MECA0010 “Uncertainty quantification and stochastic modeling” Assignment 1

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- This assignment is a numerical project that involves the application of a Markov chain Monte Carlo method to explore the solution to a probabilistic inverse problem.
- You may work on this assignment either alone or in a small group of up to 3 students. If you work in a small group of up to 3 students, only a single report must be turned in.
- At the latest on Wednesday 16 October, send Nicolas (n.leclere@uliege.be) an email with the composition of your group (1, 2 or 3 students).
- Document your work in a report with a length of at most 10 pages. Submit your report, along with your code, to the instructors (maarten.arnst@uliege.be, n.leclere@uliege.be) at the latest on Wednesday 6 November.
- A seismic source was activated at time $t = 0$ at an unknown location on the surface of the Earth. The seismic waves produced by the explosion were recorded by a network of six seismic stations whose coordinates in a rectangular system are

$$
\begin{aligned}
(x_1, y_1) &= (3\,\mathrm{km}, 15\,\mathrm{km}), & (x_2, y_2) &= (3\,\mathrm{km}, 16\,\mathrm{km}),\\
(x_3, y_3) &= (4\,\mathrm{km}, 15\,\mathrm{km}), & (x_4, y_4) &= (4\,\mathrm{km}, 16\,\mathrm{km}),\\
(x_5, y_5) &= (5\,\mathrm{km}, 15\,\mathrm{km}), & (x_6, y_6) &= (5\,\mathrm{km}, 16\,\mathrm{km}).
\end{aligned}
$$

  The observed arrival times of the seismic waves at these stations were

$$
\begin{aligned}
t_1^{\mathrm{obs}} &= 3.12\,\mathrm{s}, & t_2^{\mathrm{obs}} &= 3.26\,\mathrm{s},\\
t_3^{\mathrm{obs}} &= 2.98\,\mathrm{s}, & t_4^{\mathrm{obs}} &= 3.12\,\mathrm{s},\\
t_5^{\mathrm{obs}} &= 2.84\,\mathrm{s}, & t_6^{\mathrm{obs}} &= 2.98\,\mathrm{s}.
\end{aligned}
$$

  It is assumed that the potential impact of experimental uncertainties on these data can be modeled in terms of independent additive Gaussian random variables with a mean value of 0 and a standard deviation of $\sigma = 0.10\,\mathrm{s}$. It is assumed that the seismic waves travel at a velocity of $v = 5\,\mathrm{km/s}$. Finally, it is assumed that Earth's surface can be approximated as a flat surface and that the coordinates are Cartesian. With this problem setting, the goal of the exercice is to estimate the epicentral coordinates of the seismic source.

- The parameters to be identified are the coordinates of the epicenter of the seismic source:

$$\boldsymbol{\theta} = (x, y).$$

  The data are the arrival times at the seismic network:

$$\boldsymbol{x}^{\mathrm{obs}} = (t_1^{\mathrm{obs}}, \ldots, t_6^{\mathrm{obs}}).$$

  The coordinates of the seismic stations and the velocity of the seismic waves are assumed to be perfectly known.

  For a given (x, y), the arrival times of the seismic waves at the seismic stations are

$$t_j = \frac{1}{v}\sqrt{(x_j - x)^2 + (y_j - y)^2}, \quad j = 1, \ldots, 6.$$

Because there is no prior information on the epicentral coordinates, the prior probability density function in the Bayesian formulation of the inverse problem is taken to be a uniform probability density function:

$$\pi(x, y) \propto c,$$

with c a constant.

Because the experimental uncertainties are assumed to be representable in terms of independent additive Gaussian random variables with a mean value of 0 and a standard deviation of σ, the likelihood function in the Bayesian formulation of the inverse problem is given by

$$l(x, y) = \prod_{j=1}^{6} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(t_j - t_j^{\mathrm{obs}})^2}{2\sigma}\right) = \prod_{j=1}^{6} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\sqrt{(x_j - x)^2 + (y_j - y)^2}/v - t_j^{\mathrm{obs}})^2}{2\sigma^2}\right).$$

Thus, the posterior probability density function is

$$\pi(x, y | t_1^{\mathrm{obs}}, \ldots, t_6^{\mathrm{obs}}) \propto c \prod_{j=1}^{6} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\sqrt{(x_j - x)^2 + (y_j - y)^2}/v - t_j^{\mathrm{obs}})^2}{2\sigma^2}\right).$$

In the Bayesian formulation of the inverse problem, this posterior probability density function describes the state of knowledge we have about the epicentral coordinates as it follows from the prior information and the data.

- The assignment is then as follows:

  1. Please plot the posterior probability density function (up to a multiplicative constant) in the region of the plane where its probability mass is concentrated. You could use a contour plot or a 3D surface plot. You could consider the region $[0, 20\,\mathrm{km}] \times [0, 15\,\mathrm{km}]$. Interpret your results.
  2. Please implement a Markov chain Monte Carlo method to sample from the Bayesian posterior probability density function. Use the random-walk Metropolis–Hastings method. As the covariance matrix $[C]$ of the multivariate Gaussian probability density function involved in this method, use a covariance matrix of the form $[C] = (2.4^2/m)[\Sigma]$, with $m = 2$ the dimensionality and $[\Sigma]$ an approximation to the covariance matrix of the Bayesian posterior probability density function. Use a diagonal approximation $[\Sigma]$. Use your implementation to generate a trajectory $\{(x^{(\ell)}, y^{(\ell)}),\ \ell = 1, \ldots\}$ of a Markov chain that samples from the Bayesian posterior probability density function. Visualize your results in several insightful ways. You could consider plotting the trajectory on top of a contour plot of the Bayesian posterior probability density function. You could consider making plots of $x^{(\ell)}$ and $y^{(\ell)}$ as a function of the index ℓ, plotting histograms of the values $\{x^{(\ell)},\ \ell = 1, \ldots\}$ and $\{y^{(\ell)},\ \ell = 1, \ldots\}$, plotting estimates $\overline{x}^{(\ell)} = \frac{1}{\ell}\sum_{k=1}^{\ell} x^{(k)}$ and $\overline{y}^{(\ell)} = \frac{1}{\ell}\sum_{k=1}^{\ell} y^{(k)}$ of the posterior mean values and estimates $\sigma_X^{(\ell)} = \sqrt{\frac{1}{\ell}\sum_{k=1}^{\ell}(x^{(k)} - \overline{x}^{(\ell)})^2}$ and $\sigma_Y^{(\ell)} = \sqrt{\frac{1}{\ell}\sum_{k=1}^{\ell}(y^{(k)} - \overline{y}^{(\ell)})^2}$ of the posterior standard deviation as a function of the index ℓ, and so on.
  3. Please repeat your work under 1 and 2, using, this time, however, a value of $\sigma = 0.25\,\mathrm{s}$ as the standard deviation representing the significance of the experimental uncertainties. Interpret your results.
  4. Formulate yourself an additional question that adds to the previous questions and suscites your interest. Provide an answer to your question. (An example could be exploring what happens if sensor positions are changed. Another example could be exploring a more advanced MCMC algorithm that you may have found in a reference. Etc.). Discuss at least once about the additional question with one of the instructors.

Good luck!