MATH0024 PDEs Homework 1

Maarten Arnst and Romain Boman.

1 Organization

- Two homework assignments are planned. Together, your solutions to all the exercises will count for 1/3 of your grade.
- Guidelines for the preparation of your report for this homework:
  - The report should collect your solutions to all the exercises that you worked on.
  - This homework is an individual assignment. Although you are strongly encouraged to discuss and exchange ideas about this homework with each other, you may not copy solutions from each other. You are required to prepare your own report that must contain your own solutions and reflect your own understanding of the material.
  - The report must be neat, well organized, and professionally presented. All figures must be computer plots. Please choose adequate ranges, ticks, and scales; please label all figure axes; and please include proper units where appropriate.
  - If you should consult any references, please list them at the end of your report.
  - The report must be sent in PDF format by email to both R. Boman (r.boman@uliege.be) and M. Arnst (maarten.arnst@uliege.be) before/on Thursday 14 November. Late homework will not be accepted unless the lateness can be appropriately justified or prior arrangements were made. Please attach your Python or other code.
- If you should need some help:
  - Concise introduction to programming, Python, and report writing as part of the Discussion session on Wednesday 23 October, 8h30–12h30.
  - Discussion session on Wednesday 30 October, 8h30–12h30.
  - Your participation to both discussion sessions is mandatory.
  - You are welcome to email R. Boman or M. Arnst to ask questions as well as to ask for feedback on one or more intermediate versions of your report and/or your code. It is proposed that Louis Basset, Fanny Bodart, Maxime Borbouse, Elise Boury Schmidt, Julien Brandoit, Ioan-Catalin Chirita and Adrien Daloze-Tilman contact R. Boman, and Alessandro Demoors, Yann Paul Fotsing Takam Fonkou, Henri Gérard, Constant Gobron, Lydia Leruth, Guillaume Maertens de Noordhout, Arnaud Radermecker and Alejandro Sior contact M. Arnst; we'll switch groups next time. If you submit an intermediate version at the latest one week before the due date, R. Boman and M. Arnst guarantee that you will receive a detailed feedback. You may submit an intermediate version multiple times. Your grade is based only on the final version that you submit.
  - You are welcome to contact R. Boman or M. Arnst by email to make an appointment.
- Please check the errata list to correct errors in the slides sent out before the lectures (if any).

2 Assignment

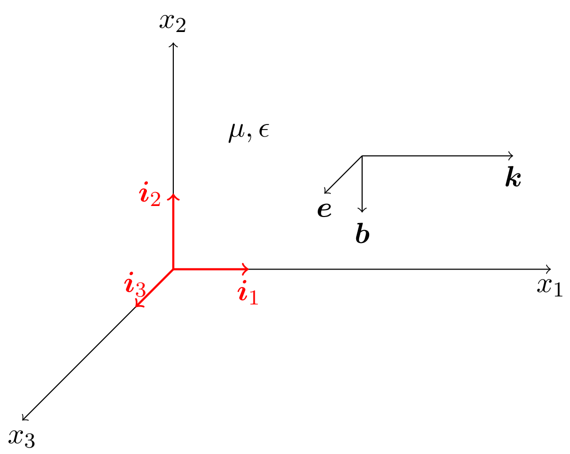
1. **Lecture 1**: For the following differential equations, please state the order and the best classification as either a linear, semilinear, quasilinear, or fully nonlinear differential equation:

   (a) Burgers's equation:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

   (b) Inviscid Burgers's equation:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

   (c) Van der Pol equation:
$$\frac{d^2u}{dt^2} - \mu(1-u^2)\frac{du}{dt} + u = 0.$$

2. **Lecture 2**:



Maxwell's equations for a dielectric medium with homogeneous electric permittivity ϵ and magnetic permittivity μ, with no charge density, and with no current density, read as

$$\begin{cases} \dfrac{\partial \boldsymbol{b}}{\partial t} + \mathbf{curl}_{\boldsymbol{x}}\boldsymbol{e} = \mathbf{0}, \\ \mathrm{div}_{\boldsymbol{x}}\boldsymbol{d} = 0, \\ \dfrac{\partial \boldsymbol{d}}{\partial t} - \mathbf{curl}_{\boldsymbol{x}}\boldsymbol{h} = \mathbf{0}, \\ \mathrm{div}_{\boldsymbol{x}}\boldsymbol{b} = 0, \end{cases} \tag{1}$$

with the constitutive equations

$$\begin{cases} \boldsymbol{d} = \epsilon\,\boldsymbol{e}, \\ \boldsymbol{b} = \mu\,\boldsymbol{h}. \end{cases} \tag{2}$$

From the first and third equations, and with the constitutive equations, we obtain

$$\mathbf{curl}_{\boldsymbol{x}}\left(\frac{1}{\mu}\frac{\partial \boldsymbol{b}}{\partial t}\right) + \mathbf{curl}_{\boldsymbol{x}}\left(\frac{1}{\mu}\mathbf{curl}_{\boldsymbol{x}}\boldsymbol{e}\right) = \mathbf{0}, \tag{3}$$

hence,

$$\frac{\partial}{\partial t}\mathbf{curl}_{\boldsymbol{x}}\boldsymbol{h} + \mathbf{curl}_{\boldsymbol{x}}\left(\frac{1}{\mu}\mathbf{curl}_{\boldsymbol{x}}\boldsymbol{e}\right) = \mathbf{0}, \tag{4}$$

and therefore,

$$\frac{\partial^2 \boldsymbol{d}}{\partial t^2} + \mathbf{curl}_{\boldsymbol{x}} \left(\frac{1}{\mu} \mathbf{curl}_{\boldsymbol{x}} \boldsymbol{e} \right) = \mathbf{0}, \tag{5}$$

so that

$$\epsilon \frac{\partial^2 \boldsymbol{e}}{\partial t^2} + \mathbf{curl}_{\boldsymbol{x}} \left(\frac{1}{\mu} \mathbf{curl}_{\boldsymbol{x}} \boldsymbol{e} \right) = \mathbf{0}, \tag{6}$$

Thus, we get a wave equation for the electric field. Analogously, a similar wave equation for the magnetic field can also be established.

Among other solutions, such wave equations admit so-called plane-wave solutions. Here, we are interested in plane-wave solutions with a linear polarization, with the electric field with a polarization vector along $\boldsymbol{i}_3$, and propagating along $\boldsymbol{i}_1$. Such solutions are of the form

$$\begin{cases} \boldsymbol{e}(\boldsymbol{x}, t) = e_0 \, \boldsymbol{i}_3 \, \exp\big(i(k_1 x_1 - \omega t)\big), \\ \boldsymbol{b}(\boldsymbol{x}, t) = b_0 \, \boldsymbol{i}_2 \, \exp\big(i(k_1 x_1 - \omega t)\big); \end{cases} \tag{7}$$

here, e_0 and b_0 are constants, k_1 is the wavenumber, ω is the circular frequency, and t is the time. The wavenumber k_1 and the circular frequency ω are related by the dispersion relation $c = 1/\sqrt{\mu\epsilon} = \omega/k_1$, in which $c = 1/\sqrt{\mu\epsilon}$ is the speed of propagation.

Let us look for the equation that governs the spatial component of the electric field in such plane-wave solutions. Let us thus look for an electric field of the form

$$\boldsymbol{e}(\boldsymbol{x}, t) = e_3(x_1) \, \boldsymbol{i}_3 \, \exp(-i\omega t), \tag{8}$$

in which e_3 is a function of the spatial coordinate x_1.

Injecting this equation into the wave equation for the electric field, we obtain:

$$\epsilon \, e_3 \, \boldsymbol{i}_3 \, \exp(-i\omega t)(-\omega^2) + \mathbf{curl}_{\boldsymbol{x}} \left(\frac{1}{\mu} \frac{\partial e_3}{\partial x_1} \, \exp(-i\omega t)(-\boldsymbol{i}_2) \right) = \mathbf{0}, \tag{9}$$

hence,

$$\epsilon \, e_3 \, \boldsymbol{i}_3 \, \exp(-i\omega t)(-\omega^2) + \frac{\partial}{\partial x_1} \left(\frac{1}{\mu} \frac{\partial e_3}{\partial x_1} \right) \exp(-i\omega t)(-\boldsymbol{i}_3) = \mathbf{0}, \tag{10}$$

and therefore

$$\frac{\partial}{\partial x_1} \left(\frac{1}{\mu} \frac{\partial e_3}{\partial x_1} \right) + \epsilon \, \omega^2 \, e_3 = 0. \tag{11}$$

With the dispersion relation $c = 1/\sqrt{\mu\epsilon} = \omega/k_1$, we ultimately obtain

$$\frac{\partial^2 e_3}{\partial x_1^2} + k_1^2 e_3 = 0. \tag{12}$$

This equation is called the Helmholtz equation.

In this exercise, we'll be interested in the fundamental solution to the Helmholtz equation. Dropping the subscripts, the question is as follows: Please show that the function

$$E(x) = -\frac{i \exp(ik|x|)}{2k} \tag{13}$$

is a fundamental solution to the Helmholtz equation, that is, that this function satisfies

$$\frac{\partial^2 E}{\partial x^2} + k^2 E = \delta_0 \quad \text{in } \mathbb{R}. \tag{14}$$

Hint: Proceed in a manner that follows how we determined the first-order derivative and the second-order derivative in the sense of the distributions of $E_{1\mathrm{D}}(x) = \frac{1}{2}|x|$.

3. **Lecture 3**: It can be shown that the distribution associated with the function E_{3D} from $\mathbb{R}^3$ into $\mathbb{R}$ given by $E_{3D}(\boldsymbol{x}) = \frac{-1}{4\pi}\frac{1}{\|\boldsymbol{x}\|}$ and the distribution associated with the function $\widehat{E}_{3D}$ from $\mathbb{R}^3$ into $\mathbb{C}$ given by $\widehat{E}_{3D}(\boldsymbol{\xi}) = \frac{-1}{\|\boldsymbol{\xi}\|^2}$ are tempered distributions. Accepting this property, please show that $\widehat{E}_{3D}$ is the Fourier transform of E_{3D} in the sense of the distributions.

   HINT: The classical Fourier transform of the function from $\mathbb{R}^3$ into $\mathbb{R}$ that maps $\boldsymbol{x}$ onto $\exp(-\pi a\|\boldsymbol{x}\|^2)$ is the function from $\mathbb{R}^3$ into $\mathbb{C}$ that maps $\boldsymbol{\xi}$ onto $a^{-3/2}\exp(-\frac{\|\boldsymbol{\xi}\|^2}{4\pi a})$; here, a is a strictly positive constant. As the Fourier transform in the sense of the distributions is a generalization of the classical Fourier transform, it follows that

   $$\int_{\mathbb{R}^3} a^{-3/2}\exp\left(-\frac{\|\boldsymbol{\xi}\|^2}{4\pi a}\right)\varphi(\boldsymbol{\xi})d\boldsymbol{\xi} = \int_{\mathbb{R}^3}\exp(-\pi a\|\boldsymbol{x}\|^2)\hat{\varphi}(\boldsymbol{x})d\boldsymbol{x}, \quad \text{for all Schwartz functions } \varphi.$$

   Multiply both sides of this expression with $a^{-1/2}$ and integrate with respect to a from 0 to $+\infty$. Simplify the expression thus obtained by using the definition of the gamma function $\Gamma(z) = \int_0^{+\infty} t^{z-1}\exp(-t)dt$ and hence $\Gamma(z)/b^z = \int_0^{+\infty} t^{z-1}\exp(-bt)dt$. Conclude by using that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.

4. **Lectures 4**: Please write a weak formulation for the boundary-value problem

   $$\begin{cases} -\dfrac{d^2u}{dx^2}(x) = \alpha(L-x)^2, & \text{in }]0, L[, \\ u(0) = u_0, & \text{at } \{x = 0\}, \\ \dfrac{\partial u}{\partial x}(L) = g_L, & \text{at } \{x = L\}, \end{cases}$$

   with α, L, u_0, and g_L real constants. Provide enough details to show your understanding of each step in obtaining the weak formulation and handling the boundary conditions.

5. **Lecture 5**: Let us consider the following BVP for the Helmholtz equation:

   $$\begin{cases} \dfrac{d^2u}{dx^2} + k^2u = 0 & \text{in }]0, 1[, \\ u(0) = 1 & \text{at } \{x = 0\}, \\ \dfrac{du}{dx} - iku = 0 & \text{at } \{x = 1\}. \end{cases}$$

   The boundary condition at $x = 1$ is a so-called mixed boundary condition, also called Robin boundary condition. It is called mixed because it involves both the derivative of the solution and the solution itself. An interpretation of this boundary condition can be obtained by differentiating the plane-wave solution for the electric field in (7) with respect to the spatial coordinate:

   $$\frac{\partial \boldsymbol{e}}{\partial x_1}(\boldsymbol{x}, t) = (ik_1)e_0\boldsymbol{i}_3\exp\big(i(k_1x_1 - \omega t)\big) = (ik_1)\boldsymbol{e}(\boldsymbol{x}, t). \tag{15}$$

   Thus, with a positive wavenumber, it is a condition satisfied by a plane wave propagating to the right.

   (a) Determine the exact solution. Interpret your result. Interpret why the mixed boundary condition in this BVP is often also called an absorbing boundary condition.

   (b) Implement the FE method to approximate the solution to this BVP using a mesh with elements of equal size h and the piecewise linear polynomials

   $$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & \text{for } x_{j-1} \le x \le x_j, \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & \text{for } x_j \le x \le x_{j+1}, \\ 0 & \text{otherwise} \end{cases}$$

as basis functions; here, $x_0 = 0, x_1, x_2, \ldots, x_{\mu_h} = 1$ denote the nodes of the mesh. Use appropriate Python functions to exploit the sparsity of the system matrix. Make your code sufficiently general so that the user can easily change the mesh spacing h. Carefully describe in your report how you proceeded in your implementation, including how you deal with the mixed boundary condition. Use your finite element implementation to investigate the approximation error as you refine the mesh.

Good luck!