# INTRODUCTION TO STRING THEORY: ASSIGNMENT 3

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## 1. Mapping the Infinite Cylinder to the Entire Complex Plane

A general string can be described as the surface parametrized by $-\infty < \tau < \infty$ and $0 < \sigma < 2\pi$ where $\sigma = 0$ is associated to $\sigma = 2\pi$. In order to make problems that arise in string theory more manageable, it is of great interest to map this space to the entire complex plane. We define $z = \tau + i\sigma$. Any such transformation must be conformal, as in satisfying the Cauchy-Riemann relations. The way to solve this problem is to imagine oneself at the center of the cylinder. Looking in the positive $\tau$ direction it is clear that the radii of slices of the cylinder in $\tau$ are equal to $e^\tau$. To describe every point on these infinitely many concentric circles with radius $R = e^\tau$ where $-\infty < \tau < \infty$, the transformation is multiplied by $e^{i\sigma}$, such that the complete transformation from the infinite cylinder to the entire complex plane becomes:

$$(\tau, \sigma) \to e^{\tau + i\sigma} = e^z \tag{1}$$

Note that this transformation maps to the entire complex plane minus its origin.

## 2. General Mapping of Points in The Complex Plane

We want to determine a general conformal mapping of the entire complex plane (including points at infinity) onto itself. This is a well known class of transformations known as *linear fractional transformations* (LFT). They have the form,

$$f(z) = \frac{az + b}{cz + d} \tag{2}$$

We want to extend this transformation to the entire complex plane $\mathbb{C} = \mathbb{C} \cup \infty$. In order to do that we define $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$. We are also interested in the number of points that must be fixed in order to fully specify a mapping. To do so we multiply $f(z)$ by 1 in the form of $\frac{1/c}{1/c}$, so that $f(z) = \frac{\frac{a}{c} + \frac{b}{c}}{z + \frac{d}{c} z}$. Without loss of generality, we can set $c = 1$. Therefore, it is clear that in order to solve for a,b, and d we must specify the maps of three points that preserve angles. Linear fractional transformations are often used in string theory to fix three locations of vertex operators, such that one must only integrate over the possible positions of the remaining ones in order to determine the scattering amplitude off a string.

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## 3. Electrostatic Energy of Four Charges in the Complex Plane

We assume we have four particles entering the complex world sheet at positions $z_1, z_2, z_3$, and $z_4$ with respective charges $Q_1, Q_2, Q_3$, and $Q_4$. In order to determine the electrostatic energy we need to solve the 2-D Laplace equation:

$$\frac{\partial^2 V}{\partial \tau^2} + \frac{\partial^2 V}{\partial \sigma^2} = 0 \tag{3}$$

This can be done by obtaining the Green's function that solves the following equation:

$$\Delta_z G(z, z_i) = 2\pi Q_i(z)\delta^2(z - z_i) \tag{4}$$

Equation (4) has the solution $G(z, z_i) = Q_i log|z - z_i|$, which yields an electrostatic potential,

$$V = \sum_{i=1}^{4} Q_i log|z - z_i| \tag{5}$$

In order to determine the electrostatic energy given by the general formula $E = \frac{1}{2}\int dz(\nabla V)^2$. After taking the gradient of (5) and finding it's magnitude squared, the energy of these particles with different charges entering the world sheet at respective z positions is given by the above stated integral (NOTE: we only sum over indices not equal to each other because the self energy of particles does not contribute to the electrostatic energy):

$$(\nabla V)^2 = \sum_{i \neq j}^{4} Q_i Q_j \left( \frac{\overline{(z - z_i)}(z - z_j)}{|z - z_i|^2 |z - z_j|^2} \right) \tag{6}$$

Integrating this equation over all z yields the electrostatic energy:

$$E(z_i, ...) = -\frac{1}{2}\sum_{i \neq j}^{4} Q_i Q_j log(|z_i - z_j|) \tag{7}$$

## 4. Relevance to String Theory

Computing the electrostatic energy of four particles with respective charges $Q_i$ at positions $z_i$ is of great relevance to string theory, particularly in computing the scattering amplitude. In the infinite momentum frame where the problem can be considered static, the scattering amplitude is given by:

$$A = \int dz_1 ... e^{-E(z_1,...)} \tag{8}$$

Therefore, having computed the electrostatic energy of particles (in our case in non-compact dimensions), we can then compute the scattering amplitude, which is often left in integral form.