Juliette Bruce's Research Statement

My research interests lie in algebraic geometry, commutative algebra, and arithmetic geometry. In particular, I am interested in using homological and combinatorial methods to study the geometry of zero loci of systems of polynomials (i.e. algebraic varieties). I am also interested in studying the arithmetic properties of varieties over finite fields. Further, I am passionate about promoting inclusivity, diversity, and justice in the mathematics community. Broadly speaking my current research follows these ideas in two directions.

- **Homological Algebra on Toric Varieties:** A classical story in algebraic geometry is that homological methods and tools like minimal free resolutions and Caastelunouvo–Mumford regularity capture the geometry of subavarieties of projective space in nuanced ways. Much of my work has sought to generalize this story by developing ways homological algebra can be used to study the geometry of toric varieties (i.e., "nice' compactifications of the torus $(\mathbb{C}^{\times})^n$). Such generalizations are closely connected to multigraded commutative algebra and representation theory.
- **Cohomology of Moduli Spaces and Arithmetic Groups:** One of the most classically studied objects in algebraic geometry is he moduli space of abelian varieties of dimension g, which we denote $\mathcal{A}_g$. Recently my work has focused on developing ways to compute a canonical "part" of the cohomology of $\mathcal{A}_g$, which we call the top-weight cohomology. This turns out to be closely connected to the study of cohomology of various arithmetic groups like $\mathrm{GL}_g(\mathbb{Z})$, $\mathrm{Sp}_{2g}(\mathbb{Z})$, and $\mathrm{GL}_g(\mathbb{Z})(m)$, as well as the study of automorphic forms.

1. Homological Algebra on Toric Varieties

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_i is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring R with a twisted grading, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j} is the graded piece of degree $d-j$. The $\beta_{i,j}$'s are the *Betti numbers* of M, and they count the number of i-syzygies of M of degree j. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [Voi02, Voi05, AFP+19]. See also [Eis05, Conjecture 9.6] and [Sch86, BE91, FP05, Far06, AF11, FK16, FK17].

Much of my work can be viewed as understanding how minimal graded free resolutions capture the geometry when the role of $\mathbb{P}^r$ is replaced by another variety Y. In particular, I have focused on the case when Y is a toric variety, i.e. a compactification of the torus $(\mathbb{C}^n)^r$ where the action of the torus extends to the boundary. Toric varieties are a paritulcarly nice class of varieites to look for such generalizations

1.1 Asymptotic Syzygies Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84a, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question: In what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set,

$$\rho_q\left(X, L_d\right) \;:=\; \frac{\#\left\{p \in \mathbb{N} \mid \big| \beta_{p,p+q}\left(X, L_d\right) \neq 0\right\}}{r_d},$$

which is the percentage of degrees in which non-zero syzygies appear [Eis05, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84a] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that $\deg L_d = d$ then*

$$\lim_{d\to\infty} \rho_2\left(X; L_d\right) = 0.$$

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q\left(X; L_d\right) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q\left(X; \mathcal{O}\left(d_1, d_2\right)\right) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\, j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.}\\ \textit{terms}\end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [Lem18] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right)$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [EL12, Conjectures 7.1, 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld. Such a generalization is complicated by the difference between the Cox ring and homogenous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$. A central theme in this work is to exploit the fact that a key regular sequence I use has a number of non-trivial symmetries.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety and fix an index $1 \leq q \leq \dim X$. Let $(L_d)_{d\in\mathbb{N}}$ be a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample, can one compute $\lim_{d\to\infty} \rho_q(X; L_d)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru22a].

1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. My co-authors and I exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. A follow-up project used similar computational approaches to compute the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$ in over 200 new examples. This data provided support for several existing conjectures, and led to a number of new conjectures [BEGY20, BEGY21, BCE+22].

1.3 Multigraded Castelnuovo–Mumford Regularity Introduced by Mumford, the Castelnuovo–Mumford Regularity of a projective variety $X \subset \mathbb{P}^n$ is a measure of the complexity of X given in terms of the vanishing of certain cohomology groups of X. Roughly speaking one should think about Castelnuovo–Mumford regularity as being a numerical measure of geometric complexity. Mumford was interested in such a measure as it plays a key role in constructing Hilbert and Quot schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. However, Eisenbud and Goto showed that regularity is also closely connected to interesting homological properties.

Theorem 1.5. [EG84] *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $M = \bigoplus_{e\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent:*

(1) M is d-regular;
(2) $\beta_{p,q}(M) = 0$ for all $p \geq 0$ and $q > d + i$;
(3) $M_{\geq d}$ has a linear resolution.

Maclagan and Smith introduced a generalized notion of Castelnuovo–Mumford regularity, which they call multigraded Castelnuovo–Mumford regularity, where $\mathbb{P}^n$ can be replaced by any toric variety. Similarly to the definition in the classical setting multigraded Castelnuovo–Mumford regularity is defined in terms of the vanishing of certain cohomology groups. Crucially, however, the multigraded Castelnuovo–Mumford regularity of a subvariety or module is not a single number, but instead an infinite subset of $\mathbb{Z}^r$.

As an example, let us consider the case of products of projective spaces. Fixing a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ we let $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ and $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\mathrm{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$. Fixing some notation given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let:

$$L_i(\mathbf{d}) := \bigcup_{\substack{\mathbf{v}\in\mathbb{N} \\ |\mathbf{v}|=i}} (\mathbf{d} - \mathbf{v}) + \mathbb{N}^r.$$

Note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i + 1)$-corners. Roughly speaking we define regularity by requiring the i-th cohomology of certain twists of $\mathcal{F}$ to vanish on L_i.

Definition 1.6. [MS04, Definition 6.1] *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if*

$$H^i(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})) = 0 \qquad \text{for all } \mathbf{e} \in L_i(\mathbf{d}).$$

The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:

$$\operatorname{reg}(\mathcal{F}) \coloneqq \left\{ \mathbf{d} \in \mathbb{Z}^r \;\middle|\; \mathcal{F} \textit{ is } \mathbf{d}\textit{-regular} \right\} \subset \mathbb{Z}^r.$$

The obvious approaches to generalize Theorem 1.5 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity [BCHS21, Example 5.1] Despite this we show that part (3) of Theorem 1.5 can be generalized. To do so we introduce the following generalization of linear resolutions.

Definition 1.7. *A complex $F_\bullet$ of $\mathbb{Z}^r$-graded free S-modules is $\mathbf{d}$-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d}-\mathbf{1})$.*

Theorem 1.8. [BCHS21, Theorem A] *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$:*

$$M \textit{ is } \mathbf{d}\textit{-regular} \iff M_{\geq \mathbf{d}} \textit{ has a } \mathbf{d}\textit{-quasilinear resolution.}$$

The proof of Theorem 1.8 is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel. Precisely, if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$ we prove the that

$$\dim_{\mathbb{C}} \operatorname{Tor}^S_j(M_{\geq \mathbf{d}}, \mathbb{C})_{\mathbf{a}} = h^{|\mathbf{a}|-j}\left(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{O}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\right) \quad \text{for } |\mathbf{a}| \geq j \geq 0,$$

where the $\mathcal{O}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The result then follows from showing that M being $\mathbf{d}$-regular is equivalent to certain vanishings of the right-hand side above.

Recent breakthroughs using symplectic geometry to understand exceptional connections and resolutions of the diagonal on arbitrary toric varieties mean that there is hope one may be able to generalize the above argument to arbitrary toric varieties. With this in mind, an interesting question I am pursing is the following.

Question 1.9. *How can Theorem 1.8 be generalized to an arbitrary smooth projective toric varieties? in particular, what is the correct definition of quasilinear resolutions?*

1.3.1 Multigraded Regularity of Powers of Ideals Building on the work of many people [BEL91, Cha97], Cutkosky, Herzog, Trung [CHT99] and independently Kodiyalam [?kodiyalam00] showed the Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^n$ has surprisingly predictable asymptotic behavior. In particular, given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$, there exist constants $d, e \in \mathbb{Z}$ such that $\operatorname{reg}(I^t) = dt + e$ for $t \gg 0$.

Building upon our work discussed above, my collaborators and I generalized this result to arbitrary toric varieties. In particular, Definition 1.6 can be extended to all toric varieties by letting S be Cox ring of the toric variety X, replacing $\mathbb{Z}^r$ with the Picard group of X, and replacing $\mathbb{N}^r$ with the nef cone of X. My collaborators and I show that the multigraded regularity of powers of ideals is bounded and translates in a predictable way. In particular, the regularity of I^t essentially translates within $\operatorname{Nef} X$ in fixed directions at a linear rate.

Theorem 1.10. [BCHS22, Theorem 4.1] *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on I, such that for each integer $t > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators f_i of I, we have*

$$t\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}\left(I^t\right) \subseteq t\mathbf{q}_2 + \operatorname{Nef} X.$$

A key aspect of the proof of this theorem is showing that the multigraded regularity of an ideal is finitely generated, in the sense that there exists vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^r$ such that $\mathbf{v} + \operatorname{Nef} X \subset \operatorname{reg}(I) \subset \mathbf{w} + \operatorname{Nef} X$. Perhaps somewhat surprisingly, my co-authors and I showed that this can fail for arbitrary modules [BCHS22].

Question 1.11. *Let X be a smooth projective toric variety. Can one characterize when* $\operatorname{reg}(M)$ *is finitely generated for a module M over the Cox ring of X?*

An first toy case of this question that I think is particular, interesting and feel would make a lovely first research project for a student is to attempt to answer Question 1.11 when M is the Cox ring of a torus fixed-point. In this special case, the question seems to reduce down to a delicate combinatorial question about counting lattice points and vector partition functions.

2. Cohomology of Moduli Spaces and Arithmetic Groups

Some of the most classical objects in algebraic geometry are moduli spaces, i.e., spaces which parameterize a given collection of geometric objects. Example of such include: the moduli space of (smooth) genus g curves which we denote by $\mathcal{M}_g$, the moduli space of abelian varieties of dimension g which we denote by $\mathcal{A}_g$. Despite their classical nature much remains unknown about the geometry of both of these spaces. For example, the rational cohomology of $\mathcal{M}_g$ is only known for $g \leq 4$. Moreover, classical results suggest that $\mathcal{M}_g$ should have a lot cohomology because the Euler characteristic of $\mathcal{M}_g$ grows super exponentially. Recent groundbreaking work of Chan, Galatius, and Payne has shed the first direct light on this phenomena by not only constructing new non-trivial cohomology classes, but also showing that the dimension of certain cohomology groups of $\mathcal{M}_g$ grow at least exponentially.

Theorem 2.1. *For $g \geq 2$ the dimension of $H^{4g-6}(\mathcal{M}_g;\mathbb{Q})$ grows at least exponentially. More precisely,*

$$\dim H^{4g-6}\left(\mathcal{M}_g;\mathbb{Q}\right) > \beta^g + \textit{constant}$$

for any real number $\beta < \beta_0$ where $\beta_0 \approx 1.3247\ldots$ is the real solution of $t^3 - t - 1 = 0$.

2.1 Cohomology of $\mathcal{A}_g$ The moduli space of (principally polarized) abelian varieties of dimension g, is a smooth variety $\mathcal{A}_g$ (truthfully a smooth Delinge–Mumford stack) whose points are in one to one correspondence with isomorpism classes of principally polarized abelian varieties of dimension g. Concretely, we may view it as the quotient $[\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})]$ where H_g is the Siegel upper half-space. Notice this means that $\mathcal{A}_g$ can also be viewed is a rational classifying space for the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Similar to the moduli space of curves $\mathcal{A}_g$ has long been studied, but much remains unknown about its geometry. For example the (singular) cohomology is $\mathcal{A}_g$ is only fully know when $g = 0$ in which case A_g is a single point, $g = 1$ blah blah, $g = 2$ which is a classical result of NEDEDED and $g = 3$ due to work of Hain. In fact the cohomology of $\mathcal{A}_g$ is so mysterious until recently it was not know whether $H^i(\mathcal{A}_g;\mathbb{Q}) \neq 0$ for some g and some odd i. This was a question posed by Gruskestkky that my recent work answered. Building upon the work of Chan, Galatius, and Payne, my co-authors and I developed new methods for understanding a certain canonical quotient of the cohomology of $\mathcal{A}_g$. In particular, our results construct non-trivial cohomology class for $H^k(\mathcal{A}_g;\mathbb{Q})$ in a large number of new cases.

Theorem 2.2. *The rational cohomology $H^k\left(\mathcal{A}_g;\mathbb{Q}\right) \neq 0$ for:*

$$(g,k) = (5,15), (5,20), (6,30), (7,28), (7,33), (7,37), \textit{ and } (7,42).$$

For broader context, since $\mathcal{A}_g$ is a rational classifying space for $\mathrm{Sp}_{2g}(\mathbb{Z})$ there is natural isomorphism $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$. In particular, the above results provide new non-vanishing results for $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$. In fact, our methods show an even more mysterious NEDEDEDEDDEDE

The situation is analogous to that of the moduli space of curves $\mathcal{M}_g$, which is a rational classifying space for the mapping class group Mod_g via its action on Teichmüller space. Moreover, in both cases, we find ourselves in the advantageous situation that $\mathcal{M}_g$ and $\mathcal{A}_g$ are smooth and separated Deligne

Mumford stacks with coarse moduli spaces which are algebraic varieties, permitting Deligne's mixed Hodge theory to be applied to study the rational cohomology of these groups.

In particular, the rational cohomology of a complex algebraic variety X of dimension d admits a weight filtration with graded pieces $\mathrm{Gr}_j^W H^k(X;\mathbb{Q})$. As $\mathrm{Gr}_j^W H^k(X;\mathbb{Q})$ vanishes whenever $j > 2d$, $\mathrm{Gr}_{2d}^W H^k(X;\mathbb{Q})$ is referred to as the *top weight* part of $H^k(X;\mathbb{Q})$.) Theorem NEDED above is actually based upon a more prcisely NEDEDEDED

2.2 Cohomology of $\mathcal{A}_g$ and $\mathrm{GL}_g(\mathbb{Z})(m)$ Note that the cohomology of $\mathrm{Sp}(2g,\mathbb{Z})(m)$ – and hence $\mathcal{A}_g(m)$ – is closely connected to study of automorphic forms. Thus it is natural to wonder whether our methods for computing the top-weight cohomology of $\mathcal{A}_g(m)$ shed new light on automorphic forms. In particular, the top-weight cohomology of $\mathcal{A}_g(m)$ comes from understanding the boundary of a locally symmetric space one may hope it is related to Siegel–Eisenstein series. An ongoging project and conversations with Melody Chan and Peter Sarnak hopes to address this question.

Question 2.3. *What is the relationship between the top-weight cohomology of $\mathcal{A}_g(m)$ and Siegel Eisenstein series?*

2.3 Matroid Complexes and Cohomology of $\mathcal{A}_g^{\mathbf{mat}}$ Together with my co-authors I identified a subcomplex $R_g^\bullet$ with even richer combinatorics EDEDED

Goal Theorem 2.4. *Compute the homology of the matroid complex $R_\bullet^{(g)}$ for all $g \geq 14$.*

This is ongoing work with three graduate students, where currently we combined theoretical results and large-scale computations to compute the cohomology for all $g \leq 9$.

Computing the homology of the regular matroid complexe is interesting, not only because it provides a new approach for studying the combinatorics of matroids, but also because it is closely related to the cohomology of partial compactification of $\mathcal{A}_g$. x

Goal Theorem 2.5. *The top-weigh cohomology of $\mathcal{A}_g^{\mathrm{mat}}$ can be computed from the cohomlogy of $R_\bullet^{(g)}$.*

Work of Whitwaker and Kontsevich on graph complexe also lea to the following natural question. If answered in the affirmative this would likely allow for any known cohomlogy classes to be spread out and converted in statements about the dimension of top-weight cohomology for more g.

Question 2.6. *Does the complex $\mathcal{R}_\bullet$ carry a natural Lie bracket, endowing it with the structure of a differentially graded Lie algebra?*

Constructing such a Lie bracket likely relies on developing a new understanding of the ways one can combine two matroids. Ongoing work with the graduate students mentioned above is studying this problem in the special cases of graphic and co-graphic matroids, however, for more general matroids such a constriction remains mysterious.

3. Varieties over Finite Fields

Over a finite field, a number of classical statements from algebraic geometry no longer hold. For example, if $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension n over $\mathbb{C}$, Bertini's theorem states that, if $H \subset \mathbb{P}^r$ is a generic hyperplane, then $X \cap H$ is smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Using an ingenious probabilistic sieving argument, Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree, then a version of Bertini's theorem is true [Poo04]. More specifically Poonen showed that as, $d \to \infty$, the percentage of hypersurfaces $H \subset \mathbb{P}^r_{\mathbf{F}_q}$ of degree d such that $X \cap H$ is smooth is determined by the Hasse-Weil zeta function of X. Below we write $\mathbf{F}_q[x_0, \ldots, x_r]_d$ for the $\mathbf{F}_q$-vector space of homogenous polynomials of degree d.

Theorem 3.1. [Poo04, Theorem 1.1] *Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth variety of dimension n. Then:*

$$\lim_{d\to\infty} \text{Prob}\left(\begin{matrix} f \in \mathbf{F}_q[x_0, x_1, \ldots, x_r]_d \\ X \cap \mathbb{V}(f) \textit{ is smooth of dimension } n-1 \end{matrix}\right) = \zeta_X(n+1)^{-1} > 0. \tag{1}$$

3.1 A Probabilistic Study of Systems of Parameters Given an n dimensional projective variety $X \subset \mathbb{P}^r$, a collection of homogenous polynomials $f_0, f_1, \ldots, f_k$ of degree d is a (partial) system of parameters if $\dim X \cap \mathbb{V}(f_0, f_1, \ldots, f_k) = \dim X - (k+1)$. Systems of parameters are closely tied to Noether normalization, as the existence of a finite (i.e. surjective with finite fibers) map $X \to \mathbb{P}^n$ is equivalent to the existence of a system of parameters of length $n+1$.

Inspired by work of Poonen [Poo04] and Bucur and Kedlaya [BK12], Daniel Erman and I computed the asymptotic probability that randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ over $\mathbf{F}_q$ form a system of parameters. By adapting Poonen's closed point sieve to sieve over higher dimensional varieties, we showed that, when $k < n$, the probability that randomly chosen $f_0, f_1, \ldots, f_k$ form a partial system of parameters is controlled by a zeta-function-like power series that enumerates higher dimensional varieties instead of closed points. In the following, $|Z|$ denotes the number of irreducible components of Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

Theorem 3.2. [BE19, Theorem 1.4] *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\text{Prob}\left(\begin{matrix} f_0, f_1, \ldots, f_k \textit{ of degree } d \\ \textit{are parameters on } X \end{matrix}\right) = 1 - \sum_{\substack{Z \subseteq X \textit{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)\binom{n-k+d}{n-k}}\right).$$

From this we proved the first explicit bound for Noether normalization over $\mathbf{F}_q$ and gave a new proof of recent results on Noether normalizations of families over $\mathbb{Z}$ and $\mathbf{F}_q[t]$ [GLL15, CMBPT17].

3.2 Jacobians Covering Abelian Varieties Over an infinite field, it is a classic result that every abelian variety is covered by a Jacobian variety of bounded dimension. Building upon work of Bucur and Kedlaya [BK12], Li and I proved an analogous result for abelian varieties over finite fields. We did so by first proving an effective version of Poonen's Bertini theorem over finite fields.

Theorem 3.3. [BL20, Theorem A] *Fix $r, n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbf{F}_q$ be a finite field of characteristic p. There exists an explicit constant $C_{r,q}$ such that if $A \subset \mathbb{P}^r_{\mathbf{F}_q}$ is a non-degenerate abelian variety of dimension n, then for any $d \in \mathbb{N}$ satisfying*

$$C_{r,q}\zeta_A\left(n + \tfrac{1}{2}\right)\deg(A) \leq \frac{q^{\overline{\frac{d}{\max\{n+1,p\}}}}\, d}{d^{n+1} + d^n + q^d},$$

there exists a smooth curve over $\mathbf{F}_q$ whose Jacobian J maps surjectively onto A, where

$$\dim J \leq \mathrm{O}\left(\deg(A)^2 d^{2(n-1)} r^{-1}\right).$$

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