Mathematical Models in Social Sciences – Notes

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Chapter 1

Introduction

The human society and human history:
– communication between humans, basis for the organization of a society
– resources and economic framework

The human society is shaped by intrinsic and extrinsic forces. Extrinsic forces are for example the available natural resources. An intrinsic force is communication between humans, that is necessary to establish social norms. In-between are economics and aspects of techniques: Techniques as agriculture and craft, or applied sciences converts the extrinsic, natural resources into available resources, economy distributes these resources within the society.

• Sociophysics/sociamathematics/socioinformatics
• models, descriptive.
• Understanding the mechanisms → influence the dynamics
not science, normative questions arise
dangerous, internet, social media, bots, Cambridge analytics.

Chapter 2

Communication and Democratic Elections

Communication is for sure one of the most important mechanisms that shape and structure society. We target here not only the simple spread of information, but rather the formation of social groups and social norms. As we want to validate tour ideas using data – at least at a simple level – we focus on a communication process, where tons of data are easily accessible: democratic elections.

Elections are central devices in modern democracies (in contrast to several democratic systems in the past, where a lottery has been an additional element to voting; it is an actual discussion if such a component could help to stabilize a democratic system [**?**]). In a democratic election, mature citizens think about the tasks a society is faced with, and decide, which candidate/party offers the best concepts to deal with these tasks. If this is picture of elections is complete, we cannot expect any statistical patterns in election results. However, as we will see below, a simple data analysis shows a lot of different, stable patterns. Those patterns are a reminiscence of the discussion and communication process within the society. In that, we are able to study one of the most fundamental forces that shape a society: communication and the formation of a basic agreement.

2.1 Votes per candidate within a party

To warm up, we look at a remarkable result. Consider a party that has several candidates. All candidates can be voted for in several election districts. How is the distribution of the vote share XXXX (wie geht das eigenltich???)

ACHTUNG! BAUSTELLE! ACHTUNG! BAUSTELLE! ACHTUNG! BAUSTELLE!

We focus on one party. This party not only has a set of candidates, but also a set of voters. These voters are undecided in the sense that we will vote for the party at hand, but do not know yet which candidate to select. Fortunato [15] proposes a branching process model for that process.

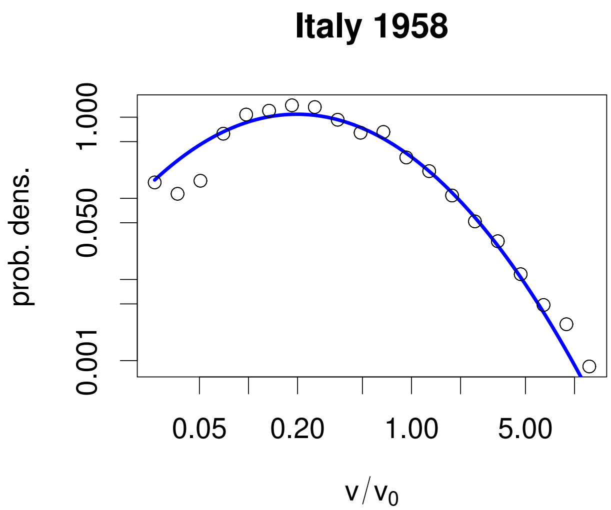

Figure 2.1: Distribution of v/v_0 in logarithmic scale, together with log-normal distribution ($\mu = -0.54$, $\sigma = -2\mu$).

Model 2.1 *The candidate-attracts-voter process consist of a group of undecided voters Y_t, and K groups of voters that already did decide for one out of a group with K candidates, $X_t^1, \ldots, X_K^t$. We start with $Y_0 = N$ (at time $t = 0$ we have $N \in \mathbb{N}$ undecided voters), and $X_0^i = 1$ (each group contains one single individual - the candidate his/herself).*
The dynamics of the process is described by the transitions

$$(Y_t, X_t^i) \to (Y_t - 1, X_t^i + 1) \;\; \textit{at rate } Y_t\, X_t^i$$

(where we suppressed the groups X_j with $j \neq i$ ti simplify the notation). The distribution of the groups for $Y_t = 0$ yields the distribution of votes for the candidates.

Obviously, the set of absorbing states is given by the elements

$$(0, X_1, \ldots, X_K)$$

where $X_i \geq 1$, and $\sum_{i=1}^K X_i = N + K$.

Excursion: Data processing
xxxxx
End excursion.

characteristic distribution and model.

[39] Another model for the same fact.

2.2 The voter model

The most basic model that describes the change of opinions in a population is the voter model. In its basic version, we only have two opinions; any person "re-thinks" his or her opinion at rate μ. This process is most simple: the person contacts a randomly selected person in the population (including his/herself, so-called selfing), and just copies the opinion of the corresponding individual.

Model 2.2 [The voter model] *Let $N \in \mathbb{N}$ be the population size, and X_t denote the number of supporters of opinion A at time t (whilst the B-supporters are $N - X_t$ in number). Then,*

$$\begin{aligned} X_t \to X_t + 1 \quad &\textit{at rate} \quad \nu(N - X_t)\,(X_t/N) \\ X_t \to X_t - 1 \quad &\textit{at rate} \quad \nu X_t\,(1 - X_t/N). \end{aligned}$$

It is interesting to note that the voter model appears in the literature about population genetics under the name Moran model [13, 45, 49, **?**]. The process of re-thinking an opinion corresponds to the death of an individual, while the adoption of a (new/old) opinion corresponds to the birth of an individual, where all individuals (including selfing - that is, also including the dead individual) have the same chance to produce offspring. In that, this model is termed neutral model – fitness differences play no role, in the same way, as in the present interpretation of the model the content of the opinion does not affect the chance of an opinion to spread.

Proposition 2.3 *For the voter process X_t, the state X_t is a Martingale. That is, for $t > s \geq 0$, we find*

$$E(X_t|X_s) = X_s.$$

Proof: Let $P_i(t) = P(X_t = i)$. The master equations read

$$\dot{p}_i(t) = -2i(1 - i/N)\,p_i + (i-1)(1-(i-1)/N)\,p_{i-1} + (i+1)(1-(i+1)/N)\,p_{i+1}$$

where we formally define $p_{-1} = p_{N+1} = 0$. Then, for $\Delta > 0$,

$$\begin{aligned} E(X_{t+\Delta t}) \;&=\; E(E(X_{t+\Delta t}|X_t)) = E(X_t) + 1\sum_{i=0}^{N} P(X_{t+\Delta t} = i+1|X_t = i)P(X_t = i) \\ &\qquad - 1\sum_{i=0}^{N} P(X_{t+\Delta t} = i-1|X_t = i)P(X_t = i) \\ &=\; E(X_t) + \sum_{i=0}^{N} \mu\,(N-i)\,i\;P(X_t = i)/N - \sum_{i=0}^{N} \mu\,i\,(N-i)\;P(X_t = i)/N = E(X_t). \end{aligned}$$

Hence, $E(X_t)$ is constant. If we know that $X_s = k$, $k \in \{0, \ldots, N\}$, then $E(X_t) = E(X_s) = k$. Hence,

$$E(X_t|X_s = k) = k = X_s.$$

Or, in a more condensed equation,

$$E(X_t|X_s) = X_s.$$

□

Note that the voter model possesses two absorbing states: $X_t = 0$ and $X_t = N$. As we only have a finite number of states, and the Voter model is irreducible, any realization eventually jumps into one of the absorbing states a.s. The Martingale property of X_t allows to simply compute the probability to jump to 0 (resp. N).

Theorem 2.4 *For the voter model with population size N and initial number of supporters for opinion A given by $X_0 \in \{0, \dots, N\}$, we find*

$$\lim_{t\to\infty} P(X_t = N) = X_0/N, \qquad \lim_{t\to\infty} P(X_t = 0) = 1 - X_0/N.$$

Proof: We know that the expected value does not change. As the Markov process is irreducible, all realizations eventually jump into an absorbing state. That is, for $t \to \infty$, only the states $X_t = 0$ and $X_t = N$ have a positive probability mass. If q is that for $X = 0$, we find

$$X_0 = E(X_t) = \lim_{t\to\infty} E(X_t) = q\,0 + (1-q)\,N.$$

Hence, $q = 1 - X_0/N$.

□

It is straight forward to generalize these results to a Voter model with K opinions (see exercise 2.1-2.3). It is less simple to assume a certain contact graph - not every individual communicates with any other individual, but an individual only will communicate with neighbors. If, moreover, the number of individuals (nodes in that graph) tends to infinity, the asymptotic behavior (survival of only one or several opinions) depends on the topology of the graph [30, 12]. However, at least for now, we stay with a finite, homogeneously mixing population.

The central result so far fas been that all opinions die out but one. This is for sure not in line with our every-day experience. A multitude of opinions are present. Our model misses some effects. One point could be, that the model is completely neutral. All opinions are identical, there is no rational arguing about opinions going on. However, for large parts of the present considerations, we will keep this neutrality assumptions. Another point is, that not everyone will change his/her opinion readily. We all know stubborn persons who are completely persuaded by their rather ridiculous opinion, and no argument will affect their deep persuasion. That is, we could divide the population in one class that readily change their mind, and one class that only seldom change their opinion. If we take the time scales into the extreme, we have one flexible group, and one group that has frozen opinions. We obtain the so-called zealot model, that we discuss next.

Exercises

Exercise 2.1: Generalize the voter model from 2 to K opinions.

Solution of Exercise 2.1 The state at a given time is characterized by $X_1, \dots, X_K$. Transitions always affect the group size of 2 groups: one is increased, and one decreased by 1. The transition probabilities read (where we suppress all groups that do not change in this transition)

$$(X_i, X_j) \to (X_i + 1, X_j - 1) \quad \text{at rate} \quad \nu X_j \, \frac{X_i}{N}$$

where N is the (constant) total population size, and ν the rate at which one individual changes his/her mind.

Exercise 2.2: Show for the K-opinion voter model that the expected number of individuals that adopt a given opinion does not change in time.

Solution of Exercise 2.2 We chose X_1 as a focal group. We might define a marginal process: Let $Y_1 = X_1$. Then, the probability for $Y_1 \to Y_1 + 1$ is given by

$$\sum_{j=2}^{K} \nu X_j \; \frac{X_1}{N} = \nu Y_1 \, Y_2/N = \nu \, \frac{Y_1(N_Y - 1)}{N}$$

while that for $Y_1 \to Y_1 - 1$ reads

$$\sum_{j=2}^{K} \nu X_1 \; \frac{X_j}{N} = \nu Y_1 \, Y_2/N = \nu \, \frac{Y_1(N_Y - 1)}{N}.$$

Hence, this marginal model is a 2-opinion the voter process, where we know that the expected value is constant. Therefore,

$$E(X_1) = E(Y_1) \equiv \text{const}.$$

For symmetry reasons, $E(X_i) \equiv$ const for $i = 1, \ldots, K$.

Exercise 2.3: Determine for the K-opinion voter model the absorbing states. Determine the probability to eventually jump into a given absorbing state, in dependence on the initial condition.

Solution of Exercise 2.3 Absorbing states are given if all individuals are member of a singe goup only.
We can use the argument of exercise 2.2: If we select a focal group, and only distinguish between focal and non-focal group, we find back a 2-opinion voter model. Hence,

$$P(\text{eventually are all individuals in group } i) = \frac{\text{initial number of individual in group } i}{N}.$$

2.3 The Zealot model

The voter model is perhaps the most simple model to describe opinion formation. However, it has unrealistic consequences: In finite (or randomly mixing) populations, only one opinion will persist in the long run (a.s.), while all other opinions die out. Obviously, this model is too simple to capture reality. Variants of the model are introduced that prevent this uniformity. Perhaps the most important extension in the context of democratic elections is the zealot model, which comes in various flavors [34, 7, 6, 35, 5, 41, 27]. Interestingly enough, that model has been also published in the context of population genetics [7], which emphasize the similarity of the models and methods for population genetics and democratic elections.

2.3.1 Two-party zealot model

We start with the two-opinion zealot model as formulated by Aguiar and coworkers [7, 6, 5]. It is interesting to note that this model first appeared in the context of population genetics [7], to describe foraging strategy of insects [25]. and finance markets [25], and only later it was also used to describe social processes, as finance markets [5] or elections.

Model 2.5 [Two-party Zealot Model] *Let $N \in \mathbb{N}$ denote the population size of a well mixed population. Each individual either supports party A or party B. The stochastic process $X_t \in \{0, \ldots, N\}$ represent the number of individuals supporting A, such that $N - X_t$ is the number of supporters of B. Additionally, there are pseudo-voters (zealots) with number $N^a, N^b \in \mathbb{R}$ ($N^* > 0$, $* \in \{a, b\}$) who have a fixed opinion, and always support A (N^a) resp. B (N^b). These zealots represent external influences as the mass media: newspaper, radio, television, or the internet.*
At rate ν, a person changes his/her opinion; he/she selects a voter or a pseudo-voter (with selfing) randomly, and copies the opinion of the selected person. Each of the $N + N^a + N^b$ individuals have the same probability to be chosen. That is, the transition rates defining the stochastic process read

$$X_t \to X_t + 1 \quad \text{at rate} \quad \nu\,(N - X_t)\,\frac{X_t + N^a}{N + N^a + N^b} \tag{2.1}$$

$$X_t \to X_t - 1 \quad \text{at rate} \quad \nu\, X_t\,\frac{N - X_t + N^b}{N + N^a + N^b}. \tag{2.2}$$

We aim to obtain insight into the effect of the communication effect within the society. We identified external forces, the zealots. This concept is kind of cheating, as most likely the zealots are parts of the society. There are some rare cases where zealots indeed stand outside of the society. E.g., if a state uses internet trolls, bots, and fake news to influence the public opinion of another state. Mostly, the zealots (newspaper, television stations) will not be outside the community. However, we might assume that the time scale at which the zealots change their opinion and the time scale the individuals change their opinion are different. If the zealots are much slower than the citizens, then the model might be appropriate.

The next proposition is a direct consequence of the model definition.

Proposition 2.6 *With $p_i(t) = P(X_t = i)$, the master equations of model 2.5 reads*

$$\begin{aligned}\frac{d}{dt}p_i \;&=\; -\nu\left(\frac{i + N^a}{N^a + N^b + N}\;\frac{N-i}{N} + \frac{N - i + N^b}{N^a + N^b + N}\;\frac{i}{N}\right)p_i \\ &\quad +\nu\left(\frac{i - 1 + N^a}{N^a + N^b + N}\;\frac{N-i+1}{N}\right)p_{i-1} + \nu\left(\frac{N - i - 1 + N^b}{N^a + N^b + N}\;\frac{i+1}{N}\right)p_{i+1}.\end{aligned} \tag{2.3}$$

As we will see below, it is possible to explicitly elaborate the invariant measure of the process X_t, that is, the stationary states of the ODE (2.3). For now, we choose a different route to obtain an idea about the long term behavior, and use the normal (or diffusion) approximation. Before we start with the diffusion approximation itself, we first obtain an idea about the long term behavior of the process. Thereto, we derive an ODE for $E(X_t)$. We do not use the master equations (which would be possible) but use the observation

$$\begin{aligned}E(X_{t+\Delta t}|X_t) &= X_t + \Delta t \left(\nu\,\frac{(N-X_t)X_t + (N-X_t)N^a}{N+N^a+N^b} - \nu\,\frac{X_t\,(N-X_t) + X_t\,N^b}{N+N^a+N^b}\right) + \mathcal{O}(\Delta t^2)\\ &= X_t + \Delta t\;\nu\bigg(\frac{(N-X_t)\,N^a}{N+N^a+N^b} - \frac{X_t\,N^b}{N+N^a+N^b}\bigg) + \mathcal{O}(\Delta t^2).\end{aligned}$$

Using the law of iterated expectations, and rearranging the resulting equation yields

$$\frac{E(X_{t+\Delta t}) - E(X_t)}{\Delta t} = \nu\bigg(\frac{(N-E(X_t))\,N^a}{N+N^a+N^b} - \frac{E(X_t)\,N^b}{N+N^a+N^b}\bigg) + \mathcal{O}(\Delta t).$$

It is remarkable that the interactions between voters cancel each other. For the voter model without zealots ($N^a = N^b = 0$), we recover the fact that X_t is a Martingale: $E(X_t|X_{t'}) = X_{t'}$ for $t > t' \geq 0$. The symmetry of the model without zealots prevents the expected value of the supporters of a party to change. Only the zealots represent external forces that are able to influence the expectations. As dependencies between voters do not play a role, we obtain a proper ordinary differential equation for $E(X_t)$: Taking the limit $\Delta t \to 0$, we obtain the following corollary.

Corollary 2.7 *The expected value of X_t satisfies the ODE*

$$\frac{d}{dt}E(X_t) = \nu\bigg(\frac{(N-E(X_t))\,N^a}{N+N^a+N^b} - \frac{E(X_t)\,N^b}{N+N^a+N^b}\bigg). \tag{2.4}$$

For $t \to \infty$ we obtain

$$\lim_{t\to\infty} E(X_t) = N\,\frac{N^a}{N^a+N^b}. \tag{2.5}$$

In the long run, the expectation of our process is determined by the fraction of A-zealots among all zealots. However, the shape of the invariant distribution can be very different, as the simulations presented in Fig. 2.2 indicates. We know that the original voter model (no zealots) tends to an absorbing state. If there are only few zealots, $(N^a + N^b)/N \ll 1$, then the voter model is only weakly perturbed. The invariant measure is still localized around the states $X = 0$ and $X = N$, which are the absorbing states for the voter model (Fig. 2.2, panel (c)). Only if N^a/N, $N^b/N \gg 0$, the process X_t/N will be attracted by $N^a/(N^a + N^b)$ (Fig. 2.2, panel (a)), and performs a random walk close to this point.

We are particularly interested in the invariant distribution of the model, assuming that changes due to external influences are rather slow, and the voter dynamics is rather large, s.t. the population is in equilibrium. That assumption is for sure not (always) given, but as we will see, in many cases it is yields a reasonable approximation. In order to investigate and characterize the invariant measure, we use three different approaches: (a) We directly compute the invariant measure for the model as it is. In the two further approaches, we take into account that the number fo voters per election district is large. Hence, a continuum limit gives a handy approximation of the individual-based model. (b) In the first continuum limit (the limit with strong-effects) we assume that the number of zealots increases with the population size N. (c) In the last approach (the limit with weak effects), the number of zealots are assumed to be constant while the population size tends to infinity.

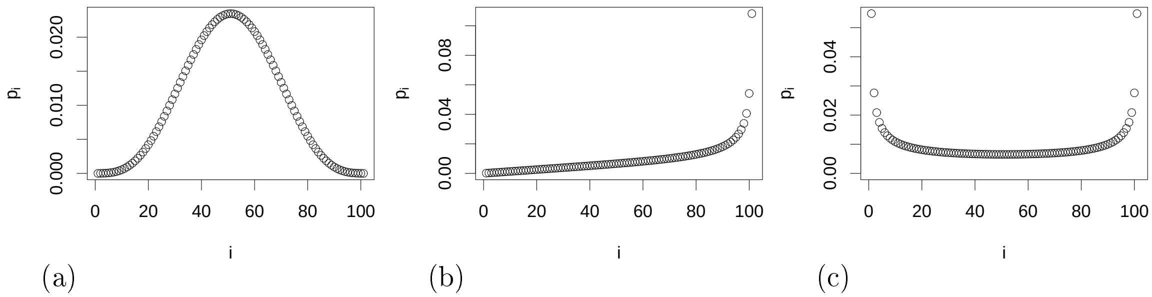

Figure 2.2: Simulations of the master equations. $N = 100$, $\nu = 1$, initial condition is the uniform distribution, simulated time $t = 8000$ (s.t. we have an approximately constant solution). (a) $N^a = 5$, $N^b = 5$, (b) $N^a = 2$, $N^b = 0.5$, (c) $N^a = 0.5$, $N^b = 0.5$.

Invariant measure of the zealot model

We are interested in the invariant measure of the zealot model. We follow [41], and use the detailed balance equation, which we introduce next.

Theorem 2.8 *Consider a time-continuous Markov process on a finite state space I. The transition rate from $\sigma \to \sigma'$ for $\sigma, \sigma' \in I$ given by $\rho(\sigma \to \sigma')$. A probability measure P on I is a stationary distribution, if*

$$\rho(\sigma \to \sigma')\, P(\sigma) = \rho(\sigma' \to \sigma)\, P(\sigma'). \tag{2.6}$$

Equation (2.6) is known as the detailed balance equation.
Proof: (of theorem 2.8). Let X_t the I-valued stochastic process, and $p_\sigma(t) = P(X_t = \sigma)$. The master equation of this stochastic process is given by

$$\begin{aligned} \frac{d}{dt} p_\sigma(t) &= -\Big(\sum_{\sigma' \in I \setminus \{\sigma\}} \rho(\sigma \to \sigma') \Big) p_\sigma(t) + \sum_{\sigma' \in I \setminus \{\sigma\}} \rho(\sigma' \to \sigma)\, p_{\sigma'}(t) \\ &= \sum_{\sigma' \in I \setminus \{\sigma\}} \Big(\rho(\sigma' \to \sigma)\, p_{\sigma'}(t) - \rho(\sigma \to \sigma') p_\sigma(t) \Big). \end{aligned}$$

From (2.6) we obtain that

$$\rho(\sigma \to \sigma')\, P(\sigma) = \rho(\sigma' \to \sigma)\, P(\sigma').$$

Therefore, $P(\sigma)$ is a stationary solution of the master equation, and thus an invariant distribution.

$\square$

Before we start with defining the invariant measure, we introduce the Pochhammer symbol, and prove an important property.

Definition 2.9 *For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we define the Pochhammer symbol of x by $(x)_{(0)} = 1$, and for $n > 1$*

$$(x)_{(n)} = x\,(x+1)\,\ldots\,(x+n-1). \tag{2.7}$$

It is remarkable that the well known binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

extends to the Pochammer symbol.

Proposition 2.10 *For $x, y \in \mathbb{R}$, $n \in \mathbb{N}_0$, we have*

$$(x+y)_{(n)} = \sum_{k=0}^{n} \binom{n}{k} x_{(k)} y_{(n-k)}. \tag{2.8}$$

Proof: For $n = 0$, the equation is trivially satisfied as both sides of the equation collapse to 1. To show formula (2.19) for $n \in \mathbb{N}$, we use induction over n. For $n = 1$, we find (recall $x_{(0)} = 1$)

$$(x+y)_{(1)} = (x+y) = x_{(1)}\,y_{(0)} + x_{(0)}\,y_{(1)} = \sum_{k=0}^{1} \binom{n}{k} x_{(k)} y_{(n-k)}.$$

Now assume that the formula is true for $n \in \mathbb{N}$. Then,

$$\begin{aligned}
(x+y)_{(n+1)} = (x+y+n)\,(x+y)_{(n)} &= (x+y+n) \sum_{k=0}^{n} \binom{n}{k} x_{(k)} y_{(n-k)} \\
= \quad & \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\,y_{(n-k)}(\,(x+k) + (y+n-k)\,) \\
= \quad & \sum_{k=0}^{n} \binom{n}{k} x_{(k+1)}\,y_{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\,y_{(n-k+1)} \\
= \quad & \sum_{k=1}^{n+1} \binom{n}{k-1} x_{(k)}\,y_{(n-k+1)} + \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\,y_{(n-k+1)} \\
= \quad & \binom{n}{0} x_{(0)}\,y_{(n+1)} + \sum_{k=1}^{n+1} \left\{ \binom{n}{k-1} + \binom{n}{k} \right\} x_{(k)}\,y_{(n-k+1)} + \binom{n}{n} x_{(n+1)}\,y_{(0)} \\
= \quad & \binom{n}{0} x_{(0)}\,y_{(n+1)} + \sum_{k=1}^{n+1} \binom{n+1}{k} x_{(k)}\,y_{(n+1-k)} + \binom{n}{n} x_{(n+1)}\,y_{(0)} \\
= \quad & \sum_{k=0}^{n+1} \binom{n+1}{k} x_{(k)} y_{(n+1-k)}
\end{aligned}$$

□

Note that the detailed balance equation checks that the flow between any of two states is balanced, and not only the overall probability flow. Hence, this equation is rather strict. The detailed balance equation is a sufficient but not a necessary condition for an invariant probability measure.

Theorem 2.11 *Let $N^a, N^b > 0$, and $N \in \mathbb{N}$, and X the number of voters for party A in the invariant distribution of the zealot model.*

$$P(X=n) \quad = \quad \binom{N}{n} \frac{(N^a)_{(n)}\ (N^b)_{(N-n)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}}. \tag{2.9}$$

where $x_{(k)} = x\,(x+1)\,\cdots\,(x+k-1)$ is the Pochhammer symbol; we define $x_{(0)} = 1$.

Proof. We have two taks to accomplish: We show that the probabilities are well defined (non-negative and sum up to one), and we show that the detailed balance equation holds truw. We start with the second task.
As transitions in the zealot model either increase or decrease X by one, we only need to check if

$$T_{n\to n+1} := P(X=n)\,\rho(n\to n+1) = P(X=n+1)\,\rho(n+1\to n) =: T_{n+1\to n}.$$

Now,

$$\begin{aligned}
T_{n\to n+1} \quad &= \quad \frac{N!}{n!\,(N-n)!}\,\frac{(N^a)_{(n)}\ (N^b)_{(N-n)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}}\ \nu\,(N-n)\ \frac{n+N^a}{N+N^a+N^b}\\
&= \quad \frac{N!}{n!\,(N-(n+1))!}\,\frac{(N^a)_{(n+1)}\ (N^b)_{(N-n)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}}\ \frac{\nu}{N+N^a+N^b}
\end{aligned}$$

and

$$\begin{aligned}
T_{n+1\to n} \quad &= \quad \frac{N!}{(n+1)!\,(N-(n+1))!}\,\frac{(N^a)_{(n+1)}\ (N^b)_{(N-(n+1))}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}}\ \nu\,(n+1)\ \frac{N-(n+1)+N^b}{N+N^a+N^b}\\
&= \quad \frac{N!}{n!\,(N-(n+1))!}\,\frac{(N^a)_{(n+1)}\ (N^b)_{(N-n)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}}\ \frac{\nu}{N+N^a+N^b}.
\end{aligned}$$

That is, we indeed find $T_{n\to n+1} := P(X=n)\,\rho(n\to n+1) = P(X=n+1)\,\rho(n+1\to n) =: T_{n+1\to n}$. Additionally, we show that the terms $P(X=n)$ indeed define a probability measore on $\{0,\ldots,N\}$. Clearly, $P(X=n)\geq 0$. Furthermore, the probabilities sum up to 1:

$$\sum_{n=0}^{N} P(X=n) \quad = \quad \frac{\sum_{n=0}^{N}\binom{N}{n}\ (N^a)_{(n)}\ (N^b)_{(N-n)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}} = \frac{(N^a)_{(N)}\ +\ (N^b)_{(N)}}{(N^a)_{(N)}\ +\ (N^b)_{(N)}} = 1.$$

where we used formula (2.19) to simplify the sum.

□

Limit with weak effects

Theorem 2.11 yields a handy characterization of the individual based zealot model. However, often the number of voters is large, and rather the vote shares than the absolute number of voters of a party are investigated. Thereto we develop two different limits of the distribution of $N\to\infty$ that differ in the scaling of the zealot's number with the population size N. We start with the assumption that the number of zealots is constant, and does not increase with the population size N. In that, the relative number of zealots become smaller and smaller, which satisfies the term "weak effects" for that case. As the relative frequency of floating voters tend to zero, naively we expect that also the effect of zealots tend to zero. However, the voter model (which is the limiting model if we take the number of zealots to zero) is completely undecided w.r.t. the two opinions. Hence, even the minimal (and in the limit $N \to \infty$ zero) influence by zealots still affect the invariant measure. Mobilia [34] perhaps was the first who did note that result in the setting of a spatially structured model. In population genetics, weak effects are a commonly used and convenient tool to investigate models with large population size. We consider two different ways to derive that limes: Either to directly scale the invariant distribution given in Theorem 2.11, or via a Fokker-Planck equation and the stationary solution of that equation.

Before we state the theorem we recall the definition of the (real) Γ-function, and some properties. The real Γ- function is defined for $\mathbb{R}_+$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt.$$

Partial integration yields $\Gamma(1+x) = x\Gamma(x)$, which implies for $n \in \mathbb{N}$, $a \in \mathbb{R}_+$ that

$$n! = \Gamma(n+1), \qquad a_{(n)} = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.10}$$

There is a handy approximation of the Γ-function for large, real arguments: the Stirling's formula,

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\, \left(\frac{x}{e}\right)^x\; e^{\zeta(x)}$$

where $\zeta(x)$ is a function with $0 \leq \zeta(x) \leq 1/(12\,x)$.

Theorem 2.12 *Consider the vote shares $x = X/N$ for the invariant measure of the zealot model. For $N \to \infty$, the distribution of x converges in distribution to a Beta distribution Beta(N^a, N^b).*

Proof: We may rewrite the invariant distribution for the individual-based zealot model as

$$P(X=n) = \binom{N}{n}\,\frac{(N^a)_{(n)}\;(N^b)_{(N-n)}}{(N^a)_{(N)}\;+\;(N^b)_{(N)}} = C(M, N^a, N^b)\;\frac{(N^a)_{(n)}}{n!}\;\frac{(N^b)_{(N-n)}}{(N-n)!}$$

were the constant $C(N, N^a, N^b)$ (which depends on the parameters N, N^a and N^b) ensures that the probabilities sum up to 1. We are interested in the limit

$$\lim_{N\to\infty} P(x < a) = \lim_{N\to\infty} P(X < a\,N)$$

for $a \in [0,1]$. Note that all multiplicative terms that only depend on N, N^a, and N^b can be absorbed by the global normalization constant $\hat{C}(N, N^a, N^b)$. That is, we are only interested in equality up to multiplicative functions that (asymptotically) do only depend on those model parameters. In order to emphasize this fact, we write $\equiv_a$ if we neglect multiplicative factors.
If we express the factorial and the Pochhammer symbol by Γ-functions,

$$\frac{(N^a)_{(n)}}{n!} \quad = \quad \frac{\Gamma(N^a+n)\Gamma(n)}{\Gamma(N^a)\,\Gamma(n+1)\Gamma(n)} = \frac{1}{n}\;\frac{\Gamma(N^a+n)}{\Gamma(N)\,\Gamma(n)}.$$

Furthermore, we consider the limit fo $(N^a)_{(n)}/n!$ for $N \to \infty$ and $n \to \infty$, where $z = n/N$ is constant,

$$
\begin{aligned}
\frac{(N^a)_{(n)}}{n!} &\equiv_a \frac{1}{z}\frac{\Gamma(N^a+N\,z)}{\Gamma(N^a)\;\Gamma(z)} \equiv_a z^{-1}\;\frac{\sqrt{\frac{2\pi}{N^a+N\,z}}\;\left(\frac{N^a+N\,z}{e}\right)^{N^a+N\,z}}{\sqrt{\frac{2\pi}{N^a}}\;\left(\frac{N^a}{e}\right)^{N^a}\;\sqrt{\frac{2\pi}{N\,z}}\;\left(\frac{N\,z}{e}\right)^{N\,z}} \\
&\equiv_a z^{-1}\;\underbrace{\sqrt{\frac{N\,z}{N^a+N\,z}}}_{\to 1}\;(N^a+N\,z)^{N^a}\;\left\{\underbrace{\left(1+\frac{1}{N}\;\frac{N^a}{z}\right)^N}_{\to e^{N^a/z}}\right\}^z \\
&\equiv_a z_\ell^{-1}\;\underbrace{(N^a/N+z)^{N^a}}_{\to z^{N^a}}\;e^{(N^a/z)\,z} \equiv_a z^{N^a-1}.
\end{aligned}
$$

Similarly, we obtain

$$
\frac{(N^b)_{(N-n)}}{(N-n)!} \quad \equiv_a \quad (1-z)^{N^b-1}.
$$

Therewith,

$$
\begin{aligned}
\lim_{N\to\infty} P(x<a) &= \lim_{N\to\infty} \tilde{C}(N,N^a,N^b)\sum_{i\leq aN}\frac{1}{N}(i/N)^{N_a-1}\,(1-i/N)^{N^b-1}i/N \\
&= C\int_0^a z^{N_a-1}\,(1-z)^{N^b-1}\,dz.
\end{aligned}
$$

That is, the probability distribution of x is proportional to $z^{N_a-1}\,(1-z)^{N^b-1}$, which gives us a Beta distribution with parameters N^a and N^b.

□

As discussed above, we might also use a different route to the invariant measure: We can go via the Fokker-Planck equation. The stationary solution of that equation is the invariant measure of the Zealot-model under the given weak-effects scaling.

Theorem 2.13 *The Fokker-Planck equation for the scaled zealot model reads*

$$
u_T(T,x) \quad = \quad -\,\partial_x\Big((N^a - x(N^a+N^b))\,u(T,x)\Big) + \partial_x^2\Big(x(1-x)\,u(T,x)\Big) \tag{2.11}
$$

and posses the stationary solution

$$
u(x) \quad = \quad C\,x^{N_a-1}\,(1-x)^{N_b-1}. \tag{2.12}
$$

Proof: We start off with the master equations, as stated in proposition 2.6,

$$
\begin{aligned}
\frac{d}{dt}p_i &= -\nu\left(\frac{i+N^a}{N^a+N^b+N}\;\frac{N-i}{N}+\frac{N-i+N^b}{N^a+N^b+N}\;\frac{i}{N}\right)p_i \\
&\quad +\nu\left(\frac{i-1+N^a}{N^a+N^b+N}\;\frac{N-i+1}{N}\right)p_{i-1}+\nu\left(\frac{N-i-1+N^b}{N^a+N^b+N}\;\frac{i+1}{N}\right)p_{i+1}
\end{aligned}
$$

and assume that there is a smooth function $u(t,x)$ which approximates the p_i, $p_i \approx h\,u(t,x)$ for $x = i\,h$ and $h = 1/N$. We find

$$
\begin{aligned}
\partial_t h\,u(t,x) \;\approx\;& -\nu\left(\frac{N\,x+N^a}{N^a+N^b+N}\,(1-x)+\frac{N(1-x)+N^b}{N^a+N^b+N}\,x\right)h\,u(t,x)\\
&+\nu\left(\frac{N(x-h)+N^a}{N^a+N^b+N}\,(x-h)\right)h\,u(t,x-h)\\
&+\nu\left(\frac{N(1-(x+h))+N^b}{N^a+N^b+N}\,(1-(x+h))\right)h\,u(t,x+h)\\
=\;& -\nu\left(\frac{x+h\,N^a}{1+h\,N^a+h\,N^b}\,(1-x)+\frac{(1-x)+h\,N^b}{1+h\,N^a+h\,N^b}\,x\right)h\,u(t,x)\\
&+\nu\left(\frac{(x-h)+h\,N^a}{1+h\,N^a+h\,N^b}\,(1-(x-h))\right)h\,u(t,x-h)\\
&+\nu\left(\frac{(1-(x+h))+h\,N^b}{1+h\,N^a+h\,N^b}\,(x+h)\right)h\,u(t,x+h)\\
=\;& -\nu\Big(2\,x(1-x)\Big)\,h\,u(t,x)+\nu\Big((x-h)(1-(x-h))\Big)\,h\,u(t,x)\\
&+\nu\Big((1-(x+h))(x+h)\Big)\,h\,u(t,x)\\
&-\nu\,h\left(\frac{N^a-x\,(N^a+N^b)}{1+h\,N^a+h\,N^b}\,(1-x)+\frac{N^b-(1-x)(N^a+N^b)}{1+h\,N^a+h\,N^b}\,x\right)h\,u(t,x)\\
&+\nu\,h\left(\frac{N^a-(x-h)(N^a+N^b)}{1+h\,N^a+h\,N^b}\,(1-(x-h))\right)h\,u(t,x-h)\\
&+\nu\,h\left(\frac{N^b-(1-(x+h))(N^a+N^b)}{1+h\,N^a+h\,N^b}\,(x+h)\right)h\,u(t,x+h).
\end{aligned}
$$

Taylor expansion yields

$$
\begin{aligned}
&-\nu\Big(2\,x(1-x)\Big)\,h\,u(t,x)+\nu\Big((x-h)(1-(x-h))\Big)\,h\,u(t,x)\\
&+\nu\Big((1-(x+h))(x+h)\Big)\,h\,u(t,x)=\nu\,h^2\,\partial_x^2\Big(x(1-x)\,h\,u(t,x)\Big)+\mathcal{O}(h^4).
\end{aligned}
$$

The second part of our formula is more subtle.

$$
\begin{aligned}
& -\nu\, h\, \left(\frac{N^a - x\,(N_a+N_b)}{1+h\,N^a+h\,N^b}\;(1-x) + \frac{N^b-(1-x)(N^a+N^b)}{1+h\,N^a+h\,N^b}\;x\right)\;h\,u(t,x)\\
& +\nu\, h\, \left(\frac{N^a-(x-h)(N^a+N^b)}{1+h\,N^a+h\,N^b}\;(1-(x-h))\right)\;h\,u(t,x-h)\\
& +\nu\, h\, \left(\frac{N^b-(1-(x+h))(N^a+N^b)}{1+h\,N^a+h\,N^b}\;(x+h)\right)\;h\,u(t,x+h)\\
= \;& \frac{-\,\nu\,h^2}{(1+h(N^a+N^b))}\,\partial_x\bigg((N^a-x(N^a+N^b))\;(1-x)\,h\,u(t,x)\\
& \qquad\qquad -(N^b-(1-x)(N^a+N^b))\,x\,h\,u(t,x)\bigg)+\mathcal{O}(h^4)\\
= \;& -\,\nu\,h^2\;\;\partial_x\bigg((N^a-x(N^a+N^b))\,h\,u(t,x)\bigg)+\mathcal{O}(h^4)
\end{aligned}
$$

If we neglect the terms of order $\mathcal{O}(h^4)$, devide by h, and scale time, $T = t\,\nu\,h^2$, we obtain the Fokker-Planck equation

$$
u_T(T,x) \;\;=\;\; -\,\partial_x\bigg((N^a-x(N^a+N^b))\,u(T,x)\bigg)+\partial_x^2\bigg(x(1-x)\,u(T,x)\bigg).
$$

If we plug the function $u(x) = Cx^{N^a-1}(1-x)^{N^b-1}$ into that equation, we find that this function is a stationary solution of the PDE.

$\square$

Limit with strong effects

In the second limit, we assume that the zealots scale with the number of individuals,

$$
N^a = N\,n^a,\quad N^b = N\,n^b
$$

where n^a and n^b are fixed parameters. While in the weak effects limit the relative number of zealots, and therewith their effect, becomes smaller and smaller with increasing population size N, in the present case, the relative strength of that effect stays constant. As random fluctuations are likely to cancel in large populations, the noise tends to zero and we approach (for N large) a normal distribution with small variance, and (for $N\to\infty$) a deterministic situation.

We derive the Fokker-Planck equation for that scaling.

Proposition 2.14 *Let $x_t = X_t/N$, $n^a = N^a/N$, and $n^b = N^b/N$. For N large, the distri-*

bution $u(x,t)$ of x_t is well approximated by the Fokker-Planck equation

$$\begin{aligned} u_t(x,t) &= -\nu\,\partial_x\Big\{\left(\frac{-n^b\,x+n^a\,(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\} \\ &\quad +\frac{\nu}{2N}\partial_x^2\Big\{\left(\frac{(1-x\,+n^b)x+(x\,+n^a)(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\}. \end{aligned} \tag{2.13}$$

Proof: Let $h=1/N$, $x=h\,i$, and $u(t,x)=h\,p_i(t)$. We have $N^{a/b}=N\,n^{(a/b)}$. Taylor expansion yields

$$\begin{aligned} u_t(t,x) &= -\nu\left(\frac{x\,+n^a}{n^a+n^b+1}\,(1-x)+\frac{(1-x)\,+n^b}{n^a+n^b+1}\,x\right)u(t,x) \\ &\quad +\nu\left(\frac{(x-h)+n^a}{n^a+n^b+1}\,(1-x+h)\right)u(t,x-h)+\nu\left(\frac{1-(x+h)+n^b}{n^a+n^b+1}\,(x+h)\right)u(t,x+h) \\ &= \nu\partial_x\Big\{\left(\frac{(1-x\,+n^b)x-(x\,+n^a)(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\} \\ &\quad +\frac{\nu}{2N}\partial_x^2\Big\{\left(\frac{(1-x\,+n^b)x+(x\,+n^a)(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\}+\mathcal{O}(h^2) \\ &= \nu\,\partial_x\Big\{\left(-\frac{-n^b\,x+n^a\,(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\} \\ &\quad +\frac{\nu}{2N}\partial_x^2\Big\{\left(\frac{(1-x\,+n^b)x+(x\,+n^a)(1-x)}{n^a+n^b+1}\right)u(t,x)\Big\}+\mathcal{O}(h^2). \end{aligned}$$

The result follows if we drop the term $\mathcal{O}(h^2)$.

□

Basically, we now follow methods of Grasman and Herwaarden [20], and consider the equilibrium distribution of the Fokker-Planck equation (2.13). The drift part of the Fokker-Planck equation yields the ODE

$$\dot{x}=\nu\,\partial_x\left(\frac{-n^b\,x+n^a\,(1-x)}{n^a+n^b+1}\right),$$

which corresponds to the ODE (2.4). We find that

$$x(t)\to\frac{n_a}{n_a+n_b}\qquad\text{for } t\to\infty.$$

That is, if we formally take $N\to\infty$ in the Fokker-Planck equation, then the distribution $u(x,t)$ approximates a point mass at $n_a/(n_a+n_b)$,

$$u(x)\to\delta_\mu,\qquad \mu=\frac{n^a}{n^a+n^b}.$$

If N is finite but large, the delta peak is replaced by a normal distribution with mean μ; the variance of that normal distribution does vanish for $N\to\infty$. To characterize this normal distribution completely, we determine the variance explicitly. We know that for N large,

$$u(x)\approx Ce^{-(x-\mu)^2/(2\sigma^2)}$$

If we plug in this ansatz in the flux of the Fokker-Planck equation (2.13), we find

$$
\begin{aligned}
0 &\approx (n^a+n^b)(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)} + \frac{1}{2N}\frac{d}{dx}\left\{\left(2(1-x)x\ +n^bx+n^a(1-x)\right)e^{-(x-\mu)^2/(2\sigma^2)}\right\}\\
&= (n^a+n^b)(x-\mu) + \frac{1}{2N}\left\{\left(2(1-2x)\ +n^b-n^a\right)+\left(2(1-x)x\ +n^bx+n^a(1-x)\right)\ \frac{-(x-\mu)}{\sigma^2}\right\}\\
\sigma^2 &= \frac{[2(1-x)x\ +n^bx+n^a(1-x)]\,(x-\mu)/(2N)}{-[2(1-x)x\ +n^bx+n^a(1-x)]/(2N)+(n^a+n^b)(x-\mu)}\\
&= \frac{1}{2N}\frac{2(1-x)x\ +n^bx+n^a(1-x)}{(n^a+n^b)}+\mathcal{O}(N^{-2})\\
&= \frac{(1-x)x}{N}\ \frac{1+n^b/(2(1-x))+n^a/(2x)}{(n^a+n^b)}+\mathcal{O}(N^{-2}).
\end{aligned}
$$

We evaluate this term at $x=\mu$ and obtain for the leading order in N (recall that $\mu=n^a/(n^a+n^b)$, $1-\mu=n^b/(n^a+n^b)$)

$$
\begin{aligned}
\sigma^2 &= \frac{\mu(1-\mu)}{N}\ \frac{1\ +n^b/(2(1-\mu))+n^a/(2\mu)}{(n^a+n^b)}\\
&= \frac{\mu(1-\mu)}{N}\ \frac{1\ +(n^a+n^b)}{(n^a+n^b)} = \frac{\mu(1-\mu)}{N}\ \left(1+\frac{1}{n^a+n^b}\right)
\end{aligned}
$$

Particularly, the variance diverges if n_a+n_b tends to zero. If we set $n^a+n^b=0$, we recover the voter model on a complete graph (or a Moran model), and here the invariant measure is concentrated on $x=0$ and $x=1$; in this case, the approximation by a normal distribution simply breaks down. We have the following corollary.

Corollary 2.15 *If N is large, the invariant measure of X_t/N is well approximated by a normal distribution with mean μ and variance σ^2 given by*

$$
\mu = \frac{n_a}{n_a+n_b} \tag{2.14}
$$

$$
\sigma^2 = \frac{\mu(1-\mu)}{N}\ \left(1+\frac{1}{n^a+n^b}\right). \tag{2.15}
$$

In [5], the authors use the invariant distribution of X_t directly, and obtain

$$
\sigma^2 = \frac{\mu(1-\mu)}{N}\ \left(\frac{N^a+N^b}{N^a+N^b+1}+\frac{N}{N^a+N^b+1}\right)
$$

which coincides with our result for $N\to\infty$ under the scaling of N^a, N^b that we did assume.

Social interaction coefficient

Our main aim is the identification of consequences of communication within the population. We have the external forces given, the zealots. Without the internal communication, the zealots determine the opinion of an individual completely. This is the case for $n^a+n^b\to\infty$, which corresponds to $N^a + N^b \gg N$. An individual will always (a.s) select randomly a zealot, and never a fellow individual, to copy the opinion. The distribution of the A-supporter's number is simply given by a binomial distribution $\text{Bin}(N, \mu)$. For N large, the fraction of individuals that adopt opinion A approximates a normal distribution $\mathcal{N}(\mu, \sigma_0)$,

$$\mu = \frac{n^a}{n^a + n^b}, \qquad \sigma_0 = \frac{\mu(1-\mu)}{N}$$

If we compare the two distributions, we find that the communication process within the population increases the variance. This excess variance can be attributed to local interactions within the population. In that, it is possible to quantify the effect of social interactions.

Definition 2.16 *The social interaction coefficient (SI) for the two-opinion zealot model is defined by*

$$SI = \sigma/\sigma_0 = 1 + \frac{1}{n^a + n^b}.$$

Then, SI = 1 indicates no social interactions, SI > 1 refers to social interactions. This expression can be easily estimated form data, s.t. we can use data to measure the strength of social interactions (see Sect. 2.3.2).

The Sano model

Until now, the zealots have been external forces. They do not take part in the election, but only influence the election. Sano [41] proposed to consider zealots as part of the voters. That is, there are floating voters (these persons we addressed till now), and fixed voters (the zealots). The number of individuals in those two classes amount to N, the total population size. Furthermore, Sano assumes that not all zealots are indeed actively interact with the floating voters; some of the zealots are completely persuaded by their opinion, but stay only at home as couch-potatoes. The Sano-model has two additional parameters ϑ_a, ϑ_b which denote the fraction of zealots that actively influence the dynamics: Part of the zealots are thoght to be fixed in their opinion but do not take part in the political discussions. Only a fraction of zealots, $\vartheta_a N^a$ resp. $\vartheta_b N^b$, will influence floating voters.
In that, the Sano model is not really different from the zealot model, but we need to re-interpret the results we obtained till now. Also in the Sano model, we still have N floating voters that follow a zealot model, where $X_t \in \{0, \ldots, N\}$ denote supporters of party A among the floating voters. Therewith, the vote share x of party A is given by

$$x = \frac{N^a + X_t}{N + N^a + N^b}$$

while the vote share for party B reads

$$1 - x = \frac{N^b + N - X_t}{N + N^a + N^b}.$$

We are interested in the weak effects limit. Thereto, we slightly rewrite the vote share,

$$\begin{aligned} x &= \frac{N^a}{N + N^a + N^b} + \frac{X_t}{N} \frac{N}{N + N^a + N^b} \\ \Rightarrow \quad x &\sim \frac{N^a}{N + N^a + N^b} + \frac{N}{N + N^a + N^b} \text{Beta}(\vartheta_a N^a, \vartheta_b N^b) \end{aligned}$$

where we note that the offset $N^a/(N+N^a+N^b)$ and the factor $N/(N+N^a+N^b)$ are model parameters and non-random. The weak-effects limit of the vote share according to the Sano-model is hence well described by a shifted and scaled Beta distribution.

Exercises

Before we state the exercises, we introduce the noisy voter model.

Model 2.17 [Noisy voter model] *The noisy voter model with population size $N \in \mathbb{N}$ (2 groups A and B, and no spatial structure) has the state $X_t \in \{0, \ldots, N\}$ that counts the number of supporters of group A, while $N - X_t$ is the number of supporters of opinion B. At rate ν individuals potentially change their opinion. Is that the case, the focal individual copies with probability u the opinion of a randomly chosen individual (including selfing). With probability $1-u$, the individual adopts some opinion, independently of the state of the population. An individual that has been group A before, might either become a B-individual (probability $\pi_{A,B}$) or remains an A-individual (probability $\pi_{A,A}$, where $\pi_{A,A}+\pi_{A,B}=1$). Similarly, a B-individual may either become an A-individual ($\pi_{B,A}$) or remains a B-individual ($\pi_{B,B}$, where $\pi_{B,A}+\pi_{B,B}=1$).*

Exercise 3.1: What are the transition rates $X_t \to X_t+1$ and $X_t \to X_t-1$ for the noisy voter model?

Solution of Exercise 3.1 According to the description of the noisy voter model we have a transition from group B to group A, if an individual of group B rethinks his/her opinion (rate $\nu(N-X_t)$), and he/she copies faithfully the opinion of a group A-individual (probability uX_t/N), respectively chooses independently of all individual the opinion A (probability $(1-u)\pi_{B,A}$). In the decrease of X_t, group A and group B exchange their role. Hence,

$$\begin{aligned} X_t \to X_t+1 \quad &\text{at rate} \quad \nu\,(N-X_t)\left(u\,\frac{X_t}{N} + (1-u)\,\pi_{B,A}\right) \\ X_t \to X_t-1 \quad &\text{at rate} \quad \nu\,X_t\left(u\frac{N-X_t}{N} + (1-u)\,\pi_{A,B}\right). \end{aligned}$$

Exercise 3.2: (a) Show that the zealot model 2.5 is a noisy voter model (that is, we can find u and $\pi_{i,\ell}$ such that the zealot model and the noisy voter model coincide).
(b) Show that not every noisy voter model is a zealot model.

Solution of Exercise 3.2 (a) Given a zealot model with population size N, and N_A (N_B) fixed voters for opinion A (opinion B), we define $u = N/(N+N_A+N_B)$, $\pi_{A,A} = \pi_{B,A} = N_A/(N_A+N_B)$, and $\pi_{A,B} = \pi_{B,B} = N_B/(N_A+N_B)$. Then,

$$\begin{aligned} &\nu\,(N-X_t)\left(u\,\frac{X_t}{N} + (1-u)\,\pi_{B,A}\right) \\ =\;& \nu\,(N-X_t)\left(\frac{N}{N+N_A+N_B}\;\frac{X_t}{N} \;+\; \frac{N_A+N_B}{N+N_A+N_B}\;\frac{N_A}{N_A+N_B}\right) \\ =\;& \nu\,(N-X_t)\,\frac{X_t+N_A}{N+N_A+N_B} \end{aligned}$$

and similarly,

$$\begin{aligned}
&\nu\, X_t\left(u\,\frac{N-X_t}{N}+(1-u)\,\pi_{A,B}\right)\\
=\;&\nu\, X_t\left(\frac{N}{N+N_A+N_B}\;\frac{N-X_t}{N}\;+\;\frac{N_A+N_B}{N+N_A+N_B}\;\frac{N_B}{N_A+N_B}\right)\\
=\;&\nu\, X_t\,\frac{N-X_t+N_B}{N+N_A+N_B}.
\end{aligned}$$

Hence, the noisy voter model with the parameters u, $\pi_{9i,j}$ as defined above is identical with the given zealot model.
(b) Note that $\pi_{i,A}+\pi_{i,B}=1$ ($i\in\{A,B\}$), but$\pi_{A,B}$ and $\pi_{B,A}$ do not necessarily add up to one. For any zealot model, given that a person does not ask another floating voter, the probability for group A always is $N_A/(N_A+N_B)$, while that for group B reads $N_B/(N_A+N_B)$, such that

$$\pi_{A,B}+\pi_{B,A}=\frac{N_B}{N_A+N_B}+\frac{N_A}{N_A+N_B}=1.$$

That is, only noisy voter models with $\pi_{A,B}+\pi_{B,A}=1$, can be written as zealot models.

Exercise 3.3: Show that a noisy voter model can be written as a zealot model if and only if its invariant measure satisfies the detailed balance equation.

Solution of Exercise 3.2 **Beweisen!!**

2.3.2 Zealot models and data

Concerning the comparison of our theory with empirical data, the theory particularly implies three interesting results: In an election with a dichotomous outcome, many voters, and no spatial influence, we predict (a) that the election results follow a beta distribution (zealot model) or (b) a scaled beta distribution (Sano model). Furthermore, we have a tool (c) to investigate the effect of within-population communication by means of the social interaction coefficient (SI).

Zealot model and election results

If we inspect the fitted data (Fig. 2.3), at the first glance, the zealot model does not perform badly. The overall shape of the data always is met. A closer look reveals that in several cases the center of the data is underestimated, and the tails are overestimated. Seemingly, there is a mechanism that concentrates the election results around the center, at least in some elections, that the zealot model does not cover. If we inspect the results of the Kolmogorov-Smirnov test for beta distribution, in half of the cases (FRG 1983, US 2008, NL 2017), the hypothesis that the data are beta-distributed cannot be rejected, in the other half (US 2016, Brexit, France 2017) it is rejected. For the results in France 2017, this result seems to contradict the graphical representation: The beta distribution fits the data very well. However, as there are many data used, even small derivations can be detected – the result of the Kolmogorov-Smirnov test is not too important in this case. It would be more informative to compare different models, proposing different mechanisms.
One aspect of importance has been neglected in all the data: the spatial structure. It is very well possible that the number or the effect of the zealots undergo a spatial variation. Particularly for

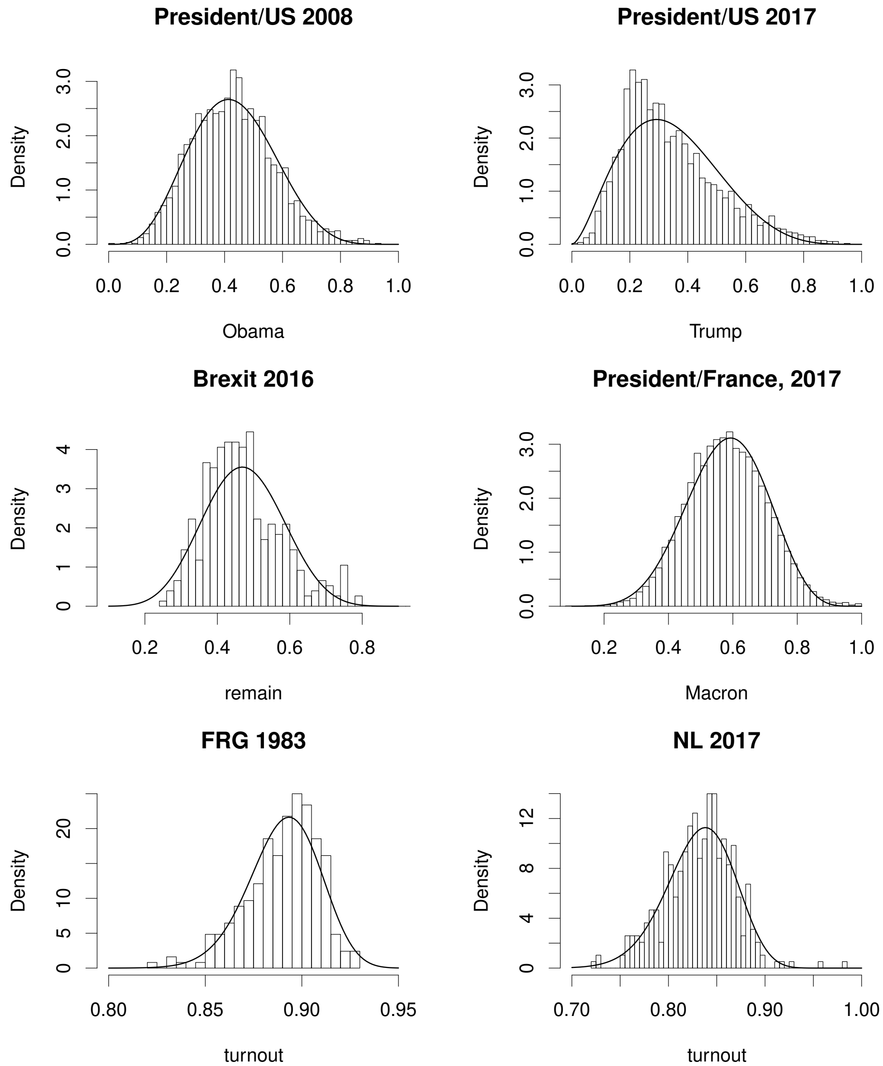

Figure 2.3: Dichotomous elections or dichotomous aspects of elections. Together with the p-values of the Kolmogorov-Smirnov test for beta distribution: US presidential elections 2008 ($p = 0.15$) US presidential elections 2017 ($p = 5.32e$-07) Brexit referendum ($p = 0.0035$) US presidential France 2017 ($p = 3.42e$-10) turnout rates, FRG, 2017 ($p = 0.482$) turnout rates, NL, 2017 ($p = 0.91$).

the Brexit data, this effect is likely to be present – different regions (Scotland and the Midlands, say) have a completely different voting behavior. In consequence, we observe the superposition of several beta distributions, which could explain the bad performance of the zealot model with respect to the Kolmogorov-Smirnov test. Another possibility are additional mechanisms that are not considered in the strict version of the zealot model, as those discussed in the Sano model.

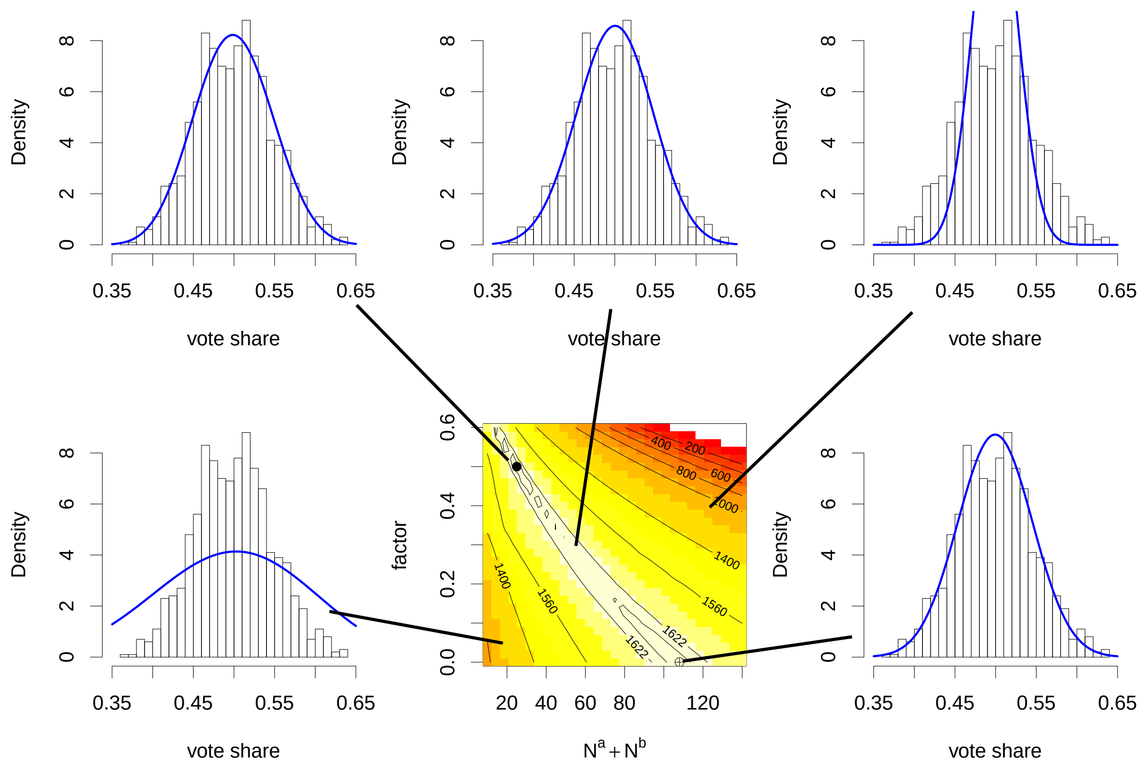

Figure 2.4: Analysis of simulation data (offset = 0.3, factor = 0.5, Beta(10,15) with 1000 realizations). Center: Heat- and contour plot of the log-likelihood (factor and $N^a + N^b$ fixed, the other parameters are used to maximize the likelihood). Additionally, histograms of the simulated data set are shown, superimposed by the probability density for the indicated parameters. Two local maxima are tagged, one by a bullet (that are the parameters used to generate the data), and another one by a crossed circle (which corresponds to a pure zealot model).

The Sano model is a refinement of the zealot model in that more effects are taken into account: As the zealots are part of the population, and only part of the zealots are active, we do not have a pure beta distribution to describe the data, but a scaled beta distribution. The estimation of the parameters is kind fo subtle.

$\boxed{\textit{Excursion:}}$ <u>Maximum-Likelihood estimation for the Sano model</u>
We solve the formula

$$x \sim \frac{N^a}{N + N^a + N^b} + \frac{N}{N + N^a + N^b}\,\mathrm{Beta}(\vartheta_a\, N^a, \vartheta_b\, N^b)$$

for the beta distribution, and obtain

$$\frac{(x - \text{offset})}{\text{factor}} \sim \text{Beta}(\hat{N}^a, \hat{N}^b),$$

where

$$\begin{aligned}
\text{offset} &= \frac{N^a}{N + N^a + N^b} \in (0,1) \\
\text{faktor} &= \frac{N}{N + N^a + N^b} \in (0,1) \\
\hat{N}^* &= \vartheta_* N^* \qquad \text{with } * \in \{a, b\}.
\end{aligned}$$

We have the natural restriction $\text{offset} + \text{factor} < 1$. The log-likelihood $LL(x)$ for a data point x is given by (see exercise 3.4)

$$\begin{aligned}
\mathsf{LL}(x) &= (\hat{N}^a - 1)\ln\left(\frac{(x - \text{offset})}{\text{factor}}\right) + (\hat{N}^b - 1)\,\ln\left(1 - \frac{(x - \text{offset})}{\text{factor}}\right) \\
&\quad + \ln\left(\frac{\Gamma(\hat{N}^a)\,\Gamma(\hat{N}^b)}{\Gamma(\hat{N}^a + \hat{N}^b)}\right) - \ln(\text{factor}).
\end{aligned} \tag{2.16}$$

It is interesting to note that we obviously cannot identify all parameters from the estimation. We estimate 4 parameters ($\hat{N}^a$, $\hat{N}^b$, factor, and offset), but we have 5 model parameters (N^a, N^b, N, ϑ_1, and ϑ_2). In any case, the parameter factor is precisely the fraction of floating voters in the system, and hence is of special interest.

In order to produce Fig. 2.5, we only focused on a single party, and constructed a dichotomous situation in dividing the votes into votes for that party, and votes fro any other party. Then we estimated the parameters to a pure zealot model. These estimates yield an upper bound for $N^a + N^b$. Last we transformed the data linearly, s.t. the maximum was mapped to 1, the minimum was mapped to 0. The zealot model for the transformed data yielded a lower bound for $N^a + N^b$. Last, we sampled the interval between the two bound equidistantly, fixed $N^a + N^b$ to one of the values, and maximized the likelihood using the other three parameters (offset, factor, and $N^a/(N^a + N^b)$). We did choose that value for $N^a + N^b$ that did maximize the likelihood along the sampled interval. Fig. 2.5 shows factor, which corresponds to the fraction of floating voters, for each of the parties analyzed.

$\boxed{\textit{End excursion.}}$

Let's consider a simulation study (Fig. 2.4). Data have been created according to the Sano model. In order to produce one point in the central chard in Fig. 2.4, first the factor and $N^a + N^b$ have been fixed at the corresponding values. The other parameters (offset and $N^a/(N^a + N^b)$) have been used to maximize the likelihood. These likelihood values are depicted in the central diagram. To better understand that diagram, we inspect the data (histograms). The data look like as they are distributed according to a beta distribution with a small variance. This small variance can be generated by the original zealot model if we choose a large number of zealots ($N^a + N^b$ large). It is also possible to have a factor distinctively smaller 1 (which squeezes the data) and a small number of zealots. It is hard to tell the two mechanisms apart; both create a small variance. If we consider the log-likelihood presented in Fig. 2.4, we find an elongated ridge where the likelihood is more or less maximal and constant. To the left or to the right of that ridge the variance of the distribution is either too small or too large. On the ridge, the data are well met. In the

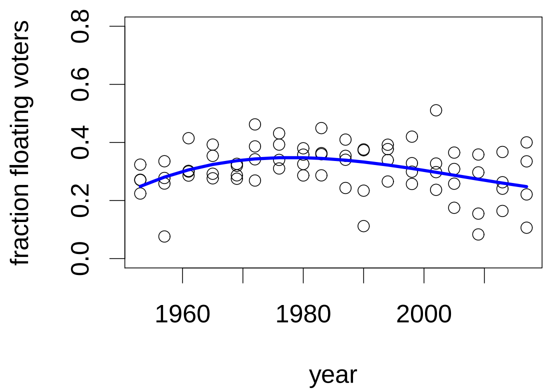

Figure 2.5: Fraction of floating voters in Germany over year for the four largest parties of an election. The bold line is the result of a third-order polynomial regression model.

particular example, there are two local (weak) maxima, one with a zero factor (such that we fall back to the original zealot model), and one with a factor ≈ 0.5, and $N^a + N^b \approx 25$ (parameters used to generate the data). Though the number of data is large (1000 realizations), it is difficult to clearly distinguish between the zealot and the Sano model.

However, a naive estimation of the fraction of floating voters for the German elections between 1950 and 2017 yields a reasonable result. The fraction of floating voters is around 0.3 (see Fig. 2.5), which is in agreement with results of polls, that yielded 31% floating voters in 2009 and around 33% in 2013 [?].

SI and election results

Excursion: Estimation of the Social Index

In a given election, we have m election districts. The number of valid votes is N_i, and the number of votes for choice A is X_i in district i. From these data, we aim at an estimator for the SI. Thereto, we slightly adapt the procedure proposed by de Braha et al. [5].

For one election district we might assume that the number of valid votes N is fixed and known beforehand (the number of valid votes are assumed to meet the number of eligible voters, where we neglect the possibility not to vote, or the possibility for non-valid votes). Then, according to the zealot model, the number of votes in favor of choice A X is beta-distributed. As indicated above, the vote share distribution $x = X/N$ can be approximated by a normal distribution $\mathcal{N}(\mu, \sigma)$, where $\mu = E(X/N)$. Note that σ is proportional to $1/N$.

The reference variance for the SI is that of a Bernoulli distribution, $\sigma_0 = \mu(1-\mu)/N$. Also σ_0 is proportional to $1/N$. We solve the defining equation for the SI for σ,

$$\mathsf{SI} = \sigma/\sigma_0 \quad \Leftrightarrow \quad \sigma = \mathsf{SI}\,\sigma_0 = \mathsf{SI}\,\mu(1-\mu)/N.$$

Therewith,

$$\frac{(X/N-\mu)}{\sqrt{\sigma}} \sim \mathcal{N}(0,1) \quad \Rightarrow \quad \frac{(X/N-\mu)^2}{\sigma} = N\,\frac{(X/N-\mu)^2}{\mathsf{SI}\,\mu(1-\mu)} \sim \chi_1^2.$$

As the expected value of a χ^2 distribution with one degree of freedom is 1, we find

$$1 = E\bigg(N\,\frac{(X/N-\mu)^2}{\mathsf{SI}\,\mu(1-\mu)}\bigg) \quad \Rightarrow \quad \mathsf{SI} = E\bigg(N\,\frac{(X/N-\mu)^2}{\mu(1-\mu)}\bigg).$$

Now we return to the data. First, we pool all data (assuming that the election districts are homogeneous) to estimate μ

$$\hat{\mu} = \frac{\sum_{i=1}^m X_i}{\sum_{i=1}^m N_i}.$$

Then we forget that we did estimate $\hat{\mu}$, but use that value as it would be known beforehand. The empirical mean value of $N\,\frac{(X/N-\mu)^2}{\mu(1-\mu)}$ is defined as the estimator for SI,

$$\widehat{\mathsf{SI}} = \frac{1}{m}\sum_{i=1}^m N_i\,\frac{(X_i/N_i-\hat{\mu})^2}{\hat{\mu}(1-\hat{\mu})}. \tag{2.17}$$

Two more remarks: The estimation of the SI developed here requires a certain homogeneity in the election districts. This homogeneity is not always given. Therefore it is wise to focus on smaller regions, and to estimate the SI for each region separately.
We are not interested in the absolute values of SI, but rather in the change over time. For a given reference region we determine the SI for several elections, and get a time series. To combine several reference regions, it turned out that it is a good idea to first normalize each of the time series: We apply the Z-transformation [5, ?] (normalize the mean to 0 and the variance to 1 in subtracting the mean and dividing by the variance). Only then, the time series corresponding to different regions are averaged.
End excursion.

The Social Interaction coefficient aims to estimate the within-population dynamics. Intrinsically, the time course of this parameter for different states is of interest. We focus here on turnout rates, as the turnout is a dichotomous choice.
Moreover, the absence from election might express dissatisfaction with the election or the governmental system, an information of utmost interest. In NL and Germany, not much is visible on the pooled data (Fig. 2.6). However, if we focus on the largest cities, we find clearly an increase in the SI since the year 2000. Similarly, in France (even in the pooled data), something happens in the year 2000, and the SI suddenly jumps to large values. These findings are line line with the similar analysis of de Barha et al. [5] for the US: also there, the SI starts to increase in the year 2000.
One can speculate that this onset of a trend is due to the internet. On the other hand, social networks as Facebook and twitter became active only around 2004, s.t. the increase could stared earlier than the social networks. In any case, the increase did happen before the bank crises (2009) and the refugee crises in Europe (2015). Anyway, the European refugee crisis did not affect the united states, where we find a similar increase in the SI. These relatively recent events did not trigger the change in the SI.

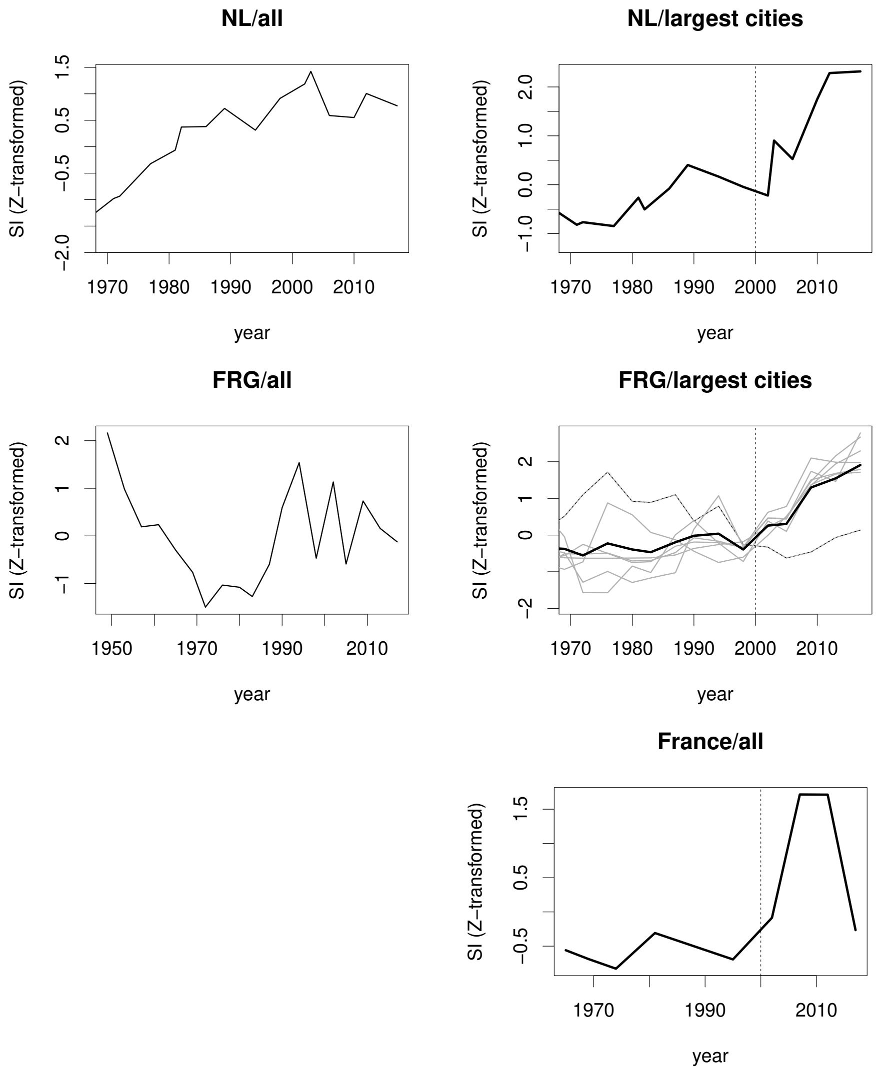

Figure 2.6: Time course of the SI for NL, FRG, and France (presidential election, first round). For NL and FRG, we have the pooled data (left) and filtered data (for NL: the 1% largest election districts, corresponding to the largest cities, for FRG the largest cities – Hamburg, Bremen, Munich (gray, dashed), Cologne and Bonn (pooled), Frankfurt, Stuttgart, Ruhr area – are estimated separately, and averaged afterwards).

Exercises

Exercise 3.4: Demonstrate formula (2.16).

Solution of Exercise 3.4 If an $\mathbb{R}$-valued random variable Y has the distribution function $\varphi_y(y)$, and $X = aY + b$, then

$$X \sim \varphi_x(x) = \frac{1}{a}\ \varphi_y((x-b)/a).$$

The factor $1/a$ is due to the fact that we need not only to transform the argument of the function $\varphi_y(.)$, but also the differential. Only then,

$$\int_{-\infty}^{ay+b} \varphi_x(t)\,dt \;=\; \int_{-\infty}^{y} \varphi_y((t-b)/a)\,dt/a \;=\; \int_{-\infty}^{y} \varphi_y(s)\,ds.$$

If we return to the situation at hand, we know that $(x-\text{offset})/\text{factor}$ is $\text{Beta}(\hat{N}^a, \hat{N}^b)$ distributed, and thus the distribution of x is given by

$$\varphi(x) = \frac{1}{\text{factor}}\,\frac{\Gamma(\hat{N}^a)\,\Gamma(\hat{N}^b)}{\Gamma(\hat{N}^a+\hat{N}^b)}\left(\frac{x-\text{offset}}{\text{faktor}}\right)^{N^a-1}\left(1-\frac{x-\text{offset}}{\text{faktor}}\right)^{N^b-1}$$

for x in the interval (offset, offset+faktor). The logarithm of this equation yields the log-likelihood of a given data point.

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IS it possible to reformulate the results (data as rank-order stats) by Fenner/Levene/Loizou (at least 2 papers, UK election results, arXiv 1609.04282, 1703.10548) in terms of the beta distribution???

2.3.3 Elections with multiple candidates or parties

We extend the model for a dichotomic election discussed above to a multy-party (-candidate) system, as given in the US presidential pre-elections, the first round of presidential elections in France, and the parliamentary election in the Netherlands, Germany, or many other countries. The straight generalization we discuss now has been proposed by Sano [41] and, in a formulation closer to the noisy voter model, by Kononovicius [27]. The model of Sano refines the idea of the Zealot model as described above. Sano takes the zealots serious. They are – in principle – parts of the voters community, and will take part in an election. Before, zealots represented e.g. mass media,that influence elections but do not vote. In the present model, they still will influence the swing voters. However, though the zealots are determined to vote for a certain party, only part of them will influence the swing voters. That is, we basically have three groups in the voters community: (a) the zealots who have a frozen, fixed opinion but do not influence someone, the zealots that also have a frozen opinion but do influence the swing voters, and the swing voters themselves. Only that latter class is addressed by the dynamic (stochastic) model, the zealots and the fraction of politically active zealots are model parameters. Below, we address only the swing voters as "voters", as we are interested in their behavior, neglecting that also zealots may vote.

Notation: In the present section, we use the definition

$$\mathcal{S}_N^K = \{n \in \mathbb{N}_0^k \mid \sum_{\ell=1}^{K} n_0 = N\}.$$

Model 2.18 Sano model. *Consider an election with K groups (parties or candidates). The total number of (floating) voters is N, and at a given time the group sizes are $X_1,\ldots,X_K$, such that $N=\sum_{\ell=1}^{K} X_\ell$. Additionally, we have zealots – individuals that do not change their opinion. The number of zealots that support group i is N_i, $i=1,\ldots,K$.*

Only a fraction θ of the zealots influence the floating voters. Among floating voters, we do allow for selfing. Floating voters as well as zealots will vote for his/her group, s.t. $X_i + N_i$ is the number of voters for group i. Let $N_F = \sum_{\ell=1}^K N_\ell$ the total number of zealots, and $M = N + \theta \sum_{\ell=1}^K N_\ell$ the total population size of politically active individuals (floating voters or influencing zealots).
Any transition affects the sizes of two groups only. The transition rates are given by

$$(X_i, X_j) \mapsto (X_i + 1, X_j - 1) \text{ at rate } \mu X_j \, \frac{X_i + \theta N_i}{M}.$$

Invariant distribution

CHECK: WAS IST, WENN $N_{i_0} = 0$(keine fixed voters fuer eine partie). Haelt der satz immer noch, oder muessen wir das ausschliessen????

Theorem 2.19 *Let $(X_1, \ldots, X_n)$ be random variables distributed according to the invariant measure of model 2.18. Let furthermore $n \in \mathcal{S}_N^K$. Assume $N_j > 0$ for all $j = 1, \ldots, K$. Then,*

$$P(X_1 = n_1, \ldots, X_K = n_k) \quad = \quad \frac{N!}{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} \tag{2.18}$$

where $x_{(k)} = x\,(x+1)\,\cdots\,(x+k-1)$ is the Pochhammer symbol; we define $x_{(0)} = 1$.

Before we demonstrate this theorem, we show a neat equation that parallels the well known binomial formula

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Let $\mathcal{S}_N^K$ denote the set of all K-tuples $n = (n_1, \ldots, n_K)$, where $n_i \in \mathbb{N}_0$ and $\sum_{i=1}^K n_i = N$.

Proposition 2.20 *For $x, y \in \mathbb{R}$, $n \in \mathbb{N}$, we have*

$$(x+y)_{(n)} = \sum_{k=0}^n \binom{n}{k} x_{(k)} y_{(n-k)}. \tag{2.19}$$

Furthermore, for $x_1, \ldots x_K \in \mathbb{R}$, $N \in \mathbb{N}$, we find

$$(\sum_{\ell=1}^K x_\ell)_{(N)} = \sum_{n \in \mathcal{S}_N^K} \binom{N}{n_1, \ldots, n_K} \prod_{\ell=1}^K (x_\ell)_{(n_\ell)}. \tag{2.20}$$

Proof: To show formula (2.19), we use induction over n. For $n = 1$, we find (recall $x_{(0)} = 1$)

$$(x+y)_{(1)} = (x+y) = x_{(1)}\, y_{(0)} + x_{(0)}\, y_{(1)} = \sum_{k=0}^1 \binom{n}{k} x_{(k)} y_{(n-k)}.$$

Now assume that the formula is true for $n \in \mathbb{N}$. Then,

$$
\begin{aligned}
& (x+y)_{(n+1)} = (x+y+n)\,(x+y)_{(n)} = (x+y+n)\sum_{k=0}^{n}\binom{n}{k}x_{(k)}y_{(n-k)} \\
= \;& \sum_{k=0}^{n}\binom{n}{k}x_{(k)}\,y_{(n-k)}(\,(x+k)+(y+n-k)\,) \\
= \;& \sum_{k=0}^{n}\binom{n}{k}x_{(k+1)}\,y_{(n-k)}+\sum_{k=0}^{n}\binom{n}{k}x_{(k)}\,y_{(n-k+1)} \\
= \;& \sum_{k=1}^{n+1}\binom{n}{k-1}x_{(k)}\,y_{(n-k+1)}+\sum_{k=0}^{n}\binom{n}{k}x_{(k)}\,y_{(n-k+1)} \\
= \;& \binom{n}{0}x_{(0)}\,y_{(n+1)}+\sum_{k=1}^{n+1}\left\{\binom{n}{k-1}+\binom{n}{k}\right\}x_{(k)}\,y_{(n-k+1)}+\binom{n}{n}x_{(n+1)}\,y_{(0)} \\
= \;& \binom{n}{0}x_{(0)}\,y_{(n+1)}+\sum_{k=1}^{n+1}\binom{n+1}{k}x_{(k)}\,y_{(n+1-k)}+\binom{n}{n}x_{(n+1)}\,y_{(0)} \\
= \;& \sum_{k=0}^{n+1}\binom{n+1}{k}x_{(k)}y_{(n+1-k)}
\end{aligned}
$$

For the second formula, we again use induction. This time, we use finite induction over K.

$$
\begin{aligned}
(\sum_{\ell=1}^{K}x_\ell)_{(N)} &= (x_1+\sum_{\ell=2}^{K}x_\ell)_{(N)} = \sum_{n_1=0}^{N}\binom{N}{n_1,\ N-n_1}(x_1)_{(n_1)}(\sum_{\ell=2}^{K}x_\ell)_{(N-n_1)} \\
&= \sum_{n_1=0}^{N}\sum_{n_2=0}^{N-n_1}\binom{N}{n_1,\ n_2,\ N-n_1-n_2}(x_1)_{(n_1)}\,(x_2)_{(n_2)}(\sum_{\ell=3}^{K}x_\ell)_{(N-n_1-n_2)} \\
&= \ldots = \sum_{n\in\mathcal{S}_N^K}\binom{N}{n_1,\ldots,n_K}\prod_{\ell=1}^{K}(x_\ell)_{(n_\ell)}.
\end{aligned}
$$

□

Proof: (of theorem 2.19) We aim to use the detailed balance equation. Consider two states n^1 and n^2, between that the Markov process might jump forth and back. That is, there are two groups i and j, such that

$$n_i^1 = n_i^2 - 1, \qquad n_j^1 = n_j^2 + 1.$$

The rate at which the process jumps from n^1 to n^2 is given by

$$\lambda_{1\to 2} = \mu n_i^1\ \frac{n_j^1+\theta N_j}{M}$$

while the rate to jump back from n^2 to n^1 reads

$$\lambda_{2\to 1} = \mu(n_j^1+1)\ \frac{n_i^1-1+\theta N_i}{M}.$$

We aim to show the detailed balance equation, that is, we aim to prove the equation

$$\lambda_{1\to 2}\, P(X_1 = n_1^1, \ldots, X_K = n_k^1) = \lambda_{2\to 1}\, P(X_1 = n_1^2, \ldots, X_K = n_k^2).$$

We replace $P(X_1 = n_1^*, \ldots, X_K = n_1^*)$ using (2.18). As that probability has a product structure, most terms cancel out. It is sufficient to show that

$$T_{1\to 2} := \lambda_{1\to 2}\, \frac{(\theta\, N_i)_{(n_i)}}{n_i!}\, \frac{(\theta\, N_j)_{(n_j)}}{n_j!} = \lambda_{2\to 1}\, \frac{(\theta\, N_i)_{(n_i-1)}}{(n_i-1)!}\, \frac{(\theta\, N_j)_{(n_j+1)}}{(n_j+1)!} =: T_{2\to 1}.$$

Let us first consider $T_{1\to 2}$,

$$T_{1\to 2} \;=\; \mu n_i^1\, \frac{n_j^1 + \theta N_j}{M}\, \frac{(\theta\, N_i)_{(n_i)}}{n_i!}\, \frac{(\theta\, N_j)_{(n_j)}}{n_j!} = \frac{\mu}{M}\, \frac{(\theta\, N_i)_{(n_i)}}{(n_i-1)!}\, \frac{(\theta\, N_j)_{(n_j+1)}}{n_j!}.$$

Similarly, we find

$$T_{2\to 1} \;=\; \mu(n_j^1 + 1)\; \frac{n_i^1 - 1 + \theta N_i}{M}\, \frac{(\theta\, N_i)_{(n_i-1)}}{(n_i-1)!}\, \frac{(\theta\, N_j)_{(n_j+1)}}{(n_j+1)!} = \frac{\mu}{M}\, \frac{(\theta\, N_i)_{(n_i)}}{(n_i-1)!}\, \frac{(\theta\, N_j)_{(n_j+1)}}{n_j!}.$$

Hence, $T_{1\to 2} = T_{2\to 1}$, as required. The last step to show is

$$\sum_{n\in\mathcal{S}_N^K} P(X_1 = n_1, \ldots, X_k = n_K) = \frac{N!}{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}} \sum_{n\in\mathcal{S}_N^K} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} = 1.$$

In order to do so, we rewrite the sum and then use formula (2.20),

$$\sum_{n\in\mathcal{S}_N^K} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} = \frac{1}{N!} \sum_{n\in\mathcal{S}_N^K} \binom{N}{n_1, \ldots, n_K} \prod_{\ell=1}^K (\theta\, N_j)_{(n_\ell)} = \frac{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}}{N!}.$$

If we multiply that equation by $N!/(\sum_{\ell=1}^K \theta N_\ell)_{(N)}$, we find that the probabilities indeed sum up to one.

□

Limit of the invariant distribution

We are now prepared to show our asymptotic result. We consider a sequence of models with more and more voters. That is, N tends to infinity, while N_j are constant (see also [1]).
Before we state the theorem we recall the definition of the (real) Γ-function, and some properties. The real Γ- function is defined for $\mathbb{R}_+$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt.$$

Partial integration yields $\Gamma(1+x) = x\Gamma(x)$, which implies for $n \in \mathbb{N}$, $a \in \mathbb{R}_+$ that

$$n! = \Gamma(n+1), \qquad a_{(n)} = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.21}$$

There is a handy approximation of the Γ-function for large, real arguments: the Striling's formula,

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\, \left(\frac{x}{e}\right)^x\, e^{\zeta(x)}$$

where $\zeta(x)$ is a function with $0 \le \zeta(x) \le 1/(12\, x)$.

Theorem 2.21 *Let $\mu_\ell = N_\ell / N_F$. Define furthermore the random variables x_ℓ by*

$$x_\ell = X_\ell / N.$$

In the limit $N \to \infty$, we obtain a Dirichlet distribution for $(x_1, \ldots, x_K)$,

$$(x_1, \ldots, x_K) \sim Dir(\theta\, N_F\, \mu_1, \ldots, \theta\, N_F\, \mu_K). \tag{2.22}$$

Recall that the Dirichlet distribution with parameter $\alpha = (\alpha_1, \ldots, \alpha_K)$ is given by

$$\varphi(z_1, \ldots, z_K) = B(\alpha) \prod_{\ell=1}^{K} z_\ell^{\alpha_\ell - 1} \tag{2.23}$$

if the sum of the z_ℓ is one, and zero else, where

$$B(\alpha) = \frac{\prod_{\ell=1}^{K} \Gamma(\alpha_\ell)}{\Gamma(\sum_\ell \alpha_\ell)}.$$

Proof: We rewrite (2.18) as

$$P(X_1 = n_1, \ldots, X_K = n_k) = \hat{C}(N, \theta, N_\ell) \prod_{\ell=1}^{K} \frac{(\theta\, N_\ell)_{(n_\ell)}}{n_\ell!}.$$

We focus on one (generic) factor in that product, where we use $n_\ell = z_\ell N$,

$$\frac{(\theta\, N_\ell)_{(n_\ell)}}{n_\ell!}.$$

We are interested in the limit $N \to \infty$. Note that all multiplicative terms that only depend on N, θ, and N_j can be absorbed by the global normalization constant $\hat{C}(N, \theta, N_\ell)$. That is, we are only interested in equalities up to multiplicative functions that (asymptotically) do only depend on the model parameters N, θ, and N_j. We only keep the n_ℓ (z_ℓ)-dependent terms. In order to emphasize this fact, we write $\equiv_a$ if we neglect multiplicative factors.
If we express the factorial and the Pochhammer symbol by Γ-functions, we find

$$\begin{aligned}
\frac{(\theta\, N_\ell)_{(n_\ell)}}{n_\ell!} &= \frac{\Gamma(\theta\, N_\ell + n_\ell)\Gamma(n_\ell)}{\Gamma(\theta\, N_\ell)\,\Gamma(n_\ell + 1)\Gamma(n_\ell)} = \frac{1}{n_\ell}\, \frac{\Gamma(\theta\, N_\ell + n_\ell)}{\Gamma(\theta\, N_\ell)\,\Gamma(n_\ell)} \equiv_a \frac{1}{z_\ell} \frac{\Gamma(\theta\, N_\ell + N\, z_\ell)}{\Gamma(\theta\, N_\ell)\; \Gamma(N\, z_\ell)} \\
&\equiv_a \quad z_\ell^{-1} \; \frac{\sqrt{\frac{2\pi}{\theta\, N_\ell + N\, z_\ell}} \; \left(\frac{\theta\, N_\ell + N\, z_\ell}{e}\right)^{\theta\, N_\ell + N\, z_\ell}}{\sqrt{\frac{2\pi}{\theta\, N_\ell}} \; \left(\frac{\theta\, N_\ell}{e}\right)^{\theta\, N_\ell} \; \sqrt{\frac{2\pi}{N\, z_\ell}} \; \left(\frac{N\, z_\ell}{e}\right)^{N\, z_\ell}} \\
&\equiv_a \quad z_\ell^{-1} \; \underbrace{\sqrt{\frac{N\, z_\ell}{\theta\, N_\ell + N\, z_\ell}}}_{\to 1} \; (\theta\, N_\ell + N\, z_\ell)^{\theta\, N_\ell} \; \left\{ \underbrace{\left(1 + \frac{1}{N}\, \frac{\theta\, N_\ell}{z_\ell}\right)^{N}}_{\to e^{\theta\, N_\ell / z_\ell}} \right\}^{z_\ell} \\
&\equiv_a \quad z_\ell^{-1} \; \underbrace{(\theta N_\ell / N + z_\ell)^{\theta\, N_\ell}}_{\to z_\ell^{\theta\, N_\ell}} \; e^{(\theta N_\ell / z_\ell)\, z_\ell} \equiv_a z_\ell^{\theta\, N_\ell - 1}.
\end{aligned}$$

As $\theta\, N_\ell = \theta N_F\, \mu_\ell$, the result follows: The vector of vote shares $(x_1, \ldots, x_K)$ follows a Dirichlet distribution.

$\square$

Exercises

Exercise 3.5: The noisy voter model with population size $N > 1$ and K groups (and without spatial structure) has the set

$$\mathcal{S} = \left\{ (x_1, \ldots, x_K) \in \mathbb{N}^K \;\middle|\; \sum_{i=1}^{K} x_i = N \right\}$$

as state space, where x_i is the number of supporters of opinion i. In each time step, a randomly chosen individual changes potentially his/her opinion. With probability u, he/she copies the opinion of a randomly chosen individual. With probability $1-u$, the individual adopts some opinion, independently on the state of the population. In that case, an individual belonging to group i will switch to group ℓ with probability $\pi_{i,\ell}$.
Show that the noisy voter model for $K = 2$ covers the zealot model 2.5 but is more general.

Solution of Exercise 3.5 (a) Given a zealot model with population size N, and N_A (N_B) fixed voters for opinion A (opinion B), we define $p = N/(N+N_A+N_B)$, $\pi_{1,1} = \pi_{2,1} = N_A/(N_A+N_B)$, and $\pi_{2,1} = \pi_{2,2} = N_B/(N_A + N_B)$.
(b) Note that $\pi_{i,1} + \pi_{i,2} = 1$, but it is not necessary that $\pi_{1,2} + \pi_{2,1}$ add up to one. For any zealot model, given that a person does not ask another floating voter, the probability for group A always is $N_A/(N_A + N_B)$, while that for group B reads $N_B/(N_A + N_B)$, such that

$$P(\text{jump from group 1 to group 2}) + P(\text{jump form group 2 to group 1}) = 1.$$

That is, only noisy voter models with $\pi_{1,\ell} = \pi_{2,\ell}$, $\ell \in \{1, 2\}$, can be written as zealot models.

2.3.4 Large population limit for the multiparty model - revised

We obtained the Dirichlet distribution for the vote share of parties. In this limit, we took the number of floating voters, N, to infinity, while the number of zealots, N_F, stays finite. This limit seems to be strange – it might be more reasonable to assume that a certain fraction of the population are zealots, and the remaining fraction are floating voters, s.t. both groups – zealots and floating voters – tend to infinity in the same way if the population size tends to infinity. We approach th question of the appropriate scaling by investigating the diffusion limit of the model. In that, we basically extend proposition 2.3.1 and corollary 2.15 to the K-party model.

Thereto, we start with the master equations, as developed in exercise 3.6: Let $p_{i_1,\ldots,i_K}(t) =$

$P(X_1 = i_1, \ldots, X_K = i_K)$. Then,

$$\begin{aligned}
\dot{p}_{i_1,\ldots,i_K} &= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right)\Big)\, p_{i_1,\ldots,i_K} \\
&\quad +\mu \sum_{j=1}^{K} \left(\sum_{k\neq j} (i_k+1) \left(\frac{i_j - 1 + N_j}{M}\right) p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k + 1} \right) \\
&= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right)\Big)\, p_{i_1,\ldots,i_K} \\
&\quad +\mu \sum_{j=1}^{K} \left(\frac{i_j - 1 + \theta N_j}{M} \sum_{k\neq j} (i_k+1)\; p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k + 1} \right)
\end{aligned}$$

From here, we go to th Fokker-Planck equation. Let $u(x_1, \ldots, x_K, t)$ probability density to find the sate $(x_1, \ldots x_K)$ at time t. Let $h = 1/N$, then

$$x_\ell \approx i_\ell/N = i_\ell\, h, \qquad u(x_1, \ldots, x_K, t) \approx h\, p_{i_1,\ldots,i_K}(t)$$

and

$$n_i = \theta N_i/N, \quad m = M/N = 1 + \theta N_F/N.$$

NOte that always $\sum_{\ell=1}^{K} x_\ell = 1$, that is, in all formulas we need to understand that

$$x_K = 1 - \sum_{i=1}^{K-1} x_i, \qquad \partial_{x_K} = -\sum_{i=1}^{K-1} \partial_{x_i}.$$

Therewith,

$$\begin{aligned}
u_t &= h\,\dot{p} \\
&= h\Big[-\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right)\Big)\, p_{i_1,\ldots,i_K} \\
&\quad +\mu \sum_{j=1}^{K} \left(\frac{i_j - 1 + \theta N_j}{M} \sum_{k\neq j} (i_k+1)\; p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1,\; i'_k = i_k + 1} \right) \Big] \\
&= h^{-1}\Big[-\mu\Big(\sum_{j=1}^{K} x_j \left(1 - \frac{x_j + n_j}{m}\right)\Big)\, u(x_1, \ldots, x_K, t) \\
&\quad +\frac{\mu}{m} \sum_{j=1}^{K} \left((x_j - h + n_j) \sum_{k\neq j} (x_k + h)\; u(x'_1, \ldots, x'_K, t)\Big|_{x'_j = x_j - h,\; x'_k = x_k + h} \right) \Big]
\end{aligned}$$

Let us consider the first term. We write $u(x)$ instead of $u(x_1, \ldots, x_K, t)$ and obtain,

$$
\begin{aligned}
& -\mu\Big(\sum_{j=1}^{K} x_j \,\left(1 - \frac{x_j + n_j}{m}\right)\Big)\; u(x) = -\mu\, u(x) \left(\sum_{j=1}^{K} \left(x_j - \frac{x_j(x_j+n_j)}{m}\right)\right) \\
= \;& -\mu\, u(x) \left(1 - \sum_{j=1}^{K} \frac{x_j(x_j+n_j)}{m}\right).
\end{aligned}
$$

Now we turn to the second term. We focus one arbitrary term in the last sum, and expand this term w.r.t. h. Let $j, k \in \{1, \ldots, K\}$, and $j \neq k$. We suppress all arguments of u expect x_j and x_k. Then, Taylor expansion yields

$$
\begin{aligned}
& (x_j - h + n_j)\,(x_k + h)\; u(x_j - h, x_k + h) \\
= \;& \Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) + \; h\,(-\partial_{x_j} + \partial_{x_k})\Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) \\
& + \frac{h^2}{2}(\partial_{x_j}^2 - 2\partial_{x_k}\partial_{x_j} + \partial_{x_k}^2)\Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) + \mathcal{O}(h^3) \\
= \;& T_0 + h\, T_1 + \frac{h^2}{2} T_2 + \mathcal{O}(h^3).
\end{aligned}
$$

Note that the argument does not depend on the indices any more (before, we did increase/decrease some entry in the argument of u by h, now in the argument of u, h is zero). We can write $u(x)$, again in the understanding that $u(x) = u(x_1, \ldots, x_K, t)$.
<u>Let us return to the full sum, and only consider the zero order term T_0:</u>

$$
\begin{aligned}
\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k\neq j} T_0 \;&=\; \frac{\mu}{m} \sum_{j=1}^{K} \sum_{k\neq j} \Big((x_j + n_j)\, x_k\; u(x)\Big) \\
&=\; u(x) \frac{\mu}{m} \left(\sum_{j=1}^{K} (x_j + n_j) \sum_{k\neq j} x_k \right) = u(x) \frac{\mu}{m} \; \sum_{j=1}^{K} (x_j + n_j)(1 - x_j) \\
&=\; \mu\, u(x)\, \frac{\sum_{j=1}^{K}(x_j + n_j) - \sum_{j=1}^{K}(x_j(x_j + n_j))}{m} = \mu\, u(x)\, \frac{m - \sum_{j=1}^{K} x_j(x_j + n_j)}{m} \\
&=\; \mu\, u(x) \left(1 - \sum_{j=1}^{K} \frac{x_j(x_j+n_j)}{m}\right).
\end{aligned}
$$

Now we proceed to term T_1:

$$\begin{aligned}
\frac{\mu}{m}\sum_{j=1}^{K}\sum_{k\neq j}T_1 &= \frac{\mu\,h}{m}\sum_{j=1}^{K}\sum_{k\neq j}(-\partial_{x_j}+\partial_{x_k})\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&= \frac{\mu\,h}{m}\Big[\sum_{j=1}^{K}\sum_{k\neq j}\partial_{x_k}\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\sum_{k\neq j}\partial_{x_j}\Big((x_j+n_j)\,x_k\;u(x)\Big)\Big]\\
&= \frac{\mu\,h}{m}\Big[\sum_{k=1}^{K}\sum_{j\neq k}\partial_{x_k}\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\;\Big(\sum_{k\neq j}x_k\Big)\;u(x)\Big)\Big]\\
&= \frac{\mu\,h}{m}\Big[\sum_{j=1}^{K}\sum_{\ell\neq j}\partial_{x_j}\Big((x_\ell+n_\ell)\,x_j\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\,(1-x_j)\;u(x)\Big)\Big]\\
&= \frac{\mu\,h}{m}\Big[\sum_{j=1}^{K}\partial_{x_j}\Big((m-x_j-n_j)\,x_j\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\,(1-x_j)\;u(x)\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\mu\,h}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big((x_j+n_j)\,(1-x_j)-(m-x_j-n_j)\,x_j\Big)u(x)\Big\}\\
&= \frac{-\mu\,h}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big(n_j-(m-1)x_j\Big)u(x)\Big\}.
\end{aligned}$$

Remark: The deterministic drift is given by

$$\begin{aligned}
\dot{x}_j &= (x_j+n_j)\,(1-x_j)-(m-x_j-n_j)\,x_j && (2.24)\\
&= (x_j+n_j)\;-(x_j+n_j)\,x_j-mx_j\;+(x_j+n_j)\,x_j && (2.25)\\
&= n_j-(m-1)x_j && (2.26)
\end{aligned}$$

Hence, without noise,

$$\lim_{t\to\infty}x_j(t)=\frac{n_j}{m-1}=\frac{n_j}{\sum_i n_i}=\frac{N_j}{\sum_i N_i}.$$

The equilibrium is given by the relative fractions of zealots.

Last we handle term T_2.

T_2 has three (second) derivatives. We first consider $\partial^2_{x_j} + \partial^2_{x_k}$ (let us call this term $T_2^{(a)}$), and consider the mixed derivative (term $T_2^{(b)}$) only later.

$$\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(a)} \;=\; \frac{\mu\, h}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} (\partial^2_{x_j} + \partial^2_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big)$$

Up to a sign, this term parallels that of T_2, and hence

$$\frac{\mu}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(a)} \;=\; \frac{\mu\, h}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big((x_j + n_j)\, (1 - x_j) + (m - x_j - n_j)\, x_j \Big) u(x) \Big\}.$$

Last term:

$$\begin{aligned}
\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(b)} \;&=\; \frac{-2\mu\, h^2}{2m} \sum_{j=1}^{K} \sum_{k \neq j} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big) \\
&=\; \frac{-\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big) \\
&\quad + \frac{\mu\, h^2}{m} \sum_{j=1}^{K} \partial_{x_j^2} \Big((x_j + n_j)\, x_j\; u(x) \Big)
\end{aligned}$$

and therefore,

$$\begin{aligned}
\frac{\mu}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} T_2 \;&=\; \frac{\mu\, h^2}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big((x_j + n_j)\, (1 + x_j) + (m - x_j - n_j)\, x_j \Big) u(x) \Big\} \\
&\quad - \frac{\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big) \\
&=\; \frac{\mu\, h^2}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big(n_j + (m + 1)\, x_j \Big) u(x) \Big\} \\
&\quad - \frac{\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big)
\end{aligned}$$

Therewith, we have the Fokker-Planck equation

$$\begin{aligned}
u_t \;&=\; \frac{-\mu}{m} \sum_{j=1}^{K} \partial_{x_j} \Big\{ \Big(n_j - (m - 1) x_j \Big) u(x) \Big\} \\
&\quad + \frac{\mu}{2\, m\, N} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big(n_j + (m + 1)\, x_j \Big) u(x) \Big\} \qquad (2.27) \\
&\quad - \frac{\mu}{m\, N} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big)
\end{aligned}$$

In an alternative form:

Corollary 2.22 *The Fokker-Planck equation for Sano's multiparty-zealots model reads:*

$$\begin{aligned}
u_t &= \frac{-\mu}{m}\sum_{j=1}^{K}\partial_{x_j}\left\{\Big(n_j-(m-1)x_j\Big)u(x)\right\}\\
&\quad +\frac{\mu}{2\,m\,N}\sum_{j=1}^{K}\partial^2_{x_j}\left\{\Big((x_j+n_j)\,(1-x_j)+(m-x_j-n_j)\,x_j\Big)u(x)\right\} \qquad (2.28)\\
&\quad -\frac{\mu}{m\,N}\sum_{j,k=1,j\neq k}^{K}(\partial_{x_j}\,\partial_{x_k})\Big((x_j+n_j)\,x_k\ u(x)\Big)
\end{aligned}$$

Remark 2.23 *From here – and that is the central insight in all these lengthy calculations – we have two different possibilities to proceed:*
- *We can return from n_i to $n_i = \theta N_i/N$ again. Thatis, we assume that the numbr of zealots stays constant, even if the population size tends to infinity. In this case, we will find (at least formally) our original result back again - the Dirichlet distribution as stated in theorem 2.21.*
- *Or, we can stay with the assumption that the number of zealots increase witht he population size in the same way as the population size. In that case, as usual, the noise term scales wit $1/N$, and becomes weaker and weaker for increasing N. The distribution will concentrate closely around the expected value, which is given by the zero of the drift term (eqn. (2.24)). In this case, a normal distribution with a variance that vanishes for N to infinity is an appropriate approximation.*

All in all, we have two different assumptions: Either the absolute number of zealots stay constant, or scales with the population size. The expectation of the distribution stays the same.
In the first case, however, the variance for the vote shares tend to some positive value, as the zealots are not sufficiently many to control the floating voters opinion. In the second case, the variance is of order $1/N$ and vanishes for $N \to \infty$.

Possibility 1: Weak limit

In this first approach, that will yield the Dirichlet distribution as before, in theorem 2.21, we choose

$$n_i = \theta N_i/N, \quad m = M/N = 1 + \theta N_F/N$$

That is, we assume that the number fo zealots do not scale with the population size, but the effects of the zealots become weak (in the sense that they tend to zero with $1/N$). We ignore terms of order $\mathcal{O}(1/N^2)$ in the Fokker-Planck equation, and obtain

$$
\begin{aligned}
u_t &= \frac{-\mu}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big(n_j-(m-1)x_j\Big)u(x)\Big\}\\
&\quad+\frac{\mu}{2\,m\,N}\sum_{j=1}^{K}\partial_{x_j}^2\Big\{\Big((x_j+n_j)\,(1-x_j)+(m-x_j-n_j)\,x_j\Big)u(x)\Big\}\\
&\quad-\frac{\mu}{m\,N}\sum_{j,k=1,j\neq k}^{K}(\partial_{x_j}\,\partial_{x_k})\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&= \frac{-\mu}{N}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big(\theta N_j-\theta N_F x_j\Big)u(x)\Big\}\\
&\quad+\frac{\mu}{N}\sum_{j=1}^{K}\partial_{x_j}^2\Big\{\Big(x_j\,(1-x_j)\Big)u(x)\Big\}-\frac{\mu}{N}\sum_{j,k=1,j\neq k}^{K}(\partial_{x_j}\,\partial_{x_k})\Big(x_j\,x_k\;u(x)\Big)+\text{higher order terms}
\end{aligned}
$$

We can remove the population size N from the equation by resealing the time. For the stationary case, (it is sufficient that) the flux of the equation becomes zero, s.t. for $j\in\{1,\ldots,K\}$,

$$
0 = \Big(\theta N_j-\theta N_F x_j\Big)u(x)+\partial_{x_j}\Big\{\Big(x_j\,(1-x_j)\Big)u(x)\Big\}-\sum_{k=1,k\neq j}^{K}\partial_{x_k}\Big\{x_j\,x_k\;u(x)\Big\}.
$$

At that point, we need to recall that $x_K=\sum_{\ell=1}^{K-1}x_\ell$, and $\partial_{x_K}=-\sum_{\ell=1}^{K-1}\partial_{x_\ell}$. Therewith, for $j=0,\ldots,K-1$,

$$
\begin{aligned}
0 &= \Big(\theta N_j-\theta N_F x_j\Big)u(x)+\partial_{x_j}\Big\{\Big(x_j\,(1-x_j)\Big)u(x)\Big\}-\sum_{k=1,k\neq j}^{K-1}\partial_{x_k}\Big\{x_j\,x_k\;u(x)\Big\}\\
&\quad-\sum_{\ell=1}^{K-1}\partial_{x_\ell}\Big\{x_j\Big(1-\sum_{i=1}^{K-1}x_i\Big)\;u(x)\Big\}
\end{aligned}
$$

We claim that $u(x_1,\ldots,x_K)=c\prod_{\ell=1}^{K}x_\ell^{a_\ell}$ is a solution (given that $\sum_j x_j=1$), if we choose $a_j\in\mathbb{R}$ accordingly:

$$
\begin{aligned}
&-\Big(\theta N_j-\theta N_F x_j\Big)\prod_{\ell=1}^{K}x_\ell^{a_\ell}+\partial_{x_j}\Big\{\Big(x_j\,(1-x_j)\Big)\prod_{\ell=1}^{K}x_\ell^{a_\ell}\Big\}-\sum_{k=1,j\neq k}^{K}\partial_{x_k}\Big\{x_j\,x_k\;\prod_{\ell=1}^{K}x_\ell^{a_\ell}\Big\}\\
&= -\theta N_j\prod_{\ell=1}^{K}x_\ell^{a_\ell}+\theta N_F x_j^{1+a_j}\prod_{\ell\neq j}^{K}x_\ell^{a_\ell}+(1+a_j)\prod_{\ell=1}^{K}x_\ell^{a_\ell}-(2+a_j)x_j^{1+a_j}\prod_{\ell\neq j}^{K}x_\ell^{a_\ell}\\
&\quad-\sum_{k=1,j\neq k}^{K}(1+a_k)x_j^{1+a_j}\prod_{\ell\neq j}^{K}x_\ell^{a_\ell}
\end{aligned}
$$

This term is zero, if

$$
-\theta N_j+(1+a_j)=0\quad\Rightarrow\quad a_j=\theta N_j-1
$$

and the second condition reads

$$0 = \theta N_F - (2 + a_j) - \sum_{k=1,k\neq j}^{K} (1 + a_k).$$

If we use our choice for a_j, we have

$$\theta N_F - (2 + a_j) - \sum_{k=1,k\neq j}^{K} (1 + a_k) = -1 + \theta \sum_{i=1}^{K} N_i - \sum_{j=k}^{K} (1 + a_k) = -1.$$

FAST RICHTIG – WIR HABEN EINE 1 ZU VIEL IN DER ZWEITEN GLEICHUNG!!!!!!!
PROBLEM: WIR BERUECKSICHTIGEN NICHT, DASS summs $x_i = 1$!!!!!!

Possibility 2: Normal approximation and deterministic limit

If we do not rescale n_i but assume that these parameters are independent of N, we can follow [20], and use asymptotic method to solve this equation (that is the same approach we used before in the two-party situation, in corollary 2.15). Note that

$$q_j = \frac{n_j}{\sum_{j=1}^{K} n_j} = \frac{n_j}{m-1}$$

are the stationary states for the flux; consequently, we introduce ($\varepsilon^2 = 1/N$)

$$x_i = q_i + \varepsilon y_i, \qquad \partial_{x_i} = \varepsilon^{-1} \partial_{y_i}.$$

Then,

$$\begin{aligned} n_j - (m-1)x_j &= -\varepsilon(m-1)y_j \\ x_j + n_j &= q_j + \varepsilon y_j + (m-1)q_j = mq_j + \varepsilon y_j \\ m - x_j - n_j &= m - (mq_j + \varepsilon y_j) = m(1-q_j) - \varepsilon y_j. \end{aligned}$$

Consider the stationary solution of the equation, and define $w(y) = u(q + \varepsilon y)$, then

$$\begin{aligned} 0 &= \sum_{j=1}^{K} \partial_{y_j} \Big\{ (m-1) y_j \, w \Big\} \\ &\quad + \frac{1}{2} \sum_{j=1}^{K} \partial_{y_j^2} \Big\{ \Big((mq_j + \varepsilon y_j)(1 - q_j - \varepsilon y_j) + (m(1-q_j) - \varepsilon y_j)(q_i + \varepsilon y_i) \Big) w(y) \Big\} \\ &\quad - \sum_{j\neq k} \partial_{y_j} \partial_{y_k} \Big\{ (mq_j + \varepsilon y_j)\,(q_k + \varepsilon y_k)\, w(y) \Big\} \end{aligned} \tag{2.29}$$

In zero'th order, this equation yields

$$\begin{aligned} 0 &= \sum_{j=1}^{K} \partial_{y_j} \Big\{ (m-1) y_j \, w \Big\} \\ &\quad + \frac{1}{2} \sum_{j=1}^{K} \partial_{y_j^2} \Big\{ \Big(mq_j\,(1 - q_j) + m(1-q_j)\, q_i \Big) w(y) \Big\} \\ &\quad - \sum_{j\neq k} \partial_{y_j} \partial_{y_k} \Big\{ mq_j \, q_k \, w(y) \Big\} \end{aligned}$$

$$0 = \left(1-\frac{1}{m}\right)\sum_{j=1}^K \partial_{y_j}\left\{y_j\, w\right\} + \sum_{j=1}^K \partial_{y_j^2}\left\{q_j\,(1-q_j) w(y)\right\} - \sum_{j\neq k}\partial_{y_j}\partial_{y_k}\left\{q_j\, q_k\, w(y)\right\}$$

If we define the matrix Q as the diagonal matrix with q_i in the i'th entry, and $q = (q_1,\ldots,q_K)^T$, with

$$A = Q - q\, q^T$$

we may write

$$0 = \left(1-\frac{1}{m}\right)\sum_{j=1}^K \partial_{y_j}\left\{y_j\, w\right\} + \sum_{j,k}\partial_{y_j}\partial_{y_k}\left\{Q_{j,k}\, w(y)\right\}$$

NACHSTER SCHRITT: SIEHE BACHELOR ARBEIT KITILAKIS!!!! AUSFUEHREN!!!!
Standard calculations (see e.g. [26] or the WKB ansatz) show that for an Ansatz

$$w(y) = \exp\left(-\frac{1}{2}\sum_{i,j} B_{i,j} y_i y_j\right)$$

the PDE superimposes a condition on B^{-1},

$$B^{-1}\left(1-\frac{1}{m}\right) + \left(1-\frac{1}{m}\right)B^{-1} = 2\,A$$

Here we need to take this equation with *grano salis*, as A is not invertible ($\mathbf{e}^T A = 0$), and so is B (consequence of mass conservation, that is, $\sum y_j = 0$). In any case, we find that a normal distribution with variance

$$\Sigma = \left(1-\frac{1}{m}\right)^{-1}\,[Q - q\, q^T]$$

satisfies the PDE. Hence,

$$\frac{1}{N}(X_1,\ldots,X_K) \approx_a q + N^{-1/2}\mathcal{N}(0, \mathrm{SI}\,[Q - q\, q^T]), \qquad \mathrm{SI} = (1-\frac{1}{m})^{-1}. \tag{2.30}$$

respectively

$$N^{1/2}\,(X_1/N - q_1,\ldots,X_K/N - q_K) \sim_a \mathcal{N}(0, \mathrm{SI}\,[Q - q\, q^T]).$$

If we introduce $\widetilde{m} = \sum_i N^i/N$, then $m = 1 + \widetilde{m}$, and

$$\mathrm{SI} = (1-\frac{1}{m})^{-1} = (1-\frac{1}{1+\widetilde{m}})^{-1} = \frac{\widetilde{m}+1}{\widetilde{m}} = 1 + \frac{1}{\widetilde{m}}.$$

In particular, if $1/\widetilde{m} = 0$ we obtain the normal approximation of the multinomial distribution. We can identify the strength of the zealot's effect by the size of $1/\widetilde{m}$.

Parameter estimation

We return to the Dirichlet distribution for the vote shares. How can we estimate the parameter from the data? We observe in the i'th district (from m districts, $i = 1, \ldots, m$) a certain number of voters V^i, structured by parties

$$V_1^i, \ldots, V_K^i, \qquad V^i = \sum_{\ell=1}^{K} V_\ell^i.$$

The model assumes that the voters of party ℓ, $\ell = 1, \ldots, K$, in district i can be separated in fixed voters N_ℓ^i and swing voters X_ℓ^i,

$$V_\ell^i = N_\ell^i + X_\ell^i.$$

Let $N^i = \sum_{\ell=1}^{K} X_\ell^i$ the total number of swing voters in that district, and $N_F^i = \sum_{\ell=1} K N_\ell^i$ that of fixed voters.
We assume that the fraction of fixed voters/swing voters are stable over the districts. That is, N^i/V^i is (approximately) independent on the district (on i). Let p denote the fraction of floating voters,

$$p = N^i/V^i, \qquad \Rightarrow \qquad 1 - p = N_F^i/V^i.$$

Furthermore, we assume that $N_\ell^i/V^i = \mu_\ell$ is independent of the vote district. With these assumptions, we can write the vote shares as

$$\begin{aligned} z_\ell^i &:= V_\ell^i/V^i = N_\ell^i/V^i + X_\ell^i/V^i = N_\ell^i/V^i + X_\ell^i/V^i \\ &= (N_F^i/V^i)\ N_\ell/N_F \ + \ (N^i/V^i)\ X_\ell^i \\ &= (1-p)\mu_\ell \ + \ p\ x_\ell^i \end{aligned}$$

where (note that $N_F^i = (1-p)V^i$)

$$(x_1^i, \ldots, x_K^i) \sim \mathrm{Dir}(\theta N_F^i \mu_1, \ldots, \theta N_F^i \mu_K) = \mathrm{Dir}(\theta(1-p)V^i\mu_1, \ldots, \theta(1-p)V^i\mu_K).$$

All in all, we have
<u>Measurements</u>:

z_ℓ^i	votes share of party ℓ in district i, where $\ell = 1, \ldots, K$, and $i = 1, \ldots m$
V^i	number of (active) voters in district i

<u>Parameters</u>:

$p \in [0,1]$	fraction of floating voters, identical for all districts
$\theta \in [0,1]$	fraction of active/influencing zealots, identical for all districts
$\mu_\ell \in [0,1]$	fraction of zealots supporting party ℓ, $\ell = 1, \ldots, K$, identical

We have, all in all, $m \times (K-1)$ data points (the vote shares ad up to 1), and $K - 1 + 2 = K + 1$ parameters. We aim at a maximum likelihood estimation,

$$(x_1, \ldots, x_K) = (z_\ell^1 - (1-p)\mu_1, \ldots, z_\ell^K - (1-p)\mu_K)/p \sim \mathrm{Dir}(\theta(1-p)V^i\mu_1, \ldots, \theta(1-p)V^i\mu_K).$$

That is,

$$\begin{aligned} \mathcal{L}(z_\ell^i | V^i, p, \theta, \mu_\ell) &= \prod_{i=1}^{m} \frac{\prod_{\ell'=1}^{K} \Gamma(\theta(1-p)V^i\,\mu_{\ell'})}{\Gamma(\theta(1-p)V^i)} \prod_{\ell=1}^{K} \left(\frac{z_\ell^i - (1-p)\mu_\ell}{p}\right)^{\theta(1-p)V^i\mu_\ell - 1} \\ &= \prod_{i=1}^{m} \frac{\prod_{\ell'=1}^{K} \Gamma(\theta(1-p)V^i\,\mu_{\ell'})}{\Gamma(\theta(1-p)V^i)\ p^{\theta(1-p)V^i - K}} \prod_{\ell=1}^{K} \left(z_\ell^i - (1-p)\mu_\ell\right)^{\theta(1-p)V^i\mu_\ell - 1} \end{aligned}$$

Exercises

Exercise 3.6: Determine the master equations for Sano's model.
Solution of Exercise 3.6 Let $p_{i_1,\ldots,i_K}(t) = P(X_1 = i_1, \ldots, X_K = i_K)$. Then,

$$
\begin{aligned}
\dot{p}_{i_1,\ldots,i_K} &= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right)\Big)\, p_{i_1,\ldots,i_K} \\
&\quad +\mu \sum_{j=1}^{K}\left(\sum_{k\neq j}(i_k+1)\left(\frac{i_j - 1 + \theta N_j}{M}\right) p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k + 1}\right) \\
&= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right)\Big)\, p_{i_1,\ldots,i_K} \\
&\quad +\mu \sum_{j=1}^{K}\left(\frac{i_j - 1 + \theta N_j}{M}\sum_{k\neq j}(i_k+1)\; p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k + 1}\right).
\end{aligned}
$$

2.3.5 Mauro Mobilia

Three party voter model - how long does it take to equilibrium? Possibility to eliminate an opinion. Again, how long does it take???
Uses the method by Grasman + Herwarden.
Do we learn someting interestign?????
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX hier weiter XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

2.4 Election and the Infinity-allele Moran model

2.4.1 Model

Tellier/Hoesel/Mueller

2.4.2 Island-model

We consider a variant of the noisy voter model. In contrast to the standard noisy voter model, where an individual assumes with a positive probability one of the possible opinions (independently of the number of supporters of that opinion), in the variant considered, there is a probability to create a new opinion. This model parallels the infinite allele model.
We consider a rudimentary spatial structure: there are two voting districts (demes, islands), where communication in possible within the election districts as well as between the election districts (gene flow between the islands is possible).

State of the model:
Each individual supports a certain opinion.

Dynamics of the model:
An individual changes his/her opinion at rate μ. With probability u, he/she creates a completely new opinion (group, party). With probability $1-u$ he/she contacts an individual our of the $2\,N$ individual (including his/herself “selfing”). Fist, he/she decides if a person from the own deme is contacted (probability q), or an individual from the other deme is the contactee (probability $1-q$). Consequently, the focal individual copies the opinion of a randomly chosen member in the selected deme.

Analysis

We focus on the probability to be IBD for two individuals. These individuals are (deliberately) selected to be both in deme 1, in deme 2, or in different demes. We might symbolize the configuration by a two-tupel, where the entries counts the number of lineages in the firs/second deme, s.t. $(2,0)$ corresponds to: all lineages in deme 1, or $(1,1)$ indicates that one lineage is in deme 1, and the other one is in deme 2.

For symmetry reasons, we have two different probabilities to be IBD:

$$\begin{aligned} q_{within} &= P(IBD|\text{ initial configuration }(2,0)) \qquad (2.31)\\ &= P(IBD|\text{ initial configuration }(0,2)),\\ q_{between} &= P(IBD|\text{ initial configuration }(1,1)). \qquad (2.32)\end{aligned}$$

Communication probability between districts does not scale with size

Theorem 2.24 *Let $q\in(0,1)$ be fixed, and assume that u scales with N,*

$$\theta = 2\,N\,u$$

where θ is given. Then, the probability that two particles are IBD is given by

$$q_{within} = \frac{1}{1+\theta} + \frac{(q-1/2)}{1-q}\,\frac{\theta^2}{2(1+\theta)^2}N^{-1} + \mathcal{O}(N^{-2}).$$

Furthermore,

$$q_{between} = \frac{1}{1+\theta} - \frac{(q-1/2)}{1-q}\,\frac{\theta(2+\theta)}{2(1+\theta)^2}N^{-1} + \mathcal{O}(N^{-2}).$$

Remark 2.25 *In lowest order, the probabilities do not depend on the configuration (both particles from one deme, or the particles are from different demes). Idea: The particles jump back and forth between the demes on a time scale $\mathcal{O}(1)$, but coalesce/mutate on a time scale $\mathcal{O}(1/N)$.*

Proof: (of the theorem).
Note by $c_{2,0}$ ($c_{0,2}$) the situation that both lineages are within deme 1 (deme 2), and by $c_{1,1}$ the situation that the two lineages are in different demes. To obtain $P(IBD)$ we aim at the probability that coalescence happens before a mutation takes place. As the model is symmetric

(both demes have the same size etc.), $P(IBD|c_{2,0}) = P(IBD|c_{0,2})$. For shortness of notation, we write $q_{2,0}$ instead of q_{within} and $q_{1,1}$ instead of $q_{between}$. The rates of the possible events are given

event	state $c_{2,0}$	state $c_{1,1}$
jump	$2\mu(1-q)$	$2\mu\,(1-q)(1-1/N)$
mutation	$2\mu u$	$2\mu u$
coalescence	$2\mu q\,(1/N)$	$2\mu(1-q)\,(1/N)$

If we use $\theta = 2Nu$, then the table reads

event	state $c_{2,0}$	state $c_{1,1}$
jump	$2\mu(1-q)$	$2\mu\,(1-q)(1-1/N)$
mutation	$2\mu\theta/(2N)$	$2\mu\theta/(2N)$
coalescence	$2\mu q/N$	$2\mu(1-q)/N$

We use the embedded jump chain, and introduce two additional states: A if a coalescence event did happen, and B if a mutation took place. Let $q_{2,0}$ ($q_{1,1}$) denote the probability that we eventually jump into state A if we start in state $c_{2,0}$ ($c_{1,1}$). Furthermore, we define the probabilities for the jump chain

event	state $c_{2,0}$	state $c_{1,1}$
jump	$p_2^j = \frac{1-q}{1-q+(q+\theta/2)/N}$	$p_1^j = \frac{(1-q)(1-1/N)}{1-q+(\theta/2)/N}$
mutation	$p_2^m = \frac{\theta/(2N)}{1-q+(q+\theta/2)/N}$	$p_1^m = \frac{\theta/(2N)}{1-q+(\theta/2)/N}$
coalescence	$p_2^c = \frac{q/N}{1-q+(q+\theta/2)/N}$	$p_1^c = \frac{(1-q)/N}{1-q+(\theta/2)/N}$

Then,

$$\begin{aligned}
q_{2,0} &= P(c_{2,0} \to \ldots \to A) \\
&= P(c_{2,0} \to \ldots \to A|\text{no jump})\,(1-p_2^j) + P(c_{2,0} \to \ldots \to A|\text{jump})\,p_2^j \\
&= \frac{p_2^c}{p_2^c + p_2^m}\,(1-p_2^j) + q_{1,1}\,p_2^j
\end{aligned}$$

Since $p_2^j + p_2^m + p_2^c = 1$, we find for $q_{2,0}$ (and in a parallel equation for $q_{1,1}$)

$$\begin{aligned}
q_{2,0} &= p_2^c + p_2^j\,q_{1,1} \\
q_{1,1} &= p_1^c + p_1^j\,q_{2,0}.
\end{aligned}$$

Hence,

$$q_{2,0} = \frac{p_2^c + p_2^j p_1^c}{1 - p_1^j p_2^j}, \qquad q_{1,1} = \frac{p_1^c + p_1^j p_2^c}{1 - p_1^j p_2^j}.$$

Using the probabilities defined above, we have

$$\begin{aligned}
q_{2,0} &= \frac{\frac{q/N}{1-q+(q+\theta/2)/N} + \frac{1-q}{1-q+(q+\theta/2)/N}\;\frac{(1-q)/N}{1-q+(\theta/2)/N}}{1 - \frac{(1-q)(1-1/N))}{1-q+(\theta/2)/N}\;\frac{1-q}{1-q+(q+\theta/2)/N}} \\
&= \frac{1}{1+\theta} + \frac{(q-1/2)}{1-q}\;\frac{\theta^2}{2(1+\theta)^2}N^{-1} + \mathcal{O}(N^{-2}).
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
q_{1,1} &= \frac{\frac{(1-q)/N}{1-q+(\theta/2)/N} + \frac{(1-q)(1-1/N)}{1-q+(\theta/2)/N}\ \frac{q/N}{1-q+(q+\theta/2)/N}}{1-\frac{(1-q)(1-1/N)}{1-q+(\theta/2)/N}\ \frac{1-q}{1-q+(q+\theta/2)/N}} \\
&= \frac{1}{1+\theta} - \frac{(q-1/2)}{1-q}\ \frac{\theta(2+\theta)}{2(1+\theta)^2} N^{-1} + \mathcal{O}(N^{-2}).
\end{aligned}
$$

□

Application to election data

From that result, we can find an estimator for q.

Proposition 2.26

$$
\frac{(q-1/2)}{1-q} = N\ \frac{2(\,P(IBD|c_{2,0}) - P(IBD|c_{1,1})\,)}{2-(\,P(IBD|c_{2,0}) + P(IBD|c_{1,1})\,)} + \mathcal{O}(N^{-1}). \tag{2.33}
$$

Proof: Note

$$
P(IBD|c_{2,0}) - P(IBD|c_{1,1}) = \frac{(q-1/2)}{1-q}\ \frac{\theta}{1+\theta} N^{-1} + \mathcal{O}(N^{-2}).
$$

Furthermore,

$$
1 - \frac{1}{2}(P(IBD|c_{2,0}) + P(IBD|c_{1,1})) = \frac{\theta}{1+\theta} + \frac{(q-1/2)}{1-q}\ \frac{\theta}{2(1+\theta)^2} N^{-1} + \mathcal{O}(N^{-2}).
$$

Hence,

$$
\frac{2(\,P(IBD|c_{2,0}) - P(IBD|c_{1,1})\,)}{2-(\,P(IBD|c_{2,0}) + P(IBD|c_{1,1})\,)} = \frac{(q-1/2)}{1-q}\ N^{-1} + \mathcal{O}(N^{-2}).
$$

□

Communication probability between districts does scale with size

In this section, we assume that local communication prevails if an election district is large.

Theorem 2.27 *Let $q = 1-(r\,N^{\alpha})^{-1}$, $\alpha \in [1/2, 1)$, $\theta = 2\,N\,u$, where r, α, and θ are given parameters. Then, the probability that two particles are IBD is given by*

$$
\begin{aligned}
q_{2,0} &= \frac{1}{1+\theta} + r\,\frac{\theta^2}{4(\theta+1)^2}\,N^{-\alpha} + \mathcal{O}(N^{-1}) \\
q_{11} &= \frac{1}{1+\theta} - r\,\frac{\theta\,(\theta+2)}{4(\theta+1)^2}\,N^{-\alpha} + \mathcal{O}(N^{-1}).
\end{aligned}
$$

Proof: With the same reasoning as in theorem 2.24, we obtain

$$q_{2,0} = \frac{\frac{q/N}{1-q+(q+\theta/2)/N} + \frac{1-q}{1-q+(q+\theta/2)/N}\,\frac{(1-q)/N}{1-q+(\theta/2)/N}}{1 - \frac{(1-q)(1-1/N))}{1-q+(\theta/2)/N}\,\frac{1-q}{1-q+(q+\theta/2)/N}}$$

$$q_{1,1} = \frac{\frac{(1-q)/N}{1-q+(\theta/2)/N} + \frac{(1-q)(1-1/N)}{1-q+(\theta/2)/N}\,\frac{q/N}{1-q+(q+\theta/2)/N}}{1 - \frac{(1-q)(1-1/N)}{1-q+(\theta/2)/N}\,\frac{1-q}{1-q+(q+\theta/2)/N}}.$$

Replacing q by $1-(r\,N^{\alpha})^{-1}$, and using Taylor expansion yields the result.

□

We again combine these two equations to find an estimator for r. If we add the equations for $q_{1,1}$ and $q_{2,0}$ we obtain

$$\frac{\theta}{1+\theta} = 1 - \frac{1}{1+\theta} = 1 - \frac{1}{2}(q_{1,1}+q_{2,0}) + \text{h.o.t.}$$

If we subtract the two equation, we have

$$q_{2,0} - q_{1,1} + \text{h.o.t.} = r\left(\frac{\theta^2+\theta(\theta+2)}{4(\theta+1)^2}\right) = r\left(\frac{\theta}{2(\theta+1)}\right) = \frac{1}{2}\left(1-\frac{1}{2}(q_{1,1}+q_{2,0})\right) + \text{h.o.t.}.$$

and hence

$$r = \frac{2(q_{2,0}-q_{1,1})}{1-\frac{1}{2}(q_{1,1}+q_{2,0})} + \text{h.o.t.}$$

Analysis of election data

How do we estimate q in practice?

We select two election districts, and compute the vector of vote shares x_i (length K in case of K parties) for district $i \in \{1,2\}$, and determine the total number of valid votes N_i. Then, we define

$$\hat{q}_{2,0} = \frac{1}{2}\left(\sum_{k=1}^{K}(x_1)_k\,\frac{N_1(x_1)_k-1}{N_1-1} + \sum_{k=1}^{K}(x_2)_k\,\frac{N_1(x_2)_k-1}{N_1-1}\right)$$

resp.

$$\hat{q}_{2,2} = \sum_{k=1}^{K}(x_1)_k(x_2)_k.$$

Finally, we have the estimation

$$\hat{r} = \frac{2(\hat{q}_{2,0}-\hat{q}_{1,1})}{1-\frac{1}{2}(\hat{q}_{1,1}+\hat{q}_{2,0})}$$

We decompose all election districts into regions, and only compute $\hat{r}$ for each combination within a region. At the end, we take the mean from all combinations, and obtain one within-island communication coefficient per election.
Using this estimator, we analyze data. For Germany, we use all combinations of election districts, provided that the ℓ^1 -distance of the election results is below or equal to the median of all distances (to ensure a certain homogeneity). For the Netherlands, we only take into account the

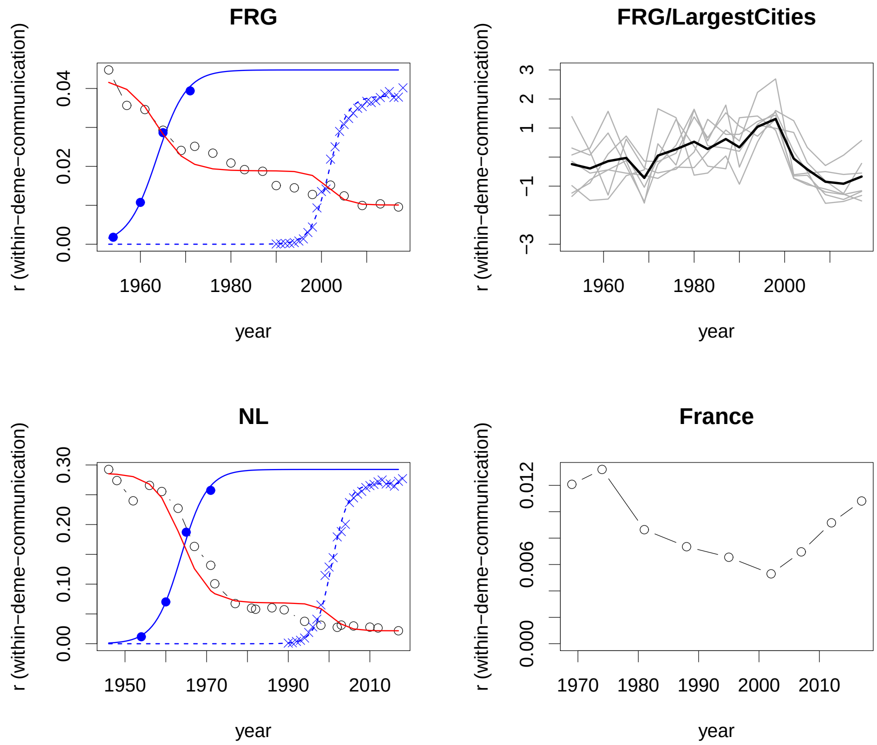


Figure 2.7: Analysis of the communication coefficient r (note: the smaller r, the larger the communication between islands) for whole Germany, the large cities in Germany (black: averaged), the Netherlands and France. In whole Germany, and the Netherlands, we add the distribution of television and internet (blue curves), and the fit of r by a linear combination of TV and internet (red line).)

80% largest election districts, as the size of election district heavily differ.
We correlate the results with the penetration of the population with TV and internet (see also Tab. **??**). The linear model shows a strong correlation with TV availability (the inter-deme communication increases as r decreases); also the internet may have an influence, but that influence is less clear. Particularly, in the Netherlands, there is a recent decrease of r starting around 1995, which seems to be too early to be caused by the internet (where we used data for the availability of TV and internet in Germany to explain the data in The Netherlands).

Interestingly enough, the within-island communication decrease, that is, the global communication increases. We cannot find traces of localization by filer-bubbles.

Germany		
variable	point estimate	p-value
intercept	0.04	8e-13
TV	-0.02	4.3e-8
internet	-0.01	0.0003
the Netherlands		
variable	point estimate	p-value
intercept	0.28	2e-16
TV	-0.21	1.12e-12
internet	-0.05	0.0019

Table 2.1: Estimation of the effect of TV and internet o the intra-island communication.

2.5 Elections and the Potts model

In the present section we present a statistical physics approach to describe election results. The approach originated in the investigation of spin systems: An atom may assume the state "spin up" or "spin down". The spins will change according to the influence of an external magnetic field, and by the influence of neighboring spins. Several authors [33, 17, 37] point out that this model bears some similarities with the situation of voters, who are influenced by other voters, and perhaps are also influenced by some global information, broadcast by mass media. Let us find out how the stochastic physics approach works, and what we can learn form it. The model formulation, however, is completely different from the formulation of the models used above: E.g. in the voter model, we started off with mechanisms that have been re-formulated as a stochastic process. The resulting process has an invariant measure. We used that invariant measure to analyze data. In the statistical physics approach, we directly start with a distribution. No dynamics is required. Only later, we aim to establish mechanisms formulated in terms of a stochastic process, which has that distribution as an invariant measure.
We follow Nicolao et al. [37], who constructs a q-state Potts model that allows to analyze election data. In the Potts model, the notion of entropy is central. Therefore, let us briefly discuss some basic facts related to entropy.

2.5.1 Prelude: Shannon Entropy

The concept of entropy first appeared in the context of thermodynamics, similar to the definition of the temperature. Only slowly it became clear that temperature and entropy are different of nature (though both are connected with the disorder of a system [23]).

For us it is sufficient to consider a discrete system. Our system has K states, named by $s_1, \ldots, s_K$. There is a probability measure R that describes the state of the system. We find state s_i with probability $R(s_i)$.

Definition 2.28 *Let the real constant k be strictly positive. The Shannon entropy for the probability measure R is given by*

$$S(R) = -k \sum_{i=1}^{K} R(s_i)\ \ln(R(s_i)). \tag{2.34}$$

Note that in mathematics and mathematical physics, usually $k = 1$, while in thermodynamics, $k = k_B$ is the Boltzmann's constant. Luckily, in the present context, we do not need the Boltzmann's constant but use $k = 1$.

Since $R(s_i) \in [0, 1]$, we have $S(R) \geq 0$.

Choosing the definition $0\ \ln(0) := \lim_{x\to 0} x \ln(x) = 0$, we find that the entropy is zero if $R(s_{i_0}) = 1$ for some $i_0 \in \{1, \ldots, K\}$, and $P(s_i) = 0$ for $i \neq i_0$. If the state is (almost) sure and not random, the entropy is zero. If we have $R(s_i) = 1/K$ for all possible states, we find $S = \ln(K)$. All other distributions have a smaller entropy than $S = \ln(K)$ (see exercise 5.1).

As an intuitive example, considere a classroom with N classmates. Each of them can sit on his/her chair and concentrate, or can be jumped up and babble. If the teacher is there, then (hopefully) all children concentrate. The state is known. Entropy is zero. If the teacher leaves

the classroom, some children will be still on his/her chair, and some have jumped up, it will be noisy - the state is unknown, the entropy is large. The less we know the larger the entropy: If the teacher is there, we know the state of all children, if the teacher is gone we have no idea what the children will do.
The entropy quantifies the degree of uncertainty about the system. The less we know about the system the larger the entropy. We can also say: the less ordered is the system.

The following result can give us some further intuition for this a interpretation of the entropy as a measuring uncertainty. We play a game with a partner. The partner selects a state s_{i_0} from our K states according to the random measure R. It is our task to find out which s_{i_0} he/she did choose. We are only allowed to ask yes/no questions. And, of course, we want to be clever, that is, the expected number of questions to ask should be minimal.

One possibility is to go through all single states. That is, we can ask "did you choose s_1? Did you choose s_2? etc. At the end, we get the desired answer. The expected number of questions is in that case

$$E(\text{number of questions}) = \sum_{i=1}^{K} i\, P(\text{selected state is } s_i) = \sum_{i=1}^{K} i\, R(s_i).$$

This strategy cannot be really clever - particularly, re-numbering the states s_i yields to different results. Clearly, a good strategy to ask questions should not depend on the order in which the states are numbered.
We can do better by a bi-partition algorithm. Let $\Omega = \{s_1, \ldots, s_K\}$ denote the state space. We divide Ω in two disjoint subsets of equal probability,

$$\Omega = A_1 \dot\cup A_2, \qquad R(A_1) = R(A_2) = 1/2.$$

As we are in a discrete situation, the equality $R(A_1) = R(A_2) = 1/2$ may be impossible to fulfill. For simplicity, we assume that we can always divide sets in two sets of equal probability.
In any case, we now can ask "is your state in set A_1?". According to the answer, we can focus on A_1 or A_2 only. We multiply all probabilities within the focal set by 2, and are in the same situation as before: The sum of the probabilities in the focal set is 1. It is clear how to proceed: We again divide the interesting subset into two subsets of equal probability, and divide, while multiplying the probabilities by 2, and divide, while multiplying the probabilities by 2,. . . , until we are left with a subset of size one only. This subset contains the desired state.
How many question do we need to ask? Let us assume our partner did choose the state s_{i_0}. The probability for this state is $R(s_{i_0})$. If we had to divide the original set n times, we did multiply the probability n times by 2 to always guarantee that the sum of all probabilities in the sequence of subsets is always 1. Hence, $2^n R(s_{i_0}) = 1$, and we required $n \approx -\log_2(R(s_{i_0}))$ steps. The expected number of questions thus reads

$$-\sum_{i=1}^{K} R(s_i)\, \log_2(R(s_i)) = -\sum_{i=1}^{K} R(s_i)\, \ln(R(s_i))/\ln(2) = S(R)/\ln(2).$$

The entropy is proportional to the number of questions necessary to determine the unknown state that our partner did choose, and in that, a measure about our uncertainty.

Exercises

Exercise 5.1: Consider a probability measure on K discrete states $s_1,\dots,s_K$. Show that $R(s_i) = 1/K$ maximizes the entropy.
Hint: Optimization with constrains; use Lagrange multipliers.

Solution of Exercise 5.1 Let $p_i = R(s_i)$. The entropy is a function on $p_1, \dots, p_K$ given by

$$S(p_1, \dots, p_K) = -\sum_{i=1}^{K} p_i \ln(p_i).$$

To find the maximizing probabilities, we use a Lagrange multiplier λ to ensure the condition $\sum_{i=1}^{K} p_i = 1$, and maximize

$$f(p_1, \dots, p_K, \lambda) = -\sum_{i=1}^{K} p_i \ln(p_i) + \lambda(1 - \sum_{i=1}^{K} p_i).$$

Hence,

$$0 = \frac{\partial}{\partial p_i} f(p_1, \dots, p_K, \lambda) = \ln(p_i) + 1 - \lambda$$

and thus

$$p_i = p_j = e^{\lambda - 1}.$$

All probabilities are the same. As the probabilities sum up to one, they are given by

$$p_i = 1/K.$$

The probability measure that maximizes the entropy assigns an equal probability for each state. This probability function gives the least information about an randomly selected item.

2.5.2 Abstract framework: Potts model

The Potts model was proposed by the mathematician Renfrey Potts in his dissertation theses in 1952 as a simplification of the Ising spin model [40]. Indeed, it is simple enough to allow for some analytical results, but complicated enough to allow for applications to real world problems. We introduce the Potts model in the way Nicolau et al. proposed in [37].

Model 2.29 **q-state Potts model.** *Consider a population of N individuals/voters. Each of these individuals can be in favor of one party/candidate out of q parties/candidates. Let*

$$\Sigma_N = \{1, \dots, q\}^N.$$

If we would know the opinion of each individuals, which we number by $1, \dots, N$, then $\sigma \in \Sigma_N$ would characterize the state of the population: Individual i would have opinion σ_i. As we have only vague information, the state of the population is given by some probability function R on Σ_N. However, even R is unknown. What we do know is the result of e.g. polls, that give us some information about observable $F_\ell : \Sigma_N \to \mathbb{R}$. That is, the expectations

$$E_R(F_\ell(.)) \stackrel{def}{=} \sum_{\sigma \in \Sigma_N} F_\ell(\sigma)\, R(\sigma)$$

are known for M (independent) functions $F_1, \ldots, F_M : \Sigma_N \to \mathbb{R}$.
(1) We construct a probability measure Q on the power set of Σ_N, that approximates/interpolates the unknown R. If Q is a good approximation, we can compute all magnitudes of interest using Q instead of R. Of course, Q should give us the same values for the observables as R, that is,

$$E_Q(F_\ell(.)) \stackrel{def}{=} \sum_{\sigma\in\Sigma_N} F_\ell(\sigma)\, Q(\sigma) \stackrel{!}{=} E_R(F_\ell(.)).$$

In the last formula, the first equal sign is just the definition of the expectation, the second equal sign is a condition/requirement for Q.
(2) There are, of course, many probability measures that satisfy that condition. If the expectations of the M observables is all we know, we should use the probability measure with the maximal disorder, that is, Q should maximize an entropy. The classical entropy to use is the Shannon entropy,

$$S(Q) = -\sum_{\sigma\in\Sigma_N} Q(\sigma)\, \ln(Q(\sigma)).$$

We call the probability measure Q that satisfies these two conditions the Boltzmann distribution.

Below, we apply that model to election data. The observables represent the results of polls. E.g., the expected vote share of a party are known. Perhaps also correlations between opinions can be visible. In that, we construct a probability measure for the current opinions of the target group.

Notation:
Later we will denote by σ elements in Σ_N. That is, σ is a vector that assigns to each individual i, $i = 1, \ldots, N$, a preference $\sigma_i \in \{1, \ldots, q\}$. In contrast, $\hat\sigma$ denotes a random variable, described by a certain probability measure Q on Σ_N,

$$P_Q(\hat\sigma = \sigma) = Q(\sigma).$$

Similarly, $\hat\sigma_i$ is the i'th entry of the random vector $\hat\sigma$, which represents the (uncertain) state of individual i; the distribution of $\hat\sigma_i$ is the corresponding marginal distribution of $\hat\sigma$.
In that, $E_Q(F_\ell(\hat\sigma))$ is the expected value of $F_\ell(\hat\sigma)$, where the distribution of $\hat\sigma$ is Q. As we will only consider one probability distribution denoted by Q, we drop the index Q and write $E(F_\ell(\hat\sigma))$ instead of $E_Q(F_\ell(\hat\sigma))$, or $P(\hat\sigma = \sigma)$ instead of $P_Q(\hat\sigma = \sigma)$, etc. Wit this notation, we find for example

$$E(F(\hat\sigma)) \;=\; \sum_{\sigma\in\Sigma_N} F(\sigma)P(\hat\sigma = \sigma) \;=\; \sum_{\sigma\in\Sigma_N} F(\sigma)Q(\sigma).$$

As we find out in the next theorem, the distribution Q for the Potts Model can be explicitly determined.

Theorem 2.30 *The Boltzmann distribution Q that maximizes the Shannon entropy under the restriction that $E(F_\ell(\hat\sigma)) = f_\ell$ for $\ell = 1, \ldots, M$, where $F_\ell : \Sigma_N \to \mathbb{R}$, and $f_\ell \in \mathbb{R}$ are* *prescribed, is given by*

$$Q(\sigma) = \frac{e^{-H(\sigma)}}{Z}, \qquad Z = \sum_{\sigma \in \Sigma_N} e^{-H(\sigma)} \tag{2.35}$$

with

$$H(\sigma) = -\sum_{\ell=1}^{M} \lambda_\ell \, F_\ell(\sigma), \tag{2.36}$$

where the Lagrange multipliers $\lambda_1, \ldots, \lambda_M$ *are determined by the conditions*

$$\sum_{\sigma \in \Sigma_N} F_\ell(\sigma) \, \frac{e^{\sum_{\ell=1}^{M} \lambda_\ell \, F_\ell(\sigma)}}{Z} = f_\ell, \qquad \ell = 1, \ldots, M. \tag{2.37}$$

Note that it is not clear if the conditions (2.37) can be satisfied. If, for example, f_ℓ is not in the range of F_ℓ, then this condition cannot be realized. Furthermore, it is also not clear if there is a unique solution. In any case, if there is a Boltzmann distribution, we have a way to explicitly construct it.

Proof: (of theorem 2.30) Maximization under constrains can be done using Lagrange multipliers; we consider the functional

$$\mathcal{S}(Q, \lambda_1, \ldots, \lambda_{M+1}) = S(Q) + \sum_{i=1}^{M} \lambda_\ell \Big(E(F_\ell(\hat{\sigma})) - f_\ell \Big) + \lambda_{M+1} \Big(\sum_{\sigma \in \Sigma_N} Q(\sigma) - 1 \Big).$$

The last term guarantees that $\sum_{\sigma \in \Sigma_N} Q(\sigma) = 1$. Next we use that Q is defined by the real number $Q(\sigma) \in [0,1]$ for each given $\sigma \in \Sigma_N$. If we take the derivative w.r.t. a specific component $Q(\sigma_0)$ (that is, we select $\sigma_0 \in \Sigma_N$ and aim to obtain the value of Q for this specific state), we find (recall $E(F(\hat{\sigma})) = \sum_{\sigma \in \Sigma_N} F(\sigma) Q(\sigma)$, that is, $\frac{\partial}{\partial Q(\sigma_0)} E_Q(F(\sigma)) = F(\sigma_0)$)

$$\begin{aligned}
\frac{\partial}{\partial Q(\sigma_0)} \mathcal{S}(Q, \lambda_1, \ldots, \lambda_{M+1}) &= -\ln(Q(\sigma_0)) - 1 + \sum_{i=1}^{M} \lambda_\ell \, F_\ell(\sigma_0) + \lambda_{M+1} \\
&= -\ln(Q(\sigma_0)) - 1 - H(\sigma_0) + \lambda_{M+1}.
\end{aligned}$$

That is, $\frac{\partial}{\partial Q(\sigma_0)} \mathcal{S} = 0$ implies

$$Q(\sigma_0) = e^{\lambda_{M+1} - 1 - H(\sigma_0)} = e^{\lambda_{M+1} - 1} \; e^{-H(\sigma_0)}.$$

λ_{M+1} is defined by the requirement $\sum_{\sigma \in \Sigma_N} Q(\sigma) = 1$, that is,

$$(e^{-\lambda_M + 1})^{-1} = \sum_{\sigma \in \Sigma_N} e^{-H(\sigma)} = Z.$$

□

As the background of the model is quantum physics and the Ising model, mostly $H(\sigma)$ is referred to as "Hamiltonian" and Z is named "partition function". We will discuss some intuitive interpretations of the partition function below.

2.5.3 First example: Expectations known

As a first example, we assume that the poll/our foreknowledge only addresses the opinion of each person. For each person, the information obtained by the poll is the number of a candidate, that is, a number between 1 and q. We measure the frequency of individuals voting for candidate i. We translate the result into observables. Instead of asking: "For which candidate will you vote"? We ask q questions: "Will you vote for candidate 1?", "Will you vote for candidate 2?",... We do not store information about individuals but only memorize the overall fraction of votes. That is, we introduce

$$F_\ell(\sigma) = \frac{1}{N}\sum_{i=1}^{N}\delta(\sigma_i,\ell), \qquad \ell = 1,\dots,q.$$

Here, $\delta(\sigma_i,\ell)$ denotes the Kronecker-function, which is 1 if $\sigma_i = \ell$ and 0 otherwise. That is, we obtain the empirical value $\overline{x}_\ell$ for $E(F_\ell(\hat\sigma))$, which can be reformulated as the probability that a randomly selected individual will vote for candidate $\ell \in \{1,\dots,q\}$. These questions/observables yield no idea about correlations. As our Q maximizes the entropy (the disorder), we might expect that the states $\hat\sigma_i$ are independent w.r.t. the random measure Q. That is, we might expect that Q yields the multinomial distribution with probability $\overline{x}_\ell$ for outcome ℓ. The next proposition shows indeed that this intuitive idea leads to the correct insight.

Proposition 2.31 *Let $F_\ell(\sigma) = \frac{1}{N}\sum_{i=1}^N \delta(\sigma_i,\ell)$ and require $E(F_\ell(\hat\sigma)) = \overline{x}_\ell$ for $\ell = 1,\dots,q$. Define $N_\ell(\sigma) = \sum_{i=1}^N \delta(\sigma_i,\ell)$. Then*

$$Q(\sigma) = \prod_{\ell=1}^{q} \overline{x}_\ell^{N_\ell(\sigma)}$$

and

$$P(N_1(\hat\sigma) = n_1,\dots,N_q(\hat\sigma) = n_q) = N!\prod_{\ell=1}^{q}\frac{\overline{x}_\ell^{n_\ell}}{n_\ell!}.$$

Proof: According to proposition 2.30 we have (with the notation $N_\ell(\sigma)$ introduced above)

$$H(\sigma) = -\sum_{\ell=1}^{q}\lambda_\ell\,F_\ell(\sigma) = -\,\frac{1}{N}\,\sum_{\ell=1}^{q}\lambda_\ell\,N_\ell(\sigma)$$

Hence,

$$Q(\sigma) = \frac{1}{Z}\,e^{-H(\sigma)} = \frac{1}{Z}\,e^{\frac{1}{N}\,\sum_{\ell=1}^q \lambda_\ell\,N_\ell(\sigma)} = \frac{1}{Z}\,\prod_{\ell=1}^{q}\left(e^{\frac{1}{N}\,\lambda_\ell}\right)^{N_\ell(\sigma)} = \frac{1}{Z}\,\prod_{\ell=1}^{q}\rho_\ell^{N_\ell(\sigma)}$$

where $\rho_\ell := e^{\frac{1}{N}\,\lambda_\ell} \geq 0$. In this notation, we have

$$Z = \sum_{\sigma\in\Sigma_N}\prod_{\ell=1}^{q}\rho_\ell^{N_\ell(\sigma)}.$$

As the number of $\sigma \in \Sigma_N$ with given $N_1(\sigma) = n_1,\dots,N_q(\sigma) = n_q$ reads $\binom{N}{n_1,\dots,n_q}$, we have

$$Z = \sum_{(n_1,\dots,n_q)}\binom{N}{n_1,\dots,n_q}\prod_{\ell=1}^{q}\rho_\ell^{n_\ell}$$

where the sum extends over all tuples $(n_1, \ldots, n_q)$ with $\sum_{\ell=1}^q n_\ell = N$. That is, $Z = \left(\sum_{\ell=1}^q \rho_\ell\right)^N$, and (using $N = \sum_{\ell=1}^q N_\ell(\sigma)$)

$$Q(\sigma) = \prod_{\ell=1}^q \left(\frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}\right)^{N_\ell(\sigma)}.$$

If we only consider the summary statistics $N_1(\sigma), \ldots, N_q(\sigma)$, we multiply the probability $Q(\sigma)$ by the number of possible combinations, and obtain

$$P(N_1(\hat\sigma) = n_1, \ldots, N_q(\hat\sigma) = n_q) \;=\; N! \prod_{\ell=1}^q \frac{1}{n_\ell!} \left(\frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}\right)^{n_\ell}.$$

That is, $(N_1(\sigma), \ldots, N_q(\sigma))$ is distributed according to a multinomial distribution. Therefore,

$$E(N_\ell(\hat\sigma)) = N \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}} \quad \Rightarrow \quad E(F_\ell(\hat\sigma)) = E(N_\ell(\hat\sigma)/N) = \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}.$$

In order to determine $\rho_\ell / \sum_{\ell'=1}^q \rho'_\ell$, we inspect the conditions for the Lagrange multipliers

$$\overline{x}_\ell = E(F_\ell(\hat\sigma)) = \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}.$$

An individual assumes state $\ell \in \{0, \ldots, q\}$ with probability $p_\ell = \rho_\ell / \sum_{\ell'=1}^q \rho_{\ell'} = \overline{x}_\ell$.

$\square$

Remark 2.32 *Note that we did not assume that the σ_i are independent, but we exclusively assumed that the only information we have is information about the expectations. The Potts-machinery then implied that we have a multinomial distribution, that is, that the σ_i's are independent. This fact is caused by the maximization of the entropy, or equivalently, by the maximization of the disorder.*

We did derive the first moments of Q. The next lemma addresses the second moments of Q.

Lemma 2.33 *For $1 \leq \ell, \ell' \leq q$, we have*

$$\begin{aligned} & E(F_\ell(\hat\sigma)\, F_{\ell'}(\hat\sigma))) - E(F_\ell(\hat\sigma))\, E(F_{\ell'}(\hat\sigma)) \\ = \; & \frac{1}{N}\Big[\delta(\ell,\ell')E(F_\ell(\hat\sigma)) - E(F_\ell(\hat\sigma))E(F_{\ell'}(\hat\sigma))\Big]. \end{aligned} \tag{2.38}$$

Proof: As the states of different individuals are uncorrelated, we find

$$
\begin{aligned}
& E(N_\ell(\hat\sigma)\, N_{\ell'}(\hat\sigma)) - E(N_\ell(\hat\sigma))\, E(N_{\ell'}(\hat\sigma)) \\
= \;& E\Bigg(\left(\sum_{i=1}^{N} \delta(\hat\sigma_i,\ell)\right)\left(\sum_{j=1}^{N} \delta(\hat\sigma_j,\ell')\right)\Bigg) - \left(\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell))\right)\left(\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell'))\right) \\
= \;& \sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell)\,\delta(\hat\sigma_i,\ell')) + \sum_{i,j=1,i\neq j}^{N} E(\delta(\hat\sigma_i,\ell)\,\delta(\hat\sigma_j,\ell')) \\
& - \left(\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell))\right)\left(\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell'))\right) \\
= \;& \sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell)\,\delta(\ell,\ell')) \\
& + \sum_{i,j=1,i\neq j}^{N} E(\delta(\hat\sigma_i,\ell))\, E(\delta(\hat\sigma_j,\ell')) - \sum_{i,j=1}^{N} E(\delta(\hat\sigma_i,\ell))\, E(\delta(\hat\sigma_j,\ell')) \\
= \;& \delta(\ell,\ell') \sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell)) - \sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell))\, E(\delta(\hat\sigma_i,\ell')).
\end{aligned}
$$

We now make a second time use of the fact that states of different individuals are i.i.d. Hence, $E(\delta(\hat\sigma_i,\ell)) = E(\delta(\hat\sigma_j,\ell))$, and $E(\delta(\hat\sigma_i,\ell)) = \frac{1}{N}\sum_{k=1}^{N} E(\delta(\hat\sigma_k,\ell))$. Therefore,

$$
\begin{aligned}
\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell))\, E(\delta(\hat\sigma_i,\ell')) &= \sum_{i=1}^{N}\left(\frac{1}{N}\sum_{k=1}^{N} E(\delta(\hat\sigma_k,\ell))\right)\left(\frac{1}{N}\sum_{k=1}^{N} E(\delta(\hat\sigma_k,\ell'))\right) \\
&= \frac{1}{N}\left(\sum_{k=1}^{N} E(\delta(\hat\sigma_k,\ell))\right)\left(\sum_{k=1}^{N} E(\delta(\hat\sigma_k,\ell'))\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E(N_\ell(\hat\sigma)\, N_{\ell'}(\hat\sigma)) - E(N_\ell(\hat\sigma))\, E(N_{\ell'}(\hat\sigma)) \\
= \;& \delta(\ell,\ell') \sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell)) - \frac{1}{N}\left(\sum_{i=1}^{N} E(\delta(\hat\sigma_i,\ell))\right)\left(\sum_{j=1}^{N} E(\delta(\hat\sigma_j,\ell'))\right) \\
= \;& \delta(\ell,\ell') E(N_\ell(\hat\sigma)) - \frac{1}{N} E(N_\ell(\hat\sigma)) E(N_{\ell'}(\hat\sigma)).
\end{aligned}
$$

If we divide this equation by N^2, the result follows.

$\square$

2.5.4 Second example: Curie-Weiss model

We now add information about the overall correlations in the population. That is, we not only measure expectations as before,

$$
N_\ell(\sigma) = \sum_{i=1}^{N} \delta(\sigma_i,\ell), \qquad E(N_\ell(\hat\sigma)) = N\,\overline{x}_\ell, \qquad \ell = 1,\dots,q,
$$

(where $\overline{x}_\ell$ is known from measurements) but we also measure

$$G(\sigma) = \sum_{i<j}^{N} \delta(\sigma_i, \sigma_j), \qquad E(G(\hat{\sigma})) = \overline{C}.$$

This version of the Potts model is also called Curie-Weiss model [16].

It is straightforward to determine the Hamiltonian $H(\sigma)$. With the notation introduced above,

$$H(\sigma) \quad = \quad -\sum_{\ell=1}^{q} \lambda_\ell N_\ell(\sigma) - \lambda_{q+1} G(\sigma).$$

We now rename $\lambda_\ell = h_\ell$ of $\ell = 1, \ldots, q$, and $\lambda_{q+1} = J/N$. Note that this model assumes a certain scaling of the Lagrange multiplier λ_{q+1} with population size N. Therefore, we choose $\lambda_{q+1} = J/N$. If the population size becomes large, we implicitly assume that the correlations (as determined by the function $G(.)$) become small. To be clear: This is an assumption that might or might not be met by a real world problem. Mostly, we will find that this assumption is appropriate, as the interaction between two given individuals will be weaker if interactions between many individuals (N large) are possible. Therewith,

$$\begin{aligned}
H(\sigma) \quad &= \quad -\frac{J}{N} \sum_{i<j}^{N} \delta(\sigma_i, \sigma_j) - \sum_{\ell=1}^{q} h_\ell \sum_{i=1}^{N} \delta(\sigma_i, \ell) \\
&= \quad -\frac{J}{2N} \sum_{i,j=1}^{N} \delta(\sigma_i, \sigma_j) + \frac{J}{2N} \sum_{i=1}^{N} \delta(\sigma_i, \sigma_i) - \sum_{\ell=1}^{q} h_\ell\, N_\ell(\sigma) \\
&= \quad -\frac{J}{2N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 + \frac{J}{2} - \sum_{\ell=1}^{q} h_\ell\, N_\ell(\sigma).
\end{aligned}$$

As the probabilities $Q(\sigma)$ are given by $e^{-H(\sigma)}/Z$, the additive term $J/2$ just cancels out (appears in the Hamilton as well as in the partition function). We can drop that constant.

Corollary 2.34 *Assume that $E(F_\ell(\hat{\sigma})) = \overline{x}_\ell$, and $E(G(\hat{\sigma})) = \overline{C}$. Then, $Q(\sigma) = \exp(-H(\sigma))/Z$, where the Hamiltonian for the Curie-Weiss model is given by*

$$H(\sigma) = -\frac{J}{2\,N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma). \tag{2.39}$$

Rescaling the parameters J and h_i, we can easily work out a weak limit for the Curie-Weiss model. Let $q = 2$, and

$$x = N_1(\sigma)/N, \quad \tilde{J} = J\,N, \quad \tilde{h}_i = h_i\,N.$$

Then, using $N_2(\sigma)/N = 1 - N_1(\sigma)/N$, we find

$$\begin{aligned}
&\frac{J}{2N}\left(N_1(\sigma)^2 + N_2(\sigma)^2\right) - h_1 N_1(\sigma) - h_(\sigma) = \frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) - \tilde{h}_1 x - \tilde{h}_2(1-x) \\
= \quad &\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) - (\tilde{h}_1 - \tilde{h}_2)x - \tilde{h}_2.
\end{aligned}$$

Corollary 2.35 *In the weak limit, $\tilde{J} = J\,N$, $\tilde{h}_i = h_i\,N$, $h = \tilde{h}_1 - \tilde{h}_2$, and $x = N_1(\sigma)/N$, the limiting distribution for the Curie-Weiss model reads*

$$\varphi(x) = C\,\exp\left[-\,\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) + \tilde{h}\,x\right]. \tag{2.40}$$

The (local) maxima of this distribution coincide with local maxima of the exponent. To identify these points, we equate the derivative to zero, and find

$$0 \;=\; \frac{d}{dx}\left[-\,\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) + \tilde{h}\,x\right] = -\frac{\tilde{J}}{2}\left(2x - 2(1-x)\right) + \tilde{h} \quad\Rightarrow\quad x = \frac{\tilde{J} + \tilde{h}}{2\tilde{J}}.$$

That is, the distribution of the weak limit of the Curie-Weiss model always is unimodal. We will see below, that the original scaling of the Curie-Weiss model allows for more complex pattern.

We return to the original scaling, given by J and h_i. We reformulate the Hamilton (and the partition function) as an integral. Thereto, we use the following formula for a q-dimensional Gaussian integral: Let $A \in \mathbb{R}^{n\times n}$ denote a positive definite and symmetric matrix, and $y \in \mathbb{R}^n$ a vector. Then (see exercise 5.2),

$$\frac{1}{(2\pi)^{q/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^TAx + x^Ty}\,dx_1\ldots\,dx_q = \frac{e^{\frac{1}{2}y^TA^{-1}y}}{\sqrt{\det(A)}} \tag{2.41}$$

This formula is equivalent with

$$e^{\frac{1}{2}y^TA^{-1}y} = \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^TAx + x^Ty}\,dx_1\ldots\,dx_q.$$

This formula is used to perform a so-called Hubbard-Stratonovich transformation [28]: The quadratic terms in $e^{-H(\sigma)}$ can be expressed as an integral. After some more calculations (straightforward but lengthy, see exercise 5.3) we are led to the following proposition.

Theorem 2.36 *Up to negligible terms for $N \to \infty$ we have*

$$H(\sigma) = \kappa\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\,dx_1\ldots\,dx_q \tag{2.42}$$

and

$$Z = \kappa\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{\ell=1}^{q}\exp\left\{-N\left[\sum_{\ell=1}^{q}\frac{1}{2}Jx_\ell^2 - \ln\left(\sum_{\ell=1}^{q} e^{Jx_\ell + h_\ell}\right)\right]\right\}\,dx_1\ldots\,dx_q \tag{2.43}$$

with

$$\kappa = \left(\frac{J\,N}{2\,\pi}\right)^{q/2}.$$

J and h_ℓ are Lagrange multipliers determined by $E(F_\ell(\hat{\sigma})) = \overline{x}_\ell$, and $E(G(\hat{\sigma})) = \overline{C}$.

Proof: Exercise 5.3.

Remark 2.37 *We find that*

$$Q(\sigma) = \frac{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-\frac{1}{2} N J \sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q} (h_\ell + J x_\ell) N_\ell(\sigma)} \, dx_1 \ldots dx_q}{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\left\{ -N \left[\frac{1}{2} J \sum_{\ell=1}^{q} x_\ell^2 - \ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J x_\ell} \right) \right] \right\} dx_1 \ldots dx_q},$$

that is, the factor $\kappa = \left(\frac{J N}{2 \pi}\right)^{q/2}$ *cancels out in the probability measure.*

As the theory is established in statistical physics, the nomenclature comes from that field: h is called external field, J couplings. Furthermore, the term

$$f(x_1, \ldots, x_q) = -\ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J x_\ell} \right) + \frac{1}{2} J \sum_{\ell=1}^{q} x_\ell^2 \tag{2.44}$$

is called the free energy and assumes its minimum at some value x_ℓ^{sp}.

The result of theorem 2.34 is also called a functional integral [28]. At the time being it is not clear why we want to represent the Hamiltonian in such a complex way. At the first glance, it seems that we do not gain anything. However, it is indeed true that this reformulation yields additional insight and is in many cases simpler to handle. In the remaining part of the section, we do some elementary calculations to become familiar with this functional integral. Based on the results we obtain in these investigations, we will reconsider the Hamilton, the partition function, and the free energy.

We discuss why the values x_ℓ^{sp} play a special role. Before, we state a useful lemma.

Lemma 2.38 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a smooth and positive function with a unique minimum* $x_0 \in \mathbb{R}^n$ *and* $f(x) > c_1 |x|^2 + c_2$*, where* $c_1 > 0$*. Then,*

$$\varphi_N(x) = \frac{e^{-N f(x)}}{\int_{\mathbb{R}^n} e^{-N f(x)} \, dx}$$

tends for $N \to \infty$ *to* $\delta_{x_0}(x)$*, that is, for all bounded* $\psi \in C^0(\mathbb{R}^n, \mathbb{R})$ *we find*

$$\lim_{N \to \infty} \int_{\mathbb{R}^n} \psi(x) \, \varphi_N(x) \, dx = \psi(x_0).$$

Proof: Exercise 5.4.

Proposition 2.39 *The consistency conditions for the Lagrange multipliers* h_ℓ *become for* $N \to \infty$

$$\overline{x}_k = \frac{e^{h_k + J x_k^{sp}}}{\sum_{\ell=1}^{q} e^{h_\ell + J x_\ell^{sp}}} \tag{2.45}$$

where x_ℓ^{sp} *minimize the free energy* $f(x_1, \ldots, x_q) = -\ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J x_\ell} \right) + \frac{1}{2} J \sum_{\ell=1}^{q} x_\ell^2$.

Proof:

$$E(N_k(\hat\sigma)/N) \quad = \quad \frac{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2}\ \sum_{\sigma\in\Sigma_N}\frac{1}{N}\sum_{i=1}^{N}\delta(\sigma_i,k)\, e^{\sum_{\ell=1}^{q}(h_\ell+J\,x_\ell)N_\ell(\sigma)}\,dx_1\dots dx_q}{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^{q}x_\ell^2-\ln\left(\sum_{\ell=1}^{q}e^{h_\ell+J\,x_\ell}\right)\right]\right\}\,dx_1\dots dx_q}$$

The sum in the numerator reads

$$\frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{i=1}^{N}\delta(\sigma_i,k)\,e^{\sum_{\ell=1}^{q}(h_\ell+J\,x_\ell)N_\ell(\sigma)} = \frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{j=1}^{N}\prod_{\ell=1}^{q}\delta(\sigma_j,k)\exp\left\{(h_\ell+J\,x_\ell)N_\ell(\sigma)\right\}$$

$$= \quad \frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{j=1}^{N}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\delta(\sigma_j,k)\exp\left\{(h_\ell+J\,x_\ell)\delta(\sigma_i,\ell)\right\}$$

$$= \quad \frac{1}{N}\sum_{j=1}^{N}\sum_{s_1=1}^{q}\dots\sum_{s_N=1}^{q}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\delta(s_j,k)\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}$$

$$= \quad \frac{1}{N}\sum_{j=1}^{N}\prod_{i=1}^{N}\left(\sum_{s_i=1}^{q}\delta(s_j,k)\prod_{\ell=1}^{q}\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}\right)$$

$$= \quad \frac{1}{N}\sum_{j=1}^{N}\exp(h_k+J\,x_k)\prod_{i=1,i\neq j}^{N}\left(\sum_{s_i=1}^{q}\prod_{\ell=1}^{q}\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}\right)$$

$$= \quad \frac{1}{N}\sum_{j=1}^{N}\exp(h_k+J\,x_k)\prod_{i=1,j\neq j}^{N}\left(\sum_{s_i=1}^{q}\exp(h_{s_i}+J\,x_{s_i})\right)$$

$$= \quad \frac{1}{N}\sum_{j=1}^{N}\exp(h_k+J\,x_k)\prod_{i=1,i\neq j}^{N}\left(\sum_{\ell=1}^{q}\exp(h_\ell+J\,x_\ell)\right)$$

$$= \quad \exp\{(h_k+J\,x_k)\}\left(\sum_{\ell=1}^{q}\exp(h_\ell+J\,x_\ell)\right)^{N-1} = \exp\left\{h_k+J\,x_k+(N-1)\ln\left(\sum_{\ell=1}^{q}e^{h_\ell+J\,x_\ell}\right)\right\}.$$

If we use the fact that $N-1\approx N$ for N large, we have in eading order

$$E(N_k(\hat\sigma)/N) \quad = \quad \frac{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{(h_k+J\,x_k)}\ \exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^{q}x_\ell^2-\ln\left(\sum_{\ell=1}^{q}e^{h_\ell+J\,x_\ell}\right)\right]\right\}\,dx_1\dots dx_q}{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^{q}x_\ell^2-\ln\left(\sum_{\ell=1}^{q}e^{h_\ell+J\,x_\ell}\right)\right]\right\}\,dx_1\dots dx_q}$$

According to the lemma 2.38, for $N\to\infty$ this integral becomes a point mass at the minimum of $f(x_1,\dots,x_q)$ defined above, and

$$\lim_{N\to\infty}E(N_k(\hat\sigma)/N) = e^{h_k+Jx_k^{sp}}.$$

We may rewrite this equation by

$$\overline{x}_k = \lim_{N\to\infty}E(N_k(\hat\sigma)/N) = \lim_{N\to\infty}\frac{E(N_k(\hat\sigma)/N)}{1} = \lim_{N\to\infty}\frac{E(N_k(\hat\sigma)/N)}{\sum_{\ell=1}^{q}E(N_\ell(\hat\sigma)/N)} = \frac{e^{h_k+Jx_k^{sp}}}{\sum_{\ell=1}^{q}e^{h_\ell+Jx_\ell^{sp}}}.$$

This last expression has the advantage, that the normalization of $\overline{x}_k$ (sum becomes 1) is made explicit.

$\square$

Exercises

Exercise 5.2: (a) Let $y \in \mathbb{R}$, $a \in \mathbb{R}$, and $a > 0$. Show the identity (where you might want to use, without proof, that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\, a\, x^2 + xy}\, dx = \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}}.$$

(b) Show the equation for the Gaussian integral (2.41).

Solution of Exercise 5.2 (a) As $a\, x^2 - 2xy = (\sqrt{a}x - y/\sqrt{a})^2 - y^2/a$, we have

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\, a\, x^2 + xy}\, dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,(\sqrt{a}x - y/\sqrt{a})^2}\, dx \;\; e^{\frac{1}{2}y^2/a} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,(\sqrt{a}x - y/\sqrt{a})^2}\, \sqrt{a}dx \;\; \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\tilde{x}^2}\, d\tilde{x} \;\; \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}} = \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}},
\end{aligned}$$

where we used the transformation $\tilde{x} = \sqrt{a}x - y/\sqrt{a}$.

(b) The idea is the very same as in part (a), only that we need to work in q dimensions instead of one dimension. As A is symmetric and positive definite, we find a regular (orthogonal) matrix M such that

$$A = MDM^T$$

where D is the diagonal matrix with the (positive) eigenvalues of A on the diagonal. If we denote by $\sqrt{D}$ the diagonal matrix with the square root of the eigenvalues on the diagonal, then $\sqrt{D}\,\sqrt{D} = D$, and with $B = M\,\sqrt{D}$, we have

$$A = B\,B^T.$$

Furthermore, B is regular, as M and $\sqrt{D}$ are regular. With these prerequisites, we proceed:

$$\begin{aligned}
& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^T y}\, dx_1 \ldots dx_q \\
= \;& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx)^T(Bx) + (Bx)^T B^{-1} y - \frac{1}{2}((B^{-1}y)^T(B^{-1}y)) + \frac{1}{2}((B^{-1}y)^T(B^{-1}y))}\, dx_1 \ldots dx_q \\
= \;& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx + B^{-1}y)^T(Bx + B^{-1}y)}\, dx_1 \ldots dx_q \;\; e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))} \\
= \;& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx + B^{-1}y)^T(Bx + B^{-1}y)}\, \det(B)\, dx_1 \ldots dx_q \;\; \frac{e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))}}{\det(B)} \\
= \;& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{\ell=1} q\tilde{x}_\ell^2}\, d\tilde{x}_1 \ldots d\tilde{x}_q \;\; \frac{e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))}}{\det(B)} = \frac{e^{\frac{1}{2}y^T A^{-1} y}}{\det(B)}.
\end{aligned}$$

Paralleling the one-dimensional case, we used the transformation $\tilde{x} = Bx + B^{-1}y$. Recall that the determinant is just the product of all eigenvalues. The result follows with the identity

$$\det(B)^2 = \det(B)\det(B^T) = \det(B\,B^T) = \det(A) \quad \Rightarrow \quad \det(B) = \sqrt{\det(A)}.$$

Exercise 5.3: Prove Theorem 2.36.

Solution of Exercise 5.3 In order to find the expression for Z, we use a Hubbard-Stratonovich transformation [28]. The following formula for a q-dimensional Gaussian integral is useful. Let $A \in \mathbb{R}^{n\times n}$ denote a positive definite matrix, and $y \in \mathbb{R}^n$ a vector. Then,

$$\frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T Ax + x^T y}\, dx_1 \ldots dx_q = \frac{e^{\frac{1}{2}y^T A^{-1} y}}{\sqrt{\det(A)}},$$

that is,

$$e^{\frac{1}{2}y^T A^{-1} y} = \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T Ax + x^T y}\, dx_1 \ldots dx_q.$$

For $\sigma \in \Sigma_N$ given, we define a vector $y = y(\sigma) = (y_1, \ldots, y_q)^T$ by $y_\ell = N_\ell(\sigma)$, the vector $h = (h_1, \ldots, h_q)^T$, and the matrix $A = (N/J)I$, such that (note that $y = y(\sigma)$)

$$\begin{aligned}
Z &= \sum_{\sigma\in\Sigma_N} e^{-H(\sigma)} = \sum_{\sigma\in\Sigma_N} \exp\left\{\frac{J}{2N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 + \sum_{\ell=1}^{q} h_\ell\, N_\ell(\sigma)\right\} \\
&= \sum_{\sigma\in\Sigma_N} e^{\frac{1}{2}y^T A^{-1} y}\ e^{h^T y} = \sum_{\sigma\in\Sigma_N} \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T Ax + x^t y}\, dx_1 \ldots dx_q\ e^{h^T y} \\
&= \sum_{\sigma\in\Sigma_N} \left\{ \left(\frac{N}{2\,J\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T Ax + x^T y}\, dx_1 \ldots dx_q\ e^{h^T y} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{N}{2\,J\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,x^T x/J} \sum_{\sigma\in\Sigma_N} e^{(h^T + x^T)y}\, dx_1 \ldots dx_q \\
&= \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q}(x_\ell/J)^2} \sum_{\sigma\in\Sigma_N} e^{\sum_{\ell=1}^{q}(h_\ell + J\,(x_\ell/J))N_\ell(\sigma)}\, dx_1/J \ldots dx_q/J \\
&= \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2} \sum_{\sigma\in\Sigma_N} e^{\sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\, dx_1 \ldots dx_q
\end{aligned}$$

where we used the transformation $x_\ell \mapsto x_\ell/J$ of the integral variables in the last step. Before we proceed, we note that $H(\sigma)$ can be transformed in a similar way,

$$H(\sigma) = \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\, dx_1 \ldots dx_q.$$

Now we turn to the sum within the integral,

$$
\begin{aligned}
& \sum_{\sigma\in\Sigma_N} \exp\left\{\sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)\right\} = \sum_{\sigma\in\Sigma_N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)N_\ell(\sigma)\right\} \\
= & \sum_{\sigma\in\Sigma_N}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(\sigma_i,\ell)\right\} = \sum_{s_1=1}^{q}\cdots\sum_{s_N=1}^{q}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(s_i,\ell)\right\} \\
= & \prod_{i=1}^{N}\left(\sum_{s_i=1}^{q}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(s_i,\ell)\right\}\right) = \prod_{i=1}^{N}\left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right) \\
= & \left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right)^N = \exp\left\{N\ln\left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right)\right\}.
\end{aligned}
$$

Hence,

$$
Z = \left(\frac{1}{2\,N\,J\,\pi}\right)^{q/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\sum_{\ell=1}^{q}x_\ell^2 - \ln\left(\sum_{\ell=1}^{q}e^{h_\ell + J\,x_\ell}\right)\right]\right\}dx_1\ldots dx_q.
$$

Exercise 5.4: Demonstrate Lemma 2.38.

Solution of Exercise 5.4 Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be continuous. As the integral over $\varphi_N(.)$ is 1, we have

$$
\Big|\int_{\mathbb{R}^n}\psi(x)\,\varphi_N(x)\,dx - \psi(x_0)\Big| = \Big|\int_{\mathbb{R}^n}(\psi(x)-\psi(x_0))\,\varphi_N(x)\,dx\Big| \le \int_{\mathbb{R}^n}|\psi(x)-\psi(x_0)|\,\varphi_N(x)\,dx.
$$

Let $\tilde\psi(x) = |\psi(x)-\psi(x_0)|$. $\tilde\psi$ is a continuous, bounded, and non-negative function with $\psi(x_0) = 0$. If for such a function it is true that $\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx \to 0$ for $N\to\infty$, the lemma is established.

Let $\varepsilon > 0$, arbitrary, fixed. We show that $\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx < \varepsilon$ for N large.

Therefore we first find $r > 0$, s.t.

$$
\max_{|x-x_0|\le r}\tilde\psi(x) < \varepsilon/2.
$$

Next we note that for this given $r > 0$

$$
\lim_{N\to\infty}\int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx \le \|\tilde\psi(x)\|_\infty \lim_{N\to\infty}\int_{|x-x_0|>r}\varphi_N(x)\,dx = 0.
$$

Therefore, there is $N_0 > 0$ s.t. for all $N > N_0$

$$
\int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx < \varepsilon/2.
$$

Together, we have for $N > N_0$

$$
\begin{aligned}
\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx &= \int_{|x-x_0|\le r}\tilde\psi(x)\,\varphi_N(x)\,dx + \int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx \\
&< \int_{|x-x_0|\le r}\varphi_N(x)\,dx\ \max_{|x-x_0|\le r}\tilde\psi(x) + \varepsilon/2 \\
&\le \int_{\mathbb{R}^n}\varphi_N(x)\,dx\ \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

2.5.5 Interlude: Free energy and the partition function

Remark 2.40 *There is an appealing heuristic short-cut for the proof of Proposition 2.39.* *(a) We go back to the general setting given in theorem 2.30. That is, we consider general observables* $F_\ell(\sigma)$ *with Lagrange multipliers* λ_ℓ, $H(\sigma) = -\sum_{\ell=1}^{M} \lambda_\ell F_\ell(\sigma)$, *and* $Z = \sum_{\sigma\in\Sigma_N} e^{-H(\sigma)}$. *Then,*

$$\begin{aligned} E(F_\ell(\hat\sigma)) &= \sum_{\sigma\in\Sigma_N} F_\ell(\sigma) e^{-H(\sigma)}/Z = \sum_{\sigma\in\Sigma_N} F_\ell(\sigma) e^{\sum_{\ell'=1}^{M}\lambda_{\ell'}F_{\ell'}(\sigma))}/Z \\ &= \frac{\frac{\partial}{\partial\lambda_\ell}\sum_{\sigma\in\Sigma_N} e^{-H(\sigma))}}{\sum_{\sigma\in\Sigma_N} e^{-H(\sigma)}} = \frac{\partial}{\partial\lambda_\ell}\ln(Z). \end{aligned}$$

(b) We now return to the setting at hand. Since x_ℓ^{sp} *minimize the free energy, we have for* N *large that* Z *is approximately proportional to*

$$Z \sim e^{-N\,f(x_1^{sp},\dots,x_q^{sp})}$$

and hence

$$\ln(Z) \sim -N\,f(x_1^{sp},\dots,x_q^{sp}).$$

If we combine the two observations (a) and (b), we obtain

$$E(N_\ell(\hat\sigma)) \;=\; -N\,\frac{\partial}{\partial h_\ell}\,f(x_1^{sp},\dots,x_q^{sp}) = N\,\frac{e^{h_k+Jx_k^{sp}}}{\sum_{\ell=1}^{q} e^{h_\ell+Jx_\ell^{sp}}}.$$

In that, equation (2.45) can be re-written as

$$\overline{x}_\ell = -\frac{\partial}{\partial h_\ell}\,f(x_1^{sp},\dots,x_q^{sp}). \tag{2.46}$$

In a similar way, we can obtain information about correlation.

Proposition 2.41 *Let* $\ell \neq \ell'$ *and assume that the partition function* Z *is proportional to* $e^{-Nf(x_1^{sp},\dots,x_q^{sp})}$,

$$Z \sim e^{-Nf(x_1^{sp},\dots,x_q^{sp})}.$$

Then,

$$\begin{aligned} -N\,\frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} f(x_1^{sp},\dots,x_q^{sp}) &= -N\,\frac{\partial}{\partial h_{\ell'}}\,x_\ell^{sp} \\ = \quad E\Big(N_\ell(\hat\sigma)\,N_{\ell'}(\hat\sigma)\Big) &- E(N_\ell(\hat\sigma))\,E(N_{\ell'}(\hat\sigma)). \end{aligned} \tag{2.47}$$

Proof: We know that

$$\frac{\partial}{\partial h_\ell}\,\ln(Z) = \frac{1}{Z}\sum_{\sigma\in\Sigma_N} N_\ell(\sigma) e^{-H(\sigma)}$$

and hence for $\ell \neq \ell'$

$$
\begin{aligned}
\frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} \ln(Z) &= \frac{1}{Z} \sum_{\sigma \in \Sigma_N} N_\ell(\sigma)\, N_{\ell'}(\sigma))(\sigma) e^{-H(\sigma)} \\
&\quad - \left(\frac{1}{Z} \sum_{\sigma \in \Sigma_N} N_\ell(\sigma) e^{-H(\sigma)} \right) \left(\frac{1}{Z} \sum_{\sigma \in \Sigma_N} N_{\ell'}(\sigma) e^{-H(\sigma)} \right)
\end{aligned}
$$

With $x_\ell^{sp} = E(F_\ell(\hat\sigma)) = \frac{\partial}{\partial h_\ell} \ln(Z)$ the result follows.

$\square$

Remark 2.42 *Note that we can rewrite the equation (2.47) as*

$$
-\frac{1}{N} \frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} f(x_1^{sp}, \ldots, x_q^{sp}) \quad = \quad E\bigg(F_\ell(\hat\sigma)\, F_{\ell'}(\hat\sigma) \bigg) - E(F_\ell(\hat\sigma))\, E_Q(F_{\ell'}(\hat\sigma)).
$$

2.5.6 Phase transitions.

The task we are faced with is to find the value of the Lagrange multipliers h_ℓ and J, given the data $\overline{x}_\ell$ and $\overline{C}$. This task is rather difficult. What we do is to reverse this question: We fix h_ℓ and J, and ask which values for $\overline{x}_\ell$ are possible. As we obtain non-trivial answers from considering $\overline{x}_\ell$ only we do not also discuss $\overline{C}$. In that, the Lagrange multipliers gain importance at its own. This step resembles the definition of a (physical) temperature in statistical physics. Also the temperature is a Lagrange multiplier, which comes into life. In this context, h_k is called the external field, and J the couplings.

Proposition 2.43 *If h_ℓ and J are prescribed, then $\overline{x}_\ell = x_\ell^{sp}$, and the consistency equations for the external fields become*

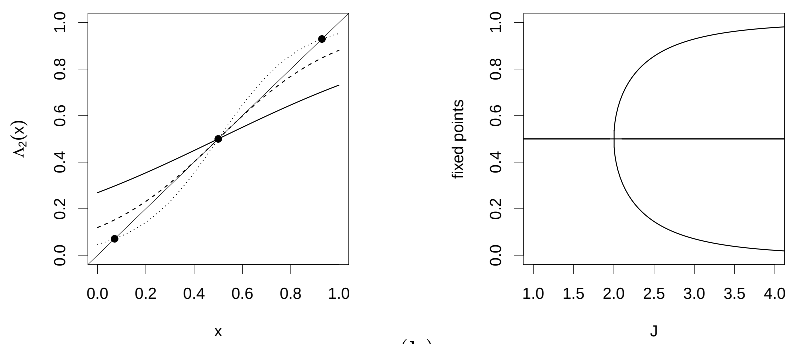
$$
x_k^{sp} = \frac{e^{h_k + J x_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + J x_\ell^{sp}}} \tag{2.48}
$$

Proof: We consider two equations: First of all, Equation (2.45) reads

$$
\overline{x}_k = \frac{e^{h_k + J x_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + J x_\ell^{sp}}}.
$$

The second condition is given by the fact that x_ℓ^{sp} minimizes the free energy. If we take the partial derivative of $f(x_1, \ldots, x_q)$ w.r.t. x_k we find

$$
\begin{aligned}
0 &= \frac{\partial}{\partial x_k} f(x_, \ldots, x_q) = \frac{\partial}{\partial x_k} \bigg\{ -\ln\left(\sum_{\ell=1}^q e^{h_\ell + J\, x_\ell} \right) + \frac{1}{2} J \sum_{\ell=1}^q x_\ell^2 \bigg\} \\
&= -\frac{J\, e^{h_k + J\, x_k}}{\sum_{\ell=1}^q e^{h_\ell + J\, x_\ell}} + J\, x_k.
\end{aligned}
$$



(a) (b)

Figure 2.8: Left panel: $q = 2$, $J = 1$ (solid line), $J = 2$ (dashed line) $J = 3$ (dotted line). Fixed points (indicated by bullets) (intersections of the graph of $\Lambda_2(x)$ with the the line $y = x$, indicated by the straight solid line) are possible values for x_ℓ^{sp}. Right panel: Fixed points of $\Lambda_2(x)$ over parameter J.

Hence, the minimizing values x_k^{sp} satisfy

$$x_k^{sp} = \frac{e^{h_k + Jx_k^{sp}}}{\sum_{\ell=1}^{q} e^{h_\ell + Jx_\ell^{sp}}} = \overline{x}_k.$$

The solutions x_k^{sp} are the possible values of the observables $\overline{x}_k$, given the Lagrange multipliers.

□

We focus on simple cases where the consistency conditions can be explicitly solved. The most simple assumption seems to be that all are equal, $x_\ell^{sp} = x$. As x_ℓ^{sp} sum up to 1, we conclude $x = 1/q$. We will find below, that this is a sensible solution if all external fields are identical. However, the restriction assumed (all x_ℓ^{sp} are equal) is too strong to allow for all or only for more interesting solutions.

Proposition 2.44 *Assume that the external fields are identical, $h_i = h_j$. For $q = 2$, there is always the trivial solution $x_1^{sp} = x_2^{sp} = 1/2$. At $J = 2$, a second branch with $x_1^{sp} \neq x_2^{sp}$ crosses the trivial solution.*

Proof: As we assume $h_1 = h_2$, the consistency equation becomes

$$x_\ell^{sp} = \frac{e^{Jx_k^{sp}}}{\sum_{\ell=1}^{q} e^{Jx_\ell^{sp}}}$$

where $x_1^{sp} + x_2^{sp} = 1$. That is, we have for x_1^{sp} the equation

$$x_1^{sp} = \Lambda_2(x_1^{sp}) = \frac{e^{Jx_1^{sp}}}{e^{Jx_1^{sp}} + e^{J(1-x_1^{sp})}} = \frac{1}{1 + e^{J(1-2x_1^{sp})}}.$$

Note that a solution for that equation x_1^{sp} is in the interval $(0,1)$. For the corresponding $x_2^{sp} = 1 - x_1^{sp}$ we find

$$x_2^{sp} = 1 - x_1^{sp} = 1 - \Lambda_2(x_1^{sp}) = 1 - \frac{e^{Jx_1^{sp}}}{e^{Jx_1^{sp}} + e^{J(1-x_1^{sp})}} = \frac{e^{Jx_2^{sp}}}{\sum_{\ell=1}^{q} e^{Jx_\ell^{sp}}}.$$

That is, any fixed point of $\Lambda_1(.)$ defines a valid solution of the consistency equation. We find at once that $\Lambda_2(1/2) = 1/2$. Furthermore,

$$\Lambda_2'(x) = \frac{2\,J\;e^{J(1-2x)}}{(1+e^{J(1-2x)})^2}.$$

That is, $\Lambda_2'(1/2) = J/2$. At $J = 2$, the slope is exactly one. As $\Lambda_2''(1/2) \neq 0$, a second branch of solution crosses the trivial solution.

$\square$

Remark 2.45 *A more refined analysis of the fixed points of $\Lambda_2(.)$ reveals, that a Pitchfork bifurcation happens at $q = 2$: If $J \leq 2$, there only is the trivial solution $x_\ell^{sp} = 1/2$; for $J > 2$, a symmetric, second solutions branches away from that trivial solution (see Fig. 2.8).*

We prove a similar result for $q > 2$. First of all, we determine $q - 1$ different functions which have fixed points that correspond to solutions of the consistency equations. All of these functions have the trivial solution $x_\ell^{sp} = 1/q$ as fixed point, independently of the value fo J. If J is large enough, additional fixed points appear. The definition of these functions is based on the idea to have the similar value in the first m and the last $q - m$ components in $(x_1^{sp}, \ldots, x_q^{sp})$.

Proposition 2.46 *Assume that the external fields are identical, $h_i = h_j$. Let furthermore x be a fixed point of*

$$\Lambda_{m,q}(x) := \frac{1}{m + (q-m)e^{J\,\frac{1-qx}{q-m}}}, \qquad m = 1, \ldots, q-1. \tag{2.49}$$

Define y by

$$y = \frac{1 - mx}{q - m}. \tag{2.50}$$

Then, $x_1^{sp} = \ldots = x_m^{sp} = x$ and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$ is a solution of the consistency equation 2.48.

Proof: Heuristics: Assume that $x_1^{sp} = \ldots = x_m^{sp} = x$ and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$ is a solution of the consistency equation. As the components of x_ℓ^{sp} add up to 1, we have $1 = m\,x + (q-m)\,y$. The consistency equation 2.48 for $x = x_1^{sp}$ (and all external fields equal) reads

$$x = \frac{e^{Jx}}{\sum_{\ell=1}^{q} e^{Jx_\ell^{sp}}} = \frac{e^{Jx}}{me^{Jx} + (q-m)e^{Jy}} = \frac{1}{m + (q-m)e^{J(y-x)}} = \Lambda_{m,q}(x).$$

Now we forget the heuristics, and assume that x is a fixed point of $\Lambda_{m,q}(.)$, y is defined according to eqn. (2.50), $x_1^{sp} = \ldots = x_m^{sp} = x$, and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$. Due to the construction,

we know that x_1^{sp} satisfies the consistency equation, and for symmetry reasons the consistency condition is also established for $x_2^{sp},\ldots,x_m^{sp}$.
It remains to show that y satisfies the consistency equation of $x_{m+1}^{sp}=y$. If this is given, also $x_{m+1}^{sp},\ldots,x_{m+1}^{sp}$ satisfy their consistency equations, and we are done. We find by straightforward calculations

$$\begin{aligned}
y &= \frac{1-mx}{q-m} = \frac{1-m\Lambda_{m,q}(x)}{q-m} = \frac{1}{q-m}\left(1-\frac{m}{m+(q-m)e^{J\,\frac{1-qx}{q-m}}}\right)\\
&= \frac{1}{q-m}\left(1-\frac{me^{Jx}}{me^{Jx}+(q-m)e^{Jy}}\right) = \frac{1}{q-m}\left(\frac{(q-m)e^{Jy}}{me^{Jx}+(q-m)e^{Jy}}\right)\\
&= \frac{e^{Jy}}{me^{Jx}+(q-m)e^{Jy}},
\end{aligned}$$

which is precisely the consistency condition for y.

□

Remark 2.47 *(a) If we have a solution x, and if $y\neq x$, we have $\binom{q}{m}$ different solutions of the consistency equations, as we can assign x to arbitrarily chosen components of x_ℓ^{sp}.*
(b) Simple calculations show that $\lambda_{m,q}(x)=x$ implies $\Lambda_{q-m,q}(y)=y$. In this sense, the functions $\Lambda_{q-m,q}(y)$ do not produce independent solutions, but the fixed point equations come in pairs.
(c) We always have (independently of J) the trivial solution

$$\Lambda_{m,q}(1/q)=1/q.$$

That fixed point corresponds to the solution with maximal symmetry, $x_\ell^{sp}=1/q$. A straightforward calculations shows that

$$J=q \quad\Rightarrow\quad \Lambda'_{m,q}(1/q)=1.$$

We expect non-trivial branch(es) to appear at $J=q$ (see also Fig. 2.9, panel (d)). However, if we inspect the numerical construction of the consistency equations' solution, we find that there are non-trivial branches of solutions for $J<q$. They come into life by saddle-node bifurcations. It turns out that this is always the case for $q>2$.
Note that the map $x\mapsto y=\frac{1-mx}{q-m}$ either leaves the branches invariant (the symmetric branches in Fig. 2.9, panel (d)) or maps one branch to another one.

transition

That was the analysis in the lmimt $N\to\infty$. Can we tell more, about the way the invariant measure approximates the delta-distribution?

Lagrange parameters

We return to the problem to directly infer the Lagrange multipliers h_ℓ and J, given expectations $\overline{x}_k$ and $\overline{x_k x_{k'}}$ (NOTATION????).

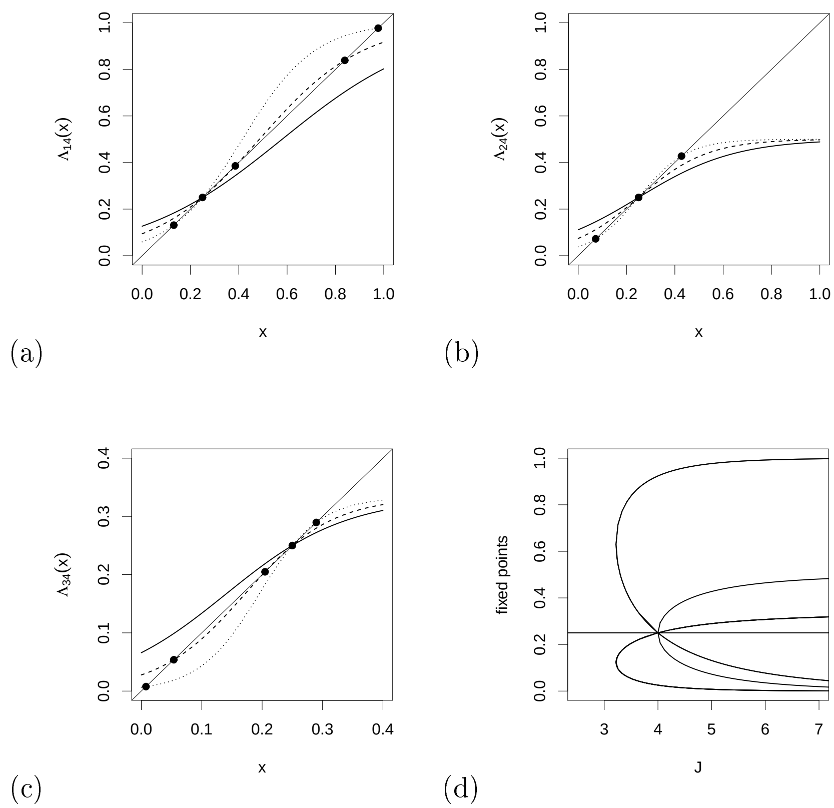


Figure 2.9: (a)-(c): $q = 4$, $J = 2.5$ (solid line), $J = 3.5$ (dashed line) $J = 5$ (dotted line). Intersection of the graph of $\Lambda(x)$ with $y = x$ correspond to possible solutions of the consistency equations. We have $m = 1$ (a), $m = 2$ (b), $m = 3$ (c). (d) Fixed points of $\Lambda_{m,4}(x)$, $m = 1, \ldots, q$, over parameter J.

2.5.7 Estimation of external field and couplings from data

We now return to our original aim: Given a time series of fractions $\vec{x}_t$, and the assumption that the dynamics is in an stationary state, can we estimate the Lagrange multipliers, that is, the external fields h_ℓ and the coupling J? First of all, the time series gives us at each time point information t about the the fraction of supporters of candidate ℓ, $\ell = 1, \ldots, q$. We do not have information in continuous time, but at T fixed time points, $t_1, \ldots, t_T$. Moreover, the number of participants in each of the T polls are not constant, but a time-dependent number N_t. We estimate $\overline{x}_\ell$ by the weighted mean,

$$\overline{x}_\ell = \sum_{t=1}^{T} \frac{N_t}{\sum_{t'=1^T} N_{t'}} \, (\vec{x}(t))_\ell.$$

In a similar way, we estimate the pairwise correlation functions

$$C_{\ell,\ell'} = \sum_{t=1}^{T} \frac{N_t}{\sum_{t'=1}^{T} N_{t'}} \, (\vec{x}(t))_\ell \; (\vec{x}(t))_{\ell'}.$$

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX hier weiter XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

2.6 Dynamic models with phase transitions

Most models we considered so far have been based on variants of the voter model. Instead, in the the last section we investigated an application of the Potts model. Immediately the question comes to mind if these models are similar, or in which aspects they are fundamentally different. Some differences are obvious the constructions: The (noisy) voter model is defined as a mechanical model. It is formulated as a stochastic process. This process has an invariant measure, and mostly we either investigate the transient dynamics, or the properties of the invariant measure as given by the process, e.g., by means of the detailed balance equation.
The Potts model, instead, is defined as a probability measure only, without an accompanying mechanistic stochastic process that produces the Potts probability measure as its invariant measure.

In the present section, we embed the Potts model into a stochastic process. Therewith, we better understand the mechanisms of the Potts model that lead to the phase transition (in particular, the differences to the Voter model). Equipped with that insight, we develop a model for reinforcement; as we will see, also that model shows phase transitions.

2.6.1 Potts model versus voter model

In order to compare the variants of the voter model and the Potts model, we augment the Potts model with a stochastic process. This process is by no means unique – the Potts model is the equivalent of a stationary state of an unknown dynamical system, and there are many different dynamical systems with the very same stationary state. However, statistical physics proposes a witty method to construct the stochastic process. One general idea to construct a stochastic process around a given distribution is the Metropolis-Hastings algorithm [21]. That algorithm, though of utmost importance in, e.g., Bayesian statistics, is less convenient to interpret in mechanistic terms. For the Curie Weiss model, an appealing construction is the Glauber dynamics [19, 29]. The starting point of the Glauber dynamics is the detailed balance equation – if the combination of given transition rates and a given distribution satisfy this equation, then that distribution is already the invariant distribution of the process. The idea of the Glauber dynamics is to choose most simple transition rates that do satisfy the detailed balance equation. The result of this process is, of course, dependent on the special form of the Hamiltonian in the Potts model.

It is possible to better understand the Potts (equilibrium) distribution using the Glauber dynamics, as it is more intuitive to investigate a stochastic process than a given, static probability measure. Particularly, as we shall see, the comparison with the transition rates of the zealot model is possible. We consider the two-state situation. For mathematical convenience we choose $q \in \{\pm 1\}$ instead of $q \in \{1, 2\}$.

Model 2.48 Glauber dynamics. *Consider the state space for the 2-state Potts model: Each of N individuals assumes the states 1 or -1,*

$$\Sigma_N = \{-1, 1\}^N.$$

The individuals are numbered by $1, \ldots, N$, the state of the system is given by $\sigma \in \Sigma_N$. The Glauber process (for $q = 2$) is given by a Σ_N-valued, time-continuous Markov process $\hat{\sigma}_t$. The state changes a.s. in only one component at a time, that is, the opinion of only one individual flips. Let flip : $\{-1, 1\} \to \{-1, 1\}$ with flip$(1) = -1$ and flip$(-1) = 1$. Let furthermore $i_0 \in \{1, \ldots, N\}$, $\sigma \in \Sigma_N$, and $\sigma' \in \Sigma_N$ the state that only differs from σ in i_0,

$$\sigma'_{i_0} = \mathit{flip}(\sigma_{i_0}), \qquad \sigma'_i = \sigma_i \text{ for } i \neq i_0.$$

Then, the transition rate from σ to σ' is defined as

$$\sigma \to \sigma' \quad \text{at rate} \quad \frac{\mu}{2}\left[1 - \sigma_i \tanh\left(h + \frac{J}{N}\sum_{j \neq i} \sigma_j\right)\right]. \tag{2.51}$$

As mentioned before, if the Hamming distance between two states is larger than 1, the transition rates between those states are zero.

Theorem 2.49 *The invariant measure for the Glauber-process reads*

$$Q(\sigma) = \frac{e^{-H(\sigma)}}{Z}, \qquad H(\sigma) = -\frac{J}{2N}\sum_{i=1}^N \sum_{j=1}^N \sigma_i \sigma_j - h \sum_{i=1}^N \sigma_i,$$

where Z is the corresponding partition function.

Proof: As a sufficient condition we show the detailed balance equation,

$$Q(\sigma)\ \mathrm{rate}(\sigma \to \sigma') = Q(\sigma')\ \mathrm{rate}(\sigma' \to \sigma).$$

If transitions are not possible between two states, the rates are zero, and the equation is fulfilled. In case of non-zero transition rates, we write

$$\frac{\mathrm{rate}(\sigma \to \sigma')}{\mathrm{rate}(\sigma' \to \sigma)} = \frac{Q(\sigma')}{Q(\sigma)}.$$

As only transitions between states that differ in exactly one component are possible, and all individuals are exchangeable, we may w.l.o.g. assume that $\sigma_1 = -1$, $\sigma'_1 = 1$, and $\sigma_i = \sigma'_i$ for

$i = 2, \ldots, N$. Then,

$$\begin{aligned}
\frac{Q(\sigma')}{Q(\sigma)} &= \frac{e^{-H(\sigma')}}{e^{-H(\sigma)}} = \frac{\exp\left(\frac{J}{2N}\sum_{i=1}^N\sum_{j=1}^N \sigma_i'\sigma_j' + h\,\sum_{i=1}^N \sigma_i'\right)}{\exp\left(\frac{J}{2N}\sum_{i=1}^N\sum_{j=1}^N \sigma_i\sigma_j + h\,\sum_{i=1}^N \sigma_i\right)} \\
&= \frac{\exp\left(\frac{J}{N}\sigma_1'\sum_{i=2}^N \sigma_i' + h\sigma_1'\right)\ \exp\left(\frac{J}{2N} + \frac{J}{2N}\sum_{i=2}^N\sum_{j=2}^N \sigma_i'\sigma_j' + h\sum_{i=2}^N \sigma_i'\right)}{\exp\left(\frac{J}{N}\sigma_1\sum_{i=2}^N \sigma_i + h\sigma_1\right)\ \exp\left(\frac{J}{2N} + \frac{J}{2N}\sum_{i=2}^N\sum_{j=2}^N \sigma_i\sigma_j + h\sum_{i=2}^N \sigma_i\right)} \\
&= \frac{\exp\left(\frac{J}{N}\sigma_1'\sum_{i=2}^N \sigma_i' + h\sigma_1'\right)}{\exp\left(\frac{J}{N}\sigma_1\sum_{i=2}^N \sigma_i + h\sigma_1\right)}.
\end{aligned}$$

The identity (which is given for $\alpha \in \mathbb{R}$ and $x = \pm 1$)

$$e^{\alpha x} = \cosh(\alpha\, x) + \sinh(\alpha\, x) = \cosh(\alpha) + x\sinh(\alpha) = \cosh(\alpha)(1 + x\tanh(\alpha))$$

and the observation $\sigma_i = \sigma_i'$ for $i > 1$, and $\sigma_1' = -\sigma_1$ implies ($i_0 := 1$)

$$\frac{Q(\sigma')}{Q(\sigma)} = \frac{1 + \sigma_{i_0}'\ \tanh\left(\frac{J}{N}\sum_{i=2}^N \sigma_i' + h\right)}{1 + \sigma_{i_0}\ \tanh\left(\frac{J}{N}\sum_{i=2}^N \sigma_i + h\right)} = \frac{1 - \sigma_{i_0}\ \tanh\left(\frac{J}{N}\sum_{i\neq i_0}^N \sigma_i + h\right)}{1 - \sigma_{i_0}'\ \tanh\left(\frac{J}{N}\sum_{i\neq i_0}^N \sigma_i' + h\right)} = \frac{\text{rate}(\sigma \to \sigma')}{\text{rate}(\sigma' \to \sigma)}.$$

□

Let us compare the transition probabilities of the Zealot model 2.5 (two opinions) and the Glauber dynamics. In the notation of the Zealot mode we used type A and type B to refer to the two opinions. In order to have the very same state space $\Sigma_N = \{-1, 1\}^N$ in both processes, we rename type A individual into type -1 individuals, and type B individuals into type 1 individuals. Accordingly, the number of type ± 1 zealots is denoted by $N_{\pm 1}$. With that change of notation, we have the following proposition.

Proposition 2.50 *Let $\sigma \in \Sigma_N$, $\ell \in \{\pm 1\}$, and $\overline{\ell} = flip(\ell)$. For the zealot model, the rate at which a given type-ℓ individual changes its type, given the state $\sigma \in \Sigma_N$, reads*

$$r_z(\ell, \sigma) \quad = \quad \mu\,\frac{\sum_{i=0}^N \delta(\sigma_i, \overline{\ell}) + N_{\overline{\ell}}}{N + N_{-1} + N_{+1}}. \tag{2.52}$$

For the Glauber dynamics, the rate at which a given individual with state ℓ jumps to state $\overline{\ell}$ is given by

$$r_G(\ell, \sigma) \quad = \quad \frac{\mu}{2}\left[1\ -\ \tanh\left(2\,J\left(\frac{1}{2} - \frac{\sum_{i=1}^N \delta(\sigma_i, \overline{\ell})}{N}\right) + h\ell\right)\right] + \mathcal{O}(N^{-1}). \tag{2.53}$$

With

$$J = \frac{N}{N+N_{-1}+N_{+1}}, \quad h = \frac{N_{-1}-N_{+1}}{N+N_{-1}+N_{+1}} \tag{2.54}$$

we find

$$r_z(\ell,\sigma) = r_G(\ell,\sigma) + \mathcal{O}(J,h,N^{-1}).$$

Proof: *Zealot model.* According to the model description of the zealot model 2.5, an individual reconsiders at rate μ his/her opinion. He/she contacts a randomly selected individual (including him/herself and all zealots), and copies the opinion of that individual. As we have $\sum_{i=0}^N \delta(\sigma_i,\overline{\ell})$ individuals and $N_{\overline{\ell}}$ zealots with the opposite opinion, the result follows.

Glauber dynamics. Assume the number of the focal ℓ-individual is $i_0 \in \{1,\dots,N\}$. Using the identity for the product of two integers $\ell, m \in \{\pm 1\}$

$$\ell\, m = 2\delta(\ell,m) - 1,$$

the rate for that individual to swap his/her state, as stated in eqn. (2.51) can be written as

$$\begin{aligned}
r_G(\ell,\sigma) &= \frac{\mu}{2}\left[1-\ell\,\tanh\left(h+\frac{J}{N}\sum_{j\neq i_0}\sigma_j\right)\right] = \frac{\mu}{2}\left[1-\,\tanh\left(h\,\ell+\frac{J}{N}\sum_{j\neq i_0}\ell\sigma_j\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(\ell h+\frac{J}{N}\sum_{j\neq i_0}(2\delta(\sigma_j,\ell)-1)\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(2\,J\left(\frac{\sum_{i=1}^N\delta(\sigma_i,\ell)}{N}-\frac{1}{N}-\frac{(N-1)}{2N}\right)+h\ell\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(2\,J\left(\frac{1}{2}-\frac{\sum_{i=1}^N\delta(\sigma_i,\overline{\ell})}{N}\right)+h\ell\right)\right]+\mathcal{O}(N^{-1}),
\end{aligned}$$

where we used $\sum_{i=1}^N\delta(\ell,\sigma_i)+\sum_{i=1}^N\delta(\overline{\ell},\sigma_i)=N$ in the last step.

Comparison of r_z and r_G. As the first order Taylor approximation of $\tanh(x)$ at $x=0$ reads

$$\tanh(x) = x + \text{h.o.t.},$$

where h.o.t. represents the error term $\mathcal{O}(x^2)$, we find for J and h small that

$$r_G(\ell,\sigma) \;=\; \mu\left[\frac{1-J}{2}+\,J\,\frac{\sum_{i=1}^N\delta(\sigma_i,\overline{\ell})}{N}-\frac{h}{2}\,\ell\right]+\text{h.o.t.}$$

Note in particular that the parameter J determines the zero'th order term as well as the coefficient of the term $\sum_{i=1}^N\delta(\sigma_i,\ell)/N$. We will come back to this point.
We also rewrite $v_z(\ell,\sigma)$. Thereto we use that ($\ell\in\{\pm1\}$)

$$N_{\overline{\ell}} = \frac{(1-\ell)}{2}N_{+1}+\frac{(1+\ell)}{2}N_{-1} = \frac{1}{2}(N_{+1}+N_{-1})+\ell\,\frac{1}{2}(N_{-1}-N_{+1}).$$

Therewith,

$$\begin{aligned}
r_z(\ell,\sigma) &= \mu\,\frac{\sum_{i=0}^N\delta(\sigma_i,\overline{\ell})+N_{\overline{\ell}}}{N+N_{-1}+N_{+1}} = \mu\left[\frac{N}{N+N_{-1}+N_{+1}}\,\frac{\sum_{i=0}^N\delta(\sigma_i,\overline{\ell})}{N}+\frac{N_{\overline{\ell}}}{N+N_{-1}+N_{+1}}\right]\\
&= \mu\left[\frac{N_{+1}+N_{-1}}{2(N+N_{-1}+N_{+1})}+\frac{N}{N+N_{-1}+N_{+1}}\,\frac{\sum_{i=0}^N\delta(\sigma_i,\overline{\ell})}{N}+\frac{N_{-1}-N_{+1}}{2(N+N_{-1}+N_{+1})}\,\ell\right]
\end{aligned}$$

If we now define

$$J = \frac{N}{N + N_{-1} + N_{+1}}, \qquad h = \frac{N_{-1} - N_{+1}}{N + N_{-1} + N_{+1}} \qquad \Rightarrow \qquad 1 - J = \frac{N_{-1} + N_{+1}}{N + N_{-1} + N_{+1}}.$$

We obtain

$$r_z(\ell, \sigma) \quad = \quad \mu \left[\frac{1-J}{2} + J \frac{\sum_{i=0}^{N} \delta(\sigma_i, \overline{\ell})}{N} + \frac{h}{2}\,\ell \right] = r_G(\ell, \sigma) + \mathcal{O}(J, h, N^{-1}).$$

□

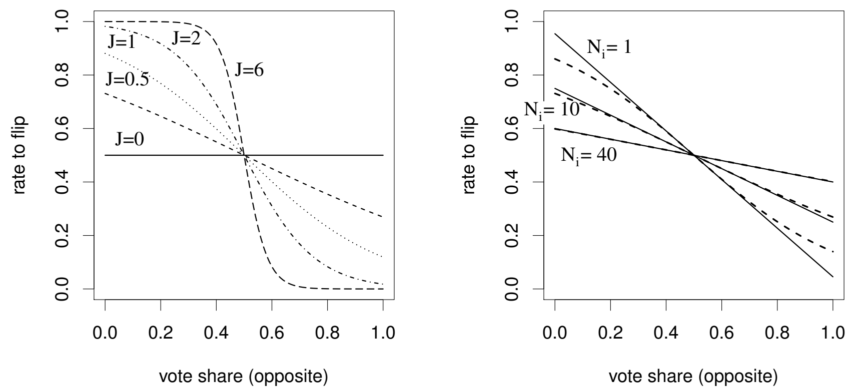


Figure 2.10: Left panel: Flip rates for the Glauber dynamics, $h = 0$, and J as indicated. Right panel: Comparison of flip rates Glauber dynamics (dashed) and zealot model (solid line) for three cases: We always did choose $h = 0$, and $N = 20$. Furthermore, $N_1 = N_2 = 1$; $N_1 = N_{-1} = 10$; $N_1 = N_{-1} = 40$; J is chosen according to $J = (N/(N + N_1 + N_{-1}))$. The correspondence of $J = 1$ and the voter model (proposition 2.50) is nicely visible.

Remark 2.51 *(a) Please note that the correspondence of the two models (for J and h small, and N large) is not trivial. The zealot model has two parameters ($N_{\pm 1}$), as well as the Glauber model (J and h). It might appear to be natural that these parameters can be identified. However, in the proof we find that we an affine-linear expression in $\sum_{i=0}^{N} \delta(\sigma_i, \overline{\ell})/N$ and ℓ needs to coincide in the two models to establish a first order approximation, s.t. three conditions are to be fulfilled. It is more than a coincidence that the three conditions can be satisfied by the choice of two parameters only.*
(b) The choices of J and h are natural: J represents the coupling of individuals in the Glauber dynamics, that is, the intensity individuals do interact. In the zealot model, $N/(N + N_{+1} + N_{-1})$ is the probability that a focal individual does not interact with a zealot, but with another individual. In that, this fraction also expresses the intensity of interactions in the voter model.
The external field h in the Glauber model indicates the advantage one opinion $\in \{0, 1\}$ above

the other state due to external forces and independent on the actual state of the system. In the very same way, $(N_{-1} - N_{+1})/(N + N_{-1} + N_{+1})$ quantifies the normalized advantage of group -1 above group $+1$ in the zealot model.
(c) Due to the definition of J, the correspondence between Glauber dynamics and voter model is only possible for $J \leq 1$. That condition shows that a Glauber dynamics resembling a zealot model is far away from the phase transition/bifurcation at $J = 2$.

In the Glauber dynamics as well as in the voter processes, individuals orient themselves at the fractions of individuals with the opposite opinion. While the voter model indicates that the probability to change the mind increases linear with the fraction of individuals with the opposite opinion, in the Glauber dynamics we have a nonlinear function (see figure 2.10). For $J = 0$, the individuals flip their mind independently of all other individuals. If J increases, the Glauber rate approaches a threshold function: In the limit $J \to \infty$ (the so-called zero temperature limit), all individuals jump to the majority opinion. The voter model is based on pairwise interactions, while in the Glauber model (for J large), individuals orient themselves at the majority. The latter mechanism allows for the phase transition. If J is sufficiently large, the equilibrium situation, where both opinions are equally present, is destabilized in favor of one of states where either the one opinion is predominating.

Corollary 2.52 *If J is small, the Glauber dynamics resembles a zealot model, in the low temperature limit, $J \to \infty$, the Glauber dynamics approximates a majority rule.*

We reformulate the Gaubers dynamics in a way that is close to the formulation of the zealots model as stated in model 2.5.

Remark 2.53 *Let X_t denote the number of individuals with state 1 at time t, s.t. we have $N - X_t$ individuals with state -1. Then,*

$$X_t \to X_t + 1 \quad \textit{at rate} \quad \mu\,(N - X_t)\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{X_t}{N} - \frac{N+1}{2N}\right)\right) + h\right] \tag{2.55}$$

$$X_t \to X_t - 1 \quad \textit{at rate} \quad \mu\,X_t\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{N-1}{2N} - \frac{X_t}{N}\right)\right) + h\right]. \tag{2.56}$$

Carry out!!!
The deterministic limit for large population size addresses $x(t) = X_t/N$, and is given by

$$\frac{d}{dt}x = \mu(1-x)\frac{1}{2}\left[1 - \tanh\left(2\,J\left(x - \frac{1}{2}\right)\right) + h\right] - \mu\,x\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{1}{2} - x\right)\right) + h\right] \tag{2.57}$$

For $h = 0$, this ODE undergoes a Pitchfork bifurcation at $x = 1/2$, and $J = 2$.

2.6.2 Glauber dynamics for the q-opinion Curie-Weiss model

Recall that

$$N_\ell(\sigma) = \sum_{i=1}^{N} \delta(\sigma_i, \ell).$$

Model 2.54 Glauber dynamics for the q-Potts model. *Let $\Sigma_N = \{1,\ldots,q\}^N$, $\sigma,\sigma' \in \Sigma_N$. If the Hamming distance between σ and σ' is unequal 1, the transition rates for $\sigma \to \sigma'$ and $\sigma' \to \sigma$ are zero. Let the Hamming distance be 1, and $i_0 \in \{1,\ldots,N\}$ denote the single individual/site where the two states disagree: $\sigma_{i_0} = k \in \{1,\ldots,q\}$, and $\sigma_{i_0} = k' \in \{1,\ldots,q\}$, where $k \neq k'$. Then,*

$$\text{transition rate } \sigma \to \sigma' \quad \text{is} \quad \mu\,\delta(\sigma_{i_0},k)\exp\left(J\left(\frac{N_{k'}(\sigma)}{N}-1\right)+h_{k'}\right). \tag{2.58}$$

Theorem 2.55 *The invariant measure of the Glauber process for the q-Potts model reads $Q(\sigma) = e^{-H(\sigma)}/Z$ with*

$$H(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma). \tag{2.59}$$

Z denotes the partition function.

Proof: We first note that

$$H(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q}\sum_{i,j=1}^{N}\delta(\sigma_i,\ell)\delta(\sigma_j,\ell) - \sum_{\ell=1}^{q} h_\ell \sum_{i=1}^{N}\delta(\sigma_i,\ell).$$

The proof parallels that of theorem 2.49: We use the detailed balance equation. Assume that σ,σ' only differ in component/individual i_0, and $\sigma_{i_0} = k$, $\sigma'_{i_0} = k'$ (where $k,k' \in \{1,\ldots,q\}$, $k \neq k'$). Then,

$$\begin{aligned}
\frac{Q(\sigma)}{Q(\sigma')} &= \frac{\exp\left(\frac{J}{2N}\sum_{\ell=1}^{q}\sum_{i,j=1}^{N}\delta(\sigma_i,\ell)\delta(\sigma_j,\ell) + \sum_{\ell=1}^{q} h_\ell \sum_{i=1}^{N}\delta(\sigma_i,\ell)\right)}{\exp\left(\frac{J}{2N}\sum_{\ell=1}^{q}\sum_{i,j=1}^{N}\delta(\sigma'_i,\ell)\delta(\sigma'_j,\ell) + \sum_{\ell=1}^{q} h_\ell \sum_{i=1}^{N}\delta(\sigma'_i,\ell)\right)}\\
&= \frac{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k) + \frac{J}{2N} + h_k\right)}{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k') + \frac{J}{2N} + h_{k'}\right)} = \frac{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k) + h_k\right)}{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k') + h_{k'}\right)}\\
&= \frac{\delta(\sigma'_{i_0},k')\,\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k) + h_k - J\right)}{\delta(\sigma_{i_0},k)\,\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k') + h_{k'} - J\right)}\\
&= \frac{\text{rate } \sigma' \to \sigma}{\text{rate } \sigma \to \sigma'}.
\end{aligned}$$

We used in that computation that $\sigma_i = \sigma'_i$ for $i \neq i_0$, and $\sigma_{i_0} = k \neq k'$: Therewith, we have $\delta(\sigma_{i_0},k) = 1$ and $\sum_{j\neq i_0}\delta(\sigma_j,k') = N_{k'}(\sigma)$.

□

We proceed to show an asymptotic relation between the multi-party zealot model and the multi-opinion Glauber model.

Remark 2.56 *Consider the situation with q parties, $\Sigma_N = \{1,\dots,q\}^N$. Let $\vec{X} = (X_1,\dots,X_q)$ denote the number of individuals structured by the q groups. The number of individuals is given by $N = \sum_{i=1}^q X_i$. For the Sano resp. the Glauber model, any transition affects two groups only – one group decreases by one, the other increases by one.*
Recall the Sano model 2.18: The number of zealots of group i are N_i. Only a fraction θ of zealots influences the voters, and we define $M = N + \theta\sum_{i=1}^q N_i$. The transition rate $(\dots,X_i,\dots,X_j,\dots) \to (\dots,X_i+1,\dots,X_j-1,\dots)$ reads

$$r_z(i,j,\vec{X}) \;=\; \mu X_j \frac{X_i+\theta N_i}{M}. \tag{2.60}$$

According to eqn. (2.58), the transition rate for $(\dots,X_i,\dots,X_j,\dots) \to (\dots,X_i+1,\dots,X_j-1,\dots)$ in the Glauber dynamics is given by

$$r_G(i,j,\vec{X}) \;=\; \mu X_j \exp\left(J\left(\frac{X_i}{N}-1\right)+h_i\right). \tag{2.61}$$

Proposition 2.57 *With the choice*

$$J = \frac{N}{M}, \qquad h_i = -\theta\sum_{j\neq i}^q \frac{N_j}{M} \qquad \text{for } i = 1,\dots,q \tag{2.62}$$

we find

$$r_z(i,j,\vec{X}) = r_G(i,j,\vec{X}) + \mathcal{O}(J,h_1,\dots,h_q).$$

Proof: As in the Sano model as well as the Glauber dynamics, μX_j appears as an multiplicative factor, we investigate the remaining part of the transition rates.
Sano model:

$$r_z(i,j,\vec{X}) = \frac{X_i+\theta N_i}{M} = \frac{N}{M}\,\frac{X_i}{N} + \theta\frac{N_i}{M}.$$

Glauber Dynamics: First order expansion of the exponential function yields

$$r_G(i,j,\vec{X}) \;=\; \exp\left(J\left(\frac{X_i}{N}-1\right)+h_i\right) = h_i + 1 - J + J\frac{X_i}{N} + \text{h.o.t.}$$

Comparison suggests the choice $J = \frac{N}{M}$ and $h_i = -\theta\sum_{j\neq i}^q \frac{N_j}{M}$ as

$$1 + h_i - J \;=\; 1 - \theta\sum_{j\neq i}^q \frac{N_j}{M} - \frac{N}{M} = \theta\sum_{j=1}^q \frac{N_j}{M} - \theta\sum_{j\neq i}^q \frac{N_j}{M} = \theta\frac{N_i}{M}$$

s.t. $r_z(i,j,\vec{X}) = r_G(i,j,\vec{X}) + \mathcal{O}(J,h_1,\dots,h_q)$.

□

Carry out!!!
Deterministic limit, Saddle-Node bifurcations for $q > 2$.

2.6.3 Echo chambers and reinforcement

Echo chambers and epistemic bubbles not only exist since the appearance of the internet, but even before electronic mass media became abundant, newspapers targeting on a certain clientele, clubs, or simply the social environment provided echo chambers. Though the existence of echo chambers are statistically confirmed [18, 9], the strength of their effect is under debate [11]. In the present section, we aim to modify the voter model to cover basic effects of echo chambers and reinforcement. Persons who "life" in an echo chamber will not interact with a representative sample of the population, but the sample of persons who are more likely to share his/her opinion. We model that fact by adapting the two-opinion zealot model in weighting the subpopulation with the opposite population by the probability $\vartheta_i < 1$ to interact with those individuals.

Model 2.58 Zealot model with reinforcement. *Let N denote the total population size, N_i the number of zealots for opinion $i \in \{1, 2\}$, and ϑ_i the weights for the opposite opinion. If X_t is the number of supporters for opinion 1, then*

$$X_t \to X_t + 1 \quad \text{at rate} \quad \mu(N - X_t)\,\frac{\vartheta_1(X_t + N_1)}{\vartheta_1(X_t + N_1) + (N - X_t + N_2)}, \tag{2.63}$$

$$X_t \to X_t - 1 \quad \text{at rate} \quad \mu X_t\,\frac{\vartheta_2(N - X_t + N_2)}{(X_t + N_1) + \vartheta_2(N - X_t + N_2)}. \tag{2.64}$$

Note that ϑ_1 is the probability og group-2-individuals to interact with group 1, and ϑ_2 that of group-1-members to interact with group 2. Obviously, this model agrees with the zealot model in case of $\vartheta_1 = \vartheta_2 = 1$.
In the discussion of the zealot model, we have seen that two different scalings of the zealot's numbers N_i are possible (Section 2.3.4): Either N_i are constant in N (weak limit), or they scale linearly with N, s.t. $N_i = n_i N$ (deterministic limit). Only the latter case yields for $N \to \infty$ an ODE - the first case resulted in the Dirichlet/beta distribution. In order to better understand the consequences of the mechanism proposed, we first consider the deterministic limit.

Deterministic limit

Proposition 2.59 *Let $N_i = n_i N$. Then, the deterministic limit for $x(t) = X_t/N$ reads*

$$\dot{x} \;=\; -\mu x\,\frac{\vartheta_2(1 - x + n_2)}{(x + n_1) + \vartheta_2(1 - x + n_2)} + \mu(1 - x)\,\frac{\vartheta_1(x + n_1)}{\vartheta_1(x + n_1) + (1 - x + n_2)}. \tag{2.65}$$

For $n_1 = n_2 = n$ and $\vartheta_1 = \vartheta_2$, $x = 1/2$ always is a stationary point; this stationary point undergoes a pitchfork bifurcation at $\vartheta_1 = \vartheta_2 = \vartheta_p$, where

$$\vartheta_p \;=\; \frac{1 - 2n}{1 + 2n}. \tag{2.66}$$

Proof: The rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1 - x)\,\frac{\vartheta_1(x + n_1)}{\vartheta_1(x + n_1) + (1 - x + n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1 - x + n_2)}{(x + n_1) + \vartheta_2(1 - x + n_2)}.$$

Therewith, the Fokker-Planck equation for the large population size (Kramers-Moyal expansion) reads

$$\partial_t u(x,t) = -\partial_x((f_+(x) - f_-(x))\, u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x) + f_-(x))\, u(x,t))$$

and the ODE due to the drift term in case of $N \to \infty$ reads

$$\frac{d}{dt}x = f_+(x) - f_-(x).$$

This result establishes eqn. (2.65). For the following, let $\vartheta_1 = \vartheta_2 = \vartheta$. If we also choose $n_1 = n_2 = n$, we have a neutral model, and $x = 1/2$ is a stationary point for all $\vartheta \geq 0$, $n > 0$. We find the Taylor expansion of the r.h.s. at $x = 1/2$ (using the computer algebra package maxima [32])

$$\begin{aligned} \mu^{-1}\frac{d}{dt}x &= -x\,\frac{\vartheta(1-x+n_2)}{(x+n_1)+\vartheta(1-x+n_2)} + (1-x)\,\frac{\vartheta(x+n_1)}{\vartheta(x+n_1)+(1-x+n_2)} \\ = \quad & -2\,\vartheta\,\frac{(2\,n+1)\,\vartheta+(2\,n-1)}{(2\,n+1)\,(\vartheta+1)^2}\left(x-\frac{1}{2}\right) + \frac{32\,\vartheta\,(\vartheta+n-\vartheta^2(n+1))}{(2\,n+1)^3\,(\vartheta+1)^4}\left(x-\frac{1}{2}\right)^3 + \mathcal{O}((x-1/2)^4) \end{aligned}$$

For $\vartheta \in (0,1)$, $n > 0$, the coefficient in front of the third order term always is non-zero, while the coefficient in front of the linear term becomes zero at $\vartheta = \vartheta_p$. Hence, we have a pitchfork bifurcation at that parameter.

□

The pitchfork bifurcation is unstable against any perturbation that breaks the symmetry $x \mapsto 1-x$ (Fig. 2.11). In panel (a), we have the symmetric case, and find the proper pitchfork bifurcation. Panel (b) shows the result if the number of zealots only differs slightly, where the reinforcement-parameter for both groups are assumed to be identical. We still find a reminiscent of the pitchfork bifurcation: The stable branches in (b) are close to the stable branches in (a), and also the unstable branches correspond to each other. For the limit $n_2 \to n_1$, panel (b) converges to panel (a). However, the branches are not connected any more but dissolve in two unconnected parts, and the pitchfork bifurcation is replaced by a saddle-node bifurcation.
In panel (c) and (d), the upper branch visible in panel (b) did vanish, and only the lower branch is present. As ϑ_2 is kept constant ($\vartheta_2 = 0.5$ in panel (c) and $\vartheta_2 = 0$ in panel (d)) and only ϑ_1 does vary, there is no continuous transition to panel (a).

The effect of reinforcement for a given group is similar to an increase of the number of that group's zealots. If the reinforcement becomes strong, the fraction of the group that performes reinforcement is able to become large. In panel (d), the second group has only 1/10 of the zealots of the first group, but is able to take over if the members of that group do an extreme reinforcement ($\vartheta_1 \ll 1$). However, if both groups reinforce themselves, the mechanism is kind of symmetrical, with a bistable setting as the consequence.

Weak limit

We now turn to the second scaling – the effect of zealots, and also the effect of the echo chambers, are taken to be weak. Under these circumstances, it is possible to find a limiting distribution.

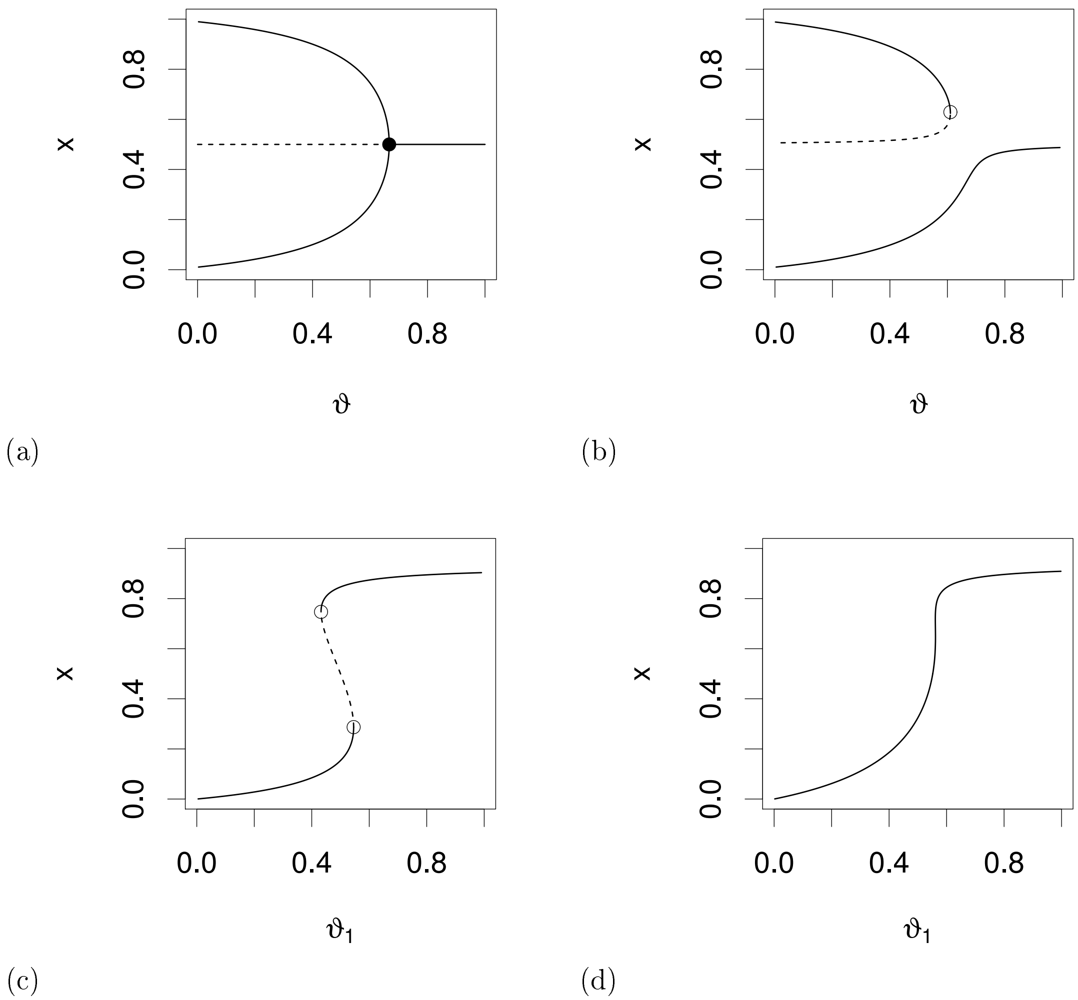


Figure 2.11: Stationary points of the reinforcement model over ϑ. The pitchfork bifurcation in (a) is indicated by a bullet, the saddle-node bifurcations in (b) and (c) are indicated by open circles. Stable branches of stationary points are represented by solid lines, unstable branches by dotted lines. (a) $n_1 = n_2 = 0.1$, $\vartheta_1 = \vartheta_2 = \vartheta$, (b) $n_1 = 0.1$, $n_2 = 0.105$, $\vartheta_1 = \vartheta_2 = \vartheta$, (c) $n_1 = n_2 = 0.1$, $\vartheta_2 = 0.5$, (d) $n_1 = 0.2$, $n_2 = 0.02$, $\vartheta_1 = 1.0$.

Theorem 2.60 *Let N_i denote the number of zealots for group i, N the population size, and $\vartheta_i = 1 - \theta_i/N$ the parameter describing reinforcement.In the limit $N \to \infty$, the density of the invariant measure for the random variable $z_t = X_t/N$ is given by*

$$\varphi(x) = C\, e^{\frac{1}{2}(\theta_1+\theta_2)x^2 - \theta_1 x}\; x^{N_1-1}\,(1-x)^{N_2-1}, \tag{2.67}$$

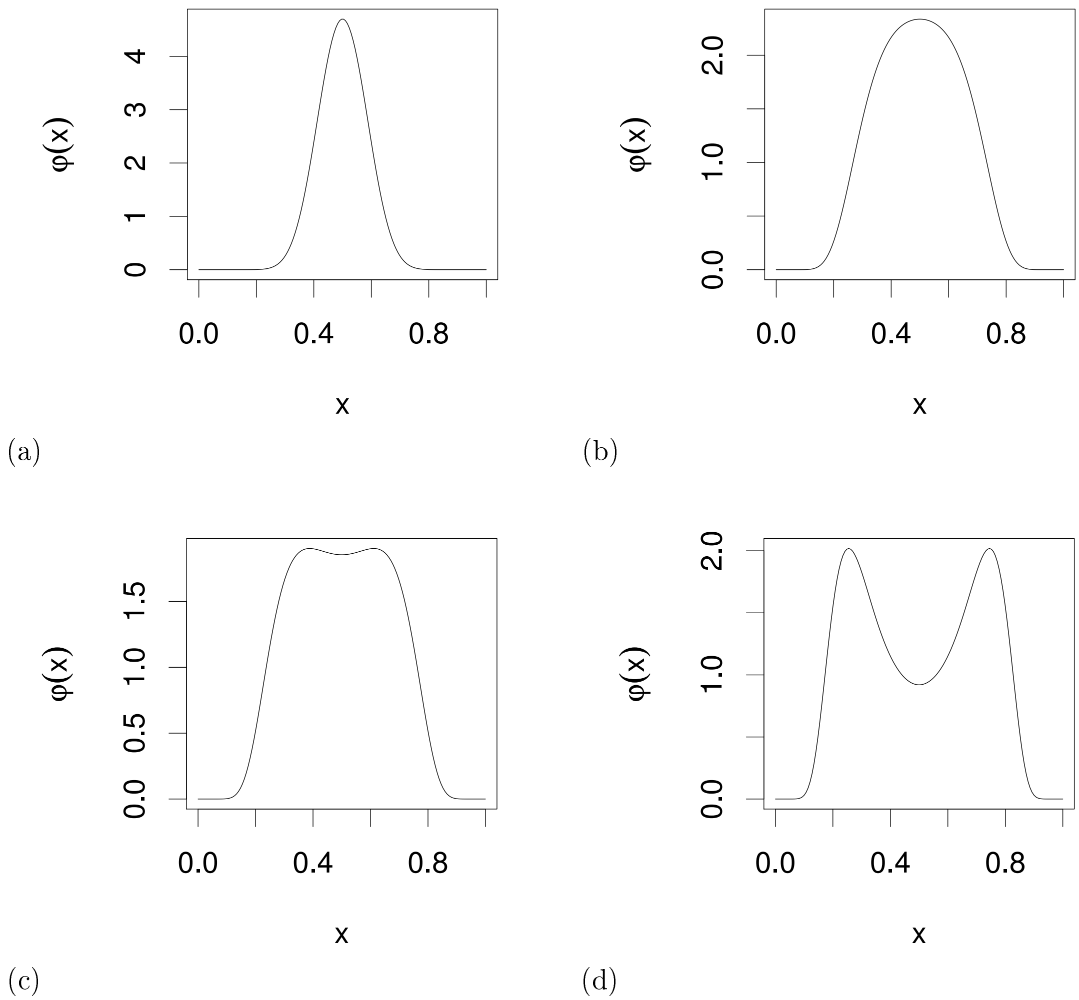


Figure 2.12: Invariant distribution, given in eqn. (2.67) for $N_1 = N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 70$, (c) $\theta_1 = \theta_2 = 80$, (d) $\theta_1 = \theta_2 = 100$.

where C is determined by the condition $\int_0^1 \varphi(x)\,dx = 1$.

Proof: We again start off with the Fokker-Planck equation, obtained by the Kramers-Moyal expansion, where we use the scaling $n_i = N_i/N$, and ϑ_i constant in N. Only afterwards, we proceed to the desired scaling.
As seen above, the rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1-x)\,\frac{\vartheta_1(x+n_1)}{\vartheta_1(x+n_1)+(1-x+n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1-x+n_2)}{(x+n_1)+\vartheta_2(1-x+n_2)}.$$

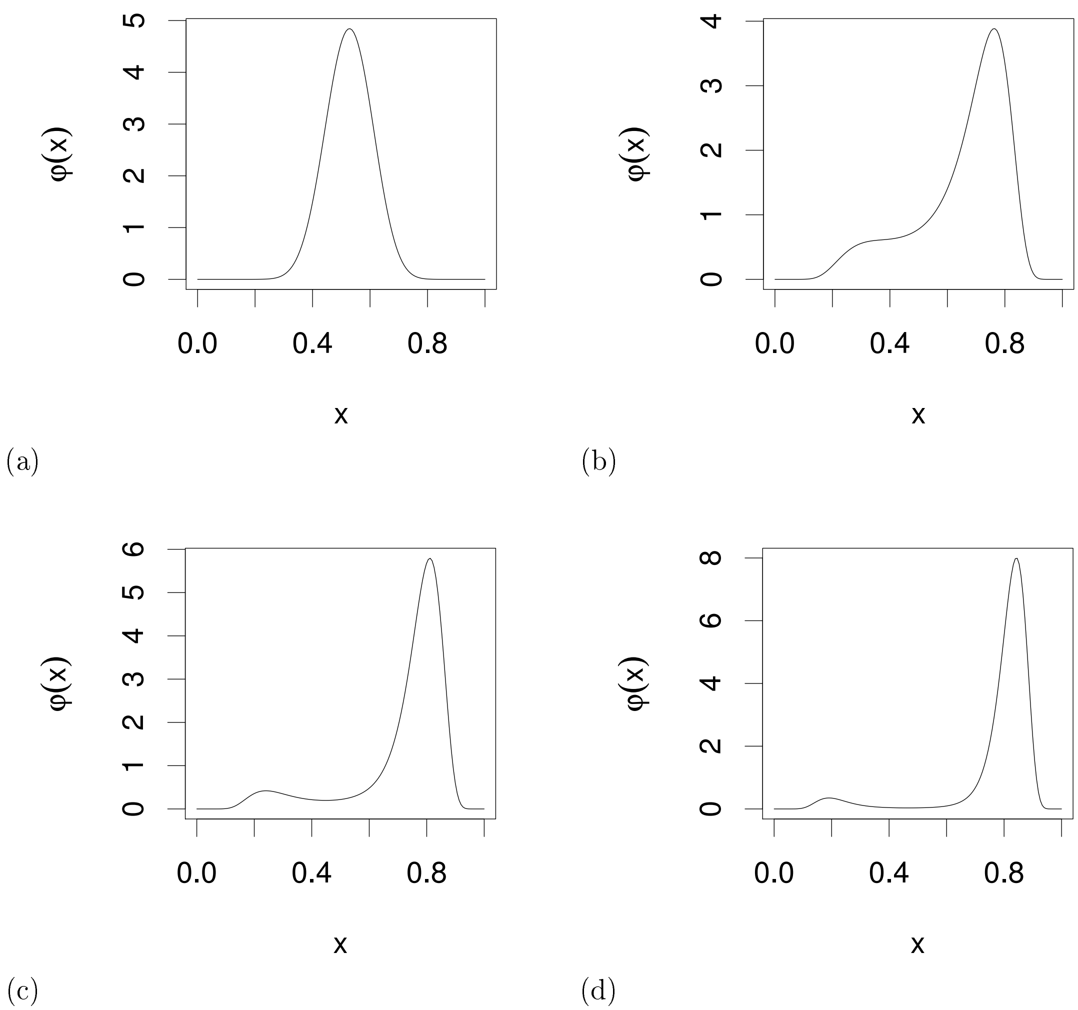


Figure 2.13: Invariant distribution, given in eqn. (2.67) for $N_1 = 22$, $N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 100$, (c) $\theta_1 = \theta_2 = 120$, (d) $\theta_1 = \theta_2 = 140$.

Therewith, the limiting Fokker-Planck equation reads

$$\partial_t u(x,t) = -\partial_x((f_+(x) - f_-(x))\, u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x) + f_-(x))\, u(x,t))$$

Now we rewrite drift and noise term with the new scaling $n_i = N_i/N$, $\vartheta_i = 1 - \theta_i/N$, where we neglect terms of order $\mathcal{O}(N^{-2})$. We find (using maxima [32]) that ($h := 1/N$)

$$\begin{aligned}
& f_+(x) - f_-(x)\\
= \;& \mu(1-x)\,\frac{(1-h\,\theta_1)(x+h\,N_1)}{(1-h\,\theta_1)(x+h\,N_1)+(1-x+h\,N_2)} - \mu x\,\frac{(1-h\,\theta_2)(1-x+h\,N_2)}{(x+h\,N_1)+(1-h\,\theta_2)(1-x+h\,N_2)}\\
= \;& \mu\bigg([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x) - (N_1+N_2)\,x + N_1\bigg)\,h + \mathcal{O}(h^2),
\end{aligned}$$

while $h(f_+(x)+f_-(x)) = h\,2\,\mu x(1-x) + \mathcal{O}(h^2)$. If we rescale time, $T = \mu\,h\,t$, the Fokker-Planck equation becomes

$$\partial_T u(x,T) = -\,\partial_x\bigg\{\,\bigg([(\theta_1{+}\theta_2)x{-}\theta_1]\,x\,(1{-}x){-}(N_1{+}N_2)\,x{+}N_1\bigg)\;u(x,T)\,\bigg\}{+}\partial_x^2\bigg\{x\,(1{-}x)\,u(x,T)\bigg\}.$$

For the invariant distribution $\varphi(x)$, the flux of that rescaled Fokker-Planck equation is zero, that is,

$$-\bigg([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x) - (N_1+N_2)\,x + N_1\bigg)\varphi(x) + \frac{d}{dx}\bigg(x(1-x)\,\varphi(x)\bigg) = 0.$$

With $v(x) = x(1-x)u(x)$, we have

$$v'(x) = \bigg([(\theta_1+\theta_2)x-\theta_1]\;+\frac{N_1}{x} - \frac{N_2}{1-x}\,x\bigg)v(x)$$

and hence

$$v(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1}\,(1-x)^{N_2}$$

resp.

$$\varphi(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1-1}\,(1-x)^{N_2-1}$$

□

For $\theta_1 = \theta_2 = 0$, we obtain the beta distribution, as we fall back to the zealot model without reinforcement. In the given scaling, the reinforcement is expressed by the exponential multiplicative factor. As $\vartheta_i = 1 - h\,\theta_i$, and h small, one could be tempted to assume that we are in the subcritical parameter range of the reinforcement model only, s.t. the distribution does not show a phase transition. As we see next, this idea is wrong.

Let us first consider the symmetric case, $N_1 = N_2 = \underline{N}$, and $\theta_1 = \theta_2 = \underline{\theta}$ (see Fig. 2.12). In that case, the distribution is given by

$$\varphi(x) = C\,e^{\underline{\theta}\,x\,(1-x)}\;x^{\underline{N}-1}\,(1-x)^{\underline{N}-1}.$$

The function always is symmetric w.r.t. $x = 1/2$. If $\underline{\theta}$ is small, and $\underline{N} > 0$, we find an unimodal function, with a maximum at 1/2. If, however, $\hat{\theta}$ is increased, eventually a bimodal distribution appears – we find back the pitchfork bifurcation that we already known from the deterministic limit of the model (Fig. 2.11, panel a).

As soon as $N_1 \neq N_2$, the symmetry is broken (Fig. 2.13), and we have an a situation resembling Fig. 2.11, panel (b). In the stochastic setting, however, we have more information: the second branch concentrates only little probability mass, and will play in practice only a minor role (if any at all). Only if N_1 and N_2 are distinctively unequal, this second branch ias able to concentrate sufficient probability mass to gain visibility in empirical data.

Remark 2.61 *We can use the zealot model or we can use the reinforcement model to fit and interpret election data. The zealot model for two parties yields the beta distribution. The density of the reinforcement model basically consist of a product, where one term is identical with the beta distribution,*

$$x^{N_1-1}\,(1-x)^{N_2-1}$$

while the second term expresses the influence of reinforcement

$$e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}.$$

Only if the data have a shape that is different from that of a beta distribution, the reinforcement component leads to a significantly improved fit. This is given, e.g., in case of a bimodal shape of the data (where at least one maximum is in the interior of the interval $(0,1)$*), or if the data have heavy tails. Both properties hint to the fact that the election districts are of two different types: one, where the party under consideration is relatively strong, and one where it is relatively weak. This difference, in turn, can be interpreted as the effect of reinforcement: In some election districts voters agree that the given party is preferable, in others they agree that the party is to avoid. The population is not (spatially) homogeneous, but some segregation - most likely caused by social mechanisms - take place. In that, the data analysis of spatially structured election data (results structured by election districts) based on the reinforcement model is able to detect spatial segregation and the consequences thereof.*

carry out Glauber/Reinf formulas!

Remark 2.62 *We compare the reinforcement model with the Glauber dynamics. For* $\vartheta_\ell = 1$*, we fall back to the zealot model. In that, we have proposition 2.50: The rates of Glauber dynamics and zealot model agree up to higher order (if the parameters* J *and* h *of the Glauber model are chosen appropriately). If we have* $\vartheta_\ell = 1-\omega\,\theta_\ell$*, and* ω *small, we can use Taylor expansion in* ω *to find a connection between Glauber dynamics and the reinforcement model. It turns out that the zero order terms in* ω *do not carry any information about the reinforcement, and these terms can be approximated by the Glauber dynamics (proposition 2.50). In contrast, the first order terms have a structure that is incompatible with the Glauber rates. Though the effect of reinforcement resembles that of the Curie-Weiss model (phase transitions), the mechanism is different. In the Curie-Weiss model, a majority rule leads to the phase transition. In the reinforcement model, the ignorance of the opposite group is the driving force that induces phase transitions. Apart of that, the reinforcement model still is a variant of the voter model.*

2.6.4 Spatial model and weak effects limit

Election districts clearly identify a spatial structure. In order to better understand the spatial effects, we develop a spatially structured model and determine also for that the probability density i the weak effects limit.
Let us consider election districts ordered in a two-dimensional lattice (torus) $\mathbb{Z}_n\times\mathbb{Z}_m$. In what follows, we always work modulo n resp. modulo m in the spatial indices. Let furthermore N denote the total population size in one election district, N_i the number of zealots for opinion $i\in\{1,2\}$, and ϑ_i the weights for the opposite opinion. The parameters N_i, θ_i are isotropic in space, that is, independent on the election district. If $X_t^{(k,\ell)}$ is the number of supporters for

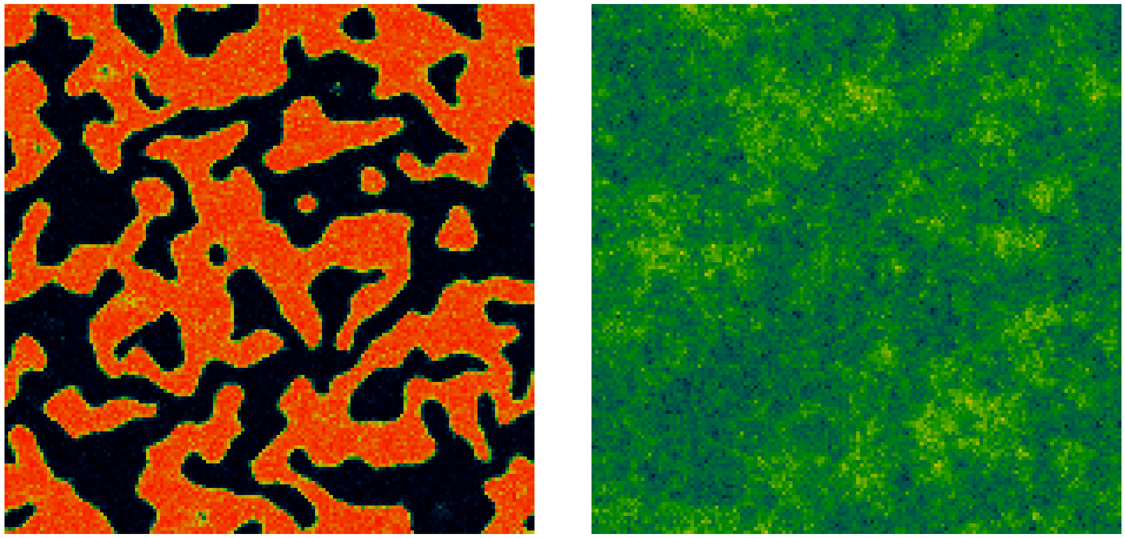


Figure 2.14: Simulation of the particle-based spatial reinforcement model given by eqns (2.68), (2.69). In both panels: grid size 150×150, $N = 100$. Left: $N_a = N_b = 6$, $\vartheta_1 = \vartheta_2 = 109$, $\gamma = 200$. Right: $Na = 16$, $N_b = 26$, $\vartheta_1 = \vartheta_2 = 110$, $\gamma = 200$.

opinion 1 in district $(k,\ell) \in \mathbb{Z}_n \times \mathbb{Z}_m$, while $N - X_t^{(k,\ell)}$ is that for opinion 2 (in the corresponding election district), then

$$X_t^{(k,\ell)} \to X_t^{(k,\ell)} + 1 \quad \text{at rate} \quad \mu(N - X_t^{(k,\ell)})\, \frac{\vartheta_1(\hat{X}_t^{(k,\ell)} + N_1)}{\vartheta_1(\hat{X}_t^{(k,\ell)} + N_1) + (N - \hat{X}_t^{(k,\ell)} + N_2)}, \tag{2.68}$$

$$X_t^{(k,\ell)} \to X_t^{(k,\ell)} - 1 \quad \text{at rate} \quad \mu X_t^{(k,\ell)}\, \frac{\vartheta_2(N - \hat{X}_t^{(k,\ell)} + N_2)}{(\hat{X}_t^{(k,\ell)} + N_1) + \vartheta_2(N - \hat{X}_t^{(k,\ell)} + N_2)}. \tag{2.69}$$

where $X_t^{(k,\ell)}$ denotes the weighted average of opinion-1-supporters in the Moore neighborhood of (k,ℓ),

$$\hat{X}_t^{(k,\ell)} = (1-\tau) X_t^{(k,\ell)} + \tau \check{X}^{(k,\ell)}, \qquad \check{X}^{(k,\ell)} := \frac{1}{8} \sum_{(k',\ell') \sim (k,\ell)} X_t^{(k',\ell')}$$

and $(k',\ell') \sim (k,\ell)$ if $|k-k'| \le 1$, $|\ell - \ell'| \le 1$, and $(k',\ell') \neq (k,\ell)$ (neighboring sites according to the Moore neighborhood). The parameter $\tau \in [0,1]$ plays the role of the spatial interaction strength: If $\tau = 0$, the sites are independent, if $\tau = 1$, individuals in an election district only communicate with individuals of neighboring election districts.

Theorem 2.63 *Let N_i denote the number of zealots for group i, N the population size, and $\vartheta_i = 1 - \theta_i/N$ the parameter describing reinforcement. We also scale the spacial interaction strength $\tau = \sigma/N$. In the limit $N \to \infty$, the density of the invariant measure for the random* *variable $z_t = X_t/N$ is given by*

$$\psi(x^{(\cdot,\cdot)}) = C \prod_{k,\ell} \Bigg(\varphi(x^{(k,\ell)}) \quad \exp\Bigg\{ -\frac{\gamma}{32} \sum_{(k',\ell')\sim(k,\ell)} (x^{(k,\ell)} - x^{(k',\ell')})^2 \Bigg\} \Bigg), \tag{2.70}$$

where $\varphi(.)$ is the homogeneous-population distribution defined in eqn. (2.67), and C is determined by the condition that the integral over $\psi(.)$ is 1.

Proof: We again start off with the Fokker-Planck equation, obtained by the Kramers-Moyal expansion, where we use the scaling $n_i = N_i/N$, and ϑ_i constant in N. Only afterwards, we proceed to the desired scaling.

As seen above, the rates to increase/decrease the state in site (k,ℓ) can be written as $f_+^{(k,\ell)}(X_t^{(\cdot,\cdot)}/N)$ resp. $f_-^{(k,\ell)}(X_t^{(\cdot,\cdot)}/N)$, where (recall that $n_i = N_i/N$)

$$\begin{aligned} f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) &= \frac{[\mu(1-x^{(k,\ell)})]\ [\vartheta_1((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1)]}{\vartheta_1((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1) + (1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)}, \\ f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) &= \frac{[\mu x^{(k,\ell)}]\ [\vartheta_2(1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)]}{((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1) + \vartheta_2(1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)}. \end{aligned}$$

Here, $\hat{x}^{(k,\ell)}$ is the average of $x^{(\cdot,\cdot)}$ at the Moore neighborhood of (k,ℓ).

Therewith, the flux $j^{(k,\ell)}(x^{(\cdot,\cdot)})$ for the limiting Fokker-Planck equation is defined by

$$\begin{aligned} j^{(k,\ell)}(x^{(\cdot,\cdot)}) &= -\Big(f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) - f_-^{(k,\ell)}(x^{(\cdot,\cdot)})\Big)u(x^{(\cdot,\cdot)}) \\ &\quad + \frac{1}{2N}\,\partial_{x^{(k,\ell)}}\Big\{\Big(f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) + f_-^{(k,\ell)}(x^{(\cdot,\cdot)})\Big)u(x^{(\cdot,\cdot)})\Big\} \end{aligned}$$

and the Fokker-Planck equation itself reads

$$\partial_t u(x^{(\cdot,\cdot)}) = \sum_{k,\ell} \partial_{x^{(k,\ell)}} j^{(k,\ell)}(x^{(\cdot,\cdot)}).$$

Now we rewrite drift and noise term with the new scaling $n_i = N_i/N$, $\vartheta_i = 1 - \theta_i/N$, $\tau = \gamma/N$, where we neglect terms of order $\mathcal{O}(N^{-2})$. We find (using maxima [32]) that ($h := 1/N$)

$$\begin{aligned} & f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) - f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) \\ =\ & \frac{\mu\,(1-x^{(k,\ell)})\ [\vartheta_1((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1)]}{\vartheta_1((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1) + (1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)} \\ & - \frac{\mu\,x^{(k,\ell)}\ [\vartheta_2(1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)]}{((1-\tau)\,x^{(k,\ell)} + \tau\check{x}^{(k,\ell)} + n_1) + \vartheta_2(1-(1-\tau)\,x^{(k,\ell)} - \tau\check{x}^{(k,\ell)} + n_2)} \\ =\ & \mu\Big((\check{x}^{(k,\ell)} - x^{(k,\ell)})\,\gamma + x^{(k,\ell)}\,(1-x^{(k,\ell)})\,(\theta_2\,x^{(k,\ell)} - \theta_1\,(1-x^{(k,\ell)})) - (N_2+N_1)x^{(k,\ell)} + N_1\Big)\,h + \mathcal{O}(h^2), \end{aligned}$$

while $f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) + f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) = 2\,\mu\,x^{(k,\ell)}\,(1-x^{(k,\ell)}) + \mathcal{O}(h)$. Hence, in lowest order, $j^{(k,\ell)}(x^{(\cdot,\cdot)}) = 0$ reads

$$\begin{aligned} & \partial_{x^{(k,\ell)}}\Big\{\Big(x^{(k,\ell)}(1-x^{(k,\ell)})\Big)u(x^{(\cdot,\cdot)})\Big\} \\ =\ & \Big((\check{x}^{(k,\ell)} - x^{(k,\ell)})\,\gamma + x^{(k,\ell)}\,(1-x^{(k,\ell)})\,(\theta_2\,x^{(k,\ell)} - \theta_1\,(1-x^{(k,\ell)})) - (N_2+N_1)x^{(k,\ell)} + N_1\Big)u(x^{(\cdot,\cdot)}). \end{aligned}$$

For $\gamma = 0$, this equation collapse to the equation for the homogeneous case. We therefore define $v(x^{(\cdot,\cdot)})$ by

$$u(x^{(\cdot,\cdot)}) = v(x^{(\cdot,\cdot)}) \prod_{k,\ell} e^{\frac{1}{2}(\theta_1+\theta_2)(x^{(k,\ell)})^2 - \theta_1 x^{(k,\ell)}} (x^{(k,\ell)})^{N_1-1} (1 - x^{(k,\ell)})^{N_2-1}$$

and obtain

$$\partial_{x^{(k,\ell)}} v(x^{(\cdot,\cdot)}) \quad = \quad \gamma\, (\check{x}^{(k,\ell)} - x^{(k,\ell)})\ v(x^{(\cdot,\cdot)}).$$

That system of equations has the solution

$$v(x^{(\cdot,\cdot)}) = C \exp\left\{\gamma \sum_{(k,\ell)} \left(\frac{1}{16} \sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - \frac{1}{2}\,(x^{(k,\ell)})^2\right)\right\}.$$

The factor $1/16 = (1/8)/2$ is due to symmetry reasons: Each pair (k_1,ℓ_1),(k_2,ℓ_2) with $(k_1,\ell_1) \sim (k_2,\ell_2)$ appears twice in the sum. We rewrite the sum as follows

$$\begin{aligned}
&\sum_{(k,\ell)} \left(\frac{1}{16} \sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - \frac{1}{2}\,(x^{(k,\ell)})^2\right)\\
= \;& \frac{1}{16} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - 8\,(x^{(k,\ell)})^2\right)\\
= \;& -\frac{1}{16} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} \left(- x^{(k,\ell)} x^{(k',\ell')} + (x^{(k,\ell)})^2\right)\right)\\
= \;& -\frac{1}{16} \sum_{(k,\ell)} \frac{1}{2}\left(\sum_{(k',\ell')\sim(k,\ell)} \left((x^{(k',\ell')})^2 - 2x^{(k,\ell)} x^{(k',\ell')} + (x^{(k,\ell)})^2\right)\right)\\
= \;& -\frac{1}{32} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} \left(x^{(k,\ell)} - x^{(k',\ell')}\right)^2\right)
\end{aligned}$$

$\square$

The distribution consist of a multiplicative structure, where the for each site one part is related to the local dynamics ($\varphi(x^{(k,\ell)})$), while the second term punishes differences in the state between neighboring sites. The assumption of the weak coupling prevents the neighboring sites to directly influence the internal dynamics.

We aim to use the Metropolis-Hastings algorithm [**?**] to draw samples from that measure. One ingredient in that algorithm is computation of the quotient of the likelihood of two states that only differ in one single site. The following proposition is an immediate consequence of the product structure of $\psi(.)$. Note the change of the prefactor in front of the sum form 32 to 16 due to symmetry reasons.

Proposition 2.64 *Consider two states $x^{(\cdot,\cdot)}$ and $y^{(\cdot,\cdot)}$, where $\forall (k,\ell) \neq (k_0,\ell_0) : x^{(k,\ell)} =$*

$y^{(k,\ell)}$. *Then,*

$$\frac{\psi(x^{(\cdot,\cdot)})}{\psi(y^{(\cdot,\cdot)})} \;=\; \frac{\varphi(x^{(k_0,\ell_0)})}{\varphi(y^{(k_0,\ell_0)})}\;\frac{\exp\left\{-\frac{\gamma}{16}\sum_{(k',\ell')\sim(k_0,\ell_0)}\left(x^{(k_0,\ell_0)}-x^{(k',\ell')}\right)^2\right\}}{\exp\left\{-\frac{\gamma}{16}\sum_{(k',\ell')\sim(k_0,\ell_0)}\left(y^{(k_0,\ell_0)})-(y^{(k',\ell')}\right)^2\right\}}.$$

Simulations by means of the Metropolis-Hastings algorithm reveals that the spatial reinforcement model, for an appropriate parameter set, behaves as the Ising model (Fig. 2.14). Particularly, we find homogeneous regions, paralleling the magnetic domains. Within those domains, the stateis rather homogeneously distributed around one of the two bistable states. Within such a region, the resulting distributions resembles a non-reinforcement, zealot model. Only if the spatial domain that we consider is large enough to contain several of those homogeneous regions, the reinforcement becomes clearly visible. As the range of those domains depend on the parameter of the models (particularly the parameters that characterize the communication), it might be that a whole country looks homogeneously, and reinforcement is difficult to detect in data.

Data analysis

Parameter estimation: The maximum likelihood parameter estimation is somewhat subtle, as the distribution

$$\varphi(x) = C\, e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\, x}\; x^{N_1-1}\,(1-x)^{N_2-1},$$

incorporates exponential terms - in particular, if the parameters become large, the integral $\int_0^1 e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\, x}\; x^{N_1-1}\,(1-x)^{N_2-1}\,dx$ becomes numerically unstable. Therefore, we re-parameterize the distribution, defining $\hat\nu$, $\hat s$, $\hat\theta$, and $\hat\psi$ by

$$\theta_1 = \hat s\,\hat\theta\,\hat\psi,\quad \theta_2 = \hat s\,\hat\theta\,(1-\hat\psi),\quad N_1+1 = \hat s\,(1-\hat\theta)\,\hat\nu,\quad N_2+1 = \hat s\,(1-\hat\theta)\,(1-\hat\nu), \tag{2.71}$$

where $\hat\theta, \hat\psi, \hat\nu \in [0,1]$, and $\hat s > 0$, with the restriction $\hat s\,(1-\hat\theta)\,\hat\nu > 1$, and $\hat s\,(1-\hat\theta)\,(1-\hat\nu) > 1$. Therewith, the distribution becomes

$$\varphi(x) = \hat C\,\exp\left[\;\hat s\;\left(\hat\theta(x^2/2-\hat\psi\,x)\;+(1-\hat\theta)\,\hat\nu\,\ln(x)+(1-\hat\theta)\,(1-\hat\nu)\,\ln(1-x)+A\right)\;\right].$$

Here, A is a constant that can be chosen in dependency on the parameters and the data at hand. In practice, it is used to avoid an exponent that has a very large absolute number. The constant $\hat C$ is, as before, determined by the fact that the integral is one. This form allows for a reasonable maximum likelihood estimation, given appropriate election data.

Numerical issues: The model assumes continuous data, while the election data are discrete. Therefore, a vote share of 0 or 1 is possible in the empirical data, but the distribution may have poles for those values. We replace all empirical vote shares below 10^{-10} by 10^{-10}, and similarly, all data above $1-10^{-10}$ by $1-10^{-10}$. In order to determine the normalization constant of $\varphi(x)$, we do not integrate from 0 to 1, but only from 0.001 to 0.999. Furthermore, for numerical reasons, we restrict $\hat s$ by an upper limit, that we mostly define as 1800.

Test for reinforcement: The zealot model and the reinforcement model are nested. In that, we can use the likelihood-ratio test to check for the significance of the reinforcement component: If

$\mathcal{LL}_0$ is the log-likelihood for the restricted model ($\theta_1 = \theta_2 = 0$, resp. $\hat\theta = 0$), and $\mathcal{LL}$ is that for the full reinforcement model, we have asymptotically, for a large sample size

zitat li lihood tio test

$$2(\mathcal{LL} - \mathcal{LL}_0) \sim \chi^2_2$$

That is, twice the difference in the log-likelihoods is asymptotically χ^2 distributed, where the degree of freedom is the number of the surplus parameters (here: θ_1 and θ_2, resp. $\hat\theta$ and $\hat\psi$, that is, the degree of freedom is 2).
Additionally, we use the Kolmogorov-Smirnov test to find out if the model-distribution of either the full reinforcement mode, or the zealot model ($\theta_1 = \theta_2 = 0$, resp. $\hat\theta = 0$) agrees with the empirical distribution. If both models are in line with the data, then the reinforcement component will rather not add to the interpretation of the data, if only the reinforcement model approximates the data well (or, at least, much better than the zealot model), we can expect that it is sensible to take the reinforcement component into account.

Graphical representation: Apart of the histograms with the empirical distributions of the vote shares, and the distribution according to the reinforcement model, we will focus on the reinforcement parameter of the group at hand θ_2. As $\vartheta_2 = 1 - \theta_2/N$ is the decrease in the probability that an individual of the focal group interacts with an individual of the opposite group, θ_2 is a measure for the reinforcement for the focal group. Furthermore, we will consider $\theta_1 + \theta_2$, which is the total amount of reinforcement in both groups. Last, we investigate

$$\overline{\theta} = \sum_{party} \text{vote share of the party } \times \ \theta_2(party).$$

$\overline{\theta}$ is a measure for the overall degree of reinforcement in the population. Herein, we dismiss parties with a vote share below 0.1%, and parties that have a non-zero vote share in less than 20 districts.

US presidential elections If we investigate the recent US presidential elections, we find a good agreement of model and data for 2004-2106 (Fig. 2.15). The Kolmogorov-Smirnov test yields p-values larger or equal 0.12, indicating that this model cannot be rejected (Tab. 2.2), while the Kolmogorov-Smirnov test for the zealot model performs worse (either in a gradual way for the 2004 elections, or indeed with the result that the zealot model can be rejected at a confidence level $p < 0.003$ for all elections later than 2004).
Though we estimate the parameters for republicans and democrats independently, the reinforcement parameters $\hat\psi$ approximately add up to 1 (apart from the 2000 election). If no further candidate is in the game (the vote shares for democrats and republicans add up to 1 in each election district), this observation is a logical consequence of the model structure. However, we do have further candidates. Our results indicates that these further candidates only play a minor role.
The 2000-election is different. The Kolmogorov-Smirnov-test clearly shows (in line with the graphical representation) that the data do not follow the model-distribution. Here, the candidate for the green party had a significant influence on the election results (Fig. 2.17). Our model has difficulties to get on with the interplay of three groups, resulting in a rather bad fit of the data.

If we inspect the reinforcement parameters for the two parties (Fig. 2.16), we find that the reinforcement for the democrats is rather unimportant (in comparison to that of the republicans)

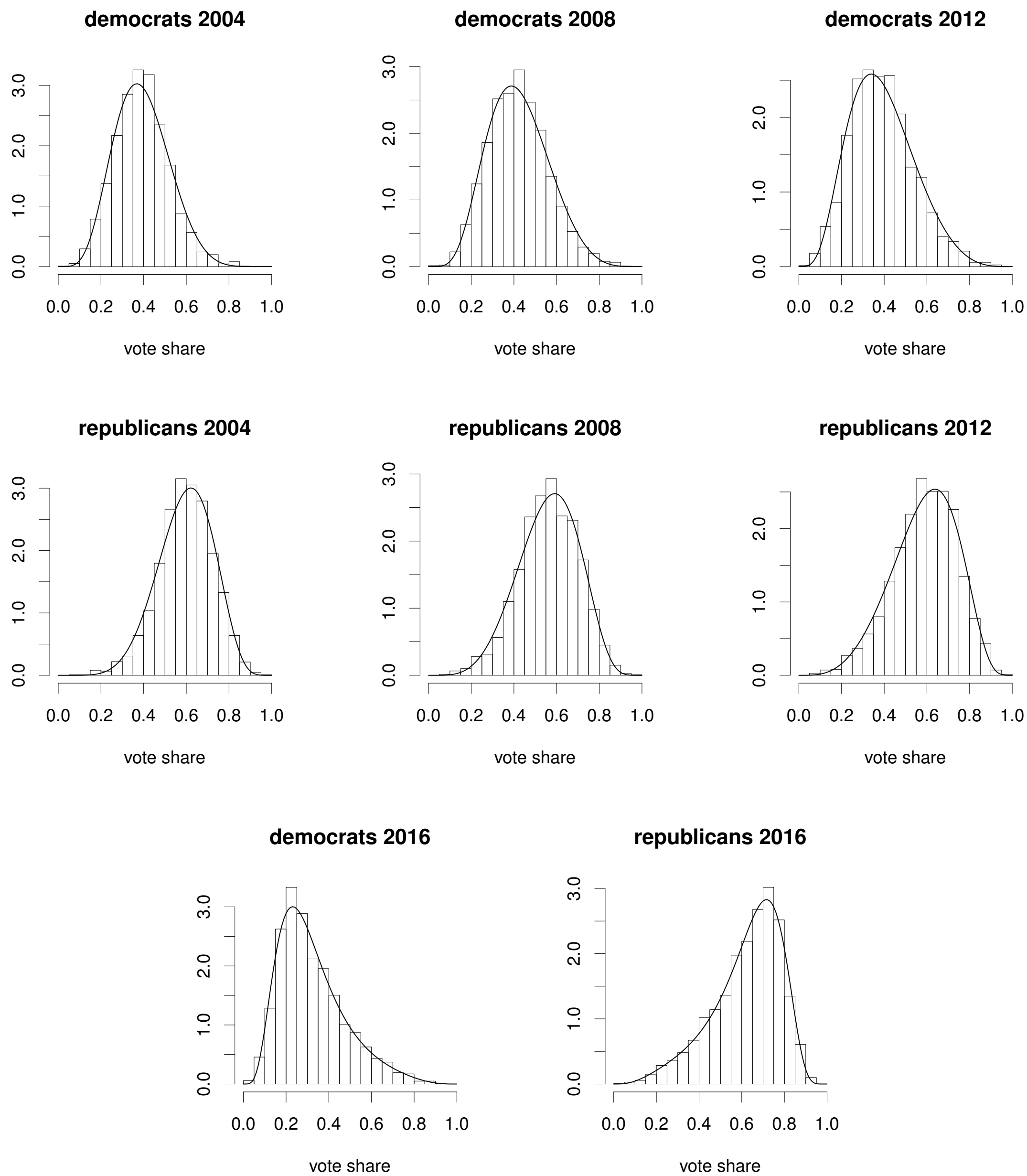


Figure 2.15: Distribution of the vote share of republicans/democrats in 2004, 2008, 2012 and 2016 presidential elections. We find a good agreement (Kolmogorov-Smirnov-test, $p \geq 0.12$ for all elections shown here). Only from 2008 on, the reinforcement model is significantly better ($p < 0.005$) than the zealot model.

before the 2016 elections. In the 2016 election, the reinforcement of the democrats jumps to the value of the republicans before 2016. Furthermore, in the 2016 elections, the reinforcement of the republicans suddenly shows a threefold increase. We detect here the consequences of the

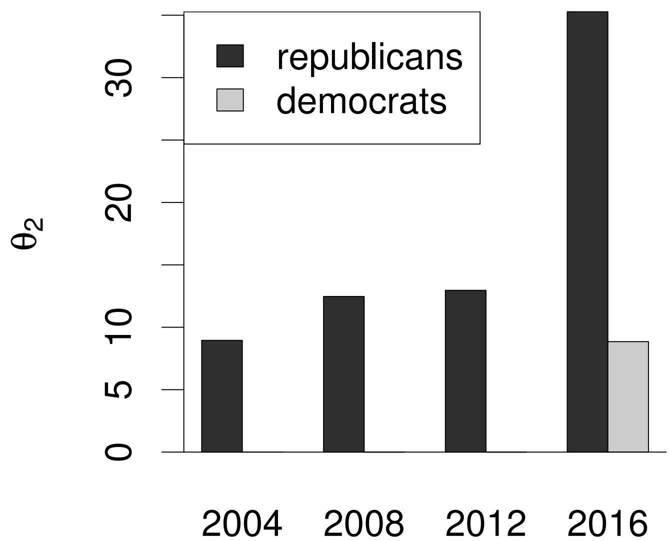


Figure 2.16: Plot of the reinforcement parameter. Note that in 2004-2012, the parameter for the democrats is negligible small, and only in 2016 it becomes visible.

year	party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf)	p_{ks} (beta
2000	republicans	0.0180	0.820	6.611e-05	89	72.99446	<1e-10	<1e-10	<1e-10
2000	democrats	0.190	4.889e-05	6.61e-05	2.24	0.0001	1	<1e-10	<1e-10
2000	green	0.0004	0.81	6.61e-05	193.5	156.9	<1e-10	<1e-10	<1e-10
2004	republicans	0.519	0.412	6.611e-05	21.7	8.95	0.15	0.15	0.092
2004	democrats	0.440	0.3253	0.99993	18.4	0.0004	0.60	0.12	0.1
2008	republicans	0.435	0.522	6.611e-05	23.9	12.5	0.0026	0.12	0.066
2008	democrats	0.544	0.5167	0.99993	22.5	0.0008	0.0023	0.14	0.070
2012	republicans	0.429	0.587	6.611e-05	22.1	13.0	1.5e-06	0.24	0.0076
2012	democrats	0.535	0.569	0.99993	20.2	0.00076	7.1e-06	0.23	0.011
2016	republicans	0.324	0.7860	0.19	55.5	35.3	<1e-10	0.66	7.4e-10
2016	democrats	0.563	0.7861	0.74	43.3	8.85	<1e-10	0.14	6.77e-10

Table 2.2: Estimated parameters for the two parties in the eight elections. p_{ll} is the result of the likelihood ration test for the significance of the reinforcement component; p_{ks} is the result of the Kolmogorov-Smirnov-test for the question of the empirical cumulative distribution differs significantly from the cumulative distribution of the model (either the reinforcement model, or zealot model with the beta distribution).

populist attitude of the republican's candidate.

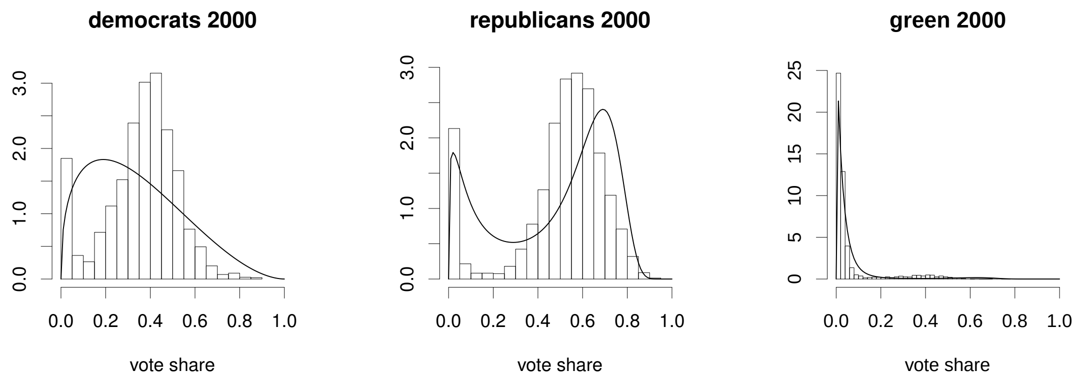


Figure 2.17: Distribution for the 2000 presidential elections (democrats/republicans/green). We did fit the model using the focal group versus the pooled remaining groups.

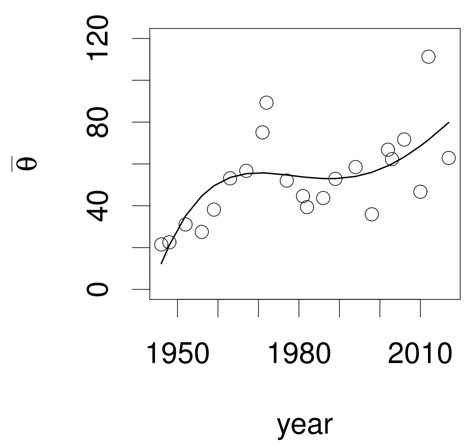


Figure 2.18: Average reinforcement parameter $\overline{\theta}$ for The Netherlands, together with the forth order polynomial fitting theses estimates.

The Netherlands As the election system of The Netherlands is base on proportional representation, but the theory developed so far describes a binary choice, we consider the result of one party versus the rest. Note that the data bear some kind of randomness due to the size of the election districts: The election district sizes vary from several 100 to over a million voters. We neglect that difference but focus on the vote share of a certain party.
We start off with the winner of the elections after the second world war: the "Katholieke Volkspartij (KVP)", the Catholic People's Party. After the second world war, this party played a major role in The Netherlands with a vote share of about 0.33; only after 1967, the vote share dropped

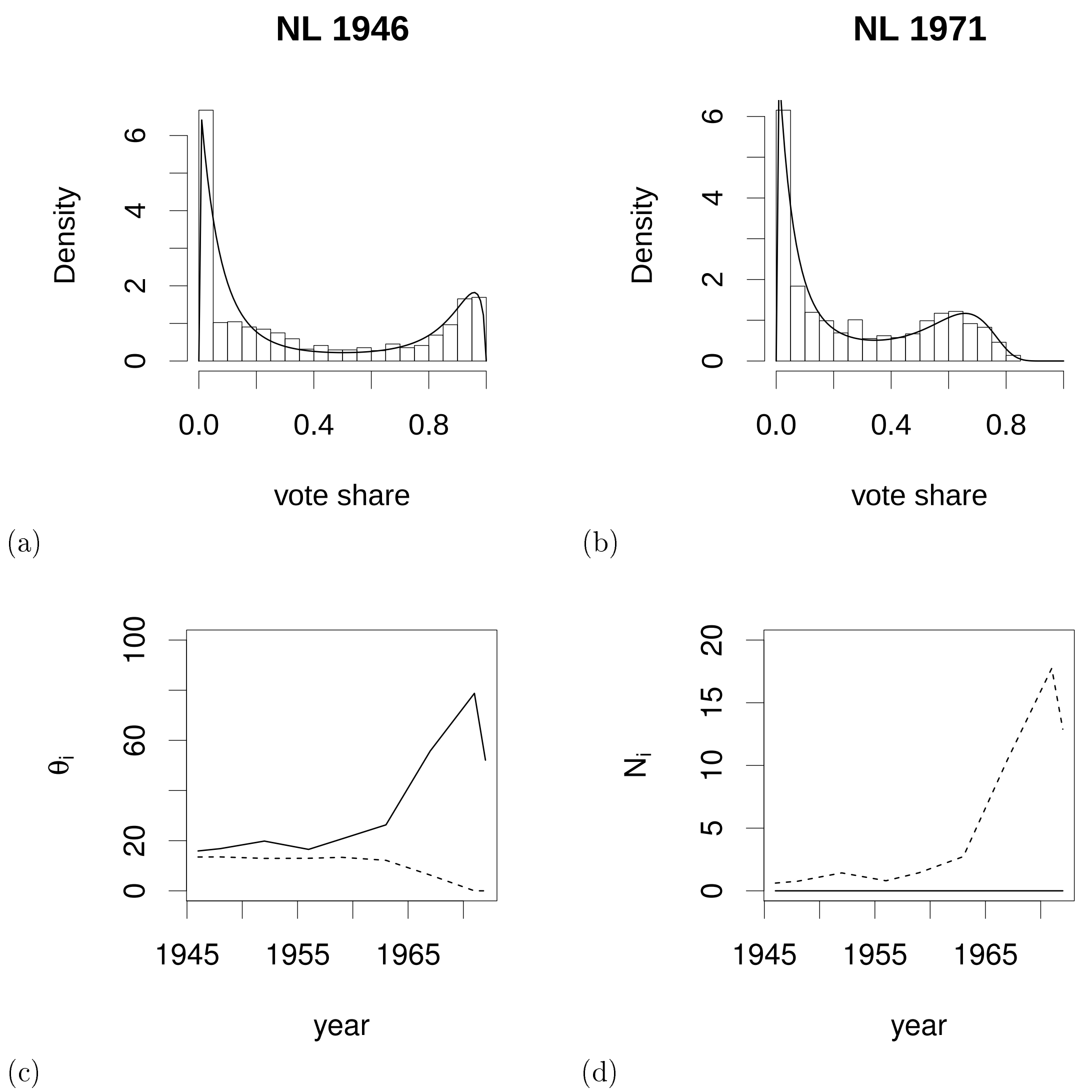


Figure 2.19: Vote share and fit of the reinforcement model to data for the Catholic People's Party. (a) and (b) Vote share with density for 1948 and 1971; (c) θ_i over time (bold: θ_2, dashed: θ_1; note that θ_2 is the reinforcement parameter of the supporters of the Catholic People's Party); (d) N_i over time (bold: N_1, dashed: N_2; note that N_1, the number of zealots for the Catholic People's Party, is estimated as $N_1 \ll 1$.)

to around 0.25. The party did merge with two other parties 1980, and did already before, from 1977 on, appeared in a coalition with two other religions parties.

During the active time of the party, the reinforcement model is highly superior to the zealot model (likelihood-ratio-test, $p < 10^{-10}$). We clearly find a bimodal distribution (Fig. 2.19), where one peak is at close to zero, the other decreasing in the years 1948 to 1971 from about 0.95 to 0.6.

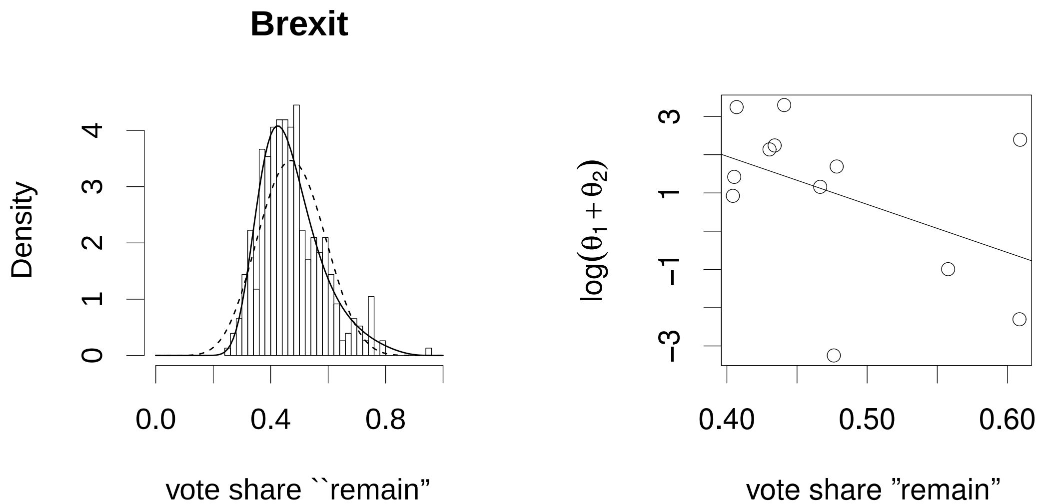
year	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	θ_1	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
1946	6.61e-05	0.98	0.46	30.01	15.9	13.5	2.11e-216	8.55e-15	<1e-20
1948	6.61e-05	0.98	0.45	31.11	16.8	13.5	1.72e-214	4.40e-12	<1e-20
1952	6.61e-05	0.96	0.39	34.21	19.8	12.9	2.63e-191	1.40e-13	<1e-20
1956	6.61e-05	0.97	0.44	30.31	16.5	13.0	3.9e-195	1.03e-11	<1e-20
1959	6.61e-05	0.96	0.39	35.51	20.7	13.4	8.82e-196	4.35e-11	<1e-20
1963	6.61e-05	0.93	0.32	41.21	26.3	12.2	7.16e-163	6.86e-06	<1e-20
1967	6.61e-05	0.86	0.10	72.51	55.7	6.3	2.21e-120	0.00015	<1e-20
1971	6.61e-05	0.82	6.61e-05	96.51	78.7	0.005	7.96e-95	0.00067	<1e-20
1972	6.61e-05	0.80	6.61e-05	65.01	52.11	0.003	5.15e-43	8.96e-06	<1e-20

Table 2.3: Parameter for the Christian People's Party.

Obviously, this party did divide the population. The religious segregation at that time also did lead to a certain spatial segregation: Catholic and Protestant population tended to separate. Spatial or social segregation is for sure one of the major driver for reinforcement, as this segregation creates a homogeneous environment that minimizes the contact with different opinions. This observation is also reflected by the parameter estimation: the number of zealots of that party is estimated by a value close to zero; the strength of the party is solely explained by a strong reinforcement component, that was increasing over time. One can speculate that the religious basis of the Catholic People's Party did strongly promote this reinforcement.

To obtain an impression about the trend in the overall reinforcement, we considered $\overline{\theta}$ as defined above (Fig. 2.18). We find that after a phase with rather constant enforcement, in recent years the enforcement seems to grow. One can speculate that this growth is due to the internet and social media. However, as the noise in the estimate is rather large, it is difficult to decide about the exact timing of that increase.

Brexit The reinforcement-model is statistically clearly superior to the zealot model to describe the Brexit data (likelihood-ratio test $p = 1.3 \cdot 10^{-11}$). Also the Kolmogorov-Smirnov test clearly indicates that the reinforcement-model cannot be rejected ($p = 0.75$), but the zealot model that does not incorporate reinforcement is not appropriate ($p = 0.0009$). Interestingly, the point estimate for $\hat{\psi}$, which describes the relative weight for the reinforcement of one group versus the other group, indicates that the brexitiers are responsible for over 99% of the reinforcement. In order to investigate this finding more thoroughly, we use the likelihood-ratio test to compare a restricted model where both groups do have the same reinforcement parameters ($\hat{\psi} = 0.5$) and the model where the reinforcement parameter for both groups are arbitrary $\hat{\psi} \in [0, 1]$). It turns out that there is no significant superiority of the second model over the restricted model. That is, though the point estimate seems to hint that the brexitiers are way more prone to reinforcement in comparison with the remainers, there is no statistically significant signal supporting that finding. To further investigate the question which of the two groups are deeper involved in reinforcement, we estimate the model-parameters for the different regions (assuming equal reinforcement for both groups), and draw the estimation for $\theta_1 + \theta_2$ over the vote share for "remain". It turns out, that there is a negative (but non-significant) correlation of the reinforcement and the vote share for remain (Fig. 2.20). Also that results hints to the fact that the brexitiers are involved



(a) (b)

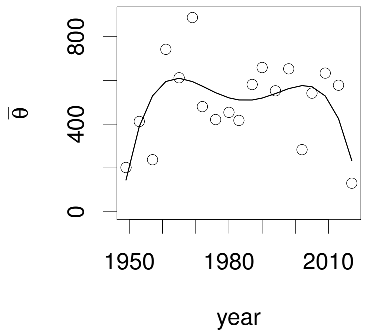
Figure 2.20: (a) Vote share for “remain” in the Brexit vote, together with the reinforcement model (solid line) and the zealot model (dashed line). (b) Estimation for $\theta_1 + \theta_2$ over vote share for “remain”, estimated separately for the different regions, together with a linear fit o the logarithmic scale.

to reinforcement with a higher degree.
In any case, we find a high influence of reinforcement, which could be considered to be in line with the overall perception of the Brexit process.

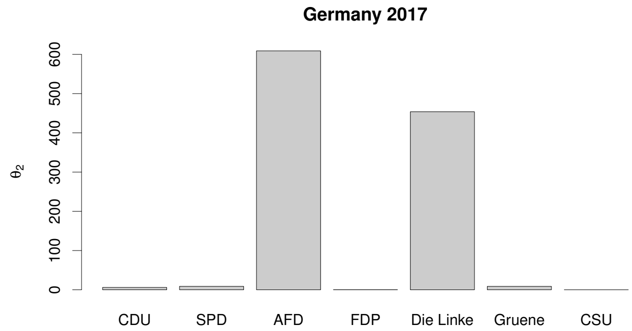
Germany The overall degree of reinforcement in Germany seems to be rather stable, but very noisy (Fig. 2.21). If we investigate the detailed election results from 2017 for the 7 parties that are present in th parliament, we find that reinforcement plays a role for particularly two parties: the “AFD” and “Die Linke”. Both parties are known as populist parties, one fat the political left- and one at the right wing. For all other parties, reinforcement is not statistical significant for the data at hand. If we compare the results for whole Germany and the “old” states only (tables 2.5 and 2.6), we find that the reinforcement of the AFD is strongly connected with the “new” states,

$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
0.87	0.76	0.99996	254.5	0.0084	1.3e-11	0.75	0.0009

Table 2.4: Parameter for the Brexit referendum, “remainers”. For symmetry reasons, the parameters for “leave” are identical, but $\hat{\psi}_{leave} = 1 - \hat{\psi}_{remain}$, and accordingly $\theta_{2,leave} = 192.97$. We have the result of the likelihood-ratio test for $\hat{\theta} = 0$ ($p_{ll} = 1.3e - 11$), the result for the Kolmogorovff-Smirnov test if the data are in line with the reinforcement-distribution (p_{ks}(Reinf) $= 0.75$), and the result of Kolmogorov-Smirnov, if the data are in line with the zealot (beta) distribution (p_{ks}(beta) $= 0.0009$).



(a)



(b)

Figure 2.21: (a) Average reinforcement parameter $\overline{\theta}$ for Germany, together with the forth order polynomial fitting theses estimates. (b) Reinforcement parameter for several parties, from the election data (2017) for whole Germany.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.39	0.39	0.86	112.5	6.02	1	0.68	0.78
SPD	0.19	0.17	6.61e-05	50.5	8.80	0.99	0.20	0.20
AFD	0.07	0.77	6.61e-05	792.5	608.91	8.6e-11	0.29	0.0009
FDP	0.31	0.73	0.998	182.5	0.28	1	0.52	0.54
Die Linke	0.054	0.78	6.61e-05	583.5	453.7	3.44e-09	2.1e-06	2.93e-08
Gruene	0.070	0.15	6.61e-05	58	8.7	1	0.85	0.84
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.004	0.004

Table 2.5: Results for whole Germany, 2017.

while the reinforcement of "die Linke" seems to be independent of the classification in "old" and "new" states.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.52	0.67	0.76	294.5	46.74	0.33	0.50	0.78
SPD	0.21	4.95e-05	6.61e-05	47.2	0.0023	1	0.17	0.17
AFD	0.30	0.71	0.99996	189.5	0.0055	1	0.86	0.92
FDP	0.29	0.68	0.99991	251.5	0.015	0.70	0.62	0.68
Die Linke	0.088	0.83	0.14	1159.5	834.22	0.00086	0.00068	0.00016
Gruene	0.22	0.86	0.38	289.5	154.36	0.16	0.112	0.075
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.0038	0.0038

Table 2.6: Results for Germany, only the "old" states ,2017.

2.7 Size of the parliament and population size

Universal size effects for populations in group-outcome decision-making problems [4]; two models in the paper:
(1) Simple Argument
A parliament should minimize the number of voters er representative, N/n_r. At the same time, it should maximize the efficiency of the parliament; the number of pairwise interactions, roughly n_r^2, should be minimal. A weighted sum of both expressions should be minimizes,

$$\frac{N}{n_r} + A\, n_r^2 = \min$$

which leads to

$$n_r \sim N^{1/3}$$

which is in accordance to empirical data.
(2) Kind of a hierarchical Galam-Model.

2.8 Have a look - more models and approaches

[22]: A STD based on the derivative of a Brownian motion. Claims universality in social processes, and finance systems. Scale-free growth behavior.

[44, 46, 43] Sznajd model for opinion evolution.

Rumors and infection. ODE rumor models. [47]: Opinions, Conflicts, and Consensus: Modeling Social Dynamics in a Collaborative Environment (Wikipedia)

[29], Phenotypical model to explain the Rank-order relation in elections

2.9 Spatial valence model

Consider the following scenario: You are living in a long street. In this street, there are also two bakeries. They have an identical offer to an identical price. To which bakery you will go? .... exactly. You will choose the bakery that is closer to you. Now, take the bakery's point of view. Each of the bakeries want to have as many customers as possible. Where should they localize themselves? The first at the beginning of the street, the second as far away from the fist one as possible, at the end of the street? In this way, each of them would have the half street as customers. However, if one of the bakeries now move to the middle of the street, this bakery suddenly gets more than half of the inhabitants as customers. We see, at the end, both bakeries should be located at the very same spot: In the middle of the street.

These very ideas can be reformulated to describe the relationship between parties and voters. Parties offer a political program, voters have certain opinions and interests. A voter will vote for that party that is closest to his/her opinion. However, also parties react on the interests of voters and adapt their program (up to certain degree) to obtain the maximum number of votes possible. There is a whole bunch of models addressing the question how parties will develop under this process.

2.9.1 Hotelling model

Hotelling described 1929 [24] the very same setting for parties and voters: We have voters and parties. The political opinions can be localizes in a one-dimensional space (from left-wing to right-wing, say). Also parties, characterized by their program, can be localized in this one-dimensional feature space. How do parties adapt their program to maximize the number of votes? We state his model and the central theorem – the convergence theorem – in the most simple version to understand the basic ingredients that lead to the predicted behaviour.

Model 2.65 Hotelling model. *Let N be an odd number of voters, the opinion of the i'th voter is characterized by $z_i \in \mathbb{R}$. We assume the generic condition $z_i \neq z_j$ for $i \neq j$. Let $\mu_A, \mu_B \in \mathbb{R}$ characterize the location of two parties A and B. A voter will vote for the party closest to his/her opinion. If $p_i = 1$ if individual i votes for A, $p_i = 0$ if he/she votes for B, and $p_i = 1/2$ if he/she is undecided, then*

$$p = p(\mu_A, \mu_B; z_i) = \begin{cases} 1 & \text{if} \quad |z_i - \mu_A| < |z_i - \mu_B| \\ 0 & \text{if} \quad |z_i - \mu_A| > |z_i - \mu_B| \\ 1/2 & \text{if} \quad |z_i - \mu_A| = |z_i - \mu_B|. \end{cases}$$

The number of votes for party A reads

$$U_A(\mu_A, \mu_B, z_1, \dots, z_N) = \sum_{i=1}^{N} p(\mu_A, \mu_B; z_i),$$

and that for party A is given by

$$U_B(\mu_A, \mu_B; z_1, \dots, z_N) = N - U_A(\mu_A, \mu_B; z_1, \dots, z_N)$$

The equilibrium strategies (μ_A^*, μ_B^*) are defined by the Nash equilibrium, that is

$$\forall \mu_A \in \mathbb{R}: \; U_A(\mu_A, \mu_B^*, z_1, \ldots, z_N) \leq U_A(\mu_A^*, \mu_B^*, z_1, \ldots, z_N),$$
$$\forall \mu_B \in \mathbb{R}: \; U_B(\mu_A^*, \mu_B, z_1, \ldots, z_N) \leq U_B(\mu_A^*, \mu_B^*, z_1, \ldots, z_N).$$

vielleicht besser in definition oben Nash gleichg. wie normal einfueren?

Proposition 2.66 (Convergence theorem, Median Voting Theorem) *The only equilibrium strategy is given by $(\mu_A^*, \mu_B^*) = (z^*, z^*)$, where z^* is the median of $(z_i)_{i=1,\ldots,N}$.*

Proof: *(a) $(\mu_A^*, \mu_B^*) = (z^*, z^*)$ is an equilibrium strategy.*
For symmetry reasons, we only need to consider party A. As $\mu_A^* = \mu_B^*$, $p(\mu_A, \mu_B; z_i) = 1/2$, and

$$U_A(\mu_A, \mu_B^*, z_1, \ldots, z_N) = N/2.$$

If $\mu_A > \mu_B$, we increase then

$$\forall z_i \geq \mu_A : \; p(\mu_A, \mu_B, z_i) = 1, \quad \forall z_i \leq \mu_B : \; p(\mu_A, \mu_B, z_i) = 0.$$

Since we assume that N is odd, if $\mu_B^* = z^*$ is the median of the points z_i, then $U_B(\mu_A, \mu_B^*; z_1, \ldots, z_N) \geq (N+1)/2$, and hence

$$U_A(\mu_A, \mu_B^*; z_1, \ldots, z_N) \leq N - (N+1)/2 < N/2 = U_A(\mu_A^*, \mu_B^*; z_1, \ldots, z_N).$$

Symmetry reasons show that U_A is also deceased if we choose $\mu_A < \mu_A^*$.
(b) $(\mu_A^*, \mu_B^*) = (z^*, z^*)$ is the only equilibrium strategy.
Assume $\mu_A \neq \mu_B\ \not{0}z^*$ is an equilibrium. Then, the gain for both parties is $N/2$. In this case, moving μ_A, say, to z^* increases the gain of A to at least $(N+1)/2 > N/2$. Hence we have no equilibrium.
Now assume $\mu_A \neq \mu_B$. If it is not the case that both parties receive $N/2$ votes, one party receives less and hence moves into z^*, where it gains at least $N/2$.
If, however, both parties obtain $N/2$, the median individual(s) are undecided, and the distance of μ_A and μ_B are identical from z^*. Since $\mu_A \neq \mu_B$ we are faced with a symmetrical situation: $\mu_A = z_0 + \varepsilon$, and $\mu_B = z_0 - \varepsilon$ (or vice versa). Hence if one party moves towards the median point (while the other stays), this party increases its gain. We have no equilibrium strategy.

□

This theorem predicts that in the given setup, parties eventually converge towards the same program. This predictions unexpected - the general feeling is, that parties distinguish from each other and have their regular voters. However, Hotelling formulated his model not in the setting of political parties, but rather for more general cases of competition. E.g., consider a street, where two bakeries compete. If inhabitants in the street only decide based on the distance to a bakery which of the two shops they will visit, the theorem tells us that the bakeries should both be in the middle of the street, abreast. And indeed, such a situation is not often to find in cities. Down [10] adapted Hotellings ideas to investigate voting behaviour.

Literaturhinweise einbauen!!!

A whole bunch of literature exploits how to modify the model, such that the convergence theorem still holds, or eventually non-symmetric equilibrium distributions appear. The range of modifications ranges from relatively small steps (what happens if N is even? What happens if ties for z_i are allowed?) or adaptation of the gain functions (instead to maximize the number of voters, the aim could be to win). The dimension of the trai space is increased ($z_i, \mu_A, \mu_B \in \mathbb{R}^n$ instead of $z_i, \mu_A, \mu_B \in \mathbb{R}$), and stochasticity is introduced to ensure that the equilibrium strategies are stable (in the same spirit as trembling hand equilibria are introduced in game theory). Below we discuss in particular the n-dimensional deterministic version, and a stochastic version of the Hotelling model.

2.9.2 Stochastic valence model in higher dimensions

The Hotelling model has a striking implication, the median voter theorem. However, a brief look into the newspapers clearly shows that parties do take different positions. What's wrong with the Hotelling model?

Three extensions are of particular interest: First, the world is not one-dimensional. The simple left-right scheme does not adequately represent reality. There are more dimensions, as social issues, economics, security, education etc. The position of a vote (and the position of a party) is better represented by a point in $\mathbb{R}^n$.

Second, the world is not deterministic. Te model should incorporate a certain amount of stochasticity to take into account our lack of complete knowledge about a given individual.

A third observation is that parties, or, better, leading candidates of parties, have a different electoral perception. This perception breaks the intrinsic symmetry between parties. We will use real numbers $\lambda_j \in \mathbb{R}$ to describe the overall perception of party j. This perception can be caused, e.g. by their past performance in the government. The paper by Schofield [42] investigates the consequences of these two extensions.

Model 2.67 *Let $x_1, \dots, x_m \in \mathbb{R}^n$ denote the positions of voters, and $z_1, \dots, z_k$ the position of parties. The position of voters are given and fixed, while parties may choose their positions. We define the utility $u_{i,j}$ of party j for voter i by*

$$u_{i,j}(z_j) = \lambda_j - \beta \|x_i - z_j\|^2 + e_{i,j}$$

where λ_j is the valence of party j (identical to all voters), $\|x_i - z_j\|^2$ is the euclidean distance between the voter's and the party's position, and $e_{i,j}$ are i.i.d. random variables that express our lack of knowledge. We might assume that $e_{i,j}$ follow an extreme value distribution of type 1, defined by $P(e_{i,j} < h) = e^{-e^{-h}}$. The parameter $\beta \in \mathbb{R}$ does scale the importance of the distance in comparison to noise and valences.

Voter i selects party j if for him/her the utility of party j is maximal. The probability for voter i to vote for party j reads

$$p_{i,j}(z_1, \dots, z_k) = P(\, u_{i,j}(z_j) > u_{i,\ell}(z_\ell) \quad \text{for all } \ell = 1, \dots, k,\ \ell \neq j),$$

and consequently, the expected vote share for party j is then defined by

$$V_j(z_1, \dots, z_k) = \frac{1}{m} \sum_{i=1}^{m} p_{i,j}(z_1, \dots, z_k).$$

The position of the voters are assumed to be given. The parties, however, are assumed to opportunistically aim to maximize their vote share. It might be, that parties are driven by zealaots that are persuaded by a certain position. In this case, the model will fail. However,

parties for sure respond to polls and aim to come up with popular positions. Let us assume that they have no intrinsic program, but only want to win as many votes as possible. We can use that as a working hypothesis and observe the predictions that come out of this assumption.
In that, it is near at hand to use the notion of Nash equilibria for the given situation. We might expect that the positions of the parties will approximate these Nash equilibria.

Definition 2.68 *We first define the local best response of party j given the position of all other parties. Let $\hat{z}_\ell$ for $\ell \neq j$ the positions of the other parties. As we only work locally, we also consider a reference value $\hat{z}_j$ for party j, and introduce the notation*

$$\hat{z}_\bullet = (\hat{z}_1, \ldots, \hat{z}_k).$$

The set of ε-close, local best responses is defined by

$$U_{j,\varepsilon}(\hat{z}_\bullet) = \{z_j \in B_\varepsilon(\hat{z}_j)\,:\,\forall \tilde{z}_j \in B_\varepsilon(\hat{z}_j)\setminus\{z_j\} : V_j(\hat{z}_1, \ldots, \tilde{z}_j, \ldots \hat{z}_k) < V_j(\hat{z}_1, \ldots, z_j, \ldots \hat{z}_k)\}.$$

The vector of party locations $z^*_\bullet = (z^*_1, \ldots, z^*_k)$ is a (strict) local Nash equilibrium, if there is $\varepsilon > 0$ s.t.

$$z^*_j \in U_{j,\varepsilon}(z^*_\bullet) \quad \text{for } j = 1, \ldots, k.$$

The next proposition characterizes local Nash equilibria. To clarify the notation we note that for a function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, the expression

$$\nabla f(x,y) = \begin{pmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{pmatrix}$$

is the gradient, and

$$\nabla\,\nabla^T f(x,y) = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} (\partial_x, \partial_y) f(x,y) = \begin{pmatrix} \partial_{xx} f(x,y) & \partial_{xy} f(x,y) \\ \partial_{yx} f(x,y) & \partial_{yy} f(x,y) \end{pmatrix}$$

establishes the Hessian matrix. In the following, ∇_{z_j} indicates the gradient w.r.t. the vector z_j.

Proposition 2.69 *Assume that $V_j(z_1, \ldots, z_k)$ are twice differential functions. Sufficient criteria that $(z^*_1, \ldots, z^*_k)$ is a local strict Nash equilibrium are the frist and second order conditions:*

$$(1) \qquad \nabla_{z_j} V_j(z^*_1, \ldots, z_j, \ldots, z^*_k)\Big|_{z_j = z^*_j} = 0 \tag{2.72}$$

$$(2) \qquad \sigma\left(\nabla_{z_j}\,\nabla^T_{z_j} V_j(z^*_1, \ldots, z_j, \ldots, z^*_k)\Big|_{z_j = z^*_j}\right) \subset \mathbb{C}^- \tag{2.73}$$

for $j = 1, \ldots, k$.

Proof: The two conditions imply that $z^*_\bullet = (z^*_1, \ldots, z^*_k)$ is a local maximum in the sense that varying locally one vector component z_j in $z^*_\bullet$, while keeping all other components constant, (strictly) decreases the function V_j. This property establishes that $z^*_\bullet$ is a local strict Nash equilibrium. □

In order to investigate the local Nash equilibria of our model, we first represent p_{ij} in a more handy way.

Proposition 2.70 *Let $u^*_{i,j}(z_j) = \lambda_j + \beta\|x_i - z_j\|$. Then, under the conditions given by the model definition,*

$$p_{i,j}(z_1,\ldots,z_k) = \frac{e^{u^*_{i,j}(z_j)}}{\sum_{\ell=1}^k e^{u^*_{i,\ell}(z_\ell)}}. \tag{2.74}$$

Proof: Let $e_0,\ldots,e_\ell$ denote i.i.d. random variables, with $P(e_i < h) = e^{-e^{-h}}$. Let furthermore $\varphi(h) = \frac{d}{dh}\mathbf{e}^{-e^{-h}}$ the probability density of e_0, and let $a_i \in \mathbb{R}$ denote constant real values. Then,

$$\begin{aligned}
& P(e_0 > \max\{a_1+e_1,\ldots,a_\ell+e_\ell\}\,) \\
= & \int_{-\infty}^{\infty} P(h > \max\{a_1+e_1,\ldots,a_\ell+e_\ell\}\,|\,e_0 = h\,)\,\varphi(h)\,dh \\
= & \int_{-\infty}^{\infty} P(\,(h-a_1 > e_1)\wedge\ldots\wedge(h-a_\ell > e_\ell)\,)\,\varphi(h)\,dh \\
= & \int_{-\infty}^{\infty} \prod_{i=1}^{\ell} P(h-a_i > e_i)\,\varphi(h)\,dh = \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{\ell} e^{-h+a_i}}\frac{d}{dh}e^{-e^{-h}}\,dh \\
= & \int_{-\infty}^{\infty} e^{-\left(1+\sum_{i=1}^{\ell} e^{a_i}\right)e^{-h}} e^{-h}\,dh
\end{aligned}$$

Since $\frac{d}{dx}\frac{1}{\alpha}e^{-\alpha e^{-x}} = e^{-\alpha e^{-x}}e^{-x}$, we have

$$P(e_0 > \max\{a_1+e_1,\ldots,a_\ell+e_\ell\}\,) = \frac{1}{1+\sum_{i=1}^{\ell} e^{a_i}}.$$

Therewith we find (note that we replace e_i by $e_{i,j}$, the random variables that appear in the definition of $u_{ij}(z_j) = u^*(z_j) + e_{i,j}$)

$$\begin{aligned}
p_{i,j}(z_1,\ldots,z_k) &= P(u^*_{i,j}(z_j) + e_{i,j} > u^*_{i,\ell}(z_\ell) + e_{i,\ell} \quad \text{for } \ell \neq j) \\
&= P(e_{i,j} > u^*_{i,\ell}(z_\ell) - u^*_{i,j}(z_j) + e_{i,\ell} \quad \text{for } \ell \neq j) \\
&= \frac{1}{1+\sum_{\ell\neq j} e^{u^*_{i,\ell}(z_\ell) - u^*_{i,j}(z_j)}} = \frac{e^{u^*_{i,j}(z_j)}}{\sum_{\ell=1}^k e^{u^*_{i,\ell}(z_\ell)}}.
\end{aligned}$$

□

Note that we obtain a multinomial logist model. For $k = 2$ we have the classic logist model, with the well known methods to estimate parameters. Due to the intrinsic independency assumptions on $e_{i,j}$, the abilities of the model are limited.

The best candidate for a local Nash equilibrium is the mean value of the voters.

Theorem 2.71 *Let $\overline{x} = \frac{1}{m}\sum_{i=1}^m x_i$. Let $z^*_j = \overline{x}$ for $j = 1,\ldots,k$. Then, the first order equilibrium condition*

$$\nabla_{z_j} V_j(z^*_1,\ldots,z_j,\ldots,z^*_k)\Big|_{z_j = z^*_j} = 0$$

is given. Let

$$cov(x) := \frac{1}{m}\sum_{i=1}^{m}(x_i - \overline{x})\,(x_i - \overline{x})^T. \tag{2.75}$$

and $p_j = \frac{e^{\lambda_j}}{\sum_{i=1}^k e^{\lambda_i}}$. *The second order equilibrium condition will are satisfied, if all eigenvalues of the symmetric matrices (for* $j = 1, \ldots, k$*)*

$$C \;=\; \beta(1-2p_j)\,cov(x) - I \tag{2.76}$$

are negative.

Proof: We can shift all voter states x_i and all party positions z_i by $\overline{x}$, $\tilde{x}_i = x_i - \overline{x}$, $\tilde{z}_i = z_i - \overline{x}$. As the vectors x_i, z_i only enter the model vial the utility function, and there in form of $\|x_i - z_j\|^2$, the model is shift invariant. That is, without restriction, $\overline{x} = 0$. In $\tilde{x}_i$, $\tilde{z}_i$, we drop the tilde again, and proceed with x_i and z_i.

We show that the first order condition is true for $z_\ell = 0 = \overline{x}$. We consider in the next computation $\rho_{i,j}$ as a function of $(u^*_{i,1}, \ldots, u^*_{i,j})$, and the scores $u^*_{i,j}$ as a function of $z_1, \ldots, z_k$ in the sense that $u^*_{i,j} = u^*_{i,j}(z_j)$.

$$\begin{aligned}
\nabla_{z_j} V_j(z_1, \ldots, z_k) &= \frac{1}{m}\sum_{i=1}^{m}\nabla_{z_j} p_{i,j}(u^*_{i,1}, \ldots, u^*_{i,k})\\
&= \frac{1}{m}\sum_{i=1}^{m}\frac{\partial}{\partial u^*_{i,j}} p_{i,j}(u^*_{i,1}, \ldots, u^*_{i,k})\;\left(\nabla_{z_j} u^*_{i,j}(z_j)\right)\\
&= \frac{1}{m}\sum_{i=1}^{m}\frac{\partial}{\partial u^*_{i,j}} p_{i,j}(u^*_{i,1}, \ldots, u^*_{i,k})\,\nabla_{z_j}(\lambda_j + \beta\|x_i - z_j\|^2)\\
&= \frac{2\beta}{m}\;\sum_{i=1}^{m}\frac{\partial}{\partial u^*_{i,j}} p_{i,j}(u^*_{i,1}, \ldots, u^*_{i,k})\,(x_i - z_j).
\end{aligned}$$

The derivative $\frac{\partial}{\partial u^*_{i,j}} p_{i,j}$ is given by (we suppress the argument in $p_{i,j}$)

$$\frac{\partial}{\partial u^*_{i,j}} p_{i,j} = \frac{\partial}{\partial u^*_{i,j}}\left(\frac{1}{1 + \sum_{\ell\neq j} e^{u^*_{i,\ell} - u^*_{i,j}}}\right) = p^2_{i,j}(1/p_{i,j} - 1) = p_{i,j} - p^2_{i,j} = p_{i,j}(1 - p_{i,j}). \tag{2.77}$$

All in all, we obtain

$$\nabla_{z_j} V_j(z_1, \ldots, z_k) \;=\; \frac{2\beta}{m}\;\sum_{i=1}^{m} p_{i,j}(1 - p_{i,j})\,(x_i - z_j). \tag{2.78}$$

At $z_1 = \ldots = z_k = 0$ we have $u^*_{i,\ell} - u^*_{i,j} = \lambda_\ell + \beta\|x_i - 0\| - \lambda_j - \beta\|x_i - 0\| = \lambda_\ell - \lambda_j$, independently on i, the expression $p_{i,j} - p^2_{i,j}$ does not depend on i, and we can write

$$p_{i,j} - p^2_{i,j} = p_j - p^2_j.$$

Therewith, and with $\overline{x} = 0$, we obtain

$$\nabla_{z_j} V_j(0,\ldots,0) = (p_j - p_j^2)\,\frac{2\beta}{m}\sum_{i=1}^{m} x_i = 0.$$

We proceed with the second order condition, and inspect the Hessian. From (2.78), $\nabla_{z_j}^T V_j(z_1,\ldots,z_k) = \frac{2\beta}{m}\ \sum_{i=1}^m (p_{i,j} - p_{i,j}^2)\,(x_i - z_j)^T$. Hence,

$$\nabla_{z_j}\,\nabla_{z_j}^T V_j(z_1,\ldots,z_k) \;=\; \frac{2\beta}{m}\ \sum_{i=1}^{m}\Big(\nabla_{z_j}(p_{i,j} - p_{i,j}^2)\Big)\,(x_i - z_j)^T + \frac{2\beta}{m}\ \sum_{i=1}^{m}(p_{i,j} - p_{i,j}^2)\,\nabla_{z_j}(x_i - z_j)^T.$$

In (2.77), we already worked out the derivative of $p_{i,j}$. Therewith we proceed

$$\begin{aligned}&\sum_{i=1}^{m}\Big(\nabla_{z_j}\,[p_{i,j} - p_{i,j}^2]\Big)\,(x_i - z_j)^T\\ =\;& 2\beta\sum_{i=1}^{m}\Big([(p_{i,j} - p_{i,j}^2) - 2p_{i,j}(p_{i,j} - p_{i,j}^2)]\Big)\,(x_i - z_j)\,(x_i - z_j)^T\end{aligned}$$

and thus at $z_1 = \ldots = z_k = 0$

$$\sum_{i=1}^{m}\Big(\nabla_{z_j}\,[p_{i,j} - p_{i,j}^2]\Big)\,(x_i - z_j)^T\bigg|_{z_1=\ldots=z_k=0} = p_j(1-p_j)(1-2p_j)\sum_{i=1}^{m}\,x_i\,x_i^T.$$

Furthermore,

$$\sum_{i=1}^{m}(p_{i,j} - p_{i,j}^2)\,\nabla_{z_j}(x_i - z_j)^T\bigg|_{z_1=\ldots=z_k=0} = -p_j(1-p_j)I.$$

All in all,

$$\nabla_{z_j}\,\nabla_{z_j}^T V_j(0,\ldots,0) \;=\; 2\beta\,p_j(1-p_j)\bigg(2\beta\,(1-2p_j)\,\frac{1}{m}\sum_{i=1}^{m}\,x_i\,x_i^T - I\bigg).$$

With the definition $\mathrm{cov}(x,x) = \frac{1}{m}\sum_{i=1}^m\,x_i\,x_i^T$, the matrix $\nabla_{z_j}\,\nabla_{z_j}^T V_j(0,\ldots,0)$ has eigenvalues with negative real parts (or, as the matrix is symmetric, only negative eigenvalues) if and only if the matrix

$$2\beta(1-2p_j)\mathrm{cov}(x) - I$$

has negative eigenvalues. We specify p_j using $u_j^*(0) = \lambda_j$,

$$p_j = \frac{e^{\lambda_j}}{\sum_{i=1}^k e^{\lambda_i}}.$$

The result follows if we note that we need to shift the coordinate system back by adding $-\overline{x}$, which in particular modifies $\mathrm{cov}(x)$. □

The following proposition is a consequence of this theorem. The proposition tells us when $z_1 = \ldots = z_k = 0$ is a Nash equilibrium. Recall that n is the dimension of the feature space s.t. $x_i, z_j \in \mathbb{R}^n$, and the trace of a matrix is the sum of the diagonal elements.

Proposition 2.72 *If $z_1 = \ldots = z_k = \overline{x}$ is a local strict local Nash equilibrium, then necessarily*

$$2\beta(1-2p_j)\, tr(cov(x)) < n \qquad for\ j = 1, \ldots, k. \tag{2.79}$$

In case of $n = 2$, stricter condition

$$2\beta(1-2p_j)\, tr(cov(x)) < 1 \qquad for\ j = 1, \ldots, k. \tag{2.80}$$

is sufficient.

Proof: (a) Necessary condition: Let $A_j = 2\beta(1-2p_j)\text{cov}(x) - I$. As the trace is the sum of the eigenvalues, the condition that all eigenvalues are negative implies $2\beta(1-2p_j)\text{tr}(\text{cov}(x)) - n < 0$. Sufficiency for $n = 2$: The Ruth-Hurwitz criterium indicates that the trace of A_j is negative, and the determinant is positive. As the sufficient condition is more strict than the necessary condtion, it implies that the trace is negative. Let $c_{i,j}$ denote the entries of $\text{cov}(x)$. Then,

$$\det(A_j) \quad = \quad 4\beta^2(1-\rho_j)^2(c_{11}c_{22} - c_{12}c_{21}) - 2\beta(1-2p_j)(c_{11}+c_{22}) + 1.$$

The matrix $\text{cov}(x)$ is positive (semi-)definite, as

$$\forall u \in \mathbb{R}^n : \quad u^T \text{cov}(x) u = \frac{1}{m}\sum_{i=1}^m u^T(x_i - \overline{x})(x_i - \overline{x})^T u = \frac{1}{m}\sum_{i=1}^m (\,(x_i - \overline{x})^T u\,)^2 \geq 0.$$

Hence, the eigenvalues (and therefore also the determinant) of $\text{cov}(x)$ is non-negative.
As the sufficient condition states $2\beta(1-2p_j)(c_{11}+c_{22}) < 1$, the determinant is positive, and we are done. □

This last result allows to understand the effects of the three extensions of the Hotelling model we consider in the stochastic valence model. Let us focus on the case $n = 2$.

First we assume that all valences are zero, $\lambda_j = 0$. Our sufficient condition for $z_1 = \ldots = z_k = \overline{x}$ to be locally stable Nash equilibrium (that is, the condition that the mean voting theorem holds) reads

$$2\beta(1-2/n)\text{tr}(\text{cov}(x)) < 1.$$

That is, in case of $n = 2$, the Mean Voting Theorem (as an obvious variant of the Median Voting Theorem for symmetric voter distributions) holds true. If all parties choose the location of the average voter, we always have a Nash equilibrium. If $n > 2$, $z_1 = \ldots = z_k = \overline{x}$ is not necessarily a Nash equilibrium. For example the parameter β may destroy this property if it becomes too large.
β balances the weight of the deterministic part of the scoring (or utility) function $u^*_{i,j}(z_j) = -\|x_i - z_j\|$ on the one hand, and the noise introduced by $e_{i,j}$ on the other hand. If β is small, the noise is important and the condition is true, while the condition will break down (and also the necessary condition becomes wrong) if β is large.
Hence, the mean voting theorem is not true in 23 (or higher) dimension without noise. Noise is able to stabilize the mean voter as a local Nash equilibrium for the parties. Higher dimensions, in turn, destabilize this equilibrium.

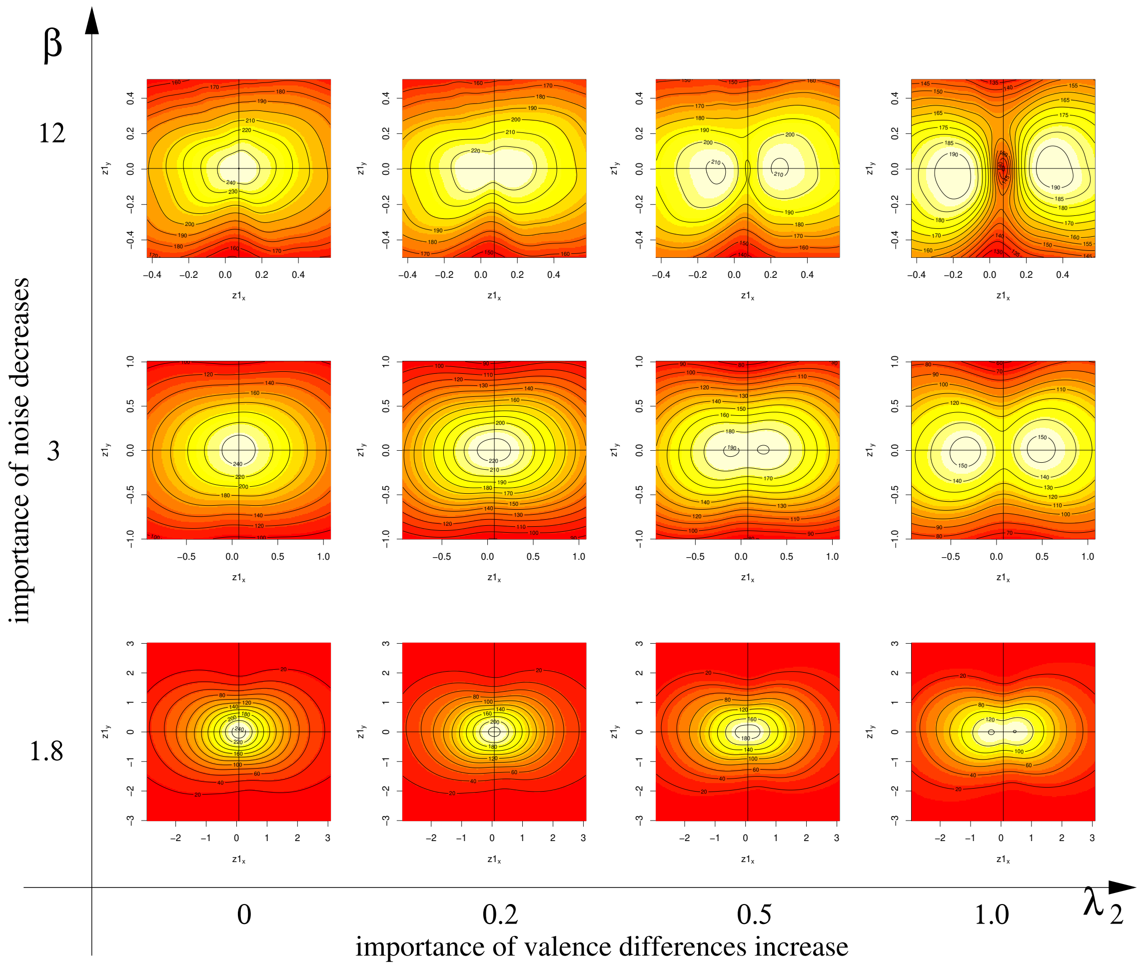


Figure 2.22: $V_1(z_1, z_2)$ for $z_j \in \mathbb{R}^2$, $z_2 = 0$, and x/y component of z_1 as given in the coordinate system. x_i are 500 realizations; first and second component of x_i are independent, first component is just $\mathcal{N}(0,1)$ distributed, second component distributed according to $\mathcal{N}(0, 1/\sqrt{2})$. $\lambda_1 = 0$, λ_2 and β as stated in the figure. The horizontal/vertical line indicates the mean value of x_i (x- and y-component). Since $z_2 = \overline{x}$, then $z_1 = \overline{x}$ is a Nash equilibrium if the corresponding figure shows a clear maximum at $z_1 = \overline{x}$. Else, it is better for z_1 to go away from $z_1 = \overline{z}$, s.t. $z_1 = z_2 = \overline{x}$ is no Nash equilibrium any more.

Last we discuss the effect of the valences. In case of unbalanced valences, we might have an index j_0 s.t. p_{j_0} become small (if λ_{j_0} becomes much larger than all other valences). Then, the l.h.s. of the inequality tends to $2\beta\mathrm{tr}(\mathrm{cov}(x))$, which is harder to archive. That is, also differences in the valences are able to de-stabilize $\overline{x}$ as a Nash equilibrium, even in case of $n = 2$. The noise, however, can counteract. It depends on the balance between variation of valences and importance of noise, if we find the trivial Nash equilibrium predicted by the Mean Voting Theorem, or if we find non-trivial, more realistic Nash equilibria.

2.10 Strategic Voting

Voters do not always vote according to their preferences ("honest voting"), instead, they may try to optimize the impact of their vote ("strategic voting"). An example in Germany, where a 5% threshold is implemented in the election for the German parliament, is the vote share of the "ecologic-democratic party" (Oekologisch-Demokratische Partei). To large extends, the ecologic-democratic party and the Green party (Bündnis 90/die Grünen) have similar goals, where the ecologic-democratic party is slightly more conservative, and particularly, was up to now always below the 5% threshold. Therefore it is to expect that citizens who would prefer the ecologic-democratic party will vote for the Green party, as the latter is member in the parliament and in this is able to influence political decisions in a more powerful way than the ecologic-democratic party. In the present section, we aim to develop a theory for such mechanisms.

2.10.1 Basic Game Theory?

2.10.2 Poisson Games

(did introduce) War das wirklich Myerson? Lit Research!!

Uncertainty in the number of co-players may influence the strategy a given player does choose. Classical game theory assumes a fixed (and known) population of players. Myerson [36] did introduce games with population uncertainty. In view of our application, elections, we might have an idea about the number of eligible voters, but we cannot know the number of active voters. However, only those influence the election result. And indeed, elections are the main application field of games with population uncertainty [36, **?**].

Let us introduce the central ingredients of Myerson's approach. We have an unknown (random) number of players. This number is known *a posteriori*, but there is only a limited knowledge *a priorie*. A given player is characterized by his/her type (e.g., the social status), which is one from a finite set of possible types. As we do not know the number of players, we also have only limited knowledge about the number of players of a given type. Furthermore, he/she selects a strategy out of a finite number of strategies (e.g., which party to vote for). Players are assumed to act purely rational, and aim to maximize an utility function. For a focal player, this function depends on his/her own type, on the strategy he/she did choose (as this strategy influences the outcome), and the strategy all other players do select (again, as also those strategies do influence the outcome).

According to game theory, voters act rationally. Therefore, a voter only cares if he/she is able to influence the voting result, that is, if the decision depends on his/her own choice. If he/she is not able to change the result, it is even questionable if he/she takes the effort to vote at all. The so-called "paradox of not voting" [10] is widely discussed on literature, and researchers who are in favor of the purely-rational-individuals-paradigm aim to prove that voting indeed pays [14], see also Section 2.10.3.

Model 2.73 Games with population uncertainty
Type-structured population size: *We consider a finite population of players. The population size is a random number N. That is, N is an $\mathbb{N}$-valued random number, not known to the participants of the game. Each player has one type out of a finite set of possible types; let T denote this finite set. All individuals that are of identical type are assumed to be indistinguishable. We furthermore assume that the a priorie knowledge about the type-structured*

population of players, given by the random vector

$$(N_t)_{t\in T} \in \mathbb{N}_0^{|T|}$$

(where N_t is the number of players with type t that actually take part) is characterized by a given random measure Q on $\mathbb{N}_0^{|T|}$.
Strategies: *Also the set of possible pure strategies S is assumed to be finite. Let $\Delta(S)$ denote the set of all possible probability measures on S. With this notion, $\sigma \in \Delta(S)$ denotes a mixed strategies. As the individuals of the same type are indistinguishable, the strategy a person opts for only depends on the persons' type,*

$$\sigma : T \to \Delta(S).$$

Utility function: *In order to define and to evaluate the outcome of the utility function, the information unknown to the players beforehand is required: In a realization, we assume that we know how many individuals did choose which strategy. That is, we have a (realization of a random) vector, called the action profile,*

$$Z = (N_s)_{s\in S},$$

where N_s denote the (random) number of individuals that opt for strategy $s \in S$. Therefore, $\sum_{s\in S} N_s = N$. Then, the utility function for our target individual (with type t and - in the given realization - the pure strategy s that might be selected as a realization of a mixed strategy) reads

$$U : \mathbb{N}_0^{|S|} \times S \times T \to \mathbb{R}, \qquad (Z, s, t) \mapsto U_t(Z, s).$$

We assume that U is globally bounded.
We call the tuple (T, Q, S, U) a game with population uncertainty.

Let us exemplify this model structure with a tangible (toy) example: A teacher offers her class to play a game. Those who don't feel like it can take a book from the school library and read. In the game, the players either assume the role of hunters, or the role of prey. Each prey child gets a ribbon. The predator-children do hunt the prey-children in order to chase the ribbon, the prey-children try to hide and to escape. Who collects the most ribbons is winner among the hunters, who hides the longest time is winner among the prey.

Better example, where the utility function is clearly defined???

In this setting, the number N of children that take part in the game is a random number. Depending on the way we want to model the scene, there is only one type of children (all children in the class are equivalent w.r.t. the game). Alternatively, we could distinguish between girls and boys, say. In that case, we have two types of players,

$$T = \{\text{girls}, \text{boys}\}.$$

Furthermore, we have two strategies: hunter and prey,

$$S = \{\text{hunter}, \text{prey}\}.$$

Any participant may chose a mixed strategy, but needs to define his/her choice at the beginning of the game in selecting a realization of the random strategy $\sigma(t)$ he/she did decide for. Note that this strategy only depends on the type. So, all girls are assumed to select the same mixed strategy and all boys select the same mixed strategy; the boys' strategy may, of course, be different from the girls' strategy. In general, the utility function depends on the type of a focal individual. If, e.g., boys run faster than girls, girls may have an disadvantage as hunters if the prey mainly consist of boys. There are for sure many other aspects to consider that give an advantage to girls. However, considerations as those above imply that the utility function does not only depend on the own strategy of a player, and the strategy of all other players, but also of the focal players' type.

Poisson game

We will focus on Poisson games, which we introduce next. In that it is useful to recall the aggregation property (additivity) of Poisson variables:

$$\mathrm{Pois}(\lambda_1) + \mathrm{Pois}(\lambda_2) = \mathrm{Pois}(\lambda_1 + \lambda_2).$$

The sum of two Poissonion random variables again is a Poissonian random variables, where the expectations just sum up.

Definition 2.74 *Let $\nu \in \mathbb{R}$, $\nu > 0$, and assign to each $t \in T$ a real number $r_t > 0$, where $(r_t)_{t\in T}$ sums up to 1,*

$$\sum_{t\in T} r_t = 1$$

A Poisson game is a game with population uncertainty, where the number N_t of payers of a given type $t \in T$ follows a Poisson distribution $Pois(\nu\, r_t)$, independently of the number of players who are of a different type.

With other words, the total number of players follows a Poisson random variable with expectation ν. Furthermore, a randomly selected player has with probability r_t type $t \in T$. This observation allows to state the probability measure Q and the distribution of the total number of participating players:

Corollary 2.75 *Consider a Poisson game (with the notation introduced above). The probability that the population structure $Y = (N_t)_{t\in T}$ equals a given vector $y = (n_t)_{t\in T} \in \mathbb{N}_0^{|T|}$ reads*

$$\begin{aligned} Q(y) &:= P((N_t)_{t\in T} = (n_t)_{t\in T}) \\ &= \prod_{t\in T} \frac{1}{n_t!}(\nu\, r_t)^{n_t} e^{-r_t\,\nu} = \left(\prod_{t\in T} \frac{1}{n_t!}(\nu\, r_t)^{n_t}\right) e^{-\nu}. \end{aligned} \tag{2.81}$$

The probability that the total population size $N = \sum_{t\in T} N_t$ equals $n \in \mathbb{N}_0$ follows a Poissonian distribution,

$$P(N = n) = \frac{1}{n!}\nu^n e^{-\nu}. \tag{2.82}$$

Next we do not consider the number of individual of a given type, but the (random number) of individuals that decide for a given strategy $s \in S$. For Poisson games, we again find that these random numbers are independent and follow a Poissonian distribution.

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Theorem 2.76 *Let (T, Q, S, U) be a Poisson game. Let X_s denote the number of individuals that select in a realization strategy $s \in S$. Then, X_s are independent Poisson variables with mean μ_s given by*

$$\mu_s = \nu \sum_{t\in T} r_t\, \sigma_t(s). \tag{2.83}$$

Proof: Let X be the random vector which characterizes the number of players structured by strategy, that is, $(X)_s = X_s$. We aim to understand the probability $P(X = x)$ for $x \in \mathbb{N}_0^{|S|}$. In general, the players that choose strategy s consist of different types. It is necessary to disentangle types and strategy: For a given strategy s, we write $x_s = \sum_{t\in T} w_{t,s}$. That is, the players that actually choose a certain strategy are structured according to their type; the number of type-t-players hat select strategy s is $w_{t,s}$. The set $W(x)$ contains all possible combinations that yield $x \in \mathbb{N}_0^{|S|}$,

$$W(x) = \{w = (w_{t,s})_{t\in T, s\in S} \in \mathbb{N}_0^{|T|\times|S|} \mid \sum_{t\in T} w_{t,s} = x_s\}.$$

For $w \in W$, we define two different operators, that yield on the one hand all individuals that play a certain strategy and on the other all individuals of a certain type,

$$\Pi_S : W(x) \to \mathbb{N}_0^{|S|},\;\; w \mapsto (\sum_{t\in T} w_{t,s})_{s\in S}, \qquad \Pi_T : W(x) \to \mathbb{N}_0^{|T|},\;\; w \mapsto (\sum_{s\in S} w_{t,s})_{t\in T}.$$

Then, of course, $\Pi_S(w) = x$ for all $w \in W(x)$. However, Π_T contains the useful information about the number of types. A given type-t-player will chose a given strategy s with probability $\sigma_{t_0}(s)$. As the players select independently the strategy, the type-t_0-players (with number $(\Pi_T w))_{t_0}$) are distributed to the strategies in S according to a multinomial distribution,

$$(\Pi_T(w))_{t_0}! \prod_{s\in S}\left(\frac{\sigma_{t_0}(s)^{w_{t_0,s}}}{w_{t_0,s}!}\right).$$

Since the distribution of type-t-individuals to the possible strategies in S follows a multinational distribution, we have

$$P(X = x) \;=\; \sum_{w\in W(x)} Q(\Pi_T(w)) \prod_{t\in T}\left((\Pi_T(w))_t! \prod_{s\in S}\left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right).$$

As we have a Poisson game, the probability function Q is determined by eqn. (2.81),

$$\begin{aligned}
P(X = x) \;&=\; \sum_{w\in W(x)}\left(\prod_{t\in T}\frac{1}{(\Pi_T(w))_t!}(n\, r_t)^{(\Pi_T(w))_t}\right)e^{-\nu}\prod_{t\in T}\left((\Pi_T(w))_t! \prod_{s\in S}\left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right)\\
&=\; \sum_{w\in W(x)}\prod_{t\in T}\left((\nu\, r_t)^{(\Pi_T(w))_t}\prod_{s\in S}\left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right)e^{-\nu}\\
&=\; \sum_{w\in W(x)}\prod_{t\in T}\prod_{s\in S}\left(\frac{(\nu\, r_t\,\sigma_t(s))^{w_{t,s}}}{w_{t,s}!}\right)e^{-\nu} = \sum_{w\in W(x)}\prod_{t\in T}\prod_{s\in S}\left(\frac{(\nu\, r_t\,\sigma_t(s))^{w_{t,s}}}{w_{t,s}!}e^{-\nu\, r_t\,\sigma_t(s)}\right)
\end{aligned}$$

where we used that $\sum_s \sigma_t(s) = 1$, and $\sum_t r_t = 1$. We aim to write this expression as a product over terms depending on s. Therefore, we note that the different s-components of $w_{t,s}$ in $W(x)$ are independent; we have for a given strategy $s_0 \in S$ that

$$\{(w_{s_0,t})_{t\in T} \mid w \in W(x)\} = \{v \in N_0^{|T|} \mid \sum_{t\in T} v_t = x_s\}.$$

In that sense, we define

$$V(x_s) = \{w \in N_0^{|T|} \mid \sum_{t\in T} w_t = x_s\}$$

and have

$$W(x) = \bigoplus_{s\in S} V(x_s).$$

In a slightly abuse of notation, we write for $w_s \in V(x_s)$, that $(w_s)_t = w_{t,s}$. Then,

$$\begin{aligned}
P(X = x) &= \sum_{(w_s)_{s\in S}\in\bigoplus_{s\in S} V(x_s)} \prod_{t\in T}\prod_{s\in S}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right)\\
&= \sum_{(w_s)_{s\in S}\in\bigoplus_{s\in S} V(x_s)} \prod_{s\in S}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right)\\
&= \prod_{s\in S}\sum_{w_s\in V(x_s)}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right).
\end{aligned}$$

We observe that the different components of X are independent. The probability for $X = x$ is the product of the marginal probabilities

$$P(X_s = x_s) = \sum_{w_s\in V(x_s)}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right).$$

We find that X_s is the sum of $|T|$ independent Poisson random variables, which have expectation $\nu\, r_t\, \sigma_t(s)$. Therefore, also X_s is Poisson distributed, with expectation $\nu \sum_{t\in T} r_t\, \sigma_t(s)$.

□

The property that the components of X_s are independent simplifies the analysis. Therefore we introduce a name for this property.

Definition 2.77 *A game with population uncertainty (T, Q, S, U) has the independent-action property, if for all $\sigma : T \to \Delta(S)$ the random number of players that select a given strategy $s \in S$ is independent of the number of players that choose a different strategy $s' \in S$ (for all $s' \neq s$).*

Note that games with a fixed number of layers cannot have the independent-action property, as the number of players structured by the strategy sum up to a given, fixed total number of players. The next theorem shows that the independent-action property forces the game to be a Poisson-game.

Theorem 2.78 *Let (T, Q, S, U) a game with population uncertainty that has the independent-action property. Assume $|S| > 1$. Then, (T, Q, S, U) is a Poisson-game, and X_s is a Poisson random variable with expectation μ_s, where*

$$\mu_s \quad = \quad \sum_{t \in T} \sigma_t(s) \sum_{y \in \mathbb{N}_0^{|T|}} Q(y)\, y_t. \qquad (2.84)$$

The total number of players also follows a Poisson distribution.

Proof: A voter of a given type t_0 decides independently of all other individuals with probability $\sigma_{t_0}(s)$ for strategy $s \in S$, where $P(X_{s_a} = 0) > 0$. That is a key feature of the model that we intend to use.
We have a strategy function $\sigma : T \to \Delta(S)$ given. We focus on two different strategies $s_a, s_b \in S$, $s_a \neq s_b$, and define $\sigma^* : T \to \Delta(S)$ by

$$\forall t \in T : \sigma_t^*(s_a) = \sigma_t(s_a) + \frac{1}{2}\sigma_t(s_b), \quad \sigma_t^*(s_b) = \frac{1}{2}\sigma_t(s_b), \quad \forall s \in S \setminus \{s_a, s_b\} :, \sigma_t^*(s) = \sigma_t(s).$$

We focus on X_{s_a} and X_{s_b}. We are interested in the probability that $X_{s_a} = \ell$ and $X_{s_b} = k$; to be more precise, in the relation of these probabilities for σ and σ^*. We denote $P_\sigma(X_{s_a} = \ell, X_{s_b} = k)$ resp. $P_{\sigma^*}(X_{s_a} = \ell, X_{s_b} = k)$ to indicate which strategy function we have in mind. If we only consider the individuals of one type t_0 we find

$$\begin{aligned}
&P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k \,|\, \text{ only type } t_0 \text{ individuals}) = \left(\frac{\sigma_{t_0}^*(s_b)}{\sigma_{t_0}^*(s_a) + \sigma_{t_0}^*(s_b)}\right)^k \\
= \quad &\left(\frac{\sigma_{t_0}(s_b)}{2(\sigma_{t_0}(s_a) + \sigma_{t_0}(s_b))}\right)^k = \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 0, X_{s_b} = k \,|\, \text{ only type } t_0 \text{ individuals}).
\end{aligned}$$

As this relation is linear and identical for all types, we have

$$P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k) \quad = \quad \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 0, X_{s_b} = k).$$

Similarly, we have (first for a given type, and then for all type distributions)

$$\begin{aligned}
&P_{\sigma^*}(X_{s_a} = 1, X_{s_b} = k) = \binom{k+1}{1} \frac{\sigma_t^*(s_a)^1 \; \sigma_t^*(s_b)^k}{(\sigma_t^*(s_a) + \sigma_t^*(s_b))^{k+1}} \\
= \quad &\binom{k+1}{1} \frac{(\sigma_t(s_a) + \frac{1}{2}\sigma_t(s_b))^1 \; (\frac{1}{2}\sigma_t(s_b))^k}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} \\
= \quad &\left(\frac{1}{2}\right)^k \binom{k+1}{1} \frac{\sigma_t(s_a)\, \sigma_t(s_b)^k}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} + \left(\frac{1}{2}\right)^{k+1} \binom{k+1}{1} \frac{(\sigma_t(s_a)^0 \sigma_t(s_b))^{k+1}}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} \\
= \quad &\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 1, X_{s_b} = k) + \left(\frac{1}{2}\right)^{k+1} \binom{k+1}{1} P_\sigma(X_{s_a} = 0, X_{s_b} = k+1) \\
= \quad &\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 1, X_{s_b} = k) + \left(\frac{1}{2}\right)^{k+1} (k+1)\, P_\sigma(X_{s_a} = 0, X_{s_b} = k+1).
\end{aligned}$$

With these preparations, consider

$$\frac{P_{\sigma^*}(X_{s_a}=1, X_{s_b}=k)}{P_{\sigma^*}(X_{s_a}=0, X_{s_b}=k)}.$$

As the components of X are independent, we have that

$$\frac{P_{\sigma^*}(X_{s_a}=1, X_{s_b}=k)}{P_{\sigma^*}(X_{s_a}=0, X_{s_b}=k)} = \frac{P_{\sigma^*}(X_{s_a}=1)}{P_{\sigma^*}(X_{s_a}=0)}$$

is independent of k. On the other hand, we can use the computations from above, and find

$$\begin{aligned}
&\frac{P_{\sigma^*}(X_{s_a}=1, X_{s_b}=k)}{P_{\sigma^*}(X_{s_a}=0, X_{s_b}=k)} \\
= \ &\frac{\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a}=1, X_{s_b}=k) + \left(\frac{1}{2}\right)^{k+1} (k+1)\, P_\sigma(X_{s_a}=0, X_{s_b}=k+1)}{\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a}=0, X_{s_b}=k)} \\
= \ &\frac{P_\sigma(X_{s_a}=1)}{P_\sigma(X_{s_a}=0)} + \frac{(k+1)\, P_\sigma(X_{s_a}=0, X_{s_b}=k+1)}{2P_\sigma(X_{s_a}=0, X_{s_b}=k)} \\
= \ &\frac{P_\sigma(X_{s_a}=1)}{P_\sigma(X_{s_a}=0)} + \frac{(k+1)\, P_\sigma(X_{s_b}=k+1)}{2P_\sigma(X_{s_b}=k)}.
\end{aligned}$$

If we compare the two results, we find that $\lambda(b) = (k+1)\, P_\sigma(X_{s_b}=k+1)/P_\sigma(X_{s_b}=k)$ is independent of k. Hence,

$$P_\sigma(X_{s_b}=k+1) = \frac{\lambda(b)}{(k+1)}\, P_\sigma(X_{s_b}=k) = \frac{\lambda(b)^2}{(k+1)k}\, P_\sigma(X_{s_b}=k) = \ldots = \frac{\lambda(b)^{k+1}}{(k+1)!}\, P_\sigma(X_{s_b}=0).$$

Since $\sum_{k=0}^\infty P_\sigma(X_{s_b}=k)=1$, we have $X_{s_b} \sim \text{Pois}(\lambda(b))$.
The expectation of X_{s_b} is given by

$$\mu_{s_b} \quad = \quad \sum_{t\in T} \sigma_t(s_b) \sum_{y\in\mathbb{N}_0^{|T|}} Q(y)\, y_t. \tag{2.85}$$

Furthermore, the total population size is the sum of all components of X_s, that is, a sum of independent Poisson variables, and hence again a Poisson variable.

□

Environment of a player and probability equivalence property

In order to decide rationally for a strategy, the knowledge of the environment (who many players of which type) is decisive. A player does know the overall setting of the game (T, Q, S, U). However, the player does know one mode information: He/she is part of the game. How can he/she utilize this information?
The following example, that clarifies up to a certain degree the implication of that fact is given in [36]: Let us assume that a game may have either 300 or 600 players. According to the probability distribution Q, each outcome has the same probability. Let us assume that there are

600 potential players; that is, either all or half of the potential players actually become players. Each player is assumed to behave alike. For this example we assume that the game is repeated again and again. For an external game theorist, the average population size is thus

$$\frac{1}{2}\,600+\frac{1}{2}\,300=450.$$

Now we consider a focal person who is a potential player. This person is interested in his/her environment: If he/she takes part in the game, who many individuals will also take part? Importantly, we are not interested in all realizations of the game, but only in realizations when the focal individual is part of the game. Only is this is the case, we consider the population size of a game.

In realizations where only 300 players take part, a given person has probability 1/2 to be in the game. If we have 600 players, the probability to be "in" is 1.

In half of all realizations (probability 1/2) we have 600 participants, and our focal individual is in for sure. In the other half of the realizations, we only have 300 individuals, and for those realizations, we have only probability 1/2 that our focal individual is "in". That is, the probability to be part of a 300-person-game is only 1/4. Hence, the expected number of players in games our focal individual takes part is

$$\frac{\frac{1}{2}\,600+\frac{1}{4}\,300}{\frac{1}{4}+\frac{1}{4}}=500.$$

The average number of players a focal individual finds in his/her environment is hence 499, much larger than the average number of players an independent person outside of the setting recognizes (450).

The environment of an individual is the number of fellow players (to be more precise: the number of fellow players, structured by their type). We have the expected number of players, determined by the expectation according to the probability measure Q. This number is not the environment of a player. On the one hand, this number is decreased (by one), as the player himself/herself is part of the game, but does not count to the environment. On the other hand, this number is increased, as the fact that a focal individual takes part in the game indicates that we rather have a larger than a smaller population size. In the following, we discuss how these two effects balance.

We first think about the environment. To be more precise, in the first step, we think about the total number of players that are in the environment of a randomly chosen person. The probability measure Q defines the probability $P(N=n)$, the probability that there are $n\in\mathbb{N}_0$ players in the game. We already know that one player (the focal individual) is in the game. $N=0$ hence no valid realization. Obviously, the knowledge that the focal player is present changes the probability distribution of N. How can we define the environment of our focal player? That is, how likely is it that our focal individual has n co-players? Let us denote this probability $P(N-1=n|1)$ (with this notation we emphasize that we condition on the fact that there is the focal player, and that we ant ti know the number of additional players).

The idea to determine this probability is the following: Create a lot of (100, say) realizations of the game (see Fig. 2.23). If we now select randomly one realization, we recover the probability distribution $P(N=n)$ for the population size. However, we are interested in a focal individual. Therefore, we randomly select one *individual* from all individuals within any of the realizations. The probability to select a realization with a high population size is thus larger than a realization

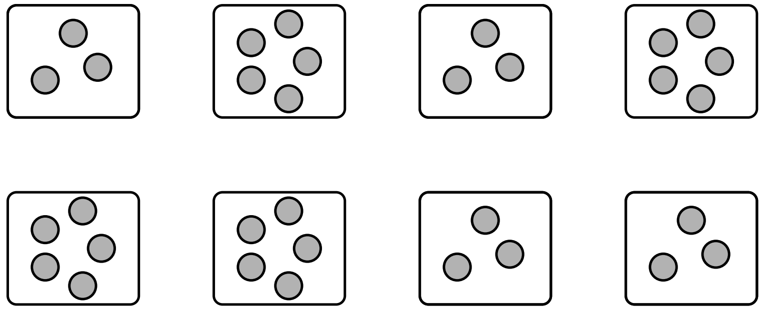

Figure 2.23: 8 realization of a game with population uncertainty. The population size is either 3 or 5, with popularity 1/2 for each. If we randomly select a *realization* from these 8 realizations, we obtain probability 1/2 for a population size 3, and probability 1/2 for population size 5. If we randomly select an *individual* out of the 32 individuals, we obtain a probability of $20/32 = 5/8$ for population size 5, and only $12/32 = 3/8$ for population size 3.

with a small size. To be more precise, the probability is proportional to the population size and the probability to create a game with this size. This consideration yields the following definition.

Definition 2.79 *The distribution of the number of co-players of a randomly chosen focal player is given by*

$$P(N-1=n|1) = \frac{(n+1)P(N=n+1)}{\sum_{k=1}^{\infty} k\, P(N=k)}. \tag{2.86}$$

We say that a probability distribution for the number of players satisfies the environmental equivalence property, if the number of co-players follows the same probability distribution as the number of players,

$$P(N=n) = P(N-1=n|1).$$

We emphasize that this is a definition, based on the construction described. As we will see below, this idea is connected with an augmented model that includes the way players are recruited. For now, we stay with the definition and investigate the consequences. We are particularly interested in the distributions that have the probability equivalence property for the population size.
The equivalent property does mean, that the external game theorist and a (potential) player of the game, both "see" the same number of (co)players. The next proposition shows, that the equivalence property forces the game to be a Poisson game.

Proposition 2.80 *The population size of a game, where the distribution of the population size has the equivalence property, follows a Poisson distribution.*

Proof: We know that

$$P(N-1=n|1) = C\,(n+1)\,P(N=n+1)$$

where C is a normalization constant. Furthermore, the equivalence property indicates $P(N = n) = P(N-1=n|1)$. Hence,

$$P(N=n) = C\,(n+1)\,P(N=n+1)$$

respectively, by finite induction,

$$P(N=n) = \frac{1}{n!}\,C^{-n} P(N=0).$$

As the probabilities have to sum up to one, N is a Poissonian random variable with expectation $1/C$.

□

For the Poissonian population size, we can augment the Poisson game model by the recruitment process. We have a large population (size M) of potential players. Each player has a probability p to become recruited, s.t. the number of players is distributed according to the binomial distribution $\mathrm{Bin}(M,p)$. If M tends to infinity and p to zero, s.t. $M\,p \to r > 0$, we obtain the Poisson distribution. If we now consider a focal individual how is a player, then there are only $M-1$ potential players left. The number of co-players is distributed according to $\mathrm{Bin}(M-1,p)$. For M finite, we do not find the equivalence property. However, if M tends to infinity, while $(M-1)p \to r$ also $\mathrm{Bin}(M-1,p)$ tends to the Poisson distribution with expectation r. In this case, we establish the equivalence distribution.

We can refine this result: The environment of an individual is not only defined by the total population size, but by the population size, structured according to the types. Thereto, we return to the population structure $Y = (N_t)_{tT}$. We introduce δ_{t_0} to denote the Kronecker-vector

$$(\delta_{t_0})_t = 1 \text{ if } t = t_0 \quad \text{ and } (\delta_{t_0})_t = 0 \text{ else.}$$

We parallel the notation from above, and write $P(Y - \delta_{t_0} = y|\delta_{t_0})$ to denote the probability that a t_0-individual find the environment y.

Definition 2.81 *The type-structured environment of a t_0-individual has the distribution*

$$(Y - \delta_{t_0} = y|\delta_{t_0}) = C\,Q(y+\delta_{t_0})\,(y_{t_0}+1) \tag{2.87}$$

where C is determined by the fact that the probabilities $P(Y - \delta_{t_0} = y|\delta_{t_0})$ sum up to one. We say that a random game satisfies the environmental equivalence property, if the distribution of the structured population size and that for the structured environment of a t-type (for all types $t \in T$) coincide,

$$P(Y - \delta_{t_0} = y|\delta_{t_0}) = Q(Y=y).$$

Theorem 2.82 *A game with population uncertainty that has the environmental equivalence property is a Poisson random game.*

Proof: We have for $t_0 \in T$

$$(Y - \delta_{t_0} = y|\delta_{t_0}) = C\,Q(y+\delta_{t_0})\,(y_{t_0}+1) = Q(y).$$

Hence, by finite induction,

$$Q(y) = C \prod_{t\in T} \frac{(c_t)^{y_t}}{y_t!}$$

where c_t and C are suited constants. The entries of the random vector Y follow independent Poisson distributions, and hence the game is a Poisson game.

□

Equilibrium strategy

We briefly touch the question if there are equilibrium strategies. As we only use Poisson games later on, we only investigate Poisson games. We will not rigorously prove the existence, but we connect the equilibrium strategy with the notion of a Nash equilibrium [8].

Given the strategy σ, the distribution of the number of strategy-structured number of players is given by some random vector $Z = (N_s)_{s\in S}$ that has the independent-action property. The entries follow a Poisson distribution, with parameters μ_s stated in Theorem 2.78. The expected value utility function of a type t_0 player who selects the pure strategy $s_0 \in S$ is given by

$$\overline{U}_{t_0}(s_0) = \sum_{z\in\mathbb{N}_0^{|S|}} P(Z = z)\, U_t(Z, s_0).$$

If this player (note, only this focal player, not all players of type t_0) plays the random strategy $\theta \in \Delta(S)$, then the expected outcome reads

$$\tilde{U}_{t_0}(\theta) = \sum_{s\in S} \theta(s)\overline{U}_{t_0}(s).$$

The set of the best pure response of the single t_0-player is

$$\mathrm{BPR}_{t_0}R(\sigma) = \{s_0 \in S \,|\, \forall s \in S : \overline{U}_{t_0}(s) \leq \overline{U}_{t_0}(s_0)\,\},$$

and the optimal mixed strategies are all randomized strategies on the set $\mathrm{BPR}_{t_0}(\sigma)$,

$$\mathrm{BR}_{t_0} = \Delta(\mathrm{BPR}_{t_0}(\sigma)).$$

With this notation, we define the Nash equilibrium.

Definition 2.83 *σ^* is a Nash equilibrium, if for all types $t \in T$,*

$$\sigma_t^* \in BR_t(\sigma^*).$$

The existence of Nash equilibria can be shown, as usual, based on Kakutanis Fixed point theorem. Instead of a proof, we consider a toy example.

The Really Bad Gang versus Gentleman thugs.
We have two groups of organized criminal gangs, the Really Bad Guys (RBG) and the Gentleman Thugs (GT). The RBGs have exactly 100 members, and all work together. We only focus on the GTs, such that our type set is

$$T = \{\mathrm{GT}\}.$$

The distribution Q is defined via

$$N_{GT} \sim \mathrm{Poiss}(r_{GT})$$

where r_{GT} denote the expected group sizes of the two gangs.
Each member of the GTs either go alone on a raid, or cooperate together and with the RBGs. In case they work alone, they catch 1 unit. Alternatively, the two groups can choose to cooperate. Then, their loot per member is much higher, 100 units. However, the gang which has more members involved will take the complete loot. In case of equal member size, each individual

receives 50 units.
We have two strategies: "alone" (a), and "cooperate", (c)

$$S = \{a, c\}.$$

The utility function is given by ($t =$GT)

$$U_t(y, a) = 1, \quad U_{RGB}(y, c) = \begin{cases} 100 & \text{if} \quad y_c > 100 \\ 50 & \text{if} \quad y_c = 100 \\ 0 & \text{if} \quad y_c < 100. \end{cases}.$$

Let y denote the environment of an GT. If the number of (other) cooperating GT-individuals equals 99 or are larger or equal 100, the focal GT-person wants to cooperate, as in that case, either both gangs have the equal size (payoff 50 for our focal individual), or the GTs has even more cooperators (payoff 100). Only if the cooperating GT-individuals are less than 99, our focal GT want to work alone to earn 1 unit.
However, our focal individual has to decide before he/she knowns the exact population size. All he/she does know is Q, that is, the parameter r_{GT}. Let us assume that he population plays strategy σ. In this simple case (only one type, and only two pure strategies), the strategy σ is defined by the probability p to cooperate. In this case, the number of cooperating GTs is distributed according to the Poissonian distribution, with expectation $p\, r_{GT}$. Let N_c denote the random number of cooperating individuals, that is, a Poissonian Random variable with expectation $p\, r_{GT}$. Then,

$$\overline{U}_t(a) = 1, \qquad \overline{U}_t(c) = 0\, P(N_c < 99) + 50\, P(N_c = 100) + 100\, P(N_c > 100).$$

Note that we used here the environmental equivalence relation. The probability that our focal individual finds a certain number of fellow cooperators is the same as that for an external observer to find this number of cooperators. The optimal pure response for the focal individual is

check!! Here we use the environmental equivalence cond, but the result/Nash equil is independent on this condition

$$\begin{aligned} \text{BR}(\sigma) &= \{a\} \text{ if } \overline{U}_t(c) < 1 = \overline{U}_t(a), \\ \text{BR}(\sigma) &= \{c\} \text{ if } \overline{U}_t(c) > 1 = \overline{U}_t(a), \\ \text{BR}(\sigma) &= \{a, c\} \text{ if } \overline{U}_t(c) = 1 = \overline{U}_t(a). \end{aligned}$$

We have only one case, where an individual is willing to randomize his/her strategy: If $U_t(c) = U_t(a)$. ELse, the decision is deterministic. We have several Nash equilibria.
(1) All work alone.
If $\sigma = a$, then $\overline{U}_t(c) = 0$. The Nash equilibrium is hence $\sigma_t \in \text{BR}_t(\sigma)$.
(2) All cooperate.
This Nash equilibrium only exist, if $\overline{U}_t(c) > 1$ for $\sigma = c$, i.e. if all individuals cooperate ($p = 1$).
(3) Randomized strategy.
Assume that $\overline{U}_t(c) > 1$ for $\sigma = c$ (the randomized strategy will only work under that condition). If all individual cooperate, the utility function is positive. We know that $\overline{U}_t(c) = 0$ if $p = 0$. Thus, there is $\hat{p} \in (0, 1)$, s.t. $\overline{U}_t(c) = 1$ if all individuals play the strategy σ that randomize with probability $\hat{p}$ for cooperation. Then,

$$\sigma_t \in \text{BR}_t(\sigma),$$

and we have a mixed Nash equilibrium.
Note that we did not use the environmental equivalence condition in the three cases, that did

determine the possible Nash equilibria. The environmental equivalence condition allowed us to consider a focal individual, but the considerations yield conditions, that are independent on this point of view.

2.10.3 Strategic Voting – turnout rate

We apply Poisson games to the question how large the turnout rate of an democratic election in case of a two-party system will be. We observe that the participation in elections mostly decreases in the recent years. One central question is hence, why do voters vote? Palfrey and Rosenthal developed an approach, based on rational voting [38]. Myrson [36] did reformulate these ideas in the context of Poisson games.

Model 2.84 *Consider a population of voters, divided into two types, leftist and rightist voters. The total number of voters is N; the probability for a voter to be rightist is p_r, and $p_\ell = 1 - p_r$ for a leftist.*
Two candidates stand for election, a right and a left candidate. A voting is connected to a price: a vote "costs" a voter 0.05 units (if the favorite candidate does not make it). If "his/her" candidate wins, he/she wins 1 unit. In case of a tie, the winner is determined by a fair coin toss, that gives each candidate the probability $1/2$ to wind.
A rational citizen aims to maximize his/her expected gain in that elections.

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Note that this model bears some reasonable aspects: It is well known that the turnout rate depends on the weather – if the weather is nice, less citizen will vote. This is a clear argument for the fact that citizen indeed assign some costs to the voting process: E.g., if I go to vote, I cannot do some spare time activities during that time. On the other hand, we are happy if "our" favorite candidate wins. In that, this model seems to describe reality appropriately. On the other hand, citizens will not rationally quantify and weight their gain and loss, and decide on that basis. However, we can see where we are led to from the idea of a rational voter.

Chapter 3

Economic models

3.1 Measure inequalities: Lorenz curve and Gini coefficient

The economic power clearly is not uniformly distributed in the society. A small fraction of the society holds the majority of the wealth. In recent years, that inequality seems to grow and to become more prevalent in the world. In order to justify a statement like that, tools to measure the inequality in the society are required. The Italian statistician Corrado Gini proposed the most frequently used measure for inequality, based on the Lorenz curve. We briefly discuss the Lorenz curve, and then give the definition of the Gini coefficient.

Assume that we have a population of size N. Each of individual holds the wealth W_i. We do not consider depths, that is, $W_i \geq 0$. We order these values,

$$0 \leq W_1 \leq W_2 \leq \cdots \leq W_N,$$

and normalize the sum of all of them to 1,

$$w_i = \frac{W_i}{\sum_{j=1}^N W_j}.$$

Note that we allow for ties, that is, $w_i = w_{i+1}$ for some indices is possible. For the Lorenz curve, named after Max O. Lorenz, an American economist [31], we take a the poorer fraction $f \in (0,1)$ of individuals within the society, and determine the total wealth that his fraction does hold,

$$L(f) = \sum_{i \leq N f} w_i. \tag{3.1}$$

The Lorenz curve is the graph of $L(f)$. That is, $L(0-) = 0$ (left handed limit), and $L(1) = 1$. Furthermore, we have always $\mathcal{L}(w) \leq w$ (see exercise 3.1).

How is the Lorenz curve related to inequality in the population? Let us consider two possibilities. (a) The most unequal society: one persons owns everything, all other persons own nothing. That is, $w_i = 0$ for $i < N$, and $w_N = 1$. Hence, $L(f) = 0$ for $f < 1$, while $L(1) = 1$. The largest possible concentration of wealth leads to an extreme Lorenz curve, that is zero for $f \in [0,1)$, and becomes 1 for $f = 1$.
After we understand the effect of the maximal wealth concentration, we investigate the perfectly

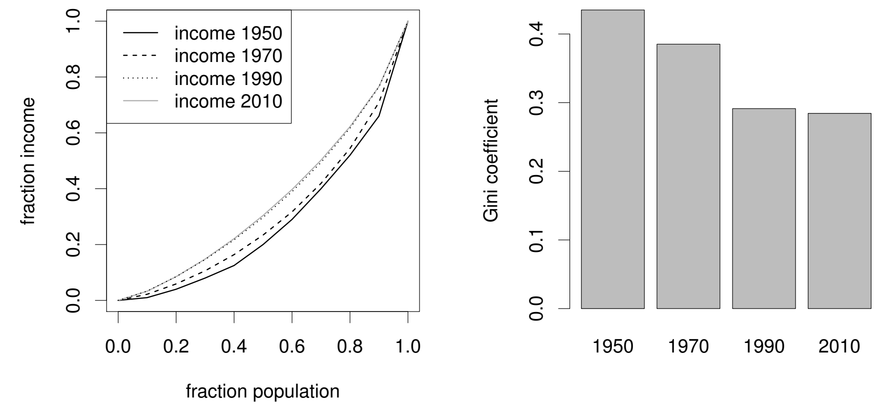

Figure 3.1: Lorenz curve and Gini coefficients for the income in Germany in the years 1950, 1970, 1990, 2010. Data from [48].

equal distribution of wealth in the population: all individuals hold the very same wealth. Hence $w_i = 1/N$, and $L(f) = \lfloor fN \rfloor /N$. The Lorenz curve approximates a straight line connecting $(0,0)$ with $(1,1)$.

The Gini index now measures twice the deviation of the Lorenz curve from that for the uniform wealth distribution,

$$G = G((w_i)_{i=1,\dots,N}) = \frac{2}{N}\sum_{i=1}^{N}(i/N - L(i/N+)). \tag{3.2}$$

We have some elementary properties of G - as they are of importance, we formulate them as a theorem.

Theorem 3.1 *We have* $0 \le G \le (1-1/N)(1-2/N)$*, where*

$$G((w_i)_{i=1,\dots,N}) = 0 \quad \Rightarrow \quad \forall i \in \{1,\dots,N\}: \; w_i = 1/N$$

and

$$G((w_i)_{i=1,\dots,N}) = (1-1/N)(1-2/N) \quad \Rightarrow \quad \forall i \in \{1,\dots,N-1\}: \; w_i = 0, \; \textit{and } w_N = 1.$$

Proof: Exercise 3.2.
We see that in these two extreme situations – G maximal or G minimal – the information about G is already sufficient to determine the wealth distribution as given by $(w_i)_{i=1,\dots,N}$. In-between, if G is neither maximal nor minimal, (w_i) has a certain freedom and is not completely characterized by G.

In Fig. 3.1, the development of the Lorenz curve and the Gini coefficients for the incomes in Germany are depicted. We find that the inequalities after the second world war decrease until

1990; since then, the inequalities are more or less constant. The wealth distribution, however, looks very dissimilar. The Lorenz curve and the Gini coefficient can be applied to many different aspects of inequality: income before or after taxes, wealth, GRUNDBESITZ etc. Furthermore, that data analysis only addresses inequalities, not absolute values. The GIPs of Germany from 1950 and 2010 are complete different, while the Lorenz curves rather look similar. It is necessary to keep that in one's mind in interpreting time series of the Gini coefficient.

As the population size N is large, it is sensible to consider the continuum limit. In that case, the vector $(w_i)_{i=1,\dots,N}$ is replaced by a wealth distribution $\varphi(w)$. The support of this distribution is $\mathbb{R}_+$ (as we do not consider depths) and the probability that a randomly chosen person has wealth w in the interval $[w_0, w_1]$ is given by $\int_{w_0}^{w_1} \varphi(w')\,dw'$. In another interpretation,

$$\int_0^w w'\,\varphi(w')\,dw'$$

is the total amount of wealth in the population that is hold by persons with a wealth less or equal w, while

$$N(w) = \int_0^w \varphi(w')\,dw'$$

is the fraction of persons within the population with a wealth less or equal w. As φ is a distribution, we know that φ is normalized s.t. $\lim_{w\to\infty} N(w) = 1$, corresponding to a normalization of the total population size N to 1. We also want to normalize the total amount of wealth in the population to 1, and hence we define

$$L(w) := \frac{\int_0^w w'\,\varphi(w')\,dw'}{\int_0^\infty w'\,\varphi(w')\,dw'}.$$

The Lorenz curve $\mathcal{L}$ has $N(w)$ as the abscissa, and $L(w)$ as ordinate coordinate. The Gini coefficient is twice the integral of the area between the Lorenz curve and the diagonal. If we assume that $\varphi(x) > 0$ for $x > 0$, and $\varphi(x)$ is smooth, then $N(w)$ is strictly increasing an w, and $N^{-1}(w)$ is well defined. We can write

$$G = 2\int_0^\infty (\,x - L(N^{-1}(x))\,)\,dx.$$

In order to rewrite this formula in a more accessible way, we use the transformation $y = N^{-1}(x)$, s.t.

$$\frac{dy}{dx} = \frac{1}{N'(N^{-1}(x))} = \frac{1}{\varphi(y)} \qquad \Rightarrow dx = \varphi(y)\,dy$$

and the boundaries of the integral are mapped to 0 and ∞, respectively. Therefore,

$$\begin{aligned} G &= 2\int_0^\infty (N(y) - L(y))\,\varphi(y)\,dy = 2\int_0^\infty \frac{1}{2}\frac{d}{dy}N^2(y)\,dy - 2\int_0^\infty L(y)\,\varphi(y)\,dy \\ &= 1 - 2\int_0^\infty L(y)\,dN(y). \end{aligned}$$

These considerations yield the following definitions, where we focus on welath distributions with nice properties only: We assume that the distributions are continuous and strictly positive for $x \geq 0$. Furthermore, as the Lorenz curve is invariant against a linear scaling (choice of units for wealth, see exercise **??**), we assume without restriction that the expected wealth in the populaiton is 1.

Definition 3.2 *Consider the set $\mathfrak{D}$ that consist of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are zero for $x < 0$, continuous, positive on $[0, \infty)$, and satisfy $\int_{-\infty}^{\infty} \varphi(x)\, dx = \int_{-\infty}^{\infty} x\varphi(x)\, dx = 1$. For $\varphi \in \mathfrak{D}$, define*

$$N(w) = \int_0^w \varphi(w')\, dw', \qquad L(w) := \int_0^w w'\, \varphi(w')\, dw'. \tag{3.3}$$

The Lorenz curve for φ is defined as the graph of

$$\mathcal{L} : [0, 1] \to [0, 1], \qquad y \mapsto L(N^{-1}(y)). \tag{3.4}$$

We introduce the Lorenz operator

$$\Lambda : \mathcal{D} \to L^1(0, 1), \quad \varphi \mapsto L(N^{-1})(y).$$

The Gini coefficient is given by

$$G(\varphi) = 1 - 2 \int_0^\infty L(w)\, dN(w) = 2 \int_0^1 [y - \Lambda(\varphi)(y)]\, dy. \tag{3.5}$$

Note that $N(w)$ is strictly increasing for $w \geq 0$ as we assume that φ is strictly positive on $x \geq 0$. Therefore, the function $\mathcal{L}(y)$ is well defined.
Next we lift some of the conclusions of Theorem 3.1, that we did prove in the discrete situation, to the infinite population limit.

Theorem 3.3 *Let $\varphi \in \mathfrak{D}$. Then, $\Lambda(\varphi) \in C^2(0, 1)$, $\Lambda(\varphi)(0) = 0$, $\Lambda(\varphi)(1) = 1$, $\frac{d}{dy}\Lambda(\varphi)(0) = 0$, $\lim_{y \to 1-} \frac{d}{dy}\Lambda(\varphi)(y) = \infty$, and th function $\Lambda(\varphi)$ is strictly concave. Particularly,*

$$0 < \Lambda(\varphi(y) < y \text{ for } y \in [0, 1]$$

and $0 < G(\varphi) < 1$.

Proof: According to the conditions on φ, we have $\varphi \in C^0(\mathbb{R})$, $\varphi(x) = 0$ for $x \leq 0$, and $\varphi(x) > 0$ for $x > 0$.

Lorenz curve is below or equal the diagonal:
As $N'(w) = \varphi(w) > 0$ for $x > 0$, $N^{-1}(x)$ is well defined, and the Lorenz curve is given by the graph of

$$\mathcal{L}(x) := L(N^{-1}(x)).$$

Hence,

$$\frac{d}{dx}\mathcal{L}(x) = \frac{d}{dx}L(N^{-1}(x)) = \frac{d}{dy}L(y)\bigg|_{y=N^{-1}(x)} \frac{1}{N'(N^{-1}(x))} = \frac{N^{-1}(x)\,\varphi(N^{-1}(x))}{\varphi(N^{-1}(x))} = N^{-1}(x)$$

and

$$\frac{d^2}{dx^2}\mathcal{L}(x) = \frac{1}{N'(N^{-1}(x))} = \frac{1}{\varphi(N^{-1}(x))} > 0.$$

That is, $\mathcal{L}(x) \in C^2((0, 1))$ and $\mathcal{L}(x)$ is strictly concave, $\mathcal{L}(0) = 0$ and $\mathcal{L}(1) = 1$. Therefore, $\mathcal{L}(x) < x$ in $(0, 1)$.

Furthermore, $\frac{d}{dx}\mathcal{L}(0) = N^{-1}(0)$. Since $N(0) = 0$, we have $\frac{d}{dx}\mathcal{L}(0) = 0$. Furthermore, $\varphi(x)$ is strictly positive for all $x \geq 0$, s.t. $\int_0^w \varphi(x)\, dx = 1$ is only given for $w = \infty$. Hence,

$$\lim_{y\to 1} \frac{d}{dx}\mathcal{L}(y) = \lim_{y\to 1} N^{-1}(1) = \infty.$$

$G(\varphi) \in (0, 1)$:
Then, the function $\mathcal{L}(x)$ is well defined, we can use $G = 2\int_0^1 (x - \mathcal{L}(x))\, dx$. As $x > \mathcal{L}(x) > 0$ in $x \in (0, 1)$, we have $0 < G < 1$.

□

We did use strong conditions on $\varphi(x)$ in the theorem above. For practical purposes, these assumptions might be sufficient, as the wealth distribution for real-world data can be well approximated by a continuous, positive distribution. However, for theoretical investigations, important cases are missing. For example, an equal distribution of wealth is represented by a Dirac point measure $\delta_{w_0}(w)$, $w_0 > 0$. If we naively use Definition 3.2 for that case, we have $N(w) = L(w) = 0$ for $w < w_0$, and $N(w) = L(w) = 1$ for $w > w_0$ (where $N(w)$, $L(w)$ are undefined in $w = w_0$). That is, we know that the Lorentz curve hits $(0, 0)$ and $(1, 1)$, but we cannot see if and how these two points are connected. We aim to extend our definition. Thereto, we start with the reverse problem: Given a Lorenz curve, can we decide if there is a continuous, positive distribution that creates that curve?

In order to understand how to approach that problem, we consider a Lorenz curve $\mathcal{L}(y)$ that is derived by a nice distribution $\varphi(x)$ (which is smooth, strictly positive in $\mathbb{R}_+$, satisfies $\int x\varphi(x)\, dy = 1$, and everything we want to have). Then, according to the proof of the theorem above, we have

$$\mathcal{L}(y) = L(N^{-1}(y)), \quad \frac{d}{dy}\mathcal{L}(y) = \mathcal{L}'(y) = N^{-1}(y).$$

From the last equation, we conclude $N(x) = (\mathcal{L}')^{-1}(x)$ and

$$\varphi(x) = \left[\frac{d}{dx}\Big(\mathcal{L}'(.)\Big)^{-1}\right](x).$$

This observations motivate the following proposition and its proof.

Theorem 3.4 *Let $\mathcal{L} : [0, 1] \to [0, 1]$ be a function in $C^2([0, 1])$, with $\mathcal{L}(0) = 0$, $\mathcal{L}(1) = 1$, that is strictly increasing and concave. If furthermore $\mathcal{L}'(0) = 0$ and $\lim_{x\to 1} \mathcal{L}'(x) = \infty$, then there is a unique $\varphi \in \mathfrak{D}$, s.t. the Lorenz curve associated with φ coincides with $\mathcal{L}$.*

Proof: As $\mathcal{L}$ is strictly concave,

$$\frac{d^2}{dx^2}\mathcal{L}(x) > 0,$$

the function $\mathcal{L}'(x) := \frac{d}{dx}\mathcal{L}(x)$ has an inverse on its range. According to the assumptions, the range of $\mathcal{L}'$ is $\mathbb{R}_+ = \{x \geq 0\}$. We define

$$\varphi : \mathbb{R} \to \mathbb{R}, \qquad w \mapsto \varphi(w)$$

where

$$\varphi(w) := \frac{d}{dw}\Big((\mathcal{L}')^{-1}(w)\Big) \quad \text{for } w \geq 0, \qquad \varphi(w) = 0 \quad \text{for } w < 0.$$

This function is our candidate for which we aim to show the properties mentioned in the proposition.

Property 1: $\varphi(x)$ is continuous and positive in $\mathbb{R}_+$.
As $\mathcal{L}'(y)$ is strictly increasing ($\mathcal{L}$ is strictly concave), also its inverse $(\mathcal{L}')^{-1}(w)$ is strictly increasing (in $\mathbb{R}_+$), and thus the derivative of this function is strictly positive for $x \geq 0$. The function is also continuous, as $\mathcal{L} \in C^2$.

Property 2: $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} \varphi(w)\,dw = \int_{0}^{\infty} \varphi(w)\,dw = \int_{0}^{\infty} \frac{d}{dw}\Big((\mathcal{L}')^{-1}(w)\Big)\,dw = (\mathcal{L}')^{-1}(\infty) - (\mathcal{L}')^{-1}(0).$$

As $\mathcal{L}'(0) = 0$ and $\mathcal{L}'(1) = \infty$, we conclude

$$\int_{-\infty}^{\infty} \varphi(w)\,dw = 1 - 0 = 1.$$

Property 3: $\int_{-\infty}^{\infty} x\varphi(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} w\,\varphi(w)\,dw = \int_{0}^{\infty} w\,\varphi(w)\,dw = \int_{0}^{\infty} w\,\frac{d}{dw}\Big((\mathcal{L}')^{-1}(w)\Big)\,dw.$$

Transformation:

$$y = (\mathcal{L}')^{-1}(w) \quad \Rightarrow \quad \frac{dy}{dw} = \frac{d}{dw}(\mathcal{L}')^{-1}(w) \quad \Rightarrow \quad dy = \frac{d}{dw}(\mathcal{L}')^{-1}(w)\,dw$$

and $w = \mathcal{L}'(y)$. As the boundaries are maped to 0 and 1 by $(\mathcal{L}')^{-1}(w)$, we have

$$\int_{-\infty}^{\infty} w\,\varphi(w)\,dw = \int_{0}^{1} \mathcal{L}'(y)\,dy = \mathcal{L}(1) - \mathcal{L}(0) = 1.$$

Property 4: $\varphi(x)$ generates the Lorenz curve we started with.
As in the proof of property 2, we find

$$N(w) = \int_{0}^{w} \varphi(x)\,dx = (\mathcal{L}')^{-1}(w)$$

and hence $N^{-1}(y) = \mathcal{L}'(y)$. Furthermore, with the same transformation as in the proof of property 3, we obtain

$$L(w) = \int_{0}^{w} x\,\varphi(x), dx = \int_{0}^{(\mathcal{L}')^{-1}(w)} \mathcal{L}'(y)\,dy = \mathcal{L}((\mathcal{L}')^{-1}(w)).$$

Therewith,

$$L(N^{-1}(y)) = \mathcal{L}((\mathcal{L}')^{-1}(w))\Big|_{w=N^{-1}(y)} = \mathcal{L}((\mathcal{L}')^{-1}(w))\Big|_{w=\mathcal{L}'(y)} = \mathcal{L}(y).$$

$\square$

Corollary 3.5 *Let $\mathfrak{L}$ denote the set of Lorenz curves, i.e. functions $\mathcal{L}(y)$ in $C^2([0,1])$ with $\mathcal{L}(0) = 0$, $\mathcal{L}(1) = 1$, that are strictly increasing and concave. Furthermore $\mathcal{L}'(0) = 0$ and $\lim_{x\to 1} \mathcal{L}'(x) = \infty$.*
Then, the operator

$$\Lambda : \mathfrak{D} \to \mathfrak{L}$$

that assigns to a distribution in $\mathfrak{D}$ its Lorenz curve is well defined and one-to-one.

We can use that fact to define a new topology on $\mathfrak{D}$: The distance of two distributions is defined as the L^1 distance of the corresponding Lorenz curves.

Proposition 3.6 *Define*

$$d : \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}, \quad d(\varphi_1, \varphi_2) = \|\Lambda(\varphi_1)(y) - \Lambda(\varphi_2)(y)\|_{L^1}.$$

Then, $(\mathfrak{D}, d)$ is a metric space.

Therewith, we now are able to extend the operator Λ to a large set of Lorenz curves, which contain all kind of singular elements we wish for.

Proposition 3.7 *Let $\overline{\mathfrak{D}}$ be defined as the closure of $\mathfrak{D}$ under the distance $d(.,.)$. Then, the operator Λ has a continuous extension on $\overline{\mathfrak{D}}$,*

$$\overline{\Lambda} : \overline{\mathfrak{D}} \to L^1(0,1).$$

The Gini coefficient can also be continuously extended to $\overline{\mathfrak{D}}$ via

$$\overline{G} : \overline{\mathfrak{D}} \to \mathbb{R}, \quad \overline{G}(\phi) = 2\int_0^1 (x - \overline{\Lambda}(\phi)(x))\, dx.$$

The next theorem extends Theorem 3.3 res. Theorem 3.1 to $\overline{\mathfrak{D}}$. Note that we define the minimal element f_* of the Lorenz curves not as a function that is 0 on $[0,1)$ and 1 in $y = 1$, but as $F_*(y) = 0$. As we work in L^1, we can change functions at single points. Functions in L^1 are equivalent classes.

Theorem 3.8 *Let $\phi \in \overline{\mathfrak{D}}$. Then, the Lorenz curve $\Lambda(\phi)$ is below or equal the diagonal, non-negative (both in the L^1-sense), and $0 \leq G(\varphi) \leq 1$. Furthermore, for $f^*(y) = y$ and $f_*(y) \equiv 0$, we find $\phi_*, \phi^* \in \overline{\mathfrak{D}}$ s.t. $\overline{\Lambda}(\phi_*) = f_*$, and $\overline{\Lambda}(\phi^*) = f^*$.*

Proof: For $\varphi \in \mathfrak{D}$, we know that

$$f_*(y) \leq \Lambda(\varphi)(y) \leq f^*(y).$$

Hence, any limit $\phi \in \overline{\mathfrak{D}}$ of a sequence $\varphi_n \in \mathfrak{D}$ that is a Cauchy sequence in $(\mathfrak{D}, d)$ (defined by the fact that $\Lambda(\varphi_n)$ converges in L^1) satisfies

$$f_*(y) \leq \Lambda(\phi)(y) \leq f^*(y).$$

Furthermore, the image of $\overline{\mathfrak{D}}$ under $\overline{\Lambda}$ is subset L^1, s.t. $\overline{G}$ is well defined, and due to the lower and upper bound f_* and f^*, we have $\overline{G}(\phi) \in [0,1]$.

The existence of ϕ_* and ϕ^* that are mapped to f_* and f^*, respectively, is shown in exercises 3.6 and 3.7.

$\square$

In what follows, we are interested in local maxima of G in $\check{\mathcal{D}}'$.

Theorem 3.9 *The set $\overline{\Lambda}(\overline{\mathfrak{D}}))$ is a closed, convex subset of $L^1(0,1)$.*
The function $\overline{G} : \overline{\mathfrak{D}}$ has only one global minimum and one global maximum, with values 0 and 1, where the corresponding wealth distributions are given by ϕ_* and ϕ^*.

Proof: *Step 1:* Recall the conditions for functions $\mathcal{L} \in \Lambda(\mathfrak{D})$ (theorem 3.3 and 3.4)
(1) $\mathcal{L} \in C^2([0,1],[0,1])$,
(2) $\mathcal{L}(0) = 0$, $\mathcal{L}(1) = 1$,
(3) $\frac{d}{dy}\mathcal{L}(y)|_{y=0} = 0$, $\lim_{y\to\infty} \frac{d}{dy}\mathcal{L}(y) = \infty$,
(4) $\mathcal{L}$ is strictly increasing and concave.
If $\mathcal{L}_1$ and $\mathcal{L}_2$ have these properties, then also $\tau\mathcal{L}_1 + (1-\tau)\mathcal{L}_2$. Thus, $\Lambda(\mathfrak{D})$ is convex. Since $\overline{\Lambda}(\overline{\mathfrak{D}})$ is the closure of $\Lambda(\mathfrak{D})$ in L^1, the set $\overline{\Lambda}(\overline{\mathfrak{D}})$ is convex and closed.

Step 2: Consider $\phi \in \overline{\mathfrak{D}}$. If $G(\phi) \not\in \{0,1\}$, then the Gini index will decrease over the family Lorenz curves (if τ runs form 1 to 0)

$$\tau\overline{\Lambda}(\phi) + (1-\tau)\overline{\Lambda}(\phi^*) = \tau\overline{\Lambda}(\phi) + (1-\tau)\overline{f}^*.$$

Similarly, the Gini coefficient will increase over the family Lorenz curves (if τ runs form 1 to 0)

$$\tau\overline{\Lambda}(\phi) + (1-\tau)\overline{\Lambda}(\phi_*) = \tau\overline{\Lambda}(\phi) + (1-\tau)\overline{f}_*.$$

Since $\overline{\Lambda}(\overline{\mathfrak{D}})$ is convex, these family fo Lorenz curves are indeed feasible Lorenz curves. hence, a local maximum/minimum of thew Gini coefficient in $\overline{\Lambda}(\overline{\mathfrak{D}})$ does not exist.

Step 3: If $G(\phi) = 0$, then $\overline{\Lambda}(\phi)(y) = f^*(y)$ (in L^1), and hence $\phi = \phi^*$. Similarly, $G(\phi) = 1$, then $\overline{\Lambda}(\phi)(y) = f_*(y)$ (in L^1), and hence $\phi = \phi_*$.

$\square$

3.2 Yard-Sale Model

The Yard Sale Model [2, 3] is a most simple description of fundamental aspects of trading. Two persons exchange values, and they do that in a fair manner: before and after the trading event, both individuals have the same wealth. However, by mistake, it might be that the exchanged items are not of precisely the same value, but one of the items has a slightly higher price than the other. That is, one individual makes a small loss, while the other one gains a small amount of money (or values in terms of items). In any case, that event happened aimlessly, and in that, it is pure chance who will gain and who will loose.

Model 3.10 Yard Sale Model. *Consider N individuals, where the i'th person has wealth w_i. At rate μ a person trades with a randomly chosen other person. Let j denote the index of that person. Their wealth will change in that interaction,*

$$\begin{pmatrix} w_i \\ w_j \end{pmatrix} \mapsto \begin{pmatrix} w_i \\ w_j \end{pmatrix} + \eta\, r\, \min\{w_i, w_j\} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

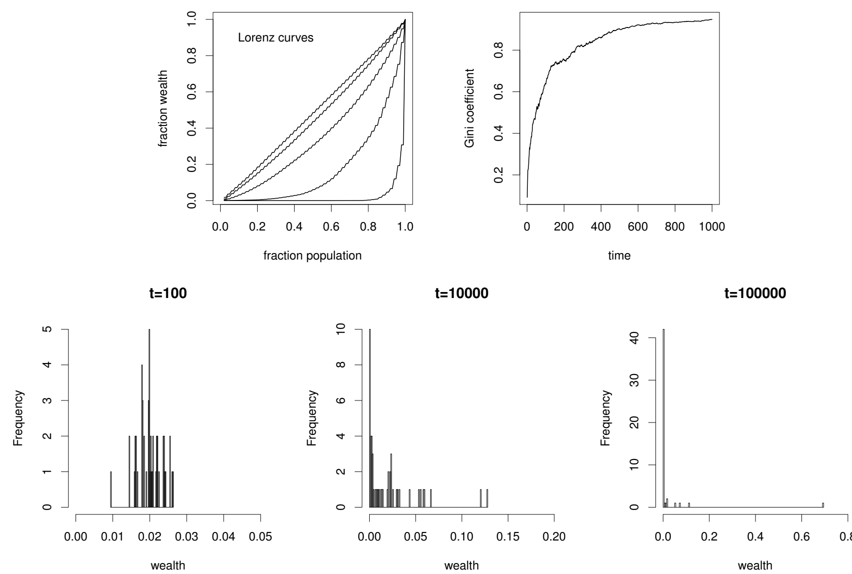

Figure 3.2: Simulation of the Yard Sale Model (population size 50, $r = 0.1$, initialization: each individual holds the wealth 1/50). The Lorenz curves are given (from top to down) for the times 10, 100, 1000, 10000, and 100000. The histograms (note the different scale of the x-axis) are given at the indicated time points.

where η is a family of i.i.d. random variables (one for each transaction) assuming values ± 1 with $P(\eta = 1) = P(\eta = -1) = 1/2$, and $r \in (0, 1)$ is a small parameter. Note that w_i are non-negative random variables, but in general not bounded.

The simulations (see Fig. 3.2) indicate, that in that completely fair and symmetric model, with completely equal initial conditions, the wealth concentrates step by step in one person only. In order to better understand that finding, we aim to work out a Fokker-Planck equation for that process. We will not do that rigorously, but by heuristic arguments only. Therefore, we start with a time-discrete stochastic process X_t; The random variables Y_t are i.i.d. with $E(Y_t) = y$ and $\mathrm{var}(Y_t) = \sigma^2$, and $\varepsilon > 0$ is a parameter that we will sue to scale the system. Let $X_{t+\Delta t}$ defined by

$$X_{t+\Delta t} = X_t + Y_t.$$

If we wait for N steps, we have

$$X_{t+N\Delta t} = X_t + \varepsilon \sum_{i=0}^{N-1} Y_{t+i\,\Delta t} = X_t + \varepsilon\, N\, y + \sigma\, \varepsilon\, N \left(\frac{1}{N} \sum_{i=0}^{N-1} \frac{Y_{t+i\,\Delta t} - y}{\sqrt{\mathrm{var}(Y_t)}} \right).$$

According to the law of large numbers,

$$\frac{1}{N} \sum_{i=0}^{N-1} \frac{Y_{t+i\,\Delta t} - y}{\sqrt{\mathrm{var}(Y_t)}} \sim \mathcal{N}(0, 1)$$

and (in a somewhat sloppy notation)

$$X_{t+N\Delta t} = X_t + \varepsilon\, N\, y + \sigma\, \varepsilon\, N \mathcal{N}(0,1) = X_t + \varepsilon\, N\, y + \sigma\, \mathcal{N}(0, \sqrt{\varepsilon N}).$$

If we define $\varepsilon = \delta t$ and $\tilde{\Delta} t = N\Delta t$, we have the Euler-Maruyama formula

$$X_{t+\tilde{\Delta} t} = X_t + y\, \tilde{\Delta} t + \sigma\, \mathcal{N}(0, \sqrt{\tilde{\Delta} t}).$$

For $\tilde{\Delta} t \to 0$ we obtain (in an appropriate sense) the SDE

$$dX_t = y\, dt + \sigma dW_t.$$

If the process X_t does not change too much in the time interval $\tilde{\Delta} t$, we can allow Y_t to depend explicitly on X_t and another, given random variable Z_t

$$Y_t = F(X_t, Z_t)$$

s.t.

$$dX_t = E(F(X_t, Z_t))\, dt + \sqrt{\mathrm{var}(F(X_t, Z_t))}\, dW_t.$$

In particular, if $E(F(X_t, Z_t)) = 0$, we have

$$\mathrm{var}(F(X_t, Z_t)) = E(F^2(X_t, Z_t))$$

and

$$dX_t = \sqrt{E(F^2(X_t, Z_t)))}\, dW_t.$$

After these remarks, let us return to the Yard Sale Model. We have a focal individual with wealth W_t, that selects randomly a trading partner with wealth $\tilde{W}$ from the pool of trading partners. Here, we neglect any correlation (which is appropriate if the population is large enough), and assume that the probability distribution of partners is given by some given density $u(w,t)$. Then, (recall that $\eta = \pm 1$ with equal probability)

$$E(\eta\, r\, \min\{W_t, \tilde{W}\}) = \frac{1}{2} \int (1\, r\, \min\{W_t, \tilde{w}\}) u(w,t)\, dw + \frac{1}{2} \int (-1\, r\, \min\{W_t, \tilde{w}\}) u(w,t)\, dw = 0.$$

The variance can be computed as follows (with $\eta^2 = 1$)

$$\begin{aligned}
\mathrm{var}(v) &= E(\,(\eta\, r\, \min\{W_t, \tilde{W}\})^2\,) = r^2 E(\eta^2\, \min\{W_t, \tilde{W}\}^2) \\
&= r^2 \Big(\int_0^{W_t} w^2\, u(w,t)\, dw + \int_{W_t}^{\infty} W_t^2\, u(w,t)\, dw \Big) \\
&= r^2 \Big(\int_0^{W_t} w^2\, u(w,t)\, dw + W_t^2 \int_{W_t}^{\infty} u(w,t)\, dw \Big) \\
&= 2\, r^2 \left[\frac{1}{2} W_t^2\, A(W_t) + B(W_t) \right]
\end{aligned}$$

with

$$A(W_t, t) = \int_{W_t}^{\infty} u(w,t)\, dw, \qquad B(W_t, t) = \frac{1}{2} \int_0^{W_t} w^2\, u(w,t)\, dw.$$

Hence,

$$dW_t = \sqrt{2}\, r \sqrt{\frac{1}{2} W_t^2\, A(W_t) + B(W_t)}\, dW_t.$$

As W_t is a typical representative of the population, the probability density $u(w,t)$ of the population agrees with that of W_t. Then, the corresponding Fokker-Planck equation reads

$$\begin{aligned}
\partial_t u(w,t) &= \frac{1}{2}\partial_w^2\Big(2\, r^2 \Big[\sqrt{\frac{1}{2} W_t^2\, A(W_t,t) + B(W_t,t)}\Big]^2 u(w,t)\Big)\\
&= r^2 \partial_w^2 \Big(\Big[w^2\, A(w) + B(w)\Big]\, u(w,t)\Big).
\end{aligned}$$

Note that A and B also depend on $u(w,t)$, s.t. in the present case the Fokker-Planck equation is no linear but a non-linear PDE. We summarize our heuristic consideration in the following model definition.

Model 3.11 *The Fokker-Planck equation for the Yard Sale Model is given by*

$$\partial_t u(w,t) = r^2 \partial_w^2 \Big(\Big[\frac{w^2}{2}\, A(w,t) + B(w,t)\Big]\, u(w,t)\Big) \tag{3.6}$$

with no-flux boundary conditions,

$$\partial_w \Big(\Big[\frac{w^2}{2}\, A(w,t) + B(w,t)\Big]\, u(w,t)\Big)\Big|_{w=0} = 0,$$

where A and B also depend on $u(w,t)$, and are defined by

$$A(w,t) = \int_w^\infty u(x,t)\, dx, \qquad B(w,t) = \frac{1}{2}\int_0^w x^2\, u(x,t)\, dx. \tag{3.7}$$

This Fokker-Planck equation has been derived by various arguments. The present route was published in [2], but there is also a route via a Boltzmann-Equation published [3]. Anyhow, we now investigate the solution of the that Fokker-Planck equation.

The Yard Sale Model has a constant population and a constant total wealth. We find these two facts back in the following proposition for the solution of the Fokker-Planck equation.

Proposition 3.12 *Assume that a classical solution $u(w,t)$ of eqn. (3.6) exists, and tends exponentially fast to zero for $w \to \infty$. Then, the zeroth and first moments are constant in time,*

$$\frac{d}{dt}\int_0^\infty u(w,t)\, dw = 0, \qquad \frac{d}{dt}\int_0^\infty w\, u(w,t)\, dw = 0.$$

Proof: As we assume that the asymptotic of the solution for $w \to \infty$ is well behaved, we can exchange integral and time derivative to find

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty u(w,t)\, dw &= \int_0^\infty u_t(w,t)\, dw = r^2 \int_0^\infty \partial_{w^2}\Big\{ \Big(\frac{w^2}{2} A(w) + B(w)\Big) u(w,t)\Big\}\, dw\\
&= -r^2 \partial_w \Big\{ \Big(\frac{w^2}{2} A(w) + B(w)\Big) u(w,t)\Big\}\Big|_{w=0} = 0.
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty w\,u(w,t)\,dw = \int_0^\infty w\,u_t(w,t)\,dw = r^2\int_0^\infty w\,\partial_{w^2}\Big\{\Big(\frac{w^2}{2}A(w)+B(w)\Big)u(w,t)\Big\}\,dw\\
= \;& r^2\,w\,\partial_w\Big\{\Big(\frac{w^2}{2}A(w)+B(w)\Big)u(w,t)\Big\}\Big|_{w=0}^{w=\infty} - r^2\int_0^\infty \partial_w\Big\{\Big(\frac{w^2}{2}A(w)+B(w)\Big)u(w,t)\Big\}\,dw\\
= \;& r^2\left\{\Big(\frac{w^2}{2}A(w)+B(w)\Big)u(w,t)\right\}\Big|_{w=0} = 0
\end{aligned}
$$

as $w^2A(w)/2|_{w=0} = 0$ and $B(0) = 0$.

□

In the Yard-Sale Model,the Gini coefficient is non-decreasing.

Theorem 3.13 *Let $u(w,t)$ denote the classical solution of the Fokker-Planck equation 3.6. Then, for all $t > 0$,*

$$\frac{d}{dt}G(u(.,t)) > 0.$$

Proof: We have

$$L(w;t) = \frac{\int_0^w w'\,\varphi(w')\,dw'}{\int_0^\infty w'\,\varphi(w')\,dw'}, \qquad N(w;t) = \int_0^w w(w',t)\,dw'.$$

Then, $G(u(.,t)) = 1 - 2\int_0^\infty L(w;t)\,dN(w;t)$ and, since we have a smooth density (a solution of a parabolic equation)

$$
\begin{aligned}
G(u(.,t)) \;&=\; 1-2\int_0^\infty L(w;t)\,dN(w;t) = 1-2\int_0^\infty L(w;t)\,u(w,t)\,dw\\
&=\; 1-2\,\frac{\int_0^\infty\int_0^w w'\,u(w',t)\,dw'\;u(w,t)\,dw}{\int_0^\infty wu(w,t)\,dw} = 1-2\,\frac{\int_0^\infty\int_{w'}^\infty u(w,t)\,dw\;w'\,u(w',t)\,dw'}{\int_0^\infty wu(w,t)\,dw}.
\end{aligned}
$$

We know that $\int_0^\infty wu(w,t)\,dw =: E$ is independent of time (see proposition 3.12). Hence,

$$
\begin{aligned}
&\frac{-E}{2}\,\frac{d}{dt}G(u(.,t)) = \int_0^\infty\int_{w'}^\infty u_t(w,t)\,dw\;w'\,u(w',t)\,dw' + \int_0^\infty\int_{w'}^\infty u(w,t)\,dw\;w'\,u_t(w',t)\,dw'\\
= \;& \int_0^\infty\int_0^w w'\,u(w',t)\,dw'\;u_t(w,t)\,dw + \int_0^\infty\Big(1-\int_0^{w'} u(w,t)\,dw\Big)\,w'\,u_t(w',t)\,dw'\\
= \;& \int_0^\infty\Big\{\int_0^w w'\,u(w',t)\,dw'\;u_t(w,t) + \Big(1-\int_0^w u(w',t)\,dw'\Big)\,w\,u_t(w,t)\Big\}\,dw\\
= \;& \int_0^\infty\Big(w-\int_0^w (w-w')\,u(w',t)\,dw'\Big)\,u_t(w,t)\;dw
\end{aligned}
$$

We replace $u_t(w,t)$ using the PDE, and obtain by partial integration

$$
\begin{aligned}
& \frac{d}{dt} G(u(.,t)) \\
= \; & \frac{-2}{E} \int_0^\infty \left(w - \int_0^w (w - w')\, u(w',t)\, dw' \right) \partial_w^2 \left(\left[\frac{w^2}{2} A(w,t) + B(w,t) \right] u(w,t) \right) dw \\
= \; & \frac{2}{E} \int_0^\infty \left(1 - \int_0^w u(w',t)\, dw' \right) \partial_w \left(\left[\frac{w^2}{2} A(w,t) + B(w,t) \right] u(w,t) \right) dw \\
= \; & \frac{2}{E} \int_0^\infty u^2(w,t) \left[\frac{w^2}{2} A(w,t) + B(w,t) \right] dw
\end{aligned}
$$

Next we investigate the expression $\frac{w^2}{2} A(w,t) + B(w,t)$

$$
\begin{aligned}
\frac{w^2}{2} A(w,t) + B(w,t) &= \frac{w^2}{2} \int_w^\infty u(x,t)\, dx + \frac{1}{2} \int_0^w x^2\, u(x,t)\, dx \\
&= \frac{w^2}{2} \left(1 - \int_0^w u(x,t)\, dx + \int_0^w \frac{x^2}{w^2}\, u(x,t)\, dx \right) \\
&= \frac{w^2}{2} \left(1 - \int_0^w \left(1 - \frac{x^2}{w^2} \right) u(x,t)\, dx \right) \\
&= \frac{w^2}{2} \left(1 - \int_0^w \left(1 - \frac{x^2}{w^2} \right) u(x,t)\, dx \right) \geq 0.
\end{aligned}
$$

Hence, $G'(t) \geq 0$. According to our assumptions, $u(w,t)$ is a classical solution for any finite time t. Furthermore, $w^2\, A(w,t)/2 + B(w,t) = 0$ implies

$$
\forall w \in \mathbb{R}_+ : \int_0^w \left(1 - \frac{x^2}{w^2} \right) u(x,t)\, dx = 1.
$$

This identity is impossible for any continuous, non-negative function u with $\int_0^\infty u(w,t)\, dw = 1$. Therefore, the Fokker-Planck equation is parabolic. If we start with a continuous solution, the parabolic character of the PDE forces the solution to be continuous for any finite time $t \in \mathbb{R}_+$. In that case, $w^2\, A(w,t)/2 + B(w,t) > 0$, and $G'(t) > 0$.

□

$G(u(.\,,t))$ is a kind of Lyapunov-function for the Fokker-Planck equation of the Yard-Sale-Model. The function G is bounded, and the values of G increase over all classical trajectories. We are inclinded to conclude that any trajectory tends to a local maximum of G, s.t. G cannot increase any more.
To support this conclusion requires some more work, as we only consider classical solutions (basically elements in $\mathfrak{D}$), and we know that there are many reasonable non-classical wealth distributions in $\overline{\mathfrak{D}}$. It is likely that the limit of a classical solution for $t \to \infty$ tends to a non-classical wealth distributions. We even cannot exclude at that point that the solution does not converge at all (and becomes periodic or even worse).

In any case we did prove (under the assumption that the solution is classical for any finite times) that the Yard Sale Model increases monotonically. Numerically, we find our conjecture confirmed that the Gini coefficient tends to 1. We can use Theorem 3.9 to express that observation in the following way.

Theorem 3.14 *If $G(t)$ increases to an local maximum, then $\lim_{t\to\infty} G(t) = 1$, and the wealth distribution tends to the maximal inequality, $u(w,t) \to \phi^*(w)$.*

3.3 Yard Sale Model with Redistribution

In real world economies, there are mechanisms to redistribute wealth. Particularly the tax system and support to the economically weaker part of the society is a mechanism to counteract the effects of the Yard sale model.

The tax is assumed to bee simply proportional to the wealth, and thee money is shared uniformly within the population. That is, the rich people have no higher tax rate, nor do they get a smaller share. This simple assumptions allows to represent redistribution by one single parameter only.

Model 3.15 Yard Sale Model with redistribution. *Consider N individuals, where the i'th person has wealth w_i. At rate μ a person trades with a randomly chosen other person. Let j denote the index of that person. Their wealth will change in that interaction,*

$$\begin{pmatrix} w_i \\ w_j \end{pmatrix} \mapsto \begin{pmatrix} w_i \\ w_j \end{pmatrix} + \eta\, r\, \min\{w_i, w_j\} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where η is a family of i.i.d. random variables (one for each transaction) assuming values ± 1 with $P(\eta = 1) = P(\eta = -1) = 1/2$, and $r \in (0,1)$ is a small parameter. Note that w_i are non-negative random variables, but in general not bounded.
Additionally, we introduce redistribution: at rate κ, a fraction τ of the wealth is transferred to a central fond which is fairly redistributed at once. That is,

$$w_i \mapsto \kappa\Big(- w_i\,\tau + \frac{\tau}{N}\sum_{i=1}^{N} w_i\Big).$$

We introduce the limed parameter $\zeta = \kappa\tau/\mu$.

As we can scale time (choose time units) s.t. μ becomes 1 without changing the invariant distribution of the model, the system only depends on η and ζ.

Exercises

Exercise 3.1: Assume that for the finite sequence $(w_i)_{i=1,\dots,N})$ that $0 \le w_1 \le w_2 \le \dots \le w_N$, and that the sequence adds up to one, $\sum_{i=1}^N w_i = 1$. Show that

$$\forall j = 0,\dots,N:\ \sum_{i=1}^{j} w_i \le j/N.$$

Solution of Exercise 3.1 Assume for $j_0 \in \{0,\dots,N\}$ that $\sum_{i=1}^{j_0} w_i > j_0/N$, while $\sum_{i=1}^{j} w_i \le j/N$ for $j = 0,\dots,j_0-1$. It is impossible that $j_0 = N$, as the values w_i add up to 1. Also $j_0 = 0$

is impossible, as, *per definitionem*, $\sum_{i=1}^{0} w_i = 0$.
We conclude that $w_{j_0} > 1/N$: As $\sum_{i=1}^{j_0-1} w_i \leq (j_0 - 1)/N$, and $\sum_{i=1}^{j_0} w_i > j_0/N$, we have

$$w_{j_0} = \sum_{i=1}^{j_0} w_i - \sum_{i=1}^{j_0-1} w_i > \frac{j_0}{N} - \frac{j_0 - 1}{N} = 1/N.$$

Furthermore, w_i is non-decreasing, which implies $w_i > 1/N$ for $i > j_0$. Hence,

$$\sum_{i=1}^{N} w_i = \sum_{i=1}^{j_0} w_i + \sum_{i=j_0+1}^{N} w_i > \frac{j_0}{N} + (N - j_0)\frac{1}{N} = 1$$

contradicting the fact that the total sum only amounts to 1.

Exercise 3.2: Let N denote the population size, and $(w_i)_{i=1,\ldots,N}$ the wealth distribution. Show that (a) $0 \leq G \leq 1$, that (b) $G = 0$ implies that $w_i = 1/N$ for all i, and that (c) $G = 1$ implies $w_i = 0$ for $i = 1, \ldots, N-1$, and $w_N = 1$.
Solution of Exercise 3.2 For (a), we know that $0 \leq L(i/N) \leq i/N$ (exercise 3.1), and hence $0 \leq G = (2/N)\sum_{i=1}^{N} i/N - L(i/N)$. Furthermore, w_i are monotoneously non-decreasing, s.t. the maximal G is given by $w_i = 0$ for $i < N$ and $w_N = 1$, which implies that

$$G \leq \frac{2}{N}\sum_{i=1}^{N-1} i/N = \frac{2(N-1(N-2)}{2N^2} = (1 - 1/N)(1 - 2/N).$$

(b) As $G = 0$, we have $L(i/N) = i/N$, which implies that $w_i = 1/N$ for all $i = 1, \ldots, N$.
(c) We know that $w_i = 0$ for $i < N$ and $w_N = 1$ implies $G = (1 - 1/N)(1 - 2/N)$. Assume that $w_{i_0} > 0$ for $i_0 < N$. Then, $G < (1 - 1/N)(1 - 2/N)$ as w_i are non-decreasing.

Exercise 3.3: Compute the Lorenz curve $\mathcal{L}(x)$ for the exponential distribution $\varphi(x) = \mu e^{-\mu x}$.
Solution of Exercise 3.3 We have

$$\begin{aligned}
\int_0^x \varphi(t)\,dt &= \int_0^x \mu e^{-\mu t}\,dy = 1 - e^{-\mu x} \\
\int_0^x t\,\varphi(t)\,dt &= -\int_0^x t\frac{d}{dt}e^{-\mu t}\,dt = -te^{-\mu t}\Big|_0^x + \int_0^x e^{-\mu t}\,dt = -xe^{-\mu x} + \frac{1}{\mu}\left(1 - e^{-\mu x}\right)
\end{aligned}$$

and in particular, $E = \int_0^\infty t\varphi(t)\,dt = 1/\mu$. We solve $y = \int_0^x \varphi(t)\,dt$ for x, and obtain

$$y = 1 - e^{-\mu x} \quad \Leftrightarrow \quad x = \frac{-1}{\mu}\ln(1 - y).$$

Therewith,

$$\mathcal{L}(y) = \frac{\mu^{-1}(1 - y)\ln(1 - y)\ + y\mu^{-1}}{E} = y + (1 - y)\ln(1 - y).$$

Note that the curve is independent of μ. The Lorenz curve is non-responsive to the absolute total wealth, but only the shape of the wealth distribution has an effect on $\mathcal{L}(y)$. Linear scaling of the wealth (that is, to choose the unit in which we measure wealth) does not affect the Lorenz curve.

Exercise 3.4: Let $\varphi_1(x)$ and $\varphi_2(x)$ be related by

$$\varphi_2(x) = \zeta\,\varphi_1(\zeta\,x)$$

with $\zeta > 0$. Show that these two distributions create the identical Lorenz curve.

Solution of Exercise 3.4 Let $N_i(x) := \int_0^x \varphi_i(t)\,dt, \quad i = 1, 2$. Then,

$$N_2(x) = \int_0^x \varphi_2(t)\,dt = \int_0^x \varphi_1(\zeta\,t)\,\zeta\,dt = \int_0^{x/\zeta} \varphi_1(t)\,dt = N_1(x/\zeta)$$

and with $L_i(x) := \int_0^x t\,\varphi_i(t)\,dt$, $i = 1, 2$,

$$L_2(x) = \int_0^x t\,\varphi_2(t)\,dt = \int_0^x t\,\varphi_1(\zeta\,t)\,\zeta\,dt = \zeta^{-1}\int_0^{x/\zeta} t\,\varphi_1(t)\,dt = L_1(x/\zeta)\,/\zeta.$$

In particular,

$$\int_0^\infty t\,\varphi_2(t)\,dt = \zeta^{-1}\int_0^\infty t\,\varphi_1(t)\,dt.$$

$x_2 = N_2^{-1}(y)$ is defined by $N_2(x_2) = y$. Similarly, $x_1 = N_1^{-1}(y)$ is equivalent with $N_1(x_1) = y$. Furthermore, $y = N_1(x_1) = N_2(x_1\zeta)$, and therefore $x_2 = x_1\,\zeta$,

$$N_2^{-1}(y) = N_1^{-1}(y)\,\zeta.$$

Equipped with those identities, we find for $\mathcal{L}_i(y)$, the Lorenz curves generated by the φ_i, that

$$\mathcal{L}_2(y) = \frac{L_2(N_2^{-1}(y))}{\int_0^\infty t\,\varphi_2(t)\,dt} = \frac{L_1(\,(N_1^{-1}(y)\,\zeta)\,/\zeta)\,/\,\zeta}{\int_0^\infty t\,\varphi_1(t)\,dt/\zeta} = \frac{L_1(N_1^{-1}(y))}{\int_0^\infty t\,\varphi_1(t)\,dt} = \mathcal{L}_1(y).$$

Exercise 3.5: Let $\phi : \mathbb{R} \to \mathbb{R}$ in C^0, $\phi(x) \geq 0$, $\mathrm{supp}(\phi) \subset [-1/2, 1/2]$, $\int_{-\infty}^\infty \phi(x)\,dx = 1$, $\int_{-\infty}^\infty x\,\phi(x)\,dx = 0$, $\int_{-\infty}^\infty |x|\,\phi(x)\,dx =: C$ finite. Let $\lambda > 0$, $k \in \mathbb{R}$.
(a) Show that $\tilde\phi(x) = \lambda\,\phi(\lambda\,(x-k))$ satisfies: $\tilde\phi$ in C^0, $\tilde\phi(x) \geq 0$, $\mathrm{supp}(\tilde\phi) \subset [k-1/(2\lambda), k+1/(2\lambda)]$, $\int_{-\infty}^\infty \tilde\phi(x)\,dx = 1$, $\int_{-\infty}^\infty x\,\tilde\phi(x)\,dx = k$, $\int_{-\infty}^\infty |x|\,\tilde\phi(x)\,dx \leq |k| + C/\lambda$.
(b) Explicitly construct a $\phi(x)$ that satisfies the requirements given above.

Solution of Exercise 3.5 (a) $\tilde\phi$ in C^0, $\tilde\phi(x) \geq 0$ is clear. If $\tilde\phi(x) > 0$, then

$$|\lambda(x-k)| < 1/2 \;\Leftrightarrow\; |x-k| < 1/(2\lambda) \;\Leftrightarrow\; x \in [k - 1/(2\lambda), k + 1/(2\lambda)].$$

Furthermore, with $y = \lambda(x-k)$, we have

$$\int_{-\infty}^\infty \tilde\phi(x)\,dx = \int_{-\infty}^\infty \phi(\lambda\,(x-k))\,\lambda dx = \int_{-\infty}^\infty \phi(y)\,dy = 1.$$

Next,

$$\int_{-\infty}^\infty x\,\tilde\phi(x)\,dx = \int_{-\infty}^\infty (x-k)\,\phi(\lambda\,(x-k))\,\lambda dx + k\int_{-\infty}^\infty \tilde\phi(x)\,dx = \frac{1}{\lambda}\int_{-\infty}^\infty y\,\phi(y)\,dy + k = k.$$

Last,

$$\int_{-\infty}^{\infty} |x|\,\tilde{\phi}(x)\,dx \le \frac{1}{\lambda}\int_{-\infty}^{\infty} \lambda\,|x-k|\,\phi(\lambda\,(x-k))\,\lambda dx + |k|\int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx$$
$$= \frac{1}{\lambda}\int_{-\infty}^{\infty} |y|\,\phi(y)\,dy + |k| = |k| + C/\lambda.$$

(b) Define, for example,

$$\phi(x) = \begin{cases} 0 & \text{for} \quad x > 1/2 \\ 4\,(1/2-x) & \text{for} \quad 1/2 \ge x > 0 \\ 4\,(x+1/2) & \text{for} \quad 0 \ge x > -1/2 \\ 0 & \text{for} \quad -1/2 \ge x \end{cases}$$

It is straightforward to check that $\phi(x)$ satisfies everything we looked for.

Exercise 3.6: Let ϕ satisfy the assumptions of exercise 3.5, and define for $\lambda \in (0,1)$ the family of distributions

$$\varphi_\lambda(x) = \left(\lambda\,e^{-x} + (1-\lambda)\;\frac{1}{\lambda}\,\phi((x-1)/\lambda)\right)\chi_{[0,\infty)}(x).$$

(a) Show that $\varphi_\lambda \in \mathfrak{D}$.
(b) Show that the family of Lorenz curves $\mathcal{L}_\lambda(y)$ that correspond to $\varphi_\lambda(x)$ tend to $f^*(y) = y$ in L^1 for $\lambda \to 0$,

$$\lim_{\lambda\to 0} \|\mathcal{L}_\lambda(y) - f^*(y)\|_{L^1(0,1)} = 0.$$

(c) Conclude that there is a unique $\phi^* \in \overline{\mathfrak{D}}$ with

$$\overline{\Lambda}(\phi^*)(y) = f^*(y).$$

Solution of Exercise 3.6 (a) φ_λ is positive for $x > 0$, continuous for $x > 0$. Furthermore, $\int_0^\infty e^{-x}\,dx = \int_0^\infty xe^{-x}\,dx = 1$. For $\lambda \in (0,1)$, $\mathrm{supp}(\phi((x-1)/\lambda)) \subset \mathbb{R}_+$. Due to exercise 3.5 (a), we also have $\int_0^\infty \phi((x-1)/\lambda)/\lambda\,dx = \int_0^\infty x\phi((x-1)/\lambda)/\lambda\,dx = 1$. As φ_λ is a convex combination of e^{-x} and $\phi((x-1)/\lambda)/\lambda\chi_{\mathbb{R}_+}(x)$, the result follows.
(b) We start with the computation of $N_\lambda(w)$,

$$N_\lambda(w) = \int_0^w \varphi_\lambda(x)\,dx = \lambda(1-e^{-w}) + (1-\lambda)\int_0^w \frac{1}{\lambda}\phi((x-1)/\lambda)\,dx.$$

Next we turn to $L_\lambda(w)$:

$$\begin{aligned} L_\lambda(w) &= \int_0^w x\left(\lambda\,e^{-x} + (1-\lambda)\;\frac{1}{\lambda}\,\phi((x-1)/\lambda)\right)dx \\ &= \lambda\Big((1-e^{-w}) - we^{-w}\Big) \\ &\quad +(1-\lambda)\int_0^w \frac{1}{\lambda}\,\phi((x-1)/\lambda)\,dx + (1-\lambda)\int_0^w \frac{1}{\lambda}\,(x-1)\phi((x-1)/\lambda)\,dx \\ &= N_\lambda(w) - \lambda\Bigg(\underbrace{we^{-w} - (1-\lambda)\int_0^{(w-1)/\lambda} y\phi(y)\,dy}_{=:G_\lambda(w)}\Bigg). \end{aligned}$$

Note that $|G(w)|$ is uniformly bounded in w and λ. The Lorenz curve now reads

$$\mathcal{L}_\lambda(y) = L_\lambda(N_\lambda^{-1}(y)) = y - \lambda G_\lambda(N_\lambda^{-1}(y)).$$

Hence,

$$\begin{aligned}\lim_{\lambda\to 0} \|\mathcal{L}_\lambda(y) - f^*(y)\|_{L^1(0,1)} &= \lim_{\lambda\to 0} \|y - \lambda G_\lambda(N_\lambda^{-1}(y)) - y\|_{L^1(0,1)} = \lim_{\lambda\to 0} \lambda \int_0^1 |G_\lambda(N_\lambda^{-1}(y))|\, dy \\ &= = \lim_{\lambda\to 0} \lambda \int_0^1 |G_\lambda(w)|\, \varphi_\lambda(w)\, dw.\end{aligned}$$

As $|G_\lambda|$ is bounded, and the integral over φ_λ is 1, the integral $\int_0^1 |G_\lambda(w)|\, \varphi_\lambda(w)\, dw$ is bounded, and the limit becomes zero.
(c) As $\mathcal{L}_{1/n}(y)$ is a Cauchy sequence in L^1, so is $\varphi_{1/n}(y)$ in $\overline{\mathfrak{D}}$. Therefore, φ_λ converges in $\overline{\mathfrak{D}}$ to some $\phi^* \in \overline{\mathfrak{D}}$, and the limit of the Lorenz curvea ia the Lorenz curves of the limit. That establishes $\overline{\Lambda}(\phi^*)(y) = f^*(y)$. Since the distance between two elements in $\overline{\mathfrak{D}}$, there can be only one element in $\overline{\mathfrak{D}}$ that is mapped to $f^*(y)$. We have uniqueness.
If we consider φ_λ, we find that

$$\lim_{\lambda\to 0} \varphi_\lambda(w) = \delta_1(w).$$

A representation of ϕ^* is a point mass at 1: all individuals hold the same wealth, which is located at 1 as we require that the expected value for the wealth is 1.

Exercise 3.7: Let ϕ satisfy the assumptions of exercise 3.5, and define for $\lambda \in (0,1)$ the family of distributions

$$\varphi_n(x) = \left\{\frac{1}{n}\, e^{-x} + \left(1 - \frac{1}{n}\right) \left(\frac{n}{n+1}\, n\, \phi((x-1/n)\, n) + \frac{1}{n+1}\, n\phi((x-n)\, n)\right)\right\}\chi_{[0,\infty)}(x).$$

(a) Show that $\varphi_n \in \mathfrak{D}$.
(b) Let $N_n(w) = \int_0^w \phi_n(x)\, dx$, and $y_n := N_n(n - 1/n)$. Consider the family of Lorenz curves $\mathcal{L}_n(y)$ that correspond to $\varphi_n(x)$ and show that

$$\int_0^{y_n} \mathcal{L}_n(y)\, dy \le 2/n.$$

Also show that $y_n \to 1$ for $n \to \infty$, and conclude that $\|\mathcal{L}_n(y) - f_*(y)\|_{L^1} = \|\mathcal{L}_n(y)\|_{L^1} \to 0$ for $n \to \infty$.
(c) Conclude that there is a unique $\phi_* \in \overline{\mathfrak{D}}$ with

$$\overline{\Lambda}(\phi_*)(y) = f_*(y).$$

Solution of Exercise 3.7 (a) $\phi_n(x)$ are zero for $x < 0$. As $\mathrm{supp}(n\, \phi((x-1/n)\, n)) \subset [0, 2/n]$, we have that $\int_0^\infty \varphi_n(x)\, dx = 1$. Continuity of $x \ge 0$ is also given. Furthermore,

$$\int_{-\infty}^\infty x\, \varphi_n(x)\, dx = \frac{1}{n} + \left(1 - \frac{1}{n}\right) \left(\frac{n}{n+1}\, \frac{1}{n} + \frac{1}{n+1}\, n\right) = 1.$$

(b) The value y_n is defined by $y_n := N_n(n-1/n)$. As the support of $n\phi((x-n)\,n)$ is subset of $[n-1/n, n+1/n]$, we have

$$\begin{aligned} y_n &= N_n(n-1/n) = \int_0^{n-1/n} \frac{1}{n}\, e^{-x} + \left(1-\frac{1}{n}\right)\, \frac{n}{n+1}\, n\,\phi((x-1/n)\,n)\,dx \\ &= \frac{1}{n}\,\left(1-e^{-n+1/n}\right) + \left(1-\frac{1}{n}\right)\, \frac{n}{n+1} \end{aligned}$$

and $y_n \to 1$ for $n \to \infty$.
We have $\mathcal{L}_n(y_n)$ given by (again, note that $\operatorname{supp}(n\phi((x-n)\,n)) \subset [n-1/n, n+1/n]$ is outside the integration interval, and the expected value of $n\phi(n(x-1/n))$ is given by $1/n$, and $\int_0^x xe^{-x}\,dx = 1-e^{-x}-xe^{-x}$),

$$\begin{aligned} &\int_0^{n-1/n} x\,\varphi_n(x)\,dx = \frac{1}{n}\left(1-e^{-n+1/n} - (n-1/n)e^{-n+1/n}\right) + \left(1-\frac{1}{n}\right)\,\frac{n}{n+1}\,\frac{1}{n} \\ \leq\ & \frac{2}{n}. \end{aligned}$$

As $\mathcal{L}_n(x)$ is monotonously increasing, we thus know that $\mathcal{L}_n(y) < 2/n$ for $y \in [0, y_n]$, and $\mathcal{L}_n(y) \leq 1$ for $y \in [y_n, 1]$. Therefore,

$$\|\mathcal{L}_n(y) - f_*(y)\|_{L^1} \leq \frac{2}{n}\, y_n + (1-y_n)$$

s.t. $\|\mathcal{L}_n(y) - f_*(y)\|_{L^1} \to 0$ for $n \to \infty$ (since $y_n \to 1$ for $n \to \infty$).
(c) As $\mathcal{L}_n(y)$ converges in L^1, φ_n converge in $\overline{\mathfrak{D}}$ to some ϕ_*. Hence, $\overline{\Lambda}(\phi_*) = f_* \equiv 0$ in $L^1(0,1)$. Due to the identification of distribution and Lorenz curve via $\overline{\Lambda}$, the distribution ϕ_* us unique.

Exercise 3.8: Compute the Lorenz curve $\mathcal{L}(x)$ Define the family of distribution

$$\varphi_\lambda(x) = \lambda\, e^{-\lambda(x-(1-1/\lambda))}\,\chi_{[1-1/\lambda,\infty)}(x)$$

where $\lambda \geq 1$.
(a) Show $\int_{-\infty}^{\infty} \varphi_\lambda(x)\,dx = \int_{-\infty}^{\infty} x\,\varphi_\lambda(x)\,dx = 1$.
(b) Compute the Lorenz curve $\mathcal{L}_\lambda(y) = \overline{\Lambda}(\varphi_\lambda)(y)$.
(c) Investigate the convergence of $\overline{\Lambda}(\varphi_\lambda)(y)$ and φ_λ for $\lambda \to \infty$.

Solution of Exercise 3.8 (a)

$$\int_{-\infty}^{\infty} \varphi_\lambda(x)\,dx = \lambda \int_{1-1/\lambda}^{\infty} e^{-\lambda(x-(1-1/\lambda))}\,dx = -e^{-\lambda(x-(1-1/\lambda))}\Big|_{1-1/\lambda}^{\infty} = 1.$$

Furthermore,

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_\lambda(x)\,dx = \lambda \int_{1-1/\lambda}^{\infty} x\,e^{-\lambda(x-(1-1/\lambda))}\,dx = -\int_{1-1/\lambda}^{\infty} x\,\frac{d}{dx}e^{-\lambda(x-(1-1/\lambda))}\,dx \\ =\ & -x\,e^{-\lambda(x-(1-1/\lambda))}\Big|_{1-1/\lambda}^{\infty} + \frac{1}{\lambda}\,\lambda \int_{1-1/\lambda}^{\infty} e^{-\lambda(x-(1-1/\lambda))}\,dx \\ =\ & (1-1/\lambda)\ +\ 1/\lambda = 1. \end{aligned}$$

(b) φ_λ is strictly positive and continuous for $x > 1 - 1/\lambda$. We find for $x \geq 1 - 1/\lambda$

$$N(w) = \int_{1-1/\lambda}^{w} e^{-\lambda(x-(1-1/\lambda))}\,dx = -e^{-\lambda(x-(1-1/\lambda))}\Big|_{1-1/\lambda}^{w} = 1 - e^{-\lambda(w-(1-1/\lambda))}$$

and

$$\begin{aligned}
L(w) &= \int_{-\infty}^{w} \varphi_\lambda(x)\,dx = \lambda \int_{1-1/\lambda}^{w} x\,e^{-\lambda(x-(1-1/\lambda))}\,dx = -\int_{1-1/\lambda}^{w} x\,\frac{d}{dx} e^{-\lambda(x-(1-1/\lambda))}\,dx \\
&= -x\,e^{-\lambda(x-(1-1/\lambda))}\Big|_{1-1/\lambda}^{w} + \frac{1}{\lambda}\,\lambda \int_{1-1/\lambda}^{w} e^{-\lambda(x-(1-1/\lambda))}\,dx \\
&= \Big(-w\,e^{-\lambda(w-(1-1/\lambda))} + (1-1/\lambda)\Big) + N(w) = \Big(-w\,(1-N(w)) + (1-1/\lambda)\Big) + N(w) \\
&= \Big(-\Big(1 - \frac{1}{\lambda} + \frac{1}{\lambda}\ln(1-N(w))\Big)(1-N(w)) + (1-1/\lambda)\Big) + N(w) \\
&= N(w) + \frac{1}{\lambda}\Big(\Big(1-\frac{1}{\lambda}\Big)N(w) - (1-N(w))\,\ln(1-N(w))\Big).
\end{aligned}$$

Hence, with $y = N^{-1}(w)$, we have

$$\mathcal{L}(y) = y + \frac{1}{\lambda}\Big(\Big(1-\frac{1}{\lambda}\Big)y - (1-y)\,\ln(1-y)\Big).$$

(c) For $\lambda \to \infty$, we find $\mathcal{L}(y) \to y = f^*(y)$, while (in the sense of generalized functions) $\lim_{\lambda\to\infty} \varphi_\lambda(x) = \delta_1(x)$.

Appendix A

Data files and particularities

A.1 Zealot model

Figure 2.3

• US presidential elections 2008
`https://doi.org/10.7910/DVN/VOQCHQ`
file: countypres_2000-2016.csv
download: 2019-12-27
Na in the data. Removed.
Election district with no votes for the democratic and republican candidate. removed.

• US presidential elections 2017
`https://doi.org/10.7910/DVN/VOQCHQ`
file: countypres_2000-2016.csv
download: 2019-12-27
Na in the data. Removed.
Election district with no votes for the democratic and republican candidate. removed.

• Brexit referendum
`https://www.electoralcommission.org.uk/who-we-are-and-what-we-do/elections-and-referendums/past-elections-and-referendums/eu-referendum/results-and-turnout-eu-referendum`
Download: 2019-8-30

• US presidential France 2017
`https://www.data.gouv.fr/fr/posts/les-donnees-des-elections/` file: Presidentielle_2017_Resultats_Communes_Tour_2_c.xls
Download: 2020-1-13
2017 presidential election - second round"
There are NA in the data - they have been removed.

- turnout rates, FRG, 2017

- turnout rates, NL, 2017

https://www.verkiezingsuitslagen.nl/verkiezingen/detail/TK19770525
File: uitslag_alle_gemeenten_TK20170315.csv
Download: 2019-9-29
In some election districts, more active than eligible voters are reported (inconsistent!). These election districts have been removed.
Two election districts with a vote share smaller than 0.6 are removed as outliers.

Figure 2.5

- election results for the German Parliament, 1950-2017

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