Poncelet Polygons

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1 Introduction

Poncelet's Closure Theorem states

Theorem 1. *Let C be an ellipse in $\mathbb{D}$. If there is a polygon $\mathcal{P}_0$ inscribed in $\mathbb{T}$ that circumscribes C, then for any complex point $z \in \mathbb{T}$, there exists a polygon $\mathcal{P}$ inscribed in $\mathbb{T}$ and circumscribed about C such that z is a vertex of $\mathcal{P}$. The ellipse is called a Poncelet-n ellipse.*

For higher degrees of n, for instance a hexagon, connecting subsets of the vertexes, to make a triangle or intersecting diagonals, can circumscribe new ellipses. Thus there is an infinite family of $\mathcal{P}$ that circumscribes a package of ellipses.

2 OPUC

A orthogonal polynomial on the unit circle (OPUC) is a polynomial whose zeros exist on $\mathbb{D}$ and are orthogonal with respect to integration on the unit circle in the complex plane. The OPUCs used are defined by the product

$$\Phi_n(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} z - f_j, \quad f_j \in \mathbb{D}$$

or the sum

$$\Phi_n(z) = \sum_{j=0}^{n} c_j z^j, \quad c_n = 1$$

Each ellipse in the package have their own foci. These foci are the zeros of an OPUC. The inverse function $\Phi_n^*(z)$ is defined by switching and conjugating the coefficients of the monomial $z - f_j$

$$\Phi_n^*(z; f_1, f_2, ..., f_n) = \prod_{j=1}^{n} 1 - z\overline{f_j}$$

or

$$\Phi_n^*(z) = \sum_{j=0}^{n} \overline{c_j} z^{n-j} = z^n \overline{\Phi_n(1/\overline{z})}$$

3 Szegő Recursion

Using the functions $\Phi_n(z)$ and $\Phi_n^*(z)$, $\Phi_{n+1}(z)$ and $\Phi_{n+1}^*(z)$ can be defined recursively.

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z)$$
$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z)$$

α_n is known as the Verblunsky coefficient. Szegő recursion can be inverted,

$$\Phi_{n-1}(z) = (1-|\alpha_n|^2)^{-1}(\Phi_n(z) + \overline{\alpha_n}\Phi_n^*(z))$$
$$\Phi_{n-1}^*(z) = (1-|\alpha_n|^2)^{-1}(\alpha_n z\Phi_n(z) + z\Phi_n^*(z))$$

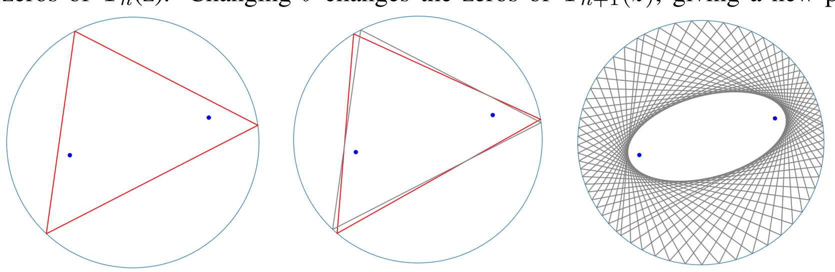
allowing for the recovery of smaller degrees of orthogonal polynomials from larger ones.

4 POPUC

Replacing $\overline{\alpha_n}$ with $e^{i\theta}$ for $\Phi_n(z)$ whose zeros are the foci gives

$$\Phi_{n+1}(z) = z\Phi_n(z) - e^{i\theta}\Phi_n^*(z)$$

Because $e^{i\theta}$ has a magnitude of one the zeros of $\Phi_{n+1}(z)$ are pushed to $\mathbb{T}$. With all of the zeros on $\mathbb{T}$, this OPUC is known as a paraorthogonal polynomial on the unit circle (POPUC). Connecting the zeros gives a polygon that circumscribes the ellipse for the zeros of Φ_n(z). Changing θ changes the zeros of $\Phi_{n+1}(z)$, giving a new polygon.



5 CMV Matrix

A matrix Θ_j is defined using the Verblunsky coefficients as such,

$$\Theta_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad \rho_j = \sqrt{1-|\alpha_j|^2}$$

Summing the even and odd indexed Θ_n into

$$\mathcal{L} := \Theta_0 \bigoplus \Theta_2 \bigoplus \Theta_4 \bigoplus ..., \quad \mathcal{M} := 1 \bigoplus \Theta_1 \bigoplus \Theta_3 \bigoplus ...$$

such that

$$\mathcal{L}_2 = \begin{pmatrix} \overline{\alpha}_0 & \rho_0 & 0 & 0 \\ \rho_0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & \overline{\alpha}_2 & \rho_2 \\ 0 & 0 & \rho_2 & -\alpha_2 \end{pmatrix}$$

Multiplying the infinite sums $\mathcal{L} * \mathcal{M}$ gives the matrix

$$\mathcal{G} = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

The a nxn matrix, A, can be cut from the top left of $\mathcal{G}$. For example, $n = 2$

$$A_2 = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 \\ \rho_0 & -\overline{\alpha_1}\alpha_0 \end{pmatrix}$$

The characteristic polynomial of A is $\Phi_n(z)$ and the eigenvalues of A are the foci of the package of ellipses.

Because $\mathcal{L}$ is always $2k$x$2k$ and $\mathcal{M}$ is $2j{+}1$x$2j{+}1$ they cannot be directly multiplied. $\mathcal{L}$ or $\mathcal{M}$ must be dilated with $e^{i\theta}$ to give them equal dimension. For example,

$$\mathcal{L}_0\mathcal{M}_1 = \begin{pmatrix} \overline{\alpha}_0 & \rho_0 & 0 \\ \rho_0 & -\alpha_0 & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\alpha}_1 & \rho_1 \\ 0 & \rho_1 & -\alpha_1 \end{pmatrix}$$

The characteristic polynomial of the resulting matrix is the desired POPUC. The eigenvalues are the vertices of the Poncelet's polygon.

6 Blaschke Products

Blaschke products are defined as

$$B_n(z) = \prod_{j=1}^{n} \frac{z - f_j}{1 - \overline{f_j}z}$$

This is equivalent to the $\Phi_n(z)/\Phi_n^*(z)$. When $f_{n+1} = 0$, $B_{n+1}(z) = zB_n$. z is the preimage of $e^{i\theta}$ under $B_{n+1}(z)$. The zeros of $B_{n+1}(z) - e^{i\theta} = 0$ are the zeros of the POPUC.

Using $B_2(z)$ given f_1 and f_2, higher degrees of f_n for the OPUC can be found. For $n = 3$

$$B_2(f_3; f_1, f_2) = f_1 f_2 = \frac{(f_3 - f_1)(f_3 - f_2)}{(1 - f_3\overline{f_1})(1 - f_3\overline{f_2})}$$

This is known as Mirman's iteration. In the above case, the iteration gives one solution. However, as n grows, so too does the number of equations in the iteration and the number of solutions. This makes satisfying the iteration necessary but not sufficient. That is, a set solutions will satisfy the iteration, but may not satisfy the OPUC. The solutions to the OPUC will satisfy the iteration.

7 Computational Construction

The scripts used were built using Sage and Python with the Numpy and Matplotlib packages. They were built to replace previously used Mathmatica code. The code for the triangle case was completed by using Inverse Szegő Recursion to recover the Verblunsky coefficients used to build the CMV matrix. The heptagon case is yet to be solved. Currently, a Mirman iteration with four functions is being used to find possible sets of f_3, f_4, f_5, and f_6. These solutions will have impossible sets removed before following a similar procedure to the triangle case to determine the correct set.