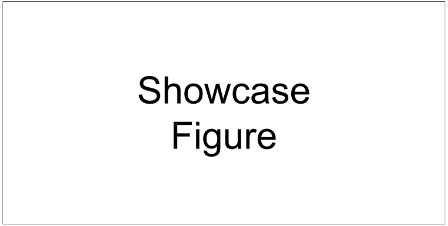
Your Title

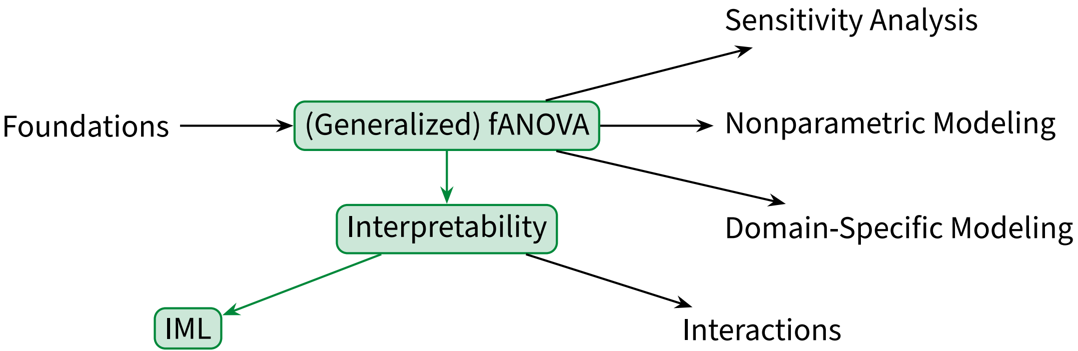
Your Name

July 31, 2025



Outline

Overview of the fANOVA Research Field



References: [1, 2, 5, 7, 6, 4, 3]

Outline

Classical fANOVA Decomposition

General Form

$$y(\boldsymbol{X}) = \sum_{u \subseteq \{1,\ldots,N\}} y_u(\boldsymbol{X}_u)$$

- y : Model output
- y_u : Component functions for subset u

Conditions for Classical fANOVA

Strong Annihilating Conditions

$$\int_{\mathbb{R}} y_u(\boldsymbol{x}_u) f_{\{i\}}(x_i)\, d\nu(x_i) = 0 \quad \text{for } i \in u \neq \emptyset.$$

- Ensures unique component functions
- Applies under independent (product-type) input distributions

Key Properties

$$\mathbb{E}[y_u(\boldsymbol{X}_u)] = 0$$

$$\mathbb{E}[y_u(\boldsymbol{X}_u)y_v(\boldsymbol{X}_v)] = 0 \quad (u \neq v)$$

- Zero mean components
- Mutual orthogonality

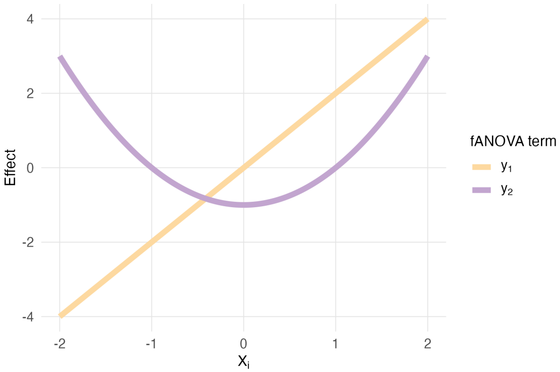
Component Construction

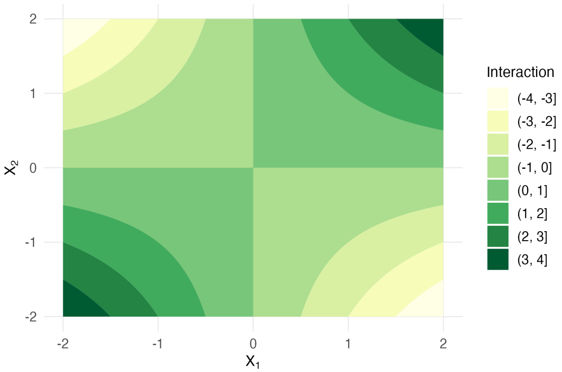
$$y_\emptyset = \int_{\mathbb{R}^N} y(\boldsymbol{x}) \prod_{i=1}^{N} f_{\{i\}}(x_i)\, d\nu(x_i) = \mathbb{E}[y(\boldsymbol{X})].$$

$$y_u(x_u) = \int y(x) f_{-u}(x_{-u}) dx_{-u} - \sum_{v \subset u} y_v(x_v)$$

- f_{-u} : marginal density of variables not in u
- Components solved sequentially by increasing order

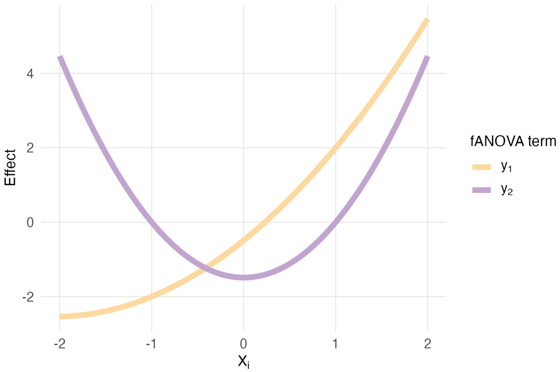
Example: 2D Function

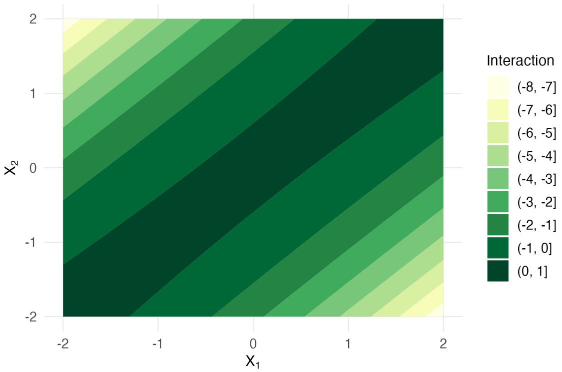




Outline

Example with Dependent Inputs ($\rho = 0.8$)





Weaker Annihilating Conditions

Weak Annihilating Conditions

$$\int_{\mathbb{R}} y_{u,G}(\boldsymbol{x}_u) f_{\boldsymbol{X}_u}(\boldsymbol{x}_u) d\nu(x_i) = 0 \quad \text{for} \quad i \in u \neq \emptyset.$$

- Allows dependent input distributions
- Leads to hierarchical orthogonality

Key Properties (Generalized)

$$\mathbb{E}[y_{u,G}(\boldsymbol{X}_u)] := \int_{\mathbb{R}^N} y_{u,G}(\boldsymbol{x}_u) f_{\boldsymbol{X}}(\boldsymbol{x}) \, d\nu(\boldsymbol{x}) = 0.$$

$$\mathbb{E}[y_{u,G}(\boldsymbol{X}_u) y_{v,G}(\boldsymbol{X}_v)] := \int_{\mathbb{R}^N} y_{u,G}(\boldsymbol{x}_u) y_{v,G}(\boldsymbol{x}_v) f_{\boldsymbol{X}}(\boldsymbol{x}) \, d\nu(\boldsymbol{x}) = 0.$$

- Zero mean components remain
- Orthogonality is weaker: hierarchical

Component Definition (Coupled System)

$$y_{\emptyset,G} = \int_{\mathbb{R}^N} y(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x})\, d\nu(\boldsymbol{x}) \tag{1}$$

$$\begin{aligned} y_{u,G}(\boldsymbol{X}_u) = &\int_{\mathbb{R}^{N-|u|}} y(\boldsymbol{X}_u, \boldsymbol{x}_{-u}) f_{-u}(\boldsymbol{x}_{-u})\, d\nu(\boldsymbol{x}_{-u}) - \sum_{v \subsetneq u} y_{v,G}(\boldsymbol{X}_v) \\ &- \sum_{\substack{\emptyset \neq v \subseteq \{1,\ldots,N\} \\ v \cap u \neq \emptyset,\ v \not\subset u}} \int_{\mathbb{R}^{|v \cap -u|}} y_{v,G}(\boldsymbol{X}_{v\cap u}, \boldsymbol{x}_{v\cap -u}) f_{v\cap -u}(\boldsymbol{x}_{v\cap -u})\, d\nu(\boldsymbol{x}_{v\cap -u}). \end{aligned} \tag{2}$$

- All components solved simultaneously
- Depends on marginal densities and coupling terms

How to Construct the Components

- Coupled system $\rightarrow$ difficult to obtain analytical solutions
- Use alternative method via Fourier Polynomial ([5])
- Building blocks: mutually orthogonal, zero-mean basis functions $\psi_{i,j}$, coefficients $c_{i,j}$

Basis Representation of a Polynomial

$$
\begin{aligned}
y(x_1, x_2) &= a_0 + a_1 x_1 + a_2 x_2 + a_{11} x_1^2 + a_{22} x_2^2 + a_{12} x_1 x_2 \\
&= c_0 + c_{1,1}\,\psi_{1,1}(x_1) + c_{2,1}\,\psi_{2,1}(x_2) \\
&\quad + c_{1,2}\,\psi_{1,2}(x_1) + c_{2,2}\,\psi_{2,2}(x_2) + c_{12,11}\,\psi_{12,11}(x_1, x_2) \\
&= \underbrace{c_0}_{y_0} + \underbrace{\left(c_{1,1}\,\psi_{1,1}(x_1) + c_{1,2}\,\psi_{1,2}(x_1)\right)}_{y_1(x_1)} \\
&\quad + \underbrace{\left(c_{2,1}\,\psi_{2,1}(x_2) + c_{2,2}\,\psi_{2,2}(x_2)\right)}_{y_2(x_2)} \\
&\quad + \underbrace{c_{12,11}\,\psi_{12,11}(x_1, x_2)}_{y_{12}(x_1, x_2)}.
\end{aligned}
$$

Basis Functions proposed by Rahman (2014)[5]

$$
\begin{aligned}
\psi_{\emptyset}(x_1, x_2) &= 1, \\
\psi_{1,1}(x_1) &= x_1, \\
\psi_{2,1}(x_2) &= x_2, \\
\psi_{1,2}(x_1) &= x_1^2 - 1, \\
\psi_{2,2}(x_2) &= x_2^2 - 1, \\
\psi_{12,11}(x_1, x_2) &= \frac{\rho(x_1^2 + x_2^2)}{1 + \rho^2} - x_1 x_2 + \frac{\rho(\rho^2 - 1)}{1 + \rho^2},
\end{aligned}
$$

Alternative Generalization of fANOVA, [2]

$$\{y_{u,G}(\boldsymbol{x}_u) \mid u \subseteq d\} = \arg \min_{\{g_u \in \mathcal{L}^2(\mathbb{R}^{|u|})\}} \int_{\mathbb{R}^N} \left(\sum_{u \subseteq d} g_u(\boldsymbol{x}_u) - y(\boldsymbol{x}) \right)^2 f_{\boldsymbol{X}}(\boldsymbol{x}) \, d\nu(\boldsymbol{x}),$$

subject to hierarchical orthogonality conditions:

$$\forall v \subseteq u, \; \forall g_v : \int_{\mathbb{R}^N} y_u(\boldsymbol{x}_u) g_v(\boldsymbol{x}_v) f_{\boldsymbol{X}}(\boldsymbol{x}) \, d\nu(\boldsymbol{x}) = 0.$$

Variance Decomposition, [6]

$$\mu := \mathbb{E}[y(\boldsymbol{X})] = y_{\emptyset,G}.$$

$$\begin{aligned}
\sigma^2 &:= \mathbb{E}\left[\left(y(\boldsymbol{X}) - \mu_G\right)^2\right] \\
&= \mathbb{E}\left[\left(y_{\emptyset,G} + \sum_u y_{u,G}(\boldsymbol{X}_u) - y_{\emptyset,G}\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_u y_{u,G}(\boldsymbol{X}_u)\right)^2\right] \\
&= \sum_u \mathbb{E}\left[y_{u,G}^2(\boldsymbol{X}_u)\right] + \sum_{u \not\subseteq v,\, v \not\subseteq u} \mathbb{E}\left[y_{u,G}(\boldsymbol{X}_u) y_{v,G}(\boldsymbol{X}_v)\right],
\end{aligned}$$

Outline

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