RESEARCH STATEMENT

JULIETTE BRUCE

My research interests lie in using homological, combinatorial, and explicit methods to study the geometry of zero locus of systems of polynomials, that is the geometry of algebraic varieties, over both fields of characteristic zero and $p > 0$.

2. Bridges Between Geometry and Syzygies

Given a graded module M over a ring R, a helpful tool for understanding the structure of M is its minimal free resolution. In essence, a minimal free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *free resolution* of a graded R-module M is an exact sequence

$$\cdots \to F_k \to F_{k-1} \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_k is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{b_{ij}}$. Note that $R(-j)$ is the ring R with its grading twisted, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j}, is the graded piece of degree $d-j$. Useful numerical invariants of this graded free resolution are the b_{ij}, which are called the *Betti numbers* of this resolution. These Betti numbers are often arranged into a matrix called a *Betti table*.

An interesting application of the theory of free resolutions is to algebraic varieties. Given a projective variety X embedded in $\mathbb{P}^n$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_n]$ and I_X is the ideal of homogenous polynomials vanishing on X. Now, S_X is naturally a graded S-module, and so we may consider its minimal free resolution. The minimal graded free resolution of X is often closely related to the extrinsic and intrisic geometry of X. As example of this phenomena, consider the following theorem, a special case of Green's Conjecture [?avramov_lectutres_2011].

Theorem 2.1 ([?voisin_greens_2002], [?voisin_greens_2005])**.** *Let C be a generic smooth projective curve of genus g over a characteristic zero field embedded in $\mathbb{P}^{g-1}$ by the complete canonical series. Then the length of the first linear strand of the minimal free resolution of I_X is $g-3-$Cliff(C).*

2.1 Liaison Theory via Virtual Resolutions

Generically if one picks two polynomials $f, g \in \mathbb{C}[x,y,z]$ their common zero locus $\mathbb{V}(f,g) \subset \mathbb{P}^3$ will be one dimensional (i.e. $\mathbb{V}(f,g)$ is an algebraic curve). Curves arriving in such a way are called complete intersections. While in a sense curves which are complete intersections are generic, much of the interesting geometry of curves in $\mathbb{P}^3$ comes from curves which are not complete intersections. Classically one approach to understanding the geometry of space curves in $\mathbb{P}^3$ is by asking when the union of two (or more) curves forms a particular nice variety. Two curves $C, C' \subset \mathbb{P}^3$ are said to be linked via a complete intersection if $C \cup C'$ is a complete intersection.

While the liaison theory of curves in $\mathbb{P}^3$ is well understood the same theory for curves in other 3-folds (even $\mathbb{P}^1 \times \mathbb{P}^2$ remains quite mysterious. One reason for this stark difference is that the minimal graded free resolution of a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ is much less understood than for curves in $\mathbb{P}^3$. For example, classical theorems implies that curves in $\mathbb{P}^3$ ♠♠♠ Juliette: [FINISH THOUGHTS]

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In an ongoing program with Christine Berkesch and Patricia Klein I hope to use the newly developed theory of virtual resolutions to better understand the linkage theory curves in $\mathbb{P}^1 \times \mathbb{P}^2$.

Goal Theorem 2.2. *Let C and C' be linked curves in $\mathbb{P}^2 \times \mathbb{P}^1$ then C is virtually Cohen-Macaulay if and only if C' is virtually Cohen-Macaualy.*

Goal Theorem 2.3. *The even liaison classes of curves in $\mathbb{P}^1 \times \mathbb{P}^2$ satisfying some technical assumptions are in bijection with* ♠♠♠ Juliette: [finish]

2.2 Syzygies of Rational Curves via Maximal Rank

3. Bridges to Arithmetic Geometry

Over a finite field a number of classical statements from algebraic geometry no longer hold. For example, if X is a smooth projective variety of dimension n over $\mathbb{C}$ then Bertini's Theorem states that a generic hyperplane section of X (i.e. $X \cap H$ for a generic hyperplane $H \subset \mathbb{P}^n$) will be smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field. ♠♠♠ Juliette: [insert example]. Similar statements for connectedness, irreducibility, and other properties also fail over finite fields.

Historically, the lack of such Bertini theorems over finite fields has made many results in algebraic geometry over finite fields more complicated. (Bertini theorems are extremely useful as they often provide a basis for induction on dimension.) However, using an ingenious probabilistic argument Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of possibly arbitrarily large degree a version of Bertini's Theorem for Smoothness (highlighted above) is true. More specifically Poonen showed that as $d \to \infty$ the percentage of hypersurfaces $H \subset \mathbb{P}^n$ of degree d such that $X \cap H$ is smooth is

In recent years much work has gone into extending these result via NEDEDE

Theorem 3.1. *[Poonen] Let $X \subset \mathbb{P}^r_{\mathbf{F}_q}$ be a smooth projective variety of dimension n,....*

$$\lim_{d\to\infty} \operatorname{Prob}\left(\begin{matrix} f \in H^0(X, dA) \\ X \cap H_f \text{ is smooth of dimension } n-1 \end{matrix}\right) = \zeta_X(n+1)^{-1} > 0 \tag{1}$$

Corollary 3.2. *Let $X \subseteq \mathbb{P}^r_{\mathbf{F}_q}$ be a n-dimensional closed subscheme and let $k < n$. Then*

$$\lim_{d\to\infty} \frac{\operatorname{Prob}\left(\begin{matrix} (f_0,\dots,f_k) \text{ of degree } d \\ \text{are } \underline{\text{not}} \text{ parameters on } X \end{matrix}\right)}{q^{-(k+1)\binom{n-k+d}{n-k}}} = \#\left\{\begin{matrix} (n-k)\text{-planes } L \subseteq \mathbb{P}^r_{\mathbf{F}_q} \\ \text{such that } L \subseteq X \end{matrix}\right\}.$$

3.1 Uniform Bertini

Notice that in the statement of Poonen's Bertini theorem while the left hand side of equation ♠♠♠ Juliette: [CITE] is dependent of the embedding of X into projective space (i.e. it depends on our choice of very ample line bundle A) while the overall limit is itself independent of the embedding of X. This suggests that there may be a more general and uniform statement of Poonen's Bertini Theorem. That is one might hope that the analogous limit along any sequence $(L_d)_{d\in\mathbb{N}}$ of line bundles growing in positivity may limit to $\zeta_X(n+1)^{-1}$.

Work of Erman and Wood on semi-ample Bertini Theorems shows that such an analogue of Theorem 3.1 fails for sequences of line bundles, which do not grow in an ample fashion. Working with Isabel Vogt I have begun a project attempting to formalize and establish such a result.

Goal Theorem 3.3. *Let $X/\mathbf{F}_q$ be a smooth variety of dimension n. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of line bundles on X going to infinity in all directions then*

$$\lim_{d\to\infty} \operatorname{Prob}\left(\begin{matrix} f \in H^0(X, L_d) \\ X \cap H_f \text{ is smooth of dimension } n-1 \end{matrix}\right) = \zeta_X(n+1)^{-1} \tag{2}$$

4. Broader Impacts

4.1 Organizing

In the Spring of 2017 I organized *Math Careers Beyond Academia* (50 participants), a one day professional development conference on STEM careers outside of academia, and how graduate students should prepare for these careers. In April 2018 I organized a four day workshop focused on creating new packages for Macaulay2 – an open source computer algebra system – by bringing together over 45 developers and users of all skill levels and experience. Further in February 2019 I organized *Geometry and Arithmetic of Surfaces* (40 participants), a workshop providing a diverse group of early career researchers the opportunity to learn about interesting cutting edge topics in the arithmetic and algebraic geometry of surfaces from a diverse set of prominent active researchers. In April 2019 I organized the *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants) focused on forming a community of women and non-binary researchers interested in commutative algebra. In Fall 2019 I organized a *Special Session on Combinatorial Algebraic Geometry* at the AMS Fall Central Sectional.

4.2 Mentoring

Through Wisconsin's Directed Reading Program I have done reading courses with three undergraduates, one of whom I have been working with for over a year. I have been a mentor to two undergraduate women studying math via the AWM's Mentoring Network program. Further, through I have lead directed reading courses with three undergraduates. During the Fall 2018 semester, working with Girls' Math Night Out I mentored two high school aged girls leading them through a project exploring RSA cryptography.

4.3 A More Inclusive Learning Community.

During the Fall of 2016 in response to a growing climate of hate, bias, and discrimination on campus I pushed the Mathematics Department to form a committee on inclusivity and diversity. As a member of this committee I took the lead in drafting a statement on the department's commitment to inclusivity and non-discrimination that was accepted by the faculty at a department meeting. I also worked to create template syllabi statements that let students know about these department polices, and that inform them of other campus resources that may be helpful. All teachers within the department are now encouraged to use these statements.

Over the summer of 2017 I co-founded Out in Science, Technology, Engineering, and Mathematics at UW (oSTEM@UW) as a resource for LGBTQ+ students in STEM. During my time as vice president oSTEM@UW grew dramatically eventually having over fifty active members. The efficacy and importance of such a group has been made clear by the numerous student comments indicating how helpful and encouraging oSTEM@UW is to them. For example, after a meeting, a student emailed me to say "It made me very happy to see other friendly LGBTQ+ faces around, and I got to meet two people who were already in classes of mine! Thanks so much for organizing this stuff – it's really helpful for me personally, and I believe it was encouraging for the others attending as well."

This semester as the vice president of oSTEM@UW I organized for eleven members – including multiple undergraduates – to attend the annual national oSTEM Inc. conference. This four day conference with participants from around the world is intended to help individuals learn to build community and unity within the diverse LGBTQ+ family. It also has opportunities for participants to present their research, which a few of our members will be doing. For a couple of those UW -Madison students going, this is their first opportunity to talk about their research. I secured grants from on-campus and off-campus sources to defer the cost of attendance, and give these eleven students this amazing educational and social experience.

Since 2017 I have been the organizer of the campus social organization for LGBTQ+ graduate students and post-graduate students at the University of Wisconsin - Madison. In this role I have co-organized a weekly coffee social hour intended to give LGBTQ+ graduate and post-graduates students a place to relax, make friends, and discussion the challenges of being LGBTQ+ at the UW - Madison.

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