} - b_{ij}]$$

- ▶ **Transposition** of A gives a matrix $A^T = \mathcal{M}_{nm}$ with

$$A^T = [a_{ji}], \quad j = 1, \dots, n, \quad i = 1, \dots, m$$

Matrix multiplication

The (matrix) **product** of A and B , AB , requires the “inner dimensions” to match, i.e., the number of columns in A must equal the number of rows in B

Suppose that is the case, i.e., let $A \in \mathcal{M}_{mn}$, $B \in \mathcal{M}_{np}$. Then the i,j entry in $C := AB$ takes the form

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Recall that the matrix product is not commutative, i.e., in general, $AB \neq BA$ (when both those products are defined, i.e., when $A, B \in \mathcal{M}_n$)

Special matrices

Definition 20 (Zero and identity matrices)

The **zero** matrix is the matrix 0_{mn} whose entries are all zero. The **identity** matrix is a square $n \times n$ matrix \mathbb{I}_n with all entries on the main diagonal equal to one and all off diagonal entries equal to zero

Definition 21 (Symmetric matrix)

A square matrix $A \in \mathcal{M}_n$ is **symmetric** if $\forall i, j = 1, \dots, n$, $a_{ij} = a_{ji}$. In other words, $A \in \mathcal{M}_n$ is symmetric if $A = A^T$

Properties of symmetric matrices

Theorem 22

1. If $A \in \mathcal{M}_n$, then $A + A^T$ is symmetric
2. If $A \in \mathcal{M}_{mn}$, then $AA^T \in \mathcal{M}_m$ and $A^TA \in \mathcal{M}_n$ are symmetric

X symmetric $\iff X = X^T$, so use X = the matrix whose symmetric property you want to check

1. True if $A + A^T = (A + A^T)^T$. We have

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

2. AA^T symmetric if $AA^T = (AA^T)^T$. We have

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

A^TA works similarly

Determinants

Definition 23 (Determinant)

Let $A \in \mathcal{M}_n$ with $n \geq 2$. The **determinant** of A is the scalar

$$\det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij}$$

where $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is the (i, j) -**cofactor** of A and A_{ij} is the submatrix of A from which the i th row and j th column have been removed

This is a cofactor expansion along the i th row

This is a recursive formula: it gives result in terms of $n - 1$ \mathcal{M}_{n-1} matrices, to which it must in turn be applied, all the way down to

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Two special matrices and their determinants

Definition 24

$A \in \mathcal{M}_n$ is **upper triangular** if $a_{ij} = 0$ when $i > j$, **lower triangular** if $a_{ij} = 0$ when $j > i$, **triangular** if it is either upper or lower triangular and **diagonal** if it is both upper and lower triangular

When A diagonal, we often write $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$

Theorem 25

Let $A \in \mathcal{M}_n$ be triangular or diagonal. Then

$$\det(A) = \prod_{i=1}^n a_{ii} = a_{11}a_{22} \cdots a_{nn}$$

Inversion/Singularity

Definition 26 (Matrix inverse)

$A \in \mathcal{M}_n$ is **invertible** (or **nonsingular**) if $\exists A^{-1} \in \mathcal{M}_n$ s.t.

$$AA^{-1} = A^{-1}A = \mathbb{I}$$

A^{-1} is the **inverse** of A . If A^{-1} does not exist, A is **singular**

Theorem 27

Let $A \in \mathcal{M}_n$, $\mathbf{x}, \mathbf{b} \in \mathbb{F}^n$. Then

- ▶ A invertible $\iff \det(A) \neq 0$
- ▶ If A invertible, A^{-1} is unique
- ▶ If A invertible, then $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

Revisiting matrix arithmetic

With addition, subtraction, scalar multiplication, multiplication, transposition and inversion, you can perform arithmetic on matrices essentially as on scalar, if you bear in mind a few rules

- ▶ The sizes have to be compatible
- ▶ The order is important since matrix multiplication is not commutative
- ▶ Transposition and inversion change the order of products:

$$(AB)^T = B^T A^T \text{ and } (AB)^{-1} = B^{-1} A^{-1}$$

Eigenvalues / Eigenvectors / Eigenpairs

Definition 28

Let $A \in \mathcal{M}_n$. A vector $\mathbf{x} \in \mathbb{F}^n$ such that $\mathbf{x} \neq \mathbf{0}$ is an **eigenvector** of A if $\exists \lambda \in \mathbb{F}$ called an **eigenvalue**, s.t.

$$A\mathbf{x} = \lambda\mathbf{x}$$

A couple (λ, \mathbf{x}) with $\mathbf{x} \neq \mathbf{0}$ s.t. $A\mathbf{x} = \lambda\mathbf{x}$ is an **eigenpair**

If (λ, \mathbf{x}) eigenpair, then for $c \neq 0$, $(\lambda, c\mathbf{x})$ also eigenpair since $A(c\mathbf{x}) = cA\mathbf{x} = c\lambda\mathbf{x}$ and dividing both sides by c .

Similarity

Definition 29 (Similarity)

$A, B \in \mathcal{M}_n$ are **similar** ($A \sim B$) if $\exists P \in \mathcal{M}_n$ invertible s.t.

$$P^{-1}AP = B$$

Theorem 30 (\sim is an equivalence relation)

$A, B, C \in \mathcal{M}_n$, then

- ▶ $A \sim A$ (\sim **reflexive**)
- ▶ $A \sim B \implies B \sim A$ (\sim **symmetric**)
- ▶ $A \sim B$ and $B \sim C \implies A \sim C$ (\sim **transitive**)

Similarity (cont.)

Theorem 31

$A, B \in \mathcal{M}_n$ with $A \sim B$. Then

- ▶ $\det A = \det B$
- ▶ A invertible $\iff B$ invertible
- ▶ A and B have the same eigenvalues

Diagonalisation

Definition 32 (Diagonalsability)

$A \in \mathcal{M}_n$ is **diagonalsable** if $\exists D \in \mathcal{M}_n$ diagonal s.t. $A \sim D$

In other words, $A \in \mathcal{M}_n$ is diagonalsable if there exists a diagonal matrix $D \in \mathcal{M}_n$ and a nonsingular matrix $P \in \mathcal{M}_n$ s.t.

$$P^{-1}AP = D$$

Could of course write $PAP^{-1} = D$ since P invertible, but $P^{-1}AP$ makes more sense for computations

Theorem 33

$A \in M_n$ diagonalisable $\iff A$ has n linearly independent eigenvectors

Corollary 34 (Sufficient condition for diagonalisability)

$A \in M_n$ has all its eigenvalues distinct $\implies A$ diagonalisable

For $P^{-1}AP = D$: in P , put the linearly independent eigenvectors as columns and in D , the corresponding eigenvalues

Linear combination and span

Definition 35 (Linear combination)

Let V be a vector space. A **linear combination** of a set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors in V is a *vector*

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

where $c_1, \dots, c_k \in \mathbb{F}$

Definition 36 (Span)

The set of all linear combinations of a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the **span** of $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$,

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \dots, c_k \in \mathbb{F}\}$$

Finite/infinite-dimensional vector spaces

Theorem 37

The span of a set of vectors in V is the smallest subspace of V containing all the vectors in the set

Definition 38 (Set of vectors spanning a space)

If $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = V$, we say $\mathbf{v}_1, \dots, \mathbf{v}_k$ **spans** V

Definition 39 (Dimension of a vector space)

A vector space V is **finite-dimensional** if some set of vectors in it spans V . A vector space V is **infinite-dimensional** if it is not finite-dimensional

Linear (in)dependence

Definition 40 (Linear independence/Linear dependence)

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors in a vector space V is **linearly independent** if

$$(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0) \Leftrightarrow (c_1 = \cdots = c_k = 0),$$

where $c_1, \dots, c_k \in \mathbb{F}$. A set of vectors is **linearly dependent** if it is not linearly independent.

If linearly dependent, assume w.l.o.g. that $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \cdots - \frac{c_k}{c_1}\mathbf{v}_k$$

i.e., \mathbf{v}_1 is a linear combination of the other vectors in the set

Theorem 41

*Let V be a finite-dimensional vector space. Then the **cardinal** (number of elements) of every linearly independent set of vectors is less than or equal to the number of elements in every spanning set of vectors*

E.g., in \mathbb{R}^3 , a set with 4 or more vectors is automatically linearly dependent

Basis

Definition 42 (Basis)

Let V be a vector space. A **basis** of V is a set of vectors in V that is both linearly independent and spanning

Theorem 43 (Criterion for a basis)

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of vectors in a vector space V is a basis of V
 $\iff \forall \mathbf{v} \in V, \mathbf{v}$ can be written uniquely in the form

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k,$$

where $c_1, \dots, c_k \in \mathbb{F}$

Plus/Minus Theorem

Theorem 44 (Plus/Minus Theorem)

S a nonempty set of vectors in vector space V

- ▶ If S is linearly independent and $V \ni \mathbf{v} \notin \text{span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent
- ▶ If $\mathbf{v} \in S$ is linear combination of other vectors in S , then $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$

More on bases

Theorem 45 (Basis of finite-dimensional vector space)

Every finite-dimensional vector space has a basis

Theorem 46

Any two bases of a finite-dimensional vector space have the same number of vectors

Definition 47 (Dimension)

The **dimension** $\dim V$ of a finite-dimensional vector space V is the number of vectors in any basis of the vector space

Theorem 48 (Dimension of a subspace)

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V . Then $\dim U \leq \dim V$

Constructing bases

Theorem 49

Let V be a finite-dimensional vector space. Then every linearly independent set of vectors in V with $\dim V$ elements is a basis of V

Theorem 50

Let V be a finite-dimensional vector space. Then every spanning set of vectors in V with $\dim V$ elements is a basis of V

To finish: the “famous” “growing result”

Theorem 51

Let $A \in \mathcal{M}_n$. The following statements are equivalent (TFAE)

1. The matrix A is invertible
2. $\forall \mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$)
3. The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$
4. $RREF(A) = \mathbb{I}_n$
5. The matrix A is equal to a product of elementary matrices
6. $\forall \mathbf{b} \in \mathbb{F}^n$, $A\mathbf{x} = \mathbf{b}$ has a solution
7. There is a matrix $B \in \mathcal{M}_n$ such that $AB = \mathbb{I}_n$
8. There is an invertible matrix $B \in \mathcal{M}_n$ such that $AB = \mathbb{I}_n$
9. $\det(A) \neq 0$
10. 0 is not an eigenvalue of A