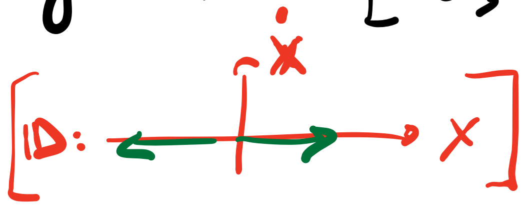
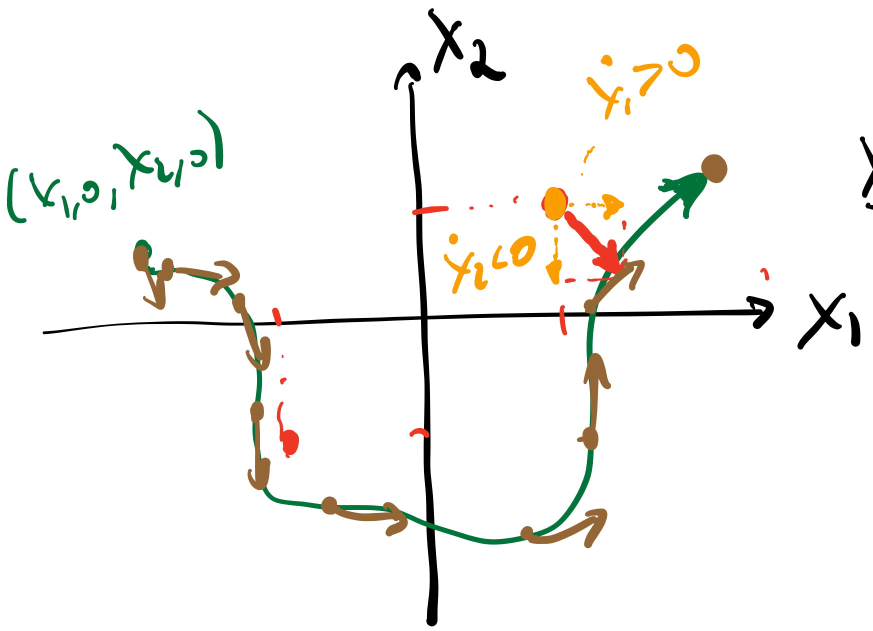
SYS0002 - LECTURE #2 $\underbrace{\text{ESPACE D'ÉTAT}}_{\text{ODE}}$ EN $\underbrace{\text{2D}}_{\text{2 VARIABLES}}$.

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \quad \underline{\text{OU}} \quad \dot{\underline{x}} = f(\underline{x}), \; \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; \dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$



↳ PORTRAIT DE PHASE



$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ POSITION DANS LE PLAN.

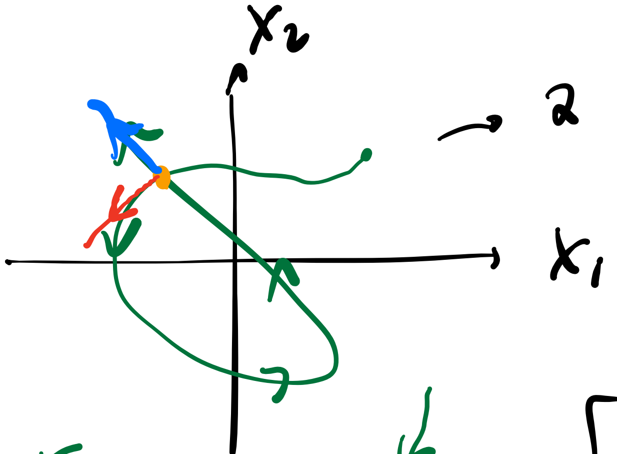
$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ "VECTEUR VITESSE"

↳ CHAMP DE VECTEUR : ON ASSOCIE UN VECTEUR "VITESSE" $\dot{\underline{x}}$ A CHAQUE POSITION $\underline{x}$.

→ TRAJECTOIRE ≡ "FLUX" LE LONG DU CHAMP DE VECTEUR. ⇒ SOLUTION.

① TOUT LE PORTRAIT DE PHASE EST REMPLI DE TRAJECTOIRES.

② TRAJECTOIRES NE SE CROISENT JAMAIS!



→ 2 VECTEURS VITESSE ≠ AU MÊME POINT.

⇓

IMPOSSIBLE! CAR

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{bmatrix}.$$

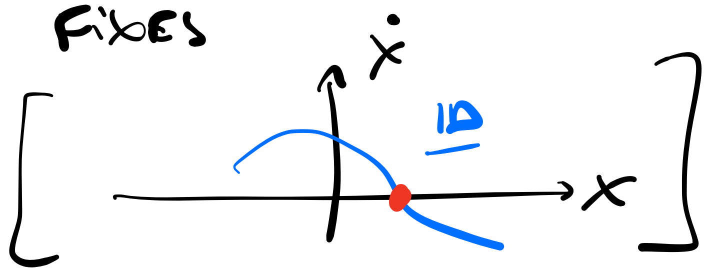
POINT FIXE.

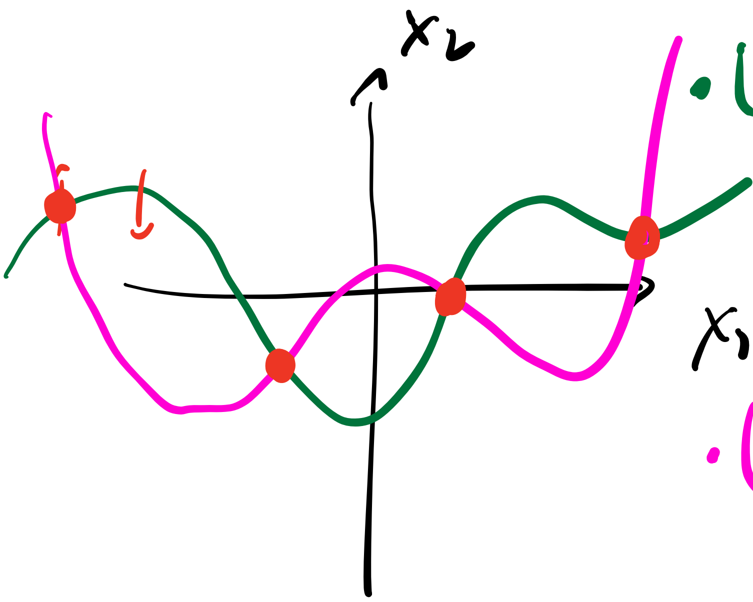
↳ STABILISATION SUR POINT FIXE.

[OU DIVERGENCE VERS ±∞]

① DÉTECTER LES POINTS FIXES

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) = 0 \\ \dot{x}_2 = f_2(x_1, x_2) = 0 \end{cases}$$





• $(\tilde{x}_1, \tilde{x}_2)$ t.q. $\dot{x}_1 = f_1(x_1, x_2) = 0$

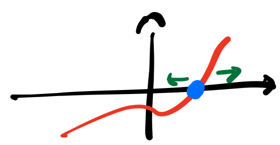
- ISOCLINE
- NULLCLINE - x_1

• $(\tilde{x}_1, \tilde{x}_2)$ t.q. $\dot{x}_2 = f_2(x_1, x_2) = 0$

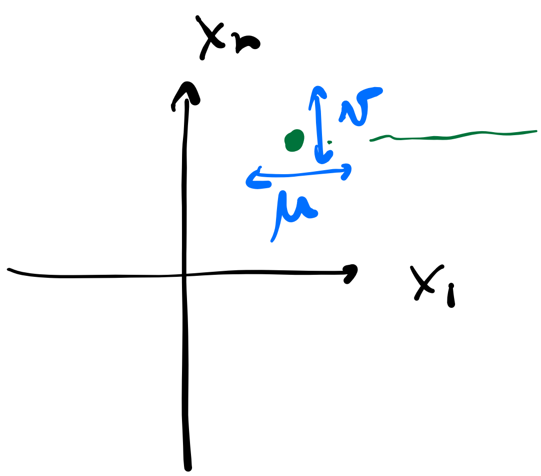
- NULLCLINE - x_2

↳ POINTS FIXES : INTERSECTIONS DES NULLCLINES.

② STABILITÉ DES POINTS FIXES ?



↳ PETITES PERTURBATIONS EN 2D



$(\bar{x}_1, \bar{x}_2)$: POINT FIXE

$$\begin{cases} \mu(t) = x_1(t) - \bar{x}_1 \\ \nu(t) = x_2(t) - \bar{x}_2 \end{cases}$$

$$\begin{cases} \dot{\mu} = \dot{x}_1 = f_1(\bar{x}_1+\mu, \bar{x}_2+\nu) \\ \dot{\nu} = \dot{x}_2 = f_2(\bar{x}_1+\mu, \bar{x}_2+\nu) \end{cases}$$

TAYLOR:

$$\dot{\mu} = f_1(\bar{x}_1+\mu, \bar{x}_2+\nu)$$

$$\approx \underbrace{\cancel{f_1(\bar{x}_1, \bar{x}_2)}}_{=0} + \mu \cdot \frac{\partial f_1}{\partial x_1}\Big|_{(\bar{x}_1, \bar{x}_2)}$$

$$+ \nu \frac{\partial f_1}{\partial x_2}\Big|_{(\bar{x}_1, \bar{x}_2)} + \cancel{O(\mu^2, \nu^2, \mu\cdot\nu \ldots)}$$

NEGLIGEABLE SI

$$\frac{\partial f_1}{\partial x_1}\Big|_{(\bar{x}_1, \bar{x}_2)} \neq 0$$

OU $$\frac{\partial f_1}{\partial x_2}\Big|_{(\bar{x}_1, \bar{x}_2)} \neq 0.$$

$$\dot{\nu} = f_2(\bar{x}_1 + \mu, \bar{x}_2 + \nu)$$

$$\simeq \underbrace{\cancel{f_2(\bar{x}_1, \bar{x}_2)}}_{=0} + \mu \cdot \left.\frac{\partial f_2}{\partial x_1}\right|_{(\bar{x}_1, \bar{x}_2)} + \nu \left.\frac{\partial f_2}{\partial x_2}\right|_{(\bar{x}_1, \bar{x}_2)} \cancel{+ O(\nu^2, \mu^2, \mu \cdot \nu, \ldots)}$$

$$\hookrightarrow \begin{cases} \dot{\mu} \simeq \mu \cdot \left.\dfrac{\partial f_1}{\partial x_1}\right|_{(\bar{x}_1, \bar{x}_2)} + \nu \cdot \left.\dfrac{\partial f_1}{\partial x_2}\right|_{(\bar{x}_1, \bar{x}_2)} \\ \dot{\nu} \simeq \mu \cdot \left.\dfrac{\partial f_2}{\partial x_1}\right|_{(\bar{x}_1, \bar{x}_2)} + \nu \cdot \left.\dfrac{\partial f_2}{\partial x_2}\right|_{(\bar{x}_1, \bar{x}_2)} \end{cases}$$

$$\begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix} = \left.\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{(\bar{x}_1, \bar{x}_2)} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

MATRICE JACOBIENNE ÉVALUÉE AU POINT FIXE $(\bar{x}_1, \bar{x}_2)$

[1D $\equiv$ $\dot{\eta} = a \cdot \eta$, $a : \left.\dfrac{\partial f}{\partial x}\right|_{\bar{x}}$ PENTE DE $f(x)$ EN $\bar{x}$]

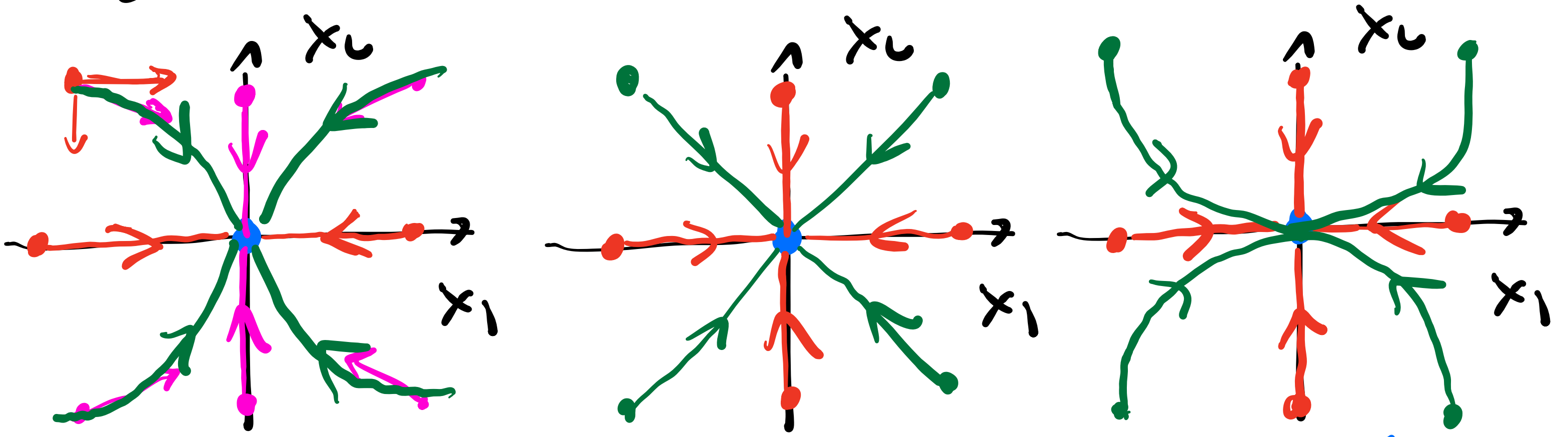
$$\hookrightarrow \begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

COMMENT CARACTÉRISER LA STABILITÉ D'UN POINT FIXE À PARTIR DE $J|_{(\bar{x}_1, \bar{x}_2)}$?

$\rightarrow \begin{cases} \dot{x}_1 = a \cdot x_1 \\ \dot{x}_2 = -x_2 \end{cases} \xrightarrow{1D} \begin{cases} x_1(t) = x_{1,0} \cdot e^{at} \\ x_2(t) = x_{2,0} \cdot e^{-t} \end{cases}$

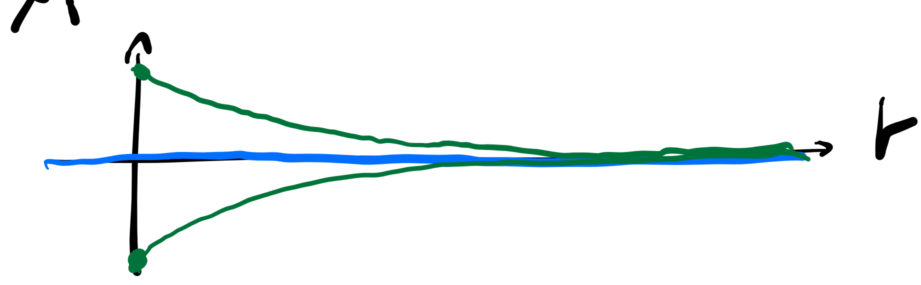
① POINT FIXE ? $(\bar{x}_1, \bar{x}_2) = (0,0)$

$a < -1$ $\quad$ $a = -1$ $\quad$ $-1 < a < 0$

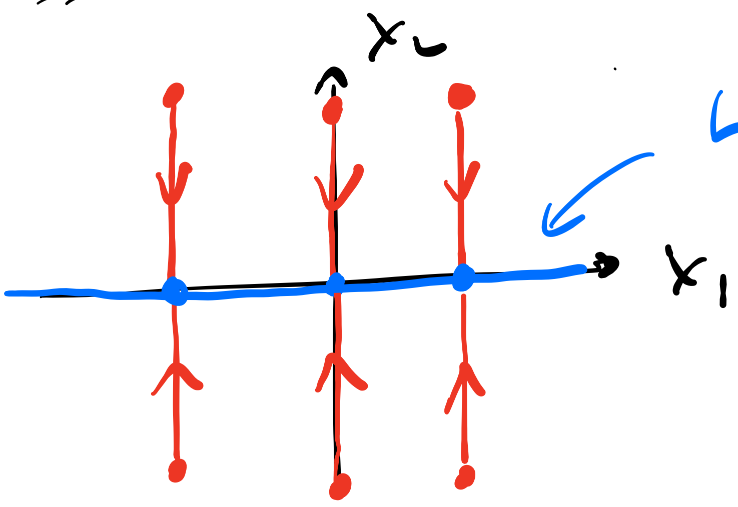


STABLE. $\quad$ STABLE $\quad$ STABLE.

NOEUD STABLE.



$\underline{a=0}$ $\quad x_1(t) = x_{1,0} \cdot e^{at} = x_{1,0}.$

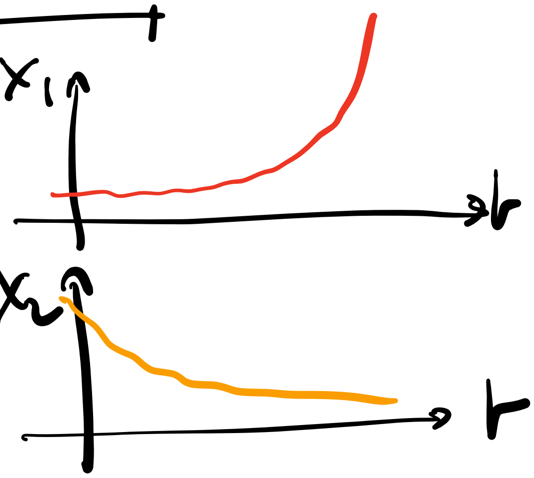


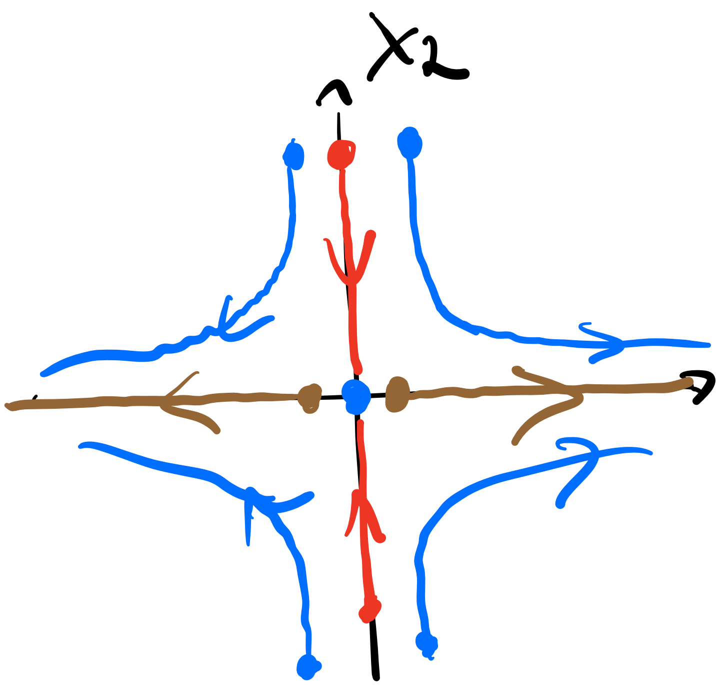
LIGNE DE POINTS FIXES STABLES.

$\underline{a>0}$

$x_1(t) = x_{1,0} \cdot e^{at}$

$x_2(t) = x_{2,0} \cdot e^{-t}$





$\dot{x}_1 = a x_1$

$a > 0$

ATTRACTIF SELON x_2

REPULSIF SELON x_1

↳ POINT DE SELLE.

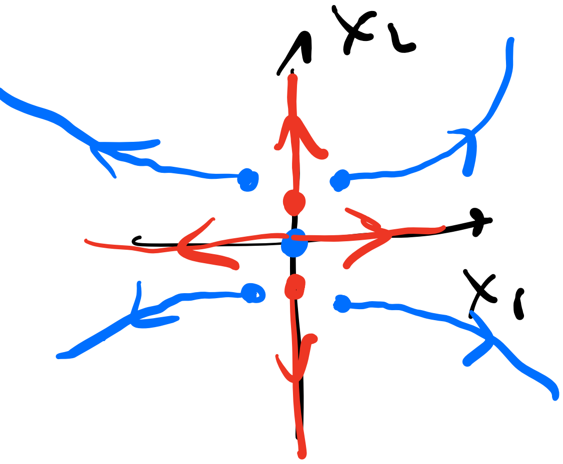
INSTABLE MAIS ATTIRE TRANSITOIREMENT LES TRAJECTOIRES.

$$x_1(t) = x_{1,0} \cdot e^{at} \qquad a > 0$$

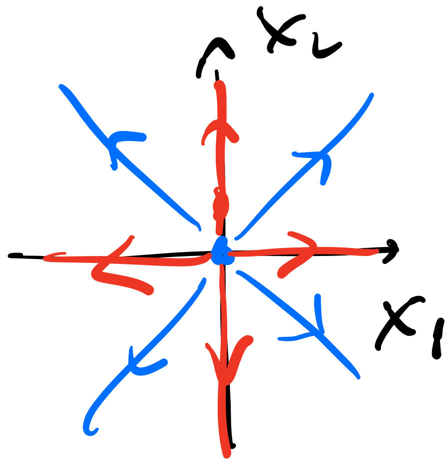
$$x_2(t) = x_{2,0} \cdot e^{t}$$

$$\begin{cases} \dot{x}_1 = a x_1 \\ \dot{x}_2 = x_2 \end{cases}$$

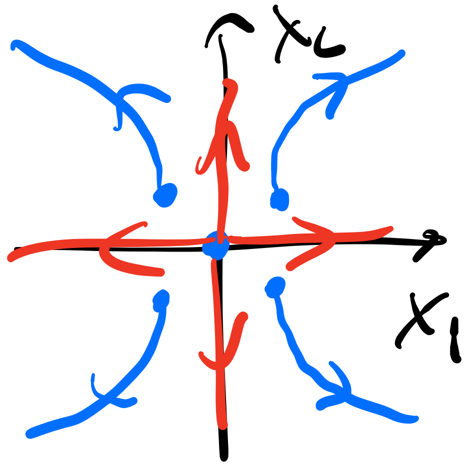
$a > 1$



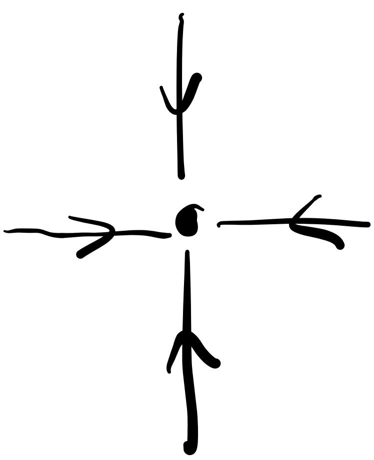
$a = 1$



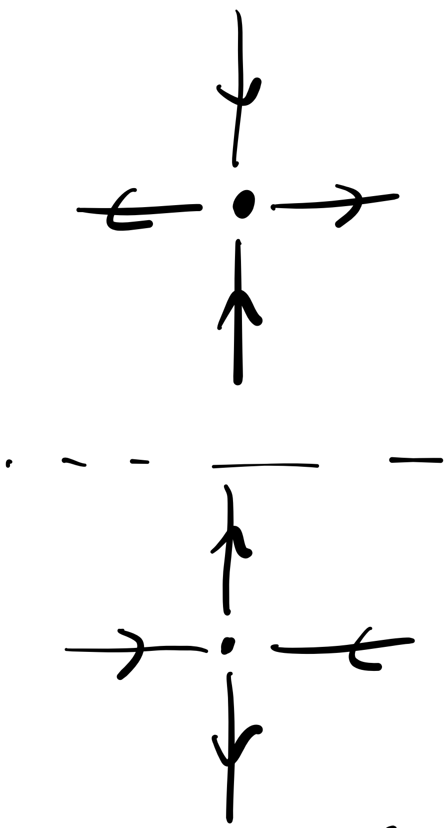
$0 < a < 1$



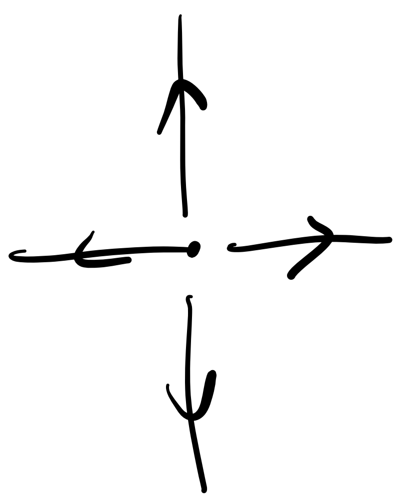
↳ NOEUD INSTABLE.



NOEUD STABLE



POINT DE SELLE



NOEUD INSTABLE.

$\hookrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

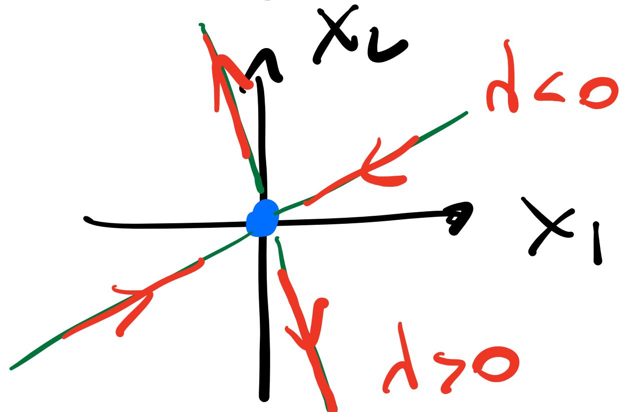
RAPPEL : $\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{(\bar{x}_1, \bar{x}_2)}$

→ COUPLÉ : ① TRAJECTOIRES RECTILIGNES ?
② EVOLUTION TEMPORELLE SUR CHAQUE TRAJECTOIRE ?

① $J|_{(\bar{x}_1, \bar{x}_2)} = A$ → VECTEURS PROPRES DE A
→ TRAJECTOIRES RECTILIGNES

② EVOLUTION TEMPORELLE SUR CHAQUE V.P.
↳ VALEUR PROPRE ASSOCIÉE.

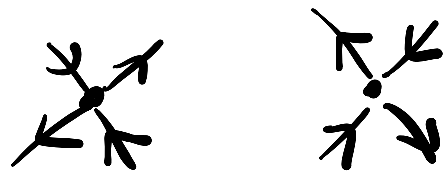
$\lambda < 0 \rightarrow e^{\lambda t} \rightarrow$ ATTRACTIF
$\lambda > 0 \rightarrow e^{\lambda t} \rightarrow$ REPULSIF



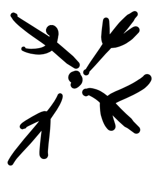
$\hookrightarrow \lambda_1, \lambda_2$ V.P. $J|_{(\bar{x}_1, \bar{x}_2)}$

① $\lambda_1 > 0, \lambda_2 > 0$: NOEUD INSTABLE

② $\lambda_1 > 0, \lambda_2 < 0$
$\lambda_1 < 0, \lambda_2 > 0$ POINT DE SELLE



③ $\lambda_1 < 0, \lambda_2 < 0$. NOEUD STABLE



Ex :
$$\begin{cases} \dot{x}_1 = x_1 + 5x_2 \\ \dot{x}_2 = -x_1 + 3x_2 \end{cases} \qquad \dot{\underline{x}} = A \cdot \underline{x}$$

$$A = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$$

↳ ① POINT FIXE : $(\bar{x}_1, \bar{x}_2) = (0,0)$

② VALEURS PROPRES DE A

$$\tau = 1 + 3 = 4$$
$$\Delta = 3 + 5 = 8$$

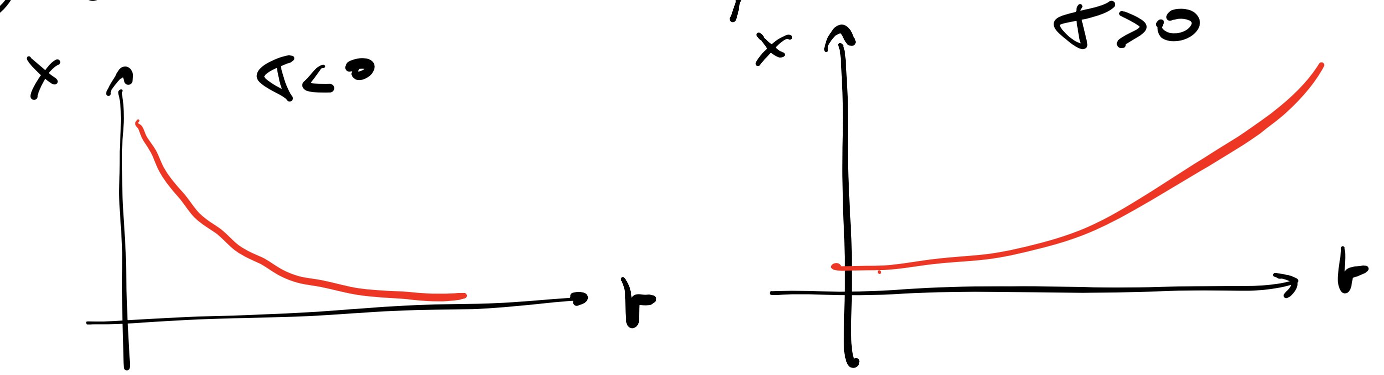
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$
$$= \frac{4 \pm \sqrt{16 - 4.8}}{2}$$
$$= \frac{4 \pm \sqrt{-16}}{2}$$
$$= \frac{4 \pm 4.j}{2} = 2 \pm 2j$$

$$\boxed{j = \sqrt{-1}}$$

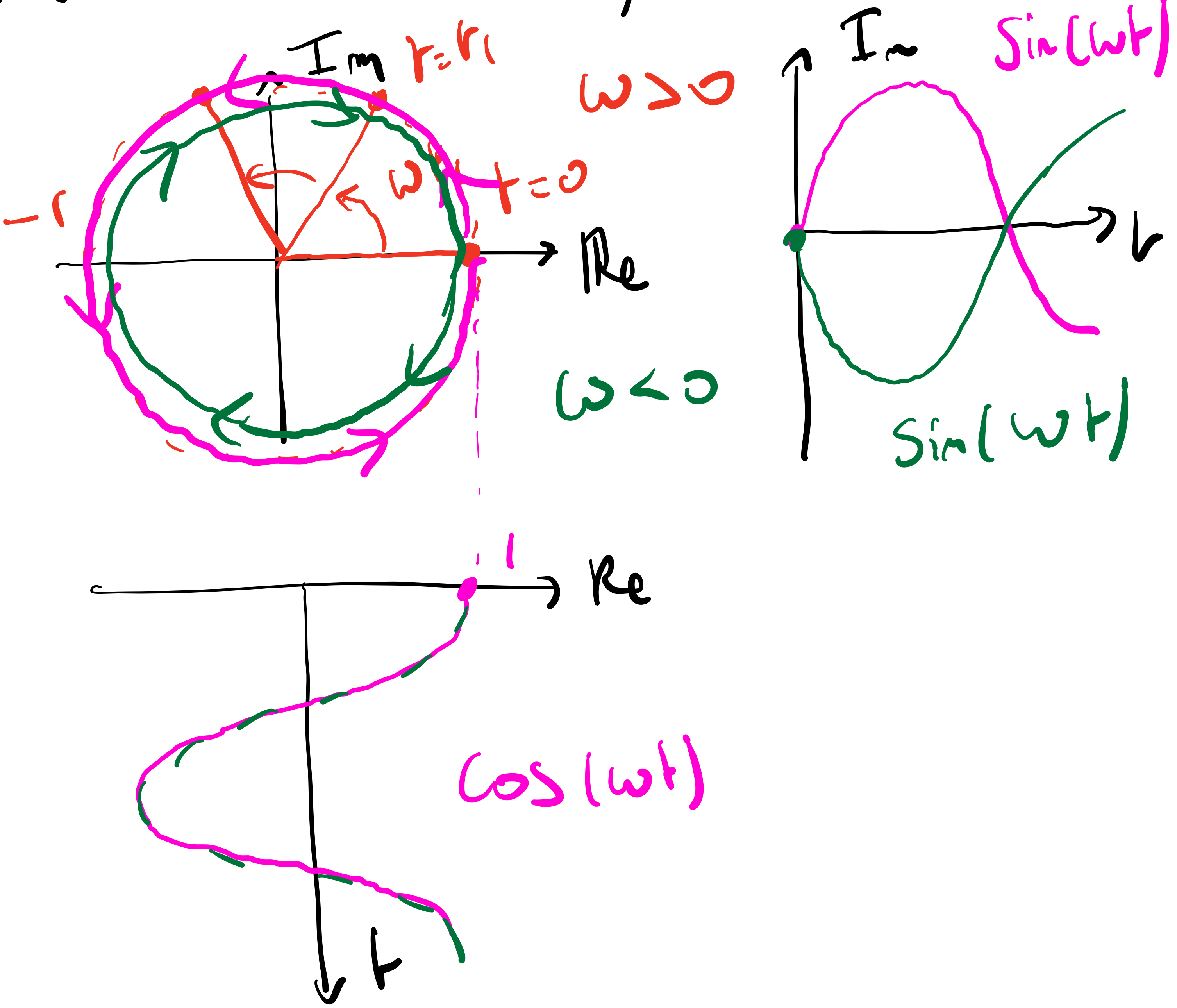
COMPLEXES CONJUGUÉES !

$\hookrightarrow \quad x(t) = X \cdot e^{\lambda t} \quad ; \quad \lambda = \sigma + j\omega$

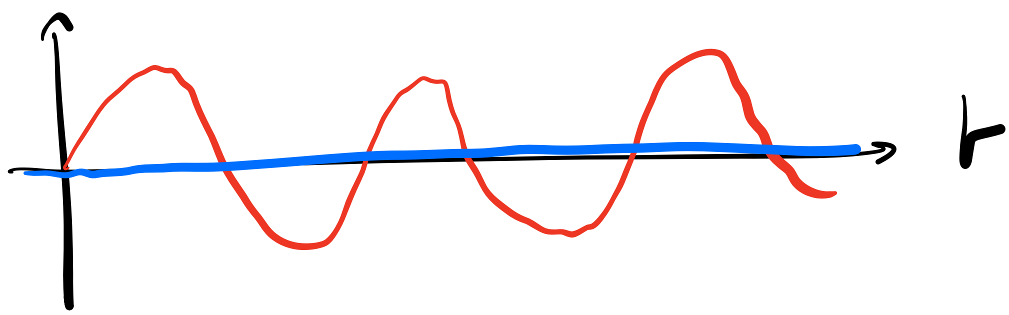
Ⓐ $\omega = 0 \longrightarrow x(t) = e^{\sigma t} \quad ; \quad \sigma \in \mathbb{R}$



Ⓑ $\sigma = 0 \longrightarrow x(t) = e^{j\omega t} \quad ; \quad \omega \in \mathbb{R}$



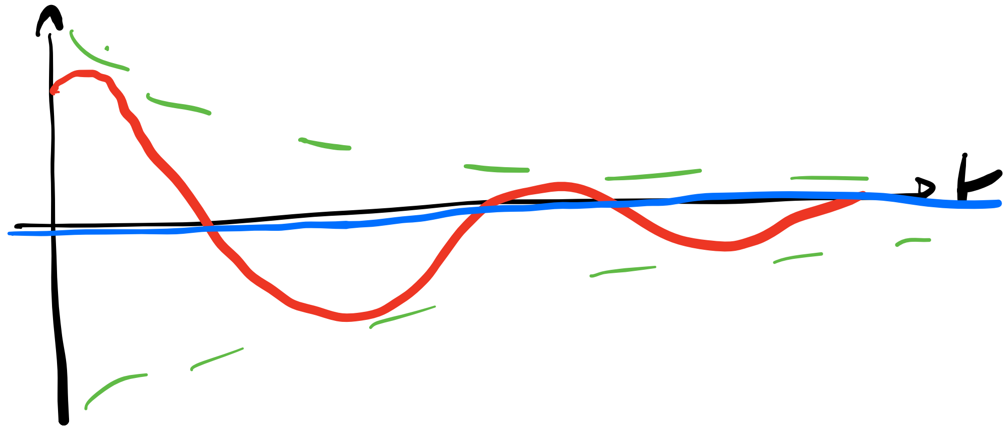
↳



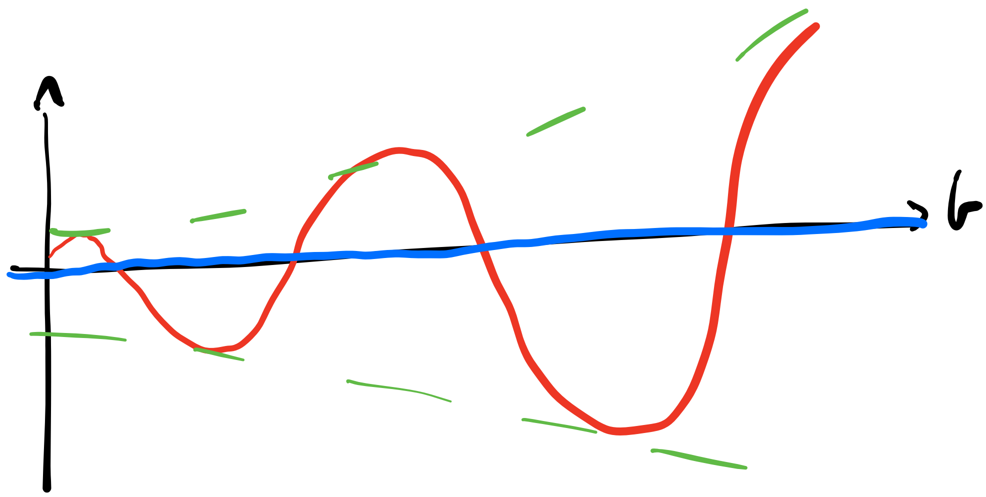
Ⓒ $\sigma \neq 0, \ \omega \neq 0$ $\quad x(t) = e^{\lambda t} = e^{(\sigma + j\omega)t}$

$= e^{\sigma t} \cdot e^{j\omega t}$

↳ $\sigma < 0$



↳ $\sigma > 0$



$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \longrightarrow \begin{array}{l} (\bar{x}_1, \bar{x}_2) \text{ POINT FIXE} \\ f_1(\bar{x}_1, \bar{x}_2) = f_2(\bar{x}_1, \bar{x}_2) = 0 \end{array}$$

↳ LINEARISATION AUTOUR DE $(\bar{x}_1, \bar{x}_2)$:

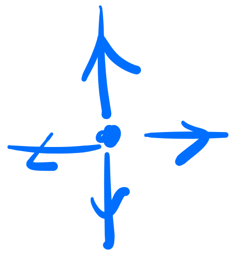
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

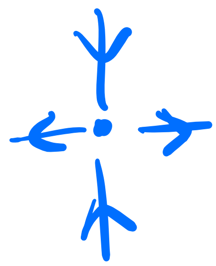
$$A \equiv J\Big|_{(\bar{x}_1, \bar{x}_2)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{(\bar{x}_1, \bar{x}_2)}$$

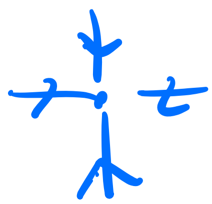
→ STABILITÉ / TYPE DE POINT FIXE

↳ λ_1, λ_2 : VALEURS PROPRES DE $J|_{(\bar{x}_1, \bar{x}_2)}$

Ⓐ λ_1, λ_2 RÉELES.

- $\lambda_1, \lambda_2 > 0$ →  NOEUD INSTABLE

- $\lambda_1 > 0, \lambda_2 < 0$
  $\lambda_1 < 0, \lambda_2 > 0$ →  POINT DE SELLE

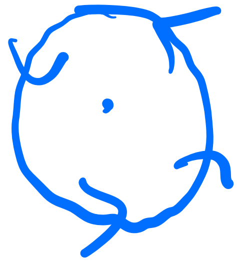
- $\lambda_1, \lambda_2 < 0$ →  NOEUD STABLE

Ⓑ λ_1, λ_2 COMPLEXES CONJUGUÉS

$$\lambda_1, \lambda_2 = \sigma \pm j\omega$$

- $\sigma > 0$ →  SPIRALE INSTABLE.

- $\sigma < 0$ →  SPIRALE STABLE.

$\left[\ \cdot\ \nabla = 0 \longrightarrow$  CENTRE $\right]$