# Juliette Bruce's Proposal Narrative

My research interests lie in algebraic geometry, commutative algebra, and arithmetic geometry. In particular, I am interested in using homological and combinatorial methods to study the geometry of zero loci of systems of polynomials (i.e. algebraic varieties). Using the Burke Award I would continue my research in the following two directions.

- **Homological Algebra on Toric Varieties:** A classical story in algebraic geometry is that homological methods and tools like minimal free resolutions and Castelnuovo–Mumford regularity capture the geometry of subvarieties of projective space in nuanced ways. My work looks to generalize this story by developing ways homological algebra can be used to study the geometry of toric varieties (i.e., "nice' compactifications of the torus $(\mathbb{C}^{\times})^n$).
- **Cohomology of Moduli Spaces and Arithmetic Groups:** Despite its importance in algebraic geometry and number theory much remains unknown about the topology of $\mathcal{A}_g$, the moduli space of abelian varieties of dimension $g$. I have been working to study a canonical "part" of the cohomology of $\mathcal{A}_g$, called the top-weight cohomology. This turns out to be closely connected to the study of cohomology of various arithmetic groups like $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$, as well as the study of automorphic forms.

My plans in each of these directions is described in more detail below. The Burke Award would be extremely helpful in my continued research in these areas as it would allow me to remain connected to colleagues and collaborators via research visits, present at research conferences and seminars, and purchase necessary materials for my research (computing, books, etc.). I hope to supplement the funding provided by the Burke Award by applying for research funding from the National Science Foundation, American Mathematical Society, and the Association for Women in Mathematics.

## 1. Homological Algebra on Toric Varieties

Given a graded module $M$ over a graded ring $R$, a helpful tool for understanding the structure of $M$ is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating $M$ by a sequence of free $R$-modules. Formally, a *graded free resolution* of a module $M$ is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each $F_i$ is a graded free $R$-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring $R$ with a twisted grading, so that $R(-j)_d$ is equal to $R_{d-j}$ where $R_{d-j}$ is the graded piece of degree $d-j$. The $\beta_{i,j}$'s are the *Betti numbers* of $M$, and they count the number of $i$-syzygies of $M$ of degree $j$. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety $X$ embedded in $\mathbb{P}^r$, we associate to $X$ the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and $I_X$ is the ideal of homogenous polynomials vanishing on $X$. As $S_X$ is naturally a graded $S$-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of $X$. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [2, 16, 17]. See also [10, Conjecture 9.6] and [3, 11, 12, 15].

Much of my work can be viewed as understanding how minimal graded free resolutions capture the geometry when the role of $\mathbb{P}^r$ is replaced by another variety $Y$. In particular, I have focused on the case when $Y$ is a toric variety, i.e., a compactification of the torus $(\mathbb{C}^{\times})^r$ where the action of the torus extends to the boundary. Examples of toric varieties include projective space, products of projective spaces, and Hirzebruch surfaces. Work of Cox shows there is a correspondence between (toric) subvarieties of a fixed toric variety and quotients of a polynomial ring similar to the story discussed above for $\mathbb{P}^r$ [8]. As such recent years have seen substantial work looking to use homological algebra and to better understand the geometry of toric varieties [1, 4, 6, 9, 13, 14]. Using the Burke Award I would love to help organize a conference/seminar to help promote and grow this field and work. Further, it would be helpful in having computational resources to explore these new ideas.

## 2. Cohomology of Moduli Spaces and Arithmetic Groups

Some of the most classical objects in algebraic geometry are moduli spaces, i.e., spaces that parameterize a given collection of geometric objects. The quintessential example of a moduli space is $\mathcal{M}_g$, the moduli space of (smooth) genus $g$ curves, also known as the moduli space of compact Riemann surfaces of genus $g$. Despite their classical nature, much remains unknown about the geometry of many moduli spaces. For example, the rational cohomology of $\mathcal{M}_g$ is only known for $g \leq 4$. However, classical results suggest that $\mathcal{M}_g$ should have a lot of cohomology because its Euler characteristic grows super exponentially. Recent groundbreaking work of Chan, Galatius, and Payne has shed the first direct light on this phenomenon by constructing new non-trivial cohomology classes [7]. Much of my recent work has sought to build up the groundwork laid by Chan, Galatius, and Payne to study the rational cohomology of other moduli spaces. Of particular, interest to me has been the moduli space of abelian varieties and various generalizations. This work has deep connections to the cohomology of various arithmetic groups like $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$.

The moduli space $\mathcal{A}_g$ actually a special instance of the moduli space of (principally polarized) abelian varieties of dimension $g$ with level $m$-structure. Denoted by $\mathcal{A}_g(m)$, we may view it as the quotient $[\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})](m)$ where $\mathrm{Sp}_{2g}(\mathbb{Z})(m)$ is the principal congruence subgroup $\ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$. From this perspective, one may hope to study the rational cohomology of $\mathcal{A}_g(m)$ and $\mathrm{Sp}_{2g}(\mathbb{Z})(m)$ by generalizing the ideas of Chan, Galatius, and Payne. In ongoing work, Melody Chan and I are working to prove the following.

**Goal Theorem 2.1.** *Let $d = \binom{g+1}{2}$ be the dimension of $\mathcal{A}_g(m)$. For any integers $m \geq 1$ and $g \geq 0$ there exists a cellular complex $LA_g(m)^{\mathrm{trop}}$ such that for all $i \geq 0$ there is a natural isomorphism*

$$\tilde{H}_{i-1}\left(LA_g(m)^{\mathrm{trop}}; \mathbb{Q}\right) \cong \mathrm{Gr}^W_{2d} H^i\left(\mathcal{A}_g(m); \mathbb{Q}\right),$$

The methods behind Goal Theorem 2.1 show new connections between the cohomology of $\mathcal{A}_g(m)$ and the cohomology of $\mathrm{GL}_g(\mathbb{Z})(m)$. The cohomology of $\mathrm{Sp}(2g, \mathbb{Z})(m)$ – and hence $\mathcal{A}_g(m)$ – and $\mathrm{GL}_g(\mathbb{Z})(m)$ are closely connected to automorphic forms. The Burke Award will greatly help this and other collaborations continue and start by providing funds for travel.

## References


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