RESEARCH STATEMENT

JULIETTE BRUCE

My research interests lie in algebraic geometry, commutative algebra, and algebraic geometry. In particular, I am interest in using homological and combinatorial methods to study the geometry of zero locus of systems of polynomials, that is the geometry of algebraic varieties, and studying the arithmetic properties of varieties over finite fields.

2. Bridges Between Geometry and Syzygies

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal free resolution. In essence, a minimal free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *free resolution* of a graded R-module M is an exact sequence

$$\cdots \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

where each F_p is a graded free R-module, and hence can be written as $\bigoplus_q R(-q)^{\beta_{p,q}}$. Note that $R(-q)$ is the ring R with its grading twisted, so that $R(-q)_d$ is equal to R_{d-q} where R_{d-q}, is the graded piece of degree $d-q$. The $\beta_{p,q}$'s are called the *Betti numbers* of M, and they are often a useful numerical invariant when studying M.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0,\ldots,x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal free resolution. The minimal graded free resolution of X is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomena is the Green's Conjecture, which relates the Clifford index of a canonical curve with the vansihing of certain $\beta_{p,q}$. The generic case of this conjecture was resolved by Voisin in characteriztic zero [Voi02, Voi05], and was recently shown to be true over fields of characteristic $p > 0$ for sufficiently large p ♠♠♠ Juliette: [CITATION].

2.1 Asymptotic Syzygies

Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways this perspective dates back to classical work on the defining equations of curves of high degree and of projective normality. However, the modern take viewpoint arrose from the pioneering work of Green [Gre84a], [Gre84b].

In order to give a flavor of the results of asymptotic syzygies we will focus on the question of in what degrees do non-zero syzygies occur. Towards this we set,

$$\rho_q(X,L) \;:=\; \frac{\#\left\{p \in \mathbb{N} \,\middle|\, \beta_{p,p+q}(X,L) \neq 0\right\}}{r_d},$$

which by the Hilbert Syzygy Theorem is the percentage of degrees in which non-zero syzygies appear [Eis05]Theorem 1.1.

Now for any particular, X, L_d, and q it is often the case that computing $\rho_q(X;L_d)$ is quite difficult. The asymptotic perspective thus, shifts the question slightly, and asks to instead consider a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ and ask how $\rho_q(X;L_d)$ behaves along the sequence of $(L_d)_{d\in\mathbb{N}}$.

2010 *Mathematics Subject Classification.* 13D02, 14M25.

The author was partially supported by the NSF GRFP under Grant No. DGE-1256259 and NSF grant DMS-1502553.

Theorem 2.1. *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is sequence of very ample line bundles on X such that $\deg L_d = d$ then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Theorem 2.2. *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq n$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q(X; L_d) = 1.$$

Theorem 2.3 (Juliette Bruce)**.** *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exists constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q(X; \mathcal{O}(d_1, d_2)) \geq 1 - \sum_{\substack{i+j=q\\0\leq i\leq n\\0\leq j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}} \right) - O\begin{pmatrix}\text{lower ord.}\\ \text{terms}\end{pmatrix}.$$

2.2 Syzygies via Highly Distributed Computing While the use of syzygies to study the geometry of projective varieties has proved very fruitful, it turns out to be quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only know for $d \leq 5$. That said it turns out that the problem computing syzygies can be reduced to computing the ranks of a number of matrices. However, the reduction has generally proven intractable because the number of matrices and their sizes quickly become extremely large. For example, in order to compute the syzygies of $\mathbb{P}^2$ embedded by the 6-uple Veronese embedding there are well over 6,000 relevant matrices with the around 2,000 of them being on the order of $4,000,000 \times 12,000,000$.

In a project my co-authors and I exploited recent advances in numerical linear algebra and high throughput high performance computing to overcome these challenges and compute a number of new examples of Veronese syzygies. This data provided support for a number of existing conjectures, as well as led us to conjecture previously unseen relationships between the representation theory and syzygies of Veronese embeddings. The data resulting from this project has been made publicly available via the website Syzygy-Data.com, as well as, a package for the computer algebra system Macaualy2.

Recently I have begun working to use similar computational techniques to compute the syzygies for various Hirzebruch surfaces including $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, by exploiting the fact that there exists semi-ample line bundles on Hirzebruch surfaces we have managed to compute the syzygies in approximately 100 new examples. We hope that these new examples will lead to new conjectures regarding the syzygies of Hirzebruch surfaces. In particular, we hope to find ways to generalize ♠♠♠ Juliette: [NEED] to other Hirzebruch surfaces.

2.3 Liaison Theory via Virtual Resolutions Generically if one picks two polynomials $f, g \in \mathbb{C}[x,y,z]$ their common zero locus $\mathbb{V}(f,g) \subset \mathbb{P}^3$ will be one dimensional (i.e. $\mathbb{V}(f,g)$ is an algebraic curve). Curves arriving in such a way are called complete intersections. While in a sense curves which are complete intersections are generic, much of the interesting geometry of curves in $\mathbb{P}^3$ comes from curves which are not complete intersections. Classically one approach to understanding the geometry of space curves in $\mathbb{P}^3$ is by asking when the union of two (or more) curves is a complete intersection. Such curves are said to be linked, and the general idea so to try and identify properties that are preserved under linkage.

While the liaison theory of curves in $\mathbb{P}^3$ is well understood the same theory for curves in other 3-folds (even $\mathbb{P}^1 \times \mathbb{P}^2$) remains quite mysterious. One reason for this stark difference is that while minimal graded free resolutions are an extremely useful tool to study quasicoherent sheaves on projective space, when working over other varieties – like smooth toric varieties – minimal graded free resolutions tend to be less useful. For example, minimal graded free resolutions over the Cox ring of a smooth toric variety seem overly burdened by algebraic structure that is often irrelevant to the geometry.

In an ongoing program with Christine Berkesch and Patricia Klein I hope to use the newly developed theory of virtual resolutions to better understand the linkage theory curves in toric 3-folds. Broadly a virtual resolution is a homological representation of a finitely graded module over the Cox ring of a smooth toric variety that attempts to overcome the challenges mentioned in the previous paragraph by allowing a certain amount of homology.

Broadly, a virtual resolution is a homological representation of a finitely generated graded module over the Cox ring of a smooth projective toric variety. While graded minimal free resolutions are useful for studying quasicoherent sheaves over projective space, when working over a product of projective spaces or, more generally, over smooth projective toric vari- eties, they are somewhat less useful. In particular, graded minimal free resolutions over the Cox ring seem too burdened by algebraic structure that is, in some sense, unimportant geometrically

Goal Theorem 2.4. *Let C and C' be linked curves in $\mathbb{P}^2 \times \mathbb{P}^1$ then C is virtually Cohen-Macaulay if and only if C' is virtually Cohen-Macaualy.*

Goal Theorem 2.5. *The even liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$ satisfying some technical assumptions are in bijection with* ♠♠♠ Juliette: [finish]

We have begun exploring these

3. Bridges to Arithmetic Geometry

Over a finite field a number of classical statements from algebraic geometry no longer hold. For example, if X is a smooth projective variety of dimension n over $\mathbb{C}$ then Bertini's Theorem states that a generic hyperplane section of X (i.e. $X \cap H$ for a generic hyperplane $H \subset \mathbb{P}^n$) will be smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field. ♠♠♠ Juliette: [insert example]. Similar statements for connectedness, irreducibility, and other properties also fail over finite fields.

Historically, the lack of such Bertini theorems over finite fields has made many results in algebraic geometry over finite fields more complicated. (Bertini theorems are extremely useful as they often provide a basis for induction on dimension.) However, using an ingenious probabilistic argument Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of possibly arbitrarily large degree a version of Bertini's Theorem for Smoothness (highlighted above) is true. More specifically Poonen showed that as $d \to \infty$ the percentage of hypersurfaces $H \subset \mathbb{P}^n$ of degree d such that $X \cap H$ is smooth is

In recent years much work has gone into extending these result via NEDEDE

Theorem 3.1. *[Poonen] Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth projective variety of dimension n,....*

$$\lim_{d\to\infty} \operatorname{Prob}\left(\begin{matrix} f \in H^0(X, dA) \\ X \cap H_f \text{ is smooth of dimension } n-1 \end{matrix} \right) = \zeta_X(n+1)^{-1} > 0 \tag{1}$$

3.1 A Probabilistic Study of Systems of Parameters Over an infinite field a Noether normalization states that if $X \subset \mathbb{P}^r$ is a projective variety of dimension n then there exists a finite (i.e. surjective with finite fibres) map $X \to \mathbb{P}^n$. The existence of such a map is equivalent to the existence of a collection of homogenous polynomials $f_0, \ldots, f_n$ such that $X \cap \mathbb{V}(f_0, \ldots, f_n) = \varnothing$. Such a collection of polynomials is called a system of parameters.

Adapting Poonen's closed point sieve to sieve over higher dimensional varieties

Corollary 3.2. *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a n-dimensional closed subscheme and let $k < n$. Then*

$$\lim_{d\to\infty} \frac{\operatorname{Prob}\left(\begin{matrix} (f_0, \ldots, f_k) \text{ of degree } d \\ \text{are } \underline{\text{not}} \text{ parameters on } X \end{matrix} \right)}{q^{-(k+1)\binom{n-k+d}{n-k}}} = \# \left\{ \begin{matrix} (n-k)\text{-planes } L \subseteq \mathbb{P}^r_{\mathbf{F}_q} \\ \text{such that } L \subseteq X \end{matrix} \right\}.$$

3.2 Explicit Bertini Theorems and Jacobians Covering Abelian Varieties

Over an infinite field, it is a classic result that every abelian variety is covered by the Jacobian variety of a smooth connected curve. In fact, by using Bertini's theorem if $A \subset \mathbb{P}^r$ is an abelian variety of dimension n over an infinite field there exists a hyperplane H such that $A \cap H$ is a smooth curve whose Jacobian covers A. Moreover, one can even provide an effective upper bound on the dimension of the $\mathrm{Jac}(A \cap H)$ in terms of the dimension and degree of A (see [Mil08, Section III]).

Over a finite field the failure of Bertini's theorem makes things substantially more subtle. For example, while Poonen's Bertini theorem shows that every abelian variety over a finite field is covered by a Jacobian variety it is not enough to effectively bound the dimension of such covers. This is a result of the fact Poonen's Bertin

3.3 Uniform Bertini

Notice that in the statement of Poonen's Bertini theorem while the left hand side of equation ♠♠♠ Juliette: [CITE] is dependent of the embedding of X into projective space (i.e. it depends on our choice of very ample line bundle A) while the overall limit is itself independent of the embedding of X. This suggests that there may be a more general and uniform statement of Poonen's Bertini Theorem. That is one might hope that the analogous limit along any sequence $(L_d)_{d\in\mathbb{N}}$ of line bundles growing in positivity may limit to $\zeta_X(n+1)^{-1}$. I am working on a project, joint with Isabel Vogt, attempting to formalize and prove such a theorem.

Work of Erman and Wood on semi-ample Bertini Theorems shows that such an analogue of Theorem 3.1 fails for sequences of line bundles, which do not grow in an ample fashion. Isabel and I believe that this failure can be fixed by introducing a technical assumption on the the sequence of lines bundles grows in positivity we call going to infinity in all directions. Formally a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ goes to infinity in all directions if for every ample line bundle A there exists an $N \in \mathbb{N}$ such that $L_i - A$ is ample for all $i \geq N$. In particular, we are working to prove the following uniform version of Poonen's Bertini theorem.

Goal Theorem 3.3. *Let $X/\mathbf{F}_q$ be a projective variety of dimension n. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of line bundles on X going to infinity in all directions then*

$$\lim_{d\to\infty} \mathrm{Prob}\left(\begin{array}{c} f \in H^0(X, L_d) \\ X \cap H_f \text{ is smooth of dimension } n-1 \end{array} \right) = \zeta_X(n+1)^{-1} \tag{2}$$

Thus far we have managed to prove this theorem in a number of examples ($X = \mathbb{P}^1 \times \mathbb{P}^1$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$), and we are hopefully that similar methods might extend to all products of projective spaces or possibly whenever the nef cone fo X is finitely generate polyhedral cone. That said it appears the more general case when $\mathrm{Nef}(X)$ is not finitely generated will require new techniques.

Viewing Poonen's Bertini Theorem as a statement on the existence of smooth sections for line bundles sufficiently twisted by ample line bundles it is natural to ask the analogous question when twisting by arbitrary nef line bundles. In particular, inspired by Fujita's vanishing theorem and work of Erman and Wood the following conjecture is another natural version of Poonen's Bertini Theorem.

Conjecture 3.4. *Let $X/\mathbf{F}_q$ be a projective variety of dimension n and $\mathcal{L}$ be an ample line bundle on X. If $\mathcal{F}$ is any coherent sheaf on X there exists an integer $m(\mathcal{F},\mathcal{L})$ such that*

$$\left\{ f \in H^0(X, \mathcal{F} \otimes \mathcal{L}(k) \otimes \mathcal{D}) \;\middle|\; \begin{array}{c} X \cap H_f \text{ is smooth} \\ \text{of dimension } n-1 \end{array} \right\} \tag{3}$$

for all $k \geq m(\mathcal{F},\mathcal{L})$ and all $\mathcal{D} \in \mathrm{Nef}(X)$.

I hope that my work with Isabel Vogt on the Goal Theorem 3.3 to shed light on this conjecture, and allow us to prove at least parts of it.

4. Broader Impacts

4.1 Organizing In the Spring of 2017 I organized *Math Careers Beyond Academia* (50 participants), a one day professional development conference on STEM careers outside of academia, and how graduate students should prepare for these careers. In April 2018 I organized a four day workshop focused on creating new packages for Macaulay2 – an open source computer algebra system – by bringing together over 45 developers and users of all skill levels and experience. Further in February 2019 I organized *Geometry and Arithmetic of Surfaces* (40 participants), a workshop providing a diverse group of early career researchers the opportunity to learn about interesting cutting edge topics in the arithmetic and algebraic geometry of surfaces from a diverse set of prominent active researchers. In April 2019 I organized the *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants) focused on forming a community of women and non-binary researchers interested in commutative algebra. In Fall 2019 I organized a *Special Session on Combinatorial Algebraic Geometry* at the AMS Fall Central Sectional. At the 2019 Joint Mathematics Meetings I am organizing a Spectra Panel entitled *Supporting Transgender and Non-binary Students*.

4.2 Math Circles I began volunteering with the MMC in Fall 2014; at the time, the circle's main programming was a weekly on-campus lecture given by a member of the math department. After volunteering with the MMC for roughly a year I stepped into the role of student organizer overseeing much of the administrative needs of the circle including scheduling and overseeing speakers. Over my time as organizer I worked hard to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations especially those focused on underrepresented groups. These ties helped the weekly attendance of the MMC more than double during my time as organizer. Additionally, I led the creation of a new outreach arm of the MMC, which visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This program has dramatically expanded the reach of the circle. For example, during my last full yeas as organizer I planned over 10 visits to high schools around the state of Wisconsin reaching over 250 students.

4.3 Mentoring Since the winter of 2018 I have led reading courses with three undergraduates through Wisconsin's Directed Reading Program. One of these students, and undergraduate woman, worked with me for over a year, during which time I helped her through the process of applying for Research Experiences for Undergrads and researching options for graduate school. I have been a mentor to two undergraduate women studying math via the AWM's Mentoring Network program. Further, through I have lead directed reading courses with three undergraduates. Working with Girls' Math Night Out I mentored two high school aged girls leading them through a project exploring RSA cryptography. Additionally, during the 2018-2019 academic year I served as a mentor to 6 first year graduate students (all women or non-binary students).

4.4 A More Inclusive Community. During the Fall of 2016 in response to a growing climate of hate, bias, and discrimination on campus I pushed the Mathematics Department to form a committee on inclusivity and diversity. As a member of this committee I took the lead in drafting a statement on the department's commitment to inclusivity and non-discrimination that was accepted by the faculty at a department meeting. I also worked to create template syllabi statements that let students know about these department polices, and that inform them of other campus resources that may be helpful. All teachers within the department are now encouraged to use these statements.

Over the summer of 2017 I co-founded Out in Science, Technology, Engineering, and Mathematics at UW (oSTEM@UW) as a resource for LGBTQ+ students in STEM. During my time leading oSTEM@UW the group grew dramatically eventually having over fifty active members. The efficacy and importance of such a group was made clear by the numerous student comments indicating how helpful and encouraging oSTEM@UW is to them. For example, after a meeting, a student emailed me to say "It made me very happy to see other friendly LGBTQ+ faces around, and I got to meet two people who were already in classes of mine! Thanks so much for organizing this stuff – it's really helpful for me personally, and I believe it was encouraging for the others attending as well." Additionally, while leading oSTEM@UW I organized and obtained travel grants for eleven members – including multiple undergraduates – to attend the annual national oSTEM Inc. conference.

Since 2017 I have been the organizer of the campus social organization for LGBTQ+ graduate and post-graduate students, which currently has over 350 members. In this role I have co-organized a weekly coffee social hour intended to give LGBTQ+ graduate and post-graduates students a place to relax, make friends, and discussion the challenges of being LGBTQ+ at the UW - Madison.

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