Linear ODEs

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Definition (Linear ODE)

A *linear* ODE is a differential equation taking the form

$$\frac{d}{dt}x = A(t)x + B(t), \tag{LNH}$$

where $A(t) \in \mathcal{M}_n(\mathbb{R})$ with continuous entries, $B(t) \in \mathbb{R}^n$ with real valued, continuous coefficients, and $x \in \mathbb{R}^n$. The associated IVP takes the form

$$\begin{aligned} \frac{d}{dt}x &= A(t)x + B(t) \\ x(t_0) &= x_0. \end{aligned} \tag{1}$$

Types of systems

- $x' = A(t)x + B(t)$ is linear nonautonomous ($A(t)$ depends on t) nonhomogeneous (also called *affine* system).
- $x' = A(t)x$ is linear nonautonomous homogeneous.
- $x' = Ax + B$, that is, $A(t) \equiv A$ and $B(t) \equiv B$, is linear autonomous nonhomogeneous (or affine autonomous).
- $x' = Ax$ is linear autonomous homogeneous.

- If $A(t + T) = A(t)$ for some $T > 0$ and all t, then linear periodic.

Existence and uniqueness of solutions

Theorem (Existence and Uniqueness)

Solutions to (1) *exist and are unique on the whole interval over which A and B are continuous.*
In particular, if A, B are constant, then solutions exist on $\mathbb{R}$.

The vector space of solutions

Theorem

Consider the homogeneous system

$$\frac{d}{dt}x = A(t)x, \tag{LH}$$

with $A(t)$ defined and continuous on an interval J. The set of solutions of (LH) *forms an n-dimensional vector space.*

Fundamental matrix

Definition

A set of n linearly independent solutions of (LH) on J, $\{\phi_1, \ldots, \phi_n\}$, is called a fundamental set of solutions of (LH) and the matrix

$$\Phi = [\phi_1 \; \phi_2 \; \ldots \; \phi_n]$$

is called a fundamental matrix of (LH).

Fundamental matrix solution

Let $X \in \mathcal{M}_n(\mathbb{R})$ with entries $[x_{ij}]$. Define the derivative of X, X' (or $\frac{d}{dt}X$) as

$$\frac{d}{dt}X(t) = [\frac{d}{dt}x_{ij}(t)].$$

The system of n^2 equations

$$\frac{d}{dt}X = A(t)X$$

is called a *matrix differential equation*.

Theorem

A fundamental matrix Φ *of* (LH) *satisfies the matrix equation* $X' = A(t)X$ *on the interval* J.-

Abel's formula

Theorem

If Φ is a solution of the matrix equation $X' = A(t)X$ on an interval J and $\tau \in J$, then

$$\det\Phi(t) = \det\Phi(\tau)\exp\left(\int_\tau^t \mathrm{tr}A(s)ds\right)$$

for all $t \in J$.

The resolvent matrix

Definition (Resolvent matrix)

Let $t_0 \in J$ and $\Phi(t)$ be a fundamental matrix solution of (LH) on J. Since the columns of Φ are linearly independent, it follows that $\Phi(t_0)$ is invertible. The *resolvent* (or *state transition matrix*, or *principal fundamental matrix*) of (LH) is then defined as

$$\mathcal{R}(t, t_0) = \Phi(t)\Phi(t_0)^{-1}.$$

Proposition

The resolvent matrix satisfies the Chapman-Kolmogorov identities

1. $\mathcal{R}(t,t) = I$,
2. $\mathcal{R}(t,s)\mathcal{R}(s,u) = \mathcal{R}(t,u)$,

as well as the identities

3. $\mathcal{R}(t,s)^{-1} = \mathcal{R}(s,t)$,
4. $\frac{\partial}{\partial s}\mathcal{R}(t,s) = -\mathcal{R}(t,s)A(s)$,
5. $\frac{\partial}{\partial t}\mathcal{R}(t,s) = A(t)\mathcal{R}(t,s)$.

Proposition

$\mathcal{R}(t, t_0)$ is the only solution in $\mathcal{M}_n(\mathbb{K})$ of the initial value problem

$$\frac{d}{dt}M(t) = A(t)M(t)$$
$$M(t_0) = \mathbb{I},$$

with $M(t) \in \mathcal{M}_n(\mathbb{K})$.

Theorem

The solution to the IVP consisting of the linear homogeneous nonautonomous system (LH) *with initial condition* $x(t_0) = x_0$ *is given by*

$$\phi(t) = \mathcal{R}(t, t_0)x_0.$$

A variation of constants formula

Theorem (Variation of constants formula)

Consider the IVP

$$x' = A(t)x + g(t, x) \tag{2a}$$

$$x(t_0) = x_0, \tag{2b}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *a smooth function, and let* $\mathcal{R}(t, t_0)$ *be the resolvent associated to the homogeneous system* $x' = A(t)x$*, with* $\mathcal{R}$ *defined on some interval* $J \ni t_0$*. Then the solution* ϕ *of* (2) *is given by*

$$\phi(t) = \mathcal{R}(t, t_0)x_0 + \int_{t_0}^{t} \mathcal{R}(t, s)g(\phi(s), s)ds, \tag{3}$$

on some subinterval of J*.*

Autonomous linear systems

Consider the autonomous affine system

$$\frac{d}{dt}x = Ax + B, \tag{A}$$

and the associated homogeneous autonomous system

$$\frac{d}{dt}x = Ax. \tag{L}$$

Exponential of a matrix

Definition (Matrix exponential)

Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of A, denoted e^{At}, is a matrix in $\mathcal{M}_n(\mathbb{K})$, defined by

$$e^{At} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k,$$

where $\mathbb{I}$ is the identity matrix in $\mathcal{M}_n(\mathbb{K})$.

Properties of the matrix exponential

- $\Phi(t) = e^{At}$ is a fundamental matrix for (L) for $t \in \mathbb{R}$.
- The resolvent for (L) is given for $t \in J$ by
$$\mathcal{R}(t, t_0) = e^{A(t-t_0)} = \Phi(t - t_0).$$
- $e^{At_1} e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \in \mathbb{R}$. 1
- $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.
- $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.
- The unique solution ϕ of (L) with $\phi(t_0) = x_0$ is given by
$$\phi(t) = e^{A(t-t_0)} x_0.$$

Computing the matrix exponential

Let P be a nonsingular matrix in $\mathcal{M}_n(\mathbb{R})$. We transform the IVP

$$\begin{aligned} \frac{d}{dt}x &= Ax \\ x(t_0) &= x_0 \end{aligned} \tag{L\_IVP}$$

using the transformation $x = Py$ or $y = P^{-1}x$.

The dynamics of y is

$$\begin{aligned} y' &= (P^{-1}x)' \\ &= P^{-1}x' \\ &= P^{-1}Ax \\ &= P^{-1}APy \end{aligned}$$

The initial condition is $y_0 = P^{-1}x_0$.

We have thus transformed IVP (L_IVP) into

$$\begin{aligned} \frac{d}{dt}y &= P^{-1}APy \\ y(t_0) &= P^{-1}x_0 \end{aligned} \tag{L\_IVP\_y}$$

From the earlier result, we then know that the solution of (L_IVP_y) is given by

$$\psi(t) = e^{P^{-1}AP(t-t_0)}P^{-1}x_0,$$

and since $x = Py$, the solution to (L_IVP) is given by

$$\phi(t) = Pe^{P^{-1}AP(t-t_0)}P^{-1}x_0.$$

So everything depends on $P^{-1}AP$.

Diagonalizable case

Assume P nonsingular in $\mathcal{M}_n(\mathbb{R})$ such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with all eigenvalues $\lambda_1, \ldots, \lambda_n$ different.

We have

$$e^{P^{-1}AP} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k$$

For a (block) diagonal matrix M of the form

$$M = \begin{pmatrix} m_{11} & & 0 \\ & \ddots & \\ 0 & & m_{nn} \end{pmatrix}$$

there holds

$$M^k = \begin{pmatrix} m_{11}^k & & 0 \\ & \ddots & \\ 0 & & m_{nn}^k \end{pmatrix}$$

Therefore,

$$
\begin{aligned}
e^{P^{-1}AP} &= \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda_n^k \end{pmatrix} \\
&= \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}
\end{aligned}
$$

And so the solution to (L_IVP) is given by

$$\phi(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0.$$

Nondiagonalizable case

The Jordan canonical form is

$$P^{-1}AP = \begin{pmatrix} J_0 & & 0 \\ & \ddots & \\ 0 & & J_s \end{pmatrix}$$

so we use the same property as before (but with block matrices now), and

$$e^{P^{-1}APt} = \begin{pmatrix} e^{J_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_s t} \end{pmatrix}$$

The first block in the Jordan canonical form takes the form

$$J_0 = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

and thus, as before,

$$e^{J_0 t} = \begin{pmatrix} e^{\lambda_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}$$

Other blocks J_i are written as

$$J_i = \lambda_{k+i}\mathbb{I} + N_i$$

with $\mathbb{I}$ the $n_i \times n_i$ identity and N_i the $n_i \times n_i$ nilpotent matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$\lambda_{k+i}\mathbb{I}$ and N_i commute, and thus

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}$$

Since N_i is nilpotent, $N_i^k = 0$ for all $k \geq n_i$, and the series $e^{N_i t}$ terminates, and

$$e^{J_i t} = e^{\lambda_{k+i} t} \begin{pmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & & \\ 0 & & & 1 \end{pmatrix}$$

Theorem

For all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$*, there is a unique solution* $x(t)$ *to* (L_IVP) *defined for all* $t \in \mathbb{R}$*. Each coordinate function of* $x(t)$ *is a linear combination of functions of the form*

$$t^k e^{\alpha t} \cos(\beta t) \quad and \quad t^k e^{\alpha t} \sin(\beta t)$$

where $\alpha + i\beta$ *is an eigenvalue of* A *and* k *is less than the algebraic multiplicity of the eigenvalue.*

Fixed points (equilibria)

Definition

A *fixed point* (or *equilibrium point*, or *critical point*) of an autonomous differential equation

$$x' = f(x)$$

is a point p such that $f(p) = 0$. For a nonautonomous differential equation

$$x' = f(t, x),$$

a fixed point satisfies $f(t, p) = 0$ for all t.

A fixed point is a solution.

Orbits, limit sets

Orbits and limit sets are defined as for maps.

For the equation $x' = f(x)$, the subset $\{x(t), t \in I\}$, where I is the maximal interval of existence of the solution, is an *orbit*.

If the maximal solution $x(t, x_0)$ of $x' = f(x)$ is defined for all $t \geq 0$, where f is Lipschitz on an open subset V of $\mathbb{R}^n$, then the omega limit set of x_0 is the subset of V defined by

$$\omega(x_0) = \bigcap_{\tau=0}^{\infty} \left(\overline{\{x(t, x_0) : t \geq \tau\}} \cap V\} \right).$$

Proposition

A point q is in $\omega(x_0)$ iff there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k, x_0) = q \in V$.

Definition (Liapunov stable orbit)

The orbit of a point p is *Liapunov stable* for a flow ϕ_t if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, p) < \delta$ implies that $d(\phi_t(x), \phi_t(p)) < \varepsilon$ for all $t \geq 0$. If p is a fixed point, then this is written $d(\phi_t(x), p) < \varepsilon$.

Definition (Asymptotically stable orbit)

The orbit of a point p is *asymptotically stable* (or *attracting*) for a flow ϕ_t if it is Liapunov stable, and there exists $\delta_1 > 0$ such that $d(x, p) < \delta_1$ implies that $\lim_{t\to\infty} d(\phi_t(x), \phi_t(p)) = 0$. If p is a fixed point, then it is asymptotically stable if it is Liapunov stable and there exists $\delta_1 > 0$ such that $d(x, p) < \delta_1$ implies that $\omega(x) = \{p\}$.

Contracting linear equation

Theorem

Let $A \in \mathcal{M}_n(\mathbb{R})$, and consider the equation (L). *Then the following conditions are equivalent.*

1. *There is a norm $\| \ \|_A$ on $\mathbb{R}^n$ and a constant $a > 0$ such that for any $x_0 \in \mathbb{R}^n$ and all $t \geq 0$,*

$$\|e^{At}x_0\|_A \leq e^{-at}\|x_0\|_A.$$

2. *There is a norm $\| \ \|_B$ on $\mathbb{R}^n$ and constants $a > 0$ and $C \geq 1$ such that for any $x_0 \in \mathbb{R}^n$ and all $t \geq 0$,*

$$\|e^{At}x_0\|_B \leq Ce^{-at}\|x_0\|_B.$$

3. *All eigenvalues of A have negative real parts.*

In that case, the origin is a *sink* or *attracting*, the flow is a *contraction* (antonyms *source*, *repelling* and *expansion*).

Hyperbolic linear equation

Definition

The linear differential equation (L) is *hyperbolic* if A has no eigenvalue with zero real part.

Definition (Stable eigenspace)

The *stable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^s = \mathsf{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) < 0\}$$

Definition (Center eigenspace)

The *center eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^c = \mathsf{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) = 0\}$$

Definition (Unstable eigenspace)

The *unstable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^u = \mathsf{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) > 0\}$$

We can write

$$\mathbb{R}^n = E^s \oplus E^u \oplus +E^c,$$

and in the case that $E^c =$, then $\mathbb{R}^n = E^s \oplus E^u$ is called a *hyperbolic splitting*.

The symbol $\oplus$ stands for *direct sum*.

Definition (Direct sum)

Let U, V be two subspaces of a vector space X. Then the span of U and V is defined by $u + v$ for $u \in U$ and $v \in V$. If U and V are disjoint except for 0, then the span of U and V is called the *direct sum* of U and V, and is denoted $U \oplus V$.

Trichotomy

Define

$$V^s = \{v : \text{there exists } a > 0 \text{ and } C \geq 1 \text{ such that}$$
$$\|e^{At}v\| \leq Ce^{-at}\|v\| \text{ for } t \geq 0\}.$$
$$V^u = \{v : \text{there exists } a > 0 \text{ and } C \geq 1 \text{ such that}$$
$$\|e^{At}v\| \leq Ce^{-a|t|}\|v\| \text{ for } t \leq 0\}.$$
$$V^c = \{v : \text{for all } a > 0, \|e^{At}v\|e^{-a|t|} \to 0 \text{ as } t \to \pm\infty\}.$$

Theorem

The following are true.

1. *The subspaces E^s, E^u and E^c are invariant under the flow e^{At}.*
2. *There holds that $E^s = V^s$, $E^u = V^u$ and $E^c = V^c$, and thus $e^{At}|_{E^u}$ is an exponential expansion, $e^{At}|_{E^s}$ is an exponential contraction, and $e^{At}|_{E^c}$ grows subexponentially as $t \to \pm\infty$.*

Topologically conjugate linear ODEs

Definition (Topologically conjugate flows)

Let ϕ_t and ψ_t be two flows on a space M. ϕ_t and ψ_t are *topologically conjugate* if there exists an homeomorphism $h : M \to M$ such that

$$h \circ \phi_t(x) = \psi_t \circ h(x),$$

for all $x \in M$ and all $t \in \mathbb{R}$.

Definition (Topologically equivalent flows)

Let ϕ_t and ψ_t be two flows on a space M. ϕ_t and ψ_t are *topologically equivalent* if there exists an homeomorphism $h : M \to M$ and a function $\alpha : \mathbb{R} \times M \to \mathbb{R}$ such that

$$h \circ \phi_{\alpha(t+s,x)}(x) = \psi_t \circ h(x),$$

for all $x \in M$ and all $t \in \mathbb{R}$, and where $\alpha(t,x)$ is monotonically increasing in t for each x and onto all of $\mathbb{R}$.

Theorem

Let $A, B \in \mathcal{M}_n(\mathbb{R})$*.*

1. *If all eigenvalues of* A *and* B *have negative real parts, then the linear flows* e^{At} *and* e^{Bt} *are topologically conjugate.*
2. *Assume that the system is hyperbolic, and that the dimension of the stable eigenspace of* A *is equal to the dimension of the eigenspace of* B*. Then the linear flows* e^{At} *and* e^{Bt} *are topologically conjugate.*

Theorem

Let $A, B \in \mathcal{M}_n(\mathbb{R})$. Assume that e^{At} and e^{Bt} are linearly conjugate, i.e., there exists M with $e^{Bt} = Me^{At}M^{-1}$. Then A and B have the same eigenvalues.