PHYSICS 153A: ASSIGNMENT 1

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1. Problem 1: Matrix Exponentials in Quantum Mechanics

2. Problem 2: The Creation Operator

It is of great interest to solve the quantum harmonic oscillator problem in the energy basis. To do so, we first introduce the following operators written in terms of position and momentum operators X and P respectively.

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} X + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P \tag{1}$$

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} X - i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} P \tag{2}$$

We then note that the operator $a^\dagger a = \frac{H}{\hbar\omega} - 1/2$ from which we note that $H = (a^\dagger a + 1/2)\hbar\omega$. We now define $\hat{H} = \frac{H}{\hbar\omega}$, and attempt to determine how $a^\dagger$ acts on an eigenstate $|\epsilon\rangle$ of $\hat{H}$ with eigenvalue ϵ.

We will now use the relation that $[a^\dagger, \hat{H}] = [a^\dagger, a^\dagger a + 1/2] = -a^\dagger$, in order to determine the behavior of operators (1) and (2) on an eigenstate of the Hamiltonian. Consider the following:

$$\begin{aligned}
\hat{H}a^\dagger|\epsilon\rangle &= (a^\dagger\hat{H} - [a^\dagger, \hat{H}])|\epsilon\rangle \\
&= (a^\dagger\hat{H} + a^\dagger)|\epsilon\rangle \\
&= (\epsilon + 1)a^\dagger|\epsilon\rangle
\end{aligned}$$

This implies that $a^\dagger|\epsilon\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $\epsilon + 1$. What this is telling us is that

$$a^\dagger|\epsilon\rangle = \alpha_\epsilon|\epsilon + 1\rangle \tag{3}$$

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Clearly this operator is called the *creation* or *raising operator* because it acts on an state of the Hamiltonian and returns the next state. Clearly, by continuing action of the raising operator on an eigenstate of the Hamiltonian, one can increase the current state ad infinitum. The behavior of the *lowering operator* is similar, but must obviously cease to lower the state at a certain point, since the eigenvalues of the Hamiltonian must be nonnegative. It turns out that eigenvalues of $\hat{H}$ are $\epsilon_n = n + 1/2$ for $n = 0, 1, 2, ...$ We proceed to label each eigenstate $|n\rangle$. Let us use now use equation (3) to determine the constant α_n.

The adjoint of equation (3) is

$$a \langle n| = \alpha_n^* \langle n+1| \tag{4}$$

We will need to use the result that $aa^\dagger = \hat{H} + 1/2$.
Combining equations (3) and (4), we obtain

$$\begin{aligned} \langle n| a^\dagger a |n\rangle &= \langle n+1|n+1\rangle \alpha_n^* \alpha_n \\ \langle n| \hat{H} + 1/2 |n\rangle &= |\alpha_n|^2 \\ \langle n| n+1 |n\rangle &= |\alpha_n|^2 \\ \alpha_n &= (n+1)^{1/2} \end{aligned}$$

where we have used the following:

1) $|n+1\rangle$ is normalized
2) $\hat{H}|n\rangle = (n+1/2)|n\rangle$

Now that we have determined how the raising operator acts on the eigenstates of the Hamiltonian, we can easily determine the result of $(a^\dagger)^n |0\rangle$.

$$\begin{aligned} (a^\dagger)^n |0\rangle &= (a^\dagger)^{n-1} (1)^{1/2} |1\rangle \\ &= (a^\dagger)^{n-2} (1 \times 2)^{1/2} |2\rangle \\ &= (a^\dagger)^{n-n} (1 \times 2 \times 3 \times ... \times n)^{1/2} |n\rangle \\ &= (n!)^{1/2} |n\rangle \end{aligned}$$

3. Problem 3: The non-existence of Eigenstates of the Creation Operator

Following directly from the result we derived above we would like to show that the *creation operator* has no eigenfunctions. If we postulate a general function that is a linear combination of the eigenfunctions of harmonic oscillator Hamiltonian:

$$|\psi\rangle = \sum_{n=0}^{\infty} \beta_n |n\rangle \tag{5}$$

where β_n are some constants multiplying each eigenfunction of H. Let us see what happens when we act on $|\psi\rangle$ with $a^\dagger$.

$$\begin{aligned} a^\dagger |\psi\rangle &= \sum_{n=0}^{\infty} \beta_n a^\dagger |n\rangle \\ &= \sum_{n=1}^{\infty} \beta_n (n+1)^{1/2} |n+1\rangle \end{aligned}$$

4. Problem 4: The Eigenstates of the Annihilation Operator