Rappels physiques:

en cartésien : $\overrightarrow{dl_e} = dx\,\vec{x} + dy\,\vec{y} + dz\,\vec{z},\ \overrightarrow{dS_e} = dydz\,\vec{x} + dxdz\,\vec{y} + dxdy\,\vec{z},\ dV_e = dxdydz$
en cylindrique : $\overrightarrow{dl_e} = d\rho\,\vec{\rho} + \rho d\varphi\,\vec{\varphi} + dz\,\vec{z},\ \overrightarrow{dS_e} = \rho d\varphi dz\,\vec{\rho} + d\rho dz\,\vec{\varphi} + \rho d\varphi dz\,\vec{z},\ dV_e = \rho d\rho d\varphi dz$
en sphérique : $\overrightarrow{dl_e} = dr\,\vec{r} + rd\theta\,\vec{\theta} + rsin(\theta)d\varphi\,\vec{\varphi}, \overrightarrow{dS_e} = r^2\sin(\theta)\,d\theta d\varphi\,\vec{r} + r\sin(\theta)\,drd\varphi\,\vec{\theta} + rdrd\theta\,\vec{\varphi},\ dV_e = r^2\sin(\theta)\,drd\theta d\varphi$

Gradient

For a function $f(x, y, z)$ in three-dimensional Cartesian coordinate variables, the gradient is the vector field:

$$\overrightarrow{\mathbf{grad}}f = \vec{\nabla}f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{pmatrix}$$

More generally, for a function of *n* variables $\psi(x_1, \dots, x_n)$, also called a scalar field, the gradient is the vector field:

$$\vec{\nabla}\psi = \begin{pmatrix}\frac{\partial\psi}{\partial x_1} & \dots & \frac{\partial\psi}{\partial x_n}\end{pmatrix}.$$

For a vector field $\vec{A} = (A_1, \dots, A_n)$ written as a 1 × *n* row vector, also called a tensor field of order 1, the gradient or covariant derivative is the *n* × *n* Jacobian matrix:

$$\vec{\nabla}\vec{A} = \vec{J}_{\vec{A}} = \left(\frac{\partial A_i}{\partial x_j}\right)_{ij}.$$

For a tensor field $\vec{A}$ of any order *k*, $\overrightarrow{\mathrm{grad}}(\vec{A}) = \vec{\nabla}\vec{A}$ is a tensor field of order *k* + 1.

Divergence

In Cartesian coordinates, the divergence of a continuously differentiable vector field $\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ is the scalar-valued function:

$$\mathbf{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\cdot\begin{pmatrix}F_x & F_y & F_z\end{pmatrix} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The divergence of a tensor field $\vec{A}$ of non-zero order *k* is written as $\mathrm{div}(\vec{A}) = \vec{\nabla}\cdot\vec{A}$, a contraction to a tensor field of order *k* − 1. Specifically, the divergence of a vector is a scalar. The divergence of a higher order tensor field may be found by decomposing the tensor field into a sum of outer products and using the identity,

$$\vec{\nabla}\cdot\left(\vec{B}\otimes\hat{\vec{A}}\right) = \hat{\vec{A}}(\vec{\nabla}\cdot\vec{B}) + (\vec{B}\cdot\vec{\nabla})\hat{\vec{A}}$$

where $\vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{\nabla}}$ is the directional derivative in the direction of $\vec{B}$ multiplied by its magnitude. Specifically, for the outer product of two vectors,

$$\vec{\nabla}\cdot\left(\vec{b}\vec{a}^{\mathsf{T}}\right) = \vec{a}\left(\vec{\nabla}\cdot\vec{b}\right) + \left(\vec{b}\cdot\vec{\nabla}\right)\vec{a}.$$

Curl

In Cartesian coordinates, for $\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ the curl is the vector field:

$$\overrightarrow{\mathbf{curl}}\vec{F} = \vec{\nabla}\wedge\vec{F} = \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix} = \begin{vmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{matrix}\wedge\begin{vmatrix}F_x\\ F_y\\ F_z\end{matrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{k}$$

In Einstein notation, the vector field $\vec{F} = \begin{pmatrix}F_1 & F_2 & F_3\end{pmatrix}$ has curl given by:

$$\vec{\nabla} \wedge \vec{F} = \varepsilon^{ijk} \frac{\partial F_k}{\partial x^j}$$

where ε = ±1 or 0 is the Levi-Civita parity symbol.

Laplacian

In **Cartesian coordinates**, the Laplacian of a function $f(x, y, z)$ is

$$\boldsymbol{\Delta} f = \vec{\nabla}^2 f = (\vec{\nabla} \cdot \vec{\nabla}) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

For a tensor field, $\vec{A}$, the Laplacian is generally written as:

$$\boldsymbol{\Delta} \vec{A} = \vec{\nabla}^2 \vec{A} = (\vec{\nabla} \cdot \vec{\nabla}) \vec{A}$$

and is a tensor field of the same order.

In *Feynman subscript notation*,

$$\vec{\nabla}_{\vec{B}}\left(\overline{A \cdot B}\right) = \vec{A} \wedge \left(\vec{\nabla} \wedge \vec{B}\right) + \left(\vec{A} \cdot \vec{\nabla}\right) \vec{B}$$

where the notation $\vec{\nabla}$ means the subscripted gradient operates on only the factor .

First derivative identities

For scalar fields ψ, ϕ and vector fields $\vec{A}$, $\vec{B}$, we have the following derivative identities.

Distributive properties

$$\begin{aligned}
\vec{\nabla}(\psi + \phi) &= \vec{\nabla}\psi + \vec{\nabla}\phi \\
\vec{\nabla}(\vec{A} + \vec{B}) &= \vec{\nabla}\vec{A} + \vec{\nabla}\vec{B} \\
\vec{\nabla} \cdot (\vec{A} + \vec{B}) &= \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B} \\
\vec{\nabla} \wedge (\vec{A} + \vec{B}) &= \vec{\nabla} \wedge \vec{A} + \vec{\nabla} \wedge \vec{B}
\end{aligned}$$

Product rule

We have the following generalizations of the product rule in single variable calculus.

$$\begin{aligned}
\vec{\nabla}(\psi\phi) &= \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi \\
\vec{\nabla}(\psi\vec{A}) &= (\vec{\nabla}\psi)^T \vec{A} + \psi\vec{\nabla}\vec{A} = \vec{\nabla}\psi \otimes \vec{A} + \psi\, \vec{\nabla}\vec{A} \\
\vec{\nabla} \cdot (\psi\vec{A}) &= \psi\vec{\nabla} \cdot \vec{A} + (\vec{\nabla}\psi) \cdot \vec{A} \\
\vec{\nabla} \wedge (\psi\vec{A}) &= \psi\vec{\nabla} \wedge \vec{A} + (\vec{\nabla}\psi) \wedge \vec{A} \\
\vec{\nabla}^2(fg) &= \left(\vec{\nabla}^2(f)\right)g + 2(\vec{\nabla}(f)) \cdot (\vec{\nabla}(g)) + f\vec{\nabla}^2(g)
\end{aligned}$$

In the second formula, the transposed gradient $(\vec{\nabla}\psi)^T$ is an *n* × 1 column vector, $\vec{A}$ is a 1 × *n* row vector, and their product is an *n* × *n* matrix: this may also be considered as the tensor product ⊗ of two vectors, or of a covector and a vector.

Quotient rule

$$\begin{aligned}
\vec{\nabla}\left(\frac{\psi}{\phi}\right) &= \frac{\phi\, \vec{\nabla}\psi - (\vec{\nabla}\phi)\psi}{\phi^2} \\
\vec{\nabla} \cdot \left(\frac{\vec{A}}{\phi}\right) &= \frac{\phi\, \vec{\nabla} \cdot \vec{A} - (\vec{\nabla}\phi) \cdot \vec{A}}{\phi^2} \\
\vec{\nabla} \wedge \left(\frac{\vec{A}}{\phi}\right) &= \frac{\phi\, \vec{\nabla} \wedge \vec{A} - (\vec{\nabla}\phi) \wedge \vec{A}}{\phi^2}
\end{aligned}$$

Chain rule

Let $f(x)$ be a one-variable function from scalars to scalars, $\vec{r}(t) = (r_1(t), \dots, r_n(t))$ a parametrized curve, and $F: \mathbb{R}^n \to \mathbb{R}$ a function from vectors to scalars. We have:

$$\begin{aligned}\vec{\nabla}(f \circ F) &= (f' \circ F)\,\vec{\nabla}F \\ (F \circ \vec{r})' &= (\vec{\nabla}F \circ \vec{r}) \cdot \vec{r}' \\ \vec{\nabla}(F \circ \vec{A}) &= (\vec{\nabla}F \circ \vec{A})\,\vec{\nabla}\vec{A}\end{aligned}$$

For a coordinate parametrization $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ we have:

$$\vec{\nabla} \cdot (\vec{A} \circ \Phi) = \mathrm{tr}\big((\vec{\nabla}\vec{A} \circ \Phi) J_\Phi\big)$$

Here we take the trace of the product of two *n* × *n* matrices: the gradient of and the Jacobian of Φ.

Dot product rule

$$\begin{aligned}\vec{\nabla}(\vec{A} \cdot \vec{B}) &= (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A} + \vec{A} \wedge (\vec{\nabla} \wedge \vec{B}) + \vec{B} \wedge (\vec{\nabla} \wedge \vec{A}) \\ &= \vec{A}\,\vec{J}_{\vec{B}} + \vec{B}\,\vec{J}_{\vec{A}} = \vec{A}\,\vec{\nabla}\vec{B} + \vec{B}\,\vec{\nabla}\vec{A}\end{aligned}$$

where $\vec{J}_{\vec{A}} = \vec{\nabla}\vec{A} = (\partial A_i/\partial x_j)_{ij}$ denotes the Jacobian matrix of the vector field $\vec{A} = (A_1, \ldots, A_n)$.

Alternatively, using Feynman subscript notation,

$$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{\nabla}_{\vec{A}}(\vec{A} \cdot \vec{B}) + \vec{\nabla}_{\vec{B}}(\vec{A} \cdot \vec{B})\ .$$

As a special case, when A = B,

$$\frac{1}{2}\vec{\nabla}\big(\vec{A} \cdot \vec{A}\big) = (\vec{A} \cdot \vec{\nabla})\vec{A} + \vec{A} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{A}\,\vec{J}_{\vec{A}} = \vec{A}\,\vec{\nabla}\vec{A}.$$

The generalization of the dot product formula to Riemannian manifolds is a defining property of a Riemannian connection, which differentiates a vector field to give a vector-valued 1-form.

Cross product rule

$$\begin{aligned}\vec{\nabla} \cdot (\vec{A} \wedge \vec{B}) &= (\vec{\nabla} \wedge \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \wedge \vec{B}) \\ \vec{\nabla} \wedge (\vec{A} \wedge \vec{B}) &= \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} \\ &= (\vec{\nabla} \cdot \vec{B} + \vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{\nabla})\vec{B} \\ &= \vec{\nabla} \cdot \big(\vec{B}\vec{A}^{\mathrm{T}}\big) - \vec{\nabla} \cdot \big(\vec{A}\vec{B}^{\mathrm{T}}\big) \\ &= \vec{\nabla} \cdot \big(\vec{B}\vec{A}^{\mathrm{T}} - \vec{A}\vec{B}^{\mathrm{T}}\big)\end{aligned}$$

Second derivative identities

The curl of the gradient of *any* continuously twice-differentiable scalar field ϕ is always the zero vector:

$$\vec{\nabla} \wedge (\vec{\nabla}\phi) = \vec{0}$$

The divergence of the curl of *any* vector field is always zero:

$$\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0$$

Laplacian identities

$$\vec{\nabla}^2\psi = \vec{\nabla} \cdot (\vec{\nabla}\psi)$$

$$\vec{\nabla} \wedge \big(\vec{\nabla} \wedge \vec{A}\big) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2\vec{A}$$

Here $\vec{\nabla}^2$ is the vector Laplacian operating on the vector field .

Summary of important identities

Vector calculus identities

$\vec{a} \cdot \big(\vec{b} \wedge \vec{c}\big) = \vec{b} \cdot (\vec{c} \wedge \vec{a}) = \vec{c} \cdot \big(\vec{a} \wedge \vec{b}\big) = \det\big(\vec{a}, \vec{b}, \vec{c}\big)$

$\vec{a} \wedge \big(\vec{b} \wedge \vec{c}\big) = \big(\vec{c} \wedge \vec{b}\big) \wedge \vec{a} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}\big(\vec{a} \cdot \vec{b}\big)$

$\big(\vec{a} \wedge \vec{b}\big) \cdot \big(\vec{c} \wedge \vec{d}\big) = (\vec{a} \cdot \vec{c})\big(\vec{b} \cdot \vec{d}\big) - \big(\vec{a} \cdot \vec{d}\big)\big(\vec{b} \cdot \vec{c}\big)$ (Binet-Cauchy Identity)

Gradient

$\vec{\nabla}(\psi + \phi) = \vec{\nabla}\psi + \vec{\nabla}\phi$

$\vec{\nabla}(\psi\phi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$

$\vec{\nabla}(\psi\vec{A}) = \vec{\nabla}\psi \otimes \vec{A} + \psi\vec{\nabla}\vec{A}$
$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\wedge(\vec{\nabla}\wedge\vec{B}) + \vec{B}\wedge\left(\vec{\nabla}\wedge\vec{A}\right)$

Divergence

$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$
$\vec{\nabla}\cdot\left(\psi\vec{A}\right) = \psi\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\psi$
$\vec{\nabla}\cdot\left(\vec{A}\wedge\vec{B}\right) = (\vec{\nabla}\wedge\vec{A})\cdot\vec{B} - (\vec{\nabla}\wedge\vec{B})\cdot\vec{A}$

Curl

$\vec{\nabla}\wedge(\vec{A}+\vec{B}) = \vec{\nabla}\wedge\vec{A} + \vec{\nabla}\wedge\vec{B}$
$\vec{\nabla}\wedge\left(\psi\vec{A}\right) = \psi\,(\vec{\nabla}\wedge\vec{A}) + \vec{\nabla}\psi\wedge\vec{A}$
$\vec{\nabla}\wedge\left(\psi\vec{\nabla}\phi\right) = \vec{\nabla}\psi\wedge\vec{\nabla}\phi$
$\vec{\nabla}\wedge\left(\vec{A}\wedge\vec{B}\right) = \vec{A}\left(\vec{\nabla}\cdot\vec{B}\right) - \vec{B}\left(\vec{\nabla}\cdot\vec{A}\right) + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} - \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}$

Second derivatives

$\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{A}) = 0$ et en physique $\vec{\nabla}\cdot\vec{B} = 0 \Leftrightarrow \exists\vec{A}\ \ \vec{B} = \vec{\nabla}\wedge\vec{A}$ (potentiel vecteur champ magnétique)
$\vec{\nabla}\wedge(\vec{\nabla}\psi) = \vec{0}$ et en physique $\vec{\nabla}\wedge\vec{E} = \vec{0} \Leftrightarrow\ \exists V\ \ \vec{E} = \vec{\nabla}\,V$ (potentiel électrique)
$\vec{\nabla}\cdot(\vec{\nabla}\psi) = \vec{\nabla}^2\psi$ (scalar Laplacian)
$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}\wedge\left(\vec{\nabla}\wedge\vec{A}\right) = \vec{\nabla}^2\vec{A}$ (vector Laplacian)
$\vec{\nabla}\cdot(\psi\vec{\nabla}\varphi) = \psi\vec{\nabla}^2\varphi + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi$ (Green's first identity)
$\psi\vec{\nabla}^2\varphi - \varphi\vec{\nabla}^2\psi = \vec{\nabla}\cdot\left(\psi\vec{\nabla}\varphi - \varphi\vec{\nabla}\psi\right)$ (Green's second identity)
$\vec{\nabla}^2(\phi\psi) = \phi\vec{\nabla}^2\psi + 2(\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi) + \left(\vec{\nabla}^2\phi\right)\psi$
$\vec{\nabla}^2(\psi\vec{A}) = \vec{A}\vec{\nabla}^2\psi + 2(\vec{\nabla}\psi\cdot\vec{\nabla})\vec{A} + \psi\vec{\nabla}^2\vec{A}$
$\vec{\nabla}^2(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\vec{\nabla}^2\vec{B} - \vec{B}\cdot\vec{\nabla}^2\vec{A} + 2\vec{\nabla}\cdot((\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{B}\wedge(\vec{\nabla}\wedge\vec{A}))$ (Green's vector identity)

Integration

Surface–volume integrals

Pour un ouvert borné régulier de R^d, et un champ de vecteur $\vec{A}\in C^1\left(\overline{\Omega}\right)$, avec $d\vec{S} = \vec{n}dS$ avec $dS = \vec{n}\cdot d\overrightarrow{S_e}$ intégrale triple = dimension d, double = $d-1$

$\oiint_{\partial V}\vec{A}\cdot d\vec{S}\ =\ \iiint_V\left(\vec{\nabla}\cdot\vec{A}\right)dV$ (Green-Ostrogradsky/Stokes divergence theorem)

$$\oiint_{\partial V}\psi\ d\vec{S}\ =\ \iiint_V\vec{\nabla}\,\psi\,dV$$

$$\oiint_{\partial V}\vec{A}\wedge d\vec{S}\ =\ -\iiint_V\vec{\nabla}\wedge\vec{A}\,dV$$

$\oiint_{\partial V}\psi\,\vec{\nabla}\varphi\cdot d\vec{S}\ =\ \iiint_V\left(\psi\vec{\nabla}^2\varphi + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi\right)dV$ (Green's first identity) (par G.O. sur l'identité locale)

$\oiint_{\partial V}\left(\psi\vec{\nabla}\varphi - \varphi\vec{\nabla}\psi\right)\cdot d\vec{S} = \oiint_{\partial V}\left(\psi\frac{\partial\varphi}{\partial\vec{n}} - \varphi\frac{\partial\psi}{\partial\vec{n}}\right)dS = \iiint_V\left(\psi\vec{\nabla}^2\varphi - \varphi\vec{\nabla}^2\psi\right)dV$ (Green's second identity) (par G.O. sur l'identité locale)

$\iiint_V\vec{A}\cdot\vec{\nabla}\psi\,dV = \oiint_{\partial V}\psi\,\vec{A}\cdot d\vec{S} - \iiint_V\psi\,\vec{\nabla}\cdot\vec{A}\,dV$ (integration by parts) (par G.O. sur $\vec{\nabla}\cdot\left(\psi\vec{A}\right)$)

Pour $i\in\{1,\dots,d\}$ et pour $u,v\in C^1\left(\overline{\Omega},R\right)$, avec $\vec{A} = v\vec{e_i}$ on a (integration par parties projetée)

$\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx = \int_{\partial\Omega}u(x)v(x)n_i(x)dS(x) - \int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)\,dx$

Curve–surface integrals

In the following curve–surface integral theorems, S denotes a 2d open surface with a corresponding 1d boundary $C = \partial S$ (a closed curve):

$\oint_{\partial S} \vec{A} \cdot d\vec{l} = \iint_S (\vec{\nabla} \wedge \vec{A}) \cdot d\vec{S}$ (Stokes' theorem) (cas particulier de Green-Ostrogradski en 2d)

$$\oint_{\partial S} \psi \, d\vec{l} = -\iint_S \vec{\nabla} \psi \wedge d\vec{S}$$

Integration around a closed curve in the clockwise sense is the negative of the same line integral in the counterclockwise sense.