Economic Growth, Unemployment and Mortality: Computational Tools

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1 Linear Equations

1.1 Singular Value Decomposition

The basic model for this paper is the Lee-Carter model. It is based on modeling the logarithm of the force of mortality (μ_{xt}), at age x during year t, as a linear function of an age effect constant (α_x) and the interaction of a time (κ_t) and age-time (β_x) parameter:

$$\ln(\mu_{xt}) = \alpha_x + \beta_x \kappa_t + \epsilon_{xt}. \tag{1}$$

where $\epsilon_{xt} \sim I.I.D.(0, \sigma_\epsilon^2)$. Usually, constraints are imposed on the parameters κ_t and β_x in order to identify them.

The most common way to estimate α_x, β_x, and κ_t is to use Singular Value Decomposition (SVD).

The basis of SVD is to take an X by T matrix M and decompose it in the following way

$$M = UDV^T \tag{2}$$

where U and V are unitary matrix and D is a diagonal matrix of the singular values of M. Note that the more general form of the SVD decomposition takes the conjugate transpose instead of the transpose of V.

The algorithm to find U, D, and V is based on a two-step procedure where, first the matrix M is reduced to a bidiagonal form using for example Householder reflections, and second the SVD is computed on the bidiagonal reduced form by using a variant of the algorithm used for QR-decomposition.

In the case of the Lee-Carter model, we can take M as $\ln(\mu_{xt}) - \frac{1}{T}\ln(\mu_{xt})$. Then, decomposing the M with SVD yields the following

$$M = UDV^T = D_1 U_{x1} V_{t1}^T + ... + D_X U_{xX} V_{tX}^T \tag{3}$$

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where D_1 is the max entry of D. Using this we can approximate M by setting $\hat{\beta}_x = U_{x1}$ and $\hat{\kappa}_t = D_1 V_{t1}^T$. Finally, let $\hat{\alpha}_x = \ln(\mu_{xt}) - \hat{\beta}_x \hat{\kappa}_t$. Note that in the paper, we did not used this technique to solve the Lee-Carter model, but it is mentioned

2 Finite-Dimensional Optimization

2.1 Newton-Raphson Method

In order to solve some log-likelihood functions described in the paper, the Newton-Raphson method was used. The method is simply an particular instance of the Netwon's method for root-finding applied to optimization problems of the following form

$$\max_x F(x) \tag{4}$$

where $F(x)$ is a twice differentiable function. If x^* is the solution of this optimization problem, we have $F'(x^*) = 0$.

To find an iteration method to find x^*, we can compute the Taylor expansion of $F(x)$ around some $x^{(k)}$

$$F(x) \approx F(x^{(k)}) + F'(x^{(k)})(x - x^{(k)}) + \frac{1}{2}(x - x^{(k)})^T F''(x^{(k)})(x - x^{(k)}) \tag{5}$$

If we plug in the first-order condition in (5), the Taylor expansion reduces to

$$F'(x^{(k)}) + F''(x^{(k)})(x - x^{(k)}) = 0 \tag{6}$$

Then, we get the following updating rule for $x^{(k)}$

$$x^{(k+1)} \leftarrow x^{(k)} - [F''(x^{(k)})]^{-1} F'(x^{(k)}$$

which is known as the Netwon-Raphson method.

The method is usually well-behaved on globally concave(convex) functions, but if a function has multiple local maxima(minima) it might run into some problems if $x^{(0)}$ is too far from the global maxima(minima).

In this paper for example, we estimate the following log-likelihood function:

$$L(\alpha, \beta, \kappa) = \sum_{x,t} \{D_{xt}(\alpha_x + \beta_x \kappa_t) - E_{xt} exp(\alpha_x + \beta_x \kappa_t)\} + const. \tag{7}$$

where D_{xt} be the observed number of deaths, D_{xt} is the exposure-to-risk and α_x, β_x and κ_t are the define as in the Lee-Carter model.

To maximizes this log-likelihood function, we use the Newton-Raphson approach in the following way

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{\frac{\partial L^{(v)}}{\partial \theta}}{\frac{\partial^2 L^{(v)}}{\partial \theta}} \tag{8}$$

where $L^{(v)} = L^{(v)}(\hat{\theta}^{(v)})$

This give the following iterative procedure for α_x, β_x and κ_t

$$\hat{\alpha}_x^{(v+1)} = \hat{\alpha}_x^{(v)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(v)})}{-\sum_t \hat{D}_{xt}^{(v)}},\ \hat{\beta}_x^{(v+1)} = \hat{\beta}_x^{(v)},\ \hat{\kappa}_t^{(v+1)} = \hat{\kappa}_t^{(v)}, \tag{9}$$

$$\tilde{\kappa}_t^{(v+2)} = \hat{\kappa}_t^{(v+1)} - \frac{\sum_x (D_{xt} - \hat{D}_{xt}^{(v+1)})\hat{\beta}_x^{(v+1)}}{-\sum_x \hat{D}_{xt}^{(v+1)}(\hat{\beta}_x^{(v+1)})^2},\ \hat{\alpha}_x^{(v+2)} = \hat{\alpha}_x^{(v+1)},\ \hat{\beta}_x^{(v+2)} = \hat{\beta}_x^{(v+1)}, \tag{10}$$

$$\hat{\kappa}_t^{(v+2)} = \tilde{\kappa}_t^{(v+2)} - (\frac{1}{T}\sum_{s=1}^{T} \tilde{\kappa}_s^{(v+2)}), \tag{11}$$

$$\hat{\beta}_x^{(v+3)} = \hat{\beta}_x^{(v+2)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(v+2)})\hat{\kappa}_t^{(v+2)}}{-\sum_t \hat{D}_{xt}^{(v+2)}(\hat{\kappa}_t^{(v+2)})^2},\ \hat{\alpha}_x^{(v+3)} = \hat{\alpha}_x^{(v+2)},\ \hat{\kappa}_t^{(v+3)} = \hat{\kappa}_t^{(v+2)}, \tag{12}$$

where $\hat{D}_{xt} = E_{xt}exp(\hat{\alpha}_x + \hat{\beta}_x\hat{\kappa}_t)$. Note that (11) is an added step to insure identification, and it does not come from the Netwon-Raphson method.