Introduced by Mumford, the Castelnuovo–Mumford Regularity of a projective variety $X \subset \mathbb{P}^n$ is a measure of the complexity of X given in terms of the vanishing of certain cohomology groups of X. Roughly speaking one should think about Castelnuovo–Mumford regularity as being a measure of geometric complexity. Such a measure can be easily extended to modules over a standard graded polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ by requiring the analogous vanishing conditions for local cohomology.

Mumford was interested in such a measure as it plays a key role in constructing Hilbert and Quot schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. However, Eisenbud and Goto showed that regularity is also closely connected to interesting homological properties.

Theorem 0.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $M = \bigoplus_{e \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent:*

(1) M is d-regular;

(2) $\beta_{p,q}(M) = 0$ for all $p \geq 0$ and $q > d + i$;

(3) $M_{\geq d}$ has a linear resolution.

My collaborators and I have worked to generalize this result to the multigraded setting, i.e. from coherent sheaves on a single projective space to sheaves on a product of projective spaces. In particular, fixing a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ we let $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ and $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\mathrm{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$.

Maclagan and Smith generalized Castelnuovo–Mumford regularity to this setting in terms of certain cohomology vanishing. Fixing some notation given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let:

$$L_i(\mathbf{d}) := \bigcup_{\substack{\mathbf{v} \in \mathbb{N} \\ |\mathbf{v}| = i}} (\mathbf{d} - \mathbf{v}) + \mathbb{N}^r.$$

Note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i+1)$-corners. Roughly speaking we define regularity by requiring the i-th cohomology of certain twists of $\mathcal{F}$ to vanish on L_i.

Definition 0.2. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if*

$$H^i(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})) = 0 \qquad \text{for all } \mathbf{e} \in L_i(\mathbf{d}).$$

The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:

$$\mathrm{reg}(\mathcal{F}) := \left\{ \mathbf{d} \in \mathbb{Z}^r \;\middle|\; \mathcal{F} \text{ is } \mathbf{d}\text{-regular} \right\} \subset \mathbb{Z}^r.$$

The obvious approaches to generalize Theorem 0.1 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity Example 5.1]bruceHellerSayrafi21 Despite this we show that part (3) of Theorem 0.1 can be generalized. To do so we introduce the following generalization of linear resolutions.

Definition 0.3. *A complex $F_\bullet$ of $\mathbb{Z}^r$-graded free S-modules is $\mathbf{d}$-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d} - \mathbf{1})$.*

Theorem 0.4. **Theorem A Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$:*

$$M \text{ is } \mathbf{d}\text{-regular} \iff M_{\geq \mathbf{d}} \text{ has a } \mathbf{d}\text{-quasilinear resolution.}$$

The proof of Theorem 0.4 is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel. Precisely, if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$ we prove the that

$$\dim_{\mathbb{C}} \operatorname{Tor}_j^S(M_{\geq \mathbf{d}}, \mathbb{C})_{\mathbf{a}} = h^{|\mathbf{a}|-j}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{O}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) \quad \text{for } |\mathbf{a}| \geq j \geq 0,$$

where the $\mathcal{O}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The result then follows from showing that M being $\mathbf{d}$-regular is equivalent to certain vanishings of the right-hand side above.

Theorem 0.5. *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on I, such that for each integer $t > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators f_i of I, we have*

$$t\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}\big(I^t\big) \subseteq t\mathbf{q}_2 + \operatorname{Nef} X.$$

Building on the work of many people , Cutkosky, Herzog, Trung and independently Kodiyalam showed the Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^n$ has surprisingly predictable asymptotic behavior. In particular, given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$, there exist constants $d, e \in \mathbb{Z}$ such that $\operatorname{reg}\big(I^t\big) = dt + e$ for $t \gg 0$.

Building upon our work discussed above, my collaborators and I generalized this result to arbitrary toric varieties. In particular, Definition 0.2 can be extended to all toric varieties by letting S be Cox ring of the toric variety X, replacing $\mathbb{Z}^r$ with the Picard group of X, and replacing $\mathbb{N}^r$ with the nef cone of X. My collaborators and I show that the multigraded regularity of powers of ideals is bounded and translates in a predictable way. In particular, the regularity of I^t essentially translates within $\operatorname{Nef} X$ in fixed directions at a linear rate.

Relevance of Visit. Being a postdoctoral fellow at MSRI would be extremely beneficially towards my career goals as it would provide me with a stimulating mathematical environment, and the opportunity to engage and work with a number of leading researchers in commutative algebra. My research interest align extremely well with topics the of semester long program. For example, one ongoing project I am working on concerns better understanding the homological implications of certain generalizations of Castelnuovo–Mumford regularity to toric varieties, and in a separate project I am beginning to explore ways Green's conjecture may be generalized to describe the syzygies of canonical stacky curves. Both of these projects touch upon many of the topics included in the program description. As such begin a postdoc at MSRI would likely prove valuable toward progressing both of these projects, as well as a fantastic opprotunity to being new collaborations with other participants. Moreover, given the the ways my research interest align with the program I feel I would be able to find numerous ways to myself contribute to the program.

Overall, being a postdoctoral fellow for this semester program in commutative algebra presents an amazing opportunity to connect and build relationships within the research area that has long felt like my research community/home. And such connections would significantly advance my goals of being a math professor at a research university.

Building Mathematical Community. As an LGBTQ+ woman, I have worked hard to promote diversity, inclusivity, and justice in the mathematical community by mentoring students, supporting women and LGBTQ+ people in mathematics, and organizing conferences and outreach programs.

As a postdoc, I have put significant effort into mentoring, advising, and working with students, especially those from underrepresented groups. During Summer 2021 in order to help fill gaps caused by the COVID-19 pandemic I organized a virtual summer undergraduate research program for 6 undergraduates from around the world. In Summer 2022 I advised an undergraduate student on a research project related to my work on asymptotic syzygies. This student is now applying to graduate schools in math. I began research projects with multiple graduate students, in which I played a substantial mentoring and guiding role. In the Spring and Summer of 2022, I did a reading course with a first-year graduate woman who is now working in commutative algebra.

Since Fall 2020 I have organized an annual virtual conference promoting the work of transgender and non-binary mathematicians. Highlighting the importance of such conferences one participant said, "I've been really considering leaving mathematics. [Trans Math Day 2020] reminded me why I'm here and why I want to stay. ... If a conference like this had been around for me five years ago, my life would have been a lot better." Trans Math Day regularly has 50 participants.

For over the last two years I have served as a board member for *Spectra: The Association for LGBTQ+ Mathematicians*. As a board member I have overseen the growth and formalization of the organization, including the creation and adoption of bylaws, the creation of an invited lecture at the Joint Mathematics Meetings, and a fundraising campaign that has raised over $20,000 to support LGBTQ+ students and mathematicians. I am currently the inaugural president of Spectra, and as of right now we have over 500 people on our mailing/membership lists.

In response to the COVID-19 pandemic and the shift of many mathematical activities to virtual formats, I worked to find ways for these online activities to reach those often at the periphery. During Summer and Fall 2020, I helped with Ravi Vakil's *Algebraic Geometry in the Time of Covid* project. This massive online open-access course in algebraic geometry brought together $\sim 2,000$ participants from around the world. In Spring 2021, I organized an 8-week virtual reading course for undergraduates in algebraic geometry and commutative algebra.

I have organized over 15 conferences, special sessions, and workshops including: *M2@UW* (45 participants), *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Minority Genders* (35 participants), *CAZoom* (70 participants), *Western Algebraic Geometry Symposium* (100 participants)x, and Spec($\overline{\mathbb{Q}}$) (50 participants). When organizing these conferences I have given paid special attention to promoting women and other underrepresented groups in mathematics. For example, *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* and *GEMS in Combinatorics* focused on forming communities of women and non-binary researchers in commutative algebra and combinatorics respectively. Further, Spec($\overline{\mathbb{Q}}$) highlighted the research of LGBTQ+ mathematicians in algebra, geometry, and number theory.