

ECON 6140 - Problem Set # 3

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Open Economy with Durable Goods

(1) The Langrangian of the problem:

$$\begin{aligned}\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [& u(c_t) + v(z_{t+1}) + \lambda_t(R^*b_t + af(k_t) - c_t - x_{kt} - qx_{zt} - b_{t+1} - \frac{d}{2}(z_{t+1} - z_t)^2) \\ & + \mu_t(x_{zt} + (1 - \delta_z)z_t - z_{t+1}) \\ & + \nu_t(x_{kt} + (1 - \delta_k)k_t - k_{t+1})]\end{aligned}$$

KKT:

$$\begin{aligned}c_t : c_t^{-\eta} - \lambda_t &= 0 \\ z_{t+1} : z_{t+1}^{-\eta} - \lambda_t d(z_{t+1} - z_t) + \beta \lambda_{t+1} d(z_{t+2} - z_{t+1}) - \mu_t + \beta \mu_{t+1} (1 - \delta_z) &= 0 \\ b_{t+1} : \beta \lambda_{t+1} R^* - \lambda_t &= 0 \\ k_{t+1} : \beta \lambda_{t+1} af'(k_{t+1}) + \beta \nu_{t+1} (1 - \delta_k) - \nu_t &= 0 \\ x_{zt} : \mu_t - q \lambda_t &= 0 \\ x_{kt} : \nu_t - \lambda_t &= 0 \\ &: R^*b_t + af(k_t) \geq c_t + x_{kt} + qx_{zt} + b_{t+1} + \frac{d}{2}(z_{t+1} - z_t)^2 \\ &: x_{zt} + (1 - \delta_z)z_t \geq z_{t+1} \\ &: x_{kt} + (1 - \delta_k)k_t \geq k_{t+1} \\ TVC_1 : \lim_{T \rightarrow \infty} \beta^T u_c(c_T) k_{T+1} &= 0 \\ TVC_2 : \lim_{T \rightarrow \infty} \beta^T u_c(c_T) b_{T+1} &= 0\end{aligned}$$

Note that we need an extra transversality condition for capital.

Note that since $R^* = \frac{1}{\beta}$, then $\lambda_t = \lambda_{t+1}$.

The steady state FOCs are given by

$$\begin{aligned}\left(\frac{c}{z}\right)^\eta &= q(1 - \beta(1 - \delta_z)) \\ \beta(af'(k) + (1 - \delta_k)) &= 1\end{aligned}$$

and the steady state constraint yield

$$\begin{aligned}(R^* - 1)b + af(k) &= c + x_k + qx_z \\ x_z &= \delta_z z \\ x_k &= \delta_k k\end{aligned}$$

Note that if $f'(\cdot)$ is decreasing, k is uniquely determined by $(\beta, a, \delta_k, f(\cdot))$.

Since d doesn't enter these equations, hence it is irrelevant for the steady states.

If $q \uparrow$ or $\delta_z \uparrow$, then $\frac{c}{z} \uparrow$. Since k is only determined by the coefficient, we need b to adjust such that $(R^* - 1)b + af(k) = c + q\delta_z z + \delta_k k$ holds for the new steady state.

- (2) Recall that since $R^* = \frac{1}{\beta}$, we have $\lambda_t = \lambda_{t+1} = \lambda$. This in turns implies that $\mu_t = q\lambda$ and $\nu_t = \lambda$, i.e. every Lagrange multiplier is constant. This implies that c_t and z_{t+1} are also constant for $t \geq 0$. Note that z_0 is given, therefore $z_t = z$ for $t \geq 1$ which might not be equal to z_0 . In fact, z is equal to the steady state value of

$$z = c[q(1 - \beta(1 - \delta_z))]^{-\frac{1}{\eta}}$$

Moreover, we get

$$\beta(af'(k_{t+1}) + (1 - \delta_k)) = 1$$

i.e. k_{t+1} is also constant if $f'(\cdot)$ is decreasing and k is given by $\beta(af'(k) + (1 - \delta_k)) = 1$.

Since k_{t+1} is constant $x_{kt} + (1 - \delta_k)k_t = k_{t+1}$ becomes $x_{kt} = \delta_k k$ for $t \geq 1$ and $x_{k0} = k - (1 - \delta_k)k_0$ for $t = 0$. Since z_{t+1} is constant $x_{zt} + (1 - \delta_z)z_t = z_{t+1}$ becomes $x_{zt} = \delta_z z$ for $t \geq 1$ and $x_{z0} = z - (1 - \delta_z)z_0$ for $t = 0$.

Now, we look at the dynamics of b_t . Note that it is fully determined by

$$b_{t+1} = R^*b_t + af(k_t) - c - x_{kt} - qx_{zt}$$

Since we know b_0 , the whole sequence of x_{kt} and x_{zt} , the only thing missing is c . As shown previously, c_t is constant. Hence, to pin down b_t we need to find $c_0 = c$.

Let's look at the phase diagram of b and z with $t \geq 1$. Note that we get the following equation

$$\begin{aligned}z_{t+1} &= z_t = z \\ b_{t+1} &= R^*b_t + af(k) - c - \delta_k k - q\delta_z z_t\end{aligned}$$

Hence, the locus for b is

$$b_t = -\frac{a\beta}{1-\beta}f(k) + \frac{c\beta}{1-\beta} + \frac{\delta_k\beta}{1-\beta}k + \frac{q\delta_z\beta}{1-\beta}z_t$$

Hence, the steady state is $b = -\frac{a\beta}{1-\beta}f(k) + \frac{c\beta}{1-\beta} + \frac{\delta_k\beta}{1-\beta}k + \frac{q\delta_z\beta}{1-\beta}z$. Since z_t is constant, to get to the steady state, we need $b_t = b$. Therefore, we need c to be such that $b_1 = -\frac{a\beta}{1-\beta}f(k) + \frac{c\beta}{1-\beta} + \frac{\delta_k\beta}{1-\beta}k + \frac{q\delta_z\beta}{1-\beta}z$.

Hence, we choose $c_0 = c$ such that $b_1 = R^*b_0 + af(k_0) - c_0 - x_{k0} - qx_{z0}$ holds and voilà!

(3) Again, every Lagrange multiplier are constant. Therefore, we get

$$\beta(af'(k_{t+1}) + (1 - \delta_k)) = 1$$

i.e. k_{t+1} is constant if $f'(\cdot)$ is decreasing. Since k_{t+1} is constant $x_{kt} + (1 - \delta_k)k_t = k_{t+1}$ becomes $x_{kt} = \delta_k k$ for $t \geq 1$ and $x_{k0} = k - (1 - \delta_k)k_0$ for $t = 0$.

Moreover, the FOC of z_{t+1} is now

$$z_{t+1}^{-\eta} = \lambda[d(z_{t+1} - z_t) - \beta d(z_{t+2} - z_{t+1}) + q(1 - \beta(1 - \delta_z))]$$

Since capital is constant, we can focus on a phase diagram to see what happens to z_t only. Let $y_t = z_{t+1} - z_t$. Hence, we have the two following dynamic equations

$$\begin{aligned} z_{t+1} &= y_t + z_t \\ y_{t+1} &= \frac{1}{\beta}y_t + \frac{q}{\beta d}(1 - \beta(1 - \delta_z)) - \frac{1}{\lambda\beta d}(y_t + z_t)^{-\eta} \end{aligned}$$

Take the loci where z is constant. This yields $y = 0$, i.e. the y -axis.

For the second loci its shape is not too important for our analysis. We simply note that for $y = 0$, we get

$$z^{-\eta} = \lambda q(1 - \beta(1 - \delta_z))$$

Since, $\lambda = c^{-\eta}$, we have $z^{-\eta} = c^{-\eta}q(1 - \beta(1 - \delta_z))$, i.e. the steady state derived previously.

To describe the dynamics, take $y_t < 0$. This implies that $z_{t+1} < z_t$. Therefore on the steady path, if we start over the steady state, z_t will decrease until it reaches z . The reverse can be said for $y_t > 0$. The speed of converge is describe in part 4).

- (4) Note that as derived previously, capital goods are constant for period $t \geq 1$. Hence to converge to the steady state, we only one period.

For durable goods, our analysis implies that regardless if we start over or under the second locus, the y_t will converge to the steady state $y = 0$ over time. This in turns implies that the difference between z_{t+1} and z_t will decrease every period until $z_t = z$. Hence, the convergence of durable goods is not “instantaneous” like capital.

Transition paths in the one sector growth model

- (1) With the parameters defined in the problem set, we get that the steady state is $k^* = 10.03$ and $c^* = 1.639$. Thus, our starting value is $k_0 = 9.027599$. Figure 1 shows the transition of k_0 on the steady path. It takes 53 periods to converge to the steady state when starting from $k_0 = .9k^*$.

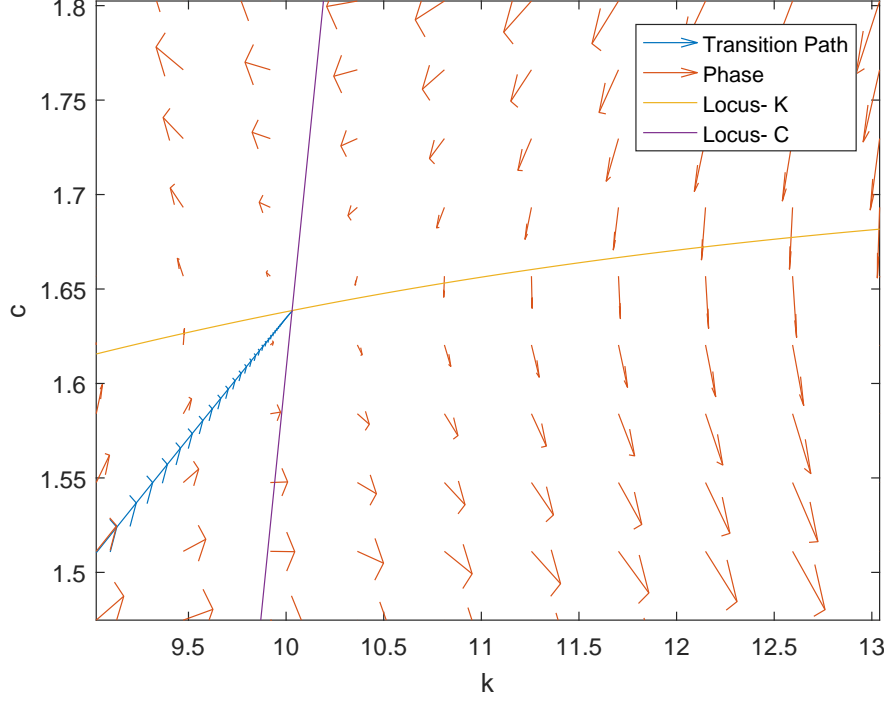


Figure 1: Transition path for growth model starting at $k_0 = .9k^*$

Note that we also plot the loci of our phase diagram on Figure 1. Recall that they are given by

$$\begin{aligned} k : c_t &= Ak_t^\alpha - \delta k_t \\ c : c_t &= k_t^\alpha + (1 - \delta)k_t - k^* \end{aligned}$$

where k^* is the steady state value equal to $\left(\frac{\alpha\beta A}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}$.

(2) See Figure 2.

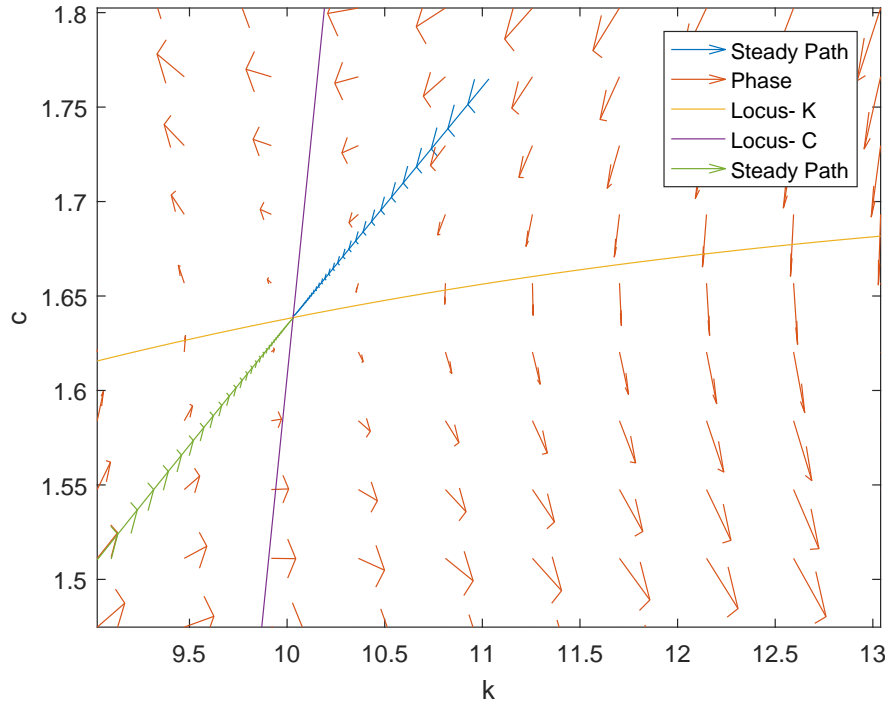


Figure 2: Phase diagram for growth model

Note that the scales on this figure are not 1-1.

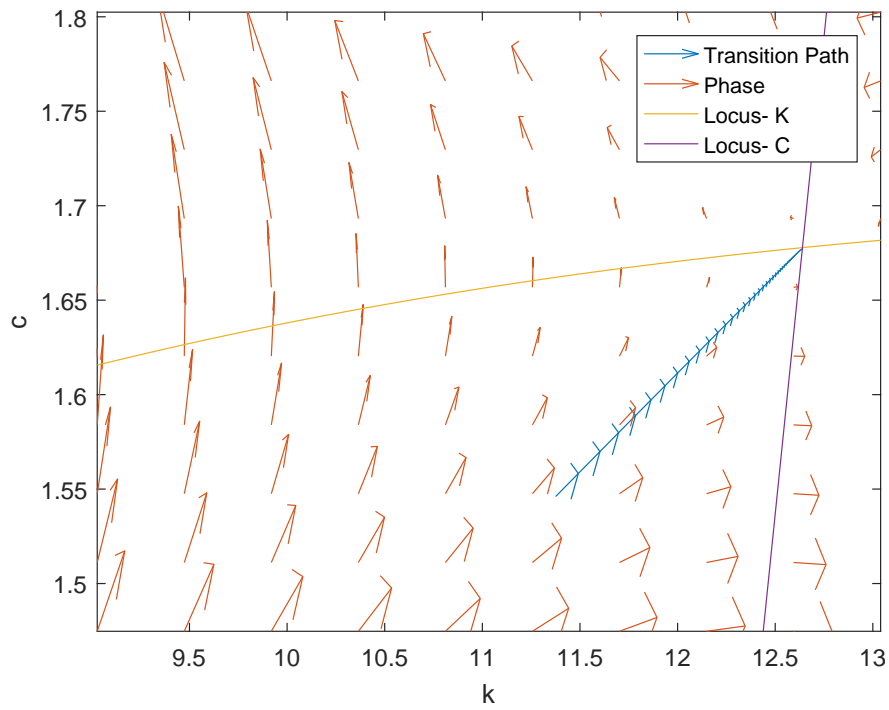


Figure 3: Phase diagram for growth model with increase of 1% in β

- (3) By increasing β by 1% the second locus shift to the right and we get that the new steady state is $k^* = 12.640$ and $c^* = 1.678$. Therefore the starting value is $k_0 = 11.376$. Note that since k^* is bigger, k_0 is a bit farther in absolute terms. This results in a longer convergence term of 59 periods. The transition path is plotted in Figure 3.
- (4) Note that if A goes up, the first locus also goes up. This implies that the new steady path also shifts up and the new steady state will be one where $k^* \uparrow$ and $c^* \uparrow$.

In this problem, we let the increase in A happen at $t = 1$. Moreover, let's assume that it was an unexpected increase permanent increase in A , i.e. consumer don't preventively adjust to a future steady path.

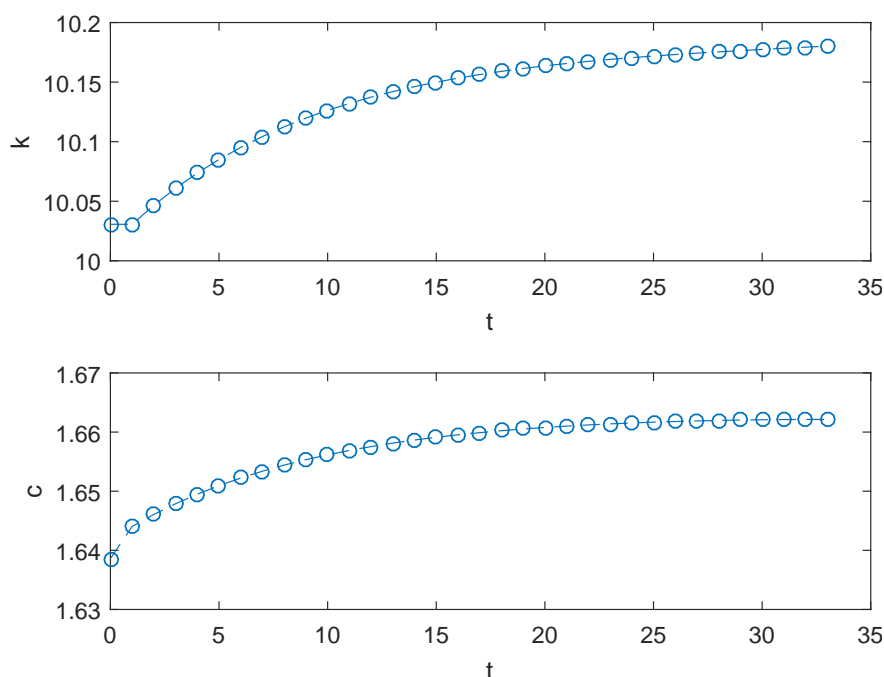


Figure 4: Time path for consumption and capital with increase of 1% in A

Now, since the point (k_0, c_0) is the old steady , it might not be on the new steady path. To adjust to that the consumer will instantaneously change it consumption at time $t = 1$ to get on the new steady path unlike k_0 that can't act reactively. From there on on out, both k_t and c_t will monotonically increase until it reaches the new steady path as shown in Figure 4.

Code

```

1 %% Shooting Algorithm
2 % Course: ECON 6140
3 % Version: 1.0
4 % Author: Julien Neves

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5
6 %% Question
7 tol = .001; % tolerance
8 N = 100000; % grid size
9 T = 600; % time periods
10
11 alpha = 0.33; % labor share
12 delta = 0.05; % depreciation of capital
13 sigma = 0.5; % CRRA
14 beta = .98; % discount factor
15 A = 1; % technology
16
17 k = zeros(1,T+1); % initial k path vector
18 c = zeros(1,T+1); % initial c path vector
19
20 kstar = ((alpha*beta*A)/(1-beta+beta*delta))^(1/(1-alpha)); % k
    steady state
21 cstar = A*kstar^alpha+(1-delta)*kstar-kstar; % c steady state
22
23 k0 = 0.9*kstar; % starting k value
24
25 lb_k = 0.9*kstar; % lower bound of k axis
26 ub_k = 1.3*kstar; % upper bound of k axis
27
28 lb_c = 0.9*cstar; % lower bound of c axis
29 ub_c = 1.1*cstar; % lower bound of c axis
30
31 axis_k = lb_k : (ub_k-lb_k)/(N-1) : ub_k; % k axis
32 axis_c = lb_c : (ub_c-lb_c)/(N-1) : ub_c; % c axis
33
34 crit = 1; % initialize tolerance criteria
35 ite = 1; % initialize iteration
36
37 while (crit>tol && ite<=length(axis_c))
38     k(1) = k0; % set starting k0
39     c(1) = axis_c(ite); % pick c0
40     for t = 1:T
41         k(t+1) = A*k(t)^alpha+(1-delta)*k(t)-c(t); % compute k(t
            +1)
42         c(t+1) = c(t)*(beta*alpha*A*k(t+1)^(alpha-1)+beta*(1-delta
            ))^(1/sigma); %compute c(t+1)
43         crit = max(abs(kstar-k(t+1)),abs(cstar-c(t+1))); %
            deviation from steady state
44         if crit<=tol
45             % if close to steady state stop algorithm

```

```

46         k = k(1:t+1); % cut path after convergences
47         c = c(1:t+1); % cut path after convergences
48         break
49     else
50         continue
51     end
52 end
53 ite = ite + 1; % update iteration
54 end
55
56 u = gradient(k); % compute k gradient of steady path
57 v = gradient(c); % compute c gradient of steady path
58
59 loci_k = A*axis_k.^alpha - delta*axis_k; % compute k loci
60 loci_c = axis_k.^alpha +(1-delta)*axis_k-kstar; % compute c loci
61
62 [K,C]= meshgrid(lb_k : (ub_k-lb_k)/(10-1) : ub_k,lb_c : (ub_c-lb_c
63 )/(10-1) : ub_c); % create (k,c) grid
64
65 dK = A*K.^alpha-delta.*K-C; % compute k gradient of every point in
66 grid
67 dC = C.*(beta*alpha*A*(dK+K).^(alpha-1)+beta*(1-delta)).^(1/sigma)
68 -C; % compute c gradient of every point in grid
69
70 % plot steady path, phase diagram, and locus
71 figure(1)
72 quiver(k,c,u,v,0)
73 axis([lb_k ub_k lb_c ub_c])
74 xlabel('k') % x-axis label
75 ylabel('c') % y-axis label
76 hold on
77 quiver(K,C,dK,dC,0)
78 plot(axis_k,loci_k)
79 plot(axis_k,loci_c)
80 legend('Transition Path', 'Phase','Locus- K','Locus- C')
81 hold off
82 print -depsc fig1.eps
83
84 fprintf('Question 1 \nStarting - K : %f \nSteady state - K : %f \
85         \nSteady state - C : %f \nConvergence time : %d.\n\n',k0,kstar ,
86         cstar ,t);
87
88 kstar_old = kstar; % store original k steady state
89 cstar_old = cstar; % store original c steady state
90 k_old = k; % store transition path for k

```



```

86 c_old = c; % store transition path for c
87 u_old = u; % store transition path for k
88 v_old = v; % store transition path for c
89
90 %% Question 2
91 k = zeros(1,T+1); % initial k path vector
92 c = zeros(1,T+1); % initial c path vector
93
94 k0 = 1.1*kstar; % starting k value
95
96 crit = 1; % initialize tolerance criteria
97 ite = 1; % initialize iteration
98
99 while (crit>tol && ite<=length(axis_c))
100     k(1) = k0; % set starting k0
101     c(1) = axis_c(ite); % pick c0
102     for t = 1:T
103         k(t+1) = A*k(t)^alpha+(1-delta)*k(t)-c(t); % compute k(t
+1)
104         c(t+1) = c(t)*(beta*alpha*A*k(t+1)^(alpha-1)+beta*(1-delta
))^(1/sigma); %compute c(t+1)
105         crit = max(abs(kstar-k(t+1)),abs(cstar-c(t+1))); %
deviation from steady state
106         if crit<=tol
107             % if close to steady state stop algorithm
108             k = k(1:t+1); % cut path after convergences
109             c = c(1:t+1); % cut path after convergences
110             break
111         else
112             continue
113         end
114     end
115     ite = ite + 1 ; % update iteration
116 end
117
118 u = gradient(k); % compute k gradient of steady path
119 v = gradient(c); % compute c gradient of steady path
120
121 % plot steady path, phase diagram, and locus
122 figure(2)
123 quiver(k,c,u,v,0)
124 hold on
125 quiver(K,C,dK,dC,0)
126 plot(axis_k, loci_k)
127 plot(axis_k, loci_c)

```

```

128 quiver(k_old,c_old,u_old,v_old,0)
129 hold off
130 axis([lb_k ub_k lb_c ub_c])
131 xlabel('k') % x-axis label
132 ylabel('c') % y-axis label
133 legend('Steady Path', 'Phase','Locus- K','Locus- C','Steady Path')
134 print -depsc fig2.eps
135
136
137 %% Question 3
138 alpha = 0.33; % labor share
139 delta = 0.05; % depreciation of capital
140 sigma = 0.5; % CRRA
141 beta = .9898; % discount factor
142 A = 1; % technology
143
144 k = zeros(1,T+1); % initial k path vector
145 c = zeros(1,T+1); % initial c path vector
146
147 kstar = ((alpha*beta*A)/(1-beta+beta*delta))^(1/(1-alpha)); % k
    steady state
148 cstar = A*kstar^alpha+(1-delta)*kstar-kstar; % c steady state
149
150 k0 = 0.9*kstar; % starting k value
151
152 % note that we use the axis define in question 1
153 axis_k = lb_k : (ub_k-lb_k)/(N-1) : ub_k; % k axis
154 axis_c = lb_c : (ub_c-lb_c)/(N-1) : ub_c; % c axis
155
156 crit = 1; % initialize tolerance criteria
157 ite = 1; % initialize iteration
158
159 while (crit>tol && ite<=length(axis_c))
160     k(1) = k0; % set starting k0
161     c(1) = axis_c(ite); % pick c0
162     for t = 1:T
163         k(t+1) = A*k(t)^alpha+(1-delta)*k(t)-c(t); % compute k(t
            +1)
164         c(t+1) = c(t)*(beta*alpha*A*k(t+1)^(alpha-1)+beta*(1-delta
            ))^(1/sigma); %compute c(t+1)
165         crit = max(abs(kstar-k(t+1)),abs(cstar-c(t+1))); %
            deviation from steady state
166         if crit<=tol
167             % if close to steady state stop algorithm
168             k = k(1:t+1); % cut path after convergences

```

```

169         c = c(1:t+1); % cut path after convergences
170         break
171     else
172         continue
173     end
174 end
175     ite = ite + 1 ; % update iteration
176 end
177
178 u = gradient(k); % compute k gradient of steady path
179 v = gradient(c); % compute c gradient of steady path
180
181 loci_k = A*axis_k.^alpha - delta*axis_k; % compute k loci
182 loci_c = axis_k.^alpha +(1-delta)*axis_k-kstar; % compute c loci
183
184 [K,C]= meshgrid(lb_k : (ub_k-lb_k)/(10-1) : ub_k,lb_c : (ub_c-lb_c
185 )/(10-1) : ub_c); % create (k,c) grid
186
187 dK = A*K.^alpha-delta.*K-C; % compute k gradient of every point in
188 grid
189 dC = C.*(beta*alpha*A*(dK+K).^(alpha-1)+beta*(1-delta)).^(1/sigma)
190 -C; % compute c gradient of every point in grid
191
192 % plot steady path, phase diagram, and locus
193 figure(3)
194 quiver(k,c,u,v,0)
195 axis([lb_k ub_k lb_c ub_c])
196 xlabel('k') % x-axis label
197 ylabel('c') % y-axis label
198 hold on
199 quiver(K,C,dK,dC,0)
200 plot(axis_k,loci_k)
201 plot(axis_k,loci_c)
202 hold off
203 legend('Transition Path', 'Phase', 'Locus- K', 'Locus- C')
204 print -depsc fig3.eps
205
206 fprintf('Question 3 \nStarting - K : %f \nSteady state - K : %f \
207 nSteady state - C : %f \nConvergence time : %d.\n\n',k0,kstar ,
208 cstar ,t);
209
210 %% Question 4
211 alpha = 0.33; % labor share
212 delta = 0.05; % depreciation of capital

```

```

209 sigma = 0.5;      % CRRA
210 beta = .98;       % discount factor
211 A = 1.01;         % technology
212
213 k = zeros(1,T+1);  % initial k path vector
214 c = zeros(1,T+1);  % initial c path vector
215
216 kstar = ((alpha*beta*A)/(1-beta+beta*delta))^(1/(1-alpha)); % k
    steady state
217 cstar = A*kstar^alpha+(1-delta)*kstar-kstar; % c steady state
218
219 k0 = kstar_old; % starting k value
220
221 lb_k = kstar_old; % lower bound of k axis
222 ub_k = 1.1*kstar; % upper bound of k axis
223
224 lb_c = 0.9*cstar; % lower bound of c axis
225 ub_c = 1.1*cstar; % lower bound of c axis
226
227 axis_k = lb_k : (ub_k-lb_k)/(N-1) : ub_k; % k axis
228 axis_c = lb_c : (ub_c-lb_c)/(N-1) : ub_c; % c axis
229
230 crit = 1; % initialize tolerance criteria
231 ite = 1; % initialize iteration
232
233 while (crit>tol && ite<=length(axis_c))
234     k(1) = k0; % set starting k0
235     c(1) = axis_c(ite); % pick c0
236     for t = 1:T
237         k(t+1) = A*k(t)^alpha+(1-delta)*k(t)-c(t); % compute k(t
            +1)
238         c(t+1) = c(t)*(beta*alpha*A*k(t+1)^(alpha-1)+beta*(1-delta
            ))^(1/sigma); %compute c(t+1)
239         crit = max(abs(kstar-k(t+1)),abs(cstar-c(t+1))); %
            deviation from steady state
240         if crit<=tol
241             % if close to steady state stop algorithm
242             k = k(1:t+1); % cut path after convergences
243             c = c(1:t+1); % cut path after convergences
244             break
245         else
246             continue
247         end
248     end
249     ite = ite + 1 ; % update iteration

```

```

250 end
251
252 k = [kstar_old , k]; % add k0
253 c = [cstar_old , c]; % add c0
254
255 % plot time path of k and c
256 figure(4)
257 subplot(2,1,1)
258 plot(0:length(k)-1,k, '—o');
259 xlabel('t') % x-axis label
260 ylabel('k') % y-axis label
261 subplot(2,1,2)
262 plot(0:length(c)-1,c, '—o');
263 xlabel('t') % x-axis label
264 ylabel('c') % y-axis label
265 print -depsc fig4.eps
266
267 fprintf('Question 4 \nStarting - K : %f \nSteady state - K : %f \nSteady state - C : %f \nConvergence time : %d.\n\n',k0,kstar ,
        cstar ,t);

```