$X \sim N(m, \sigma^2)$ $\quad m ? \quad \sigma^2 ?$

moyenne $\begin{cases} E(X) = m \end{cases}$ $\Rightarrow \left(\hat{m}_{MO} = \frac{1}{n}\sum_{i=1}^{n} X_i \right)$ ①

moment d'ordre 2 $\quad E(X^2) = \underbrace{\mathrm{var}\, X}_{\sigma^2} + \underbrace{E^2[X]}_{m^2} = m^2 + \sigma^2$ ②

$$\begin{cases} E(X) = m \\ E[X^2] = m^2 + \sigma^2 \end{cases} \Leftrightarrow \begin{cases} m = E[X] \\ \sigma^2 = E[X^2] - E^2[X] \end{cases}$$

$$\boxed{\begin{aligned} \hat{m}_{MO} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ \hat{\sigma}^2_{MO} &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \end{aligned}}$$

$$p(\theta \mid x_1, \dots, x_n) = \frac{p(x_1, \dots, x_n \mid \theta)\, p(\theta)}{p(x_1, \dots, x_n)}$$

loi a posteriori

vraisemblance

$$\hat{\theta}_{MAP} = \arg\max_{\theta}\ p(\theta \mid x_1, \dots, x_n)$$

maximum de la loi a posteriori

$$\hat{\theta}_{MMSE} = E[\theta \mid x_1, \dots, x_n]$$

moyenne de la loi a posteriori

$X_i \sim \mathcal{E}(\lambda)$ $\quad p(x_i ; \lambda) = \begin{cases} \lambda e^{-\lambda x_i} & x_i > 0 \\ 0 & x_i < 0 \end{cases}$

$\boxed{\theta = \lambda}$

$$\boxed{p(x_1, \dots, x_n \mid \theta) = \lambda^n e^{-\lambda \sum x_i}}$$

$p(\lambda)$
loi a priori sur λ

$$\boxed{p(\lambda) = \begin{cases} e^{-\lambda} & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}}$$

$$\frac{p(x_1 \dots x_n \mid \lambda)\, p(\lambda)}{p(x_1, \dots, x_n)} = \frac{\lambda^{n} e^{-\lambda(\sum x_i + 1)}}{p(x_1, \dots, x_n)}$$

$$\propto \lambda^n e^{-\lambda \underbrace{(\sum x_i + 1)}}$$

$$\lambda \mid X_1 \dots X_n \sim \Gamma\left(\overset{\nu}{\overline{n+1}}, \overset{\theta}{\overline{\sum x_i + 1}}\right)$$

$$\boxed{E[\lambda \mid X_1 \dots X_n] = \frac{n+1}{\sum x_i + 1}}$$

$$\frac{\theta^{\nu}}{\Gamma(\nu)} = \boxed{\frac{(\sum x_i + 1)^{n+1}}{\Gamma(n+1)}}$$

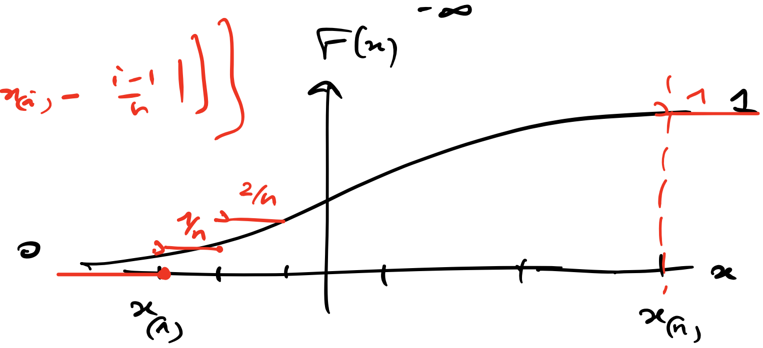
Lois continues

Kolmogorov H_0 $X_1, \dots, X_n \sim N(m, \sigma^2)$

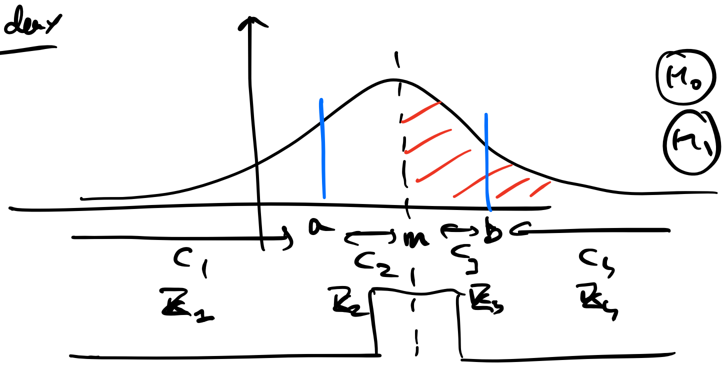
H_1 non H_0

$$F(x) = P[X < x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(w-m)^2}{2\sigma^2}\, dw$$

$$\max_i \left\{ \max \left\{ \left| F(x_{(i)}) - \frac{i}{n} \right|, \left| F(x_{(i)}) - \frac{i-1}{n} \right| \right\} \right\}$$



chi-deux



H_0 $X_1, \dots, X_n \sim N(m, \sigma^2)$

H_1 non H_0

$$\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \frac{1}{4}$$

m

$$\int_{-\infty}^{\frac{a-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \frac{1}{4}$$

$$b - m = m - a$$

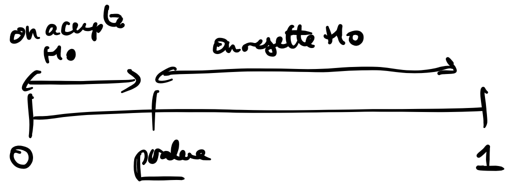
$$\boxed{b = 2m - a}$$

$$F_{N(0,1)}\left(\frac{a-m}{\sigma}\right) = \frac{1}{4}$$

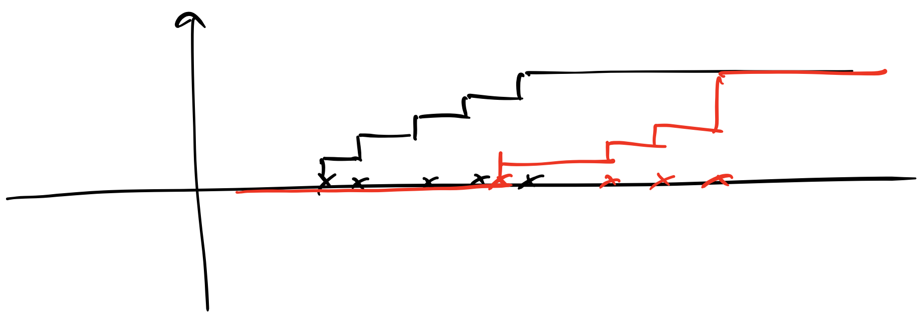
$$\frac{a-m}{\sigma} = F^{-1}\left(\frac{1}{4}\right)$$

$$\boxed{a = \sigma F^{-1}\left(\frac{1}{4}\right) + m}$$

$$\sum_{i=1}^{k} \frac{(z_i - np_i)^2}{np_i}$$



0.001



$U = 15$
$W = 51$

H_0 : $F = G$
H_1 : $G < F$

On rejette H_0 si $\boxed{U > S_\alpha}$ ou $\boxed{W > S_\alpha}$

$\boxed{\text{Recherche du seuil } S_\alpha}$

Méthode 1 : on se fixe $\alpha = P[\text{Rejeter } H_0 \mid H_0 \text{ vraie}] = 0.01$ ou 0.05

$$\alpha = P[U > S_\alpha \mid F = G]$$

U suit la loi de Mann-Whitney $m < 8$, $n < 8$

ou

ou U suit la loi normale $N(m_u, \sigma_u^2)$

$m \geq 8$
$n \geq 8$

$m_U = \frac{nm}{2}$

$\sigma_U^2 = \frac{nm(n+m+1)}{2}$

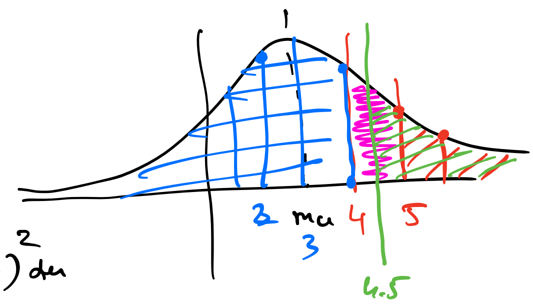
- **Sans correction de continuité**

$$\alpha = P\left[\underbrace{\frac{U - m_U}{\sigma_U}}_{X} > \frac{S_\alpha - m_U}{\sigma_U} \;\middle|\; X \sim N(0,1)\right]$$

$$\alpha = 1 - F_{N(0,1)}\left(\frac{S_\alpha - m_U}{\sigma_U}\right)$$

$$\Rightarrow \boxed{S_\alpha = F^{-1}_{N(0,1)}(1-\alpha)\,\sigma_U + m_U}$$

- **Avec correction de continuité**



$$\alpha = P[U > S_\alpha \mid F = G]$$

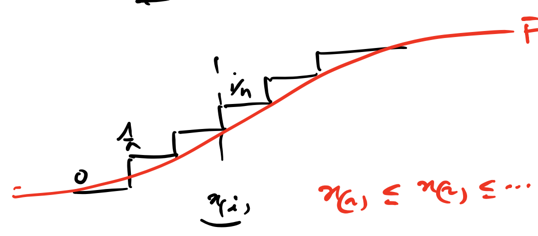
$$\approx \int_{S_\alpha}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_U^2}} \exp -\frac{(u - m_U)^2}{2\sigma_U^2}\, du$$

$$\approx \int_{S_\alpha - \frac{1}{2}}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_U^2}} \exp -\frac{(u - m_U)^2}{2\sigma_U^2}\, du$$

$\boxed{v = u + \frac{1}{2}}$

$$\approx \int_{S_\alpha}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_U^2}} \exp -\left[\frac{\left(v - (m_U + \frac{1}{2})\right)^2}{2\sigma_U^2}\right]$$

$\boxed{m_U \rightarrow m_U + \frac{1}{2}}$



$x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$

Tests de Normalité

loi Normale

H_0 : $(x_1, \dots, x_n) \sim N(m, \sigma^2)$

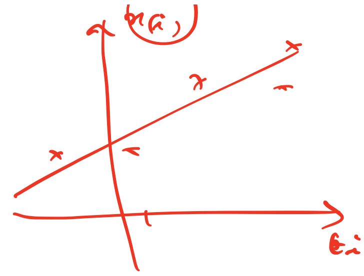
H_1 : non H_0

$$\boxed{P[X \leq x_i]} = \Phi\left(\frac{x_i - m}{\sigma}\right)$$

$$P[X \leq x_{(i)}] = \Phi\left(\frac{x_{(i)} - m}{\sigma}\right) \approx \frac{i}{n}$$

$$\frac{x_{(i)} - m}{\sigma} = \Phi^{-1}\left(\frac{i}{n}\right)$$

$$\boxed{x_{(i)} = \sigma \underbrace{\phi^{-1}\left(\frac{i}{n}\right)}_{t_i} + m}$$

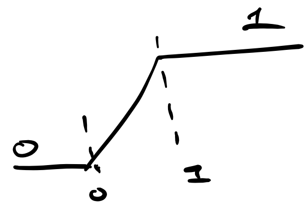


$\boxed{\text{Q.Q plot}}$

Loi uniforme

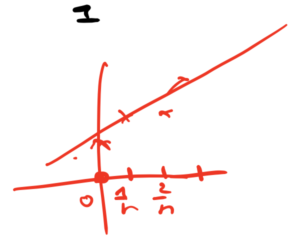
H_0 : $x_1, \dots, x_n \sim U]a,b[$

H_1 : non H_0



$$P[X \leq x_i] = \frac{x_i - a}{b-a}$$

$$P[X \leq x_{(i)}] = \frac{x_{(i)} - a}{b-a} \approx \frac{i}{n}$$



$$\boxed{x_{(i)} = (b-a) \underbrace{\frac{i}{n}}_{t_i} + a}$$

Données ordonnées $\Rightarrow$ $\boxed{Y_i \sim N(0,1)}$

$$\boxed{Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}}$$

$\alpha_i = E[Y_{(i)}]$ moyenne et la matrice de covariance

$$\boxed{B} = \begin{pmatrix} Var[Y_{(1)}] & & \\ & \ddots & \\ & & Var[Y_{(n)}] \end{pmatrix}$$

$cov[Y_{(i)}, Y_{(j)}]$

$$Y_{(1)} = \min(Y_1, \dots, Y_n)$$

$$P[Y_{(1)} < u] = 1 - P[Y_{(1)} \geq u]$$
$$= 1 - P[Y_1 \geq u, \dots, Y_n \geq u]$$
$$= 1 - \underbrace{[1 - F(u)]^n}_{\text{loi du minimum}}$$

$$\boxed{F(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}$$

$$E[Y_{(1)}] = \int u\, p(u)\, du = \dots$$

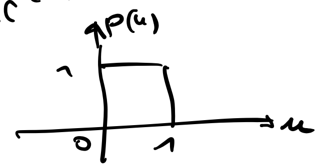
$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ vecteur des moyens des données ordonnées

$B = \begin{bmatrix} \ddots \end{bmatrix}$ matrice de covariance des données ordonnées

pour un échantillon $(Y_1, \dots, Y_n)$ de loi normale $N(0,1)$

Exo 1 : Fonction de répartition d'une loi ~~no~~ uniforme $]0,1[$

$$F(x) = \int_{-\infty}^{x} p(u)\,du = \int_{-\infty}^{x} \mathbb{1}_{]0,1[}(u)\,du$$



Rappel

$$\text{donc} \quad F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1\,du = x & \text{si } x \in]0,1[\\ 1 & \text{si } x > 1 \end{cases}$$

donc $F(x) = x \quad \forall x \in]0,1[$

$x_{(1)} \leq x_{(2)} \leq x_{(3)} \dots \leq x_{(10)}$

	$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$	$x_{(6)}$	$x_{(7)}$	$x_{(8)}$	$x_{(9)}$	$x_{(10)}$
$x_{(i)}$	0,15	0,18	0.22	0.26	0.39	0.53	0.62	0.7	0.96	0.99
E_i^-	0,15	0.08	~~0.07~~	~~0.04~~	~~0.01~~	~~0.03~~	~~0.02~~	~~0~~	0.16	0.09
E_i^+	~~0.05~~	~~0.02~~	0.08	0.14	0.11	0.07	0.08	0.10	~~0.06~~	~~0.01~~

$\max_i \{E_i^-, E_i^+\} = 0.16$

$$E_i^- = \left| F_0(x_{(i)}) - \frac{i-1}{n} \right| = \left| x_{(i)} - \frac{i-1}{n} \right|$$

Donc $\boxed{D_n = 0.16}$

$$E_i^+ = \left| F_0(x_{(i)}) - \frac{i}{n} \right| = \left| x_{(i)} - \frac{i}{n} \right|$$

$\boxed{\text{On rejette } H_0 \text{ si } D_n > S_\alpha}$

$$P\left[\sqrt{n}\,D_n < y\right] \xrightarrow[n\to\infty]{} K(y)$$

On se fixe α (= 0.05 par ex) $= P\left[D_n > s_\alpha \mid D_n \text{ suit la loi de Kolmogorov}\right]$

donc $\alpha = 1 - P\left[D_n \leq s_\alpha \mid D_n \sim K\right]$

$= 1 - P\left[\sqrt{n}\,D_n \leq \sqrt{n}\,s_\alpha \mid -\right]$

$$\alpha = 1 - K(\sqrt{n}\,s_\alpha)$$

$$\Rightarrow \sqrt{n}\,s_\alpha = K^{-1}(1-\alpha)$$

$$\boxed{s_\alpha = \frac{1}{\sqrt{n}} K^{-1}(1-\alpha)} = 0.40925$$

↑ Table

$\boxed{D_n = 0.16 < s_\alpha \text{ donc on accepte } H_0 \text{ avec } \alpha = 0.05}$

Exo2

$U = 2 + 2 + 6 = \boxed{10}$ $\qquad W = 1+2+3+9+8+9+14 = \boxed{46}$

$U = W - \frac{8\times 9}{2}$ $\quad 46 - 36 = \boxed{10}$

$y_2 = 360 < y_7 = 380 < y_8 = 400 < y_1 = y_3 = 430 < x_5 = 458 < x_6 = 478 < y_6 = 480 < \dots$

(rangs : 1, 2, 3, 4, 5, 6, 7, 8 ; $r=1$, $r=2$, $r=3$, $\boxed{r = 4.5}$, $r=5$, $r=\cancel{7}$, $r=8$)

$\dots < y_5 = 490 < x_6 = 538 < x_3 = x_8 = 548 < x_2 = 568 < y_4 = 570 < x_1 = 578 < x_7 = 618$

(rangs : 9, 10, 11, 12, 13, 14, 15, 16 ; $r=9$, 10, 11.5, 13, 14, 15, 16)

$\alpha_1 = 1 \quad \alpha_2 = 1 \quad \alpha_3 = 1 \quad \alpha_4 = \alpha_5 = 1 \quad \alpha_6 = 0 \quad \alpha_7 = 0 \quad \alpha_8 = 1$

$\alpha_9 = 1 \quad \alpha_{10} = 0 \quad \alpha_{11} = \alpha_{12} = 0 \quad \alpha_{13} = 0 \quad \alpha_{14} = 1 \quad \alpha_{15} = 0 \quad \alpha_{16} = 0$

$\sum_{k=1}^{j} \alpha_k$	1	2	3	4	5	5	5	6	7	7	7	7
$\lvert 2\sum_{k=1}^{j} \alpha_k - j\rvert$	1	2	3	4	5	4	3	4	5	4	3	2
	1	2	3	4	5	6	7	8	9	10	11	12

7	8	8	8
1	2	1	0
13	14	15	16

$\boxed{D_n = \frac{5}{8}}$

$$\boxed{\text{On rejette } H_0 \text{ si } D_n > S_{n,m,\alpha}}$$

H_0 $F=G$
(les 2 échantillons ont la même loi)

H_1 non H_0

$$\alpha = P(D_n > S_{n,\alpha} \mid D_n \sim \text{Kolmogorov\_Smirnov})$$

$\underset{n=m}{\uparrow}$

$$= 1 - F(S_{n,\alpha})$$

$$\boxed{S_{n,\alpha} = F^{-1}(1-\alpha)}$$

$$S_{8,8,0.01} = \frac{56}{8\times 8} = \frac{56}{64} = \boxed{\frac{7}{8}}$$

$$S_{8,8,0.05} = \frac{48}{8\times 8} = \frac{6}{8} = \boxed{\frac{3}{4}} = \frac{6}{8}$$

Dans les deux cas $D_n <$ seuil $\Rightarrow$ on accepte H_0 avec les risques $\alpha = 0.01$ et $\alpha = 0.05$
Donc les deux soporifiques ont la même efficacité

Test #2 : Mann_Whitney

H_0 $F=G$

H_1

hypothèse H_0 : même efficacité $F=G$

hypothèse H_1 : $x_i > y_j$, i.e $F < G$

Calcul de la statistique de Mann-Whitney

Méthode 1 $$U = \sum_{j=1}^{m} \sum_{i=1}^{n} \mathbb{1}_{y_j > x_i}$$

$U = 10$

On rejette H_0 si $U <$ Seuil.

$\alpha = P[\,U <$ Seuil $\mid U \sim$ Mann-Whitney$]$ Table

$\alpha = F[S_\alpha] \Rightarrow \boxed{S_\alpha = F^{-1}(\alpha)} =$ $n_1 = 8$, $n_2 - n_1 = 0$

$\approx \boxed{15}$ Attention, utiliser les tables du test unilatéral

$U <$ Seuil $= 15$ donc on rejette l'hypothèse H_0 avec le seuil $\alpha = 0.05$

le agrofonique a une meilleure efficacité que le produit inactif avec $\alpha = 0.05$

Exo3 Q2 $Z_i = Y_i - X_i$ $\quad Z_i \sim N(m, \sigma^2)$.

H_0 $m = 0$ ~~si m et σ^2 connu~~

H_1 $m \neq 0$ si σ^2 connu Rejet de H_0 si $\frac{1}{n}\sum_{i=1}^{n} x_i >$ Seuil

On est dans le contexte des tests paramétriques.

Test de Student

$U = \frac{1}{n}\sum_{i=1}^{n} x_i \sim N\left(m, \frac{\sigma^2}{n}\right)$

$\alpha = P[\,U >$ seuil $\mid m = 0]$

$= P\left[\frac{U}{\sigma/\sqrt{n}} > \frac{\text{seuil}}{\sigma/\sqrt{n}} \;\middle|\; \frac{U}{\sigma/\sqrt{n}} \sim N(0,1)\right]$

$= 1 - F\left[\frac{\text{seuil}}{\sigma/\sqrt{n}}\right]$

$$\text{Seuil} = F^{-1}(1-\alpha)\frac{\sigma}{\sqrt{n}}$$

Test de Student

Rejet de H_0 si $\left| \frac{\sqrt{n} \sum_{i=1}^{n} x_i}{s_n} \right| > \text{Seuil}$

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

estimateur de la variance