

# HIGH TEMPERATURE PARTITION FUNCTION OF THE OPEN BOSONIC STRING

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The partition function allows the direct computation of macroscopic variables, therefore, computing such a statistical measure for a string could allow us to compute thermodynamic quantities of string black holes, modeling these as strings enclosed in a certain volume  $V_{BH}$  and in contact with a heat reservoir at temperature  $T$ . As usual, we proceed to work in “natural units,” for which  $c = \hbar = 1$ .

The partition function of any system involves the sum over all of the allowable energies of the system. Therefore, determination of the partition function of a relativistic string requires that we first compute the number of ways of constructing  $E(N)$  by acting on the vacuum state, known as the *number of partitions of  $N$* ,  $p(N)$ . To do this, we first analyze the quantum non-relativistic string, to aid analysis of the relativistic case.

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## 1. THE QUANTUM NON-RELATIVISTIC STRING

A quantum non-relativistic string can be viewed as a collection of quantum simple harmonic oscillators (QSHO), each possessing their own creation and annihilation operators. Therefore, the following commutation relation holds:

$$(1) \quad [a_m, a_n^\dagger] = \delta_{mn}$$

where the subscript denotes the oscillator on which the operator acts. We also work in the convention that the basic unit of frequency of the oscillator  $\omega_0 = 1$ . We note at this point that the Hamiltonian of the quantum string is,

$$(2) \quad \hat{H} = \sum_{l=1}^{\infty} l a_l^\dagger a_l = \hat{N}$$

which is clearly just the number operator, which returns the number of quanta of oscillator energy units. Note at this point that acting upon the vacuum state  $|\Omega\rangle$  with  $a_l$ , for any  $l$ , will return 0. Therefore, a quantum state of the string is defined by  $|\Psi\rangle = (a_1^\dagger)^{n_1} \cdot (a_2^\dagger)^{n_2} \dots (a_l^\dagger)^{n_l} |\Omega\rangle$ , where the set  $\{n_1, n_2, \dots, n_l\}$ , known as the set of *occupation numbers*, fully specifies the state of the string. The eigenvalue of the number operator, and the

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<sup>1</sup>Motivated by *Zwiebach B.* “A First Course in String Theory”

Hamiltonian, is therefore specified by  $N = n_1 + n_2 + \dots = \sum_{l=1}^{\infty} l n_l$ , which in our chosen units, is equal to the energy  $E$ . It is here that we encounter the statistical problem of the number of ways that this energy can be constructed by acting on a state with any combination of the creation operators. This value, called  $p(N)$ , is the *partition of  $N$*  that we wish to compute. It can be defined as  $p(N) \equiv (\text{the number of sets of positive integers that add up to } N)$ , which corresponds to the number of states with  $\hat{N}$  eigenvalue equal to  $N$ .

However, we cannot yet determine a formula for  $p(N)$  applicable to a string. We have to generalize this energy, since up to now we have assumed the string can only vibrate in 1 transverse dimension. We define  $m$  to be the number of transverse directions that the string can vibrate in. For each frequency we now imagine that we have  $m$  harmonic oscillators representing the possible ways in which the strings motion could be polarized. Therefore, each occupation number acquires a superscript  $m$  such that the possible number and energy eigenvalues be  $N = E = \sum_{l=0}^{\infty} \sum_{q=1}^m l n_l^{(m)}$ . We now move to the statistical problem of determining  $p(N)$ .

**1.1. Determining  $p(N)$ .** We can determine an approximation for  $\ln p(N)$ , given large  $N$ , by noting that the entropy of the string is simply given by

$$(3) \quad S(E) = k \ln p(N) = k \ln p(E)$$

such that we can determine  $p(N)$  by determining the entropy of the string, which we arrive at by first determining the partition function  $Z$  of the string.  $Z$  is given by the following equation:

$$(4) \quad Z = \sum_{n_k^{(1)}, \dots, n_k^{(m)}} \exp \left( -\beta \sum_{l=0}^{\infty} \sum_{g=1}^m l n_l^{(g)} \right)$$

where  $\beta = \frac{1}{kT}$ , where  $T$  is the temperature of the thermal reservoir that the string is in contact with. At this juncture, it is important to note that the sum over  $n_k^{(m)}$  can be factorized as follows,

$$(5) \quad Z = \sum_{n_k^{(1)}} \exp \left( -\beta \sum_{l=0}^{\infty} l n_l^{(1)} \right) \cdots \sum_{n_k^{(m)}} \exp \left( -\beta \sum_{l=0}^{\infty} l n_l^{(m)} \right)$$

since we are summing over each dimension of vibration independently, each containing the same occupation numbers. We note that this factorized partition function is equivalent to

$$(6) \quad Z = (Z_m)^m$$

where  $Z_m$  is the partition function for a string vibrating in a single dimension. Note from this simple relation that the total free energy is

$$(7) \quad F = -1/\beta \ln Z_m = -1/\beta m \ln Z_m = mF_m.$$

The entropy also acquires a simple factor of  $m$  due to differentiation,

$$(8) \quad S = \frac{\partial F}{\partial T} = mS_m$$

and so does the energy

$$(9) \quad E = \frac{\partial \ln Z}{\partial \beta} = mE_m.$$

Both  $S$  and  $E$  are necessary since we care to determine the entropy as a function of the energy. Therefore, in order to compute the total  $p(N)$  for the quantum non-relativistic string, we decouple the dimensions of vibration and solve them one at a time, and since we have imposed no distinction between the dimensions, we need only solve for the partition function and entropy in one of them.

**1.2. General Solution starting from A Single Dimension.** We will determine  $p(N)$  for a string vibrating in  $m$  dimensions by obtaining the partition function of a string vibrating in 1 dimension. In determining the 1 dimensional partition function  $Z_m$ , note that in summing over all possible energy states we are summing over the set of each occupation numbers, where each occupation number ranges from 0 to infinity. The partition function is

$$(10) \quad Z_m = \sum_i \exp(-\beta E_i) = \sum_{n_1, n_2, \dots} \exp(-\beta(n_1 + 2n_2 + 3n_3 + \dots)).$$

Each occupation number can be summed over independently by expanding the exponential, allowing us to obtain

$$(11) \quad Z_m = \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} \exp(-ln_l\beta),$$

and noting that each sum over  $n_l$  is simply a geometric series this becomes:

$$(12) \quad Z_m = \prod_{l=1}^{\infty} (1 - \exp(-l\beta)).$$

We now use the fact that  $F = -1/\beta \ln Z$ . It becomes convenient to work in the high temperature limit such that  $\beta \ll 1$ , and the summation can be approximated by an integral. We also make the coordinate substitution  $x = l\beta$ , which yields

$$(13) \quad F_m \approx 1/\beta^2 \int_0^\infty dx \ln(1 - \exp(-x)).$$

We are now also free to expand the integrand as a power series, valid for any  $0 \leq e^{-x} < 1$ , as

$$(14) \quad \ln(1 - e^{-x}) = -(e^{-x} + \frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x} + \dots)$$

which, when substituted back into  $F$  gives us

$$(15) \quad F_m = -1/\beta^2 \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) = -\frac{1}{\beta^2} \frac{\pi^2}{6}$$

The sum in parentheses was the well known zeta function (in this case with argument two) which evaluated to  $\zeta(2) = \frac{\pi^2}{6}$ . We are now ready to solve for the *number of partitions of  $N$*  in a non-relativistic string vibrating in  $m$  transverse dimensions. Using (8), (9), and (15), we can obtain the high temperature approximation of the entropy and energy for the entire string:

$$(16) \quad S(\beta) = -\frac{\partial(mF_m)}{\partial T} = mk \frac{\pi^2}{3} \frac{1}{\beta}$$

$$(17) \quad E(\beta) = \frac{\partial \ln Z}{\partial \beta} = m \frac{\pi^2}{6} \frac{1}{\beta^2}.$$

Combining (13) and (14), and using the fact that  $E = N$  in our chosen coordinates, we obtain

$$(18) \quad S(E) = k2\pi \left( \frac{mN}{6} \right)^{1/2}$$

which, by direct comparison with equation (3) reveals that

$$(19) \quad \ln p(N) \simeq 2\pi \left( \frac{mN}{6} \right)^{1/2}$$

in the limit that  $N$  is very large, since this is required at high  $T$  in order to evaluate  $F$ . Though this is an approximation, for  $N > 10,000$  it is within 1% of the exact value. We have therefore used common techniques in statistical thermodynamics to approximate the solution to a complex combinatoric problem such that it asymptotically approaches the exact solution as  $N \rightarrow \infty$ .

## 2. THE RELATIVISTIC STRING PARTITION FUNCTION

Having computed the *number of partitions of  $N$* , we are ready to approach the problem of computing the partition function of a relativistic string placed in a box of volume  $V$ , in contact with a reservoir at temperature  $T$ .

**2.1. Enumeration of States.** In order to approach this problem, we have to enumerate the states of the bosonic string. To do this, it is effective to utilize the basis of states determined for the open string in light-cone coordinates. The light cone coordinates are  $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$  and  $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1)$ . We select the light-cone gauge such that these coordinates are fixed by the gauge, and therefore the string can only oscillate in  $m = D - 2$  transverse dimensions, where  $D$  is the total number of dimensions, which for the case of the boson is 26. Therefore, the number of transverse dimensions in which the string vibrates is  $m = 24$ . Quantities along these dimensions are indexed by  $I$ . In light-cone coordinates, the complete set of basis of states for the string Hilbert space is given by

$$(20) \quad |\gamma, p\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I\dagger})^{\gamma_{n,I}} |p^+, \vec{p}_T\rangle$$

where  $|p^+, \vec{p}_T\rangle$  denote the momentum eigenstates and  $(a_n^{I\dagger})^{\gamma_{n,I}}$  is the light-cone coordinate creation operator. This set of states is obtained by acting on the momentum eigenstates in the light-cone coordinates with the light-cone basis creation operators. The states are fully enumerated by the set of occupation numbers  $\{\gamma_{n,I}\}$  and the momenta.  $\vec{p}_T$  denotes the set of momenta in the transverse dimensions in which the string can vibrate.

In order to enumerate the energies of all the basis states of the relativistic string, we must make use of the mass-squared of the string, which in light-cone coordinates is given by

$$(21) \quad M^2(\{\gamma_{n,I}\}) = -p^2 = 2p^+p^- - p^I p^I = \frac{1}{\alpha'}(N^\perp - 1).$$

Here,  $\alpha'$  is the slope parameter relating the angular momentum and energy squared of a string by  $J = \alpha' E^2$ , and  $N^\perp = \sum_{n=1}^{\infty} n \gamma_{n,I}$ , known as the number operator for a relativistic open string.  $N^\perp$  is the sum of the standard number operators, each associated to a harmonic oscillator in the string. It acts on the states of the string as follows:

$$(22) \quad N^\perp |\gamma, p\rangle = N_\gamma^\perp |\gamma, p\rangle$$

where

$$(23) \quad N_\gamma^\perp = \sum_{n=1}^{\infty} \sum_{I=2}^{25} n \gamma_{n,I}.$$

Therefore, it returns the sum of the mode numbers of the creation operators appearing in the state. We can now write the set of energies of the possible string states as

$$(24) \quad E(\{\gamma_{n,I}\}, \vec{p}) = (M^2(\{\gamma_{n,I}\}) + \vec{p}^2)^{1/2}$$

and proceed to determine the partition function for the open bosonic string.

**2.2. The Partition Function.** We denote the partition function of the relativistic string  $Z_s$ , which is as usual a summation over all the possible energy states. We note that for the relativistic string, this can be represented by a double summation over all the values of the spatial momenta  $\vec{p}$  and the set of all occupation numbers  $\gamma_{n,I}$ :

$$(25) \quad Z_s = \sum_{\gamma_{n,I}} \sum_{\vec{p}} \exp(-\beta(M^2 + \vec{p}^2)^{1/2}) =$$

$$(26) \quad = \sum_{\gamma_{n,I}} Z(M^2)$$

$Z(M^2)$  is the partition function for the single relativistic particle with mass-squared given by  $M^2$ . The transition from (25) to (26) is a matter of notation. Finally, we are in a position to apply the equation for  $p_{24}(N)$  valid in the high  $N$  and  $T$  limit, where the subscript has been added to indicate that it is the *number of partitions of  $N$*  for a string vibrating in  $m = 24$  transverse dimensions. The summation over the occupation numbers in (26) can be converted to a sum over  $N$ , where  $N^\perp \equiv N$ , by noting that there are  $p_{24}(N)$  states with number eigenvalue  $N$ . This allows us to write

$$(27) \quad Z_s = \sum_{N=0}^{\infty} p_{24}(N) Z(M^2(N))$$

for which the only approximations made were in computing  $p(N)$ . At this point, an approximation needs to be made in order to evaluate this partition function. We denote an integer  $N_0$  such that  $N \geq N_0$  and  $p_{24}(N_0)$  is well approximated, and then rewrite the partition function of the bosonic string as

$$(28) \quad Z_s = Z_0 + \sum_{N=N_0}^{\infty} p_{24}(N) Z(M^2(N))$$

where the introduced  $Z_0$  represents the component of the partition function summed up to  $N = N_0 - 1$ . The partition function is divided into two components because  $Z_0$  is very challenging to evaluate and given an appropriate choice of  $N_0$ , it should be negligible compared to the term being summed from  $N_0$  to infinity. Note that there is a limiting temperature for which this approximation becomes valid, known as the Hagedorn temperature, given by

$$(29) \quad \frac{1}{\beta_H} = kT_H = \frac{1}{4\pi(\alpha')^{1/2}}.$$

In the high energy limit, the energy of the string can be arbitrarily increased, while the temperature of the string remains fixed at  $T_H$ . We therefore focus on evaluating the second sum on the right of equation (28) in the limit that we approach  $T_H$ , which we can further approximate by an integral which we denote as

$$(30) \quad Z_{s'} \simeq \int_{N_0}^{\infty} dN p_{24}(N) Z(M^2(N)).$$

In order to compute this integral we first seek to rewrite the density of states as a function of the mass  $M$  by rewriting  $p_{24}(N)dN = \rho(M)dM$ , where the density can be made into a function of  $M$  by using equation (21), which approximates to  $\alpha'M^2 \simeq N$ . A small amount of algebra, and combination of equations (19) and (29) convert the integral to

$$(31) \quad Z_{s'} \simeq \sqrt{2} \int_{M_0}^{\infty} d(\sqrt{\alpha'}M) (\sqrt{\alpha'}M)^{-25/2} \exp(\beta_H M) Z(M^2).$$

We then rewrite the single relativistic particle partition function <sup>2</sup> in terms of  $T_H$ . In addition, we make the coordinate substitution  $x = \sqrt{\alpha'}M$ . When substituted into the integral, these changes yield

$$(32) \quad Z_{s'} \simeq 2^{13} V (kT kT_H)^{25/2} \int_{\sqrt{N_0}}^{\infty} dx \exp(-4\pi x [T_H/T - 1]).$$

An additional reason for rewriting everything in terms of  $T_H$  becomes evident: the integral  $Z_{s'}$  diverges if  $T$  is smaller than the Hagedorn temperature. Evaluating the integral subject to this constraint, and adding it to  $Z_0$  gives us

$$(33) \quad Z_s \simeq Z_0 + \frac{2^{11}}{\pi} V (kT kT_H)^{25/2} \left( \frac{T}{T_H - T} \right) \exp \left( -4\pi \sqrt{N_0} \left[ \frac{T_H}{T} - 1 \right] \right).$$

Here we make  $T \rightarrow T_H$ , causing the exponential to tend to zero, and the term outside to diverge and grossly outweigh  $Z_0$ . Therefore, for an open bosonic string contained in a box of volume  $V$  in contact with a reservoir at temperature  $T$  approaching  $T_H$ , its partition function as  $N \rightarrow \infty$  is well approximated by

$$(34) \quad Z_s \simeq \frac{2^{11}}{\pi} V (kT_H)^{25} \left( \frac{T_H}{T_H - T} \right).$$

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<sup>2</sup>The derivation of the single particle partition function can be found in Zwiebach B. "A First Course in String Theory"

### 3. CONCLUSION

The analysis of the quantum non-relativistic string allowed us to well approximate a complex combinatoric problem (the determination of  $p(N)$ ) in the high  $T$  and  $N$  limit. Utilizing this result, we obtained a reasonable approximation of the partition function of an open bosonic string in contact with a heat reservoir at a temperature very near  $T_H$ . Though only valid in this regime, the partition function allows us to compute various macroscopic variables of interest, such as the average energy of the string:

$$(35) \quad E_s = -\frac{\partial \ln Z_s}{\partial \beta} \simeq \frac{1}{\beta - \beta_H} \simeq kT_H \left( \frac{T_H}{T_H - T} \right).$$

From this equation, the energy appears to diverge as the temperature approaches  $T_H$ .

There are still many limitations to our understanding of the thermodynamics of strings, as even this derivation is only approximately valid given very specific constraints.