ECON 6140 - Problem Set # 2

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Government expenditure, corruption and output

(1) The social planner problem is the following:

$$\begin{aligned}
\max_{c_t,l_t,n_t,k_{t+1},c^g_t,x_t} & \sum_{t=0}^{\infty} \beta^t \left\{u(c_t,l_t) + v(c^g_t)\right\} \\
\text{s. t. } c_t + x_t + g_t &\leq zf(n_t,k_t) \\
k_{t+1} &\leq (1-\delta_k)k_t + x_{kt} \\
n_t + l_t &\leq 1 \\
c^g_t &= \theta g_t \\
c_t, l_t, n_t, k_{t+1}, c^g_t, x_t &\geq 0
\end{aligned}$$

Hence, the feasibility constraints are the following:

(i) $c_t + x_t + g_t \leq zf(n_t,k_t)$ - Total production has to be bigger than private consumption, investment and government spending.

(ii) $k_{t+1} \leq (1-\delta_k)k_t + x_{kt}$ - Investment constraint for capital.

(iii) $n_t + l_t \leq 1$ - Time constraint for labor/leisure.

(iv) $c^g_t = \theta g_t$ - Fraction of government spending not wasted and used to purchase the public consumption good.

(v) $c_t, l_t, n_t, k_{t+1}, c^g_t, x_t \geq 0$ - Non-negativity constraints for the variables.

To describe the interior solution, we need the first order conditions, the resource constraint and a transversality condition. Note that for this problem every constraint is binding. Moreover, if g is small enough, we have that $c_t, l_t, n_t, k_{t+1}, c^g_t, x_t$ are strictly positive. Hence, these conditions can be summarized in the following way

$$\begin{aligned}
c_t &: u_c(c_t, l_t) - \lambda_t = 0 \\
c_t^g &: v_c(c_t^g) - \eta_t = 0 \\
n_t &: \lambda_t z f_n(n_t, k_t) - \psi_t = 0 \\
l_t &: u_l(c_t, l_t) - \psi_t = 0 \\
x_t &: \mu_t - \lambda_t = 0 \\
k_{t+1} &: \beta\lambda_{t+1} z f_k(n_{t+1}, k_{t+1}) + \beta\mu_{t+1}(1-\delta_k) - \mu_t = 0 \\
&: c_t + x_t + g_t \leq z f(n_t, k_t) \\
&: k_{t+1} \leq (1-\delta_k)k_t + x_{kt} \\
&: n_t + l_t \leq 1 \\
&: c_t^g = \theta g_t \\
TVC &: \lim_{T\to\infty} \beta^T u_c(c_T, l_T) k_{T+1} = 0
\end{aligned}$$

This can be simplified to

$$\begin{aligned}
u_c(c_t, 1-n_t) &= \lambda_t \\
v_c(\theta g_t) &= \eta_t \\
\lambda_t z f_n(n_t, k_t) &= \psi_t \\
u_l(c_t, 1-n_t) &= \psi_t \\
\mu_t &= \lambda_t \\
\beta\lambda_{t+1} z f_k(n_{t+1}, k_{t+1}) + \beta\mu_{t+1}(1-\delta_k) &= \mu_t \\
c_t + k_{t+1} - (1-\delta_k)k_t + g_t &= z f(n_t, k_t) \\
\lim_{T\to\infty} \beta^T u_c(c_T, 1-n_T) k_{T+1} &= 0
\end{aligned}$$

(3) In steady state, our conditions are the following

$$\begin{aligned}
u_c(c, 1-n) &= \lambda \\
v_c(\theta g) &= \eta \\
\lambda z f_n(n, k) &= u_l(c, 1-n) \\
\beta[z f_k(n, k) + (1-\delta_k)] &= 1 \\
c + \delta_k k + g &= z f(n, k)
\end{aligned}$$

For $\beta[z f_k(n, k) + (1-\delta_k)] = 1$ to hold, we need the following conditions

$$\begin{aligned}
\lim_{k_t\to 0} z f_k(n_t, k_t) &> \frac{1}{\beta} - (1-\delta_k) \\
\lim_{k_t\to\infty} z f_k(n_t, k_t) &< \frac{1}{\beta} - (1-\delta_k)
\end{aligned}$$

Now, we need to show that $c > 0$. Let $g = 0$. The fourth condition implies that

$$zf(n,k) > \delta_k k$$

This holds if we have

$$zf_k(n,k) > \delta_k$$

Since at the steady state we have $zf_k(n_t, k_t) = \frac{1}{\beta} - (1-\delta_k)$, we simply need the condition that

$$\frac{1}{\beta} - (1-\delta_k) > \delta_k$$

This is true for $\beta \in (0,1)$. For g small enough, we can simply extend the previous analysis due to continuity.

(4) (a) For f homogeneous of degree 1, we have f_k homogeneous of degree 0. Therefore,

$$\beta \left[zf_k \left(1, \frac{k}{n}\right) + (1-\delta_k) \right] = 1$$

Since g does not enter this condition, we need $\frac{k}{n}$ to remain constant.

(b) Let $g \uparrow$, then the budget constraint implies that $zf(n,k) - g$ the amount of disposable income decreases. By the characterization of normality defined in the problem set, we have $c \downarrow$ and $l \downarrow$. Therefore, $n \uparrow$.

(c) Let $g \uparrow$. For $\frac{k}{n}$ to be constant, we need $k \uparrow$ since $n \uparrow$. Note that $f(\cdot)$ is increasing in both k and n, therefore $zf(k,n) \uparrow$, i.e. the output per worker increases if $g \uparrow$.

(5) (a) For f homogeneous of degree 1, we have f_k homogeneous of degree 0. Therefore,

$$\beta \left[zf_k \left(1, \frac{k}{n}\right) + (1-\delta_k) \right] = 1$$

Since θ does not enter this condition, we need $\frac{k}{n}$ to remain constant.

(b) Note that θ does not affect the choice of n, since it only appears to affect c^g which is separable from l and c. Hence, n will remain constant.

(c) Since n is constant for any movement in θ and $\frac{k}{n}$ is also constant, we have that $f(k,n)$ stays the same. Hence, the output per worker is also constant.

Skill-biased technical change

(1) First, we derived the marginal products of capital and skilled labor, i.e.

$$f_k(\cdot) = \mu\lambda z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}-1}(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}-1} k_t^{\rho-1}$$
$$f_{n_s}(\cdot) = (1-\mu)\lambda z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}-1}(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}-1} n_s^{\rho-1}$$

Then, we get

$$
\begin{aligned}
\frac{f_k}{f_{n_s}} &= \left(\frac{\mu}{1-\mu}\right)\left(\frac{k}{n_s}\right)^{\rho-1} \\
\ln\left(\frac{f_k}{f_{n_s}}\right) &= \ln\left(\frac{\mu}{1-\mu}\right) + (1-\rho)\ln\left(\frac{n_s}{k}\right) \\
\ln\left(\frac{n_s}{k}\right) &= \frac{1}{1-\rho}\ln\left(\frac{f_k}{f_{n_s}}\right) - \frac{1}{1-\rho}\ln\left(\frac{\mu}{1-\mu}\right)
\end{aligned}
$$

Therefore, the elasticity of substitution between capital and skilled labor is equal to

$$
\epsilon = \frac{\partial \ln\left(\frac{n_s}{k}\right)}{\partial \ln\left(\frac{f_k}{f_{n_s}}\right)} = \frac{1}{1-\rho}
$$

Note that ϵ depends only ρ and it is increasing in ρ.

(2) The interior point conditions can be summarized in the following way

$$
\begin{aligned}
c_t &: u_c(c_t, l_t) - \nu_t = 0 \\
n_u &: \nu_t f_{n_u}(z, k_t, n_u, n_s) - \psi_t = 0 \\
n_s &: \nu_t f_{n_s}(z, k_t, n_u, n_s) - \psi_t = 0 \\
l_t &: u_l(c_t, l_t) - \psi_t = 0 \\
x_t &: \mu_t - \nu_t = 0 \\
k_{t+1} &: \beta\nu_{t+1} z f_n(n_{t+1}, k_{t+1}) + \beta\mu_{t+1}(1-\delta_k) - \mu_t = 0 \\
&: c_t + x_t + g_t \leq f(z, k_t, n_u, n_s) \\
&: k_{t+1} \leq (1-\delta_k)k_t + x_{kt} \\
&: n_u + n_s + l_t \leq 1 \\
TVC &: \lim_{T\to\infty} \beta^T u_c(c_T, l_T) k_{T+1} = 0
\end{aligned}
$$

In steady state, our conditions are the following we have

$$
\begin{aligned}
u_c(c, 1-n_u-n_s) &= \nu \\
f_{n_u}(z, k, n_u, n_s) = f_{n_s}(z, k, n_u, n_s) &= \frac{u_l(c, 1-n_u-n_s)}{u_c(c, 1-n_u-n_s)} \\
\beta[f_k(z, k, n_u, n_s) + (1-\delta_k)] &= 1 \\
c + \delta_k k &= f(z, k, n_u, n_s)
\end{aligned}
$$

For $\beta[f_k(z, k, n_u, n_s) + (1-\delta_k)] = 1$ to hold, we need the following conditions

$$
\begin{aligned}
\lim_{k\to 0} f_k(z, k, n_u, n_s) &> \frac{1}{\beta} - (1-\delta_k) \\
\lim_{k\to \infty} f_k(z, k, n_u, n_s) &< \frac{1}{\beta} - (1-\delta_k)
\end{aligned}
$$

Now, we need to show that $c > 0$. Let $g = 0$. The fourth condition implies that

$$f(z, k, n_u, n_s) > \delta_k k$$

This holds if we have

$$f_k(z, k, n_u, n_s) > \delta_k$$

Since at the steady state we have $f_k(z, k, n_u, n_s) = \frac{1}{\beta} - (1 - \delta_k)$, we simply need the condition that

$$\frac{1}{\beta} - (1 - \delta_k) > \delta_k$$

This is true for $\beta \in (0, 1)$. For g small enough, we can simply extend the previous analysis due to continuity.

(3) First, we derived the marginal products of unskilled and skilled labor, i.e.

$$f_{n_s}(\cdot) = (1-\mu)\lambda z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}-1}(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}-1} n_s^{\rho-1}$$
$$f_{n_u}(\cdot) = (1-\lambda)z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}-1} n_u^{\sigma-1}$$

In competitive equilibrium, we know the price of labor is equal to its marginal product. Hence, $\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)}$ and

$$\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)} = (1-\mu)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{n_s^{\rho-1}}{n_u^{\sigma-1}}\right)(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}-1}$$

We can express this in terms of ratio and log-linearize the skill-premium in the following way

$$\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)} = (1-\mu)\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{n_u}{n_s}\right)^{1-\sigma}\left(\mu\left(\frac{k}{n_s}\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}$$
$$\ln\left(\frac{w_s}{w_u}\right) \approx (1-\sigma)\ln\left(\frac{n_u}{n_s}\right) + \mu\left(\frac{\sigma}{\rho} - 1\right)\left(\frac{k}{n_s}\right)^\rho$$

For the sake of argument, I will assume that σ is defined in a region such that $(1-\sigma)$ is positive. This implies that an increase in $\frac{n_u}{n_s}$ will increase $\frac{w_s}{w_u}$. For $\frac{k}{n_s}$ it is a bit more ambiguous than that. If $\rho > 0$ then $\frac{k}{n_s} \uparrow$ will imply an increase in $\frac{w_s}{w_u}$ and vice versa.

(4) (a) Let $h(k, n_u, n_s) = (\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}}$. This implies $f(z, k, n_u, n_s) = zh(k, n_u, n_s)$ and that $h(\cdot)$ is homogeneous of degree one.

Looking at the third condition for the steady state, we have

$$\beta\left[zh_k\left(\frac{k}{n_s}, \frac{n_u}{n_s}, 1\right) + (1-\delta_k)\right] = 1$$

If $z \uparrow$, we need $h_k\left(\frac{k}{n_s}, \frac{n_u}{n_s}, 1\right) \downarrow$ to offset this increase. Since, in equilibrium $\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)} = 1$ is constant, we need that any change in $\frac{k}{n_s}$ must be match by $\frac{n_u}{n_s}$.

If $\rho > 0$, then $\frac{k}{n_s} \uparrow$ and $\frac{n_u}{n_s} \downarrow$ satisfy both conditions. If $\rho < 0$ the situation is a bit more ambiguous, since $\frac{n_u}{n_s} \downarrow$ and $\frac{k}{n_s} \uparrow$ both decrease $\frac{w_s}{w_u}$. Hence, we need either $\frac{k}{n_s} \downarrow$ or $\frac{n_u}{n_s} \uparrow$. Let's assume for the moment that $\rho > 0$, i.e. $\frac{k}{n_s} \uparrow$ for $z \uparrow$.

(b) If $z \uparrow$ and $\rho > 0$, we have $\frac{k}{n_s} \uparrow$ and $\frac{n_u}{n_s} \downarrow$ which implies that $h(\cdot) \uparrow$. In turns, this yield that $zh(\cdot) \uparrow$, i.e. output per worker increase for an increase in z.

(c) Note that the skill premium is constant and equal to 1. Therefore, it does not change.

(5) (a) Looking at the third condition for the steady state, we have

$$\beta \left[z h_k \left(\frac{k}{n_s}, \frac{n_u}{n_s}, 1 \right) + (1 - \delta_k) \right] = 1$$

If $\frac{n_u}{n_s} \uparrow$, we need $\frac{k}{n_s}$ to offset the increase in $h_k \left(\frac{k}{n_s}, \frac{n_u}{n_s}, 1 \right)$, i.e $\frac{k}{n_s} \uparrow$. Therefore an increase in relative supply of skill to unskill workers will imply an increase in $\frac{k}{n_s}$.

(b) Note that an increase in both $\frac{n_u}{n_s} \uparrow$ and $\frac{k}{n_s} \uparrow$ implies that $f(\cdot) \uparrow$. Thus, the output per worker will also increase.

(c) Note that the skill premium is constant and equal to 1. Therefore, it does not change. Now, this seems to be a mistake since by the previous discussion

$$\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)} = (1-\mu) \left(\frac{\lambda}{1-\lambda} \right) \left(\frac{n_u}{n_s} \right)^{1-\sigma} \left(\mu \left(\frac{k}{n_s} \right)^{\rho} + (1-\mu) \right)^{\frac{\sigma}{\rho}-1}$$
$$\ln \left(\frac{w_s}{w_u} \right) \approx (1-\sigma) \ln \left(\frac{n_u}{n_s} \right) + \mu \left(\frac{\sigma}{\rho} - 1 \right) \left(\frac{k}{n_s} \right)^{\rho}$$

should govern the movement of the skill-premium with changes in $\frac{n_u}{n_s}$ and $\frac{k}{n_s}$. I'm therefore unsure, if $\frac{w_s}{w_u} = \frac{f_{n_s}(\cdot)}{f_{n_u}(\cdot)} = (1-\mu) \left(\frac{\lambda}{1-\lambda} \right) \left(\frac{n_u}{n_s} \right)^{1-\sigma} \left(\mu \left(\frac{k}{n_s} \right)^{\rho} + (1-\mu) \right)^{\frac{\sigma}{\rho}-1}$ applies or $\frac{w_s}{w_u} = 1$.

(6) To explore the effect of λ on $\frac{w_s}{w_u}$, we take the equation derived previously and compute its derivate with respect to λ, i.e.

$$\frac{\partial \frac{w_s}{w_u}}{\partial \lambda} = \frac{1}{(1-\lambda)^2} > 0$$

Therefore, as $\lambda \uparrow$, we have $\frac{w_s}{w_u} \uparrow$.