Juliette Bruce's Research Statement

My research interests lie in pure mathematics, more specifically, in algebraic geometry, commutative algebra, and arithmetic geometry. Broadly these fields make use of deep connections between geometry and algebra to study the solutions of systems of polynomial equations. Further, I am passionate about promoting inclusivity, diversity, and justice in the mathematics community.

1. Syzygies in Algebraic Geometry

One might recall that given a list of numbers $p_1, p_2, \ldots, p_d$ there always exists a polynomial of degree less than or equal to d, $a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, such that the solutions to the equation $a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 = 0$ is exactly $p_1, p_2, \ldots, p_d$. For example, if $d = 2$ and the list of numbers is $4, 9$ then the polynomial $x^2 + 13x + 26$ has zeros exactly when $x = 4$ and $x = 9$.

It is natural to then wonder what happens in higher dimensions, that is, given a list of points $p_1, p_2, \ldots, p_d$ in $\mathbb{R}^r$ for what D can we guarantee that there exists a multivariate polynomial $F(\mathbf{x})$ of degree less than or equal to D such that $F(p_1) = F(p_2) = \cdots = F(p_d) = 0$. Note that in higher dimensions the set of solutions to the equation $F(\mathbf{x}) = 0$ will be higher higher dimensional, and so we can no longer insist that the only solutions to $F(\mathbf{x}) = 0$ are our points $p_1, p_2, \ldots, p_d$.

that given d points in $\mathbb{C}^r$ there is a polynomial of degree $< D$ passing through each of the points? – it turns out that the answer depends not just on the number of points, but also on the geometry of how the points are arranged in space. However, fascinatingly, the way to realize this connection with geometry, and answer the question, is through algebra by studying the polynomial relations amongst collections of polynomials.

While this result is well know, in general polynomial relations amongst a collection of polynomials – which are called syzygies – are extremely mysterious. For example, once one moves beyond points (0 dimensional) and curves (1 dimensional), and begins to study surfaces (2 dimensional) not only is little known, but only a handful of examples have ever been computed. This is because the complexity of such computations grows rapidly and quickly overwhelms our computational powers.

I will discuss how the emergence of highly efficient numerical linear algebra algorithms together with high performance, massively distributed computing has provided a new approach for such computations. In particular, by making use of these tools I – with several coauthors and several thousand computers – have managed to compute previously unknown examples. These examples have begun shedding light on these higher dimensional syzygies, and led us to put forth several new, and perhaps surprising, conjectures.

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_i is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring R with a twisted grading, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j} is the graded piece of degree $d-j$. The $\beta_{i,j}$'s are the *Betti numbers* of M, and they count the number of i-syzygies of M of degree j. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [Voi02, Voi05, AFP+19]. See also [Eis05, Conjecture 9.6] and [Sch86, BE91, FP05, Far06, AF11, FK16, FK17].

1.1 Asymptotic Syzygies Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question: In what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set,

$$\rho_q(X, L_d) \;\; := \;\; \frac{\#\left\{p \in \mathbb{N} \;\middle|\; \beta_{p,p+q}(X, L_d) \neq 0\right\}}{r_d},$$

which is the percentage of degrees in which non-zero syzygies appear [Eis05, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that* $\deg L_d = d$ *then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety,* $\dim X \geq 2$, *and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q(X; L_d) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q(X; \mathcal{O}(d_1, d_2)) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\, j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.}\\ \textit{terms}\end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [Lem18] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2))$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [EL12, Conjectures 7.1, 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld. Such a generalization is complicated by the difference between the Cox ring and homogenous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$. A central theme in this work is to exploit the fact that a key regular sequence I use has a number of non-trivial symmetries.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety and fix an index $1 \leq q \leq \dim X$. Let $(L_d)_{d\in\mathbb{N}}$ be a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample, can one compute $\lim_{d\to\infty} \rho_q(X; L_d)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru19b].

1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. My co-authors and I exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. This data provided support for several existing conjectures, as well as led us to make a number of new conjectures [BEGSY18]. The resulting data has been made publicly available via SyzygyData.com as well as, a package for Macaualy2 [BEGSY19, M2].

Recently I have begun using similar computational techniques to compute the syzygies for Hirzebruch surfaces. Thus far, we have computed the syzygies in ~ 100 new examples. It is our hope that these examples will lead to new conjectures regarding the syzygies of Hirzebruch surfaces. In particular, we believe our data will be useful in addressing Question 1.4.

1.3 Liaison Theory via Virtual Resolutions One approach to studying the geometry of curves is by asking when the union of two (or more) curves is a complete intersection. Such curves are said to be linked, and liaison theory studies linkage equivalence classes [PS74, Hun84, HU87, HU88]. The liaison theory of curves in $\mathbb{P}^3$ is well understood, but the same theory for curves in other 3-folds is mysterious. In work with Christine Berkesch and Patricia Klein, I hope to use the newly developed theory of virtual resolutions to better understand the liaison theory of curves in toric 3-folds.

Broadly, a virtual resolution is a homological representation of a graded module over the Cox ring of a smooth toric variety that attempts to better capture the relevant geometric information by allowing a limited amount of homology [BES17]. In particular, they seem capable of being able to capture the subtle differences between ideal and set theoretic complete intersections that arise when studying liaison on toric varieties.

Ambitiously, we hope to use virtual resolutions to find a way to classify liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$ analogous to Rao modules [PR78]. We would like to answer the following question.

Question 1.5. *What invariant classifies even liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$?*

Using virtual resolutions we have managed to answer this question under a fairly restrictive hypothesis. We hope to explore whether this hypothesis can be weakened.

For any homological property associated to free resolutions (e.g. Cohen-Macaulay), it is possible to define an analogous virtual property associated to virtual resolutions (e.g. virtual Cohen-Macaulay). We would like to prove the following virtual version of a result of Peskine and Szpiro [PS74].

Conjecture 1.6. *Let C and C' be linked curves in $\mathbb{P}^1 \times \mathbb{P}^2$. Then, C is virtually Cohen-Macaulay if and only if C' is virtually Cohen-Macaualy.*

A helpful tool in approaching these questions is the ability to compute interesting examples via the `VirtualResolutions` Macaulay2 package, which I co-authored [ABLS19].

2. Varieties over Finite Fields

Over a finite field, a number of classical statements from algebraic geometry no longer hold. For example, if $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension n over $\mathbb{C}$, Bertini's theorem states that, if $H \subset \mathbb{P}^r$ is a generic hyperplane, then $X \cap H$ is smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Using an ingenious probabilistic sieving argument, Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree, then a version of Bertini's theorem is true [Poo04]. More specifically Poonen showed that as, $d \to \infty$, the percentage of hypersurfaces $H \subset \mathbb{P}^r_{\mathbf{F}_q}$ of degree d such that $X \cap H$ is smooth is determined by the Hasse-Weil zeta function of X. Below we write $\mathbf{F}_q[x_0, \ldots, x_r]_d$ for the $\mathbf{F}_q$-vector space of homogenous polynomials of degree d.

Theorem 2.1. [Poo04, Theorem 1.1] *Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth variety of dimension n. Then:*

$$\lim_{d\to\infty} \operatorname{Prob}\begin{pmatrix} f \in \mathbf{F}_q[x_0, x_1, \ldots, x_r]_d \\ X \cap \mathbb{V}(f) \textit{ is smooth of dimension } n-1 \end{pmatrix} = \zeta_X(n+1)^{-1} > 0. \tag{1}$$

2.1 A Probabilistic Study of Systems of Parameters Given an n dimensional projective variety $X \subset \mathbb{P}^r$, a collection of homogenous polynomials $f_0, f_1, \ldots, f_k$ of degree d is a (partial) system of parameters if $\dim X \cap \mathbb{V}(f_0, f_1, \ldots, f_k) = \dim X - (k+1)$. Systems of parameters are closely tied to Noether normalization, as the existence of a finite (i.e. surjective with finite fibers) map $X \to \mathbb{P}^n$ is equivalent to the existence of a system of parameters of length $n+1$.

Inspired by work of Poonen [Poo04] and Bucur and Kedlaya [BK12], Daniel Erman and I computed the asymptotic probability that randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ over $\mathbf{F}_q$ form a system of parameters. By adapting Poonen's closed point sieve to sieve over higher dimensional varieties, we showed that, when $k < n$, the probability that randomly chosen $f_0, f_1, \ldots, f_k$ form a partial system of parameters is controlled by a zeta-function-like power series that enumerates higher dimensional varieties instead of closed points. In the following, $|Z|$ denotes the number of irreducible components of Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

Theorem 2.2. [BE, Theorem 1.4] *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\begin{pmatrix} f_0, f_1, \ldots, f_k \textit{ of degree } d \\ \textit{are parameters on } X \end{pmatrix} = 1 - \sum_{\substack{Z \subseteq X \textit{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)\binom{n-k+d}{n-k}}\right).$$

From this we proved the first explicit bound for Noether normalization over $\mathbf{F}_q$ and gave a new proof of recent results on Noether normalizations of families over $\mathbb{Z}$ and $\mathbf{F}_q[t]$ [GLL15, CMBPT17].

2.2 Jacobians Covering Abelian Varieties Over an infinite field, it is a classic result that every abelian variety is covered by a Jacobian variety of bounded dimension. Building upon work of Bucur and Kedlaya [BK12], Li and I proved an analogous result for abelian varieties over finite fields. We did so by first proving an effective version of Poonen's Bertini theorem over finite fields.

Theorem 2.3. [BL, Theorem A] *Fix $r, n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbf{F}_q$ be a finite field of characteristic p. There exists an explicit constant $C_{r,q}$ such that if $A \subset \mathbb{P}^r_{\mathbf{F}_q}$ is a non-degenerate abelian variety of dimension n, then for any $d \in \mathbb{N}$ satisfying*

$$C_{r,q}\zeta_A\left(n+\tfrac{1}{2}\right)\deg(A) \leq \frac{q^{\overline{\frac{d}{\max\{n+1,p\}}}}d}{d^{n+1}+d^n+q^d},$$

there exists a smooth curve over $\mathbf{F}_q$ whose Jacobian J maps surjectively onto A, where

$$\dim J \leq \mathrm{O}\left(\deg(A)^2 d^{2(n-1)} r^{-1}\right).$$

2.3 Uniform Bertini Notice that in the statement of Poonen's Bertini theorem, while the left-hand side of equation (1) is dependent of the embedding of X into projective space (i.e. the choice of very ample line bundle), the overall limit is itself independent of the embedding of X. This suggests that there may be a more general and uniform statement of Poonen's Bertini theorem. One might hope that the analogous limit along any sequence $(L_d)_{d\in\mathbb{N}}$ of line bundles growing in positivity equals $\zeta_X(n+1)^{-1}$. I am working with Isabel Vogt to formalize and prove such a theorem.

Work of Erman and Wood on semi-ample Bertini theorems shows that a naive analogue of Theorem 2.1 fails [EW15]. Vogt and I believe that this can be fixed by introducing an assumption on how the sequence of lines bundles grows in positivity. We say a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ *goes to infinity in all directions* if for every ample line bundle A there exists $N \in \mathbb{N}$ such that $L_i - A$ is ample for all $i \geq N$. We are working to prove the following uniform version of Theorem 2.1.

Conjecture 2.4. *Let $X/\mathbf{F}_q$ be a smooth projective variety of dimension n. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of line bundles on X going to infinity in all directions then*

$$\lim_{d\to\infty} \mathrm{Prob}\left(f \in H^0(X, L_d) \;\middle|\; \begin{array}{c} X \cap \mathbb{V}(f) \text{ is smooth} \\ \text{of dimension } n-1 \end{array}\right) = \zeta_X(n+1)^{-1}. \tag{2}$$

We have verified this conjecture in a number of examples ($X = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$). We are hopeful that similar methods will extend to whenever the nef cone of X is a finitely generated.

3. Broader Impacts

3.1 Organizing In the Spring of 2017 I organized *Math Careers Beyond Academia* (50 participants), a one-day professional development conference on STEM careers outside of academia. In April 2018 I organized *M2@UW* (45 participants), a four-day workshop focused on creating new packages for Macaulay2. In February 2019 I organized *Geometry and Arithmetic of Surfaces* (40 participants), a workshop providing a diverse group of early-career researchers the opportunity to learn about interesting topics in the arithmetic and algebraic geometry. In April 2019 I organized the *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants) focused on forming a community of women and non-binary researchers interested in commutative algebra. I organized a *Special Session on Combinatorial Algebraic Geometry* at the AMS Fall 2019 Central Sectional. At the 2020 Joint Mathematics Meetings, I am organizing a panel titled *Supporting Transgender and Non-binary Students.*

3.2 Math Circles I began volunteering with the Madison Math Circle in Fall 2014 at the time, the circle's main programming was a weekly on-campus lecture given by a member of the math department. After roughly a year I stepped into the role of organizer overseeing the administrative needs of the circle. As an organizer, I worked to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations focused on underrepresented groups. These ties helped the weekly attendance of the circle to more than double. I also led the creation of a new outreach arm of the circle, which visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This dramatically expanded the reach of the Madison Math Circle, and during my final year as an organizer the circle reached over 300 students.

3.3 Mentoring Since the Winter of 2018 I have led reading courses with three undergraduates. One of these students, an undergraduate woman, worked with me for over a year, during which time I helped her through the process of applying for summer research projects. Working with *Girls' Math Night Out* I lead two girls in high school through a project exploring RSA cryptography. During 2018-2019, I mentored 6 first-year graduate students (all women or non-binary students). I am currently mentoring two undergraduate women via the AWM's Mentoring Network.

3.4 A More Inclusive Community. In Fall of 2016, I pushed the Mathematics Department to form a committee on inclusivity and diversity. As a member of this committee, I drafted a statement on the department's commitment to inclusivity and non-discrimination that was accepted by the faculty at a department meeting. I also created template syllabi statements that let students know about these department policies. Everyone within the department is encouraged to use these.

Over the summer of 2017, I co-founded oSTEM@UW as a resource for LGBTQ+ students in STEM. During my time leading oSTEM@UW, the group grew to over fifty active members. As one member said, "It made me very happy to see other friendly LGBTQ+ faces around... Thanks so much for organizing this stuff – it's really helpful". Additionally, I organized and obtained a travel grant for 11 members to attend the national oSTEM Inc. conference.

Since 2017 I have been the organizer of the campus social organization for LGBTQ+ graduate and post-graduate students, which currently has over 350 members. In this role, I have co-organized a weekly coffee social hour intended to give LGBTQ+ graduate and post-graduate students a place to relax, make friends, and discuss the challenges of being LGBTQ+ at the UW - Madison.

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