ECON 6140 - Problem Set # 2

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Problem 1

(1) Figure 1 shows the possible endowment paths.

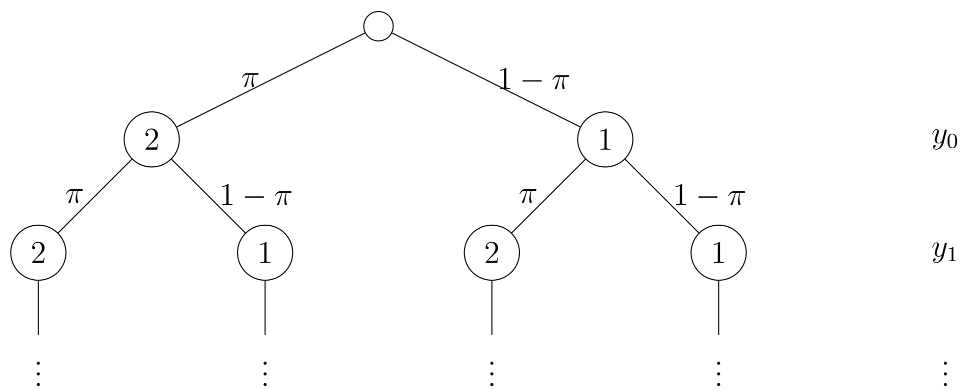

Figure 1: Possible Endowment Paths

(2) For $t \geq 1$, the Bellman equation is given by

$$\begin{aligned} V(a,y) = \max_{c,a'} & \ln(c) + \beta V(a',y') \\ \text{s.t. } & a' = (1+r)(a-c) + y' \\ & a \geq c \end{aligned}$$

Note that for $t \geq 1$, $y' = y$ and therefore $c = \frac{y-a'}{1+r} + a$, thus

$$V(a,y) = \max_{a' \geq y} \ln\left(\frac{y-a'}{1+r} + a\right) + \beta V(a',y)$$

Hence, the Euler equation for this problem is given by

$$\frac{1}{c} \geq \beta(1+r)\frac{1}{c'}$$

where it holds with equality if $a' \geq y$ ($a \geq c$) is not binding.

Since $\beta(1+r) = 1$, we get that Euler yields

$$c' \geq c$$

Let $c' = c = c^*$. If we can show that $a \geq c$ is not binding, then we are right to assume that $c' = c = c^*$. Looking at the budget constraint and imposing the no Ponzi scheme condition, we get that for $t \geq 1$,

$$\begin{aligned}
a_t &= \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \left(c_{t+j} - \frac{y_{t+1+j}}{1+r}\right) \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \left(c^* - \frac{y_1}{1+r}\right) \\
a_t &= \frac{1+r}{r} c^* - \frac{1}{r} y_1
\end{aligned}$$

Hence, the assets holdings are constant. From the budget constraint, we have

$$a_t = a_1 = (1+r)(a_0 - c_0) + y_1 = (1+r)(y_0 - c_0) + y_1$$

and therefore

$$\begin{aligned}
a_1 &= \frac{1+r}{r} c^* - \frac{1}{r} y_1 \\
c^* &= \frac{1}{1+r} y_1 + \frac{r}{1+r} a_1 \\
&= \frac{1}{1+r} y_1 + \frac{r}{1+r} ((1+r)(y_0 - c_0) + y_1) \\
\Rightarrow c^* &= y_1 + r(y_0 - c_0) < a_t
\end{aligned}$$

i.e. $a \geq c$ is not binding.

We can plug our results back into our Bellman equation to get the following value function

$$\begin{aligned}
V(a, y) &= \ln(y_1 + r(y_0 - c_0)) + \beta V(a, y) \\
V(a, y) &= \frac{1}{1-\beta} \ln(y_1 + r(y_0 - c_0)) \\
\Rightarrow V(a, y) &= \frac{1+r}{r} \ln(y_1 + r(y_0 - c_0))
\end{aligned}$$

Thus, the problem at $t = 0$ is the following

$$\begin{aligned}
w(y_0) &= \max_{c_0 \geq y_0} \ln(c_0) + \beta \,\mathsf{E} \left\{ \left(\frac{1+r}{r}\right) \ln(y_1 + r(y_0 - c_0)) \right\} \\
&= \max_{c_0 \geq y_0} \ln(c_0) + \frac{1}{r} \left[\pi \ln(2 + r(y_0 - c_0)) + (1-\pi) \ln(1 + r(y_0 - c_0))\right]
\end{aligned}$$

Then, the FOCs yield

$$\frac{1}{c_0} \geq \frac{\pi}{2 + r(y_0 - c_0)} + \frac{1-\pi}{1 + r(y_0 - c_0)}$$

where it holds with equality if $y_0 = c_0$

Now, let $y_0 = 2$. Can $y_0 = c_0$? If $y_0 = c_0 = 2$, then $\frac{1}{c_0} \geq \frac{\pi}{2+r(y_0-c_0)} + \frac{1-\pi}{1+r(y_0-c_0)}$ holds with inequality and

$$\frac{1}{2} > \frac{\pi}{2} + \frac{1-\pi}{1} = \frac{2-\pi}{2} \geq \frac{1}{2}$$

i.e. contradiction. Thus, we know that $a_0 = y_0$ and we can solve for c_0 with

$$\frac{1}{c_0} = \frac{\pi}{2 + r(y_0 - c_0)} + \frac{1-\pi}{1 + r(y_0 - c_0)}$$

Finally, for $t \geq 1$, $a_t = (1+r)(a_0 - c_0) + y_1 = (1+r)(y_0 - c_0) + y_1$ and $c_t = r(a_0 - c_0) + y_1 = r(y_0 - c_0) + y_1$.

(3) Let $y_0 = 1$. Can $y_0 > c_0$? If $y_0 > c_0$, then $\frac{1}{c_0} \geq \frac{\pi}{2+r(y_0-c_0)} + \frac{1-\pi}{1+r(y_0-c_0)}$ holds with equality and

$$1 < \frac{1}{c_0} = \frac{\pi}{2 + r(1 - c_0)} + \frac{1-\pi}{1 + r(1 - c_0)} < \frac{2-\pi}{2} \leq 1$$

which is a contradiction.

Hence, the constraint binds and $y_0 = c_0 = 1$. In turn, we have for $t \geq 1$, $c_t = y_1$.

(4) For $y_0 = 2$, we have

$$u'(c^*) = \begin{cases} \frac{1}{r(2-c_0)+2} & \text{with prob. } \pi \\ \frac{1}{r(2-c_0)+1} & \text{with prob. } (1-\pi) \end{cases}$$

For $y_0 = 1$, we have

$$u'(c^*) = \begin{cases} \frac{1}{2} & \text{with prob. } \pi \\ 1 & \text{with prob. } (1-\pi) \end{cases}$$

(5) In LS, we have the following proposition from Chamberlain and Wilson

Proposition 1. *If there is an $\epsilon > 0$ such that for any $\alpha \in \mathbb{R}_+$*

$$P(\alpha \leq y_t \leq \alpha + \epsilon \mid I_t) < 1 - \epsilon$$

for all I_t and $t \geq 0$, then $P(\lim_{t\to\infty} c_t = \infty) = 1$.

Let $\alpha = y_1$. Note that for any $t \geq 1$ and $\epsilon > 0$, we have

$$P(\alpha \leq y_t \leq \alpha + \epsilon \mid I_t) = 1 \not< 1 - \epsilon$$

Hence, the conditions are not satisfied.

Problem 2

(1) Let $y_t = y_1$ and $a_t = -\phi$. Then,

$$\begin{aligned} c_t + a_{t+1} &= (1+r)a_t + y_t \\ c_t &= -(1+r)\frac{y_1}{r} + y_1 - a_{t+1} \\ c_t &= -\phi - a_{t+1} \leq 0 \end{aligned}$$

Hence, to prevent a situation where $a_{t+1} \geq -\phi$, we need $c_t > 0$. Having the following Inada condition, $\lim_{c\to 0} u'(c) = \infty$, prevents the borrowing constraint when $y_t = y_1$.

Now, let $y_{t-1} = y_i \in Y$. If there's a chance that in the next period we go from $i \to 1$, i.e. $y_t = y_1$, then the borrowing constraint can't hold with equality due to the previous argument. Thus, we require $\pi_{i1} > 0$, i.e. a positive probability that from any i we can go to 1 in the next period.

(2) First, we carry over the previous conditions on u and π to this new problem.

By iterating, the budget constraint and imposing a no Ponzi-scheme conditions, we get

$$\begin{aligned} a_t &= \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (c_{t+j} - y_{t+j}) \\ \Rightarrow a_t &= \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (\mathsf{E}_t\{c_{t+j}\} - \mathsf{E}_t\{y_{t+j}\}) \end{aligned}$$

Similarly, we have

$$a_{t+1} = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (\mathsf{E}_t\{c_{t+1+j}\} - \mathsf{E}_t\{y_{t+1+j}\})$$

Hence, given $a_0 > 0$, if we can show $a_{t+1} \geq a_t \geq \cdots \geq a_0 > 0$.

With no borrowing constraint, we have that the Euler equation holds with equality, i.e.

$$u'(c_t) = \beta(1+r)\,\mathsf{E}_t\{u'(c_{t+1})\}$$

First, we impose that $\beta(1+r) \geq 1$, hence

$$u'(c_t) \geq \mathsf{E}_t\{u'(c_{t+1})\}$$

Next, we impose that u is convex so that we can apply Jensen's inequality and get

$$u'(c_t) \geq u'(\mathsf{E}_t\{c_{t+1}\})$$

Finally, imposing that u' is decreasing, we have

$$E_t\{c_{t+1}\} \geq c_t$$

This in turns, implies that c_t is sub-martingale and therefore

$$\mathsf{E}_t\left\{c_{t+1+j}\right\} = \mathsf{E}_t\left\{\mathsf{E}_{t+j}\left\{c_{t+1+j}\right\}\right\} \geq \mathsf{E}_t\left\{c_{t+j}\right\}$$

If have conditions on π such that y_t is a super martingale, we would get the following

$$\mathsf{E}_t\left\{y_{t+1+j}\right\} = \mathsf{E}_t\left\{\mathsf{E}_{t+j}\left\{y_{t+1+j}\right\}\right\} \leq \mathsf{E}_t\left\{y_{t+j}\right\}$$

and

$$\begin{aligned}
\mathsf{E}_t\left\{c_{t+1+j}\right\} - \mathsf{E}_t\left\{y_{t+1+j}\right\} &\geq \mathsf{E}_t\left\{c_{t+j}\right\} - \mathsf{E}_t\left\{y_{t+j}\right\} \\
\left(\frac{1}{1+r}\right)^j \left(\mathsf{E}_t\left\{c_{t+1+j}\right\} - \mathsf{E}_t\left\{y_{t+1+j}\right\}\right) &\geq \left(\frac{1}{1+r}\right)^j \left(\mathsf{E}_t\left\{c_{t+j}\right\} - \mathsf{E}_t\left\{y_{t+j}\right\}\right) \\
\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(\mathsf{E}_t\left\{c_{t+1+j}\right\} - \mathsf{E}_t\left\{y_{t+1+j}\right\}\right) &\geq \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(\mathsf{E}_t\left\{c_{t+j}\right\} - \mathsf{E}_t\left\{y_{t+j}\right\}\right) \\
a_{t+1} &\geq a_t
\end{aligned}$$

i.e. the desired result.

Problem 3

(1) First, note that

$$\begin{aligned}
\ln c_{t+1} &\sim N(\mu_t, \nu_t) \\
\Rightarrow -\gamma \ln c_{t+1} &\sim N(-\gamma\mu_t, \gamma^2\nu_t)
\end{aligned}$$

This, implies that $\mathsf{E}_t\left\{c_{t+1}^{-\gamma}\right\} = e^{-\gamma u_t + \frac{1}{2}\gamma^2\nu_t}$.

Then, the Euler equation turns out to be

$$\begin{aligned}
u'(c_t) &= \beta R\, \mathsf{E}_t\left\{u'(c_{t+1})\right\} \\
c_t^{-\gamma} &= \beta R\, \mathsf{E}_t\left\{c_{t+1}^{-\gamma}\right\} \\
c_t^{-\gamma} &= \beta R\, \mathsf{E}_t\left\{e^{-\gamma \ln c_{t+1}}\right\} \\
c_t^{-\gamma} &= \beta R e^{-\gamma\mu_t + \frac{1}{2}\gamma^2\nu_t} \\
-\gamma \ln c_t &= \ln \beta R - \gamma\mu_t + \frac{1}{2}\gamma^2\nu_t \\
-\gamma \ln c_t &= \ln \beta R - \gamma\, \mathsf{E}_t\left\{\ln c_{t+1}\right\} + \frac{1}{2}\gamma^2\nu_t \\
\gamma\, \mathsf{E}_t\left\{\ln c_{t+1} - \ln c_t\right\} &= \ln \beta R + \frac{1}{2}\gamma^2\nu_t \\
\Rightarrow \mathsf{E}_t\left\{\Delta \ln c_{t+1}\right\} &= \frac{1}{\gamma}\ln \beta R + \frac{1}{2}\gamma\nu_t
\end{aligned}$$

(2) Note that if we increase volatility, then $\mathsf{E}_t\left\{\Delta \ln c_{t+1}\right\} \uparrow$. This means that the expected consumption grows, implying that current saving must increase, i.e. precautionary saving behavior.

(3) First, we look at the regression

$$\begin{aligned}\Rightarrow \mathsf{E}\left\{\mathsf{E}_t\left\{\Delta \ln c_{t+1}\right\} \mid y_t\right\} &= \mathsf{E}\left\{\mathsf{E}_t\left\{\alpha_0 + \alpha_1 y_t + \epsilon_{t+1}\right\} \mid y_t\right\} \\ &= \alpha_0 + \alpha_1 y_t + \mathsf{E}\left\{\epsilon_{t+1} \mid y_t\right\}\end{aligned}$$

Note that for the estimate of the previous regression to be consistent, we need $\mathsf{E}\left\{\epsilon_{t+1} \mid y_t\right\} = 0$.

Looking at the previous model, where PIH holds, we have

$$\begin{aligned}\Rightarrow \mathsf{E}\left\{\mathsf{E}_t\left\{\Delta \ln c_{t+1}\right\} \mid y_t\right\} &= \mathsf{E}\left\{\frac{1}{\gamma}\ln \beta R + \frac{1}{2}\gamma \nu_t \mid y_t\right\} \\ \mathsf{E}_t\left\{\Delta \ln c_{t+1}\right\} &= \underbrace{\frac{1}{\gamma}\ln \beta R}_{\alpha_0} + \underbrace{0}_{\alpha_1 y_t} + \underbrace{\frac{1}{2}\gamma\, \mathsf{E}\left\{\nu_t \mid y_t\right\}}_{\mathsf{E}\{\epsilon_{t+1} \mid y_t\}}\end{aligned}$$

Thus, if we have $\mathsf{E}\left\{\nu_t \mid y_t\right\} \neq 0$, the implied regression $\Delta \ln c_{t+1} = \alpha_0 + \alpha_1 y_t + \epsilon_{t+1}$ has $\mathsf{E}\left\{\epsilon_{t+1} \mid y_t\right\} = 0$. Hence, it would be possible to have α_1 statistically different from 0, while the PIH still holds.

Problem 4

First, we start with the budget constraint,

$$\begin{aligned}c_t + a_{t+1} &= Ra_t + y_t \\ \Rightarrow a_{t+1} &= Ra_t + y_t - B(Ra_t + y_t) - D \\ &= (1-B)(Ra_t + y_t) - D\end{aligned}$$

Ignoring borrowing limits and the fact that income is i.i.d., the Euler equation reduce to the following:

$$\begin{aligned}u'(c_t) &= \beta R\, \mathsf{E}_t\left\{u'(c_{t+1})\right\} \\ e^{-\sigma c_t} &= \beta R\, \mathsf{E}_t\left\{e^{-\sigma c_{t+1}}\right\} \\ e^{-\sigma(B(Ra_t+y_t)+D)} &= \beta R\, \mathsf{E}_t\left\{e^{-\sigma(B(Ra_{t+1}+y_{t+1})+D)}\right\} \\ e^{-\sigma(B(Ra_t+y_t)+D)} &= \beta R\, \mathsf{E}_t\left\{e^{-\sigma(B(R((1-B)(Ra_t+y_t)-D)+y_{t+1})+D)}\right\} \\ e^{-\sigma(B(Ra_t+y_t)+D)} &= \beta R\, \mathsf{E}_t\left\{e^{-\sigma B y_{t+1}}\right\} e^{-\sigma(B(R((1-B)(Ra_t+y_t)-D))+D)} \\ e^{-\sigma(B(Ra_t+y_t)+D)} &= \beta R\, \mathsf{E}\left\{e^{-\sigma B y_{t+1}}\right\} e^{-\sigma(BR(1-B)(Ra_t+y_t)+(1-RB)D)} \\ e^{-\sigma B((1-R(1-B))(Ra_t+y_t)+RD)} &= \beta R\, \mathsf{E}\left\{e^{-\sigma B y_{t+1}}\right\}\end{aligned}$$

Hence, since the right side is a constant, we need

$$(1 - R(1 - B)) = 0 \Leftrightarrow B = 1 - \frac{1}{R}$$

Hence,

$$\begin{aligned}
e^{-\sigma B((1-R(1-B))(Ra_t+y_t)+RD)} &= \beta R\,\mathsf{E}\left\{e^{-\sigma B y_{t+1}}\right\} \\
e^{-\sigma(R-1)D} &= \beta R\,\mathsf{E}\left\{e^{-\sigma\left(\frac{R-1}{R}\right)y_{t+1}}\right\} \\
-\sigma(R-1)D &= \ln\beta R + \ln\mathsf{E}\left\{e^{-\sigma\left(\frac{R-1}{R}\right)y_{t+1}}\right\} \\
\Rightarrow D &= -\frac{1}{\sigma(R-1)}\left[\ln\beta R + \ln\mathsf{E}\left\{e^{-\sigma\left(\frac{R-1}{R}\right)y_{t+1}}\right\}\right]
\end{aligned}$$

Thus, since R is constant, we have that B and D are constant and identical across agents. Recall that

$$a_{t+1} = (1 - B)(Ra_t + y_t) - D$$

How does a_{t+1} change if we have mean preserving spread? Then $\mathsf{E}\left\{e^{-\sigma\left(\frac{R-1}{R}\right)y_{t+1}}\right\}$ will increase and therefore $D \downarrow$ and finally $a_{t+1} \uparrow$. Since D is the same for everyone, the change in a_{t+1} will be identical across the distribution of a_t and y_t.