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The noisy voter model

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Abstract

The noisy voter model is a spin system on a graph which may be obtained from the basic voter model by adding spontaneous flipping from 0 to 1 and from 1 to 0 at each site. Using duality, we obtain exact formulas for some important time-dependent and equilibrium functionals of this process. By letting the spontaneous flip rates tend to zero, we get the basic voter model, and we calculate the exact critical exponents associated with this "phase transition". Finally, we use the noisy voter model to present an alternate view of a result due to Cox and Griffeath on clustering in the two-dimensional basic voter model.

Keywords: Voter model; Noisy voter model; Graph; Duality; Moran model; Transient behaviour; Random walk; Green function; Critical exponents; Scaling

1. Introduction

The voter model is a particle system that was originally introduced to model the interaction of two distinct populations that are competing for territory (Clifford and Sudbury, 1973). Variants of this model include the stepping-stone model in genetics (e.g., Sawyer, 1977) and the Williams–Bjerknes model for tumor growth (e.g., Bramson and Griffeath, 1981), also called the "biased voter model" (see Durrett, 1988).

Briefly, the basic voter model is a Markov process $\{\eta_t : t \geqslant 0\}$ which takes values in the set of all $\{0,1\}$-valued functions on the d-dimensional integer lattice $\boldsymbol{Z}^d$ (i.e. it is a "spin system" on $\boldsymbol{Z}^d$). Thus, for each x in $\boldsymbol{Z}^d$, $\eta_t(x)$ is either 0 or 1, and it randomly flips back and forth over time according to the following rules. If $\eta_t(x)$ is

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1 (respectively, 0), then it flips to 0 (respectively, 1) at rate equal to the number of neighbours of x which are 0 (respectively, 1) at time t. We use the term "neighbour of x" to denote a point y of $\boldsymbol{Z}^d$ whose Euclidean distance from x is 1. In this paper we shall examine the "noisy voter model", whose flip rate from 0 to 1 (respectively, from 1 to 0) equals the corresponding rate for the basic voter model plus a constant $\beta \geqslant 0$ (respectively, $\delta \geqslant 0$). This added term may be viewed as representing an additional source of flipping that is independent of the states of the neighbouring sites. General theory tells us that the noisy voter model is exponentially ergodic if and only if either β or δ is nonzero, but there are still some interesting things to be said about this model.

The purpose of our study is twofold. Firstly, we obtain exact formulas for some important quantities of the noisy voter model as a function of time, as well as their values in equilibrium. Secondly, by showing that the basic voter model is a weak limit of noisy models, we find the exact values of certain critical exponents and also recover some known results about the basic voter model. This demonstrates how a properly chosen ergodic model can serve as a tool for studying interesting physical phenomena in a nonergodic model.

To provide an opening context, let us consider a more general family of nearest-neighbour spin systems on a graph G (with no loops). Let $S(G)$ denote the set of sites of G. For each site $x \in S(G)$, let $N(x)$ denote the set of neighbours of x. We shall assume that G is *r-regular*, i.e. that every site has exactly r neighbours. Also, let $|A|$ denote the cardinality of a set A. Assume that there are constants $\lambda_k \geqslant 0$ and $\mu_k \geqslant 0$ ($k = 0, 1, \ldots, r$) such that the spin system $\{\eta_t : t \geqslant 0\}$ satisfies

$$\begin{aligned} &\Pr\{\eta_{t+h}(x) = 1 \,\big|\, \eta_t(x) = 0,\ |\{y \in N(x) : \eta_t(y) = 1\}| = k\} = \lambda_k h + \mathrm{o}(h), \\ &\Pr\{\eta_{t+h}(x) = 0 \,\big|\, \eta_t(x) = 1,\ |\{y \in N(x) : \eta_t(y) = 1\}| = k\} = \mu_k h + \mathrm{o}(h) \end{aligned} \tag{1}$$

for every $x \in S(G)$, $t \geqslant 0$, and $k = 0, 1, \ldots, r$. Then the basic voter model on G satisfies (1) with $\lambda_k = k$ and $\mu_k = r - k$. The noisy voter model satisfies (1) with $\lambda_k = \beta + k$ and $\mu_k = \delta + r - k$, $k = 0, \ldots, r$.

In the special case in which G is the complete graph on N sites (a complete graph is a graph in which every site is a neighbour of every other site), the noisy voter model corresponds to the Moran model of population genetics (Moran, 1958; Karlin and McGregor, 1962; Donnelly, 1984).

If G is a finite graph, then a quantity of considerable interest in certain chemical applications (see for example Granovsky et al. (1989)) is the *mean coverage function*

$$M^A(t) := E(|\eta_t| \,\big|\, \eta_0 = I_A\}, \quad t \geqslant 0, \tag{2}$$

where A is an arbitrary subset of $S(G)$ and I_A is the indicator function of A (that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$). Granovsky and Rozov (1994) gave the following characterization of noisy voter models. Let G be a finite, r-regular graph with no cycles of length 3. Then the mean coverage function of a spin system given by (1) satisfies a first-order linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}M^A(t) + aM^A(t) = b, \quad \text{for all } A \subset S(G), \quad t \geqslant 0, \tag{3}$$

for some constants a and $b \geqslant 0$ *if and only if* the rates of the system satisfy $\lambda_k = \lambda_0 + vk$ and $\mu_k = \mu_r + v(r-k)$ for every $k = 0, \ldots, r$, where v is a constant. If $v > 0$, then these processes are essentially just noisy voter models. The parameter v simply reflects a change of time scale; however, when $v = 0$ we get the elementary case of independent flipping at every site. Solving (3), Granovsky and Rozov (1994) show that

$$\begin{aligned}\Pr\{\eta_t(x) = 1 \mid \eta_0 \equiv 0\} &= \frac{\lambda_0}{\lambda_0 + \mu_r}\left(1 - \exp[-(\lambda_0 + \mu_r)t]\right) \\ &= \frac{\beta}{\beta+\delta}\left(1 - \exp[-(\beta+\delta)t]\right), \quad t \geqslant 0,\end{aligned} \tag{4}$$

for the noisy voter model on any r-regular graph, finite or infinite.

Terminology. We say that a random walk (in discrete or continuous time) on an r-regular graph is *simple* if each jump is made to each of the neighbouring sites with probability $1/r$. Also, we say that a continuous-time random walk W_t has rate $\alpha > 0$ if

$$\Pr\{W_{t+h} = x \,\big|\, W_t = x\} = 1 - \alpha h + \mathrm{o}(h), \quad \text{for every } x \in S(G)$$

(i.e. if the time between consecutive jumps is exponentially distributed with mean $1/\alpha$).

We now give an outline of the present paper. Our approach to the analysis of the transient behaviour of the noisy voter model will be through *duality*, which is described fully in Section 2. It is well known that the basic voter model has a system of coalescing random walks as its dual; it turns out that the addition of noise (i.e. the parameters β and δ) to the voter model adds random births and deaths to the dual. Ultimately, we express marginal distributions (Proposition 3) and two-point covariances (Proposition 5) of the noisy voter model in terms of random walk quantities. The results include the basic voter model as a special case.

Section 3 deals with the equilibrium ($t \to \infty$) behaviour. When $\beta + \delta > 0$, we know that the noisy voter model has a unique equilibrium distribution. We shall write $\eta_{[\kappa,\theta]}$ to denote a random field having this limiting distribution, where we use the parametrization

$$\kappa = \beta + \delta \quad \text{and} \quad \theta = \frac{\beta}{\beta+\delta}. \tag{5}$$

We derive an expression for the equilibrium covariances in terms of the Green function of simple random walk (Proposition 9). In contrast to the noisy case, the basic voter model ($\beta = \delta = 0$) is not ergodic because the states of "all 1's" and "all 0's" are absorbing states. However, for each $\theta \in [0,1]$, if the initial state of the basic voter model is chosen according to a product Bernoulli measure with density $p = \theta$, then the model converges weakly (as $t \to \infty$) to a limit, denoted $\eta_{\text{voter}[\theta]}$, which satisfies $E(\eta_{\text{voter}[\theta]}(x)) = \theta$ for all $x \in S(G)$. We prove in Proposition 10 that the noisy voter model $\eta_{[\kappa,\theta]}$ converges weakly to $\eta_{\text{voter}[\theta]}$ as $\kappa \to 0$ and θ remains fixed. By analogy with phase transitions in statistical mechanics, we can view 0 as a "critical value" of the parameter κ, as we shall now describe.

A general theorem (Griffeath, 1979, p. 21) tells us that in the case $G = \boldsymbol{Z}^d$, the covariance of $\eta_{[\kappa,\theta]}(x)$ and $\eta_{[\kappa,\theta]}(y)$ decays exponentially in distance between x and y whenever $\kappa > 0$. In the context of equilibrium statistical mechanics, the rate of decay of the covariances determines a characteristic length scale $\xi = \xi_{[\kappa,\theta]}$ for the model, called the *correlation length*, which typically satisfies

$$\operatorname{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)) \approx \exp[-||x||_{[\kappa,\theta]}/\xi], \tag{6}$$

as $x \to \infty$ (where $||\cdot||_{[\kappa,\theta]}$ is some norm on $\boldsymbol{R}^d$ which may depend on κ and θ). Moreover, as κ approaches its critical value (where the system is no longer ergodic), the correlation length typically diverges according to a power law:

$$\xi \sim \text{constant } \kappa^{-\nu}, \quad \text{as } \kappa \to 0. \tag{7}$$

In the physics literature, the exponent $\nu > 0$ is a *critical exponent.* Another critical exponent of interest is related to the *susceptibility*, defined by

$$\chi = \chi_{[\kappa,\theta]} = \sum_{x \in \boldsymbol{Z}^d} \operatorname{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)). \tag{8}$$

This is finite for $\kappa > 0$, since the covariances decay exponentially in $||x||$, but as $\kappa \to 0$ the susceptibility diverges according to a critical exponent $\gamma > 0$:

$$\chi \sim \text{constant } \kappa^{-\gamma}. \tag{9}$$

(For a brief description of the significance of these critical exponents in statistical mechanics, see Chapter 3 of Fernández, Fröhlich and Sokal (1992)). The values of ν and γ are generally not known; moreover, it is often difficult even to prove that they exist, i.e. that the equations (7) and (9) are correct. Thanks to our representations of covariances in terms of the Green function of simple random walk, we are able to prove that the critical exponents γ and ν do indeed exist for the noisy voter model in $\boldsymbol{Z}^d$, and we obtain their exact values (Theorem 12 and Corollary 13).

In Section 4 we obtain exact formulas for the marginal distributions and covariances of the noisy voter model on $\boldsymbol{Z}^d$ and on some other special graphs. As a byproduct, we obtain a representation of the Green function of simple random walk on $\boldsymbol{Z}^d$ as a Laplace transform of a product of Bessel functions, which we have been unable to find in the literature.

We believe that the noisy voter model can also be a useful tool for studying scaling behaviour of the basic voter model. As an example, Section 5 presents an alternate view of a result due to Cox and Griffeath (1986) on clustering in the two-dimensional basic voter model. We show that their space–time rescaling of the basic voter model has the same limit as an appropriate spatial rescaling of equilibrium noisy voter models with the noise parameters β and δ tending to zero.

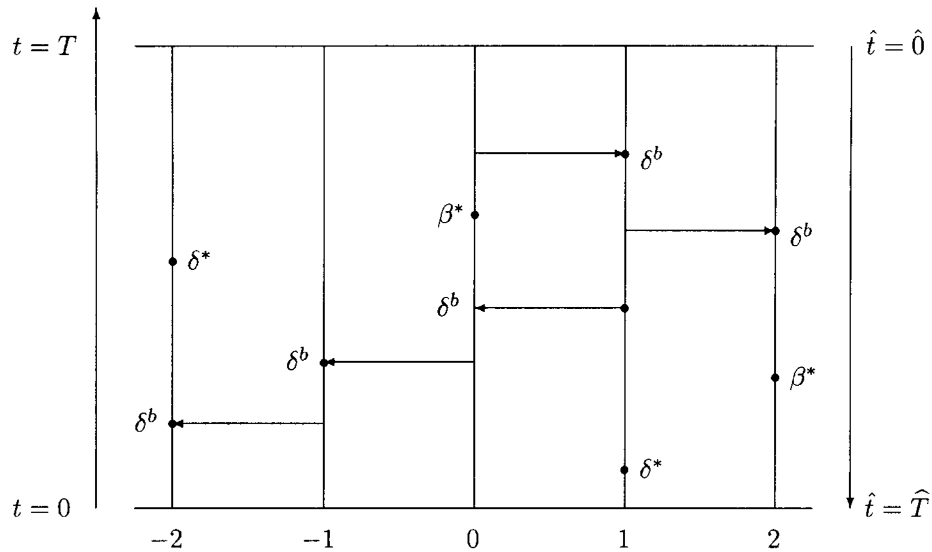

Fig. 1. A realization of the percolation substructure $\mathcal{P}$ for the noisy voter model on ^{1}. The dual substructure $\widehat{\mathcal{P}}_T$ is obtained by reversing the direction of all arrows in $\mathcal{P}$. Observe that $\eta^{\emptyset}_T(x) = 1$ for $x = 0, 1$; and $\eta^{Z}_T(x) = 1$ for $x = -1, 0, 1$. For the dual, we have $\hat{\phi}^{-2,T}_{\widehat{T}} = I_{\{-1\}}$, and $\hat{\phi}^{0,T}_{\widehat{T}} = \hat{\phi}^{1,T}_{\widehat{T}} = \hat{\phi}^{2,T}_{\widehat{T}} = I_{\{1\}}$. Also, $m^{1,T} = \beta^*$ and $m^{2,T} = \delta^*$.

2. Graphical representation and transient behaviour

We now describe the graphical construction of the noisy voter model, and then use it to get a dual relationship. We shall express much of the following in terms of a graph G that is r-regular; of course, everything will apply directly to the integer lattice (with $S(G) = Z^d$ and $r = 2d$).

Our construction follows the standard graphical representation of interacting particle systems due to Harris (see Griffeath (1979) for a general description). However, our dual structure differs from that of Griffeath (1979) in its treatment of birth marks, and this facilitates our calculations of the transient behaviour of the present model (Proposition 5). A similar duality structure was discussed in Donnelly (1984) for the Moran model (the noisy voter model on a complete graph).

For each site $x \in S(G)$, and for each neighbour $y \in N(x)$, let $\mathcal{T}_1^{x,y}$ be a Poisson process on $[0, \infty)$ of rate 1. Similarly, for each site $x \in S(G)$, let $\mathcal{T}_2^x$ and $\mathcal{T}_3^x$ be Poisson processes on $[0, \infty)$ of rates $\beta \geqslant 0$ and $\delta \geqslant 0$ respectively. Let all of these Poisson processes be independent. Now for each event τ in $\mathcal{T}_2^x$, (respectively, $\mathcal{T}_3^x$), put a "birth mark" β^* (respectively, "death mark" δ^*) at the point (x, τ) in $S(G) \times [0, \infty)$. Also, for each event τ in $\mathcal{T}_1^{x,y}$, draw an arrow from (x, τ) to (y, τ), and put a "blockage mark" δ^b at (y, τ). This is the *percolation substructure* $\mathcal{P}$ of the noisy voter model (see Fig. 1).

Suppose that $x, y \in S(G)$ and that $0 \leqslant s \leqslant t$. We say that there is a *V-path from (x, s) to (y, t)* if there is an $n \geqslant 0$, a sequence of times $s_0 = s < s_1 < s_2 < \ldots <$

$s_n \leqslant s_{n+1} = t$, and a sequence of sites $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ such that
(i) There is an arrow from (x_{i-1}, s_i) to (x_i, s_i) for every $i = 1, 2, \ldots, n$, and
(ii) There is no δ^b in the segment $\{x_i\} \times (s_i, s_{i+1})$ for every $i = 0, 1, \ldots, n$;
if in addition
(iii) There is no δ^* in the segment $\{x_i\} \times (s_i, s_{i+1}]$ for every $i = 0, 1, \ldots, n$,
then we say that there is a $\overline{V}$*-path from* (x, s) *to* (y, t). The $n = 0$ case should be understood as saying that there is a V-path from (x, s) to (x, t) whenever $s \leqslant t$. If U and W are subsets of $S(G) \times [0, \infty)$, then we say that there is a V-path (respectively, $\overline{V}$-path) from U to W if there exist $(x, s) \in U$ and $(y, t) \in W$ with $s \leqslant t$ such that there is a V-path (respectively, $\overline{V}$-path) from (x, s) to (y, t).

Now let A be a subset of $S(G)$. Define the spin system $\{\eta_t^A, t \geqslant 0\}$ as follows. For each $t \geqslant 0$ and each $y \in S(G)$, let $\eta_t^A(y)$ equal 1 if there is a $\overline{V}$-path either from $A \times \{0\}$ to (y, t) or from some birth mark β^* in $S(G) \times (0, t]$ to (y, t), and let it equal 0 otherwise. Then η_t^A is precisely the noisy voter model with parameters β and δ started from the state A (i.e., $\eta_0^A = I_A$). If A is a singleton, say $A = \{z\}$, then we shall usually write η_t^z instead of $\eta_t^{\{z\}}$; the same convention will hold for other processes that we shall define below.

Next, fix a time $T > 0$. Let $\widehat{\mathcal{P}}_T$ be the *dual substructure* obtained by restricting the percolation substructure $\mathcal{P}$ to $S(G) \times [0, T]$ and reversing the direction of all arrows in $\mathcal{P}$ (while keeping all of the β^*, δ^*, and δ^b marks unchanged). When working with the dual, we shall use reversed time $\hat{t} = T - t$ $(0 \leqslant t \leqslant T)$. We shall now use $\widehat{\mathcal{P}}_T$ to define a system of coalescing random walks over the reversed time interval $[\hat{0}, \widehat{T}]$. (In fact, this system is the dual of the basic voter model; see Liggett (1985) or Durrett (1988)).

For any $B \subset S(G)$ and any $\hat{t} \in [\hat{0}, \widehat{T}]$, define the process $\hat{\phi}_{\hat{t}}^{B,T}$ so that $\hat{\phi}_{\hat{t}}^{B,T}(y)$ equals 1 if: (i) there is a V-path in $\widehat{\mathcal{P}}_T$ from $B \times \{\hat{0}\}$ to $(y, \hat{t})$; and (ii) there is no arrow from $(y, \hat{t})$ to any other $(y', \hat{t})$, and $\hat{\phi}_{\hat{t}}^{B,T}(y)$ equals 0 otherwise. Observe that for any $z \in S(G)$ and any $\hat{t} \in [\hat{0}, \widehat{T}]$, there is a unique site $y \in S(G)$ such that $\hat{\phi}_{\hat{t}}^{z,T}(y) = 1$. If we interpret this unique site as the position of a particle at time $\hat{t}$, then it is apparent that $\hat{\phi}_{\hat{t}}^{z,T}$ $(\hat{t} \in [\hat{0}, \widehat{T}])$ represents a continuous-time simple random walk on G of rate r that starts at z. (We shall often abuse our notation by using $\hat{\phi}_{\hat{t}}^{z,T}$ to denote this $S(G)$-valued Markov process as well.) Moreover, the process $\hat{\phi}_{\hat{t}}^{\{y,z\},T}$ may be viewed as two random walks that evolve independently until the first time that they meet, and after that they evolve identically, i.e. they *coalesce*. So we see that the process $\hat{\phi}_{\hat{t}}^{B,T}$ is a system of coalescing random walks that start from the sites of B at time $\hat{0}$.

Next, define $\sigma^{[B,T]}$ to be the smallest $s \in [0, T]$ such that there is a β^* or δ^* at some point of $\{(y, \hat{s}) : \hat{\phi}_{\hat{s}}^{B,T}(y) = 1\}$, and define $m^{[B,T]}$ to be β^* or δ^* according to the identity of this mark at time $\sigma^{[B,T]}$. (If no such s exists, then define $\sigma^{[B,T]} = +\infty$ and keep $m^{[B,T]}$ undefined).

We are now in a position to formulate the following basic duality relationship for events on $\mathcal{P}$ and $\widehat{\mathcal{P}}_T$:

$$\{\eta_T^A(x) = 1, \quad \text{for at least one } x \text{ in } B\}$$

$$= \bigcup_{x \in B} \{\sigma^{[x,T]} \leqslant T, m^{[x,T]} = \beta^*\} \ \cup$$
$$\{\sigma^{[B,T]} = +\infty, \hat{\phi}^{B,T}_{\widehat{T}}(z) = 1, \quad \text{for some } z \text{ in } A\}. \tag{10}$$

Definition 1. Fix $p \in [0,1]$. We denote by $\eta^{(p)}_t$ $(t \geqslant 0)$ the noisy voter model with the initial state chosen according to a product Bernoulli measure with density p: that is, $\{\eta^{(p)}_0(z) : z \in S(G)\}$ are independent random variables, equal to 1 with probability p and equal to 0 with probability $1 - p$. Moreover, this initial state is independent of the percolation substructure.

Definition 2. Let $Q_t(\cdot,\cdot)$ be the transition probability kernel of a continuous-time simple random walk of rate r on G. Thus we have

$$Q_t(x, A) = \Pr\{\hat{\phi}^{x,T}_{\hat{t}} \in A\}, \quad \text{for every } x \in S(G), \quad A \subset S(G),$$

and $0 \leqslant t \leqslant T$.

Specializing (10) to the case where B is a singleton, we obtain the following result.

Proposition 3. Using the notation (5), assume that $\kappa > 0$ and $\theta \in [0,1]$. For any $A \subset S(G)$, $x \in S(G)$, $T \geqslant 0$, and $p \in [0,1]$:
(a) $\Pr\{\eta^A_T(x) = 1\} = \left(1 - e^{-\kappa T}\right)\theta + e^{-\kappa T} Q_T(x, A)$; and
(b) $\Pr\{\eta^{(p)}_T(x) = 1\} = \theta + e^{-\kappa T}\left(p - \theta\right)$.

Proof. For part (a), we take $B = \{x\}$ in (10). Recalling that the process $\hat{\phi}^{x,T}_{\hat{t}}$ represents a single random walk, we see from elementary Poisson process calculations that

$$\Pr\{\sigma^{[x,T]} \leqslant T \text{ and } m^{[x,T]} = \beta^*\} = \left(1 - e^{-(\beta+\delta)T}\right)\frac{\beta}{\beta + \delta}.$$

Next, since $\sigma^{[x,T]}$ is independent of the trajectory of the random walk $\hat{\phi}^{x,T}_{\hat{t}}$ $(\hat{t} \in [\hat{0}, \widehat{T}])$,

$$\begin{aligned}
&\Pr\{\sigma^{[x,T]} = +\infty \text{ and } \hat{\phi}^{x,T}_{\widehat{T}}(z) = 1, \quad \text{for some } z \text{ in } A\} \\
&= e^{-(\beta+\delta)T}\Pr\{\hat{\phi}^{x,T}_{\widehat{T}}(z) = 1, \quad \text{for some } z \text{ in } A\} \\
&= e^{-(\beta+\delta)T} Q_T(x, A).
\end{aligned} \tag{11}$$

Part (a) follows by adding the above two displayed equations.

The proof of part (b) is similar to part (a), and is left to the reader. □

Remark 4. (i) Part (b) of Proposition 3 holds for any initial distribution μ on configurations $\eta \in \{0,1\}^{S(G)}$ such that $\mu\{\eta(z) = 1\} = p$ for every $z \in S(G)$. Also, the results of both (a) and (b) hold for $\kappa = 0$ as well, even though θ is undefined, if we interpret $(1 - e^{-\kappa T})\theta$ to be 0.

(ii) Part (a) trivially implies Theorem 6 of Cox and Durrett (1991), which says that if $\beta = 0$ then $\Pr\{\eta^A_T(x) = 1\} \leqslant e^{-\delta T}$ for every A, x, and T.

(iii) Taking A to be the empty set in part (a) gives (4).

Before we state the next result, we need some more notation. Recall that for any two different sites x and y in $S(G)$, the process $\hat{\phi}_{\hat{t}}^{\{x,y\},T}$ ($\hat{t} \in [\hat{0}, \widehat{T}]$), which is defined on $\widehat{\mathcal{P}}_T$, may be viewed as a pair of coalescing random walks. With this in mind, we shall define the "coalescing time" $\Psi \equiv \Psi^{\{x,y\},T}$ to be equal to the smallest $u \in [0,T]$ (if any exists) such that there exists a $\Theta \in S(G)$ with a V-path from $(x, \hat{0})$ to $(\Theta, \hat{u})$ and another V-path from $(y, \hat{0})$ to $(\Theta, \hat{u})$ (both paths in $\widehat{\mathcal{P}}_T$). Observe that in this case, the "coalescing place" Θ is an unambiguously defined random variable. If no such u exists, then we set $\Psi = +\infty$ and leave Θ undefined.

Proposition 5. Fix $\kappa > 0$ and $\theta \in [0,1]$. For any $x \neq y \in S(G)$, $p \in [0,1]$, and any $T \geqslant 0$,

$$\mathrm{Cov}\left(\eta_T^{(p)}(x), \eta_T^{(p)}(y)\right) = \int_0^T v_{T-u}^{(p)} e^{-2\kappa u}\, \Pr\{\Psi^{\{x,y\},T} \in \mathrm{d}u\},$$

where

$$\begin{aligned} v_t^{(p)} &:= \left(\theta + e^{-\kappa t}\,[p-\theta]\right) \cdot \left(1 - \theta - e^{-\kappa t}\,[p-\theta]\right) \\ &= \mathit{Var}(\eta_t^{(p)}(z)), \quad \text{for every } z \in S(G), \quad t \geqslant 0. \end{aligned} \tag{12}$$

Proof. Firstly, the second equality of (12) is immediate from Proposition 3(b).

We shall use the following elementary conditioning property: if X and Y are random variables and $\mathcal{S}$ is a sigma-algebra, then

$$\mathrm{Cov}(X,Y) = E(\mathrm{Cov}(X,Y|\mathcal{S})) + \mathrm{Cov}(E(X|\mathcal{S}), E(Y|\mathcal{S})), \tag{13}$$

where by definition

$$\mathrm{Cov}(X,Y|\mathcal{S}) = E(XY|\mathcal{S}) - E(X|\mathcal{S})E(Y|\mathcal{S}).$$

We first apply this property to the case $X = \eta_T^{(p)}(x)$ and $Y = \eta_T^{(p)}(y)$ and $\mathcal{S}$ the sigma-algebra generated by the random variable $\Psi \equiv \Psi^{\{x,y\},T}$.

First, recalling (10), we observe that X is independent of Ψ, so $E(X|\mathcal{S}) = E(X)$. Similarly, $E(Y|\mathcal{S}) = E(Y)$, and therefore

$$\mathrm{Cov}(E(X|\mathcal{S}), E(Y|\mathcal{S})) = 0. \tag{14}$$

Furthermore, the random variables X and Y are conditionally independent given the event $\{\Psi = +\infty\}$, and so

$$\mathrm{Cov}(X, Y|\Psi = +\infty) = 0. \tag{15}$$

We now have to evaluate $\mathrm{Cov}(X,Y|\Psi = u)$ for $0 \leqslant u \leqslant T$. To do this we shall apply (13) again in a slightly different form. This time, fix $u \in [0,T]$ and let $\mathcal{Q}_u$ be the sigma-algebra generated by the two events $H := \{\sigma^{[x,T]} \leqslant u\}$ and $K := \{\sigma^{[y,T]} \leqslant u\}$. Then we have

$$\operatorname{Cov}(X,Y|\Psi=u)=E(\operatorname{Cov}(X,Y|\Psi=u,\mathcal{Q}_u)) \\ +\operatorname{Cov}(E(X|\Psi=u,\mathcal{Q}_u),E(Y|\Psi=u,\mathcal{Q}_u)). \tag{16}$$

To simplify this equation, we shall first show that

$$\operatorname{Cov}(E(X|\Psi=u,\mathcal{Q}_u),E(Y|\Psi=u,\mathcal{Q}_u))=0. \tag{17}$$

To see this, observe that by (10) $E(X|\Psi=u,\mathcal{Q}_u)$ equals

$$\beta/(\beta+\delta) \text{ on } \{\Psi=u\}\cap H$$

and equals

$$pe^{-(\beta+\delta)(T-u)}+(1-e^{-(\beta+\delta)(T-u)})\beta/(\beta+\delta) \text{ on } \{\Psi=u\}\cap H^c.$$

Therefore we can write

$$E(X|\Psi=u,\mathcal{Q}_u)=aI_H+b \text{ on } \{\Psi=u\},$$

where a and b are constants (depending on β, δ, p, and $T-u$) and I_H is the indicator function of the event H. Similarly, we can write

$$E(Y|\Psi=u,\mathcal{Q}_u)=aI_K+b \text{ on } \{\Psi=u\}.$$

Eq. (17) now follows from the fact that H and K are conditionally independent given $\{\Psi=u\}$.

Thus the right hand side of (16) reduces to $E(\operatorname{Cov}(X,Y|\Psi=u,\mathcal{Q}_u))$. Conditional independence tells us that

$$\operatorname{Cov}(X,Y|\Psi=u,\mathcal{Q}_u)=0 \text{ on } \{\Psi=u\}\cap(H\cup K),$$

so we are left with $\operatorname{Cov}(X,Y|\{\Psi=u\}\cap H^c\cap K^c)$. But

$$\{\Psi=u\}\cap H^c\cap K^c=\{\Psi=u\}\cap\{\sigma^{[\{x,y\},T]}>u\},$$

so

$$\operatorname{Cov}(X,Y)=\int_0^T \operatorname{Cov}(X,Y|\Psi=u,\sigma^{[\{x,y\},T]}>u) \\ \times\Pr\{\sigma^{[\{x,y\},T]}>u,\Psi\in \mathrm{d}u\}.$$

Moreover, on the event $\{\Psi=u\}\cap\{\sigma^{[\{x,y\},T]}>u\}$, we have $X=Y=\eta^{(p)}_{T-u}(\Theta)$; so

$$\operatorname{Cov}(X,Y)=\int_0^T \operatorname{Var}(\eta^{(p)}_{T-u}(\Theta)|\Psi=u,\sigma^{[\{x,y\},T]}>u)e^{-2(\beta+\delta)u} \\ \times\Pr\{\Psi\in \mathrm{d}u\}.$$

Finally, we use the independence of $\eta^{(p)}_{T-u}(z)$ and $\{\Psi=u,\,\Theta=z,\,\sigma^{[\{x,y\},T]}>u\}$ for every $z\in G$, as well as (12), to conclude

$$\operatorname{Cov}(X,Y)=\int_0^T v^{(p)}_{T-u}e^{-2(\beta+\delta)u}\Pr\{\Psi\in \mathrm{d}u\}.$$

This completes the proof. $\square$

Remark 6. It is easy to see that for a fixed finite $T > 0$, the noisy voter model converges weakly to the basic voter model (with the same initial state) as β and δ tend to 0. Accordingly, letting κ tend to 0 in Proposition 5 recovers the following known result for the basic voter model $\bar{\eta}_t$:

$$\mathrm{Cov}(\bar{\eta}_T^{(p)}(x), \bar{\eta}_T^{(p)}(y)) = p(1-p)\Pr\{\Psi^{\{x,y\},T} \leqslant T\}$$

(Durrett, 1988, p. 25).

3. Equilibrium properties

Our main motivation for this section is the case $G = \mathbf{Z}^d$, although the first half of the section only requires that G have a group structure. More precisely, we shall assume throughout this section that the set of sites $S(G)$ is an Abelian group under addition, and that the neighbourhoods conform to the group structure, i.e. $N(x) = \{y + x : y \in N(0)\}$ for every $x \in S(G)$ (here 0 denotes the identity element of the group).

When $\beta + \delta > 0$, we know that the noisy voter model is exponentially ergodic by Theorem 4.1 in Chapter I of Liggett (1985). In fact, Granovsky and Rozov (1994) observed that the rate of convergence to equilibrium is exactly $\kappa = \beta + \delta$, which equals the $\epsilon - M$ bound of this theorem. As described in Section 1, we write $\eta_{[\kappa,\theta]}$ to denote a random field having the model's unique equilibrium distribution.

In this section we will be interested in letting $T \to \infty$, and so we need a dual substructure which is not restricted to a finite time interval $[0, T]$. Let $\widehat{\mathcal{P}}_\infty$ be a copy of $\mathcal{P}$ on $S(G) \times [0, \infty)$ in which all the arrows are reversed, while all the marks remain unchanged. We shall work only with ordinary time (not reversed time) in $\widehat{\mathcal{P}}_\infty$. Define a *V-path* in $\widehat{\mathcal{P}}_\infty$ exactly as in $\mathcal{P}$. For any $B \subset S(G)$ and any $t \geqslant 0$, define $\phi_t^B(y)$ to be equal to 1 if there is a V-path in $\widehat{\mathcal{P}}_\infty$ from $B \times \{0\}$ to (y, t) (and there is no arrow from (y, t) to any other (y', t)), and to be equal to 0 otherwise. Then the process ϕ_t^B $(t \geqslant 0)$ is a system of coalescing random walks (in $\widehat{\mathcal{P}}_\infty$) that start from the sites of B. Furthermore it is clear that for every $T > 0$, the process $\{\phi_t^B : 0 \leqslant t \leqslant T\}$ has the same law as the process $\{\hat{\phi}_{\hat{t}}^{B,T} : \hat{0} \leqslant \hat{t} \leqslant \widehat{T}\}$ which was defined in Section 2.

We will also need the following pieces of notation for $\widehat{\mathcal{P}}_\infty$, which are the obvious analogues of notation for $\widehat{\mathcal{P}}_T$ as defined in Section 2:

- For $x, y \in S(G)$, let $\Psi^{\{x,y\}}$ be the coalescing time for the two random walks ϕ_t^x and ϕ_t^y;
- For $B \subset S(G)$, let σ^B be the smallest time $s \geqslant 0$ such that there is a β^* or a δ^* mark at some point of the set $\{(y, s) : \phi_s^B(y) = 1\}$, and let m^B be β^* or δ^* according to the identity of this mark. (Observe that $\sigma^B < \infty$ a.s. for every nonempty B.)

Proposition 7. For $x \in S(G)$, let W_t^x $(t \geqslant 0)$ be a continuous-time simple random walk of rate $2r$ on G which starts at x. Define the random variable τ^x to be the first time that the random walk W_t^x hits the origin. For any $x, y \in S(G)$, $\kappa > 0$, $\theta \in [0,1]$, and any initial distribution μ on $\{0,1\}^{S(G)}$, we have

$$\lim_{T\to\infty} \mathrm{Cov}\,(\eta_T^\mu(x), \eta_T^\mu(y)) = \theta(1-\theta) \int_0^\infty e^{-2\kappa u}\,\Pr\{\tau^{x-y} \in du\}$$
$$= \theta(1-\theta) E\left(\exp\left[-2\kappa\tau^{x-y}\right]\right).$$

Proof. Since the noisy voter model is ergodic, it suffices to prove the result for a particular starting distribution, namely the product Bernoulli measure with density p for some $p \in [0,1]$.

Observe that the difference in positions between the random walks ϕ_t^x and ϕ_t^y is itself a random walk that starts at $x-y$ and jumps at rate $2r$ up to the coalescing time $\Psi^{\{x,y\}}$. Therefore $\Psi^{\{x,y\}}$ and τ^{x-y} have the same distribution. The result is now immediate from Proposition 5 and the dominated convergence theorem. □

Proposition 7 tells us that for $\kappa > 0$ and $x \in S(G)$,

$$\mathrm{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)) = \theta(1-\theta) E\left(\exp\left[-2\kappa\tau^x\right]\right). \tag{18}$$

The next result gives an alternate expression for the covariances in terms of the Green function of simple random walk.

Definition 8. For $n = 0, 1, \ldots$ and $x, y \in S(G)$, let $p_n(x,y)$ be the probability that a simple random walk on G arrives at y at the nth jump, given that it starts from x. Also, for s real, let

$$G(x,y;s) := \sum_{n=0}^{\infty} p_n(x,y) s^n$$

be the Green function of the random walk (observe that this converges if $|s| < 1$).

Proposition 9. Let $q = r/(\kappa + r)$. For $x, y \in S(G)$ and $\kappa > 0$,

$$\mathrm{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)) = \theta(1-\theta) \frac{G(x,0;q)}{G(0,0;q)}.$$

Proof. Let J^x be the number of jumps of the random walk W_t^x (defined in Proposition 7) up to and including the hitting time τ^x. Let α_i $(i \geqslant 1)$ be the time between the $(i-1)$th and ith jumps of W_t^x. Then $\alpha_1, \alpha_2, \ldots$ are i.i.d. exponential random variables with parameter $2r$, and $\tau^x = \alpha_1 + \cdots + \alpha_{J^x}$. Therefore

$$E\left(\exp\left[-2\kappa\tau^x\right]\right) = \sum_{n=0}^{\infty} \left(\frac{2r}{2\kappa + 2r}\right)^n \Pr\{J^x = n\}. \tag{19}$$

Now the relationship

$$p_n(x,0) = \sum_{k=0}^{n} \Pr\{J^x = k\} p_{n-k}(0,0), \quad \text{for all } x \neq 0, \quad n \geqslant 0$$

implies

$$\begin{aligned} G(x,0;q) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \Pr\{J^x = k\} p_{n-k}(0,0) q^n \\ &= \sum_{k=0}^{\infty} \Pr\{J^x = k\} q^k \sum_{n=k}^{\infty} p_{n-k}(0,0) q^{n-k} \\ &= E(\exp[-2\kappa\tau^x]) G(0,0;q) \qquad \text{(by (19))} \end{aligned}$$

Combining this with (18) completes the proof. □

We next consider what happens to the noisy voter model as the noise disappears, i.e. as β and δ tend to 0. As we remarked at the end of the previous section, for each finite $t > 0$ the noisy voter model converges weakly to the basic voter model. The next result says that this is also true at $t = +\infty$, if the basic voter model starts from product Bernoulli measure and if θ converges to this Bernoulli density.

Proposition 10. Suppose that β and δ approach 0 in such a way that $\kappa = \beta + \delta > 0$ and $\theta \to \overline{\theta} \in [0,1]$. Then $\eta_{[\kappa,\theta]}$ converges weakly to $\eta_{\text{voter}[\overline{\theta}]}$ (defined in Section 1).

Proof. It suffices to show convergence on events of the form $\{\eta_{[\kappa,\theta]}(x) = 0$ for every x in $B\}$ for finite sets $B \subset S(G)$ (c.f. Durrett, 1988, p. 26).

Fix a finite set $B \subset S(G)$. Let $|\phi_t^B|$ be the number of sites x such that $\phi_t^B(x) = 1$. Observe that $|\phi_t^B|$ does not change except at times when two of the random walks that started from B coalesce, and at these times $|\phi_t^B|$ decreases by 1 (unless more than two walks coalesce at once, but this has probability 0, so we ignore this possibility). Let $L \equiv L^B$ be the random time $(0 \leqslant L < \infty)$ at which the last coalescence occurs among the walks started from B; then $|\phi_t^B|$ remains constant for $t \geqslant L$.

Write $P_{[\kappa,\theta]}$ to denote the probability measure on the percolation substructure $\mathcal{P}$ (and hence on $\widehat{\mathcal{P}}_T$ for every $T > 0$) with parameters κ and θ. Then, by Eq. (10),

$$\begin{aligned} &P_{[\kappa,\theta]}\{\eta_T^{S(G)}(x) = 0, \quad \text{for all } x \text{ in } B\} \\ &\quad = P_{[\kappa,\theta]}\{\sigma^{[x,T]} \leqslant T \text{ and } m^{[x,T]} = \delta^*, \quad \text{for all } x \text{ in } B\}. \end{aligned}$$

As $T \to \infty$, the times $\{\sigma^{[x,T]} : x \in B\}$ converge jointly in distribution to the times $\{\sigma^x : x \in B\}$, which are finite a.s. Therefore the $T \to \infty$ limit of the above gives

$$\begin{aligned} &\Pr\{\eta_{[\kappa,\theta]}(x) = 0, \quad \text{for all } x \text{ in } B\} \\ &\quad = P_{[\kappa,\theta]}\{m^x = \delta^*, \quad \text{for all } x \text{ in } B\} \end{aligned} \tag{20}$$

(we have abused our notation slightly by also using $P_{[\kappa,\theta]}$ to refer to the probability measure on $\widehat{\mathcal{P}}_\infty$). Using the random time L defined in the previous paragraph, we write the right hand side of (20) as

$$P_{[\kappa,\theta]}\{m^x = \delta^*, \quad \text{for all } x \text{ in } B, \text{ and } \sigma^B \leqslant L\}$$
$$+ P_{[\kappa,\theta]}\{m^x = \delta^*, \quad \text{for all } x \text{ in } B, \text{ and } \sigma^B > L\}. \tag{21}$$

Observe that $P_{[\kappa,\theta]}\{\sigma^B \leqslant L\} \to 0$ as $\kappa \to 0$. Therefore the first term in (21) tends to 0, and the second term tends to

$$\sum_{k=1}^{|B|}(1-\bar{\theta})^k \Pr\{|\phi_L^B| = k\},$$

which is precisely

$$\Pr\{\eta_{\text{voter}[\bar{\theta}]}(x) = 0, \quad \text{for all } x \text{ in } B\}$$

(Durrett, 1988, p. 26).

This completes the proof. □

Remark 11. Applying Proposition 10 to (18) recovers the well-known result for the basic voter model (Clifford and Sudbury, 1973, p. 586) that

$$\operatorname{Cov}(\eta_{\text{voter}[\theta]}(x), \eta_{\text{voter}[\theta]}(0)) = \theta(1-\theta)\Pr\{\tau^x < \infty\}, \quad x \in S(G).$$

Our final results of this section will only concern the case $\boldsymbol{Z}^d$ (as well as some other periodic lattices in $\boldsymbol{R}^d$). The correlation length ξ and the susceptibility χ were defined in Section 1, along with their associated critical exponents ν and γ (see Eqs. (6)–(9)).

Theorem 12. Let G be the integer lattice $\boldsymbol{Z}^d$.

(a) Define $\xi_{[\kappa,\theta]}$ by

$$\frac{1}{\xi_{[\kappa,\theta]}} = \lim_{n\to\infty} \frac{-\log \operatorname{Cov}(\eta_{[\kappa,\theta]}(x_{(n)}), \eta_{[\kappa,\theta]}(0))}{n}, \tag{22}$$

where $x_{(n)} = (n, 0, \ldots, 0) \in \boldsymbol{Z}^d$. This limit exists for all $\kappa > 0$ and satisfies

$$\xi_{[\kappa,\theta]} \sim \kappa^{-1/2}, \quad \text{as } \kappa \to 0. \tag{23}$$

(b) The susceptibility $\chi = \chi_{[\kappa,\theta]}$ satisfies

$$\chi_{[\kappa,\theta]} = \frac{\theta(1-\theta)}{(1-q)G(0,0;q)} \tag{24}$$

and, as $\kappa \to 0$ for fixed θ,

$$\chi_{[\kappa,\theta]} \sim \begin{cases} \dfrac{constant}{\kappa^{1/2}}, & \text{if } d = 1, \\ \dfrac{constant}{\kappa\log(1/\kappa)}, & \text{if } d = 2, \\ \dfrac{constant}{\kappa}, & \text{if } d \geqslant 3. \end{cases} \tag{25}$$

Corollary 13. The critical exponents ν and γ are well defined for the noisy voter model. We have $\nu = 1/2$ in every dimension d, while γ is $1/2$ in one dimension and 1 in two or more dimensions (with a logarithmic correction in two dimensions).

Proof of Theorem 12.

(a) By Proposition 9, the right hand side of (22) equals

$$\lim_{n\to\infty} \frac{-\log G(x_{(n)}, 0; q)}{n},$$

which is known to exist by part (a) of Theorem A.2 in Madras and Slade (1993). Also, part (b) of that theorem says that $\xi \sim [2d(1-q)]^{-1/2}$ as $q \to 1$, which implies (23) (recall $1 - q = \kappa/(\kappa + 2d)$).

(b) Eq. (24) follows from the definition of χ, Proposition 9, and the elementary relations

$$\sum_{x\in \boldsymbol{Z}^d} G(x,0;q) = \sum_{x\in \boldsymbol{Z}^d} \sum_{n=0}^{\infty} p_n(x,0) q^n = \sum_{n=0}^{\infty} q^n.$$

Thus the desired asymptotic behaviour of χ is given by the behaviour of $G(0,0;q)$ as $q \to 1$, which is easy to get since $p_{2n}(0,0) \sim \text{const.} n^{-d/2}$ as $n \to \infty$.

□

4. Exact formulas

In this section we shall present some closed form expressions for the general formulas that were derived in the preceding sections. Note that in all expressions the case $\kappa = 0$ yields the result for the basic voter model.

First we discuss the marginal distributions and (for finite graphs G) the mean coverage functions

$$M^A(t) := E|\eta_t^A| = \sum_{x\in S(G)} \Pr\{\eta_t^A(x) = 1\}, \qquad (A \subset S(G), \quad t \geqslant 0)$$

and

$$M^{(p)}(t) := E|\eta_t^{(p)}| = \sum_{x\in S(G)} \Pr\{\eta_t^{(p)}(x) = 1\}, \qquad (0 \leqslant p \leqslant 1, \quad t \geqslant 0).$$

For $\eta_t^{(p)}$, Proposition 3(b) shows that the marginal distributions are independent of all details of the graph G, and in fact coincide with those for a system with no interaction (i.e., each site is an independent two-state Markov chain with flip rates β and δ). For the case of η_t^A, Proposition 3(a) shows that the marginal distributions depend on the transition probabilities $Q_t(x, A)$, which depend strongly on the structure of the graph. However, we do obtain a general expression for the mean coverage function:

$$M^A(t) = |S(G)|\theta + e^{-\kappa t}\left(|A| - |S(G)|\theta\right), \quad (t \geqslant 0). \tag{26}$$

This follows from Proposition 3(a) because Q_t is doubly stochastic. This formula was originally obtained in Granovsky and Rozov (1994) in a different way. The transient behaviour of the basic voter model on finite graphs has been studied by Donnelly and Welsh (1983).

In some special cases, we can obtain exact formulas for the marginal distributions:

Example I. The complete graph on N sites. In this case, G is $(N-1)$-regular. We have

$$Q_t(y, \{x\}) = \frac{1}{N} + \left(1_{\{x\}}(y) - \frac{1}{N}\right) e^{-Nt}, \quad (x, y \in S(G), \quad t \geqslant 0),$$

and hence

$$\Pr\{\eta_t^A(y) = 1\} = (1 - e^{-\kappa t})\theta + \frac{|A|}{N} e^{-\kappa t} + e^{-(\kappa + N)t}\left(1_A(y) - \frac{|A|}{N}\right)$$
$$(y \in S(G), \quad A \subset S(G), \quad t \geqslant 0).$$

As we have mentioned, the noisy voter model on a complete graph corresponds to the Moran model of population genetics. Explicit formulas for the transient probabilities of the Moran model were obtained in the celebrated paper of Karlin and McGregor (1962). Donnelly (1984) derived explicit expressions for moments of some other functionals of the process.

Example II. The integer lattice $\mathbf{Z}^d$ $(d \geqslant 1)$. This graph is $2d$-regular. For $x = (x_1, \ldots, x_d) \in \mathbf{Z}^d$, we know (see Feller, 1971, II.7) that for the continuous time random walk ϕ_t^0,

$$Q_t(0, \{x\}) = e^{-2dt} \prod_{j=1}^{d} I_{x_j}(2t), \quad (t \geqslant 0), \tag{27}$$

where I_n is the modified Bessel function of order n (Gray and Mathews, 1966) which satisfies

$$I_n(2t) = \sum_{k=0}^{\infty} \frac{t^{2k+|n|}}{k!(|n|+k)!}, \quad (n \in \mathbf{Z}, \quad t \geqslant 0).$$

Therefore, for $y = (y_1, \ldots, y_d) \in \mathbf{Z}^d$ and $t \geqslant 0$,

$$\Pr\{\eta_t^A(y) = 1\} = (1 - e^{-\kappa t})\theta + e^{-(\kappa + 2d)t} \sum_{x \in A} \prod_{j=1}^{d} I_{x_j - y_j}(2t).$$

We now turn to the calculation of covariances. Proposition 5 reduces this to the problem of computing (functionals of) the distribution of the coalescing times

$\Psi^{\{x,y\}}$ (or equivalently, if G has a group structure, the distribution of the hitting times τ^{x-y}.)

Example I (continued): The complete graph on N sites. In this case, it is not hard to see that for distinct $x, y \in S(G)$, the distribution of $\Psi^{\{x,y\}}$ is exponential with rate 2, and so

$$\mathrm{Cov}\left(\eta_t^{(p)}(x), \eta_t^{(p)}(y)\right) = 2\int_0^t v_{t-u}^{(p)} e^{-2(\kappa+1)u}\, \mathrm{d}u,$$

for any $x \neq y \in S(G)$, $p \in [0,1]$, and $t \geqslant 0$, where $v_t^{(p)}$ is given explicitly by (12). We note that this expression is independent of N. The integral can be evaluated by simple calculus, and one finds

$$\mathrm{Cov}\left(\eta_t^{(p)}(x), \eta_t^{(p)}(y)\right) = \frac{\theta(1-\theta)}{\kappa+1} + \mathrm{O}(e^{-\kappa t}).$$

Example II (continued): The integer lattice $\boldsymbol{Z}^d$. We begin with the simplest case, $d = 1$. For $x \in \boldsymbol{Z}^1$, $x \neq 0$, it is known that the density $f_x(t)$ of τ^x (defined in Proposition 7) is

$$f_x(t) = e^{-4t}\frac{|x|}{t} I_x(4t), \quad (t \geqslant 0)$$

(Feller, 1971, II.7). Therefore

$$\mathrm{Cov}\left(\eta_t^{(p)}(x), \eta_t^{(p)}(0)\right) = |x| \int_0^t v_{t-u}^{(p)} e^{-2(\kappa+2)u} \frac{I_x(4u)}{u}\, \mathrm{d}u,$$

for any $x \neq 0$, $p \in [0,1]$, and $t \geqslant 0$, where $v_t^{(p)}$ is given by (12). The equilibrium covariances are

$$\mathrm{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)) = \theta(1-\theta)\left(\frac{2+\kappa-(\kappa^2+4\kappa)^{1/2}}{2}\right)^{|x|},$$

(for $\kappa \geqslant 0$, $x \in \boldsymbol{Z}^d$) by (18) and Feller (1971, XIV.6). Therefore the correlation length is

$$\xi = -\left[\log\left(\frac{2+\kappa-(\kappa^2+4\kappa)^{1/2}}{2}\right)\right]^{-1}$$

and the susceptibility is

$$\chi = \theta(1-\theta)\left(\frac{\kappa+4}{\kappa}\right)^{1/2}, \quad \kappa > 0.$$

For general d, the answers are more complicated. Since

$$G(x,0;q) = (\kappa+r)\int_0^\infty e^{-\kappa t} Q_t(x, \{0\})\, \mathrm{d}t$$

(recall $q = r/(r+\kappa)$), we can use (27) to obtain

$$\mathrm{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0)) = \theta(1-\theta)\frac{\int_0^\infty e^{-(2d+\kappa)t} \prod_{j=1}^d I_{x_j}(2t)\ \mathrm{d}t}{\int_0^\infty e^{-(2d+\kappa)t}[I_0(2t)]^d\ \mathrm{d}t},$$

for $x = (x_1, \ldots, x_d) \in \boldsymbol{Z}^d$ (by Proposition 9) and

$$\chi = \frac{\theta(1-\theta)}{\kappa \int_0^\infty e^{-(2d+\kappa)t}[I_0(2t)]^d\ \mathrm{d}t}$$

(by (24)). Note that for the cases $\boldsymbol{Z}^2$ and $\boldsymbol{Z}^3$, this representation of the Green function as a Laplace transform of a product of Bessel functions appears in Eq. (II,19) of Montroll (1964) and Eq. (14) of Katsura et al. (1971) respectively. (The latter paper also discusses numerical evaluation of these integrals, motivated by their applications in physics.)

Example III: The infinite r-regular tree. This is the connected r-regular graph which contains no cycles. Consider a random walk on the tree that starts from a site $x \neq 0$, where 0 is the root of the tree. Observe that, until the time τ^x, the distance of this walk from 0 behaves exactly like an asymmetric nearest-neighbour random walk on the integers with probability $1/r$ (respectively, $1 - 1/r$) of jumping to the left (respectively, right). By (18) and Feller (1971, XIV.6), the equilibrium covariances are

$$\mathrm{Cov}(\eta_{[\kappa,\theta]}(x), \eta_{[\kappa,\theta]}(0))$$
$$= \theta(1-\theta)\left(\frac{r + 2\kappa - [(r+2\kappa)^2 - 4(r-1)]^{1/2}}{2(r-1)}\right)^{|x|}.$$

5. Scaling in two dimensions

Cox and Griffeath (1986) considered the basic voter model $\overline{\eta}_t^\theta$ in $\boldsymbol{Z}^2$, starting from Bernoulli product measure with density $\theta \in [0,1]$. (In this section we shall use $\overline{\eta}_t$ to denote the basic voter model.) Among other things, they proved that for any $\alpha \in [0,1]$, a spatial rescaling by $t^{\alpha/2}$ converges to a limiting random field $\overline{\eta}_\infty^{\theta,\alpha}$, i.e.,

$$\{\overline{\eta}_t^\theta(xt^{\alpha/2}), x \in \boldsymbol{Z}^2\} \Rightarrow \{\overline{\eta}_\infty^{\theta,\alpha}(x), x \in \boldsymbol{Z}^2\}, \quad \text{as } t \to \infty \tag{28}$$

(using the convention that the point $xt^{\alpha/2}$ of $\boldsymbol{R}^2$ is identified with the nearest point of $\boldsymbol{Z}^2$). They also identify the limiting random field, showing in particular that

$$\lim_{t\to\infty} \mathrm{Cov}(\overline{\eta}_t^\theta(xt^{\alpha/2}), \overline{\eta}_t^\theta(yt^{\alpha/2})) = \theta(1-\theta)(1-\alpha), \tag{29}$$

for every $y \neq x \in \boldsymbol{Z}^2$. Suppose that $0 < \alpha < 1$ and that we rescale the equilibrium noisy voter model $\eta_{[\kappa,\theta]}$ spatially by $\kappa^{-\alpha/2}$, and then let $\kappa \to 0$ (keeping θ fixed). Then we shall show that: (i) the rescaled covariances tend to the right hand side of (29); and (ii) this is not a coincidence, because the rescaled fields converge to the limit $\overline{\eta}_\infty^{\theta,\alpha}$ as defined by (28). Of course, (i) is a consequence of (ii), but it is interesting to see how (i) can be derived by a direct calculation, independently of the argument for (ii).

We shall prove (i) using our Proposition 9 and some classical results of Spitzer (1976) for two-dimensional simple random walk. Let $X \equiv X(\kappa) = x\kappa^{-\alpha/2}$, where $x \neq 0$ is a fixed point of $\boldsymbol{Z}^2$. Then

$$\begin{aligned}
&\operatorname{Cov}(\eta_{[\kappa,\theta]}(X), \eta_{[\kappa,\theta]}(0)) \\
&= \theta(1-\theta)\frac{G(X,0;q)}{G(0,0;q)} \\
&= \theta(1-\theta)\left(\frac{Z(X;q)}{G(0,0;q)} - \frac{A(X,0)}{G(0,0;q)} + 1\right), \qquad (30)
\end{aligned}$$

where $A(\cdot,\cdot)$ is the *potential kernel* for simple random walk in $\boldsymbol{Z}^2$ (see Section 12 of Spitzer (1976)), and we define

$$Z(X;q) := A(X,0) - G(0,0;q) + G(X,0;q).$$

We know that $A(y,0) \sim (2/\pi)\log||y||$ as $||y|| \to \infty$ (Proposition 12.3 of Spitzer (1976)), and so we have $A(X,0) \sim (\alpha/\pi)\log(1/\kappa)$ as $\kappa \to 0$. Also, since $p_{2n}(0,0) \sim 1/(\pi n)$ as $n \to \infty$ for simple random walk in $\boldsymbol{Z}^2$, we know that

$$G(0,0;q) \sim (1/\pi)\log[1/(1-q)] \sim (1/\pi)\log(1/\kappa), \quad \text{as } \kappa \to 0$$

(i.e., as $q \to 1$). Therefore, to prove (i), it suffices to show that

$$\lim_{\kappa \to 0} \frac{Z(X;q)}{G(0,0;q)} = 0. \qquad (31)$$

For $u \equiv (u_1, u_2) \in \boldsymbol{R}^2$, let $\Phi(u) \equiv (\cos u_1 + \cos u_2)/2$ denote the characteristic function for simple random walk in $\boldsymbol{Z}^2$. Then

$$G(y,0;q) = (2\pi)^{-2} \int \frac{e^{\mathrm{i}u\cdot y}}{1 - q\Phi(u)}\,\mathrm{d}u, \quad \text{for every } y \in \boldsymbol{Z}^2,$$

where the integration is over the square $[-\pi,\pi]^2$ in $\boldsymbol{R}^2$ (see Proposition 6.3 of Spitzer (1976)). Using this and Eq. (3) from p. 122 of Spitzer (1976), we have

$$\begin{aligned}
Z(X;q) &= (2\pi)^{-2} \int \left(\frac{1 - e^{\mathrm{i}u\cdot X}}{1 - \Phi(u)} - \frac{1 - e^{\mathrm{i}u\cdot X}}{1 - q\Phi(u)}\right)\mathrm{d}u \\
&= (2\pi)^{-2} \int \frac{(1 - e^{\mathrm{i}u\cdot X})\Phi(u)(1-q)}{(1 - q\Phi(u))(1 - \Phi(u))}\,\mathrm{d}u.
\end{aligned}$$

Since the imaginary part of the integral is 0, we have

$$\begin{aligned}
|Z(X;q)| &\leqslant (2\pi)^{-2} \int \frac{\operatorname{Re}(1 - e^{\mathrm{i}u\cdot X})(1-q)}{(1 - q\Phi(u))(1 - \Phi(u))}\,\mathrm{d}u \\
&\leqslant (2\pi)^{-2} \int \frac{||u||^2||X||^2(1-q)}{(1 - q\Phi(u))(1 - \Phi(u))}\,\mathrm{d}u.
\end{aligned}$$

Now observe that $||u||^2/(1-\Phi(u))$ is bounded on $[-\pi,\pi]^2$, so the last integral is bounded by a constant times $||X||^2(1-q)G(0,0;q)$. Finally, (31) follows from this and the fact that $||X||^2(1-q) \to 0$ as $\kappa \to 0$, for any $\alpha \in (0,1)$.

The proof of (ii) is similar to our proof of Proposition 10. Fix a finite set $B \subset \mathbf{Z}^2$ and a number $\alpha \in (0,1)$. For $\kappa > 0$, let

$$B(\kappa) \equiv B(\kappa,\alpha) = \{x\kappa^{-\alpha/2} : x \in B\}$$

be the rescaled version of B. Let $L_1(\kappa) = 1/(\kappa \log(1/\kappa))$ and $L_2(\kappa) = \log(1/\kappa)/\kappa$. Let $\sigma_1(\kappa)$ (respectively, $\sigma_2(\kappa)$) be the smallest (respectively, largest) of $\{\sigma^x : x \in B(\kappa)\}$. Observe that $P_{[\kappa,\theta]}\{\sigma_1(\kappa) \leqslant L_1(\kappa)\}$ and $P_{[\kappa,\theta]}\{\sigma_2(\kappa) \geqslant L_2(\kappa)\}$ both tend to 0 as $\kappa \to 0$. Therefore

$$\begin{aligned} &\lim_{\kappa\to 0} \Pr\{\eta_{[\kappa,\theta]}(x) = 0, \quad \text{for every } x \text{ in } B(\kappa)\} \\ &\quad = \lim_{\kappa\to 0} P_{[\kappa,\theta]}\{m^x = \delta^*, \quad \text{for every } x \text{ in } B(\kappa), \\ &\qquad\qquad \text{and} \quad L_1(\kappa) \leqslant \sigma_1(\kappa) \leqslant \sigma_2(\kappa) \leqslant L_2(\kappa)\}. \end{aligned} \tag{32}$$

Let $|B| = n$. Theorem 3 of Cox and Griffeath (1986) tells us that for each $j = 1,\ldots,n$ there exists a continuous function $p_{n,j}(\cdot)$ (which is the same for every α) such that for any $\rho \in [\alpha,\infty)$ we have

$$\lim_{t\to\infty} \Pr\{|\phi_{t^\rho}^{B(1/t)}| = j\} = p_{n,j}(\alpha/\rho).$$

(Notice that $\sum_{j=1}^{n} p_{n,j}(\alpha/\rho) = 1$.) We will use this to show that for each $j = 1,\ldots,n$,

$$\lim_{\kappa\to 0} \Pr\{|\phi_t^{B(\kappa)}| = j, \quad \text{for all } t \in [L_1(\kappa), L_2(\kappa)]\} = p_{n,j}(\alpha). \tag{33}$$

This will imply our result (ii), because it is now apparent that the right side of (32) equals

$$\sum_{j=1}^{n} (1-\theta)^j p_{n,j}(\alpha),$$

which Cox and Griffeath show is exactly

$$\Pr\{\bar\eta_\infty^{\theta,\alpha}(x) = 0, \quad \text{for every } x \in B\}.$$

To prove (33), fix $j \in \{1,\ldots,n\}$. Choose $\epsilon > 0$ such that $\alpha < 1-\epsilon$. Then $L_1(\kappa) \geqslant \kappa^{-(1-\epsilon)}$ for all sufficiently small κ. Therefore

$$\begin{aligned} &\liminf_{\kappa\to 0} \Pr\left\{|\phi_{L_1(\kappa)}^{B(\kappa)}| \leqslant j\right\} \\ &\quad \geqslant \lim_{\kappa\to 0} \Pr\left\{|\phi_{\kappa^{-(1-\epsilon)}}^{B(\kappa)}| \leqslant j\right\} = \sum_{i=1}^{j} p_{n,i}(\alpha/1-\epsilon). \end{aligned}$$

Similarly,

$$\begin{aligned} &\limsup_{\kappa\to 0} \Pr\left\{|\phi_{L_2(\kappa)}^{B(\kappa)}| \leqslant j\right\} \\ &\quad \leqslant \lim_{\kappa\to 0} \Pr\left\{|\phi_{\kappa^{-(1+\epsilon)}}^{B(\kappa)}| \leqslant j\right\} = \sum_{i=1}^{j} p_{n,i}(\alpha/1+\epsilon). \end{aligned}$$

Since the functions $p_{n,i}$ are continuous, we can let $\epsilon \to 0$ and deduce

$$\lim_{\kappa\to 0} \Pr\left\{|\phi^{B(\kappa)}_{L_1(\kappa)}| = i\right\} = \lim_{\kappa\to 0} \Pr\left\{|\phi^{B(\kappa)}_{L_2(\kappa)}| = i\right\} = p_{n,i}(\alpha),$$

for every $i = 1, \ldots, n$. Since $|\phi^{B(\kappa)}_t|$ is noncreasing in t, this implies (33).

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