Mathematical Models in Social Sciences – Notes

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Contents

Chapter 1

Introduction

The human society and human history:
– communication between humans, basis for the organization of a society
– resources and economic framework

The human society is shaped by intrinsic and extrinsic forces. Extrinsic forces are for example the available natural resources. An intrinsic force is communication between humans, that is necessary to establish social norms. In-between are economics and aspects of techniques: Techniques as agriculture and craft, or applied sciences converts the extrinsic, natural resources into available resources, economy distributes these resources within the society.

• Sociophysics/sociamathematics/socioinformatics
• models, descriptive.
• Understanding the mechanisms $\rightarrow$ influence the dynamics
not science, normative questions arise
dangerous, internet, social media, bots, Cambridge analytics.

Chapter 2

Communication and Democratic Elections

Communication is for sure one of the most important mechanisms that shape and structure society. We target here not only the simple spread of information, but rather the formation of social groups and social norms. As we want to validate tour ideas using data – at least at a simple level – we focus on a communication process, where tons of data are easily accessible: democratic elections.

Elections are central devices in modern democracies (in contrast to several democratic systems in the past, where a lottery has been an additional element to voting; it is an actual discussion if such a component could help to stabilize a democratic system [**?**]). In a democratic election, mature citizens think about the tasks a society is faced with, and decide, which candidate/party offers the best concepts to deal with these tasks. If this is picture of elections is complete, we cannot expect any statistical patterns in election results. However, as we will see below, a simple data analysis shows a lot of different, stable patterns. Those patterns are a reminiscence of the discussion and communication process within the society. In that, we are able to study one of the most fundamental forces that shape a society: communication and the formation of a basic agreement.

2.1 The voter model

→ ODE for the voter model?
→ we have something like an epidemic process for the spread of rumors. In which relation does that concept to the voter model, zealot model etc.?

2.2 Votes per candidate within a party

[14] characteristic distribution and model.

[35] Another model for the same fact.

2.3 The Zealot model

The voter model is perhaps the most simple model to describe opinion formation: Not arguments, but imitation is the considered as the driving force of the dynamics. In its basic form, single individuals copy from time to time the opinion of a randomly chosen neighbor [28]. Clearly, if only one opinion is present in the population, the state of the system will not change. The everyone-agrees-with-everyone-state is absorbing. Consequently, for finite populations, only one opinion will persist in the long run (a.s.), while all other opinions die out. Obviously, this model is too simple to capture reality. Variants of the model are introduced that prevent this uniformity. Perhaps the most important extension is the noisy voter model [**?**], which we discuss next in form of the zealot model [7, 5, 31, 4, 37, 25].

2.3.1 BESSERE STRUKTUR!!!!

(a) Theorie zunaechst fuer 2 Parteien entwickeln:
- Zeloten als Einfluss von ausserhalb
- Zeloen als Teil der Waelhler, nur θbbeeinflussr (Sano)
- Invariante Verteilunf, N endlich
- zwei Large Population limits (Zeloten nicht skalieren/Zeloten skalieren): deterministic limit/weak limit

(b) dann eine Zusammenfassung fuer das K-Parteien-System (hier zumindest Teil der eher technischen Beweise in einen Appendix)

(c) Dann Datenanalyse durch das Zealot-Model
- Verteilung der Wahlerbegnisse Ergebnisse (eine Partei)
- SI, Vote share
- SI, groesste Partei gegen rest
- SI, Mahalnobis
- Anteil von beeinflussenden Zeloten (Sano)

2.3.2 Two-party zealot model

We start with the two-opinion zealot model as formulated by Aguiar and coworkers [7, 5, 4]. It is interesting to note that this model first appeared in the context of population genetics [7], to describe foraging strategy of insects [24]. and finance markets [24], and only later it was also used to describe social processes, as finance markets [4] or elections.

Model 2.1 Two-party Zealot Model. *Let $N \in \mathbb{N}$ denote the population size of a well mixed population. Each individual either supports party A or party B. The stochastic process $X_t \in \{0, \ldots, N\}$ represent the number of individuals supporting A, such that $N - X_t$ is the number of supporters of B. Additionally, there are pseudo-voters (zealots) with number $N^a, N^b \in \mathbb{R}$ ($N^* > 0$, $* \in \{a, b\}$) who have a fixed opinion, and always support A (N^a) resp. B (N^b). These zealots represent external influences as the mass media: newspaper, radio, television, or the internet.*
At rate ν, a person changes his/her opinion; he/she selects a voter or a pseudo-voter (with selfing) randomly, and copies the opinion of the selected person. Each of the $N + N^a + N^b$

individuals have the same probability to be chosen. That is, the transition rates defining the stochastic process read

$$X_t \to X_t + 1 \quad \text{at rate} \quad \nu\,(N - X_t)\,\frac{X_t + N^a}{N + N^a + N^b} \tag{2.1}$$

$$X_t \to X_t - 1 \quad \text{at rate} \quad \nu\, X_t\,\frac{N - X_t + N^b}{N + N^a + N^b}. \tag{2.2}$$

We aim to obtain insight into the effect of the communication effect within the society. We identified external forces, the zealots. This concept is kind of cheating, as most likely the zealots are parts of the society. There are some rare cases where zealots indeed stand outside of the society. E.g., if a state uses internet trolls, bots, and fake news to influence the public opinion of another state. Mostly, the zealots (newspaper, television stations) will not be outside the community. However, we might assume that the time scale at which the zealots change their opinion and the time scale the individuals change their opinion are different. If the zealots are much slower than the citizens, then the model might be appropriate.

The next proposition is a direct consequence of the model definition.

Proposition 2.2 *With $p_i(t) = P(X_t = i)$, the master equations of model 2.1 reads*

$$\begin{aligned}\frac{d}{dt}p_i &= -\nu\left(\frac{i + N^a}{N^a + N^b + N}\;\frac{N-i}{N} + \frac{N - i + N^b}{N^a + N^b + N}\;\frac{i}{N}\right)p_i \\ &\quad +\nu\left(\frac{i - 1 + N^a}{N^a + N^b + N}\;\frac{N - i + 1}{N}\right)p_{i-1} + \nu\left(\frac{N - i - 1 + N^b}{N^a + N^b + N}\;\frac{i+1}{N}\right)p_{i+1}.\end{aligned} \tag{2.3}$$

As we will see below, it is possible to explicitly elaborate the invariant measure of the process X_t, that is, the stationary states of the ODE (2.3). For now, we choose a different route to obtain an idea about the long term behavior, and use the normal (or diffusion) approximation. Before we start with the diffusion approximation itself, we first obtain an idea about the long term behavior of the process. Thereto, we derive an ODE for $E(X_t)$. We do not use the master equations (which would be possible) but use the observation

$$\begin{aligned}E(X_{t+\Delta}|X_t) &= X_t + \Delta\Big(\nu\,\frac{(N - X_t)X_t + (N - X_t)N^a}{N + N^a + N^b} - \nu\,\frac{X_t\,(N - X_t) + X_t\,N^b}{N + N^a + N^b}\Big) + \mathcal{O}(\Delta^2) \\ &= X_t + \Delta\,\nu\Big(\frac{(N - X_t)\,N^a}{N + N^a + N^b} - \frac{X_t\,N^b}{N + N^a + N^b}\Big) + \mathcal{O}(\Delta^2).\end{aligned}$$

Using the law of iterated expectations, and rearranging the resulting equation yields

$$\frac{E(X_{t+\Delta}) - E(X_t)}{\Delta} = \nu\Big(\frac{(N - E(X_t))\,N^a}{N + N^a + N^b} - \frac{E(X_t)\,N^b}{N + N^a + N^b}\Big) + \mathcal{O}(\Delta).$$

Taking the limit $\Delta \to 0$, we obtain the following corollary.

Corollary 2.3 *The expected value of X_t satisfies the ODE*

$$\frac{d}{dt}E(X_t) = \nu\Big(\frac{(N - X_t)\,N^a}{N + N^a + N^b} - \frac{X_t\,N^b}{N + N^a + N^b}\Big). \tag{2.4}$$

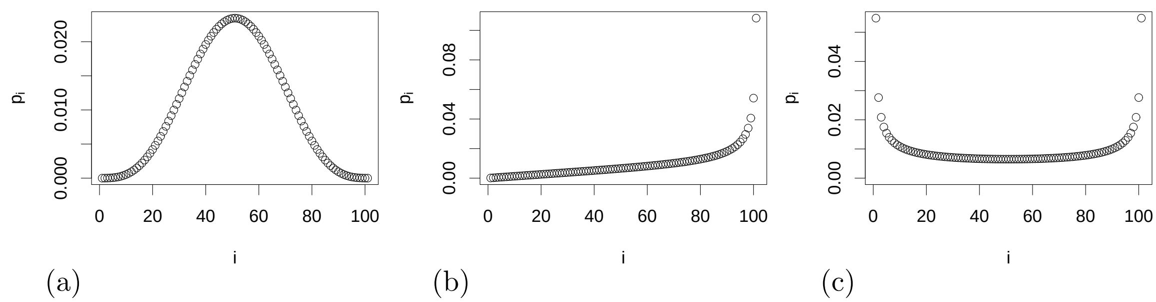


Figure 2.1: Simulations of the master equations. $N = 100$, $\nu = 1$, initial condition is the uniform distribution, simulated time $t = 8000$ (s.t. we have an approximately constant solution). (a) $N^a = 5$, $N^b = 5$, (b) $N^a = 2$, $N^b = 0.5$, (c) $N^a = 0.5$, $N^b = 0.5$.

For $t \to \infty$ we obtain

$$\lim_{t\to\infty} E(X_t) = N\,\frac{N^a}{N^a + N^b}. \tag{2.5}$$

In the long run, the expectation of our process is determined by the fraction of A-zealots among all zealots. However, the shape of the invariant distribution can be very different, as the simulations presented in Fig. 2.1 indicates. We know that the original voter model (no zealots) tends to an absorbing state. If there are only few zealots, $(N^a + N^b)/N \ll 1$, then the voter model is only weakly perturbed. The invariant measure is still localized around the states $X = 0$ and $X = N$, which are the absorbing states for the voter model (Fig. 2.1, panel (c)). Only if N^a/N, $N^b/N \gg 0$, the process X_t/N will be attracted by $N^a/(N^a + N^b)$ (Fig. 2.1, panel (a)), and performs a random walk close to this point.

The last case $(N^a/N, N^b/N \gg 0)$ is exactly the scenario suited for the normal approximation. If N becomes large, and the model is rescaled appropriately, the resulting stochastic process is well described by a diffusion process. The distribution of that diffusion process, in turn, is described by the corresponding Fokker-Planck equation.

Proposition 2.4 *Let $x_t = X_t/N$, $n^a = N^a/N$, and $n^b = N^b/N$. For N large, the distribution $u(x,t)$ of x_t is well approximated by the Fokker-Planck equation*

$$\begin{aligned} u_t(x,t) &= -\nu\,\partial_x\left\{\left(\frac{-n^b\,x + n^a\,(1-x)}{n^a + n^b + 1}\right) u(t,x)\right\} \\ &\quad + \frac{\nu}{2N}\partial_x^2\left\{\left(\frac{(1-x+n^b)x + (x+n^a)(1-x)}{n^a+n^b+1}\right) u(t,x)\right\}. \end{aligned} \tag{2.6}$$

Proof: Let $h = 1/N$, $x = h\,i$, and $u(t,x) = h\,p_i(t)$. We have $N^{a/b} = N\,n^{(a/b)}$. Taylor expansion yields

$$
\begin{aligned}
u_t(t,x) &= -\nu \left(\frac{x\,+n^a}{n^a+n^b+1}\,(1-x) + \frac{(1-x)\,+n^b}{n^a+n^b+1}\,x \right) u(t,x) \\
&\quad +\nu \left(\frac{(x-h)+n^a}{n^a+n^b+1}\,(1-x+h) \right) u(t,x-h) + \nu \left(\frac{1-(x+h)+n^b}{n^a+n^b+1}\,(x+h) \right) u(t,x+h) \\
&= \nu\partial_x \Big\{ \left(\frac{(1-x\,+n^b)x - (x\,+n^a)(1-x)}{n^a+n^b+1} \right) u(t,x) \Big\} \\
&\quad + \frac{\nu}{2N}\partial_x^2 \Big\{ \left(\frac{(1-x\,+n^b)x + (x\,+n^a)(1-x)}{n^a+n^b+1} \right) u(t,x) \Big\} + \mathcal{O}(h^2) \\
&= \nu\,\partial_x \Big\{ \left(-\frac{-n^b\,x + n^a\,(1-x)}{n^a+n^b+1} \right) u(t,x) \Big\} \\
&\quad + \frac{\nu}{2N}\partial_x^2 \Big\{ \left(\frac{(1-x\,+n^b)x + (x\,+n^a)(1-x)}{n^a+n^b+1} \right) u(t,x) \Big\} + \mathcal{O}(h^2).
\end{aligned}
$$

The result follows if we drop the term $\mathcal{O}(h^2)$.

□

Basically, we now follow methods of Grasman and Heuwarden [**?**], and consider the equilibrium distribution of the Fokker-Planck equation (2.6). The drift part of the Fokker-Planck equation yields the ODE

$$\dot{x} = \nu\,\partial_x \left(\frac{-n^b\,x + n^a\,(1-x)}{n^a+n^b+1} \right),$$

which corresponds to the ODE (2.4). We find that

$$x(t) \to \frac{n_a}{n_a+n_b} \qquad \text{for } t \to \infty.$$

That is, if we formally take $N \to \infty$ in the Fokker-Planck equation, then the distribution $u(x,t)$ approximates a point mass at $n_a/(n_a+n_b)$,

$$u(x) \to \delta_\mu, \qquad \mu = \frac{n^a}{n^a+n^b}.$$

If N is finite but large, the delta peak is replaced by a normal distribution with mean μ; the variance of that normal distribution does vanish for $N \to \infty$. To characterize this normal distribution completely, we determine the variance explicitly. We know that for N large,

$$u(x) \approx Ce^{-(x-\mu)^2/(2\sigma^2)}$$

If we plug in this ansatz in the flux of the Fokker-Planck equation (2.6), we find

$$
\begin{aligned}
0 &\approx (n^a+n^b)(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)} + \frac{1}{2N}\frac{d}{dx}\left\{ \left(2(1-x)x\ + n^b x + n^a(1-x)\right) e^{-(x-\mu)^2/(2\sigma^2)}\right\} \\
&= (n^a+n^b)(x-\mu) + \frac{1}{2N}\left\{ \left(2(1-2x)\ + n^b - n^a\right) + \left(2(1-x)x\ + n^b x + n^a(1-x)\right)\ \frac{-(x-\mu)}{\sigma^2}\right\} \\
\sigma^2 &= \frac{[2(1-x)x\ + n^b x + n^a(1-x)]\,(x-\mu)/(2N)}{-[2(1-x)x\ + n^b x + n^a(1-x)]/(2N) + (n^a+n^b)(x-\mu)} \\
&= \frac{1}{2N}\frac{2(1-x)x\ + n^b x + n^a(1-x)}{(n^a+n^b)} + \mathcal{O}(N^{-2}) \\
&= \frac{(1-x)x}{N}\ \frac{1+n^b/(2(1-x))+n^a/(2x)}{(n^a+n^b)} + \mathcal{O}(N^{-2}).
\end{aligned}
$$

We evaluate this term at $x=\mu$ and obtain for the leading order in N (recall that $\mu = n^a/(n^a+n^b)$, $1-\mu = n^b/(n^a+n^b)$)

$$
\begin{aligned}
\sigma^2 &= \frac{\mu(1-\mu)}{N}\ \frac{1\ + n^b/(2(1-\mu)) + n^a/(2\mu)}{(n^a+n^b)} \\
&= \frac{\mu(1-\mu)}{N}\ \frac{1\ + (n^a+n^b)}{(n^a+n^b)} = \frac{\mu(1-\mu)}{N}\ \left(1+\frac{1}{n^a+n^b}\right)
\end{aligned}
$$

Particularly, the variance diverges if n_a+n_b tends to zero. If we set $n^a+n^b=0$, we recover the voter model on a complete graph (or a Moran model), and here the invariant measure is concentrated on $x=0$ and $x=1$; in this case, the approximation by a normal distribution simply breaks down. We have the following corollary.

Corollary 2.5 *If N is large, the invariant measure of X_t/N is well approximated by a normal distribution with mean μ and variance σ^2 given by*

$$
\mu = \frac{n_a}{n_a+n_b} \tag{2.7}
$$

$$
\sigma^2 = \frac{\mu(1-\mu)}{N}\ \left(1+\frac{1}{n^a+n^b}\right). \tag{2.8}
$$

In [4], the authors use the invariant distribution of X_t directly, and obtain

$$
\sigma^2 = \frac{\mu(1-\mu)}{N}\ \left(\frac{N^a+N^b}{N^a+N^b+1} + \frac{N}{N^a+N^b+1}\right)
$$

which coincides with our result for $N\to\infty$, under the scaling of N^a, N^bwe did assume. We note that the variance given by de Braha allows a second limit: If we take N^a, N^b to be independent of N, we have for $N\to\infty$ that $x\sigma^2 \to \mu(1-\mu)\ (N^a+N^b+1)^{-1}$. We will discuss these two possible scalings later, in section 2.3.4.

Our main aim is the identification of consequences of communication within the population. We have the external forces given, the zealots. Without the internal communication, the zealots determine the opinion of an individual completely. This is the case for $n^a+n^b\to\infty$, which corresponds to $N^a + N^b \gg N$. An individual will always (a.s) select randomly a zealot, and never a fellow individual, to copy the opinion. The distribution of the A-supporters' number is simply given by a binomial distribution $\mathrm{Bin}(N, \mu)$. For N large, the fraction of individuals that adopt opinion A approximates again a normal distribution $\mathcal{N}(\mu, \sigma_0)$,

$$\mu = \frac{n^a}{n^a + n^b}, \qquad \sigma_0 = \frac{\mu(1-\mu)}{N}$$

If we compare the two distributions, we find that the communication process within the population increases the variance. This excess variance can be attributed to local interactions within the population. In that, it is possible to quantify the effect of social interactions.

Definition 2.6 *The social interaction coefficient (SI) for the two-opinion zealot model is defined by*

$$SI = \sigma/\sigma_0 = 1 + \frac{1}{n^a + n^b}.$$

Then, SI $= 1$ indicates no social interactions, SI > 1 refers to social interactions. This expression can be easily estimated form data, s.t. we can use data to measure the strength of social interactions (see Sect. 2.3.5).

Exercises

Exercise 3.1: The noisy voter model with population size $N > 1$ and K groups (and without spatial structure) has the set

$$\mathcal{S} = \left\{ (x_1, \ldots, x_K) \in \mathbb{N}^K \;\middle|\; \sum_{i=1}^{K} x_i = N \right\}$$

as state space, where x_i is the number of supporters of opinion i. In each time step, a randomly chosen individual changes potentially his/her opinion. With probability u, he/she copies the opinion of a randomly chosen individual. With probability $1 - u$, the individual adopts some opinion, independently on the state of the population. In that case, an individual belonging to group i will switch to group ℓ with probability $\pi_{i,\ell}$.
Show that the noisy voter model for $K = 2$ covers the zealot model 2.1 but is more general.

Solution of Exercise 3.1 (a) Given a zealot model with population size N, and N_A (N_B) fixed voters for opinion A (opinion B), we define $p = N/(N+N_A+N_B)$, $\pi_{1,1} = \pi_{2,1} = N_A/(N_A+N_B)$, and $\pi_{2,1} = \pi_{2,2} = N_B/(N_A + N_B)$.
(b) Note that $\pi_{i,1} + \pi_{i,2} = 1$, but it is not necessary that $\pi_{1,2} + \pi_{2,1}$ add up to one. For any zealot model, given that a person does not ask another floating voter, the probability for group A always is $N_A/(N_A + N_B)$, while that for group B reads $N_B/(N_A + N_B)$, such that

$$P(\text{jump form grop 1 to group 2}) + P(\text{jump form group 2 to group 1}) = 1.$$

That is, only noisy voter models with $\pi_{1,\ell} = \pi_{2,\ell}$, $\ell \in \{1, 2\}$, can be written as zealot models,

2.3.3 Elections with multiple candidates or parties

We extend the model for a dichotomic election discussed above to a multy-party (-candidate) system, as given in the US presidential pre-elections, the first round of presidential elections in France, and the parliamentary election in the Netherlands, Germany, or many other countries. The straight generalization we discuss now has been proposed by Sano [37] and, in a formulation closer to the noisy voter model, by Kononovicius [25]. The model of Sano refines the idea of the Zealot model as described above. Sano takes the zealots serious. They are – in principle – parts of the voters community, and will take part in an election. Before, zealots represented e.g. mass media,that influence elections but do not vote. In the present model, they still will influence the swing voters. However, though the zealots are determined to vote for a certain party, only part of them will influence the swing voters. That is, we basically have three groups in the voters community: (a) the zealots who have a frozen, fixed opinion but do not influence someone, the zealots that also have a frozen opinion but do influence the swing voters, and the swing voters themselves. Only that latter class is addressed by the dynamic (stochastic) model, the zealots and the fraction of politically active zealots are model parameters. Below, we address only the swing voters as "voters", as we are interested in their behavior, neglecting that also zealots may vote.

Notation: In the present section, we use the definition

$$\mathcal{S}_N^K = \{n \in \mathbb{N}_0^k \,|\, \sum_{\ell=1}^{K} n_0 = N\}.$$

Model 2.7 Sano model. *Consider an election with K groups (parties or candidates). The total number of (floating) voters is N, and at a given time the group sizes are $X_1, \dots, X_K$, such that $N = \sum_{\ell=1}^K X_\ell$. Additionally, we have zealots – individuals that do not change their opinion. The number of zealots that support group i is N_i, $i = 1, \dots, K$. Only a fraction θ of the zealots influence the floating voters. Among floating voters, we do allow for selfing. Floating voters as well as zealots will vote for his/her group, s.t. $X_i + N_i$ is the number of voters for group i. Let $N_F = \sum_{\ell=1}^K N_\ell$ the total number of zealots, and $M = N + \theta \sum_{\ell=1}^K N_\ell$ the total population size of politically active individuals (floating voters or influencing zealots).*
Any transition affects the sizes of two groups only. The transition rates are given by

$$(X_i, X_j) \mapsto (X_i + 1, X_j - 1) \text{ at rate } \mu X_j \,\frac{X_i + \theta N_i}{M}.$$

Invariant distribution

We are interested in the invariant measure of this model. We follow [37], and use the detailed balance equation, which we introduce next.

Theorem 2.8 *A Markov process with finite index set I, and transition rate from $\sigma \to \sigma'$*

for $\sigma, \sigma' \in I$ given by $\rho(\sigma \to \sigma')$ has the probability $P(\sigma)$ as stationary distribution, if

$$\frac{\rho(\sigma \to \sigma')}{\rho(\sigma' \to \sigma)} = \frac{P(\sigma')}{P(\sigma)}. \tag{2.9}$$

Equation (2.9) is known as the detailed balance equation.
Proof: (of theorem 2.8). Let X_t the I-valued stochastic process, and $p_\sigma(t) = P(X_t = \sigma)$. The master equation of this stochastic process is given by

$$\begin{aligned} \frac{d}{dt} p_\sigma(t) &= -\Big(\sum_{\sigma' \in I \setminus \{\sigma\}} \rho(\sigma \to \sigma') \Big) p_\sigma(t) + \sum_{\sigma' \in I \setminus \{\sigma\}} \rho(\sigma' \to \sigma)\, p_{\sigma'}(t) \\ &= \sum_{\sigma' \in I \setminus \{\sigma\}} \Big(\rho(\sigma' \to \sigma)\, p_{\sigma'}(t) - \rho(\sigma \to \sigma') p_\sigma(t) \Big). \end{aligned}$$

From (2.9) we obtain that

$$\rho(\sigma \to \sigma')\, P(\sigma) = \rho(\sigma' \to \sigma)\, P(\sigma').$$

Therefore, $P(\sigma)$ is a stationary solution of the master equation, and thus an invariant distribution.

$\square$

Note that the detailed balance equation checks that the flow between any of two states is balanced, and not only the overall probability flow. Hence, this equation is rather strict. The detailed balance equation is a sufficient but not a necessary condition for an invariant probability measure.

CHECK: WAS IST, WENN $N_{i_0} = 0$(keine fixed voters fuer eine partie). Haelt der satz immer noch, oder muessen wir das ausschliessen????

Theorem 2.9 *Let $(X_1, \ldots, X_n)$ be random variables distributed according to the invariant measure of model 2.7. Let furthermore $n \in \mathcal{S}_N^K$. Assume $N_j > 0$ for all $j = 1, \ldots, K$. Then,*

$$P(X_1 = n_1, \ldots, X_K = n_k) = \frac{N!}{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} \tag{2.10}$$

where $x_{(k)} = x\,(x+1)\,\cdots\,(x+k-1)$ is the Pochhammer symbol; we define $x_{(0)} = 1$.

Before we demonstrate this theorem, we show a neat equation that parallels the well known binomial formula

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Let $\mathcal{S}_N^K$ denote the set of all K-tuples $n = (n_1, \ldots, n_K)$, where $n_i \in \mathbb{N}_0$ and $\sum_{i=1}^K n_i = N$.

Proposition 2.10 *For $x, y \in \mathbb{R}$, $n \in \mathbb{N}$, we have*

$$(x+y)_{(n)} = \sum_{k=0}^{n} \binom{n}{k} x_{(k)} y_{(n-k)}. \tag{2.11}$$

Furthermore, for $x_1, \ldots x_K \in \mathbb{R}$, $N \in \mathbb{N}$, we find

$$(\sum_{\ell=1}^{K} x_\ell)_{(N)} = \sum_{n \in \mathcal{S}_N^K} \binom{N}{n_1, \ldots, n_K} \prod_{\ell=1}^{K} (x_\ell)_{(n_\ell)}. \tag{2.12}$$

Proof: To show formula (2.11), we use induction over n. For $n=1$, we find (recall $x_{(0)} = 1$)

$$(x+y)_{(1)} = (x+y) = x_{(1)}\, y_{(0)} + x_{(0)}\, y_{(1)} = \sum_{k=0}^{1} \binom{n}{k} x_{(k)} y_{(n-k)}.$$

Now assume that the formula is true for $n \in \mathbb{N}$. Then,

$$\begin{aligned}
&(x+y)_{(n+1)} = (x+y+n)\,(x+y)_{(n)} = (x+y+n) \sum_{k=0}^{n} \binom{n}{k} x_{(k)} y_{(n-k)} \\
= & \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\, y_{(n-k)} (\,(x+k) + (y+n-k)\,) \\
= & \sum_{k=0}^{n} \binom{n}{k} x_{(k+1)}\, y_{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\, y_{(n-k+1)} \\
= & \sum_{k=1}^{n+1} \binom{n}{k-1} x_{(k)}\, y_{(n-k+1)} + \sum_{k=0}^{n} \binom{n}{k} x_{(k)}\, y_{(n-k+1)} \\
= & \binom{n}{0} x_{(0)}\, y_{(n+1)} + \sum_{k=1}^{n+1} \left\{ \binom{n}{k-1} + \binom{n}{k} \right\} x_{(k)}\, y_{(n-k+1)} + \binom{n}{n} x_{(n+1)}\, y_{(0)} \\
= & \binom{n}{0} x_{(0)}\, y_{(n+1)} + \sum_{k=1}^{n+1} \binom{n+1}{k} x_{(k)}\, y_{(n+1-k)} + \binom{n}{n} x_{(n+1)}\, y_{(0)} \\
= & \sum_{k=0}^{n+1} \binom{n+1}{k} x_{(k)} y_{(n+1-k)}
\end{aligned}$$

For the second formula, we again use induction. This time, we use finite induction over K.

$$\begin{aligned}
(\sum_{\ell=1}^{K} x_\ell)_{(N)} &= (x_1 + \sum_{\ell=2}^{K} x_\ell)_{(N)} = \sum_{n_1=0}^{N} \binom{N}{n_1,\ N-n_1} (x_1)_{(n_1)} (\sum_{\ell=2}^{K} x_\ell)_{(N-n_1)} \\
&= \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \binom{N}{n_1,\ n_2,\ N-n_1-n_2} (x_1)_{(n_1)}\, (x_2)_{(n_2)} (\sum_{\ell=3}^{K} x_\ell)_{(N-n_1-n_2)} \\
&= \ldots = \sum_{n \in \mathcal{S}_N^K} \binom{N}{n_1, \ldots, n_K} \prod_{\ell=1}^{K} (x_\ell)_{(n_\ell)}.
\end{aligned}$$

$\square$

Proof: (of theorem 2.9) We aim to use the detailed balance equation. Consider two states n^1 and n^2, between that the Markov process might jump forth and back. That is, there are two groups i and j, such that

$$n_i^1 = n_i^2 - 1, \qquad n_j^1 = n_j^2 + 1.$$

The rate at which the process jumps from n^1 to n^2 is given by

$$\lambda_{1\to 2} = \mu n_i^1 \, \frac{n_j^1 + \theta N_j}{M}$$

while the rate to jump back from n^2 to n^1 reads

$$\lambda_{2\to 1} = \mu(n_j^1 + 1) \, \frac{n_i^1 - 1 + \theta N_i}{M}.$$

We aim to show the detailed balance equation, that is, we aim to prove the equation

$$\lambda_{1\to 2}\, P(X_1 = n_1^1, \dots, X_K = n_k^1) = \lambda_{2\to 1}\, P(X_1 = n_1^2, \dots, X_K = n_k^2).$$

We replace $P(X_1 = n_1^*, \dots, X_K = n_1^*)$ using (2.10). As that probability has a product structure, most terms cancel out. It is sufficient to show that

$$T_{1\to 2} := \lambda_{1\to 2} \frac{(\theta\, N_i)_{(n_i)}}{n_i!} \frac{(\theta\, N_j)_{(n_j)}}{n_j!} = \lambda_{2\to 1} \frac{(\theta\, N_i)_{(n_i-1)}}{(n_i-1)!} \frac{(\theta\, N_j)_{(n_j+1)}}{(n_j+1)!} =: T_{2\to 1}.$$

Let us first consider $T_{1\to 2}$,

$$T_{1\to 2} \;=\; \mu n_i^1 \, \frac{n_j^1 + \theta N_j}{M} \frac{(\theta\, N_i)_{(n_i)}}{n_i!} \frac{(\theta\, N_j)_{(n_j)}}{n_j!} = \frac{\mu}{M} \frac{(\theta\, N_i)_{(n_i)}}{(n_i-1)!} \frac{(\theta\, N_j)_{(n_j+1)}}{n_j!}.$$

Similarly, we find

$$T_{2\to 1} \;=\; \mu(n_j^1 + 1) \, \frac{n_i^1 - 1 + \theta N_i}{M} \frac{(\theta\, N_i)_{(n_i-1)}}{(n_i-1)!} \frac{(\theta\, N_j)_{(n_j+1)}}{(n_j+1)!} = \frac{\mu}{M} \frac{(\theta\, N_i)_{(n_i)}}{(n_i-1)!} \frac{(\theta\, N_j)_{(n_j+1)}}{n_j!}.$$

Hence, $T_{1\to 2} = T_{2\to 1}$, as required. The last step to show is

$$\sum_{n\in\mathcal{S}_N^K} P(X_1 = n_1, \dots, X_k = n_K) = \frac{N!}{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}} \sum_{n\in\mathcal{S}_N^K} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} = 1.$$

In order to do so, we rewrite the sum and then use formula (2.12),

$$\sum_{n\in\mathcal{S}_N^K} \prod_{\ell=1}^K \frac{(\theta\, N_j)_{(n_\ell)}}{n_\ell!} = \frac{1}{N!} \sum_{n\in\mathcal{S}_N^K} \binom{N}{n_1, \dots, n_K} \prod_{\ell=1}^K (\theta\, N_j)_{(n_\ell)} = \frac{(\sum_{\ell=1}^K \theta N_\ell)_{(N)}}{N!}.$$

If we multiply that equation by $N!/(\sum_{\ell=1}^K \theta N_\ell)_{(N)}$, we find that the probabilities indeed sum up to one.

$\square$

Limit of the invariant distribution

We are now prepared to show our asymptotic result. We consider a sequence of models with more and more voters. That is, N tends to infinity, while N_j are constant (see also [1]).
Before we state the theorem we recall the definition of the (real) Γ-function, and some properties. The real Γ- function is defined for $\mathbb{R}_+$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt.$$

Partial integration yields $\Gamma(1+x) = x\Gamma(x)$, which implies for $n \in \mathbb{N}$, $a \in \mathbb{R}_+$ that

$$n! = \Gamma(n+1), \qquad a_{(n)} = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.13}$$

There is a handy approximation of the Γ-function for large, real arguments: the Striling's formula,

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\, \left(\frac{x}{e}\right)^x\, e^{\zeta(x)}$$

where $\zeta(x)$ is a function with $0 \le \zeta(x) \le 1/(12\,x)$.

Theorem 2.11 *Let $\mu_\ell = N_\ell/N_F$. Define furthermore the random variables x_ℓ by*

$$x_\ell = X_\ell/N.$$

In the limit $N \to \infty$, we obtain a Dirichlet distribution for $(x_1, \ldots, x_K)$,

$$(x_1, \ldots, x_K) \sim Dir(\theta\, N_F\, \mu_1, \ldots, \theta\, N_F\, \mu_K). \tag{2.14}$$

Recall that the Dirichlet distribution with parameter $\alpha = (\alpha_1, \ldots, \alpha_K)$ is given by

$$\varphi(z_1, \ldots, z_K) = B(\alpha) \prod_{\ell=1}^K z_\ell^{\alpha_\ell - 1} \tag{2.15}$$

if the sum of the z_ℓ is one, and zero else, where

$$B(\alpha) = \frac{\prod_{\ell=1}^K \Gamma(\alpha_\ell)}{\Gamma(\sum_\ell \alpha_\ell)}.$$

Proof: We rewrite (2.10) as

$$P(X_1 = n_1, \ldots, X_K = n_k) = \hat{C}(N, \theta, N_\ell) \prod_{\ell=1}^K \frac{(\theta\, N_\ell)_{(n_\ell)}}{n_\ell!}.$$

We focus on one (generic) factor in that product, where we use $n_\ell = z_\ell N$,

$$\frac{(\theta\, N_\ell)_{(n_\ell)}}{n_\ell!}.$$

We are interested in the limit $N\to\infty$. Note that all multiplicative terms that only depend on N, θ, and N_j can be absorbed by the global normalization constant $\hat{C}(N,\theta,N_\ell)$. That is, we are only interested in equalities up to multiplicative functions that (asymptotically) do only depend on the model parameters N, θ, and N_j. We only keep the n_ℓ (z_ℓ)-dependent terms. In order to emphasize this fact, we write $\equiv_a$ if we neglect multiplicative factors.
If we express the factorial and the Pochhammer symbol by Γ-functions, we find

$$\begin{aligned}
\frac{(\theta\,N_\ell)_{(n_\ell)}}{n_\ell!} \quad &= \quad \frac{\Gamma(\theta\,N_\ell+n_\ell)\Gamma(n_\ell)}{\Gamma(\theta\,N_\ell)\,\Gamma(n_\ell+1)\Gamma(n_\ell)} = \frac{1}{n_\ell}\,\frac{\Gamma(\theta\,N_\ell+n_\ell)}{\Gamma(\theta\,N_\ell)\,\Gamma(n_\ell)} \equiv_a \frac{1}{z_\ell}\frac{\Gamma(\theta\,N_\ell+N\,z_\ell)}{\Gamma(\theta\,N_\ell)\;\Gamma(N\,z_\ell)}\\
&\equiv_a \quad z_\ell^{-1}\;\frac{\sqrt{\frac{2\pi}{\theta\,N_\ell+N\,z_\ell}}\;\left(\frac{\theta\,N_\ell+N\,z_\ell}{e}\right)^{\theta\,N_\ell+N\,z_\ell}}{\sqrt{\frac{2\pi}{\theta\,N_\ell}}\;\left(\frac{\theta\,N_\ell}{e}\right)^{\theta\,N_\ell}\;\sqrt{\frac{2\pi}{N\,z_\ell}}\;\left(\frac{N\,z_\ell}{e}\right)^{N\,z_\ell}}\\
&\equiv_a \quad z_\ell^{-1}\;\underbrace{\sqrt{\frac{N\,z_\ell}{\theta\,N_\ell+N\,z_\ell}}}_{\to 1}\;(\theta\,N_\ell+N\,z_\ell)^{\theta\,N_\ell}\;\left\{\underbrace{\left(1+\frac{1}{N}\,\frac{\theta\,N_\ell}{z_\ell}\right)^{N}}_{\to e^{\theta\,N_\ell/z_\ell}}\right\}^{z_\ell}\\
&\equiv_a \quad z_\ell^{-1}\;\underbrace{(\theta N_\ell/N+z_\ell)^{\theta\,N_\ell}}_{\to z_\ell^{\theta\,N_\ell}}\;\;e^{(\theta N_\ell/z_\ell)\,z_\ell}\equiv_a z_\ell^{\theta\,N_\ell-1}.
\end{aligned}$$

As $\theta\,N_\ell=\theta N_F\,\mu_\ell$, the result follows: The vector of vote shares $(x_1,\ldots,x_K)$ follows a Dirichlet distribution.

□

2.3.4 Large population limit for the multiparty model - revised

We obtained the Dirichlet distribution for the vote share of parties. In this limit, we took the number of floating voters, N, to infinity, while the number of zealots, N_F, stays finite. This limit seems to be strange – it might be more reasonable to assume that a certain fraction of the population are zealots, and the remaining fraction are floating voters, s.t. both groups – zealots and floating voters – tend to infinity in the same way if the population size tends to infinity. We approach th question of the appropriate scaling by investigating the diffusion limit of the model. In that, we basically extend proposition 2.3.2 and corollary 2.5 to the K-party model.

Thereto, we start with the master equations, as developed in exercise 3.2: Let $p_{i_1,\ldots,i_K}(t)=$

$P(X_1 = i_1, \ldots, X_K = i_K)$. Then,

$$
\begin{aligned}
\dot{p}_{i_1,\ldots,i_K} &= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right) \Big)\, p_{i_1,\ldots,i_K} \\
&\quad + \mu \sum_{j=1}^{K} \left(\sum_{k\neq j} (i_k+1) \left(\frac{i_j - 1 + N_j}{M}\right) p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k+1} \right) \\
&= -\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right) \Big)\, p_{i_1,\ldots,i_K} \\
&\quad + \mu \sum_{j=1}^{K} \left(\frac{i_j - 1 + \theta N_j}{M} \sum_{k\neq j} (i_k+1)\ p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1, i'_k = i_k+1} \right)
\end{aligned}
$$

From here, we go to th Fokker-Planck equation. Let $u(x_1, \ldots, x_K, t)$ probability density to find the sate $(x_1, \ldots x_K)$ at time t. Let $h = 1/N$, then

$$
x_\ell \approx i_\ell / N = i_\ell\, h, \qquad u(x_1, \ldots, x_K, t) \approx h\, p_{i_1,\ldots,i_K}(t)
$$

and

$$
n_i = \theta N_i / N, \quad m = M/N = 1 + \theta N_F / N.
$$

NOte that always $\sum_{\ell=1}^{K} x_\ell = 1$, that is, in all formulas we need to understand that

$$
x_K = 1 - \sum_{i=1}^{K-1} x_i, \qquad \partial_{x_K} = -\sum_{i=1}^{K-1} \partial_{x_i}.
$$

Therewith,

$$
\begin{aligned}
u_t &= h\,\dot{p} \\
&= h\Big[-\mu\Big(\sum_{j=1}^{K} i_j \left(1 - \frac{i_j + \theta N_j}{M}\right) \Big)\, p_{i_1,\ldots,i_K} \\
&\qquad + \mu \sum_{j=1}^{K} \left(\frac{i_j - 1 + \theta N_j}{M} \sum_{k\neq j} (i_k+1)\ p_{i'_1,\ldots,i'_K}\Big|_{i'_j = i_j - 1,\ i'_k = i_k+1} \right) \Big] \\
&= h^{-1}\Big[-\mu\Big(\sum_{j=1}^{K} x_j \left(1 - \frac{x_j + n_j}{m}\right) \Big)\, u(x_1, \ldots, x_K, t) \\
&\qquad + \frac{\mu}{m} \sum_{j=1}^{K} \left((x_j - h + n_j) \sum_{k\neq j} (x_k + h)\ u(x'_1, \ldots, x'_K, t)\Big|_{x'_j = x_j - h,\ x'_k = x_k + h} \right) \Big]
\end{aligned}
$$

Let us consider the first term. We write $u(x)$ instead of $u(x_1, \ldots, x_K, t)$ and obtain,

$$
\begin{aligned}
& -\mu\Big(\sum_{j=1}^{K} x_j \left(1 - \frac{x_j + n_j}{m}\right)\Big)\, u(x) = -\mu\, u(x) \left(\sum_{j=1}^{K} \left(x_j - \frac{x_j(x_j+n_j)}{m}\right)\right) \\
= \quad & -\mu\, u(x) \left(1 - \sum_{j=1}^{K} \frac{x_j(x_j+n_j)}{m}\right).
\end{aligned}
$$

Now we turn to the second term. We focus one arbitrary term in the last sum, and expand this term w.r.t. h. Let $j, k \in \{1, \ldots, K\}$, and $j \neq k$. We suppress all arguments of u expect x_j and x_k. Then, Taylor expansion yields

$$
\begin{aligned}
& (x_j - h + n_j)\,(x_k + h)\; u(x_j - h, x_k + h) \\
= \quad & \Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) + h\,(-\partial_{x_j} + \partial_{x_k})\Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) \\
& + \frac{h^2}{2}(\partial_{x_j}^2 - 2\partial_{x_k}\partial_{x_j} + \partial_{x_k}^2)\Big((x_j + n_j)\, x_k\; u(x_j, x_k)\Big) + \mathcal{O}(h^3) \\
= \quad & T_0 + h\, T_1 + \frac{h^2}{2} T_2 + \mathcal{O}(h^3).
\end{aligned}
$$

Note that the argument does not depend on the indices any more (before, we did increase/decrease some entry in the argument of u by h, now in the argument of u, h is zero). We can write $u(x)$, again in the understanding that $u(x) = u(x_1, \ldots, x_K, t)$.
<u>Let us return to the full sum, and only consider the zero order term T_0:</u>

$$
\begin{aligned}
\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} T_0 \quad &= \quad \frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} \Big((x_j + n_j)\, x_k\; u(x)\Big) \\
&= \quad u(x) \frac{\mu}{m} \left(\sum_{j=1}^{K} (x_j + n_j) \sum_{k \neq j} x_k\right) = u(x) \frac{\mu}{m} \sum_{j=1}^{K} (x_j + n_j)(1 - x_j) \\
&= \quad \mu\, u(x)\, \frac{\sum_{j=1}^{K} (x_j + n_j) - \sum_{j=1}^{K} (x_j(x_j + n_j))}{m} = \mu\, u(x)\, \frac{m - \sum_{j=1}^{K} x_j(x_j + n_j)}{m} \\
&= \quad \mu\, u(x) \left(1 - \sum_{j=1}^{K} \frac{x_j(x_j + n_j)}{m}\right).
\end{aligned}
$$

Now we proceed to term T_1:

$$\begin{aligned}
\frac{\mu}{m}\sum_{j=1}^{K}\sum_{k\neq j} T_1 &= \frac{\mu\,h}{m}\sum_{j=1}^{K}\sum_{k\neq j}(-\partial_{x_j}+\partial_{x_k})\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&= \frac{\mu\,h}{m}\Bigg[\sum_{j=1}^{K}\sum_{k\neq j}\partial_{x_k}\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\sum_{k\neq j}\partial_{x_j}\Big((x_j+n_j)\,x_k\;u(x)\Big)\Bigg]\\
&= \frac{\mu\,h}{m}\Bigg[\sum_{k=1}^{K}\sum_{j\neq k}\partial_{x_k}\Big((x_j+n_j)\,x_k\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\,\Big(\sum_{k\neq j}x_k\Big)\;u(x)\Big)\Bigg]\\
&= \frac{\mu\,h}{m}\Bigg[\sum_{j=1}^{K}\sum_{\ell\neq j}\partial_{x_j}\Big((x_\ell+n_\ell)\,x_j\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\,(1-x_j)\;u(x)\Big)\Bigg]\\
&= \frac{\mu\,h}{m}\Bigg[\sum_{j=1}^{K}\partial_{x_j}\Big((m-x_j-n_j)\,x_j\;u(x)\Big)\\
&\qquad -\sum_{j=1}^{K}\partial_{x_j}\Big((x_j+n_j)\,(1-x_j)\;u(x)\Big)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= \frac{-\mu\,h}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big((x_j+n_j)\,(1-x_j)-(m-x_j-n_j)\,x_j\Big)u(x)\Big\}\\
&= \frac{-\mu\,h}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big(n_j-(m-1)x_j\Big)u(x)\Big\}.
\end{aligned}$$

Remark: The deterministic drift is given by

$$\begin{aligned}
\dot{x}_j &= (x_j+n_j)\,(1-x_j)-(m-x_j-n_j)\,x_j && (2.16)\\
&= (x_j+n_j)\;-(x_j+n_j)\,x_j-mx_j\;+(x_j+n_j)\,x_j && (2.17)\\
&= n_j-(m-1)x_j && (2.18)
\end{aligned}$$

Hence, without noise,

$$\lim_{t\to\infty}x_j(t)=\frac{n_j}{m-1}=\frac{n_j}{\sum_i n_i}=\frac{N_j}{\sum_i N_i}.$$

The equilibrium is given by the relative fractions of zealots.

Last we handle term T_2.

T_2 has three (second) derivatives. We first consider $\partial^2_{x_j} + \partial^2_{x_k}$ (let us call this term $T_2^{(a)}$), and consider the mixed derivative (term $T_2^{(b)}$) only later.

$$\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(a)} \;=\; \frac{\mu\, h}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} (\partial^2_{x_j} + \partial^2_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big)$$

Up to a sign, this term parallels that of T_2, and hence

$$\frac{\mu}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(a)} \;=\; \frac{\mu\, h}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \bigg\{ \Big((x_j + n_j)\,(1 - x_j) + (m - x_j - n_j)\, x_j \Big) u(x) \bigg\}.$$

Last term:

$$\begin{aligned}
\frac{\mu}{m} \sum_{j=1}^{K} \sum_{k \neq j} T_2^{(b)} \;&=\; \frac{-2\mu\, h^2}{2m} \sum_{j=1}^{K} \sum_{k \neq j} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big) \\
&=\; \frac{-\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big) \\
&\quad + \frac{\mu\, h^2}{m} \sum_{j=1}^{K} \partial_{x_j^2} \Big((x_j + n_j)\, x_j \; u(x) \Big)
\end{aligned}$$

and therefore,

$$\begin{aligned}
\frac{\mu}{2\, m} \sum_{j=1}^{K} \sum_{k \neq j} T_2 \;&=\; \frac{\mu\, h^2}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \bigg\{ \Big((x_j + n_j)\,(1 + x_j) + (m - x_j - n_j)\, x_j \Big) u(x) \bigg\} \\
&\quad - \frac{\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big) \\
&=\; \frac{\mu\, h^2}{2\, m} \sum_{j=1}^{K} \partial^2_{x_j} \bigg\{ \Big(n_j + (m+1)\, x_j \Big) u(x) \bigg\} \\
&\quad - \frac{\mu\, h^2}{m} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big)
\end{aligned}$$

Therewith, we have the Fokker-Planck equation

$$\begin{aligned}
u_t \;&=\; \frac{-\mu}{m} \sum_{j=1}^{K} \partial_{x_j} \bigg\{ \Big(n_j - (m-1) x_j \Big) u(x) \bigg\} \\
&\quad + \frac{\mu}{2\, m\, N} \sum_{j=1}^{K} \partial^2_{x_j} \bigg\{ \Big(n_j + (m+1)\, x_j \Big) u(x) \bigg\} \\
&\quad - \frac{\mu}{m\, N} \sum_{j,k=1}^{K} (\partial_{x_j}\, \partial_{x_k}) \Big((x_j + n_j)\, x_k \; u(x) \Big)
\end{aligned} \tag{2.19}$$

In an alternative form:

Corollary 2.12 *The Fokker-Planck equation for Sano's multiparty-zealots model reads:*

$$\begin{aligned} u_t &= \frac{-\mu}{m}\sum_{j=1}^{K}\partial_{x_j}\Big\{\Big(n_j-(m-1)x_j\Big)u(x)\Big\} \\ &\quad +\frac{\mu}{2\,m\,N}\sum_{j=1}^{K}\partial^2_{x_j}\Big\{\Big((x_j+n_j)\,(1-x_j)+(m-x_j-n_j)\,x_j\Big)u(x)\Big\} \\ &\quad -\frac{\mu}{m\,N}\sum_{j,k=1,j\neq k}^{K}(\partial_{x_j}\,\partial_{x_k})\Big((x_j+n_j)\,x_k\;u(x)\Big) \end{aligned} \tag{2.20}$$

Remark 2.13 *From here – and that is the central insight in all these lengthy calculations – we have two different possibilities to proceed:*

- *We can return from n_i to $n_i = \theta N_i/N$ again. Thatis, we assume that the numbr of zealots stays constant, even if the population size tends to infinity. In this case, we will find (at least formally) our original result back again - the Dirichlet distribution as stated in theorem 2.11.*
- *Or, we can stay with the assumption that the number of zealots increase witht he population size in the same way as the population size. In that case, as usual, the noise term scales wit $1/N$, and becomes weaker and weaker for increasing N. The distribution will concentrate closely around the expected value, which is given by the zero of the drift term (eqn. (2.16)). In this case, a normal distribution with a variance that vanishes for N to infinity is an appropriate approximation.*

All in all, we have two different assumptions: Either the absolute number of zealots stay constant, or scales with the population size. The expectation of the distribution stays the same.
In the first case, however, the variance for the vote shares tend to some positive value, as the zealots are not sufficiently many to control the floating voters opinion. In the second case, the variance is of order $1/N$ and vanishes for $N\to\infty$.

Possibility 1: Weak limit

In this first approach, that will yield the Dirichlet distribution as before, in theorem 2.11, we choose

$$n_i = \theta N_i/N, \quad m = M/N = 1+\theta N_F/N$$

That is, we assume that the number fo zealots do not scale with the population size, but the effects of the zealots become weak (in the sense that they tend to zero with $1/N$). We ignore terms of order $\mathcal{O}(1/N^2)$ in the Fokker-Planck equation, and obtain

$$
\begin{aligned}
u_t &= \frac{-\mu}{m} \sum_{j=1}^{K} \partial_{x_j} \Big\{ \Big(n_j - (m-1) x_j \Big) u(x) \Big\} \\
&\quad + \frac{\mu}{2\,m\,N} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big((x_j + n_j)\,(1 - x_j) + (m - x_j - n_j)\, x_j \Big) u(x) \Big\} \\
&\quad - \frac{\mu}{m\,N} \sum_{j,k=1, j\neq k}^{K} (\partial_{x_j}\,\partial_{x_k}) \Big((x_j + n_j)\, x_k\; u(x) \Big) \\
&= \frac{-\mu}{N} \sum_{j=1}^{K} \partial_{x_j} \Big\{ \Big(\theta N_j - \theta N_F x_j \Big) u(x) \Big\} \\
&\quad + \frac{\mu}{N} \sum_{j=1}^{K} \partial^2_{x_j} \Big\{ \Big(x_j\,(1 - x_j) \Big) u(x) \Big\} - \frac{\mu}{N} \sum_{j,k=1, j\neq k}^{K} (\partial_{x_j}\,\partial_{x_k}) \Big(x_j\, x_k\; u(x) \Big) + \text{higher order terms}
\end{aligned}
$$

We can remove the population size N from the equation by resealing the time. For the stationary case, (it is sufficient that) the flux of the equation becomes zero, s.t. for $j \in \{1, \ldots, K\}$,

$$
0 = \Big(\theta N_j - \theta N_F x_j \Big) u(x) + \partial_{x_j} \Big\{ \Big(x_j\,(1 - x_j) \Big) u(x) \Big\} - \sum_{k=1, k\neq j}^{K} \partial_{x_k} \Big\{ x_j\, x_k\; u(x) \Big\}.
$$

At that point, we need to recall that $x_K = \sum_{\ell=1}^{K-1} x_\ell$, and $\partial_{x_K} = -\sum_{\ell=1}^{K-1} \partial_{x_\ell}$. Therewith, for $j = 0, \ldots, K-1$,

$$
\begin{aligned}
0 &= \Big(\theta N_j - \theta N_F x_j \Big) u(x) + \partial_{x_j} \Big\{ \Big(x_j\,(1 - x_j) \Big) u(x) \Big\} - \sum_{k=1, k\neq j}^{K-1} \partial_{x_k} \Big\{ x_j\, x_k\; u(x) \Big\} \\
&\quad - \sum_{\ell=1}^{K-1} \partial_{x_\ell} \Big\{ x_j \Big(1 - \sum_{i=1}^{K-1} x_i \Big)\; u(x) \Big\}
\end{aligned}
$$

We claim that $u(x_1, \ldots, x_K) = c \prod_{\ell=1}^{K} x_\ell^{a_\ell}$ is a solution (given that $\sum_j x_j = 1$), if we choose $a_j \in \mathbb{R}$ accordingly:

$$
\begin{aligned}
&-\Big(\theta N_j - \theta N_F x_j \Big) \prod_{\ell=1}^{K} x_\ell^{a_\ell} + \partial_{x_j} \Big\{ \Big(x_j\,(1 - x_j) \Big) \prod_{\ell=1}^{K} x_\ell^{a_\ell} \Big\} - \sum_{k=1, j\neq k}^{K} \partial_{x_k} \Big\{ x_j\, x_k\; \prod_{\ell=1}^{K} x_\ell^{a_\ell} \Big\} \\
&= -\theta N_j \prod_{\ell=1}^{K} x_\ell^{a_\ell} + \theta N_F x_j^{1+a_j} \prod_{\ell\neq j}^{K} x_\ell^{a_\ell} + (1 + a_j) \prod_{\ell=1}^{K} x_\ell^{a_\ell} - (2 + a_j) x_j^{1+a_j} \prod_{\ell\neq j}^{K} x_\ell^{a_\ell} \\
&\quad - \sum_{k=1, j\neq k}^{K} (1 + a_k) x_j^{1+a_j} \prod_{\ell\neq j}^{K} x_\ell^{a_\ell}
\end{aligned}
$$

This term is zero, if

$$
-\theta N_j + (1 + a_j) = 0 \quad \Rightarrow \quad a_j = \theta N_j - 1
$$

and the second condition reads

$$0 = \theta N_F - (2 + a_j) - \sum_{k=1,k\neq j}^{K} (1 + a_k).$$

If we use our choice for a_j, we have

$$\theta N_F - (2 + a_j) - \sum_{k=1,k\neq j}^{K} (1 + a_k) = -1 + \theta \sum_{i=1}^{K} N_i - \sum_{j=k}^{K} (1 + a_k) = -1.$$

FAST RICHTIG – WIR HABEN EINE 1 ZU VIEL IN DER ZWEITEN GLEICHUNG!!!!!!!
PROBLEM: WIR BERUECKSICHTIGEN NICHT, DASS summs $x_i = 1$!!!!!!

Possibility 2: Normal approximation ands deterministic limit

If we do not rescale n_i but assume that these parameters are independent of N, we can follow [**?**], and use asymptotic method to solve this equation (that is the same approach we used before in the two-party situation, in corollary 2.5). Note that

$$q_j = \frac{n_j}{\sum_{j=1}^K n_j} = \frac{n_j}{m-1}$$

are the stationary states for the flux; consequently, we introduce ($\varepsilon^2 = 1/N$)

$$x_i = q_i + \varepsilon y_i, \qquad \partial_{x_i} = \varepsilon^{-1} \partial_{y_i}.$$

Then,

$$\begin{aligned} n_j - (m-1)x_j &= -\varepsilon(m-1)y_j \\ x_j + n_j &= q_j + \varepsilon y_j + (m-1)q_j = mq_j + \varepsilon y_j \\ m - x_j - n_j &= m - (mq_j + \varepsilon y_j) = m(1-q_j) - \varepsilon y_j. \end{aligned}$$

Consider the stationary solution of the equation, and define $w(y) = u(q + \varepsilon y)$, then

$$\begin{aligned} 0 &= \sum_{j=1}^{K} \partial_{y_j} \Big\{ (m-1) y_j \, w \Big\} \\ &+ \frac{1}{2} \sum_{j=1}^{K} \partial_{y_j^2} \Big\{ \Big((mq_j + \varepsilon y_j)(1 - q_j - \varepsilon y_j) + (m(1-q_j) - \varepsilon y_j)(q_i + \varepsilon y_i) \Big) w(y) \Big\} \\ &- \sum_{j\neq k} \partial_{y_j} \partial_{y_k} \Big\{ (mq_j + \varepsilon y_j)\,(q_k + \varepsilon y_k)\, w(y) \Big\} \end{aligned} \tag{2.21}$$

In zero'th order, this equation yields

$$\begin{aligned} 0 &= \sum_{j=1}^{K} \partial_{y_j} \Big\{ (m-1) y_j \, w \Big\} \\ &+ \frac{1}{2} \sum_{j=1}^{K} \partial_{y_j^2} \Big\{ \Big(mq_j\,(1 - q_j) + m(1-q_j)\, q_i \Big) w(y) \Big\} \\ &- \sum_{j\neq k} \partial_{y_j} \partial_{y_k} \Big\{ mq_j\, q_k \, w(y) \Big\} \end{aligned}$$

$$0 \quad = \quad \left(1-\frac{1}{m}\right)\sum_{j=1}^{K}\partial_{y_j}\left\{y_j\,w\right\}+\sum_{j=1}^{K}\partial_{y_j^2}\left\{q_j\,(1-q_j)w(y)\right\}-\sum_{j\neq k}\partial_{y_j}\partial_{y_k}\left\{q_j\,q_k\,w(y)\right\}$$

If we define the matrix Q as the diagonal matrix with q_i in the i'th entry, and $q=(q_1,\ldots,q_K)^T$, with

$$A = Q - q\,q^T$$

we may write

$$0 \quad = \quad \left(1-\frac{1}{m}\right)\sum_{j=1}^{K}\partial_{y_j}\left\{y_j\,w\right\}+\sum_{j,k}\partial_{y_j}\partial_{y_k}\left\{Q_{j,k}\,w(y)\right\}$$

NACHSTER SCHRITT: SIEHE BACHELOR ARBEIT KITILAKIS!!!! AUSFUEHREN!!!!
Standard calculations (see e.g. [**?**] or the WKB ansatz) show that for an Ansatz

$$w(y) = \exp\left(-\frac{1}{2}\sum_{i,j}B_{i,j}y_iy_j\right)$$

the PDE superimposes a condition on B^{-1},

$$B^{-1}\left(1-\frac{1}{m}\right)+\left(1-\frac{1}{m}\right)B^{-1} = 2\,A$$

Here we need to take this equation with *grano salis*, as A is not invertible ($\mathbf{e}^TA=0$), and so is B (consequence of mass conservation, that is, $\sum y_j = 0$). In any case, we find that a normal distribution with variance

$$\Sigma = \left(1-\frac{1}{m}\right)^{-1}\,[Q-q\,q^T]$$

satisfies the PDE. Hence,

$$\frac{1}{N}(X_1,\ldots,X_K)\approx_a q + N^{-1/2}\mathcal{N}(0,\mathrm{SI}\,[Q-q\,q^T]), \qquad \mathrm{SI}=(1-\frac{1}{m})^{-1}. \tag{2.22}$$

respectively

$$N^{1/2}\,(X_1/N-q_1,\ldots,X_K/N-q_K)\sim_a\mathcal{N}(0,\mathrm{SI}\,[Q-q\,q^T]).$$

If we introduce $\widetilde{m}=\sum_i N^i/N$, then $m = 1+\widetilde{m}$, and

$$\mathrm{SI} = (1-\frac{1}{m})^{-1} = (1-\frac{1}{1+\widetilde{m}})^{-1} = \frac{\widetilde{m}+1}{\widetilde{m}} = 1+\frac{1}{\widetilde{m}}.$$

In particular, if $1/\widetilde{m}=0$ we obtain the normal approximation of the multinomial distribution. We can identify the strength of the zealot's effect by the size of $1/\widetilde{m}$.

Parameter estimation

We return to the Dirichlet distribution for the vote shares. How can we estimate the parameter from the data? We observe in the i'th district (from m districts, $i = 1, \ldots, m$) a certain number of voters V^i, structured by parties

$$V_1^i, \ldots, V_K^i, \qquad V^i = \sum_{\ell=1}^{K} V_\ell^i.$$

The model assumes that the voters of party ℓ, $\ell = 1, \ldots, K$, in district i can be separated in fixed voters N_ℓ^i and swing voters X_ℓ^i,

$$V_\ell^i = N_\ell^i + X_\ell^i.$$

Let $N^i = \sum_{\ell=1}^K X_\ell^i$ the total number of swing voters in that district, and $N_F^i = \sum_{\ell=1} K N_\ell^i$ that of fixed voters.

We assume that the fraction of fixed voters/swing voters are stable over the districts. That is, N^i/V^i is (approximately) independent on the district (on i). Let p denote the fraction of floating voters,

$$p = N^i/V^i, \qquad \Rightarrow \qquad 1 - p = N_F^i/V^i.$$

Furthermore, we assume that $N_\ell^i/V^i = \mu_\ell$ is independent of the vote district. With these assumptions, we can write the vote shares as

$$\begin{aligned} z_\ell^i &:= V_\ell^i/V^i = N_\ell^i/V^i + X_\ell^i/V^i = N_\ell^i/V^i + X_\ell^i/V^i \\ &= (N_F^i/V^i)\ N_\ell/N_F \ + \ (N^i/V^i)\ X_\ell^i \\ &= (1-p)\mu_\ell \ + \ p\ x_\ell^i \end{aligned}$$

where (note that $N_F^i = (1-p)V^i$)

$$(x_1^i, \ldots, x_K^i) \sim \mathrm{Dir}(\theta N_F^i \mu_1, \ldots, \theta N_F^i \mu_K) = \mathrm{Dir}(\theta(1-p)V^i\mu_1, \ldots, \theta(1-p)V^i\mu_K).$$

All in all, we have

Measurements:

z_ℓ^i	votes share of party ℓ in district i, where $\ell = 1, \ldots, K$, and $i = 1, \ldots m$
V^i	number of (active) voters in district i

Parameters:

$p \in [0,1]$	fraction of floating voters, identical for all districts
$\theta \in [0,1]$	fraction of active/influencing zealots, identical for all districts
$\mu_\ell \in [0,1]$	fraction of zealots supporting party ℓ, $\ell = 1, \ldots, K$, identical

We have, all in all, $m \times (K-1)$ data points (the vote shares ad up to 1), and $K - 1 + 2 = K + 1$ parameters. We aim at a maximum likelihood estimation,

$$(x_1, \ldots, x_K) = (z_\ell^1 - (1-p)\mu_1, \ldots, z_\ell^K - (1-p)\mu_K)/p \sim \mathrm{Dir}(\theta(1-p)V^i\mu_1, \ldots, \theta(1-p)V^i\mu_K).$$

That is,

$$\begin{aligned} \mathcal{L}(z_\ell^i | V^i, p, \theta, \mu_\ell) &= \prod_{i=1}^m \frac{\prod_{\ell'=1}^K \Gamma(\theta(1-p)V^i\,\mu_{\ell'})}{\Gamma(\theta(1-p)V^i)} \prod_{\ell=1}^K \left(\frac{z_\ell^i - (1-p)\mu_\ell}{p}\right)^{\theta(1-p)V^i\mu_\ell - 1} \\ &= \prod_{i=1}^m \frac{\prod_{\ell'=1}^K \Gamma(\theta(1-p)V^i\,\mu_{\ell'})}{\Gamma(\theta(1-p)V^i)\ p^{\theta(1-p)V^i - K}} \prod_{\ell=1}^K \left(z_\ell^i - (1-p)\mu_\ell\right)^{\theta(1-p)V^i\mu_\ell - 1} \end{aligned}$$

Exercises

Exercise 3.2: Determine the master equations for Sano's model.
Solution of Exercise 3.2 Let $p_{i_1,\ldots,i_K}(t) = P(X_1 = i_1,\ldots,X_K = i_K)$. Then,

$$
\begin{aligned}
\dot p_{i_1,\ldots,i_K} &= -\mu\Big(\sum_{j=1}^K i_j\,\left(1-\frac{i_j+\theta N_j}{M}\right)\Big)\,p_{i_1,\ldots,i_K}\\
&\quad +\mu\sum_{j=1}^K\left(\sum_{k\neq j}(i_k+1)\left(\frac{i_j-1+\theta N_j}{M}\right)p_{i'_1,\ldots,i'_K}\Big|_{i'_j=i_j-1,i'_k=i_k+1}\right)\\
&= -\mu\Big(\sum_{j=1}^K i_j\,\left(1-\frac{i_j+\theta N_j}{M}\right)\Big)\,p_{i_1,\ldots,i_K}\\
&\quad +\mu\sum_{j=1}^K\left(\frac{i_j-1+\theta N_j}{M}\sum_{k\neq j}(i_k+1)\;p_{i'_1,\ldots,i'_K}\Big|_{i'_j=i_j-1,i'_k=i_k+1}\right).
\end{aligned}
$$

2.3.5 Zealot models and data

We compare the prediction of the zealot model with election data. We address the outcome of parties, as well as the turnout rates. Theorem 2.11 can be used to obtain the (equilibrium) distribution of the vote share for a given party by dividing the voters into supporters of that party, and supporters of the other parties. That is, we focus on the marginal distribution of the focal party's vote share. Also the turnout rate can be considered – here the two groups are voters and non-voters. The Dirichlet distribution in case of two groups coincides with the beta distribution. We have the following corollary, for which we use the notation of Theorem 2.11.

Corollary 2.14 *According to the large population limit of the zealot model, the vote share of a given party A follows a beta distribution with mean μ and variance σ,*

$$\mu = \frac{n_A}{n_A+n_B},\qquad \sigma^2 = \frac{\mu(1-\mu)}{N}\,\left(1+\frac{1}{n^A+n^B}\right). \tag{2.23}$$

Several papers investigate the vote share or turnout rate distribution,e.g. in the US [13, 4], France [3], Brazil [6], Mexico [20], with different methods. One striking result for some approaches [3] is the appearance of characteristic shapes, that resemble each other after a Z-transformation (subtract the mean, divide by the standard deviation). These findings are a first hint, that indeed a fixed family of distribution describes the votes share. In a certain range of parameters (given by μ and σ), it is indeed possible to overlay different beta-distributions by a Z-transformation. If, however, the parameters are too extreme, beta distributions will look completely different.

We use the German election results to compare our theoretical predictions with data. Here we assume that different election districts are independent repetitions of the same probability experiment. That assumption is, of course, only approximately appropriate, as different election districts have different social structures, that in turn influence the election results. We expect

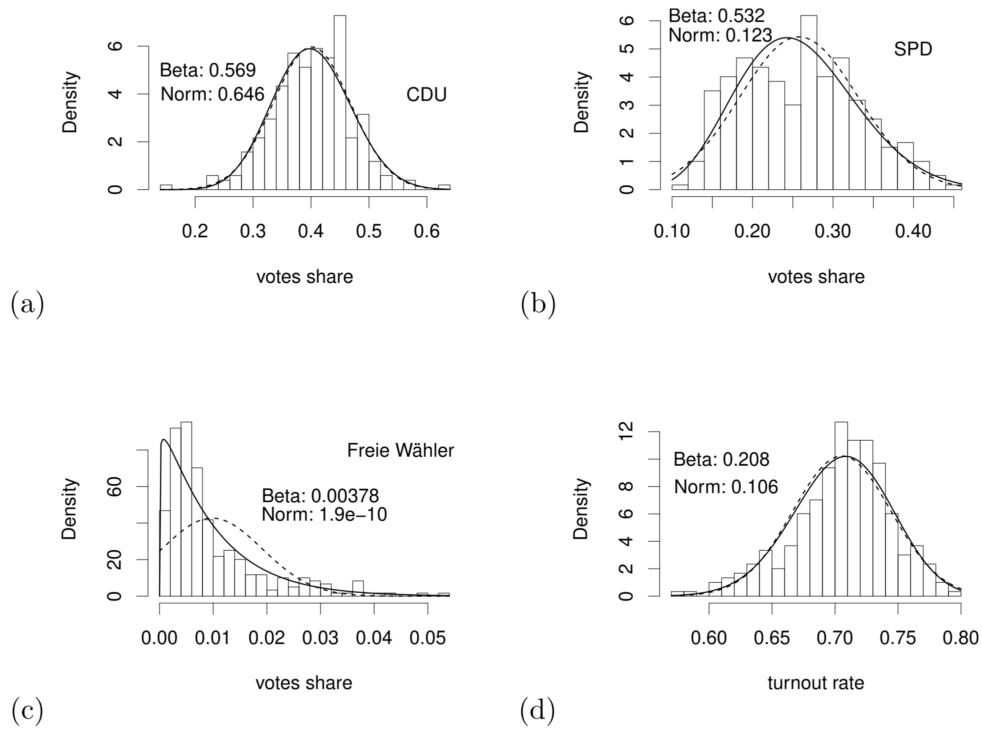


Figure 2.2: Histogram of votes shares with density of beta-distribution (solid lines) and normal distribution (dashed line). The p-values for the Kolmogorov-Smirnov test on beta distribution and normal distributions are given. (a) Christ democrats (CDU, 2013), (b) Socialists (SPD, 2013), (c) Freie Wähler, a small conservative party, (2013), (d) turnout rate (2013),

that this effect adds to the variance between the election districts. If the variance introduced by the voter dynamics is reasonably larger than that by the social background, we can nevertheless hope to find back the beta distribution. And indeed, as we can see in Figure 2.2 (panels a,b), the election results for some big tent parties agree well with a beta distribution. This visual impression is supported by the Kolmogorov-Smirnov-test, that clearly indicates that the hypothesis (data are beta distributed) cannot be rejected. On the other hand, the data basically resemble a normal distribution, with given mean and variance. As the beta and the normal distribution resemble each other in that range of parameters, we could argue that we basically see the consequence of the law of large numbers. And indeed, the Kolmogorov-Smirnov test suggests that both distribution describe the data equally well. The superiority of the model over the generic normal distribution is visible for a small party (Figure 2.2, panel c). The results of that party are localized at zero. The beta distribution rather suited to characterize also these data, while the normal distribution is not. Also the turnout-rate can be described by the beta-distribution, that is, by the zealot-model.

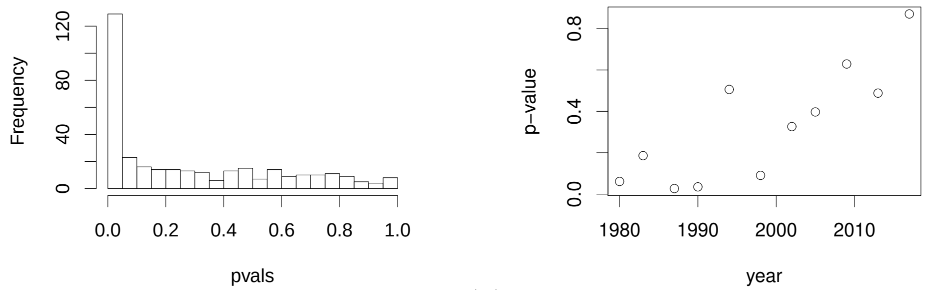
If we inspect the histogram of all p-values (Kolmogorov-Smirnov test, Figure 2.3, panel a) we get the impression that there are two different populations, one that indeed follows the beta-distribution, and one with small p-values, that distinctively are not beta-distributed. This second population indicates that the model is not complete. It might be, that neutral models are too simple to cover all aspects, we might have strategic thinking in the voting behavior (we will discuss this aspect later, in Section **??**), or it might be, that we observe transient behavior; the prediction of beta-distribution is only valid for the invariant state. The last conjecture is supported by the corresponding time series for the Green party, which started with a small p-value that increased later on (Figure 2.3, panel b).
Also the size of the party influence the p-value: The linear model

$$\text{p-value} \sim \text{vote share}$$

indicates a significal influence (point estimate: 0.44, significance $2.9 \cdot 10^{-5}$); however, this factor does only compare a small amount of the variability ($R^2 = 0.05$).
Perhaps a more pronounced reason is spatial heterogeneity (as reported in, e.g., [25]). To better understand this idea, we check the result of “Die Linke” for the election in Germany from 2017. The overall p-value for the beta-distribution was $6.2 \cdot 10^{-6}$. If we restrict to the 16 states, we have a mean p-value of 0.6 (minimum is 0.007, variance is 0.11). To confirm our impression that the p-values on the states are indeed higher than the overall p-value (that the effect is not simply the smaller number of data), we resample for each state the same number of election districts, and predict the resampled p-values by the p-value of the state (linear model without intercept). The point estimate for the proportionality constant is 0.79,, and the 95% confidence interval $[0.78, 0.8]$. That is, the proportionality ocnstant is less one, indicating that the qwithin-state p-value is indeed larger than the p-value we obtain by resampling the corresponding number of data form the overall election results. We clearly can explain the small p-values for some party by spatial effects.

Time course of the social interaction coefficient In Definition 2.6, we introduced the social interaction coefficient (SI) by

$$SI = 1 + \frac{1}{n^a + n^b}.$$



(a) (b)

Figure 2.3: Test of the vote-shares for all beta-distribution (all parties and all elections 1953-2017 in the FRG) by the Kolmogorov-Smirnov test. (a) Histogram of all p-values (b) p-values of the Green Party over time.

The larger SI, the smaller is the effect of the zealots in comparison with internal communication effects. We aim to estimate SI using the outcome of elections in several districts. As it is, the theory developed for the SI is based on a division of the population into two groups A and B. We consider the turnout rate (the two groups are voters – group A – and non-voters – group B) and the vote share for a given party (here the reference population is only the voting subpopulation, which is divided into supporters of the given party – group A – versus non-supporters – group B). We can simply estimate the population size N within one district, the expected fraction μ of the population that belong to group A, and the variance σ^2 between the different districts. To be more precise, consider the turnout rate, and let X_i denote the number of group-A individuals (voters) and Y_i th number of group-B individuals (non-voters) for districts $1,\dots,\ell$. Then,

$$\hat{N} = \frac{1}{\ell}\sum_{i=1}^{\ell}(X_i+Y_i), \quad \hat{\mu}_i = \frac{X_i}{X_i+Y_i} \quad \hat{\mu} = \frac{1}{\ell}\sum_{i=1}^{\ell}\hat{\mu}_i, \quad \hat{\sigma}^2 = \frac{1}{\ell-1}\ell\sum_{i=1}^{\ell}(\hat{\mu}_i-\hat{\mu})^2.$$

As the variance is given (see Corollary 2.5)

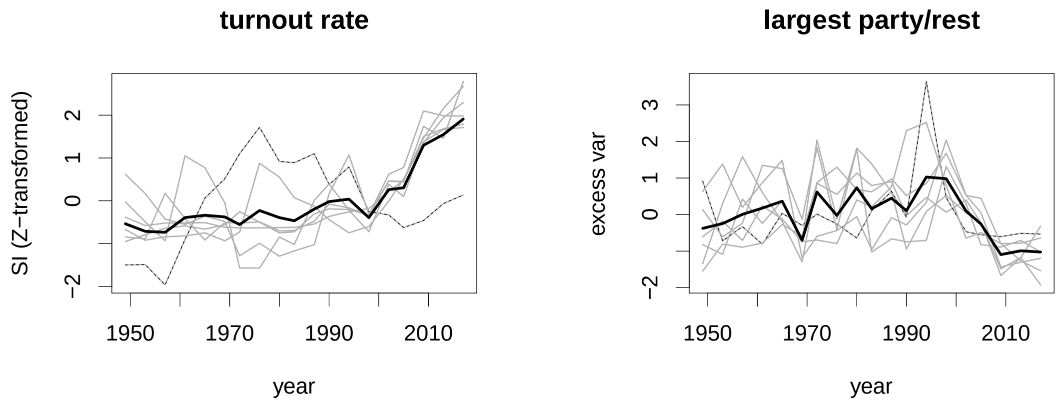
$$\sigma = \frac{\mu(1-\mu)}{N}\left(1+\frac{1}{n^a+n^b}\right)$$

we find the estimator for SI

$$\widehat{SI} = \frac{\hat{\sigma}^2\,\hat{N}}{\hat{\mu}(1-\hat{\mu})}. \tag{2.24}$$

XXXXXXXXXXXXXXXXXXXXXXXXX
here the results form simul/noisyVoter
chanage of CI over time, increase since 2000. XXXXXXXXXXXXXXXXXXXXXXXXXXX

IS it possible to reformulate the results (data as rank-order stats) by Fenner/Levene/Loizou (at least 2 papers, UK election results, arXiv 1609.04282, 1703.10548) in terms of the beta distribution???



(a) (b)

Figure 2.4: The SI estimated for the turnout rate (a), and the vote share of the of the dominating party versus the rest (b). Grey lines: SI for larger cities (Hamburg, Bremen, Munich, Cologne and Bonn (pooled), Frankfurt, Stuttgart, Ruhr area) separated, normalized by the z-transformation. Bold line: average of the gray lines; Munich marked by the dash line.

2.3.6 Mauro Mobilia

Three party voter model - how long does it take to equilibrium? Possibility to eliminate an opinion. Again, how long does it take???
Uses the method by Grasman + Herwarden.
Do we learn someting interestign?????
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX hier weiter XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

2.4 Election and the Infinity-allele Moran model

2.4.1 Model

Tellier/Hoesel/Mueller

2.4.2 Deme-model

computations.

2.5 Elections and the Potts model

In the present section we present a statistical physics approach to describe election results. The approach originated in the investigation of spin systems: An atom may assume the state "spin up" or "spin down". The spins will change according to the influence of an external magnetic

field, and by the influence of neighboring spins. Several authors [30, 16, 33] point out that this model bears some similarities with the situation of voters, who are influenced by other voters, and perhaps are also influenced by some global information, broadcast by mass media. Let us find out how the stochastic physics approach works, and what we can learn form it. The model formulation, however, is completely different from the formulation of the models used above: E.g. in the voter model, we started off with mechanisms that have been re-formulated as a stochastic process. The resulting process has an invariant measure. We used that invariant measure to analyze data. In the statistical physics approach, we directly start with a distribution. No dynamics is required. Only later, we aim to establish mechanisms formulated in terms of a stochastic process, which has that distribution as an invariant measure.
We follow Nicolao et al. [33], who constructs a q-state Potts model that allows to analyze election data. In the Potts model, the notion of entropy is central. Therefore, let us briefly discuss some basic facts related to entropy.

2.5.1 Prelude: Shannon Entropie

The concept of entropy first appeared in the context of thermodynamics, similar to the definition of the temperature. Only slowly it became clear that temperature and entropy are different of nature (though both are connected with the disorder of a system [22]).

For us it is sufficient to consider a discrete system. Our system has K states, named by $s_1, \ldots, s_K$. There is a probability measure R that describes the state of the system. We find state s_i with probability $R(s_i)$.

Definition 2.15 *Let the real constant k be strictly positive. The Shannon entropy for the probability measure R is given by*

$$S(R) = -k \sum_{i=1}^{K} R(s_i)\, \ln(R(s_i)). \tag{2.25}$$

Note that in mathematics and mathematical physics, usually $k = 1$, while in thermodynamics, $k = k_B$ is the Boltzmann's constant. Luckily, in the present context, we do not need the Boltzmann's constant but use $k = 1$.

Since $R(s_i) \in [0, 1]$, we have $S(R) \geq 0$.

Choosing the definition $0\, \ln(0) := \lim_{x\to 0} x \ln(x) = 0$, we find that the entropy is zero if $R(s_{i_0}) = 1$ for some $i_0 \in \{1, \ldots, K\}$, and $P(s_i) = 0$ for $i \neq i_0$. If the state is (almost) sure and not random, the entropy is zero. If we have $R(s_i) = 1/K$ for all possible states, we find $S = \ln(K)$. All other distributions have a smaller entropy than $S = \ln(K)$ (see exercise 5.1).

As an intuitive example, considere a classroom with N classmates. Each of them can sit on his/her chair and concentrate, or can be jumped up and babble. If the teacher is there, then (hopefully) all children concentrate. The state is known. Entropy is zero. If the teacher leaves the classroom, some children will be still on his/her chair, and some have jumped up, it will be noisy - the state is unknown, the entropy is large. The less we know the larger the entropy: If the teacher is there, we know the state of all children, if the teacher is gone we have no idea what the children will do.

The entropy quantifies the degree of uncertainty about the system. The less we know about the system the larger the entropy. We can also say: the less ordered is the system.

The following result can give us some further intuition for this a interpretation of the entropy as a measuring uncertainty. We play a game with a partner. The partner selects a state s_{i_0} from our K states according to the random measure R. It is our task to find out which s_{i_0} he/she did chose. We are only allowed to ask yes/no questions. And, of course, we want to be clever, that is, the expected number of questions to ask should be minimal.

One possibility is to go through all single states. That is, we can ask "did you choose s_1? Did you choose s_2? etc. At the end, we get the desired answer. The expected number of questions is in that case

$$E(\text{number of questions}) = \sum_{i=1}^{K} i\, P(\text{selected state is } s_i) = \sum_{i=1}^{K} i\, R(s_i).$$

This strategy cannot be really clever - particularly, re-numbering the states s_i yields to different results. Clearly, a good strategy to ask questions should not depend on the order in which the states are numbered.
We can do better by a bi-partition algorithm. Let $\Omega = \{s_1, \ldots, s_K\}$ denote the state space. We divide Ω in two disjoint subsets of equal probability,

$$\Omega = A_1 \dot\cup A_2, \qquad R(A_1) = R(A_2) = 1/2.$$

As we are in a discrete situation, the equality $R(A_1) = R(A_2) = 1/2$ may be impossible to fulfill. For simplicity, we assume that we can always divide sets in two sets of equal probability.
In any case, we now can ask "is your state in set A_1?". According to the answer, we can focus on A_1 or A_2 only. We multiply all probabilities within the focal set by 2, and are in the same situation as before: The sum of the probabilities in the focal set is 1. It is clear how to proceed: We again divide the interesting subset into two subsets of equal probability, and divide, while multiplying the probabilities by 2, and divide, while multiplying the probabilities by 2,..., until we are left with a subset of size one only. This subset contains the desired state.
How many question do we need to ask? Let us assume our partner did choose the state s_{i_0}. The probability for this state is $R(s_{i_0})$. If we had to divide the original set n times, we did multiply the probability n times by 2 to always guarantee that the sum of all probabilities in the sequence of subsets is always 1. Hence, $2^n\, R(s_{i_0}) = 1$, and we required $n \approx -\log_2(R(s_{i_0}))$ steps. The expected number of questions thus reads

$$-\sum_{i=1}^{K} R(s_i)\, \log_2(R(s_i)) = -\sum_{i=1}^{K} R(s_i)\, \ln(R(s_i))/\ln(2) = S(R)/\ln(2).$$

The entropy is proportional to the number of questions necessary to determine the unknown state that our partner did chose, and in that, a measure about our uncertainty.

Exercises

Exercise 5.1: Consider a probability measure on K discrete states $s_1,\ldots,s_K$. Show that $R(s_i) = 1/K$ maximizes the entropy.
Hint: Optimization with constrains; use Lagrange multipliers.

Solution of Exercise 5.1 Let $p_i = R(s_i)$. The entropy is a function on $p_1, \ldots, p_K$ given by

$$S(p_1, \ldots, p_K) = -\sum_{i=1}^{K} p_i \ln(p_i).$$

To find the maximizing probabilities, we use a Lagrange multiplier λ to ensure the condition $\sum_{i=1}^{K} p_i = 1$, and maximize

$$f(p_1, \ldots, p_K, \lambda) = -\sum_{i=1}^{K} p_i \ln(p_i) + \lambda(1 - \sum_{i=1}^{K} p_i).$$

Hence,

$$0 = \frac{\partial}{\partial p_i} f(p_1, \ldots, p_K, \lambda) = \ln(p_i) + 1 - \lambda$$

and thus

$$p_i = p_j = e^{\lambda - 1}.$$

All probabilities are the same. As the probabilities sum up to one, they are given by

$$p_i = 1/K.$$

The probability measure that maximizes the entropy assigns an equal probability for each state. This probability function gives the least information about an randomly selected item.

2.5.2 Abstract framework: Potts model

The Potts model was proposed by the mathematician Renfrey Potts in his dissertation theses in 1952 as a simplification of the Ising spin model [36]. Indeed, it is simple enough to allow for some analytical results, but complicated enough to allow for applications to real world problems. We introduce the Potts model in the way Nicolau et al. proposed in [33].

Model 2.16 ***q*-state Potts model.** *Consider a population of N individuals/voters. Each of these individuals can be in favor of one party/candidate out of q parties/candidates. Let*

$$\Sigma_N = \{1, \ldots, q\}^N.$$

If we would know the opinion of each individuals, which we number by $1, \ldots, N$, then $\sigma \in \Sigma_N$ would characterize the state of the population: Individual i would have opinion σ_i. As we have only vague information, the state of the population is given by some probability function R on Σ_N. However, even R is unknown. What we do know is the result of e.g. polls, that give us some information about observable $F_\ell : \Sigma \to \mathbb{R}$. That is, the expectations

$$E_R(F_\ell(.)) \stackrel{def}{=} \sum_{\sigma \in \Sigma_N} F_\ell(\sigma)\, R(\sigma)$$

are known for M (independent) functions $F_1, \ldots, F_M : \Sigma_N \to \mathbb{R}$.
(1) We construct a probability measure Q on the power set of Σ_N, that approximates/interpolates the unknown R. If Q is a good approximation, we can compute all magnitudes of interest using Q instead of R. Of course, Q should give us the same values

for the observables as R, that is,

$$E_Q(F_\ell(.)) \overset{def}{=} \sum_{\sigma\in\Sigma_N} F_\ell(\sigma)\, Q(\sigma) \overset{!}{=} E_R(F_\ell(.)).$$

In the last formula, the first equal sign is just the definition of the expectation, the second equal sign is a condition/requirement for Q.
(2) There are, of course, many probability measures that satisfy that condition. If the expectations of the M observables is all we know, we should use the probability measure with the maximal disorder, that is, Q should maximize an entropy. The classical entropy to use is the Shannon entropy,

$$S(Q) = -\sum_{\sigma\in\Sigma_N} Q(\sigma)\, \ln(Q(\sigma)).$$

We call the probability measure Q that satisfies these two conditions the Boltzmann distribution.

Notation:
Later we will denote by σ elements in Σ_N. That is, σ is a vector that assigns to each individual i, $i = 1,\ldots,N$, a preference $\sigma_i \in \{1,\ldots,q\}$. In contrast, $\hat{\sigma}$ denotes a random variable, described by a certain probability measure Q on Σ_N,

$$P_Q(\hat{\sigma} = \sigma) = Q(\sigma).$$

Similarly, $\hat{\sigma}_i$ is the i'th entry of the random vector $\hat{\sigma}$, which represents the (uncertain) state of individual i; the distribution of $\hat{\sigma}_i$ is the corresponding marginal distribution of $\hat{\sigma}$.
In that, $E_Q(F_\ell(\hat{\sigma}))$ is the expected value of $F_\ell(\hat{\sigma})$, where the distribution of $\hat{\sigma}$ is Q. As we will only consider one probability distribution denoted by Q, we drop the index Q and write $E(F_\ell(\hat{\sigma}))$ instead of $E_Q(F_\ell(\hat{\sigma}))$, or $P(\hat{\sigma} = \sigma)$ instead of $P_Q(\hat{\sigma} = \sigma)$, etc. Wit this notation, we find for example

$$E(F(\hat{\sigma})) \;=\; \sum_{\sigma\in\Sigma_N} F(\sigma)P(\hat{\sigma} = \sigma) \;=\; \sum_{\sigma\in\Sigma_N} F(\sigma)Q(\sigma).$$

As we find out in the next theorem, the distribution Q for the Potts Model can be explicitly determined.

Theorem 2.17 *The Boltzmann distribution Q that maximizes the Shannon entropy under the restriction that $E(F_\ell(\hat{\sigma})) = f_\ell$ for $\ell = 1,\ldots,M$, where $F_\ell : \Sigma_N \to \mathbb{R}$, and $f_\ell \in \mathbb{R}$ are prescribed, is given by*

$$Q(\sigma) = \frac{e^{-H(\sigma)}}{Z}, \qquad Z = \sum_{\sigma\in\Sigma_N} e^{-H(\sigma)} \tag{2.26}$$

with

$$H(\sigma) = -\sum_{\ell=1}^{M} \lambda_\ell\, F_\ell(\sigma), \tag{2.27}$$

where the Lagrange multipliers $\lambda_1, \ldots, \lambda_M$ are determined by the conditions

$$\sum_{\sigma \in \Sigma_N} F_\ell(\sigma) \frac{e^{\sum_{\ell=1}^M \lambda_\ell F_\ell(\sigma)}}{Z} = f_\ell, \qquad \ell = 1, \ldots, M. \tag{2.28}$$

Note that it is not clear if the conditions (2.28) can be satisfied. If, for example, f_ℓ is not in the range of F_ℓ, then this condition cannot be realized. Furthermore, it is also not clear if there is a unique solution. In any case, if there is a Boltzmann distribution, we have a way to explicitly construct it.

Proof: (of theorem 2.17) Maximization under constrains can be done using Lagrange multipliers; we consider the functional

$$\mathcal{S}(Q, \lambda_1, \ldots, \lambda_{M+1}) = S(Q) + \sum_{i=1}^M \lambda_\ell \Big(E(F_\ell(\hat\sigma)) - f_\ell \Big) + \lambda_{M+1} \Big(\sum_{\sigma \in \Sigma_N} Q(\sigma) - 1 \Big).$$

The last term guarantees that $\sum_{\sigma \in \Sigma_N} Q(\sigma) = 1$. Next we use that Q is defined by the real number $Q(\sigma) \in [0, 1]$ for each given $\sigma \in \Sigma_N$. If we take the derivative w.r.t. a specific component $Q(\sigma_0)$ (that is, we select $\sigma_0 \in \Sigma_N$ and aim to obtain the value of Q for this specific state), we find (recall $E(F(\hat\sigma)) = \sum_{\sigma \in \Sigma_N} F(\sigma) Q(\sigma)$, that is, $\frac{\partial}{\partial Q(\sigma_0)} E_Q(F(\sigma)) = F(\sigma_0)$)

$$\begin{aligned} \frac{\partial}{\partial Q(\sigma_0)} \mathcal{S}(Q, \lambda_1, \ldots, \lambda_{M+1}) &= -\ln(Q(\sigma_0)) - 1 + \sum_{i=1}^M \lambda_\ell F_\ell(\sigma_0) + \lambda_{M+1} \\ &= -\ln(Q(\sigma_0)) - 1 - H(\sigma_0) + \lambda_{M+1}. \end{aligned}$$

That is, $\frac{\partial}{\partial Q(\sigma_0)} \mathcal{S} = 0$ implies

$$Q(\sigma_0) = e^{\lambda_{M+1} - 1 - H(\sigma_0)} = e^{\lambda_{M+1} - 1}\, e^{-H(\sigma_0)}.$$

λ_{M+1} is defined by the requirement $\sum_{\sigma \in \Sigma_N} Q(\sigma) = 1$, that is,

$$(e^{-\lambda_M + 1})^{-1} = \sum_{\sigma \in \Sigma_N} e^{-H(\sigma)} = Z.$$

$\square$

As the background of the model is quantum physics and the Ising model, mostly $H(\sigma)$ is referred to as "Hamiltonian" and Z is named "partition function". We will discuss some intuitive interpretations of the partition function below.

2.5.3 First example: Expectations known

As a first example, we assume that the poll/our foreknowledge only addresses the opinion of each person. For each person, the information obtained by the poll is the number of a candidate, that is, a number between 1 and q. We measure the frequency of individuals voting for candidate i. We translate the result into observables. Instead of asking: "For which candidate will you vote"? We ask q questions: "Will you vote for candidate 1?", "Will you vote for candidate

2?",... We do not store information about individuals but only memorize the overall fraction of votes. That is, we introduce

$$F_\ell(\sigma) = \frac{1}{N}\sum_{i=1}^{N} \delta(\sigma_i, \ell), \qquad \ell = 1, \dots, q.$$

Here, $\delta(\sigma_i, \ell)$ denotes the Kronecker-function, which is 1 if $\sigma_i = \ell$ and 0 otherwise. That is, we obtain the empirical value $\overline{x}_\ell$ for $E(F_\ell(\hat\sigma))$, which can be reformulated as the probability that a randomly selected individual will vote for candidate $\ell \in \{1, \dots, q\}$. These questions/observables yield no idea about correlations. As our Q maximizes the entropy (the disorder), we might expect that the states $\hat\sigma_i$ are independent w.r.t. the random measure Q. That is, we might expect that Q yields the multinomial distribution with probability $\overline{x}_\ell$ for outcome ℓ. The next proposition shows indeed that this intuitive idea leads to the correct insight.

Proposition 2.18 *Let $F_\ell(\sigma) = \frac{1}{N}\sum_{i=1}^N \delta(\sigma_i, \ell)$ and require $E(F_\ell(\hat\sigma)) = \overline{x}_\ell$ for $\ell = 1, \dots, q$. Define $N_\ell(\sigma) = \sum_{i=1}^N \delta(\sigma_i, \ell)$. Then*

$$Q(\sigma) = \prod_{\ell=1}^{q} \overline{x}_\ell^{N_\ell(\sigma)}$$

and

$$P(N_1(\hat\sigma) = n_1, \dots, N_q(\hat\sigma) = n_q) = N! \prod_{\ell=1}^{q} \frac{\overline{x}_\ell^{n_\ell}}{n_\ell!}.$$

Proof: According to proposition 2.17 we have (with the notation $N_\ell(\sigma)$ introduced above)

$$H(\sigma) = -\sum_{\ell=1}^{q} \lambda_\ell\, F_\ell(\sigma) = -\frac{1}{N}\sum_{\ell=1}^{q} \lambda_\ell\, N_\ell(\sigma)$$

Hence,

$$Q(\sigma) = \frac{1}{Z}\, e^{-H(\sigma)} = \frac{1}{Z}\, e^{\frac{1}{N}\sum_{\ell=1}^q \lambda_\ell\, N_\ell(\sigma)} = \frac{1}{Z}\prod_{\ell=1}^{q}\left(e^{\frac{1}{N}\lambda_\ell}\right)^{N_\ell(\sigma)} = \frac{1}{Z}\prod_{\ell=1}^{q} \rho_\ell^{N_\ell(\sigma)}$$

where $\rho_\ell := e^{\frac{1}{N}\lambda_\ell} \geq 0$. In this notation, we have

$$Z = \sum_{\sigma\in\Sigma_N}\prod_{\ell=1}^{q} \rho_\ell^{N_\ell(\sigma)}.$$

As the number of $\sigma \in \Sigma_N$ with given $N_1(\sigma) = n_1, \dots, N_q(\sigma) = n_q$ reads $\binom{N}{n_1,\dots,n_q}$, we have

$$Z = \sum_{(n_1,\dots,n_q)} \binom{N}{n_1, \dots, n_q} \prod_{\ell=1}^{q} \rho_\ell^{n_\ell}$$

where the sum extends over all tuples $(n_1, \ldots, n_q)$ with $\sum_{\ell=1}^q n_\ell = N$. That is, $Z = \left(\sum_{\ell=1}^q \rho_\ell\right)^N$, and (using $N = \sum_{\ell=1}^q N_\ell(\sigma)$)

$$Q(\sigma) = \prod_{\ell=1}^q \left(\frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}\right)^{N_\ell(\sigma)}.$$

If we only consider the summary statistics $N_1(\sigma), \ldots, N_q(\sigma)$, we multiply the probability $Q(\sigma)$ by the number of possible combinations, and obtain

$$P(N_1(\hat\sigma) = n_1, \ldots, N_q(\hat\sigma) = n_q) \;=\; N! \prod_{\ell=1}^q \frac{1}{n_\ell!} \left(\frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}\right)^{n_\ell}.$$

That is, $(N_1(\sigma), \ldots, N_q(\sigma))$ is distributed according to a multinomial distribution. Therefore,

$$E(N_\ell(\hat\sigma)) = N \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}} \quad \Rightarrow \quad E(F_\ell(\hat\sigma)) = E(N_\ell(\hat\sigma)/N) = \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}.$$

In order to determine $\rho_\ell / \sum_{\ell'=1}^q \rho'_\ell$, we inspect the conditions for the Lagrange multipliers

$$\overline{x}_\ell = E(F_\ell(\hat\sigma)) = \frac{\rho_\ell}{\sum_{\ell'=1}^q \rho_{\ell'}}.$$

An individual assumes state $\ell \in \{0, \ldots, q\}$ with probability $p_\ell = \rho_\ell / \sum_{\ell'=1}^q \rho_{\ell'} = \overline{x}_\ell$.

$\square$

Remark 2.19 *Note that we did not assume that the σ_i are independent, but we exclusively assumed that the only information we have is information about the expectations. The Potts-machinery then implied that we have a multinomial distribution, that is, that the σ_i's are independent. This fact is caused by the maximization of the entropy, or equivalently, by the maximization of the disorder.*

We did derive the first moments of Q. The next lemma addresses the second moments of Q.

Lemma 2.20 *For $1 \leq \ell, \ell' \leq q$, we have*

$$\begin{aligned} & E(F_\ell(\hat\sigma)\, F_{\ell'}(\hat\sigma))) - E(F_\ell(\hat\sigma))\, E(F_{\ell'}(\hat\sigma)) \\ = \; & \frac{1}{N}\Big[\delta(\ell,\ell')E(F_\ell(\hat\sigma)) - E(F_\ell(\hat\sigma))E(F_{\ell'}(\hat\sigma))\Big]. \end{aligned} \tag{2.29}$$

Proof: As the states of different individuals are uncorrelated, we find

$$\begin{aligned}
& E(N_\ell(\hat\sigma)\,N_{\ell'}(\hat\sigma)) - E(N_\ell(\hat\sigma))\,E(N_{\ell'}(\hat\sigma)) \\
= \;& E\Bigg(\Bigg(\sum_{i=1}^N \delta(\hat\sigma_i,\ell)\Bigg)\Bigg(\sum_{j=1}^N \delta(\hat\sigma_j,\ell')\Bigg)\Bigg) - \Bigg(\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell))\Bigg)\Bigg(\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell'))\Bigg) \\
= \;& \sum_{i=1}^N E(\delta(\hat\sigma_i,\ell)\,\delta(\hat\sigma_i,\ell')) + \sum_{i,j=1,i\neq j}^N E(\delta(\hat\sigma_i,\ell)\,\delta(\hat\sigma_j,\ell')) \\
& - \Bigg(\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell))\Bigg)\Bigg(\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell'))\Bigg) \\
= \;& \sum_{i=1}^N E(\delta(\hat\sigma_i,\ell)\,\delta(\ell,\ell')) \\
& + \sum_{i,j=1,i\neq j}^N E(\delta(\hat\sigma_i,\ell))\,E(\delta(\hat\sigma_j,\ell')) - \sum_{i,j=1}^N E(\delta(\hat\sigma_i,\ell))\,E(\delta(\hat\sigma_j,\ell')) \\
= \;& \delta(\ell,\ell')\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell)) - \sum_{i=1}^N E(\delta(\hat\sigma_i,\ell))\,E(\delta(\hat\sigma_i,\ell')).
\end{aligned}$$

We now make a second time use of the fact that states of different individuals are i.i.d. Hence, $E(\delta(\hat\sigma_i,\ell)) = E(\delta(\hat\sigma_j,\ell))$, and $E(\delta(\hat\sigma_i,\ell)) = \frac{1}{N}\sum_{k=1}^N E(\delta(\hat\sigma_k,\ell))$. Therefore,

$$\begin{aligned}
\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell))\,E(\delta(\hat\sigma_i,\ell')) &= \sum_{i=1}^N \Bigg(\frac{1}{N}\sum_{k=1}^N E(\delta(\hat\sigma_k,\ell))\Bigg)\Bigg(\frac{1}{N}\sum_{k=1}^N E(\delta(\hat\sigma_k,\ell'))\Bigg) \\
&= \frac{1}{N}\Bigg(\sum_{k=1}^N E(\delta(\hat\sigma_k,\ell))\Bigg)\Bigg(\sum_{k=1}^N E(\delta(\hat\sigma_k,\ell'))\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
& E(N_\ell(\hat\sigma)\,N_{\ell'}(\hat\sigma)) - E(N_\ell(\hat\sigma))\,E(N_{\ell'}(\hat\sigma)) \\
= \;& \delta(\ell,\ell')\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell)) - \frac{1}{N}\Bigg(\sum_{i=1}^N E(\delta(\hat\sigma_i,\ell))\Bigg)\Bigg(\sum_{j=1}^N E(\delta(\hat\sigma_j,\ell'))\Bigg) \\
= \;& \delta(\ell,\ell')E(N_\ell(\hat\sigma)) - \frac{1}{N}E(N_\ell(\hat\sigma))E(N_{\ell'}(\hat\sigma)).
\end{aligned}$$

If we divide this equation by N^2, the result follows.

□

2.5.4 Second example: Curie-Weiss model

We now add information about the overall correlations in the population. That is, we not only measure expectations as before,

$$N_\ell(\sigma) = \sum_{i=1}^N \delta(\sigma_i,\ell), \qquad E(N_\ell(\hat\sigma)) = N\,\overline{x}_\ell, \qquad \ell = 1,\ldots,q,$$

(where $\overline{x}_\ell$ is known from measurements) but we also measure

$$G(\sigma) = \sum_{i<j}^{N} \delta(\sigma_i, \sigma_j), \qquad E(G(\hat{\sigma})) = \overline{C}.$$

This version of the Potts model is also called Curie-Weiss model [15].

It is straightforward to determine the Hamiltonian $H(\sigma)$. With the notation introduced above,

$$H(\sigma) \;\; = \;\; -\sum_{\ell=1}^{q} \lambda_\ell N_\ell(\sigma) - \lambda_{q+1} G(\sigma).$$

We now rename $\lambda_\ell = h_\ell$ of $\ell = 1, \ldots, q$, and $\lambda_{q+1} = J/N$. Note that this model assumes a certain scaling of the Lagrange multiplier λ_{q+1} with population size N. Therefore, we choose $\lambda_{q+1} = J/N$. If the population size becomes large, we implicitly assume that the correlations (as determined by the function $G(.)$) become small. To be clear: This is an assumption that might or might not be met by a real world problem. Mostly, we will find that this assumption is appropriate, as the interaction between two given individuals will be weaker if interactions between many individuals (N large) are possible. Therewith,

$$\begin{aligned}
H(\sigma) &= -\frac{J}{N} \sum_{i<j}^{N} \delta(\sigma_i, \sigma_j) - \sum_{\ell=1}^{q} h_\ell \sum_{i=1}^{N} \delta(\sigma_i, \ell) \\
&= -\frac{J}{2N} \sum_{i,j=1}^{N} \delta(\sigma_i, \sigma_j) + \frac{J}{2N} \sum_{i=1}^{N} \delta(\sigma_i, \sigma_i) - \sum_{\ell=1}^{q} h_\ell \, N_\ell(\sigma) \\
&= -\frac{J}{2N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 + \frac{J}{2} - \sum_{\ell=1}^{q} h_\ell \, N_\ell(\sigma).
\end{aligned}$$

As the probabilities $Q(\sigma)$ are given by $e^{-H(\sigma)}/Z$, the additive term $J/2$ just cancels out (appears in the Hamilton as well as in the partition function). We can drop that constant.

Corollary 2.21 *Assume that* $E(F_\ell(\hat{\sigma})) = \overline{x}_\ell$, *and* $E(G(\hat{\sigma})) = \overline{C}$. *Then,* $Q(\sigma) = \exp(-H(\sigma))/Z$, *where the Hamiltonian for the Curie-Weiss model is given by*

$$H(\sigma) = -\frac{J}{2\,N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma). \tag{2.30}$$

Rescaling the parameters J and h_i, we can easily work out a weak limit for the Curie-Weiss model. Let $q = 2$, and

$$x = N_1(\sigma)/N, \quad \tilde{J} = J\,N, \quad \tilde{h}_i = h_i\,N.$$

Then, using $N_2(\sigma)/N = 1 - N_1(\sigma)/N$, we find

$$\begin{aligned}
&\frac{J}{2N}\left(N_1(\sigma)^2 + N_2(\sigma)^2\right) - h_1 N_1(\sigma) - h_(\sigma) = \frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) - \tilde{h}_1 x - \tilde{h}_2(1-x) \\
= \;\; &\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) - (\tilde{h}_1 - \tilde{h}_2)x - \tilde{h}_2.
\end{aligned}$$

Corollary 2.22 *In the weak limit, $\tilde{J} = J\,N$, $\tilde{h}_i = h_i\,N$, $h = \tilde{h}_1 - \tilde{h}_2$, and $x = N_1(\sigma)/N$, the limiting distribution for the Curie-Weiss model reads*

$$\varphi(x) = C\,\exp\left[-\,\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) + \tilde{h}\,x\right]. \tag{2.31}$$

The (local) maxima of this distribution coincide with local maxima of the exponent. To identify these points, we equate the derivative to zero, and find

$$0 \;=\; \frac{d}{dx}\left[-\,\frac{\tilde{J}}{2}\left(x^2 + (1-x)^2\right) + \tilde{h}\,x\right] = -\frac{\tilde{J}}{2}\left(2x - 2(1-x)\right) + \tilde{h} \quad\Rightarrow\quad x = \frac{\tilde{J} + \tilde{h}}{2\tilde{J}}.$$

That is, the distribution of the weak limit of the Curie-Weiss model always is unimodal. We will see below, that the original scaling of the Curie-Weiss model allows for more complex pattern.

We return to the original scaling, given by J and h_i. We reformulate the Hamilton (and the partition function) as an integral. Thereto, we use the following formula for a q-dimensional Gaussian integral: Let $A \in \mathbb{R}^{n\times n}$ denote a positive definite and symmetric matrix, and $y \in \mathbb{R}^n$ a vector. Then (see exercise 5.2),

$$\frac{1}{(2\pi)^{q/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^TAx + x^Ty}\,dx_1\ldots\,dx_q = \frac{e^{\frac{1}{2}y^TA^{-1}y}}{\sqrt{\det(A)}} \tag{2.32}$$

This formula is equivalent with

$$e^{\frac{1}{2}y^TA^{-1}y} = \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^TAx + x^Ty}\,dx_1\ldots\,dx_q.$$

This formula is used to perform a so-called Hubbard-Stratonovich transformation [26]: The quadratic terms in $e^{-H(\sigma)}$ can be expressed as an integral. After some more calculations (straightforward but lengthy, see exercise 5.3) we are led to the following proposition.

Theorem 2.23 *Up to negligible terms for $N \to \infty$ we have*

$$H(\sigma) = \kappa\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\,dx_1\ldots\,dx_q \tag{2.33}$$

and

$$Z = \kappa\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{\ell=1}^{q}\exp\left\{-N\left[\sum_{\ell=1}^{q}\frac{1}{2}Jx_\ell^2 - \ln\left(\sum_{\ell=1}^{q} e^{Jx_\ell + h_\ell}\right)\right]\right\}\,dx_1\ldots\,dx_q \tag{2.34}$$

with

$$\kappa = \left(\frac{J\,N}{2\,\pi}\right)^{q/2}.$$

J and h_ℓ are Lagrange multipliers determined by $E(F_\ell(\hat{\sigma})) = \overline{x}_\ell$, and $E(G(\hat{\sigma})) = \overline{C}$.

Proof: Exercise 5.3.

Remark 2.24 *We find that*

$$Q(\sigma) = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\, dx_1 \ldots dx_q}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^{q} x_\ell^2 - \ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J\,x_\ell}\right)\right]\right\} dx_1 \ldots dx_q},$$

that is, the factor $\kappa = \left(\frac{J\,N}{2\,\pi}\right)^{q/2}$ *cancels out in the probability measure.*

As the theory is established in statistical physics, the nomenclature comes from that field: h is called external field, J couplings. Furthermore, the term

$$f(x_1, \ldots, x_q) = -\ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J\,x_\ell}\right) + \frac{1}{2}\,J\,\sum_{\ell=1}^{q} x_\ell^2 \tag{2.35}$$

is called the free energy and assumes its minimum at some value x_ℓ^{sp}.

The result of theorem 2.21 is also called a functional integral [26]. At the time being it is not clear why we want to represent the Hamiltonian in such a complex way. At the first glance, it seems that we do not gain anything. However, it is indeed true that this reformulation yields additional insight and is in many cases simpler to handle. In the remaining part of the section, we do some elementary calculations to become familiar with this functional integral. Based on the results we obtain in these investigations, we will reconsider the Hamilton, the partition function, and the free energy.

We discuss why the values x_ℓ^{sp} play a special role. Before, we state a useful lemma.

Lemma 2.25 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a smooth and positive function with a unique minimum* $x_0 \in \mathbb{R}^n$ *and* $f(x) > c_1|x|^2 + c_2$*, where* $c_1 > 0$*. Then,*

$$\varphi_N(x) = \frac{e^{-\,N\,f(x)}}{\int_{\mathbb{R}^n} e^{-\,N\,f(x)}\, dx}$$

tends for $N \to \infty$ *to* $\delta_{x_0}(x)$*, that is, for all bounded* $\psi \in C^0(\mathbb{R}^n, \mathbb{R})$ *we find*

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} \psi(x)\,\varphi_N(x)\,dx = \psi(x_0).$$

Proof: Exercise 5.4.

Proposition 2.26 *The consistency conditions for the Lagrange multipliers* h_ℓ *become for* $N \to \infty$

$$\overline{x}_k = \frac{e^{h_k + J x_k^{sp}}}{\sum_{\ell=1}^{q} e^{h_\ell + J x_\ell^{sp}}} \tag{2.36}$$

where x_ℓ^{sp} *minimize the free energy* $f(x_1, \ldots, x_q) = -\ln\left(\sum_{\ell=1}^{q} e^{h_\ell + J\,x_\ell}\right) + \frac{1}{2}\,J\,\sum_{\ell=1}^{q} x_\ell^2$.

Proof:

$$E(N_k(\hat\sigma)/N) \;=\; \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^q x_\ell^2}\;\sum_{\sigma\in\Sigma_N}\frac{1}{N}\sum_{i=1}^N \delta(\sigma_i,k)\,e^{\sum_{\ell=1}^q (h_\ell+J\,x_\ell)N_\ell(\sigma)}\,dx_1\ldots dx_q}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^q x_\ell^2-\ln\left(\sum_{\ell=1}^q e^{h_\ell+J\,x_\ell}\right)\right]\right\}dx_1\ldots dx_q}$$

The sum in the numerator reads

$$\frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{i=1}^N\delta(\sigma_i,k)\,e^{\sum_{\ell=1}^q(h_\ell+J\,x_\ell)N_\ell(\sigma)} = \frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{j=1}^N\prod_{\ell=1}^q\delta(\sigma_j,k)\exp\left\{(h_\ell+J\,x_\ell)N_\ell(\sigma)\right\}$$

$$\begin{aligned}
&= \frac{1}{N}\sum_{\sigma\in\Sigma_N}\sum_{j=1}^N\prod_{i=1}^N\prod_{\ell=1}^q\delta(\sigma_j,k)\exp\left\{(h_\ell+J\,x_\ell)\delta(\sigma_i,\ell)\right\}\\
&= \frac{1}{N}\sum_{j=1}^N\sum_{s_1=1}^q\cdots\sum_{s_N=1}^q\prod_{i=1}^N\prod_{\ell=1}^q\delta(s_j,k)\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}\\
&= \frac{1}{N}\sum_{j=1}^N\prod_{i=1}^N\left(\sum_{s_i=1}^q\delta(s_j,k)\prod_{\ell=1}^q\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}\right)\\
&= \frac{1}{N}\sum_{j=1}^N\exp(h_k+J\,x_k)\prod_{i=1,i\neq j}^N\left(\sum_{s_i=1}^q\prod_{\ell=1}^q\exp\left\{(h_\ell+J\,x_\ell)\delta(s_i,\ell)\right\}\right)\\
&= \frac{1}{N}\sum_{j=1}^N\exp(h_k+J\,x_k)\prod_{i=1,j\neq j}^N\left(\sum_{s_i=1}^q\exp(h_{s_i}+J\,x_{s_i})\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N}\sum_{j=1}^N\exp(h_k+J\,x_k)\prod_{i=1,i\neq j}^N\left(\sum_{\ell=1}^q\exp(h_\ell+J\,x_\ell)\right)\\
&= \exp\{(h_k+J\,x_k)\}\left(\sum_{\ell=1}^q\exp(h_\ell+J\,x_\ell)\right)^{N-1} = \exp\left\{h_k+J\,x_k+(N-1)\ln\left(\sum_{\ell=1}^q e^{h_\ell+J\,x_\ell}\right)\right\}.
\end{aligned}$$

If we use the fact that $N-1\approx N$ for N large, we have in eading order

$$E(N_k(\hat\sigma)/N) \;=\; \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{(h_k+J\,x_k)}\;\exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^q x_\ell^2-\ln\left(\sum_{\ell=1}^q e^{h_\ell+J\,x_\ell}\right)\right]\right\}dx_1\ldots dx_q}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\,\sum_{\ell=1}^q x_\ell^2-\ln\left(\sum_{\ell=1}^q e^{h_\ell+J\,x_\ell}\right)\right]\right\}dx_1\ldots dx_q}$$

According to the lemma 2.25, for $N\to\infty$ this integral becomes a point mass at the minimum of $f(x_1,\ldots,x_q)$ defined above, and

$$\lim_{N\to\infty}E(N_k(\hat\sigma)/N) = e^{h_k+Jx_k^{sp}}.$$

We may rewrite this equation by

$$\overline{x}_k = \lim_{N\to\infty}E(N_k(\hat\sigma)/N) = \lim_{N\to\infty}\frac{E(N_k(\hat\sigma)/N)}{1} = \lim_{N\to\infty}\frac{E(N_k(\hat\sigma)/N)}{\sum_{\ell=1}^q E(N_\ell(\hat\sigma)/N)} = \frac{e^{h_k+Jx_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell+Jx_\ell^{sp}}}.$$

This last expression has the advantage, that the normalization of $\overline{x}_k$ (sum becomes 1) is made explicit.

$\square$

Exercises

Exercise 5.2: (a) Let $y \in \mathbb{R}$, $a \in \mathbb{R}$, and $a > 0$. Show the identity (where you might want to use, without proof, that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,a\,x^2 + xy}\,dx = \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}}.$$

(b) Show the equation for the Gaussian integral (2.32).

Solution of Exercise 5.2 (a) As $a\,x^2 - 2xy = (\sqrt{a}x - y/\sqrt{a})^2 - y^2/a$, we have

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,a\,x^2 + xy}\,dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,(\sqrt{a}x - y/\sqrt{a})^2}\,dx\;\; e^{\frac{1}{2}y^2/a} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,(\sqrt{a}x - y/\sqrt{a})^2}\,\sqrt{a}dx\;\; \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\tilde{x}^2}\,d\tilde{x}\;\; \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}} = \frac{e^{\frac{1}{2}y^2/a}}{\sqrt{a}},
\end{aligned}$$

where we used the transformation $\tilde{x} = \sqrt{a}x - y/\sqrt{a}$.
(b) The idea is the very same as in part (a), only that we need to work in q dimensions instead of one dimension. As A is symmetric and positive definite, we find a regular (orthogonal) matrix M such that

$$A = MDM^T$$

where D is the diagonal matrix with the (positive) eigenvalues of A on the diagonal. If we denote by $\sqrt{D}$ the diagonal matrix with the square root of the eigenvalues on the diagonal, then $\sqrt{D}\,\sqrt{D} = D$, and with $B = M\,\sqrt{D}$, we have

$$A = B\,B^T.$$

Furthermore, B is regular, as M and $\sqrt{D}$ are regular. With these prerequisites, we proceed:

$$\begin{aligned}
& \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^T y}\,dx_1 \ldots dx_q \\
= & \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx)^T(Bx) + (Bx)^T B^{-1} y - \frac{1}{2}((B^{-1}y)^T(B^{-1}y)) + \frac{1}{2}((B^{-1}y)^T(B^{-1}y))}\,dx_1 \ldots dx_q \\
= & \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx + B^{-1}y)^T(Bx + B^{-1}y)}\,dx_1 \ldots dx_q\;\; e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))} \\
= & \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(Bx + B^{-1}y)^T(Bx + B^{-1}y)} \det(B)\,dx_1 \ldots dx_q\;\; \frac{e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))}}{\det(B)} \\
= & \frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{\ell=1} q\tilde{x}_\ell^2}\,d\tilde{x}_1 \ldots d\tilde{x}_q\;\; \frac{e^{\frac{1}{2}((B^{-1}y)^T(B^{-1}y))}}{\det(B)} = \frac{e^{\frac{1}{2}y^T A^{-1} y}}{\det(B)}.
\end{aligned}$$

Paralleling the one-dimensional case, we used the transformation $\tilde{x} = Bx + B^{-1}y$. Recall that the determinant is just the product of all eigenvalues. The result follows with the identity

$$\det(B)^2 = \det(B)\det(B^T) = \det(B\,B^T) = \det(A) \quad \Rightarrow \quad \det(B) = \sqrt{\det(A)}.$$

Exercise 5.3: Prove Theorem 2.23.

Solution of Exercise 5.3 In order to find the expression for Z, we use a Hubbard-Stratonovich transformation [26]. The following formula for a q-dimensional Gaussian integral is useful. Let $A \in \mathbb{R}^{n\times n}$ denote a positive definite matrix, and $y \in \mathbb{R}^n$ a vector. Then,

$$\frac{1}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^T y}\, dx_1 \ldots dx_q = \frac{e^{\frac{1}{2}y^T A^{-1} y}}{\sqrt{\det(A)}},$$

that is,

$$e^{\frac{1}{2}y^T A^{-1} y} = \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^T y}\, dx_1 \ldots dx_q.$$

For $\sigma \in \Sigma_N$ given, we define a vector $y = y(\sigma) = (y_1, \ldots, y_q)^T$ by $y_\ell = N_\ell(\sigma)$, the vector $h = (h_1, \ldots, h_q)^T$, and the matrix $A = (N/J)I$, such that (note that $y = y(\sigma)$)

$$\begin{aligned}
Z &= \sum_{\sigma\in\Sigma_N} e^{-H(\sigma)} = \sum_{\sigma\in\Sigma_N} \exp\left\{\frac{J}{2N} \sum_{\ell=1}^{q} N_\ell(\sigma)^2 + \sum_{\ell=1}^{q} h_\ell\, N_\ell(\sigma)\right\} \\
&= \sum_{\sigma\in\Sigma_N} e^{\frac{1}{2}y^T A^{-1} y}\; e^{h^T y} = \sum_{\sigma\in\Sigma_N} \frac{\sqrt{\det(A)}}{(2\pi)^{q/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^t y}\, dx_1 \ldots dx_q\; e^{h^T y} \\
&= \sum_{\sigma\in\Sigma_N} \left\{ \left(\frac{N}{2\,J\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^T A x + x^T y}\, dx_1 \ldots dx_q\; e^{h^T y} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{N}{2\,J\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,x^T x/J} \sum_{\sigma\in\Sigma_N} e^{(h^T + x^T)y}\, dx_1 \ldots dx_q \\
&= \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q}(x_\ell/J)^2} \sum_{\sigma\in\Sigma_N} e^{\sum_{\ell=1}^{q}(h_\ell + J\,(x_\ell/J))N_\ell(\sigma)}\, dx_1/J \ldots dx_q/J \\
&= \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2} \sum_{\sigma\in\Sigma_N} e^{\sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\, dx_1 \ldots dx_q
\end{aligned}$$

where we used the transformation $x_\ell \mapsto x_\ell/J$ of the integral variables in the last step. Before we proceed, we note that $H(\sigma)$ can be transformed in a similar way,

$$H(\sigma) = \left(\frac{J\,N}{2\,\pi}\right)^{q/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,N\,J\,\sum_{\ell=1}^{q} x_\ell^2 + \sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)}\, dx_1 \ldots dx_q.$$

Now we turn to the sum within the integral,

$$
\begin{aligned}
&\sum_{\sigma\in\Sigma_N} \exp\left\{\sum_{\ell=1}^{q}(h_\ell + J\,x_\ell)N_\ell(\sigma)\right\} = \sum_{\sigma\in\Sigma_N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)N_\ell(\sigma)\right\}\\
= &\sum_{\sigma\in\Sigma_N}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(\sigma_i,\ell)\right\} = \sum_{s_1=1}^{q}\cdots\sum_{s_N=1}^{q}\prod_{i=1}^{N}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(s_i,\ell)\right\}\\
= &\prod_{i=1}^{N}\left(\sum_{s_i=1}^{q}\prod_{\ell=1}^{q}\exp\left\{(h_\ell + J\,x_\ell)\delta(s_i,\ell)\right\}\right) = \prod_{i=1}^{N}\left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right)\\
= &\left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right)^N = \exp\left\{N\ln\left(\sum_{\ell=1}^{q}\exp(h_\ell + J\,x_\ell)\right)\right\}.
\end{aligned}
$$

Hence,

$$
Z = \left(\frac{1}{2\,N\,J\,\pi}\right)^{q/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-N\left[\frac{1}{2}\,J\sum_{\ell=1}^{q}x_\ell^2 - \ln\left(\sum_{\ell=1}^{q}e^{h_\ell + J\,x_\ell}\right)\right]\right\}dx_1\ldots dx_q.
$$

Exercise 5.4: Demonstrate Lemma 2.25.

Solution of Exercise 5.4 Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be continuous. As the integral over $\varphi_N(.)$ is 1, we have

$$
\left|\int_{\mathbb{R}^n}\psi(x)\,\varphi_N(x)\,dx - \psi(x_0)\right| = \left|\int_{\mathbb{R}^n}(\psi(x) - \psi(x_0))\,\varphi_N(x)\,dx\right| \le \int_{\mathbb{R}^n}|\psi(x) - \psi(x_0)|\,\varphi_N(x)\,dx.
$$

Let $\tilde\psi(x) = |\psi(x) - \psi(x_0)|$. $\tilde\psi$ is a continuous, bounded, and non-negative function with $\psi(x_0) = 0$. If for such a function it is true that $\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx \to 0$ for $N \to \infty$, the lemma is established.

Let $\varepsilon > 0$, arbitrary, fixed. We show that $\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx < \varepsilon$ for N large.

Therefore we first find $r > 0$, s.t.

$$
\max_{|x-x_0|\le r}\tilde\psi(x) < \varepsilon/2.
$$

Next we note that for this given $r > 0$

$$
\lim_{N\to\infty}\int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx \le \|\tilde\psi(x)\|_\infty\lim_{N\to\infty}\int_{|x-x_0|>r}\varphi_N(x)\,dx = 0.
$$

Therefore, there is $N_0 > 0$ s.t. for all $N > N_0$

$$
\int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx < \varepsilon/2.
$$

Together, we have for $N > N_0$

$$
\begin{aligned}
\int_{\mathbb{R}^n}\tilde\psi(x)\,\varphi_N(x)\,dx &= \int_{|x-x_0|\le r}\tilde\psi(x)\,\varphi_N(x)\,dx + \int_{|x-x_0|>r}\tilde\psi(x)\,\varphi_N(x)\,dx\\
&< \int_{|x-x_0|\le r}\varphi_N(x)\,dx\,\max_{|x-x_0|\le r}\tilde\psi(x) + \varepsilon/2\\
&\le \int_{\mathbb{R}^n}\varphi_N(x)\,dx\,\varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

2.5.5 Interlude: Free energy and the partition function

Remark 2.27 *There is an appealing heuristic short-cut for the proof of Proposition 2.26. (a) We go back to the general setting given in theorem 2.17. That is, we consider general observables* $F_\ell(\sigma)$ *with Lagrange multipliers* λ_ℓ, $H(\sigma) = -\sum_{\ell=1}^M \lambda_\ell F_\ell(\sigma)$, *and* $Z = \sum_{\sigma\in\Sigma_N} e^{-H(\sigma)}$. *Then,*

$$\begin{aligned} E(F_\ell(\hat\sigma)) &= \sum_{\sigma\in\Sigma_N} F_\ell(\sigma) e^{-H(\sigma)}/Z = \sum_{\sigma\in\Sigma_N} F_\ell(\sigma) e^{\sum_{\ell'=1}^M \lambda_{\ell'} F_{\ell'}(\sigma))}/Z \\ &= \frac{\frac{\partial}{\partial\lambda_\ell}\sum_{\sigma\in\Sigma_N} e^{-H(\sigma))}}{\sum_{\sigma\in\Sigma_N} e^{-H(\sigma)}} = \frac{\partial}{\partial\lambda_\ell}\ln(Z). \end{aligned}$$

(b) We now return to the setting at hand. Since x_ℓ^{sp} *minimize the free energy, we have for* N *large that* Z *is approximately proportional to*

$$Z \sim e^{-N\, f(x_1^{sp},\ldots,x_q^{sp})}$$

and hence

$$\ln(Z) \sim -N\, f(x_1^{sp},\ldots,x_q^{sp}).$$

If we combine the two observations (a) and (b), we obtain

$$E(N_\ell(\hat\sigma)) = -N\,\frac{\partial}{\partial h_\ell} f(x_1^{sp},\ldots,x_q^{sp}) = N\,\frac{e^{h_k + Jx_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + Jx_\ell^{sp}}}.$$

In that, equation (2.36) can be re-written as

$$\overline{x}_\ell = -\frac{\partial}{\partial h_\ell} f(x_1^{sp},\ldots,x_q^{sp}). \tag{2.37}$$

In a similar way, we can obtain information about correlation.

Proposition 2.28 *Let* $\ell \neq \ell'$ *and assume that the partition function* Z *is proportional to* $e^{-Nf(x_1^{sp},\ldots,x_q^{sp})}$,

$$Z \sim e^{-Nf(x_1^{sp},\ldots,x_q^{sp})}.$$

Then,

$$\begin{aligned} &-N\,\frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} f(x_1^{sp},\ldots,x_q^{sp}) = -N\,\frac{\partial}{\partial h_{\ell'}}\, x_\ell^{sp} \\ &= E\Big(N_\ell(\hat\sigma)\, N_{\ell'}(\hat\sigma)\Big) - E(N_\ell(\hat\sigma))\, E(N_{\ell'}(\hat\sigma)). \end{aligned} \tag{2.38}$$

Proof: We know that

$$\frac{\partial}{\partial h_\ell}\ln(Z) = \frac{1}{Z}\sum_{\sigma\in\Sigma_N} N_\ell(\sigma) e^{-H(\sigma)}$$

and hence for $\ell \neq \ell'$

$$
\begin{aligned}
\frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} \ln(Z) \;\;&=\;\; \frac{1}{Z}\sum_{\sigma\in\Sigma_N} N_\ell(\sigma)\, N_{\ell'}(\sigma))(\sigma)e^{-H(\sigma)} \\
&\quad - \left(\frac{1}{Z}\sum_{\sigma\in\Sigma_N} N_\ell(\sigma)e^{-H(\sigma)}\right)\left(\frac{1}{Z}\sum_{\sigma\in\Sigma_N} N_{\ell'}(\sigma)e^{-H(\sigma)}\right)
\end{aligned}
$$

With $x_\ell^{sp} = E(F_\ell(\hat\sigma)) = \frac{\partial}{\partial h_\ell}\ln(Z)$ the result follows.

$\square$

Remark 2.29 *Note that we can rewrite the equation (2.38) as*

$$
-\frac{1}{N}\frac{\partial^2}{\partial h_\ell \partial h_{\ell'}} f(x_1^{sp},\dots,x_q^{sp}) \;\;=\;\; E\Big(F_\ell(\hat\sigma)\, F_{\ell'}(\hat\sigma)\Big) - E(F_\ell(\hat\sigma))\, E_Q(F_{\ell'}(\hat\sigma)).
$$

2.5.6 Phase transitions.

The task we are faced with is to find the value of the Lagrange multipliers h_ℓ and J, given the data $\overline{x}_\ell$ and $\overline{C}$. This task is rather difficult. What we do is to reverse this question: We fix h_ℓ and J, and ask which values for $\overline{x}_\ell$ are possible. As we obtain non-trivial answers from considering $\overline{x}_\ell$ only we do not also discuss $\overline{C}$. In that, the Lagrange multipliers gain importance at its own. This step resembles the definition of a (physical) temperature in statistical physics. Also the temperature is a Lagrange multiplier, which comes into life. In this context, h_k is called the external field, and J the couplings.

Proposition 2.30 *If h_ℓ and J are prescribed, then $\overline{x}_\ell = x_\ell^{sp}$, and the consistency equations for the external fields become*

$$
x_k^{sp} = \frac{e^{h_k + Jx_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + Jx_\ell^{sp}}} \tag{2.39}
$$

Proof: We consider two equations: First of all, Equation (2.36) reads

$$
\overline{x}_k = \frac{e^{h_k + Jx_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + Jx_\ell^{sp}}}.
$$

The second condition is given by the fact that x_ℓ^{sp} minimizes the free energy. If we take the partial derivative of $f(x_1,\dots,x_q)$ w.r.t. x_k we find

$$
\begin{aligned}
0 \;\;&=\;\; \frac{\partial}{\partial x_k} f(x_,\dots,x_q) = \frac{\partial}{\partial x_k}\left\{ -\ln\left(\sum_{\ell=1}^q e^{h_\ell + J\,x_\ell}\right) + \frac{1}{2}\,J\sum_{\ell=1}^q x_\ell^2\right\} \\
&=\;\; -\frac{J\,e^{h_k + J\,x_k}}{\sum_{\ell=1}^q e^{h_\ell + J\,x_\ell}} + J\,x_k.
\end{aligned}
$$

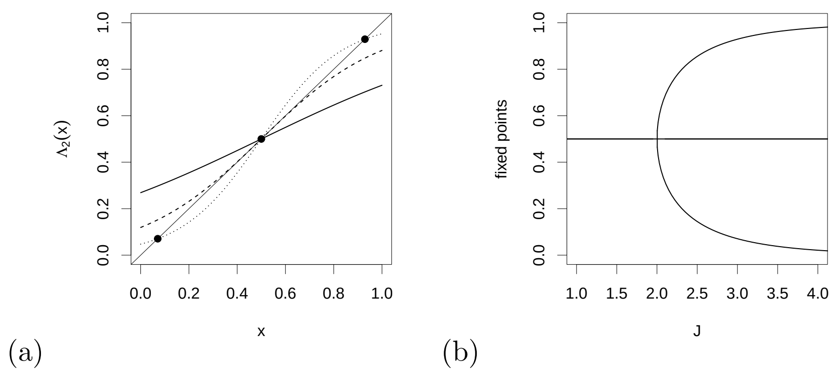


Figure 2.5: Left panel: $q = 2$, $J = 1$ (solid line), $J = 2$ (dashed line) $J = 3$ (dotted line). Fixed points (indicated by bullets) (intersections of the graph of $\Lambda_2(x)$ with the the line $y = x$, indicated by the straight solid line) are possible values for x_ℓ^{sp}. Right panel: Fixed points of $\Lambda_2(x)$ over parameter J.

Hence, the minimizing values x_k^{sp} satisfy

$$x_k^{sp} = \frac{e^{h_k + Jx_k^{sp}}}{\sum_{\ell=1}^q e^{h_\ell + Jx_\ell^{sp}}} = \overline{x}_k.$$

The solutions x_k^{sp} are the possible values of the observables $\overline{x}_k$, given the Lagrange multipliers.

□

We focus on simple cases where the consistency conditions can be explicitly solved. The most simple assumption seems to be that all are equal, $x_\ell^{sp} = x$. As x_ℓ^{sp} sum up to 1, we conclude $x = 1/q$. We will find below, that this is a sensible solution if all external fields are identical. However, the restriction assumed (all x_ℓ^{sp} are equal) is too strong to allow for all or only for more interesting solutions.

Proposition 2.31 *Assume that the external fields are identical, $h_i = h_j$. For $q = 2$, there is always the trivial solution $x_1^{sp} = x_2^{sp} = 1/2$. At $J = 2$, a second branch with $x_1^{sp} \neq x_2^{sp}$ crosses the trivial solution.*

Proof: As we assume $h_1 = h_2$, the consistency equation becomes

$$x_\ell^{sp} = \frac{e^{Jx_k^{sp}}}{\sum_{\ell=1}^q e^{Jx_\ell^{sp}}}$$

where $x_1^{sp} + x_2^{sp} = 1$. That is, we have for x_1^{sp} the equation

$$x_1^{sp} = \Lambda_2(x_1^{sp}) = \frac{e^{Jx_1^{sp}}}{e^{Jx_1^{sp}} + e^{J(1-x_1^{sp})}} = \frac{1}{1 + e^{J(1-2x_1^{sp})}}.$$

Note that a solution for that equation x_1^{sp} is in the interval $(0,1)$. For the corresponding $x_2^{sp} = 1 - x_1^{sp}$ we find

$$x_2^{sp} = 1 - x_1^{sp} = 1 - \Lambda_2(x_1^{sp}) = 1 - \frac{e^{Jx_1^{sp}}}{e^{Jx_1^{sp}} + e^{J(1-x_1^{sp})}} = \frac{e^{Jx_2^{sp}}}{\sum_{\ell=1}^{q} e^{Jx_\ell^{sp}}}.$$

That is, any fixed point of $\Lambda_1(.)$ defines a valid solution of the consistency equation.
We find at once that $\Lambda_2(1/2) = 1/2$. Furthermore,

$$\Lambda_2'(x) = \frac{2\,J\,e^{J(1-2x)}}{(1+e^{J(1-2x)})^2}.$$

That is, $\Lambda_2'(1/2) = J/2$. At $J = 2$, the slope is exactly one. As $\Lambda_2''(1/2) \neq 0$, a second branch of solution crosses the trivial solution.

□

Remark 2.32 *A more refined analysis of the fixed points of $\Lambda_2(.)$ reveals, that a Pitchfork bifurcation happens at $q = 2$: If $J \leq 2$, there only is the trivial solution $x_\ell^{sp} = 1/2$; for $J > 2$, a symmetric, second solutions branches away from that trivial solution (see Fig. 2.5).*

We prove a similar result for $q > 2$. First of all, we determine $q-1$ different functions which have fixed points that correspond to solutions of the consistency equations. All of these functions have the trivial solution $x_\ell^{sp} = 1/q$ as fixed point, independently of the value fo J. If J is large enough, additional fixed points appear. The definition of these functions is based on the idea to have the similar value in the first m and the last $q-m$ components in $(x_1^{sp}, \ldots, x_q^{sp})$.

Proposition 2.33 *Assume that the external fields are identical, $h_i = h_j$. Let furthermore x be a fixed point of*

$$\Lambda_{m,q}(x) := \frac{1}{m + (q-m)e^{J\,\frac{1-qx}{q-m}}}, \qquad m = 1, \ldots, q-1. \tag{2.40}$$

Define y by

$$y = \frac{1-mx}{q-m}. \tag{2.41}$$

Then, $x_1^{sp} = \ldots = x_m^{sp} = x$ and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$ is a solution of the consistency equation 2.39.

Proof: Heuristics: Assume that $x_1^{sp} = \ldots = x_m^{sp} = x$ and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$ is a solution of the consistency equation. As the components of x_ℓ^{sp} add up to 1, we have $1 = m\,x + (q-m)\,y$. The consistency equation 2.39 for $x = x_1^{sp}$ (and all external fields equal) reads

$$x = \frac{e^{Jx}}{\sum_{\ell=1}^{q} e^{Jx_\ell^{sp}}} = \frac{e^{Jx}}{me^{Jx} + (q-m)e^{Jy}} = \frac{1}{m + (q-m)e^{J(y-x)}} = \Lambda_{m,q}(x).$$

Now we forget the heuristics, and assume that x is a fixed point of $\Lambda_{m,q}(.)$, y is defined according to eqn. (2.41), $x_1^{sp} = \ldots = x_m^{sp} = x$, and $x_{m+1}^{sp} = \ldots = x_q^{sp} = y$. Due to the construction, we know

that x_1^{sp} satisfies the consistency equation, and for symmetry reasons the consistency condition is also established for $x_2^{sp}, \ldots, x_m^{sp}$.
It remains to show that y satisfies the consistency equation of $x_{m+1}^{sp} = y$. If this is given, also $x_{m+1}^{sp}, \ldots, x_{m+1}^{sp}$ satisfy their consistency equations, and we are done. We find by straightforward calculations

$$\begin{aligned}
y &= \frac{1-mx}{q-m} = \frac{1-m\Lambda_{m,q}(x)}{q-m} = \frac{1}{q-m}\left(1 - \frac{m}{m+(q-m)e^{J\,\frac{1-qx}{q-m}}}\right)\\
&= \frac{1}{q-m}\left(1 - \frac{me^{Jx}}{me^{Jx}+(q-m)e^{Jy}}\right) = \frac{1}{q-m}\left(\frac{(q-m)e^{Jy}}{me^{Jx}+(q-m)e^{Jy}}\right)\\
&= \frac{e^{Jy}}{me^{Jx}+(q-m)e^{Jy}},
\end{aligned}$$

which is precisely the consistency condition for y.

□

Remark 2.34 *(a) If we have a solution x, and if $y \neq x$, we have $\binom{q}{m}$ different solutions of the consistency equations, as we can assign x to arbitrarily chosen components of x_ℓ^{sp}.*
(b) Simple calculations show that $\lambda_{m,q}(x) = x$ implies $\Lambda_{q-m,q}(y) = y$. In this sense, the functions $\Lambda_{q-m,q}(y)$ do not produce independent solutions, but the fixed point equations come in pairs.
(c) We always (independently of J have the trivial solution

$$\Lambda_{m,q}(1/q) = 1/q.$$

That fixed point corresponds to the solution with maximal symmetry, $x_\ell^{sp} = 1/q$. A straightforward calculations shows that

$$J = q \quad \Rightarrow \quad \Lambda'_{m,q}(1/q) = 1.$$

We expect non-trivial branch(es) to appear at $J = q$ (see also Fig. 2.6, panel (d)). However, if we inspect the numerical construction of the consistency equations' solution, we find that there are non-trivial branches of solutions for $J < q$. They come into life by saddle-node bifurcations. It turns out that this is always the case for $q > 2$.
Note that the map $x \mapsto y = \frac{1-mx}{q-m}$ either leaves the branches invariant (the symmetric branches in Fig. 2.6, panel (d)) or maps one branch to another one.

transition

That was the analysis in the lmimt $N \to \infty$. Can we tell more, about the way the invariant measure approximates the delta-distribution?

Lagrange parameters

We return to the problem to directly infer the Lagrange multipliers h_ℓ and J, given expectations $\overline{x}_k$ and $\overline{x_k x_{k'}}$ (NOTATION????).

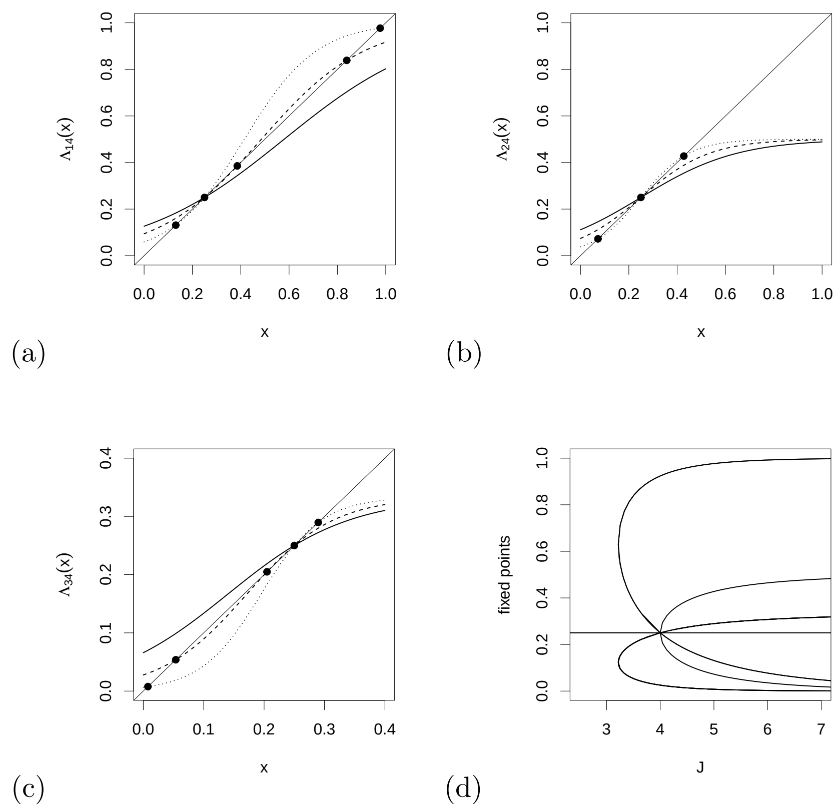


Figure 2.6: (a)-(c): $q = 4$, $J = 2.5$ (solid line), $J = 3.5$ (dashed line) $J = 5$ (dotted line). Intersection of the graph of $\Lambda(x)$ with $y = x$ correspond to possible solutions of the consistency equations. We have $m = 1$ (a), $m = 2$ (b), $m = 3$ (c). (d) Fixed points of $\Lambda_{m,4}(x)$, $m = 1, \dots, q$, over parameter J.

2.5.7 Estimation of external field and couplings from data

We now return to our original aim: Given a time series of fractions $\vec{x}_t$, and the assumption that the dynamics is in an stationary state, can we estimate the Lagrange multipliers, that is, the external fields h_ℓ and the coupling J? First of all, the time series gives us at each time point information t about the the fraction of supporters of candidate ℓ, $\ell = 1, \dots, q$. We do not have information in continuous time, but at T fixed time points, $t_1, \dots, t_T$. Moreover, the number of participants in each of the T polls are not constant, but a time-dependent number N_t. We estimate $\overline{x}_\ell$ by the weighted mean,

$$\overline{x}_\ell = \sum_{t=1}^{T} \frac{N_t}{\sum_{t'=1^T} N_{t'}} \, (\vec{x}(t))_\ell.$$

In a similar way, we estimate the pairwise correlation functions

$$C_{\ell,\ell'} = \sum_{t=1}^{T} \frac{N_t}{\sum_{t'=1}^{T} N_{t'}} \, (\vec{x}(t))_\ell \, (\vec{x}(t))_{\ell'}.$$

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2.6 Dynamic models with phase transitions

Most models we considered so far have been based on variants of the voter model. Instead, in the the last section we investigated an application of the Potts model. Immediately the question comes to mind if these models are similar, or in which aspects they are fundamentally different. Some differences are obvious the constructions: The (noisy) voter model is defined as a mechanical model. It is formulated as a stochastic process. This process has an invariant measure, and mostly we either investigate the transient dynamics, or the properties of the invariant measure as given by the process, e.g., by means of the detailed balance equation.
The Potts model, instead, is defined as a probability measure only, without an accompanying mechanistic stochastic process that produces the Potts probability measure as its invariant measure.

In the present section, we embed the Potts model into a stochastic process. Therewith, we better understand the mechanisms of the Potts model that lead to the phase transition (in particular, the differences to the Voter model). Equipped with that insight, we develop a model for reinforcement, that also will develop phase transitions.

2.6.1 Potts model versus voter model

In order to compare the variants of the voter model and the Potts model, we augment the Potts model with a stochastic process. This process is by no means unique – the Potts model is the equivalent of a stationary state of an unknown dynamical system, and there are many different dynamical systems with the very same stationary state. However, statistical physics proposes a witty method to construct the stochastic process. One general idea to construct a stochastic process around a given distribution is the Metropolis-Hastings algorithm [19]. That algorithm, though of utmost importance in, e.g., Bayesian statistics, is less convenient to interpret in mechanistic terms. For the Curie Weiss model, an appealing construction is the Glauber dynamics [18, 27]. The starting point of the Glauber dynamics is the detailed balance equation – if the combination of given transition rates and a given distribution satisfy this equation, then that distribution is already the invariant distribution of the process. The idea of the Glauber dynamics is to chose most simple transition rates that do satisfy the detailed balance equation. The result of this process is, of course, dependent on the special form of the Hamiltonian in the Potts model.

It is possible to better understand the Potts (equilibrium) distribution using the Glauber dynamics, as it is more intuitive to investigate a stochastic process than a given, static probability measure. Particularly, as we shall see, the comparison with the transition rates of the zealot model is possible. We consider the two-state situation. For mathematical convenience we chose $q \in \{\pm 1\}$ instead of $q \in \{1, 2\}$.

Model 2.35 Glauber dynamics. *Consider the state space for the 2-state Potts model: Each of N individuals assumes the states 1 or -1,*

$$\Sigma_N = \{-1, 1\}^N.$$

The individuals are numbered by $1, \dots, N$, the state of the system is given by $\sigma \in \Sigma_N$. The Glauber process (for $q = 2$) is given by a Σ_N-valued, time-continuous Markov process $\hat{\sigma}_t$. The state changes a.s. in only one component at a time, that is, the opinion of only one individual flips. Let flip : $\{-1, 1\} \to \{-1, 1\}$ with flip$(1) = -1$ and flip$(-1) = 1$. Let furthermore $i_0 \in \{1, \dots, N\}$, $\sigma \in \Sigma_N$, and $\sigma' \in \Sigma_N$ the state that only differs from σ in i_0,

$$\sigma'_{i_0} = \mathit{flip}(\sigma_{i_0}), \qquad \sigma'_i = \sigma_i \text{ for } i \neq i_0.$$

Then, the transition rate from σ to σ' is defined as

$$\sigma \to \sigma' \quad \text{at rate} \quad \frac{\mu}{2}\left[1 - \sigma_i \tanh\left(h + \frac{J}{N}\sum_{j \neq i} \sigma_j\right)\right]. \tag{2.42}$$

As mentioned before, if the Hamming distance between two states is larger than 1, the transition rates between those states are zero.

Theorem 2.36 *The invariant measure for the Glauber-process reads*

$$Q(\sigma) = \frac{e^{-H(\sigma)}}{Z}, \qquad H(\sigma) = -\frac{J}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i \sigma_j - h\sum_{i=1}^{N} \sigma_i,$$

where Z is the corresponding partition function.

Proof: As a sufficient condition we show the detailed balance equation,

$$Q(\sigma)\ \mathrm{rate}(\sigma \to \sigma') = Q(\sigma')\ \mathrm{rate}(\sigma' \to \sigma).$$

If transitions are not possible between two states, the rates are zero, and the equation is fulfilled. In case of non-zero transition rates, we write

$$\frac{\mathrm{rate}(\sigma \to \sigma')}{\mathrm{rate}(\sigma' \to \sigma)} = \frac{Q(\sigma')}{Q(\sigma)}.$$

As only transitions between states that differ in exactly one component are possible, and all individuals are exchangeable, we may w.l.o.g. assume that $\sigma_1 = -1$, $\sigma'_1 = 1$, and $\sigma_i = \sigma'_i$ for

$i = 2, \ldots, N$. Then,

$$\begin{aligned}
\frac{Q(\sigma')}{Q(\sigma)} &= \frac{e^{-H(\sigma')}}{e^{-H(\sigma)}} = \frac{\exp\left(\frac{J}{2N}\sum_{i=1}^N\sum_{j=1}^N \sigma_i'\sigma_j' + h\sum_{i=1}^N \sigma_i'\right)}{\exp\left(\frac{J}{2N}\sum_{i=1}^N\sum_{j=1}^N \sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right)} \\
&= \frac{\exp\left(\frac{J}{N}\sigma_1'\sum_{i=2}^N \sigma_i' + h\sigma_1'\right)\ \exp\left(\frac{J}{2N} + \frac{J}{2N}\sum_{i=2}^N\sum_{j=2}^N \sigma_i'\sigma_j' + h\sum_{i=2}^N \sigma_i'\right)}{\exp\left(\frac{J}{N}\sigma_1\sum_{i=2}^N \sigma_i + h\sigma_1\right)\ \exp\left(\frac{J}{2N} + \frac{J}{2N}\sum_{i=2}^N\sum_{j=2}^N \sigma_i\sigma_j + h\sum_{i=2}^N \sigma_i\right)} \\
&= \frac{\exp\left(\frac{J}{N}\sigma_1'\sum_{i=2}^N \sigma_i' + h\sigma_1'\right)}{\exp\left(\frac{J}{N}\sigma_1\sum_{i=2}^N \sigma_i + h\sigma_1\right)}.
\end{aligned}$$

The identity (which is given for $\alpha \in \mathbb{R}$ and $x = \pm 1$)

$$e^{\alpha x} = \cosh(\alpha\, x) + \sinh(\alpha\, x) = \cosh(\alpha) + x\sinh(\alpha) = \cosh(\alpha)(1 + x\tanh(\alpha))$$

and the observation $\sigma_i = \sigma_i'$ for $i > 1$, and $\sigma_1' = -\sigma_1$ implies ($i_0 := 1$)

$$\frac{Q(\sigma')}{Q(\sigma)} = \frac{1 + \sigma_{i_0}'\ \tanh\left(\frac{J}{N}\sum_{i=2}^N \sigma_i' + h\right)}{1 + \sigma_{i_0}\ \tanh\left(\frac{J}{N}\sum_{i=2}^N \sigma_i + h\right)} = \frac{1 - \sigma_{i_0}\ \tanh\left(\frac{J}{N}\sum_{i\neq i_0}^N \sigma_i + h\right)}{1 - \sigma_{i_0}'\ \tanh\left(\frac{J}{N}\sum_{i\neq i_0}^N \sigma_i' + h\right)} = \frac{\mathrm{rate}(\sigma\to\sigma')}{\mathrm{rate}(\sigma'\to\sigma)}.$$

□

Let us compare the transition probabilities of the Zealot model 2.1 (two opinions) and the Glauber dynamics. In the notation of the Zealot mode we used type A and type B to refer to the two opinions. In order to have the very same state space $\Sigma_N = \{-1, 1\}^N$ in both processes, we rename type A individual into type -1 individuals, and type B individuals into type 1 individuals. Accordingly, the number of type ± 1 zealots is denoted by $N_{\pm 1}$. With that change of notation, we have the following proposition.

Proposition 2.37 *Let $\sigma \in \Sigma_N$, $\ell \in \{\pm 1\}$, and $\overline{\ell} = flip(\ell)$. For the zealot model, the rate at which a given type-ℓ individual changes its type, given the state $\sigma \in \Sigma_N$, reads*

$$r_z(\ell, \sigma) = \mu\,\frac{\sum_{i=0}^N \delta(\sigma_i, \overline{\ell}) + N_{\overline{\ell}}}{N + N_{-1} + N_{+1}}. \tag{2.43}$$

For the Glauber dynamics, the rate at which a given individual with state ℓ jumps to state $\overline{\ell}$ is given by

$$r_G(\ell, \sigma) = \frac{\mu}{2}\left[1 - \tanh\left(2\,J\left(\frac{1}{2} - \frac{\sum_{i=1}^N \delta(\sigma_i, \overline{\ell})}{N}\right) + h\ell\right)\right] + \mathcal{O}(N^{-1}). \tag{2.44}$$

With

$$J = \frac{N}{N+N_{-1}+N_{+1}}, \quad h = \frac{N_{-1}-N_{+1}}{N+N_{-1}+N_{+1}} \tag{2.45}$$

we find

$$r_z(\ell,\sigma) = r_G(\ell,\sigma) + \mathcal{O}(J,h,N^{-1}).$$

Proof: *Zealot model.* According to the model description of the zealot model 2.1, an individual reconsiders at rate μ his/her opinion. He/she contacts a randomly selected individual (including him/herself and all zealots), and copies the opinion of that individual. As we have $\sum_{i=0}^{N}\delta(\sigma_i,\overline{\ell})$ individuals and $N_{\overline{\ell}}$ zealots with the opposite opinion, the result follows.

Glauber dynamics. Assume the number of the focal ℓ-individual is $i_0 \in \{1,\ldots,N\}$. Using the identity for the product of two integers $\ell, m \in \{\pm 1\}$

$$\ell\, m = 2\delta(\ell,m) - 1,$$

the rate for that individual to swap his/her state, as stated in eqn. (2.42) can be written as

$$\begin{aligned}
r_G(\ell,\sigma) &= \frac{\mu}{2}\left[1-\ell\,\tanh\left(h+\frac{J}{N}\sum_{j\neq i_0}\sigma_j\right)\right] = \frac{\mu}{2}\left[1-\,\tanh\left(h\,\ell+\frac{J}{N}\sum_{j\neq i_0}\ell\sigma_j\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(\ell h+\frac{J}{N}\sum_{j\neq i_0}(2\delta(\sigma_j,\ell)-1)\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(2\,J\left(\frac{\sum_{i=1}^{N}\delta(\sigma_i,\ell)}{N}-\frac{1}{N}-\frac{(N-1)}{2N}\right)+h\ell\right)\right]\\
&= \frac{\mu}{2}\left[1-\,\tanh\left(2\,J\left(\frac{1}{2}-\frac{\sum_{i=1}^{N}\delta(\sigma_i,\overline{\ell})}{N}\right)+h\ell\right)\right]+\mathcal{O}(N^{-1}),
\end{aligned}$$

where we used $\sum_{i=1}^{N}\delta(\ell,\sigma_i)+\sum_{i=1}^{N}\delta(\overline{\ell},\sigma_i) = N$ in the last step.

Comparison of r_z and r_G. As the first order Taylor approximation of $\tanh(x)$ at $x=0$ reads

$$\tanh(x) = x + \text{h.o.t.},$$

where h.o.t. represents the error term $\mathcal{O}(x^2)$, we find for J and h small that

$$r_G(\ell,\sigma) \;=\; \mu\left[\frac{1-J}{2}+J\,\frac{\sum_{i=1}^{N}\delta(\sigma_i,\overline{\ell})}{N}-\frac{h}{2}\,\ell\right]+\text{h.o.t.}$$

Note in particular that the parameter J determines the zero'th order term as well as the coefficient of the term $\sum_{i=1}^{N}\delta(\sigma_i,\ell)/N$. We will come back to this point.
We also rewrite $v_z(\ell,\sigma)$. Thereto we use that ($\ell\in\{\pm1\}$)

$$N_{\overline{\ell}} = \frac{(1-\ell)}{2}N_{+1}+\frac{(1+\ell)}{2}N_{-1} = \frac{1}{2}(N_{+1}+N_{-1})+\ell\,\frac{1}{2}(N_{-1}-N_{+1}).$$

Therewith,

$$\begin{aligned}
r_z(\ell,\sigma) &= \mu\,\frac{\sum_{i=0}^{N}\delta(\sigma_i,\overline{\ell})+N_{\overline{\ell}}}{N+N_{-1}+N_{+1}} = \mu\left[\frac{N}{N+N_{-1}+N_{+1}}\,\frac{\sum_{i=0}^{N}\delta(\sigma_i,\overline{\ell})}{N}+\frac{N_{\overline{\ell}}}{N+N_{-1}+N_{+1}}\right]\\
&= \mu\left[\frac{N_{+1}+N_{-1}}{2(N+N_{-1}+N_{+1})}+\frac{N}{N+N_{-1}+N_{+1}}\,\frac{\sum_{i=0}^{N}\delta(\sigma_i,\overline{\ell})}{N}+\frac{N_{-1}-N_{+1}}{2(N+N_{-1}+N_{+1})}\,\ell\right]
\end{aligned}$$

If we now define

$$J = \frac{N}{N + N_{-1} + N_{+1}}, \quad h = \frac{N_{-1} - N_{+1}}{N + N_{-1} + N_{+1}} \quad \Rightarrow \quad 1 - J = \frac{N_{-1} + N_{+1}}{N + N_{-1} + N_{+1}}.$$

We obtain

$$r_z(\ell, \sigma) \quad = \quad \mu \left[\frac{1 - J}{2} + J \frac{\sum_{i=0}^{N} \delta(\sigma_i, \overline{\ell})}{N} + \frac{h}{2} \ell \right] = r_G(\ell, \sigma) + \mathcal{O}(J, h, N^{-1}).$$

$\square$

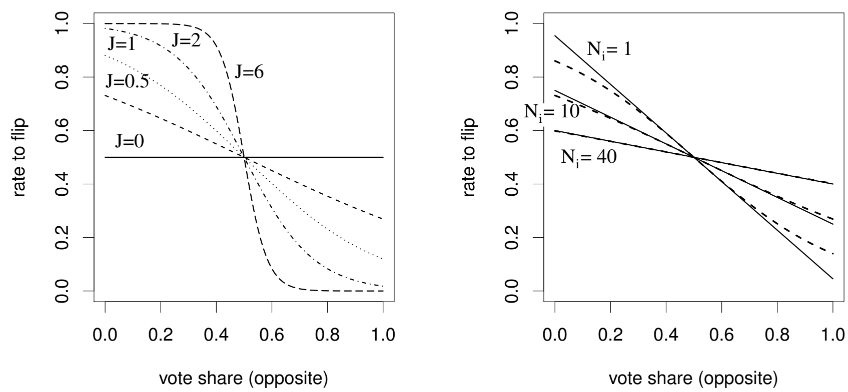


Figure 2.7: Left panel: Flip rates for the Glauber dynamics, $h = 0$, and J as indicated. Right panel: Comparison of flip rates Glauber dynamics (dashed) and zealot model (solid line) for three cases: We always did choose $h = 0$, and $N = 20$. Furthermore, $N_1 = N_2 = 1$; $N_1 = N_{-1} = 10$; $N_1 = N_{-1} = 40$; J is chosen according to $J = (N/(N + N_1 + N_{-1}))$. The correspondence of $J = 1$ and the voter model (proposition 2.37) is nicely visible.

Remark 2.38 *(a) Please note that the correspondence of the two models (for J and h small, and N large) is not trivial. The zealot model has two parameters ($N_{\pm 1}$), as well as the Glauber model (J and h). It might appear to be natural that these parameters can be identified. However, in the proof we find that we an affine-linear expression in $\sum_{i=0}^{N} \delta(\sigma_i, \overline{\ell})/N$ and ℓ needs to coincide in the two models to establish a first order approximation, s.t. three conditions are to be fulfilled. It is more than a coincidence that the three conditions can be satisfied by the choice of two parameters only.*
(b) The choices of J and h are natural: J represents the coupling of individuals in the Glauber dynamics, that is, the intensity individuals do interact. In the zealot model, $N/(N + N_{+1} + N_{-1})$ is the probability that a focal individual does not interact with a zealot, but with another individual. In that, this fraction also expresses the intensity of interactions in the voter model.
The external field h in the Glauber model indicates the advantage one opinion $\in \{0, 1\}$ above

the other state due to external forces and independent on the actual state of the system. In the very same way, $(N_{-1} - N_{+1})/(N + N_{-1} + N_{+1})$ quantifies the normalized advantage of group -1 above group $+1$ in the zealot model.
(c) Due to the definition of J, the correspondence between Glauber dynamics and voter model is only possible for $J \leq 1$. That condition shows that a Glauber dynamics resembling a zealot model is far away from the phase transition/bifurcation at $J = 2$.

In the Glauber dynamics as well as in the voter processes, individuals orient themselves at the fractions of individuals with the opposite opinion. While the voter model indicates that the probability to change the mind increases linear with the fraction of individuals with the opposite opinion, in the Glauber dynamics we have a nonlinear function (see figure 2.7). For $J = 0$, the individuals flip their mind independently of all other individuals. If J increases, the Glauber rate approaches a threshold function: In the limit $J \to \infty$ (the so-called zero temperature limit), all individuals jump to the majority opinion. The voter model is based on pairwise interactions, while in the Glauber model (for J large), individuals orient themselves at the majority. The latter mechanism allows for the phase transition. If J is sufficiently large, the equilibrium situation, where both opinions are equally present, is destabilized in favor of one of states where either the one opinion is predominating.

Corollary 2.39 *If J is small, the Glauber dynamics resembles a zealot model, in the low temperature limit, $J \to \infty$, the Glauber dynamics approximates a majority rule.*

We reformulate the Gaubers dynamics in a way that is close to the formulation of the zealots model as stated in model 2.1.

Remark 2.40 *Let X_t denote the number of individuals with state 1 at time t, s.t. we have $N - X_t$ individuals with state -1. Then,*

$$X_t \to X_t + 1 \quad \text{at rate} \quad \mu\,(N - X_t)\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{X_t}{N} - \frac{N+1}{2N}\right)\right) + h\right] \tag{2.46}$$

$$X_t \to X_t - 1 \quad \text{at rate} \quad \mu\,X_t\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{N-1}{2N} - \frac{X_t}{N}\right)\right) + h\right]. \tag{2.47}$$

Carry out!!!
The deterministic limit for large population size addresses $x(t) = X_t/N$, and is given by

$$\frac{d}{dt}x = \mu(1-x)\frac{1}{2}\left[1 - \tanh\left(2\,J\left(x - \frac{1}{2}\right)\right) + h\right] - \mu\,x\,\frac{1}{2}\left[1 - \tanh\left(2\,J\left(\frac{1}{2} - x\right)\right) + h\right] \tag{2.48}$$

For $h = 0$, this ODE undergoes a Pitchfork bifurcation at $x = 1/2$, and $J = 2$.

2.6.2 Glauber dynamics for the q-opinion Curie-Weiss model

Recall that

$$N_\ell(\sigma) = \sum_{i=1}^{N} \delta(\sigma_i, \ell).$$

Model 2.41 Glauber dynamics for the q-Potts model. *Let $\Sigma_N = \{1,\ldots,q\}^N$, $\sigma,\sigma' \in \Sigma_N$. If the Hamming distance between σ and σ' is unequal 1, the transition rates for $\sigma \to \sigma'$ and $\sigma' \to \sigma$ are zero. Let the Hamming distance be 1, and $i_0 \in \{1,\ldots,N\}$ denote the single individual/site where the two states disagree: $\sigma_{i_0} = k \in \{1,\ldots,q\}$, and $\sigma_{i_0} = k' \in \{1,\ldots,q\}$, where $k \neq k'$. Then,*

$$\textit{transition rate } \sigma \to \sigma' \qquad \textit{is} \qquad \mu\,\delta(\sigma_{i_0},k)\exp\left(J\left(\frac{N_{k'}(\sigma)}{N}-1\right)+h_{k'}\right). \tag{2.49}$$

Theorem 2.42 *The invariant measure of the Glauber process for the q-Potts model reads $Q(\sigma) = e^{-H(\sigma)}/Z$ with*

$$H(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma). \tag{2.50}$$

Z denotes the partition function.

Proof: We first note that

$$H(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q} N_\ell(\sigma)^2 - \sum_{\ell=1}^{q} h_\ell N_\ell(\sigma) = -\frac{J}{2N}\sum_{\ell=1}^{q}\sum_{i,j=1}^{N}\delta(\sigma_i,\ell)\delta(\sigma_j,\ell) - \sum_{\ell=1}^{q} h_\ell\sum_{i=1}^{N}\delta(\sigma_i,\ell).$$

The proof parallels that of theorem 2.36: We use the detailed balance equation. Assume that σ,σ' only differ in component/individual i_0, and $\sigma_{i_0} = k$, $\sigma'_{i_0} = k'$ (where $k,k' \in \{1,\ldots,q\}$, $k \neq k'$). Then,

$$\begin{aligned}
\frac{Q(\sigma)}{Q(\sigma')} &= \frac{\exp\left(\frac{J}{2N}\sum_{\ell=1}^q\sum_{i,j=1}^N\delta(\sigma_i,\ell)\delta(\sigma_j,\ell)+\sum_{\ell=1}^q h_\ell\sum_{i=1}^N\delta(\sigma_i,\ell)\right)}{\exp\left(\frac{J}{2N}\sum_{\ell=1}^q\sum_{i,j=1}^N\delta(\sigma'_i,\ell)\delta(\sigma'_j,\ell)+\sum_{\ell=1}^q h_\ell\sum_{i=1}^N\delta(\sigma'_i,\ell)\right)}\\
&= \frac{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k)+\frac{J}{2N}+h_k\right)}{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k')+\frac{J}{2N}+h_{k'}\right)} = \frac{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k)+h_k\right)}{\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k')+h_{k'}\right)}\\
&= \frac{\delta(\sigma'_{i_0},k')\,\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma'_j,k)+h_k-J\right)}{\delta(\sigma_{i_0},k)\,\exp\left(\frac{J}{N}\sum_{j\neq i_0}\delta(\sigma_j,k')+h_{k'}-J\right)}\\
&= \frac{\text{rate }\sigma'\to\sigma}{\text{rate }\sigma\to\sigma'}.
\end{aligned}$$

We used in that computation that $\sigma_i = \sigma'_i$ for $i \neq i_0$, and $\sigma_{i_0} = k \neq k'$: Therewith, we have $\delta(\sigma_{i_0},k) = 1$ and $\sum_{j\neq i_0}\delta(\sigma_j,k') = N_{k'}(\sigma)$.

□

We proceed to show an asymptotic relation between the multi-party zealot model and the multi-opinion Glauber model.

Remark 2.43 *Consider the situation with* q *parties,* $\Sigma_N = \{1,\ldots,q\}^N$. *Let* $\vec{X} = (X_1,\ldots,X_q)$ *denote the number of individuals structured by the* q *groups. The number of individuals is given by* $N = \sum_{i=1}^q X_i$. *For the Sano resp. the Glauber model, any transition affects two groups only – one group decreases by one, the other increases by one.*
Recall the Sano model 2.7: The number of zealots of group i *are* N_i. *Only a fraction* θ *of zealots influences the voters, and we define* $M = N + \theta\sum_{i=1}^q N_i$. *The transition rate* $(\ldots,X_i,\ldots,X_j,\ldots) \to (\ldots,X_i+1,\ldots,X_j-1,\ldots)$ *reads*

$$r_z(i,j,\vec{X}) \;=\; \mu X_j \frac{X_i+\theta N_i}{M}. \tag{2.51}$$

According to eqn. (2.49), the transition rate for $(\ldots,X_i,\ldots,X_j,\ldots) \to (\ldots,X_i+1,\ldots,X_j-1,\ldots)$ *in the Glauber dynamics is given by*

$$r_G(i,j,\vec{X}) \;=\; \mu X_j \exp\left(J\left(\frac{X_i}{N}-1\right)+h_i\right). \tag{2.52}$$

Proposition 2.44 *With the choice*

$$J = \frac{N}{M}, \qquad h_i = -\theta\sum_{j\neq i}^q \frac{N_j}{M} \qquad \text{for } i = 1,\ldots,q \tag{2.53}$$

we find

$$r_z(i,j,\vec{X}) = r_G(i,j,\vec{X}) + \mathcal{O}(J,h_1,\ldots,h_q).$$

Proof: As in the Sano model as well as the Glauber dynamics, μX_j appears as an multiplicative factor, we investigate the remaining part of the transition rates.
Sano model:

$$r_z(i,j,\vec{X}) = \frac{X_i+\theta N_i}{M} = \frac{N}{M}\,\frac{X_i}{N} + \theta\frac{N_i}{M}.$$

Glauber Dynamics: First order expansion of the exponential function yields

$$r_G(i,j,\vec{X}) \;=\; \exp\left(J\left(\frac{X_i}{N}-1\right)+h_i\right) = h_i + 1 - J + J\frac{X_i}{N} + \text{h.o.t.}$$

Comparison suggests the choice $J = \frac{N}{M}$ and $h_i = -\theta\sum_{j\neq i}^q \frac{N_j}{M}$ as

$$1 + h_i - J \;=\; 1 - \theta\sum_{j\neq i}^q \frac{N_j}{M} - \frac{N}{M} = \theta\sum_{j=1}^q \frac{N_j}{M} - \theta\sum_{j\neq i}^q \frac{N_j}{M} = \theta\frac{N_i}{M}$$

s.t. $r_z(i,j,\vec{X}) = r_G(i,j,\vec{X}) + \mathcal{O}(J,h_1,\ldots,h_q)$.

□

Carry out!!!
Deterministic limit, Saddle-Node bifurcations for $q > 2$.

2.6.3 Echo chambers and reinforcement

Echo chambers and epistemic bubbles not only exist since the appearance of the internet, but even before electronic mass media became abundant, newspapers targeting on a certain clientele, clubs, or simply the social environment provided echo chambers. Though the existence of echo chambers are statistically confirmed [17, 9], the strength of their effect is under debate [11]. In the present section, we aim to modify the voter model to cover basic effects of echo chambers and reinforcement. Persons who "life" in an echo chamber will not interact with a representative sample of the population, but the sample of persons who are more likely to share his/her opinion. We model that fact by adapting the two-opinion zealot model in weighting the subpopulation with the opposite population by the probability $\vartheta_i < 1$ to interact with those individuals.

Model 2.45 Zealot model with reinforcement. *Let N denote the total population size, N_i the number of zealots for opinion $i \in \{1, 2\}$, and ϑ_i the weights for the opposite opinion. If X_t is the number of supporters for opinion 1, then*

$$X_t \to X_t + 1 \quad \text{at rate} \quad \mu(N - X_t)\,\frac{\vartheta_1(X_t + N_1)}{\vartheta_1(X_t + N_1) + (N - X_t + N_2)}, \tag{2.54}$$

$$X_t \to X_t - 1 \quad \text{at rate} \quad \mu X_t\,\frac{\vartheta_2(N - X_t + N_2)}{(X_t + N_1) + \vartheta_2(N - X_t + N_2)}. \tag{2.55}$$

Note that ϑ_1 is the probability og group-2-individuals to interact with group 1, and ϑ_2 that of group-1-members to interact with group 2. Obviously, this model agrees with the zealot model in case of $\vartheta_1 = \vartheta_2 = 1$.

In the discussion of the zealot model, we have seen that two different scalings of the zealot's numbers N_i are possible (Section 2.3.4): Either N_i are constant in N (weak limit), or they scale linearly with N, s.t. $N_i = n_i\,N$ (deterministic limit). Only the latter case yields for $N \to \infty$ an ODE - the first case resulted in the Dirichlet/beta distribution. In order to better understand the consequences of the mechanism proposed, we first consider the deterministic limit.

Deterministic limit

Proposition 2.46 *Let $N_i = n_i\,N$. Then, the deterministic limit for $x(t) = X_t/N$ reads*

$$\dot{x} \;=\; -\mu x\,\frac{\vartheta_2(1 - x + n_2)}{(x + n_1) + \vartheta_2(1 - x + n_2)} + \mu(1 - x)\,\frac{\vartheta_1(x + n_1)}{\vartheta_1(x + n_1) + (1 - x + n_2)}. \tag{2.56}$$

For $n_1 = n_2 = n$ and $\vartheta_1 = \vartheta_2$, $x = 1/2$ always is a stationary point; this stationary point undergoes a pitchfork bifurcation at $\vartheta_1 = \vartheta_2 = \vartheta_p$, where

$$\vartheta_p \;=\; \frac{1 - 2n}{1 + 2n}. \tag{2.57}$$

Proof: The rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1 - x)\,\frac{\vartheta_1(x + n_1)}{\vartheta_1(x + n_1) + (1 - x + n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1 - x + n_2)}{(x + n_1) + \vartheta_2(1 - x + n_2)}.$$

Therewith, the Fokker-Planck equation for the large population size (Kramers-Moyal expansion) reads

$$\partial_t u(x,t) = -\partial_x((f_+(x) - f_-(x))\, u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x) + f_-(x))\, u(x,t))$$

and the ODE due to the drift term in case of $N \to \infty$ reads

$$\frac{d}{dt}x = f_+(x) - f_-(x).$$

This result establishes eqn. (2.56). For the following, let $\vartheta_1 = \vartheta_2 = \vartheta$. If we also choose $n_1 = n_2 = n$, we have a neutral model, and $x = 1/2$ is a stationary point for all $\vartheta \geq 0$, $n > 0$. We find the Taylor expansion of the r.h.s. at $x = 1/2$ (using the computer algebra package maxima [29])

$$\begin{aligned} \mu^{-1}\frac{d}{dt}x &= -x\,\frac{\vartheta(1-x+n_2)}{(x+n_1)+\vartheta(1-x+n_2)} + (1-x)\,\frac{\vartheta(x+n_1)}{\vartheta(x+n_1)+(1-x+n_2)} \\ = \quad & -2\,\vartheta\,\frac{(2\,n+1)\,\vartheta+(2\,n-1)}{(2\,n+1)\,(\vartheta+1)^2}\left(x-\frac{1}{2}\right) + \frac{32\,\vartheta\,(\vartheta+n-\vartheta^2(n+1))}{(2\,n+1)^3\,(\vartheta+1)^4}\left(x-\frac{1}{2}\right)^3 + \mathcal{O}((x-1/2)^4) \end{aligned}$$

For $\vartheta \in (0,1)$, $n > 0$, the coefficient in front of the third order term always is non-zero, while the coefficient in front of the linear term becomes zero at $\vartheta = \vartheta_p$. Hence, we have a pitchfork bifurcation at that parameter.

□

The pitchfork bifurcation is unstable against any perturbation that breaks the symmetry $x \mapsto 1-x$ (Fig. 2.8). In panel (a), we have the symmetric case, and find the proper pitchfork bifurcation. Panel (b) shows the result if the number of zealots only differs slightly, where the reinforcement-parameter for both groups are assumed to be identical. We still find a reminiscent of the pitchfork bifurcation: The stable branches in (b) are close to the stable branches in (a), and also the unstable branches correspond to each other. For the limit $n_2 \to n_1$, panel (b) converges to panel (a). However, the branches are not connected any more but dissolve in two unconnected parts, and the pitchfork bifurcation is replaced by a saddle-node bifurcation.
In panel (c) and (d), the upper branch visible in panel (b) did vanish, and only the lower branch is present. As ϑ_2 is kept constant ($\vartheta_2 = 0.5$ in panel (c) and $\vartheta_2 = 0$ in panel (d)) and only ϑ_1 does vary, there is no continuous transition to panel (a).

The effect of reinforcement for a given group is similar to an increase of the number of that group's zealots. If the reinforcement becomes strong, the fraction of the group that performes reinforcement is able to become large. In panel (d), the second group has only 1/10 of the zealots of the first group, but is able to take over if the members of that group do an extreme reinforcement ($\vartheta_1 \ll 1$). However, if both groups reinforce themselves, the mechanism is kind of symmetrical, with a bistable setting as the consequence.

Weak limit

We now turn to the second scaling – the effect of zealots, and also the effect of the echo chambers, are taken to be weak. Under these circumstances, it is possible to find a limiting distribution.

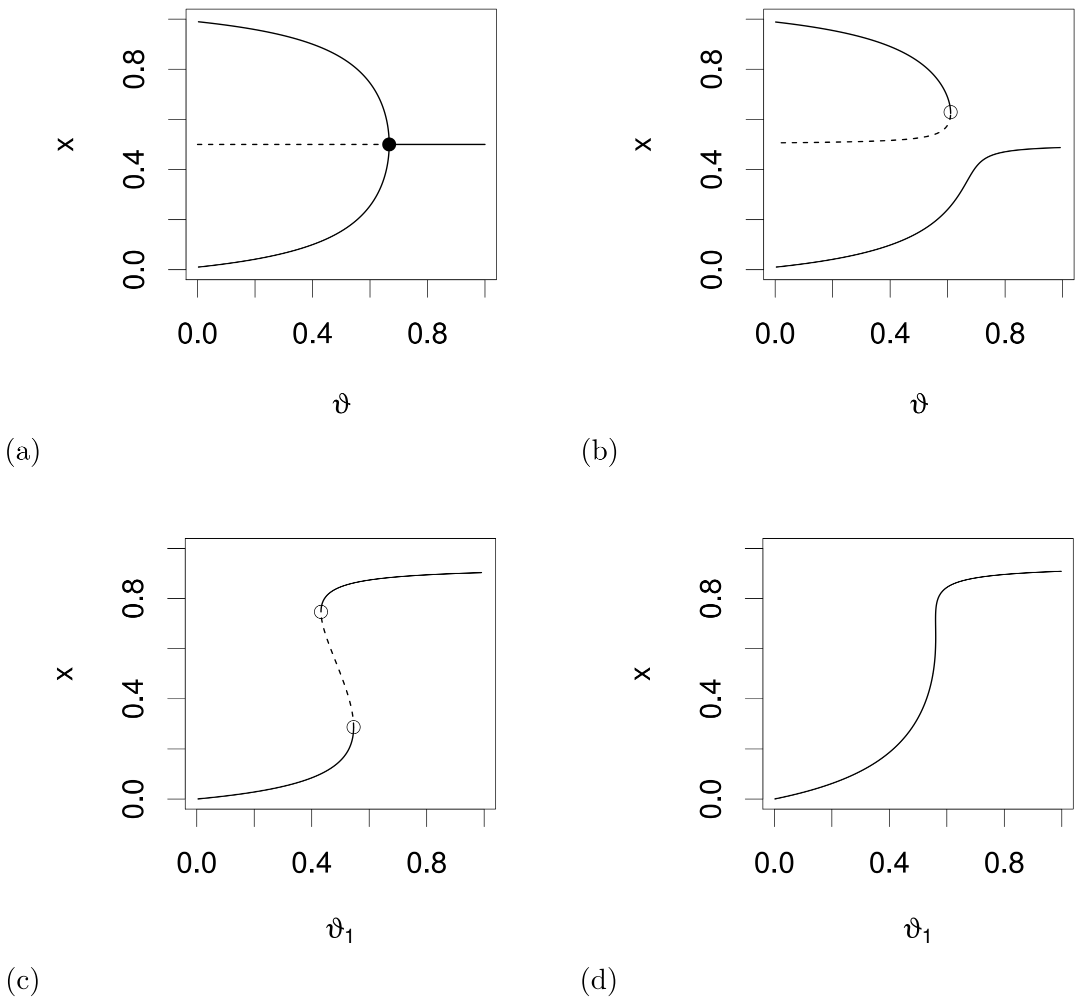


Figure 2.8: Stationary points of the reinforcement model over ϑ. The pitchfork bifurcation in (a) is indicated by a bullet, the saddle-node bifurcations in (b) and (c) are indicated by open circles. Stable branches of stationary points are represented by solid lines, unstable branches by dotted lines. (a) $n_1 = n_2 = 0.1$, $\vartheta_1 = \vartheta_2 = \vartheta$, (b) $n_1 = 0.1$, $n_2 = 0.105$, $\vartheta_1 = \vartheta_2 = \vartheta$, (c) $n_1 = n_2 = 0.1$, $\vartheta_2 = 0.5$, (d) $n_1 = 0.2$, $n_2 = 0.02$, $\vartheta_1 = 1.0$.

Theorem 2.47 *Let N_i denote the number of zealots for group i, N the population size, and $\vartheta_i = 1 - \theta_i/N$ the parameter describing reinforcement.In the limit $N \to \infty$, the density of the invariant measure for the random variable $z_t = X_t/N$ is given by*

$$\varphi(x) = C\, e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1 x}\; x^{N_1-1}\,(1-x)^{N_2-1}, \tag{2.58}$$

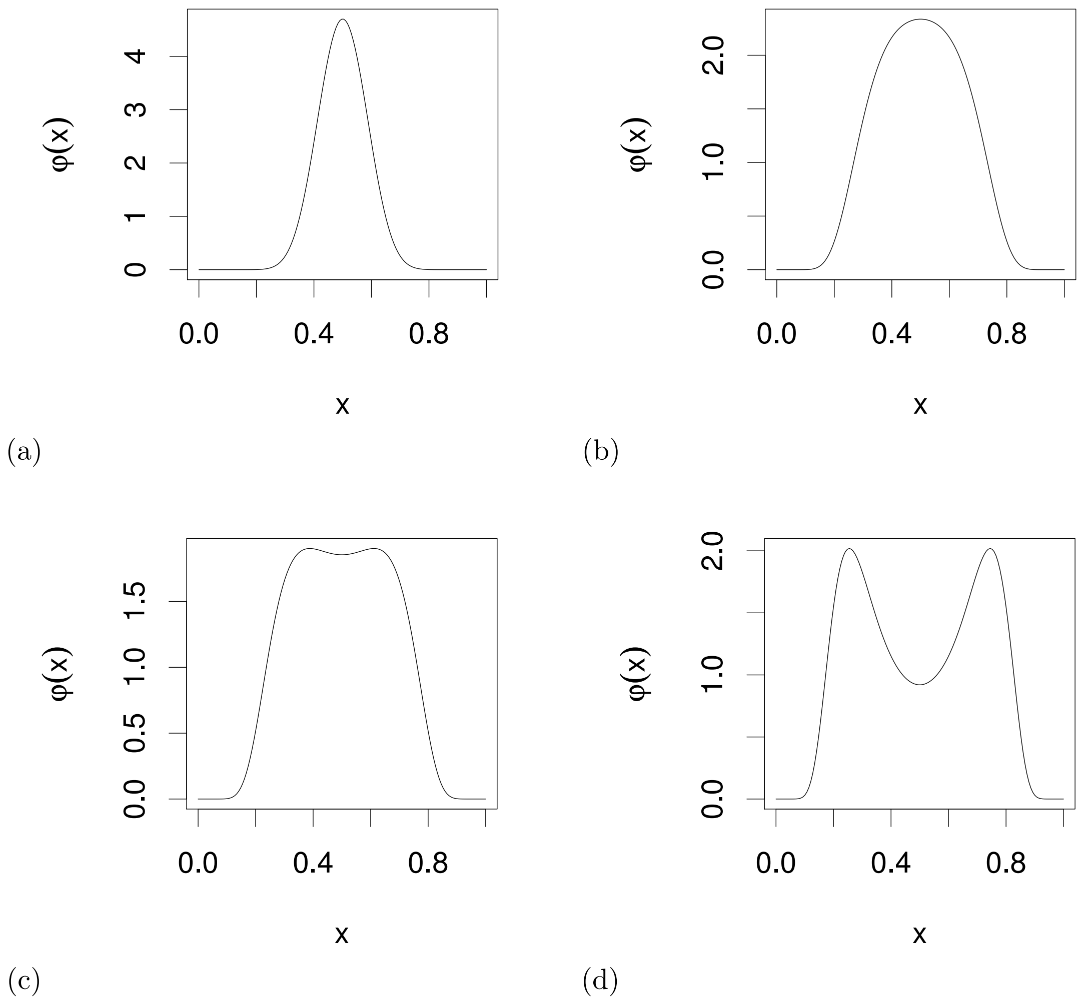


Figure 2.9: Invariant distribution, given in eqn. (2.58) for $N_1 = N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 70$, (c) $\theta_1 = \theta_2 = 80$, (d) $\theta_1 = \theta_2 = 100$.

where C is determined by the condition $\int_0^1 \varphi(x)\,dx = 1$.

Proof: We again start off with the Fokker-Planck equation, obtained by the Kramers-Moyal expansion, where we use the scaling $n_i = N_i/N$, and ϑ_i constant in N. Only afterwards, we proceed to the desired scaling.
As seen above, the rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1-x)\,\frac{\vartheta_1(x+n_1)}{\vartheta_1(x+n_1)+(1-x+n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1-x+n_2)}{(x+n_1)+\vartheta_2(1-x+n_2)}.$$

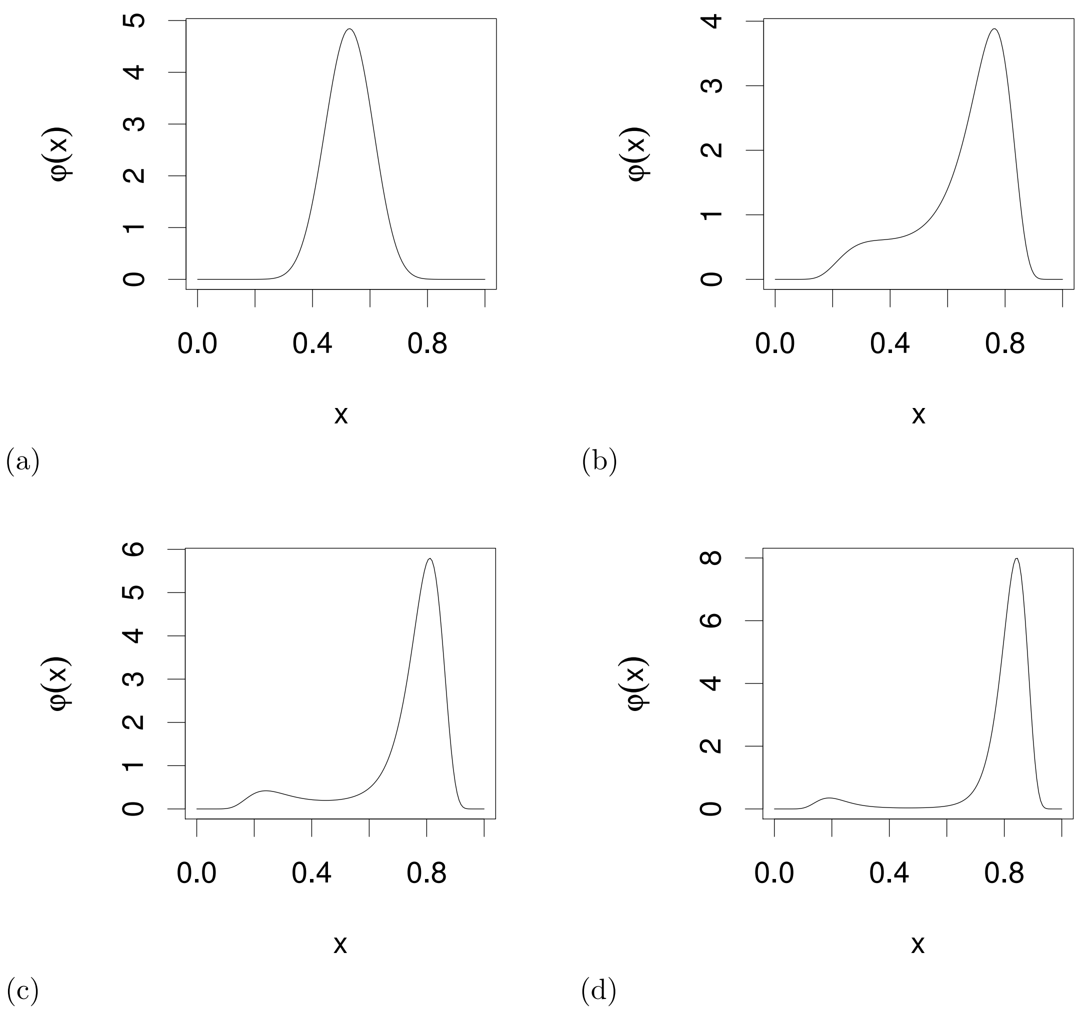


Figure 2.10: Invariant distribution, given in eqn. (2.58) for $N_1 = 22$, $N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 100$, (c) $\theta_1 = \theta_2 = 120$, (d) $\theta_1 = \theta_2 = 140$.

Therewith, the limiting Fokker-Planck equation reads

$$\partial_t u(x,t) = -\partial_x((f_+(x) - f_-(x))\, u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x) + f_-(x))\, u(x,t))$$

Now we rewrite drift and noise term with the new scaling $n_i = N_i/N$, $\vartheta_i = 1 - \theta_i/N$, where we neglect terms of order $\mathcal{O}(N^{-2})$. We find (using maxima [29]) that ($h := 1/N$)

$$\begin{aligned}
& f_+(x) - f_-(x) \\
= \;& \mu(1-x)\,\frac{(1-h\,\theta_1)(x+h\,N_1)}{(1-h\,\theta_1)(x+h\,N_1)+(1-x+h\,N_2)} - \mu x\,\frac{(1-h\,\theta_2)(1-x+h\,N_2)}{(x+h\,N_1)+(1-h\,\theta_2)(1-x+h\,N_2)} \\
= \;& \mu\bigg([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x) - (N_1+N_2)\,x + N_1\bigg)\,h + \mathcal{O}(h^2),
\end{aligned}$$

while $h(f_+(x)+f_-(x)) = h\,2\,\mu x(1-x) + \mathcal{O}(h^2)$. If we rescale time, $T = \mu\,h\,t$, the Fokker-Planck equation becomes

$$\partial_T u(x,T) = -\,\partial_x\bigg\{\,\bigg([(\theta_1{+}\theta_2)x{-}\theta_1]\,x\,(1{-}x){-}(N_1{+}N_2)\,x{+}N_1\bigg)\;u(x,T)\;\bigg\}{+}\partial_x^2\bigg\{x\,(1{-}x)\,u(x,T)\bigg\}.$$

For the invariant distribution $u(x)$, the flux of that rescaled Fokker-Planck equation is zero, that is,

$$-\bigg([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x) - (N_1+N_2)\,x + N_1\bigg)u(x) + \frac{d}{dx}\bigg(x(1-x)\,u(x)\bigg) = 0.$$

With $v(x) = x(1-x)u(x)$, we have

$$v'(x) = \bigg([(\theta_1+\theta_2)x-\theta_1]\; + \frac{N_1}{x} - \frac{N_2}{1-x}\,x\bigg)v(x)$$

and hence

$$v(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1}\,(1-x)^{N_2}$$

resp.

$$u(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1-1}\,(1-x)^{N_2-1}$$

$\square$

For $\theta_1 = \theta_2 = 0$, we obtain the beta distribution, as we fall back to the zealot model without reinforcement. In the given scaling, the reinforcement is expressed by the exponential multiplicative factor. As $\vartheta_i = 1 - h\,\theta_i$, and h small, one could be tempted to assume that we are in the subcritical parameter range of the reinforcement model only, s.t. the distribution does not show a phase transition. As we see next, this idea is wrong.

Let us first consider the symmetric case, $N_1 = N_2 = \underline{N}$, and $\theta_1 = \theta_2 = \underline{\theta}$ (see Fig. 2.9). In that case, the distribution is given by

$$\varphi(x) = C\,e^{\underline{\theta}\,x\,(1-x)}\;x^{\underline{N}-1}\,(1-x)^{\underline{N}-1}.$$

The function always is symmetric w.r.t. $x = 1/2$. If $\underline{\theta}$ is small, and $\underline{N} > 0$, we find an unimodal function, with a maximum at 1/2. If, however, $\hat{\theta}$ is increased, eventually a bimodal distribution appears – we find back the pitchfork bifurcation that we already known from the deterministic limit of the model (Fig. 2.8, panel a).

As soon as $N_1 \neq N_2$, the symmetry is broken (Fig. 2.10), and we have an a situation resembling Fig. 2.8, panel (b). In the stochastic setting, however, we have more information: the second branch concentrates only little probability mass, and will play in practice only a minor role (if any at all). Only if N_1 and N_2 are distinctively unequal, this second branch ias able to concentrate sufficient probability mass to gain visibility in empirical data.

Remark 2.48 *We can use the zealot model or we can use the reinforcement model to fit and interpret election data. The zealot model for two parties yields the beta distribution. The density of the reinforcement model basically consist of a product, where one term is identical with the beta distribution,*

$$x^{N_1-1}\,(1-x)^{N_2-1}$$

while the second term expresses the influence of reinforcement

$$e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}.$$

Only if the data have a shape that is different from that of a beta distribution, the reinforcement component leads to a significantly improved fit. This is given, e.g., in case of a bimodal shape of the data (where at least one maximum is in the interior of the interval $(0,1)$), or if the data have heavy tails. Both properties hint to the fact that the election districts are of two different types: one, where the party under consideration is relatively strong, and one where it is relatively weak. This difference, in turn, can be interpreted as the effect of reinforcement: In some election districts voters agree that the given party is preferable, in others they agree that the party is to avoid. The population is not (spatially) homogeneous, but some segregation - most likely caused by social mechanisms - take place. In that, the data analysis of spatially structured election data (results structured by election districts) based on the reinforcement model is able to detect spatial segregation and the consequences thereof.

carry out Glauber/Reinf formulas!

Remark 2.49 *We compare the reinforcement model with the Glauber dynamics. For $\vartheta_\ell = 1$, we fall back to the zealot model. In that, we have proposition 2.37: The rates of Glauber dynamics and zealot model agree up to higher order (if the parameters J and h of the Glauber model are chosen appropriately). If we have $\vartheta_\ell = 1-\omega\,\theta_\ell$, and ω small, we can use Taylor expansion in ω to find a connection between Glauber dynamics and the reinforcement model. It turns out that the zwero order terms in ω do not carry any information about the reinforcement, and these terms can be approximated by the Glauber dynamics (proposition 2.37). In contrast, the first order terms have a structure that is incompatible with the Glauber rates. Though the effect of reinforcement resembles that of the Curie-Weiss model (phase transitions), the mechanism is different. In the Curie-Weiss model, a majority rule leads to the phase transition. In the reinforcement model, the ignorance of the opposite group is the driving force that induces phase transitions. Apart of that, the reinforcement model still is a variant of the voter model.*

Data analysis

Parameter estimation: The maximum likelihood parameter estimation is somewhat subtle, as the distribution

$$\varphi(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1-1}\,(1-x)^{N_2-1},$$

incorporates exponential terms - in particular, if the parameters become large, the integral $\int_0^1 e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1-1}\,(1-x)^{N_2-1}\,dx$ becomes numerically unstable. Therefore, we reparameterize the distribution, defining $\hat{\nu}$, $\hat{s}$, $\hat{\theta}$, and $\hat{\psi}$ by

$$\theta_1 = \hat{s}\,\hat{\theta}\,\hat{\psi}, \quad \theta_2 = \hat{s}\,\hat{\theta}\,(1-\hat{\psi}), \quad N_1 + 1 = \hat{s}\,(1-\hat{\theta})\,\hat{\nu}, \quad N_2 + 1 = \hat{s}\,(1-\hat{\theta})\,(1-\hat{\nu}), \tag{2.59}$$

where $\hat{\theta}, \hat{\psi}, \hat{\nu} \in [0,1]$, and $\hat{s} > 0$, with the restriction $\hat{s}\,(1-\hat{\theta})\,\hat{\nu} > 1$, and $\hat{s}\,(1-\hat{\theta})\,(1-\hat{\nu}) > 1$. Therewith, the distribution becomes

$$\varphi(x) = \hat{C}\,\exp\left[\;\hat{s}\;\left(\hat{\theta}(x^2/2 - \hat{\psi}\,x)\; + (1-\hat{\theta})\,\hat{\nu}\,\ln(x) + (1-\hat{\theta})\,(1-\hat{\nu})\,\ln(1-x) + A\right)\;\right].$$

Here, A is a constant that can be chosen in dependency on the parameters and the data at hand. In practice, it is used to avoid an exponent that has a very large absolute number. The constant $\hat{C}$ is, as before, determined by the fact that the integral is one. This form allows for a reasonable maximum likelihood estimation, given appropriate election data.

Numerical issues: The model assumes continuous data, while the election data are discrete. Therefore, a vote share of 0 or 1 is possible in the empirical data, but the distribution may have poles for those values. We replace all empirical vote shares below 10^{-10} by 10^{-10}, and similarly, all data above $1 - 10^{-10}$ by $1 - 10^{-10}$. In order to determine the normalization constant of $\varphi(x)$, we do not integrate from 0 to 1, but only from 0.001 to 0.999. Furthermore, for numerical reasons, we restrict $\hat{s}$ by an upper limit, that we mostly define as 1800.

Test for reinforcement: The zealot model and the reinforcement model are nested. In that, we can use the likelihood-ratio test to check for the significance of the reinforcement component: If $\mathcal{LL}_0$ is the log-likelihood for the restricted model ($\theta_1 = \theta_2 = 0$, resp. $\hat{\theta} = 0$), and $\mathcal{LL}$ is that for the full reinforcement model, we have asymptotically, for a large sample size

zitat likelihood ratio test

$$2(\mathcal{LL} - \mathcal{LL}_0) \sim \chi^2_2$$

That is, twice the difference in the log-likelihoods is asymptotically χ^2 distributed, where the degree of freedom is the number of the surplus parameters (here: θ_1 and θ_2, resp. $\hat{\theta}$ and $\hat{\psi}$, that is, the degree of freedom is 2).
Additionally, we use the Kolmogorov-Smirnov test to find out if the model-distribution of either the full reinforcement mode, or the zealot model ($\theta_1 = \theta_2 = 0$, resp. $\hat{\theta} = 0$) agrees with the empirical distribution. If both models are in line with the data, then the reinforcement component will rather not add to the interpretation of the data, if only the reinforcement model approximates the data well (or, at least, much better than the zealot model), we can expect that it is sensible to take the reinforcement component into account.

Graphical representation: Apart of the histograms with the empirical distributions of the vote shares, and the distribution according to the reinforcement model, we will focus on the reinforcement parameter of the group at hand θ_2. As $\vartheta_2 = 1 - \theta_2/N$ is the decrease in the probability that an individual of the focal group interacts with an individual of the opposite group, θ_2 is a measure for the reinforcement for the focal group. Furthermore, we will consider $\theta_1 + \theta_2$, which is the total amount of reinforcement in both groups. Last, we investigate

$$\overline{\theta} = \sum_{party} \text{vote share of the party } \times\; \theta_2(party).$$

$\overline{\theta}$ is a measure for the overall degree of reinforcement in the population. Herein, we dismiss parties with a vote share below 0.1%, and parties that have a non-zero vote share in less than 20 districts.

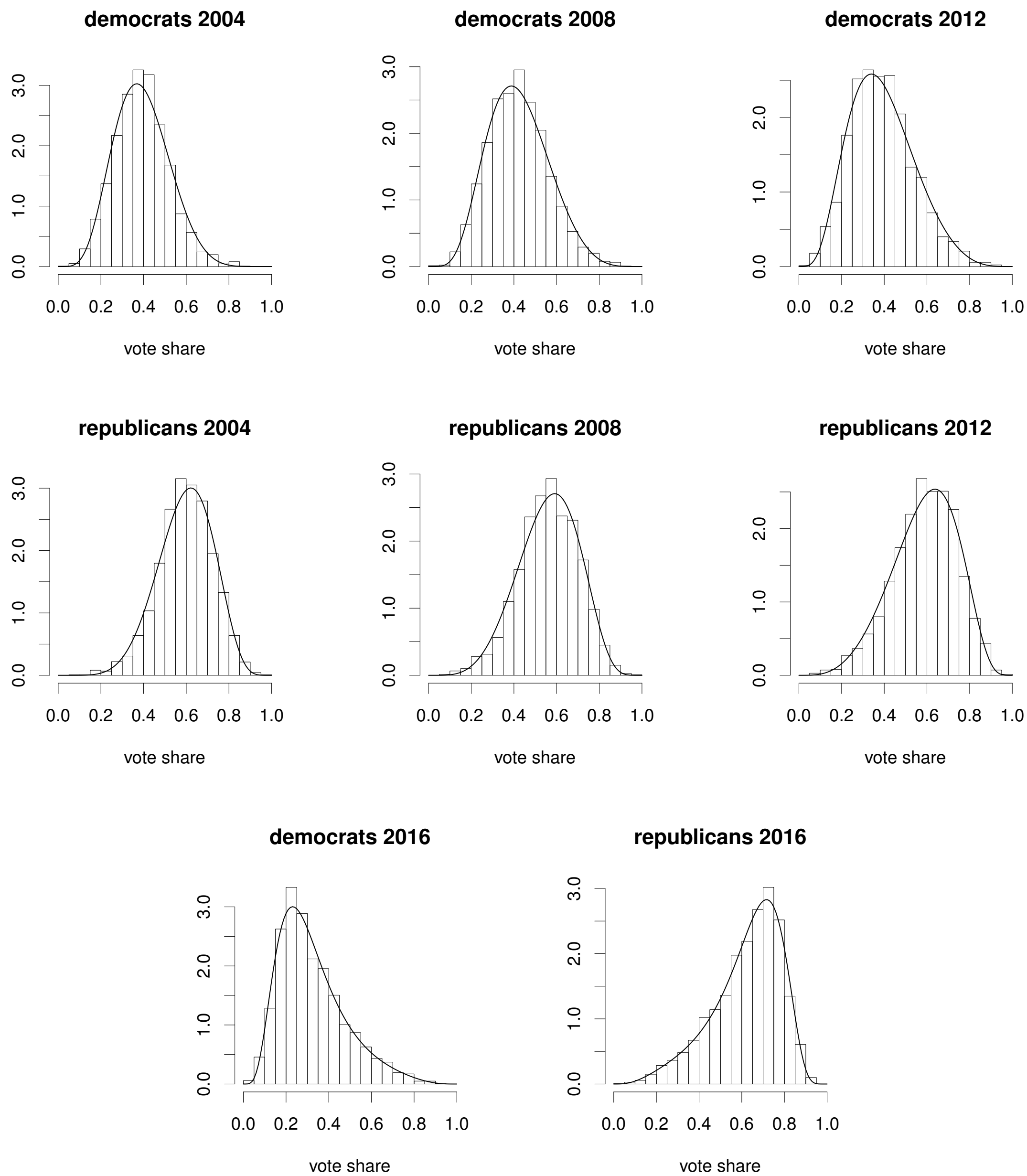


Figure 2.11: Distribution of the vote share of republicans/democrats in 2004, 2008, 2012 and 2016 presidential elections. We find a good agreement (Kolmogorov-Smirnov-test, $p \geq 0.12$ for all elections shown here). Only from 2008 on, the reinforcement model is significantly better ($p < 0.005$) than the zealot model.

US presidential elections If we investigate the recent US presidential elections, we find a good agreement of model and data for 2004-2106 (Fig. 2.11). The Kolmogorov-Smirnov test yields p-values larger or equal 0.12, indicating that this model cannot be rejected (Tab. 2.1),

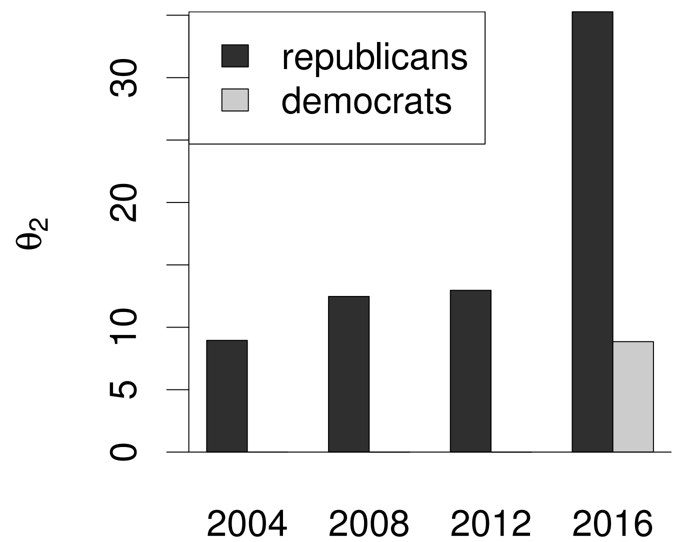


Figure 2.12: Plot of the reinforcement parameter. Note that in 2004-2012, the parameter for the democrats is negligible small, and only in 2016 it becomes visible.

while the Kolmogorov-Smirnov test for the zealot model performs worse (either in a gradual way for the 2004 elections, or indeed with the result that the zealot model can be rejected at a confidence level $p < 0.003$ for all elections later than 2004).
Though we estimate the parameters for republicans and democrats independently, the reinforcement parameters $\hat{\psi}$ approximately add up to 1 (apart from the 2000 election). If no further candidate is in the game (the vote shares for democrats and republicans add up to 1 in each election district), this observation is a logical consequence of the model structure. However, we do have further candidates. Our results indicates that these further candidates only play a minor role.
The 2000-election is different. The Kolmogorov-Smirnov-test clearly shows (in line with the graphical representation) that the data do not follow the model-distribution. Here, the candidate for the green party had a significant influence on the election results (Fig. 2.13). Our model has difficulties to get on with the interplay of three groups, resulting in a rather bad fit of the data.

If we inspect the reinforcement parameters for the two parties (Fig. 2.12), we find that the reinforcement for the democrats is rather unimportant (in comparison to that of the republicans) before the 2016 elections. In the 2016 election, the reinforcement of the democrats jumps to the value of the republicans before 2016. Furthermore, in the 2016 elections, the reinforcement of the republicans suddenly shows a threefold increase. We detect here the consequences of the populist attitude of the republican's candidate.

year	party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf)	p_{ks} (beta)
2000	republicans	0.0180	0.820	6.611e-05	89	72.99446	<1e-10	<1e-10	<1e-10
2000	democrats	0.190	4.889e-05	6.61e-05	2.24	0.0001	1	<1e-10	<1e-10
2000	green	0.0004	0.81	6.61e-05	193.5	156.9	<1e-10	<1e-10	<1e-10
2004	republicans	0.519	0.412	6.611e-05	21.7	8.95	0.15	0.15	0.092
2004	democrats	0.440	0.3253	0.99993	18.4	0.0004	0.60	0.12	0.1
2008	republicans	0.435	0.522	6.611e-05	23.9	12.5	0.0026	0.12	0.066
2008	democrats	0.544	0.5167	0.99993	22.5	0.0008	0.0023	0.14	0.070
2012	republicans	0.429	0.587	6.611e-05	22.1	13.0	1.5e-06	0.24	0.0076
2012	democrats	0.535	0.569	0.99993	20.2	0.00076	7.1e-06	0.23	0.011
2016	republicans	0.324	0.7860	0.19	55.5	35.3	<1e-10	0.66	7.4e-10
2016	democrats	0.563	0.7861	0.74	43.3	8.85	<1e-10	0.14	6.77e-10

Table 2.1: Estimated parameters for the two parties in the eight elections. p_{ll} is the result of the likelihood ration test for the significance of the reinforcement component; p_{ks} is the result of the Kolmogorov-Smirnov-test for the question of the empirical cumulative distribution differs significantly from the cumulative distribution of the model (either the reinforcement model, or zealot model with the beta distribution).

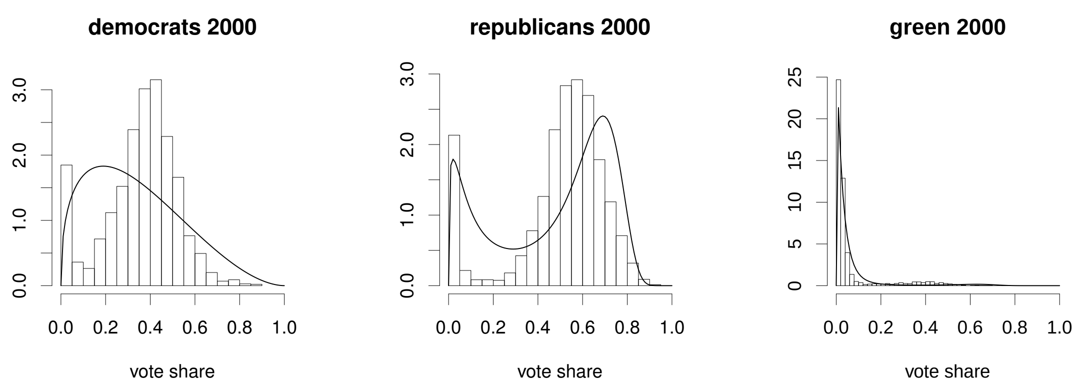


Figure 2.13: Distribution for the 2000 presidential elections (democrats/republicans/green). We did fit the model using the focal group versus the pooled remaining groups.

The Netherlands As the election system of The Netherlands is base on proportional representation, but the theory developed so far describes a binary choice, we consider the result of one party versus the rest. Note that the data bear some kind of randomness due to the size of the election districts: The election district sizes vary from several 100 to over a million voters. We neglect that difference but focus on the vote share of a certain party.
We start off with the winner of the elections after the second world war: the "Katholieke Volkspartij (KVP)", the Catholic People's Party. After the second world war, this party played a major role in The Netherlands with a vote share of about 0.33; only after 1967, the vote share dropped to around 0.25. The party did merge with two other parties 1980, and did already before, from 1977 on, appeared in a coalition with two other religions parties.

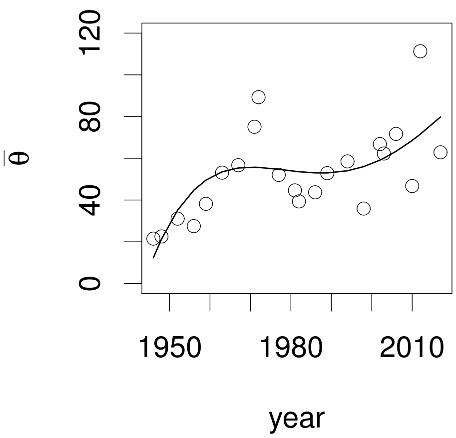


Figure 2.14: Average reinforcement parameter $\overline{\theta}$ for The Netherlands, together with the forth order polynomial fitting theses estimates.

During the active time of the party, the reinforcement model is highly superior to the zealot model (likelihood-ratio-test, $p < 10^{-10}$). We clearly find a bimodal distribution (Fig. 2.15), where one peak is at close to zero, the other decreasing in the years 1948 to 1971 from about 0.95 to 0.6.
Obviously, this party did divide the population. The religious segregation at that time also did lead to a certain spatial segregation: Catholic and Protestant population tended to separate. Spatial or social segregation is for sure one of the major driver for reinforcement, as this segregation creates a homogeneous environment that minimizes the contact with different opinions. This observation is also reflected by the parameter estimation: the number of zealots of that party is estimated by a value close to zero; the strength of the party is solely explained by a strong reinforcement component, that was increasing over time. One can speculate that the religious basis of the Catholic People's Party did strongly promote this reinforcement.

To obtain an impression about the trend in the overall reinforcement, we considered $\overline{\theta}$ as defined above (Fig. 2.14). We find that after a phase with rather constant enforcement, in recent years the enforcement seems to grow. One can speculate that this growth is due to the internet and social media. However, as the noise in the estimate is rather large, it is difficult to decide about the exact timing of that increase.

Brexit The reinforcement-model is statistically clearly superior to the zealot model to describe the Brexit data (likelihood-ratio test $p = 1.3 \cdot 10^{-11}$). Also the Kolmogorov-Smirnov test clearly indicates that the reinforcement-model cannot be rejected ($p = 0.75$), but the zealot model that does not incorporate reinforcement is not appropriate ($p = 0.0009$). Interestingly, the point estimate for $\hat{\psi}$, which describes the relative weight for the reinforcement of one group versus the other group, indicates that the brexitiers are responsible for over 99% of the reinforcement. In

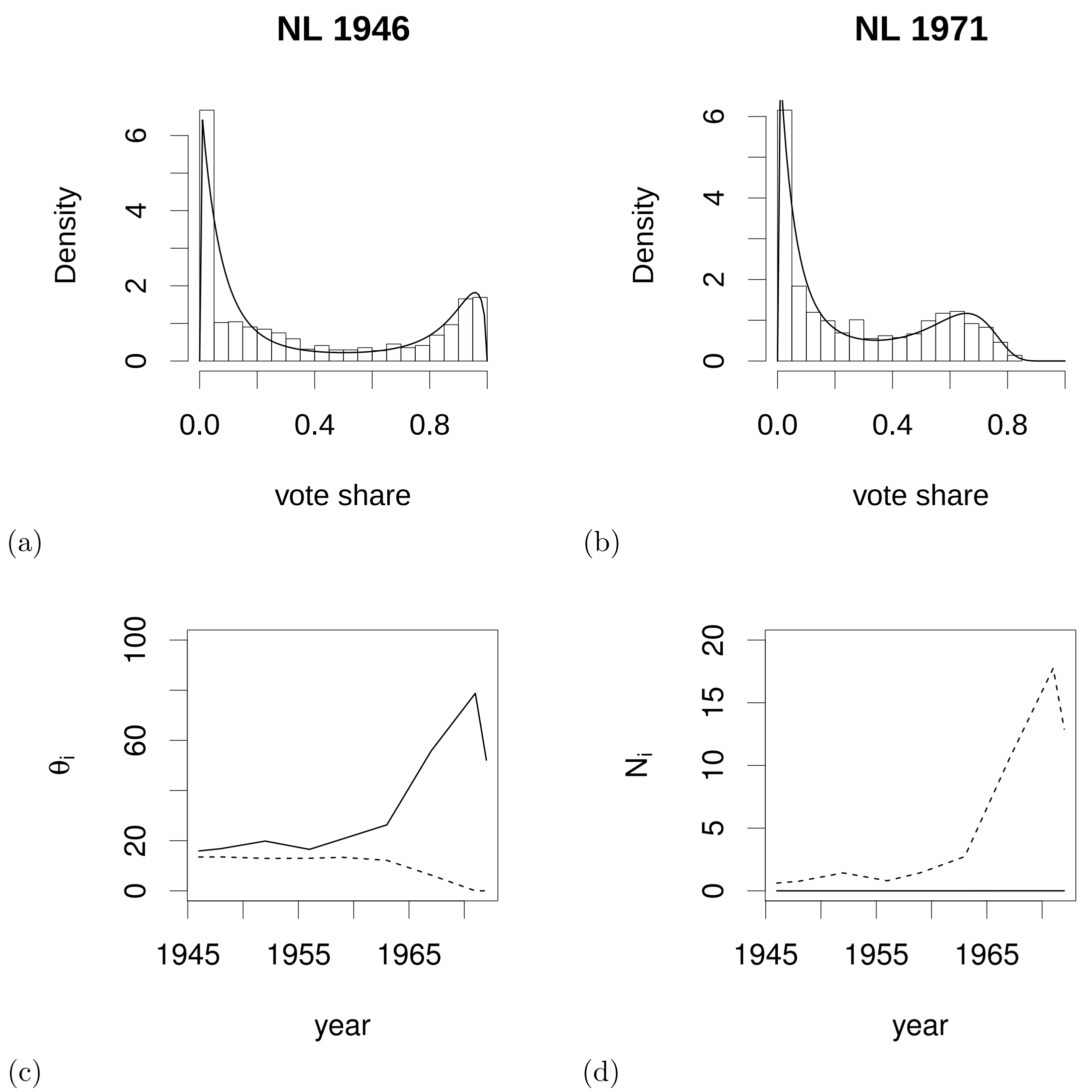
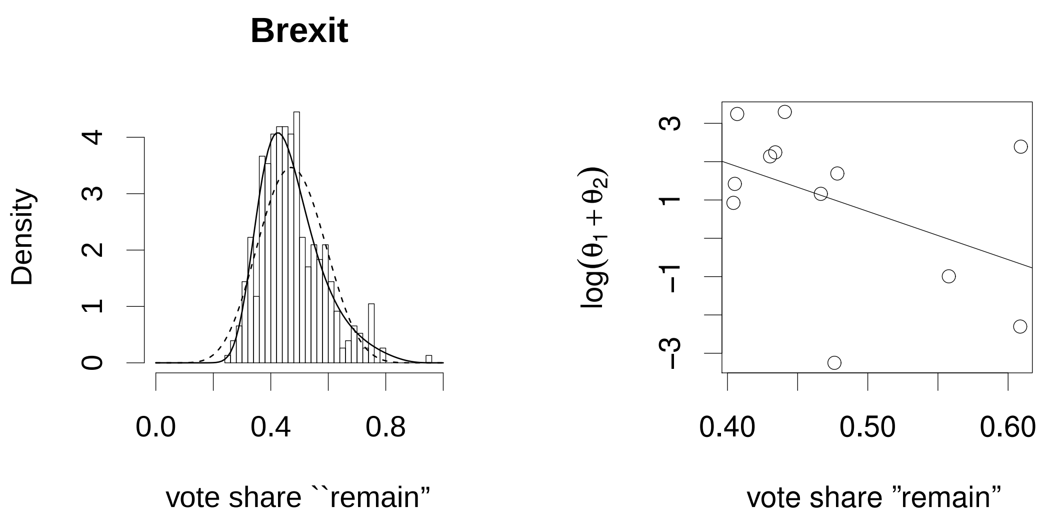


Figure 2.15: Vote share and fit of the reinforcement model to data for the Catholic People's Party. (a) and (b) Vote share with density for 1948 and 1971; (c) θ_i over time (bold: θ_2, dashed: θ_1; note that θ_2 is the reinforcement parameter of the supporters of the Catholic People's Party); (d) N_i over time (bold: N_1, dashed: N_2; note that N_1, the number of zealots for the Catholic People's Party, is estimated as $N_1 \ll 1$.)

order to investigate this finding more thoroughly, we use the likelihood-ratio test to compare a restricted model where both groups do have the same reinforcement parameters ($\hat{\psi} = 0.5$) and the model where the reinforcement parameter for both groups are arbitrary $\hat{\psi} \in [0, 1]$). It turns out that there is no significant superiority of the second model over the restricted model. That is, though the point estimate seems to hint that the brexitiers are way more prone to reinforcement in comparison with the remainers, there is no statistically significant signal

year	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	θ_1	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
1946	6.61e-05	0.98	0.46	30.01	15.9	13.5	2.11e-216	8.55e-15	<1e-20
1948	6.61e-05	0.98	0.45	31.11	16.8	13.5	1.72e-214	4.40e-12	<1e-20
1952	6.61e-05	0.96	0.39	34.21	19.8	12.9	2.63e-191	1.40e-13	<1e-20
1956	6.61e-05	0.97	0.44	30.31	16.5	13.0	3.9e-195	1.03e-11	<1e-20
1959	6.61e-05	0.96	0.39	35.51	20.7	13.4	8.82e-196	4.35e-11	<1e-20
1963	6.61e-05	0.93	0.32	41.21	26.3	12.2	7.16e-163	6.86e-06	<1e-20
1967	6.61e-05	0.86	0.10	72.51	55.7	6.3	2.21e-120	0.00015	<1e-20
1971	6.61e-05	0.82	6.61e-05	96.51	78.7	0.005	7.96e-95	0.00067	<1e-20
1972	6.61e-05	0.80	6.61e-05	65.01	52.11	0.003	5.15e-43	8.96e-06	<1e-20

Table 2.2: Parameter for the Christian People's Party.



(a) (b)

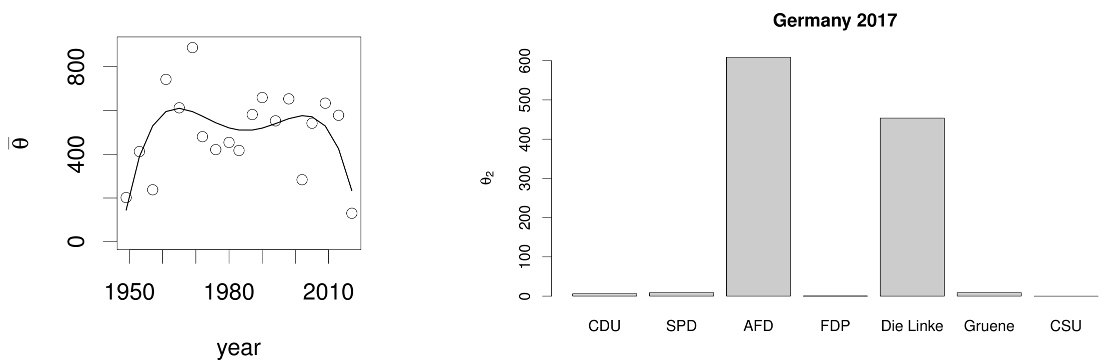
Figure 2.16: (a) Vote share for "remain" in the Brexit vote, together with the reinforcement model (solid line) and the zealot model (dashed line). (b) Estimation for $\theta_1 + \theta_2$ over vote share for "remain", estimated separately for the different regions, together with a linear fit o the logarithmic scale.

supporting that finding. To further investigate the question which of the two groups are deeper involved in reinforcement, we estimate the model-parameters for the different regions (assuming equal reinforcement for both groups), and draw the estimation for $\theta_1 + \theta_2$ over the vote share for "remain". It turns out, that there is a negative (but non-significant) correlation of the reinforcement and the vote share for remain (Fig. 2.16). Also that results hints to the fact that the brexitiers are involved to reinforcement with a higher degree.

In any case, we find a high influence of reinforcement, which could be considered to be in line with the overall perception of the Brexit process.

$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
0.87	0.76	0.99996	254.5	0.0084	1.3e-11	0.75	0.0009

Table 2.3: Parameter for the Brexit referendum, "remainers". For symmetry reasons, the parameters for "leave" are identical, but $\hat{\psi}_{leave} = 1 - \hat{\psi}_{remain}$, and accordingly $\theta_{2,leave} = 192.97$. We have the result of the likelihood-ratio test for $\hat{\theta} = 0$ ($p_{ll} = 1.3e - 11$), the result for the Kolmogorovff-Smirnov test if the data are in line with the reinforcement-distribution (p_{ks}(Reinf) $= 0.75$), and the result of Kolmogorov-Smirnov, if the data are in line with the zealot (beta) distribution (p_{ks}(beta) $= 0.0009$).



(a) (b)

Figure 2.17: (a) Average reinforcement parameter $\overline{\theta}$ for Germany, together with the forth order polynomial fitting theses estimates. (b) Reinforcement parameter for several parties, from the election data (2017) for whole Germany.

Germany The overall degree of reinforcement in Germany seems to be rather stable, but very noisy (Fig. 2.17). If we investigate the detailed election results from 2017 for the 7 parties that are present in th parliament, we find that reinforcement plays a role for particularly two parties: the "AFD" and "Die Linke". Both parties are known as populist parties, one fat the political left- and one at the right wing. For all other parties, reinforcement is not statistical significant for the data at hand. If we compare the results for whole Germany and the "old" states only (tables 2.4 and 2.5), we find that the reinforcement of the AFD is strongly connected with the "new" states, while the reinforcement of "die Linke" seems to be independent of the classification in "old" and "new" states.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.39	0.39	0.86	112.5	6.02	1	0.68	0.78
SPD	0.19	0.17	6.61e-05	50.5	8.80	0.99	0.20	0.20
AFD	0.07	0.77	6.61e-05	792.5	608.91	8.6e-11	0.29	0.0009
FDP	0.31	0.73	0.998	182.5	0.28	1	0.52	0.54
Die Linke	0.054	0.78	6.61e-05	583.5	453.7	3.44e-09	2.1e-06	2.93e-08
Gruene	0.070	0.15	6.61e-05	58	8.7	1	0.85	0.84
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.004	0.004

Table 2.4: Results for whole Germany, 2017.

party	$\hat{\nu}$	$\hat{\theta}$	$\hat{\psi}$	$\hat{s}$	θ_2	p_{ll}	p_{ks} (Reinf.)	p_{ks} (beta)
CDU	0.52	0.67	0.76	294.5	46.74	0.33	0.50	0.78
SPD	0.21	4.95e-05	6.61e-05	47.2	0.0023	1	0.17	0.17
AFD	0.30	0.71	0.99996	189.5	0.0055	1	0.86	0.92
FDP	0.29	0.68	0.99991	251.5	0.015	0.70	0.62	0.68
Die Linke	0.088	0.83	0.14	1159.5	834.22	0.00086	0.00068	0.00016
Gruene	0.22	0.86	0.38	289.5	154.36	0.16	0.112	0.075
CSU	0.37	6.75e-05	6.61e-05	35.6	0.0024	1	0.0038	0.0038

Table 2.5: Results for Germany, only the “old” states ,2017.

2.7 Size of the parliament and population size

Universal size effects for populations in group-outcome decision-making problems [2]; two models in the paper:
(1) Simple Argument
A parliament should minimize the number of voters er representative, N/n_r. At the same time, it should maximize the efficiency of the parliament; the number of pairwise interactions, roughly n_r^2, should be minimal. A weighted sum of both expressions should be minimizes,

$$\frac{N}{n_r} + A\, n_r^2 = \min$$

which leads to

$$n_r \sim N^{1/3}$$

which is in accordance to empirical data.
(2) Kind of a hierarchical Galam-Model.

2.8 Have a look - more models and approaches

[21]: A STD based on the derivative of a Brownian motion. Claims universality in social processes, and finance systems. Scale-free growth behavior.

[40, 41, 39] Sznajd model for opinion evolution.

Rumors and infection. ODE rumor models. [42]: Opinions, Conflicts, and Consensus: Modeling Social Dynamics in a Collaborative Environment (Wikipedia)

[27], Phenotypical model to explain the Rank-order relation in elections

2.9 Spatial valence model

Consider the following scenario: You are living in a long street. In this street, there are also two bakeries. They have an identical offer to an identical price. To which bakery you will go? .... exactly. You will choose the bakery that is closer to you. Now, take the bakery's point of view. Each of the bakeries want to have as many customers as possible. Where should they localize themselves? The first at the beginning of the street, the second as far away from the fist one as possible, at the end of the street? In this way, each of them would have the half street as customers. However, if one of the bakeries now move to the middle of the street, this bakery suddenly gets more than half of the inhabitants as customers. We see, at the end, both bakeries should be located at the very same spot: In the middle of the street.

These very ideas can be reformulated to describe the relationship between parties and voters. Parties offer a political program, voters have certain opinions and interests. A voter will vote for that party that is closest to his/her opinion. However, also parties react on the interests of voters and adapt their program (up to certain degree) to obtain the maximum number of votes possible. There is a whole bunch of models addressing the question how parties will develop under this process.

2.9.1 Hotelling model

Hotelling described 1929 [23] the very same setting for parties and voters: We have voters and parties. The political opinions can be localizes in a one-dimensional space (from left-wing to right-wing, say). Also parties, characterized by their program, can be localized in this one-dimensional feature space. How do parties adapt their program to maximize the number of votes? We state his model and the central theorem – the convergence theorem – in the most simple version to understand the basic ingredients that lead to the predicted behaviour.

Model 2.50 Hotelling model. *Let N be an odd number of voters, the opinion of the i'th voter is characterized by $z_i \in \mathbb{R}$. We assume the generic condition $z_i \neq z_j$ for $i \neq j$. Let $\mu_A, \mu_B \in \mathbb{R}$ characterize the location of two parties A and B. A voter will vote for the party closest to his/her opinion. If $p_i = 1$ if individual i votes for A, $p_i = 0$ if he/she votes for B, and $p_i = 1/2$ if he/she is undecided, then*

$$p = p(\mu_A, \mu_B; z_i) = \begin{cases} 1 & \textit{if} \quad |z_i - \mu_A| < |z_i - \mu_B| \\ 0 & \textit{if} \quad |z_i - \mu_A| > |z_i - \mu_B| \\ 1/2 & \textit{if} \quad |z_i - \mu_A| = |z_i - \mu_B|. \end{cases}$$

The number of votes for party A reads

$$U_A(\mu_A, \mu_B, z_1, \dots, z_N) = \sum_{i=1}^{N} p(\mu_A, \mu_B; z_i),$$

and that for party A is given by

$$U_B(\mu_A, \mu_B; z_1, \dots, z_N) = N - U_A(\mu_A, \mu_B; z_1, \dots, z_N)$$

The equilibrium strategies (μ_A^*, μ_B^*) *are defined by the Nash equilibrium, that is*

$$\forall \mu_A \in \mathbb{R}: \; U_A(\mu_A, \mu_B^*, z_1, \dots, z_N) \leq U_A(\mu_A^*, \mu_B^*, z_1, \dots, z_N),$$
$$\forall \mu_B \in \mathbb{R}: \; U_B(\mu_A^*, \mu_B, z_1, \dots, z_N) \leq U_B(\mu_A^*, \mu_B^*, z_1, \dots, z_N).$$

vielleicht besser in definition oben Nash gleichg. wie normal einfueren?

Proposition 2.51 (Convergence theorem, Median Voting Theorem) *The only equilibrium strategy is given by* $(\mu_A^*, \mu_B^*) = (z^*, z^*)$*, where* z^* *is the median of* $(z_i)_{i=1,\dots,N}$*.*

Proof: *(a)* $(\mu_A^*, \mu_B^*) = (z^*, z^*)$ *is an equilibrium strategy.*
For symmetry reasons, we only need to consider party A. As $\mu_A^* = \mu_B^*$, $p(\mu_A, \mu_B; z_i) = 1/2$, and

$$U_A(\mu_A, \mu_B^*, z_1, \dots, z_N) = N/2.$$

If $\mu_A > \mu_B$, we increase then

$$\forall z_i \geq \mu_A: \; p(\mu_A, \mu_B, z_i) = 1, \quad \forall z_i \leq \mu_B: \; p(\mu_A, \mu_B, z_i) = 0.$$

Since we assume that N is odd, if $\mu_B^* = z^*$ is the median of the points z_i, then $U_B(\mu_A, \mu_B^*; z_1, \dots, z_N) \geq (N+1)/2$, and hence

$$U_A(\mu_A, \mu_B^*; z_1, \dots, z_N) \leq N - (N+1)/2 < N/2 = U_A(\mu_A^*, \mu_B^*; z_1, \dots, z_N).$$

Symmetry reasons show that U_A is also deceased if we choose $\mu_A < \mu_A^*$.
(b) $(\mu_A^*, \mu_B^*) = (z^*, z^*)$ *is the only equilibrium strategy.*
Assume $\mu_A \neq \mu_B \not0 z^*$ is an equilibrium. Then, the gain for both parties is $N/2$. In this case, moving μ_A, say, to z^* increases the gain of A to at least $(N+1)/2 > N/2$. Hence we have no equilibrium.
Now assume $\mu_A \neq \mu_B$. If it is not the case that both parties receive $N/2$ votes, one party receives less and hence moves into z^*, where it gains at least $N/2$.
If, however, both parties obtain $N/2$, the median individual(s) are undecided, and the distance of μ_A and μ_B are identical from z^*. Since $\mu_A \neq \mu_B$ we are faced with a symmetrical situation: $\mu_A = z_0 + \varepsilon$, and $\mu_B = z_0 - \varepsilon$ (or vice versa). Hence if one party moves towards the median point (while the other stays), this party increases its gain. We have no equilibrium strategy.

□

This theorem predicts that in the given setup, parties eventually converge towards the same program. This predictions unexpected - the general feeling is, that parties distinguish from each other and have their regular voters. However, Hotelling formulated his model not in the setting of political parties, but rather for more general cases of competition. E.g., consider a street, where two bakeries compete. If inhabitants in the street only decide based on the distance to a bakery which of the two shops they will visit, the theorem tells us that the bakeries should both be in the middle of the street, abreast. And indeed, such a situation is not often to find in cities. Down [10] adapted Hotellings ideas to investigate voting behaviour.

A whole bunch of literature exploits how to modify the model, such that the convergence theorem still holds, or eventually non-symmetric equilibrium distributions appear. The range of modifications ranges from relatively small steps (what happens if N is even? What happens if ties for z_i are allowed?) or adaptation of the gain functions (instead to maximize the number of voters, the aim could be to win). The dimension of the trai space is increased ($z_i, \mu_A, \mu_B \in \mathbb{R}^n$ instead of $z_i, \mu_A, \mu_B \in \mathbb{R}$), and stochasticity is introduced to ensure that the equilibrium strategies are stable (in the same spirit as trembling hand equilibria are introduced in game theory). Below we discuss in particular the n-dimensional deterministic version, and a stochastic version of the Hotelling model.

Literaturhinweise einbauen!!!

2.9.2 Stochastic valence model in higher dimensions

The Hotelling model has a striking implication, the median voter theorem. However, a brief look into the newspapers clearly shows that parties do take different positions. What's wrong with the Hotelling model?

Three extensions are of particular interest: First, the world is not one-dimensional. The simple left-right scheme does not adequately represent reality. There are more dimensions, as social issues, economics, security, education etc. The position of a vote (and the position of a party) is better represented by a point in $\mathbb{R}^n$.

Second, the world is not deterministic. Te model should incorporate a certain amount of stochasticity to take into account our lack of complete knowledge about a given individual.

A third observation is that parties, or, better, leading candidates of parties, have a different electoral perception. This perception breaks the intrinsic symmetry between parties. We will use real numbers $\lambda_j \in \mathbb{R}$ to describe the overall perception of party j. This perception can be caused, e.g. by their past performance in the government. The paper by Schofield [38] investigates the consequences of these two extensions.

Model 2.52 *Let $x_1, \ldots, x_m \in \mathbb{R}^n$ denote the positions of voters, and $z_1, \ldots, z_k$ the position of parties. The position of voters are given and fixed, while parties may choose their positions. We define the utility $u_{i,j}$ of party j for voter i by*

$$u_{i,j}(z_j) = \lambda_j - \beta\|x_i - z_j\|^2 + e_{i,j}$$

where λ_j is the valence of party j (identical to all voters), $\|x_i - z_j\|^2$ is the euclidean distance between the voter's and the party's position, and $e_{i,j}$ are i.i.d. random variables that express our lack of knowledge. We might assume that $e_{i,j}$ follow an extreme value distribution of type 1, defined by $P(e_{i,j} < h) = e^{-e^{-h}}$. The parameter $\beta \in \mathbb{R}$ does scale the importance of the distance in comparison to noise and valences.

Voter i selects party j if for him/her the utility of party j is maximal. The probability for voter i to vote for party j reads

$$p_{i,j}(z_1, \ldots, z_k) = P(\, u_{i,j}(z_j) > u_{i,\ell}(z_\ell) \quad \text{for all } \ell = 1, \ldots, k,\ \ell \neq j),$$

and consequently, the expected vote share for party j is then defined by

$$V_j(z_1, \ldots, z_k) = \frac{1}{m} \sum_{i=1}^{m} p_{i,j}(z_1, \ldots, z_k).$$

The position of the voters are assumed to be given. The parties, however, are assumed to opportunistically aim to maximize their vote share. It might be, that parties are driven by zealaots that are persuaded by a certain position. In this case, the model will fail. However, parties for sure respond to polls and aim to come up with popular positions. Let us assume that they have no intrinsic program, but only want to win as many votes as possible. We can use that as a working hypothesis and observe the predictions that come out of this assumption.
In that, it is near at hand to use the notion of Nash equilibria for the given situation. We might expect that the positions of the parties will approximate these Nash equilibria.

Definition 2.53 *We first define the local best response of party j given the position of all other parties. Let $\hat{z}_\ell$ for $\ell \neq j$ the positions of the other parties. As we only work locally, we also consider a reference value $\hat{z}_j$ for party j, and introduce the notation*

$$\hat{z}_\bullet = (\hat{z}_1, \ldots, \hat{z}_k).$$

The set of ε-close, local best responses is defined by

$$U_{j,\varepsilon}(\hat{z}_\bullet) = \{z_j \in B_\varepsilon(\hat{z}_j) \,:\, \forall \tilde{z}_j \in B_\varepsilon(\hat{z}_j) \setminus \{z_j\} : V_j(\hat{z}_1, \ldots, \tilde{z}_j, \ldots \hat{z}_k) < V_j(\hat{z}_1, \ldots, z_j, \ldots \hat{z}_k)\}.$$

The vector of party locations $z^*_\bullet = (z^*_1, \ldots, z^*_k)$ is a (strict) local Nash equilibrium, if there is $\varepsilon > 0$ s.t.

$$z^*_j \in U_{j,\varepsilon}(z^*_\bullet) \quad \text{for } j = 1, \ldots, k.$$

The next proposition characterizes local Nash equilibria. To clarify the notation we note that for a function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, the expression

$$\nabla f(x, y) = \begin{pmatrix} \partial_x f(x, y) \\ \partial_y f(x, y) \end{pmatrix}$$

is the gradient, and

$$\nabla\, \nabla^T f(x, y) = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} (\partial_x, \partial_y) f(x, y) = \begin{pmatrix} \partial_{xx} f(x, y) & \partial_{xy} f(x, y) \\ \partial_{yx} f(x, y) & \partial_{yy} f(x, y) \end{pmatrix}$$

establishes the Hessian matrix. In the following, ∇_{z_j} indicates the gradient w.r.t. the vector z_j.

Proposition 2.54 *Assume that $V_j(z_1, \ldots, z_k)$ are twice differential functions. Sufficient criteria that $(z^*_1, \ldots, z^*_k)$ is a local strict Nash equilibrium are the frist and second order conditions:*

$$(1) \qquad \nabla_{z_j} V_j(z^*_1, \ldots, z_j, \ldots, z^*_k)\Big|_{z_j = z^*_j} = 0 \tag{2.60}$$

$$(2) \qquad \sigma\left(\nabla_{z_j}\, \nabla^T_{z_j} V_j(z^*_1, \ldots, z_j, \ldots, z^*_k)\Big|_{z_j = z^*_j}\right) \subset \mathbb{C}^- \tag{2.61}$$

for $j = 1, \ldots, k$.

Proof: The two conditions imply that $z^*_\bullet = (z^*_1, \ldots, z^*_k)$ is a local maximum in the sense that varying locally one vector component z_j in $z^*_\bullet$, while keeping all other components constant, (strictly) decreases the function V_j. This property establishes that $z^*_\bullet$ is a local strict Nash equilibrium. □

In order to investigate the local Nash equilibria of our model, we first represent p_{ij} in a more handy way.

Proposition 2.55 *Let $u^*_{i,j}(z_j) = \lambda_j + \beta\|x_i - z_j\|$. Then, under the conditions given by the model definition,*

$$p_{i,j}(z_1, \ldots, z_k) = \frac{e^{u^*_{i,j}(z_j)}}{\sum_{\ell=1}^k e^{u^*_{i,\ell}(z_\ell)}}. \tag{2.62}$$

Proof: Let $e_0, \ldots, e_\ell$ denote i.i.d. random variables, with $P(e_i < h) = e^{-e^{-h}}$. Let furthermore $\varphi(h) = \frac{d}{dh}\mathbf{e}^{-e^{-h}}$ the probability density of e_0, and let $a_i \in \mathbb{R}$ denote constant real values. Then,

$$\begin{aligned}
& P(e_0 > \max\{a_1 + e_1, \ldots, a_\ell + e_\ell\}\,) \\
= & \int_{-\infty}^{\infty} P(h > \max\{a_1 + e_1, \ldots, a_\ell + e_\ell\}\,|\, e_0 = h\,)\,\varphi(h)\,dh \\
= & \int_{-\infty}^{\infty} P(\,(h - a_1 > e_1) \wedge \ldots \wedge (h - a_\ell > e_\ell)\,)\,\varphi(h)\,dh \\
= & \int_{-\infty}^{\infty} \prod_{i=1}^{\ell} P(h - a_i > e_i)\,\varphi(h)\,dh = \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{\ell} e^{-h+a_i}} \frac{d}{dh} e^{-e^{-h}}\,dh \\
= & \int_{-\infty}^{\infty} e^{-\left(1+\sum_{i=1}^{\ell} e^{a_i}\right)e^{-h}} e^{-h}\,dh
\end{aligned}$$

Since $\frac{d}{dx}\frac{1}{\alpha}e^{-\alpha e^{-x}} = e^{-\alpha e^{-x}}e^{-x}$, we have

$$P(e_0 > \max\{a_1 + e_1, \ldots, a_\ell + e_\ell\}\,) = \frac{1}{1 + \sum_{i=1}^{\ell} e^{a_i}}.$$

Therewith we find (note that we replace e_i by $e_{i,j}$, the random variables that appear in the definition of $u_{ij}(z_j) = u^*(z_j) + e_{i,j}$)

$$\begin{aligned}
p_{i,j}(z_1, \ldots, z_k) &= P(u^*_{i,j}(z_j) + e_{i,j} > u^*_{i,\ell}(z_\ell) + e_{i,\ell} \quad \text{for } \ell \neq j) \\
&= P(e_{i,j} > u^*_{i,\ell}(z_\ell) - u^*_{i,j}(z_j) + e_{i,\ell} \quad \text{for } \ell \neq j) \\
&= \frac{1}{1 + \sum_{\ell \neq j} e^{u^*_{i,\ell}(z_\ell) - u^*_{i,j}(z_j)}} = \frac{e^{u^*_{i,j}(z_j)}}{\sum_{\ell=1}^k e^{u^*_{i,\ell}(z_\ell)}}.
\end{aligned}$$

□

Note that we obtain a multinomial logist model. For $k = 2$ we have the classic logist model, with the well known methods to estimate parameters. Due to the intrinsic independency assumptions on $e_{i,j}$, the abilities of the model are limited.

The best candidate for a local Nash equilibrium is the mean value of the voters.

Theorem 2.56 *Let* $\overline{x} = \frac{1}{m}\sum_{i=1}^m x_i$*. Let* $z_j^* = \overline{x}$ *for* $j = 1, \ldots, k$*. Then, the first order equilibrium condition*

$$\nabla_{z_j} V_j(z_1^*, \ldots, z_j, \ldots, z_k^*)\Big|_{z_j = z_j^*} = 0$$

is given. Let

$$cov(x) := \frac{1}{m}\sum_{i=1}^m (x_i - \overline{x})\,(x_i - \overline{x})^T. \tag{2.63}$$

and $p_j = \frac{e^{\lambda_j}}{\sum_{i=1}^k e^{\lambda_i}}$*. The second order equilibrium condition will are satisfied, if all eigenvalues of the symmetric matrices (for* $j = 1, \ldots, k$*)*

$$C \;=\; \beta(1 - 2p_j)\,cov(x) - I \tag{2.64}$$

are negative.

Proof: We can shift all voter states x_i and all party positions z_i by $\overline{x}$, $\tilde{x}_i = x_i - \overline{x}$, $\tilde{z}_i = z_i - \overline{x}$. As the vectors x_i, z_i only enter the model vial the utility function, and there in form of $\|x_i - z_j\|^2$, the model is shift invariant. That is, without restriction, $\overline{x} = 0$. In $\tilde{x}_i$, $\tilde{z}_i$, we drop the tilde again, and proceed with x_i and z_i.

We show that the first order condition is true for $z_\ell = 0 = \overline{x}$. We consider in the next computation $\rho_{i,j}$ as a function of $(u_{i,1}^*, \ldots, u_{i,j}^*)$, and the scores $u_{i,j}^*$ as a function of $z_1, \ldots, z_k$ in the sense that $u_{i,j}^* = u_{i,j}^*(z_j)$.

$$\begin{aligned}
\nabla_{z_j} V_j(z_1, \ldots, z_k) &= \frac{1}{m}\sum_{i=1}^m \nabla_{z_j} p_{i,j}(u_{i,1}^*, \ldots, u_{i,k}^*) \\
&= \frac{1}{m}\sum_{i=1}^m \frac{\partial}{\partial u_{i,j}^*} p_{i,j}(u_{i,1}^*, \ldots, u_{i,k}^*)\,\left(\nabla_{z_j} u_{i,j}^*(z_j)\right) \\
&= \frac{1}{m}\sum_{i=1}^m \frac{\partial}{\partial u_{i,j}^*} p_{i,j}(u_{i,1}^*, \ldots, u_{i,k}^*)\,\nabla_{z_j}(\lambda_j + \beta\|x_i - z_j\|^2) \\
&= \frac{2\beta}{m}\sum_{i=1}^m \frac{\partial}{\partial u_{i,j}^*} p_{i,j}(u_{i,1}^*, \ldots, u_{i,k}^*)\,(x_i - z_j).
\end{aligned}$$

The derivative $\frac{\partial}{\partial u_{i,j}^*} p_{i,j}$ is given by (we suppress the argument in $p_{i,j}$)

$$\frac{\partial}{\partial u_{i,j}^*} p_{i,j} = \frac{\partial}{\partial u_{i,j}^*}\left(\frac{1}{1 + \sum_{\ell \neq j} e^{u_{i,\ell}^* - u_{i,j}^*}}\right) = p_{i,j}^2(1/p_{i,j} - 1) = p_{i,j} - p_{i,j}^2 = p_{i,j}(1 - p_{i,j}). \tag{2.65}$$

All in all, we obtain

$$\nabla_{z_j} V_j(z_1, \ldots, z_k) \;=\; \frac{2\beta}{m}\sum_{i=1}^m p_{i,j}(1 - p_{i,j})\,(x_i - z_j). \tag{2.66}$$

At $z_1 = \ldots = z_k = 0$ we have $u^*_{i,\ell} - u^*_{i,j} = \lambda_\ell + \beta\|x_i - 0\| - \lambda_j - \beta\|x_i - 0\| = \lambda_\ell - \lambda_j$, independently on i, the expression $p_{i,j} - p^2_{i,j}$ does not depend on i, and we can write

$$p_{i,j} - p^2_{i,j} = p_j - p^2_j.$$

Therewith, and with $\overline{x} = 0$, we obtain

$$\nabla_{z_j} V_j(0, \ldots, 0) = (p_j - p_j^2)\, \frac{2\beta}{m} \sum_{i=1}^m x_i = 0.$$

We proceed with the second order condition, and inspect the Hessian. From (2.66), $\nabla^T_{z_j} V_j(z_1, \ldots, z_k) = \frac{2\beta}{m}\ \sum_{i=1}^m (p_{i,j} - p^2_{i,j})\,(x_i - z_j)^T$. Hence,

$$\nabla_{z_j} \nabla^T_{z_j} V_j(z_1, \ldots, z_k) \;=\; \frac{2\beta}{m}\ \sum_{i=1}^m \left(\nabla_{z_j} (p_{i,j} - p^2_{i,j})\right) (x_i - z_j)^T + \frac{2\beta}{m}\ \sum_{i=1}^m (p_{i,j} - p^2_{i,j})\, \nabla_{z_j} (x_i - z_j)^T.$$

In (2.65), we already worked out the derivative of $p_{i,j}$. Therewith we proceed

$$\begin{aligned} &\sum_{i=1}^m \left(\nabla_{z_j} \left[p_{i,j} - p^2_{i,j}\right]\right) (x_i - z_j)^T \\ =\;& 2\beta \sum_{i=1}^m \left(\left[(p_{i,j} - p^2_{i,j}) - 2p_{i,j}(p_{i,j} - p^2_{i,j})\right]\right) (x_i - z_j)\,(x_i - z_j)^T \end{aligned}$$

and thus at $z_1 = \ldots = z_k = 0$

$$\sum_{i=1}^m \left(\nabla_{z_j} \left[p_{i,j} - p^2_{i,j}\right]\right) (x_i - z_j)^T \Big|_{z_1 = \ldots = z_k = 0} = p_j(1 - p_j)(1 - 2p_j) \sum_{i=1}^m x_i\, x_i^T.$$

Furthermore,

$$\sum_{i=1}^m (p_{i,j} - p^2_{i,j})\, \nabla_{z_j} (x_i - z_j)^T \Big|_{z_1 = \ldots = z_k = 0} = -p_j(1 - p_j) I.$$

All in all,

$$\nabla_{z_j} \nabla^T_{z_j} V_j(0, \ldots, 0) \;=\; 2\beta\, p_j(1 - p_j) \left(2\beta\, (1 - 2p_j)\, \frac{1}{m} \sum_{i=1}^m x_i\, x_i^T - I \right).$$

With the definition $\mathrm{cov}(x, x) = \frac{1}{m} \sum_{i=1}^m x_i\, x_i^T$, the matrix $\nabla_{z_j} \nabla^T_{z_j} V_j(0, \ldots, 0)$ has eigenvalues with negative real parts (or, as the matrix is symmetric, only negative eigenvalues) if and only if the matrix

$$2\beta(1 - 2p_j)\mathrm{cov}(x) - I$$

has negative eigenvalues. We specify p_j using $u^*_j(0) = \lambda_j$,

$$p_j = \frac{e^{\lambda_j}}{\sum_{i=1}^k e^{\lambda_i}}.$$

The result follows if we note that we need to shift the coordinate system back by adding $-\overline{x}$, which in particular modifies $\mathrm{cov}(x)$. □

The following proposition is a consequence of this theorem. The proposition tells us when $z_1 = \ldots = z_k = 0$ is a Nash equilibrium. Recall that n is the dimension of the feature space s.t. $x_i, z_j \in \mathbb{R}^n$, and the trace of a matrix is the sum of the diagonal elements.

Proposition 2.57 *If $z_1 = \ldots = z_k = \overline{x}$ is a local strict local Nash equilibrium, then necessarily*

$$2\beta(1-2p_j)\, tr(cov(x)) < n \qquad for\ j = 1, \ldots, k. \tag{2.67}$$

In case of $n = 2$, stricter condition

$$2\beta(1-2p_j)\, tr(cov(x)) < 1 \qquad for\ j = 1, \ldots, k. \tag{2.68}$$

is sufficient.

Proof: (a) Necessary condition: Let $A_j = 2\beta(1-2p_j)\mathrm{cov}(x) - I$. As the trace is the sum of the eigenvalues, the condition that all eigenvalues are negative implies $2\beta(1-2p_j)\mathrm{tr}(\mathrm{cov}(x)) - n < 0$.
Sufficiency for $n = 2$: The Ruth-Hurwitz criterium indicates that the trace of A_j is negative, and the determinant is positive. As the sufficient condition is more strict than the necessary condtion, it implies that the trace is negative. Let $c_{i,j}$ denote the entries of $\mathrm{cov}(x)$. Then,

$$\det(A_j) \;=\; 4\beta^2(1-\rho_j)^2(c_{11}c_{22} - c_{12}c_{21}) - 2\beta(1-2p_j)(c_{11}+c_{22}) + 1.$$

The matrix $\mathrm{cov}(x)$ is positive (semi-)definite, as

$$\forall u \in \mathbb{R}^n: \;\; u^T\mathrm{cov}(x)u = \frac{1}{m}\sum_{i=1}^{m} u^T(x_i-\overline{x})(x_i-\overline{x})^T u = \frac{1}{m}\sum_{i=1}^{m}(\,(x_i-\overline{x})^T u\,)^2 \geq 0.$$

Hence, the eigenvalues (and therefore also the determinant) of $\mathrm{cov}(x)$ is non-negative.
As the sufficient condition states $2\beta(1-2p_j)(c_{11}+c_{22}) < 1$, the determinant is positive, and we are done. $\square$

This last result allows to understand the effects of the three extensions of the Hotelling model we consider in the stochastic valence model. Let us focus on the case $n = 2$.

First we assume that all valences are zero, $\lambda_j = 0$. Our sufficient condition for $z_1 = \ldots = z_k = \overline{x}$ to be locally stable Nash equilibrium (that is, the condition that the mean voting theorem holds) reads

$$2\beta(1-2/n)\mathrm{tr}(\mathrm{cov}(x)) < 1.$$

That is, in case of $n = 2$, the Mean Voting Theorem (as an obvious variant of the Median Voting Theorem for symmetric voter distributions) holds true. If all parties choose the location of the average voter, we always have a Nash equilibrium. If $n > 2$, $z_1 = \ldots = z_k = \overline{x}$ is not necessarily a Nash equilibrium. For example the parameter β may destroy this property if it becomes too large.
β balances the weight of the deterministic part of the scoring (or utility) function $u^*_{i,j}(z_j) = -\|x_i - z_j\|$ on the one hand, and the noise introduced by $e_{i,j}$ on the other hand. If β is small, the noise is important and the condition is true, while the condition will break down (and also the necessary condition becomes wrong) if β is large.
Hence, the mean voting theorem is not true in 23 (or higher) dimension without noise. Noise is able to stabilize the mean voter as a local Nash equilibrium for the parties. Higher dimensions, in turn, destabilize this equilibrium.

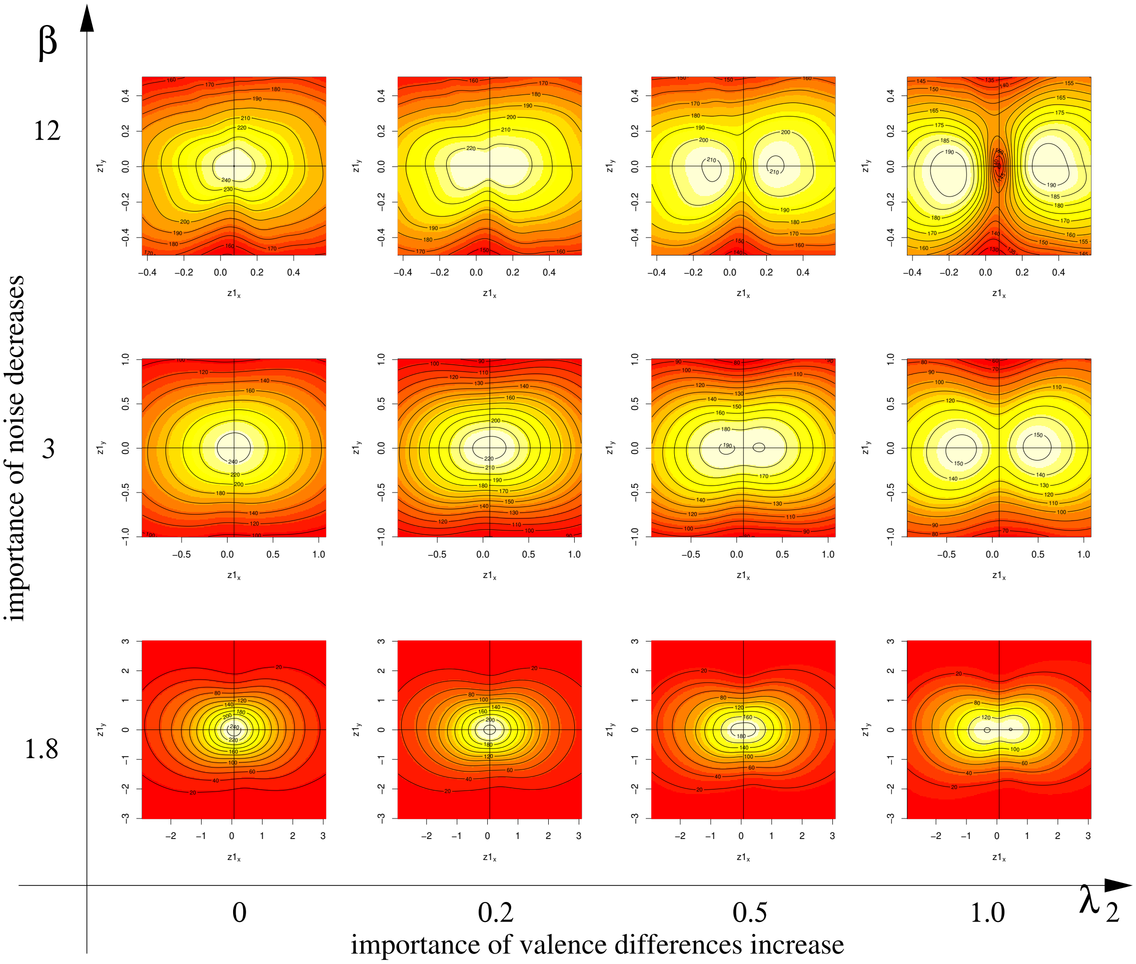


Figure 2.18: $V_1(z_1, z_2)$ for $z_j \in \mathbb{R}^2$, $z_2 = 0$, and x/y component of z_1 as given in the coordinate system. x_i are 500 realizations; first and second component of x_i are independent, first component is just $\mathcal{N}(0,1)$ distributed, second component distributed according to $\mathcal{N}(0, 1/\sqrt{2})$. $\lambda_1 = 0$, λ_2 and β as stated in the figure. The horizontal/vertical line indicates the mean value of x_i (x- and y-component). Since $z_2 = \overline{x}$, then $z_1 = \overline{x}$ is a Nash equilibrium if the corresponding figure shows a clear maximum at $z_1 = \overline{x}$. Else, it is better for z_1 to go away from $z_1 = \overline{z}$, s.t. $z_1 = z_2 = \overline{x}$ is no Nash equilibrium any more.

Last we discuss the effect of the valences. In case of unbalanced valences, we might have an index j_0 s.t. p_{j_0} become small (if λ_{j_0} becomes much larger than all other valences). Then, the l.h.s. of the inequality tends to $2\beta \mathrm{tr}(\mathrm{cov}(x))$, which is harder to archive. That is, also differences in the valences are able to de-stabilize $\overline{x}$ as a Nash equilibrium, even in case of $n = 2$. The noise, however, can counteract. It depends on the balance between variation of valences and importance of noise, if we find the trivial Nash equilibrium predicted by the Mean Voting Theorem, or if we find non-trivial, more realistic Nash equilibria.

2.10 Strategic Voting

Voters do not always directly vote according to their preferences ("honest voting"), but they may try to optimize the impact of their vote. If, for example, their favorite party is too small to have a notable impact, a vooter might choose to vote for another party that is similar (or at least closer to the voters' opinion than all other parties). An example could be the "ecologic-democratic party" (Oekologisch-Demokratische Partei) and the "green party" (Grüne) in Germany. Both parties focus on ecology, but the ecologic-democratic party is more to the right than the green party.

2.10.1 Basic Game Theory?

2.10.2 Poisson Games

(did introduce) War das wirklich Myerson? Lit Research!!

Uncertainty in the number of co-players may influence the strategy a given player does choose. Classical game theory assumes a fixed (and known) population of players. Myerson [32] did introduce games with population uncertainty. In view of our application, elections, we might know the number of eligible voters, but we do not know the number of active voters. However, only those influence the election result. And indeed, elections are one of the main examples for games with population uncertainty [32, **?**].

Let us introduce the central ingredients of Myerson's approach. We have an unknown number of players (a random number of players). This number will be known *a posteriori*, but there is only a limited knowledge *a priorie*. A given player is characterized be his/her type (e.g., the social status), which is one from a finite set of possible types. As we do not know the number of players, we also have only limited knowledge about the number of players of a given type. Furthermore, he/she selects a strategy out of a finite number of strategies (e.g., which party to vote for). The players are assumed to act purely rational, and aim to maximize an utility function. For a focal player, this function depends on his/her own type, on the strategy he/she did choose (as this strategy influences the outcome), and the strategy all other players do select (again, as these strategies do influence the outcome).

For elections, one interesting consequence of the assumption of a rational voter is that a voter only cares if he/she is able to influence the result, that is, if the decision depends on his/her strategy. If he/she is not able to change the result, it is even questionable if he/she takes the effort to vote at all. The so-called "paradox of not voting" [10] is discussed, and researches who are in favor of purely rational individuals aim to prove that voting indeed pays [12], see also Section 2.10.3.

Model 2.58 Game with population uncertainty *We consider a finite population of players. The population size is a random number N. That is, N is an $\mathbb{N}$-valued random number, not known to the participants of the game. The players have different types. Only a finite number of types are possible. Let T denote the finite set of the possible types. All individuals of the same type are basically indistinguishable. Also the set of possible pure strategies S is finite. Let $\Delta(S)$ denote the set of all possible probability measures on S. With this notion, $\sigma \in \Delta(S)$ denotes all possible mixed strategies. As the individuals of the same type are indistinguishable, the strategy a person opts for only depends on the persons'*

type, $\sigma : T \to \Delta(S)$.
In order to define the utility function for an individual, we need to know his/her type (recall the indistinguishably of individuals of the same type; different types may have different preferences), and the pure strategy $s \in S$ chosen by that individual. Furthermore, we need to know the strategy of all other players. In order to evaluate the utility function, the information which is not known by he player beforehand, is necessary: In a realization, we assume that we know how many individuals did choose which strategy. That is, we have a vector, called the action profile,

$$Z = (N_s)_{s\in S},$$

where N_s denote the (random) number of individuals that opt for strategy $s \in S$. Therefore, $\sum_{s\in S} N_s = N$. Then, the utility function for our target individual reads

$$U : \mathbb{N}_0^{|S|} \times S \times T \to \mathbb{R}, \qquad (Z, s, t) \mapsto U_t(Z, s).$$

We assume that U is globally bounded.
For the analysis, we go from realizations to expectations. Thereto we already introduced $\sigma(t)$ as the random strategy of type $t \in T$. We furthermore introduce a probability measure on $\mathbb{N}_0^k$ that assigns a probability to each distributions of players to the possible strategies (and hence implicitly also for the random number N of the total number fo players). This probability function Q is known to the players beforehand.
Individuals choose their strategy in order to maximize their utility.
We call the tuple (T, Q, S, U) a game with population uncertainty.

Let us exemplify this model structure with a tangible (toy) example: A teacher offers her class to play a game. Who is not in the mood may take a book from the school library and read. The players either take the role of hunters, or the role of prey. Each prey child gets a ribbon. The predator-children have to hunt the prey-children in order to chase the ribbon, the prey-children have to hide and escape. Who collects the most ribbons is winner among the hunters, who hides the longest time is winner among the prey.
In this setting, the number N of children that take part in the game is a random number. Depending on the way we want to model the scene, there is only one type of children (all children in the class are equivalent w.r.t. the game). Alternatively, we could distinguish between girls and boy, say. In that case, we have two types of players, $T = \{\text{girls}, \text{boys}\}$. Furthermore, we have two strategies: hunter and prey. Any participant may chose a mixed strategy, but needs to define his/her choice at the beginning of the game in selecting a realization of the random strategy $\sigma(t)$ he/she did decide for. Note that this strategy only depends on the types. So, all girls are assumed to select an identical mixed strategy and all boys select an identical mixed strategy; the boys' strategy may, of course, be different from the girls' strategy. The utility function can depend on the types. If, e.g., boys run faster than girls, girls may have an disadvantage as hunters if the prey mainly consist of boys. There are for sure many other aspects to consider, that give an advantage to girls. However, considerations as those above imply that the utility function does not only depend on the own strategy of a player, and the strategy of all other players, but also of the focal players' type.

Poisson game

We will focus on Poisson games, which we introduce next. In that it is useful to recall the aggregation property (additivity) of Poisson variables:

$$\mathrm{Pois}(\lambda_1) + \mathrm{Pois}(\lambda_2) = \mathrm{Pois}(\lambda_1 + \lambda_2).$$

The sum of two Poissonion random variables is again a Poissonian random variables, where the expectations just sum up.

Definition 2.59 *Let $\nu \in \mathbb{R}$, $\nu > 0$, and assign to each $t \in T$ number $r_t > 0$, s.t. $\sum_{t\in T} r_t = 1$. A Poisson game is a game with population uncertainty, where the number N_t of payers of type $t \in T$ is given by a Poisson distribution Pois($\nu\, r_t$), independently of the number of players who are of a different type.*

With other words, the total number of players follows a Poisson random variable with expectation ν. These individuals are distributed to the different types, where the probability for a given type $t \in T$ is r_t. As the total number of players is not given, but the number fo players of the different types are defined, these random variables (they are $|T|$ random variables) can be assumed to be independent.

Corollary 2.60 *Given a Poisson game with the notation introduced above, we find for the probability that the population structure $Y = (N_t)_{t\in T}$ equals a given vector $y = (n_t)_{t\in T} \in \mathbb{N}_0^{|T|}$,*

$$\begin{aligned} Q(Y = y) &= Q((N_t)_{t\in T} = (n_t)_{t\in T}) \\ &= \prod_{t\in T} \frac{1}{n_t!}(\nu\, r_t)^{n_t} e^{-r_t\, n} = \left(\prod_{t\in T} \frac{1}{n_t!}(\nu\, r_t)^{n_t}\right) e^{-\nu}. \end{aligned} \tag{2.69}$$

The probability that the total population size $N = \sum_{t\in T} N_t$ equals $n \in \mathbb{N}_0$ reads

$$P(N = n) = \frac{1}{n!}\nu^n e^{-\nu}. \tag{2.70}$$

Next we do not consider the number of individual of a given type, but the (random number) of individuals that decide for a given strategy $s \in S$. For Poisson games, we again have that these random numbers are independent and follow a Poissonian distribution.

Theorem 2.61 *Let (T, Q, S, U) be a Poisson game. Let X_s denote the number of individuals that select in a realization strategy $s \in S$. Then, X_s are independent Poisson variables with mean μ_s given by*

$$\mu_s = \nu \sum_{t\in T} r_t\, \sigma_t(s). \tag{2.71}$$

Proof: Let X be the random vector which characterizes the number of players structured by strategy, that is, $(X)_s = X_s$. We aim to understand the probability $P(X = x)$ for $x \in \mathbb{N}_0^{|S|}$. In general, the players that choose strategy s consist of different types. It is necessary to disentangle types and strategy: For a given strategy s, we write $x_s = \sum_{t\in T} w_{t,s}$. That is, the players that actually choose a certain strategy are structured according to their type; the number of type-t-players hat select strategy s is $w_{t,s}$. The set $W(x)$ contains all possible combinations that yield $x \in \mathbb{N}_0^{|S|}$,

$$W(x) = \{w = (w_{t,s})_{t\in T, s\in S} \in \mathbb{N}_0^{|T|\times|S|} \mid \sum_{t\in T} w_{t,s} = x_s\}.$$

For $w \in W$, we define two different operators, that yield on the one hand all individuals that play a certain strategy and on the other all individuals of a certain type,

$$\Pi_S : W(x) \to \mathbb{N}_0^{|S|},\ \ w \mapsto (\sum_{t\in T} w_{t,s})_{s\in S}, \qquad \Pi_T : W(x) \to \mathbb{N}_0^{|T|},\ \ w \mapsto (\sum_{s\in S} w_{t,s})_{t\in T}.$$

Then, of course, $\Pi_S(w) = x$ for all $w \in W(x)$. However, Π_T contains the useful information about the number of types. A given type-t-player will chose a given strategy s with probability $\sigma_{t_0}(s)$. As the players select independently the strategy, the type-t_0-players (with number $(\Pi_T w))_{t_0}$) are distributed to the strategies in S according to a multinomial distribution,

$$(\Pi_T(w))_{t_0}! \prod_{s\in S} \left(\frac{\sigma_{t_0}(s)^{w_{t_0,s}}}{w_{t_0,s}!}\right).$$

Since the distribution of type-t-individuals to the possible strategies in S follows a multinational distribution, we have

$$P(X = x) \;\; = \;\; \sum_{w\in W(x)} Q(\Pi_T(w)) \prod_{t\in T} \left((\Pi_T(w))_t! \prod_{s\in S} \left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right).$$

As we have a Poisson game, the probability function Q is determined by eqn. (2.69),

$$\begin{aligned}
P(X = x) \;\; &= \;\; \sum_{w\in W(x)} \left(\prod_{t\in T} \frac{1}{(\Pi_T(w))_t!}(n\, r_t)^{(\Pi_T(w))_t}\right) e^{-\nu} \prod_{t\in T}\left((\Pi_T(w))_t! \prod_{s\in S}\left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right)\\
&= \;\; \sum_{w\in W(x)} \prod_{t\in T} \left((\nu\, r_t)^{(\Pi_T(w))_t} \prod_{s\in S}\left(\frac{\sigma_t(s)^{w_{t,s}}}{w_{t,s}!}\right)\right) e^{-\nu}\\
&= \;\; \sum_{w\in W(x)} \prod_{t\in T} \prod_{s\in S} \left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!}\right) e^{-\nu} = \sum_{w\in W(x)} \prod_{t\in T} \prod_{s\in S} \left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right)
\end{aligned}$$

where we used that $\sum_s \sigma_t(s) = 1$, and $\sum_t r_t = 1$. We aim to write this expression as a product over terms depending on s. Therefore, we note that the different s-components of $w_{t,s}$ in $W(x)$ are independent; we have for a given strategy $s_0 \in S$ that

$$\{(w_{s_0,t})_{t\in T} \mid w \in W(x)\} = \{v \in N_0^{|T|} \mid \sum_{t\in T} v_t = x_s\}.$$

In that sense, we define

$$V(x_s) = \{w \in N_0^{|T|} \mid \sum_{t\in T} w_t = x_s\}$$

and have

$$W(x) = \bigoplus_{s\in S} V(x_s).$$

In a slightly abuse of notation, we write for $w_s \in V(x_s)$, that $(w_s)_t = w_{t,s}$. Then,

$$\begin{aligned}
P(X = x) &= \sum_{(w_s)_{s\in S}\in\bigoplus_{s\in S} V(x_s)} \prod_{t\in T}\prod_{s\in S}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right)\\
&= \sum_{(w_s)_{s\in S}\in\bigoplus_{s\in S} V(x_s)} \prod_{s\in S}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right)\\
&= \prod_{s\in S}\sum_{w_s\in V(x_s)}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right).
\end{aligned}$$

We observe that the different components of X are independent. The probability for $X = x$ is the product of the marginal probabilities

$$P(X_s = x_s) = \sum_{w_s\in V(x_s)}\prod_{t\in T}\left(\frac{(\nu\, r_t\, \sigma_t(s))^{w_{t,s}}}{w_{t,s}!} e^{-\nu\, r_t\, \sigma_t(s)}\right).$$

We find that X_s is the sum of $|T|$ independent Poisson random variables, which have expectation $\nu\, r_t\, \sigma_t(s)$. Therefore, also X_s is Poisson distributed, with expectation $\nu\sum_{t\in T} r_t\, \sigma_t(s)$.

□

The property that the components of X_s are independent simplifies the analysis. Therefore we introduce a name for this property.

Definition 2.62 *A game with population uncertainty (T, Q, S, U) has the independent-action property, if for all $\sigma : T \to \Delta(S)$ the random number of players that select a given strategy $s \in S$ is independent of the number of players that choose a different strategy $s' \in S$ (for all $s' \neq s$).*

Note that games with a fixed number of layers cannot have the independent-action property, as the number of players structured by the strategy sum up to a given, fixed total number of players. The next theorem shows that the independent-action property forces the game to be a Poisson-game.

Theorem 2.63 *Let (T, Q, S, U) a game with population uncertainty that has the independent-action property. Assume $|S| > 1$. Then, (T, Q, S, U) is a Poisson-game, and X_s is a Poisson random variable with expectation μ_s, where*

$$\mu_s = \sum_{t\in T}\sigma_t(s)\sum_{y\in\mathbb{N}_0^{|T|}} Q(y)\, y_t. \tag{2.72}$$

The total number of players also follows a Poisson distribution.

Proof: A voter of a given type t_0 decides independently of all other individuals with probability $\sigma_{t_0}(s)$ for strategy $s \in S$, where $P(X_{s_a} = 0) > 0$. That is a key feature of the model that we

intend to use.
We have a strategy function $\sigma : T \to \Delta(S)$ given. We focus on two different strategies $s_a, s_b \in S$, $s_a \neq s_b$, and define $\sigma^* : T \to \Delta(S)$ by

$$\forall t \in T : \sigma^*_t(s_a) = \sigma_t(s_a) + \frac{1}{2}\sigma_t(s_b), \quad \sigma^*_t(s_b) = \frac{1}{2}\sigma_t(s_b), \quad \forall s \in S \setminus \{s_a, s_b\} :, \sigma^*_t(s) = \sigma_t(s).$$

We focus on X_{s_a} and X_{s_b}. We are interested in the probability that $X_{s_a} = \ell$ and $X_{s_b} = k$; to be more precise, in the relation of these probabilities for σ and σ^*. We denote $P_\sigma(X_{s_a} = \ell, X_{s_b} = k)$ resp. $P_{\sigma^*}(X_{s_a} = \ell, X_{s_b} = k)$ to indicate which strategy function we have in mind. If we only consider the individuals of one type t_0 we find

$$\begin{aligned} P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k \,|\text{ only type } t_0 \text{ individuals}) &= \left(\frac{\sigma^*_{t_0}(s_b)}{\sigma^*_{t_0}(s_a) + \sigma^*_{t_0}(s_b)}\right)^k \\ = \left(\frac{\sigma_{t_0}(s_b)}{2(\sigma_{t_0}(s_a) + \sigma_{t_0}(s_b))}\right)^k &= \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 0, X_{s_b} = k \,|\text{ only type } t_0 \text{ individuals}). \end{aligned}$$

As this relation is linear and identical for all types, we have

$$P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k) \quad = \quad \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 0, X_{s_b} = k).$$

Similarly, we have (first for a given type, and then for all type distributions)

$$\begin{aligned} P_{\sigma^*}(X_{s_a} = 1, X_{s_b} = k) &= \binom{k+1}{1} \frac{\sigma^*_t(s_a)^1 \; \sigma^*_t(s_b)^k}{(\sigma^*_t(s_a) + \sigma^*_t(s_b))^{k+1}} \\ &= \binom{k+1}{1} \frac{(\sigma_t(s_a) + \frac{1}{2}\sigma_t(s_b))^1 \; (\frac{1}{2}\sigma_t(s_b))^k}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} \\ &= \left(\frac{1}{2}\right)^k \binom{k+1}{1} \frac{\sigma_t(s_a)\,\sigma_t(s_b)^k}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} + \left(\frac{1}{2}\right)^{k+1} \binom{k+1}{1} \frac{(\sigma_t(s_a)^0 \sigma_t(s_b))^{k+1}}{(\sigma_t(s_a) + \sigma_t(s_b))^{k+1}} \\ &= \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 1, X_{s_b} = k) + \left(\frac{1}{2}\right)^{k+1} \binom{k+1}{1} P_\sigma(X_{s_a} = 0, X_{s_b} = k+1) \\ &= \left(\frac{1}{2}\right)^k P_\sigma(X_{s_a} = 1, X_{s_b} = k) + \left(\frac{1}{2}\right)^{k+1} (k+1)\, P_\sigma(X_{s_a} = 0, X_{s_b} = k+1). \end{aligned}$$

With these preparations, consider

$$\frac{P_{\sigma^*}(X_{s_a} = 1, X_{s_b} = k)}{P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k)}.$$

As the components of X are independent, we have that

$$\frac{P_{\sigma^*}(X_{s_a} = 1, X_{s_b} = k)}{P_{\sigma^*}(X_{s_a} = 0, X_{s_b} = k)} = \frac{P_{\sigma^*}(X_{s_a} = 1)}{P_{\sigma^*}(X_{s_a} = 0)}$$

is independent of k. On the other hand, we can use the computations from above, and find

$$\begin{aligned}
&\frac{P_{\sigma^*}(X_{s_a}=1, X_{s_b}=k)}{P_{\sigma^*}(X_{s_a}=0, X_{s_b}=k)}\\
=\;& \frac{\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a}=1, X_{s_b}=k) + \left(\frac{1}{2}\right)^{k+1} (k+1)\, P_\sigma(X_{s_a}=0, X_{s_b}=k+1)}{\left(\frac{1}{2}\right)^k P_\sigma(X_{s_a}=0, X_{s_b}=k)}\\
=\;& \frac{P_\sigma(X_{s_a}=1)}{P_\sigma(X_{s_a}=0)} + \frac{(k+1)\, P_\sigma(X_{s_a}=0, X_{s_b}=k+1)}{2P_\sigma(X_{s_a}=0, X_{s_b}=k)}\\
=\;& \frac{P_\sigma(X_{s_a}=1)}{P_\sigma(X_{s_a}=0)} + \frac{(k+1)\, P_\sigma(X_{s_b}=k+1)}{2P_\sigma(X_{s_b}=k)}.
\end{aligned}$$

If we compare the two results, we find that $\lambda(b) = (k+1)\, P_\sigma(X_{s_b} = k+1)/P_\sigma(X_{s_b} = k)$ is independent of k. Hence,

$$P_\sigma(X_{s_b}=k+1) = \frac{\lambda(b)}{(k+1)}\, P_\sigma(X_{s_b}=k) = \frac{\lambda(b)^2}{(k+1)k}\, P_\sigma(X_{s_b}=k) = \ldots = \frac{\lambda(b)^{k+1}}{(k+1)!}\, P_\sigma(X_{s_b}=0).$$

Since $\sum_{k=0}^\infty P_\sigma(X_{s_b}=k) = 1$, we have $X_{s_b} \sim \mathrm{Pois}(\lambda(b))$.
The expectation of X_{s_b} is given by

$$\mu_{s_b} \;=\; \sum_{t\in T} \sigma_t(s_b) \sum_{y\in\mathbb{N}_0^{|T|}} Q(y)\, y_t. \tag{2.73}$$

Furthermore, the total population size is the sum of all components of X_s, that is, a sum of independent Poisson variables, and hence again a Poisson variable.

□

Environment of a player and probability equivalence property

In order to decide rationally for a strategy, the knowledge of the environment (who many players of which type) is decisive. A player does know the overall setting of the game (T, Q, S, U). However, the player does know one mode information: He/she is part of the game. How can he/she utilize this information?
The following example, that clarifies up to a certain degree the implication of that fact is given in [32]: Let us assume that a game may have either 300 or 600 players. According to the probability distribution Q, each outcome has the same probability. Let us assume that there are 600 potential players; that is, either all or half of the potential players actually become players. Each player is assumed to behave alike. For this example we assume that the game is repeated again and again. For an external game theorist, the average population size is thus

$$\frac{1}{2}\,600 + \frac{1}{2}\,300 = 450.$$

Now we consider a focal person who is a potential player. This person is interested in his/her environment: If he/she takes part in the game, who many individuals will also take part?

Importantly, we are not interested in all realizations of the game, but only in realizations when the focal individual is part of the game. Only is this is the case, we consider the population size of a game.
In realizations where only 300 players take part, a given person has probability 1/2 to be in the game. If we have 600 players, the probability to be "in" is 1.
In half of all realizations (probability 1/2) we have 600 participants, and our focal individual is in for sure. In the other half of the realizations, we only have 300 individuals, and for those realizations, we have only probability 1/2 that our focal individual is "in". That is, the probability to be part of a 300-person-game is only 1/4. Hence, the expected number of players in games our focal individual takes part is

$$\frac{\frac{1}{2}\,600+\frac{1}{4}\,300}{\frac{1}{4}+\frac{1}{4}}=500.$$

The average number of players a focal individual finds in his/her environment is hence 499, much larger than the average number of players an independent person outside of the setting recognizes (450).

The environment of an individual is the number of fellow players (to be more precise: the number of fellow players, structured by their type). We have the expected number of players, determined by the expectation according to the probability measure Q. This number is not the environment of a player. On the one hand, this number is decreased (by one), as the player himself/herself is part of the game, but does not count to the environment. On the other hand, this number is increased, as the fact that a focal individual takes part in the game indicates that we rather have a larger than a smaller population size. In the following, we discuss how these two effects balance.

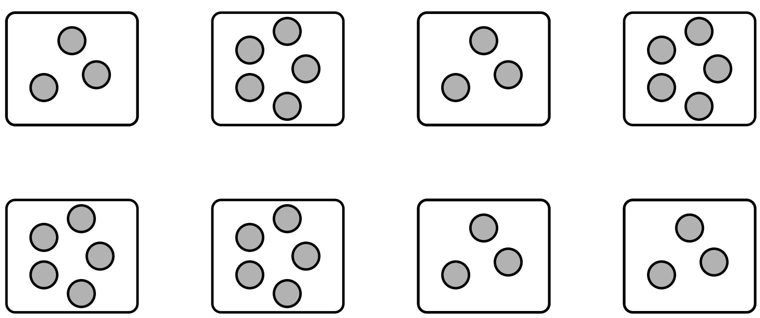


Figure 2.19: 8 realization of a game with population uncertainty. The population size is either 3 or 5, with popularity 1/2 for each. If we randomly select a *realization* from these 8 realizations, we obtain probability 1/2 for a population size 3, and probability 1/2 for population size 5. If we randomly select an *individual* out of the 32 individuals, we obtain a probability of $20/32 = 5/8$ for population size 5, and only $12/32 = 3/8$ for population size 3.

We first think about the environment. To be more precise, in the first step, we think about the total number of players that are in the environment of a randomly chosen person. The probability measure Q defines the probability $P(N = n)$, the probability that there are $n \in \mathbb{N}_0$ players in the game. We already know that one player (the focal individual) is in the game.

$N = 0$ hence no valid realization. Obviously, the knowledge that the focal player is present changes the probability distribution of N. How can we define the environment of our focal player? That is, how likely is it that our focal individual has n co-players? Let us denote this probability $P(N-1=n|1)$ (with this notation we emphasize that we condition on the fact that there is the focal player, and that we ant ti know the number of additional players).
The idea to determine this probability is the following: Create a lot of (100, say) realizations of the game (see Fig. 2.19). If we now select randomly one realization, we recover the probability distribution $P(N = n)$ for the population size. However, we are interested in a focal individual. Therefore, we randomly select one *individual* from all individuals within any of the realizations. The probability to select a realization with a high population size is thus larger than a realization with a small size. To be more precise, the probability is proportional to the population size and the probability to create a game with this size. This consideration yields the following definition.

Definition 2.64 *The distribution of the number of co-players of a randomly chosen focal player is given by*

$$P(N-1=n|1) = \frac{(n+1)P(N=n+1)}{\sum_{k=1}^{\infty} k\, P(N=k)}. \tag{2.74}$$

We say that a probability distribution for the number of players satisfies the environmental equivalence property, if the number of co-players follows the same probability distribution as the number of players,

$$P(N=n) = P(N-1=n|1).$$

We emphasize that this is a definition, based on the construction described. As we will see below, this idea is connected with an augmented model that includes the way players are recruited. For now, we stay with the definition and investigate the consequences. We are particularly interested in the distributions that have the probability equivalence property for the population size.
The equivalent property does mean, that the external game theorist and a (potential) player of the game, both "see" the same number of (co)players. The next proposition shows, that the equivalence property forces the game to be a Poisson game.

Proposition 2.65 *The population size of a game, where the distribution of the population size has the equivalence property, follows a Poisson distribution.*

Proof: We know that

$$P(N-1=n|1) = C\,(n+1)\,P(N=n+1)$$

where C is a normalization constant. Furthermore, the equivalence property indicates $P(N = n) = P(N-1=n|1)$. Hence,

$$P(N=n) = C\,(n+1)\,P(N=n+1)$$

respectively, by finite induction,

$$P(N=n) = \frac{1}{n!}\,C^{-n}P(N=0).$$

As the probabilities have to sum up to one, N is a Poissonian random variable with expectation $1/C$.

□

For the Poissonian population size, we can augment the Poisson game model by the recruitment process. We have a large population (size M) of potential players. Each player has a probability p to become recruited, s.t. the number of players is distributed according to the binomial distribution $\mathrm{Bin}(M,p)$. If M tends to infinity and p to zero, s.t. $M\,p \to r > 0$, we obtain the Poisson distribution. If we now consider a focal individual how is a player, then there are only $M-1$ potential players left. The number of co-players is distributed according to $\mathrm{Bin}(M-1,p)$. For M finite, we do not find the equivalence property. However, if M tends to infinity, while $(M-1)p \to r$ also $\mathrm{Bin}(M-1,p)$ tends to the Poisson distribution with expectation r. In this case, we establish the equivalence distribution.

We can refine this result: The environment of an individual is not only defined by the total population size, but by the population size, structured according to the types. Thereto, we return to the population structure $Y = (N_t)_{tT}$. We introduce δ_{t_0} to denote the Kronecker-vector

$$(\delta_{t_0})_t = 1 \text{ if } t = t_0 \quad \text{ and } (\delta_{t_0})_t = 0 \text{ else.}$$

We parallel the notation from above, and write $P(Y - \delta_{t_0} = y|\delta_{t_0})$ to denote the probability that a t_0-individual find the environment y.

Definition 2.66 *The type-structured environment of a t_0-individual has the distribution*

$$(Y - \delta_{t_0} = y|\delta_{t_0}) = C\,Q(y + \delta_{t_0})\,(y_{t_0} + 1) \tag{2.75}$$

where C is determined by the fact that the probabilities $P(Y - \delta_{t_0} = y|\delta_{t_0})$ sum up to one. We say that a random game satisfies the environmental equivalence property, if the distribution of the structured population size and that for the structured environment of a t-type (for all types $t \in T$) coincide,

$$P(Y - \delta_{t_0} = y|\delta_{t_0}) = Q(Y = y).$$

Theorem 2.67 *A game with population uncertainty that has the environmental equivalence property is a Poisson random game.*

Proof: We have for $t_0 \in T$

$$(Y - \delta_{t_0} = y|\delta_{t_0}) = C\,Q(y + \delta_{t_0})\,(y_{t_0} + 1) = Q(y).$$

Hence, by finite induction,

$$Q(y) = C \prod_{t \in T} \frac{(c_t)^{y_t}}{y_t!}$$

where c_t and C are suited constants. The entries of the random vector Y follow independent Poisson distributions, and hence the game is a Poisson game.

□

Equilibrium strategy

We briefly touch the question if there are equilibrium strategies. As we only use Poisson games later on, we only investigate Poisson games. We will not rigorously prove the existence, but we connect the equilibrium strategy with the notion of a Nash equilibrium [8].

Given the strategy σ, the distribution of the number of strategy-structured number of players is given by some random vector $Z = (N_s)_{s\in S}$ that has the independent-action property. The entries follow a Poisson distribution, with parameters μ_s stated in Theorem 2.63. The expected value utility function of a type t_0 player who selects the pure strategy $s_0 \in S$ is given by

$$\overline{U}_{t_0}(s_0) = \sum_{z\in\mathbb{N}_0^{|S|}} P(Z = z)\, U_t(Z, s_0).$$

If this player (note, only this focal player, not all players of type t_0) plays the random strategy $\theta \in \Delta(S)$, then the expected outcome reads

$$\tilde{U}_{t_0}(\theta) = \sum_{s\in S} \theta(s)\overline{U}_{t_0}(s).$$

The set of the best pure response of the single t_0-player is

$$\mathrm{BPR}_{t_0}R(\sigma) = \{s_0 \in S \,|\, \forall s \in S : \overline{U}_{t_0}(s) \leq \overline{U}_{t_0}(s_0)\,\},$$

and the optimal mixed strategies are all randomized strategies on the set $\mathrm{BPR}_{t_0}(\sigma)$,

$$\mathrm{BR}_{t_0} = \Delta(\mathrm{BPR}_{t_0}(\sigma)).$$

With this notation, we define the Nash equilibrium.

Definition 2.68 *σ^* is a Nash equilibrium, if for all types $t \in T$,*

$$\sigma_t^* \in BR_t(\sigma^*).$$

The existence of Nash equilibria can be shown, as usual, based on Kakutanis Fixed point theorem. Instead of a proof, we consider a toy example.

The Really Bad Gang versus Gentleman thugs.
We have two groups of organized criminal gangs, the Really Bad Guys (RBG) and the Gentleman Thugs (GT). The RBGs have exactly 100 members, and all work together. We only focus on the GTs, such that our type set is

$$T = \{\mathrm{GT}\}.$$

The distribution Q is defined via

$$N_{GT} \sim \mathrm{Poiss}(r_{GT})$$

where r_{GT} denote the expected group sizes of the two gangs.
Each member of the GTs either go alone on a raid, or cooperate together and with the RBGs. In case they work alone, they catch 1 unit. Alternatively, the two groups can choose to cooperate. Then, their loot per member is much higher, 100 units. However, the gang which has more members involved will take the complete loot. In case of equal member size, each individual

receives 50 units.
We have two strategies: "alone" (a), and "cooperate", (c)

$$S = \{a, c\}.$$

The utility function is given by (t =GT)

$$U_t(y, a) = 1, \quad U_{RGB}(y, c) = \begin{cases} 100 & \text{if} \quad y_c > 100 \\ 50 & \text{if} \quad y_c = 100 \\ 0 & \text{if} \quad y_c < 100. \end{cases} .$$

Let y denote the environment of an GT. If the number of (other) cooperating GT-individuals equals 99 or are larger or equal 100, the focal GT-person wants to cooperate, as in that case, either both gangs have the equal size (payoff 50 for our focal individual), or the GTs has even more cooperators (payoff 100). Only if the cooperating GT-individuals are less than 99, our focal GT want to work alone to earn 1 unit.
However, our focal individual has to decide before he/she knowns the exact population size. All he/she does know is Q, that is, the parameter r_{GT}. Let us assume that he population plays strategy σ. In this simple case (only one type, and only two pure strategies), the strategy σ is defined by the probability p to cooperate. In this case, the number of cooperating GTs is distributed according to the Poissonian distribution, with expectation $p\, r_{GT}$. Let N_c denote the random number of cooperating individuals, that is, a Poissonian Random variable with expectation $p\, r_{GT}$. Then,

$$\overline{U}_t(a) = 1, \qquad \overline{U}_t(c) = 0\, P(N_c < 99) + 50\, P(N_c = 100) + 100\, P(N_c > 100).$$

Note that we used here the environmental equivalence relation. The probability that our focal individual finds a certain number of fellow cooperators is the same as that for an external observer to find this number of cooperators. The optimal pure response for the focal individual is

check!! Here we use the environmental equivalence cond, but the result/Nash equil is independent on this condition

$$\begin{aligned} \text{BR}(\sigma) &= \{a\} \text{ if } \overline{U}_t(c) < 1 = \overline{U}_t(a), \\ \text{BR}(\sigma) &= \{c\} \text{ if } \overline{U}_t(c) > 1 = \overline{U}_t(a), \\ \text{BR}(\sigma) &= \{a, c\} \text{ if } \overline{U}_t(c) = 1 = \overline{U}_t(a). \end{aligned}$$

We have only one case, where an individual is willing to randomize his/her strategy: If $U_t(c) = U_t(a)$. ELse, the decision is deterministic. We have several Nash equilibria.
(1) All work alone.
If $\sigma = a$, then $\overline{U}_t(c) = 0$. The Nash equilibrium is hence $\sigma_t \in \text{BR}_t(\sigma)$.
(2) All cooperate.
This Nash equilibrium only exist, if $\overline{U}_t(c) > 1$ for $\sigma = c$, i.e. if all individuals cooperate ($p = 1$).
(3) Randomized strategy.
Assume that $\overline{U}_t(c) > 1$ for $\sigma = c$ (the randomized strategy will only work under that condition). If all individual cooperate, the utility function is positive. We know that $\overline{U}_t(c) = 0$ if $p = 0$. Thus, there is $\hat{p} \in (0, 1)$, s.t. $\overline{U}_t(c) = 1$ if all individuals play the strategy σ that randomize with probability $\hat{p}$ for cooperation. Then,

$$\sigma_t \in \text{BR}_t(\sigma),$$

and we have a mixed Nash equilibrium.
Note that we did not use the environmental equivalence condition in the three cases, that did determine the possible Nash equilibria. The environmental equivalence condition allowed us to consider a focal individual, but the considerations yield conditions, that are independent on this point of view.

2.10.3 Strategic Voting – turnout rate

We apply Poisson games to the question how large the turnout rate of an democratic election in case of a two-party system will be. We observe that the participation in elections mostly decreases in the recent years. One central question is hence, why do voters vote? Palfrey and Rosenthal developed an approach, based on rational voting [34]. Myrson [32] did reformulate these ideas in the context of Poisson games.

Model 2.69 *Consider a population of voters, divided into two types, leftist and rightist voters. The total number of voters is N; the probability for a voter to be rightist is p_r, and $p_\ell = 1 - p_r$ for a leftist.*
Two candidates stand for election, a right and a left candidate. A voting is connected to a price: a vote "costs" a voter 0.05 *units (if the favorite candidate does not make it). If "his/her" candidate wins, he/she wins 1 unit. In case of a tie, the winner is determined by a fair coin toss, that gives each candidate the probability* 1/2 *to wind.*
A rational citizen aims to maximize his/her expected gain in that elections.

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Note that this model bears some reasonable aspects: It is well known that the turnout rate depends on the weather – if the weather is nice, less citizen will vote. This is a clear argument for the fact that citizen indeed assign some costs to the voting process: E.g., if I go to vote, I cannot do some spare time activities during that time. On the other hand, we are happy if "our" favorite candidate wins. In that, this model seems to describe reality appropriately. On the other hand, citizens will not rationally quantify and weight their gain and loss, and decide on that basis. However, we can see where we are led to from the idea of a rational voter.

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