Numerical stability HPSC

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Introduction This document is a synthesis of the stability analysis of the acoustic wave propagation equation within the framework of the HPSC course. This analysis is based on the references [2, 1].

Our system of equations

$$\begin{cases} \dfrac{\partial P}{\partial t} = -\rho c^2 \nabla \cdot \mathbf{v} \\ \dfrac{\partial \mathbf{v}}{\partial t} = -\dfrac{1}{\rho} \nabla P \end{cases} \tag{1}$$

can be equivalently written as:

$$\frac{\partial^2 P}{\partial t^2} = c^2 \Delta P \tag{2}$$

Under certain assumptions (c, ρ constants). Equation 2 is a wave equation, and c is the wave propagation speed. The operator Δ denotes the Laplacian.

By using discretization on a rectangular grid, with δ as the spatial step (considered constant and identical across the grid) and Δ_t as the temporal step (also considered constant), we can obtain a discrete version of equation 2. The spatial grids of velocity and pressure are shifted by $\frac{1}{2}$, allowing the use of centered differences in the approximation. Let $P^{\tau}_{m,n,p} = P(m\delta, n\delta, p\delta, \tau\Delta_t)$. The update equation of our numerical scheme is obtained as follows:

$$\begin{aligned} P^{\tau+1}_{m,n,p} + P^{\tau-1}_{m,n,p} = \lambda^2 (P^{\tau}_{m+1,n,p} &+ P^{\tau}_{m-1,n,p} \\ &+ P^{\tau}_{m,n+1,p} + P^{\tau}_{m,n-1,p} \\ &+ P^{\tau}_{m,n,p+1} + P^{\tau}_{m,n,p-1}) + \left(2 - 6\lambda^2\right) P^{\tau}_{m,n,p} \end{aligned}$$

with $\lambda^2 = \left(\dfrac{c}{v_0}\right)^2$ symbolizing the ratio of the physical velocity c to the 'numerical velocity' $v_0 = \dfrac{\delta}{\Delta_t}$.

We can analyze the stability of this numerical scheme through a Von Neumann analysis. Since the initial equation is linear, we can study the evolution of the error using the same equation. In particular, the update equation for P is also the update equation for the error ϵ. Let $\epsilon^{\tau}_{m,n,p}$ be the discrete version of the error.

To perform the Von Neumann analysis, we focus on the spatial Fourier transform of ϵ. We know that ϵ is a superposition of different modes: $\hat{\epsilon}(\mathbf{k}, t)e^{j\mathbf{k}\cdot\mathbf{x}}$, where j is the imaginary unit, $\mathbf{k}$ is the vector of spatial wave numbers, $\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$, $\mathbf{x}$ is the position vector, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The Von Neumann stability criterion is that it is sufficient for each error mode to be 'stable' for the total error to be stable. By stable mode, we mean that

$$\forall\mathbf{k}, \exists C \text{ such that } \quad \lim_{t\to\infty} |\hat{\epsilon}(\mathbf{k}, t)|^2 \leq C|\hat{\epsilon}(\mathbf{k}, t_0)|^2 \tag{3}$$

where t_0 refers to the initial error.

Criterion 3 can be translated into the discrete version of the equation:

$$|\xi| \leq 1 \Longleftrightarrow |\xi|^2 \leq 1 \tag{4}$$

with ξ as the error amplification factor. Alternatively, in the case of a second-order update equation:

$$|\xi_1| \leq 1 \quad \& \quad |\xi_2| \leq 1 \Longleftrightarrow |\xi_1|^2 \leq 1 \quad \& \quad |\xi_2|^2 \leq 1 \tag{5}$$

where $\xi_{1,2}$ are the roots of the characteristic polynomial of the update equation.

In the following, we denote $\hat{\epsilon}(\mathbf{k}, \tau\Delta_t)$ as $\hat{\epsilon}_\tau$, i.e., we drop the explicit reference to wave numbers for readability.

General Analysis Suppose an update equation has the form

$$\hat{\epsilon}_{\tau+2} + F(\mathbf{k})\hat{\epsilon}_{\tau+1} + \hat{\epsilon}_\tau = 0 \tag{6}$$

The corresponding characteristic polynomial is:

$$\xi^2 + F\xi + 1 = 0 \Leftrightarrow \xi_{1,2} = \frac{1}{2}(-F \pm \sqrt{F^2 - 4}) \tag{7}$$

If there exists a factorization of $F(\mathbf{k}) = -2\lambda^2\Gamma(\mathbf{k}) - 2$, where Γ is a function independent of λ, then the stability criterion can be simplified to:

$$|\xi_1| \leq 1 \quad \& \quad |\xi_2| \Leftrightarrow |F(\mathbf{k})| \leq 2 \tag{8}$$

Analysis of Our Scheme It can be observed that after injecting different error modes into the update equation, we obtain the characteristic polynomial:

$$\xi^2 + F(\mathbf{k})\xi + 1 = 0,$$
$$F(\mathbf{k}) = -2\left(1 + \lambda^2 \left[\cos(k_x\delta) + \cos(k_y\delta) + \cos(k_z\delta) - 3\right]\right)$$

By analyzing the extrema, we can conclude on stability. Indeed:

$$\max_{\mathbf{k}}|F(\mathbf{k})| \leq 2 \implies F(\mathbf{k}) \leq 2, \forall \mathbf{k} \tag{9}$$

In extreme cases, all cosine terms evaluate strictly to 1 ($F = -2$, okay criterion satisfied: stable scheme) or strictly to -1 ($F = -2 - 2\lambda^2(-6) = -2 + 12\lambda^2$). For this extremum, we find the criterion (as $\lambda^2 > 0$):

$$\begin{aligned}
-2 + 12\lambda^2 &\leq 2 \\
\lambda^2 &\leq \frac{4}{12} \\
\lambda &\leq \sqrt{\frac{1}{3}} \\
\lambda &\leq \frac{1}{\sqrt{3}}
\end{aligned}$$

Conclusion The numerical scheme used is stable as long as the following condition is met:

$$\begin{aligned}
\lambda &\leq \frac{1}{\sqrt{3}} \\
\frac{c}{v_0} &\leq \frac{1}{\sqrt{3}} \\
\frac{c\Delta_t}{\delta} &\leq \frac{1}{\sqrt{3}}
\end{aligned}$$

However, note that to obtain this result, certain assumptions were made. In particular, for a grid where c and ρ are not constants, it is advisable to consider the worst-case scenario. Using the maximum value of c in the grid is a good practice.

Physical Interpretation This result can be interpreted as follows: the speed of information propagation in the scheme must be greater than the physical wave propagation speed; otherwise, instability is introduced. In particular, as the scheme only involves the 6 direct neighbors of a specific node, information can only propagate at a speed of 1 node per time step. In 1D, this would not have been a problem if the speed of the numerical scheme were at least as large as the physical speed. However, in 3D, physical information travels in 3D and can cover a distance $\sqrt{3}$ times greater than the numerical information in one iteration (physical information also travels diagonally between nodes of a cubic grid). The appearance of the factor $\sqrt{3}$ in the stability criterion is then understandable.

References

[1] Julius O. Smith III et al. *Electrical Engineering*. 2006. URL: `https://ccrma.stanford.edu/~bilbao/master/`.

[2] Allen Taflove and Morris E. Brodwin. “Numerical Solution of Steady-state Electromagnetic Scattering Problems Using the Time-Dependent Maxwell’s Equations”. In: *IEEE Transactions on Microwave Theory and Techniques* nrrr-23.8 (Aug. 1975).