THE SCHURVERONESE PACKAGE IN MACAULAY2

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ABSTRACT. This note introduces the Macaulay2 package SchurVeronese, which gathers together data about Veronese syzygies and makes it readily accessible in Macaulay2. In addition to standard Betti tables, the package includes information about the Schur decompositions of the various spaces of syzygies. The package also includes a number of functions useful for manipulating and studying this data.

In [BEGY18] the authors used a combination of high-throughput and high-performance computation and numerical techniques to compute the Betti tables of $\mathbb{P}^2$ under the d-uple Veronese embedding, as well as the Betti tables of the pushforwards of line bundles $\mathcal{O}_{\mathbb{P}^2}(b)$ under that embedding, for a number of values of b and d. These computations resulted in new data, such as Betti tables, multigraded Betti numbers, and Schur Betti numbers. (When $b = 0$, most the cases had been previously computed in [CCDL].) This note introduces the *SchurVeronese* package for Macaulay2, which makes this data readily accessible via Macaulay2 for further experimentation and study.

1. Veronese Syzygies

Throughout this section we fix $n \in \mathbb{N}$ and let $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$ be the polynomial ring with the standard grading. The dth Veronese module of S twisted by b is

$$S(b; d) := \bigoplus_{i \in \mathbb{Z}} S_{di+b}.$$

If $b = 0$, then $S(0; d)$ is the Veronese subring and if $b \neq 0$ then $S(b; d)$ is an $S(0; d)$-module. Moreoever, if we set $R = \mathrm{Sym}(S_d)$ to be the symmetric algebra on S_d, then we may consider $S(b; d)$ as a graded R-module. Geometrically, if $b = 0$ this corresponds to the homogenous coordinate ring of $\mathbb{P}^n$ under the d-uple embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$, and for other b it corresponds to the pushforward of $\mathcal{O}_{\mathbb{P}^n}(b)$ under the d-uple embedding.

Our interest is in studying the syzygies of $S(b; d)$. See the introduction of [BEGY18] for background on Veronese syzygies including a summary of known results. Throughout, we set $K_{p,q}(\mathbb{P}^n, b; d) := \mathrm{Tor}_p^R(S(b; d), \mathbb{C})_{p+q}$ which is isomorphic to the vector space of degree $p + q$ syzygies of $S(b; d)$ of homological degree p. Using the standard conventions for graded Betti numbers, the rank of the vector space $K_{p,q}$ corresponds to the Betti number $\beta_{p,p+q}$, and

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we write $\beta_{p,q}(S(b;d)) = \dim K_{p,q}(\mathbb{P}^n, b; d)$. The Betti table of $S(b;d)$ will be the table where $\beta_{p,p+q}(S(b;d))$ is placed in the (p,q)-spot.

Outside of the case $n=1$, the Betti tables of $S(b;d)$ are unknown even for modest values of d. For instance, there is not even a conjecture about what the Betti table of $S(b;d)$ should be for $n=2$ and $d \geq 7$.

This package provides an array of computed data about $S(b;d)$ in the case $n=2$ and for $0 \leq b < d \leq 8$ (though the data are incomplete for some of the larger values of d). While computing this data, including the Schur functor decompositions, took substantial time, the output data are rather concise and easy to work with in Macaulay2. The bulk of this package thus consists of these output data, which is included as auxiliary files. The front-end of the package calls up and presents the data in a user-friendly format. Our hope is that this will allow those interested in Veronese syzygies to make headway on formulating conjectures and proving results in this area. Moreover, as new cases of Veronese syzygies computed, these can easily be incorporated into future versions of the package.

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2. An overview of the data

When computing data for $S(b;d)$ we always work under the hypothesis that $0 \leq b < d$, as the Betti table of $S(b;d)$ and $S(b+d;d)$ only differ by a vertical shift. We have included data for the cases $n=1$ and $d \leq 10$, although this can also easily be computed using the Eagon-Northcott complex. The main data is for the cases $n=2$ and $0 \leq b < d \leq 8$. In [BEGY18], we obtained full computations for $d \leq 6$; moreover since those algorithms worked in parallel with respect to multidegrees, we obtained incomplete data for some cases where $d = 7, 8$, and we have included that partial data in this package as well.

The algorithms in [BEGY18] are a mix of symbolic and numeric algebra. Thus some entries in the data are not provably correct, while others are. One can determine precisely when $K_{p,q} \neq 0$ by combining [EL12, Remark 6.5], [Gre84b, Theorem 2.2], and [Gre84, Theorem 2.c.6]. Our computation of a $K_{p,q}$-group (and all related data such as the Schur functor decomposition) will be provably correct if and only if $K_{p+1,q-1}$ and $K_{p-1,q+1}$ both vanish; in cases where this does not occur, the data for $K_{p,q}$ may have been computed numerically, and thus may not be provably correct. For a longer discussion of potential numerical error issues, see [BEGY18, §5.2].

3. Total Betti Tables

The Betti table for $S(b;d)$ can be called up using the `totalBettiTally` command. For example, the Betti table of $S(2;4)$ when $n=2$ is produced below.

```
i6 : totalBettiTally(4,2,0)

             0  1   2    3    4    5    6    7    8    9  10 11 12
o6 = total: 1 75 536 1947 4488 7095 7920 6237 3344 1089 175 24  3
         0: 1  .   .    .    .    .    .    .    .    .   .  .  .
         1: . 75 536 1947 4488 7095 7920 6237 3344 1089 120  .  .
         2: .  .   .    .    .    .    .    .    .    .  55 24  3
```

```
o6 : BettiTally
```

Note that this is purely numeric: the package does not produce a minimal free resolution; the function simply returns the Betti numbers obtained by a previous computation. The command `totalBetti` is similar, but expresses the Betti numbers simply as a hash table.

There is also a distinction between the indexing conventions. When working with hash tables, we follow the more concise $K_{p,q}$ indexing conventions, instead of the $\beta_{p,p+q}$ indexing conventions used for Betti tallies. Thus for instance, in the above example, the Betti number $\beta_{2,3}$ would correspond to key $(2, \{3\}, 3)$ in the Betti tally, but in the hash table `totalBetti` it corresponds to key $(2, 1)$:

```
i4 :E =  totalBetti(4,2,0);

i5 : E#(2,1)
o5 = 536
```

If one tries to call a Betti table outside of the acceptable range of n, b, d, we return a message explaining the problem.

```
i10 : totalBettiTally(4,3,0)
o10 = Need n = 1 or 2
```

As noted above, there were instances where we were able to partially compute Betti tables, for instance in the case of the 7-uple embedding of $\mathbb{P}^2$. In those cases, we have recorded the entries that we know, and we mark the unknown entries with "infinity". For example:

```
i14 : B = totalBetti(7,2,0);

i15 : B#(4,1)
o15 = 1031184

i16 : B#(20,1)
o16 = infinity
o16 : InfiniteNumber
```

Thus, in this case, we have that $\dim K_{4,1}(\mathbb{P}^2, 2; 7) = 1031184$ but we were unable to compute $\dim K_{20,1}(\mathbb{P}^2, 2; 7)$.

4. Schur Decomposition

When $n = 2$ and $d \geq 5$ the Betti tables of $S(b; d)$ are often unwieldy to work with as they and their entries tend to be quite large. For example, the Betti table of $S(0; 6)$ has 26 columns and many of the entries are on the order of 10^7.

A more concise way of recording the syzygies would be to take into account the symmetries coming from representation theory. The natural linear action of $\mathbf{GL}_{n+1}(\mathbb{C})$ on S induces an action on each vector space $K_{p,q}(\mathbb{P}^n, b; d)$. We can thus decompose this as a direct sum of Schur functors of total weight $d(p + q) + b$ i.e.

$$K_{p,q}(\mathbb{P}^n, b; d) = \bigoplus_{|\lambda|=d(p+q)+b} \mathbf{S}_\lambda(\mathbb{C}^{n+1})^{\oplus m_{p,\lambda}(\mathbb{P}^n,b;d)},$$

with $m_{p,\lambda}(\mathbb{P}^n, b; d)$ being the Schur Betti numbers and $\mathbf{S}_\lambda$ being the Schur functor corresponding to the partition λ [FH91, p. 76]. The Schur Betti numbers can be accessed via the `schurBetti` command, which outputs a hash table whose keys correspond to pairs (p, q) for which $K_{p,q}(\mathbb{P}^n, b; d) \neq 0$, and whose values are lists corresponding to the Schur decomposition of this syzygy module.

For example, let us consider $K_{2,1}(\mathbb{P}^2, 0; 4)$, which is a vector space of dimension 536. As a representation of $\mathbf{GL}_3(\mathbb{C})$, it turns out to be the sum of 9 distinct Schur functors, each appearing with multiplicity 1:

$$K_{2,1}(\mathbb{P}^2, 0; 4) = \mathbf{S}_{(9,2,1)} \oplus \mathbf{S}_{(8,4,0)} \oplus \mathbf{S}_{(8,3,1)} \oplus \mathbf{S}_{(7,5,0)} \oplus \mathbf{S}_{(7,4,1)} \oplus \mathbf{S}_{(7,3,2)} \oplus \mathbf{S}_{(6,5,1)} \oplus \mathbf{S}_{(6,4,2)} \oplus \mathbf{S}_{(5,4,1)}.$$

```
i26 : (schurBetti(4,2,0))#(2,1)

o26 = {({9, 2, 1}, 1), ({8, 4, 0}, 1), ({8, 3, 1}, 1), ({7, 5, 0}, 1),
      -----------------------------------------------------------------
 ({7, 4, 1}, 1), ({7, 3, 2}, 1), ({6, 5, 1}, 1), ({6, 4, 2}, 1), ({5, 4, 3}, 1)}

o8 : List
```

From this, it is easy to compute statistics such as the number of representations and the number of distinct representations appearing in the Schur decomposition of $K_{p,q}(n, b; d)$. The *SchurVeronese* package provides commands for these. For instance, in our example from above we see that:

```
i11 : (numDistinctRepsBetti(4,2,0))#(2,1)
o11 = 9
```

We can also display the number of representations appearing in each entry of the Betti table. In the following example, the first table counts distinct Schur functors and the second counts the number of Schur functors with multiplicity.

```
i29 : makeBettiTally numDistinctRepsBetti(4,2,0)

             0 1 2  3  4  5  6  7  8  9 10 11 12
o29 = total: 1 2 9 17 23 23 26 25 21 13  3  1  1
          0: 1 . .  .  .  .  .  .  .  .  .  .  .
          1: . 2 9 17 23 23 26 25 21 13  1  .  .
          2: . . .  .  .  .  .  .  .  .  2  1  1

i30 : makeBettiTally numRepsBetti(4,2,0)

             0 1 2  3  4  5  6  7  8  9 10 11 12
o30 = total: 1 2 9 28 55 79 86 69 38 14  3  1  1
          0: 1 . .  .  .  .  .  .  .  .  .  .  .
          1: . 2 9 28 55 79 86 69 38 14  1  .  .
          2: . . .  .  .  .  .  .  .  .  2  1  1
```

Thus, $K_{4,1}(\mathbb{P}^2, 0; 4)$ is the sum of 55 irreducible representations, 23 of which are distinct.

5. Multigraded Betti Numbers

One can also specialize the action of $\mathbf{GL}_{n+1}(\mathbb{C})$ to the torus action via $(\mathbb{C}^*)^{n+1}$. This gives a decomposition of $K_{p,q}(\mathbb{P}^n, b; d)$ into a sum of $\mathbb{Z}^{n+1}$-graded vector spaces of total weight $d(p+q)+b$. Specifically, writing $\mathbb{C}(-\mathbf{a})$ for the vector space $\mathbb{C}$ together with the $(\mathbb{C}^*)^{n+1}$-action given by $(\lambda_0, \lambda_1, \ldots, \lambda_n) \cdot \mu = \lambda_0^{a_0}\lambda_1^{a_1}\cdots\lambda_n^{a_n}\mu$ we have

$$K_{p,q}(\mathbb{P}^n, b; d) = \bigoplus_{\substack{\mathbf{a}\in\mathbb{Z}^{n+1} \\ |\mathbf{a}|=d(p+q)+b}} \mathbb{C}(-\mathbf{a})^{\oplus\beta_{p,\mathbf{a}}(\mathbb{P}^n,b;d)}$$

as $\mathbb{Z}^{n+1}$-graded vector spaces, or equivalently as $(\mathbb{C}^*)^{n+1}$ representations. The *SchurVeronese* package produces these multigraded Betti numbers for a number of examples via the `multiBetti` command. Similar to the `schurBetti` this command outputs a hash table whose keys correspond to pairs (p,q) for which $K_{p,q}(\mathbb{P}^n, b; d) \neq 0$, and whose values are multigraded Hilbert polynomials encoding the multigraded decomposition of $K_{p,q}(n, b; d)$. More specifically the value of `(multiBetti(d,n,b))#(p,q)` is the polynomial $\sum_{\substack{\mathbf{a}\in\mathbb{Z}^{n+1} \\ |\mathbf{a}|=d(p+q)+b}} \beta_{p,\mathbf{a}}(n,b;d)\mathbf{t}^{\mathbf{a}}$. where $\mathbf{t}^{\mathbf{a}}$ denotes $t_0^{a_0}t_1^{a_1}\cdots t_n^{a_n}$.

For example, $K_{12,2}(2,0;4)$ is the following 3-dimensional $\mathbb{Z}^3$-graded vector space:

$$K_{12,2}(2,0;4) \cong \mathbb{C}(-(19,19,18)) \oplus \mathbb{C}(-(19,18,19)) \oplus \mathbb{C}(-(18,19,19)).$$

The following code computes this, illustrating that the multigraded Hilbert function for $K_{12,2}(2,0;4)$ is $t_0^{19}t_1^{19}t_2^{18} + t_0^{19}t_1^{18}t_2^{19} + t_0^{18}t_1^{19}t_2^{19}$.

```
i4 : (multiBetti(4,2,0))#(12,2)

      19 19 18    19 18 19    18 19 19
o4 = t  t  t   + t  t  t   + t  t  t
      0  1  2     0  1  2     0  1  2

o4 : QQ[t , t , t ]
         0   1   2
```

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