UNIVERSITY OF MANITOBA

LINEAR ALGEBRA AND MATRIX THEORY

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Chapter contents

Foreword

Purpose of the course

This course introduces some notions of linear algebra and, more specifically, matrix analysis. Matrix analysis is a vast subject and presenting the topic in a one semester course necessarily implies making choices as to the specific material covered. These choices are obviously tainted by the personal experience of the person making them. I am a mathematical biologist and most of my work involves the analysis of dynamical systems of various forms formulated in this context: ordinary, delayed or partial differential equations, discrete-time systems, discrete-time or continuous-time Markov chains. I also dabble in applied graph theory. This will be apparent here: for me, the most important type of problems in matrix analysis consists in obtaining results concerning the location of eigenvalues. However, this is not a course on spectral matrix analysis but on matrix analysis altogether, so I will try to be agnostic in the matter and present other interesting topics.

Note that a lot of proofs will be omitted. I will however, as much as possible, point to the location of the proofs in relevant publications, so that you can find read them if you are interested. (We will prove a certain number of results during lectures.)

Credits

These notes were originally typeset by Adriana-Stefania Ciupeanu as her MATH 4370/7370 final project in Fall 2018. The latex template was adapted by Adriana from the Big Orange Book. Since then, I have revisited and edited the notes. Any remaining mistakes or typos are my fault. If you find any, please let me know.

Most of these notes draw from three main references: Fiedler [**?**], Horn & Johnson [**?**] and Zhang [**?**]. I have tried, as much as possible, to harmonise notation.

Notation

In these lecture notes, we use a field of real or complex numbers. We write $\mathbb{F}$ for that field (when unspecified); otherwise, we use $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We sometimes denote $\mathbb{F}^\star = \mathbb{F} \setminus \{0\}$ the field without its zero element. The identity matrix is denoted $\mathbb{I}$ and is assumed to be of the proper size for the notation where it arises to make sense. If this leads to ambiguity, we denote $\mathbb{I}_n$ the $n \times n$ identity matrix. The vector of all ones is denoted $\mathbb{1} = (1, \ldots, 1)^T$ (it is assumed to be a column vector). As for the identity matrix, size is only indicated when needed, with $\mathbb{1}_n$ being the vector with n ones. We denote $\mathbb{J}_{mn}$ the $m \times n$-matrix of all ones, dropping the size indication if it is not needed. Other specific notation is introduced where needed.

CHAPTER 1

Matrices are everywhere

The purpose of this chapter is to give a few examples that illustrate how ubiquitous matrices are in mathematics. In the process, we introduce some concepts that are used again later. Note, however, that precise definitions are given in subsequent chapter; here the concepts are introduced with very few explanations.

Chapter contents

1.1 Linear systems of difference equations

This section is loosely based on [**?**] and [**?**].

1.1.1 General theory

Let $x(t) \in \mathbb{R}^n$ be a *state variable* and $t \in \mathbb{N}$ be an *independent variable*, typically thought of as *time*. Let $A \in \mathcal{M}_n(\mathbb{R})$. An *autonomous homogeneous linear system of difference equations* is a sequence defined by

$$x(t+1) = Ax(t) \tag{1.1a}$$
$$x(0) = x_0 \in \mathbb{R}^n, \tag{1.1b}$$

where x_0 is called the initial condition (IC). (Autonomous: A is a constant that does not depend on t. Homogeneous: the system is not of the form $x(t+1) = Ax(t) + b$.) Given the initial condition x_0, we have from (1.1a) that

$$x(1) = Ax(0)$$
$$x(2) = Ax(1) = A^2x(0).$$

By induction and using (1.1b),

$$x(t) = A^t x(0) = A^t x_0,$$

for all $t \in \mathbb{N}$.

Thus the behaviour of $x(t)$ as $t \to \infty$ depends on A^t. In order to understand what this behaviour could be, the following two questions should be answered. What is (if it exists) $\lim_{t\to\infty} x(t)$? What is (if it exists) $\lim_{t\to\infty} A^t$?

Let us make a first observation. Let v be an eigenvector associated to the eigenvalue λ of A, i.e., λ be such that $v \neq 0$ satisfies $Av = \lambda v$. Then we have

$$\begin{aligned} A^2 v &= A(Av) \\ &= A(\lambda v) \\ &= \lambda A v \\ &= \lambda^2 v, \end{aligned}$$

i.e., v is also an eigenvector of A^2 and is associated to the eigenvalue λ^2. By induction,

$$A^k v = \lambda^k v,$$

i.e., v is an eigenvector of the matrix A^k associated to the eigenvalue λ^k. $A^k v$ is a vector in $\mathbb{F}^n$; we can thus take its norm $\|A^k v\|$, where $\|\cdot\|$ is some norm on $\mathbb{F}^n$. It follows that if $|\lambda| < 1$, then $\|A^k v\| = |\lambda|^k \|v\|$ goes to zero as $k \to \infty$. This provides the basis for the following result.

Theorem 1.1.1 Let $A \in \mathcal{M}_n(\mathbb{R})$, consider the map Ax. Then the following statements are equivalent:

1. There exists a norm $\| \; \|_\alpha$ on $\mathbb{R}^n$ and a constant $0 < \mu < 1$ such that for any $x \in \mathbb{R}^n$, the iterates satisfy, for all $k \geq 0$,
$$\|A^k x\|_\alpha \leq \mu^k \|x\|_\alpha.$$
2. For any norm $\| \; \|_\beta$ on $\mathbb{R}^n$, there exists constants $0 < \mu < 1$ and $C \geq 1$ such that all $x \in \mathbb{R}^n$ and all $k \geq 0$,
$$\|Ax\|_\beta \leq C\mu^k \|x\|_\beta.$$
3. All the eigenvalues λ of A satisfy $|\lambda| < 1$.

R Thus it is important to be able to locate the eigenvalues. For instance, to be able to use the result above, we need to ensure they all lie inside the unit disk in $\mathbb{C}$. We will see later in the course that for certain classes of matrices, this can be achieved without explicitly computing the eigenvalues.

A linear map corresponding to a matrix with all eigenvalues of modulus less than 1 is a *linear contraction*, with the origin a *linear sink* or *attracting fixed point*. If all eigenvalues have modulus larger than 1, then the map induced by A is a *linear expansion*, and the origin is a *linear source* or *repelling fixed point*. The map Ax is a *hyperbolic linear map* if all eigenvalues of A have modulus different of 1.

Theorem 1.1.1 and the discussion above can be summarized using the following definition and theorem.

Definition 1.1.2 Let $\mathcal{M}_n$ be the set of square $n \times n$ matrices. For $A \in \mathcal{M}_n$, denote $\sigma(A)$ the set of its eigenvalues, i.e.,
$$\sigma(A) = \{\lambda \in \mathbb{C}; \exists v \neq 0, \; Av = \lambda v\},$$
which we call the **spectrum** of A. We call **spectral radius** of A the real number
$$\rho(A) = \max_{\lambda \in \sigma(A)} \{|\lambda|\}.$$

We then have the following result concerning the long term behaviour of $x(t)$ in the difference equation above.

Theorem 1.1.3 If $\rho(A) < 1$, then $\lim_{t\to\infty} x(t) = 0$.

1.1.2 Example – Leslie matrices

Leslie matrices arise when considering the dynamics of populations reproducing every year, such as most fish. Let $x(t) = (x_1(t), \dots, x_n(t))^T$ be the vector of distribution of the population in ages, i.e., $x_i(t)$ is the population of fish of age a between $i-1$ and i at time t. (We say the model is *age-structured*.) Assume n is the maximum age observed for that species. Time t is taken in years and is *discrete*, because for instance one would conduct yearly surveys of the population to determine abundance in each age class.

A proportion $s_i \in [0, 1]$ of individuals of age $i - 1 \le a < i$ survive to the next year. As years progress at the same rate as age, this means that $x_{i+1}(t + 1) = s_i x_i(t)$. Individualsalso reproduce. When they do, they give birth to f_i individuals in the first age class. This means that birth in the first age class takes the form

$$x_1(t + 1) = f_1 x_1(t) + f_2 x_2(t) + \cdots + f_n x_n(t).$$

Altogether, we have the following system:

$$\begin{aligned}
x_1(t + 1) &= f_1 x_1(t) + f_2 x_2(t) + \cdots + f_n x_n(t) \\
x_2(t + 1) &= s_1 x_1(t) \\
x_3(t + 1) &= s_2 x_2(t) \\
&\vdots \\
x_n(t + 1) &= s_{n-1} x_{n-1}(t).
\end{aligned}$$

Suppose that on the first year counts are taken, we observe an initial age distribution $x(0) = (x_1(0), \dots, x_n(0))^T$. Assume the survival rates s_i and fecundity constants f_i are known. How can we expect the population to evolve over the course of time? In particular, is the species likely to survive or will it become extinct?

To consider this, we first write the system in vector form. We have

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ \vdots \\ x_n(t+1) \end{pmatrix} = \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & & s_{n-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \tag{1.2}$$

which we summarize as

$$x(t + 1) = Lx(t), \tag{1.3}$$

with L the matrix in (1.2), i.e., a *nonnegative* matrix with only the first row and the first sub-diagonal nonzero. By Theorem 1.1.3, if $\rho(L) < 1$ then $x(t) \to 0$ at $t \to \infty$, so in this case, the population would become extinct.

We will return to this matrix in Chapter 4, when we consider the properties of nonnegative matrices.

1.2 Ordinary differential equation

An *ordinary differential equation* (ODE) is an equation of the form

$$\frac{d}{dt}x(t) = f(x(t)), \tag{1.4}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. An *initial value problem* (IVP) is the consideration of (1.4) together with an initial condition $x(t_0) = x_0 \in \mathbb{R}^n$. A solution to (1.4) is a function $\phi(t)$ that satisfies (1.4). A solution to the IVP with $x(t_0) = x_0$ associated to (1.4) is, among all solutions to (1.4), the one that additionally satisfies the initial condition, i.e., such that $\phi(t_0) = x_0$.

1.2.1 Linear systems

Consider the autonomous linear system

$$x'(t) = Ax(t), \tag{1.5}$$

where $A \in \mathcal{M}_n$ is a constant.

Let $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ be the spectrum of A. Let $w_j = u_j + iv_j$ be a generalized eigenvector of A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$, with $v_j = 0$ if $b_j = 0$, and

$$B = \{u_1, \dots, u_k, u_{k+1}, v_{k+1}, \dots, u_m, v_m\}$$

be a basis of $\mathbb{R}^n$, with $n = 2m - k$, where k is the number of real eigenvalues in $\sigma(A)$.

Definition 1.2.1 — Stable, unstable and center subspaces. The *stable*, *unstable* and *center* subspaces of the linear system (1.5) are given, respectively, by

$$E^s = \text{Span}\{u_j, v_j : a_j < 0\},$$

$$E^u = \text{Span}\{u_j, v_j : a_j > 0\}$$

and

$$E^c = \text{Span}\{u_j, v_j : a_j = 0\}.$$

Definition 1.2.2 The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called the *flow* of the linear system (1.5).

The term *flow* is used since e^{At} describes the motion of points $x_0 \in \mathbb{R}^n$ along trajectories of (1.5).

Definition 1.2.3 If all eigenvalues of A have nonzero real part, i.e., if $E^c = \emptyset$, then the flow e^{At} of system (1.5) is a *hyperbolic flow* and the system (1.5) is a *hyperbolic linear system*.

Definition 1.2.4 A subspace $E \subset \mathbb{R}^n$ is *invariant with respect to the flow* e^{At}, or *invariant under the flow* of (1.5), if $e^{At}E \subset E$ for all $t \in \mathbb{R}$.

Theorem 1.2.5 Let E be the generalized eigenspace of A associated to the eigenvalue λ. Then $AE \subset E$.

Theorem 1.2.6 Let $A \in \mathcal{M}_n(\mathbb{R})$. Then

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c.$$

Furthermore, if the matrix A is the matrix of the linear autonomous system (1.5), then E^s, E^u and E^c are invariant under the flow of (1.5), *i.e.*, let $x_0 \in E^S$, $y_0 \in E^C$ and $z_0 \in E^U$, then $e^{At}x_0 \in E^S$, $e^{At}y_0 \in E^C$ and $e^{At}z_0 \in E^U$.

1.2.2 Nonlinear systems

There is no general theory allowing to obtain explicit solutions of a nonlinear IVP. Instead of seeking explicit solutions, we thus use *qualitative analysis*, which consists in using results from analysis to establish properties of the solutions without needing to actually find their explicit expression.

Suppose you are given a system of ordinary differential equations

$$x' = f(x), \tag{1.6}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. A standard step when studying (1.6) *qualitatively* is to seek *equilibria* of (1.6), i.e., points $x^\star \in \mathbb{R}^n$ such that

$$f(x^\star) = 0. \tag{1.7}$$

At such a point, $x' = 0$, meaning that system (1.6) is at rest. If you were to consider solutions to (1.6) with an initial condition $x(0) = x^\star$, then there would hold that $x(t) = x^\star$ for all $t \geq 0$.

What would happen if instead of starting at $x^\star$, you were to choose an initial condition $x(0)$ close to but distinct from $x^\star$?

Brief elements of stability theory

The question we just asked can be attacked using two extremely important theorems, the Hartman-Grobman Theorem and the Stable Manifold Theorem. In order to understand these theorems, the following definitions are required.

Definition 1.2.7 — Homeomorphism. Let X be a metric space and let A and B be subsets of X. A **homeomorphism** $h : A \to B$ of A onto B is a continuous injective (one-to-one) map of A onto B such that $h^{-1} : B \to A$ is continuous. The sets A and B are called **homeomorphic** or **topologically equivalent** if there is a homeomorphism of A onto B.

Definition 1.2.8 — Differentiable manifold. An n-dimensional **differentiable manifold** M (or a manifold of class C^k) is a connected metric space with an open covering $\{U_\alpha\}$ (*i.e.*, $M = \bigcup_\alpha U_\alpha$) such that

1. for all α, U_α is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, *i.e.*,

for all α there exists a homeomorphism of U_α onto B, $h_\alpha : U_\alpha \to B$,

2. if $U_\alpha \cap U_\beta \neq \emptyset$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms, then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$h = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

is differentiable (or of class C^k) and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, the determinant of the Jacobian, $\det Dh(x) \neq 0$.

Let us first state Hartman-Grobman in a much simplified version. An equilibrium point $x^\star$ is **hyperbolic** if the Jacobian matrix Df of (1.6) evaluated at $x^\star$, denoted $Df(x^\star)$, has no eigenvalues with zero real part, i.e., is invertible.

Theorem 1.2.9 — Hartman-Grobman. Let $x^\star$ be a hyperbolic equilibrium point of (1.6). Then in some neighbourhood $\mathcal{N}(x^\star)$ of $x^\star$, the flow of (1.6) is topologically equivalent to the flow of the linear system

$$x' = Df(x^\star)(x - x^\star), \tag{1.8}$$

where $Df(x^\star)$ is the Jacobian matrix Df of f evaluated at $x^\star$.

Theorem 1.2.9 says that in the vicinity of a hyperbolic equilibrium point, the behaviour of the nonlinear system (1.6) is similar to that of the linear system $Df(x^\star)(x - x^\star)$.

Because their slightly different formulation helps better understand the result, we give two more statements of the Hartman-Grobman Theorem. For simplicity and without loss of generality since both results are local results, we assume hereforth that $x^\star = 0$, *i.e.*, that a change of coordinates has been performed translating $x^\star$ to the origin. We also assume that $t_0 = 0$. The first statement is adapted from [1, p. 311].

Theorem 1.2.10 — Hartman-Grobman. Suppose that 0 is an equilibrium point of the nonlinear system (1.6). Let φ_t be the flow of (1.6), and ψ_t be the flow of the linearized system $x' = Df(0)x$. If 0 is a hyperbolic equilibrium, then there exists an open subset $\mathcal{D}$ of $\mathbb{R}^n$ containing 0, and a homeomorphism G with domain in $\mathcal{D}$ such that $G(\varphi_t(x)) = \psi_t(G(x))$ whenever $x \in \mathcal{D}$ and both sides of the equation are defined.

Another form comes from [4].

Theorem 1.2.11 — Hartman-Grobman. Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1.6). Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part.

Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $\mathcal{I}_0 \subset \mathbb{R}$ containing 0 such that for all $x_0 \in U$ and $t \in \mathcal{I}_0$,

$$H \circ \phi_t(x_0) = e^{At} H(x_0);$$

i.e., H maps trajectories of (1.6) near the origin onto trajectories of $x' = Df(0)x$ near the origin and preserves the parametrization by time.

So "all" we need to do is to study said linear system. This is done using the next result.

Theorem 1.2.12 — Stable manifold theorem. Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1.6). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (1.8) at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable subspace E^u of (1.8) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(x_0) = 0$$

There are several approaches to the proof of this result. Hale [3] gives a proof using functional analysis. Another proof comes from [4, p. 108-111], who derives it from [2, p. 330-335]. It consists in showing that there exists a real nonsingular constant matrix C such that if $y = C^{-1}x$ then there are $n-k$ real continuous functions $y_j = \psi_j(y_1, \dots, y_k)$ defined for small $|y_i|$, $i \leq k$, such that

$$y_j = \psi_j(y_1, \dots, y_k) \quad (j = k+1, \dots, n)$$

define a k-dimensional differentiable manifold $\tilde{S}$ in y-space. The stable manifold in S space is obtained by applying the transformation P^{-1} to y so that $x = P^{-1}y$ defines S in terms of k curvilinear coordinates $y_1, \dots, y_k$.

1.2.3 Example – A chemostat model

To illustrate the use of these results, we take the example of a nonlinear system consisting of two nonlinear differential equations modeling a biological device called a *chemostat*. Without going into details, the system is the following:

$$\frac{dS}{dt} = D\left(S^0 - S\right) - \mu(S)x \tag{1.9a}$$
$$\frac{dx}{dt} = (\mu(S) - D)x. \tag{1.9b}$$

Parameters S^0 and D, respectively the *input concentration* and the *dilution* rate, are real and positive. The function μ is the *growth* function. It is generally assumed to satisfy $\mu(0) = 0$, $\mu' > 0$ and $\mu'' < 0$.

To be complete, one should verify that the positive quadrant is positively invariant under the flow of (1.9), *i.e.*, that for $S(0) \geq 0$ and $x(0) \geq 0$, solutions remain nonnegative for all positive times, and similar properties. But since we are here only interested in applications of the stable manifold theorem, we proceed to a very crude analysis, and will not deal with this point.

Write the system in vector form as

$$\xi' = f(\xi),$$

with $\xi = (S, x)^T$ and

$$f(\xi) = \begin{pmatrix} D(S^0 - S) - \mu(S)x \\ (\mu(S) - D)x \end{pmatrix}.$$

Equilibria of the system are found by solving $f(\xi) = 0$. We find two, the first one situated on one of the boundaries of the positive quadrant,

$$\xi_T^\star = (S_T^\star, x_T^\star) = \left(S^0, 0\right),$$

the second one in the interior of $\mathbb{R}_+^2$,

$$\xi_I^\star = (S^\star, x^\star) = \left(\lambda, S^0 - \lambda\right).$$

where λ is such that $\mu(\lambda) = D$. Note that this implies that if $\lambda \geq S^0$, $\xi_T^\star$ is the only equilibrium of the system since in that case, $\xi_I^\star \ngeq 0$, which is not biologically realistic. (Concentrations/counts cannot be negative.)

At an arbitrary point $\xi = (S, x)$, the Jacobian matrix of (1.9) is given by

$$Df(\xi) = \begin{pmatrix} -D - \mu'(S)x & -\mu(S) \\ \mu'(S)x & \mu(S) - D \end{pmatrix}.$$

Thus, at the trivial equilibrium $\xi_T^\star$,

$$Df(\xi_T^\star) = \begin{pmatrix} -D & -\mu(S^0) \\ 0 & \mu(S^0) - D \end{pmatrix}.$$

We have two eigenvalues, $-D$ and $\mu(S^0) - D$. Let us suppose that $\mu(S^0) - D < 0$. Note that this implies that $\xi_T^\star$ is the only equilibrium, since, as we have seen before, $\xi_I^\star$ is not feasible if $\lambda > S^0$.

As the system has dimensionality 2, and that the matrix $Df(\xi_T^\star)$ has two negative eigenvalues, the stable manifold theorem (Theorem 1.2.12) states that there exists a 2-dimensional differentiable manifold $\mathcal{M}$ such that

- $\phi_t(\mathcal{M}) \subset \mathcal{M}$,
- for all $\xi_0 \in \mathcal{M}$, $\lim_{t\to\infty} \phi_t(\xi_0) = \xi_T^\star$.
- At $\xi_T^\star$, $\mathcal{M}$ is tangent to the stable subspace E^S of the linearized system $\xi' = Df(\xi_T^\star)(\xi - \xi_T^\star)$.

Since there are no eigenvalues with positive real part, there does not exist an unstable manifold in this case. Let us now characterize the nature of the stable subspace E^S. It is obtained by studying the linear system

$$
\begin{aligned}
\xi' &= Df(\xi_T^\star)(\xi - \xi_T^\star) \\
&= \begin{pmatrix} -D & -\mu(S^0) \\ 0 & \mu(S^0) - D \end{pmatrix} \begin{pmatrix} S - S^0 \\ x \end{pmatrix} \\
&= \begin{pmatrix} -D(S - S^0) - \mu(S^0)x \\ (\mu(S^0) - D)x \end{pmatrix}.
\end{aligned} \tag{1.10}
$$

Of course, the Jacobian matrix associated to this system is the same as that of the nonlinear system (at $\xi_T^\star$). Associated to the eigenvalue $-D$ is the eigenvector $v_1 = (1,0)^T$, to $\mu(S^0) - D$ is $v_2 = (-1,1)^T$.

The stable subspace is thus given by Span (v_1, v_2), *i.e.*, the whole of $\mathbb{R}^2$. In fact, the stable manifold of $\xi_T^\star$ is the whole positive quadrant, since all solutions limit to this equilibrium.

The same type of analysis can be conducted at the interior equilibrium $\xi_I^\star$. It is a little harder in this case, since $x^\star > 0$ there and therefore the Jacobian matrix $Df(\xi_I^\star)$ does not have the same upper triangular structure as $Df(\xi_T^\star)$.

1.3 Discrete-time Markov chains

1.3.1 Formulation of the chain

A **discrete-time Markov chain** is a stochastic process defined as follows. Consider a system with n **states** denoted $S_1, \dots, S_n$. The system starts in a given state. Every time step, it switches to a different state. (Transition from one state to itself is also allowed.) We assume that the system is *stochastic*, i.e., that the transitions happen at random. In discrete-time Markov chains, the instants at which transitions occur (or not) are in a discrete set, typically rescaled to be $\mathbb{N}$. (In *continuous-time* Markov chains, the time of the switches themselves is a continuous random variable, so times are in $\mathbb{R}$.)

Let $p_i(t)$ be the probability that state S_i occurs on the t^{th} time step, $1 \le i \le n$. (We use the letter t for time again, although the Markov chain literature typically thinks of generations and uses other letters.) One of the key assumptions in Markov chains is that the process is *memoryless*: the transition that occurs from time t to time $t+1$ depends only on the state of the system at time t. Since one the states S_i must occur on the t^{th} time step,

$$
p_1(t) + p_2(t) + \cdots + p_n(t) = 1.
$$

Let $p_i(t+1)$ be the probability that state S_i, $1 \le i \le n$, occurs on the $(t+1)^{th}$ repetition of the experiment. There are n ways to be in state S_i at step $t+1$:

1. Step t is S_1. The probability of getting S_1 on the t^{th} step is $p_1(t)$ and the probability of having S_i after S_1 is p_{i1}. Therefore, by the multiplication principle, $\mathbb{P}(S_i|S_1) = p_{i1}p_1(k)$.
2. We get S_2 on step t and S_i on step $t+1$. Then $\mathbb{P}(S_i|S_2) = p_{i2}p_2(t)$.

..

n. The probability of occurrence of S_i at step $t+1$ if S_n at step t is $\mathbb{P}(S_i|S_n) = p_{in}p_n(t)$.

Therefore, $p_i(t+1)$ is sum of all these,

$$\begin{aligned} p_i(t+1) &= \mathbb{P}(S_i|S_1) + \cdots + \mathbb{P}(S_i|S_n) \\ &= p_{i1}p_1(t) + \cdots + p_{in}p_n(t). \end{aligned}$$

Therefore,

$$\begin{aligned} p_1(t+1) &= p_{11}p_1(t) + p_{12}p_2(t) + \cdots + p_{1n}p_n(t) \\ &\vdots \\ p_k(t+1) &= p_{k1}p_1(t) + p_{k2}p_2(t) + \cdots + p_{kr}p_r(t) \\ &\vdots \\ p_n(t+1) &= p_{n1}p_1(t) + p_{n2}p_2(t) + \cdots + p_{nn}p_r(t) \end{aligned}$$

In matrix form,

$$p(t+1) = Pp(t), \quad t = 1, 2, 3, \ldots \tag{1.11}$$

where $p(t) = (p_1(t), p_2(t), \ldots, p_n(t))^T$ is a (column) probability vector and $P = (p_{ij})$ is a $n \times n$ **transition matrix** (or sometimes *projection matrix*),

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}. \tag{1.12}$$

It is also possible to write the system as $p(t+1) = p(t)P$, with $p(t)$ a row vector. In this case, entry p_{ij} in the transition matrix is the probability of transition from i to j, not from j to i as in system (1.11). Whatever the form, this is quite reminiscent of discrete-time dynamical systems described in Section 1.1.

Before proceeding further, let us make two remarks concerning P.

- Entries of P being probabilities, they are all in [0, 1]. So in particular, they are all nonnegative. We say P is a *nonnegative matrix*.
- The *column* sums of P all equal 1. Take for instance the first column: its entries represent the probabilities of transition from state 1. Since an event must always happen, the sum of these probabilities *must* be 1. We say P is a *stochastic matrix*.

We will return to nonnegative matrices in Chapter 4, but for now, let us focus a little on stochastic matrices.

Definition 1.3.1 — Stochastic matrix. The nonnegative $n \times n$ matrix M is **stochastic** if $\sum_{j=1}^n a_{ij} = 1$ for all $i = 1, \ldots, n$ xor $\sum_{i=1}^n a_{ij} = 1$ for all $j = 1, \ldots, n$.

In other words, M is stochastic if the sum of entries on each row equals one or (exclusively) the sum of entries on each column equals 1. Sometimes the terms row stochastic and column stochastic are used to make it clear which of the direction enjoys that property. If each row sum and each column sum equals one, we say that the matrix is **doubly stochastic**, but this will not be considered here. Stochastic matrices enjoy some very useful properties, not least of which is the following.

Theorem 1.3.2 Let M be a stochastic matrix M. Then all eigenvalues λ of M are such that $|\lambda| \leq 1$. Furthermore, $\lambda = 1$ is an eigenvalue of M.

To see that 1 is an eigenvalue, write the definition of, say, a row stochastic matrix, i.e., M has row sums 1. In vector form, $M\mathbb{1} = \mathbb{1}$. Now remember that λ is an eigenvalue of M, with associated eigenvector $v \neq 0$, if and only if $Mv = \lambda v$. So, in the expression $M\mathbb{1} = \mathbb{1}$, we read an eigenvector, $\mathbb{1}$, and an eigenvalue, 1. Proving the first conclusion of Theorem 1.3.2 involves a theorem called the Perron-Frobenius Theorem, which we will see in much detail in Chapter 4.

1.3.2 Long time behaviour

Just as we did in Section 1.1, we consider the evolution given an initial vector. Let $p(0) = (p_1(0), \dots, p_n(0))^T$ be the initial distribution vector, with $\mathbb{1}^T p(0) = 1$, i.e., such that the sum of the entries of $p(0)$ be 1; we could also write $\langle \mathbb{1}, p(0) \rangle = 1$. Then

$$\begin{aligned} p(1) &= Pp(0) \\ p(2) &= Pp(1) \\ &= P(Pp(0)) \\ &= P^2 p(0). \end{aligned}$$

Iterating, we get that for any t,

$$p(t) = P^t p(0).$$

Therefore,

$$\lim_{t\to+\infty} p(t) = \lim_{t\to+\infty} P^t p(0) = \left(\lim_{t\to+\infty} P^t\right) p(0), \tag{1.13}$$

if the latter limit exists. So if we can characterize the nature of matrix P^t and in particular, the existence of the limit $\lim_{t\to\infty} P^t$, we will know the long time behaviour of the Markov chain. It turns out that the product of two stochastic matrices is a stochastic matrix.

Theorem 1.3.3 If M, N are nonsingular stochastic matrices, then MN is a stochastic matrix.

So the product of any number of stochastic matrices is also stochastic.

Corollary 1.3.4 If M is a nonsingular stochastic matrix, then for any $t \in \mathbb{N}$, M^t is a stochastic matrix.

From this, it follows that the matrix P^t in (1.13) is stochastic; so, in particular, it is a nonnegative matrix with column sums all equal to 1.

1.3.3 Regular Markov chains

Definition 1.3.5 — Regular Markov chain. A regular Markov chain is one in which P^k is positive for some integer $k > 0$, i.e., P^k has only positive entries, no zero entries.

Definition 1.3.6 A nonnegative matrix M is primitive if, and only if, there is an integer $k > 0$ such that M^k is (entry-wise) positive.

Theorem 1.3.7 A Markov chain is regular if, and only if, the transition matrix P is primitive.

Theorem 1.3.8 If P is the transition matrix of a regular Markov chain, then

1. the powers P^t approach a stochastic matrix W,
2. each column of W is the same vector $w = (w_1, \dots, w_n)^T$,
3. the components of w are positive.

So if the Markov chain is regular,

$$\lim_{t\to+\infty} p(t) = \lim_{t\to+\infty} P^t p(0) = W p(0).$$

Let M be an $n \times n$ matrix, u, v be two column vectors, $\lambda \in \mathbb{R}$. Then, if

$$Mu = \lambda u,$$

u is the (right) eigenvector corresponding to λ, and if

$$v^T M = \lambda v^T$$

then v is the left eigenvector corresponding to λ. Note that to a given eigenvalue there corresponds one left and one right eigenvector (to multiples). The vector w is in fact the eigenvector corresponding to the eigenvalue 1 of P. (We already know that the left eigenvector corresponding to 1 is $\mathbb{1}^T$, since $\mathbb{1}^T P = \mathbb{1}$, i.e., the column sums of P all equal 1.) To see this, remark that, if $p(t)$ converges, then $p(t+1) = Pp(t)$ in the limit for large t, so w is a fixed point of the system. We thus write

$$w = Pw$$

and solve for w, which amounts to finding w as the (right) eigenvector corresponding to the eigenvalue 1.

Alternatively, we can find w as the (left) eigenvector associated to the eigenvalue 1 for the transpose of P,

$$w^T P^T = w^T.$$

This remark is important going "the other way": suppose you were to use a numerical or symbolic mathematics program to seek the left eigenvectors of a matrix. Typically, the output of functions in these programs are the right eigenvectors. To obtain the left eigenvectors, it then suffices to compute the eigenvectors of the transpose of the matrix.

Now remember that when you compute an eigenvector, you get a result that is the eigenvector, to a multiple. So the expression you obtain for w might have to be normalized (you want a probability vector). Once you obtain w, check that the norm $\|w\|$ defined by

$$\|w\| = w_1 + \cdots + w_n$$

is equal to one. If not, use

$$\frac{w}{\|w\|}.$$

1.4 Discretisation of partial differential equations

The following discussion is based on [5]. We consider a simple example of partial differential equation. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded connected region, $u = u(x)$ for $x \in \Omega$. A linear elliptic boundary value problem on Ω takes the form

$$Lu = f, \quad \Omega \tag{1.14a}$$
$$\alpha u + \beta \frac{\partial u}{\partial v} = g, \quad \partial\Omega, \tag{1.14b}$$

where L is a linear differential operator of the form

$$Lu = -\nabla(k\nabla u + bu) + qu.$$

Suppose that the scalar functions $k(x)$, $q(x)$ and the vector function $b(x) = (b_1(x), \ldots, b_d(x))^T$ for $d > 1$, otherwise a scalar function $b(x)$, are sufficiently smooth over the region Ω. Further, let $k(x) \geq k_0 = \text{const} > 0$ and $q(x) \geq 0$ for each $x \in \Omega$. Thus, L together with corresponding boundary conditions on $\partial\Omega$ is a linear elliptic differential operator.

As a particular example, suppose $\Omega = (0, 1)$ is a one-dimensional domain, i.e., an interval, and suppose

$$\bar{\omega}_h = \left\{0 = x_1 < x_2 < \ldots < x_n = 1\right\} = \omega_h + \gamma_h$$

is a discretisation of the closure of said interval, with $\gamma_h = \{x_0, x_n\}$ and $h_i = h = 1/n$ be a uniform grid, i.e., $x_i = ih$. Consider the following problem

$$Lu = -u'' + b(x)u' = 0, \quad x \in \Omega \tag{1.15a}$$
$$u(0) = u_0, \quad u(1) = u_1, \tag{1.15b}$$

with $b(x)$ bounded in Ω. Let us approximate on $\bar{\omega}$ using *centred differences* (a finite difference scheme):

$$y_0 = u_0 \tag{1.16a}$$
$$-D_+D_-y_i + b_iD_0y_i = 0, \quad i = 1, \dots, n-1 \tag{1.16b}$$
$$y_n = u_1, \tag{1.16c}$$

where $b_i = b(x_i)$. The notation D_+D_- and D_0 refer to *difference operators*:

$$D_+y_i = \frac{y_{i+1} - y_i}{h} \tag{1.17a}$$
$$D_-y_i = \frac{y_i - y_{i-1}}{h} \tag{1.17b}$$
$$D_0y_i = \frac{y_{i+1} - y_{i-1}}{2h} \tag{1.17c}$$
$$D_+D_-y_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}. \tag{1.17d}$$

These operators "encode" derivatives on the discrete grid used.

For $i = 1, \dots, n-1$, define

$$\gamma_i = \frac{hb_i}{2}.$$

Then the problem can be written in matrix form

$$Ay = f, \tag{1.18}$$

where $f = (u_0, 0, \dots, 0, u_1)^T = u_0e_1 + u_1e_{n+1}$ and

$$A = \begin{pmatrix} 1 & 0 & & & & & \\ -(1+\gamma_1) & 2 & -(1-\gamma+1) & & & & \\ & -(1+\gamma_2) & 2 & -(1-\gamma_2) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -(1+\gamma_{n-1}) & 2 & -(1-\gamma_{n-1}) \\ & & & & & 0 & 1 \end{pmatrix}.$$

Properties of the (approximate) solution then depend on the properties of the matrix A. Such matrices are sometimes called M-matrices and enjoy quite interesting properties; see Chapter 5.

CHAPTER 2

Eigenvalues, eigenvectors, similarity and Geršgorin disks

Eigenvalues and eigenvectors are central to matrix analysis, as we saw through several examples in Chapter 1. In this chapter, we define them and investigate some of their elementary properties. We also introduce a transformation, *similarity*, which has the property of not affecting eigenvalues. Finally, we discuss a very interesting result by Gershgorin allowing one to locate eigenvalues.

Chapter contents

2.1 Eigenpairs

First, we recall a well known definition.

Definition 2.1.1 Let $A \in \mathcal{M}_n(\mathbb{F})$. If $\lambda \in \mathbb{C}$ and $v \neq 0 \in \mathbb{F}^n$ are such that $AV = \lambda v$, then λ is an **eigenvalue** of A associated to the **eigenvector** v. We also say that (λ, v) form an **eigenpair**.

■ **Example 2.1** Let $D = \text{diag}(d_1, \dots, d_n)$. Consider the standard basis vectors $e_1, \dots, e_n$. Then, for

$i = 1, \dots, n$,

$$De_i = d_i e_i.$$

Thus, for $i = 1, \dots, n$, (d_i, e_i) are eigenpairs of $D = \operatorname{diag}(d_1, \dots, d_n)$. ■

The eigenpair equation takes the form $Av = \lambda v$, for $v \neq 0$. Rewriting this,

$$Av = \lambda v \iff Av - \lambda v = 0 \iff Av - \lambda \mathbb{I} v = 0 \iff (A - \lambda \mathbb{I})v = 0,$$

where $\mathbb{I}$ is the identity matrix of the appropriate size. Hence, since we seek $v \neq 0$, the homogeneous system $(A - \lambda \mathbb{I})v = 0$ must have non-trivial solutions; this implies that $(A - \lambda \mathbb{I})$ must be singular. So, if λ is an eigenvalue, there must hold that $\det(A - \lambda \mathbb{I}) = 0$.

R It is essential to remember that one seeks a *nonzero* vector v. Clearly, if $v = 0$, then $Av = \lambda v$ for any λ, since this just means that $0 = 0$.

R Definition 2.1.1 is fundamental and underlies the content of these lecture notes. Many of the proofs or solutions to problems in this course start by just writing down the eigenpair equation, $Av = \lambda v$ for $v \neq 0$.

Definition 2.1.2 The **spectrum** of $A \in \mathcal{M}_n$ is the set of all its eigenvalues and its denoted $\sigma(A)$

■ **Example 2.2** Let (λ, v) be an eigenpair of $A \in \mathcal{M}_n(\mathbb{F})$. Then for $c \neq 0$, (λ, cv) is an eigenpair of A.

$$\begin{aligned}
(\lambda, cv) \text{ eigenpair of } A &\iff Acv = \lambda cv \\
&\iff cAv = \lambda cv \\
&\iff c\lambda v = c\lambda v \\
&\iff v = v
\end{aligned}$$

Thus eigenvectors can be expressed *to a multiple*. ■

R Often, we use normalised eigenvectors, $\tilde{v} = v/||v||$, so that $\|\tilde{v}\| = 1$. Also, for eigenvectors v that have all their components nonpositive, we typically use $-v$, so that all components are nonnegative.

■ **Example 2.3** Let (λ, v) be an eigenpair of A. Then

$$\bar{A}v = \bar{A}\bar{v}$$

and

$$\bar{\lambda}v = \bar{\lambda}\bar{v}.$$

Therefore $\bar{A}\bar{v} = \bar{\lambda}\bar{v}$, which implies that $\sigma(\bar{A}) = \overline{\sigma(A)}$.
Now assume $A \in \mathcal{M}_n(\mathbb{R})$, then $\bar{A} = A$. Thus if $\lambda \in \sigma(A)$, then we have that $\bar{\lambda} \in \sigma(A)$. ■

Theorem 2.1.3 $0 \in \sigma(A) \iff A$ is singular.

Proof. ($\Longrightarrow$) Assume $0 \in \sigma(A)$. Since $\det(A - \lambda\mathbb{I}) = 0$ for all $\lambda \in \sigma(A)$, there holds in particular that $\det(A - 0\mathbb{I}) = \det(A) = 0$, so A is singular. ($\Longleftarrow$) Assume A singular. Then $\det(A) = 0$. Since $A = A - 0\mathbb{I}$, it follows that $\det(A) = \det(A - 0\mathbb{I}) = 0$ and thus $0 \in \sigma(A)$. ■

The following result shows that adding a constant to the main diagonal of a matrix "shifts" the spectrum by the same amount.

Theorem 2.1.4 $A \in \mathcal{M}_n(\mathbb{F})$, $\lambda, \mu \in \mathbb{C}$ given. Then $\lambda \in \sigma(A)$ if and only if $\lambda + \mu \in \sigma(A + \mu\mathbb{I})$.

Proof. Assume that $\lambda \in \sigma(A)$, i.e., we have that there exists $v \neq 0$ such that $Av = \lambda v$.

$$\begin{aligned}(A + \mu I)v &= Av + \mu v \\ &= \lambda v + \mu v \\ &= (\lambda + \mu)v \\ &\implies \lambda + \mu \in \sigma(A + \mu\mathbb{I}).\end{aligned}$$

Conversely,

$$\begin{aligned}\lambda + \mu \in \sigma(A + \mu\mathbb{I}) &\iff (A + \mu\mathbb{I}) = (\lambda + \mu)v \\ &\iff Av + \mu v = \lambda v + \mu v \\ &\iff \lambda \in \sigma(A).\end{aligned}$$

■

2.2 Characteristic equation and algebraic multiplicity

Definition 2.2.1 The **characteristic equation** of $A \in \mathcal{M}_n$ is

$$p_A(z) = \det(A - zI).$$

The **characteristic equation** of A is $p_A(z) = 0$

Thus, by the Fundamental Theorem of Algebra, if $p_A(z)$ has degree n, then $p_A(z)$ has n complex roots including multiplicity (or at most n roots if not counting multiplicity). This roots are the eigenvalues of A and thus $\sigma(A)$ has at most n elements in $\mathbb{C}$.

Theorem 2.2.2 Let $A \in \mathcal{M}_n$. Then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \det(A) = \prod_{i=1}^{n} \lambda_i.$$

Let $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a k-degree polynomial. By the fundamental theorem of calculus,

$$p(t) = \prod_{i=1}^{k} (t - \lambda_i), \lambda_i \in \mathbb{C}.$$

If $M \in \mathcal{M}_n$, then

$$p(A) = a_0 I + a_1 A + \ldots + a_k A^k.$$

We can factor the same way

$$p(A) = \prod_{i=1}^{k} (A - \lambda_i I).$$

The λ_i's at the eigenvalues.

Theorem 2.2.3 Let $p(T)$ be a k-degree polynomial. If (λ, v) eigenpair of $A \in \mathcal{M}_n$, then $(p(\lambda), v)$ is an eigenpair for $p(A)$.

Definition 2.2.4 Let $A \in \mathcal{M}_n$. The (algebraic) **multiplicity** of $\lambda \in \sigma(A)$ is its multiplicity as a zero of the characteristic polynomial $p_A(\lambda)$.

Definition 2.2.5 The **spectral radius** of $A \in \mathcal{M}_n$ is

$$\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}.$$

Proposition 2.2.6 For all $\lambda \in \sigma(A)$, $A \in \mathcal{M}_n$, λ lies in the closed bounded disk in $\mathbb{C}$,

$$\{z \in \mathbb{C} \mid |z| \leq \rho(A)\}.$$

■ **Example 2.4** Let A be a block upper triangular matrix, then $p_A(t) = \prod_{i=i}^{k} p_{A_{ii}}(t)$. ■

Theorem 2.2.7 Let $A \in \mathcal{M}_n$, then there exists $\delta > 0$ such that $A + \epsilon I$ is non-singular for $0 < |\epsilon| < \delta$.

Proof. We have seen that $\lambda \in \sigma(A) \iff \lambda + \epsilon \in \sigma(A + \epsilon I)$. Thus if we have $0 \in \sigma(A + \epsilon I) \iff \lambda + \epsilon = 0$ for some $\lambda \in \sigma(A)$ iff $\epsilon = -\lambda$ for some $\lambda \in \sigma(A)$.

If all eigenvalues of A are zero, then $\delta = 1$.

If some eigenvalues of A are non-zero, consider $\delta = \min\{|\lambda| \mid \lambda \in \sigma(A), \lambda \neq 0\}$. Then if we take ϵ such that $0 < |\epsilon| < \lambda$, we have that $-\epsilon \notin \sigma(A)$, i,e., $0 \notin \sigma(A + \epsilon I)$. ■

Theorem 2.2.8 $A \in \mathcal{M}_n$. Suppose that $\lambda \in \sigma(A)$ has algebraic multiplicity k. Then

$$\operatorname{rank}(A - \lambda I) \geq n - k,$$

with equality when $k = 1$.

2.3 Similarity

Definition 2.3.1 Let A, $B\mathcal{M}_n$. We say that B is **similar** to A if there exists a non-singular $S \in \mathcal{M}_n$ such that

$$B = S^{-1}AS.$$

We call the transformation $A \mapsto S^{-1}AS$ a **similarity transformation** with similarity matrix S.If $S = P$, a **permutation matrix** is such that $B = P^TAP$, the we say that A and B are **permutation similar**. We denote "A similar to B" is written as $A \sim B$.

Theorem 2.3.2 Similarity is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.

Theorem 2.3.3 Let A, $B \in \mathcal{M}_n$. If A is similar to B, then they have the same characteristic polynomial:

$$p_A(t) = p_B(t).$$

Proof. Suppose that B is similar to A. Then we have that $B = S^{-1}AS$. Then we have that

$$\begin{aligned} p_B(t) &= \det(B - tI) \\ &= \det(S^{-1}AS - tS^{-1}S) \\ &= \det(S^{-1}(A - tI)S) \\ &= \det(S^{-1})\det(A - tI)\det(S) \\ &= \det(S^{-1})\det(S)\det(A - tI) \\ &= \det(A - tI) \\ &= p_A(t). \end{aligned}$$

■

Corollary 2.3.4 Let A, $B \in \mathcal{M}_n$. If B is similar to A, then

1. A and B have the same eigenvalues.
2. If B is a diagonal matrix, then the main diagonal entries are the eigenvalues of A.
3. $B = 0$ if and only if $A = 0$.
4. $B = I$ if and only if $A = I$

Definition 2.3.5 If $A \in \mathcal{M}_n$. If A is similar to a diagonal matrix, then A is **diagonalisable**.

Theorem 2.3.6 Let $A \in \mathcal{M}_n$.

1. $$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix} \tag{2.1}$$

   with $\Lambda = diag(\lambda_1, \dots, \lambda_k\}$, $D \in \mathcal{M}_{n-k}$, $1 \leq k \leq n$ if and only if k linear independent vectors in $\mathbb{C}^n$, each of which is an eigenvector of A.
2. A diagonalisable if and only if there are n linearly independent eigenvectors, each of which is an eigenvector of A.
3. If $x^{(1)}, \dots, x^{(n)}$ are linear independent eigenvectors of A, define

   $$S = [x^{(1)} \dots x^{(n)}].$$

   Then $S^{-1}AS$ is diagonal.
4. If

   $$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix},$$

   then the diagonal entries of Λ are eigenvalues of A, if $A \sim \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, then $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A.

Look at the example on page 60 from Horn and Johnson for cases where things go wrong.

Lemma 2.3.7 Let $\lambda_1, \dots, \lambda_k$, $k \geq 2$ be k distinct eigenvalue of A. Let $x^{(i)}$ be an eigenvector associated to λ_i, $i = 1, \dots, k$. Then $x^{(1)}, \dots, x^{(k)}$ are linear independent.

Theorem 2.3.8 If $A \in \mathcal{M}_n$ has n distinct eigenvalues, then it is diagonalisable.

Lemma 2.3.9 Let $B = \bigoplus_{i=1}^{d} B_{ii}$. Then B is diagonalisable if and only if each of the B_{ii} is diagonalisable.

Definition 2.3.10 Two matrices A and B in $\mathcal{M}_n$ are **simultaneously diagonalisable** if there exists $S \in \mathcal{M}_n$ non-singular such that $S^{-1}AS$ and $S^{-1}BS$ are diagonal. (Needs to be for the same matrix S.)

Theorem 2.3.11 Let A, $B \in \mathcal{M}_n$ be diagonalisable. Then A and B commute if and only if A and B are simultaneously diagonalisable.

Proof. Exercise. Hint:

1. A, $B \in \mathcal{M}_n$, S non-singular. $AB = BA$ iff $(S^{-1}AS)(S^{-1}BS) = (S^{-1}BS)(S^{-1}AS)$.

2. A, B simultaneously diagonalisable commute

■

(R) See Definition 1.3.16 and following for commuting families and simultaneously diagonalisable families.

2.4 Left and right eigenvectors, geometric multiplicity

Theorem 2.4.1 Let $A \in \mathcal{M}_n$, then

1. $\sigma(A) = \sigma(A^T)$
2. $\sigma(A^*) = \overline{\sigma(A)}$.

Proof. Reminder: $\det A^T = \det A$ and $\det A^* = \overline{\det A}$. [Exercise: show this.]

1. We have that the characteristic polynomial of A^T is

$$\begin{aligned} p_{A^T}(\lambda) &= \det\left(A^T - \lambda \mathbb{I}\right) \\ &= \det\left(A^T - \lambda \mathbb{I}^T\right) \\ &= \det\left((A - \lambda \mathbb{I})^T\right) \\ &= \det(A - \lambda \mathbb{I}) \\ &= p_A(\lambda). \end{aligned}$$

2. We have that the characteristic polynomial of A^* at $\bar{\lambda}$ is

$$\begin{aligned} p_{A^*}(\bar{\lambda}) &= \det\left(A^* - \bar{\lambda} \mathbb{I}\right) \\ &= \det\left(A^* - (\lambda \mathbb{I})^*\right) \\ &= \det\left((A - \lambda \mathbb{I})^*\right) \\ &= \overline{\det\left(A - \lambda \mathbb{I}\right)} \\ &= \overline{p_A(\lambda)}. \end{aligned}$$

As a consequence, $p_{A^*}(\bar{\lambda}) = 0 \iff \overline{p_A(\lambda)} = 0 \iff p_A(\lambda) = 0$ and the eigenvalues of A^* are the complex conjugates of those of A. ■

Definition 2.4.2 Take $A \in \mathcal{M}_n$, for a given $\lambda \in \sigma(A)$, the set of $X \in \mathbb{C}^n$ such that $Ax = \lambda x$ is the eigenspace associated to λ. Every non-zero vector in the eigenspace associated to $\lambda \in \sigma(A)$ is an eigenvector of A associated to λ.

■ **Example 2.5** Suppose x, y are eigenvectors associated to λ. Then any non-zero combination of

x and y is also an eigenvector associated to λ. Let $\alpha,\ \beta \in \mathbb{C}$, then

$$\begin{aligned} A(\alpha x + \beta y) &= A\alpha x + A\beta y \\ &= \alpha A x + \beta A y \\ &= \alpha \lambda x + \beta \lambda y \\ &= \lambda(\alpha x + \beta y). \end{aligned}$$

Thus the set of all eigenvectors corresponding to a given eigenvalue is a subspace of $\mathbb{C}^n$. This subspace is the **nullspace** of $A - \lambda I$ (i.e., the set of x such that $(A-\lambda I)x = 0$). It has dimension $n - \operatorname{rank}(A - \lambda I)$. ■

Definition 2.4.3 Let $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. The dimension of the eigensapce associated to λ is the **geometric multiplicity** of λ. We say that λ is **simple** if its algebraic multiplicity is one, it is **semisimple** if its algebraic and geometric multiplicities are equal.

Proposition 2.4.4 Let λ be an eigenvalue of A. We have that the algebraic multiplicity is greater of equal to the geometric multiplicity. Furthermore, if the algebraic multiplicity is one the the geometric multiplicity is one as well.

Idea of Proof. Let $\lambda \in \sigma(A)$ has algebraic multiplicity k, then $\operatorname{rank}(A - \lambda I) \geq n - k$ with equality when $k = 1$. ■

■ **Example 2.6** Verify the following statements for the eigenvalue $\lambda = 1$.

1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda = 1$, the geometric multiplicity and the algebraic multiplicity are one.
2. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda = 1$, the geometric multiplicity and the algebraic multiplicity are 2.
3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\lambda = 1$, the geometric multiplicity is one and the algebraic multiplicity is 2.

Solution:

1. $p_A(\lambda) = (\lambda - 1)(\lambda - 2)$, thus $\lambda = 1$ has algebraic multiplicity one. To compute the eigenspace associated to $\lambda = 1$, we seek the nullspace of $A - \mathbb{I}$. We have
$$A - \mathbb{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Now $(A - \mathbb{I})v = 0 \iff (0 = 0, v_2 = 0)$. Thus an eigenvector associated to $\lambda = 1$ is $(1, 0)^T$. Hence the eigenspace is the x-axis and its dimension is one; hence $\lambda = 1$ has geometric multiplicity 1.
2. $p_A(\lambda) = (1 - \lambda)^2$, thus $\lambda = 1$ has algebraic multiplicity equal to two. To find the null space,

we write

$$A - \mathbb{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, eigenvectors associated to $\lambda = 1$ are $(1, 0)^T$, $(0, 1)^T$. Thus the geometric multiplicity is two.

3. $p_A(\lambda) = (1-\lambda)^2$ and thus $\lambda = 1$ has algebraic multiplicity 2. The nullspace associated to $\lambda = 1$ is found by considering the matrix

$$A - \mathbb{I} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here, we find a single eigenvector, $(1, 0)^T$. The eigenspace associated to $\lambda = 1$ is the x-axis, thus the geometric multiplicity of $\lambda = 1$ is one.

■

Definition 2.4.5 Let $A \in \mathcal{M}_n$. We say that A is

- **defective** if the geometric multiplicity is less then the algebraic multiplicity for *some* eigenvalue.
- **non-defective** if *for all* eigenvalues, the geometric multiplicity equals the algebraic multiplicity.
- **non-derogatory** if *for all* eigenvalues, the geometric multiplicity is one.
- **derogatory** otherwise.

Theorem 2.4.6 Let $A \in \mathcal{M}_n$.

1. A is diagonalisable if and only if it is nondefective.
2. A has distinct eigenvalues if and only if A is nonderogatory and non-defective.

■ **Example 2.7** In Example 2.6, the first matrix is nondefective and nonderogatory (clearly, since the eigenspace associated to $\lambda = 1$ has dimension 1, so does the eigenspace associated to the remaining eigenvalue $\lambda = 2$); the second matrix is nondefective and derogatory and the third matrix is defective and nonderogatory. Thus, by Theorem 2.4.6, the two first matrices are diagonalisable, the third is not. ■

R $\sigma(A) = \sigma(A^T)$, however they might have different spaces associated to each eigenvalue.

Definition 2.4.7 Let $0 \neq y \in \mathbb{C}^n$, then we say that y is a **left eigenvector** of $A \in \mathcal{M}_n$ associated to $\lambda \in \sigma(A)$ if $y^* A = \lambda y^*$.

Theorem 2.4.8 Let $0 \neq x \in \mathbb{C}^n$, $A \in \mathcal{M}_n$. Assume that $Ax = \lambda x$ for some λ. If $x^* A = \mu x^*$, then $\lambda = \mu$.

Proof. Assume x satisfies the conditions and that $\|x\| = 1$. Then

$$\mu = \mu x^* x = x^* A x = x^* \lambda x = \lambda x^* x = \lambda. \qquad \blacksquare$$

R y is a left eigenvector associated to λ is also a right eigenvector of A^* associated to $\bar{\lambda}$. $\bar{y}$ eigenvector of A^T associated to λ.

Let $A \in \mathcal{M}_n$ diagonalisable, S non-singular matrix, $S^{-1}AS = \Lambda$. Partition $S = [x_1, \dots, x_n]$ and $S^{-*} = [y_1, \dots, y_n]$, where x_i and y_i are the right and left eigenvectors associated to λ_i, respectively.

Theorem 2.4.9 Let $A \in \mathcal{M}_n$, $x,\ y \in \mathbb{C}^n$, $\lambda,\ \mu \in \mathbb{C}$. Assume $Ax = \lambda x$ and $y^* A = \mu y^*$.

1. If $\lambda \neq \mu$, then $y^* x = 0$, then $x \perp y$.
2. If $\lambda = \mu$ and $y^* x \neq 0$, then there exists S non-singular of the form $S = [xS_1]$ such that $S^{-*} = [y/(x^* y) Z_1]$ and $A = S \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} S^{-1}$
   Conversely, if A is similar to a block matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, B \in \mathcal{M}_{n-1},$$

then it has a non-orthogonal pair of left and right eigenvectors associated to λ.

Theorem 2.4.10 Let $A,\ B \in \mathcal{M}_n$, with $A \sim B$ with similarity matrix S. If (λ, x) is an eigenpair of B, then (λ, Sx) is an eigenpair of A. If (λ, y) is a left eigenpair of B, then $(\lambda, S^{-*}y)$ is a left eigenpair of A.

Theorem 2.4.11 Let $A \in \mathcal{M}_n$, $\lambda \in \mathbb{C}$, $x,\ y \in \mathbb{C}^n$ non-zero. Suppose that $\lambda \in \sigma(A)$ and $Ax = \lambda x$ and $y^* A = \lambda y^*$

1. If λ has algebraic multiplicity 1, then $y^* x \neq 0$
2. If λ has geometric multiplicity 1, then it has algebraic multiplicity 1 if and only if $y^* x \neq 0$.

Exercise 2.1 Show Theorems 2.4.10 and 2.4.11.

2.5 The Geršgorin Theorem

Let us now take a detour into something completely different. The previous results concern properties of eigenvalues, eigenvectors and eigenspaces. Now we ask *where are these eigenvalues?* As highlighted in Chapter 1, this problem is prevalent in matrix theory. We return to this problem in much more detail later, but let us start with a result and extensions that made a lot of noise when they were published. This section is based mostly on Varga's book *Geršgorin and His Circles* [**?**], which is highly recommended reading if you enjoy matrix theory.

Let $A \in \mathcal{M}_n(\mathbb{C})$. Denote $N = \{1, \ldots, n\}$. For $i \in N$, define

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|$$

to be the ith deleted row sums of A. Assume that $r_i(A) = 0$ if $n = 1$. Let

$$\Gamma_i(A) = \{z \in \mathbb{C} \mid |z - a_{ii}| \le r_i(A)\} \qquad i \in N$$

be the ith **Gershgorin disk** of A and

$$\Gamma(A) = \bigcup_{i \in N} \Gamma_i(A)$$

be the **Gershgorin set** of A. Γ_i and Γ are closed and bounded in $\mathbb{C}$. $\Gamma_i(A)$ is a disk centred at a_{ii} and with radius $r_i(A)$, $i \in N$.

■ **Example 2.8** Consider the matrix

$$A_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The Gershgorin disks are as shown in Figure 2.1. As a matter of fact, $\sigma(A_1) = \{0, 0\}$. ■

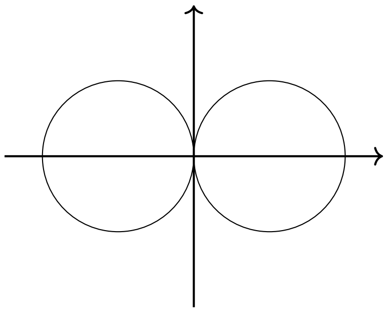


Figure 2.1: The Gershgorin disks in Example 2.8.

Theorem 2.5.1 — Gershgorin, 1931. For all $A \in \mathcal{M}_n(\mathbb{C})$ and for all $\lambda \in \sigma(A)$, there exists $k \in \mathbb{N}$ such that

$$|\lambda - a_{kk}| \le r_k(A),$$

i.e., $\lambda \in \Gamma_k(A)$ and thus $\lambda \in \Gamma(A)$. Since this is true for all λ, we have

$$\sigma(A) \subseteq \Gamma(A).$$

R This also works with deleted column sums; indeed, just consider A^T in this case. However, this typically gives different disks.

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$, let $\lambda \in \sigma(A)$ be given. Consider an associated eigenvector x ($x \neq 0$). Thus $Ax = \lambda x$. Re-write $Ax = \lambda x$ as

$$Ax = \lambda x \iff \forall i \in N, \sum_{j \in N} a_{ij} x_j = \lambda x_i.$$

Because $x \neq 0$, there exists $k \in N$ for which

$$0 < |x_k| = \max\{|x_i|, i \in N\}.$$

For this value of k, i.e., for this row, we have

$$\sum_{i \in N} a_{ki} x_i = \lambda x_k.$$

However,

$$\begin{aligned} \sum_{i \in N} a_{ki} x_i = \sum_{i \in N \setminus \{k\}} a_{ki} x_i + a_{kk} x_k &\iff \lambda x_k = \sum_{i \in N \setminus \{k\}} a_{ki} x_i + a_{kk} x_k \\ &\iff (\lambda - a_{kk}) x_k = \sum_{i \in N \setminus \{k\}} a_{ki} x_i. \end{aligned}$$

Take the modulus on both sides:

$$\begin{aligned} |\lambda - a_{kk}||x_k| &= \left| \sum_{i \in N \setminus \{k\}} a_{ki} x_i \right| \\ &\implies |\lambda - a_{kk}|\,|x_k| \leq \sum_{i \in N \setminus \{k\}} |a_{ki}|\,|x_i| \\ &\implies |\lambda - a_{kk}|\,|x_k| \leq \sum_{i \in N \setminus \{k\}} |a_{ki}|\,|x_k| \qquad (|x_k| = \max\{|x_i|, i \in \mathbb{N}\}) \end{aligned}$$

■

Corollary 2.5.2 Let $A \in \mathcal{M}_n(\mathbb{C})$, then

$$\rho(A) = \max\{|\lambda|,\ \lambda \in \sigma(A)\} \leq \max_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}|.$$

Proof. Let $\lambda \in \sigma(A)$. By Gershgorin's Theorem (Theorem 2.5.1), there exists $k \in N$ such that $|\lambda - a_{kk}| \leq r_k(A)$. Applying the triangle inequality on the left hand side

$$\begin{aligned} &|\lambda - a_{kk}| \geq |\lambda| - |a_{kk}| \\ &\implies |\lambda| - |a_{kk}| \leq |\lambda - a_{kk}| \leq r_k(A) \\ &\implies |\lambda| \leq |a_{kk}| + r_k(A) = \sum_{j \in N} |a_{kj}|. \end{aligned}$$

True for all λ, then for all $\lambda \in \sigma(A)$, $|\lambda| \leq \max_{i \in N} \sum_{j \in N} |a_{ij}|$ ■

Definition 2.5.3 Let $A \in \mathcal{M}_n(\mathbb{C})$. A is **strictly diagonally dominant** (SDD) if

$$\forall i \in N, |a_{ii}| > r_i(A).$$

Theorem 2.5.4 For all $A \in \mathcal{M}_n(\mathbb{C})$ SDD, we have that A is non-singular.

Proof. Suppose that A is SDD and singular. Then $0 \in \sigma(A)$. Then by Gershgorin's Theorem (Theorem 2.5.1), there exists $k \in N$ such that

$$|\lambda - a_{kk}| \leq r_k(A)$$

with $\lambda = 0$ here; so $|a_{kk}| \leq r_k(A)$, which is a contradiction. ■

Let $x \in \mathbb{R}^n$, $x > 0$, i.e., $x = (x_1, \dots, x_n)$ is such that $x_i > 0$ for all i. Let $X = \mathrm{diag}(x) = \mathrm{diag}(x_1, \dots, x_n)$ such that X is invertible. Let $A \in \mathcal{M}_n(\mathbb{C})$, then $X^{-1}AX = \left[\frac{a_{ij}x_j}{x_i}\right]_{i,j\in N}$. Also $X^{-1}AX$ similar to A, so $\sigma(X^{-1}AX) = \sigma(A)$.
Let $r_i^{x_i}(A) = r_i(X^{-1}AX) = \sum_{j\in N\setminus\{i\}} \frac{|a_{ij}|x_j}{x_i}$ be the ith weighted rows sums of A. Let

$$\Gamma_i^{r^x} = \{z \in \mathbb{C},\ |z - a_{ii}| \leq r_i^x(A)\}$$

and

$$\Gamma^{r^x} = \bigcup_{i\in N} \Gamma_i^{r^x}$$

be the ith **weighted Gershgorin disk** and the **weighted Gershgorin set** of A, respectively.

Corollary 2.5.5 For any matrix $A \in \mathcal{M}_n(\mathbb{C})$ and $x \in \mathbb{R}^n$, $x > 0$,

$$\sigma(A) \subset \Gamma^{r^x}(A)$$

Question: How many eigenvalues are contained in each "component"?
Assume $n \geq 2$. Let S be a proper subset of N, i.e., $\emptyset \neq S \subsetneq N$, with $|S|$ its cardinality.

Let $A \in \mathcal{M}_n(\mathbb{C})$, $x > 0$ in $\mathbb{R}^n$ and

$$F_S^{r^x} = \bigcup_{i\in S} \Gamma_i^{r^x}(A).$$

Then

$$\Gamma_S^{r^x}(A) \cap \Gamma_{N\setminus S}^{r^x}(A) = \emptyset.$$

Theorem 2.5.6 For all $A \in \mathcal{M}_m(\mathbb{C})$, for all $x \in \mathbb{R}^n$, $x > 0$ for which

$$\Gamma_S^{r^x}(A) \cap \Gamma_{N\setminus S}^{r^x}(A) = \emptyset.$$

for some proper subset S of N, then $\Gamma_S^{r^x}(A)$ contains exactly $|S|$ eigenvalues of A.

Proof. Consider the set $A(t) = [a_{i,j}(t)] \in \mathcal{M}_n(\mathbb{C})$ with $a_{ii}(t) = a_{ii}$ and $a_{ij}(t) = ta_{ij}$, $i \neq j$ and $t \in [0, 1]$. $A(0) = \operatorname{diag}(a_{11}, \dots, a_{nn})$ and $A(1) = A$.

Then

$$\begin{aligned} r_i^x(A(t)) &= \sum_{j\in N\setminus\{i\}} \frac{|a_{ij}|x_j}{x_i} \\ &= t \sum_{j\in N\setminus\{i\}} \frac{|a_{ij}|x_j}{x_i} \\ &= tr_i^x(A) \\ &\leq r_i^x(A) \qquad \forall t \in [0,\ 1]. \end{aligned}$$

Thus for all $t \in [0,\ 1]$, $\Gamma_i^{rx}(A(t)) \subset \Gamma_i^{r^x}(A)$, $i \in N$.
Consider this together with the assumption

$$\Gamma_S^{r^x}(A) \cap \Gamma_{N\setminus S}^{r^x}(A) = \emptyset.$$

This implies that

$$\Gamma_S^{r^x}(A(t)) \cap \Gamma_{N\setminus S}^{r^x}(A(t)) = \emptyset, \qquad \forall t \in [0, 1].$$

From Corollary 2.5.5, we have that

$$\sigma(A) \subset \Gamma^{r^x}(A(t)), \forall t \in [0, 1].$$

For $t = 0$, $A(0) = \operatorname{diag}(a_{11}, \dots, a_{nn})$, and thus

$$\sigma(A(0)) = \{a_{11}, \dots, a_{nn}\}.$$

Thus

$$\Gamma_S^{r^x}(A(0)) = \{a_{ii}, i \in S\}$$

contains exactly $|S|$ eigenvalues of $A(0)$. We have seen that the eigenvalues $\lambda_i(t)$ of $A(t)$ depend closely on $t \in [0, 1]$. But, $\Gamma_S^{r^x}(A(t)) \cap \Gamma_{N\setminus S}^{r^x}(A(t)) = \emptyset$ for all $t \in [0, 1]$ means that it is impossible to add or remove eigenvalues from $\Gamma_S^{r^x}(A(t))$, and thus $\Gamma_S^{r^x}(A(t))$ contains $|S|$ eigenvalues for $t \in [0, 1]$. And $A(1) = A$, so $\Gamma_s^{r^x}(A)$ contains $|S|$ eigenvalues. ∎

2.6 Extensions of Geršgorin disks using graph theory

We have seen that a matrix A that is SDD is nonsingular. Can we weaken this? What if diagonal dominance is not strict, i.e., $|a_{ii}| = r_i(A)$ for some $i \in N$, $|a_{ii}| \geq r_i(A)$ for all $i \in N$. This is not sufficient for nonsingularity. If we take the matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

is DD and singular, however,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is DD and singular.

Definition 2.6.1 $A \in \mathcal{M}_n(\mathbb{C})$ is **reducible** if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbb{R})$ and $r \in N = \{1, \dots, n\}$ such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where $A_{11} \in \mathcal{M}_r$, $A_{22} \in \mathcal{M}_{n-r}$. If there is no such P, then we say that A is **irreducible**.

R If $A \in \mathcal{M}_1$, then A irreducible if $a_{11} \neq 0$.

In the reducible case, we can continue the process and find a matrix P (permutation) such that

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ & & \ddots & \\ 0 & & \dots & R_{nm} \end{pmatrix}$$

with the diagonal block R_{ii} irreducible. This is the **normal reduced form** of A.

R Establishing irreducibility this way is *hard*. If no obvious permutation of rows and columns gives rise to a matrix in reduced form, then deciding on irreducibility requires to exhaust all possible permutation matrices to assert none exists. There are $n!$ permutation matrices of size $n \times n$.

Let $A \in \mathcal{M}_n(\mathbb{C})$. Let $\{v_1, \dots, v_n\}$ be n distinct points called vertices. For any (i,j), $i,j \in N$, for which $a_{ij} \neq 0$, connect v_j to v_j using a directed arc $\overrightarrow{v_i v_j}$. If $a_{ii} \neq 0$, there is a loop from v_i to v_j. The collection of all the directed arcs (and loops) obtained thusly is called the **directed graph** (or **digraph**) associated to A and is denoted $\mathcal{G}(A)$.

A **directed path** in $\mathcal{G}(A)$ is a collection of directed arcs from v_i to v_j, i.e.,

$$\overrightarrow{v_{i_1}v_{i_2}}, \ldots, \overrightarrow{v_{i_{n-1}}v_{i_n}}.$$

Along a directed path,

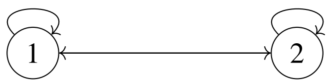
$$\prod_{k=1}^{n-1} a_{i_k} a_{i_{k+1}} \neq 0.$$

R Given a graph $\mathcal{G}$, the matrix A such that $\mathcal{G}(A) = \mathcal{G}$ is the **adjacency matrix** of $\mathcal{G}$.

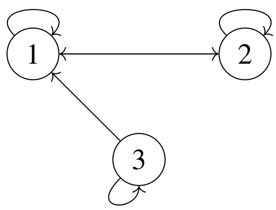
Definition 2.6.2 Let $\mathcal{G}$ be a digraph with vertex set $\{v_1, \ldots, v_n\}$. $\mathcal{G}$ is **strongly connected** if for all ordered pairs (v_i, v_j) of vertices, there is a directed path from v_i to v_j in $\mathcal{G}$.

■ **Example 2.9**

$$A_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$



$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$



not strongly connected because there is no path from 1 or 2 to 3 ■

R If $\mathcal{G}(A)$ is strongly connected, then A cannot have a row with only zero off-diagonal entries. Indeed, suppose $\mathcal{G}(A)$ is strongly connected. Without loss of generality, assume row 1 in A has only zero off-diagonal entries. Then because of the way $\mathcal{G}(A)$ is constructed, this means there are no directed arcs terminating in v_1 and as a consequence, there is no directed path terminating in v_1, contradicting strong connectedness of $\mathcal{G}(A)$.

R $\mathcal{G}(A)$ is strongly connected if and only if for any permutation matrix P, we have that $\mathcal{G}(P^T AP)$ is strongly connected. [Because permutation is a relabelling of vertices.]

The following result is of paramount importance in matrix theory, as it connects the field with graph theory and in particular, opens up brand new ways of checking Definition 2.6.1.

Theorem 2.6.3 Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is irreducible if and only $\mathcal{G}(A)$ is strongly connected.

Idea of the Proof. We show that A is reducible if and only if $\mathcal{G}(A)$ is not strongly connected.

A reducible means that A is of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

this implies that none of the vertices in A_{22} have access to $a_{11})v_i$. If $G(A)$ is not strongly connected, then there exists v_1 that is not accessible from $v_2, \ldots, v_n$. ■

Definition 2.6.4 $A \in \mathcal{M}_n(\mathbb{C})$ is **irreducibly diagonally dominant** (IDD) if A is irreducible, diagonally dominant, i.e.,

$$\forall i \in N, \quad |a_{ii}| \geq r_i(A)$$

and there exists $i \in N$ for which diagonal dominance is strict, i.e., there exists i such that $|a_{ii}| = r_i(A)$.

Theorem 2.6.5 — Taussky 1949 [?]. For any $A \in \mathcal{M}_n(\mathbb{C})$, A IDD $\Rightarrow$ A non-singular.

Proof. Case $n = 1$ trivial (by the definition). So assume $n \geq 2$.

Assume that A is IDD and singular. Since A is singular, we have that $0 \in \sigma(A)$; thus, there exists $x \neq 0$ such that $Ax = 0$. Since $x \neq 0$, we normalize x so that

$$\max_{i \in N}\{|x_i|\} = 1. \tag{2.2}$$

Let $S = \{j \in N, |x_j| = 1\}$ be the set of indices of entries of x that achieve this maximum modulus. Then, clearly, $\emptyset \neq S \subseteq N$. Then we start like in the proof of Geršgorin's Theorem (Theorem 2.5.1) and rewrite $Ax = 0$ row by row:

$$\begin{aligned} Ax = 0 \iff & \sum_{j \in N} a_{kj} x_j = 0, \qquad \forall k \in N \\ \iff & -a_{kk} x_k = \sum_{j \in N \setminus \{k\}} a_{kj} x_j, \qquad \forall k \in N. \end{aligned}$$

Now consider any $i \in S$ and set $k = i$ in the previous equation, giving

$$-a_{ii} x_i = \sum_{j \in N \setminus \{i\}} a_{ij} x_j.$$

Take the modulus and use the triangle inequality:

$$|a_{ii}|\,|x_i| = |a_{ii}| \le \sum_{j\in N\setminus\{i\}} |a_{ij}||x_j| \le \sum_{j\in N\setminus\{i\}} |a_{ij}| = r_i(A), \tag{2.3}$$

where the first equality follows from the fact that $i \in S$ and thus $|x_i| = 1$ and the second inequality follows from the normalisation (2.2). However, by the diagonal dominance hypothesis, we have $|a_{ii}| \ge r_i(A)$ for all $i \in N$; thus, for all $i \in S$, (2.3) takes the form

$$|a_{ii}| = \sum_{j\in n\setminus\{i\}} |a_{ij}|\,|x_j| = r_i(A) \qquad (i \in S). \tag{2.4}$$

Now recall that we have also assumed that there exists i such that $|a_{ii}| > r_i(A)$. As a consequence, S is a proper subset of N, since $|a_{ii}| = r_i(A)$ cannot under this assumption be true for all i.

By an earlier remark, an irreducible matrix cannot have any row with only zero off-diagonal entries. As a consequence, for all $i \in S$, by irreducibility of A, the terms in $\sum_{j\in N\setminus\{i\}} |a_{ij}|$ cannot all be zeros. Thus consider an $a_{ij} \neq 0$ for some $j \neq i \in N$. From (2.4), it follows that $|x_j| = 1$, i.e., $j \in S$.

A irreducible implies that for any (v_i, v_j), there is a directed path from v_i to v_j in $\mathcal{G}(A)$. Let us call them $a_{i_1 i_1}, \dots, a_{i_{r-1} i_n}$ are non-zero with $i_n = k$. Thus we have that $i_l \in S$ for all $1 \le l \le r$ and in particular, $i_r = k \in S$. Therefore for any $k \in N$, this is true, which means that $S = N$, which is a contradiction. ■

Here is another result of Taussky.

Theorem 2.6.6 Let $A \in \mathcal{M}_n(\mathbb{C})$ be irreducible. Suppose $\lambda \in \sigma(A)$ be such that $\forall i \in N$, $\lambda \notin \operatorname{Int} \Gamma_i(A)$. Then

$$\forall i \in N, \quad |\lambda - a_{ii}| = r_i(A). \tag{2.5}$$

In particular, if $\lambda \in \partial\Gamma(A)$ [the boundary of $\Gamma(A)$] for some $\lambda \in \sigma(A)$, then (2.5) holds for λ.

CHAPTER 3

Factorisations, canonical forms and decompositions

Chapter contents

3.1 Unitary matrices and QR factorisation

Definition 3.1.1 Let $x_1, \dots, x_k \in \mathbb{C}^n$. We say that $x_1, \dots, x_k$ is an **orthogonal list** if $x_i^* x_j = 0$ for all $i \neq j$. If, in addition, we have that $x_i^* x_i = 1$, then we say that the list is **orthonormal**.

Theorem 3.1.2 Every orthonormal list of vectors in $\mathbb{C}^n$ is linearly independent.

Proof. Let $\{x_1, \dots, x_k\}$ be an orthonormal list of vectors in $\mathbb{C}^n$. Let us assume that

$$\sum_{i=1}^{k} a_i x_i = 0,$$

for $a_1, \dots, a_k \in \mathbb{C}$. Then

$$\left(\sum_{i=1}^{k} a_i x_i\right)^* \left(\sum_{i=1}^{k} a_i x_i\right) = 0$$

$$= \sum_{i=1}^{k} \bar{a}_i a_i x_i^* x_i$$

$$= \sum_{i=1}^{k} |a_i|^2 x_i^* x_i$$

$$= \sum_{i=1}^{k} |a_i|^2.$$

This implies that $a_i = 0$ for all i's, and hence the list is linear independent. ■

R If "only" orthogonal, we need to replace "list of vectors" by "list of non-zero vectors" in the statement.

Definition 3.1.3 Let $U \in \mathcal{M}_n$, we say that U is an **unitary matrix** if $U^*U = \mathbb{I}$. Furthermore, we say that $U \in \mathcal{M}_n(\mathbb{R})$ is a **(real) orthogonal matrix** is $U^T U = \mathbb{I}$.

Theorem 3.1.4 Let $U \in \mathcal{M}_n$. Then the following are equivalent:

1. U is unitary;
2. U is non-singular and $U^* = U^{-1}$;
3. $UU^* = \mathbb{I}$;
4. U^* is unitary
5. the columns of U are orthonormal;
6. the rows of U are orthonormal;
7. for all $x \in \mathbb{C}^n$ we have $\|x\|_2 = \|Ux\|_2$

Proof. (1 $\implies$ 2) Recall that $A \in \mathcal{M}_n$ is invertible if there exists $B \in \mathcal{M}_n$ such that $BA = \mathbb{I}$, with then $A^{-1} = B$. (In full, A invertible if there exists B such that $AB = BA = \mathbb{I}$; however, it suffices for one of the two equalities to hold.) By definition, U unitary requires that $U^*U = \mathbb{I}$. As a consequence, if U unitary, then $U^* = U^{-1}$ (and U^{-1} exists so U nonsingular).

(2 $\implies$ 3) Assume U is nonsingular and such that $U^* = U^{-1}$. As U nonsingular, $UU^{-1} = U^{-1}U = \mathbb{I}$. Then substituting U^* for U^{-1} in this equality gives the result.

(3 $\implies$ 4) Assume $UU^* = \mathbb{I}$. Then U invertible with inverse U^* and since $UU^{-1} = U^{-1}U = \mathbb{I}$, it follows that $U^*U = \mathbb{I}$, i.e., U unitary.

(4 $\implies$ 1) U^* unitary $\iff$ $(U^*)^*U = \mathbb{I}$ $\iff$ $(UU^*)^* = \mathbb{I}$ $\iff$ $UU^* = \mathbb{I}$ $\iff$ U unitary.

(1 $\iff$ 5) Assume U unitary and write $U = [u_1 \cdots u_n]$. Then

$$\begin{aligned} U^*U = \mathbb{I} &\iff [u_1 \ldots u_n]^*[u_1 \ldots u_n] = \mathbb{I} \\ &\iff \begin{pmatrix} \overline{u_1}^T \\ \vdots \\ \overline{u_n}^T \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} = \mathbb{I} \\ &\iff \begin{cases} u_i^* u_i = 1 \\ u_i^* u_j = 0 & \text{for all } i \neq j. \end{cases} \end{aligned}$$

Thus the columns are orthonormal.

(1 $\iff$ 6) Similar to the previous case.

(1 $\implies$ 7) Assume that U is unitary and let $y = Ux$, for $x \in \mathbb{C}^n$. Then $y^*y = (Ux)^*(Ux) = x^*U^*Ux = x^*x$. Thus $\|y\|^2 = \|x\|^2 \implies \|y\| = \|x\| \iff \|Ux\| = \|x\|$.

(7 $\implies$ 1) Assume that for all $x \in \mathbb{C}^n$, there holds that $\|x\|_2 = \|Ux\|_2$. Let $U^*U = A = [a_{ij}] \in \mathcal{M}_n$, $z,\ w \in \mathbb{C}^n$. Let $x = z + w$, $y = Ux$.

Then $x^*x = (z+w)^*(z+w) = z^*z + w^*w + 2\operatorname{Re}(z^*w)$ [by the polarisation identity $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x,\ y\rangle$].

Now

$$\begin{aligned} y^*y &= (Ux)^*(Ux) \\ &= x^*U^*Ux \\ &= x^*Ax \\ &= (z+w)^*A(z+w) \\ &= z^*AZ + w^*Aw + 2\operatorname{Re} z^*Aw. \end{aligned}$$

$$\begin{aligned} z^*Aw &= z^*U^*Uz \\ &= (Uz)^*Uz \\ &= z^*z. \end{aligned}$$

Similary, $w^*Aw = w^*w$, this implies that

$$y^*y = z^*z + w^*w + 2\operatorname{Re}(z^*Aw),$$

which implies that $\operatorname{Re}(z^*Aw) = \operatorname{Re}(z^*w)$ for all $z,\ w$.
Take $z = e_p$ and $w = ie_q$, then

$$\operatorname{Re}(ie_p^T e_q) = 0 = \operatorname{Re}(ie_p^T Ae_q) = \operatorname{Re}(ia_{pq}) = -\mathfrak{I}(a_{pq}.$$

Thus we have that $A \in \mathcal{M}_n(\mathbb{R})$. If we take $z = e_p$ and $w = e_q$, we have that

$$e_p^T e_q = \operatorname{Re}(e_p^T e_q) = \operatorname{Re}(e_p^T A e_q) = a_{pq}.$$

Thus $A = \mathbb{I}$ and hence U is unitary. ■

Definition 3.1.5 A **linear transformation** $T : \mathbb{C}^n \to \mathbb{C}^n$ is a **Euclidean isometry** if $\|x\|_2 = \|Tx\|_2$ for all $x \in \mathbb{C}^n$.

Corollary 3.1.6 Let $U \in \mathcal{M}_n$. We say that U is a Euclidean isometry if and only if U is unitary.

R Let $U, V \in \mathcal{M}_n$ are unitary matrices (respectively real orthogonal), then UV is unitary (respectively real orthogonal).

Indeed, U, V unitary $\Leftrightarrow U^{-1}, V^{-1}$ exist and $U^{-1} = U^*, V^{-1} = V^*$. Then

$$\begin{aligned} UV \text{ unitary } &\Leftrightarrow (UV)^*UV = \mathbb{I} \\ &\Leftrightarrow V^*U^*UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I}. \end{aligned}$$

Notation: $\mathrm{GL}(n, \mathbb{F})$ is the general linear group, where the elements are non-singular matrices in $\mathcal{M}_n(\mathbb{F})$.

Theorem 3.1.7 The set of unitary (respectively real orhogonal) matrices in $\mathcal{M}_n$ forms a group, the $n \times n$ unitary (respectively real orthogonal) subgroup of $\mathrm{GL}(n, \mathbb{C})$ (respectively $\mathrm{GL}(n, \mathbb{R})$).

The set of unitary matrices is a closed and bounded subset of $\mathbb{C}^{n^2}$. Indeed, no unitary matrix has entries with modulus larger than 1. (This is because $U^*U = \mathbb{I}$ implies that each row or column of U^*U has Euclidean norm 1.) This gives us the boundedness. Let $U_k = [u_{ij}^{(k)}]$ be an infinite sequence of unitary matrices and assume that

$$\lim_{k\to\infty} u_{ij}^{(k)} = u_{ij}$$

exist for all i, j. Then since $U^*U = \mathbb{I}$ for all k, we have that

$$\lim_{k\to\infty} U_k^* U_k = U^*U = \mathbb{I}.$$

Thus the limit is also unitary, giving the closeness.

Theorem 3.1.8 — Selection Principle. Suppose that we have a sequence of unitary matrices $U_1, U_2 \ldots, \in \mathcal{M}_n$. Then there exists a subsequence $U_{k_1}, U_{k_2} \ldots$ such that the entries of U_{k_i} converge to entries of a unitary matrix as $i \to \infty$.

Lemma 3.1.9 Let $U \in \mathcal{M}_n$ be a unitary matrix partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with $U_{ii} \in \mathcal{M}_k$. Then rank U_{12} = rank U_{21} and rank U_{22} = rank $U_{11} + n - 2k$. If, furthermore, $U_{21} = 0$ and $U_{12} = 0$, then U_{11} and U_{22} are unitary.

Theorem 3.1.10 — QR factorisation. Let $A \in \mathcal{M}_{nm}$.

1. If $n \geq m$, there is a $Q \in \mathcal{M}_{nm}$ with orthogonormal columns and upper triangular $R \in \mathcal{M}_m$ with non-negative main diaginal entries such that $A = QR$.
2. If rank $A = m$ then the factors Q and R in (1) are uniquely determined and the main diagonal entries of R are all positive.
3. If $n = m$, Then the factor Q in (1) is unitary.
4. There is a unitary $Q \in \mathcal{M}_n$ and an upper triangular $R \in \mathcal{M}_{nm}$ with nonnegative diagonal entries such that $A = QR$.
5. If A is real, then Q and R are in (1), (2), (3), and (4) may be taken to be real.

3.2 Schur's Form

We require the following:

For a U unitary matrix, $U^* = U$, so the transformation $A \mapsto U^*AU$ is a similarity transformation, provided that U is unitary. This is a unitary similarity.

Definition 3.2.1 Let $A, B \in \mathcal{M}_n$. We say that A is unitary similar to B if there exists $U \in \mathcal{M}_n$ unitary such that

$$A = U^*BU.$$

If U can be taken real (i.e., if U is real orttohonal) than A is real orthogonal similar to B (if $A = U^TBU$).

R

1. Unitary similarity is an equivalence relation.
2. Unitary similarity implies similarity. However, the converse is not true.
3. Similarity is a change of bases. Unitary similarity is a change of orthonormal bases.

Definition 3.2.2 — Householder matrix. Let $0 \neq \omega \in \mathbb{C}^n$. The Householder matrix $U_\omega \in \mathcal{M}_n$ is

$$U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*.$$

R

1. If $\|\omega\| = 1$ then $U_\omega = \mathbb{I} - 2\omega\omega^*$
2. Householder matrix are unitary and Hermitian, thus $U_\omega^{-1} = U_\omega$.
3. The eigenvalues of a Householder matrix are $-1, 1, \ldots, 1$ and $|U_\omega| = 1$.

Theorem 3.2.3 Let $x, y \in \mathbb{C}^n$ and assume that $\|x\|_2 = \|y\|_2 > 0$.

- If $y = e^{i\theta}x$ for some $\theta \in \mathbb{R}$ [x, y are linearly dependent], define $U(y, x) = e^{i\theta}\mathbb{I}$.
- Otherwise, let $\phi \in [0, 2\pi)$ be such that $x^*y = e^{i\phi}|x^*y|$ (taking $\phi = 0$ if $x^*y = 0$). Let $\omega = e^{i\phi}x - y$ and define

$$U(y, x) = e^{i\phi}U_\omega,$$

where $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$ is Householder.

1. $U(y, x)$ unitary and essentially Hermitian [$e^{i\theta}U(y, x)$ Hermitian for some $\theta \in \mathbb{R}$].
2. $U(y, x)x = y$.
3. $U(y, x)z \perp y$, when $z \perp y$.
4. If $x, y \in \mathbb{R}^n$, then $U(y, x)$ is real and $U(y, x) = \mathbb{I}$ if $y = x$ and $U(y, x) = U_{x-y} \in \mathcal{M}_n(\mathbb{R})$ otherwise.

R For all $A \in \mathcal{M}_n$, $U(y, x)^*AU(y, x) = U_\omega^*AU_\omega$. This is called a Householder transformation.

Theorem 3.2.4 — Schur's Form. Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ in any prescribed order (including multiplicities). Let $x \in \mathbb{C}^n$, $\|x\| = 1$, be such that $Ax = \lambda_1 x$

1. There exists $U = [x\, u_2 \ldots u_n] \in \mathcal{M}_n$ unitary such that $U^*AU = T$, where T is upper triangular such that $t_i i = \lambda_i$, $i = 1, \ldots, n$.
2. If $A \in \mathcal{M}_n(\mathbb{R})$ and has real eigenvalues, then x can be chosen to be real and there exists

$$Q = [x\, q_2 \ldots q_n] \in \mathcal{M}_n(\mathbb{R})$$

real orthogonal and such that $Q^TAQ = T$, with T upper triangular with $t_{ii} = \lambda_1$ $i = 1, \ldots, n$.

Theorem 3.2.5 — Schur version 2. Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ (including mutiplicities). Then there esists $U \in \mathcal{M}_n$ such that

$$U^*AU = \begin{pmatrix} \lambda_1 & * & \ldots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

R The decomposition is not unique.

Proof. Let x, $\|x\| = 1 = x^*x$, such that $Ax = \lambda_1 x$.

Take U be a unitary matrix of the form $U_1 = [x\, u_2 \dots u_n]$. For this take $U_1 = U(x, , e_1)$. Take

$$\begin{aligned} U_1^* A U_1^* &= U_1^* \begin{pmatrix} Ax & Au_2 & \dots Au_n \end{pmatrix} \\ &= U_1^* \begin{pmatrix} \lambda_1 x & Au_2 & \dots Au_n \end{pmatrix} \\ &= \begin{pmatrix} x^* \\ u_2^* \\ \vdots \\ u_n^* \end{pmatrix} \begin{pmatrix} \lambda_1 x & Au_2 & \dots Au_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 x^* x & x^* Au_2 & \dots x^* AU_n \\ \lambda_1 u_2^* x & u_2^* Au_2 & \dots u_2^* Au_n \\ \vdots & & \\ \lambda_1 u_n^* x & u_n^* Au_n & \dots u^* Au_n \end{pmatrix}. \end{aligned}$$

Since $x^*x = 1$ and U_1 is unitary (thus the columns are orthogonal] we have that

$$U_1^* A U_1 = \begin{pmatrix} \lambda_1 & * \\ \mathbf{0} & A \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} x^* Au_2 & \dots & x^* AU_n \\ u_2^* Au_2 & \dots & u_2^* Au_n \\ \vdots & & \\ u_n^* Au_n & \dots & u^* Au_n. \end{pmatrix}$$

Now A_1 has eigenvalues $\lambda_2, \dots, \lambda_n$ due to the mechanism used for partition.
If $n = 2$ we are done [$A_1 \in \mathcal{M}_1$ thus $U_1^* A U_1$ is upper triangular].
If not, then let $y \in \mathbb{C}^{n-1}$ be such that $\|y\| = 1$ and $A_1 y = \lambda_2$. Let U_2 be unitary with first column. Then

$$U_2^* A_1 U_2 = \begin{pmatrix} \lambda_2 & * \\ 0 & A_2 \end{pmatrix}$$

Let $V_2 = [1] \oplus U_2$, then

$$(U_1V_2)^*AU_1V_2 = V_2^*U_1^*AU_1V_2 \tag{3.1}$$

$$= \begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{pmatrix}. \tag{3.2}$$

Continuing this process, producing unitary matrices $U_i \in \mathcal{M}_{n-i+1}$ $i = 1, \dots, n-1$ and unitary matrices $V_i \in \mathcal{M}_n$ $i = 1 \dots, n$.

Then $U = U_1V_2V_3 \dots V_{n-2}$ is unitary and U^*AU is upper triangular. ■

Theorem 3.2.6 Let $U \in \mathcal{M}_n$, $A,\ B \in \mathcal{M}_n$. Suppose A is unitarily similar to B, then

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

Corollary 3.2.7 Let $A \in \mathcal{M}_n$ have eigenvalues $\lambda_1, \dots, \lambda_n$, $T = UAU^*$ upper triangular. Then

$$\sum_{i=1}^{n} |\lambda_1|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \le \sum_{i,j=1} |a_{ij}|^2 = \operatorname{Tr} AA^*$$

with equality if T is diagonal.

3.3 Consequences of Schur's triangularisation theorem

Theorem 3.3.1 — Cayley-Hamilton. Let $A \in \mathcal{M}_n$ and $p_A(t)$ is the characteristic polynomial of A, then $p_A(A) = 0$.

Another important result that is a Consequence of Schur's is Sylvester's theorem [pole placement problem]

Theorem 3.3.2 (Every square matrix is block is block triangonalizable and diagonalizable) Assume $A \in \mathcal{M}_n$ has eigenvalues $\lambda_1, \dots, \lambda_n$ with multiplicities $n_1, \dots, n_d$ $(\sum_{i=1}^{d} n_i = n)$. Then A is unitary similar to a $d \times d$ block upper triangular matrix T, where $T_{i,j} \in M_{n_i m_j}$, $T_{ij} = 0$ if $i > i$, T_{ii} upper triangular with diagonal λ_i, $T_{ii} = \lambda \mathbb{I} + R_i$, R_i strictly upper triangular, and A is similar to a matrix to $\bigoplus_{i=1}^{d} T_{ii}$ [standard similarity, not unitary]

Theorem 3.3.3 (Every square matrix is almost diagonalisble) Let $A \in \mathcal{M}_n$ for all $\epsilon > 0$, there exists $A(\epsilon)[a_{ij}(\epsilon)] \in \mathcal{M}$ with distinct eigenvalues such that

$$\sum_{i,j} |a_{ij} - a_{ij}(\epsilon)|^2 < \epsilon$$

Theorem 3.3.4 If $A \in \mathcal{M}_n$ for all $\epsilon > 0$ there exists $S(\epsilon) \in \mathcal{M}_n$ non-singular such that

$$S^{-1}(\epsilon)AS(\epsilon) = T(\epsilon),$$

where $T(\epsilon)$ is upper triangular and $|t_{ij}(\epsilon)| < \epsilon$ for all i, j, with $i < j$.

Lemma 3.3.5 Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim_{k\to\infty} A_k = A$ (entry-wise). Then there exists $k_1 < k_2 < \dots$ and $U_{k_i} \in \mathcal{M}$ such that

1. $T_i = U_{k_i}^* A_{k_i} U_{k_i}$ upper triangular.
2. $U + \lim_{i\to\infty} U_{k_i}$ exists and is unitary.
3. $T = U^*AU$ upper triangular
4. $\lim_{i\to\infty} T_i = T$.

Theorem 3.3.6 Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim_{k\to\infty} A_k = A$ (entry-wise). Then let

$$\lambda(A) = \begin{bmatrix}\lambda_1(A) & \dots & \lambda_n(A)\end{bmatrix}^T$$

and

$$\lambda(A_k) = \begin{bmatrix}\lambda_1(A_k) & \dots & \lambda_n(A_k)\end{bmatrix}^T$$

be presentations of the eigenvalues of A and A_k. Define

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1, \dots, n\}\}.$$

Then for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N} \setminus \{0\}$ such that

$$\min_{\pi\in S_n} \max_{i=1,\dots} \{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \le \epsilon \qquad \forall k \ge N(\epsilon)$$

Recall that if x, y are two (column) vectors in $\mathbb{F}^n$, then xy^* is a rank 1 matrix in $\mathcal{M}_n(\mathbb{F})$. (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly.

Theorem 3.3.7 — Brauer. Suppose $A \in \mathcal{M}_n$ has eigenvalues $\lambda, \lambda_2, \dots, \lambda_n$. Let x be an eigenvector associated to λ. Then for every vector $v \in \mathbb{C}^n$, the eigenvalues of $A+x^*v$ are $\lambda+v^*x, \lambda_2, \dots, \lambda_n$.

3.4 Normal Matrices

Definition 3.4.1 A matrix $A \in \mathcal{M}_n$ is **normal** if $AA^* = A^*A$.

Every unitary matrix, every hermitian or skew-hermitian matrix, diagonal matrix is normal.

Theorem 3.4.2 Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, then the following are equivalent:

1. A is normal.
2. A is unitary diagonalisable.
3. $\sum_{i,j} |a_{i,j}|^2 = \sum_i |\lambda_i|^2$
4. A has n orthogonal eigenvectors.

Theorem 3.4.3 Let $A \in \mathcal{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Then

1. $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
2. A is unitary diagonalisable.
3. there exists $U \in \mathcal{M}_n$ such that $A = U\Lambda U^*$.

3.5 Jordan Canonical Form

Definition 3.5.1 A **Jordan block** $J_k(\lambda)$ is a $k \times k$ upper triagular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

Theorem 3.5.2 Let $A \in \mathcal{M}_n$ then there exists $S \in \mathcal{M}_n$ non-singular such that

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) S^{-1}$$

Theorem 3.5.3 Let $A \in \mathcal{M}_n$ with real eigenvalues. Then there exists a basis of generalised eigenvectors for $\mathbb{R}^n$, and if $\{v_1, \dots, v_n\}$ is a basis of generalised eigenvectors of $\mathbb{R}^n$, then $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ is non-singular and $A = D + N$ where $P^{-1}DP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $N = A - D$ is nilpotent[a] of order $k \leq n$, and D and N commute.

[a] nilpotent of rank k: $A^k = 0$

3.6 Singular values and the Singular value decomposition

Definition 3.6.1 Let A be a Hermitian matrix in $\mathcal{M}_n$. We say that A is **positive definite** if for all $0 \neq x \in \mathbb{C}^n$, $x^*Ax > 0$. We say that A is **positive semidefinite** if for all $x \in \mathbb{C}^n$, $x \neq 0$, $x^*Ax \geq 0$.

Theorem 3.6.2 Let $A \in \mathcal{M}_n$ be a Hermitian matrix. Then

1. for all $x \in \mathbb{C}^*$, $x^* A x \in \mathbb{R}$.
2. $\sigma(A) \subset \mathbb{R}$
3. $S^* A S$ is Hermitian for any $S \in \mathcal{M}_n$.

Proof. 1.

$$\begin{aligned}\overline{x^* AX} &= (x^* Ax)^* \\ &= x^* A^* x \\ &= X^* Ax,\end{aligned}$$

which implies that $x^* AX \in \mathbb{R}$.

2. Let $x \neq 0$ be such that $Ax = \lambda x$. Assume that $x^* x = 1$. Then

$$\begin{aligned}\lambda &= \lambda x^* x \\ &= x^* \lambda x \\ &= x^* Ax \in \mathbb{R} \qquad \text{(by 1)}\end{aligned}$$

3.

$$\begin{aligned}(S^* AS)^* &= S^* A^* S \\ &= S^* AS.\end{aligned}$$

■

Theorem 3.6.3 Each eigenvalue of a positive definite matrix (respectively positive semidefinite matrix) is positive (respectively nonnegative).

Proof. Let A be a positive semidefinite matrix and (λ, x) eigenpair of A. Then

$$\begin{aligned}x^* Ax &= x^* \lambda x \\ &= \lambda x^* x.\end{aligned}$$

Thus $\lambda = \dfrac{x^* Ax}{x^* x} \geq 0$ (and > 0 if positive definite). ■

Proposition 3.6.4 Let A be a positive semidefinite (respectively positive definite) matrix. Then $\mathrm{Tr}(A)$, $\det(A)$, the principal minors of A are all nonnegative (respectively positive). Also, $\mathrm{Tr}(A) = 0$ if and only if $A = 0$.

Theorem 3.6.5 Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix and $x \in \mathbb{C}^n$. Then

$$x^* A x = 0 \iff Ax = 0.$$

Corollary 3.6.6 Let $A \in \mathcal{M}_n$ be a positive semidefinite matrix. Then A is positive definite if and only if A is nonsingular.

Theorem 3.6.7 — Somewhat unrelated. Let $B \in \mathcal{M}_n$ be a Hermitian matrix, $y \in \mathbb{C}^n$, and $a \in \mathbb{R}$. Let

$$A = \begin{pmatrix} B & y \\ y^* & a \end{pmatrix} \in \mathcal{M}_{n+1}.$$

Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A).$$

Definition 3.6.8 The singular values of a matrix A are the (nonnegative) square roots of the eigenvalues of $A^* A$.

R $A^* A$ is positive semidefinite.

Proof.

$$\begin{aligned} x^*(A^*A)x &= (x^*A \, *)(Ax) \\ &= (Ax)^*(Ax) \\ &= \|Ax\|^2 \geq 0 \end{aligned}$$

If σ_i is a singular value of A and λ_i is an eigenvalue of $A^* A$, then

$$\sigma_i(A) = \sqrt{\lambda x_i(A^* A)}.$$

■

Theorem 3.6.9 — Zhang. Let $A \in \mathcal{M}_{mn}$ with nonzero singular values $\sigma_1, \ldots, \sigma_r$. Then there exists $U \in \mathcal{M}_n$ and $V \in \mathcal{M}_n$ unitary such that

$$A = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V,$$

where $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} \in M_{mn}$ and $D_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$.

Theorem 3.6.10 — H & J. Let $A \in \mathcal{M}_{nm}$ be a matrix, $q = \min\{m, n\}$. Assume that the rank of A is n. Then

1. there exists $V \in M_n$ and $W \in \mathcal{M}_m$ unitary matrices and $\Sigma_q \in \mathcal{M}_q = \operatorname{diag}(\sigma_1, \ldots, \sigma_q)$ diagonal such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$$

and

$$A\Sigma W,$$

where

$$\Sigma = \begin{cases} \Sigma_1, & m = n \\ \begin{pmatrix} \Sigma_q & 0 \end{pmatrix} \in \mathcal{M}_{nm}, & m > n \\ \begin{pmatrix} \Sigma_q \\ 0 \end{pmatrix} \in \mathcal{M}_{nm}, & n > m \end{cases}$$

2. The parameters $\sigma_1, \ldots, \sigma_r$ are the positive square roots of the decreasingly ordered eigenvalues of A^*A.

Proof of Theorem 3.6.9.

- Let $A \in \mathcal{M}_n$ (scalar), say $A = c$, then $|c|$ is the singular value of A and $A = |c|e^{i\theta}$ for some $\theta \in \mathbb{R}$.
- $A \in \mathcal{M}_{m1}$ or $A \in M_{1m}$. Say $A = (a_1, \ldots, a_m)$. Then $\sigma_1 = \|A\|_2$. Then let V be unitary with first row

$$\left(\frac{1}{\sigma_1} a_1, \ldots, \frac{1}{\sigma_1} a_m\right).$$

This vector has norm 1 and $A = (\sigma_1, 0, \ldots, 0)V)$.

- Let $A \in \mathcal{M}_{mn}$, $m, n > 1$, $A \neq 0$. Let u_1 be such that $\|u_1\| = 1$ and u_1 an eigenvector of A^*A associated to σ_1^2, i.e.,

$$(A^*A)u_1 = \sigma_1^2 u_1), \qquad (u_1^* u_1 = 1).$$

Let $v = \frac{1}{\sigma_1} Au_1$. Then $v_1^* v_1 = 1$ and

$$
\begin{aligned}
u_1^* A^* v_1 &= u^* A^* \frac{1}{\sigma_1} Au_1 \\
&= \frac{1}{\sigma_1} u_1^* A^* Au_1 \\
&= \frac{1}{\sigma_1} \sigma_1^2 u_1^* u_1 \\
&= \sigma_1.
\end{aligned}
$$

Let P, Q be unitary matrices and having u_1 and v_1 as first row and column, respectively.

Then, with

$$
\begin{aligned}
A^* v_1 &= \sigma_1 u_1 \\
u_1^* A^* &= (Au_1)^* = \sigma_1 v_1^*
\end{aligned}
$$

the first column of $P^* AQ$ is $(\sigma_1, 0, \ldots, 0)^T$ and the row of $P^* AQ$ is $(\sigma_1, 0, \ldots, 0)$. So

$$
P^* AQ = \begin{pmatrix} \sigma_1 & 0 \\ 0 & B^* \end{pmatrix},
$$

where $B^* \in \mathcal{M}_{m-1,n-1}$.
Repeat procedure on B^* ... .

■

R Let $A \in \mathcal{M}_{mn}$. Then $A, \overline{A}, A^T$, and A^* have the same singular values.

R Let $A \in \mathcal{M}_n$ with singular values $\sigma_1, \ldots, \sigma_n$, then

$$
\sigma_1 \ldots \sigma_n = \det(A)
$$

and

$$
\sigma_1^2 + \ldots + \sigma_n^2 = \operatorname{Tr}(A^* A).
$$

Theorem 3.6.11 Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, and $\sigma_1 \geq \cdots \geq \sigma_q$ nonincreasingly ordered singular values of A. Define

$$
\mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}
$$

to be a Hermitian matrix. Then the ordered eigenvalues of $\mathcal{A}$ are

$$-\sigma_1 \le \cdots \le -\sigma_q \le \underbrace{0 = \cdots = 0}|n-m| \le \sigma_q \le \cdots \le \sigma_1.$$

Theorem 3.6.12 — An interlacing result. Let $A \in \mathcal{M}_{nm}$, $q = \min\{m, n\}$ and $\hat{A}$ be the matrix obtained from A by deleting one row and one column. Let $\sigma_1 \ge \cdots \ge \sigma_q$ and $\hat{\sigma}_1 \ge \cdots \ge \hat{\sigma}_q$ be the nonsingular ordered singular values of A and $\hat{A}$, respectively, where $\hat{\sigma}_q = 0$ if $n \ge m$ and a column is deleted or if $n \ge m$ and a row is deleted. Then

$$\sigma_1 \ge \hat{\sigma_1} \ge \sigma_2 \ge \hat{\sigma_2} \ge \ldots \sigma_q \ge \hat{\sigma_q}.$$

Theorem 3.6.13 — von Neumann. Let A, $B \in \mathcal{M}_{mn}$, $q = \min\{m, n\}$, $\sigma_1(A) \ge \cdots \ge \sigma_q(A)$ and $\sigma_1(B) \ge \cdots \ge \sigma_q(B)$ the non-increasingly singular values of A and B, respectively. Then

$$\operatorname{Re}\operatorname{Tr}(AB^*) \le \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B).$$

Theorem 3.6.14 Let $A \in \mathcal{M}_{nm}$, $q = \min m, n$, and $\sigma_1 \ge \cdots \ge \sigma_q$ nonincreasingly ordered singular values of A, and $\alpha = \{1, \ldots, q\}$. Then

$$\operatorname{Re}\operatorname{Tr}(A) \le \sum_{i=1}^{q} \sigma_i$$

with equality if and only if $A[\alpha]$ (principal leading submatrix of A) is positive semidefinite and A has no nonzero entries outside $A[\alpha]$.

3.7 Properties of Singular Values

- Let $A \in \mathcal{M}_2$

$$\sigma_1, \sigma_2 = \frac{1}{2}\left((\operatorname{Tr} A^*A) \mp \sqrt{(\operatorname{Tr} A^*A)^2 - 4|\det A|^2}\right)$$

- The nilpotent matrix

$$A = \begin{pmatrix} 0 & a_{12} & & \\ & \ddots & & \\ & & & a_{n-1,m} \\ & & & 0 \end{pmatrix}$$

has singular values $0, |a_{12}|, \ldots, |a_{n-1,n}|$.

Theorem 3.7.1 Let $A_1, A_2, \dots \in \mathcal{M}_{nm}$ given (infinite) sequence with $\lim_{k\to\infty} A_k = A$ (entrywise). Let $q = \min(m, n)$. Let $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$ and $\sigma_1(A_k) \geq \cdots \geq \sigma_q(A_k)$ be the non-increasinly ordered singular values of A and A_k, respectively (for all k). Then

$$\lim_{k\to\infty} \sigma_i(A_k) = \sigma_i(A).$$

Theorem 3.7.2 Let $A \in \mathcal{M}_n$, $n = \operatorname{rank} A$.

1. $A = A^T$ if and only if there exists $U \in \mathcal{M}_n$ unitary and a nonegative diagonal matrix Σ such that $A = U\Sigma U^T$. Then the diagonal entries of Σ are the singular values of A.
2. If $A = -A^T$, then n is even and there exists $U \in \mathcal{M}_n$ unitary and positive real scalars $s_1, \dots, s_{r/2}$ such that

$$U\left(\begin{pmatrix} 0 & s_1 \\ -s_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_{r/2} \\ -s_{r/2} & 0 \end{pmatrix}\right)U^T.$$

   The non-zero singular values of A are $s_1, s_1, \dots, s_{r/2}, s_{r/2}$.
   Conversely, any matrix of the above form is skew-symetric.

CHAPTER 4

Nonnegative matrices

Nonnegative matrices play a very important role in a variety of applications and have been studied extensively. There are entire books devoted to their properties. Here, we review some of their most well-known properties. In particular, we present the famous Perron-Frobenius theorem. We then consider the special case of stochastic matrices, which provide the basis for Markov chains.

Chapter contents

4.1 Definitions and some preliminary results

Definition 4.1.1 A matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$ is a **nonnegative matrix** if $a_{ij} \geq 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. We write $A \geq 0$. A is a **positive matrix** if $a_{ij} > 0$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. We write $A > 0$.

R In other references, you will see:

- $A \geq 0 \iff a_{ij} \geq 0$.
- $A > 0 \iff A \geq 0$ and there exists (i, j), $a_{ij} > 0$ [positive]
- $A \gg 0 \iff a_{ij} > 0$ for all i, j [strongly positive]

I tend to favour the latter notation over the one used in these notes, but since the former is more common in matrix theory, I use the notation of Definition 4.1.1 here.

Notation

Let $A, B \in \mathcal{M}_{mn}(\mathbb{R})$. Nonnegativity and positivity are used to define partial orders on $\mathcal{M}_{mn}(\mathbb{R})$.

- $A \geq B \iff A - B \geq 0$
- $A > B \iff A - B > 0$.

The same is used for vectors $x, y \in \mathbb{R}^n$: $x \geq y$ and $x > y$ if, respectively, $x - y \geq 0$ and $x - y > 0$. Note that the order is only partial: if $A \geq 0$ and $B \geq 0$, for instance, it is not necessarily possible to decide on the ordering of A and B with respect to one another.

Theorem 4.1.2 Let A and B be nonnegative matrices of appropriate sizes. Then $A + B$ and AB are nonnegative. If $A > 0$ and $B \geq 0$, $B \neq 0$, then $AB \geq 0$ and $AB \neq 0$.

Corollary 4.1.3 Let x, $y \in \mathbb{R}^n$ be such that $x \geq y$ and $A \in \mathcal{M}_{mn}$ be nonnegative. Then $Ax \geq Ay$. Assume additionally that $x \geq y$, $x \neq y$ and $A > 0$. Then $Ax > Ay$.

Proof. Assume $A \geq 0$ and $x \geq y$. Then the ith row in Ax takes the form

$$\sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \dots, m$$

and since $x_i \geq y_i$ for all $i = 1, \dots, n$ and $a_{ij} \geq 0$ for all i, j, it follows that

$$\sum_{j=1}^{n} a_{ij} x_j \geq \sum_{j=1}^{n} a_{ij} y_j, \quad i = 1, \dots, m,$$

giving the first conclusion.

Now assume $A > 0$, i.e., $a_{ij} > 0$ for all i, j. Also, assume that $x \geq y$ and $x \neq y$, i.e., $x_i \geq y_i$ with at least one entry for which $x_i > y_i$. Without loss of generality, suppose $x_1 > y_1$. It follows that

$$\sum_{j=1}^{n} a_{ij} x_j > \sum_{j=1}^{n} a_{ij} y_j, \quad i = 1, \dots, m,$$

where the inequality is strict because $a_{i1} x_1 > a_{i1} y_1$ for each $i = 1, \dots, m$. ■

Definition 4.1.4 Let $P, Q \in \mathcal{M}_{nm}(\mathbb{F})$. P and Q have the same **zero-nonzero structure** if for all i, j, $p_{ij} \neq 0 \iff q_{ij} \neq 0$.

Zero-nonzero structure defines an equivalence relation. Therefore, as with all equivalence relations, one only needs one representative from the equivalence class. One typical representative is defined using Boolean matrices.

Definition 4.1.5 A **Boolean matrix** is a matrix whose entries are Boolean $\{0, 1\}$ and use Boolean arithmetics:

- $0 + 0 = 0$
- $1 + 0 = 0 + 1 = 1$
- $1 + 1 = 1$
- $0 \cdots 1 = 1$ and $1 = 0 = 0 \cdots 0$
- $1 \cdots 1 = 1$

Definition 4.1.6 Let $A \in \mathcal{M}_{nm}(\mathbb{F})$. Then A_B denotes the **Boolean representation** of A, defined as follows. If $A = [a_{ij}]$, then $A_B = [\alpha_{ij}]$ with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0. \end{cases}$$

Theorem 4.1.7 Let $P,], Q \in \mathcal{M}_n$ be nonnegative matrices. The zero-nonzero structure of the sum of product of P and Q is uniquely determined by the zero-nonzero structure of P and Q. Also,

$$(P + B)_B = P_B + Q_B$$
$$(PQ)_B = P_B Q_B$$

(with Boolean addition and multiplication on the right).

Theorem 4.1.8 Let $A \in \mathcal{M}_n$ be a nonnegative matrix, $k \in \mathbb{N} \setminus \{0\}$. Then the (i, j) entry in A^k is nonzero if and only if there is a directed walk of length *exactly* k from i to j in $G(A)$, the digraph associated to A.

R The walk can go thought the same vertex more than once.

Proof. We proceed by induction. The case $k = 1$ is trivial by the definition of $G(A)$.

Assume that the property is true for k, i.e., the (i, j) entry in A^k is $\neq 0$ if and only if there exists a k-length directed path from i to j in $G(A)$).

Let $A = [a_{ij}]$, $A^k := B = [b_{ij}]$ and $A^{k+1} = [c_{ij}]$. Then let us write A^{k+1} as $A^{k+1} = A^k A$, i.e., for all $i, j = 1, \ldots, n$,

$$c_{ij} = \sum_p b_{ip} a_{pj}.$$

Now, assume that there is a directed walk of length $k+1$ from i to j in $G(A)$. Take this walk to consist of the following sequence of traversed vertices: $(i,\ p_1,\ p_2, \ldots, p_k, j)$. From the induction hypothesis, the existence of a directed walk of length k from i to p_k, $(i, p_1, \ldots, p_k)$, in $G(A)$ implies that the (i, p_k) entry in A^k is positive, i.e., $b_{ip_k} > 0$. Because there is a path of length $k+1$ from i to j, $a_{p_k j} > 0$, since we need to be able to make the last transition from p_k to j. In

$$c_{ij} = \sum_p b_{ip} a_{pj},$$

all terms are nonnegative, but from what precedes, $b_{ip_k} a_{p_k j} > 0$. This implies that $c_{ij} > 0$, i.e., the (i, j) entry in A^{k+1} is positive.

Conversely, assume that the (i, j) entry in A^{k+1} is positive. Then in

$$c_{ij} = \sum_p b_{ip} a_{pj}$$

there must exists at least one entry that is positive. Suppose it is $b_{iq} a_{qj}$, i.e., $b_{iq} a_{qj} > 0$. Of course, this implies that $a_{qj} > 0$ and $b_{iq} > 0$ and in turn, $b_{iq} > 0$ if and only if A^k has entry $(i, q) > 0$. By the induction hypothesis, the latter is true if and only if there exits a walk of length k from i to q in $G(A)$. Since we also have $a_{qj} > 0$, there exists a walk of length 1 from q to j in $G(A)$. As a consequence, there exists a walk of length $k+1$ from i to j in $G(A)$. ■

■ **Example 4.1** Suppose we have a Markov chain with transition matrix:

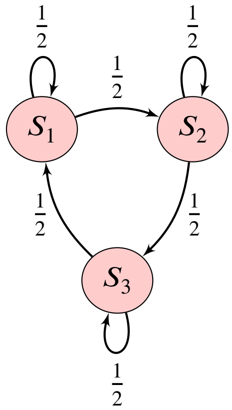
$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

This is a Markov chain with states S_1, S_2, S_3. This chain has the transition graph shown below. The Boolean representation of T is given by

$$T_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Furthermore,

$$T^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$



and

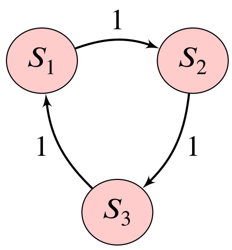
$$T_B^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$(1, 1) > 0$ if and only if there exists a path of length 2 from 1 to 1.

Now change this a little: assume the transition matrix is a circulant permutation matrix,

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The transition graph takes the form below. Here, powers of the transition matrix are



$$T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = T.$$

■

Theorem 4.1.9 Let $A \in \mathcal{M}_n$ be a nonnegative and irreducible, $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}_+ \setminus \{0\}$. Then

$$\alpha_0 \mathbb{I} + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1} > 0.$$

In particular, $(\mathbb{I}+A)^{n-1} > 0$.

Proof. Let $B := \alpha_0\mathbb{I} + \alpha_1 A + \cdots + k_{n-1}A^{n-1}$. As $A \geq 0$, it is clear from Theorem 4.1.2 that $B \geq 0$. Now consider indices (i, j). If $i = j$, then $b_{ij} \geq k_0 > 0$. Assume now $i \neq j$. As A is irreducible, there exists a path of length $k < n$ in $G(A)$ from i to j and equivalently, by Theorem 4.1.8, the (i, j) entry in A^k, $a_{ij}^{(k)}$, is positive. It follows that the (i, j) entry in B is larger than or equal to $\alpha_k a_{ij}^{(k)}$ and is positive. Thus $B > 0$.

Now expand $(\mathbb{I}+A)^{n-1}$ using the binomial formula:

$$
\begin{aligned}
(\mathbb{I}+A)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{I}^{n-1-k} A^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} A^k.
\end{aligned}
$$

So for $k = 0, \ldots, n-1$, taking

$$
\alpha_k = \binom{n-1}{k},
$$

in the first formula gives the result. ■

Definition 4.1.10 Let $A \in \mathcal{M}_n(\mathbb{F})$. The matrix $m(A) = [|a_{ij}|]$ is the **modulus** of A.

Lemma 4.1.11 $A, B \in \mathcal{M}_n(\mathbb{F})$, A potentially complex. Suppose $m(A) < B$. Then $\rho(A) \leq \rho(B)$. In particular, $\rho(A) \leq \rho(m(A))$.

Proof. See the proof of [**?**, Lemma 4.6]. ■

Lemma 4.1.12 Let $A \in \mathcal{M}_n$ be a nonnegative matrix, $0 \neq z \in \mathbb{R}^n$. If $\exists b \in \mathbb{R}$ such that $Az > bz$, then $\rho(A) > b$.

Note that the latter result also holds true if $>$ is replaced by $\geq$. See [**?**, Theorem 8.3.2] and a proof there in the latter case. Note, however, that the proof of [**?**, Theorem 8.3.2] uses Perron's Lemma. So it is better for our purpose here to use the proof in [**?**], where this result is Lemma 4.1.6.

4.2 The Perron-Frobenius theorem

There are several variations on the following result, depending on the properties of the matrix under consideration.

4.2.1 The Perron-Frobenius Theorem for irreducible matrices

Theorem 4.2.1 — **Perron-Frobenius.** Let $A \geq 0 \in \mathcal{M}_n$ be irreducible. Then the spectral radius $\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\}$ is an eigenvalue of A. It is simple (has algebraic multiplicity 1), positive and is associated with a positive eigenvector. Furthermore, there is no nonnegative eigenvector associated to any other eigenvalue of A.

R We often say that $\rho(A)$ is the **Perron root** of A; the corresponding eigenvector is the **Perron vector** of A.

4.2.2 Proof of a result of Perron for positive matrices

In order to prove the Perron-Frobenius theorem in the irreducible case (Theorem 4.2.1), we start by proving the following result of Perron for positive matrices.

Lemma 4.2.2 — Perron. Let $\mathcal{M}_n \ni A > 0$. Then $\rho(A)$ is a positive eigenvalue of A and there is only one linearly independent eigenvector associated to $\rho(A)$, which can be taken to be positive.

To show Lemma 4.2.2, we need the following result.

Lemma 4.2.3 Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+^*$ and $v_1, \dots, v_n \in \mathbb{C}$. Then

$$\left| \sum_{i=1}^{n} \alpha_i v_i \right| \leq \sum_{i=1}^{n} \alpha |v_i|, \tag{4.1}$$

with equality if and only if there exists $\eta \in \mathbb{C}$, $|\eta| = 1$, such that $\eta v_i \geq 0$ for all $i = 1, \dots, n$.

Idea of proof of Lemma 4.2.3. Let $v, w \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}_+^*$. The triangle inequality

$$|\alpha v + \beta w| \leq \alpha |v| + \beta |w|,$$

is an equality when v and w are on the same half-line starting at the origin in $\mathbb{C}$, i.e., if there exists $\eta \in \mathbb{C}$, $|\eta| = 1$, such that

$$\eta v \geq 0 \text{ and } \eta w \geq 0$$

at the same time. Generalizing gives the result. ∎

Proof of Perron's Lemma. First take $n = 1$. Then $A = [a_{11}] > 0$ if and only if $a_{11} > 0$. Thus $\rho(A) = a_{11} > 0$ and $v \neq 0$ such that $a_{11} v = \rho(A) v = a_{11} v$ can be taken as $v > 0$.

Now assume that $n > 1$. As

$$\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\},$$

there exist $\lambda \in \sigma(A)$ that achieves this maximum modulus, i.e., that is such that

$$|\lambda| = \rho(A).$$

We focus on this λ. To distinguish it from other eigenvalues, let us denote it λ_P. Let u be an eigenvector associated to λ_P, i.e., $u \neq 0$ is such that $Au = \lambda_P u$.

Use Lemma 4.2.3 on u such that $Au = \lambda_P u$ with $|\lambda_P| = \rho(A)$. We want to show that in this case, there exists $\eta \in \mathbb{C}$, $|\eta| = 1$, such that

$$\eta u_i \geq 0, \qquad i = 1, \ldots, n. \tag{4.2}$$

Assume that (4.2) does not hold. Then the inequality in (4.1) is strict. Then, for a given $k \in \{1, \ldots, n\}$, using $\lambda_P u = Au$, we get

$$|\lambda_P u_k| = |\lambda_P|\ |u_k| = \left|\sum_{j=1}^n a_{kj} u_j\right| < \sum_{j=1}^n a_{kj} |u_j|,$$

where the inequality results from the inequality (4.1) being strict and the fact that $A > 0$. This is true for all k, which implies that $|\lambda_P| m(u) < Am(u)$, where $m(u)$ is the modulus of u. Using Lemma 4.1.12 on the latter inequality, we have

$$Am(u) > |\lambda_P| m(u) \implies \rho(A) > |\lambda_P|.$$

This contradicts $\rho(A) = \lambda_P$ and as a consequence, (4.2) holds.

Since (4.2) holds, the vector

$$v = \eta u, v \neq 0$$

is nonnegative and since $Au = \lambda_P u$, $Av = \lambda_P v$.

Since $v \geq 0$ and is an eigenvector, there is $k \in \{1, \ldots, n\}$ such that $v_k \neq 0$ (actually, $v_k > 0$). Then the k-th equation in $Av = \lambda_P v$,

$$\sum_{j=1}^n a_{kj} v_j = \lambda_P v_k,$$

gives $\lambda_P > 0$. Since we have chosen λ_P such that $|\lambda_P| = \rho(A)$, this implies that $\lambda_P = \rho(A) > 0$. Since $0 \neq v \geq 0$ and $A > 0$, we have $Av > 0$, so since $\lambda_P > 0$, we also have that $v > 0$.

We have showed that if $\lambda_P \in \sigma(A)$ is such that $|\lambda_P| = \rho(A)$ and u is an associated eigenvector, then $m(u) > 0$. And we can take u to be real (and positive). Let us now show that there is a unique linearly independent eigenvector corresponding to λ_P.

Let then v, w be two linearly independent eigenvectors associated to λ_P. As $v \neq 0$, there exists k such that $v_k \neq 0$. Choosing such a k, define

$$z = w - \left(w_k v_k^{-1}\right) v.$$

Clearly, $z \neq 0$. Furthermore, z is an eigenvector of A associated to λ_P, since

$$\begin{aligned} Az &= A\left(w - w_k v_k^{-1} v\right) \\ &= Aw - A\left(w_k v_k^{-1}\right) v \\ &= Aw - \left(w_k v_k^{-1}\right) Av \\ &= \lambda_P w - \left(w_k v_k^{-1}\right) \lambda_P v \\ &= \lambda_P\left(w - \left(w_k v_k^{-1}\right) v\right) \\ &= \lambda_P z. \end{aligned}$$

However, for the k chosen above, the kth entry of z is

$$z_k = w_k - \left(\frac{w_k}{v_k}\right) v_k = w_k - w_k = 0,$$

which contradicts the fact that $m(z) > 0$ we proved earlier. As a consequence, there cannot be two linearly independent eigenvectors associated to λ_P. ∎

4.2.3 Proof of the Perron-Frobenius theorem for irreducible matrices

The following results are needed in the proof of Perron-Frobenius Theorem. They are given here without proof. See the proofs in

Theorem 4.2.4 Let $A \in \mathcal{M}_n$ and $f(x)$ a polynomial. Then

$$\sigma(f(A)) = \{f(\lambda_1), \ldots, f(\lambda_n), \lambda_i \in \sigma(A)\}.$$

If we have $g(\lambda_i) \neq 0$ for $\lambda_i \in \sigma(A)$, for some polynomial g, then the matrix $g(A)$ is non-singular and

$$\sigma\left(f(A)g(A)^{-1}\right) = \left\{\frac{f(\lambda_1)}{g(\lambda_1)}, \ldots, \frac{f(\lambda_n)}{g(\lambda_n)}, \lambda_i \in \sigma(A)\right\}.$$

If $x \neq 0$ eigenvector of A associated to $\lambda \in \sigma(A)$, then x is also an eigenvector of $f(A)$ and $f(A)g(A)^{-1}$ associated to eigenvalue $f(\lambda)$ and $f(\lambda)/g(\lambda)$, respectively.

Lemma 4.2.5 — Schur's lemma. Let $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. Then λ is simple if and only if both the following conditions are statisfied:

1. There exists only one linear independent eigenvector of A associated to λ, say u, and thus only one linear independent eigenvector of A^T associated to λ, say v.
2. Vectors u and v in (1) satisfy $v^T u \neq 0$.

Let us proceed with the proof of the Perron-Frobenius Theorem.

Proof of Theorem 4.2.1. Since A is irreducible, it follows from Theorem 4.1.9 that $(\mathbb{I} + A)^{n-1} > 0$. Of course, it is then also true that $\left((\mathbb{I} + A)^{n-1}\right)^T > 0$. We can therefore use Perron's lemma

(Lemma 4.2.2) on the latter matrix: there exists $\mathbb{R}^n \ni y > 0$ such that

$$\left((\mathbb{I} + A)^{n-1}\right)^T y = \rho\left((\mathbb{I} + A)^{n-1}\right) y,$$

or, transposing both sides,

$$y^T(\mathbb{I} + A)^{n-1} = \rho\left((\mathbb{I} + A)^{n-1}\right) y^T;$$

in other words, y^T is a left eigenvector of $(\mathbb{I} + A)^{n-1}$ associated to $\rho\left((\mathbb{I} + A)^{n-1}\right)$.

Let $\lambda \in \sigma(A)$ be such that $|\lambda| = \rho(A)$. Let $x \neq 0$ be an eigenvector associated to λ, i.e., $Ax = \lambda x$. Using the same argument as in the proof of Lemma 4.2.2, we find that

$$|\lambda| m(x) \leq Am(x),$$

where $m(x)$ is the modulus of x. In other words, since $|\lambda| = \rho(A)$,

$$\rho(A)m(x) \leq Am(x).$$

Multiplying both sides by $\rho(A) \geq 0$:

$$\rho(A)^2 m(x) \leq \rho(A)Am(x) = A(\rho(A)m(x)) \leq A(Am(X)) = A^2 m(x).$$

We can continue, getting for $k = 1, 2, \ldots$

$$\rho(A)^k m(x) \leq A^k m(x). \tag{4.3}$$

The inequality also holds for $k = 0$ (with equality in this case). Therefore, using the binomial formula and the inequality (4.3),

$$(1 + \rho(A))^{n-1} m(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \rho(A)^k m(x) \leq \sum_{k=0}^{n-1} \binom{n-1}{k} A^k m(x) = (\mathbb{I} + A)^{n-1} m(x),$$

i.e.,

$$(1 + \rho(A))^{n-1} m(x) \leq (\mathbb{I} + A)^{n-1} m(x).$$

Recall that $y > 0$ and $(\mathbb{I} + A)^{n-1} > 0$. As a consequence, by Corollary 4.1.3, left multiplication by y^T gives

$$(1 + \rho(A))^{n-1} y^T m(x) \leq y^T(\mathbb{I} + A)^{n-1} m(x).$$

We have seen that

$$y^T(\mathbb{I} + A)^{n-1} = \rho\left((\mathbb{I} + A)^{n-1}\right) y^T.$$

Thus

$$(1+\rho(A))^{n-1} y^T m(x) \leq \rho\left((\mathbb{I}+A)^{n-1}\right) y^T m(x).$$

But $y^T m(x) \in \mathbb{R}_+^*$, thus

$$(1+\rho(A))^{n-1} \leq \rho\left((\mathbb{I}+A)^{n-1}\right).$$

By Theorem 2.1.4, if $\alpha \in \sigma(A)$, then $1+\alpha \in \sigma(\mathbb{I}+A)$. Using this in conjunction with Theorem 4.2.4, it follows that the eigenvalues of $(\mathbb{I}+A)^{n-1}$ take the form $(1+\alpha)^{n-1}$, if $\alpha \in \sigma(A)$. Thus, there exists $\mu \in \sigma(A)$ such that

$$\left|(1+\mu)^{n-1}\right| = \rho\left((\mathbb{I}+A)^{n-1}\right). \tag{4.4}$$

Substitute (4.4) into $(1+\rho(A))^{n-1} \leq \rho\left((\mathbb{I}+A)^{n-1}\right)$, giving

$$(1+\rho(A))^{n-1} \leq \left|(1+\mu)^{n-1}\right|. \tag{4.5}$$

On the other hand, since $\mu \in \sigma(A)$,

$$|\mu| \leq \rho(A).$$

Take $n=2$ in (4.5). This gives $1+\rho(A) \leq |1+\mu|$. In turn, $|1+\mu| \leq 1+|\mu|$, so finally, since $|\mu| \leq \rho(A)$, it follows that $1+\rho(A) \leq |1+\mu| \leq 1+|\mu| \leq 1+\rho(A)$. As a consequence, all these quantities are equal, i.e., $1+\rho(A) = |1+\mu| = 1+|\mu|$. Thus $\mu \geq 0$. Hence $\mu = \rho(A)$.

Now substitute $\mu = \rho(A)$ into (4.3) with $k=1$:

$$\mu m(x) = Am(x)$$

we get

$$Am(x) = \rho(A)m(x) = \mu m(x).$$

By this,

$$(\mathbb{I}+A)^{n-1} m(x) = (1+\mu)^{n-1} m(x) = \rho((\mathbb{I}+A)^{n-1}) m(x).$$

By Lemma 4.2.2, giving $m(x) > 0$ and existence of only one linear independent eigenvector associated to μ. Furthermore, $\rho(A) > 0$ as A nonzero nonnegative matrix.

Only one linear independent eigenvector of A, say u, associated to $\rho(A)$. Also, $u > 0$. The same is true for A^T, let v be the associated eigenvector of A^T, $v^T > 0$ is a left eigenvector of A. Thus $v^T u > 0$.

Here we use Lemma 4.2.5, implies that $\rho(A)$ is simple.

Now, let us show that there are no nonnegative eigenvectors associated to any other eigenvalues of A. Suppose we have $Az = \xi z$, with $z \geq 0$ and $\xi \neq \rho(A)$. We know that A^T has a positive eigenvector, say $w > 0$,

$$A^T w = \rho(A)w.$$

Then, if follows that

$$w^t Az = w^t \xi z = \xi(w^t z)$$

and also,

$$w^T Az = \rho(A)(w^t z),$$

contradiction since $\xi - \rho(A) \neq 0$ by assumption. ■

4.2.4 Primitive matrices

We proved the Perron-Frobenius theorem (Theorem 4.2.1) in the case where the matrix A is irreducible, after proving Perron's lemma (Lemma 4.2.2) in the case of a positive matrix A. It is actually possible to get the same conclusions as from Perron's lemma without requiring to assume that the matrix A be positive. For this, we need the following definition.

Definition 4.2.6 Let $\mathcal{M}_n(\mathbb{R}) \ni A \geq 0$. We say that A is **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if there exists $k \in \mathbb{N}_+^*$ such that

$$A^k > 0,$$

with k the smallest integer for which this is true. We say that a matrix is **imprimitive** if it is not primitive.

R Primitivity implies irreducibility; the converse is not true. See for instance Example 4.2 below.

Theorem 4.2.7 A sufficient condition for primitivity is irreducibility with at least one positive diagonal entry.

Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d =$ gcd of all the lengths of closed walks in $G(A)$.

■ **Example 4.2** Suppose we have a Markov chain with the state transition graph shown in Figure 4.1. The associated transition matrix takes the form

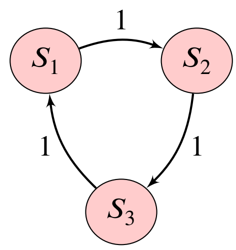


Figure 4.1: Transition graph for the discrete-time Markov chain in Example 4.2, in the irreducible imprimitive case.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The length of closed walks in $G(A)$ is as follows. $1 \to 1 : 3, 2 \to 2 : 3, 2 \to 2 : 3$. Thus the gcd is 3 and as a consequence $d = 3$ and all eigenvalues have modulus 1. Let us now consider the slightly modified chain seen in Figure 4.2. This chain has Boolean representation matrix

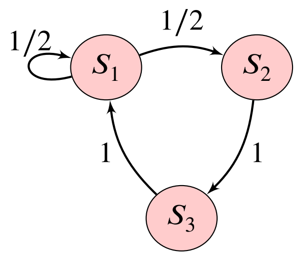


Figure 4.2: Transition graph for the discrete-time Markov chain in Example 4.2, in the primitive case.

$$A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here the gcd of the lengths of closed walks is 1; hence, A is primitive. ■

Theorem 4.2.8 Let $A \in \mathcal{M}_n$ be a non-negative matrix. If A is primitive, then $A^k > 0$ for some $0 < k \leq (n-1)n^n$.

Proof. Assume that A is a primitive matrix. Then, necessarily, A is irreducible. If so, there exists a path in $G(A)$ from p_1 to p_1, since A irreducible if and only if $G(A)$ strongly connected and thus there is a path in $G(A)$ from any vertex to any other vertex. Let k_1 be the length of the shortest such path. (Clearly, $k_1 \leq n$.) Thus, the entry (p_1, p_1) in A^{k_1} is positive. Since A is primitive, a result that A^{k_1} is irreducible, which gives us that there exists a directed path in $G(A^{k_1})$ from P_2 to P_2. This path length $k_2 \leq n$. Thus the matrix $(A^{k_1})^{k_2}$ has $(1,\ 1)$-entry is positive. Then continue on. Thus $(A^{k_1 \cdots k_n})^{n-1} > 0$. ■

Theorem 4.2.9 Let $A \geq 0$ primtive. Suppose the shortest simple directed cycle in $G(A)$ has length s, then primitivity index is $\leq n + s(n-1)$

Theorem 4.2.10 Let $A \in \mathcal{M}_n$ be a nonnegative matrix. A is primitive if and only if $A^{n^2-2n+2} > 0$.

Theorem 4.2.11 Let $A \in \mathcal{M}_n$ be a nonnegative irreducible matrix . Suppose that A has d positive entries on the diagonal. Then the primitivity index is $\leq 2n - d - 1$.

4.2.5 The Perron-Frobenius Theorem for nonnegative matrices

Theorem 4.2.12 Let $A \geq 0$ in $\mathcal{M}_n$. Then there exists $0 \neq v \geq 0$ such that $Av = \rho(A)v$.

4.2.6 The Perron-Frobenius Theorem (revamped)

Let us restate the Perron-Frobenius theorem, taking into account the different cases. The classification in the following result is inspired by the presentation in [**?**].

Theorem 4.2.13 Let $\mathcal{M}_n \ni A \geq 0$. Denote λ_P the Perron root of A, i.e., $\lambda_P = \rho(A)$, v_P and w_P the corresponding right and left Perron vectors of A, respectively. Denote d the index of imprimitivity of A (with $d = 1$ when A is primitive) and $\lambda_j \in \sigma(A)$ the spectrum of A, with $j = 2, \dots, n$ unless otherwise specified (assuming $\lambda_1 = \lambda_P$). Then conclusions of the Perron-Frobenius Theorem can be summarised as follows.

NONNEGATIVE

REDUCIBLE
- ➪ $\lambda_P \geq 0$
- ➪ $w_P \geq 0$
- ➪ $v_P \geq 0$
- ➪ $\lambda_P \geq |\lambda_j|$

IRREDUCIBLE

IMPRIMITIVE
- ➪ $\lambda_P > 0$
- ➪ $w_P > 0$
- ➪ $v_P > 0$
- ➪ $\lambda_P = |\lambda_j|$, $j = 2, \dots, d$
- ➪ $\lambda_P > |\lambda_j|$, $j > d$

PRIMITIVE
- ➪ $\lambda_P > 0$
- ➪ $w_P > 0$
- ➪ $v_P > 0$
- ➪ $\lambda_P > |\lambda_j|$, $j \neq P$

4.2.7 Application of the Perron-Frobenius Theorem

The Perron-Frobenius can be applied not only to nonnegative matrices, but also to matrices that are *essentially nonnegative*, in the sense that they are nonnegative except perhaps along the main diagonal.

Definition 4.2.14 A matrix $A \in \mathcal{M}_n$ is **essentially nonegative** (or **quasi-positive**) if there exist $\alpha \in \mathbb{R}$ such that $A + \alpha\mathbb{I} \geq 0$.

R An essentially nonnegative matrix A has non-negative off-diagonal entries. The sign of the diagonal entries is not relevant.

R Irreducibility of a matrix is not affected by the nature of its diagonal entries. Indeed, consider an essentially nonnegative matrix A. The existence of a directed path in $G(A)$ does not depend on the existence of "self-loops". The same is not true of *primitive* matrices, where the presence of negative entries on the main diagonal has an influence on the values of A^k and thus ultimately, on the capacity to find k such that $A^k > 0$.

So we can apply the "weak" versions of the Perron-Frobenius Theorem (the imprimitive cases in Theorem 4.2.13) to $A + \alpha\mathbb{I}$, which is a nonnegative matrix (potentially irreducible). One important ingredient is a result that was proved as Theorem 2.1.4. Namely, that perturbations of the entire diagonal by the same scalar lead to a shift of the spectrum; this is summarised as

$$\sigma(A + \alpha\mathbb{I}) = \{\lambda_1 + \alpha, \dots, \lambda_n + \alpha, \quad \lambda_i \in \sigma(A)\}.$$

When dealing with essentially nonnegative matrices, the focus moves from the spectral radius to the spectral abscissa, which is defined as follows.

Definition 4.2.15 Let $A \in \mathcal{M}_n$. The **spectral abscissa** of A, $s(A)$, is

$$s(A) = \max\{\mathrm{Re}(\lambda), \lambda \in \sigma(A)\}.$$

It is then a simple exercise to show that, for essentially nonnegative matrices, the Perron-Frobenius theorem can be stated as follows.

Theorem 4.2.16 Let $A \in \mathcal{M}_n(\mathbb{R})$ be essentially nonnegative. Then $s(A)$ is an eigenvalue of A and is associated to a nonnegative eigenvector. If, additionally, A is irreducible, then $s(A)$ is simple and is associated to a positive eigenvector.

Proof. Let $A \in \mathcal{M}_n$ be essentially nonnegative and let $B = A + \alpha\mathbb{I}$, where α is taken large enough that $B \geq 0$. Apply the reducible and imprimitive cases of Theorem 4.2.13 to B. In the case of B, we have $s(B) = \rho(B) = \lambda_P(B)$. Since $A = B - \alpha\mathbb{I}$, by Theorem 2.1.4, the spectrum of A is shifted left by α, in the imaginary plane. In particular, $s(B) - \alpha$ is an eigenvalue of A and is the only eigenvalue associated to a nonnegative eigenvector (positive eigenvector in the case where A is irreducible). ■

Theorem 4.2.17 Let $A \in \mathcal{M}_n$ be a nonnegative irreducible matrix and $\in \mathbb{N}_+$. Then the following ar eequivalent:

1. there exists exactly h distinct eigenvalues such that $|\lambda| = \rho(A)$.

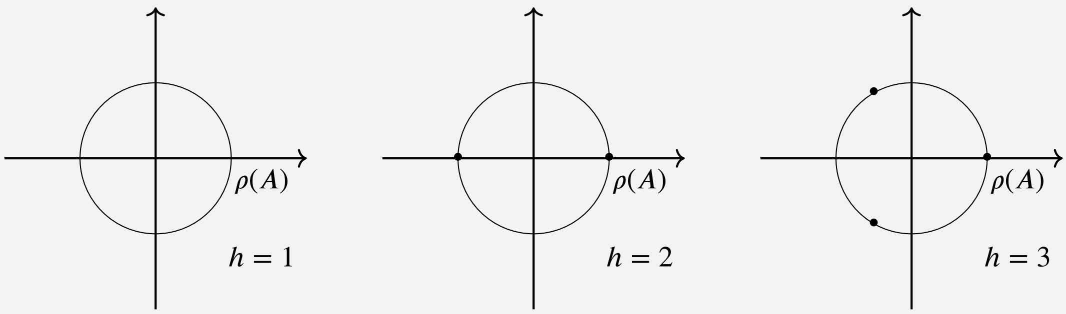
2. there exists P a permutation matrix such that

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ \vdots & & A_{23} & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 \end{pmatrix}$$

   where the diagonal blocks are square, and there does not exists other permutation matrix giving less than h horizontal blocks.
3. the greatest common divisor of the lengths of all cycles in $G(A)$ is h.
4. h is the maximal positive integer k such that

$$\sigma(e^{2\pi i/k}A) = \sigma(A)$$

Corollary 4.2.18 Let $A \in \mathcal{M}_n$, $A \geq 0$ irreducible with exactly h distinct eigenvalues of modulus $\rho(A)$. Then, we can consider this eigenvalues as points in the complex plan, the eigenvalues are the vertices of a regular polygon of h sides with centre at the origin and are of the vertices being $\rho(A)$



R For Fiedler, a primitive matrix is defines as an irreducible nonnegative matrix such that $h = 1$.

Theorem 4.2.19 Let $A \geq 0$ in $\mathcal{M}_n$, $n \geq 2$. Then the following are equivalent

1. $A^n = 0$
2. there exists $k > 0$ ($k \in \mathbb{N}$) such that $A^k = 0$
3. $G(A)$ acyclic
4. there exist a permutation matrix P such that PAP^T is upper-triangular with zeros on main diagonal
5. $\rho(A) = 0$

Theorem 4.2.20 Let $A \geq 0$ be a nonnegative matrix in $\mathcal{M}_n$. Assume that A has a positive eigenvector. Then that eigenvector is the Perron vector and is associated to $\rho(A)$.

4.3 Stochastic matrices

Stochastic matrices play a very important role in probability theory, arising most notably in Markov chains. They were briefly introduced in Chapter 1. However, let us re-explore them here now that we have a nice "hammer" in the form of the Perron-Frobenius theorem.

4.3.1 Row- and column-stochastic matrices

Definition 4.3.1 The matrix $A \in \mathcal{M}_n$ is stochastic if

- $A \geq 0$ [The matrix is nonnegative]
- $A\mathbb{1} = \mathbb{1}$, $\mathbb{1} = (1, \dots, 1)^T$. [All rows sum to 1]

Equivalently, the matrix is stochastic if its column sums all equal 1. If it is important to indicate which of the rows or columns sum to 1, we say that the matrix is *row-stochastic* or *column-stochastic*, respectively. The terms *right stochastic* and *left stochastic* are also used. Note that if both rows and columns sum to 1, then the matrix is *doubly stochastic*. We return to doubly stochastic matrices later.

As a stochastic matrix is nonnegative, at the very least the weakest version of the Perron-Frobenius theorem (Theorem 4.2.13) holds, with stronger versions holding if A is irreducible, primitive or positive. The first application of the Perron-Frobenius theorem in this context is the following.

Theorem 4.3.2 Let $A \in \mathcal{M}_n$ be stochastic. Then $\rho(A) = 1$.

Proof. Without loss of generality, assume A is row-stochastic. Since $A\mathbb{1} = \mathbb{1}$, 1 is an eigenvalue of A associated to the eigenvector $\mathbb{1}$. Since $A \geq 0$, from the Perron-Frobenius theorem (Theorem 4.2.13), there is only one nonnegative eigenvector, the eigenvector v_P associated to $\rho(A)$. As a consequence, $v_P = \mathbb{1}$ and $\lambda_P = \rho(A) = 1$. ∎

Theorem 4.3.3 Let $P \in \mathcal{M}_n$, $P \geq 0$. Assume that P has a positive eigenvector u and that $\rho(P) > 0$. Then there exists D, diagonal matrix with $\mathrm{diag}(D) > 0$, and $k > 0$, $k \in \mathbb{R}$ such that

$$A = kDPD^{-1}$$

is stochastic, with $k = \rho(P)^{-1}$

Proof. Let $Pu = \rho(P)u$ and assume that $u > 0$ and $\rho(P) > 0$. Then there exists a diagonal matrix D with positive diagonal entries,

$$D = \mathrm{diag}\left(\frac{1}{u_1}, \dots, \frac{1}{u_n}\right),$$

such that

$$Du = \mathbb{1}.$$

Set $k = \dfrac{1}{\rho(P)}$ and let $A = kDPD^{-1}$. Then

$$\begin{aligned}
A\mathbb{1} &= kDPD^{-1}\mathbb{1} \\
&= kDPu \\
&= kD\rho(P)u \\
&= Du \\
&= \mathbb{1}.
\end{aligned}$$

Also $A \geq 0$. ■

Theorem 4.3.4 Let $A,\ B \in \mathcal{M}_n$ be stochastic. Then AB is stochastic.

Proof. Assume $A, B \in \mathcal{M}_n$ are row-stochastic. Then,

$$AB\mathbb{1} = A(B\mathbb{1}) = A\mathbb{1} = \mathbb{1}.$$ ■

Theorem 4.3.5 Let A be a primitive stochastic. Then $A^k \to \mathbb{1}v^T$, $k \to \infty$, where $\mathbb{1}v^T$ has rank 1 and v is the (left) eigenvector of A^T associated to $\rho(A) = 1$ and normalised so that $v^T\mathbb{1} = 1$.

R This is a result that is used to compute the limit of a regular Markov chain.

■ **Example 4.3** Let us assume that the following (completely fictitious) daily mean temperature transition chain for Winnipeg during the winter. SC stands for *super cold*, C for *cold* and W for warm. The transition matrix corresponding to the transition graph in Figure 4.3 takes the form

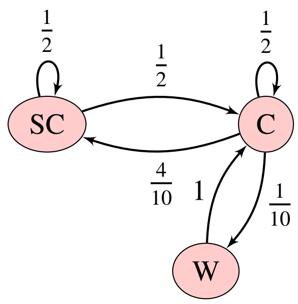


Figure 4.3: The transition graph of Example 4.3.

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{4}{10} & \frac{1}{2} & \frac{1}{10} \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly, T is primitive: the transition graph is strongly connected with loops. Therefore, the Markov chain is regular . $T^k \to \mathbb{1}v^T$ ■

4.3.2 Doubly stochastic matrices

Definition 4.3.6 The matrix $A \in \mathcal{M}_n$, $A \geq 0$ is **doubly stochastic** if $A\mathbb{1} = \mathbb{1}$ and $\mathbb{1}^T A = \mathbb{1}^T$.

R Here $\rho(A) = 1$ is associated to $\mathbb{1}$ for A and for A^T.

Consider E the Euclidean space. A set K of points in E is convex if A_1, A_2 points in K, $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $\lambda_1 + \lambda_2 = 1$, then

$$\lambda_1 A_1 + \lambda_2 A_2 \in K.$$

A convex polyhedron K is the set of all points of the form

$$\sum_{i=1}^{N} \lambda_i A_i,$$

where A_i are points in E and $\lambda_1 \in \mathbb{R}_+$.
Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$. Consider this matrix as a point in E with coordinates $[a_{11}, a_{12}, \ldots, a_{nn}]$ ($\dim E = n^2$).

Theorem 4.3.7 Let $A \in \mathcal{M}_n$, $A = [a_{ij}]$, if A is doubly stochastic, then this forms an $(n-1)^2$ dimensional subspace of $\tilde{E} = \mathbb{R}^{n^2}$.

Theorem 4.3.8 — Birkhoff. In the space $\tilde{E} = R^{n^2}$, the set of doubly stochastic matrices of order n is a convex polyhedron in E (the subspace of stochastic matrices). The vertices of the polyhedron are the points corresponding to all the permutation matrices.

CHAPTER 5

M-matrices

Not as well known as nonnegative matrices, M-matrices nonetheless enjoy interesting properties and can be very useful in applications. One of the most comprehensive references on M-matrices is the book [**?**] and I present here a famous theorem from that book. However, I follow first the presentation of [**?**], which I personally find a little more natural. Another good reference is [5]. These references were used to write this chapter, with notation adapted in some cases in order to make the content consistent.

Chapter contents

5.1 Z-matrices

Definition 5.1.1 A matrix is of class Z_n if it is in $\mathcal{M}_n(\mathbb{R})$ and such that $a_{i,j} \leq 0$, $i \neq j$, $i, j = 1, \dots, n$.

$$Z_n = \{A \in \mathcal{M}_n : a_{i,j} \leq 0, i \neq j\}.$$

We also say that $A \in Z_n$ has the **Z-sign pattern**.

Theorem 5.1.2 — [?]. Let $A \in Z_n$. Then the following are equivalent:

1. There is a nonnegative vector x such that $Ax > 0$.

2. There is a positive vector x such that $Ax > 0$.
3. There is a diagonal matrix $\text{diag}(D) > 0$ such that the entries in $AD = [w_{ik}]$ are such that

$$w_{ii} > \sum_{k \neq i} |w_{ik}| \forall i$$

4. For any $B \in Z_n$ such that $A \geq A$, then B is nonsingular.
5. Every real eigenvalue of any principal submatrix of A is positive.
6. All principal minors of A are positive.
7. For all $k = 1, \dots, n$, the sum of all principal minors is positive.
8. Every real eigenvalue of A is positive.
9. There exists a matrix $C \geq 0$ and a number $k > \rho(A)$ such that $A = k\mathbb{I} - C$.
10. There exists a splitting $A = P - Q$ of the matrix A such that $P^{-1} \geq 0$, $Q \geq 0$, and $\rho(P^{-1}Q < 1)$.
11. A is nonsingular and $A^{-1} \geq 0$
12. ...

18 The real part of any eigenvalue of A is positive.

Notation: $A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class K (or a nonsingular M-matrix).

Theorem 5.1.3 Let $A \in Z = \bigcap_{i=1,\dots} Z_n$ be symmetric. Then $A \in K$ if and only if A is positive define.

5.2 Class K_0

Theorem 5.2.1 Let $A \in Z_n$, then the following are equivalent.

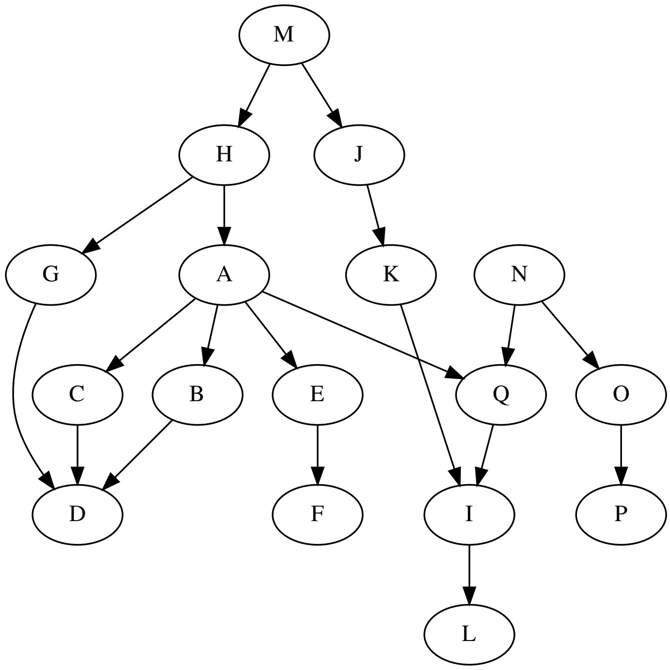
1. $A + \epsilon\mathbb{I} \in K$ for all $\epsilon > 0$.
2. Every real eigenvalue of a principal submatrix of A is nonnegative.
3. All principal minors of A are nonnegative.
4. The sum of all principal minors of order $k = 1, \dots, n$ is nonnegative.
5. Every real eigenvaue of A is nonegative.
6. There exists $C \geq 0$ and $k \geq \rho(C)$ such that $A = k\mathbb{I} - C$
7. Every eigenvalue of A has nonnegative real part.

$A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class K_0.

Theorem 5.2.2 Let $A \in Z_n$. Assume $A \in K_0$. Then $A \in K$ if and only if A is nonsingular.

The following theorem can be found in [**?**], pages 134–138. It is famous for .. well, you will understand soon.

Theorem 5.2.3 — [?]. Let $A \in \mathcal{M}_n$. Then for each fixed letter C representing one of the following conditions, conditions C_i are equivalent for each i. Moreover, letting C then represent any of the equivalent conditions C_i, the following implication tree holds:



Finally, if $A \in Z_n$, then each of the following conditions is equivalent to the statement "A is a nonsingular M-matrix".

(A_1) All of the principal minors of A are positive.

(A_2) Every real eigenvalue of each principal submatrix of A is positive.

(A_3) For each $x \neq 0$ there exists a positive diagonal matrix D such that

$$x^T ADx > 0.$$

(A_4) For each $x \neq 0$ there exists a nonnegative diagonal matrix D such that

$$x^T ADx > 0.$$

(A_5) A does not reverse the sign of any vector; that is, if $x \neq 0$ and $y = Ax$, then for some subscript i, $x_i y_i > 0$.

(A_6) For each *signature matrix* S (here S is diagonal with diagonal entries ± 1), there exists an $x \gg 0$ such that

$$SASx \gg 0.$$

(B_7) The sum of all the $k \times k$ principal minors of A is positive for $k = 1, \ldots, n$.

(C_8) A is nonsingular and all the principal minors of A are nonnegative.

(C_9) A is nonsingular and every real eigenvalue of each principal submatrix of A is nonnegative.

(C_{10}) A is nonsingular and $A + D$ is nonsingular for each positive diagonal matrix D.

(C_{11}) $A + D$ is nonsingular for each nonnegative diagonal matrix D.

(C_{12}) A is nonsingular and for each $x \neq 0$ there exists a nonnegative diagonal matrix D such that

$$x^T Dx \neq 0 \quad \text{and} \quad x^T ADx > 0.$$

(C_{13}) A is nonsingular and if $x \neq 0$ and $y = Ax$, then for some subscript i, $x_i \neq 0$ and $x_i y_i \geq 0$.

(C_{14}) A is nonsingular and for each signature matrix S there exists a vector $x > 0$ such that

$$SASx \geq 0.$$

(D_{15}) $A + \alpha I$ is nonsingular for each $\alpha \geq 0$.

(D_{16}) Every real eigenvalue of A is positive.

(E_{17}) All the leading principal minors of A are positive.

(E_{18}) There exists lower and upper triangular matrices L and U, respectively, with positive diagonals such that

$$A = LU.$$

(F_{19}) There exists a permutation matrix P such that PAP^T satisfies (E_{17}) or (E_{18}).

(G_{20}) A is *positive stable*; that is, the real part of each eigenvalue of A is positive.

(G_{21}) There exists a symmetric positive definite matrix W such that

$$AW + WA^T$$

is positive definite.

(G_{22}) $A + I$ is nonsingular and

$$G = (A + I)^{-1}(A - I)$$

is convergent.

(G_{23}) $A + I$ is nonsingular and for

$$G = (A + I)^{-1}(A - I)$$

there exists a positive definite matrix W such that

$$W - G^T WG$$

is positive definite.

(H_{24}) There exists a positive diagonal matrix D such that

$$AD + DA^T$$

is positive definite.

(H_{25}) The exists a positive diagonal matrix E such that for $B = E^{-1}AE$, the matrix

$$(B + B^T)/2$$

is positive definite.

(H_{26}) For each positive semidefinite matrix Q, the matrix QA has a positive diagonal element.

(I_{27}) A is *semipositive*; that is, there exists $x \gg 0$ with $Ax \gg 0$.

(I_{28}) There exists $x > 0$ with $Ax \gg 0$.

(I_{29}) There exists a positive diagonal matrix D such that AD has all positive row sums.

(J_{30}) There exists $x \gg 0$ with $Ax > 0$ and

$$\sum_{j=1}^{i} a_{ij}x_j > 0, \quad i = 1, \ldots, n.$$

(K_{31}) There exists a permutation matrix P such that PAP^T satisfies (J_{30}).

(L_{32}) There exists $x \gg 0$ with $y = Ax > 0$ such that if $y_{i_0} = 0$, then there exists a sequence of indices $i_1, \ldots, i_r$ with $a_{i_{j-1}i_j} \neq 0$, $j = 1, \ldots, r$ and with $y_{i_r} \neq 0$.

(L_{33}) There exists $x \gg 0$ with $y = Ax > 0$ such that the matrix $\hat{A} = [\hat{a}_{ij}]$ defined by

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \text{ or } y_i \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

is irreducible.

(M_{34}) There exists $x \gg 0$ such that for each signature matrix S,

$$SASx \gg 0.$$

(M_{35}) A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is *strictly diagonally dominant*; that is,

$$a_{ii}d_i > \sum j \neq i|a_{ij}d_j|, \qquad i = 1, \ldots, n.$$

(M_{36}) A has all positive diagonal elements and there exists a positive diagonal matrix E such that $E^{-1}AE$ is strictly diagonally dominant.

(M_{37}) A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is *lower semistrictly diagonally dominant*; that is,

$$a_{ii}d_i \geq \sum j \neq i|a_{ij}d_j|, \qquad i = 1, \ldots, n$$

and

$$a_{ii}d_i > \sum j = 1^{i-1}|a_{ij}d_j|, \qquad i = 2, \dots, n.$$

(N_{38}) A is *inverse-positive*; that is, A^{-1} exists and

$$A^{-1} \geq 0.$$

(N_{39}) A is *monotone*; that is,

$$Ax \geq 0 \Rightarrow x \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

(N_{40}) There exists inverse-positive matrices B_1 and B_2 such that

$$B_1 \leq A \leq B_2.$$

(N_{41}) There exists an inverse-positive matrix $B \geq A$ such that $I - B^{-1}A$ is convergent.
(N_{42}) There exists an inverse-positive matrix $B \geq A$ and A satisfies (I_{27}), (I_{28}) and (I_{29}).
(N_{43}) There exists an inverse-positive matrix $B \geq A$ and a nonsingular M-matrix C such that

$$A = BC.$$

(N_{44}) There exists an inverse-positive matrix B and a nonsingular M-matrix C such that

$$A = BC.$$

(N_{45}) A has a *convergent regular splitting*; that is, A has a representation

$$A = M - N, \quad M^{-1} \geq 0, \quad N \geq 0,$$

where $M^{-1}N$ is convergent.
(N_{46}) A has a *convergent weak regular splitting*; that is, A has a representation

$$A = M - N, \quad M^{-1} \geq 0, \quad M^{-1}N \geq 0,$$

where $M^{-1}N$ is convergent.
(O_{47}) Each weak regular splitting of A is convergent.
(P_{48}) Every regular splitting of A is convergent.
(Q_{49}) For each $y \geq 0$ the set

$$S_y = \{x \geq 0; A^T x \leq y\}$$

is bounded and A is nonsingular.

(Q_{50}) $S_0 = \{0\}$; that is, the inequalities $A^T x \leq 0$ and $x \geq 0$ have only the trivial solution $x = 0$ and A is nonsingular.

Proof. See [**?**]. ■

5.3 An example of application: metapopulations

We want to model how individuals move between a set of N locations. The population at time t in location $i = 1, \dots, N$ is denoted $P_i(t)$. Suppose $m_{ij} \geq 0$ is the rate at which individuals currently in location j move to location i. The evolution of the number of individuals in location i can then be modelled using a linear ordinary differential equation, in which we balance the flows toward i and out of i,

$$\frac{d}{dt}P_i(t) = \left(\sum_{j=1, j\neq i}^{N} m_{ij}P_j(t)\right) - \left(\sum_{j=1, j\neq i}^{N} m_{ji}\right)P_i(t).$$

Omitting the time dependence and writing the equations for $i = 1, \dots, N$, we have

$$\frac{d}{dt}\begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix} = \begin{pmatrix} -\sum_{j=2}^{N} m_{j1} & m_{12} & \cdots & m_{1N} \\ m_{21} & -\sum_{j=1, j\neq 2}^{N} m_{j2} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & -\sum_{j=1}^{N-1} m_{jN} \end{pmatrix}\begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix}, \tag{5.1}$$

which, when setting $\mathbf{P} = (P_1, \dots, P_N)^T$ and

$$\mathbf{M} = \begin{pmatrix} -\sum_{j=2}^{N} m_{j1} & m_{12} & \cdots & m_{1N} \\ m_{21} & -\sum_{j=1, j\neq 2}^{N} m_{j2} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & -\sum_{j=1}^{N-1} m_{jN} \end{pmatrix},$$

is written

$$\mathbf{P}' = \mathbf{MP}.$$

So the dynamical properties of the system depend on the properties of matrix $\mathbf{M}$. The first thing we notice is that the column sums of $\mathbf{M}$ all equal zero, since the diagonal entries contain the sums of the off-diagonal entries. Let us write this in matrix form:

$$\mathbb{1}^T\mathbf{M} = (0, \dots, 0).$$

However, note that $(0, \dots, 0) = 0\mathbb{1}^T$, so the previous equality takes the form

$$\mathbb{1}^T\mathbf{M} = 0\mathbb{1},$$

i.e., 0 is an eigenvalue of $\mathbf{M}$ and so $\mathbf{M}$ is singular.

CHAPTER 6

Norms and Matrix Norms

Section 6.1 should be a review for most of you. Analytic properties of norms (Section 6.2) are an interesting topic you may not have seen. Matrix norms, starting in Section 6.3, however, should be quite new for you. They have applications in a variety of domains. They are also extremely useful when proving results; for instance, [?] prove a great deal of their material on nonnegative matrices using matrix norms.

Chapter contents

6.1 Vector norms

Definition 6.1.1 — Norm. Let V be a vector space over a field $\mathbb{F}$. A function $\|\cdot\| : V \to \mathbb{R}_+$ is a **norm** if for all $x, y \in V$ and for all $c \in \mathbb{F}$,

1. $\|x\| \geq 0$ (Nonnegativity)
2. $\|x\| = 0 \iff x = 0$ (Positivity)
3. $\|cx\| = |c|\|x\|$ (Homogeneity)
4. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

R If we have 1, 3, and 4 but not 2, then we have a seminorm.

Definition 6.1.2 — Inner product. Let V be a vector space over $\mathbb{F}$. A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ is an **inner product** if for all $x,\ y,\ z \in V$ and all $c \in \mathbb{F}$,

1. $\langle x, x\rangle \geq 0$
2. $\langle x, x\rangle = 0 \iff x = 0$
3. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. $\langle cx, y\rangle = c\langle x, y\rangle$
5. $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Theorem 6.1.3 — Cauchy-Schwartz. Let $\langle\cdot,\cdot\rangle$ be an inner product on a vector space V over $\mathbb{F}$, then

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Corollary 6.1.4 If $\langle\cdot,\cdot\rangle$ is an inner product on a real or complex vector space V, then $\|\cdots\| : V \to \mathbb{R}_+$ defined by $\|x\| = \langle x,\ x\rangle^{1/2}$ is a norm on V.

R If $\langle\cdot,\cdot\rangle$ is a semi-inner product, then the resulting $\|x\| = \langle x, x\rangle^{1/2}$ is a seminorm.

Example 6.1

- ℓ_1-norm on $\mathbb{C}^n$: $\|x\|_1 = \sum_{i=1}^{n} |x\ |$
- ℓ_2-norm on $\mathbb{C}^n$: $\|x\|_2 = \left(\sum_{i=1}^{n} |x\ |^2\right)^{1/2}$
- ℓ_∞-norm on $\mathbb{C}^n$: $\|x\|_\infty = \max\{|x_i\ |;\ i = 1, \ldots, n\}$
- ℓ_p-norm on $\mathbb{C}^n$: $\|x\|_p = \left(\sum_{i=1}^{n} |x\ |^p\right)^{1/p}$

■

Theorem 6.1.5 Consider the norm $\|\cdots\|$. Then $\|\cdots\|$ is derived from an inner product if and only if it satisfies the parallelogram identity

$$\frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) = \|x\|^2 + \|y\|^2$$

Example 6.2

- The ℓ_1-norm does not satisfies the parallelogram identity and is not derived from an inner product.
- The Euclidean inner-product $\|x\|_2$ is derived from $\langle x,\ y\rangle = y^* x$.

■

Theorem 6.1.6 If $\|\cdots\|$ is a nom on $\mathbb{C}^n$ and a matrix $T \in \mathcal{M}_n$ which is non-singular. Then

$$\|x\|_T = \|Tx\|$$

is also a norm on $\mathbb{C}^n$.

6.2 Analytic properties of norms

Definition 6.2.1 Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Take a norm $\|\cdots\|$ on V. The sequence $\{x^{(k)}\}$ of vectors in V converges to $x \in V$ with respect to teh norm $\|\cdots\|$ if and only if $\|x^{(k)} - x\| \to 0$ as $k \to \infty$. We write $\lim\limits_{k\to\infty} x^{(k)} = x$ with respect to $\|\cdots\|$.

A question to asks if convergence with respect to norm $\|\cdots\|_B$ implies convergence with respect to the norm $\|\cdots\|_B$. The answer to this question is no, as we can see in the following example.

■ **Example 6.3** Let $\{f_k\}$ in $C([0,1])$ such that

$$f_k(x) = 0, 0 \le x \le \frac{1}{k}$$
$$f_k(x) = 2\left(k^{2/3} - k^{1/2}\right), \frac{1}{k} \le x \le \frac{3}{2k}$$
$$f_k(x) = 2\left(-k^{2/3} + 2k^{1/2}\right), \frac{3}{k} \le x \le \frac{4}{2k}$$
$$f_k(x) = 0, x \in \left[\frac{2}{k}, 1\right]$$

Now

$$\|f_k\|_1 = \frac{1}{2}k^{-1/2} \to 0, k \to \infty$$
$$\|f_k\|_2 = \frac{1}{\sqrt{3}}$$
$$\|f_k\|_\infty = k^{1/2} \to \infty, k \to \infty$$

■

Theorem 6.2.2 Every (vector) norm in $\mathbb{C}^n$ is uniformly continuous.

Corollary 6.2.3 Let $\|\cdots\|_\alpha$ and $\|\cdots\|_\beta$ be any two norms on a finite-dimensional vector space V. Then there exist $C_m, C_r > 0$ such that

$$C_m\|x\|_\alpha \le \|x\|_\beta \le C_r\|x\|_\alpha, \forall x \in V.$$

Corollary 6.2.4 Let $\|\cdots\|_\alpha$ and $\|\cdots\|_\beta$ norms on a finite-dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$, $\{x^{(k)}\}$ a given sequence in V, then

$$x^{(k)} \xrightarrow{\|\cdots\|_\alpha} x \iff x^{(k)} \xrightarrow{\|\cdots\|_\beta} x$$

Definition 6.2.5 Two norms are **equivalent** if whenever a sequence $\{x^{(k)}\}$ converges to x with respect to one of the norm, it converges to x in the other norm.

Theorem 6.2.6 In finite-dimensional vector spaces, all norm are equivalent.

R Remember Cauchy sequences an d complete spaces.

Definition 6.2.7 Let f be a pre-norm on $V = \mathbb{R}^n$ or $\mathbb{C}^n$ [f does not have property 2]. The function

$$f_d = (y) \max_{f(x)=1} \operatorname{Re} y^* x$$

is the dual norm of f.

R Dual norm is well defined. $\operatorname{Re} y^* x$ is a continuous function for all $y \in V$ fixed. The set $\{f(x) = 1\}$ is compact.

Equivalent definition for dual norm: $f^D(y) = \max_{f(x)=1} |y^* x|$

Lemma 6.2.8 — Extension of Cauchy-Schwartz. f a prenorm on $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for all $x, y \in V$,

$$|y^* x| \leq f(x) f^D(y)$$
$$|y^* x| \leq f^D(x) f(x)$$

R

- Dual norm of a pre-norm is a norm.
- The only norm that equals its dual norm is the Euclidean norm.

Theorem 6.2.9 Let $\|\cdots\|$ be a norm on $\mathbb{C}^n$ or $\mathbb{R}^n$, and $\|\cdots\|^D$ its dual, $c > 0$ given . Then for all $x \in V$, $\|x\| = c\|x\|^d \iff \|\cdots\| = \sqrt{c}\|\cdots\|^d$.

In particulr, $\|\cdots\| = \|\cdots\|^2 \iff \|\cdots\| = \|\cdots\|_2$.

■ **Example 6.4** Let $x, y \in \mathbb{C}^n$, note that

$$|y^* x| = |\sum_{i=1}^n \bar{y}_i x_i| \leq \sum_{i=1}^n |\bar{y}_i x_i|$$
$$\leq \left(\max_{1\leq i\leq n} |y_i|\right) \sum_j \|x_j\| = \|y\|_\infty \|x\|_1$$

$\left(\left(\max_{1\leq i\leq n} |x_i|\right) \sum_j \|y_j\| = \|x\|_\infty \|y\|_1\right)$ with equality for x such that $x_i = 1$ for a single value of i such that $|y_i\| = \|y\|_\infty$, $x_i = 0$ otherwise (with equality for x such that $x_i = \frac{y_i}{|y_i|}$ for all i such that

$y_i \neq 0$, $x_i = 0$ otherwise).

Thus $\|y\|_1^D = \max_{\|x\|_1} |y^* x| = \max_{\|x\|_1} \|y\|_\infty \|x\|_1 = \|y\|_\infty$ and
$\|y\|_\infty^D = \max_{\|x\|_\infty} |y^* x| = \max_{\|x\|_\infty} \|y\|_1 \|x\|_\infty = \|y\|_1$.
Thus $\| \cdots \|_1^D = \| \cdots \|_\infty$ and $\| \cdots \|_\infty^D = \| \cdots \|_1$.

Consider $\| \cdots \|_2$. We have $|y^* x| = \left| \sum_{i=1}^n \bar{y}_i x \right| \leq \|y\|_2 \|x\|_2$, with equality when $x = \frac{y}{\|y\|_2}$. Same reasoning as before $\| \cdots \|_2^D = \| \cdots \|_2$.

We can also show that $\| \cdots \|_q^D = \| \cdots \|_p$, with $\frac{1}{p} + \frac{1}{q} = 1$. ■

Definition 6.2.10 Let $x \in \mathbb{F}^n$. Denote $|x| = [|x_i|]$ ($| \cdots |$ entry-wise), and write that $|x| \leq |y|$ if $|x_i| \leq |y_i|$ for all $i = 1, \ldots, n$. Assume $\| \cdots \|$ is

1. monotone if $|x| \leq |y| \implies \|x\| \leq \|y\|$ for all x, y.
2. absloute if $\| |x| \|$ for all $x \in V$.

Theorem 6.2.11 Let $\| \cdots \|$ be a norm on $\mathbb{F}^n$.

1. If $\| \cdots \|$ is absolute, the

$$\|y\|^D = \max_{x \neq 0} = \frac{|y|^T |x|}{\|x\|},$$

for all $y \in V$.
2. If $\| \cdots \|$ absolute, then $\| \cdots \|^D$ is absolute and monotone.
3. $\| \cdots \|$ absolute if and only if $\| \cdots \|$.

6.3 Matrix Norms

We have that we can think of $\mathcal{M}_n$ as a n^2-dimensional vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Thus all vectors norms can be used on matrices. But that does not use all the properties of matrices.

Definition 6.3.1 Let $|||\cdots|||$ be a function from $\mathcal{M}_n \to \mathbb{R}$, $|||\cdots|||$ is a **matrix norm** if it satisfies the following, for all $A \in \mathcal{M}_n$

1. $|||A||| \geq 0$ [nonnegativity]
2. $$NtA = 0 \iff A = 0 \text{ [positivity]}$$
3. For all $c \in \mathbb{C}$

$$|||c|||\ A = |c|\ |||A|||$$

[homogeneity]
4. For all $A,\ B \in \mathcal{M}_n$ $|||A||| + B \leq |||A||| + |||B|||$ [triangular inequality]
5. For all $A,\ B \in \mathcal{M}_n$ $|||AB||| \leq |||A||| |||B|||$ [submultiplicity].

R As for norms, if 2 does not hold, $|||\cdots|||$ is a matrix semi-norm

R $|||A^2||| = |||AA||| \le |||A|||^2$ [for any matrix norm].

If $A^2 = A$, then

$$|||A^2||| = |||A||| \le |||A|||^2 \implies |||A||| \ge 1.$$

In particular, $|||I||| \ge 1$ for any matrix norm.

Assume that A is invertible, then $AA^{-1} = I$, thus

$$|||I||| = |||AA^{-1}||| \le |||A||| \, |||A^{-1}||| \tag{6.1}$$

$$|||A^{-1}||| \ge \frac{|||I|||}{|||A|||} \tag{6.2}$$

■ **Example 6.5** ℓ_1-norm for $A \in \mathcal{M}_n$, $\|A\|_1 = \sum_{i,j=1}^{n} |a_{ij}|$. This is a matrix norm (1-4 were checked before),

$$\begin{aligned}
\|AB\|_1 &= \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \\
&\le \sum_{i,j,k=1}^{n} |a_{ik} b_{kj}| \\
&\le \sum_{i,j,k,m=1}^{n} |a_{ik} b_{mj}| \\
&\left(\sum_{i,k=1}^{n} |a_{ik}| \right) \left(\sum_{j,m=1}^{n} |k_{mj}| \right) \\
&= \|A\|_1 \|B\|_1
\end{aligned}$$

Thus ℓ_1 is a matrix norm. ■

■ **Example 6.6** ℓ_2-norm (Frobenius, Schur, Hilbert-Schmidt - norm)

$$\|A\|_2 = |\operatorname{Tr} AA^*|^{1/2} = \left(\sum_{i,j=1} |a_{ij}^2 \right)^{1/2}$$

is a matrix norm. It is also an absolute norm.

We also have

$$\|A\|_2 = \sqrt{\sigma_1(A)^2 + \cdots + \sigma_n(A)^2},$$

where $\sigma_i(A)$ are the singular values of A. ■

■ **Example 6.7** The ℓ_∞ norm, $\|A\|_\infty = \max_{i,j\in\{1,\ldots n\}} |a_{ij}$ is not a matrix norm. But define $N(A) = n\|A\|_\infty$, then $N(A)$ is a matrix norm. ■

Definition 6.3.2 Let $\|\cdots\|$ be a norm on $\mathbb{C}^n$. Define $|||\cdots|||$ on $\mathcal{M}_n(\mathbb{C})$ by

$$|||A||| = \max_{\|x\|=1} \|Ax\|.$$

Then $|||\cdots|||$ is the matrix norm induced by $\|\cdots\|$.

R

$$|||A||| = \max_{\|x\|} \|Ax\| \tag{6.3}$$

$$= \max_{\|x\|leq1} \|Ax\| \tag{6.4}$$

$$= \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \tag{6.5}$$

$$= \max_{\|x\|_\alpha} \frac{\|Ax\|}{\|x\|} \tag{6.6}$$

where $\|\cdots\|$ is a norm in $\mathbb{C}^n$.

Theorem 6.3.3 The function $|||\cdots|||$ defined in the previous definition has the following properties:

1. $|||\mathbb{I}||| = 1$
2. $\|Ay\| \leq |||A||| \|y\|$ for all $A \in \mathcal{M}_n(\mathbb{C})$ and for all $y \in \mathbb{C}^n$.
3. $|||\cdots|||$ is a matrix norm on $\mathcal{M}_n(\mathbb{C})$.
4. $|||A||| = \max_{\|x\|=\|y\|^D} |y^*Ax|$.

Definition 6.3.4 $|||\cdots|||$ defined from $\|\cdots\|$ by any of the previous methods is the matrix norm induced by $\|\cdots\|$. It is also called the operator norm.

Definition 6.3.5 A norm such that $|||\mathbb{I}||| = 1$ is **unital**.

R Every induced matrix norm is unital. Every induced norm is a matrix norm.

■ **Example 6.8** if we define $|||\cdots|||_1$ as the maximal column sum:

$$|||A|||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$$

, then $|||\cdots|||$ is induced by the ℓ_1 norm on $\mathbb{C}^n$.

Write A as $[a_1 \dots a_n]$ (by columns). Then if we take

$$|||A|||_1 = \max_{1 \le i \le n} \|a_i\|_1.$$

Let $x = [x_i] \in \mathbb{C}^n$. Then

$$\begin{aligned}
\|Ax\|_1 &= \|x_1 a_1 + \dots x_n a_n\| \\
&\le \sum_i \|x_i a_i\|_1 \\
&= \sum_i |x_i| \|a_i\|_1 \\
&\le \sum_i |x_i| \left(\max_k \|a_k\|_1 \right) \\
&= \sum_i |x_i| |||A|||_1 \\
&= \|x\|_1 |||A|||_1.
\end{aligned}$$

$$\max_{\|x\|=1} \|Ax\|_1 \le |||A|||_1 \tag{6.7}$$

Now let $x = e_k$, the kth standard basis vector. For all k

$$\max_{\|x\|} \|Ax\|_1 \ge \|1 \times a_k\| = \|a_k\|.$$

So $\max_{\|x\|_1} \|Ax\|_1 \ge \max_{1 \le k \le m} \|a_k\|_1 = |||A|||_1$.

Thus from the above, Equation 6.7, and equivalence of norm:

$$|||A|||_1 = \max_{\|x\|_1} \|Ax\|.$$

■

■ **Example 6.9** if we define $||| \cdots |||_\infty$ as the maximal row sum:

$$|||A|||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|,$$

then $||| \cdots |||$ is induced by the ℓ_∞ norm on $\mathbb{C}^n$. ■

Proposition 6.3.6 For all U, V unitary matrices, we have $|||UAV|||_2 = |||A|||_2$.

Theorem 6.3.7 Let $||| \cdots |||$ matrix norm in $\mathcal{M}_n$, let $S \in \mathcal{M}_n$ nonsingular. Then for all $A \in \mathcal{M}_n$, we have that $|||A|||_S = |||SAS^{-1}|||$ is a matrix norm. Furthermore, if $||| \cdots |||$ on $\mathbb{C}^n$, then $\|x\|_S = \|Sx\|$ induces $||| \cdots |||_S$ on $\mathcal{M}_n$.

Proof. We have that 1, 2, ,3, and 4 are easy to check for $|||\cdots|||_S$. We check the submultiplicativity:

$$\begin{aligned}|||AB|||_S &= |||SABS^{-1}\\ &= |||SAS^{-1}SBS^{-1}|||\\ &\leq |||SAS^{-1}|||\,|||SBS^{-1}|||\\ &= |||A|||_S|||B|||_S.\end{aligned}$$

For the “induced part":

$$\begin{aligned}\max_{\|x\|_S} \|Ax\|_S &= \max_{\|Sx\|=1} \|SAx\|\\ &= \max_{\|y\|=1} \|SAS^{-1}y\|\\ &= |||SAS^{-1}|||\\ &= |||A|||_S.\end{aligned}$$

■

Theorem 6.3.8 Let $|||\cdots|||$ be a matrix norm on $\mathcal{M}_n$. $A \in \mathcal{M}_n$, $\lambda \in \sigma(A)$. Then

1. $|\lambda| \leq \rho(A) \leq |||A|||$.
2. If A is nonsingular, then

$$\rho(A) \geq |\lambda| \geq \frac{1}{|||A^{-1}|||}.$$

Proof.

1. Let $\lambda \in \sigma(A)$. Then $Ax = \lambda x$ for some $x \neq 0$. Define $X = [x \dots x] \in \mathcal{M}_n$ (x as columns of X). Then $AX = \lambda X$. Now define $|||\cdots|||$ a matrix norm. Then

$$\begin{aligned}|\lambda|\,|||X||| &= |||\lambda X|||\\ &= |||AX|||\\ &\leq |||A|||\,|||X|||.\end{aligned}$$

So $|\lambda| \leq |||A|||$. This is true for all λ, which implies $|\lambda x| \leq \rho(A) \leq |||A|||$.

2. A nonsingular, then for all $\lambda \in \sigma(A)$, $\lambda x^{-1} \in \sigma(A^{-1})$. Reasoning as before

$$|\lambda^{-1}| \leq |||A||| \implies |\lambda| \geq |||A^{-1}|||.$$

■

Lemma 6.3.9 Let $A \in \mathcal{M}_n$. If there exists $|||\cdots|||$ a norm on $\mathcal{M}_n$ such that $|||A||| < 1$, then $\lim_{k\to\infty} A^k = 0$ entry-wise.

Proof. Suppose that $|||\cdots|||$ is such that $|||A||| < 1$. Then, since, $|||A^k||| \leq |||A|||^k$ it follows that

$|||A||| \to 0$ as $k \to \infty$. So $A^k \to 0$ with respect to $|||\cdots|||$. Since all norms are equivalent on $\mathcal{M}_n$, the same is true for $\|A\|_\infty \implies a_{ij} \to 0$ as $k \to \infty$ for all i, j ■

R When $|||A||| < 1$ for some norm , we say that A is convergent.

Theorem 6.3.10 Let $A \in \mathcal{M}_n$, then

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1.$$

Proof. Assume $A^k \to 0$, and let $x \neq 0$ be such that $Ax = \lambda x$. Then $A^k x = \lambda^k x \to 0$ only if $|\lambda| < 1$. This is true for all λ, which implies that $\rho(A) < 1$.

Now suppose that $\rho(A) < 1$. Then, from a result that states "$A \in \mathcal{M}_n$ and $\epsilon > 0$ given. Then there exists $|||\cdots|||$ such that $\rho(A) \le |||A||| \le \rho(A) + \epsilon$", there exists $|||\cdots|||$ matrix norm such that $|||A||| < 1$. From the result before the current theorem we have that $A^k \to 0$ as $k \to 0$. ■

Theorem 6.3.11 — Gelfand Formula. Let $|||\cdots|||$ be a matrix norm on $\mathcal{M}_n$, let $A \in \mathcal{M}_n$. Then

$$\rho(A) = \lim_{k\to\infty} |||A^k|||^{1/k}.$$

Proof. If $\lambda \in \sigma(A)$, then $\lambda^k \in \sigma(A^k)$, which implies that $\rho(A)^k = \rho(A^k)$ for some $k = 1, 2, \ldots$. Thus

$$\rho(A)^k \le |||A^k||| \implies \rho(A) \le |||A^k|||^{1/k}.$$

Let $\epsilon > 0$ be given, Then let

$$\tilde{A} = \frac{1}{\rho(A) + \epsilon} A.$$

We have that $\rho(\tilde{A}) < 1$, so $\tilde{A}$ converges. So $|||\tilde{A}^k||| \to 0$ as $k \to \infty$. Thus there exists $N(\epsilon, A)$ such that

$$|||\tilde{A}||| \le 1$$

for all $k \ge N(\epsilon, A)$.
This is similar to saying that

$$\forall k \ge N, |||\tilde{A}^k||| \le (\rho(A) + \epsilon)^k$$

or, again,

$$\forall k \ge N, |||A^k|||^{1/k} \le \rho(A) + \epsilon.$$

This is true for all $\epsilon > 0$. Since this is true for all ϵ and $\rho(A) \leq |||A^k|||^{1/k}$ we have

$$\lim k \to \infty |||A^k|||^{1/k}$$

exists and it is equal to $\rho(A)$. ■

Theorem 6.3.12 Let R be the radius of convergence of the (scalar) power series $\sum_{k=0}^{\infty} a_k z^k$ and $A \in \mathcal{M}_n$. Thne the matrix power series $\sum_{k=1}^{\infty} a_k A^k$ converges if $\rho(A) < R$.

R The convergence condition for the matrix power series is equal to "there exists matrix norm $||| \cdots |||$ such that $|||A||| < R$'.

Corollary 6.3.13 Let $A \in \mathcal{M}_n$ nonsingular, if there $||| \cdots |||$ matrix norm suct that $|||\mathbb{I} - A||| \leq 1$.

Proof. Suppose that $|||\mathbb{I} - A||| < 1$, then $\sum_{k=0}^{\infty} (\mathbb{I} - A)^k)$ converges (because the radius of converges of $\sum_{k=0}^{\infty} z^k$ is 1). So let $c = \sum_{k=0}^{\infty} (\mathbb{I} - A)^k)$. We have

$$A \sum_{k=0}^{N} (\mathbb{I} - A)^k) = (\mathbb{I} - (\mathbb{I} - A)) \sum_{k=0}^{N} (\mathbb{I} - A)^k)$$
$$= \mathbb{I} - (\mathbb{I} - A)^{N+1}.$$

As $N \to \infty$, $\mathbb{I} - (\mathbb{I} - A)^{N+1} \to \mathbb{I}$ since $\mathbb{I} - A$ converges. Thus $Ac \to \mathbb{I}$ and $c = A^{-1}$. ■

Corollary 6.3.14 Let $A \in \mathcal{M}_n$ is such that $|a_{ii} > \sum_{j \neq i} |a_{ij}$ for all $i = 1, \ldots, n$. Then A is invertible.

Proof. Under strict diagonal dominance hypothesis, all the diagonal entries of A are different from 0. Let $D = (a_1 1, \ldots, a_{nn})$. Then the matrix $D^{-1}A$ has all 1's on the main diagonal, so $B = I - D^{-1}A$ has all diagonal entries equal to 0, and $b_{ij} = -\frac{a_{ij}}{a_{ii}}$ for $i \neq i$. Use the max row sum matrix norm $||| \cdots |||_\infty$. From the strict diagonal dominace hypothesis we have that

$$|||B|||_\infty < 1 \implies \mathbb{I} - B = D^{-1}A$$

is nonsingular, thus A is nonsingular. ■

6.4 Matrix norms and Singular values

Let $V = \mathcal{M}_{mn}(\mathbb{C})$ with Frobenious inner product

$$\langle A, B \rangle_F = \text{Tr}(B^* A).$$

The norm derived from the Frobenius inner product is

$$\|A\|_2 = (\mathrm{Tr}(A^*A))^{1/2}$$

is teh ℓ-2 norm (on Frobenius norm).

The spectral norm $|||\cdots|||$ defined on $\mathcal{M}_n$ by

$$|||A|||_2 = \sigma_1(A),$$

where $\sigma_1(A)$ is the largest singular value of A is induced by the ℓ-2 norm on $\mathbb{C}^n$. Inded, from the singular value decomposition theorem, let

$$A = V\Sigma W^*$$

be a singular value decomposition of A, where V, W unitary, $\Sigma = \sigma(\sigma_1, \dots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are the non-increasingly ordered singular values of A. From unitary invariance and monotonicity of the Euclidean norm, we say that

$$\begin{aligned}
\max \|Ax\|_1 &= \max_{\|x\|_1} \|V\Sigma W^*\|_2 \\
&= \max_{\|x\|_2} \|\Sigma W^* x\|_2 \\
&= \max_{\|Wy\|_2=1} \|\Sigma y\|_2 \\
&= \max_{\|y\|_2} \|\Sigma y\|_2 \\
&\leq \max_{\|y\|_2} \|\sigma_1 y\|_2 \\
&= \sigma_1 \max_{\|y\|_2} \|y\|_2 \\
&= \sigma_1
\end{aligned}$$

Since $\|\Sigma y\|_2 = \sigma_1$ for $y = e_1$,

$$\max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A).$$

We could have used

$$\begin{aligned}
\max_{\|x\|_2=1} = \|Ax\|_2^2 &= \max_{\|x\|_2=1} x^* A^* AX \\
&= \lambda_{max}(A^*A) \\
&= \sigma_1(A)
\end{aligned}$$

R For all U, V unitary $\mathcal{M}_n$ matrices, for all $A \in \mathcal{M}_n$, $|||UAV|||_2 = |||A|||_2$

APPENDIX A

Summary of required linear algebra

A.1 Fields

Definition A.1.1 — Field. A **field** is a set $\mathbb{F}$ together with two binary operations, *addition* and *multiplication*, which are required to satisfy the following *field axioms*, where $a, b, c \in \mathbb{F}$:

- **Associativity** of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$
- **Commutativity** of addition and multiplication: $a + b = b + a$ and $ab = ba$
- **Additive** and **multiplicative identity**: $\exists 0, 1 \in \mathbb{F}, 0 \neq 1$, s.t. $a + 0 = a$ and $a1 = a$
- **Additive inverses**: $\forall a \in \mathbb{F}, \exists -a \in \mathbb{F}$ s.t. $a + (-a) = 0$
- **Multiplicative inverses**: $\forall a \neq 0 \in \mathbb{F}, \exists a^{-1} \in \mathbb{F}$ s.t. $aa^{-1} = 1$
- **Distributivity** (of multiplication over addition): $a(b + c) = (ab) + (ac)$

A.2 Vector spaces

Definition A.2.1 — Vector space. A **vector space** (over $\mathbb{F}$) is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- $\forall u, v \in V, u + v = v + u$ [**commutativity**]
- $\forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}, (u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ [**associativity**]
- $\exists 0 \in V$ s.t. $\forall v \in V, v + 0 = v$ [**additive identity**]
- $\forall v \in V, \exists w \in V$ s.t. $v + w = 0$ [**additive inverse**]
- $\forall v \in V, 1v = v$ [**multiplicative identity**]
- $\forall a, b \in \mathbb{F}$ and $\forall u, v \in V, a(u + v) = au + av$ and $(a + b)v = av + bv$ [**distributivity**]

Definition A.2.2 — Subspace. Let V be a vector space over $\mathbb{F}$. Let $U \subset V$ be a subset of V. Then U is a **subspace** of V if U is a vector space over $\mathbb{F}$ for the same operations of addition and scalar multiplication as V

Theorem A.2.3 — Conditions for a subspace. $U \subset V$ is a subspace of V if and only if (iff) U satisfies the following three conditions:

- $0 \in U$ **[additive identity]**
- $\forall u, w \in U, u + w \in U$ **[closed under addition]**
- $\forall u \in U, \forall a \in \mathbb{F}, au \in U$ **[closed under scalar multiplication]**

Definition A.2.4 — Direct sum. Suppose $U_1, \dots, U_m$ are subspaces of a vector space V. The sum $U_1 + \dots + U_m$ is a **direct sum** and is then written $U_1 \oplus \dots \oplus U_m$ if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $u_1 + \dots + u_m$, where each u_j is in U_j

Theorem A.2.5 — Condition for a direct sum. Suppose $U_1, \dots, U_m$ are subspaces of a vector space V. Then $U_1 + \dots + U_m$ is a direct sum iff the only way to write 0 as a sum $u_1 + \dots + u_m$, where each $u_j \in U_j$, is by taking each u_j equal to 0

Theorem A.2.6 — Direct sum of two subspaces. Let U, W be subspaces of a vector space V. Then $U + W$ is a direct sum iff $U \cap W = \{0\}$

Definition A.2.7 — Linear combination. A **linear combination** of a list $v_1, \dots, v_m$ of vectors in V is a vector

$$a_1 v_1 + \dots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$

Definition A.2.8 — Span. The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ is the **span** of $v_1, \dots, v_m$,

$$\operatorname{Span}(v_1, \dots, v_m) = \left\{ a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbb{F} \right\}$$

The span of the empty list ( ) is $\{0\}$

Definition A.2.9 — Basis. Let V be a vector space. A **basis** of V is a list of vectors in V that is both linearly independent and spanning

Theorem A.2.10 — Criterion for a basis. A list $v_1, \dots, v_m$ of vectors in a vector space V is a basis of V if and only if $\forall v \in V$, v can be written uniquely in the form

$$v = a_1 v_1 + \dots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$

Definition A.2.11 — Dimension. The **dimension** $\dim V$ of a finite-dimensional vector space V is the length of any basis of the vector space

Definition A.2.12 — Linear map/transformation. Let V, W be vector spaces. A **linear map** (or **linear transformation**) from V to W is a function $T : V \to W$ that has the following properties:

1. **Additivity** $\forall u, v \in V, T(u+v) = T(u) + T(v)$
2. **Homogeneity** $\forall \lambda \in \mathbb{F}, \forall v \in V, T(\lambda v) = \lambda T(v)$

Definition A.2.13 — Null space. Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. The **null space** T (or **kernel** $\ker T$) of T is the subet of V consisting of those vectors that T maps to 0:

$$T = \{v \in V; Tv = 0\}$$

Theorem A.2.14 — Null space is a subspace. Let V, W be finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then T is a subspace of V

For a field $\mathbb{F}$, we denote $\mathbb{F}^n$ the n-dimensional vector space consisting of n-tuples $(x_1, \dots, x_n)$ of elements of $\mathbb{F}$.

R $\mathbb{R}^n$ is a real vector space and $\mathbb{C}^n$ is a complex vector space and a real vector space (in which case it has dimension $2n$).

Definition A.2.15 $A \subset B$, A is a **proper subset** of a set B if A is a subset of B not equal to B. We can write $A \subsetneq B$.

Definition A.2.16 — Linear independence/Linear dependence. A list $v_1, \dots, v_m$ of vectors in a vector space V is **linearly independent** if

$$\left(a_1 v_1 + \cdots + a_m v_m = 0\right) \Leftrightarrow \left(a_1 = \cdots = a_m = 0\right),$$

where $a_1, \dots, a_m \in \mathbb{F}$. A list of vectors is **linearly dependent** if it is not linearly independent.

Definition A.2.17 — Range. Let V, W be finite-dimensional vector spaces, $T : V \to W$ a function. The **range** (or **image**) of T is the subset of W defined by

$$\operatorname{range} T = \{Tv; v \in V\}$$

A.3 Matrices

Definition A.3.1 — Matrix. An m-by-n or $m \times n$ **matrix** is a rectangular array of elements of

$\mathbb{F}$ with m rows and n columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

We denote $A \in \mathcal{M}_{mn}(\mathbb{F})$, or $\mathcal{M}_{mn}(\mathbb{R})$ and $\mathcal{M}_{mn}(\mathbb{C})$ if needed. Square matrices are typically denoted by using a single index, $\mathcal{M}_n(\mathbb{F})$.

Definition A.3.2 — Submatrix. A **submatrix** is a matrix extracted from a given matrix by selecting a set of rows and columns.

R Matrices can be used to represent linear transformations.

A.3.1 Matrix Operations

1. **Zero matrix** $0_{mn} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$.
2. **The identity matrix** $\mathbb{I}_n \in \mathcal{M}_n$ is

$$\mathbb{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & 1 \end{pmatrix} = \text{diag}(1, \dots, 1).$$

   Sometimes, diag is used as a function to “extract” the diagonal of a matrix; for instance, you will sometimes see $\text{diag}(\mathbb{I}) = (1, \dots, 1)$.
3. $\mathbb{I}$ and $c\mathbb{I}$, $c \in \mathbb{F}$ commute with any $M \in \mathcal{M}_{mn}(\mathbb{F})$. (Of course, provided the sizes are compatible.)
4. **Transpose**: $A \in \mathcal{M}_{mn}(\mathbb{F})$, $A = [a_{ij}]$, then $A^T in \mathcal{M}_{nm}$ such that $A^t = [a_{ji}]$.
5. **Conjugate transpose**. $A \in \mathcal{M}_{mn}(\mathbb{F})$, $A = [a_{ij}]$, then $A^* \in \mathcal{M}_{nm}$ such that $A^* = [\ \overline{a_{ji}}\ ]$. If $A \in M_{mn}(\mathbb{R})$, then $A^* = A^T$.
6. **Reverse order law**: $(AB)^* = B^* A^*$ and $(AB)^T = B^T A^T$.

Trace: $A \in \mathcal{M}_{mn}$, $\text{Tr}(A) = \sum_{i=1}^{q} a_{ii}$, where $q = \min\{m, n\}$.

$A \in \mathcal{M}_{mn}(\mathbb{C})$ then $\text{Tr}\, AA^* = trA^*A = \sum_{i,j} |a_{ij}|^2$. In particular, $\text{Tr}\, AA^* = 0 \iff A = 0$.

Recall that $A \in \mathcal{M}_{mn}$ can be used to represent $x \mapsto Ax$ from $\mathbb{F}^n$ to $\mathbb{F}^m$. The domain of the transformation is $\mathbb{F}^n$. Range is $\mathbb{F}^m \supset \{y \in \mathbb{F}^m | y = Ax\}$. The nullspace is

$$\{x \in \mathbb{F}^n \mid Ax = 0\}.$$

Theorem A.3.3 — Rank-nullity Theorem.

$$\dim(\text{range}A) + \dim(\text{nullspace}A) = \text{rank } A + \text{nullity}A = n.$$

R In matrix product AB, left multiplication by A multiplies the columns of B, right multiplication of B multiplies the row of A.

R Range of A is aslso called the column space of A, $\{y^T A \mid y \in \mathbb{F}^m\}$ is the row space of A.

R $e_1, \ldots, e_n$ vectors of the canonical basis of $\mathbb{F}^n$ (for $\mathbb{R}^n$, $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$).

Notation: $\mathbb{J} = \mathbb{1}\mathbb{1}^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$.

Definition A.3.4 $A \in \mathcal{M}_{mn}$, the **rank** of A is the dimension of the range of A, i.e., it is the length of the largest linearly independent list of columns of A.

If $A \in \mathcal{M}_{mn}(\mathbb{C})$, then

$$\text{rank } A^* = \text{rank } A^T = \text{rank } \bar{A} = \text{rank } A.$$

Theorem A.3.5 — Singularity/nonsingularity. Let $A \in \mathcal{M}_n(\mathbb{F})$, then the following are equivalent:

1. A non-singular
2. A^{-1} exists
3. rank $A = n$
4. rows of A are linearly independent
5. columns of A are linearly independent.
6. $\det A \neq 0$
7. $\dim nullspace A = 0$
8. $Ax = b$ consistent for all $b \in \mathbb{F}^n$
9. $Ax = b$ consistent implies that $x = A^{-1}b$ uniquely.
10. $Ax = 0$ implies that $x = 0$.
11. 0 is not an eigenvalue of A.

A.3.2 A bestiary of matrices

Let $A \in M_n(\mathbb{F})$. A is

- **symmetric** if $A^T = A$
- **skew symmetric** if $A^T = -A$.

- **orthogonal** if $A^T A = \mathbb{I}$.
- A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is
  - **Hermitian** if $A^* = A$;
  - **skew Hermitian** is $A^* = -A$;
  - **essentially Hermitian** if $e^{i\theta} A$ is Hermitian for some $\theta \in \mathbb{R}$;
  - **unitary** $A^* A = \mathbb{I}$;
  - **normal** $AA^* = A^* A$.

A.4 Euclidean Inner Product and Norm

Definition A.4.1 Let $x, y \in \mathbb{C}^n$. The **Euclidean inner product** on $\mathbb{C}^n$ is

$$\langle x,\, y\rangle = y^* x.$$

The **Euclidean norm** of x is

$$\|x\|_2 = \langle x,\, x\rangle^{1/2}.$$

Proposition A.4.2 1. $\langle \alpha x_1 + \beta x_2,\, y\rangle = \alpha\langle x_1,\, y\rangle + \beta\langle x_2, y\rangle$.
2. $\langle x,\, \alpha y_1 + \beta y_2\rangle = \bar{\alpha}\langle x,\, y_1\rangle + \bar{\beta}\langle x,\, y_2\rangle$.

Inner product is linear in first argument and conjugate linear in the second (sesquilinear)

$f : V \times V \to \mathbb{C}$ defined by $\langle x,\, y\rangle \mapsto y^* x$ is a semi-inner product if it is sesquilinear and $f(x,\, x) \geq 0$ and is an inner product if it is sesquilinear and $f(x,\, x) = 0 \iff x = 0$.

Definition A.4.3 $x,\, y \in \mathbb{C}^n$ are orthogonal if $\langle x,\, y\rangle = 0$. A list $x_1, \ldots, x_l \in C^n$ is orthogonal if for all $i,\, j$ with $i \neq j$ we have that $\langle x_i,\, x_j\rangle = 0$. A list is orthonormal is it is orthogonal and $\|x_i\| = 1$ for all i.

Theorem A.4.4 — Cauchy-Schwartz. For all $x,\, y \in \mathbb{C}^n$, then

$$|\langle x,\, y\rangle| \leq \|x\|_2 \|y\|_2.$$

A.4.1 Gram-Schmidt orthonormalisation process

Every linear independent list of vectors can be transformed into an orthonormal list.

Let $v_1, \ldots, v_m$ be a linearly independent list of vectors in an inner product space V. Let

$$e_1 = \frac{v_1}{\|v_1\|}$$

For $j = 2, \ldots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\mathrm{Span}(v_1, \ldots, v_j) = \mathrm{Span}(e_1, \ldots, e_j), \quad j = 1, \ldots, m$$

A.5 Partitions sets and Matrices

A **partition** of a set S is a collection of subsets of S such that each element in S belongs to one and only one subset.

■ **Example A.1** $\{1, \ldots, n\} = \{1, 2\} \cup \{3, \ldots, n\}$. This is a sequential partition. ■

A **partition of a matrix** is a decomposition of a matrix into submatrices such that each entry in the original matrix is in exactly one of the submatrices.

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$ and let $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$. Then $A[\alpha, \beta]$ is a submatrix of A with rows those of A in α and columns those of A in β.

■ **Example A.2**

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A[\{1, 2\}, \{3\}] = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$ ■

If $\alpha = \beta$, then $A[\alpha, \alpha]$ is a principal submatrix of A. $A[\{1, \ldots, k\}, \{1, \ldots, k\}]$ $(k \leq n)$ is a leading principal submatrix.

Definition A.5.1 Let $\alpha_1, \alpha_k, \beta_1, \ldots, \beta_\ell$ be partition of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, repectively. Then the matrices

$$A[\alpha_i, \beta_j]$$

form a partition of $A \in \mathcal{M}_{mn}$.
If a matrix is partition by sequential partitions, the result is a block matrix.

■ **Example A.3** $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 0 & 10 \end{pmatrix}$ the eigenvalues are 1, 3, 6, 10. This is block lower triangular

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}.$$

The eigenvalues of A are those of the diagonal blocks A_{11} and A_{22}, A_{ii} are triangular, thus their eigenvalues are their diagonal entries. ■

■ **Example A.4 — Theorem.** Let $A \in \mathcal{M}_n(\mathbb{F})$; assume that it is non-singular. Suppose that A is written in block form as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathcal{M}_{n_1}$ and $A_{22} \in \mathcal{M}_{n_2}$, $n = n_1 + n_2$. Then

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & A_{11}^{-1}A_{12}(A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1} \\ A_{22}^{-1}A_{21}(A_{12}A_{22}^{-1}A_{21} - A_{11})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

In particular, $A_{12} = 0$ and $A_{21} = 0$ (i.e., the matrix is block diagonal). Then

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}$$

■

Let $\{1, \dots, n\}$ be an index set. Let $\alpha \subseteq \{1, \dots, n\}$, then $\alpha^c = \{1, \dots, n\} \setminus \alpha$. If $A \in \mathcal{M}_n(\mathbb{F})$, $n \geq 2$. Then

$$\mathsf{adj} A = [(-a)^{i+j} \det A[j^c, i^c]]$$

is the **adjugate** or **classical adjoint** of A.

If A is invertible, then $A^{-1} = \dfrac{1}{\det A}\mathsf{adj} A$.

Theorem A.5.2 — Cauchy's formula for the determinant of a rank one perturbation. Let $M \in M_n$ and $x,\ y \in \mathbb{F}^n$. Then

$$\det(A + xy^T) = \det A + y^T(\mathsf{adj} A)x.$$

Let $D = [d_{ij}]$, $E = [e_{ij}]$ diagonal matrices and $A \in \mathcal{M}_(\mathbb{F})$, $\det D = \prod_{i=1}^{n}$, D is non-singular if and only if $\nexists i$, $d_{ii} = 0$.

Left multiplication of A by D multiplies the rows of A by diagonal entries in D gives us

$$\begin{pmatrix} d_{11}a_1 1 & \dots & d_{nn}a_{1n} \\ \vdots & \ddots & \vdots \\ d_{11}a_{n1} & \dots & d_{nn}a_{nn} \end{pmatrix}$$

Similar multilipcation of A by E multiplies the columns of A by the entries of E

A.5.1 Block diagonal matrices and direct sums

A matrix of the form

$$A = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & \dots A_{kk}, \end{pmatrix}$$

where $A_{ii} \in \mathcal{M}_{m_i}$ is a block diagonal matrix.

We also write $A = \bigoplus_{i=1}^{k} A_{ii}$, and say the matrices are in direct sum.

We have $\det(A) = \prod_{i=1}^{k} \det(A_{ii}$.

A is non-singular if and only if none of the A_{ii}'s are singular.

rank $A = \sum_{i=1}^{k}$ rank A_{ii}

$\mathsf{adj}(A \oplus B) = (\det B)\mathsf{adj}A \oplus (detA)\mathsf{adj}B$.

The same way, we can define block upper and block lower triangular matrices.

If you can read (briefly) about special matrices in section 0.9 in Horn.

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