INTRODUCTION TO STRING THEORY: ASSIGNMENT 4

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1. Part A: Electrostatics problem for the non-compact coordinates

In determining the scattering amplitude for particles off closed strings, we must first determine the electrostatic energy of the non-compact coordinates. The equation for both the compact and non-compact coordinates, with differing boundary conditions, is:

$$\frac{\partial^2 X}{\partial \tau^2} + \frac{\partial^2 X}{\partial \sigma^2} = 0 \tag{1}$$

In order to solve this problem, we first conformally map the infinite tube-like region $(0 < \sigma \leq 2\pi)$, $(-\infty < \tau < \infty)$ with the periodic condition that identifies $\sigma = 0$ with $\sigma = 2\pi$ to the entire complex plane by the mapping $(\sigma, \tau) \to e^{\tau + i\sigma}$ where $z = \sigma + i\tau$.

We wish to determine the scalar free field X_{nc} for the non-compact coordinate. It can be computed by $\int dz G(z, z')\rho(z)$, where $G(z, z')$ is defined by the Laplace equation

$$\Delta_z G(z, z_i) = 2\pi p_i(z)\delta^2(z - z_i) \tag{2}$$

and $\rho(z) = \sum \delta(z - z_i)$. Note that the effective charges of the particles are their associated momenta p_i due to the vertex operator being $e^{ip_i x}$. Solving (2), by requiring the boundary condition to be that the field vanish strongly as $r \to \infty$, yields $G(z, z_i) = p_i log|z - z_i|$, which then yields,

$$X_{nc} = \sum_{i=1}^{N} p_i log|z - z_i| \tag{3}$$

The energy associated to particles in the non-compact coordinates will be determined subsequently, after the electrostatic problem for the compact coordinate is determined, since both are necessary to compute the scattering amplitude. Note that $p_i = \frac{n_i}{R}$ is the quantized momentum associated with the non-compact dimensions where n is the standard quantum number associated to a particle.

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2. Part B: Electrostatics problem for the compact coordinate

For the compact coordinate, the solution to equation (1) differs due to the fact that the following condition holds for a particle wrapped around the compact dimension.

$$X_c \equiv X_c + 2\pi R m_i \tag{4}$$

where m_i is the particle's winding number and R is the radius of the compact dimension. Effectively, due to branch cuts since $\oint_c X_c d\theta = 2\pi m_i R$, a particle must be wrapped around the compact dimension an integer number of times, where m_i is the integer associated with each particle. By inspecting (1) and the condition (4), we determine that the field solution for the compact dimension must be,

$$X_c = \sum_{j=1}^{n} R m_i \theta_i (z - z_i) \tag{5}$$

where $\theta = \tan^{-1}\left(\frac{Im(z-z_i)}{Re(z-z_i)}\right)$. Due to the properties of harmonic functions, the obtained solution must be unique. A particle's momentum associated with the compact coordinate is $P_i = m_i R$. Given that the solved equation is linear, we can sum the solutions to the compact and non-compact coordinates to obtain,

$$X = X_c + X_{nc} = \sum_{i=1}^{N} p_i log|z - z_i| + \sum_{j=1}^{n} R m_i \theta_i (z - z_i) \tag{6}$$

We can now determine the electrostatic energy of particles in both the compact and non-compact coordinates. In doing so, it is clear that the compact and non-compact directions do not interact since the cross terms vanish, and the total energy of the closed string is simply equal to the sum of the compact coordinate and non-compact coordinate energies. Note that since we will afterwards compute the integral representation of the scattering amplitude, we are not interested in the self energies of any field, and will therefore only be interested in cases where $i \neq j$.

The total electrostatic energy of interest will be:

$$E(z_1, ...) = \frac{1}{2} \int dz (\nabla X)^2 \tag{7}$$

We first compute the gradient of X:

$$\nabla X = \sum_{i=1}^{N} \frac{p_i}{2\pi} \frac{(z - z_i)}{|z - z_i|^2} + \sum_{j=1}^{n} R m_j \frac{\theta(z \hat{-} z_i)}{|z - z_i|} \tag{8}$$

Here we substitute $\theta(z \hat{-} z_i) = i\frac{(z-z_i)}{|z-z_i|}$ and the expressions for p_i and P_i as well as using the fact that $n = N$ (i.e. the number of particles is the same in each coordinate) to rewrite (8) as,

$$\nabla X = \sum_{l=1}^{N} \left(\frac{n_l}{R} + i m_l R\right) \frac{z - z_l}{|z - z_l|^2} \tag{9}$$

We are now in a position to compute $(\nabla X)^2 = \overline{\nabla X} \nabla X$, where the bar over the first gradient denotes the complex conjugate. Doing this will clearly cause the cross terms to cancel due to the i preceding $m_l R$ being complex conjugated. Thus we are left with the following energy,

$$(\nabla X)^2 = \sum_{l \neq k}^{N} \left(\frac{n_l n_k}{R^2} + m_l m_k R^2\right) \left(\frac{\overline{(z - z_l)}(z - z_k)}{|z - z_l|^2 |z - z_k|^2} + \frac{(z - z_l)\overline{(z - z_k)}}{|z - z_l|^2 |z - z_k|^2}\right) \tag{10}$$

In order to facilitate integrating this over the entire complex plane to obtain the total energy, we can translate the coordinate $(z - z_l)$ by setting $z_l = 0$. We also can note that $\frac{\overline{(z)}(z-z_k)}{|z|^2|z-z_k|^2} = \frac{1}{|z|}\frac{1}{(z-z_k)}$ which is simply one divided by the dot product between two complex vectors with some angle between them. We can therefore rewrite the integrand in polar coordinates and integrate over $r dr d\theta$ to obtain the final energy

$$E(z_1, ...) = -\frac{1}{2} \sum_{l \neq k}^{N} \left(\frac{n_l n_k}{R^2} + m_l m_k R^2\right) log(\overline{(|z_l - z_k|)}(|z_l - z_k|)) \tag{11}$$

3. Part C: Integral representation of the scattering amplitude

Now we must substitute this energy into the equation for the scattering amplitude off of a string. The general equation is $A = \int DX(\sigma, \tau) e^{\frac{1}{2}S}$ where S is the action, which, in our case, corresponds to the computed energy. Therefore,

$$A = \int dz_1 \overline{dz_1} ... e^{-E(z_1, \overline{z_1}, ...)} = \int dz_1 \overline{dz_1} ... \prod_{l \neq k} ((\overline{|z_l - z_k|})(|z_l - z_k|))^{\frac{1}{2} \frac{n_l n_k}{R^2} + m_l m_k R^2} \tag{12}$$

where we are integrating over all the possible points of entry of every particle. However, we can simplify this integral by virtue of the fact that 3 of the points are fixed due to the requirement of a linear fractional transformation of the form $f(z) = \frac{az+b}{cz+d}$, one must only integrate over $d-3$ of the possible z_i. Therefore, if one were to analyze a system with four incoming particles on the world sheet of the closed string, one would only have to integrate over the possible values of the position z_i at which particle 4 enters. For the case of four particles, we can conveniently select $z_1 = 0$, $z_2 = 1$, and $z_3 = \infty$ (and likewise for their complex conjugate particles) such that (12) reduces to the famous Veneziano amplitude for a closed string.

4. Part D: Duality

It is very easy to see from (12) that the amplitude is invariant to the transformation $R \to \frac{1}{R}$ and $n_i \to m_i$. The amplitude therefore exhibits T-duality, which is an assertion that there is no experiment that could be performed to distinguish between $\frac{1}{R}$ and R i.e. we would not be able to distinguish between small and large distance scales on a closed string with a compact coordinate.