Journal name manuscript No.
(will be inserted by the editor)

Euclidean distance-based skeletons: a few notes on average outward flux and ridgeness - Supplementary material

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Received: date / Accepted: date

A Appendix

A.1 Relation between AOF and ridgeness

From Eq. (5), we have

$$\mathrm{rdg}(\boldsymbol{x}, \sigma) = -\int_{\mathbb{R}^2} D(\boldsymbol{y}) \Delta G_\sigma(\boldsymbol{x} - \boldsymbol{y}) \mathrm{d}\boldsymbol{y},$$

which can be written in polar coordinates as

$$\begin{aligned}
\mathrm{rdg}(\boldsymbol{x}, \sigma) &= -\int_0^\infty \rho \Delta G_\sigma(\rho\cos\theta, \rho\sin\theta) \int_0^{2\pi} D\left(\boldsymbol{x} + \begin{bmatrix}\rho\cos\theta \\ \rho\sin\theta\end{bmatrix}\right) \mathrm{d}\theta \ \mathrm{d}\rho \\
&= \frac{1}{\pi\sigma^4} \int_0^\infty \rho\left(1 - \frac{\rho^2}{2\sigma^2}\right) \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \int_0^{2\pi} D\left(\boldsymbol{x} + \begin{bmatrix}\rho\cos\theta \\ \rho\sin\theta\end{bmatrix}\right) \mathrm{d}\theta \ \mathrm{d}\rho
\end{aligned}$$

Considering the following indefinite integral

$$\int \rho\left(1 - \frac{\rho^2}{2\sigma^2}\right) \exp\left(-\frac{\rho^2}{2\sigma^2}\right) \ \mathrm{d}\rho = \frac{\rho^2}{2} \exp\left(-\frac{\rho^2}{2\sigma^2}\right),$$

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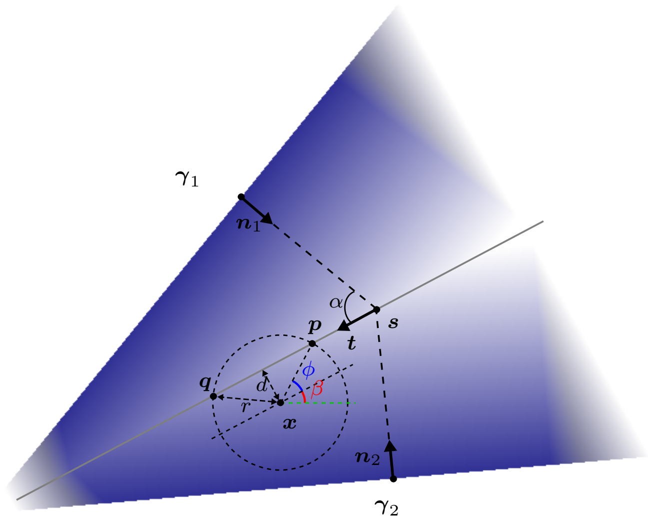

Fig. 1 Regular skeleton point at $\boldsymbol{x}$

we integrate by parts w.r.t ρ,

$$\begin{aligned}\mathrm{rdg}(\boldsymbol{x},\sigma) &= \frac{1}{\pi\sigma^4}\left\{\left[\frac{\rho^2}{2}\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\int_0^{2\pi} D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\,\mathrm{d}\theta\right]_{\rho=0}^{\rho=+\infty}\right.\\ &\left. -\int_0^{\infty}\frac{\rho^2}{2}\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\int_0^{2\pi}\nabla D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\mathrm{d}\theta\;\mathrm{d}\rho\right\}\\ &= -\frac{1}{\pi\sigma^4}\int_0^{\infty}\frac{\rho^2}{2}\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\int_0^{2\pi}\nabla D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\mathrm{d}\theta\;\mathrm{d}\rho\end{aligned}$$

Using Eq. (3), it comes

$$\mathrm{rdg}(\boldsymbol{x},\sigma) = -\frac{1}{\sigma^4}\int_0^{\infty}\rho^2\exp\left(-\frac{\rho^2}{2\sigma^2}\right)\mathrm{aof}(\boldsymbol{x},\rho)\;\mathrm{d}\rho$$

which proves Proposition 1.

A.2 Regular skeleton point - AOF

Let β be the angle formed by $\boldsymbol{t}$ and the horizontal axis, such that

$$\boldsymbol{t} = \frac{(\boldsymbol{n}_1-\boldsymbol{n}_2)^{\perp}}{\|\boldsymbol{n}_1-\boldsymbol{n}_2\|} = \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix}$$

Let $d = (\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{t}^{\perp}$ be the distance between $\boldsymbol{x}$ and the nearest point on the skeleton branch. We get rid of the trivial case $d > r$. In this case, $\mathrm{aof}_{\mathrm{regular}}(\boldsymbol{x}) = 0$. Otherwise, the circle of radius r and center $\boldsymbol{x}$ intersects the skeleton branch at points $\boldsymbol{p}$ and $\boldsymbol{q}$, as depicted in Fig. 1.

We denote by ϕ the angle formed by the line passing through $\boldsymbol{x}$ and $\boldsymbol{p}$, and the skeleton branch, such that

$$\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} = \begin{bmatrix} \sqrt{r^2-d^2} \\ d \end{bmatrix}$$

Considering Eq. (3), the circle is split into two arcs: the one from angle $\beta+\phi$ to $\beta+\pi-\phi$, on which $\nabla D = \boldsymbol{n}_1$, an the other one from angle $\beta+\pi-\phi$ to $\beta+\phi+2\pi$ on which $\nabla D = \boldsymbol{n}_2$. Hence, it comes

$$\begin{aligned}
\mathrm{aof}_{\mathrm{regular}}(\boldsymbol{x}) &= \frac{1}{2\pi}\int_0^{2\pi} \nabla D\left(\boldsymbol{x} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\left(\int_{\beta+\phi}^{\beta+\pi-\phi} \boldsymbol{n}_1\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta + \int_{\beta+\pi-\phi}^{\beta+\phi+2\pi} \boldsymbol{n}_2\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta\right) \\
&= \frac{1}{2\pi}\left(\boldsymbol{n}_1\cdot\int_{\beta+\phi}^{\beta+\pi-\phi} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta + \boldsymbol{n}_2\cdot\int_{\beta+\pi-\phi}^{\beta+\phi+2\pi} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta\right) \\
&= \frac{1}{2\pi}\left(\boldsymbol{n}_1\cdot\begin{bmatrix} -\sin(\beta-\phi)-\sin(\beta+\phi) \\ \cos(\beta-\phi)+\cos(\beta+\phi) \end{bmatrix} + \boldsymbol{n}_2\cdot\begin{bmatrix} \sin(\beta-\phi)+\sin(\beta+\phi) \\ -\cos(\beta-\phi)-\cos(\beta+\phi) \end{bmatrix}\right) \\
&= \frac{1}{2\pi}(\boldsymbol{n}_1-\boldsymbol{n}_2)\cdot\begin{bmatrix} -2\sin\beta\cos\phi \\ 2\cos\beta\cos\phi \end{bmatrix} \\
&= \frac{1}{\pi}\cos\phi\ (\boldsymbol{n}_1-\boldsymbol{n}_2)\cdot\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\
&= \frac{1}{\pi}\cos\phi\ (\boldsymbol{n}_1-\boldsymbol{n}_2)\cdot -\frac{\boldsymbol{n}_1-\boldsymbol{n}_2}{\|\boldsymbol{n}_1-\boldsymbol{n}_2\|} \\
&= -\frac{1}{\pi}\frac{\sqrt{r^2-d^2}}{r}\|\boldsymbol{n}_1-\boldsymbol{n}_2\|
\end{aligned}$$

Using the definition of object angle α in Section 3.1, we finally obtain

$$\mathrm{aof}_{\mathrm{regular}}(\boldsymbol{x}) = -\frac{2\sin\alpha}{\pi r}\sqrt{r^2-d^2}$$

which proves Proposition 2.

A.3 Regular skeleton point - Ridgeness

The Dirac delta distribution δ has the following properties:

$$\delta(ax+b) = \frac{1}{|a|}\delta\left(y-\frac{b}{a}\right) \tag{A.1}$$

$$\int_{-\infty}^{\infty}\delta(x-c)g(x)\mathrm{d}x = g(c) \tag{A.2}$$

To calculate the ridgeness at $\boldsymbol{x}$ in the vicinity of a regular skeleton point $\boldsymbol{s}$, consider Eqs. (6) and (10) simultaneously. Setting $\boldsymbol{n} = [n_x, n_y]^T = \boldsymbol{n}_1 - \boldsymbol{n}_2$ for convenience, we can write

$$\begin{aligned}
\mathrm{rdg}_{\mathrm{regular}}(\boldsymbol{x}) &= \int_{\mathbb{R}^2} -\Delta D(\boldsymbol{x}-\boldsymbol{y})G_\sigma(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^2}\frac{\|\boldsymbol{n}\|^2}{2\pi\sigma^2}\delta\left((\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{s})\cdot\boldsymbol{n}\right)\exp\left(-\frac{\|\boldsymbol{y}\|^2}{2\sigma^2}\right)\mathrm{d}\boldsymbol{y}
\end{aligned}$$

Splitting over x and y-coordinates and applying rule (A.1) with $a = -n_x$ and $b = (\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n} - yn_y$, we get

$$\mathrm{rdg}_{\mathrm{regular}}(\boldsymbol{x}) = \frac{\|\boldsymbol{n}\|^2}{2\pi\sigma^2|n_x|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left(x-\frac{(\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n}-yn_y}{n_x}\right)\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\mathrm{d}x\mathrm{d}y,$$

which is integrated w.r.t x using rule (A.2),

$$\begin{aligned}
\mathrm{rdg}_{\mathrm{regular}}(\boldsymbol{x}) &= \frac{\|\boldsymbol{n}\|^2}{2\pi\sigma^2 |n_x|} \int_{-\infty}^{\infty} \exp\left(-\frac{((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n} - y n_y)^2}{2\sigma^2 n_x^2} - \frac{y^2}{2\sigma^2}\right) \mathrm{d}y \\
&= \frac{\|\boldsymbol{n}\|^2}{2\pi\sigma^2 |n_x|} \exp\left(-\frac{((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n})^2}{2\sigma^2 n_x^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{\|\boldsymbol{n}\|^2}{2\sigma^2 n_x^2} y^2 + \frac{n_y(\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n}}{\sigma^2 n_x^2} y\right) \mathrm{d}y
\end{aligned}$$

Considering the following indefinite integral (with $a > 0$)

$$\int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) \mathrm{d}x = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

with $a = \dfrac{\|\boldsymbol{n}\|^2}{2\sigma^2 n_x^2}$ and $b = \dfrac{n_y(\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n}}{\sigma^2 n_x^2}$, we get

$$\begin{aligned}
\mathrm{rdg}_{\mathrm{regular}}(\boldsymbol{x}) &= \frac{\|\boldsymbol{n}\|^2}{2\pi\sigma^2 |n_x|} \exp\left(-\frac{((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n})^2}{2\sigma^2 n_x^2}\right) \frac{\sigma |n_x| \sqrt{2\pi}}{\|\boldsymbol{n}\|} \exp\left(\frac{n_y^2((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n})^2}{2\sigma^2 n_x^2 \|\boldsymbol{n}\|^2}\right) \\
&= \frac{\|\boldsymbol{n}\|}{\sigma\sqrt{2\pi}} \exp\left(-\frac{((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{n})^2}{2\sigma^2 \|\boldsymbol{n}\|^2}\right)
\end{aligned}$$

Using the definition of object angle α in Section 3.1, it eventually comes

$$\mathrm{rdg}_{\mathrm{regular}}(\boldsymbol{x}) = \frac{\sqrt{2\pi}\sin\alpha}{\pi\sigma} \exp\left(-\frac{((\boldsymbol{x}-\boldsymbol{s})\cdot\boldsymbol{t}^{\perp})^2}{2\sigma^2}\right)$$

which proves Proposition 3.

A.4 Complementary notes on elliptic integrals

In this section, we provide additional properties of elliptic integrals introduced in Section 3.2.1, allowing to derive the AOF and ridgeness at peaks, endpoints, ligatures and junctions. The following form will be met:

$$\int_0^{\psi} \frac{\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}} \mathrm{d}\theta$$

It can be expressed using incomplete elliptic integrals:

$$\begin{aligned}
\int_0^{\psi} \frac{\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}} \mathrm{d}\theta &= \int_0^{\psi} \frac{1}{k^2} \frac{k^2\sin^2\theta + 1 - 1}{\sqrt{1-k^2\sin^2\theta}} \mathrm{d}\theta \\
&= \frac{1}{k^2} \int_0^{\psi} \left(\frac{1}{\sqrt{1-k^2\sin^2\theta}} - \frac{1-k^2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}\right) \mathrm{d}\theta \\
&= \frac{1}{k^2} \left(\int_0^{\psi} \frac{1}{\sqrt{1-k^2\sin^2\theta}} \mathrm{d}\theta + \int_0^{\psi} \sqrt{1-k^2\sin^2\theta}\, \mathrm{d}\theta\right) \\
&= \frac{1}{k^2} \left(\mathrm{F}(\psi,k) - \mathrm{E}(\psi,k)\right)
\end{aligned} \tag{A.3}$$

Incomplete elliptic integrals with purely imaginary modulus can be transformed into elliptic integrals with real modulus as follows [3, p. 491],

$$\begin{aligned}
\mathrm{F}(\psi, ik) &= \frac{1}{\breve{k}} \left(\mathrm{K}\left(\frac{k}{\breve{k}}\right) - \mathrm{F}\left(\frac{\pi}{2}-\psi, \frac{k}{\breve{k}}\right)\right) \\
\mathrm{E}(\psi, ik) &= \breve{k} \left(\mathrm{E}\left(\frac{k}{\breve{k}}\right) - \mathrm{E}\left(\frac{\pi}{2}-\psi, \frac{k}{\breve{k}}\right)\right)
\end{aligned} \tag{A.4}$$

with $i^2 = -1$ and $\breve{k} = \sqrt{1+k^2}$. The corresponding special cases for complete elliptic integrals are

$$\mathrm{K}(ik) = \frac{1}{\breve{k}}\mathrm{K}\left(\frac{k}{\breve{k}}\right) \qquad \mathrm{E}(ik) = \breve{k}\mathrm{E}\left(\frac{k}{\breve{k}}\right) \tag{A.5}$$

The complete elliptic integrals of the first and second kind have the following power series expansions [3, p. 490]:

$$\begin{aligned}
\mathrm{K}(k) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} &&= \frac{\pi}{2} + \frac{\pi}{8}k^2 + \frac{9\pi}{128}k^4 + O\left(k^6\right)\\
\mathrm{E}(k) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{1-2n} &&= \frac{\pi}{2} - \frac{\pi}{8}k^2 - \frac{3\pi}{128}k^4 + O\left(k^6\right)
\end{aligned} \tag{A.6}$$

The incomplete elliptic integrals of the first and second kind have the following expansions around $k = 0$ [1, 2]:

$$\begin{aligned}
\mathrm{F}(\psi, k) &= \sum_{n=0}^{\infty}\frac{1}{2^{2n}n!}\left(\frac{1}{2}\right)_n\left(\psi\binom{2n}{n} + \sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{2n}{n-j}\sin(2j\psi)\right)k^{2n}\\
&= \psi + \frac{2\psi - \sin(2\psi)}{8}k^2 + \frac{3}{256}\left(12\psi - 8\sin(2\psi) + \sin(4\psi)\right)k^4 + O(k^6)\\
\mathrm{E}(\psi, k) &= \sum_{n=0}^{\infty}\frac{1}{2^{2n}n!}\left(-\frac{1}{2}\right)_n\left(\psi\binom{2n}{n} + \sum_{j=1}^{n}\frac{(-1)^j}{j}\binom{2n}{n-j}\sin(2j\psi)\right)k^{2n}\\
&= \psi - \frac{2\psi - \sin(2\psi)}{8}k^2 - \frac{1}{256}\left(12\psi - 8\sin(2\psi) + \sin(4\psi)\right)k^4 + O(k^6)
\end{aligned} \tag{A.7}$$

where $(x)_n$ is the Pochammer symbol for the rising factorial: $(x)_n = x(x+1)(x+2)...(x+n-1)$

A.5 Peak point - AOF

We switch to a polar coordinate system, centered at $\boldsymbol{x}$ s.t. $\boldsymbol{s} = \boldsymbol{x} + [\mathcal{R}\cos\beta, \mathcal{R}\sin\beta]^{\mathrm{T}}$. Consider the circle of radius r centered at $\boldsymbol{x}$. Using Eq. (16), the gradient of the distance at any point on this circle is:

$$\begin{aligned}
\nabla D\left(\boldsymbol{x} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) &= \left\|\begin{bmatrix} \mathcal{R}\cos\beta - r\cos\theta \\ \mathcal{R}\sin\beta - r\sin\theta \end{bmatrix}\right\|^{-1}\begin{bmatrix} \mathcal{R}\cos\beta - r\cos\theta \\ \mathcal{R}\sin\beta - r\sin\theta \end{bmatrix}\\
&= \frac{1}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos(\theta - \beta)}}\begin{bmatrix} \mathcal{R}\cos\beta - r\cos\theta \\ \mathcal{R}\sin\beta - r\sin\theta \end{bmatrix}
\end{aligned}$$

Plugging it into Eq (3), it comes

$$\begin{aligned}
\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos(\theta - \beta)}}\begin{bmatrix} \mathcal{R}\cos\beta - r\cos\theta \\ \mathcal{R}\sin\beta - r\sin\theta \end{bmatrix}\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\mathrm{d}\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{-r + \mathcal{R}\cos(\theta - \beta)}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos(\theta - \beta)}}\mathrm{d}\theta
\end{aligned}$$

There is translation-invariance in range $[0, 2\pi]$, so $\cos(\theta - \beta)$ can be replaced by $\cos\theta$. Moreover, since $\cos(\pi + \theta) = \cos(\pi - \theta)$,

$$\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = \frac{1}{\pi}\int_0^{\pi}\frac{-r + \mathcal{R}\cos\theta}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos\theta}}\mathrm{d}\theta$$

which proves Proposition 4. Using identity $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$ and a change of variable,

$$\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{-r + \mathcal{R} - 2\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\mathrm{d}\theta$$

The integral can be split as

$$\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = \frac{2}{\pi}\left[\frac{\mathcal{R}-r}{|\mathcal{R}-r|}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1+m\sin^2\theta}}\mathrm{d}\theta - \frac{2\mathcal{R}}{|\mathcal{R}-r|}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sqrt{1+m\sin^2\theta}}\mathrm{d}\theta\right]$$

with $m = \dfrac{4r\mathcal{R}}{(\mathcal{R}-r)^2}$. Using Eqs. (18) and (A.3), it can be expressed using the complete elliptic integrals with purely imaginary modulus:

$$\begin{aligned}
\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) &= \frac{2}{\pi}\left[\frac{\mathcal{R}-r}{|\mathcal{R}-r|}\mathrm{K}(ik) - \frac{2\mathcal{R}}{|\mathcal{R}-r|(ik)^2}(\mathrm{K}(ik) - \mathrm{E}(ik))\right] \\
&= \frac{2}{\pi}\left[\frac{\mathcal{R}-r}{|\mathcal{R}-r|}\mathrm{K}(ik) + \frac{|\mathcal{R}-r|}{2r}(\mathrm{K}(ik) - \mathrm{E}(ik))\right] \\
&= \frac{1}{\pi r}\left(\frac{\mathcal{R}^2 - r^2}{|\mathcal{R}-r|}\mathrm{K}(ik) - |\mathcal{R}-r|\mathrm{E}(ik)\right)
\end{aligned}$$

where $i^2 = -1$ and $k = \dfrac{2\sqrt{r\mathcal{R}}}{|\mathcal{R}-r|}$. We convert them to complete elliptic integrals with real modulus using rule (A.5):

$$\begin{aligned}
\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) &= \frac{1}{\pi r}\left(\frac{\mathcal{R}^2 - r^2}{\mathcal{R}+r}\mathrm{K}(k) - (\mathcal{R}+r)\mathrm{E}(k)\right) \\
&= \frac{1}{\pi r}\left((\mathcal{R}-r)\mathrm{K}(k) - (\mathcal{R}+r)\mathrm{E}(k)\right)
\end{aligned} \tag{A.8}$$

with $k = \dfrac{2\sqrt{r\mathcal{R}}}{\mathcal{R}+r}$, which proves Proposition 5.

A.6 Peak point - AOF - Asymptotical behavior

We consider the second-order Taylor approximation of D, as defined in Eq. (15), in the polar coordinate system centered at $\boldsymbol{x}$. According to Eq. (16),

$$\begin{aligned}
\nabla D(\boldsymbol{x}) &= \frac{\boldsymbol{s}-\boldsymbol{x}}{\|\boldsymbol{s}-\boldsymbol{x}\|} \\
\mathbf{H}_D(\boldsymbol{x}) &= -\frac{(\boldsymbol{s}-\boldsymbol{x})^{\perp}((\boldsymbol{s}-\boldsymbol{x})^{\perp})^{\mathrm{T}}}{\|\boldsymbol{s}-\boldsymbol{x}\|^3},
\end{aligned}$$

Since $\boldsymbol{s} = [\mathcal{R}\cos\beta, \mathcal{R}\sin\beta]^{\mathrm{T}}$, we have

$$\begin{aligned}
\nabla D(\boldsymbol{x}) &= [\cos\beta, \sin\beta]^{\mathrm{T}} \\
\mathbf{H}_D(\boldsymbol{x}) &= \frac{1}{\mathcal{R}}\begin{bmatrix} -\sin^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & -\cos^2\beta \end{bmatrix}
\end{aligned}$$

So, the approximate distance on the circle of radius r centered at $\boldsymbol{x}$ is

$$\begin{aligned}
&\widetilde{D}\left(\boldsymbol{x}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) \\
&= D(\boldsymbol{s}) - \mathcal{R} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \nabla D(\boldsymbol{x}) + \frac{1}{2}\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \mathbf{H}_D(\boldsymbol{x}) \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\
&= D(\boldsymbol{s}) - \mathcal{R} + r\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + \frac{r^2}{2\mathcal{R}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\sin^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & -\cos^2\beta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\
&= D(\boldsymbol{s}) - \mathcal{R} + r\cos(\theta-\beta) - \frac{r^2}{2\mathcal{R}}\sin^2(\theta-\beta)
\end{aligned} \tag{A.9}$$

and its gradient is

$$\begin{aligned}
\nabla\widetilde{D}\left(\boldsymbol{x}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) &= \nabla D(\boldsymbol{x}) + \mathbf{H}_D(\boldsymbol{x})\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + \frac{r}{\mathcal{R}}\begin{bmatrix} -\sin^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & -\cos^2\beta \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
\end{aligned}$$

Plugging it into Eq. (25), we obtain

$$\begin{aligned}
\widetilde{\mathrm{aof}}_{\mathrm{peak}}(\boldsymbol{x}) &= \frac{1}{2\pi}\int_0^{2\pi} \nabla\widetilde{D}\left(\boldsymbol{x}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\ \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta-\beta) - \frac{r}{\mathcal{R}}\sin^2(\theta-\beta)\ \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos\theta - \frac{r}{\mathcal{R}}\sin^2\theta\ \mathrm{d}\theta \\
&= -\frac{r}{2\mathcal{R}}
\end{aligned}$$

which proves Proposition 7. In order to prove that the AOF near a peak point is asymptotically equivalent to its second-order approximation

$$\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) \sim \widetilde{\mathrm{aof}}_{\mathrm{peak}}(\boldsymbol{x})\ (\text{as } \mathcal{R}\to+\infty),$$

we show that

$$\lim_{\mathcal{R}\to+\infty} -\frac{2\mathcal{R}}{r}\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = 1.$$

We have

$$\begin{aligned}
\mathrm{K}\left(\frac{2\sqrt{r\mathcal{R}}}{\mathcal{R}+r}\right) &= \frac{\pi}{2}\left(1+\frac{4\mathcal{R}^3r+17\mathcal{R}^2r^2+4\mathcal{R}r^3}{4(\mathcal{R}+r)^4}\right)+O\left(\frac{1}{\mathcal{R}^3}\right) \\
\mathrm{E}\left(\frac{2\sqrt{r\mathcal{R}}}{\mathcal{R}+r}\right) &= \frac{\pi(\mathcal{R}+2r)(2\mathcal{R}+r)(2\mathcal{R}^2+2r^2+\mathcal{R}r)}{8(\mathcal{R}+r)^4}+O\left(\frac{1}{\mathcal{R}^3}\right)
\end{aligned}$$

Thanks to the power series expansions of the complete elliptic integrals in Eq. (A.6), we can derive from Eq. (A.8),

$$\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = -\frac{r(2\mathcal{R}^3+23\mathcal{R}^2r+16\mathcal{R}r^2+4r^3)}{4(\mathcal{R}+r)^4}+O\left(\frac{1}{\mathcal{R}^2}\right).$$

Thus,

$$\lim_{\mathcal{R}\to+\infty} -\frac{2\mathcal{R}}{r}\mathrm{aof}_{\mathrm{peak}}(\boldsymbol{x}) = \lim_{\mathcal{R}\to+\infty} \frac{\mathcal{R}(2\mathcal{R}^3+23\mathcal{R}^2r+16\mathcal{R}r^2+4r^3)}{2(\mathcal{R}+r)^4}+O\left(\frac{1}{\mathcal{R}}\right) = 1,$$

which proves Proposition 8.

A.7 Peak point - Ridgeness

As for the AOF, using a polar transformation of the Laplacian of the distance defined in Eq. (16), we have:

$$\Delta D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)=-\left\|\begin{bmatrix}\mathcal{R}\cos\beta-\rho\cos\theta\\ \mathcal{R}\sin\beta-\rho\sin\theta\end{bmatrix}\right\|^{-1}=-\frac{1}{\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta)}}$$

Plugging it into Eq. (26), we get

$$\begin{aligned}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) &= -\int_0^\infty \rho G_\sigma(\rho)\int_0^{2\pi}\Delta D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\mathrm{d}\rho\\ &= \int_0^\infty \rho G_\sigma(\rho)\int_0^{2\pi}\frac{1}{\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta)}}\mathrm{d}\theta\mathrm{d}\rho\end{aligned}$$

Again, by translation invariance in range $[0,2\pi]$, $\cos(\theta-\beta)$ can be replaced by $\cos\theta$. Using the evenness of cos,

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=2\int_0^\infty \rho G_\sigma(\rho)\int_0^{\pi}\frac{1}{\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}}\mathrm{d}\theta\mathrm{d}\rho$$

which proves Proposition 9. Using identity $\cos\theta=1-2\sin^2\dfrac{\theta}{2}$ and a change of variable, we get

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=4\int_0^\infty \rho G_\sigma(\rho)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{(\rho-\mathcal{R})^2+4\rho\mathcal{R}\sin^2\theta}}\mathrm{d}\theta\mathrm{d}\rho$$

which, by a derivation similar to the one in Section A.5, can be expressed using the complete elliptic integral of the first kind, with complex modulus:

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=4\int_0^\infty \rho G_\sigma(\rho)\frac{1}{|\mathcal{R}-\rho|}\mathrm{K}\left(i\frac{2\sqrt{\rho\mathcal{R}}}{|\mathcal{R}-\rho|}\right)\mathrm{d}\rho$$

Using transformation (A.5), we finally obtain:

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=4\int_0^\infty \rho G_\sigma(\rho)\frac{1}{\mathcal{R}+\rho}\mathrm{K}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)\mathrm{d}\rho$$

which proves Proposition 10. Using Eqs (15) and (5) in polar coordinates, an alternative expression of the ridgeness near a peak point can be obtained:

$$\begin{aligned}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\int_0^{2\pi} D\left(\boldsymbol{x}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\ \mathrm{d}\theta\ \mathrm{d}\rho\\ &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\int_0^{2\pi} D(\boldsymbol{s})-\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta)}\ \mathrm{d}\theta\ \mathrm{d}\rho\end{aligned}$$

Since $\displaystyle\int_0^\infty \rho\Delta G_\sigma(\rho)=0$, the constant term $D(\boldsymbol{s})$ can be dropped, leading to

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=\int_0^\infty \rho\Delta G_\sigma(\rho)\int_0^{2\pi}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta)}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

Again, by translation invariance in range $[0,2\pi]$, $\cos(\theta-\beta)$ can be replaced by $\cos\theta$. Using the evenness of cos,

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x})=2\int_0^\infty \rho\Delta G_\sigma(\rho)\int_0^{\pi}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

Using identity $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$ and a change of variable, we get

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 4\int_0^{\infty} \rho \Delta G_\sigma(\rho) \int_0^{\frac{\pi}{2}} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

which, by a derivation similar to the previous one, can be expressed using the complete elliptic integral of the second kind, with complex modulus:

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 4\int_0^{\infty} \rho \Delta G_\sigma(\rho)\, |\mathcal{R}-\rho|\, \mathrm{E}\left(i\frac{2\sqrt{\rho\mathcal{R}}}{|\mathcal{R}-\rho|}\right)\ \mathrm{d}\rho$$

Using transformation (A.5), we finally obtain:

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 4\int_0^{\infty} \rho \Delta G_\sigma(\rho)(\mathcal{R}+\rho)\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)\mathrm{d}\rho \tag{A.10}$$

A.8 Peak point - Ridgeness - Asymptotical behavior

Using Eqs (31) and (A.9), the approximate ridgeness in the vicinity of a peak point is

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{peak}}(\boldsymbol{x}) &= -\int_0^{\infty}\int_0^{2\pi} \rho\Delta G_\sigma(\rho)\widetilde{D}\left(\boldsymbol{x} + \begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\mathrm{d}\rho \\
&= -\int_0^{\infty} \rho\Delta G_\sigma(\rho)\int_0^{2\pi} D(\boldsymbol{s}) - \mathcal{R} + \rho\cos(\theta-\beta) - \frac{\rho^2}{2\mathcal{R}}\sin^2(\theta-\beta)\ \mathrm{d}\theta\mathrm{d}\rho \\
&= -\int_0^{\infty} \rho\Delta G_\sigma(\rho)\int_0^{2\pi} D(\boldsymbol{s}) - \mathcal{R} + \rho\cos(\theta-\beta) - \frac{\rho^2}{2\mathcal{R}}\sin^2(\theta-\beta)\ \mathrm{d}\theta\mathrm{d}\rho
\end{aligned}$$

By translation,

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{peak}}(\boldsymbol{x}) &= -\int_0^{\infty} \rho\Delta G_\sigma(\rho)\int_0^{2\pi} D(\boldsymbol{s}) - \mathcal{R} + \rho\cos\theta - \frac{\rho^2}{2\mathcal{R}}\sin^2\theta\ \mathrm{d}\theta\mathrm{d}\rho \\
&= -\int_0^{\infty} \Delta G_\sigma(\rho)\left(2\pi\rho(D(\boldsymbol{s})-\mathcal{R}) - \rho^3\frac{\pi}{2\mathcal{R}}\right)\mathrm{d}\rho
\end{aligned}$$

Using definite integrals $\displaystyle\int_0^{\infty}\rho\Delta G_\sigma(\rho)\mathrm{d}\rho = 0$ and $\displaystyle\int_0^{\infty}\rho^3\Delta G_\sigma(\rho)\mathrm{d}\rho = \frac{2}{\pi}$, we finally obtain

$$\widetilde{\mathrm{rdg}}_{\mathrm{peak}}(\boldsymbol{x}) = \frac{1}{\mathcal{R}}$$

which proves Proposition 12. In order to prove that the ridgeness near a peak point is asymptotically equivalent to its second-order approximation

$$\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) \sim \widetilde{\mathrm{rdg}}_{\mathrm{peak}}(\boldsymbol{x})\ (\text{as } \mathcal{R}\to+\infty),$$

we show that

$$\lim_{\mathcal{R}\to+\infty}\mathcal{R}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 1.$$

From Eq. (A.10), we can write

$$\mathcal{R}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 4\int_0^{\infty}\rho\Delta G_\sigma(\rho)\mathcal{R}(\mathcal{R}+\rho)\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)\ \mathrm{d}\rho$$

Since

$$\int_0^{\infty}\rho\Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0,$$

it is also true that

$$\int_0^\infty \frac{\pi}{2}\mathcal{R}^2\rho\Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0$$

Hence,

$$\mathcal{R}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 4\int_0^\infty \rho\Delta G_\sigma(\rho)\left[\mathcal{R}(\mathcal{R}+\rho)\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right) - \frac{\pi}{2}\mathcal{R}^2\right]\ \mathrm{d}\rho$$

Thanks to the power series expansions of E in Eq. (A.6), we can derive from Eq. (A.10),

$$\begin{aligned} h(\mathcal{R},\rho) &= \mathcal{R}(\mathcal{R}+\rho)\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right) - \frac{\pi}{2}\mathcal{R}^2 \\ &= \mathcal{R}(\mathcal{R}+\rho)\left(\frac{\pi}{2} - \frac{\pi\rho\mathcal{R}}{2(\mathcal{R}+\rho)^2} - \frac{3\pi\rho^2\mathcal{R}^2}{8(\mathcal{R}+\rho)^4}\right) - \frac{\pi}{2}\mathcal{R}^2 + O\left(\frac{1}{\mathcal{R}}\right) \\ &= \frac{\pi\mathcal{R}\rho^2(\mathcal{R}^2+4\rho^2+8\mathcal{R}\rho)}{8(\mathcal{R}+\rho)^3} + O\left(\frac{1}{\mathcal{R}}\right) \end{aligned}$$

and thus,

$$\lim_{\mathcal{R}\to+\infty} h(\mathcal{R},\rho) = \frac{\pi}{8}\rho^2.$$

For every $\rho \in [0,+\infty)$, $\rho\Delta G_\sigma(\rho)h(\mathcal{R},\rho)$ is dominated by $\max(1,\rho^3)\,|\Delta G_\sigma(\rho)|$, which is L^1-integrable. Hence, by the DCT,

$$\begin{aligned} \lim_{\mathcal{R}\to+\infty} 4\int_0^\infty \rho\Delta G_\sigma(\rho)h(\mathcal{R},\rho)\ \mathrm{d}\rho &= 4\int_0^\infty \rho\Delta G_\sigma(\rho)\left(\lim_{\mathcal{R}\to+\infty} h(\mathcal{R},\rho)\right)\ \mathrm{d}\rho \\ &= \frac{\pi}{2}\int_0^\infty \rho^3\Delta G_\sigma(\rho)\ \mathrm{d}\rho \\ &= 1 \end{aligned}$$

Thus,

$$\lim_{\mathcal{R}\to+\infty} \mathcal{R}\mathrm{rdg}_{\mathrm{peak}}(\boldsymbol{x}) = 1,$$

which proves Proposition 13.

A.9 End point - AOF

The AOF being rotation-invariant, we perform the following derivations considering that $\boldsymbol{t}$ is aligned with the horizontal axis (see Fig. 6a). This simplifies the derivation and remains valid for any orientation of $\boldsymbol{t}$. Considering Eq. (33), the circle of center $\boldsymbol{s}$ and radius r is split into three arcs:

- the one from angle $-\alpha$ to α, on which $\nabla D = [-\cos\theta, -\sin\theta]^{\mathrm{T}}$,
- the one from angle α to $\pi-\alpha$, on which $\nabla D = [-\cos\alpha, -\sin\alpha]^{\mathrm{T}}$,
- and the one from angle $\pi-\alpha$ to $2\pi-\alpha$, on which $\nabla D = [-\cos\alpha, \sin\alpha]^{\mathrm{T}}$

In this case, Eq. (3) gives:

$$\begin{aligned} \mathrm{aof}_{\mathrm{end}}(\boldsymbol{s}) &= \frac{1}{2\pi}\left(\int_{-\alpha}^{\alpha}\begin{bmatrix}-\cos\theta\\-\sin\theta\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\ \mathrm{d}\theta + \int_{\alpha}^{\pi}\begin{bmatrix}-\cos\alpha\\-\sin\alpha\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\ \mathrm{d}\theta\right. \\ &\qquad \left. + \int_{\pi}^{2\pi-\alpha}\begin{bmatrix}-\cos\alpha\\\sin\alpha\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\ \mathrm{d}\theta\right) \\ &= \frac{1}{2\pi}\left(-\int_{-\alpha}^{\alpha} 1\ \mathrm{d}\theta - \int_{\alpha}^{\pi}\cos(\theta-\alpha)\ \mathrm{d}\theta - \int_{\pi}^{2\pi-\alpha}\cos(\theta+\alpha)\ \mathrm{d}\theta\right) \\ &= -\frac{1}{\pi}(\alpha+\sin\alpha) \end{aligned}$$

which proves Proposition 14.

A.10 End point - Ridgeness

Starting from a polar transformation of Eq. (5), as in Section A.9, we use the rotation invariance property of the ridgeness. The integral w.r.t to angle θ is split according to the three arcs. Using Eq. (32), we write

$$\begin{aligned}
\mathrm{rdg}_{\mathrm{end}}(\boldsymbol{s}) &= -\int_0^{\infty} \rho \Delta G_\sigma(\rho) \int_0^{2\pi} D\left(\boldsymbol{s} + \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}\right) \,\mathrm{d}\theta \,\mathrm{d}\rho \\
&= -\int_0^{\infty} \rho \Delta G_\sigma(\rho) \left(\int_{-\alpha}^{\alpha} D(\boldsymbol{s}) - \rho \,\mathrm{d}\theta + \int_{\alpha}^{\pi} D(\boldsymbol{s}) + \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\cos\alpha \\ -\sin\alpha \end{bmatrix} \mathrm{d}\theta \right. \\
&\quad \left. + \int_{\pi}^{2\pi-\alpha} D(\boldsymbol{s}) + \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\cos\alpha \\ \sin\alpha \end{bmatrix} \mathrm{d}\theta \right) \,\mathrm{d}\rho \\
&= \int_0^{\infty} \rho \Delta G_\sigma(\rho) \left(2\pi D(\boldsymbol{s}) + \int_{-\alpha}^{\alpha} \rho \,\mathrm{d}\theta + \int_{\alpha}^{\pi} \rho\cos(\theta-\alpha) \,\mathrm{d}\theta \right. \\
&\quad \left. + \int_{\pi}^{2\pi-\alpha} \rho\cos(\theta+\alpha) \,\mathrm{d}\theta \right) \,\mathrm{d}\rho
\end{aligned}$$

Using definite integrals $\int_0^{+\infty} \rho \Delta G_\sigma(\rho) \mathrm{d}\rho = 0$ and $\int_0^{+\infty} \rho^2 \Delta G_\sigma(\rho) \mathrm{d}\rho = \frac{1}{2\sigma\sqrt{2\pi}}$, we finally obtain

$$\mathrm{rdg}_{\mathrm{end}}(\boldsymbol{s}) = \int_0^{\infty} \rho^2 \Delta G_\sigma(\rho)(2\alpha + 2\sin\alpha) \,\mathrm{d}\rho = \frac{\sqrt{2\pi}}{2\pi\sigma}(\alpha + \sin\alpha)$$

which proves Proposition 15.

A.11 Ligature point - AOF

Consider Fig. 7a. Since the average outward flux is rotation-invariant, we can perform the derivation assuming that $\boldsymbol{t}$ is aligned with the horizontal axis, for convenience. Under this assumption, $\boldsymbol{p} = \boldsymbol{s} + [\mathcal{A}, \mathcal{B}]^{\mathrm{T}}$ and $\boldsymbol{q} = \boldsymbol{s} + \mathcal{R}[\mathcal{A}, -\mathcal{B}]^{\mathrm{T}}$. The distance defined in Eq. (36) now reads

$$D\left(\boldsymbol{s} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) = \begin{cases} \left\| \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta - \mathcal{B} \end{bmatrix} \right\| & \text{if } 0 \le \theta \le \pi \\ \left\| \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta + \mathcal{B} \end{bmatrix} \right\| & \text{if } \pi \le \theta \le 2\pi \end{cases} \tag{A.11}$$

and its gradient is

$$\nabla D\left(\boldsymbol{s} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) = \begin{cases} \left\| \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta - \mathcal{B} \end{bmatrix} \right\|^{-1} \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta - \mathcal{B} \end{bmatrix} & \text{if } 0 < \theta < \pi \\ \left\| \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta + \mathcal{B} \end{bmatrix} \right\|^{-1} \begin{bmatrix} r\cos\theta - \mathcal{A} \\ r\sin\theta + \mathcal{B} \end{bmatrix} & \text{if } \pi < \theta < 2\pi \end{cases}$$

Plugging it into Eq (3), it comes

$$\begin{aligned}
\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) &= \frac{1}{2\pi} \left[\int_0^{\pi} \nabla D\left(\boldsymbol{s} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \,\mathrm{d}\theta \right. \\
&\quad \left. + \int_{\pi}^{2\pi} \nabla D\left(\boldsymbol{s} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \,\mathrm{d}\theta \right] \\
&= \frac{1}{2\pi} \left[\int_0^{\pi} \frac{r - \mathcal{A}\cos\theta - \mathcal{B}\sin\theta}{\sqrt{r^2 + \mathcal{A}^2 + \mathcal{B}^2 - 2r(\mathcal{A}\cos\theta + \mathcal{B}\sin\theta)}} \,\mathrm{d}\theta \right. \\
&\quad \left. + \int_{\pi}^{2\pi} \frac{r - \mathcal{A}\cos\theta + \mathcal{B}\sin\theta}{\sqrt{r^2 + \mathcal{A}^2 + \mathcal{B}^2 - 2r(\mathcal{A}\cos\theta - \mathcal{B}\sin\theta)}} \,\mathrm{d}\theta \right]
\end{aligned}$$

Using property $\cos(\pi - x) = -\cos x$ and $\sin(\pi - x) = \sin x$, we get

$$\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\pi}\int_0^{\pi} \frac{r - \mathcal{A}\cos\theta - \mathcal{B}\sin\theta}{\sqrt{r^2 + \mathcal{A}^2 + \mathcal{B}^2 - 2r(\mathcal{A}\cos\theta + \mathcal{B}\sin\theta)}}\,\mathrm{d}\theta$$

which proves Proposition 16. From Eq (37), $\mathcal{A} = \mathcal{R}\cos\beta$ and $\mathcal{B} = \mathcal{R}\sin\beta$, which gives

$$\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\pi}\int_0^{\pi} \frac{r - \mathcal{R}\cos(\theta-\beta)}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos(\theta-\beta)}}\,\mathrm{d}\theta.$$

By translation,

$$\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\pi r}\int_{-\beta}^{\pi-\beta} \frac{r^2 - r\mathcal{R}\cos\theta}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos\theta}}\,\mathrm{d}\theta.$$

Using identity $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$ and a change of variable,

$$\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{2}{\pi r}\int_{-\frac{\beta}{2}}^{\frac{\pi}{2}-\frac{\beta}{2}} \frac{r^2 - r\mathcal{R} + 2r\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\,\mathrm{d}\theta.$$

The integral can be split as

$$\begin{aligned}
\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) &= \frac{2}{\pi r}\left(\int_{-\frac{\beta}{2}}^{0} \frac{r^2 - r\mathcal{R} + 2r\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\,\mathrm{d}\theta + \int_{0}^{\frac{\pi}{2}-\frac{\beta}{2}} \frac{r^2 - r\mathcal{R} + 2r\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\,\mathrm{d}\theta\right) \\
&= \frac{2}{\pi r}\left(\int_{0}^{\frac{\beta}{2}} \frac{r^2 - r\mathcal{R} + 2r\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\,\mathrm{d}\theta + \int_{0}^{\frac{\pi}{2}-\frac{\beta}{2}} \frac{r^2 - r\mathcal{R} + 2r\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2 + 4r\mathcal{R}\sin^2\theta}}\,\mathrm{d}\theta\right)
\end{aligned}$$

Using Eqs. (17) and (A.3), it can be expressed with incomplete elliptic integrals with purely imaginary modulus,

$$\begin{aligned}
\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\pi r}\Bigg(&|\mathcal{R}-r|\left(\mathrm{E}\left(\frac{\beta}{2}, ik\right) + \mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2}, ik\right)\right) \\
&-\frac{\mathcal{R}^2 - r^2}{|\mathcal{R}-r|}\left(\mathrm{F}\left(\frac{\beta}{2}, ik\right) + \mathrm{F}\left(\frac{\pi}{2}-\frac{\beta}{2}, ik\right)\right)\Bigg)
\end{aligned}$$

with $k = \dfrac{2\sqrt{r\mathcal{R}}}{|\mathcal{R}-r|}$. Using transformations in Eq. (A.4), we convert them to elliptic integrals with real modulus

$$\begin{aligned}
\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\pi r}\Bigg[&(\mathcal{R}+r)\left(2\mathrm{E}(k) - \mathrm{E}\left(\frac{\beta}{2}, k\right) - \mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2}, k\right)\right) \\
&-(\mathcal{R}-r)\left(2\mathrm{K}(k) - \mathrm{F}\left(\frac{\beta}{2}, k\right) - \mathrm{F}\left(\frac{\pi}{2}-\frac{\beta}{2}, k\right)\right)\Bigg]
\end{aligned} \tag{A.12}$$

with $k = \dfrac{2\sqrt{r\mathcal{R}}}{\mathcal{R}+r}$, which proves Proposition 17.

A.12 Ligature point - AOF - Asymptotical behavior

We consider the second-order Taylor approximation of D, in the polar coordinate system centered at $\boldsymbol{s}$. From Eq. (A.11), we derive the approximate distance on the circle of radius r

centered at $\boldsymbol{s}$:

$$
\begin{aligned}
&\widetilde{D}\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\\
&= D(\boldsymbol{s}) + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \nabla D(\boldsymbol{s}) + \frac{1}{2}\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \mathbf{H}_D(\boldsymbol{s}) \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\\
&= \begin{cases} \mathcal{R} + \dfrac{r}{\mathcal{R}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathcal{A} \\ -\mathcal{B} \end{bmatrix} + \dfrac{r^2}{2\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{B}^2 & -\mathcal{A}\mathcal{B} \\ -\mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \text{if } 0 \le \theta \le \pi \\ \mathcal{R} + \dfrac{r}{\mathcal{R}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathcal{A} \\ \mathcal{B} \end{bmatrix} + \dfrac{r^2}{2\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathcal{B}^2 & \mathcal{A}\mathcal{B} \\ \mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \text{if } \pi \le \theta \le 2\pi \end{cases}\\
&= \begin{cases} \mathcal{R} + \dfrac{r}{\mathcal{R}}(-\mathcal{A}\cos\theta - \mathcal{B}\sin\theta) + \dfrac{r^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta - \mathcal{A}\sin\theta)^2 & \text{if } 0 \le \theta \le \pi \\ \mathcal{R} + \dfrac{r}{\mathcal{R}}(-\mathcal{A}\cos\theta + \mathcal{B}\sin\theta) + \dfrac{r^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta + \mathcal{A}\sin\theta)^2 & \text{if } \pi \le \theta \le 2\pi \end{cases}
\end{aligned}
\tag{A.13}
$$

The approximate gradient on the circle is

$$
\nabla\widetilde{D}\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) = \begin{cases} \dfrac{1}{\mathcal{R}}\begin{bmatrix} -\mathcal{A} \\ -\mathcal{B} \end{bmatrix} + \dfrac{r}{\mathcal{R}^3}\begin{bmatrix} \mathcal{B}^2 & -\mathcal{A}\mathcal{B} \\ -\mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \text{if } 0 \le \theta \le \pi \\ \dfrac{1}{\mathcal{R}}\begin{bmatrix} -\mathcal{A} \\ \mathcal{B} \end{bmatrix} + \dfrac{r}{\mathcal{R}^3}\begin{bmatrix} \mathcal{B}^2 & \mathcal{A}\mathcal{B} \\ \mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \text{if } \pi \le \theta \le 2\pi \end{cases}
$$

Plugging it into Eq. (25), we obtain

$$
\begin{aligned}
\widetilde{\mathrm{aof}}_{\mathrm{ligature}}(\boldsymbol{x}) &= = \frac{1}{2\pi}\int_0^{2\pi} \nabla\widetilde{D}\left(\boldsymbol{x}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\\
&= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{1}{\mathcal{R}}\begin{bmatrix} -\mathcal{A} \\ -\mathcal{B} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{r}{\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathcal{B}^2 & -\mathcal{A}\mathcal{B} \\ -\mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right.\\
&\quad \left. + \int_{\pi}^{2\pi} \frac{1}{\mathcal{R}}\begin{bmatrix} -\mathcal{A} \\ \mathcal{B} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{r}{\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathcal{B}^2 & \mathcal{A}\mathcal{B} \\ \mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right]
\end{aligned}
$$

Using $\cos(x+\pi) = -\cos x$ and $\sin(x+\pi) = -\sin x$, it comes

$$
\begin{aligned}
\widetilde{\mathrm{aof}}_{\mathrm{ligature}}(\boldsymbol{x}) &= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{1}{\mathcal{R}}\begin{bmatrix} -\mathcal{A} \\ -\mathcal{B} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{r}{\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathcal{B}^2 & -\mathcal{A}\mathcal{B} \\ -\mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right.\\
&\quad \left. + \int_0^{\pi} \frac{1}{\mathcal{R}}\begin{bmatrix} \mathcal{A} \\ -\mathcal{B} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{r}{\mathcal{R}^3}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathcal{B}^2 & \mathcal{A}\mathcal{B} \\ \mathcal{A}\mathcal{B} & \mathcal{A}^2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right]\\
&= \frac{1}{2\pi}\int_0^{\pi} -\frac{2\mathcal{B}}{\mathcal{R}}\sin\theta + \frac{r}{\mathcal{R}}\,\mathrm{d}\theta\\
&= \frac{1}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}\left(\frac{r}{2} - \frac{2\mathcal{B}}{\pi}\right)
\end{aligned}
$$

which proves Proposition 18. In order to prove that the AOF at a ligature point is asymptotically equivalent to its second-order approximation

$$
\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) \sim \widetilde{\mathrm{aof}}_{\mathrm{ligature}}(\boldsymbol{s}) \text{ (as } \mathcal{A} \to +\infty),
$$

we show that

$$
\lim_{\mathcal{A}\to+\infty} \frac{2\pi\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\pi r - 4\mathcal{B}}\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = 1.
$$

Using series expansion in Eqs. (A.6) and (A.7), we derive the elliptic terms of Eq. (A.12)

$$2\mathrm{K}(k)-\mathrm{F}\left(\frac{\beta}{2},k\right)-\mathrm{F}\left(\frac{\pi}{2}-\frac{\beta}{2},k\right)=\frac{\pi}{2}+k^2\left(\frac{\pi}{8}+\frac{\sin\beta}{4}\right)+k^4\left(\frac{9\pi}{128}+\frac{3\sin\beta}{16}\right)+O(k^6)$$

$$2\mathrm{E}(k)-\mathrm{E}\left(\frac{\beta}{2},k\right)-\mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2},k\right)=\frac{\pi}{2}-k^2\left(\frac{\pi}{8}+\frac{\sin\beta}{4}\right)-k^4\left(\frac{3\pi}{128}+\frac{\sin\beta}{16}\right)+O(k^6)$$

which yields

$$\begin{aligned}\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = &\frac{1}{(\mathcal{R}+r)^4}\left(-\frac{2\sin\beta}{\pi}\mathcal{R}^4+\left(\frac{1}{2}-\frac{8\sin\beta}{\pi}\right)\mathcal{R}^3 r\right.\\ &\left.+\frac{23}{4}\mathcal{R}^2r^2+4\mathcal{R}r^3+r^4\right)+O\left(\frac{1}{\mathcal{R}^2}\right)\end{aligned}$$

From definitions of $\mathcal{R}$ and β in Eq. (37), it comes

$$\begin{aligned}\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s}) = &\frac{1}{\left(\sqrt{\mathcal{A}^2+\mathcal{B}^2}+r\right)^4}\left((\mathcal{A}^2+\mathcal{B}^2)^{\frac{3}{2}}\left(\frac{r}{2}-\frac{2\mathcal{B}}{\pi}\right)+(\mathcal{A}^2+\mathcal{B}^2)\left(\frac{23}{4}r^2-\frac{8\mathcal{B}}{\pi}r\right)\right.\\ &\left.+4r^3\sqrt{\mathcal{A}^2+\mathcal{B}^2}+r^4\right)+O\left(\frac{1}{\mathcal{A}^2}\right)\end{aligned}$$

Thus,

$$\lim_{\mathcal{A}\to+\infty}\frac{2\pi\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\pi r-4\mathcal{B}}\mathrm{aof}_{\mathrm{ligature}}(\boldsymbol{s})=\lim_{\mathcal{A}\to+\infty}\frac{(\mathcal{A}^2+\mathcal{B}^2)^2}{\left(\sqrt{\mathcal{A}^2+\mathcal{B}^2}+r\right)^4}+O\left(\frac{1}{\mathcal{A}}\right)=1$$

which proves Proposition 19.

A.13 Ligature point - Ridgeness

From Eqs (39) and (A.11), it comes

$$\begin{aligned}\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) &= -\int_0^{\infty}\rho\Delta G_\sigma(\rho)\int_0^{2\pi}D\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\mathrm{d}\rho\\ &= -\int_0^{\infty}\rho\Delta G_\sigma(\rho)\left[\int_0^{\pi}\sqrt{\rho^2+\mathcal{A}^2+\mathcal{B}^2-2\rho(\mathcal{A}\cos\theta+\mathcal{B}\sin\theta)}\mathrm{d}\theta\right.\\ &\quad\left.+\int_\pi^{2\pi}\sqrt{\rho^2+\mathcal{A}^2+\mathcal{B}^2-2\rho(\mathcal{A}\cos\theta-\mathcal{B}\sin\theta)}\mathrm{d}\theta\right]\mathrm{d}\rho\end{aligned}$$

Using properties $\cos(\pi-x)=-\cos x$ and $\sin(\pi-x)=\sin x$, we have

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s})=-2\int_0^{\infty}\rho\Delta G_\sigma(\rho)\int_0^{\pi}\sqrt{\rho^2+\mathcal{A}^2+\mathcal{B}^2-2\rho(\mathcal{A}\cos\theta+\mathcal{B}\sin\theta)}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

which proves Proposition 20. From Eq (37), $\mathcal{A}=\mathcal{R}\cos\beta$ and $\mathcal{B}=\mathcal{R}\sin\beta$, which gives

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s})=-2\int_0^{\infty}\rho\Delta G_\sigma(\rho)\int_0^{\pi}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta)}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

By translation,

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s})=-2\int_0^{\infty}\rho\Delta G_\sigma(\rho)\int_{-\beta}^{\pi-\beta}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho$$

Using transformation $\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2}$,

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho)\int_{-\frac{\beta}{2}}^{\frac{\pi}{2}-\frac{\beta}{2}} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho.$$

Splitting the integral w.r.t θ, we obtain

$$\begin{aligned}
\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) &= -4\int_0^\infty \rho\Delta G_\sigma(\rho)\left[\int_{-\frac{\beta}{2}}^{0} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\right.\\
&\qquad\left.+\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\right]\ \mathrm{d}\rho\\
&= -4\int_0^\infty \rho\Delta G_\sigma(\rho)\left[\int_0^{\frac{\beta}{2}} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\right.\\
&\qquad\left.+\int_0^{\frac{\pi}{2}-\frac{\beta}{2}} \sqrt{(\rho-\mathcal{R})^2 + 4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\right]\ \mathrm{d}\rho
\end{aligned}$$

which can be expressed with incomplete elliptic integrals of the second kind with purely imaginary modulus

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho)|\mathcal{R}-\rho|\left(\mathrm{E}\left(\frac{\beta}{2},k\right)+\mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2},k\right)\right)\ \mathrm{d}\rho$$

with $k = \dfrac{2\sqrt{\rho\mathcal{R}}}{|\mathcal{R}-\rho|}$. Using transformations in Eq. (A.4), we convert them to elliptic integrals of the second kind with real modulus

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho)(\mathcal{R}+\rho)\left(2\mathrm{E}(k)-\mathrm{E}\left(\frac{\beta}{2},k\right)-\mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2},k\right)\right)\ \mathrm{d}\rho \quad \text{(A.14)}$$

with $k = \dfrac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}$, which proves Proposition 21.

A.14 Ligature point - Ridgeness - Asymptotical behavior

From Eqs. (31) and (A.13), the approximate ridgeness for ligature point $\boldsymbol{s}$ is derived as

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{ligature}}(\boldsymbol{s}) &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\int_0^{2\pi}\widetilde{D}\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\ \mathrm{d}\theta\ \mathrm{d}\rho\\
&= -\int_0^\infty \rho\Delta G_\sigma(\rho)\left[\int_0^\pi \frac{\rho}{\mathcal{R}}(-\mathcal{A}\cos\theta-\mathcal{B}\sin\theta)+\frac{\rho^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta-\mathcal{A}\sin\theta)^2\ \mathrm{d}\theta\right.\\
&\qquad\left.+\int_\pi^{2\pi}\frac{\rho}{\mathcal{R}}(-\mathcal{A}\cos\theta+\mathcal{B}\sin\theta)+\frac{\rho^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta+\mathcal{A}\sin\theta)^2\ \mathrm{d}\theta+2\pi\mathcal{R}\right]\ \mathrm{d}\rho
\end{aligned}$$

Using $\cos(x+\pi) = -\cos x$ and $\sin(x+\pi) = -\sin x$, it comes

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{ligature}}(\boldsymbol{s}) &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\left[\int_0^\pi \frac{\rho}{\mathcal{R}}(-\mathcal{A}\cos\theta-\mathcal{B}\sin\theta)+\frac{\rho^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta-\mathcal{A}\sin\theta)^2\ \mathrm{d}\theta\right.\\
&\qquad\left.+\int_0^\pi \frac{\rho}{\mathcal{R}}(\mathcal{A}\cos\theta-\mathcal{B}\sin\theta)+\frac{\rho^2}{2\mathcal{R}^3}(\mathcal{B}\cos\theta+\mathcal{A}\sin\theta)^2\ \mathrm{d}\theta+2\pi\mathcal{R}\right]\ \mathrm{d}\rho\\
&= -\int_0^\infty \rho\Delta G_\sigma(\rho)\left(2\pi\mathcal{R}+\int_0^\pi -2\frac{\rho\mathcal{B}}{\mathcal{R}}\sin\theta+\frac{\rho^2}{2\mathcal{R}^3}(\mathcal{A}^2\sin^2\theta+\mathcal{B}^2\cos^2\theta)\ \mathrm{d}\theta\right)\ \mathrm{d}\rho\\
&= \int_0^\infty \Delta G_\sigma(\rho)\left(-2\pi\mathcal{R}\rho+4\frac{\mathcal{B}}{\mathcal{R}}\rho^2-\frac{\pi}{2\mathcal{R}}\rho^3\right)\ \mathrm{d}\rho
\end{aligned}$$

Using definite integrals $\int_0^\infty \rho \Delta G_\sigma(\rho)\mathrm{d}\rho = 0$, $\int_0^\infty \rho^2 \Delta G_\sigma(\rho)\mathrm{d}\rho = \frac{1}{2\sigma\sqrt{2\pi}}$ and $\int_0^\infty \rho^3 \Delta G_\sigma(\rho)\mathrm{d}\rho = \frac{2}{\pi}$, we finally obtain

$$\widetilde{\mathrm{rdg}}_{\mathrm{ligature}}(\boldsymbol{s}) = \frac{1}{\sqrt{\mathcal{A}^2+\mathcal{B}^2}}\left(\mathcal{B}\frac{\sqrt{2\pi}}{\pi\sigma} - 1\right)$$

which proves Proposition 22. In order to prove that the ridgeness of a ligature point is asymptotically equivalent to its second-order approximation

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) \sim \widetilde{\mathrm{rdg}}_{\mathrm{ligature}}(\boldsymbol{s}) \text{ (as } \mathcal{A} \to +\infty),$$

we show that

$$\lim_{\mathcal{A}\to+\infty} \frac{\pi\sigma\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = 1.$$

Let us consider Eq. (A.14). For convenience, we introduce

$$f(\mathcal{A},\rho) = \left(\sqrt{\mathcal{A}^2+\mathcal{B}^2}+\rho\right)\left[2\mathrm{E}(k) - \mathrm{E}\left(\frac{\beta}{2},k\right) - \mathrm{E}\left(\frac{\pi}{2}-\frac{\beta}{2},k\right)\right]$$

with $k = \frac{2\sqrt{\rho\sqrt{\mathcal{A}^2+\mathcal{B}^2}}}{\rho+\sqrt{\mathcal{A}^2+\mathcal{B}^2}}$ and $\beta = \tan^{-1}\left(\frac{\mathcal{B}}{\mathcal{A}}\right)$, so that

$$\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho) f(\mathcal{A},\rho)\ \mathrm{d}\rho.$$

We can write

$$\frac{\pi\sigma\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -\frac{4\pi\sigma}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\int_0^\infty \rho\Delta G_\sigma(\rho) f(\mathcal{A},\rho)\sqrt{\mathcal{A}^2+\mathcal{B}^2}\ \mathrm{d}\rho$$

Since

$$\int_0^\infty \rho\Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0,$$

it is also true that

$$\int_0^\infty \frac{\pi}{2}(\mathcal{A}^2+\mathcal{B}^2)\rho\Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0.$$

Hence,

$$\frac{\pi\sigma\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = -\frac{4\pi\sigma}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\int_0^\infty \rho\Delta G_\sigma(\rho)\sqrt{\mathcal{A}^2+\mathcal{B}^2}\left(f(\mathcal{A},\rho) - \frac{\pi}{2}\sqrt{\mathcal{A}^2+\mathcal{B}^2}\right)\ \mathrm{d}\rho$$

Thanks to the power series expansions in Eqs. (A.6) and (A.7), we can derive from Eq. (A.14),

$$h(\mathcal{A},\rho) = \sqrt{\mathcal{A}^2+\mathcal{B}^2}\left(f(\mathcal{A},\rho) - \frac{\pi}{2}\sqrt{\mathcal{A}^2+\mathcal{B}^2}\right) = \frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)^{\frac{3}{2}}(\pi\rho^2 - 8\mathcal{B}\rho)}{8\left(\sqrt{\mathcal{A}^2+\mathcal{B}^2}+\rho\right)^3} + O\left(\frac{1}{\mathcal{A}}\right)$$

and thus,

$$\lim_{\mathcal{A}\to+\infty} h(\mathcal{A},\rho) = \frac{\pi\rho^2 - 8\mathcal{B}\rho}{8}.$$

For every $\rho \in [0,+\infty)$, $\rho\Delta G_\sigma(\rho)h(\mathcal{A},\rho)$ is dominated by $(\max(1,\rho^3)+\mathcal{B}\max(1,\rho^2))\,|\Delta G_\sigma(\rho)|$, which is L^1-integrable. Hence, by the DCT,

$$\begin{aligned}
\lim_{\mathcal{A}\to+\infty} -\frac{4\pi\sigma}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\int_0^\infty \rho\Delta G_\sigma(\rho)h(\mathcal{A},\rho)\ \mathrm{d}\rho &= -\frac{4\pi\sigma}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\int_0^\infty \rho\Delta G_\sigma(\rho)\left(\lim_{\mathcal{A}\to+\infty} h(\mathcal{A},\rho)\right)\ \mathrm{d}\rho \\
&= -\frac{4\pi\sigma}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\int_0^\infty \left(\frac{\pi\rho^2 - 8\mathcal{B}\rho}{8}\right)\Delta G_\sigma(\rho)\ \mathrm{d}\rho \\
&= 1
\end{aligned}$$

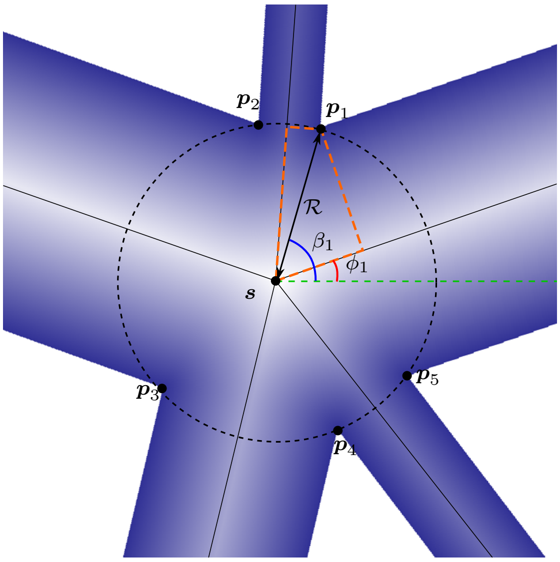

Fig. 2 Junction point with equidistant corners

Hence,

$$\lim_{\mathcal{A}\to+\infty} \frac{\pi\sigma\sqrt{\mathcal{A}^2+\mathcal{B}^2}}{\mathcal{B}\sqrt{2\pi}-\pi\sigma}\mathrm{rdg}_{\mathrm{ligature}}(\boldsymbol{s}) = 1$$

which proves Proposition 23.

A.15 Junction point - AOF

Assume that $\boldsymbol{x}_0 = \mathbf{0}$. There are n corners $\boldsymbol{p}_i$, equidistant from junction point $\boldsymbol{s}$. In the neighborhood of $\boldsymbol{x}_0$, the distance is:

$$D(\boldsymbol{x}) = \min_{i=1\dots n} D_i(\boldsymbol{x})$$

with $D_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{p}_i\|$. Switching to polar coordinates, corners are defined as $\boldsymbol{p}_i = \boldsymbol{s} + [\mathcal{R}\cos\beta_i, \mathcal{R}\sin\beta_i]^{\mathrm{T}}$, where β_i is the absolute angle formed with the x-axis. We denote by α_i the relative angle formed by $\boldsymbol{x}_0$ and two successive corners $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+1}$: $\alpha_i = \beta_{i+1} - \beta_i$. The distance is defined piecewise over radial intervals around $\boldsymbol{s}$. For a point $\boldsymbol{x}$ in the neighborhood of $\boldsymbol{s}$ s.t.

$$\boldsymbol{x} = \boldsymbol{s} + [\rho\cos\theta, \rho\sin\theta]^{\mathrm{T}},$$

assuming that $\rho \le \mathcal{R}$ and $\theta \in [\phi_i, \phi_{i+1}]$, with $\phi_i = \dfrac{\beta_{i-1}+\beta_i}{2}$, $\boldsymbol{x}$ is in the radial range of points closest to $\boldsymbol{p}_i$. This is exemplified in Fig. 8, where the area outlined with dotted orange lines is the set of points closest to $\boldsymbol{x}_1$ than any other $\boldsymbol{x}_i$. It is spanned by angle range $[\phi_1,\phi_2]$. Using Eq. (42), the gradient of the distance on the circle of radius r is

$$\begin{aligned}
\nabla D_i\left(\boldsymbol{s} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) &= \left\| \begin{bmatrix} r\cos\theta - \mathcal{R}\cos\beta_i - \\ r\sin\theta - \mathcal{R}\sin\beta_i \end{bmatrix} \right\|^{-1} \begin{bmatrix} r\cos\theta - \mathcal{R}\cos\beta_i \\ r\sin\theta - \mathcal{R}\sin\beta_i \end{bmatrix} \\
&= \frac{1}{\sqrt{r^2 + \mathcal{R}^2 - 2r\mathcal{R}\cos(\theta - \beta_i)}} \begin{bmatrix} r\cos\theta - \mathcal{R}\cos\beta_i \\ r\sin\theta - \mathcal{R}\sin\beta_i \end{bmatrix}
\end{aligned}$$

Plugging it into Eq (3), it comes

$$
\begin{aligned}
\operatorname{aof}_{\text{junction}}(\boldsymbol{s}) &= \frac{1}{2\pi}\int_0^{2\pi} \nabla D\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta\end{bmatrix}\right)\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}} \nabla D_i\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta\end{bmatrix}\right)\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\ \mathrm{d}\theta\right.\\
&\quad\left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}} \nabla D_{i+1}\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta\end{bmatrix}\right)\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\ \mathrm{d}\theta\right]\\
&= \frac{1}{2\pi}\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}} \frac{r-\mathcal{R}\cos(\theta-\beta_i)}{\sqrt{r^2+\mathcal{R}^2-2r\mathcal{R}\cos(\theta-\beta_i)}}\mathrm{d}\theta\right.\\
&\quad\left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}} \frac{r-\mathcal{R}\cos(\theta-\beta_{i+1})}{\sqrt{r^2+\mathcal{R}^2-2r\mathcal{R}\cos(\theta-\beta_{i+1})}}\mathrm{d}\theta\right]
\end{aligned}
$$

By translation, it comes

$$
\begin{aligned}
\operatorname{aof}_{\text{junction}}(\boldsymbol{s}) &= \frac{1}{2\pi}\sum_{i=1}^{n}\left[\int_0^{\frac{\alpha_i}{2}} \frac{r-\mathcal{R}\cos\theta}{\sqrt{r^2+\mathcal{R}^2-2r\mathcal{R}\cos\theta}}\mathrm{d}\theta+\int_{-\frac{\alpha_i}{2}}^{0} \frac{r-\mathcal{R}\cos\theta}{\sqrt{r^2+\mathcal{R}^2-2r\mathcal{R}\cos\theta}}\mathrm{d}\theta\right]\\
&= \frac{1}{\pi}\sum_{i=1}^{n}\int_0^{\frac{\alpha_i}{2}} \frac{r-\mathcal{R}\cos\theta}{\sqrt{r^2+\mathcal{R}^2-2r\mathcal{R}\cos\theta}}\mathrm{d}\theta
\end{aligned}
$$

which proves Proposition 24. Using identity $\cos\theta = 1-2\sin^2\dfrac{\theta}{2}$ and a change of variable,

$$
\operatorname{aof}_{\text{junction}}(\boldsymbol{s}) = \frac{2}{\pi}\sum_{i=1}^{n}\int_0^{\frac{\alpha_i}{4}} \frac{r-\mathcal{R}+2\mathcal{R}\sin^2\theta}{\sqrt{(r-\mathcal{R})^2+4r\mathcal{R}\sin^2\theta}}\mathrm{d}\theta
$$

Using Eqs. (18) and (A.3), it can be expressed using incomplete elliptic integrals with purely imaginary modulus:

$$
\operatorname{aof}_{\text{junction}}(\boldsymbol{s}) = \frac{1}{\pi r}\sum_{i=1}^{n}\frac{r^2-\mathcal{R}^2}{|r-\mathcal{R}|}\mathrm{F}\left(\frac{\alpha_i}{4}, ik_1\right)+|r-\mathcal{R}|\mathrm{E}\left(\frac{\alpha_i}{4}, ik_1\right)
$$

with $k_1 = \dfrac{2\sqrt{r\mathcal{R}}}{|\mathcal{R}-r|}$. We convert them to elliptic integrals with real modulus using rule (A.4):

$$
\operatorname{aof}_{\text{junction}}(\boldsymbol{s}) = \frac{1}{\pi r}\sum_{i=1}^{n}\left[(\mathcal{R}+r)\left(\mathrm{E}(k)-\mathrm{E}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},k\right)\right)-(\mathcal{R}-r)\left(\mathrm{K}(k)-\mathrm{F}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},k\right)\right)\right] \tag{A.15}
$$

with $k = \dfrac{2\sqrt{r\mathcal{R}}}{\mathcal{R}+r}$, which proves Proposition 25.

A.16 Junction point - AOF - Asymptotical behavior

We consider the second-order Taylor approximation of D_i, as defined in Eq. (42), in the polar coordinate system centered at $\boldsymbol{s}$. We have

$$
\begin{aligned}
\nabla D_i(\boldsymbol{s}) &= [-\cos\beta_i, -\sin\beta_i]^{\mathrm{T}}\\
\mathbf{H}_{D_i}(\boldsymbol{s}) &= \frac{1}{\mathcal{R}}\begin{bmatrix}\sin^2\beta_i & -\cos\beta_i\sin\beta_i\\ -\cos\beta_i\sin\beta_i & \cos^2\beta_i\end{bmatrix}
\end{aligned}
$$

So, the approximated distance on the arc of radius r centered at $\boldsymbol{s}$, for $\theta \in [\phi_i, \phi_{i+1}]$ is

$$\begin{aligned}
\widetilde{D}_i\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) &= \mathcal{R} + \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \nabla D_i(\boldsymbol{s}) + \frac{1}{2}\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}^{\mathrm{T}} \mathbf{H}_{D_i}(\boldsymbol{s}) \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\
&= \mathcal{R} - r\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} \\
&\quad + \frac{r^2}{2\mathcal{R}}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sin^2\beta_i & -\cos\beta_i\sin\beta_i \\ -\cos\beta_i\sin\beta_i & \cos^2\beta_i \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\
&= \mathcal{R} - r\cos(\theta-\beta_i) + \frac{r^2}{2\mathcal{R}}\sin^2(\theta-\beta_i)
\end{aligned} \tag{A.16}$$

and its gradient is

$$\begin{aligned}
\nabla\widetilde{D}_i\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right) &= \nabla D_i(\boldsymbol{s}) + \mathbf{H}_{D_i}(\boldsymbol{s})\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\
&= -\begin{bmatrix} \cos\beta_i \\ \sin\beta_i \end{bmatrix} + \frac{r}{\mathcal{R}}\begin{bmatrix} \sin^2\beta_i & -\cos\beta_i\sin\beta_i \\ -\cos\beta_i\sin\beta_i & \cos^2\beta_i \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
\end{aligned}$$

Plugging it into Eq. (25), we obtain

$$\begin{aligned}
\widetilde{\mathrm{aof}}_{\mathrm{junction}}(\boldsymbol{s}) &= \frac{1}{2\pi}\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}} \nabla\widetilde{D}_i\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right. \\
&\quad \left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}} \nabla\widetilde{D}_{i+1}\left(\boldsymbol{s}+\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}\right)\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\,\mathrm{d}\theta\right] \\
&= \frac{1}{2\pi}\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}} -\cos(\theta-\beta_i) + \frac{r}{\mathcal{R}}\sin^2(\theta-\beta_i)\,\mathrm{d}\theta\right. \\
&\quad \left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}} -\cos(\theta-\beta_{i+1}) + \frac{r}{\mathcal{R}}\sin^2(\theta-\beta_{i+1})\,\mathrm{d}\theta\right]
\end{aligned}$$

By translation,

$$\begin{aligned}
\widetilde{\mathrm{aof}}_{\mathrm{junction}}(\boldsymbol{s}) &= \frac{1}{\pi}\sum_{i=1}^{n}\int_{0}^{\frac{\alpha_i}{2}} -\cos\theta + \frac{r}{\mathcal{R}}\sin^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{\pi}\sum_{i=1}^{n} -\sin\left(\frac{\alpha_i}{2}\right) + \frac{r}{4\mathcal{R}}(\alpha_i - \sin\alpha_i) \\
&= -\frac{1}{\pi}s_2 + \frac{r}{\mathcal{R}}\left(1-\frac{1}{2\pi}s_1\right)
\end{aligned}$$

where s_1 and s_2 are defined in Eq. (46), which proves Proposition 26. In order to prove that the AOF at a junction point is asymptotically equivalent to its second-order approximation

$$\mathrm{aof}_{\mathrm{junction}}(\boldsymbol{s}) \sim \widetilde{\mathrm{aof}}_{\mathrm{junction}}(\boldsymbol{s}) \text{ (as } \mathcal{R}\to+\infty),$$

we show that

$$\lim_{\mathcal{R}\to+\infty}\frac{2\pi\mathcal{R}}{-2s_2\mathcal{R}+2\pi r - rs_1}\mathrm{aof}_{\mathrm{junction}}(\boldsymbol{s}) = 1.$$

Using series expansions in Eqs. (A.6) and (A.7), we derive the elliptic terms of Eq. (A.15)

$$\mathrm{K}(k) - \mathrm{F}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},k\right) = \frac{\alpha_i}{4} + k^2\left(\frac{1}{8}\sin\left(\frac{\alpha_i}{2}\right)+\frac{\alpha_i}{16}\right) + k^4\left(\frac{3\sin\alpha_i}{256}+\frac{3}{32}\sin\left(\frac{\alpha_i}{2}\right)+\frac{9\alpha_i}{256}\right) + O(k^6)$$

$$\mathrm{E}(k) - \mathrm{E}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},k\right) = \frac{\alpha_i}{4} - k^2\left(\frac{1}{8}\sin\left(\frac{\alpha_i}{2}\right)+\frac{\alpha_i}{16}\right) - k^4\left(\frac{\sin\alpha_i}{256}+\frac{1}{32}\sin\left(\frac{\alpha_i}{2}\right)+\frac{3\alpha_i}{256}\right) + O(k^6)$$

which yields

$$\begin{aligned}\text{aof}_{\text{junction}}(\boldsymbol{s}) &= \sum_{i=1}^{n}\left\{\frac{\alpha_i}{2\pi} - \frac{1}{\pi(\mathcal{R}+r)^4}\left[\mathcal{R}^4\left(\frac{\alpha_i}{2}+\sin\left(\frac{\alpha_i}{2}\right)\right)+\right.\right.\\ &\left.\left.+\mathcal{R}^3 r\left(\frac{7\alpha_i}{4}+4\sin\left(\frac{\alpha_i}{2}\right)+\frac{\sin\alpha_i}{4}\right)+\mathcal{R}^2r^2\left(\frac{\alpha_i}{8}-\frac{\sin\alpha_i}{8}\right)\right]\right\}+O\left(\frac{1}{\mathcal{R}^2}\right)\\ &= 1-\frac{1}{\pi(\mathcal{R}+r)^4}\left[\mathcal{R}^4(\pi+s_2)+\mathcal{R}^3 r\left(4s_2+\frac{7\pi}{2}+\frac{s_1}{4}\right)+\mathcal{R}^2r^2\left(\frac{\pi}{4}-\frac{s_1}{8}\right)\right]+O\left(\frac{1}{\mathcal{R}^2}\right)\\ &= \frac{-8s_2\mathcal{R}^4+(4\pi-2s_1-32s_2)\mathcal{R}^3r+(46\pi-s_1)\mathcal{R}^2r^2+32\pi\mathcal{R}r^3+8\pi r^4}{8\pi(\mathcal{R}+r)^4}+O\left(\frac{1}{\mathcal{R}^2}\right)\end{aligned}$$

Thus,

$$\lim_{\mathcal{R}\to+\infty}\frac{2\pi\mathcal{R}}{-2s_2\mathcal{R}+2\pi r-rs_1}\text{aof}_{\text{junction}}(\boldsymbol{s}) = \lim_{\mathcal{R}\to+\infty}\frac{\mathcal{R}^4}{(\mathcal{R}+r)^4}+O\left(\frac{1}{\mathcal{R}}\right)=1$$

which proves Proposition 27.

A.17 Junction point - Ridgeness

Using Eqs (5) and (42), the ridgeness at a junction point can be expressed in polar coordinates as

$$\begin{aligned}\text{rdg}_{\text{junction}}(\boldsymbol{s}) &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}} D_i\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\right.\\ &\left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}} D_{i+1}\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\right]\mathrm{d}\rho\\ &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\left[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta_i)}\mathrm{d}\theta\right.\\ &\left.+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos(\theta-\beta_{i+1})}\mathrm{d}\theta\right]\mathrm{d}\rho\end{aligned}$$

By translation,

$$\begin{aligned}\text{rdg}_{\text{junction}}(\boldsymbol{s}) &= -\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\left[\int_0^{\frac{\beta_{i+1}-\beta_i}{2}}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}\mathrm{d}\theta\right.\\ &\left.+\int_{-\frac{\beta_{i+1}-\beta_i}{2}}^{0}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}\mathrm{d}\theta\right]\mathrm{d}\rho\\ &= -2\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\int_0^{\frac{\alpha_i}{2}}\sqrt{\rho^2+\mathcal{R}^2-2\rho\mathcal{R}\cos\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho\end{aligned}$$

which proves Proposition 28. Using transformation $\cos\theta = 1-2\sin^2\dfrac{\theta}{2}$, it comes

$$\text{rdg}_{\text{junction}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\int_0^{\frac{\alpha_i}{4}}\sqrt{(\mathcal{R}-\rho)^2+4\rho\mathcal{R}\sin^2\theta}\ \mathrm{d}\theta\ \mathrm{d}\rho,$$

which can be expressed with incomplete elliptic integrals of the second kind with purely imaginary modulus

$$\text{rdg}_{\text{junction}}(\boldsymbol{s}) = -4\int_0^\infty \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}|\mathcal{R}-\rho|\mathrm{E}\left(\frac{\alpha_i}{4}, i\frac{2\sqrt{\rho\mathcal{R}}}{|\mathcal{R}-\rho|}\right)\ \mathrm{d}\rho.$$

Using transformations in Eq. (A.4), we obtain:

$$\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) = -4\int_0^{\infty} \rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}(\mathcal{R}+\rho)\left(\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right) - \mathrm{E}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)\right)\,\mathrm{d}\rho, \tag{A.17}$$

which proves Proposition 29.

A.18 Junction point - Ridgeness - Asymptotical behavior

Using Eqs (31) and (A.16), the approximate ridgeness at a junction point is

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{junction}}(\boldsymbol{s}) &= -\int_0^{\infty}\rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\Bigg[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}}\widetilde{D}_i\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta \\
&\quad+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}}\widetilde{D}_{i+1}\left(\boldsymbol{s}+\begin{bmatrix}\rho\cos\theta\\ \rho\sin\theta\end{bmatrix}\right)\mathrm{d}\theta\Bigg]\mathrm{d}\rho \\
&= -\int_0^{\infty}\rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\Bigg[\int_{\beta_i}^{\frac{\beta_i+\beta_{i+1}}{2}}\mathcal{R}-\rho\cos(\theta-\beta_i)+\frac{\rho^2}{2\mathcal{R}}\sin^2(\theta-\beta_i)\mathrm{d}\theta \\
&\quad+\int_{\frac{\beta_i+\beta_{i+1}}{2}}^{\beta_{i+1}}\mathcal{R}-\rho\cos(\theta-\beta_{i+1})+\frac{\rho^2}{2\mathcal{R}}\sin^2(\theta-\beta_{i+1})\mathrm{d}\theta\Bigg]\mathrm{d}\rho
\end{aligned}$$

By translation, we get

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{junction}}(\boldsymbol{s}) &= -\int_0^{\infty}\rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\Bigg[\int_0^{\frac{\beta_{i+1}-\beta_i}{2}}\mathcal{R}-\rho\cos\theta+\frac{\rho^2}{2\mathcal{R}}\sin^2\theta\mathrm{d}\theta \\
&\quad+\int_{-\frac{\beta_{i+1}-\beta_i}{2}}^{0}\mathcal{R}-\rho\cos\theta+\frac{\rho^2}{2\mathcal{R}}\sin^2\theta\mathrm{d}\theta\Bigg]\mathrm{d}\rho \\
&= -2\int_0^{\infty}\rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\int_0^{\frac{\alpha_i}{2}}\mathcal{R}-\rho\cos\theta+\frac{\rho^2}{2\mathcal{R}}\sin^2\theta\;\mathrm{d}\theta\;\mathrm{d}\rho \\
&= -2\int_0^{\infty}\rho\Delta G_\sigma(\rho)\sum_{i=1}^{n}\mathcal{R}\frac{\alpha_i}{2}-\rho\sin\left(\frac{\alpha_i}{2}\right)+\frac{\rho^2}{2\mathcal{R}}\left(\frac{\alpha_i}{4}-\frac{1}{4}\sin\alpha_i\right)\;\mathrm{d}\rho.
\end{aligned}$$

Using Eq. (46), this simplifies to

$$\begin{aligned}
\widetilde{\mathrm{rdg}}_{\mathrm{junction}}(\boldsymbol{s}) &= \int_0^{\infty}\rho\Delta G_\sigma(\rho)\left(-2\pi\mathcal{R}+2\rho s_2-\frac{\rho^2}{4\mathcal{R}}(2\pi-s_1)\right)\;\mathrm{d}\rho \\
&= \frac{1}{\sigma\sqrt{2\pi}}s_2-\frac{1}{\mathcal{R}}\left(1-\frac{1}{2\pi}s_1\right),
\end{aligned}$$

which proves Proposition 30. In order to prove that the ridgeness of a junction point is asymptotically equivalent to its second-order approximation

$$\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) \sim \widetilde{\mathrm{rdg}}_{\mathrm{junction}}(\boldsymbol{s}) \text{ (as } \mathcal{R}\to+\infty),$$

we show that

$$\lim_{\mathcal{R}\to+\infty}\frac{2\pi\sigma\mathcal{R}}{s_2\mathcal{R}\sqrt{2\pi}-2\pi\sigma+\sigma s_1}\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) = 1.$$

Let us consider Eq. (A.17). For convenience, we introduce

$$f(\mathcal{R},\rho) = (\mathcal{R}+\rho)\sum_{i=1}^{n}\left\{\mathrm{E}\left(\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)-\mathrm{E}\left(\frac{\pi}{2}-\frac{\alpha_i}{4},\frac{2\sqrt{\rho\mathcal{R}}}{\mathcal{R}+\rho}\right)\right\},$$

so that

$$\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) = -4\int_0^{\infty} \rho \Delta G_\sigma(\rho) f(\mathcal{R},\rho)\ \mathrm{d}\rho.$$

Since

$$\int_0^{\infty} \rho \Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0,$$

it is also true that

$$\int_0^{\infty} \frac{\pi}{2}\mathcal{R}\rho \Delta G_\sigma(\rho)\ \mathrm{d}\rho = 0.$$

Hence,

$$\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) = -4\int_0^{\infty} \rho \Delta G_\sigma(\rho) \left(f(\mathcal{R},\rho) - \frac{\pi}{2}\mathcal{R}\right)\ \mathrm{d}\rho$$

Let

$$h(\mathcal{R},\rho) = f(\mathcal{R},\rho) - \frac{\pi}{2}\mathcal{R}.$$

Thanks to the power series expansions in Eqs. (A.6) and (A.7), it comes

$$\begin{aligned}
h(\mathcal{R},\rho) &= \sum_{i=1}^{n} \Bigg\{ \frac{1}{(\mathcal{R}+\rho)^3} \Bigg[\left(\frac{1}{2}\sin\left(\frac{\alpha_i}{2}\right) + \frac{\alpha_i}{4}\right)\mathcal{R}^4 + \left(\sin\left(\frac{\alpha_i}{2}\right) + \frac{\alpha_i}{2}\right)\mathcal{R}^3\rho \\
&\quad + \left(\frac{\alpha_i}{16} + \frac{\sin\alpha_i}{16}\right)\mathcal{R}^2\rho^2\Bigg] - \frac{\mathcal{R}}{2}\sin\left(\frac{\alpha_i}{2}\right) + \frac{\alpha_i\rho}{4}\Bigg\} - \frac{\pi\mathcal{R}}{2} + O\left(\frac{1}{\mathcal{R}^2}\right) \\
&= \frac{1}{(\mathcal{R}+\rho)^3}\Bigg[\frac{\pi\mathcal{R}^4}{2} + \left(\frac{3\pi}{2} - \frac{s_2}{2}\right)\mathcal{R}^3\rho \\
&\quad + \left(\frac{5\pi}{8} - \frac{s_1}{16} - \frac{s_2}{2}\right)\mathcal{R}^2\rho^2 + \left(\frac{\pi}{2} - \frac{s_2}{2}\right)\mathcal{R}\rho^3 + \frac{\pi\rho^4}{2}\Bigg] - \frac{\pi\mathcal{R}}{2} + O\left(\frac{1}{\mathcal{R}^2}\right) \\
&= -\frac{\rho(8s_2\mathcal{R}^3 + (14\pi + s_1 + 8s_2)\mathcal{R}^2\rho + 8s_2\mathcal{R}\rho^2 - 8\pi\rho^3)}{16(R+\rho)^3} + O\left(\frac{1}{\mathcal{R}^2}\right)
\end{aligned}$$

and thus,

$$\lim_{\mathcal{R}\to+\infty} h(\mathcal{R},\rho) = -\frac{s_2}{2}\rho.$$

For every $\rho \in [0,+\infty)$, $\rho\Delta G_\sigma(\rho)h(\mathcal{R},\rho)$ is dominated by $s_2\max(1,\rho^3)\,|\Delta G_\sigma(\rho)|$, which is L^1-integrable. Hence, by the DCT,

$$\begin{aligned}
\lim_{\mathcal{R}\to+\infty} \mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) &= \lim_{\mathcal{R}\to+\infty} -4\int_0^{\infty} \rho\Delta G_\sigma(\rho) h(\mathcal{R},\rho)\ \mathrm{d}\rho \\
&= -4\int_0^{\infty} \rho\Delta G_\sigma(\rho)\left(\lim_{\mathcal{R}\to+\infty} h(\mathcal{R},\rho)\right)\ \mathrm{d}\rho \\
&= 2s_2\int_0^{\infty} \rho^2 \Delta G_\sigma(\rho)\ \mathrm{d}\rho \\
&= \frac{s_2\sqrt{2\pi}}{2\pi\sigma}
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{\mathcal{R}\to+\infty} \frac{2\pi\sigma\mathcal{R}}{s_2\mathcal{R}\sqrt{2\pi} - 2\pi\sigma + \sigma s_1}\mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) &= \lim_{\mathcal{R}\to+\infty} \frac{2\pi\sigma\mathcal{R}}{s_2\mathcal{R}\sqrt{2\pi} - 2\pi\sigma + \sigma s_1} \cdot \lim_{\mathcal{R}\to+\infty} \mathrm{rdg}_{\mathrm{junction}}(\boldsymbol{s}) \\
&= \frac{2\pi\sigma}{s_2\sqrt{2\pi}} \cdot \frac{s_2\sqrt{2\pi}}{2\pi\sigma} = 1
\end{aligned}$$

which proves Proposition 31.

References

1. The Wolfram Functions site: incomplete elliptic integral of the first kind. `http://functions.wolfram.com/EllipticIntegrals/EllipticF`.
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3. B.C. Carlson. Elliptic integrals. In F.W.J Olver, D.W. Lozier, R.F Boisvert, and C.W. Clark, editors, *NIST Handbook of Mathematical Functions*, chapter 19, pages 485–522. Cambridge University Press, 2010.