JULIETTE BRUCE'S RESEARCH STATEMENT

My research interests lie in algebraic geometry, commutative algebra, and arithmetic geometry. In particular, I am interested in using homological methods to study the geometry of zero loci of systems of polynomials (i.e. algebraic varieties). I am also interested in studying the arithmetic properties of varieties over finite fields. I am also passionate about promoting inclusivity, diversity, and justice in the mathematics community.

1. Syzygies in Algebraic Geometry

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \to F_{k-1} \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_p is a graded free R-module, and hence can be written as $\bigoplus_q R(-q)^{\beta_{p,q}}$. Note that $R(-q)$ is the ring R with its grading twisted, so that $R(-q)_d$ is equal to R_{d-q} where R_{d-q}, is the graded piece of degree $d-q$. The $\beta_{p,q}$'s are called the *Betti numbers* of M, and they count number of p-syzygies of M of degree q. We will often use syzygy and Betti number interchangeable throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal free resolution. The minimal graded free resolution of X is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomena is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{p,q}$ for its canonical embedding [Voi02, Voi05, AFP+19].

1.1 Asymptotic Syzygies Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84a, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question of in what degrees do non-zero syzygies occur. Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set $\rho_q(X; L_d)$ to be the percentage of degrees in which non-zero syzygies appear. For any particular, X, L_d, and q computing $\rho_q(X; L_d)$ is often quite difficult. The asymptotic perspective. thus instead, considers a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ and asks how $\rho_q(X; L_d)$ behaves along the sequence.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84a] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is sequence of very ample line bundles on X such that* $\deg L_d = d$ *then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Put differently Green showed that the syzygies of curves of high degree are as simple as possible, occurring in the lowest possible degrees. This inspired substantial work on so called N_p conditions, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq n$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q\left(X; L_d\right) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if the linear series $|kL|$ is base point free for some $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth, thus began, by proving the following non-vanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19a, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q\left(X; \mathcal{O}\left(d_1, d_2\right)\right) \geq 1 - \sum_{\substack{i+j=q \\ 0\leq i\leq n \\ 0\leq j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.}\\ \textit{terms}\end{pmatrix}.$$

Notice in the setting of ample growth, this recovers the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. In particular, if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right) \to 1$. However, in the setting of semi-ample growth, i.e. d_1 is fixed and $d_2 \to \infty$, my results bound the asymptotic percentage of non-zero syzygies away from zero. In fact, my work together with recent results of Lemmens [Lem18] has led me to conjecture that unlike in previously study cases (i.e. curves and ample growth) in the case of semi-ample growth $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right)$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case. See [EL12, Conjecture 7.1, Conjecture 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld to explicitly construct a non-trivial syzygy after quotienting by a regular sequence. Such a generalization is complicated by the fact that unlike the cases consider in their paper there are no monomial regular sequences of length $(n+m+1)$ on either the $\mathbb{Z}^2$-graded Cox ring of $\mathbb{P}^n \times \mathbb{P}^m$ or the $\mathbb{Z}$-graded homogeneous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$. Instead, my work utilizes a regular sequence first introduced by Eisenbud and Schreyer [ES09], and later used in [BEKS13, ORS19]. A central theme in my work is to exploit the fact that this regular sequence is homogeneous with respect to several non-trivial non-standard gradings. These gradings, when combined with a series of spectral sequence arguments allow me to prove Theorem 1.3.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this previous work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq n$. Let $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample compute $\lim_{d\to\infty} \rho_q\left(X; L_d\right)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru19b].

1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. This is because while the problem of computing syzygies can be reduced to computing ranks of matrices, the number of matrices and their sizes quickly become extremely large. As an example, to compute the syzygies of $\mathbb{P}^2$ embedded by the 6-uple Veronese embedding there are well over 6,000 relevant matrices with the around 2,000 of them being on the order of $4,000,000 \times 12,000,000$.

My co-authors and I exploited recent advances in numerical linear algebra and high throughput high-performance computing to compute a number of new examples of Veronese syzygies. This data provided support for several existing conjectures, as well as led us to make a number of conjectures [BEGSY18]. The data resulting from this project has been made publicly available via the website SyzygyData.com, as well as, a package for the computer algebra system Macaualy2 [BEGSY19].

Recently I have begun working to use similar computational techniques to compute the syzygies for Hirzebruch surfaces. Thus far, we have computed the syzygies in ~ 100 new examples. It is our hope that these examples will lead to new conjectures regarding the syzygies of Hirzebruch surfaces. In particular, we believe our data will be helpful in begin to address Question 1.4.

1.3 Liaison Theory via Virtual Resolutions Generically if one picks two polynomials $f, g \in \mathbb{C}[x, y, z]$ their common zero locus $\mathbb{V}(f, g) \subset \mathbb{P}^3$ will be an algebraic curve. Curves arriving in such a way are called complete intersections. Classically one approach to studying the geometry of space curves in $\mathbb{P}^3$ is by asking when the union of two (or more) curves is a complete intersection. Such curves are said to be linked, and the general idea of liaison theory is to try and identify properties that are preserved under linkage.

While the liaison theory of curves in $\mathbb{P}^3$ is well understood the same theory for curves in other 3-folds (even $\mathbb{P}^1 \times \mathbb{P}^2$) remains quite mysterious. One reason for this that while minimal graded free resolutions are an extremely useful tool to study quasicoherent sheaves on projective space when working over other varieties minimal graded free resolutions tend to be less useful. For example, minimal graded free resolutions over the Cox ring of a smooth toric variety seem overly burdened by algebraic structure that is often irrelevant to the geometry.

In ongoing work with Christine Berkesch and Patricia Klein, I hope to use the newly developed theory of virtual resolutions to better understand the linkage theory of curves in toric 3-folds. Broadly a virtual resolution is a homological representation of a finitely graded module over the Cox ring of a smooth toric variety that attempts to overcome the challenges mentioned in the previous paragraph by allowing a limited amount of homology [BES17]. Using these we hope to generalize existing results about the liaison theory of curves in $\mathbb{P}^3$ to curves in $\mathbb{P}^1 \times \mathbb{P}^2$.

Ambitiously we hope to use virtual resolutions to find a way to classify liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$ analogous to Rao modules [PR78]. We would like to answer the following question.

Question 1.5. *What invariant classifies even liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$?*

Using virtual resolutions we have managed to construct an analog of Rao modules, which answer this question under a fairly restrictive hypothesis on Betti numbers. Going forward we hope to explore whether this hypothesis can be weakened, and our construction generalized.

Further, for any homological property associated to free resolutions, it is possible to define an analogous virtual property associated to virtual resolutions. Klein, Berkesch, and I have proposed a definition of *virtual Cohen-Macaulay* for finitely generated graded modules over the Cox ring of a smooth toric variety. We would thus like to prove the following virtual version of Peskine and Szpiro's result showing that being Cohen-Macaulay is preserved under linkage [PS74].

Goal Theorem 1.6. *Let C and C' be linked curves in $\mathbb{P}^1 \times \mathbb{P}^2$ then C is virtually Cohen-Macaulay if and only if C' is virtually Cohen-Macaualy.*

A helpful tool in approaching these questions is the ability to compute interesting examples via the computer algebra system Macaualy2. These computations are made easier thanks to the `VirtualResolutions` package, which I co-authored [ABLS19].

2. Varieties over Finite Fields

Over a finite field, a number of classical statements from algebraic geometry no longer hold. For example, if X is a smooth projective variety of dimension n over $\mathbb{C}$ then Bertini's theorem states that a generic hyperplane section of X (i.e. $X \cap H$ for a generic hyperplane $H \subset \mathbb{P}^r$) will be smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Historically, the lack of such Bertini theorems over finite fields has made many results in algebraic geometry over finite fields more complicated.

Using an ingenious probabilistic sieving argument Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree a version of Bertini's theorem is true [Poo04]. More specifically Poonen showed that as $d \to \infty$ the percentage of hypersurfaces $H \subset \mathbb{P}^r_{\mathbf{F}_q}$ of degree d such that $X \cap H$ is smooth is determined by Hasse-Weil zeta function of X. Below we write $\mathbf{F}_q[x_0, \ldots, x_r]_d$ for the $\mathbf{F}_q$-vector space of homogenous polynomials of degree d.

Theorem 2.1. [Poo04, Theorem 1.1] *Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth variety of dimension n then:*

$$\lim_{d\to\infty} \operatorname{Prob}\begin{pmatrix} f \in \mathbf{F}_q[x_0, x_1, \ldots, x_r]_d \\ X \cap \mathbb{V}(f) \textit{ is smooth of dimension } n-1 \end{pmatrix} = \zeta_X(n+1)^{-1} > 0. \tag{1}$$

2.1 A Probabilistic Study of Systems of Parameters Given an n dimensional projective variety $X \subset \mathbb{P}^r$ a collection of homogenous polynomials $f_0, f_1, \ldots, f_k$ of degree d is a (partial) system of parameters if $\dim X \cap \mathbb{V}(f_0, f_1, \ldots, f_k) = \dim X - (k+1)$. Systems of parameters are closely tied to Noether normalization as the existence of a finite (i.e. surjective with finite fibers) map $X \to \mathbb{P}^n$ is equivalent to the existence of a system of parameters of length $n+1$. Over an infinite field, any collection of generic homogenous polynomials is a system of parameters. However, when working over a finite field finding systems of parameters is subtle. For example, a priori, there is no effectively bound on the degrees for which systems of parameters exist.

Inspired by the work of Poonen [Poo04] and Buccur and Kedlaya [BK12], Daniel Erman and I explored the ubiquity of (partial) systems of parameters over finite fields from a probabilistic perspective. Adapting Poonen's closed point sieve to sieve over higher dimensional varieties, we computed the asymptotic probability that randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ form a system of parameters. Doing this we showed that when $k < n$ the probability randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ form a partial system of parameters is controlled by zeta function like power series that enumerates higher dimensional varieties instead of closed points. In the following theorem, $|Z|$ denotes the number of irreducible components of a scheme Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

Theorem 2.2. [BE, Theorem 1.4] *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\begin{pmatrix} f_0, f_1, \ldots, f_k \textit{ of degree } d \\ \textit{are parameters on } X \end{pmatrix} = 1 - \sum_{\substack{Z \subseteq X \textit{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)\binom{n-k+d}{n-k}}\right).$$

As a corollary of this, we proved the first explicit bound for Noether normalization over $\mathbf{F}_q$ and gave a new proof of recent results on Noether normalizations of projective families over $\mathbb{Z}$ and $\mathbf{F}_q[t]$ [GLL15, CMBPT17].

2.2 Explicit Bertini Theorems and Jacobians Covering Abelian Varieties Over an infinite field, it is a classic result that every abelian variety is covered by the Jacobian variety of a smooth connected curve of bounded dimension. Building upon work of Bucur and Kedlaya [BK12] Li and I established an effective and explicit version of Poonen's Bertini theorem over finite fields. Using this theorem Li and I proved that every abelian variety over a finite field is covered by the Jacobian variety of a smooth connected curve whose dimension is bounded by an explicit constant.

Theorem 2.3. [BLW, Theorem A] *Fix $r, n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbf{F}_q$ be a finite field of characteristic p. There exists an explicit constant $C_{r,q}$ such that if $A \subset \mathbb{P}^r_{\mathbf{F}_q}$ is a non-degenerate abelian variety of dimension n, then for any $d \in \mathbb{N}$ satisfying*

$$C_{r,q}\zeta_A\left(n+\tfrac{1}{2}\right)\deg(A) \leq \frac{q^{\frac{d}{\max\{n+1,p\}}}d}{d^{n+1}+d^n+q^d},$$

there exists a smooth curve over $\mathbf{F}_q$ whose Jacobian J maps surjectively onto A, where

$$\dim J \leq \mathrm{O}\left(\deg(A)^2 d^{2(n-1)} r^{-1}\right).$$

2.3 Uniform Bertini Notice that in the statement of Poonen's Bertini theorem while the left-hand side of equation (1) is dependent of the embedding of X into projective space (i.e. the choice of very ample line bundle) the overall limit is itself independent of the embedding of X. This suggests that there may be a more general and uniform statement of Poonen's Bertini theorem. One might hope that the analogous limit along any sequence $(L_d)_{d\in\mathbb{N}}$ of line bundles growing in positivity equal $\zeta_X(n+1)^{-1}$. I am working with Isabel Vogt to formalize and prove such a theorem.

Work of Erman and Wood on semi-ample Bertini Theorems shows that such an analog of Theorem 2.1 fails for sequences of line bundles, which do not grow in an ample fashion [EW15]. Vogt and I believe that this failure can be fixed by introducing a technical assumption on how the sequence of lines bundles grows in positivity we call *going to infinity in all directions.* Formally a sequence of line bundles $(L_d)_{d\in\mathbb{N}}$ goes to infinity in all directions if for every ample line bundle A there exists $N \in \mathbb{N}$ such that $L_i - A$ is ample for all $i \geq N$. In particular, we are working to prove the following uniform version of Poonen's Bertini theorem.

Goal Theorem 2.4. *Let $X/\mathbf{F}_q$ be a smooth projective variety of dimension n. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of line bundles on X going to infinity in all directions then*

$$\lim_{d\to\infty} \operatorname{Prob}\left(f \in H^0(X, L_d) \;\middle|\; \begin{array}{c} X \cap \mathbb{V}(f) \textit{ is smooth} \\ \textit{of dimension } n-1 \end{array}\right) = \zeta_X(n+1)^{-1}. \tag{2}$$

We have verified this theorem in a number of examples (e.g. $X = \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$), and we are hopeful that similar methods will extend to whenever the nef cone of X is a finitely generate polyhedral cone. That said it appears the more general case will require new techniques.

Inspired by Fujita's vanishing theorem and work of Erman and Wood the following conjecture is another natural version of Poonen's Bertini Theorem, which I also hope to work on with Vogt.

Conjecture 2.5. *Let $X/\mathbf{F}_q$ be a smooth projective variety of dimension n and $\mathcal{L}$ be an ample line bundle on X. There exists an integer $m(\mathcal{L})$ such that for all $k \geq m(\mathcal{L})$ and all $\mathcal{D} \in \mathrm{Nef}(X)$.*

$$\left\{f \in H^0(X, \mathcal{L}(k) \otimes \mathcal{D}) \;\middle|\; \begin{array}{c} X \cap \mathbb{V}(f) \textit{ is smooth} \\ \textit{of dimension } n-1 \end{array}\right\} \neq \emptyset. \tag{3}$$

3. Broader Impacts

3.1 Organizing In the Spring of 2017 I organized *Math Careers Beyond Academia* (50 participants), a one-day professional development conference on STEM careers outside of academia. In April 2018 I organized, *M2@UW* (45 participants), a four-day workshop focused on creating new packages for Macaulay2. In February 2019 I organized *Geometry and Arithmetic of Surfaces* (40 participants), a workshop providing a diverse group of early-career researchers the opportunity to learn about interesting topics in the arithmetic and algebraic geometry. In April 2019 I organized the *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants) focused on forming a community of women and non-binary researchers interested in commutative algebra. I organized a *Special Session on Combinatorial Algebraic Geometry* at the AMS Fall 2019 Central Sectional. At the 2020 Joint Mathematics Meetings, I am organizing a panel titled *Supporting Transgender and Non-binary Students.*

3.2 Math Circles I began volunteering with the Madison Math Circle (MMC) in Fall 2014; at the time, the circle's main programming was a weekly on-campus lecture given by a member of the math department. After roughly a year I stepped into the role of student organizer overseeing the administrative needs of the circle. As an organizer, I worked hard to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations focused on underrepresented groups. These ties helped the weekly attendance of the MMC more than double during my time as an organizer. Additionally, I led the creation of a new outreach arm of the MMC, which visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This program has dramatically expanded the reach of the circle, and during my last year as an organizer the Madison Math Circle reach well over 250 students.

3.3 Mentoring Since the winter of 2018 I have led reading courses with three undergraduates through Wisconsin's Directed Reading Program. One of these students, an undergraduate woman, worked with me for over a year, during which time I helped her through the process of applying for summer research projects. Working with Girls' Math Night Out I lead two girls in high school through a project exploring RSA cryptography. During the 2018-2019, I mentored 6 first-year graduate students (all women or non-binary students). I am currently mentoring two undergraduate women via the AWM's Mentoring Network program.

3.4 A More Inclusive Community. In Fall of 2016, I pushed the Mathematics Department to form a committee on inclusivity and diversity. As a member of this committee, I drafted a statement on the department's commitment to inclusivity and non-discrimination that was accepted by the faculty at a department meeting. I also created template syllabi statements that let students know about these department polices. Everyone within the department is encouraged to use these.

Over the summer of 2017, I co-founded oSTEM@UW as a resource for LGBTQ+ students in STEM. During my time leading oSTEM@UW, the group grew to over fifty active members. As one member said, "It made me very happy to see other friendly LGBTQ+ faces around ... Thanks so much for organizing this stuff – it's really helpful for me personally, and I believe it was encouraging for the others attending as well." Additionally, I organized and obtained a travel grant for 11 members to attend the national oSTEM Inc. conference.

Since 2017 I have been the organizer of the campus social organization for LGBTQ+ graduate and post-graduate students, which currently has over 350 members. In this role, I have co-organized a weekly coffee social hour intended to give LGBTQ+ graduate and post-graduates students a place to relax, make friends, and discussion the challenges of being LGBTQ+ at the UW - Madison.

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