Introduction aux signaux et systèmes
Two-dimensional closed systems
Lecture 2

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Contents

Why going beyond one dimension

One-dimensional closed systems cannot oscillate

Consider the **one** dimensional closed system $\dot{x} = f(x)$. When passing through any point x in the state space, the system's trajectory is either increasing ($f(x) > 0$), decreasing ($f(x) < 0$), or at rest ($f(x) = 0$).

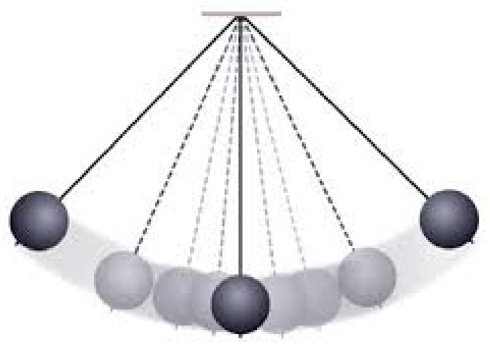
For oscillations to occur, the system's trajectory would have to pass through the same state x some times with positive speed ($f(x) > 0$) and other times with negative speed ($f(x) < 0$).

This is impossible. Indeed:

Fact: Trajectories of one-dimensional closed systems are monotone

Any trajectory of a one-dimensional closed system either increases or decreases asymptotically toward an equilibrium point or diverge monotonically to $\pm\infty$.

Even the simplest systems oscillate!



$\Rightarrow$ Need to consider systems with $n > 1$-dimensional state-space.

Two-dimensional closed systems

Two-dimensional vector fields

Consider the two-dimensional closed system (of ordinary differential equations)

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

It associates to any point $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ the vector

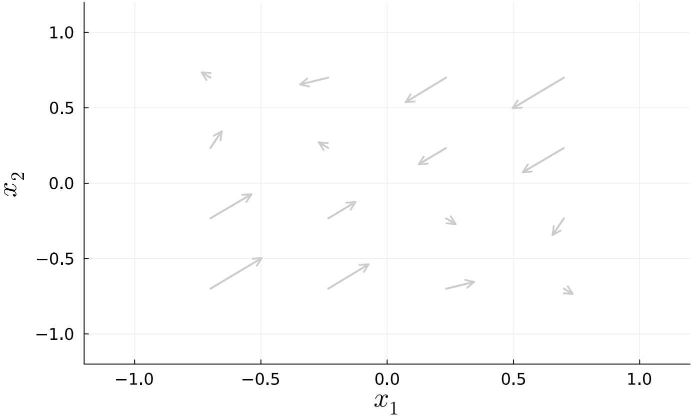
$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = f(x)$$

that dictates in which direction (increasing, decreasing, rest) and with which speed each of two states x_1 and x_2 is changing.

Two-dimensional vector fields

Example: Mutual inhibition

$$\dot{x}_1 = -x_1 - \tanh(2x_2)$$
$$\dot{x}_2 = -x_2 - \tanh(2x_1)$$



The "flow" (trajectories) of two-dimensional vector fields

Example: Mutual inhibition

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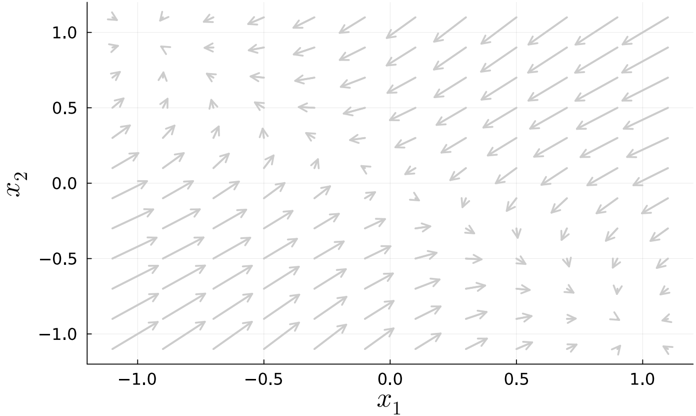

The "flow" (trajectories) of two-dimensional vector fields

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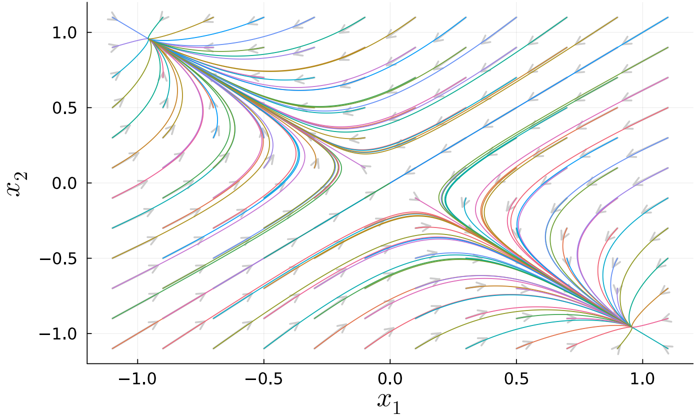
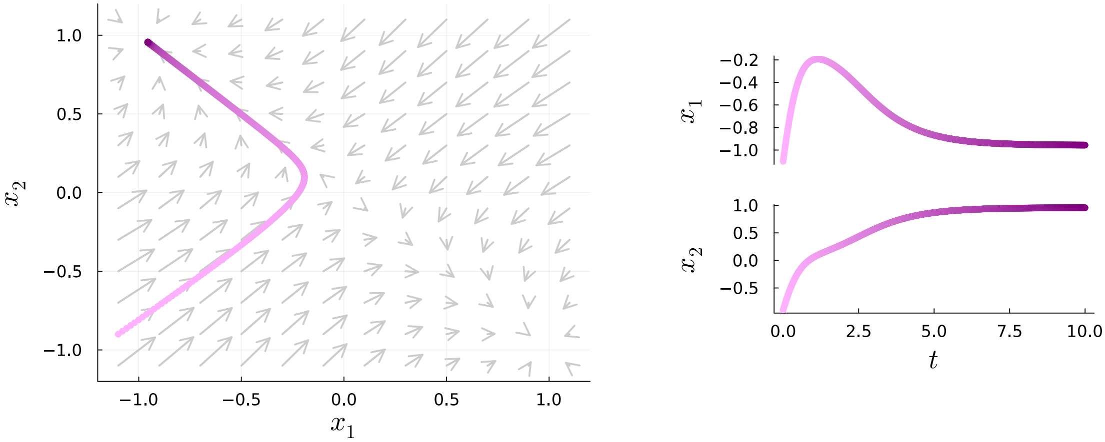


The "flow" (trajectories) of two-dimensional vector fields



The "flow" (trajectories) of two-dimensional vector fields

Fact: trajectories cannot intersect

Since the speed vector $\dot{x} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$ at any state-space point (x_1, x_2) is uniquely determined, there is one and only one trajectory passing through (x_1, x_2).

Nullclines and equilibria of two-dimensional vector fields

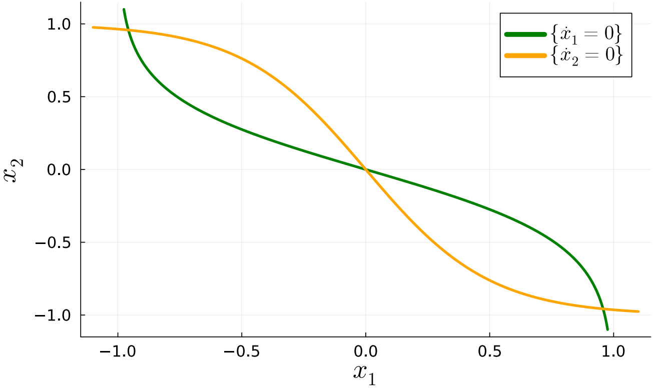
Nullclines: Multi-dimensional generalizations of equilibria

Definition: nullclines

The nullcline of variable x_i or x_i-nullcline is the set of state-space points where $\dot{x}_i = 0$:

$$\{(x_1, x_2) \ : \ \dot{x}_i = f_i(x_1, x_2) = 0\}$$

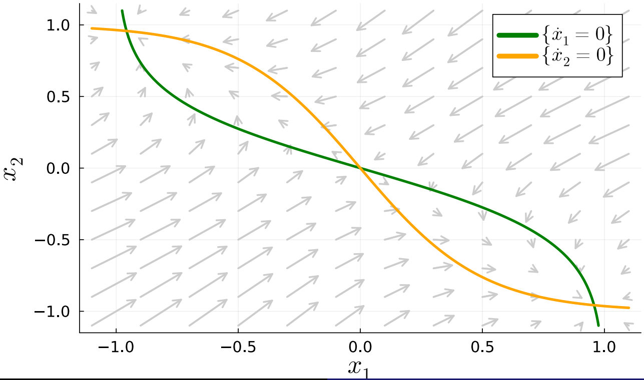
i.e., the states where variable x_i is (momentarily) at rest.



Nullclines and state-space analysis

Nullclines partition the state space in disconnected regions. Inside each region the sign of $\dot{x}_1$ and $\dot{x}_2$, hence the "direction" of the vector field, do not change.

This provides a qualitative/geometric viewpoint to understand and predicts the system's trajectories.



Equilibria of two-dimensional systems

Definition: Equilibrium points in two-dimension

A state $x^* = (x_1^*, x_2^*)$ is an equilibrium of the system

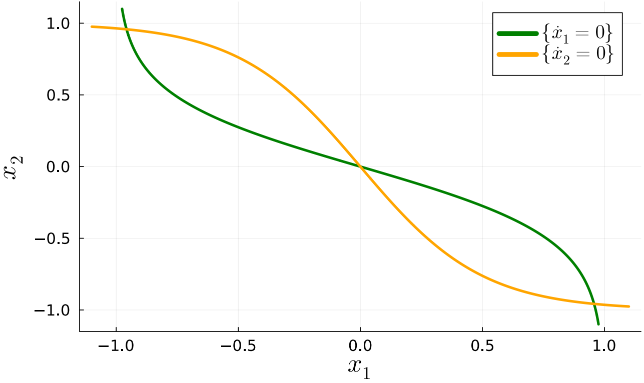
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

if both x_1 and x_2 are at rest at x^*, that is, if

$$f_1(x_1^*, x_2^*) = f_2(x_1^*, x_2^*) = 0$$

Equilibria of two-dimensional systems

Hence, equilibrium points of two-dimensional systems are exactly the points where the two nullclines intersect.



Stability of equilibria in two dimensions

Let $x^* = (x_1^*, x_2^*)$ be an equilibrium of the system $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$

Definition: Stable and unstable equilibria

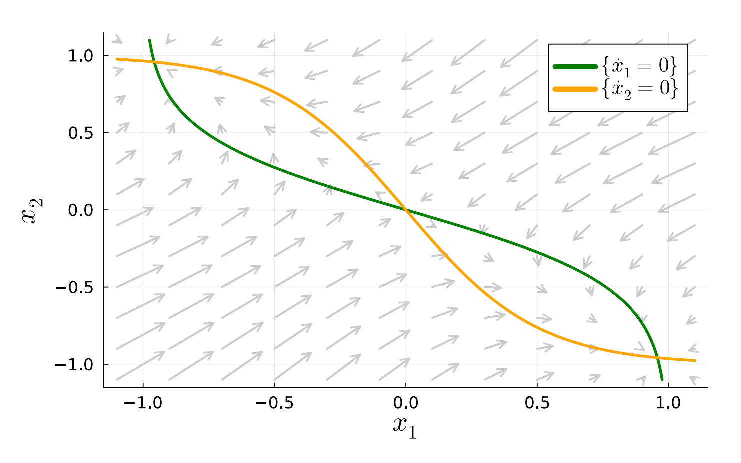
The equilibrium x^* is **stable** if:

1. it attracts all nearby trajectories, i.e., the trajectory $x(t)$ converges to x^* for all initial conditions $x(0)$ sufficiently close to x^*;
2. trajectories that start sufficiently close to x^* remain close to it for all time (*Lyapunov stability*)

An equilibrium is **unstable** if it is not stable.

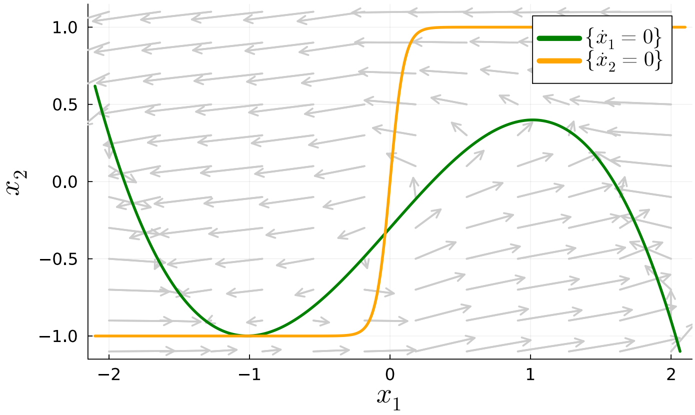
In one dimension 1 $\Rightarrow$ 2 . However, the same is not true in dimension two and above.

Example1



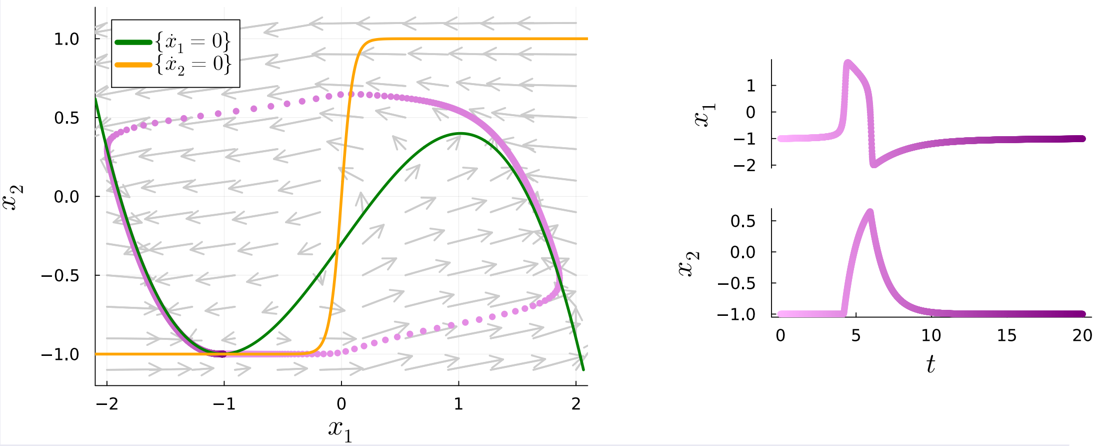
Stability of equilibria in two dimensions

Example 2: An attractive but not Lyapunov stable equilibrium



Stability of equilibria in two dimensions

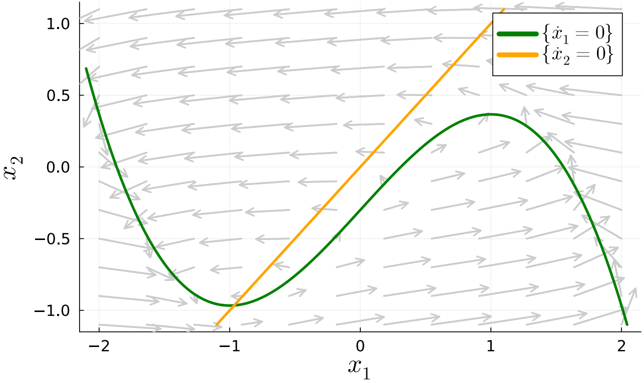
Example 2: An attractive but not Lyapunov stable equilibrium



Nullclines and state-space analysis: limitations

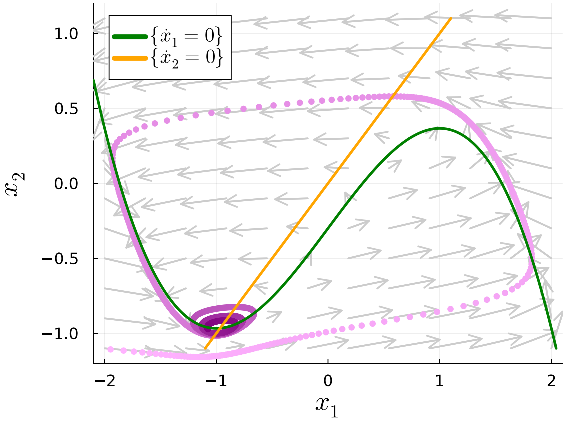
The purely qualitative approach of nullcline analysis has major limitations. For instance:

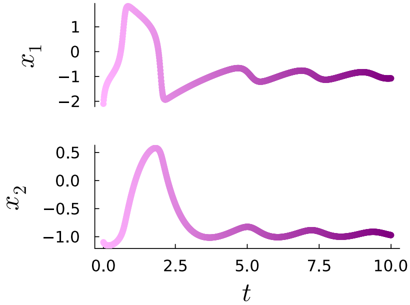
Is the equilibrium of this two-dimensional system stable or unstable?



Nullclines and state-space analysis: limitations

In hindsight, by inspecting the system behavior, we can guess it is stable but nullcline analysis falls short in explaining or predicting this observation...





Linearization of two-dimensional vector fields

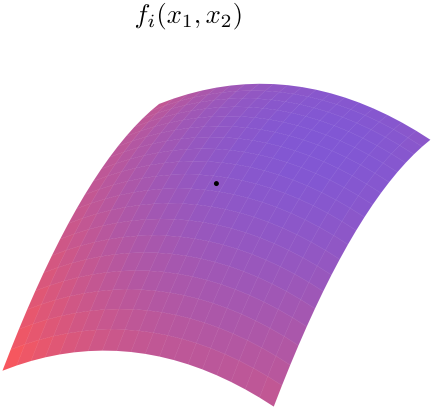
First-order Taylor expansion of two-dimensional vector fields

Similarly to what we did in one dimension, given an equilibrium $x^* = (x_1^*, x_2^*)$ of a two-dimensional vector field

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = f(x)$$

we can find a linear function that approximates f around x^* using Taylor expansion methods.

$\Rightarrow$ In other words, we will approximate $f_i(x_1, x_2)$, $i = 1, 2$, by its **tangent plane** at x^*.



First-order Taylor expansion of two-dimensional vector fields

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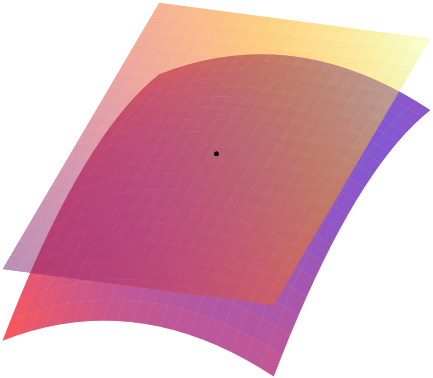
$\Rightarrow$ In other words, we will approximate $f_i(x_1, x_2)$, $i = 1, 2$, by its **tangent plane** at x^*.

$$f_i(x_1, x_2) \approx a_{i1}\,(x_1 - x_1^*) + a_{i2}\,(x_2 - x_2^*)$$



First-order Taylor expansion of two-dimensional vector fields

More rigorously, let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = x - x^*$ be the state of the system relative to the equilibrium x^*. As in the one-dimensional case, in the new coordinates η the system becomes

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x} = f(x) = f(x^* + \eta)$$

For x sufficiently close to x^*, that is, for $\|\eta\|$ sufficiently small[1], using the Taylor expansion we obtain

$$\dot{\eta} = f(x^* + \eta) = \underbrace{f(x^*)}_{\text{Order } 0} + \underbrace{\left.\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x=x^*} \eta}_{\text{Order } 1} + \underbrace{\mathcal{O}(\eta_1^2, \eta_2^2, \eta_1\eta_2)}_{\text{Higher orders}}$$

[1]The Euclidean norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$

The linearized two dimensional dynamics

Neglecting higher order terms for small enough $\|\eta\|$, we obtain the linearized two-dimensional dynamics

$$\dot{\eta} = J_{x^*}\eta, \qquad J_{x^*} = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x=x^*}$$

The matrix J_{x^*} is called the **Jacobian** of the system at x^*. Its entries, i.e., the partial derivatives $\frac{\partial f_i}{\partial x_j}$, $i, j = 1, 2$, fully determine the behavior of the system's trajectories close to x^*.

Linearization-based classification of two-dimensional equilibria

Solutions of two-dimensional linear systems

Consider the linear two-dimensional system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2, \ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Let $\tau = \mathrm{trace}(A) = a_{11} + a_{22}$ be the trace of the matrix A and $\Delta = \det(A) = a_{11}a_{22} - a_{21}a_{12}$ be its determinant. The eigenvalues λ_1, λ_2 of A are given by

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

Let v_1, v_2 be the associated eigenvectors.

- If $\tau^2 > 4\Delta$ eigenvalues and eigenvectors are real.
- If $\tau^2 < 4\Delta$ eigenvalues and eigenvectors are complex and, in particular $\lambda_2 = \bar{\lambda}_1$ and $v_2 = \bar{v}_1$. [2]

[2] $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}^n$.

Solutions of two-dimensional linear systems

Given initial conditions $x(0)$, the solution of the system is

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2, \qquad c_1 = \langle x(0), v_1 \rangle,\ c_2 = \langle x(0), v_2 \rangle$$ [3]

In other words, the solution can be split into a component $c_1 e^{\lambda_1 t} v_1$ that evolves along the first eigenvector v_1 and a component $c_2 e^{\lambda_2 t} v_2$ that evolves along the second eigenvector v_2.

Depending on whether eigenvalues and eigenvectors are real or imaginary the solution can exhibit drastically distinct behaviors.

[3] The scalar product $\langle x, z \rangle$ of two vectors $x, z \in \mathbb{R}^n$ is $\langle x, z \rangle = \sum_{i=1}^{n} x_i z_i$.

Solutions of two-dimensional linear systems: real case

When $\tau^2 > 4\Delta$ and therefore $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^2$, the solution

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2, \qquad c_1 = \langle x(0), v_1 \rangle,\ c_2 = \langle x(0), v_2 \rangle$$

has a simple geometric interpretation.

Solutions of two-dimensional linear systems: real case

In the case of real eigenvalues and eigenvectors there are three generic distinct cases

$\lambda_1 < \lambda_2 < 0$ $\qquad$ $\lambda_1 < 0 < \lambda_2$ $\qquad$ $0 < \lambda_1 < \lambda_2$

Solutions of two-dimensional linear systems: imaginary case

When $\tau^2 < 4\Delta$ and therefore $\lambda_1 = \bar{\lambda}_2 = \lambda \in \mathbb{C}$ and $v_1 = \bar{v}_2 = v \in \mathbb{C}^2$, to understand the behavior of the solution

$$x(t) = c_1 e^{(\sigma + \jmath\omega)t} v + c_2 e^{(\sigma - \jmath\omega)t} \bar{v}, \qquad c_1 = \langle x(0), v_1 \rangle,\ c_2 = \langle x(0), v_2 \rangle = \bar{c}_1$$

let $\lambda = \sigma + \jmath\omega$, $\sigma, \omega \in \mathbb{R}$, $v = \nu + \jmath\mu$, $\nu, \mu \in \mathbb{R}^2$, and observe that

$$c_2 e^{(\sigma - \jmath\omega)t} \bar{v} = \overline{c_1 e^{(\sigma + \jmath\omega)t} v}.$$

It follows that

$$x(t) = \Re\left(c_1 e^{(\sigma+\jmath\omega)t} v\right) = |c_1| e^{\sigma t} \Re\left(e^{\jmath\omega t + \angle c_1}(\nu + \jmath\mu)\right) = |c_1| e^{\sigma t}\big(\cos(\omega t + \angle c_1)\nu + \sin(\omega t + \angle c_1)\mu\big)$$

That is, $x(t)$ is the sum of two trajectories spiraling away from $(\sigma > 0)$ or toward to $(\sigma < 0)$ the origin and oscillating with the same frequency ω but $\pi/2$ phase difference along the eigenvector real and imaginary part, respectively.

Solutions of two-dimensional linear systems: imaginary case

In the case of imaginary eigenvalues and eigenvectors there are two generic distinct cases

$\sigma > 0$ $\qquad\qquad$ $\sigma < 0$

Summary of two-dimensional linear dynamical behavior