INTRODUCTION TO STRING THEORY: ASSIGNMENT 2

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1. Determination of the Path integral for Matrix Action S

We seek to determine the following integral:

$$\int dX_1 dX_2 ... e^{-S} \tag{1}$$

where,

$$S = \sum_{i,j} M_{ij} X_i X_j + \sum_i Q_i X_i \tag{2}$$

We can say that M_{ij} is a symmetric operator since any asymmetric components would vanish upon integration. In order to evaluate this integral it is necessary to make a transformation and bring it into the form of a Gaussian times another factor. We also assume M_{ij} is invertible and define its inverse M^{-1}_{jk}.

The first step in solving this problem requires us to rewrite the matrix S by completing the square:

$$S = \sum_i \beta(x_i - \alpha_i)^2 + G \tag{3}$$

where α_i and β are just constants. Noting that the minimum value of S will be when $x_i = \alpha_i$, we deduce that clearly $G = S_{min}$. By doing this we have reduced the integral in (1) to a term with quadratic dependence on X (and none of Q), and one that depends on the minimum value of the action. We determine what the minimum value of the action is by minimizing S by straightforward optimization.

$$\partial_i(\sum_{ij} M_{ij} X_i X_j + \sum_i Q_i X_i) = 0$$
$$\sum_{ij} 2M_{ij} X_j + \sum_i Q_i = 0$$
$$\therefore \vec{X}_{min} = -\frac{1}{2} M^{-1} \vec{Q}$$

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We begin by rewriting equation (2) in vector form, as we have rewritten $(X_{min})_i$ using the fact that $\sum_j M_{ij}x_j = (A\vec{x})_i$. Equation (2) becomes, substituting in $\vec{X}_{min}$, and utilizing the identity (for a symmetric M) $M^{-1}\vec{Q} = \vec{Q}^T M^{-1}$:

$$S_{min} = \vec{X}_{min} \cdot M\vec{X}_{min} + \vec{X}_{min} \cdot \vec{Q} - \frac{1}{2}(M^{-1}\vec{Q}) \cdot \vec{Q} = \tag{4}$$

$$= \frac{1}{4}(M^{-1}\vec{Q}) \cdot M(M^{-1}\vec{Q}) - \frac{1}{2}(M^{-1}\vec{Q}) \cdot \vec{Q} = \tag{5}$$

$$= \frac{1}{4}(\vec{Q}^T M^{-1} M M^{-1}\vec{Q}) - \frac{1}{2}(\vec{Q}M^{-1}\vec{Q}) = \tag{6}$$

$$= -\frac{1}{4}\vec{Q}(M^{-1})\vec{Q} = \tag{7}$$

$$= \sum_{ij} -\frac{1}{4}M_{ij}^{-1}Q_iQ_j \tag{8}$$

Therefore, we have shown that the integral in (1) reduces to,

$$\int dX_1 ... e^{\sum_i (\gamma (X_i')^2)} \cdot e^{S_{min}} \tag{9}$$

where $S_{ij} = -\frac{1}{4}M_{ij}^{-1}$, and the first term can readily be integrated and does not depend on Q since the linear terms involving Q were removed from the first term in completing the square.