Supplementary Information I: Model analysis

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1 Model analysis

Only the latter case we obtain a limiting ODE. In order to better understand the consequences of the mechanism proposed, we first consider the deterministic limit.

1.1 Deterministic limit

Proposition 1.1 *Let $N_i = n_i N$. Then, the deterministic limit for $x(t) = X_t/N$ reads*

$$\dot{x} \;=\; -\mu x\,\frac{\vartheta_2(1-x+n_2)}{(x+n_1)+\vartheta_2(1-x+n_2)} + \mu(1-x)\,\frac{\vartheta_1(x+n_1)}{\vartheta_1(x+n_1)+(1-x+n_2)}. \tag{1}$$

For $n_1 = n_2 = n$ and $\vartheta_1 = \vartheta_2$, $x = 1/2$ always is a stationary point; this stationary point undergoes a pitchfork bifurcation at $\vartheta_1 = \vartheta_2 = \vartheta_p$, where

$$\vartheta_p \;=\; \frac{1-2n}{1+2n}. \tag{2}$$

Proof: The rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1-x)\,\frac{\vartheta_1(x+n_1)}{\vartheta_1(x+n_1)+(1-x+n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1-x+n_2)}{(x+n_1)+\vartheta_2(1-x+n_2)}.$$

Therewith, the Fokker-Planck equation for the large population size (Kramers-Moyal expansion) reads

$$\partial_t u(x,t) = -\partial_x((f_+(x)-f_-(x))\,u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x)+f_-(x))\,u(x,t))$$

and the ODE due to the drift term in case of $N\to\infty$ is given by

$$\frac{d}{dt}x = f_+(x) - f_-(x).$$

This result establishes eqn. (1). For the following, let $\vartheta_1 = \vartheta_2 = \vartheta$. If we also choose $n_1 = n_2 = n$, we have a neutral model, and $x = 1/2$ is a stationary point for all $\vartheta \geq 0$, $n > 0$. We find the Taylor expansion of the r.h.s. at $x = 1/2$ (using the computer algebra package maxima [1])

$$\begin{aligned}\mu^{-1}\frac{d}{dt}x &= -x\,\frac{\vartheta(1-x+n_2)}{(x+n_1)+\vartheta(1-x+n_2)} + (1-x)\,\frac{\vartheta(x+n_1)}{\vartheta(x+n_1)+(1-x+n_2)}\\ &= -2\,\vartheta\,\frac{(2\,n+1)\,\vartheta+(2\,n-1)}{(2\,n+1)\,(\vartheta+1)^2}\left(x-\frac{1}{2}\right) + \frac{32\,\vartheta\,(\vartheta+n-\vartheta^2(n+1))}{(2\,n+1)^3\,(\vartheta+1)^4}\left(x-\frac{1}{2}\right)^3 + \mathcal{O}((x-1/2)^4)\end{aligned}$$

For $\vartheta \in (0, 1)$, $n > 0$, the coefficient in front of the third order term always is non-zero, while the coefficient in front of the linear term becomes zero at $\vartheta = \vartheta_p$. Hence, we have a pitchfork bifurcation at that parameter.

□

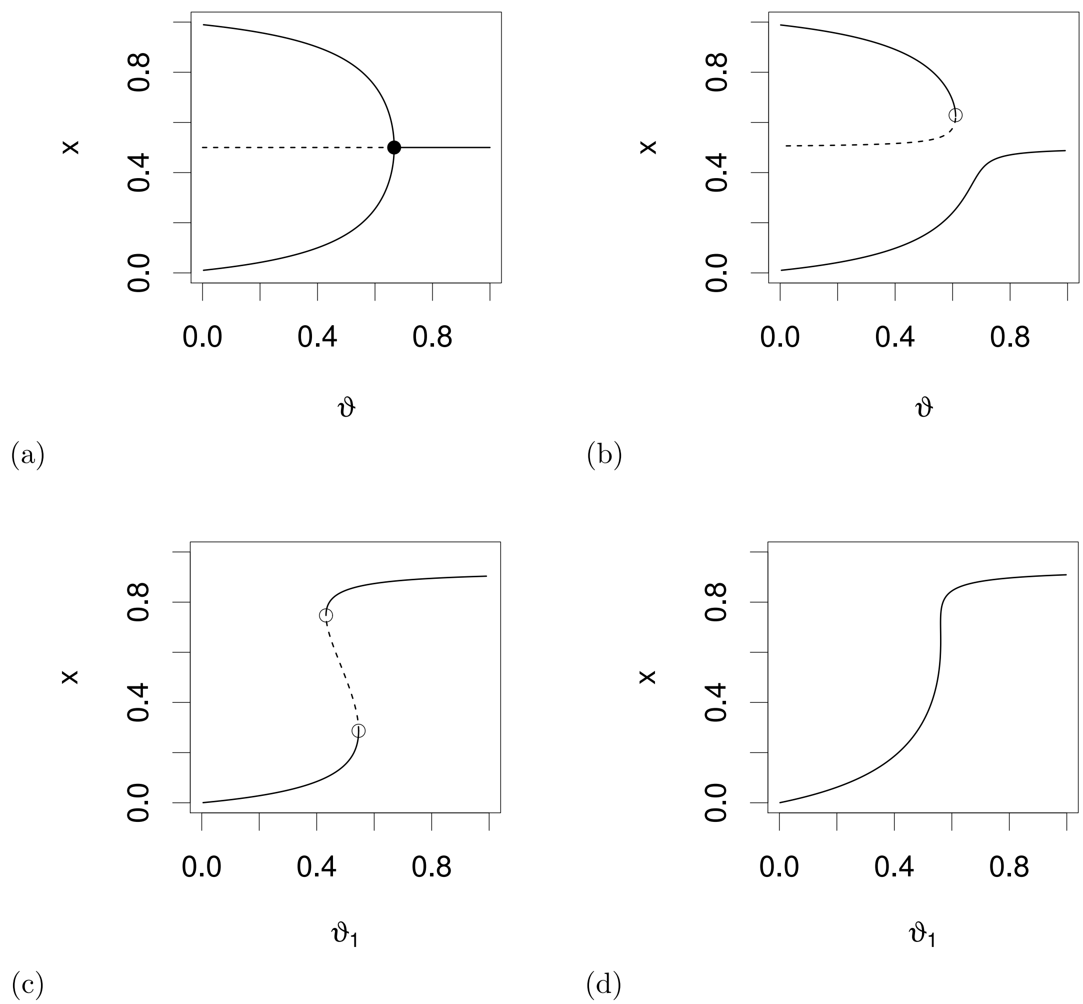


Figure 1: Stationary points of the reinforcement model over ϑ. The pitchfork bifurcation in (a) is indicated by a bullet, the saddle-node bifurcations in (b) and (c) are indicated by open circles. Stable branches of stationary points are represented by solid lines, unstable branches by dotted lines. (a) $n_1 = n_2 = 0.1$, $\vartheta_1 = \vartheta_2 = \vartheta$, (b) $n_1 = 0.1$, $n_2 = 0.105$, $\vartheta_1 = \vartheta_2 = \vartheta$, (c) $n_1 = n_2 = 0.1$, $\vartheta_2 = 0.5$, (d) $n_1 = 0.2$, $n_2 = 0.02$, $\vartheta_1 = 1.0$.

The pitchfork bifurcation is unstable against any perturbation that breaks the symmetry $x \mapsto 1-x$ (Fig. 1). In panel (a), we have the symmetric case, and find the proper pitchfork bifurcation.

Panel (b) shows the result if the number of zealots only differs slightly, where the reinforcement-parameter for both groups are assumed to be identical. We still find a reminiscent of the pitchfork bifurcation: The stable branches in (b) are close to the stable branches in (a), and also the unstable branches correspond to each other. For the limit $n_2 \to n_1$, panel (b) converges to panel (a). However, the branches are not connected any more but dissolve in two unconnected parts, and the pitchfork bifurcation is replaced by a saddle-node bifurcation.
In panel (c) and (d), the upper branch visible in panel (b) did vanish, and only the lower branch is present. As ϑ_2 is kept constant ($\vartheta_2 = 0.5$ in panel (c) and $\vartheta_2 = 0$ in panel (d)) and only ϑ_1 does vary, there is no continuous transition to panel (a).

The effect of reinforcement for a given group resembles an increase in the number of the group's zealot. Reinforcement may lead to the dominance of a group. In panel (d), the second group has only 1/10 of the zealots of the first group, but is able to take over if the members of that group do an extreme reinforcement ($\vartheta_1 \ll 1$). However, if the reinforcement of both groups is has a similar intensity and is strong, the mechanism is symmetrical, with a bistable setting as the consequence (panel (a)).

1.2 Weak effects limit

We now turn to the second scaling – the effect of zealots, and also the effect of the echo chambers, are taken to be weak. Under these circumstances, it is possible to find a limiting distribution for the invariant measure of the process.

Theorem 1.2 *Let N_i denote the number of zealots for group i, N the population size, and $\vartheta_i = 1 - \theta_i/N$ the parameter describing reinforcement. In the limit $N \to \infty$, the density of the invariant measure for the random variable $z_t = X_t/N$ is given by*

$$\varphi(x) = C\, e^{\frac{1}{2}(\theta_1+\theta_2)x^2 - \theta_1\, x}\; x^{N_1-1}\,(1-x)^{N_2-1}, \tag{3}$$

where C is determined by the condition $\int_0^1 \varphi(x)\,dx = 1$.

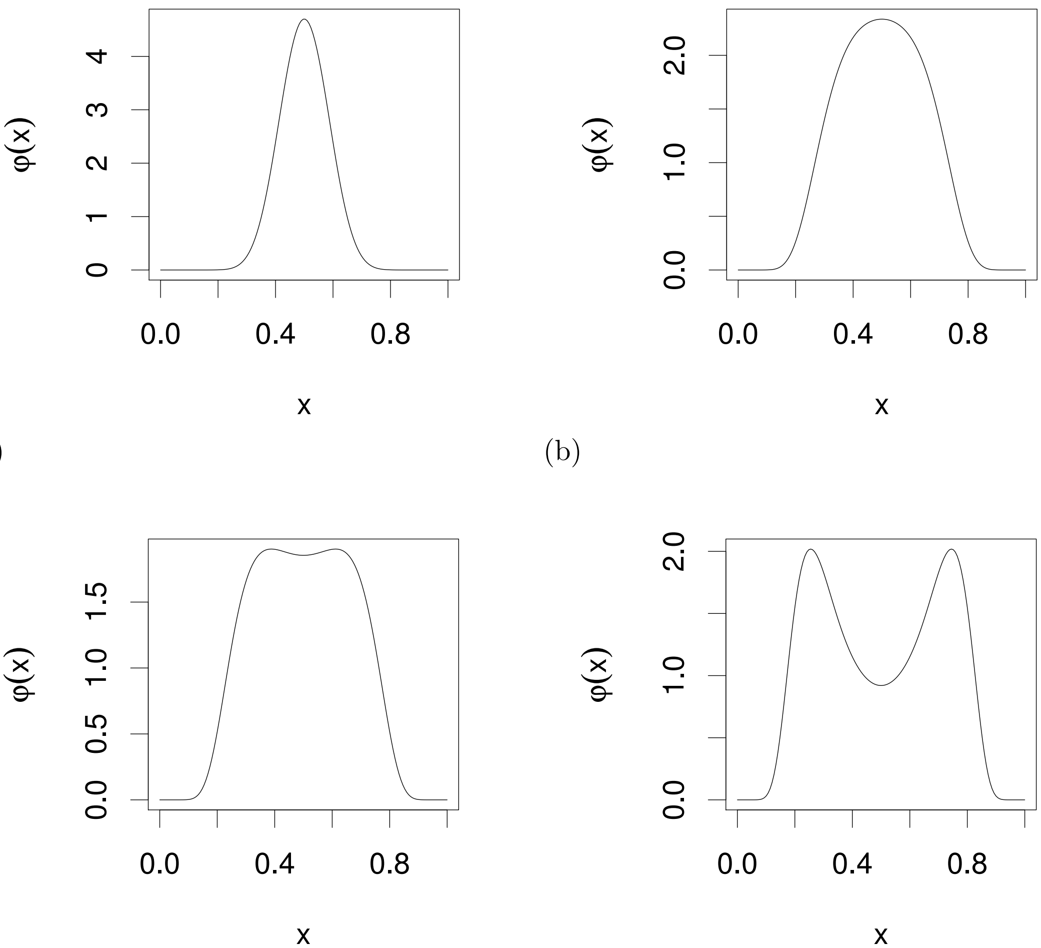
Proof: We again start off with the Fokker-Planck equation, obtained by the Kramers-Moyal expansion, where we use the scaling $n_i = N_i/N$, and ϑ_i constant in N. Only afterwards, we proceed to the desired scaling.
As seen above, the rates to increase/decrease the state can be written as $f_+(X_t/N)$ resp. $f_-(X_t/N)$, where (recall that $n_i = N_i/N$)

$$f_+(x) = \mu(1-x)\,\frac{\vartheta_1(x+n_1)}{\vartheta_1(x+n_1)+(1-x+n_2)}, \qquad f_-(x) = \mu x\,\frac{\vartheta_2(1-x+n_2)}{(x+n_1)+\vartheta_2(1-x+n_2)}.$$

Therewith, the limiting Fokker-Planck equation reads

$$\partial_t u(x,t) = -\partial_x((f_+(x)-f_-(x))\,u(x,t)) + \frac{1}{2N}\partial_x^2((f_+(x)+f_-(x))\,u(x,t))$$

Now we rewrite drift and noise term with the new scaling $n_i = N_i/N$, $\vartheta_i = 1 - \theta_i/N$, where we



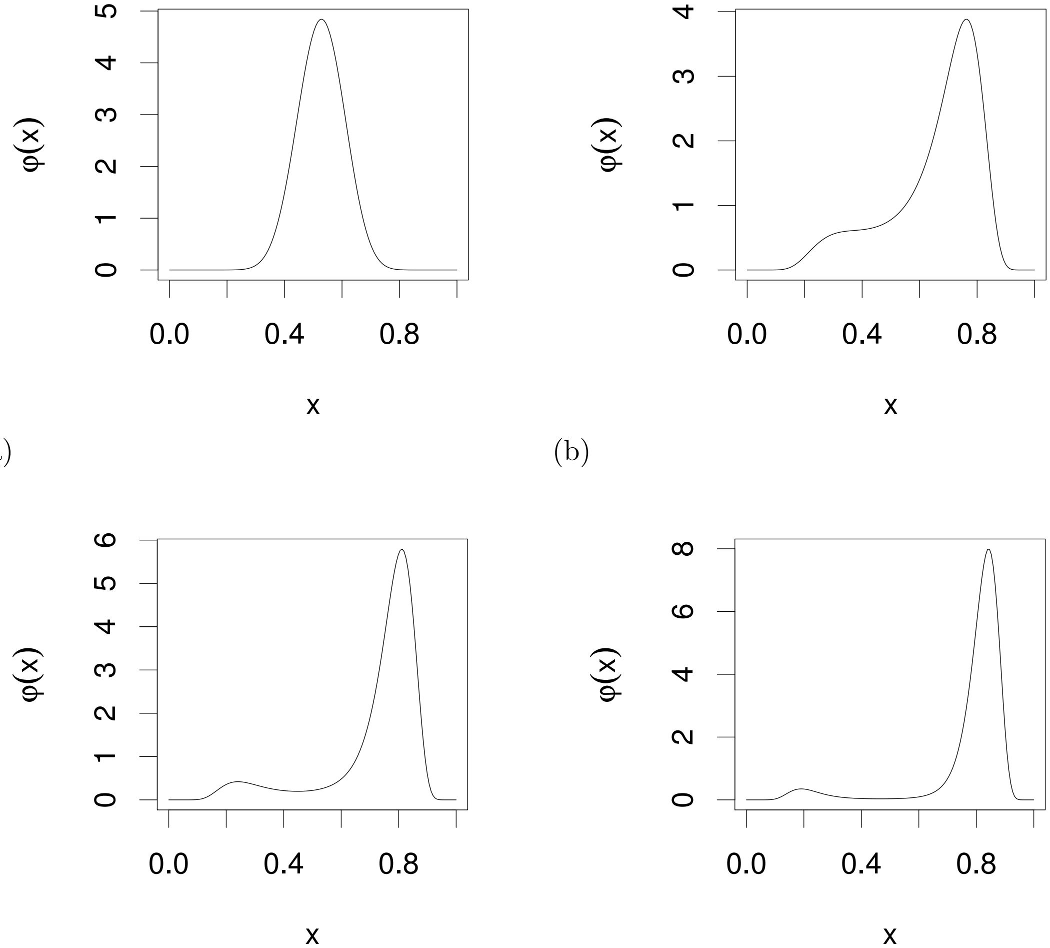
(a) (b) (c) (d)

Figure 2: Invariant distribution, given in eqn. (3) for $N_1 = N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 70$, (c) $\theta_1 = \theta_2 = 80$, (d) $\theta_1 = \theta_2 = 100$.

neglect terms of order $\mathcal{O}(N^{-2})$. We find (using maxima [1]) that ($h := 1/N$)

$$\begin{aligned}
& f_+(x) - f_-(x) \\
= \;& \mu(1-x)\,\frac{(1-h\,\theta_1)(x+h\,N_1)}{(1-h\,\theta_1)(x+h\,N_1)+(1-x+h\,N_2)} - \mu x\,\frac{(1-h\,\theta_2)(1-x+h\,N_2)}{(x+h\,N_1)+(1-h\,\theta_2)(1-x+h\,N_2)} \\
= \;& \mu\Big([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x)-(N_1+N_2)\,x+N_1\Big)\,h+\mathcal{O}(h^2),
\end{aligned}$$

while $h(f_+(x)+f_-(x)) = h\,2\,\mu x(1-x) + \mathcal{O}(h^2)$. If we rescale time, $T = \mu\,h\,t$, the Fokker-Planck



(a) (b) (c) (d)

Figure 3: Invariant distribution, given in eqn. (3) for $N_1 = 22$, $N_2 = 20$. We have $\theta_1 = \theta_2$, where (a) $\theta_1 = \theta_2 = 10$, (b) $\theta_1 = \theta_2 = 100$, (c) $\theta_1 = \theta_2 = 120$, (d) $\theta_1 = \theta_2 = 140$.

equation becomes

$$\partial_T u(x,T) = -\,\partial_x\Big\{\,\Big([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x)-(N_1+N_2)\,x+N_1\Big)\;u(x,T)\;\Big\}+\partial_x^2\Big\{x\,(1-x)\,u(x,T)\Big\}.$$

For the invariant distribution $\varphi(x)$, the flux of that rescaled Fokker-Planck equation is zero, that is,

$$-\Big([(\theta_1+\theta_2)x-\theta_1]\,x\,(1-x)-(N_1+N_2)\,x+N_1\Big)\varphi(x)+\frac{d}{dx}\Big(x(1-x)\,\varphi(x)\Big)=0.$$

With $v(x) = x(1-x)\varphi(x)$, we have

$$v'(x) = \left([(\theta_1+\theta_2)x - \theta_1] + \frac{N_1}{x} - \frac{N_2}{1-x}\,x\right)v(x)$$

and hence

$$v(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1}\,(1-x)^{N_2}$$

resp.

$$\varphi(x) = C\,e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}\;x^{N_1-1}\,(1-x)^{N_2-1}$$

$\square$

For $\theta_1 = \theta_2 = 0$, we obtain the beta distribution, as we fall back to the zealot model without reinforcement. In the given scaling, the reinforcement is expressed by the exponential multiplicative factor. As $\vartheta_i = 1 - h\,\theta_i$, and $h = 1/N$ is small, one could be tempted to assume that we are in the subcritical parameter range of the reinforcement model only, s.t. the distribution does not show a phase transition. As we see next, this idea is wrong.

Let us first consider the symmetric case, $N_1 = N_2 = \underline{N}$, and $\theta_1 = \theta_2 = \underline{\theta}$ (see Fig. 2). In that case, the distribution is given by

$$\varphi(x) = C\,e^{\underline{\theta}\,x\,(1-x)}\;x^{\underline{N}-1}\,(1-x)^{\underline{N}-1}.$$

The function always is symmetric w.r.t. $x = 1/2$. If $\underline{\theta}$ is small, and $\underline{N} > 0$, we find an unimodal function, with a maximum at 1/2. If, however, $\hat{\theta}$ is increased, eventually a bimodal distribution appears – we find back the pitchfork bifurcation that we already known from the deterministic limit of the model (Fig. 1, panel a).

As soon as $N_1 \neq N_2$, the symmetry is broken (Fig. 3), and we have an a situation resembling Fig. 1, panel (b). In the stochastic setting, however, we have more information: the second branch concentrates only little probability mass, and will play in practice only a minor role (if any at all). Only if N_1 and N_2 are distinctively unequal, this second branch ias able to concentrate sufficient probability mass to gain visibility in empirical data.

Comparison of the reinforcement model and the zealot model. We can use the zealot model or we can use the reinforcement model to fit and interpret election data. The zealot model for two parties yields the beta distribution. The density of the reinforcement model basically consist of a product, where one term is identical with the beta distribution,

$$x^{N_1-1}\,(1-x)^{N_2-1}$$

while the second term expresses the influence of reinforcement

$$e^{\frac{1}{2}(\theta_1+\theta_2)x^2-\theta_1\,x}.$$

Only if the data have a shape that is different from that of a beta distribution, the reinforcement component leads to a significantly improved fit. This is given, e.g., in case of a bimodal shape of the data (where at least one maximum is in the interior of the interval $(0,1)$), or if the data have heavy tails. Both properties hint to the fact that the election districts are of two different types: one, where the party under consideration is relatively strong, and one where it is relatively weak. This difference, in turn, can be interpreted as the effect of reinforcement: In some election districts voters agree that the given party is preferable, in others they agree that the party is to avoid. The population is not (spatially) homogeneous, but some segregation - most likely caused by social mechanisms - take place. In that, the data analysis of spatially structured election data (results structured by election districts) based on the reinforcement model is able to detect spatial segregation and the consequences thereof.

1.3 Spatial model and weak effects limit

Election districts clearly identify a spatial structure. In order to better understand the spatial effects, we develop a spatially structured model and determine also for that the probability density i the weak effects limit.

Let us consider election districts ordered in a two-dimensional lattice (torus) $\mathbb{Z}_n \times \mathbb{Z}_m$. In what follows, we always work modulo n resp. modulo m in the spatial indices. Let furthermore N denote the total population size in one election district, N_i the number of zealots for opinion $i \in \{1,2\}$, and ϑ_i the weights for the opposite opinion. The parameters N_i, θ_i are isotropic in space, that is, independent on the election district. If $X_t^{(k,\ell)}$ is the number of supporters for opinion 1 in district $(k,\ell) \in \mathbb{Z}_n \times \mathbb{Z}_m$, while $N - X_t^{(k,\ell)}$ is that for opinion 2 (in the corresponding election district), then

$$X_t^{(k,\ell)} \to X_t^{(k,\ell)} + 1 \quad \text{at rate} \quad \mu(N - X_t^{(k,\ell)}) \frac{\vartheta_1(\hat{X}_t^{(k,\ell)} + N_1)}{\vartheta_1(\hat{X}_t^{(k,\ell)} + N_1) + (N - \hat{X}_t^{(k,\ell)} + N_2)}, \tag{4}$$

$$X_t^{(k,\ell)} \to X_t^{(k,\ell)} - 1 \quad \text{at rate} \quad \mu X_t^{(k,\ell)} \frac{\vartheta_2(N - \hat{X}_t^{(k,\ell)} + N_2)}{(\hat{X}_t^{(k,\ell)} + N_1) + \vartheta_2(N - \hat{X}_t^{(k,\ell)} + N_2)}. \tag{5}$$

where $X_t^{(k,\ell)}$ denotes the weighted average of opinion-1-supporters in the Moore neighborhood of (k,ℓ),

$$\hat{X}_t^{(k,\ell)} = (1-\tau)X_t^{(k,\ell)} + \tau \check{X}^{(k,\ell)}, \qquad \check{X}^{(k,\ell)} := \frac{1}{8} \sum_{(k',\ell')\sim(k,\ell)} X_t^{(k',\ell')}$$

and $(k',\ell') \sim (k,\ell)$ if $|k-k'| \leq 1$, $|\ell - \ell'| \leq 1$, and $(k',\ell') \neq (k,\ell)$ (neighboring sites according to the Moore neighborhood). The parameter $\tau \in [0,1]$ plays the role of the spatial interaction strength: If $\tau = 0$, the sites are independent, if $\tau = 1$, individuals in an election district only communicate with individuals of neighboring election districts.

Theorem 1.3 *Let N_i denote the number of zealots for group i, N the population size, and $\vartheta_i = 1 - \theta_i/N$ the parameter describing reinforcement. We also scale the spacial interaction strength $\tau = \sigma/N$. In the limit $N \to \infty$, the density of the invariant measure for the random variable $z_t = X_t/N$ is given by*

$$\psi(x^{(\cdot,\cdot)}) = C \prod_{k,\ell} \left(\varphi(x^{(k,\ell)}) \quad \exp\left\{ -\frac{\gamma}{32} \sum_{(k',\ell')\sim(k,\ell)} (x^{(k,\ell)} - x^{(k',\ell')})^2 \right\} \right), \tag{6}$$

where $\varphi(.)$ is the homogeneous-population distribution defined in eqn. (3), and C is determined by the condition that the integral over $\psi(.)$ is 1.

Proof: We again start off with the Fokker-Planck equation, obtained by the Kramers-Moyal expansion, where we use the scaling $n_i = N_i/N$, and ϑ_i constant in N. Only afterwards, we proceed to the desired scaling.
As seen above, the rates to increase/decrease the state in site (k,ℓ) can be written as $f_+^{(k,\ell)}(X_t^{(\cdot,\cdot)}/N)$ resp. $f_-^{(k,\ell)}(X_t^{(\cdot,\cdot)}/N)$, where (recall that $n_i = N_i/N$)

$$\begin{aligned} f_+^{(k,\ell)}(x^{(\cdot,\cdot)}) &= \frac{[\mu(1-x^{(k,\ell)})]\ [\vartheta_1((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)]}{\vartheta_1((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)+(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)}, \\ f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) &= \frac{[\mu x^{(k,\ell)}]\ [\vartheta_2(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)]}{((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)+\vartheta_2(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)}. \end{aligned}$$

Here, $\hat{x}^{(k,\ell)}$ is the average of $x^{(\cdot,\cdot)}$ at the Moore neighborhood of (k,ℓ).
Therewith, the flux $j^{(k,\ell)}(x^{(\cdot,\cdot)})$ for the limiting Fokker-Planck equation is defined by

$$\begin{aligned} j^{(k,\ell)}(x^{(\cdot,\cdot)}) &= -\Big(f_+^{(k,\ell)}(x^{(\cdot,\cdot)})-f_-^{(k,\ell)}(x^{(\cdot,\cdot)})\Big)u(x^{(\cdot,\cdot)}) \\ &\quad +\frac{1}{2N}\,\partial_{x^{(k,\ell)}}\Big\{\Big(f_+^{(k,\ell)}(x^{(\cdot,\cdot)})+f_-^{(k,\ell)}(x^{(\cdot,\cdot)})\Big)u(x^{(\cdot,\cdot)})\Big\} \end{aligned}$$

and the Fokker-Planck equation itself reads

$$\partial_t u(x^{(\cdot,\cdot)}) = \sum_{k,\ell}\partial_{x^{(k,\ell)}}j^{(k,\ell)}(x^{(\cdot,\cdot)}).$$

Now we rewrite drift and noise term with the new scaling $n_i = N_i/N$, $\vartheta_i = 1-\theta_i/N$, $\tau = \gamma/N$, where we neglect terms of order $\mathcal{O}(N^{-2})$. We find (using maxima [1]) that $(h := 1/N)$

$$\begin{aligned} & f_+^{(k,\ell)}(x^{(\cdot,\cdot)})-f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) \\ =\;& \frac{\mu\,(1-x^{(k,\ell)})\ [\vartheta_1((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)]}{\vartheta_1((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)+(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)} \\ &-\frac{\mu\,x^{(k,\ell)}\ [\vartheta_2(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)]}{((1-\tau)\,x^{(k,\ell)}+\tau\check{x}^{(k,\ell)}+n_1)+\vartheta_2(1-(1-\tau)\,x^{(k,\ell)}-\tau\check{x}^{(k,\ell)}+n_2)} \\ =\;& \mu\Big((\check{x}^{(k,\ell)}-x^{(k,\ell)})\,\gamma+x^{(k,\ell)}\,(1-x^{(k,\ell)})\,(\theta_2\,x^{(k,\ell)}-\theta_1\,(1-x^{(k,\ell)}))-(N_2+N_1)x^{(k,\ell)}+N_1\Big)\,h+\mathcal{O}(h^2), \end{aligned}$$

while $f_+^{(k,\ell)}(x^{(\cdot,\cdot)})+f_-^{(k,\ell)}(x^{(\cdot,\cdot)}) = 2\,\mu\,x^{(k,\ell)}\,(1-x^{(k,\ell)})+\mathcal{O}(h)$. Hence, in lowest order, $j^{(k,\ell)}(x^{(\cdot,\cdot)}) = 0$ reads

$$\begin{aligned} & \partial_{x^{(k,\ell)}}\Big\{\Big(x^{(k,\ell)}(1-x^{(k,\ell)})\Big)u(x^{(\cdot,\cdot)})\Big\} \\ =\;& \Big((\check{x}^{(k,\ell)}-x^{(k,\ell)})\,\gamma+x^{(k,\ell)}\,(1-x^{(k,\ell)})\,(\theta_2\,x^{(k,\ell)}-\theta_1\,(1-x^{(k,\ell)}))-(N_2+N_1)x^{(k,\ell)}+N_1\Big)u(x^{(\cdot,\cdot)}). \end{aligned}$$

For $\gamma = 0$, this equation collapse to the equation for the homogeneous case. We therefore define $v(x^{(\cdot,\cdot)})$ by

$$u(x^{(\cdot,\cdot)}) = v(x^{(\cdot,\cdot)})\ \prod_{k,\ell} e^{\frac{1}{2}(\theta_1+\theta_2)(x^{(k,\ell)})^2-\theta_1\,x^{(k,\ell)}}\ (x^{(k,\ell)})^{N_1-1}\,(1-x^{(k,\ell)})^{N_2-1}$$

and obtain

$$\partial_{x^{(k,\ell)}} v(x^{(\cdot,\cdot)}) \quad = \quad \gamma\,(\check{x}^{(k,\ell)} - x^{(k,\ell)}) \;\; v(x^{(\cdot,\cdot)}).$$

That system of equations has the solution

$$v(x^{(\cdot,\cdot)}) = C \exp\left\{\gamma \sum_{(k,\ell)} \left(\frac{1}{16} \sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - \frac{1}{2}\,(x^{(k,\ell)})^2\right)\right\}.$$

The factor $1/16 = (1/8)/2$ is due to symmetry reasons: Each pair (k_1, ℓ_1),(k_2, ℓ_2) with $(k_1, \ell_1) \sim (k_2, \ell_2)$ appears twice in the sum. We rewrite the sum as follows

$$\begin{aligned}
& \sum_{(k,\ell)} \left(\frac{1}{16} \sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - \frac{1}{2}\,(x^{(k,\ell)})^2\right) \\
= \;& \frac{1}{16} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} x^{(k,\ell)} x^{(k',\ell')} - 8\,(x^{(k,\ell)})^2\right) \\
= \;& -\frac{1}{16} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} \left(- x^{(k,\ell)} x^{(k',\ell')} + (x^{(k,\ell)})^2\right)\right) \\
= \;& -\frac{1}{16} \sum_{(k,\ell)} \frac{1}{2}\left(\sum_{(k',\ell')\sim(k,\ell)} \left((x^{(k',\ell')})^2 - 2x^{(k,\ell)} x^{(k',\ell')} + (x^{(k,\ell)})^2\right)\right) \\
= \;& -\frac{1}{32} \sum_{(k,\ell)} \left(\sum_{(k',\ell')\sim(k,\ell)} \left(x^{(k,\ell)} - x^{(k',\ell')}\right)^2\right)
\end{aligned}$$

□

The distribution consist of a multiplicative structure, where the for each site one part is related to the local dynamics ($\varphi(x^{(k,\ell)})$), while the second term punishes differences in the state between neighboring sites. The assumption of the weak coupling prevents the neighboring sites to directly influence the internal dynamics.

References

[1] Maxima. Maxima, a computer algebra system. version 5.34.1, 2014.