Juliette Bruce's Research Statement

My research interests lie in algebraic geometry, commutative algebra, and arithmetic geometry. In particular, I am interested in using homological and combinatorial methods to study the geometry of zero loci of systems of polynomials (i.e. algebraic varieties). I am also interested in studying the arithmetic properties of varieties over finite fields. Further, I am passionate about promoting inclusivity, diversity, and justice in the mathematics community.

1. Syzygies in Algebraic Geometry

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_i is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring R with a twisted grading, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j} is the graded piece of degree $d-j$. The $\beta_{i,j}$'s are the *Betti numbers* of M, and they count the number of i-syzygies of M of degree j. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [Voi02, Voi05, AFP+19]. See also [Eis05, Conjecture 9.6] and [Sch86, BE91, FP05, Far06, AF11, FK16, FK17].

1.1 Asymptotic Syzygies Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84a, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question: In what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set,

$$\rho_q(X, L_d) \quad := \quad \frac{\#\left\{p \in \mathbb{N} \,\middle|\, \beta_{p,p+q}(X, L_d) \neq 0\right\}}{r_d},$$

which is the percentage of degrees in which non-zero syzygies appear [Eis05, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84a] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that* $\deg L_d = d$ *then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q\left(X; L_d\right) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19a, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q\left(X; \mathcal{O}\left(d_1, d_2\right)\right) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\ j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.}\\ \textit{terms}\end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [Lem18] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right)$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [EL12, Conjectures 7.1, 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld. Such a generalization is complicated by the difference between the Cox ring and homogenous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$. A central theme in this work is to exploit the fact that a key regular sequence I use has a number of non-trivial symmetries.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety and fix an index $1 \leq q \leq \dim X$. Let $(L_d)_{d\in\mathbb{N}}$ be a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample, can one compute $\lim_{d\to\infty} \rho_q\left(X; L_d\right)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru19b].

1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. My co-authors and I exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. This data provided support for several existing conjectures, as well as led us to make a number of new conjectures [BEGSY18]. The resulting data has been made publicly available via SyzygyData.com as well as, a package for Macaualy2 [BEGSY19, M2].

Recently I have begun using similar computational techniques to compute the syzygies for Hirzebruch surfaces. Thus far, we have computed the syzygies in ~ 100 new examples. It is our hope that these examples will lead to new conjectures regarding the syzygies of Hirzebruch surfaces. In particular, we believe our data will be useful in addressing Question 1.4.

2. Homological Algebra on Toric Varieties

3. Tropical Geometry & Moduli Spaces

Theorem 3.1. *The top-weight rational cohomology of $\mathcal{A}_g$ for $g = 5$, 6, and 7, is*

$$\operatorname{Gr}^W_{30} H^k(\mathcal{A}_5;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 15, 20, \\ 0 & \text{else,} \end{cases}$$

$$\operatorname{Gr}^W_{42} H^k(\mathcal{A}_6;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 30, \\ 0 & \text{else,} \end{cases}$$

$$\operatorname{Gr}^W_{56} H^k(\mathcal{A}_7;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 28, 33, 37, 42 \\ 0 & \text{else.} \end{cases}$$

Theorem 3.2. *The top-weight rational cohomology of $\mathcal{A}_8$, $\mathcal{A}_9$, and $\mathcal{A}_{10}$ vanish in the following ranges:*

$$\begin{aligned} \operatorname{Gr}^W_{72} H^i(\mathcal{A}_8;\mathbb{Q}) &= 0 \quad \text{for } i \geq 60, \\ \operatorname{Gr}^W_{90} H^i(\mathcal{A}_9;\mathbb{Q}) &= 0 \quad \text{for } i \geq 79, \\ \operatorname{Gr}^W_{110} H^i(\mathcal{A}_{10};\mathbb{Q}) &= 0 \quad \text{for } i \geq 99. \end{aligned}$$

4. Varieties over Finite Fields

Over a finite field, a number of classical statements from algebraic geometry no longer hold. For example, if $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension n over $\mathbb{C}$, Bertini's theorem states that, if $H \subset \mathbb{P}^r$ is a generic hyperplane, then $X \cap H$ is smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Using an ingenious probabilistic sieving argument, Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree, then a version of Bertini's theorem is true [Poo04]. More specifically Poonen showed that as, $d \to \infty$, the percentage of hypersurfaces $H \subset \mathbb{P}^r_{\mathbf{F}_q}$ of degree d such that $X \cap H$ is smooth is determined by the Hasse-Weil zeta function of X. Below we write $\mathbf{F}_q[x_0, \ldots, x_r]_d$ for the $\mathbf{F}_q$-vector space of homogenous polynomials of degree d.

Theorem 4.1. [Poo04, Theorem 1.1] *Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth variety of dimension n. Then:*

$$\lim_{d\to\infty} \operatorname{Prob}\begin{pmatrix} f \in \mathbf{F}_q[x_0, x_1, \ldots, x_r]_d \\ X \cap \mathbb{V}(f) \text{ is smooth of dimension } n-1 \end{pmatrix} = \zeta_X(n+1)^{-1} > 0. \tag{1}$$

4.1 A Probabilistic Study of Systems of Parameters Given an n dimensional projective variety $X \subset \mathbb{P}^r$, a collection of homogenous polynomials $f_0, f_1, \ldots, f_k$ of degree d is a (partial) system of parameters if $\dim X \cap \mathbb{V}(f_0, f_1, \ldots, f_k) = \dim X - (k+1)$. Systems of parameters are closely tied to Noether normalization, as the existence of a finite (i.e. surjective with finite fibers) map $X \to \mathbb{P}^n$ is equivalent to the existence of a system of parameters of length $n+1$.

Inspired by work of Poonen [Poo04] and Bucur and Kedlaya [BK12], Daniel Erman and I computed the asymptotic probability that randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ over $\mathbf{F}_q$ form a system of parameters. By adapting Poonen's closed point sieve to sieve over higher dimensional varieties, we showed that, when $k < n$, the probability that randomly chosen $f_0, f_1, \ldots, f_k$ form a partial system of parameters is controlled by a zeta-function-like power series that enumerates higher dimensional varieties instead of closed points. In the following, $|Z|$ denotes the number of irreducible components of Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

Theorem 4.2. [BE, Theorem 1.4] *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\begin{pmatrix} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{pmatrix} = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)\binom{n-k+d}{n-k}}\right).$$

From this we proved the first explicit bound for Noether normalization over $\mathbf{F}_q$ and gave a new proof of recent results on Noether normalizations of families over $\mathbb{Z}$ and $\mathbf{F}_q[t]$ [GLL15, CMBPT17].

4.2 Jacobians Covering Abelian Varieties Over an infinite field, it is a classic result that every abelian variety is covered by a Jacobian variety of bounded dimension. Building upon work of Bucur and Kedlaya [BK12], Li and I proved an analogous result for abelian varieties over finite fields. We did so by first proving an effective version of Poonen's Bertini theorem over finite fields.

Theorem 4.3. [BL, Theorem A] *Fix $r, n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbf{F}_q$ be a finite field of characteristic p. There exists an explicit constant $C_{r,q}$ such that if $A \subset \mathbb{P}^r_{\mathbf{F}_q}$ is a non-degenerate abelian variety of dimension n, then for any $d \in \mathbb{N}$ satisfying*

$$C_{r,q}\zeta_A\left(n + \tfrac{1}{2}\right)\deg(A) \leq \frac{q^{\overline{\frac{d}{\max\{n+1,p\}}}}\, d}{d^{n+1} + d^n + q^d},$$

there exists a smooth curve over $\mathbf{F}_q$ whose Jacobian J maps surjectively onto A, where

$$\dim J \leq \mathrm{O}\left(\deg(A)^2 d^{2(n-1)} r^{-1}\right).$$

4.3 Uniform Bertini Notice that in the statement of Poonen's Bertini theorem, while the left-hand side of equation (1) is dependent of the embedding of X into projective space (i.e. the choice of very ample line bundle), the overall limit is itself independent of the embedding of X. This suggests that there may be a more general and uniform statement of Poonen's Bertini theorem. One might hope that the analogous limit along any sequence $(L_d)_{d\in\mathbb{N}}$ of line bundles growing in positivity equals $\zeta_X(n+1)^{-1}$. I am working with Isabel Vogt to formalize and prove such a theorem.

Work of Erman and Wood on semi-ample Bertini theorems shows that a naive analogue of Theorem 4.1 fails [EW15]. Vogt and I believe that this can be fixed by introducing an assumption on how the sequence of lines bundles grows in positivity. We say a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ *goes to infinity in all directions* if for every ample line bundle A there exists $N \in \mathbb{N}$ such that $L_i - A$ is ample for all $i \geq N$. We are working to prove the following uniform version of Theorem 4.1.

Conjecture 4.4. *Let $X/\mathbf{F}_q$ be a smooth projective variety of dimension n. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of line bundles on X going to infinity in all directions then*

$$\lim_{d\to\infty} \operatorname{Prob}\left(f \in H^0(X, L_d) \;\middle|\; \begin{matrix} X \cap \mathbb{V}(f) \text{ is smooth} \\ \text{of dimension } n-1 \end{matrix}\right) = \zeta_X(n+1)^{-1}. \tag{2}$$

We have verified this conjecture in a number of examples ($X = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$). We are hopeful that similar methods will extend to whenever the nef cone of X is a finitely generated.

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