THE QUANTITATIVE BEHAVIOR OF ASYMPTOTIC SYZYGIES FOR HIRZEBRUCH SURFACES

JULIETTE BRUCE

Throughout this note for any real number x we let $\lfloor x \rfloor$ the integer part of x and let $\langle x \rangle$ denote the decimal part of x. The goal of this note is to prove the following

Claim 1.1. *Given a real number $a_0 > 0$, define a sequence recursively by letting $a_{n+1} = \lfloor a_n \rfloor \langle a_n \rangle + 1$ for all $n \geq 0$. For any initial value $a_0 > 0$ the sequence $\{a - n\}_{n \in \mathbb{N}}$ eventually stabilizes in the sense that there exists $N \in \mathbb{N}$ such that $a_n = a_{n+1}$ for all $n \geq N$.*

Before moving on we note two key, but straightforward observations:

(1) If $a_k = a_{k+1}$ for some $k \in \mathbb{N}$ then $a_n = a_{n+1}$ for all $n \geq k$, that is to say sequences constructed in the above way stabilize as soon as NEDEDD.

(2) If $a_k = 1$ for some $k \in \mathbb{N}$ then the sequence $\{a_n\}_{n \in \mathbb{N}}$ stabilizes and $a_n = 1$ for all $n \geq k$.

Lemma 1.2. *If $a_0 > 0$ is an integer then the sequence $\{a_n\}$ stabilizes and $a_n = 1$ for all $n \geq 1$.*

Proof. If $a_0 \in \mathbb{Z}_{>0}$ then the decimal part of a_0 is equal to zero (i.e., $\langle a_0 \rangle = 0$). Using the definition the next term in the sequence is $a_1 = \lfloor a_0 \rfloor \langle a_0 \rangle + 1 = a_0 \cdot 0 + 1 = 1$. A similar computation shows $a_2 = 1$ and so by point (1) above our sequence stabalizes as claimed. □

Lemma 1.3. *Let $0 < a_0$ be a real number.*

(1) If $a_0 \leq 1$ then the sequence $\{a_n\}_{n \in \mathbb{N}}$ stabilizes and $a_n = 1$ for all $n \geq 1$.

(2) If $1 < a_0 < 2$ then the sequence stabilizes and $a_n = 1$ for all $n \geq 2$.

Lemma 1.4. *If $a_0 \geq 2$ is not an integer there exists a number $k \in \mathbb{N}$, depending on a_0, such that $a_k < \lfloor a_0 \rfloor$*

Proof of Claim 1.1. By Lemmas **??** we know the claim is true if a_0 is an integer or $0 < a_0 < 2$ respectively. Thus, without lose of generality we may assume that $a_0 \geq 2$ and a_0 is not an integer. By Lemma **??** it is enough for us to show NEDEDED □