Numerical project, Theory and algorithmics for wave control, Theoretical Analysis track

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Control of acoustic wave by damping on a surface

The case of a plane shape wall is first considered. The motivation consists to model the noise of the source (the function g) and, for this fixed source and a fixed porous medium, to find the coefficient α in such a way that the two problems below are equivalent (see Fig. 1) :

$$\left\{\begin{array}{l} \text{in } \Omega_{air} : \\ \nabla \cdot (\eta_0 \nabla u_0) + \omega^2 \xi_0 u_0 = 0 \\ \text{in } \Omega_{wall} : \\ \nabla \cdot (\eta_1 \nabla u_1) + \omega^2 \xi_1 \left(1 + \frac{ai}{\xi_1 \omega}\right) u_1 = 0 \\ \text{on } \Gamma : \\ u_0 = u_1 \ \text{ and } \ \eta_0 \nabla u_0 \cdot \nu = \eta_1 \nabla u_1 \cdot \nu \\ \text{on the left boundary :} \\ u_0(-L, y) = g(y) \end{array}\right. \qquad \left\{\begin{array}{l} \text{in } \Omega_{air} : \\ \nabla \cdot (\eta_0 \nabla u_2) + \omega^2 \xi_0 u_2 = 0 \\ \\ \\ \text{on } \Gamma : \\ \eta_0 \nabla u_2 \cdot \nu + \alpha u_2 = 0 \\ \text{on the left boundary :} \\ u_2(-L, y) = g(y) \end{array}\right.$$

The coefficient α can be determined from the minimization problem, find α such that for two

Figure 1 – Air with boundary (left) and air with a wall composed of pourous media.

constants $A \geq 0$ and $B \geq 0$

$$A||u_0 - u_2||^2_{L^2(\Omega_{air})} + B||\nabla(u_0 - u_2)||^2_{L^2(\Omega_{air})}$$

is minimum. It can be shown that it is possible to solve this problem numerically, using the formulas of the theorem indicated in the CourseBook section 3.2.4, Theorem 3.2.2. This theorem extracted from the course is here recalled :

Theorem
Let $\Omega =]-L, L[\times]-\ell, \ell[$ be a domain with a simply connected sub-domain Ω_{air}, whose boundaries are $]-L, 0[\times \{\ell\}$, $\{-L\} \times]-\ell, \ell[$, $]-L, 0[\times \{-\ell\}$ and another boundary, denoted by Γ, which is the straight line starting in $(0, -\ell)$ and ending in $(0, \ell)$. In addition let Ω_{wall} be the supplementary domain of Ω_{air} in Ω, so that Γ is the common boundary of Ω_{air} and Ω_{wall}. The length L is supposed to be large enough.

Let the original problem (the frequency version of the wave damped problem be

$$-\nabla \cdot (\eta_0 \nabla u_0) - \omega^2 \xi_0 u_0 = 0 \text{ in } \Omega_{air}, \tag{1}$$
$$-\nabla \cdot (\eta_1 \nabla u_1) - \omega^2 \tilde{\xi}_1 u_1 = 0 \text{ in } \Omega_{wall}, \tag{2}$$

with

$$\tilde{\xi}_1 = \xi_1 \left(1 + \frac{ai}{\xi_1 \omega}\right),$$

together with boundary conditions on Γ

$$u_0 = u_1 \text{ and } \eta_0 \nabla u_0 \cdot \nu = \eta_1 \nabla u_1 \cdot \nu, \tag{3}$$

and the condition on the left boundary

$$u_0(-L, y) = g(y), \tag{4}$$

and some other boundary conditions. Let the modified problem be

$$-\nabla \cdot (\eta_0 \nabla u_2) - \omega^2 \xi_0 u_2 = 0 \text{ in } \Omega_{air} \tag{5}$$

with boundary absorption condition on Γ

$$\eta_0 \nabla u_2 \cdot \nu + \alpha u_2 = 0 \tag{6}$$

and the condition on the left boundary

$$u_2(-L, y) = g(y). \tag{7}$$

Let u_0, u_1, u_2 and g be decomposed into Fourier modes in the y direction, denoting by k the associated wave number. Then the complex parameter α, minimizing the following expression

$$A||u_0 - u_2||^2_{L_2(\Omega_{air})} + B||\nabla(u_0 - u_2)||^2_{L_2(\Omega_{air})}$$

can be found from the minimization of the error function

$$e(\alpha) := \sum_{k=\frac{n\pi}{L}, n \in \mathbb{Z}} e_k(\alpha),$$

where e_k are given by

$$\begin{aligned} e_k(\alpha) = (A + B|k|^2) \Big(\frac{1}{2\lambda_0} \Big\{ |\chi|^2 \left[1 - \exp(-2\lambda_0 L)\right] \\ +|\gamma|^2 \left[\exp(2\lambda_0 L) - 1\right] \Big\} + 2L\mathrm{Re}\left(\chi\bar{\gamma}\right) \Big) \\ + B\frac{\lambda_0}{2} \Big\{ |\chi|^2 \left[1 - \exp(-2\lambda_0 L)\right] + |\gamma|^2 \left[\exp(2\lambda_0 L) - 1\right] \Big\} - 2B\lambda_0^2 L\mathrm{Re}\left(\chi\bar{\gamma}\right) \end{aligned}$$

if $k^2 \geq \frac{\xi_0}{\eta_0}\omega^2$ or

$$\begin{aligned} e_k(\alpha) = (A + B|k|^2) \Big(L(|\chi|^2 + |\gamma|^2) + \frac{i}{\lambda_0}\mathrm{Im}\left\{\chi\bar{\gamma}\left[1 - \exp(-2\lambda_0 L)\right]\right\} \Big) \\ +BL|\lambda_0|^2 \left(|\chi|^2 + |\gamma|^2\right) + iB\lambda_0 \mathrm{Im}\left\{\chi\bar{\gamma}\left[1 - \exp(-2\lambda_0 L)\right]\right\} \end{aligned}$$

if $k^2 < \frac{\xi_0}{\eta_0}\omega^2$, in which

$$
\begin{aligned}
f(x) &= (\lambda_0\eta_0 - x)\exp(-\lambda_0 L) + (\lambda_0\eta_0 + x)\exp(\lambda_0 L),\\
\chi(k,\alpha) &= g_k\left(\frac{\lambda_0\eta_0 - \lambda_1\eta_1}{f(\lambda_1\eta_1)} - \frac{\lambda_0\eta_0 - \alpha}{f(\alpha)}\right),\\
\gamma(k,\alpha) &= g_k\left(\frac{\lambda_0\eta_0 + \lambda_1\eta_1}{f(\lambda_1\eta_1)} - \frac{\lambda_0\eta_0 + \alpha}{f(\alpha)}\right),
\end{aligned}
$$

where

$$
\begin{cases}
\lambda_0 = \sqrt{k^2 - \frac{\xi_0}{\eta_0}\omega^2} & \text{if} \quad k^2 \geq \frac{\xi_0}{\eta_0}\omega^2,\\
\lambda_0 = i\sqrt{\frac{\xi_0}{\eta_0}\omega^2 - k^2} & \text{if} \quad k^2 \leq \frac{\xi_0}{\eta_0}\omega^2.
\end{cases}
\tag{8}
$$

and

$$
\lambda_1 = \frac{1}{\sqrt{2}}\sqrt{k^2 - \frac{\xi_1}{\eta_1}\omega^2 + \sqrt{\left(k^2 - \frac{\xi_1}{\eta_1}\omega^2\right)^2 + \left(\frac{a\omega}{\eta_1}\right)^2}}
$$

$$
- \frac{i}{\sqrt{2}}\sqrt{\frac{\xi_1}{\eta_1}\omega^2 - k^2 + \sqrt{\left(k^2 - \frac{\xi_1}{\eta_1}\omega^2\right)^2 + \left(\frac{a\omega}{\eta_1}\right)^2}}.
$$

The case $A = 1$, $B = 1$ can be considered.

Question 1 :
In the theorem, the source g depends on y. Suppose that g also depends on frequencies ω. Update the statement of the theorem for this case.

Question 2 :
Choose three different problems for which it is crucial to absorb the acoustical energy and model the corresponding typical sources of the noise, denoted by $g_1(y,\omega)$, $g_2(y,\omega)$ and $g_3(y,\omega)$. The choice must be strongly motivated and explained.

Question 3 :
Program these sources and find the approximation of the coefficient α for one fixed porous material for these three sources. To plot then the real and imaginary parts of α as a function of the frequency for the three different sources, compare results, and give an explication. You can consider a material Ω_1 composed of a Melamine foam, whose characteristics are : $\phi = 0.99$ (porosity), $\sigma = 14000.0$ (resistivity), $\alpha_h = 1.02$ (tortuosity).

Question 4 :
Consider different types of material and compute the associated coefficient α for three chosen noise sources. For each material make one figure indicating the real part of α upon the wavenumber and one figure indicating the imaginary part of α upon the wavenumber. Using multiple plots, compare the efficiency of these different materials.