Supplementary Information II: Parameter estimation and test

JM, AT, VH

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1 Data analysis - details

Parameter estimation: The maximum likelihood parameter estimation is somewhat subtle, as the distribution

$$\varphi(x) = C\, e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\, x}\; x^{N_1-1}\,(1-x)^{N_2-1},$$

incorporates exponential terms - in particular, if the parameters become large, the integral $\int_0^1 e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\, x}\; x^{N_1-1}\,(1-x)^{N_2-1}\,dx$ becomes numerically unstable. Therefore, we reparameterize the distribution, defining $\hat{\nu}$, $\hat{s}$, $\hat{\theta}$, and $\hat{\psi}$ by

$$\theta_1 = \hat{s}\,\hat{\theta}\,\hat{\psi}, \quad \theta_2 = \hat{s}\,\hat{\theta}\,(1-\hat{\psi}), \quad N_1+1 = \hat{s}\,(1-\hat{\theta})\,\hat{\nu}, \quad N_2+1 = \hat{s}\,(1-\hat{\theta})\,(1-\hat{\nu}), \tag{1}$$

where $\hat{\theta}, \hat{\psi}, \hat{\nu} \in [0,1]$, and $\hat{s} > 0$, with the restriction $\hat{s}\,(1-\hat{\theta})\,\hat{\nu} > 1$, and $\hat{s}\,(1-\hat{\theta})\,(1-\hat{\nu}) > 1$. Therewith, the distribution becomes

$$\varphi(x) = \hat{C}\,\exp\left[\;\hat{s}\;\left(\hat{\theta}(x^2/2-\hat{\psi}\,x)\; + (1-\hat{\theta})\,\hat{\nu}\,\ln(x) + (1-\hat{\theta})\,(1-\hat{\nu})\,\ln(1-x) + A\right)\;\right].$$

Here, A is a constant that can be chosen in dependency on the parameters and the data at hand. In practice, it is used to avoid an exponent that has a very large absolute number. The constant $\hat{C}$ is, as before, determined by the fact that the integral is one. This form allows for a reasonable maximum likelihood estimation, given appropriate election data.

Numerical issues: The model assumes continuous data, while the election data are discrete. Therefore, a vote share of 0 or 1 is possible in the empirical data, but the distribution may have poles for those values. We replace all empirical vote shares below 10^{-10} by 10^{-10}, and similarly, all data above $1-10^{-10}$ by $1-10^{-10}$. In order to determine the normalization constant of $\varphi(x)$, we do not integrate from 0 to 1, but only from 0.001 to 0.999. Furthermore, for numerical reasons, we restrict $\hat{s}$ by an upper limit, that we mostly define as 1800.

Test for reinforcement: The zealot model and the reinforcement model are nested. In that, we can use the likelihood-ratio test to check for the significance of the reinforcement component: If $\mathcal{LL}_0$ is the log-likelihood for the restricted model ($\theta_1 = \theta_2 = 0$, resp. $\hat{\theta} = 0$), and $\mathcal{LL}$ is that for the full reinforcement model, we have asymptotically, for a large sample size

$$2(\mathcal{LL} - \mathcal{LL}_0) \sim \chi^2_2$$

That is, twice the difference in the log-likelihoods is asymptotically χ^2 distributed, where the degree of freedom is the number of the surplus parameters (here: θ_1 and θ_2, resp. $\hat{\theta}$ and $\hat{\psi}$, that

is, the degree of freedom is 2).
Additionally, we use the Kolmogorov-Smirnov test to find out if the model-distribution of either the full reinforcement mode, or the zealot model ($\theta_1 = \theta_2 = 0$, resp. $\hat{\theta} = 0$) agrees with the empirical distribution. If both models are in line with the data, then the reinforcement component will rather not add to the interpretation of the data, if only the reinforcement model approximates the data well (or, at least, much better than the zealot model), we can expect that it is sensible to take the reinforcement component into account.

1.1 Details – US

In the US, the candidates for the presidential elections are determined by the "primary elections". We do not consider them, but only the presidential elections themselves. The election of the president happens indirectly via an Electoral Collage. Each state nominates a certain number of delegates. In most states, a winner-take-all system is established. If no candidate recieves the majority of the votes, he Congress will elect a candidate.

The data set used is provided by the Havard Univ., and contains the data on county-level for the elections 2000-2016,
`https://doi.org/10.7910/DVN/VOQCHQ`, file `countypres_2000-2016.csv`.

year	party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf)	p_{ks} (beta)
2000	republicans	0.0180	0.820	6.611e-05	89	72.99446	<1e-10	<1e-10	<1e-10
2000	democrats	0.190	4.889e-05	6.61e-05	2.24	0.0001	1	<1e-10	<1e-10
2000	green	0.0004	0.81	6.61e-05	193.5	156.9	<1e-10	<1e-10	<1e-10
2004	republicans	0.519	0.412	6.611e-05	21.7	8.95	0.15	0.15	0.092
2004	democrats	0.440	0.3253	0.99993	18.4	0.0004	0.60	0.12	0.1
2008	republicans	0.435	0.522	6.611e-05	23.9	12.5	0.0026	0.12	0.066
2008	democrats	0.544	0.5167	0.99993	22.5	0.0008	0.0023	0.14	0.070
2012	republicans	0.429	0.587	6.611e-05	22.1	13.0	1.5e-06	0.24	0.0076
2012	democrats	0.535	0.569	0.99993	20.2	0.00076	7.1e-06	0.23	0.011
2016	republicans	0.324	0.7860	0.19	55.5	35.3	<1e-10	0.66	7.4e-10
2016	democrats	0.563	0.7861	0.74	43.3	8.85	<1e-10	0.14	6.77e-10

Table 1: Estimated parameters for the two parties in the eight elections. p_{ll} is the result of the likelihood ration test for the significance of the reinforcement component; p_{ks} is the result of the Kolmogorov-Smirnov-test for the question of the empirical cumulative distribution differs significantly from the cumulative distribution of the model (either the reinforcement model, or zealot model with the beta distribution).

1.2 Details – Brexit

In the Brexit referendum (23 June 2016), each voter had the choice "remain" or "leave". The election was equal, each vote was counted directly. With 51.89% (and 72.2% turnout rate), the outcome has been "leave".

The data set used is provided by the British Government, and contains data on election district level, https://www.electoralcommission.org.uk/who-we-are-and-what-we-do/elections-and-referendums/past-elections-and-referendums/eu-referendum/results-and-turnout-eu-referendum, file EU-referendum-result-data.csv.

$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
0.87	0.76	0.99996	254.5	0.0084	1.3e-11	0.75	0.0009

Table 2: Parameter for the Brexit referendum, "remainers". For symmetry reasons, the parameters for "leave" are identical, but $\hat{\psi}_{leave} = 1-\hat{\psi}_{remain}$, and accordingly $\theta_{2,leave} = 192.97$. We have the result of the likelihood-ratio test for $\hat{\theta} = 0$ ($p_{ll} = 1.3e-11$), the result for the Kolmogorovff-Smirnov test if the data are in line with the reinforcement-distribution (p_{ks}(Reinf) $= 0.75$), and the result of Kolmogorov-Smirnov, if the data are in line with the zealot (beta) distribution (p_{ks}(beta) $= 0.0009$).

1.3 Details – Germany

Each voter has two votes in the elections for the German parliament: a "first" and a "second" vote. With the first vote, a candidate in the election district can be selected. The candidate with the most votes will be send into the parliament. With the second vote, a party is selected. The vote share of the parties determines the number of seats in the parliament, where a party has to overcome a 5% threshold. IN the analysis, only the second votes are used.

The data are provided by the German Government,
https://www.bundeswahlleiter.de/en/bundeswahlleiter.html, file btw17_kerg.csv.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.39	0.39	0.86	112.5	6.02	1	0.68	0.78
SPD	0.19	0.17	6.61e-05	50.5	8.80	0.99	0.20	0.20
AFD	0.07	0.77	6.61e-05	792.5	608.91	8.6e-11	0.29	0.0009
FDP	0.31	0.73	0.998	182.5	0.28	1	0.52	0.54
Die Linke	0.054	0.78	6.61e-05	583.5	453.7	3.44e-09	2.1e-06	2.93e-08
Gruene	0.070	0.15	6.61e-05	58	8.7	1	0.85	0.84
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.004	0.004

Table 3: Results for whole Germany, 2017.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.52	0.67	0.76	294.5	46.74	0.33	0.50	0.78
SPD	0.21	4.95e-05	6.61e-05	47.2	0.0023	1	0.17	0.17
AFD	0.30	0.71	0.99996	189.5	0.0055	1	0.86	0.92
FDP	0.29	0.68	0.99991	251.5	0.015	0.70	0.62	0.68
Die Linke	0.088	0.83	0.14	1159.5	834.22	0.00086	0.00068	0.00016
Gruene	0.22	0.86	0.38	289.5	154.36	0.16	0.112	0.075
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.0038	0.0038

Table 4: Results for Germany, only the "old" states ,2017.

1.4 Details – France/Le Pen

The presidential elections in France have (in principle/mostly) two rounds, a first round and runoff elections, in case that no candidate receives more than 50% of votes in the first round. The elections are direct, each vote counts the same.
The data are provided by the France Government. We used data on the level of départements.
`https://www.data.gouv.fr/fr/posts/les-donnees-des-elections/`,
file `Presidentielle_2017_Resultats_Tour_1_c.xls` (first round) and `Presidentielle_2017_Resultats_Tour_2_c.xls` (runoff elections).

round	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
first	0.2065317	6.216446e-05	6.610696e-05	34.9	0.002	1.00	0.42	0.42
runoff	0.7032342	3.838637e-01	9.999339e-01	49.3	0.001	0.96	0.32	0.31

Table 5: Results for France, 2017 Presidential elections, candidate Le Pen (first and second round).