ECON 6140 - Problem Set # 5

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Endogenous growth and government spending

(1) The Hamiltonian for this problem is given by

$$\mathcal{H}(c(t), k(t), \lambda(t)) = e^{-\rho t}u(c(t)) + \lambda(t)[Ak(t) - c(t) - \delta k(t)]$$

Hence, we get the following sufficient conditions

$$\begin{aligned} e^{-\rho t}u'(c(t)) &= \lambda(t) \\ \dot{\lambda}(t) &= -\lambda(t)[A - \delta] \\ \lim_{t\to\infty} \lambda(t)k(t) &= 0 \\ \dot{k}(t) &= Ak(t) - c(t) - \delta k(t) \end{aligned}$$

First, we need to take the derivative of the first condition with respect to t

$$\begin{aligned} -\rho e^{-\rho t}u'(c(t)) + e^{-\rho t}u''(c(t))\dot{c}(t) &= \dot{\lambda}(t) \\ -\rho e^{-\rho t}c^{-\sigma} - e^{-\rho t}\sigma c^{-\sigma-1}\dot{c}(t) &= \dot{\lambda}(t) \end{aligned}$$

Hence,

$$\begin{aligned} \dot{\lambda}(t) &= -\lambda(t)[A - \delta] \\ -\rho e^{-\rho t}c^{-\sigma} - e^{-\rho t}\sigma c^{-\sigma-1}\dot{c}(t) &= -e^{-\rho t}c^{-\sigma}[A - \delta] \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}[A - \delta - \rho] \end{aligned}$$

Since the growth rate of $\frac{\dot{c}(t)}{c(t)}$ is constant, we are a balanced growth path. Moreover, we need k to grow at the same rate. This is straightforward if you look at the following condition

$$\frac{\dot{k}(t)}{k(t)} = A - \frac{c(t)}{k(t)} - \delta$$

(2) Solving for $\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}[A - \delta - \rho]$ yields

$$c(t) = e^{\frac{1}{\sigma}[A-\delta-\rho]}c(0)$$

If we plug this back in the utility function, we get

$$\begin{aligned} U &= \int_0^\infty e^{-\rho t} \frac{e^{\frac{(1-\sigma)}{\sigma}[A-\delta-\rho]t}}{1-\sigma} c(0)^{(1-\sigma)} dt \\ &= \int_0^\infty \frac{e^{\frac{(1-\sigma)}{\sigma}[A-\delta-\frac{\rho}{1-\sigma}]t}}{1-\sigma} c(0)^{(1-\sigma)} dt \end{aligned}$$

Thus, for U to be bounded, we need

$$\frac{(1-\sigma)}{\sigma}\left[A - \delta - \frac{\rho}{1-\sigma}\right] < 0$$

(3) Note that $MPK = \phi\left(\frac{g}{k}\right) - \frac{g}{k}\phi'\left(\frac{g}{k}\right)$

$$\begin{aligned} \eta &= -\frac{\partial y}{\partial g} \cdot \frac{g}{y} \\ &= -\phi'\left(\frac{g}{k}\right) \cdot \frac{g}{k\phi\left(\frac{g}{k}\right)} \\ &= \frac{MPK - \phi\left(\frac{g}{k}\right)}{\phi\left(\frac{g}{k}\right)} \\ &= \frac{MPK}{\phi\left(\frac{g}{k}\right)} - 1 \\ \Rightarrow MPK &= \phi\left(\frac{g}{k}\right)(1+\eta) \end{aligned}$$

(4) The Hamiltonian for this problem is given by

$$\mathcal{H}(c(t), k(t), \lambda(t)) = e^{-\rho t}u(c(t)) + \lambda(t)[f(k(t), g) - g - c(t) - \delta k(t)]$$

The new sufficient conditions are the following

$$\begin{aligned} e^{-\rho t}u'(c(t)) &= \lambda(t) \\ \dot{\lambda}(t) &= -\lambda(t)[MPK - g - \delta] \\ \lim_{t\to\infty} \lambda(t)k(t) &= 0 \\ \dot{k}(t) &= Ak(t) - c(t) - \delta k(t) \end{aligned}$$

First, we need to take the derivative of the first condition with respect to t

$$\begin{aligned} -\rho e^{-\rho t}u'(c(t)) + e^{-\rho t}u''(c(t))\dot{c}(t) &= \dot{\lambda}(t) \\ -\rho e^{-\rho t}c^{-\sigma} - e^{-\rho t}\sigma c^{-\sigma-1}\dot{c}(t) &= \dot{\lambda}(t) \end{aligned}$$

Hence,

$$
\begin{aligned}
\dot{\lambda}(t) &= -\lambda(t)[MPK - g - \delta] \\
-\rho e^{-\rho t} c^{-\sigma} - e^{-\rho t}\sigma c^{-\sigma-1}\dot{c}(t) &= -e^{-\rho t}c^{-\sigma}[MPK - g - \delta] \\
\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}[MPK - g - \delta] \\
\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}\left[\phi\left(\frac{g}{k}\right)(1+\eta) - g - \delta\right]
\end{aligned}
$$

Finally,

$$
\frac{\partial\left(\frac{\dot{c}(t)}{c(t)}\right)}{\partial g} = \frac{1}{\sigma}\left[\frac{1}{k}\phi'\left(\frac{g}{k}\right)(1+\eta) - 1\right]
$$

Note that $\frac{1}{k}\phi'\left(\frac{g}{k}\right)(1+\eta) > 0$. Therefore, the change in growth rate will depend on if the increase in g is such that $\phi'\left(\frac{g}{k}\right)(1+\eta) \geq k$ or $\phi'\left(\frac{g}{k}\right)(1+\eta) \leq k$.

The value of life

(1) The problem can be summarize by the following value function

$$
V(c) = \max_{s\in\{\text{research, stop}\}} U^s
$$

where $U^S = \begin{cases} (1-\pi)u(c(1+g)) & \text{if } s = \text{research} \\ u(c) & \text{if } s = \text{stop} \end{cases}$.

Thus, for the agent to choose "research", we need the following condition

$$
(1-\pi)u(c(1+g)) \geq u(c)
$$

(2) First order Taylor expansion and let L be the value of life.

$$
\begin{aligned}
u(c_1) &= u(c) + u'(c)(c_1 - c) \\
&= u(c) + u'(c)gc \\
\Rightarrow \frac{u(c_1)}{u'(c)} &= \frac{u(c)}{u'(c)} + gc \\
u(c_1) &= \frac{u(c)}{u'(c)}c^{-\sigma} + gc^{1-\sigma} \\
u(c_1) &= Lc^{-\sigma} + gc^{1-\sigma}
\end{aligned}
$$

Hence, the previous optimality condition reduce to

$$\begin{aligned}(1-\pi)u(c(1+g)) &\geq u(c)\\ (1-\pi)(Lc^{-\sigma}+gc^{1-\sigma}) &\geq u(c)\\ (1-\pi)L+(1-\pi)gc &\geq \frac{u(c)}{c^{-\sigma}}\\ (1-\pi)L+(1-\pi)gc &\geq L\\ (1-\pi)gc &\geq \pi L\\ \frac{(1-\pi)}{\pi}gc &\geq L\end{aligned}$$

This can be interpreted as the following: the cost of dying πL needs to be less than expected gain in consumption gc for the agent to chose research.

(3) Recall that

$$\begin{aligned}\frac{(1-\pi)}{\pi}gc &\geq L\\ g &\geq \frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\\ g &\geq \frac{\pi}{(1-\pi)}\left(\frac{\bar{u}}{c^{1-\sigma}}+\frac{1}{1-\sigma}\right)\end{aligned}$$

Note that while the growth rate g stays the same for any change in σ, any change in σ could potential affect the decision of the agent to do research or not, i.e. grow.
In fact, what's of interest for us is how $\frac{\pi}{(1-\pi)}\left(\frac{\bar{u}}{c^{1-\sigma}}+\frac{1}{1-\sigma}\right)$ changes with respect to σ

$$\begin{aligned}\frac{\partial\left(\frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\right)}{\partial\sigma} &= \frac{\pi}{(1-\pi)}\left(\frac{\bar{u}\log(c)}{c^{1-\sigma}}+\frac{1}{(1-\sigma)^2}\right)\\ &= \frac{\pi}{(1-\pi)}\left(\frac{\bar{u}\log(c)}{c^{1-\sigma}}-\frac{\bar{u}}{(1-\sigma)c^{1-\sigma}}+\frac{L}{c(1-\sigma)}\right)\\ &= \frac{\pi}{(1-\pi)}\left(\frac{\bar{u}((1-\sigma)\log(c)-1)}{(1-\sigma)c^{1-\sigma}}+\frac{L}{c(1-\sigma)}\right)\\ &= \frac{\pi}{(1-\pi)}\frac{1}{(1-\sigma)c^{1-\sigma}}\left(\bar{u}((1-\sigma)\log(c)-1)+Lc^{-\sigma}\right)\end{aligned}$$

Note that if we assume that $c \geq 1$, then $\frac{\partial\left(\frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\right)}{\partial\sigma} > 0$. If $c < 1$, then $\log(c) < 0$ and the sign of $\frac{\partial\left(\frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\right)}{\partial\sigma}$ is less straightforward. I'll assume that $c \geq 1$ for this example, i.e. $\frac{\partial\left(\frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\right)}{\partial\sigma} > 0$.
Thus, an increase in σ, implies that the right hand side of the optimality condition also increase, i.e. less likely that $g \geq \frac{\pi}{(1-\pi)}\left(\frac{\bar{u}}{c^{1-\sigma}}+\frac{1}{1-\sigma}\right)$. In other words, the less risk averse/

the higher the elasticity of inter-temporal substitution, the less likely the agent is to under take research.

Moreover, it is clear that the same can be said for L and $\bar{u}$. The more they increase, the more the agent values being alive, the less likely to take the chance of dying, and the faster $\frac{\partial\left(\frac{\pi}{(1-\pi)}\frac{u(c)}{u'(c)c}\right)}{\partial\sigma}$ increase.

(4) For sake of clarity, we enumerate all constraints faced by the social planner, i.e.

$$\begin{aligned}
&\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t) M_t dt \\
\text{s. t. } & C_t = \left(\int_0^{A_t} x_{it}^{\frac{1}{1+\alpha}} di\right)^{1+\alpha} \\
& H_t = \left(\int_0^{B_t} z_{it}^{\frac{1}{1+\alpha}} di\right)^{1+\alpha} \\
& \int_0^{A_t} x_{it} dt + \int_0^{B_t} z_{it} dt = L_{at} + L_{bt} = L_t \\
& \dot{A}_t = S_{at}^{\lambda} A_t^{\phi} \\
& \dot{B}_t = S_{bt}^{\lambda} B_t^{\phi} \\
& \dot{M}_t = -\delta_t M_t \\
& \dot{N}_t = \bar{n} N_t \\
& N_t = S_t + L_t \\
& M_0 = 1, \delta_t = -h^{\beta}, h_t = \frac{H_t}{N_t}
\end{aligned}$$

First note that for the social planner the optimal solution is such that $x_{it} = x_t$ and $z_{it} = z_t$ for all i.

This yields the following

$$
\begin{aligned}
L_{at} &= \int_0^{A_t} x_{it} dt \\
&= \int_0^{A_t} x_t dt \\
&= A_t x_t \\
L_{bt} &= \int_0^{B_t} z_{it} dt \\
&= \int_0^{B_t} z_t dt \\
&= B_t z_t \\
C_t &= \left(\int_0^{A_t} x_{it}^{\frac{1}{1+\alpha}} di \right)^{1+\alpha} \\
&= \left(x_t^{\frac{1}{1+\alpha}} \int_0^{A_t} 1 di \right)^{1+\alpha} \\
&= A_t{}^{1+\alpha} x_t \\
&= A_t{}^{\alpha} L_{at} \\
H_t &= \left(\int_0^{B_t} z_{it}^{\frac{1}{1+\alpha}} di \right)^{1+\alpha} \\
&= \left(\int_0^{B_t} z_t^{\frac{1}{1+\alpha}} di \right)^{1+\alpha} \\
&= B_t{}^{1+\alpha} z_t \\
&= B_t{}^{\alpha} L_{bt}
\end{aligned}
$$

Now, let $s_t = \frac{S_{at}}{S_t}$, $l_t = \frac{L_{at}}{L_t}$, and $\sigma_t = \frac{S_t}{N_t}$.
This implies the following

$$
\begin{aligned}
&c_t = \frac{A_t{}^{\alpha} L_{at}}{N_t} = A_t{}^{\alpha}(l_t)(1-\sigma_t) \\
&h_t = \frac{B_t{}^{\alpha} L_{bt}}{N_t} = B_t{}^{\alpha}(1-l_t)(1-\sigma_t) \\
&\dot{A}_t = S_{at}^{\lambda} A_t^{\phi} = (s_t \sigma_t N_t)^{\lambda} A_t^{\phi} \\
&\dot{B}_t = S_{bt}^{\lambda} B_t^{\phi} = ((1-s_t)\sigma_t N_t)^{\lambda} B_t^{\phi} \\
&\dot{M}_t = -(B_t{}^{\alpha}(1-l_t)(1-\sigma_t))^{-\beta} M_t \\
&\dot{N}_t = \bar{n} N_t \\
&M_0 = 1
\end{aligned}
$$

Hence, the social planner problem is given by

$$
\begin{aligned}
&\max_{c_t} \int_0^\infty e^{-\rho t} u\left(A_t{}^\alpha(l_t)(1-\sigma_t)\right) M_t dt \\
\text{s. t. } & \dot{A}_t = S_{at}^\lambda A_t^\phi = (s_t\sigma_t N_t)^\lambda A_t^\phi \\
& \dot{B}_t = S_{bt}^\lambda B_t^\phi = ((1-s_t)\sigma_t N_t)^\lambda B_t^\phi \\
& \dot{M}_t = -(B_t{}^\alpha(1-l_t)(1-\sigma_t)^{-\beta})M_t \\
& \dot{N}_t = \bar{n}N_t
\end{aligned}
$$

where $M_0 = 1$.

Finally, this yields this Hamiltonian

$$
\begin{aligned}
\mathcal{H} = \Big\{ & e^{-\rho t} u\left(A_t{}^\alpha(l_t)(1-\sigma_t)\right) M_t + \eta_t \left[(s_t\sigma_t N_t)^\lambda A_t^\phi\right] + \mu_t \left[((1-s_t)\sigma_t N_t)^\lambda B_t^\phi\right] \\
& + \nu_t \left[-(B_t{}^\alpha(1-l_t)(1-\sigma_t))^{-\beta} M_t\right] + \theta_t \bar{n} N_t \Big\}
\end{aligned}
$$

(5) See previous part.

(6) In balanced growth path, we have that

$$
\frac{\dot{A}}{A} = \gamma_A
$$

for some constant γ_A.

Hence,

$$
\begin{aligned}
\gamma_A &= \frac{\dot{A}}{A} \\
&= \frac{S_{at}^\lambda A_t^\phi}{A_t} \\
&= \frac{S_{at}^\lambda}{A_t^{\mathbf{1}-\phi}}
\end{aligned}
$$

Therefore, for γ_A to be a constant we need S_{at}^λ to grow at the same rate as $A_t^{\mathbf{1}-\phi}$.

(7) Let's look at $\delta_t = (B_t{}^\alpha(1-l_t)(1-\sigma_t))^{-\beta}$. Note that if we assume that l_t and σ_t are constant, only B_t changes with time.

On the balanced growth path, it is straightforward to see that B_t has positive growth and as such keeps increase. If $B_t \to \infty$ and $\alpha\beta > 0$, $(B_t{}^\alpha(1-l_t)(1-\sigma_t))^{-\beta} \to 0$, i.e. the mortality rate drops to 0 in the long-run.

(8) (a) Note that if $\lambda_b > \lambda_a = \lambda$, we have that producing the good H_t is faster. Hence, we get that the speed of convergence to a mortality rate of 0 will also increase.

Therefore, with $\lambda_b > \lambda_a = \lambda$, we get to the balanced growth path faster.

(b) With $\lambda_b > \lambda_a = \lambda$, we get to the steady state faster. As such the scientist will get to transfer to the consumption good side of research sooner (in the steady state), since there's no need for research of life saving good when the mortality rate is equal to 0. Thus, we get improved production of consumption sooner and as such higher welfare. In the end, everyone is happy!