ON THE TOP-WEIGHT RATIONAL COHOMOLOGY OF $\mathcal{A}_g$

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ABSTRACT. We compute the top-weight rational cohomology of $\mathcal{A}_g$ for $g = 5$, 6, and 7, and we give some vanishing results for the top-weight rational cohomology of $\mathcal{A}_8, \mathcal{A}_9$, and $\mathcal{A}_{10}$. When $g = 5$ and $g = 7$, we exhibit nonzero cohomology groups of $\mathcal{A}_g$ in odd degree, thus answering a question highlighted by Grushevsky. Our methods develop the relationship between the top-weight cohomology of $\mathcal{A}_g$ and the homology of the link of the moduli space of principally polarized tropical abelian varieties of rank g. To compute the latter we use the Voronoi complexes used by Elbaz-Vincent-Gangl-Soulé. Our computations give natural candidates for compactly supported cohomology classes of $\mathcal{A}_g$ in weight 0 that produce the stable cohomology classes of the Satake compactification of $\mathcal{A}_g$ in weight 0, under the Gysin spectral sequence for the latter space.

1. INTRODUCTION

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g. It is a smooth and separated Deligne-Mumford stack of dimension $d = \binom{g+1}{2}$, but it is not proper. The rational cohomology groups of $\mathcal{A}_g$ admit a weight filtration whose graded pieces $\mathrm{Gr}_j^W H^\bullet(\mathcal{A}_g; \mathbb{Q})$ may appear for j from d to $2d$; and we refer to the piece of weight $2d$ as the *top-weight rational cohomology* of $\mathcal{A}_g$.

The orbifold Euler characteristic and the stable cohomology of $\mathcal{A}_g$ are classically understood [Har71, Bor74]. However, the full cohomology ring $H^\bullet(\mathcal{A}_g; \mathbb{Q})$ is a mystery even for small g. The cases when $g \leq 2$ are classically known, and the case when $g = 3$ is the work of Hain [Hai02]. The full cohomology ring for $g \geq 4$ is already unknown, though when $g = 4$, much information can be determined from [HT12, HT18], where the complete Betti tables for both the Voronoi and the perfect cone compactifications of $\mathcal{A}_4$ are computed. In particular, the top-weight cohomology of $\mathcal{A}_4$ vanishes; see Remark 6.7.

We compute the top-weight rational cohomology of $\mathcal{A}_g$ for $2 \leq g \leq 7$. For $g = 3$ and 4, our computations agree with the above-mentioned results of Hain and Hulek-Tommasi, respectively. Our first main result is then the following Theorem A.

Theorem A. *The top-weight rational cohomology of $\mathcal{A}_g$ for $g = 5$, 6, and 7, is*

$$\mathrm{Gr}_{30}^W H^k(\mathcal{A}_5; \mathbb{Q}) = \begin{cases} \mathbb{Q} \ \textit{if } k = 15, 20, \\ 0 \ \textit{else}, \end{cases}$$

$$\mathrm{Gr}_{42}^W H^k(\mathcal{A}_6; \mathbb{Q}) = \begin{cases} \mathbb{Q} \ \textit{if } k = 30, \\ 0 \ \textit{else}, \end{cases}$$

$$\mathrm{Gr}_{56}^W H^k(\mathcal{A}_7; \mathbb{Q}) = \begin{cases} \mathbb{Q} \ \textit{if } k = 28, 33, 37, 42 \\ 0 \ \textit{else}. \end{cases}$$

This answers an open question of Grushevsky, who asked whether $\mathcal{A}_g$ ever has nonzero odd cohomology [Gru09, Open Problem 7].

To obtain these results, we develop methods for studying $\mathcal{A}_g$ that are analogous to those employed in [CGP18] for moduli spaces of curves. The moduli spaces $\mathcal{A}_g$ admit toroidal compactifications $\overline{\mathcal{A}_g}^\Sigma$, which are proper Deligne-Mumford stacks (see [FC90, Theorem 5.7]). The compactifications

$\overline{\mathcal{A}_g}^\Sigma$ are associated to admissible decompositions Σ of Ω_g^{rt}, the rational closure of the cone of positive definite quadratic forms in g variables (see Section 2.3). The same data is also used to construct the moduli space $A_g^{\mathrm{trop},\Sigma}$ of tropical abelian varieties of dimension g in the category of generalized cone complexes (see Section 2.6).

Then for any admissible decomposition Σ of Ω_g^{rt} and for each $i \geq 0$, and writing $LA_g^{\mathrm{trop},\Sigma}$ for the link of the cone point of $A_g^{\mathrm{trop},\Sigma}$, the following canonical identification holds:

$$\widetilde{H}_{i-1}(LA_g^{\mathrm{trop},\Sigma}; \mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g; \mathbb{Q}).$$

This statement can be deduced from [OO18, Corollary 2.9] (see pp. 24–25 of op. cit.); because the language is different and for self-containedness, we give a short proof in Theorem 3.1. Briefly, there exist admissible decompositions Σ for which $\overline{\mathcal{A}_g}^\Sigma$ is a smooth simple normal crossings compactification of $\mathcal{A}_g$ whose boundary complex is identified with $LA_g^{\mathrm{trop},\Sigma}$. However, the homeomorphism type of $LA_g^{\mathrm{trop},\Sigma}$ is independent of Σ: see Section 3 or [Oda19, A.14]. The conclusion follows by applying the generalization to Deligne-Mumford stacks, spelled out in [CGP18], of Deligne's comparison theorems in mixed Hodge theory (see Theorem 3.1).

We then compute the topology of $A_g^{\mathrm{trop},\Sigma}$ by considering the *perfect* or *first Voronoi* toroidal compactification $\overline{\mathcal{A}_g}^{\mathrm{P}}$ and its tropical version $A_g^{\mathrm{trop},\mathrm{P}}$, associated to the *perfect cone decomposition* (Fact 2.6). This decomposition is very well studied and enjoys interesting combinatorial properties, which are well-suited for our computations. We identify the homology of the link of $A_g^{\mathrm{trop},\mathrm{P}}$ with the homology of the *perfect chain complex* $P_\bullet^{(g)}$ (Definition 4.1, Proposition 4.4), using the framework of cellular chain complexes of symmetric CW-complexes due to Allcock-Corey-Payne [ACP19].

To compute the homology of the complex $P_\bullet^{(g)}$ we use a related complex $V_\bullet^{(g)}$, called the Voronoi complex. This was introduced in [EVGS13, LS78b] to compute the cohomology of the modular groups $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$. They use the perfect form cell decomposition of Ω_g^{rt}, which is invariant under the action of each of these groups, and then relate the equivariant homology of Ω_g^{rt} modulo its boundary with the cohomology of $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$, respectively. For this purpose, the homology of $V_\bullet^{(4)}$ was computed by Lee and Szczarba in [LS78a] for $\mathrm{SL}_4(\mathbb{Z})$ (and we adapt this computation to the case of $\mathrm{GL}_4(\mathbb{Z})$ in this paper), while for $g = 5, 6, 7$ the complex $V_\bullet^{(g)}$ was computed in [EVGS13] with the help of a computer program using lists of perfect forms for $g \leq 7$ by Jaquet [JC93]. In Theorem 4.13 we show that the complexes $P_\bullet^{(g)}$ and $V_\bullet^{(g)}$ sit in an exact sequence

$$0 \longrightarrow P_\bullet^{(g-1)} \longrightarrow P_\bullet^{(g)} \xrightarrow{\ \pi\ } V_\bullet^{(g)} \longrightarrow 0.$$

This sequence together with the results in [EVGS13] are then crucial to get our main result.

In Section 5 we consider a subcomplex of $P_\bullet^{(g)}$ called the inflation complex and prove that it is acyclic. Using this result, we show that $\mathrm{Gr}_{g^2+g}^W H^i(\mathcal{A}_g; \mathbb{Q}) = 0$ for $i > g^2$, which recovers the vanishing in top weight of the rational cohomology of $\mathcal{A}_g$ in degree above the virtual cohomological dimension (which for $\mathcal{A}_g$ is equal to g^2).

For $g = 8, 9$, and 10, full calculations of the top-weight cohomology of $\mathcal{A}_g$ are beyond the scope of our computations. However, our computations for $g = 7$ together with a vanishing result of [SEVKM19] allow us to deduce, in Section 6.5, the vanishing of $\mathrm{Gr}_{(g+1)g}^W H^\bullet(\mathcal{A}_g; \mathbb{Q})$ in a range slightly larger than what is implied by virtual cohomological dimension bounds.

Theorem B. *The top-weight rational cohomology of $\mathcal{A}_8$, $\mathcal{A}_9$, and $\mathcal{A}_{10}$ vanish in the following ranges:*

$$\mathrm{Gr}_{72}^W H^i(\mathcal{A}_8; \mathbb{Q}) = 0 \quad \textit{for } i \geq 60,$$
$$\mathrm{Gr}_{90}^W H^i(\mathcal{A}_9; \mathbb{Q}) = 0 \quad \textit{for } i \geq 79,$$
$$\mathrm{Gr}_{110}^W H^i(\mathcal{A}_{10}; \mathbb{Q}) = 0 \quad \textit{for } i \geq 99.$$

Relationship with the stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$. By Poincaré duality, the top-weight cohomology of $\mathcal{A}_g$ studied in this paper admits a perfect pairing with weight 0 compactly supported cohomology of $\mathcal{A}_g$. These weight 0 classes, in turn, have an interesting, not yet fully understood relationship with the stable cohomology ring of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$, whose structure was first understood by Charney-Lee [CL83].

Indeed, the stable cohomology ring of $\mathcal{A}_g^{\mathrm{Sat}}$ is freely generated by extensions of the well-known odd λ-classes and by less understood classes y_6, y_{10}, y_{14}, ... which were proven to be of weight 0 by Chen-Looijenga in [CL17]. This predicts the existence of infinitely many top-weight cohomology classes of $\mathcal{A}_g$ as g grows. More precisely, the classes found in the present paper, with Poincaré duality applied, give natural candidates for the "sources" of the y_j's in the sense of persisting in a Gysin spectral sequence relating the compactly supported cohomology groups of the space $\mathcal{A}_g^{\mathrm{Sat}}$ and those of the spaces $\mathcal{A}_{g'}$ for $g' \leq g$. See Table 4 at the end of the paper for a summary of everything that is known on the E^1 page of this spectral sequence.

This connection was explained to us by O. Tommasi and provides significant additional interest in our main results; we discuss it in detail in Section 7.

The paper is organized as follows. In Section 2, we give the necessary preliminaries. This includes a discussion of generalized cone complexes, their links, and their homology. We then discuss admissible decompositions of the rational closure of the set of positive definite quadratic forms, and focus in particular on the perfect cone decomposition. We also give a brief introduction to matroids and to perfect cones associated to matroids. Then, we give some background on the tropical moduli space $A_g^{\mathrm{trop},\Sigma}$, and on the construction of toroidal compactifications of $\mathcal{A}_g$ out of admissible decompositions.

In Section 3, we prove Theorem 3.1), which relates the top-weight cohomology of $\mathcal{A}_g$ to the reduced rational homology of the link of $A_g^{\mathrm{trop},\Sigma}$. In Section 4, we show that the perfect chain complex $P_\bullet^{(g)}$ computes the top-weight cohomology of $\mathcal{A}_g$ (Proposition 4.4). We also relate this chain complex to the Voronoi complex $V_\bullet^{(g)}$ (Theorem 4.13). In Section 5, we introduce the inflation subcomplex, which we show is acyclic in Theorem 5.15. We prove an analogous result for the coloop subcomplex $C_\bullet^{(g)}$ of the regular matroid complex $R_\bullet^{(g)}$, which may be useful for future results.

In Section 6, we put together the results obtained in Section 4 with the computations of [LS78a] for $g = 4$ and [EVGS13] in $g = 5, 6$, and 7 to describe the top-weight cohomology of $\mathcal{A}_g$ for $g = 4, 5, 6$, and 7 and to give the above mentioned bound for the vanishing of the cohomology of $\mathcal{A}_g$ in top weight for $g = 8, 9$, and 10. This proves Theorems A and B. In Section 7, we discuss the relationship with the stable cohomology of the Satake compactification, including some open questions which are partially addressed by our main results and which deserve further attention.

2. Preliminaries

In this section we give preliminaries and introduce notation.

2.1. Cones and generalized cone complexes.

A *rational polyhedral cone* σ in $\mathbb{R}^g$ (or just a *cone*, for simplicity) is the non-negative real span of a finite set of integer vectors $v_1, v_2, \ldots, v_n \in \mathbb{Z}^g$,

$$\sigma := \mathbb{R}_{\geq 0}\langle v_1, v_2, \ldots, v_n\rangle := \left\{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0}\right\}.$$

We assume all cones $\sigma \subset \mathbb{R}^g$ are *strongly convex*, meaning σ contains no nonzero linear subspaces of $\mathbb{R}^g$. The *dimension* of σ is the dimension of its linear span. The cone σ is said to be *smooth* if it is possible to choose the generating vectors $v_1, \ldots, v_n$ so that they are a subset of a $\mathbb{Z}$-basis of $\mathbb{Z}^g$. Note that some sources refer to what we call smooth cones as *basic* cones. A d dimensional cone σ is said to be *simplicial* if it is generated by d vectors, which are linearly independent over $\mathbb{R}$. A *face* of σ is any nonempty subset of σ that minimizes a linear functional on $\mathbb{R}^g$. Faces of σ are themselves rational polyhedral cones. A *facet* is a face of codimension one.

Given cones $\sigma \in \mathbb{R}^g$ and $\sigma' \in \mathbb{R}^{g'}$, a *morphism* $\sigma \to \sigma'$ is a continuous map from σ to σ' obtained as the restriction of a linear map $\mathbb{R}^g \to \mathbb{R}^{g'}$ sending $\mathbb{Z}^g$ to $\mathbb{Z}^{g'}$. A *face morphism* is a morphism of cones $\sigma \to \sigma'$ sending σ isomorphically to a face of σ'. Notice that isomorphisms of cones are examples of face morphisms. Denote with Cones the category of cones with face morphisms.

The one-dimensional faces of σ are called the *extremal rays* of σ, and there are only finitely many of these. Given an extremal ray ρ of σ, the semigroup $\rho \cap \mathbb{Z}^g$ is generated by a unique element u_ρ called the *ray generator* of ρ. An automorphism of a strongly convex cone permutes its finitely many ray generators, and is uniquely determined by this permutation. So, $\mathrm{Aut}(\sigma)$ is finite.

A *generalized cone complex* (see [ACP15]) is a topological space with a presentation as a colimit $X := \varinjlim_{i \in \mathcal{I}} \sigma_i$ of an arbitrary diagram of cones $\sigma : \mathcal{I} \to$ Cones, in which all morphisms of cones are face morphisms. A morphism $(X = \varinjlim_{i \in \mathcal{I}} \sigma_i) \to (X' = \varinjlim_{i \in \mathcal{I}} \sigma'_i)$ is a continuous map $f : X \to X'$ such that for each cone σ_i in the presentation of X, there exists a cone σ'_j in the presentation of

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X' and a morphism of cones $f_i\colon \sigma_i \to \sigma'_j$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\sigma_i & \xrightarrow{\ f_i\ } & \sigma'_j \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & X'
\end{array}
$$

We remark that the category of generalized cone complexes is equivalent to the one of stacky fans as defined in [CMV13, Def. 2.1.7].

2.2. **Links of generalized cone complexes.** For any cone $\sigma \subset \mathbb{R}^g$, let us define the *link* of σ at the origin to be the topological space $L\sigma = (\sigma - \{0\})/\mathbb{R}_{>0}$, where the action of $\mathbb{R}_{>0}$ is by scalar multiplication. Thus $L\sigma$ is homeomorphic to a closed ball of dimension $\dim \sigma - 1$. A face morphism of cones $\sigma \to \sigma'$ induces a morphism of links $L\sigma \to L\sigma'$, making L a functor from Cones to topological spaces.

Let $X = \varinjlim_{i \in \mathcal{I}} \sigma_i$ be a generalized cone complex, where $\sigma\colon \mathcal{I} \to$ Cones is a diagram of cones. We define the *link* of X as the colimit

$$LX = \varinjlim(L \circ \sigma).$$

Thus LX is a topological space, equipped with a colimit presentation as above. In fact, LX is a *symmetric CW-complex*, by [ACP19, Example 2.4]. The definition of symmetric CW-complex generalizes the *symmetric Δ-complexes* of [CGP18]. Roughly, a symmetric CW-complex is like a CW-complex, except with closed n-balls replaced by quotients thereof by finite subgroups of the orthogonal group $O(n)$.

Let X be a finite generalized cone complex, meaning that the indexing category $\mathcal{I}$ is equivalent to one with a finite number of objects and morphisms. We now write down a chain complex isomorphic to the *cellular chain complex* of LX, in the sense of [ACP19, §4], [CGP18, §3]), whose homology is identified with the singular homology of LX.

For each $p \geq -1$, let $\mathrm{Cones}_p(X)$ denote the finite groupoid whose objects are all $(p+1)$-dimensional faces of σ_i, for all $i \in \mathcal{I}$, with a morphism $\tau \to \tau'$ for each isomorphism of cones $\phi\colon \tau \xrightarrow{\cong} \tau'$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\tau & \xrightarrow{\ \phi\ } & \tau' \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

Let τ be a cone in $\mathrm{Cones}_p(X)$. We make use of three compatible notions of orientation found in the literature: (i) an orientation of τ is an orientation of $L\tau$ [LS78a], (ii) an orientation of τ is an orientation of the suspension of $L\tau$ [ACP19], and (iii) an orientation of τ is an orientation of $\mathbb{R}\tau$, the $\mathbb{R}$-linear span of τ; i.e., it is a choice of ordered basis for $\mathbb{R}\tau$, up to a change of basis with positive determinant [EVGS13]. For the first two definitions, it is clear that an orientation on τ induces an orientation on the faces of τ as well. For the third definition, given a facet τ' of τ, the induced orientation on τ' is any one such that the quantity $\epsilon(\tau', \tau)$, defined as follows, is 1.

Let $B = (v_1, \ldots, v_n, v)$ where $B' = (v_1, \ldots, v_n)$ is an orientation of τ' and v is a ray generator of a ray of τ not contained in τ'. Set $\epsilon(\tau', \tau)$ to be the sign of the orientation of B in the oriented vector space $\mathbb{R}\tau$. Note that this sign does not depend on the choice of v. These definitions are compatible, in that a choice of orientation under one definition yields a choice of orientation under the other two, and under this correspondence a cone morphism $\tau \to \sigma$ is orientation-preserving under one definition if it is orientation-preserving under all three. Say that $\tau \in \mathrm{Cones}_p(X)$ is *alternating* if all automorphisms $\tau \to \tau$ in $\mathrm{Cones}_p(X)$ are orientation-preserving on τ.

Choose a set Γ_p of representatives of isomorphism classes of alternating cones in $\mathrm{Cones}_p(X)$, and for each $\tau \in \Gamma_p$ fix an orientation of τ. If ρ' is a facet of τ, then ω_τ induces an orientation of ρ', which we denote $\omega_\tau|_{\rho'}$. Let $C_p(LX)$ be the $\mathbb{Q}$-vector space with basis Γ_p. We define a differential

$$\partial\colon C_p(LX) \to C_{p-1}(LX)$$

by extending linearly on $C_p(LX)$ the following definition: given $\tau \in \Gamma_p$ and $\rho \in \Gamma_{p-1}$, set

$$\partial(\tau)_\rho = \sum_{\rho'} \eta(\rho', \rho)$$

where ρ' ranges over the facets of τ that are isomorphic in $\mathrm{Cones}_{p-1}(X)$ to ρ, and where $\eta(\rho', \rho) = \pm 1$ according to whether an isomorphism $\phi\colon \rho' \to \rho$ in $\mathrm{Cones}_p(X)$ takes the orientation $\omega_\tau|_{\rho'}$ to ω_ρ or $-\omega_\rho$. Note that $\eta(\rho', \rho)$ is well defined, i.e., independent of choice of ϕ, precisely because ρ is alternating.

Let $C_\bullet(LX)$ denote the complex

$$\cdots \xrightarrow{\partial} C_p(LX) \xrightarrow{\partial} C_{p-1}(LX) \xrightarrow{\partial} \cdots C_{-1}(LX) \to 0.$$

The main proposition in this subsection is the following.

Proposition 2.1. Let X be a finite generalized cone complex. We have that $C_\bullet(LX)$ is a complex, i.e., $\partial^2 = 0$.

(1) If X is connected, we have, for each $p \geq 0$,

$$H_p(C_\bullet(LX)) \cong \widetilde{H}_p(LX; \mathbb{Q}).$$

(2) More generally, for each $p > 0$, we have canonical isomorphisms

$$H_p(C_\bullet(LX)) \cong H_p(LX; \mathbb{Q}),$$

and for $p = 0$ we have

$$H_0(C_\bullet(LX)) \cong \ker(H_0(LX; \mathbb{Q}) \to \mathbb{Q}\Gamma_{-1}).$$

Proposition 2.1 follows from [ACP19, Theorem 4.2], by tracing through their definition of the cellular chain complex of LX. We give a self-contained proof sketch below.

Proof sketch. Write $LX^{(p)}$ for the p-skeleton of LX, i.e., the union of the images of $L\sigma$ in X, for σ ranging over cones of dimension at most $p+1$ in X. By a standard argument analogous to [Hat02, Theorem 2.2.27], the complex

(1) $$\cdots H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \xrightarrow{\delta_p} H_{p-1}(LX^{(p-1)}, LX^{(p-2)}; \mathbb{Q}) \cdots$$

has homology canonically identified with the singular homology of LX. Moreover,

$$H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \cong \bigoplus_\tau H_p((L\tau)/\mathrm{Aut}(\tau), (\partial L\tau)/\mathrm{Aut}(\tau); \mathbb{Q}),$$

where τ ranges over a set of representatives of isomorphism classes in $\mathrm{Cones}_p(X)$. Here $\mathrm{Aut}(\tau) = \mathrm{Iso}_{\mathrm{Cones}_p(X)}(\tau, \tau)$ is the automorphism group of τ in $\mathrm{Cones}_p(X)$. Since $|\mathrm{Aut}(\tau)|$ is invertible in $\mathbb{Q}$, it follows that

$$H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \quad \cong \quad \bigoplus_\tau H_p(L\tau, \partial L\tau; \mathbb{Q})_{\mathrm{Aut}(\tau)},$$

and for each $\tau \in \mathrm{Cones}_p(X)$, we have

$$H_p(L\tau, \partial L\tau; \mathbb{Q})_{\mathrm{Aut}(\tau)} \cong \begin{cases} \mathbb{Q} & \text{if } \tau \text{ is alternating,} \\ 0 & \text{else.} \end{cases},$$

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which identifies $H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q})$ with $C_p(LX)$. $\square$

Remark 2.2. A statement analogous to Proposition 2.1 holds with $\mathbb{Q}$ replaced by a commutative ring R, if the order of $\mathrm{Aut}(\tau)$ is invertible in R for each τ.

Remark 2.3. See also the proofs in [LS78b, §3], as well as [EVGS13, §3.3], which are written in the special cases of the *Voronoi complexes* for $\mathrm{SL}_g(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$, but apply essentially verbatim to prove Proposition 2.1. The Voronoi complex of $\mathrm{GL}_g(\mathbb{Z})$ plays an important role in this paper.

2.3. **Admissible decompositions.** We now introduce admissible decompositions of the rational closure of the set of positive definite quadratic forms, which are used in the construction of toroidal compactifications of the moduli space of abelian varieties, as well as in the construction of the moduli space of tropical abelian varieties.

We denote by $\mathbb{R}^{\binom{g+1}{2}}$ the vector space of quadratic forms in $\mathbb{R}^g$, which we identify with $g \times g$ symmetric matrices with coefficients in $\mathbb{R}$. We denote by Ω_g the cone in $\mathbb{R}^{\binom{g+1}{2}}$ of positive definite quadratic forms. We define the rational closure of Ω_g to be the set Ω_g^{rt} of positive semidefinite quadratic forms whose kernel is defined over $\mathbb{Q}$. The group $\mathrm{GL}_g(\mathbb{Z})$ acts on the vector space $\mathbb{R}^{\binom{g+1}{2}}$ of quadratic forms by $h \cdot Q := hQh^t$, where $h \in \mathrm{GL}_g(\mathbb{Z})$ and h^t is its transpose. The cones Ω_g and Ω_g^{rt} are preserved by this action of $\mathrm{GL}_g(\mathbb{Z})$.

Remark 2.4. A positive semidefinite quadratic form Q in $\mathbb{R}^g$ belongs to Ω_g^{rt} if and only if there exists $h \in \mathrm{GL}_g(\mathbb{Z})$ such that

$$hQh^t = \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}$$

for some positive definite quadratic form Q' in $\mathbb{R}^{g'}$, with $0 \le g' \le g$ (see [Nam80, Sec. 8]).

The cones Ω_g and Ω_g^{rt} are not polyhedral cones. However, one can consider decompositions of these spaces into rational polyhedral cones, as in the following definition.

Definition 2.5. [[Nam80, Lemma 8.3], [FC90, Chap. IV.2]] An *admissible decomposition* of Ω_g^{rt} is a collection $\Sigma = \{\sigma_\mu\}$ of rational polyhedral cones of Ω_g^{rt} such that:
   (i) if σ is a face of $\sigma_\mu \in \Sigma$ then $\sigma \in \Sigma$,
   (ii) the intersection of two cones σ_μ and σ_ν of Σ is a face of both cones,
   (iii) if $\sigma_\mu \in \Sigma$ and $h \in \mathrm{GL}_g(\mathbb{Z})$ then $h\sigma_\mu h^t \in \Sigma$,
   (iv) the set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of cones is finite, and
   (v) $\cup_{\sigma_\mu \in \Sigma} \sigma_\mu = \Omega_g^{\mathrm{rt}}$.
We say that two cones $\sigma_\mu, \sigma_\nu \in \Sigma$ are equivalent if they are in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit.

There are three known families of admissible decompositions of Ω_g^{rt} described for all g: the perfect cone decomposition, the second Voronoi decomposition, and the central cone decomposition (see [Nam80, Chap. 8] and the references there). In this paper, we work with the perfect cone decomposition, which we now describe.

2.4. **The perfect cone decomposition.** Given a positive definite quadratic form Q, consider the set of nonzero integral vectors where Q attains its minimum,

$$M(Q) := \{\xi \in \mathbb{Z}^g \setminus \{0\} \ : \ Q(\xi) \le Q(\zeta), \forall \zeta \in \mathbb{Z}^g \setminus \{0\}\}.$$

The elements of $M(Q)$ are called the *minimal vectors* of Q. Let $\sigma[Q]$ denote the rational polyhedral subcone of Ω_g^{rt} given by the non-negative linear span of the rank one forms $\xi \cdot \xi^t \in \Omega_g^{\mathrm{rt}}$ for elements ξ of $M(Q)$, i.e.

(2) $$\sigma[Q] := \mathbb{R}_{\ge 0} \langle \xi \cdot \xi^t \rangle_{\xi \in M(Q)}.$$

The *rank* of the cone $\sigma[Q]$ is defined to be the maximum rank of an element of $\sigma[Q]$; in fact the rank of $\sigma[Q]$ is exactly the dimension of the span of $M(Q)$, see Lemma 4.8.

Fact 2.6. [Vor08] The set of cones

$$\Sigma_g^{\mathrm{P}} := \{\sigma[Q] \ : \ Q \text{ a positive definite form on } \mathbb{R}^g\}$$

is an admissible decomposition of Ω_g^{rt}, known as the *perfect cone decomposition*.

The quadratic forms Q such that $\sigma[Q]$ has maximal dimension $\binom{g+1}{2}$ are called *perfect*, hence the name of this admissible decomposition.

Example 2.7. Let us compute Σ_2^{P}. In this case, there is a unique perfect form up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, namely

$$Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

One can compute that $M(Q) = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1)\}$. Thus, up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, there is a unique perfect cone $\sigma[Q]$ of maximal dimension 3, with ray generators

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

One may check that for $i \in \{0, 1, 2\}$, all i-dimensional faces of $\sigma[Q]$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent; hence there is a unique perfect cone of each dimension up to the action of $\mathrm{GL}_2(\mathbb{Z})$.

Remark 2.8. The cones $\sigma[Q] \in \Sigma_g^{\mathrm{P}}$ need not be simplicial for $g \geq 4$ (see [Nam80, p. 93]).

2.5. **Matroidal Perfect Cones.** We now give a brief introduction to matroids and their associated orbits of perfect cones. Further background on matroids can be found in [Oxl92].

Definition 2.9. A *matroid* $M = (E, \mathcal{C})$ on a finite set E is a subset $\mathcal{C} \subset \mathcal{P}(E) \setminus \{\emptyset\}$, called the set of *circuits* of M, satisfying the following axioms:

(C1) No proper subset of a circuit is a circuit.
(C2) If $C_1, C_2 \in \mathcal{C}$ are distinct and $c \in C_1 \cap C_2$ then $(C_1 \cup C_2) - \{c\}$ contains a circuit.

A matroid $M = (E, \mathcal{C})$ is said to be *simple* if it has no circuits of length 1 or 2. A matroid $M = (E, \mathcal{C})$ is called *representable* over a field $\mathbb{F}$ if there is a matrix A over $\mathbb{F}$ such that E bijects to the columns of A with the circuits $\mathcal{C}$ of M indexing the minimal linearly dependent sets of columns of A. The matrix A is known as an $\mathbb{F}$-representation of M. An *automorphism* of a matroid is a bijection $\phi : E \to E$ such that for any subset $C \subset E$, C is a circuit of M if and only of $\phi(C)$ is a circuit of M.

Definition 2.10. A matroid is *regular* if and only if it is representable over every field.

A matroid M being regular is equivalent to M being representable over $\mathbb{R}$ by a totally unimodular matrix (i.e., a matrix such that every minor is either $-1, 0$, or 1). The *rank* of a regular matroid M is the smallest number r such that M is representable over $\mathbb{R}$ by a $r \times n$ totally unimodular matrix for some n [Oxl92, Lemma 2.2.21, page 85].

Definition 2.11. Let G be a graph. The *graphic matroid* $M(G)$ is the matroid with ground set $E(G)$ whose circuits are subsets of $E(G)$ forming a simple cycle of G.

Since graphic matroids are regular, they are representable over fields of any characteristic. This can be seen directly by constructing the following matrix representing $M(G)$. Fix an orientation

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of the edges of G. Let $A(G)$ be the $|V(G)| \times |E(G)|$ matrix with entries

$$A(G)_{ij} = \begin{cases} 0 \text{ if } v_i \notin e_j, \\ -1 \text{ if } v_i \text{ is the head of } e_j, \\ 1 \text{ if } v_i \text{ is the tail of } e_j. \end{cases}$$

This matrix represents the matroid $M(G)$ over any field.

Construction 2.12. Given a simple, regular matroid M of rank $\leq g$ choose a $g \times n$ totally unimodular matrix A that represents M over $\mathbb{R}$. Denoting the columns of A by $v_1, v_2, \ldots, v_n$ we let $\sigma_A(M) \subset \Omega_g^{\mathrm{rt}}$ be the rational polyhedral cone:

$$\sigma_A(M) := \mathbb{R}_{\geq 0} \left\langle v_1 v_1^t, v_2 v_2^t, \ldots, v_n v_n^t \right\rangle.$$

By [MV12, Theorem 4.2.1], the cone $\sigma_A(M)$ is a perfect cone in Σ_g^{P}.

The cone $\sigma_A(M)$ is uniquely determined by M up to the action of $\mathrm{GL}_g(\mathbb{Z})$. In particular, if A and A' are two different totally unimodular matrices representing M over $\mathbb{R}$ then there exists an element $h \in \mathrm{GL}_g(\mathbb{Z})$ such that $h\sigma_A(M)h^t = \sigma_{A'}(M)$ (see [MV12, Lemma 4.0.5(ii)]). We therefore denote the $\mathrm{GL}_g(\mathbb{Z})$-orbit of $\sigma_A(M)$ by $\sigma(M)$.

In the case of graphic matroids, Construction 2.12 can be made very explicit. As this is useful in Section 6, we take the time to explain it here. Fix $g > 0$. We now construct cones of Σ_g^{P} from graphs on $g + 1$ vertices. The rows of the $(g+1) \times |E(G)|$ matrix $A(G)$ as constructed above are linearly dependent. Let $A^*(G)$ be the matrix obtained from $A(G)$ by deleting the last row. The matrices $A(G)$ and $A^*(G)$ are both representations of $M(G)$. Let $v_1, \ldots, v_d$ be the columns of $A^*(G)$. Then $\sigma(M(G)) := \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_d v_d^t \rangle \in \Sigma_g^{\mathrm{P}}$ is a perfect cone (see [MV12, Theorem 4.2.1]).

Definition 2.13. The *principal cone* is $\sigma_g^{\mathrm{prin}} := \sigma(M(K_{g+1}))$, the cone corresponding to the complete graph K_{g+1}.

When $g = 2$, this is the cone discussed in Example 2.7. More generally, for arbitrary g the principal cone can be defined as the cone corresponding to the quadratic form

$$\begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{bmatrix}.$$

These two definitions agree by [BMV11, Lemma 6.1.3].

The faces of σ_g^{prin} may be understood as follows. Since $M(K_{g+1})$ is a simple matroid, the principal cone in Σ_g^{P} is simplicial by [BMV11, Theorem 4.4.4(iii)]. Therefore, a codimension i face of the principal cone comes from a graph obtained by deleting i edges from K_{g+1}.

Remark 2.14. Automorphisms of the graph G give automorphisms of the matroid $M(G)$, but not all automorphisms of $M(G)$ arise in this way. However, if G is 3-connected, then $\mathrm{Aut}(G) = \mathrm{Aut}(M(G))$ (this is proved by Whitney in [Whi32], see [HPW72, Lemmas 1 and 2]). The group $\mathrm{Aut}(M(G))$ is isomorphic to the group of permutations of the rays of $\sigma(M(G))$ induced by elements of $GL_g(\mathbb{Z})$ stabilizing $\sigma(M(G))$ [Cha12, Theorem 5.10].

2.6. **The tropical moduli space A_g^{trop}.** We now introduce the moduli space of tropical abelian varieties, which is a generalized cone complex constructed in [BMV11] and later worked out in [CMV13]. Our aim is to compute the homology of the link of A_g^{trop}, as this is canonically isomorphic to the top-weight rational cohomology of $\mathcal{A}_g$ (see Theorem 3.1).

Definition 2.15. A *principally polarized tropical abelian variety* (or, for simplicity, just *tropical abelian variety*) of dimension g is a pair $A = (\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is a positive semidefinite symmetric bilinear form on $\mathbb{R}^g$ with rational null space. We say that $A = (\mathbb{R}^g/\mathbb{Z}^g, Q)$ is *pure* if Q is positive definite.

Two tropical abelian varieties $(\mathbb{R}^g/\mathbb{Z}^g, Q)$ and $(\mathbb{R}^g/\mathbb{Z}^g, Q')$ are *isomorphic* if there is $h \in \mathrm{GL}_g(\mathbb{Z})$ such that $Q' = hQh^t$. The set of isomorphism classes of tropical abelian varieties of dimension g is in bijective correspondence with the orbits in $\Omega_g^{\mathrm{rt}}/\mathrm{GL}_g(\mathbb{Z})$.

Given an admissible decomposition Σ of Ω_g^{rt}, one can define a generalized cone complex $A_g^{\mathrm{trop},\Sigma}$ by considering the stratified quotient of Ω_g^{rt} with respect to Σ (see [CMV13, Def. 2.2.2]). Precisely, $A_g^{\mathrm{trop},\Sigma}$ is the generalized cone complex obtained as the colimit

$$A_g^{\mathrm{trop},\Sigma} := \varinjlim \{\sigma\}_{\sigma \in \Sigma}$$

with arrows given by inclusion of faces composed with the action of the group $\mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}: given two cones σ_i and $\sigma_j \in \Sigma$ and $h \in \mathrm{GL}_g(\mathbb{Z})$ with $h\sigma_i h^t$ a face of σ_j, we consider its associated lattice-preserving linear map $L_{i,j,g}: \sigma_i \hookrightarrow \sigma_j$ in the diagram. The space $A_g^{\mathrm{trop},\Sigma}$ is the *moduli space of tropical abelian varieties* of dimension g with respect to Σ.

2.7. Toroidal compactifications of the moduli space $\mathcal{A}_g$.

In this paper, $\mathcal{A}_g$ denotes the moduli stack of principally polarized abelian varieties of dimension g. It is a smooth Deligne-Mumford algebraic stack of dimension $d = \binom{g+1}{2}$, and the coarse moduli space of principally polarized abelian varieties, denoted A_g, is a quasiprojective variety.

The moduli stack $\mathcal{A}_g$ is not proper, and there are different constructions of compactifications of $\mathcal{A}_g$. In particular, it is possible to construct normal crossings compactifications of $\mathcal{A}_g$ via the theory of toroidal compactifications. Both the constructions of $\mathcal{A}_g$ and of its toroidal compactifications as algebraic stacks were achieved in [AMRT75] over the complex numbers and in [FC90] over an arbitrary base. Even though we work over the complex numbers, we often refer to the constructions in [FC90] as these are more conveniently stated within the algebraic category and specifically for moduli of abelian varieties (rather than quotients of bounded symmetric domains as in [AMRT75]).

Let Σ be an admissible decomposition of Ω_g^{rt} (in the sense of Definition 2.5). Then one may associate to Σ a *toroidal compactification* $\overline{\mathcal{A}_g}^\Sigma$ of $\mathcal{A}_g$, which is a proper Deligne-Mumford stack, although in general it is not smooth. The fact that $\mathcal{A}_g \subset \overline{\mathcal{A}_g}^\Sigma$ is toroidal means that $(\mathcal{A}_g, \overline{\mathcal{A}_g}^\Sigma)$ is étale-locally isomorphic to a torus inside a toric variety.

By construction, the toroidal compactification $\overline{\mathcal{A}_g}^\Sigma$ comes with a stratification into locally closed subsets. These are in order-reversing bijection, with respect to the order relation given by the closure, with the $\mathrm{GL}_g(\mathbb{Z})$-equivalence classes of the relative interiors of the cones in Σ. For example, the origin of Ω_g^{rt}, which is the unique zero-dimensional cone in every admissible decomposition Σ, corresponds to the open substack $\mathcal{A}_g$, which is the unique stratum of $\overline{\mathcal{A}_g}^\Sigma$ of maximal dimension d. At the other extreme, the maximal dimensional cones in Σ correspond to the zero-dimensional strata of $\overline{\mathcal{A}_g}^\Sigma$.

We study the *perfect* toroidal compactification $\mathcal{A}_g^{\mathrm{perf}} = \overline{\mathcal{A}_g}^{\Sigma_g^{\mathrm{P}}}$ of $\mathcal{A}_g$, i.e., the toroidal compactification of $\mathcal{A}_g$ associated to the perfect cone decomposition Σ_g^{P}. The geometric significance of the perfect cone compactification was highlighted in work of Shepherd-Barron [SB06], who shows that $\mathcal{A}_g^{\mathrm{perf}}$ is the canonical model of $\mathcal{A}_g$ for $g \geq 12$. For our purposes it is particularly nice because the number of strata of codimension l in the boundary of $\mathcal{A}_g^{\mathrm{perf}} \setminus \mathcal{A}_g$ is independent of g if $l \leq g$ (see [GHT18, Prop. 7.1]).

3. A comparison theorem for $\mathcal{A}_g$ and A_g^{trop}

Let Σ be any admissible decomposition of Ω_g^{rt}. As mentioned above, we can use Σ to construct both a toroidal compactification of $\mathcal{A}_g$, and the generalized cone complex $A_g^{\text{trop},\Sigma}$, the moduli space of tropical abelian varieties associated to Σ. In this section, we record the relationship between the homology of $A_g^{\text{trop},\Sigma}$ with the top-weight cohomology of $\mathcal{A}_g$, as deduced from Deligne's comparison theorems and the framework in [CGP18]. This precise relationship was already remarked by Odaka-Oshima [OO18, Corollary 2.9], as we explain further in Remark 3.2, but it is useful to have a self-contained proof, below.

Theorem 3.1. For each $i \geq 0$ and admissible decomposition Σ, we have a canonical isomorphism
$$\widetilde{H}_{i-1}(LA_g^{\text{trop},\Sigma}; \mathbb{Q}) \cong \text{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g; \mathbb{Q}),$$
where $d = \binom{g+1}{2}$ is the complex dimension of $\mathcal{A}_g$.

Proof. First, by replacing Σ with another admissible decomposition of Ω_g^{rt} that refines it, we may assume that every cone of Σ is smooth and that it enjoys the following additional property: for any $h \in \text{GL}_g(\mathbb{Z})$ and $\sigma \in \Sigma$, we have that h fixes, pointwise, the cone $h\sigma h^t \cap \sigma$. Such a refinement is well known to exist [FC90, IV.2, p. 98]. For example, one may be obtained by taking the barycentric refinement, which is simplicial, and then taking an appropriate *smooth* refinement which can be constructed as in [CLS11, Theorem 11.1.9]. The homeomorphism type of $LA_g^{\text{trop},\Sigma}$ is unchanged when passing to a refinement.

Then, by [FC90, Theorem 5.7], it follows that $\overline{\mathcal{A}_g}^\Sigma$ is a smooth, separated Deligne-Mumford stack which is a simple normal crossings compactification of $\mathcal{A}_g$ and whose boundary complex is $LA_g^{\text{trop},\Sigma}$. Now the desired result follows from the following comparison theorem: we have a canonical isomorphism
$$\widetilde{H}_{i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q}) \cong \text{Gr}_{2d}^W H^{2d-i}(\mathcal{X}; \mathbb{Q}),$$
for any normal crossings compactification $\mathcal{X} \subset \overline{\mathcal{X}}$ of smooth, separated Deligne-Mumford stacks over $\mathbb{C}$, where $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$ denotes the boundary complex of the pair $(\mathcal{X}, \overline{\mathcal{X}})$ and $d = \dim \mathcal{X}$ is the complex dimension of $\mathcal{X}$. This comparison theorem follows from Deligne's mixed Hodge theory [Del71, Del74] in the case of complex varieties; we refer to [CGP18] for the generalization to Deligne-Mumford stacks. $\square$

Remark 3.2. Let us briefly explain how Theorem 3.1 appears in [Oda19] and [OO18], since the language of those papers is somewhat different. Odaka and Odaka-Oshima study certain "hybrid" compactifications of arithmetic quotients Γ/D of Hermitian symmetric domains. The case of $\mathcal{A}_g$ is the case $\Gamma = \text{Sp}(2g, \mathbb{Z})$ and D is the "unit disc" of complex symmetric matrices Z with $Z^t\overline{Z} < \text{Id}_g$. The point is that the boundary of these compactifications is homeomorphic to $L\mathcal{A}_g^{\text{trop},\Sigma}$, so the comparison statement in [OO18, Corollary 2.9], which relies on [CGP18], combined with Theorem 2.1 of op. cit., reduces to Theorem 3.1 in this case.

It is worth emphasizing the independence of choice of the admissible decomposition of Σ, as remarked in [Oda19, A.14], that was implicit in the discussion above. More precisely, for any two admissible decompositions Σ_1 and Σ_2 of Ω_g^{rt}, we have a homeomorphism of links
$$LA_g^{\text{trop},\Sigma_1} \cong LA_g^{\text{trop},\Sigma_2}.$$

Indeed, it is well known that any two admissible decompositions Σ_1 and Σ_2 admit a common refinement $\widetilde{\Sigma}$ which is an admissible decomposition [FC90, IV.2, p. 97], and by the construction of §2.2, we have canonical homeomorphisms $LA_g^{\text{trop},\Sigma_1} \cong LA_g^{\text{trop},\tilde{\Sigma}} \cong LA_g^{\text{trop},\Sigma_2}$.

4. The Perfect and Voronoi Chain Complexes

In computing the top-weight cohomology of $\mathcal{A}_g$ there are two chain complexes that play central roles: the perfect chain complex $P_\bullet^{(g)}$ and the Voronoi chain complex $V_\bullet^{(g)}$. In this section we define both of these complexes, and show that the homology of the perfect chain complex $P_\bullet^{(g)}$ computes the top-weight cohomology of $\mathcal{A}_g$. Further, we show that the perfect and Voronoi complexes are related via a short exact sequence of chain complexes, which is useful as the Voronoi complex has seen more extensive study [EVGS13, SEVKM19]. We make use of this short exact sequence to prove our main results in Section 6.

4.1. The perfect chain complex.

We first fix some notation, most of which we adapt from Section 2.2. For $n \in \mathbb{Z}$, let $\Sigma_g^{\mathrm{P}}[n]$ be the set of perfect cones in Σ_g^{P} of dimension $n+1$, and denote the finite set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of such cones by $\Sigma_g^{\mathrm{P}}[n]/\mathrm{GL}_g(\mathbb{Z})$. We write $\sigma \sim \sigma'$ if and only if σ and σ' lie in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit. Recall that a cone $\sigma \in \Sigma_g^{\mathrm{P}}$ is *alternating* if and only if every element of $\mathrm{GL}_g(\mathbb{Z})$ stabilizing σ induces an orientation-preserving cone morphism of σ. If σ is an alternating cone then every cone in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit as σ is alternating. We call such $\mathrm{GL}_g(\mathbb{Z})$-orbits alternating. Let $\Gamma_n^{(g)} = \Gamma_n$ be a set of representatives for the alternating elements of $\Sigma_g^{\mathrm{P}}[n]/\mathrm{GL}_g(\mathbb{Z})$.

For each n and each $\sigma \in \Gamma_n$, choose an orientation ω_σ on σ; the $\mathrm{GL}_g(\mathbb{Z})$-action extends this choice to a choice of orientation on every alternating cone in Σ_g^{P}. If $\rho \subset \sigma$ is an alternating facet of σ, denote the orientation induced on ρ by $\omega_\sigma|_\rho$. Now let $\eta(\rho, \sigma)$ be 1 if the orientation on ρ agrees with the orientation induced by σ (i.e., $\omega_\rho = \omega_\sigma|_\rho$) and -1 otherwise. Finally, given $\sigma \in \Gamma_n$ and $\sigma' \in \Gamma_{n-1}$ define

$$\delta(\sigma', \sigma) := \sum_{\substack{\rho \subset \sigma \\ \rho \sim \sigma'}} \eta(\rho, \sigma) \tag{3}$$

where the sum is over all facets ρ of σ in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit as σ'. With this notation in hand we can now define the perfect chain complex.

Definition 4.1. The *perfect chain complex* $(P_\bullet^{(g)}, \partial_\bullet)$ is the rational complex defined as follows. For each n, $P_n^{(g)}$ is the $\mathbb{Q}$-vector space with basis indexed by Γ_n. The differential $\partial_n : P_n^{(g)} \to P_{n-1}^{(g)}$ is given by

$$\partial(e_\sigma) := \sum_{\sigma' \in \Gamma_{n-1}} \delta(\sigma', \sigma) e_{\sigma'}.$$

Notice that $P_n^{(g)}$ is only possibly nonzero in the range $-1 \le n \le \binom{g+1}{2} - 1$, but even within this range $P_n^{(g)}$ may be zero since alternating perfect cones do not necessarily exist in every dimension (see Example 4.2). While in many cases $\delta(\sigma', \sigma)$ is equal to $-1, 0$, or 1, this need not always be the case since a cone may have two or more facets that are $GL_g(\mathbb{Z})$-equivalent.

Example 4.2. When $g = 2$, recall from Example 2.7 that up to the action of $\mathrm{GL}_2(\mathbb{Z})$ there is precisely one cone of maximal dimension,

$$\sigma_3 := \mathbb{R}_{\ge 0} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle.$$

Since σ_3 is simplicial, its faces correspond to all subsets of the above ray generators. One can show that up to the action of $\mathrm{GL}_2(\mathbb{Z})$ there is at most one cone in each dimension:

$$\sigma_2 := \mathbb{R}_{\ge 0} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle, \quad \sigma_1 := \mathbb{R}_{\ge 0} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle, \quad \sigma_0 := \mathbb{R}_{\ge 0} \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle.$$
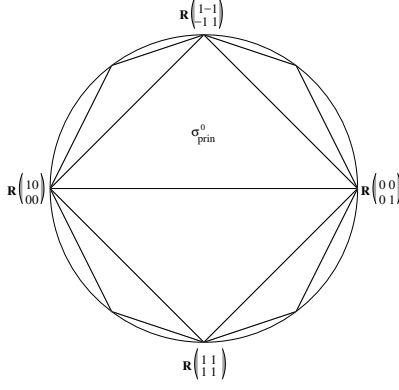
FIGURE 1. A section of Ω_2^{rt} and its perfect cone decomposition.

Thus, to determine Γ_{-1}, Γ_0, Γ_1, and Γ_2, it is enough to see which of σ_0, σ_1, σ_2, and σ_3 are alternating. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can show that A stabilizes both σ_2 and σ_3 and that the induced cone morphism is orientation-reversing. Thus, Γ_1 and Γ_2 are empty. On the other hand, since the action of $\mathrm{GL}_2(\mathbb{Z})$ fixes the cone point σ_0, both σ_0 and σ_1 are alternating. So, $\Gamma_{-1} = \{\sigma_0\}$ and $\Gamma_0 = \{\sigma_1\}$. From this we see that the complex $P_\bullet^{(2)}$ is:

$$P_2^{(2)} \qquad\qquad P_1^{(2)} \qquad\qquad P_0^{(2)} \qquad\qquad P_{-1}^{(2)} \qquad\qquad P_{-2}^{(2)}$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Q}\langle e_{\sigma_1} \rangle \xrightarrow{\ \partial_0\ } \mathbb{Q}\langle e_{\sigma_0} \rangle \longrightarrow 0$$

where ∂_0 sends e_{σ_1} to either e_{σ_0} or $-e_{\sigma_0}$ depending on the chosen orientations.

Remark 4.3. To be precise, the perfect complex $P_\bullet^{(g)}$ as constructed in Definition 4.1 is only unique up to isomorphism. In particular, the choice of representatives for Γ_n or reference orientations may result in different but isomorphic chain complexes. For instance, in Example 4.2, the differential ∂_0 is only determined up to sign.

The next proposition shows that the perfect complex $P_\bullet^{(g)}$ is isomorphic to the cellular chain complex associated to the symmetric CW complex $LA_g^{\mathrm{trop,P}}$. Thus, by Theorem 3.1, the homology of $P_\bullet^{(g)}$ computes the top-weight rational cohomology of $\mathcal{A}_g$.

Proposition 4.4. For each $i \geq 0$, there exist canonical isomorphisms

$$H_{i-1}(P_\bullet^{(g)}) \cong \widetilde{H}_{i-1}(LA_g^{\mathrm{trop,P}}; \mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g; \mathbb{Q}).$$

where $A_g^{\mathrm{trop,P}} = A_g^{\mathrm{trop},\Sigma_g^{\mathrm{P}}}$ is the tropical moduli space constructed in Section 2.6.

Proof. By construction, $P_\bullet^{(g)}$ is naturally isomorphic to the cellular chain complex of $LA_g^{\mathrm{trop,P}}$ as defined in Section 2.2. Observe that the space $A_g^{\mathrm{trop,P}}$ is connected since it deformation retracts to the cone point. Thus, the first isomorphism then follows from part (1) of Proposition 2.1 and the second isomorphism follows from Theorem 3.1. $\square$

Example 4.5. By Example 4.2, we see that $P_\bullet^{(2)}$ has trivial homology in all degrees. Thus, Proposition 4.4 recovers the fact that $\mathcal{A}_2$ has trivial top-weight cohomology [Igu62].

4.2. The Voronoi Complex. Now we introduce a closely related complex, called the Voronoi complex $V_\bullet^{(g)}$, as considered in [EVGS13, SEVKM19][1]. We shall soon see that $V_\bullet^{(g)}$ is a quotient of $P_\bullet^{(g)}$, obtained by setting to zero the generators corresponding to cones contained in the boundary of Ω_g^{rt}.

For each $n \in \mathbb{Z}$, let $\overline{\Gamma}_n^{(g)} = \overline{\Gamma}_n$ be the subset of Γ_n consisting of those cones σ such that $\sigma \cap \Omega_g \neq \emptyset$. For each $\sigma \in \overline{\Gamma}_n$, let ω_σ be an orientation of σ, and for each $\sigma \in \overline{\Gamma}_{n-1}$ define $\delta(\sigma', \sigma)$ as before in (3). With this notation, we can now define the Voronoi complex.

Definition 4.6. The *Voronoi chain complex* $(V_\bullet^{(g)}, d_\bullet)$ is the complex where $V_n^{(g)}$ is the $\mathbb{Q}$-vector space with basis indexed by $\overline{\Gamma}_n$ and the differential $d_n : V_n^{(g)} \to V_{n-1}^{(g)}$ is given by

$$d(e_\sigma) := \sum_{\sigma' \in \overline{\Gamma}_{n-1}} \delta(\sigma', \sigma) e_{\sigma'}.$$

Example 4.7. The $\mathrm{GL}_2(\mathbb{Z})$-orbits of alternating cones in Σ_2^{P} are all contained in $\Omega_2^{\mathrm{rt}} \setminus \Omega_2$. Hence the Voronoi complex $V_\bullet^{(2)}$ is zero in all degrees.

There is a natural surjection of chain complexes $P_\bullet^{(g)} \twoheadrightarrow V_\bullet^{(g)}$ given by quotienting $P_n^{(g)}$ by the subcomplex spanned by those cones contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$. Our next goal, achieved in Theorem 4.13, is to show that the kernel of the map above can naturally be identified with $P_\bullet^{(g-1)}$. We begin by noting the following two lemmas studying those cones lying in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$.

Lemma 4.8. Let $\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_n v_n^t \rangle$ with $v_1, \ldots, v_n \in \mathbb{Z}^g$ be a perfect cone. Then σ is contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ if and only if $\dim \mathrm{span}_{\mathbb{R}}\langle v_1, v_2, \ldots, v_n \rangle < g$.

Lemma 4.9. If $\sigma \in \Sigma_g^{\mathrm{P}}$ is a perfect cone and $\sigma \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ then there is a matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ and a cone $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$, where $g' < g$ and $\sigma' \cap \Omega_{g'} \neq \emptyset$, with

$$(4) \qquad A\sigma A^t = \left\{ \left(\begin{array}{c|c} Q' & 0 \\ \hline 0 & 0 \end{array} \right) \middle| Q' \in \sigma' \right\}.$$

In this situation, say σ' is a *reduction* of σ.

We now show that, in a sense that we shall make precise, the action of $\mathrm{GL}_g(\mathbb{Z})$ on σ does not depend on the ambient matrix size g. For example, given a cone $\sigma \in \Sigma_g^{\mathrm{P}}$ and a reduction $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$ of σ, we will see that σ is alternating if and only if σ' is alternating. We begin with the following definition.

Definition 4.10. Given perfect cones $\sigma_1, \sigma_2 \in \Sigma_g^{\mathrm{P}}$, let $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ denote the set of morphisms $\rho : \sigma_1 \to \sigma_2$ which are restrictions from the action of $\mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}:

$$\begin{array}{ccc} \Omega_g^{\mathrm{rt}} & \xrightarrow{X \mapsto AXA^t} & \Omega_g^{\mathrm{rt}} \\ \uparrow & & \uparrow \\ \sigma_1 & \xrightarrow{\quad \rho \quad} & \sigma_2 \end{array}.$$

Proposition 4.11. If $\sigma_1, \sigma_2 \in \Sigma_g^{\mathrm{P}}$ are perfect cones contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ and $\sigma_1', \sigma_2' \in \Sigma_{g'}^{\mathrm{P}}$ are reductions of σ_1 and σ_2 respectively, then there exists a bijection

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[1] In [EVGS13, SEVKM19] the Voronoi complex is defined as a complex of free $\mathbb{Z}$-modules, while our definition of Voronoi complex is as a complex of $\mathbb{Q}$-vector spaces.

$$\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2) \xleftrightarrow{\ \sim\ } \mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2').$$

Proof. By Lemma 4.9 we have for some $g' < g$, for $i = 1, 2$ there are $\sigma_i' \in \Sigma_{g'}^{\mathrm{P}}$ with $\sigma_i' \cap \Omega_{g'} \neq \emptyset$ and

(5)
$$\sigma_i = \left\{ \left(\begin{array}{c|c} Q' & 0 \\ \hline 0 & 0 \end{array} \right) \middle| Q' \in \sigma_i' \right\}.$$

If $\rho' \in \mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2')$ arises from the action of a matrix $A' \in \mathrm{GL}_{g'}(\mathbb{Z})$ on $\Omega_{g'}^{\mathrm{rt}}$, then

$$A = \left(\begin{array}{c|c} A' & 0 \\ \hline 0 & \mathrm{Id} \end{array} \right)$$

is in $\mathrm{GL}_g(\mathbb{Z})$ and a computation shows that A induces a cone morphism $\rho : \sigma_1 \to \sigma_2$.

In the other direction, suppose that $\rho \in \mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ comes from the action of a matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}, and let A' be the $g' \times g'$ upper left submatrix of A. Now A' induces a morphism from σ_1' to σ_2', since for all $M \in \sigma_1'$,

$$A \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \right) A^t = \left(\begin{array}{c|c} A' M A'^t & 0 \\ \hline 0 & 0 \end{array} \right)$$

where the fact that the lower right three blocks on the right hand side are zero follows from the fact that σ_2 is of the form (5).

We wish to show $\det A' = \pm 1$, that is, that $A' \in \mathrm{GL}_g'(\mathbb{Z})$. Suppose that

$$\sigma_1 = \mathbb{R}_{\geq 0} \langle v_1 v_1^t, v_2 v_2^t, \ldots, v_n v_n^t \rangle, \quad \sigma_2 = \mathbb{R}_{\geq 0} \langle w_1 w_1^t, w_2 w_2^t, \ldots, w_n w_n^t \rangle$$

where each v_i and w_j is supported only in the first g' coordinates where $g' = \dim \sigma_1 = \dim \sigma_2$. Let $\tilde{v}_i$ and $\tilde{w}_i$ be the truncations of v_i and w_i, respectively, to the first g' coordinates, so

$$\sigma_1' = \mathbb{R}_{\geq 0} \langle \tilde{v}_1 \tilde{v}_1^t, \ldots, \tilde{v}_n \tilde{v}_n^t \rangle, \quad \sigma_2' = \mathbb{R}_{\geq 0} \langle \tilde{w}_1 \tilde{w}_1^t, \ldots, \tilde{w}_n \tilde{w}_n^t \rangle.$$

After reordering, we may assume the action of A takes $v_i v_i^t$ to $w_i w_i^t$; thus $A v_i = \pm w_i$. Similarly, since each v_i and w_i are zero after the first g' coordinates, $A' \tilde{v}_i = \pm \tilde{w}_i$. Letting $B = A^{-1} \in \mathrm{GL}_g(\mathbb{Z})$ and writing B' for the $g' \times g'$ upper left submatrix of B, the same argument shows that $B' \tilde{w}_i = \pm \tilde{v}_i$ for all i. Then $A' B' \tilde{v}_i = \pm \tilde{v}_i$ for $i = 1, \ldots, n$, and since $v_1, \ldots, v_n$ span $\mathbb{R}^{g'}$, we have that $\det A' B' = \pm 1$ and $\det A' = \pm 1$.

Finally, a direct computation shows that these constructions are mutual inverses. $\qquad \square$

As a corollary of Proposition 4.11, the properties of being in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit and being alternating do not depend on g—that is, they are preserved by taking reductions.

Corollary 4.12. Two perfect cones $\sigma_1, \sigma_2 \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ are in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit if and only if there exists a $g' < g$ and reductions $\sigma_1', \sigma_2' \in \Sigma_{g'}^{\mathrm{P}}$ that are in the same $\mathrm{GL}_{g'}(\mathbb{Z})$-orbit. A perfect cone $\sigma \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ is alternating if and only if there exists a reduction $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$ which is alternating.

Proof. Two perfect cones σ_1 and σ_2 are in the same orbit if and only if $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ is nonempty. Then the claim follows from Proposition 4.11, since $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ is nonempty if and only if $\mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2')$ is nonempty. Similarly, the proof of Proposition 4.11, applied to $\sigma = \sigma_1 = \sigma_2$, shows that σ has an orientation-reversing automorphism if and only if its reduction σ' does. $\qquad \square$

Corollary 4.12 allows us to naturally identify the set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of alternating perfect cones in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ with the set of $\mathrm{GL}_{g-1}(\mathbb{Z})$-orbits of alternating perfect cones in $\Omega_{g-1}^{\mathrm{rt}}$. Thus, we have the following theorem.

Theorem 4.13. We have a short exact sequence of chain complexes

$$0 \longrightarrow P_\bullet^{(g-1)} \longrightarrow P_\bullet^{(g)} \xrightarrow{\pi} V_\bullet^{(g)} \longrightarrow 0.$$

Proof. By construction, the kernel of $\pi : P_n^{(g)} \to V_n^{(g)}$ is generated by those basis vectors e_σ where $\sigma \in \Gamma_n^{(g)} \setminus \overline{\Gamma}_n^{(g)}$. (Recall that $\Gamma_n^{(g)}$ denotes a set of representatives of alternating $\mathrm{GL}_g(\mathbb{Z})$-orbits of cones in Σ_g^P, and $\overline{\Gamma}_n^{(g)}$ denotes the subset of those that meet Ω_g.) By Corollary 4.12, such cones are in bijection with elements of $\Gamma_n^{(g-1)}$. The differentials on $P_\bullet^{(g)}$ and $P_\bullet^{(g-1)}$ are defined in the same fashion, so the result follows. $\qquad\square$

Theorem 4.13 reflects the stratification of LA_g^{trop} by the spaces $L\Omega_{g'}/\mathrm{GL}_{g'}(\mathbb{Z})$, for $g' = 1, \ldots, g$, which are rational classifying spaces for $\mathrm{GL}_{g'}(\mathbb{Z})$; this is the underlying geometric reason that it is possible to relate the cohomology of $\mathcal{A}_g$ to that of $\mathrm{GL}_{g'}(\mathbb{Z})$, as we do here. This possible relationship was suggested in the more general setting of arithmetic quotients of Hermitian symmetric domains in [OO18, §2.4].

5. The inflation complex and the coloop complex

In this section, we define a subcomplex of $P_\bullet^{(g)}$, called the inflation complex $I_\bullet^{(g)}$. We shall show in Theorem 5.15 that $I_\bullet^{(g)}$ is acyclic. This acyclicity result implies a vanishing result for $H_k(P_\bullet^{(g)})$ in low degrees, obtained in Corollary 5.16, and it is invoked in the computations in the next section for $g = 6$ and $g = 7$. In Section 5.2, we define an analogous subcomplex, the *coloop* complex, of the regular matroid complex, and prove an analogous acyclicity result. The acyclicity of the coloop complex will not be used in this paper, but should likely be useful for future study of the regular matroid complex.

5.1. The inflation complex.

Definition 5.1.

(1) Let $S \subset \mathbb{Z}^g$ be a finite set. Say $v \in S$ is a $\mathbb{Z}^g$-*coloop* of S if v is part of a $\mathbb{Z}$-basis $v, w_2, \ldots, w_g$ for $\mathbb{Z}^g$ such that any $w \in S \setminus \{v\}$ is in the $\mathbb{Z}$-linear span of $w_2, \ldots, w_g$. Equivalently, v is a $\mathbb{Z}^g$-coloop if, up to the action of $\mathrm{GL}_g(\mathbb{Z})$, we may write $v = (0, \ldots, 0, 1)$ and $w = (*, \ldots, *, 0)$ for all $w \in S \setminus \{v\}$.

(2) Now let $\sigma = \sigma[Q]$ be a perfect cone in Σ_g^P. Recall that the set $M(Q)$ of minimal vectors has the property that $v \in M(Q)$ if and only if $-v \in M(Q)$; let $M'(Q) = \{v_1, \ldots, v_n\}$ be a choice of one of $\{v, -v\}$ for each $v \in M(Q)$. So
$$\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_n v_n^t \rangle.$$
We say $v \in M'(Q)$ is a *coloop* of σ if v is a $\mathbb{Z}^g$-coloop in $M'(Q)$.

Remark 5.2. The definition of a $\mathbb{Z}^g$-coloop is inspired by the notion of a coloop of a matroid (i.e., an element not belonging to any circuit). Indeed, if $S \subset \mathbb{Z}^g$ is any finite set and $v \in S$, then v being a $\mathbb{Z}^g$-coloop of S implies that v is a coloop of S, considered as vectors in $\mathbb{R}^g$. The converse does not hold: for example, let $(v_1, v_2) = ((0, 1), (3, 2))$. Then v_1 and v_2 are coloops of the matroid $M(v_1, v_2)$ over $\mathbb{R}$, but neither is a $\mathbb{Z}^2$-coloop.

On the other hand, we prove in Lemma 5.21 that if M is a regular matroid, then M has a coloop if and only if a totally unimodular matrix A representing M has column vectors with a $\mathbb{Z}^g$-coloop, if and only if $\sigma(M)$ has a coloop in the sense of Definition 5.1(2).

Example 5.3. Consider the quadratic form defined by the positive definite matrix
$$Q = \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{pmatrix}.$$

The minimum of Q on $\mathbb{Z}^3 - \{0\}$ is 1 and

$$M(Q) = \{(0, \pm 1, 0), (0, 0, \pm 1), \pm(1, 0, -1), \pm(0, 1, -1)\} \subset \mathbb{R}^3.$$

The corresponding perfect cone $\sigma[Q]$ has a coloop, in particular, letting

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

we see that

$$A \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{pmatrix} A^t = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so $M(AQA^t)$ is $\{(\pm 1, 0, 0), (0, \pm 1, 0), \pm(1, 1, 0), (0, 0, \pm 1)\}$.

The cone $\sigma[Q]$ can also be realized as the matroidal cone $\sigma[M(G)]$ where G is the graph below. The coloop corresponds to the bridge edge of G.

Lemma 5.4. Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{Z}^g$ and suppose that $v_1 \neq v_2$ are both $\mathbb{Z}^g$-coloops for S. Then there is a $\mathbb{Z}$-basis $v_1, v_2, w_3, \ldots, w_g$ for $\mathbb{Z}^g$ such that

$$v_3, \ldots, v_n \in \mathbb{Z}\langle w_3, \ldots, w_g \rangle.$$

Proof. By restricting to $\mathbb{R}\langle v_1, \ldots, v_n \rangle$, we may assume that $\mathbb{R}\langle v_1, \ldots, v_n \rangle = \mathbb{R}^g$. Now the fact that both v_1 and v_2, being coloops, are in every basis for $\mathbb{R}^g$ chosen from $\{v_1, \ldots, v_n\}$, implies that $v_3, \ldots, v_n$ span a $(g-2)$-dimensional subspace V of $\mathbb{R}^g$. Let $w_3, \ldots, w_g$ be a $\mathbb{Z}$-basis for $V \cap \mathbb{Z}^g$. We need only verify that $v_1, v_2, w_3, \ldots, w_g$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^g$. Let $x \in \mathbb{Z}^g$. Since $v_1, v_2, w_3, \ldots, w_g$ form a $\mathbb{Q}$-basis, we have

$$x = a_1 v_1 + a_2 v_2 + a_3 w_3 + \cdots + a_g w_g, \quad \text{for some } a_i \in \mathbb{Q},$$

and it suffices to show that $a_i \in \mathbb{Z}$ for all $i = 1, 2, 3, \ldots, g$. First, we show $a_1 \in \mathbb{Z}$. Since v_1 is a $\mathbb{Z}^g$-coloop, after a change of $\mathbb{Z}$-basis, we may assume that $v_1 = (0, \ldots, 0, 1)$ and that $v_2, \ldots, v_n$ have last coordinate zero. Since $w_3, \ldots, w_g \in \text{span}_{\mathbb{R}}\langle v_3, \ldots, v_n \rangle$, each w_i also has last coordinate zero. Therefore $a_1 \in \mathbb{Z}$. By a similar argument, $a_2 \in \mathbb{Z}$. Then $a_3 w_3 + \cdots + a_g w_g \in V \cap \mathbb{Z}^g$ for $a_i \in \mathbb{Q}$. But $w_3, \ldots, w_g$ is a $\mathbb{Z}$-basis for $V \cap \mathbb{Z}^g$. Therefore it must be that $a_3, \ldots, a_g \in \mathbb{Z}$, as desired. $\square$

Corollary 5.5. Let $S \subset \mathbb{Z}^{g-1}$ be a finite set, and identify $\mathbb{Z}^{g-1}$ with its image in $\mathbb{Z}^g$ under $w \mapsto (w, 0)$. Then v is a coloop of S if and only if v is a coloop of $S \cup \{e_g\} \subset \mathbb{Z}^g$.

Proof. The forward direction is direct from the definitions. For the backward direction, suppose v is a coloop of $S \cup \{e_g\}$. Then *both* v and e_g are coloops, so by Lemma 5.4, up to the action of $\text{GL}_g(\mathbb{Z})$, we may assume that $v = e_{g-1}$ and $w = (*, \ldots, *, 0, 0)$ for all $w \in S \setminus \{v\}$. Therefore v was a coloop of $S \subset \mathbb{Z}^{g-1}$. $\square$

Corollary 5.6. A cone with two or more coloops is not alternating. That is, if $\sigma = \sigma[Q]$ where $v \neq v' \in M'(Q)$ are two distinct coloops, then σ is not alternating.

Proof. The cone σ has an orientation-reversing automorphism induced by an element of $\text{GL}_g(\mathbb{Z})$ swapping the two coloops. $\square$

We now describe two operations on cones, inflation and deflation, that add or remove coloops respectively. Inflation is described in [EVGS13, Section 6.1], and can be performed for any cone, but we shall consider it for the cones in the set $\Sigma_{g,\text{nco}}^P[n]$ defined below.

Recall that $\Sigma_g^P[n]$ denotes the set of $(n+1)$-dimensional perfect cones, and $\Sigma_g^P[n]/GL_g(\mathbb{Z})$ denotes the collection of $GL_g(\mathbb{Z})$-orbits of $(n+1)$-dimensional perfect cones.

Definition 5.7. We define two subsets of $\Sigma_g^P[n]$ as follows:

$$\Sigma_{g,\text{nco}}^P[n] := \{\sigma \in \Sigma_g^P[n] : \text{rank}(\sigma) \leq g - 1 \text{ and } \sigma \text{ has no coloop}\},$$
$$\Sigma_{g,\text{co}}^P[n] := \{\sigma \in \Sigma_g^P[n] : \sigma \text{ has exactly one coloop}\}.$$

We then define $\Sigma_{g,\text{nco}}^P[n]/GL_g(\mathbb{Z})$ and $\Sigma_{g,\text{co}}^P[n]/GL_g(\mathbb{Z})$ to be the collection of $GL_g(\mathbb{Z})$-orbits of the respective sets.

We now define inflation and deflation as operations on $\Sigma_{g,\text{nco}}^P[n]/GL_g(\mathbb{Z})$ and $\Sigma_{g,\text{co}}^P[n]/GL_g(\mathbb{Z})$, and we show these operations are well defined in Lemma 5.12.

Definition 5.8. *Inflation* is the map

$$\text{ifl} : \Sigma_{g,\text{nco}}^P[n]/GL_g(\mathbb{Z}) \longrightarrow \Sigma_{g,\text{co}}^P[n+1]/GL_g(\mathbb{Z})$$

defined as follows. Given an element of $\Sigma_{g,\text{nco}}^P[n]/GL_g(\mathbb{Z})$, choose a representative

$$\sigma = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t\rangle,$$

so that the g-th entry of the w_i are zero (see Lemma 4.9). Let

$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t, e_g e_g^t\rangle.$$

Then set $\text{ifl}([\sigma]) = [\tilde{\sigma}]$.

Remark 5.9. We check that inflation is well defined in Lemma 5.12. However, we pause to point out that $\tilde{\sigma}$ is indeed a perfect cone, as noted in [EVGS13, Section 6.1]: if $Q \in \Omega_{g-1}$ is a positive definite quadratic form such that $\sigma[Q] = \mathbb{R}_{\geq 0}\langle \tilde{w}_1 \tilde{w}_1^t, \ldots, \tilde{w}_k \tilde{w}_k^t\rangle$, where $\tilde{w}_i$ denotes the truncation of w_i by the last entry, then the inflation of σ is the cone associated to the quadratic form

$$\tilde{Q} = \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & m(Q) \end{array}\right)$$

where $m(Q)$ is the minimum value of Q on $\mathbb{Z}^{g-1} \setminus \{0\}$. Moreover, $\tilde{\sigma}$ has exactly one coloop by Corollary 5.5 and the fact that σ had no coloops.

Example 5.10. Continuing Example 5.3, we see that if Q' is the positive definite quadratic form

$$Q' = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix},$$

then $M(Q') = \{\pm(1,0,0), \pm(0,1,0), \pm(1,-1,0)\}$. Thus, the cone $\sigma[Q]$ is the inflation of $\sigma[Q']$. We may describe the cone $\sigma[Q']$ as

$$\sigma[Q'] = \left\{ \left(\begin{array}{c|c} Q'' & 0 \\ \hline 0 & 0 \end{array}\right) \ \middle| \ Q'' \in \sigma\left[\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right]\right\},$$

from which we see that $\sigma[Q']$ does not meet Ω_3, but its inflation $\sigma[Q]$ does. In general, the inflation of a perfect cone corresponding to a quadratic form of rank r will itself be a perfect cone corresponding to a quadratic form of rank $r+1$.

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Definition 5.11. We define the *deflation* operation as a map
$$\mathrm{dfl} : \Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z}) \longrightarrow \Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$$
given as follows. Given an element of $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z})$, pick a $GL_g(\mathbb{Z})$-representative
$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t, e_g e_g^t \rangle,$$
where each w_i is zero in the last coordinate. Let $\sigma = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t \rangle$. It is routine to check that σ really is a perfect cone, and moreover, it has no coloops by Corollary 5.5. Then we set $\mathrm{dfl}([\tilde{\sigma}]) = [\sigma]$. We now show that inflation and deflation are well defined.

Lemma 5.12. For each $n \in \mathbb{N}$, inflation is a well-defined operation on $\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$, and deflation is a well-defined operation on $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z})$. Furthermore, these operations are inverses of each other.

Proof. We start with inflation. Given $[\sigma] \in \Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/\mathrm{GL}_g(\mathbb{Z})$, let $\sigma_1 = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t \rangle$ and $\sigma_2 = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t \rangle$ be two $\mathrm{GL}_g(\mathbb{Z})$-representatives of $[\sigma]$ such that the g-th entry of each of the v_i, w_j is zero. By Proposition 4.11, there exist reductions σ_1' of σ_1 and σ_2' of σ_2 as well as an $A \in \mathrm{GL}_{g-1}(\mathbb{Z})$ sending σ_1' to σ_2' in $\Omega^{\mathrm{rt}}_{g-1}$. Then
$$A' = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right)$$
yields an equivalence between the two inflations.

Now let $[\sigma] \in \Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/\mathrm{GL}_g(\mathbb{Z})$. Let
$$\sigma_1 = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t, e_g e_g^t \rangle, \quad \sigma_2 = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t, e_g e_g^t \rangle$$
be two $\mathrm{GL}_g(\mathbb{Z})$ representatives of $[\sigma]$ such that the v_i, w_j have g-th coordinate zero. Then by the proof of Proposition 4.11, there exists an $A \in \mathrm{GL}_g(\mathbb{Z})$ such that
$$A v_i = \pm w_i \text{ for } i = 1, \ldots, k, \quad A e_g = \pm e_g,$$
possibly after reordering the v_i. Indeed, A must take the coloop $\pm e_g$ to $\pm e_g$. Then A gives an equivalence between the deflations $\mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t \rangle \sim \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t \rangle$.

We now have that inflation and deflation are well defined, and it is clear from the definitions that these two operations are inverses. $\square$

Lemma 5.13. Let $\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t \rangle$ be a perfect cone in Σ^{P}_g of rank $< g$ with no coloop. Then $[\sigma]$ is alternating if and only if $\mathrm{ifl}([\sigma])$ is alternating.

Proof. We may assume that $v_1, \ldots, v_n$ have last coordinate 0, so that, letting
$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t, e_g e_g^t \rangle,$$
we have $\mathrm{ifl}(\sigma) = \tilde{\sigma}$. We claim there is a natural bijection
$$(6) \qquad\qquad\qquad\qquad \mathrm{Aut}(\sigma) \longleftrightarrow \mathrm{Aut}(\tilde{\sigma}),$$
where $\mathrm{Aut}(\sigma) = \mathrm{Hom}_{\Omega^{\mathrm{rt}}_g}(\sigma, \sigma)$ (see Definition 4.10) and similarly for $\mathrm{Aut}(\tilde{\sigma})$. Moreover, we claim that (6) takes orientation-preserving/reversing automorphisms of σ to orientation-preserving/reversing automorphisms of $\tilde{\sigma}$, respectively.

Given $\rho \in \mathrm{Aut}(\sigma)$ arising from a matrix $A \in \mathrm{GL}_{g-1}(\mathbb{Z})$, the matrix
$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$
yields an automorphism $\tilde{\rho}$ of $\tilde{\sigma}$. The linear span of $\tilde{\sigma}$ is the sum of the linear span of σ and that of $e_g e_g^t$. Moreover, $\tilde{\rho}$ fixes the ray $e_g e_g^t$ of $\tilde{\sigma}$ and acts on the linear span of σ according to A; in particular, $\tilde{\rho}$ is orientation-preserving if and only if ρ was.

Next, suppose $\tilde{A} \in \mathrm{GL}_g(\mathbb{Z})$ induces $\tilde{\rho} \in \mathrm{Aut}(\tilde{\sigma})$. Recall that e_g is the only coloop of $\tilde{\sigma}$, by Corollary 5.5. Therefore $Ae_g = \pm e_g$, and hence A induces an automorphism of σ. Finally, it is routine to check that the maps constructed between $\mathrm{Aut}(\sigma)$ and $\mathrm{Aut}(\tilde{\sigma})$ are two-sided inverses. $\qquad \square$

Definition 5.14. Let $I_\bullet^{(g)}$ be the subcomplex of $P_\bullet^{(g)}$ generated in degree n by cones $\sigma \in \Gamma_n$ of rank $\leq g - 1$ and cones of rank g with a coloop.

Theorem 5.15. The chain complex $I_\bullet^{(g)}$ is acyclic.

Proof. By Lemmas 5.12 and 5.13 there is a matching of cones generating $I_\bullet^{(g)}$, given by

$$\sigma \to \begin{cases} \mathrm{ifl}(\sigma) & \text{if } \sigma \text{ has no coloop,} \\ \mathrm{dfl}(\sigma) & \text{if } \sigma \text{ has a coloop.} \end{cases}$$

Here, we have abused notation slightly, since ifl is an operation on orbits rather than orbit representatives. Thus, when we write $\mathrm{ifl}(\sigma) = \sigma'$ for $\sigma \in \Gamma_n$ we mean that σ' is the unique orbit representative in Γ_{n+1} such that $\mathrm{ifl}([\sigma]) = [\sigma']$. Similarly for deflation.

Now let σ be a generator in $I_\bullet^{(g)}$ of maximal degree; then σ must have a coloop v. We claim that $\sigma' = \mathrm{dfl}(\sigma)$ is not a facet of any other generator $\tau \neq \sigma$ of $I_\bullet^{(g)}$. Indeed, suppose

$$\tau = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t \rangle$$

is a generator of $I_\bullet^{(g)}$ containing σ' as a facet. If τ had a coloop, say v_n, then since σ' has no coloop, σ' must not contain the ray $v_n v_n^t$. But

$$\mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_{k-1} v_{k-1}^t \rangle$$

is already a facet of τ, so it must be $\sigma' = \mathrm{dfl}(\sigma) = \mathrm{dfl}(\tau)$, implying $\tau = \sigma$. So τ has no coloop. But then $\mathrm{ifl}(\tau)$ is a generator of $I_\bullet^{(g)}$ and it would have higher rank than σ.

Thus, the complex $I_\bullet^{(g)'}$ spanned by all cones except σ and $\mathrm{dfl}(\sigma)$ is a subcomplex. Then we have a short exact sequence

$$0 \longrightarrow I_\bullet^{(g)'} \longrightarrow I_\bullet^{(g)} \longrightarrow I_\bullet^{(g)}/I_\bullet^{(g)'} \longrightarrow 0,$$

where $I_\bullet^{(g)}/I_\bullet^{(g)'}$ is isomorphic to $0 \to \sigma \to \mathrm{dfl}(\sigma) \to 0$. Hence $I_\bullet^{(g)'} \to I_\bullet^{(g)}$ is a quasi-isomorphism. Repeating this, we deduce inductively that I is quasi-isomorphic to 0. $\qquad \square$

As a corollary of Theorem 5.15, we are able to prove the following vanishing result for the cohomology of $P_\bullet^{(g)}$ in low degrees.

Corollary 5.16. If $k \leq g - 2$ then $H_k\left(P_\bullet^{(g)}\right) = 0$.

Proof. Since the inflation complex $I_\bullet^{(g)}$ is acyclic by Theorem 5.15, it is enough to show that $I_k^{(g)} = P_k^{(g)}$ for all $k \leq g - 2$. For this, we simply need the well-known fact that the rank of a perfect cone is at most its dimension. Indeed, let $\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, v_2 v_2^t, \ldots, v_n v_n^t \rangle \in \Sigma_g^{\mathrm{P}}$ be an alternating cone of dimension $k + 1$. If $Q \in \sigma$ then

$$Q = \lambda_1 v_{i_1} v_{i_1}^t + \lambda_2 v_{i_2} v_{i_2}^t + \ldots + \lambda_{k+1} v_{i_{k+1}} v_{i_{k+1}}^t$$

for some $\{i_1, \ldots, i_{k+1}\} \subset \{1, \ldots, n\}$ and some $\lambda_i \in \mathbb{R}_{\geq 0}$. In particular, since $v_j v_j^t$ is a rank one quadratic form, this implies that the rank of Q is at most $k + 1$. Thus, if $k + 1 \leq g - 1$ then $\mathrm{rank}(\sigma) < g$, implying that the orbit of σ represents an element of $I_k^{(g)}$. $\qquad \square$

Remark 5.17. The proof of Corollary 5.16 shows that any cone of dimension less than or equal to $g - 1$ does not intersect Ω_g. This implies that $H_k(V_\bullet^{(g)}) = 0$ for all $k \leq g - 2$.

Remark 5.18. The virtual cohomological dimension of $\mathcal{A}_g$ is

$$\mathrm{vcd}(\mathcal{A}_g) = \mathrm{vcd}(\mathrm{Sp}(2g, \mathbb{Z})) = g^2,$$

by [BS73], see [CL17]. In particular,

$$\mathrm{Gr}_{g^2+g}^W H^i(\mathcal{A}_g; \mathbb{Q}) = 0 \quad \text{for all } i > g^2,$$

which is equivalent to, setting $i = 2\dim(\mathcal{A}_g) - j - 1 = g^2 + g - j - 1$, that

$$H_j(P^{(g)}) = 0 \quad \text{for all } j < g - 1.$$

Corollary 5.16 thus reproves, in a completely different way, the vanishing in top weight of rational cohomology of $\mathcal{A}_g$ in degree above the virtual cohomological dimension.

5.2. **The Regular Matroid Complex and Inflation.** In this section, we introduce two combinatorially defined subcomplexes, $R_\bullet^{(g)}$ and $C_\bullet^{(g)}$, of $P_\bullet^{(g)}$ coming from regular matroids and regular matroids with coloops respectively. These are not used further in this paper. Nevertheless, the matroidal cones in Σ_g^P have geometric significance: Alexeev and Brunyate, in proving the existence of a compactified Torelli map $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{perf}}$, conjectured an open locus on which $\overline{\mathcal{A}_g}^{\mathrm{perf}}$ and $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ are isomorphic and on which the two Torelli maps $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{perf}}$ and $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{Vor}}$ agree [AB12]. The fourth author and Viviani verified their conjecture, showing that the *matroidal partial compactification* $\mathcal{A}_g^{\mathrm{matr}}$, whose strata correspond to cones arising from regular matroids, is the largest such open subset [MV12]. For possible future use in studying $R^{(g)}$, we establish in this section that the complex $C_\bullet^{(g)}$, which is a matroid analogue of $I_\bullet^{(g)}$, is acyclic.

Given a cone $\sigma \in \Sigma_g^P[n]$, we say that σ is a matroidal cone if and only if there exists a simple, regular matroid M of rank at most g such that $[\sigma] = \sigma(M)$ where $\sigma(M)$ is as described in Construction 2.12. Matroidal cones are simplicial. Since the faces of a matroidal cone are themselves matroidal cones, the set of representatives of alternating cones arising from simple, regular matroids forms a subcomplex of $P^{(g)}$.

Definition 5.19. The *regular matroid complex* $R_\bullet^{(g)}$ is the subcomplex of $P_\bullet^{(g)}$ generated in degree n by cones $\sigma \in \Gamma_n$ such that σ is a matroidal cone.

Remark 5.20. When $g = 2$ and $g = 3$, the complexes $R_\bullet^{(g)}$ and $P_\bullet^{(g)}$ are in fact equal. It would be interesting to understand in general how much larger $P_\bullet^{(g)}$ is compared to $R_\bullet^{(g)}$.

Recall that an element e of a matroid M is a *coloop* if it does not belong to any of the circuits of M; equivalently, e is a coloop if it belongs to every base of M. When M is a regular matroid, this is equivalent to the existence of a totally unimodular matrix $A = [v_1, v_2, \ldots, v_n]$ representing M such that $v_i = (*, *, \ldots, *, 0)$ for $i = 1, 2, \ldots, n-1$ and $e = v_n = (0, 0, \ldots, 0, 1)$. It is worth establishing that the notions of a matroid coloop and a $\mathbb{Z}^g$-coloop agree for matroidal cones, as we show in the next lemma.

Lemma 5.21. Let M be a simple, regular matroid of rank $\leq g$. The cone $\sigma(M)$ has a $\mathbb{Z}^g$-coloop if and only if the matroid M has a coloop.

Proof. Suppose that $\sigma(M)$ has a $\mathbb{Z}^g$-coloop. By definition, there exists a quadratic form $Q \in \Omega_g$ such $[\sigma(Q)] = \sigma(M)$ and $M'(Q) = \{v_1, v_2, \ldots, v_n\}$ where $v_i = (*, *, \ldots, *, 0)$ for $i = 1, \ldots, n-1$ and $v_n = (0, 0, \ldots, 0, 1)$. Then by the construction of $\sigma(M)$, the matrix $A = [v_1, v_2, \ldots, v_n]$ is a totally unimodular matrix representing M over $\mathbb{R}$. Therefore v_n is a coloop of the matroid M.

For the other direction, suppose that the regular matroid M on the ground set $\{1, \ldots, n\}$ is represented by a full-rank totally unimodular $g' \times n$ matrix $A = [v_1, v_2, \ldots, v_n]$, for some $g' \leq g$, and that n is a coloop of M. Then n is in every base of M, so v_n is in every full-rank $g' \times g'$ submatrix of A. Reorder so that the rightmost $g' \times g'$ submatrix is full rank; call it B. Then $B \in \mathrm{GL}_{g'}(\mathbb{Z})$ by total unimodularity of A. Consider the matrix $B^{-1}A$, which still represents M. The rightmost $g' \times g'$ submatrix of $B^{-1}A$ is the identity. Moreover, each of the first $n - g'$ columns is of the form $(*, \ldots, *, 0)$, for otherwise it could replace the last column in the rightmost square submatrix to form a full rank square matrix, contradicting that n was a coloop. This shows that v_n is a $\mathbb{Z}^{g'}$-coloop of $v_1, \ldots, v_n \in \mathbb{Z}^{g'}$, and after padding by zeroes, v_n is a $\mathbb{Z}^g$-coloop of the of $v_1, \ldots, v_n$. $\qquad\square$

Definition 5.22. The *coloop complex* $C_\bullet^{(g)}$ is the subcomplex of $P_\bullet^{(g)}$ generated in degree n by cones $\sigma \in \Gamma_n$ such that σ is a matroidal cone and either:

    (1) the rank of σ is $< g$,
    (2) the rank of σ is equal to g and σ has one coloop.

By Lemma 5.21, the generators for $C_\bullet^{(g)}$ are the generators of $R_\bullet^{(g)}$ that are also generators of $I_\bullet^{(g)}$; in summary, we have inclusions of complexes

$$
\begin{array}{ccc}
C_\bullet^{(g)} & \longhookrightarrow & R_\bullet^{(g)} \\
\downarrow & & \downarrow \\
I_\bullet^{(g)} & \longhookrightarrow & P_\bullet^{(g)}
\end{array} \quad .
$$

Similar to the inflation complex, the coloop complex is acyclic.

Theorem 5.23. The chain complex $C_\bullet^{(g)}$ is acyclic.

We omit the details of the proof of Theorem 5.23. It is closely analogous to the proof of Theorem 5.15, the key step being the following lemma.

Lemma 5.24. There is a bijection of sets between

$$
\left\{ \begin{array}{c} \text{alternating, regular matroids} \\ \text{of rank} < g \text{ with } 0 \text{ coloops} \end{array} \right\} \quad \xleftrightarrow{\ \sim\ } \quad \left\{ \begin{array}{c} \text{alternating, regular matroids} \\ \text{of rank} \leq g \text{ with } 1 \text{ coloop} \end{array} \right\}.
$$

6. Computations on the cohomology of $\mathcal{A}_g$

In this section, we compute the top-weight cohomology of $\mathcal{A}_g$ for $3 \leq g \leq 7$, proving Theorem A. When $g = 3, 4$, and 5, we do this by studying the cones of Σ_g^{P} arising from matroids, from which we explicitly compute the chain complex $P_\bullet^{(g)}$. We handle the cases when $g = 6$ and $g = 7$ by utilizing the long exact sequence in homology arising from Theorem 4.13, as well as the fact that the inflation subcomplex $I_\bullet^{(g)}$ is acyclic (see Theorem 5.15). Additionally, we prove a vanishing result for the top-weight cohomology of $\mathcal{A}_g$ for $g = 8, 9$, and 10 in Theorem 6.16.

6.1. **The complex $P_\bullet^{(3)}$.** For $g = 3$, the fact that every perfect cone is matroidal allows us to compute the complex $P_\bullet^{(3)}$ directly. Using this description of $P_\bullet^{(3)}$, we then compute the top-weight cohomology of $\mathcal{A}_3$.

Proposition 6.1. The chain complex $P_\bullet^{(3)}$ is

$$
\begin{array}{ccccccc}
P_5^{(3)} & P_4^{(3)} & P_3^{(3)} & P_2^{(3)} & P_1^{(3)} & P_0^{(3)} & P_{-1}^{(3)}
\end{array}
$$

$$
0 \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\ \sim\ } \mathbb{Q} \longrightarrow 0.
$$

Proof. The only top-dimensional perfect cone of $\Sigma_3^{\mathrm{P}}/GL_3(\mathbb{Z})$ is the principal cone σ_3^{prin} coming from the complete graph K_4 [Vor08, p. 151]. The principal cone σ_3^{prin} is alternating because the automorphisms of K_4 are all alternating permutations of its edges, and every automorphism of σ_3^{prin} arises from $\mathrm{Aut}(K_4)$ by Remark 2.14. Thus, we have $P_5^{(3)} \cong \mathbb{Q}$.

The automorphisms on the codimension i faces of σ_3^{prin} arise from matroids of graphs obtained from K_4 by deleting i edges, see Figure 2. For $i = 1, \ldots, 4$, each of the matroids associated to graphs with i edges removed from K_4 has an automorphism given by an odd permutation of the edges; see Figure 3 for an example. So we have $P_j^{(3)} = 0$ for $1 \leq j \leq 4$. The single ray and vertex of $\Sigma_3^{\mathrm{P}}/GL_3(\mathbb{Z})$ are alternating, so $P_j^{(3)} \cong \mathbb{Q}$ for $j = 0$ and $j = -1$.
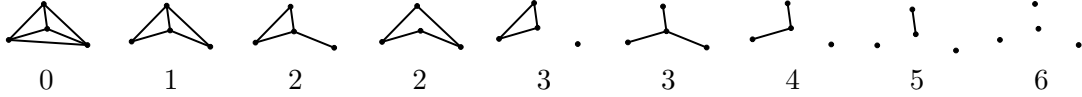
FIGURE 2. Graphs obtained by deleting the indicated number of edges from K_4, giving isomorphism classes of graphic matroids.
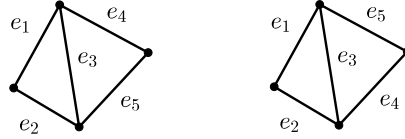
FIGURE 3. An automorphism of $M[K_4 \backslash \{e_6\}]$ interchanging e_4 and e_5.

□

Theorem 6.2. The top-weight cohomology of $\mathcal{A}_3$ is

$$
\mathrm{Gr}_{12}^W H^i(\mathcal{A}_3; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 6 \\ 0 & \text{else.} \end{cases}
$$

Proof. The top-weight cohomology of $\mathcal{A}_3$ is the homology of $P_\bullet^{(3)}$ by Theorem 3.1. □

Remark 6.3. Theorem 6.2 agrees with the work of Hain [Hai02], who computes the full cohomology ring of $\mathcal{A}_3$. Hain deduces in particular $H^6(\mathcal{A}_3; \mathbb{Q}) = E$ where E is a mixed Hodge structure that is an extension $0 \to \mathbb{Q}(-3) \to E \to \mathbb{Q}(-6) \to 0$, where $\mathbb{Q}(n)$ denotes the Tate Hodge structure of dimension 1 and weight $-2n$.

Example 6.4. While we do not need it here, we note that using the fact that all of the perfect cones in Σ_3^{P} arise from graphic matroids, one can check that the inflation complex $I_\bullet^{(3)}$ is the following:

$$
\begin{array}{ccccccc}
I_5^{(3)} & I_4^{(3)} & I_3^{(3)} & I_2^{(3)} & I_1^{(3)} & I_0^{(3)} & I_{-1}^{(3)}
\end{array}
$$

$$
0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\ \sim\ } \mathbb{Q} \longrightarrow 0.
$$

6.2. The complex $P_\bullet^{(4)}$. In this section, we explicitly compute the complex $P_\bullet^{(4)}$ by using the matroidal description of the principal cone given in Section 2.5 together with the description of a similar complex for $\mathrm{SL}_g(\mathbb{Z})$-alternating cones described in [LS78b]. We then use $P_\bullet^{(4)}$ to compute the top-weight cohomology of $\mathcal{A}_4$.

Proposition 6.5. The chain complex $P_\bullet^{(4)}$ is

$$P_9^{(4)} \quad P_8^{(4)} \quad P_7^{(4)} \quad P_6^{(4)} \quad P_5^{(4)} \quad P_4^{(4)} \quad P_3^{(4)} \quad P_2^{(4)} \quad P_1^{(4)} \quad P_0^{(4)} \quad P_{-1}^{(4)}$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \longrightarrow 0.$$

Proof. By Theorem 4.13, we have, in any degree ℓ, that $\dim P_\ell^{(4)} = \dim P_\ell^{(3)} + \dim V_\ell^{(4)}$. We have already computed $P_\bullet^{(3)}$, so we now compute $V_\bullet^{(4)}$. In [LS78b], the authors compute a complex $C_\bullet$ which is generated in degree i by the $(i+1)$-dimensional $\mathrm{SL}_4(\mathbb{Z})$-alternating perfect cones meeting Ω_g up to $\mathrm{SL}_4(\mathbb{Z})$-equivalence. Their results [LS78b, Proposition 3.1] are summarized in the first three columns of Table 1. The cone $\sigma(D_4)$ is the cone corresponding to the quadratic form

$$D_4 = \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{bmatrix}.$$

For $i \notin \{4, 5, 6, 8, 9\}$, they show that $C_i = 0$.

i	C_i	$\mathrm{SL}_4(\mathbb{Z})$-alternating cones of dim $i+1$	$\mathrm{GL}_4(\mathbb{Z})$-alternating?
4	$\mathbb{Q}$	$\sigma(\rotatebox{45}{$\triangleright$})$	no
5	$\mathbb{Q}$	$\sigma(\triangle)$	no
6	$\mathbb{Q}$	$\sigma(\boxtimes)$	yes
8	$\mathbb{Q}$	$\sigma(\boxtimes)$	no
9	$\mathbb{Q}^2$	$\sigma_4^{\mathrm{prin}}, \sigma(D_4)$	no

TABLE 1. $SL_4(\mathbb{Z})$-alternating cones of $\Sigma_4^P/SL_4(\mathbb{Z})$.

We now compute the Voronoi complex $V_\bullet^{(4)}$. As far as we know, this computation—for $\mathrm{GL}_4(\mathbb{Z})$, as opposed to $\mathrm{SL}_4(\mathbb{Z})$—constitutes a small gap in the literature, which we fill here. To obtain the complex $V_\bullet^{(4)}$, we must pass from $\mathrm{SL}_4(\mathbb{Z})$ to $\mathrm{GL}_4(\mathbb{Z})$. In doing so, two things may happen. First, two $\mathrm{SL}_4(\mathbb{Z})$-inequivalent cones may be $\mathrm{GL}_4(\mathbb{Z})$-equivalent. This does not occur by the corollary following Lemma 4.4 in [LS78b]: the $\mathrm{GL}_4(\mathbb{Z})$-orbits of cones in Σ_4^{P} are equal to the $\mathrm{SL}_4(\mathbb{Z})$-orbits. Second, a cone which is $\mathrm{SL}_4(\mathbb{Z})$-alternating may no longer be $\mathrm{GL}_4(\mathbb{Z})$-alternating. We now check whether this occurs for the cones in Table 1.

In degree 9, neither cone is alternating since transposition matrices stabilize these cones but reverse orientation, as is observed in [LS78b, p. 107]. In degrees 4, 5, and 8 the graphic matroids giving rise to each of the cones in Table 1 have an automorphism coming from an odd permutation of the ground set elements, so these cones are not alternating. Therefore $V_9^{(4)} = V_8^{(4)} = V_5^{(4)} = V_4^{(4)} = 0$. In degree 6, the cone $\sigma(\boxtimes)$ is alternating because any automorphism of $M(\boxtimes = K_4 \cup \{e\})$ fixes e and $\sigma(K_4)$ is alternating, so $V_6^{(4)} = \mathbb{Q}$.

We now explain the nonzero morphisms. Since $V_\bullet^{(4)}$ is 0 in degree less than 2, the map $P_0^{(4)} \to P_{-1}^{(4)}$ is an isomorphism. We now compute the map $P_6^{(4)} \to P_5^{(4)}$. We have that $P_6^{(4)}$ is generated by one cone $\sigma(\boxtimes = K_4 \cup \{e\})$. Its faces are the cones obtained from $K_4 \cup \{e\}$ by deleting one edge. Only

24

deleting the edge e yields a graph which gives an alternating perfect cone, and this cone generates $P_5^{(4)}$. So, this map is an isomorphism. $\square$

Theorem 6.6. The top-weight cohomology $\mathrm{Gr}_{20}^W H^i(\mathcal{A}_4; \mathbb{Q})$ of $\mathcal{A}_4$ is 0 for all i.

Proof. The top-weight cohomology of $\mathcal{A}_4$ is given by the homology of the chain complex $P_\bullet^{(4)}$ by Theorem 3.1. This chain complex has no homology. $\square$

In fact, our explicit description of $P_\bullet^{(4)}$ shows that $P_\bullet^{(4)} = I_\bullet^{(4)}$, since every nonzero generator is either of rank < 4 or has a coloop. The acyclicity of $P^{(4)}$ is then consistent with Theorem 5.15.

Remark 6.7. Theorem 6.6 can be deduced from the results in [HT12]. In particular, the weight 0 compactly supported cohomology of $\mathcal{A}_4$ is encoded in the last two columns of Table 1 in loc. cit., which describes the first page of a spectral sequence converging to the cohomology of the second Voronoi compactification $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ of $\mathcal{A}_4$. Here, these two columns contain the compactly supported cohomology of two strata of $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ whose union is exactly $\mathcal{A}_4$: the fifth column corresponds to the Torelli locus, while the sixth column corresponds to its complement in $\mathcal{A}_4$. By Poincaré duality (9), as described in Section 7, if $\mathcal{A}_4$ had top-weight cohomology it would also have compactly supported cohomology in weight 0. However, even though there are some undetermined entries in the sixth column of the aforementioned table, a close look at the table shows that the weight 0 part must vanish. Indeed, there are no weight 0 classes in the table northwest of the undetermined entries, so any weight 0 classes in the sixth column would persist in the E_∞ page of the spectral sequence and yield weight 0 classes of $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$. But this is impossible as $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ is a smooth compactification of $\mathcal{A}_4$.

6.3. **The complex $P_\bullet^{(5)}$.** By using the short exact sequence given in Theorem 4.13, we now compute the complex $P_\bullet^{(5)}$. From this we compute the top-weight cohomology of $\mathcal{A}_5$.

Proposition 6.8. The chain complex $P_\bullet^{(5)}$ is

$$
\begin{array}{cccccccc}
P_{14}^{(5)} & P_{13}^{(5)} & P_{12}^{(5)} & P_{11}^{(5)} & P_{10}^{(5)} & P_9^{(5)} & P_8^{(5)} & P_7^{(5)}
\end{array}
$$

$$
0 \longrightarrow \mathbb{Q}^3 \xrightarrow{\partial_{14}} \mathbb{Q}^2 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\partial_{11}} \mathbb{Q}^6 \xrightarrow{\partial_{10}} \mathbb{Q}^7 \xrightarrow{\partial_9} \mathbb{Q} \longrightarrow 0 \longrightarrow
$$

$$
\begin{array}{cccccccc}
P_6^{(5)} & P_5^{(5)} & P_4^{(5)} & P_3^{(5)} & P_2^{(5)} & P_1^{(5)} & P_0^{(5)} & P_{-1}^{(5)}
\end{array}
$$

$$
\mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \longrightarrow 0.
$$

Proof. By Theorem 4.13, we have in any degree ℓ that $\dim P_\ell^{(5)} = \dim P_\ell^{(4)} + \dim V_\ell^{(5)}$. We have already computed $P_\bullet^{(4)}$, so we now study $V_\bullet^{(5)}$, which was computed in [EVGS13]. Recall from Section 4.1 that Γ_n denotes the set of representatives of alternating perfect cones of dimension $n + 1$. In [EVGS13, Table 1], the cardinality of Γ_n, is given by

$\mathbf{n}$	4	5	6	7	8	9	10	11	12	13	14
$|\Gamma_n|$	0	0	0	0	1	7	6	1	0	2	3

In [EVGS13, Section 6.2], there is an explicit description of the differential maps.

Since $V_\bullet^{(5)}$ is supported in degrees > 7, while $P_\bullet^{(4)}$ is supported in degrees < 7, the differential maps $P_j^{(5)} \to P_{j-1}^{(5)}$ for $j < 7$ are inherited from $P_\bullet^{(4)}$, and likewise the differential maps $P_j^{(5)} \to P_{j-1}^{(5)}$ for $j > 7$ are inherited from $V_\bullet^{(5)}$. $\square$

Theorem 6.9. The top-weight cohomology of $\mathcal{A}_5$ is

$$\mathrm{Gr}_{30}^W H^i(\mathcal{A}_5; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 15 \text{ or } 20, \\ 0 \text{ else.} \end{cases}$$

Proof. By Proposition 6.8 and [EVGS13, Theorem 4.3] we have that $H_9(P_\bullet^{(5)}) = \mathbb{Q}$ and $H_{14}(P_\bullet^{(5)}) = \mathbb{Q}$. Then by Theorem 3.1, we obtain the desired result. □

Remark 6.10. Grushevsky asks if $\mathcal{A}_g$ ever has nonzero odd cohomology [Gru09, Open Problem 7]. Theorem 6.9 confirms that $\mathcal{A}_5$ does in degree 15. Furthermore, we will see in Theorem 6.12 that Grushevsky's question is also answered affirmatively for $\mathcal{A}_7$, where

$$\dim \mathrm{Gr}_{56}^W H^{33}(\mathcal{A}_7; \mathbb{Q}) = \dim \mathrm{Gr}_{56}^W H^{37}(\mathcal{A}_7; \mathbb{Q}) = 1.$$

6.4. **The top-weight cohomology of $\mathcal{A}_6$ and $\mathcal{A}_7$.** In [EVGS13, Theorem 4.3][2] Elbaz-Vincent, Gangl, and Soulé computed the homology of the Voronoi complex $V_\bullet^{(g)}$ for $g = 5, 6$, and 7. Combining this, together with Proposition 6.8, we are able to compute the top-weight cohomology of $\mathcal{A}_6$ and $\mathcal{A}_7$.

Theorem 6.11. The top-weight cohomology of $\mathcal{A}_6$ is

$$\mathrm{Gr}_{42}^W H^i(\mathcal{A}_6; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 30, \\ 0 \text{ else.} \end{cases}$$

Proof. By Proposition 4.4, we need to show that $H_{11}(P_\bullet^{(6)}) \cong \mathbb{Q}$ and $H_i(P_\bullet^{(6)}) = 0$ for $i \neq 11$. Consider the long exact sequence in homology arising from the short exact sequence of chain complexes given in Theorem 4.13. Combining this with the computation of the homology of $V_\bullet^{(6)}$ [EVGS13, Theorem 4.3] and the homology of $P_\bullet^{(5)}$ given in Proposition 6.8, our computation of $H_k(P_\bullet^{(6)})$ reduces to the four cases in Table 2.

i	$H_i(P_\bullet^{(5)})$	$H_i(P_\bullet^{(6)})$	$H_i(V_\bullet^{(6)})$
≥ 16	0	0	0
15	0	0	$\mathbb{Q}$
14	$\mathbb{Q}$	0	0
13	0	0	0
12	0	0	0
11	0	$\mathbb{Q}$	$\mathbb{Q}$
10	0	0	$\mathbb{Q}$
9	$\mathbb{Q}$	0	0
≤ 8	0	0	0

TABLE 2. The long exact sequence in homology for $g = 6$.

- **Case 1:** ($i \leq 8, i = 12, 13, i \geq 16$): For these values of i, both $H_i(P_\bullet^{(5)})$ and $H_i(V_\bullet^{(6)})$ are equal to zero, so $H_i(P_\bullet^{(6)}) = 0$.

---

[2]Elbaz-Vincent, Gangl, and Soulé define the Voronoi complex as a complex of free $\mathbb{Z}$-modules, and in [EVGS13, Theorem 4.3] they compute the integral homology of this complex. Our definition of the Voronoi complex $V_\bullet^{(g)}$ is a complex of $\mathbb{Q}$-vector spaces, but this causes no problems as we are only interested in the rational homology of $V_\bullet^{(g)}$.

- **Case 2:** ($i = 14, 15$): The long exact sequence in homology gives the exact sequence

$$0 \longrightarrow H_{15}(P_\bullet^{(6)}) \longrightarrow \mathbb{Q} \xrightarrow{\delta_{15}^6} \mathbb{Q} \longrightarrow H_{14}(P_\bullet^{(6)}) \longrightarrow 0.$$

  Exactness implies that the connecting homomorphism δ_{15}^6 is either an isomorphism or the zero map. By [EVGS13, Theorem 6.1] we know that inflating the cones in $V_\bullet^{(5)}$ gives an isomorphism of chain complexes $V_\bullet^{(6)} \cong V_\bullet^{(5)}[1] \oplus F_\bullet$ for some complex $F_\bullet$. Combining this with [EVGS13, Theorem 4.3] shows the nontrivial homology class in $H_{15}(V_\bullet^{(6)})$ is the inflation of a nontrivial homology class in $H_{14}(V_\bullet^{(5)})$. By Proposition 6.8, the nontrivial homology class in $H_{14}(P_\bullet^{(5)})$ is the nontrivial homology class in $H_{14}(V_\bullet^{(5)})$, so $H_{15}(V_\bullet^{(6)})$ is generated by the inflation of the nontrivial class $H_{14}(P_\bullet^{(5)})$. By the proof of the acyclicity of the inflation complex $I_\bullet^{(g)}$ (Theorem 5.15), this implies the connecting map δ_{15}^6 is an isomorphism. The exact sequence above then implies that $H_k(P_\bullet^{(6)}) = 0$ for both $k = 14$ and $k = 15$.
- **Case 3:** ($i = 11$): Since $H_{11}(P_\bullet^{(5)})$ and $H_{10}(P_\bullet^{(5)})$ vanish, the long exact sequence in homology gives us

$$0 \longrightarrow H_{11}(P_\bullet^{(6)}) \longrightarrow \mathbb{Q} \longrightarrow 0.$$

  This exactness implies that $H_{11}(P_\bullet^{(6)})$ is isomorphic to $\mathbb{Q}$.
- **Case 4:** ($i = 9, 10$): By considering the long exact sequence in homology in the range $i = 10$ to $i = 9$ we have the following exact sequence:

$$0 \longrightarrow H_{10}(P_\bullet^{(6)}) \longrightarrow \mathbb{Q} \xrightarrow{\delta_{10}^6} \mathbb{Q} \longrightarrow H_9(P_\bullet^{(6)}) \longrightarrow 0.$$

  An analysis similar to that in Case 2 shows that connecting map δ_{11}^6 is an isomorphism, implying by exactness that $H_i(P_\bullet^{(6)}) = 0$ for both $i = 9$ and $i = 10$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We now compute the top-weight rational cohomology of $\mathcal{A}_7$.

Theorem 6.12. The top-weight cohomology of $\mathcal{A}_7$ is

$$\mathrm{Gr}_{56}^W H^i(\mathcal{A}_7; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 28, 33, 37, 42 \\ 0 \text{ else.} \end{cases}$$

Proof. We compute the homology of $P_\bullet^{(7)}$ in a similar fashion to the proof of Theorem 6.11, by considering the long exact sequence in homology arising from the short exact sequence of chain complexes given in Theorem 4.13. Table 3 records the homology of $P_\bullet^{(6)}$ and $V_\bullet^{(7)}$, which are given in Table 2 and [EVGS13, Theorem 4.3] respectively.

Both **Case 1** ($i \neq 11, 12, 13, 18, 22, 27$) and **Case 2** ($i = 13, 18, 22, 27$) follow from the exactness of the long exact sequence on homology in a manner analogous to Cases 1 and 3 in the proof of Theorem 6.11.

For **Case 3** ($i = 11, 12$), the long exact sequence in homology gives the exact sequence

$$0 \longrightarrow H_{12}(P_\bullet^{(7)}) \longrightarrow \mathbb{Q} \xrightarrow{\delta_{12}^7} \mathbb{Q} \longrightarrow H_{11}(P_\bullet^{(7)}) \longrightarrow 0.$$

Now δ_{12}^7 is either an isomorphism or it is the zero map. As discussed in [EVGS13, §6.3], the nontrivial homology class in $H_{12}(V_\bullet^{(7)})$ is the inflation of a nontrivial homology class in $H_{11}(V_\bullet^{(6)})$. However, since by the proof of Theorem 6.11, the nontrivial homology class in $H_{11}(P_\bullet^{(6)})$ is the

i	$H_i(P_\bullet^{(6)})$	$H_i(P_\bullet^{(7)})$	$H_i(V_\bullet^{(7)})$
≥ 28	0	0	0
27	0	$\mathbb{Q}$	$\mathbb{Q}$
26	0	0	0
25	0	0	0
24	0	0	0
23	0	0	0
22	0	$\mathbb{Q}$	$\mathbb{Q}$
21	0	0	0
20	0	0	0
19	0	0	0
18	0	$\mathbb{Q}$	$\mathbb{Q}$
17	0	0	0
16	0	0	0
15	0	0	0
14	0	0	0
13	0	$\mathbb{Q}$	$\mathbb{Q}$
12	0	0	$\mathbb{Q}$
11	$\mathbb{Q}$	0	0
10	0	0	0
9	0	0	0
≤ 8	0	0	0

TABLE 3. The long exact sequence in homology for $g = 7$.

nontrivial homology class in $H_{11}(V_\bullet^{(6)})$, this implies that $H_{12}(V_\bullet^{(7)})$ is generated by the inflation of the nontrivial class $H_{11}(P_\bullet^{(6)})$. By the proof of the acyclicity of the inflation complex $I^{(g)}$ (Theorem 6.11), this implies the connecting map δ_{12}^7 is an isomorphism. The exact sequence above then implies $H_k(P_\bullet^{(7)}) = 0$ for $i = 11$ and $i = 12$.

$\square$

Theorem A now follows directly from Theorems 6.6, 6.9, 6.11, and 6.12. As a corollary of this we are able to deduce the top-weight Euler characteristic of $\mathcal{A}_g$ for $2 \leq g \leq 7$.

Corollary 6.13. The top-weight Euler characteristic of $\mathcal{A}_g$ for $2 \leq g \leq 7$ is

$$\chi^{\mathrm{top}}(\mathcal{A}_g) = \begin{cases} 1 & \text{if } g = 3, 5 \\ 0 & \text{if } g = 2, 4, 6, 7. \end{cases}$$

Remark 6.14. One can also deduce the top-weight Euler characteristic of $\mathcal{A}_g$ for $5 \leq g \leq 7$ directly from the numbers listed in [EVGS13, Figures 1 and 2]. It would be interesting to know whether a closed formula for the top-weight Euler characteristic of $\mathcal{A}_g$ exists in general.

Remark 6.15. We have established

(7) $$\mathrm{Gr}^W_{(g+1)g} H^{g(g-1)}(\mathcal{A}_g; \mathbb{Q}) \neq 0$$

for $g = 3, 5, 6$, and 7 ($g = 3$ also follows from [Hai02]). We ask whether (7) holds for all $g \geq 5$. Equivalently, the question is whether $H_{2g-1}(P^{(g)}) \neq 0$ for all $g \geq 5$. The connection to the stable cohomology of the Satake compactification, as summarized in Table 4, gives evidence for

this question, as explained in Section 7; see Question 7.1. We also note the possible relationship with the main theorems of [CGP18] on the rational cohomology of $\mathcal{M}_g$, which use the fact that $H_{2g-1}(G^{(g)}) \neq 0$ for $g = 3$ and $g \geq 5$ ([Bro12], [Wil15], see [CGP18, Theorem 2.7]). We leave this interesting investigation as an open question.

6.5. **Results for $g \geq 8$.** While full calculations for the top-weight cohomology of $\mathcal{A}_g$ in the range $g \geq 8$ are beyond the scope of current computations, we can nevertheless use our previous computation of the top-weight cohomology of $\mathcal{A}_7$ together with a vanishing result of [SEVKM19] to show that the top-weight cohomology of $\mathcal{A}_8, \mathcal{A}_9$, and $\mathcal{A}_{10}$ vanishes in a certain range slightly larger than what is given by the virtual cohomological dimension.

Theorem 6.16. The top-weight rational cohomology of $\mathcal{A}_8$, $\mathcal{A}_9$, and $\mathcal{A}_{10}$ vanishes in the following ranges:

$$\mathrm{Gr}_{72}^W H^i(\mathcal{A}_8; \mathbb{Q}) = 0 \quad \text{for } i \geq 60$$
$$\mathrm{Gr}_{90}^W H^i(\mathcal{A}_9; \mathbb{Q}) = 0 \quad \text{for } i \geq 79$$
$$\mathrm{Gr}_{110}^W H^i(\mathcal{A}_{10}; \mathbb{Q}) = 0 \quad \text{for } i \geq 99.$$

Proof. By Theorem 4.5 of [SEVKM19] for $g = 8, 9$, and 10 the homology $H_i(V_\bullet^{(g)}) = 0$ for $i \leq 11$, and further, $H_{12}(V_\bullet^{(8)}) = 0$. Considering the long exact sequence in homology:

$$\cdots \longrightarrow H_{i+1}\left(V_\bullet^{(g)}\right) \overset{\delta}{\longrightarrow} H_i\left(P_\bullet^{(g-1)}\right) \longrightarrow H_i\left(P_\bullet^{(g)}\right) \longrightarrow H_i\left(V_\bullet^{(g)}\right) \longrightarrow \cdots$$

coming from the short exact sequence of chain complexes given in Theorem 4.13, we see that this vanishing implies that

$$H_i(P_\bullet^{(7)}) \cong H_i(P_\bullet^{(8)}) \cong H_i(P_\bullet^{(9)}) \cong H_i(P_\bullet^{(10)})$$

for $i \leq 10$ and $H_{11}(P_\bullet^{(7)}) \cong H_{11}(P_\bullet^{(8)})$. By our computation of the homology of $P_\bullet^{(7)}$ in the proof of Theorem 6.12 we know that $H_i(P_\bullet^{(7)}) = 0$ for all $i \leq 12$, implying that for $g = 8, 9$, and 10, the homology $H_i(P_\bullet^{(g)}) = 0$ for $i \leq 10$, and further, $H_{11}(P_\bullet^{(8)}) = 0$. The result now follows from Proposition 4.4. $\qquad\square$

Remark 6.17. These vanishing bounds for $g = 8, 9, 10$ are slightly larger than the bounds provided by Corollary 5.16, equivalently, the fact that vcd $\mathcal{A}_g = g^2$ (see Remark 5.18), which imply that

$$\mathrm{Gr}_{72}^W H^i(\mathcal{A}_8; \mathbb{Q}) = 0 \quad \text{for } i \geq 65$$
$$\mathrm{Gr}_{90}^W H^i(\mathcal{A}_9; \mathbb{Q}) = 0 \quad \text{for } i \geq 82$$
$$\mathrm{Gr}_{110}^W H^i(\mathcal{A}_{10}; \mathbb{Q}) = 0 \quad \text{for } i \geq 101.$$

7. RELATIONSHIP WITH THE STABLE COHOMOLOGY OF $\mathcal{A}_g^{\mathrm{Sat}}$.

Our results on the existence of certain top-weight cohomology classes of $\mathcal{A}_g$ can be related to results of Chen-Looijenga [CL17] and Charney-Lee [CL83], which predict that, as g grows, there should be infinitely many of these classes. This connection was brought to our attention by O. Tommasi, and we thank her for explaining her ideas to us in detail.

Recall that $\mathcal{A}_g$ admits a compactification $\mathcal{A}_g^{\mathrm{Sat}}$, called the Satake or Baily-Borel compactification, first constructed as a projective variety by Baily and Borel in [BB66]. This compactification can be seen as a minimal compactification in the sense that it admits a morphism from all toroidal compactifications of $\mathcal{A}_g$. The reader interested in learning more about the vast literature on $\mathcal{A}_g$ and its compactifications can look at the very nice surveys [Gru09] and [HT18]. There are natural maps

$\mathcal{A}_g^{\text{Sat}} \to \mathcal{A}_{g+1}^{\text{Sat}}$, and the groups $H^k(\mathcal{A}_g^{\text{Sat}}; \mathbb{Q})$ stabilize for $k < g$ [CL83]. Moreover, as Charney-Lee prove, the stable cohomology ring $H^\bullet(\mathcal{A}_\infty^{\text{Sat}}; \mathbb{Q})$ of the Satake compactifications is freely generated by the classes λ_i for i odd, and the classes y_{4j+2} for $j = 1, 2, 3, \ldots$, where y_{4j+2} is in degree $4j + 2$. Here, the λ-classes extend the ith Chern class of the Hodge bundle on $\mathcal{A}_g$; in particular they are algebraic, and hence never have weight 0. But the classes y_j have weight 0, as proven recently by Chen-Looijenga [CL17]. This result is very important in the discussion that follows.

Recall also that $\mathcal{A}_g^{\text{Sat}}$ admits a stratification by locally closed substacks

$$\mathcal{A}_g^{\text{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0.$$

Thus the spectral sequence on compactly supported cohomology associated to this stratification is

$$(8) \qquad E_{p,q}^1 = H_c^{p+q}(\mathcal{A}_p; \mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{A}_g^{\text{Sat}}; \mathbb{Q}),$$

where $p = 0, \ldots, g$. This spectral sequence may be interpreted in the category of mixed Hodge structures. Passing to the weight 0 subspace, we see that the existence of the products of the y_j classes in the stable cohomology ring of the Satake compactification implies the existence of infinitely many cohomology classes in $\text{Gr}_0^W H_c^j(\mathcal{A}_g; \mathbb{Q})$ for all g, and hence by the perfect pairing

$$(9) \qquad \text{Gr}_0^W H_c^j(\mathcal{A}_g; \mathbb{Q}) \times \text{Gr}_{(g+1)g}^W H^{(g+1)g-j}(\mathcal{A}_g; \mathbb{Q}) \to \mathbb{Q}$$

provided by Poincaré duality, infinitely many classes $\text{Gr}_{(g+1)g}^W H^*(\mathcal{A}_g; \mathbb{Q})$ in top weight.

With Poincaré duality applied, all of the known results on the top-weight cohomology of $\mathcal{A}_g$, including our Theorems 6.9, 6.11, and 6.12, can thus be summarized in Table 4, which shows the weight 0 part of the E^1 page of the spectral sequence (8). (Implicit in Table 4 is the fact that all terms below the p-axis are zero. This follows from the fact that $\text{vcd}(\mathcal{A}_g) = \text{vcd}(\text{Sp}(2g, \mathbb{Z})) = g^2$.)

As explained to us by Tommasi, the classes in Theorems 6.9, 6.11, and 6.12, as well as the already-known class in $\text{Gr}_{12}^W H^6(\mathcal{A}_3; \mathbb{Q})$ [Hai02] give natural candidates for classes in $\text{Gr}_0^W H_c^{p+q}(\mathcal{A}_p; \mathbb{Q})$ that produce the classes y_{4j+2} in the spectral sequence (8), in the sense that they persist in the $E_{p,q}^\infty$ page in the Gysin spectral sequence for g sufficiently large. Indeed, looking at the $p = q$ diagonal on the E^1 page of the spectral sequence in Table 4, we are led to ask:

Question 7.1.

    (1) Is $\text{Gr}_0^W H_c^{2g}(\mathcal{A}_g; \mathbb{Q}) \neq 0$ for $g = 3$ and all $g \geq 5$?

    (2) Moreover, do these cohomology classes produce the stable cohomology classes in $\text{Gr}_0^W H^\bullet(\mathcal{A}_\infty^{\text{Sat}}; \mathbb{Q})$?

    (3) Is $\text{Gr}_0^W H_c^k(\mathcal{A}_g; \mathbb{Q}) = 0$ for $k < 2g$?

Our Theorems 6.2, 6.9, 6.11 and 6.12 verify the first and third questions for $g \leq 7$. They also verify the second question for $g = 3$ and for $g = 5$. Indeed, $\text{Gr}_0^W H_c^6(\mathcal{A}_3; \mathbb{Q})$ and $\text{Gr}_0^W H_c^{10}(\mathcal{A}_5; \mathbb{Q})$ are the only nonzero terms in the antidiagonals $p + q = 6$ and $p + q = 10$, respectively; so they produce the classes $y_6 \in \text{Gr}_0^W H^6(\mathcal{A}_\infty^{\text{Sat}}; \mathbb{Q})$ and $y_{10} \in \text{Gr}_0^W H^{10}(\mathcal{A}_\infty^{\text{Sat}}; \mathbb{Q})$, respectively. It is natural to guess that the other terms in Table 4 similarly produce products of y_j's: for example, that $\text{Gr}_0^W H_c^{12}(\mathcal{A}_6; \mathbb{Q})$ produces y_6^2, and that $\text{Gr}_0^W H_c^{14}(\mathcal{A}_7; \mathbb{Q})$ produces y_{14}, and so on.

Finally, Tommasi also remarks that the odd degree classes in weight 0 compactly supported cohomology of $\mathcal{A}_g$ detected so far, namely

$$\text{Gr}_0^W H_c^{15}(\mathcal{A}_5; \mathbb{Q}), \text{Gr}_0^W H_c^{19}(\mathcal{A}_7; \mathbb{Q}), \text{ and } \text{Gr}_0^W H_c^{23}(\mathcal{A}_7; \mathbb{Q}),$$

must of course be killed by a differential on some page of the spectral sequence, since $\mathcal{A}_g^{\text{Sat}}$ has no weight 0 stable cohomology in odd degrees. This implies the existence of some even degree classes in $\text{Gr}_0^W H_c^\bullet(\mathcal{A}_g; \mathbb{Q})$ which kill the odd degree classes and which are not related by this spectral sequence to the products of y_js. It would be very interesting to explicitly identify such classes.

	0	1	2	3	4	5	6	7	8	9	10
21	0	0	0	0	0	0	0	$\mathbb{Q}$			
20	0	0	0	0	0	0	0	0			
19	0	0	0	0	0	0	0	0			
18	0	0	0	0	0	0	0	0			
17	0	0	0	0	0	0	0	0			
16	0	0	0	0	0	0	0	$\mathbb{Q}$			
15	0	0	0	0	0	0	0	0			
14	0	0	0	0	0	0	0	0			
13	0	0	0	0	0	0	0	0			
12	0	0	0	0	0	0	0	$\mathbb{Q}$			
11	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	$\mathbb{Q}$	0	0			
9	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	0	0			
7	0	0	0	0	0	0	0	$\mathbb{Q}$			
6	0	0	0	0	0	0	$\mathbb{Q}$	0			
5	0	0	0	0	0	$\mathbb{Q}$	0	0			
4	0	0	0	0	0	0	0	0	0		
3	0	0	0	$\mathbb{Q}$	0	0	0	0	0		
2	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

TABLE 4. The page $E^1_{p,q} = \mathrm{Gr}^W_0 H^{p+q}_c(\mathcal{A}_p; \mathbb{Q}) \Rightarrow \mathrm{Gr}^W_0 H^{p+q}(\mathcal{A}^{\mathrm{Sat}}_g; \mathbb{Q})$ of the Gysin spectral sequence, for g sufficiently large. The blank entries for $p \geq 8$ are currently unknown.

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