Juliette Bruce
Graduate Student
Department of Mathematics
juliette.bruce@math.wisc.edu

October 31, 2019

To the Hiring Committee,

My name is Juliette Bruce, and I am a graduate student at the University of Wisconsin - Madison, working in pure mathematics, namely, algebraic geometry, commutative algebra, and arithmetic geometry under the guidance of my advisor Professor Daniel Erman. I expect to receive my Ph.D. in Mathematics from the University of Wisconsin - Madison in the Spring of 2020. I am writing to apply for the University Research Postdoctoral Fellowship at the University of Kentucky as a member of the Department of Mathematics. Professor Dave Jensen has agreed to be my mentor.

My application includes: a curriculum vitae, a description of my career goals, and a research statement. I will have five letters accompanying my application. Three letters of recommendation: Christine Berkesch (cberkesc@umn.edu), David Eisenbud (de@msri.org), and Daniel Erman (derman@math.wisc.edu), and two letters of support: my proposed mentor Dave Jensen (dave.jensen@uky.edu) and the chair of the Department of Mathematics Uwe Nagel (uwe.nagel@uky.edu).

I believe strongly in the importance of inclusivity, diversity, and justice, and I am passionate about promoting these values within the mathematical community. As a graduate student at the University of Wisconsin-Madison I worked hard to create a learning community that was as open and inclusive to as many people as possible. By working with outreach programs like the Madison Math Circle I expanded the reach of the university outside the bounds of campus. While on campus I have made our learning community more inclusive and welcoming of people from underrepresented groups; especially LGBTQ+ individuals, through work on the Mathematics Department's Committee on Inclusion and Diversity and by founding oSTEM@UW. Further, to promote the success of mathematicians from minority genders I organized a number of workshops and conferences. Going forward, I am excited to continue working hard to promote these values at the University of Kentucky through my research, teaching, and service.

Expanding the Learning Community. The Madison Math Circle (MMC) is an outreach program sponsored by the UW - Madison Math Department. Its goal is to kindle excitement and appreciation of math in middle and high school students. Towards the end of my first semester in graduate school, Fall 2014, I began volunteering with the MMC. At the time, the circle's main programming was a weekly on-campus lecture given by a member of the math department. After volunteering with the MMC for roughly a year, I stepped into the role of student organizer/coordinator.

During my roughly three years as organizer, I worked to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations focused on underrepresented groups. These ties helped the weekly attendance of the circle to more than double, and grow substantially more diverse. Additionally, during my time the number of women and undergraduate speakers increased. I also led the creation of a new outreach arm of the MMC, which

visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This program has dramatically expanded the reach of the circle, and during my final year as an organizer the circle reached over 300 students.

A More Inclusive Learning Community. During the Fall of 2016, in response to a growing climate of hate, bias, and discrimination on campus, I led the creation of the Mathematics Department's *Committee on Inclusivity and Diversity*. As a member of this committee I drafted a statement on the department's commitment to inclusivity and non-discrimination that was accepted by the faculty at a department meeting. I also worked to create syllabi statements that let students know about these department polices, and that inform them of other campus resources that may be helpful. Everyone within the department is now encouraged to use these statements.

More recently, my passion for creating a more inclusive campus has expanded outside of the math department to try and help address inequalities in STEM fields, more generally by founding oSTEM@UW and organizing qGrads. While a large proportion of students at UW - Madison pursue degrees in STEM adjacent fields there are few – if any – resources on campus that directly support LGBTQ+ students in STEM. This is despite the fact that many LGBTQ+ students in these fields often feel isolated, feel the need to hide their identity, or even to leave STEM altogether.

In light of this, and my own experiences as an LGBTQ+ person in STEM, over the summer of 2017 I co-founded Out in Science, Technology, Engineering, and Mathematics at UW (oSTEM@UW) as a resource for these students. During my time leading oSTEM@UW, the group grew to over fifty active members. The importance of such a group was made clear by the numerous student comments indicating how helpful and encouraging oSTEM@UW is to them. For example, after a meeting, a student emailed me to say, "It made me very happy to see other friendly LGBTQ+ faces around ...Thanks so much for organizing this stuff – it's really helpful for me personally, and I believe it was encouraging for the others attending as well." Additionally, I organized and obtained a travel grant for 11 members, including multiple undergrads, to attend the national oSTEM conference.

Since 2017 I have been the organizer of the campus social organization for LGBTQ+ graduate and post-graduate students, which currently has over 350 members. In this role, I have co-organized a weekly coffee social hour intended to give LGBTQ+ graduate and post-graduate students a place to relax, make friends, and discuss the challenges of being LGBTQ+ at UW - Madison.

Mentoring. Inspired by the mentoring that helped me navigate the challenges of being a women in mathematics, I have worked hard to mentor people from underrepresented groups. Since the Winter of 2018 I have led reading courses with three undergraduates through the *Wisconsin Directed Reading Program*. One of these students, an undergraduate woman, worked with me for over a year. During this time I helped her through the process of applying for summer research projects. This student is now applying to graduate school to pursue a Ph.D. in math. Working with *Girls' Math Night Out* I lead two girls in high school through a semester long project exploring RSA cryptography. During 2018-2019, I mentored 6 first-year graduate students (all women or non-binary students), advicing them oh how to navigte the program requirements, helping them find advisors, and organizing monthly social dinners. Since 2016 I have volunteered with the AWM's Mentoring Network, and currently I am mentoring two undergraduate women.

Organizational Service. In the Spring of 2017 I organized *Math Careers Beyond Academia* (50

participants), a one-day professional development conference on STEM careers outside of academia. In April 2018 I organized *M2@UW* (45 participants), a four-day workshop focused on creating new packages for Macaulay2. In February 2019 I organized *Geometry and Arithmetic of Surfaces* (40 participants), a workshop providing a diverse group of early-career researchers the opportunity to learn about interesting topics in the arithmetic and algebraic geometry. In April 2019 I organized the *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants) focused on forming a community of women and non-binary researchers in commutative algebra, and give young graduate students from minority genders, role models for the next stage in their careers. I organized a *Special Session on Combinatorial Algebraic Geometry* at the AMS Fall 2019 Central Sectional. At the 2020 Joint Mathematics Meetings, I am organizing a panel titled *Supporting Transgender and Non-binary Students*.

When organizing these conferences I paid particular attention to making them as inclusive of women and non-binary researchers as possible. In particular, I worked hard to make sure there was gender parity among the speakers and participants. For example, of the five speakers at G&AoS four were from generally underrepresented groups with three women and one person of color speaking. Additionally, over 30 identify as either female or non-binary researchers.

I have given substantial thought to how to make many of the smaller aspects of conferences more inclusive. For example, I designed the registration form to be thoughtful of the concerns of transgender researchers, implemented the process of putting pronouns on name tags, highlighted the locations of single occupancy and ADA compliant restrooms. The importance of such efforts was highlighted by the following comment I received from a participant, "I just wanted to thank you for making this workshop inclusive for people with all gender identifications. As a non-binary biologically female person I have always felt out of place when I participated in conferences/workshops for women when they do not specify that non-binary people are welcome or just assume I am female. I really appreciate those questions you put in the registration form. It means a lot to me."

As a graduate student I worked hard to develop programs, policies, and practices that promoted diversity, inclusion, and justice within the academic community. As I move forward in my career I hope to continue, and expand upon, this work. I believe that being a University Research Postdoctoral Fellow would give me this opportunity. In particular, if selected as a fellow, I will work hard to continue promoting these values through my research, teaching, and service.

Please do not hesitate to contact me with any questions, or if there is anything else I can provide, and thank you in advance for your consideration.

Sincerely,

Juliette E. Bruce

Juliette Bruce
Graduate Student

Department of Mathematics
University of Wisconsin–Madison 480 Lincoln Dr Madison, Wisconsin 53706
Phone: (608) 263–3054 Fax: (608) 263–8891 Web: http://www.math.wisc.edu

Juliette Bruce

October 24, 2019

Department of Mathematics
University of Wisconsin
480 Lincoln Dr.
Madison, WI 53706

juliette.bruce@math.wisc.edu
math.wisc.edu/~juliettebruce

Education

- **University of Wisconsin** — Madison, WI
  Ph.D. Mathematics — *2014 – Present*
  - Advisor: Daniel Erman
- **University of Wisconsin** — Madison, WI
  M.A. Mathematics — *2014 – 2016*
- **University of Michigan** — Ann Arbor, MI
  B.S. in Mathematics & Political Science — *2010 – 2014*

Research Interests

Algebraic Geometry, Commutative Algebra, Arithmetic Geometry, Non-linear Algebra. Specifically, homological methods in algebraic geometry, and algebraic geometry over finite fields.

Publications

10. J. Bruce and D. Erman. A probabilistic approach to systems of parameters and Noether normalization. *Algebra and Number Theory*, Accepted. E-print: arXiv:1604.01704.

9. J. Bruce and W. Li. Effective bounds on the dimensions of Jacobians covering abelian varieties. *Proc. Amer. Math. Soc.*, Accepted. E-print: arXiv:1804.11015.

8. J. Bruce, D. Erman, S. Goldstein, and J. Yang. Conjectures and computations about Veronese syzygies. *Experimental Mathematics*, To Appear. E-print: arXiv:1711.03513.

7. M. Brandt, J. Bruce, T. Brysiewicz, R. Krone, and E. Robeva. The degree of $SO(n)$. *Combinatorial Algebraic Geometry*, 207-224, Fields Inst. Commun. **80**, (2017). E-print: arXiv:1701.03200.

6. J. Bruce, M. Logue, and R. Walker. Monomial valuations, cusp singularities, and continued fractions. *Journal of Commutative Algebra*, **7** (2015) no. 4, 495-522. E-print: arXiv:1311.6493.

5. J. Bruce, P. Kao, E. Nash, B. Perez, and P. Vermeire. Betti tables of reducible algebraic curves. *Proc. Amer. Math. Soc.* **142** (2014) 4039-4051. E-print: arXiv:1210.3064.

Pre-Prints

4. J. Bruce. The Quantitative Behavior of Asymptotic Syzygies for Hirzebruch Surfaces. *Submitted.* E-Print: arXiv:1906.07333.

3. J. Bruce, D. Erman, S. Goldstein, and J. Yang. The SchurVeronese package in Macaulay2. *Submitted.* E-print: arXiv:1905.12661.

2. A. Almousa, J. Bruce, M. Loper, and M. Sayrafi. The Virtual Resolutions Package for Macaulay2. *Submitted.* E-print: arXiv:1905.07022.

1. J. Bruce. Asymptotic Syzygies in the Setting of Semi-Ample Growth. *Submitted.* E-Print: arXiv:1904.04944

Software

2. SchurVeronese, (with D. Erman, S. Goldstein, and J. Yang). Submitted for distribution with future releases of Macaulay2, a compute algebra system focused on computations in algebraic geometry and commutative algebra.

1. VirtualResolutions, (with A. Almousa, M. Loper, and M. Sayrafi). Distributed with version 1.14 of Macaulay2 (2019).

Multimedia

1. SyzygyData.com, (with D. Erman, S. Goldstein, and J. Yang). An online public database on large-scale syzygy computations.

Grants

- **Conference Grant DMS-1908799 – $15,000** March 2019
  National Science Foundation
- **Graduate Research Fellowship** 2015 – 2018
  National Science Foundation
- **Conference Grant DMS-1812462 – $15,000** February 2018
  National Science Foundation
- **Professional Development Grant – $1000** December 2016
  Graduate School – University of Wisconsin

Awards & Honors

- **Excellence in Mathematical Research Award** October 2019
  Award by the math department to a student for exceptional research in their thesis.

- **Capstone Teaching Award** October 2019
  Awarded to one student in the math dept. for an exceptional record of teaching excellence and service.
- **Elizabeth Hirschfelder Prize** October 2018
  Awarded to an outstanding female student who's demonstrated promise in their academic work.
- **Mathematics TA Service Award** April 2018
  Dept. of Mathematics - University of Wisconsin
- **Teaching Assistant Award for Exceptional Service** February 2018
  Campus-wide award recognizing TA's who perform exceptional service
- **Outstanding Achievement in Mathematics** May 2014
  Dept. of Mathematics – University of Michigan
- **Phi Beta Kappa** April 2014
  University of Michigan
- **Chancellor's Opportunity Award** April 2014
  University of Wisconsin

Seminar and Colloquium Talks

- University of Michigan - Commutative Algebra Seminar December 2019
- University of Notre Dame - Algebraic Geometry Seminar November 2019
- DePaul University - Algebra, Combinatorics, and Number Theory Seminar October 2019
- Lawerence University - Colloquium October 2019
- University of Utah - Algebraic Geometry Seminar September 2019
- Stanford University - Algebraic Geometry Seminar May 2019
- University of Kentucky - Algebra Seminar April 2019
- University of Minnesota - Commutative Algebra Seminar April 2019
- Rice University- Algebraic Geometry and Number Theory Seminar September 2018
- DePaul University - Algebra, Combinatorics, and Number Theory Seminar March 2018
- University of Michigan - Commutative Algebra Seminar December 2017

Conference Talks

- LGBTQ+Math - Fields Institute July 2020
- Foundations of Computational Mathematics - Simon Fraser University June 2020
- CA+ - Iowa State University April 2020
- Joint Math Meetings - Denver, CO January 2020
- Fall AMS Central Sectional - University of Wisconsin September 2019
- SIAM Conference on Applied Algebraic Geometry 2019 July 2019
- KUMUNUjr - University of Nebraska March 2019
- Spring AMS Southeastern Sectional - Auburn University March 2019
- Joint Math Meetings - Baltimore, MD January 2019
- Fall AMS Central Sectional - University of Michigan October 2018

- Structures on Free Resolutions - Texas Tech University — October 2017
- Midwest Algebraic Geometry Graduate Conference - University of Illinois, Chicago — April 2015

Poster Talks

- Summer School on Randomness and Learning in NLA - Max Plank Institute, Leipzig — July 2019
- 2019 AWM Research Symposium - Rice University — April 2019
- AWM Poster Session - Joint Math Meetings — January 2018
- AGNES Poster Session - Brown University — September 2018
- Lectures on Arithmetic Geometry - Rice University — February 2017
- Introductory Workshop: Combinatorial Algebraic Geometry - Fields Institute — August 2016
- Commutative Algebra and Its Interactions with Algebraic Geometry — July 2016
- Midwest Commutative Algebra and Algebraic Geometry Conference — May 2016

Conference Organizing

- **Spectra Panel: Supporting Transgender and Non-binary Students** — Joint Math Meetings
  with Christopher Goff and Greg McCarthy — *January 18, 2020*
- **Special Session on Combinatorial Algebraic Geometry** — AMS Sectional
  with Daniel Erman, Chris Eur, and Lily Silverstein — *September 14-15, 2019*
- **GWCAWMMG** — University of Minnesota
  with Christine Berkesch and Patricia Klein — *April 12-14, 2019*
- **Geometry & Arithmetic of Surfaces** — University of Wisconsin
  with Wanlin Li — *February 9-10, 2019*
- **M2@UW** — University of Wisconsin
  with Daniel Erman, Steven Sam, and Jay Yang — *April 14-17, 2018*

Outreach Activities

- **qGrads** — University of Wisconsin
  Organizer — *July 2017 – Present*
  - Campus group for LGBTQ+ graduate and post-graduate students with > 350 members.
  - Organized a weekly coffee social hour, providing a place to relax, make friends, and discussion the challenges of being LGBTQ+.
- **Supporting Transgender and Non-binary Students** — MAA Panel at the JMM
  Panelist — *January 2020*
- **Undergrad Directed Reading Program** — University of Wisconsin
  Mentor — *January 2018 – May 2019*
  - Lead two semester long reading projects on commutative algebra and algebraic geometry.

– Lead an undergraduate women on a two semester reading project, and provided guidance on applying for REU's and graduate school.

- **Graduate Peer Mentoring** — University of Wisconsin
  Mentor — *September 2018 – December 2018*
  – Mentored 5 first year graduate students from minority genders, organizing monthly dinners where the mentees could discuss issues they were facing.
- **Girls Math Night Out** — University of Wisconsin
  Mentor — *September 2018 – December 2018*
  – Lead 2 women from local high schools on a semesters long project about cryptography.
- **Madison Math Circle** — University of Wisconsin
  Lead Organizer — *January 2016 – December 2018*
  – Lead the creation of a new outreach program, which directly visits high schools around the state of Wisconsin to better serve students from underrepresented groups.
  – Expanded the total number of students reached per year from 25 to >250.
- **Madison Math Circle** — University of Wisconsin
  Student Volunteer — *January 2015 – December 2018*
- **Out in STEM (oSTEM) @ UW-Madison** — University of Wisconsin
  co-Founder — *July 2017 – Math 2018*
  – Founded, at the time, the only campus resource specifically for LGBTQ+ individuals in STEM, and grew the organization to over 50 members.
  – Secured a travel grant to help 11 members (undergraduate and graduate students) attend the national oSTEM conference.
- **Out in Math: Professional Issues Facing LGBTQ Mathematicians** — MAA Panel at the JMM
  Panelist — *January 2018*
- **Math Careers Beyond Academia** — University of Wisconsin
  Organizer — *April 14, 2017*
  – One day conference with over 50 participants.
- **Madison Mega Math Meet** — University of Wisconsin
  Graded — *May 2015*
- **Bonding Undergraduate and Graduate Students** — University of Wisconsin
  Mentor for Undergraduate — *September 2014 – December 2014*
- **Michigan Math Circle** — University of Michigan
  Organizer — *January 2013 – June 2014*

Teaching Experience

- **Math 221: Calculus and Analytic Geometry I** — University of Wisconsin
  Teaching Assistant — *Fall 2014/2018/2019*
  – Selected as a TA coordinator in 2018 and 2019, and was responsible for overseeing all other TA's and mentoring first year TA's.
  – Average score 4.9/5.0
- **Math 228: Wisconsin Emerging Scholars** — University of Wisconsin
  Instructor — *Fall 2018*

- Course providing students from underrepresented groups additional support.
- Average score: 5.0/5.0

- **Math 132: Problem Solving in Algebra, Probability & Statistics** — University of Wisconsin
  Instructor — *Spring 2015*
- **Inquiry Based Learning Courses** — University of Michigan
  Course Assistant — *2012-2014*
  - Assisted with advanced undergraduate courses on topology, analysis, and probability.
  - Facilitated inquiry based learning in the classroom, and responsible for office hours, grading, and review sessions.

Service

- **AMS Graduate Student Blog** — American Mathematical Society
  Editor — *September 2015 – September 2018*
- **AMS Graduate Student Chapter** — University of Wisconsin
  Member — *September 2014 – September 2018*
- **Graduate Student Algebraic Geometry Seminar** — UW Dept. of Mathematics
  Organizer — *March 2015 – December 2017*
- **Committee on Inclusivity and Diversity** — UW Dept. of Mathematics
  Member — *November 2016 – August 2017*
  - Created policies seeking to make the department a more welcoming, inclusive, and comfortable place. This included drafting the department's statement on inclusivity, and creating similar statements for syllabi to be used throughout the department.
- **Committee on TA Pay and Performance** — UW Dept. of Mathematics
  Member — *September 2015 – August 2017*
  - Developed and implemented a new system to evaluate TA performance, with the goal of creating a more transparent, useful, and non-biased system.
- **Instructor Excellence Program** — UW Dept. of Mathematics
  Teaching Mentor — *September 2015 – May 2016*

References

Prof. Daniel Erman
Department of Mathematics
University of Wisconsin - Madison
Madison, WI 53706-1325 USA
☎ (608)-263-3055
✉ derman@math.wisc.edu

Prof. David Eisenbud
Department of Mathematics
University of California, Berkeley
Berkeley, CA 94720-3840 USA
☎ (510)-642-0143
✉ de@math.berkeley.edu

Prof. Christine Berkesch
Department of Mathematics
University of Minnesota
Minneapolis, MN 55455-0488 USA
☎ (612)-624-6424
✉ cberkesc@umn.edu

Juliette Bruce's Career Goals

Long-term my career goal is to be a mathematics professor at a research university, using this position to continue researching algebraic geometry, mentor the next generation of researchers from underrepresented groups in math, and promoting diversity, inclusion, and justice within the mathematics community. As a way towards this goal, my short-term career goal is to have a post-doctoral position at a research university where I would be able to continue learning and refining my skills as a researcher, teacher, mentor, and organizer. Having talked with my faculty mentor Professor Dave Jensen we both feel strongly that these career goals would be significantly advanced by being a University Research Postdoctoral Fellow at the University of Kentucky.

Working with Dave Jensen would provide the opportunity to learn from one of the foremost experts in algebraic geometry. Professor Jensen has done substantial work concerning the geometry of curves and tropical geometry. Working with him would likely prove valuable in advancing my own research projects. Further, while visiting the University of Kentucky in the Spring of 2019, Professor Jensen had a number of stimulating conversations about possible research projects we might pursue. In short, working with Professor Dave Jensen would significantly advance my goals of being a math professor at a research university.

Additionally, the Department of Mathematics at the University of Kentucky has a large number of faculty in commutative algebra and algebraic geometry (Uwe Nagel, Christopher Manon, etc.). This will provide a rich mathematical environment that would help advance my research career. Further, the University of Kentucky has a number of active outreach programs (Julia Robinson Mathematics Festival, Women and Mathematics Program, Central Kentucky Mathematics Circles, Geometry Lab, etc.) through which I would be able to continue promoting inclusivity and diversity in math.

Juliette Bruce's Research Statement

My research interests lie in pure mathematics, more specifically, in algebraic geometry and commutative algebra. Broadly, these fields make use of deep connections between geometry and algebra to study the solutions of systems of polynomial equations. While these are areas of pure mathematics, the usefulness and prevalence of non-linear models means that algebraic geometry and commutative algebra have found applications in numerous other fields including biology and phylogenetics [PS05], string theory [CDH+10], chemical reaction networks [Cra15], and data science [CEYZ19] to name a few.

1. Syzygies in Algebraic Geometry

The main objects of study in algebraic geometry are the sets of solutions to systems of polynomial equations (e.g. $y - x^2 + 3x + 1 = 0, y - 2x = 0$), which are often called algebraic varieties. In particular, algebraic geometry seeks to build a dictionary between the geometry of the solution sets (i.e. varieties) and the algebra of the given equations.

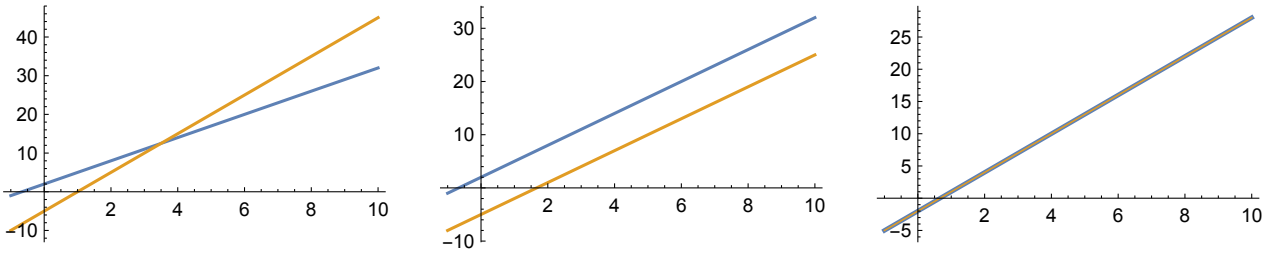

FIGURE 1. A toy example of this algebra-geometry diction is that one can analyze a system of two linear equations, $ax + by = 0, cx + dy = 0$, by considering the graphs of the corresponding lines. In particular, by studying the corresponding lines one can see that a system of two linear equations $ax + by = 0, cx + dy = 0$ has exactly one, zero, or infinitely many solutions depending on whether the corresponding lines intersect, are parallel, or are the same line.

My research focuses on furthering our understanding of how algebraic relations between polynomials affects the geometry of their solution sets. Given a collection of polynomials $f_1, f_2, \ldots, f_t$, a *syzygy* is another collection of polynomials $g_1, g_2, \ldots, g_t$ such that $f_1g_1 + f_2g_2 + \cdots + f_tg_t = 0$. Informally, a syzygy captures an algebraic relationship amoungst the polynomials $f_1, f_2, \ldots, f_t$. In my research, I have sought to understand the syzygies of a number of interesting varieties.

The study of syzygies is formalized via commutative algebra in the following way. Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_i is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring R with a twisted grading, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j} is the graded piece of degree $d - j$. The $\beta_{i,j}$'s are the *Betti numbers* of M, and they count the number of i-syzygies of M of degree j. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [Voi02, Voi05, AFP+19]. See also [Eis05, Conjecture 9.6] and [Sch86, BE91, FP05, Far06, AF11, FK16, FK17].

1.1 Asymptotic Syzygies Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84a, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question: In what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set,

$$\rho_q(X, L_d) \coloneqq \frac{\#\left\{p \in \mathbb{N} \,\middle|\, \beta_{p,p+q}(X, L_d) \neq 0\right\}}{r_d},$$

which is the percentage of degrees in which non-zero syzygies appear [Eis05, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84a] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that $\deg L_d = d$ then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q(X; L_d) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1, 0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19a, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q\left(X; \mathcal{O}\left(d_1, d_2\right)\right) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\, j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}} \right) - O\begin{pmatrix} \textit{lower ord.} \\ \textit{terms} \end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [Lem18] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right)$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [EL12, Conjectures 7.1, 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld. Such a generalization is complicated by the difference between the Cox ring and homogenous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$. A central theme in this work is to exploit the fact that a key regular sequence I use has a number of non-trivial symmetries.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety and fix an index $1 \leq q \leq \dim X$. Let $(L_d)_{d\in\mathbb{N}}$ be a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample, can one compute $\lim_{d\to\infty} \rho_q\left(X; L_d\right)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru19b].

1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. My co-authors and I exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. This data provided support for several existing conjectures, as well as led us to make a number of new conjectures [BEGSY18]. The resulting data has been made publicly available via SyzygyData.com as well as, a package for Macaualy2 [BEGSY19, M2].

Recently I have begun using similar computational techniques to compute the syzygies for Hirzebruch surfaces. Thus far, we have computed the syzygies in ~ 100 new examples. It is our hope that these examples will lead to new conjectures regarding the syzygies of Hirzebruch surfaces. In particular, we believe our data will be useful in addressing Question 1.4.

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