Poncelet Polygons

Joshua Gramm

Department of Mathematics
Dr. Taylor Poe

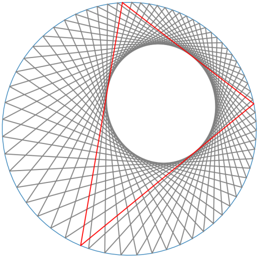
08 Dec 2022

Theorem (Poncelet's Theorem)

Let C be an ellipse in $\mathbb{D}$. If there is a polygon $\mathcal{P}_0$ inscribed in $\mathbb{T}$ that circumscribes C, then for any complex point $z \in \mathbb{T}$, there exists a polygon $\mathcal{P}$ inscribed in $\mathbb{T}$ and circumscribed about C such that z is a vertex of $\mathcal{P}$. Thus there is an infinite family of $\mathcal{P}$ that circumscribes C. The ellipse is called a Poncelet-n ellipse.

Theorem (Poncelet's Theorem)

Let C be an ellipse in $\mathbb{D}$. If there is a polygon $\mathcal{P}_0$ inscribed in $\mathbb{T}$ that circumscribes C, then for any complex point $z \in \mathbb{T}$, there exists a polygon $\mathcal{P}$ inscribed in $\mathbb{T}$ and circumscribed about C such that z is a vertex of $\mathcal{P}$. Thus there is an infinite family of $\mathcal{P}$ that circumscribes C. The ellipse is called a Poncelet-n ellipse.



OPUC

Definition (Orthogonal Polynomial on the Unit Circle)

An orthogonal polynomial on the unit circle is a polynomial whose zeros exist on $\mathbb{D}$ and are orthogonal with respect to integration on the unit circle in the complex plane.

OPUC

Definition (Orthogonal Polynomial on the Unit Circle)

An orthogonal polynomial on the unit circle is a polynomial whose zeros exist on $\mathbb{D}$ and are orthogonal with respect to integration on the unit circle in the complex plane.

- $\Phi_n(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} z - f_j,\ f_j \in \mathbb{D}$

OPUC

Definition (Orthogonal Polynomial on the Unit Circle)

An orthogonal polynomial on the unit circle is a polynomial whose zeros exist on $\mathbb{D}$ and are orthogonal with respect to integration on the unit circle in the complex plane.

- $\Phi_n(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} z - f_j,\ f_j \in \mathbb{D}$
- $\Phi_n(z) = \sum_{j=0}^{n} c_j z^j,\ c_n = 1$

OPUC

Definition (Orthogonal Polynomial on the Unit Circle)

An orthogonal polynomial on the unit circle is a polynomial whose zeros exist on $\mathbb{D}$ and are orthogonal with respect to integration on the unit circle in the complex plane.

- $\Phi_n(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} z - f_j,\ f_j \in \mathbb{D}\quad \Phi_n^*(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} 1 - z\overline{f_j}$
- $\Phi_n(z) = \sum_{j=0}^{n} c_j z^j,\ c_n = 1\quad \Phi_n^*(z) = \sum_{j=0}^{n} \overline{c_j} z^{n-j} = z^n \overline{\Phi_n(1/\overline{z})}$

Szegő Recursion

- $\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z)$
- $\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z)$

Szegő Recursion

- $\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z)$
- $\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z)$
- $\Phi_{n+1}(0) = -\overline{\alpha_n}$ called the Verblunsky coefficients, $|\alpha_n| < 1$.

POPUC

Definition (Paraorthogonal Polynomial on the Unit Circle)

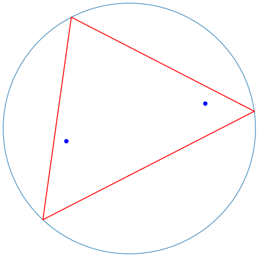
An OPUC with zeros on $\mathbb{T}$.

- $\Phi_{n+1}(z) = z\Phi_n(z) + e^{i\theta}\Phi_n^*(z)$

Definition (Paraorthogonal Polynomial on the Unit Circle)

An OPUC with zeros on $\mathbb{T}$.

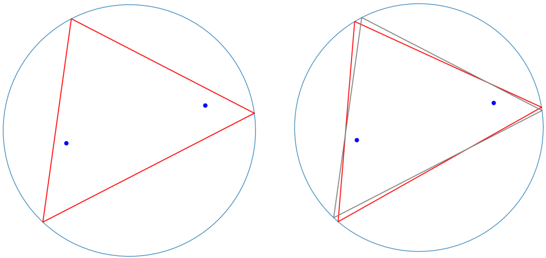
- $\Phi_{n+1}(z) = z\Phi_n(z) + e^{i\theta}\Phi_n^*(z)$



Definition (Paraorthogonal Polynomial on the Unit Circle)

An OPUC with zeros on $\mathbb{T}$.

- $\Phi_{n+1}(z) = z\Phi_n(z) + e^{i\theta}\Phi_n^*(z)$

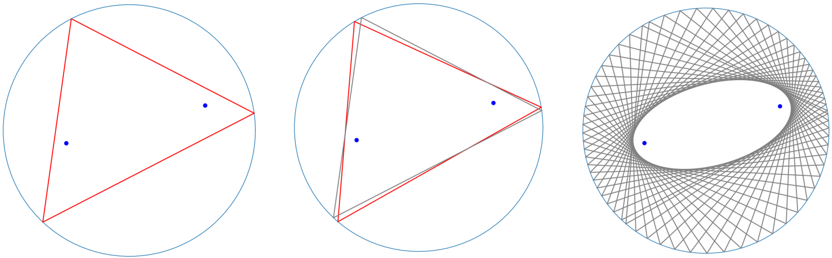


POPUC

Definition (Paraorthogonal Polynomial on the Unit Circle)

An OPUC with zeros on $\mathbb{T}$.

- $\Phi_{n+1}(z) = z\Phi_n(z) + e^{i\theta}\Phi_n^*(z)$



CMV Matrix

$$\Theta_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \rho_j = \sqrt{1 - |\alpha_j|^2}$$

CMV Matrix

$$\Theta_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \rho_j = \sqrt{1 - |\alpha_j|^2}$$

- $\mathcal{L} := \Theta_0 \bigoplus \Theta_2 \bigoplus \Theta_4 \bigoplus \ldots, \quad \mathcal{M} := 1 \bigoplus \Theta_1 \bigoplus \Theta_3 \bigoplus \ldots$

CMV Matrix

$$\Theta_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \rho_j = \sqrt{1 - |\alpha_j|^2}$$

- $\mathcal{L} := \Theta_0 \bigoplus \Theta_2 \bigoplus \Theta_4 \bigoplus \ldots, \quad \mathcal{M} := 1 \bigoplus \Theta_1 \bigoplus \Theta_3 \bigoplus \ldots$

$$\mathcal{L}_2 = \begin{pmatrix} \overline{\alpha}_0 & \rho_0 & 0 & 0 \\ \rho_0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & \overline{\alpha}_2 & \rho_2 \\ 0 & 0 & \rho_2 & -\alpha_2 \end{pmatrix}$$

CMV Matrix

- $\mathcal{G} := \mathcal{L}\mathcal{M}$

$$\mathcal{G} = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

CMV Matrix Example for Triangle

For $\Phi_2(z)$, the $2x2$ CMV matrix

$$A = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 \\ \rho_0 & -\overline{\alpha_1}\alpha_0 \end{pmatrix}, \rho_j = \sqrt{1 - |\alpha_j|^2}$$

CMV Matrix Example for Triangle

For $\Phi_2(z)$, the $2x2$ CMV matrix

$$A = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 \\ \rho_0 & -\overline{\alpha_1}\alpha_0 \end{pmatrix}, \rho_j = \sqrt{1 - |\alpha_j|^2}$$

the eigenvalues $\lambda_1, \lambda_2 = f_1, f_2$.

CMV Matrix Example for Triangle

$$\mathcal{L}_0\mathcal{M}_1 = \begin{pmatrix} \overline{\alpha}_0 & \rho_0 & 0 \\ \rho_0 & -\alpha_0 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\alpha}_1 & \rho_1 \\ 0 & \rho_1 & -\alpha_1 \end{pmatrix}$$

CMV Matrix Example for Triangle

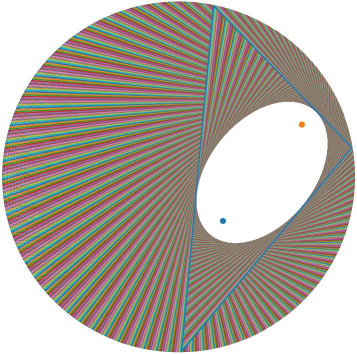
$$\mathcal{L}_0\mathcal{M}_1 = \begin{pmatrix} \overline{\alpha}_0 & \rho_0 & 0 \\ \rho_0 & -\alpha_0 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\alpha}_1 & \rho_1 \\ 0 & \rho_1 & -\alpha_1 \end{pmatrix}$$

$$\mathcal{G} = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 \\ 0 & e^{i\theta}\rho_1 & e^{i\theta}\alpha_1 \end{pmatrix}$$

which has 3 eigenvalues that satisfy the POPUC $\Phi_3(z)$.

Triangle Script

- Built in Python3:
  - numpy
  - matplitlib
- Built ground up to replace previously created Mathmatica code.
- Uses inverse Szegő Recursion to find Verblunsky coefficients for the CMV matrix.



Blaschke Product

- $$B_n(z) = \prod_{j=1}^{n} \frac{z - f_j}{1 - \overline{f_j} z}$$

Blaschke Product

$$B_n(z) = \prod_{j=1}^{n} \frac{z - f_j}{1 - \overline{f_j} z} = \frac{\Phi_n(z)}{\Phi_n^*(z)}$$

Blaschke Product

$$B_n(z) = \prod_{j=1}^{n} \frac{z - f_j}{1 - \overline{f_j} z} = \frac{\Phi_n(z)}{\Phi_n^*(z)}$$

$$B_{n+1}(z) = \frac{z\Phi_n(z)}{\Phi_n^*(z)}, f_{n+1} = 0$$

Blaschke Products

- z is the preimage of $e^{i\theta}$ under $B_{n+1}(z)$. The zeros of $B_{n+1}(z) - e^{i\theta} = 0$ are the zeros of the POPUC.

- z is the preimage of $e^{i\theta}$ under $B_{n+1}(z)$. The zeros of $B_{n+1}(z) - e^{i\theta} = 0$ are the zeros of the POPUC. $B_n(z) = B_j(B_k(z)), n = j * k$
- if and only if there exists an ellipse for n, j, k, l, m.

Mirman's Iterations

- Given f_1, f_2

$$B_2(f_3; f_1, f_2) = f_1 f_2 = \frac{(f_3 - f_1)(f_3 - f_2)}{(1 - f_3\overline{f_1})(1 - f_3\overline{f_2})}.$$

- Then $\Phi_3(z) = (z - f_1)(z - f_2)(z - f_3)$ gives the foci for the desired OPUC.

Heptagon Case

$$f_3 f_6 = \frac{(f_5 - f_1)(f_5 - f_2)}{(1 - f_5\overline{f_1})(1 - f_5\overline{f_2})} \quad f_5 f_4 = \frac{(f_6 - f_1)(f_6 - f_2)}{(1 - f_6\overline{f_1})(1 - f_6\overline{f_2})}$$

$$f_1 f_5 = \frac{(f_3 - f_1)(f_3 - f_2)}{(1 - f_3\overline{f_1})(1 - f_3\overline{f_2})} \quad f_6 f_2 = \frac{(f_4 - f_1)(f_4 - f_2)}{(1 - f_4\overline{f_1})(1 - f_4\overline{f_2})}$$

Heptagon Case

$$f_3 f_6 = \frac{(f_5 - f_1)(f_5 - f_2)}{(1 - f_5\overline{f_1})(1 - f_5\overline{f_2})} \quad f_5 f_4 = \frac{(f_6 - f_1)(f_6 - f_2)}{(1 - f_6\overline{f_1})(1 - f_6\overline{f_2})}$$

$$f_1 f_5 = \frac{(f_3 - f_1)(f_3 - f_2)}{(1 - f_3\overline{f_1})(1 - f_3\overline{f_2})} \quad f_6 f_2 = \frac{(f_4 - f_1)(f_4 - f_2)}{(1 - f_4\overline{f_1})(1 - f_4\overline{f_2})}$$

- These are necessary but not sufficient.

Heptagon Case

$$f_3 f_6 = \frac{(f_5 - f_1)(f_5 - f_2)}{(1 - f_5\overline{f_1})(1 - f_5\overline{f_2})} \quad f_5 f_4 = \frac{(f_6 - f_1)(f_6 - f_2)}{(1 - f_6\overline{f_1})(1 - f_6\overline{f_2})}$$

$$f_1 f_5 = \frac{(f_3 - f_1)(f_3 - f_2)}{(1 - f_3\overline{f_1})(1 - f_3\overline{f_2})} \quad f_6 f_2 = \frac{(f_4 - f_1)(f_4 - f_2)}{(1 - f_4\overline{f_1})(1 - f_4\overline{f_2})}$$

- These are necessary but not sufficient.

```
f3: -1.4743114174951e-13 - 1.53559015487633e-13*I, f4: 0.1098357609768773 + 0.1287819665106405*I, f5: 0.5000000000000483 + 0.09999999999995647*I, f6: 0.1098357609769365 + 0.1287819665108338*I}
f3: 0.5635484177428319 + 0.2350992288310612*I, f4: 0.4479342472417116 + 0.3121753424984741*I, f5: 0.2644660739774929 - 0.02354651909046461*I, f6: -0.02621276236397625 + 0.170239371803842*I}
f3: 0.3900907908410406 - 0.1336588372624096*I, f4: -0.2288109446178541 + 0.2789423196806355*I, f5: -0.118575491867513 + 0.4054877028041352*I, f6: 0.5891811653106882 - 0.0663500686213127*I}
f3: 9.17208680006241e-11 - 2.94434312538283e-10*I, f4: 0.7682976013299635 + 0.306585957027989*I, f5: 0.5000000004551983 + 0.0999999999299019*I, f6: 0.7682976014618198 + 0.3065859570367106*I}
f3: 0.6500964640948752 + 0.4190939680362312*I, f4: 0.7856015865810343 + 0.3287572207734939*I, f5: 0.825231227140539 + 0.4329259801035119*I, f6: 0.8542744650775125 + 0.4135638216925874*I}
f3: 1.30677678072827 - 1.056299325405623*I, f4: -0.8676412885082874 - 1.787005611954489*I, f5: -2.699439797063601 + 6.508467226306607*I, f6: 0.8492428984241508 + 3.364648653697006*I}
f3: -0.9051685443348833 - 0.01583723870469593*I, f4: -1.066985075689502 + 2.272358781569803*I, f5: 2.390524197881896 - 0.8533095837767755*I, f6: 6.657388532398144 - 2.919855697831999*I}
f3: -8.54876678857738e-17 - 2.16196370627333e-16*I, f4: 8.500564862580902 + 11.13267941374374*I, f5: 0.5000000000000057 + 0.09999999999999845*I, f6: 8.500564862581118 + 11.13267941374381*I}
f3: 0.8138440189556673 + 0.7672091365946333*I, f4: 1.424462911192785 + 0.3601298751032184*I, f5: 1.63588144989837 + 3.972022521539772*I, f6: 12.47770456612372 - 3.255859555943755*I}
f3: -(845/96677122*I + 49155/96677122)*sqrt(218196*I - 90915) + 15130395/96677122*I + 33519365/96677122, f4: 1/10*I + 1/2, f5: 3/10*I + 1/5, f6: 0}
f3: (845/96677122*I + 49155/96677122)*sqrt(218196*I - 90915) + 15130395/96677122*I + 33519365/96677122, f4: 1/10*I + 1/2, f5: 3/10*I + 1/5, f6: 0}
f3: 0, f4: 3/10*I + 1/5, f5: 1/10*I + 1/2, f6: 0}
f3: 1545/4831*I + 2915/4831, f4: 3/10*I + 1/5, f5: 1/10*I + 1/2, f6: 0}
```