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Juliette Bruce and Daniel Erman

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A probabilistic approach to systems of parameters and Noether normalization

Juliette Bruce and Daniel Erman

We study systems of parameters over finite fields from a probabilistic perspective and use this to give the first effective Noether normalization result over a finite field. Our central technique is an adaptation of Poonen's closed point sieve, where we sieve over higher dimensional subvarieties, and we express the desired probabilities via a zeta function-like power series that enumerates higher dimensional varieties instead of closed points. This also yields a new proof of a recent result of Gabber, Liu and Lorenzini (2015) and Chinburg, Moret-Bailly, Pappas and Taylor (2017) on Noether normalizations of projective families over the integers.

Given an n-dimensional projective scheme $X \subseteq \mathbb{P}^r$ over a field, Noether normalization says that we can find homogeneous polynomials that induce a finite morphism $X \to \mathbb{P}^n$. Such a morphism is determined by a system of parameters, namely by choosing homogeneous polynomials $f_0, f_1, \dots, f_n$ of degree d where $X \cap V(f_0, f_1, \dots, f_n) = \varnothing$. Such a system of polynomials $f_0, f_1, \dots, f_n$ is a system of parameters on the homogeneous coordinate ring of X. More generally, for $k \leq n$ we say that $f_0, f_1, \dots, f_k$ *are parameters on* X if

$$\dim \mathbb{V}(f_0, f_1, \dots, f_k) \cap X = \dim X - (k+1).$$

By convention, the empty set has dimension -1.

Over an infinite field any generic choice of $\leq n+1$ linear polynomials will automatically be parameters on X. Over a finite field we can ask:

Questions 1.1. Let $\mathbb{F}_q$ be a finite field and $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme:

(1) What is the probability that a random choice $f_0, f_1, \dots, f_k$ of polynomials of degree d will be parameters on X?

(2) Can one effectively bound the degrees d for which such a finite morphism exists?

We will provide new insight into these questions by studying the distribution of systems of parameters from both a geometric and probabilistic viewpoint.

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For the geometric side, we fix a field $\boldsymbol{k}$ and let $S = \boldsymbol{k}[x_0, x_1, \ldots, x_r]$ be the coordinate ring of $\mathbb{P}^r_{\boldsymbol{k}}$. We write S_d for the vector space of degree d polynomials in S. In Section 4, we define a scheme $\mathscr{D}_{k,d}(X)$ parametrizing collections that do not form parameters. The $\boldsymbol{k}$-points of $\mathscr{D}_{k,d}(X)$ are

$$\mathscr{D}_{k,d}(X)(\boldsymbol{k}) = \{(f_0, f_1, \ldots, f_k) \text{ that are } \textit{not} \text{ parameters on } X\} \subset \underbrace{S_d \times \cdots \times S_d}_{k+1 \text{ copies}}.$$

We prove an elementary bound on the codimension of these closed subschemes of the affine space $S_d^{\oplus k+1}$.

Theorem 1.2. *Let $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ be an n-dimensional closed subscheme. We have*:

$$\operatorname{codim} \mathscr{D}_{k,d}(X) = \begin{cases} \geq \binom{n-k+d}{n-k} & \text{if } k < n, \\ = 1 & \text{if } k = n. \end{cases}$$

This generalizes several results from the literature: the case $k = n$ is a classical result about Chow forms [Gelfand et al. 1994, 3.2.B]. For $d = 1$ and $k < n$, the bound is sharp, by a classical result about determinantal varieties.[1] The bound for the case $k = 0$ appears in [Benoist 2011, Lemme 3.3]. If $k < n$, then the codimension grows as $d \to \infty$ and this factors into our asymptotic analysis over finite fields. It also leads to a uniform convergence result that allows us to go from a finite field to $\mathbb{Z}$.

For the probabilistic side, we work over a finite field $\mathbb{F}_q$ and compute the asymptotic probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X. The following result, which follows from known results in the literature, shows that there is a bifurcation between the $k = n$ and $k < n$ cases, reflecting Theorem 1.2.

Theorem 1.3 [Bucur and Kedlaya 2012; Poonen 2013]. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme. The asymptotic probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\lim_{d\to\infty} \operatorname{Prob}(f_0, f_1, \ldots, f_k \textit{ of degree } d \textit{ are parameters on } X) = \begin{cases} 1 & \text{if } k < n, \\ \zeta_X(n+1)^{-1} & \text{if } k = n, \end{cases}$$

where $\zeta_X(s)$ is the arithmetic zeta function of X.

The maximal case $k = n$ follows from the $k = m + 1$ case of Bucur and Kedlaya [2012, Theorem 1.2] (though they assume that X is smooth, their proof does not need that assumption when $k = m + 1$) and is proven using Poonen's closed point sieve. Moreover, the result in both cases could be derived from a slight modification of [Poonen 2013, Proof of Theorem 2.1]. See also [Charles and Poonen 2016, Corollary 1.4] for a similar result.

The main results in our paper stem from a deeper investigation of the cases where $k < n$, as the limiting value of 1 is only the beginning of the story. In the following theorem, we use $|Z|$ to denote the number of irreducible components of a scheme Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

[1] See [Bruns and Vetter 1988, Theorem 2.5] for a modern statement and proof. That result has a complicated history, discussed in [Bruns and Vetter 1988, Section 2.E], with some cases dating as far back as [Macaulay 1916, Section 53].

Theorem 1.4. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{array}\right) = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o(q^{-e(k+1)\binom{n-k+d}{n-k}}).$$

Theorem 1.4 illustrates that the probability of finding a sequence $f_0, f_1, \ldots, f_k$ of parameters on X is intimately tied to the codimension k geometry of X. Note that, by basic properties of the Hilbert polynomial, as $d \to \infty$ we have

$$h^0(Z, \mathcal{O}_Z(d)) = \frac{\deg(Z)}{(n-k)!} d^{n-k} + o(d^{n-k}) = \deg(Z)\binom{n-k+d}{n-k} + o(d^{n-k}).$$

It follows that the term $q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))}$ lies in $o(q^{-e(k+1)\binom{n-k+d}{n-k}})$ if and only if $\deg(Z) > e$.

For instance, setting $e = 1$, the sum simplifies to $1 - N \cdot q^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}})$, where N is the number of $(n-k)$-dimensional linear subspaces lying in X. It would thus be more difficult to find parameters on a variety X containing lots of linear spaces, as illustrated in Example 8.1. More generally, the probability of finding parameters for $k < n$ depends on a power series that counts the number of $(n-k)$-dimensional subvarieties of varying degrees, in analogue with the appearance of the zeta function in the $k = n$ case.

Our approach to Theorem 1.4 is motivated by a simple observation: $f_0, f_1, \ldots, f_k$ fail to be parameters if and only if they all vanish along some $(n-k)$-dimensional subvariety of X. We thus develop an analogue of Poonen's sieve where closed points are replaced by $(n-k)$-dimensional varieties. Sieving over higher dimensional varieties presents new challenges, especially bounding the error. This error depends on the Hilbert function of these varieties, and one key innovation is a uniform lower bound for Hilbert functions given in Lemma 3.1.

This perspective also leads to our second main result: an answer to Questions 1.1.(2) where the bound is in terms of the sum of the degrees of the irreducible components. If $X \subseteq \mathbb{P}^r$ has minimal irreducible components $V_1, V_2, \ldots, V_s$ (considered with the reduced scheme structure), then we define $\widehat{\deg}(X) := \sum_{i=1}^s \deg(V_i)$ (see Definition 2.2). We set $\log_q 0 = -\infty$.

Theorem 1.5. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ where $\dim X = n$. If $\max\{d, \frac{q}{d^n}\} \geq \widehat{\deg}(X)$ and*

$$d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$$

then there exist $f_0, f_1, \ldots, f_n$ of degree d^{n+1} inducing a finite morphism $\pi : X \to \mathbb{P}^n_{\mathbb{F}_q}$.

The bound is asymptotically optimal in q. Namely, if we fix $\widehat{\deg}(X)$, then as $q \to \infty$, the bound becomes $d = 1$. Thus, a linear Noether normalization exists if $q \gg \widehat{\deg}(X)$. For a fixed q, we expect the bound could be significantly improved. (Even the case $\dim X = 0$ would be interesting, as it is related to Kakeya type problems over finite fields [Ellenberg and Erman 2016; Ellenberg et al. 2010].)

Theorem 1.5 provides the first explicit bound for Noether normalization over a finite field. (One could potentially derive an explicit bound from Nagata's argument [1962, Chapter I.14], though the inductive nature of that construction would at best yield a bound that is multiply exponential in the largest degree of a defining equation of X.)

After computing the probabilities over finite fields, we combine these analyses and characterize the distribution of parameters on projective B-schemes where $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. We use standard notions of density for a subset of a free B-module; see Definition 7.1.

Corollary 1.6. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. If $X \subseteq \mathbb{P}^r_B$ is a closed subscheme whose general fiber over B has dimension n, then*

$$\lim_{d\to\infty} \text{Density}\left\{\begin{array}{l} f_0, f_1, \dots, f_k \textit{ of degree } d \textit{ that} \\ \textit{restrict to parameters on } X_p \textit{ for all } p \end{array}\right\} = \begin{cases} 1 & \textit{if } k < n, \\ 0 & \textit{if } k = n \textit{ and all } d. \end{cases}$$

The density over B thus equals the product over all the fibers of the asymptotic probabilities over $\mathbb{F}_q$. In the case $B = \mathbb{Z}$, our proof relies on Ekedahl's infinite Chinese remainder theorem [Ekedahl 1991, Theorem 1.2] combined with Proposition 5.1, which illustrates uniform convergence in p for the asymptotic probabilities in Theorem 1.3. In the case $B = \mathbb{F}_q[t]$, we use Poonen's analogue of Ekedahl's result [Poonen 2003, Theorem 3.1].

When $k = n$, an analogue of Corollary 1.6 for smoothness is given by Poonen [2004, Theorem 5.13]. Moreover, while it is unknown if there are any smooth hypersurfaces of degree > 2 over $\mathbb{Z}$ (see for example the discussion in [Poonen 2009]), the density zero subset from Corollary 1.6 turns out to be nonempty for large d. This leads to a new proof of a recent result about uniform Noether normalizations.

Corollary 1.7. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. Let $X \subseteq \mathbb{P}^r_B$ be a closed subscheme. If each fiber of X over B has dimension n, then for some d, there exist homogeneous polynomials $f_0, f_1, \dots, f_n \in B[x_0, x_1, \dots, x_r]$ of degree d inducing a finite morphism $\pi : X \to \mathbb{P}^n_B$.*

Corollary 1.7 is a special case of a recent result of Chinburg, Moret-Bailly, Pappas and Taylor [2017, Theorem 1.2] and of Gabber, Liu and Lorenzini [2015, Theorem 8.1]. This corollary can fail when B is any of $\mathbb{Q}[t]$ or $\mathbb{Z}[t]$ or $\mathbb{F}_q[s, t]$, as in those cases, the Picard group of a finite cover of Spec B can fail to be torsion. See Section 8 for explicit examples and counterexamples and see [Chinburg et al. 2017; Gabber et al. 2015] for generalizations and applications.

There are a few earlier results related to Noether normalization over the integers. For instance [Moh 1979] shows that Noether normalizations of semigroup rings always exist over $\mathbb{Z}$; and [Nagata 1962, Theorem 14.4] implies that given a family over any base, one can find a Noether normalization over an open subset of the base. Relative Noether normalizations play a key role in [Achinger 2015, Section 5]. There is also the incorrect claim in [Zariski and Samuel 1960, page 124] that Noether normalizations exist over any infinite domain (see [Abhyankar and Kravitz 2007]). Brennan and Epstein [2011] analyze the distribution of systems of parameters from a different perspective, introducing the notion of a generic matroid to relate various different systems of parameters. In addition, after our paper was posted, work of

Charles [2017] on arithmetic Bertini theorems appeared which, under the additional hypothesis that X is integral and flat, implies a stronger version of Corollary 1.6 where one also obtains bounds on the norms of the functions.

This paper is organized as follows. Section 2 gathers background results and Section 3 involves a key lower bound on Hilbert functions. Section 4 contains our geometric analysis of parameters including a proof of Theorem 1.2. Sections 5 and 6 contain the probabilistic analysis of parameters over finite fields: Section 5 proves Theorem 1.3 and Theorem 1.5 while Section 6 gives the more detailed description via an analogue of the zeta function enumerating $(n-k)$-dimensional subvarieties, including the proof of Theorem 1.4. Section 7 contains our analysis over $\mathbb{Z}$ including proofs of Corollaries 1.6 and 1.7 and related corollaries. Section 8 contains examples.

2. Background

In this section, we gather some algebraic and geometric facts that we will cite throughout.

Lemma 2.1. *Let $\boldsymbol{k}$ be a field and let R be a $(k+1)$-dimensional graded $\boldsymbol{k}$-algebra where $R_0 = \boldsymbol{k}$. If $f_0, f_1, \ldots, f_k$ are homogeneous elements of degree d and $R/\langle f_0, f_1, \ldots, f_k\rangle$ has finite length, then the extension $\boldsymbol{k}[z_0, z_1, \ldots, z_k] \to R$ given by $z_i \mapsto f_i$ is a finite extension.*

Proof. See [Bruns and Herzog 1993, Theorem 1.5.17]. □

This lemma implies that if $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ has dimension n, and if $f_0, f_1, \ldots, f_n$ are parameters on X, then the map $\phi\colon X \to \mathbb{P}^n_{\boldsymbol{k}}$ given by sending $x \mapsto [f_0(x) : f_1(x) : \cdots : f_n(x)]$ is a finite morphism. In particular, if R is the homogeneous coordinate of X, then the ideal $\langle f_0, f_1, \ldots, f_n\rangle \subseteq R$ has finite colength, and thus the base locus of ϕ is the empty set. In other words, ϕ defines a genuine morphism. Moreover, the lemma shows that the corresponding map of coordinate rings $\phi^\sharp : R \to \boldsymbol{k}[z_0, z_1, \ldots, z_n]$ is finite, and this implies that ϕ is finite.

Definition 2.2. Let $X \subseteq \mathbb{P}^r$ be a projective scheme with minimal irreducible components $V_1, \ldots, V_s$ (considered with the reduced scheme structure). We define $\widehat{\deg}(X) := \sum_{i=1}^{s} \deg(V_i)$. For a subscheme $X' \subseteq \mathbb{A}^r$ with projective closure $\overline{X}' \subseteq \mathbb{P}^r$ we define $\widehat{\deg}(X') := \widehat{\deg}(\overline{X}')$.

This provides a notion of degree which ignores nonreduced structure but takes into account components of lower dimension. Similar definitions have appeared in the literature: for instance, in the language of [Bayer and Mumford 1993, Section 3], we would have $\widehat{\deg}(X) = \sum_{j=0}^{\dim X} \text{geom-deg}_j(X)$.

Lemma 2.3. *Let $\boldsymbol{k}$ be any field and let $X \subseteq \mathbb{A}^r_{\boldsymbol{k}}$. Let $f_0, f_1, \ldots, f_t$ be polynomials in $\boldsymbol{k}[x_1, \ldots, x_r]$. If $X' = X \cap \mathbb{V}(f_0, f_1, \ldots, f_t)$, then $\widehat{\deg}(X') \le \widehat{\deg}(X) \cdot \prod_{i=0}^{t} \deg(f_i)$.*

Proof. This follows from the refined version of Bezout's theorem [Fulton 1984, Example 12.3.1]. □

3. A uniform lower bound on Hilbert functions

For a subscheme of $\mathbb{P}^r$, the Hilbert function in degree d is controlled by the Hilbert polynomial, at least if d is very large related to some invariants of the subscheme. We analyze the Hilbert function at the other extreme, where the degree of the subscheme may be much larger than d. The following lemma, which applies to subschemes of arbitrarily high degree, provides uniform lower bounds that are crucial to bounding the error in our sieves.

Lemma 3.1. *Let $\boldsymbol{k}$ be an arbitrary field and fix some $e \geq 0$. Let $V \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ be any closed, m-dimensional subscheme of degree $> e$ with homogeneous coordinate ring R:*

(1) *We have* $\dim R_d \geq h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$ *for all d.*

(2) *For any $0 < \epsilon < 1$, there exists a constant C depending only on m and ϵ (but not on d or $\boldsymbol{k}$ or R) such that*

$$\dim R_d > (e+\epsilon)\cdot h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$$

for all $d \geq Ce^{m+1}$.

Proof. If $\boldsymbol{k}'$ is a field extension of $\boldsymbol{k}$, then the Hilbert series of R is the same as the Hilbert series of $R \otimes_{\boldsymbol{k}} \boldsymbol{k}'$. We can thus assume that $\boldsymbol{k}$ is an infinite field. For part (1), we simply take a linear Noether normalization $\boldsymbol{k}[t_0, t_1, \dots, t_m] \subseteq R$ of the ring R [Eisenbud 1995, Theorem 13.3]. This yields $\boldsymbol{k}[t_0, t_1, \dots, t_m]_d \subseteq R_d$, giving the statement about Hilbert functions.

We prove part (2) of the lemma by induction on m. Let $S = \boldsymbol{k}[x_0, x_1, \dots, x_r]$ and let $I_V \subseteq S$ be the saturated, homogeneous ideal defining V. Thus $R = S/I_V$. If $m = 0$, then we have $\dim R_d \geq \min\{d+1, \deg V\} \geq \min\{d+1, e+1\}$ which is at least $e+\epsilon$ for all $d \geq e$. This proves the case $m = 0$, where the constant C can be chosen to be 1.

Now assume the claim holds for all closed subschemes of dimension less than m. Let $V \subset \mathbb{P}^r_{\boldsymbol{k}}$ be a closed subscheme with $\dim V = m \geq 1$. Fix $0 < \epsilon < 1$. Since we are working over an infinite field, [Eisenbud 1995, Lemma 13.2(c)] allows us to choose a linear form ℓ that is a nonzero divisor in R. This yields a short exact sequence $0 \to R(-1) \xrightarrow{\cdot \ell} R \to R/\ell \to 0$. Since $R/\ell = S/(I_V + \langle \ell \rangle)$, this yields the equality

$$\dim R_i = \dim R_{i-1} + \dim(S/(I_V + \langle \ell \rangle))_i. \tag{1}$$

Letting $W = V \cap V(\ell)$ we know that $\dim W = m-1$ and $\deg W = \deg V$. Moreover, if I_V is the saturated ideal defining V and if I_W is the saturated ideal defining W, then since I_W contains $I_V + \langle \ell \rangle$, we have $\dim(S/(I_V + \langle \ell \rangle))_i \geq \dim(S/I_W)_i$. Combining with (1) yields

$$\dim R_i \geq \dim R_{i-1} + \dim(S/I_W)_i. \tag{2}$$

Now, by induction, in the case $m-1$ and $\epsilon' := \frac{1+\epsilon}{2}$, there exists C' depending on ϵ' and $m-1$ (or equivalently depending on ϵ and m) where

$$\dim(S/I_W)_i \geq (e+\epsilon')\binom{m-1+i}{m-1} \tag{3}$$

for all $i \geq C'e^m$. Now let $d \geq C'e^m$. Iteratively applying (2) for $i = d, d-1, d-2, \ldots, \lceil C'e^m \rceil$, we obtain:

$$\dim R_d \geq \dim R_{\lceil C'e^m \rceil - 1} + \sum_{i=\lceil C'e^m \rceil}^{d} \dim(S/I_W)_i.$$

By dropping the $\dim R_{\lceil C'e^m \rceil - 1}$ term and applying (3), we conclude that

$$\dim R_d \geq \sum_{i=\lceil C'e^m \rceil}^{d} (e+\epsilon')\binom{m-1+i}{m-1}.$$

The identity $\sum_{i=a}^{b} \binom{i+k}{k} = \binom{b+k+1}{k+1} - \binom{a+k}{k+1}$ implies that $\sum_{i=\lceil C'e^m \rceil}^{d} (e+\epsilon')\binom{m-1+i}{m-1}$ can be rewritten as $(e+\epsilon')\left(\binom{m+d}{m} - \binom{m-1+\lceil C'e^m \rceil}{m}\right)$. There exists a constant C depending on ϵ and m so that $(\epsilon' - \epsilon)\binom{m+d}{m} = \left(\frac{1}{2} - \frac{\epsilon}{2}\right)\binom{m+d}{m} \geq (e+\epsilon')\binom{m-1+\lceil C'e^m \rceil}{m}$ for all $d \geq \lceil Ce^{m+1} \rceil$. Thus, for all $d \geq \lceil Ce^{m+1} \rceil$ we have

$$\dim R_d \geq (e+\epsilon')\binom{m+d}{m} - (\epsilon' - \epsilon)\binom{m+d}{d} = (e+\epsilon)\binom{m+d}{m}. \qquad \square$$

Remark 3.2. Asymptotically in e, the bound of Ce^2 is the best possible for curves. For instance, let $C \subseteq \mathbb{P}^r$ be a curve of degree $(e+1)$ lying inside some plane $\mathbb{P}^2 \subseteq \mathbb{P}^r$. Let R be the homogeneous coordinate ring of C. If $d \geq e$ then the Hilbert function is given by

$$\dim R_d = (e+1)d - \tfrac{e^2-e}{2}.$$

Thus, if we want $\dim R_d \geq (e+\epsilon)(d+1)$, we will need to let $d \geq (e^2+e+2\epsilon)/(2(1-\epsilon)) \approx \frac{1}{2}e^2$. It would be interesting to know if the bound Ce^{m+1} is the best possible for higher dimensional varieties.

4. Geometric analysis

In this section we analyze the geometric picture for the distribution of parameters on X. The basic idea behind the proof of Theorem 1.2 is that $f_0, f_1, \ldots, f_k$ fail to be parameters on X if and only if they all vanish along some $(n-k)$-dimensional subvariety of X. Since the Hilbert polynomial of a $(n-k)$-dimensional variety grows like d^{n-k}, when we restrict a degree d polynomial f_j to such a subvariety, it can be written in terms of $\approx d^{n-k}$ distinct monomials. The polynomial f_j will all vanish along the subvariety if and only if all of the $\approx d^{n-k}$ coefficients vanish. This rough estimate explains the growth of the codimension of $\mathscr{D}_{k,d}(X)$ as $d \to \infty$.

We begin by constructing the schemes $\mathscr{D}_{k,d}(X)$. Fix $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ a closed subscheme of dimension n over a field $\boldsymbol{k}$. Given $k < n$ and $d > 0$, let $\mathscr{A}_{k,d}$ be the affine space $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))^{\oplus k+1}$ and $\boldsymbol{k}[c_{0,1}, \ldots, c_{k,\binom{r+d}{d}}]$ be the corresponding polynomial ring. We enumerate the monomials in $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ as $m_1, \ldots, m_{\binom{r+d}{d}}$, and then define the universal polynomial

$$F_i := \sum_{j=1}^{\binom{r+d}{d}} c_{i,j} m_j \in \boldsymbol{k}[c_{0,1}, \ldots, c_{k,\binom{r+d}{d}}] \otimes_{\boldsymbol{k}} \boldsymbol{k}[x_0, x_1, \ldots, x_r].$$

Given a closed point $c \in \mathscr{A}_{k,d}$ we can specialize $F_0, F_1, \dots, F_k$ and obtain polynomials $f_0, f_1, \dots, f_k \in \kappa(c)[x_0, x_1, \dots, x_r]$, where $\kappa(c)$ is the residue field of c. We will thus identify each element of $\mathscr{A}_{k,d}(\boldsymbol{k})$ with a collection of polynomials $\boldsymbol{f} = (f_0, f_1, \dots, f_k) \in \boldsymbol{k}[x_0, x_1, \dots, x_r]$.

Now define $\Sigma_{k,d}(X) \subseteq X \times \mathscr{A}_{k,d}$ via the equations $F_0, F_1, \dots, F_k$. Consider the second projection $p_2\colon \Sigma_{k,d}(X) \to \mathscr{A}_{k,d}$. Given a point $\boldsymbol{f} = (f_0, f_1, \dots, f_k) \in \mathscr{A}_{k,d}$, the fiber $p_2^{-1}(\boldsymbol{f}) \subseteq X$ can be identified with the points lying in $X \cap \mathbb{V}(f_0, f_1, \dots, f_k)$. For generic choices of $\boldsymbol{f}$ (after passing to an infinite field if necessary) the polynomials $f_0, f_1, \dots, f_k$ will define an ideal of codimension $k+1$, and thus the fiber $p_2^{-1}(\boldsymbol{f})$ will have dimension $n-k-1$.

There is a closed sublocus $\mathscr{D}_{k,d}(X) \subsetneq \mathscr{A}_{k,d}$ where the dimension of the fiber is at least $n-k$, and we give $\mathscr{D}_{k,d}(X)$ the reduced scheme structure. It follows that $\mathscr{D}_{k,d}(X)$ parametrizes collections $\boldsymbol{f} = (f_0, f_1, \dots, f_k)$ of degree d polynomials which fail to be parameters on X.

Remark 4.1. If we fix $X_{\mathbb{Z}} \subseteq \mathbb{P}^r_{\mathbb{Z}}$, then we can follow the same construction to obtain a scheme $\mathscr{D}_{k,d}(X_{\mathbb{Z}}) \subseteq \mathscr{A}_{k,d}$. Writing $X_{\boldsymbol{k}}$ as the pullback $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\boldsymbol{k}$, we observe that the equations defining $\Sigma_{k,d}(X_{\boldsymbol{k}})$ are obtained by pulling back the equations defining $\Sigma_{k,d}(X_{\mathbb{Z}})$. It follows that $\mathscr{D}_{k,d}(X_{\mathbb{Z}}) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\boldsymbol{k})$ has the same set-theoretic support as $\mathscr{D}_{k,d}(X_{\boldsymbol{k}})$.

Definition 4.2. We let $\mathscr{D}^{\text{bad}}_{k,d}(X)$ be the locus of points in $\mathscr{D}_{k,d}(X)$ where $f_0, f_1, \dots, f_{k-1}$ already fail to be parameters on X and let $\mathscr{D}^{\text{good}}_{k,d}(X) := \mathscr{D}_{k,d}(X) \setminus \mathscr{D}^{\text{bad}}_{k,d}(X)$. We set $\mathscr{D}^{\text{bad}}_{0,d}(X) = \varnothing$.

Remark 4.3. We have a factorization:

$$\begin{aligned} \mathscr{A}_{k,d} &\to \mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d} \\ (f_0, f_1, \dots, f_k) &\mapsto ((f_0, f_1, \dots, f_{k-1}), f_k). \end{aligned}$$

We let $\pi : \mathscr{D}_{k,d}(X) \to \mathscr{A}_{k-1,d}$ be the induced projection, which will we use to work inductively.

Proof of Theorem 1.2. First consider the case $k = n$. There is a natural rational map from $\mathscr{A}_{n,d}$ to the Grassmanian $\operatorname{Gr}(n+1, S_d)$ given by sending the point $(f_0, f_1, \dots, f_n) \in \mathscr{A}_{n,d}$ to the linear space that those polynomials span. Inside of the Grassmanian, the locus of choices of $(f_0, f_1, \dots, f_n)$ that all vanish on a point of X is a divisor in the Grassmanian defined by the Chow form; see [Gelfand et al. 1994, 3.2.B]. The preimage of this hypersurface in $\mathscr{A}_{n,d}$ is a hypersurface contained in $\mathscr{D}_{n,d}(X)$, and thus $\mathscr{D}_{n,d}(X)$ has codimension 1.

For $k < n$, we will induct on k. Let $k = 0$. A polynomial f_0 will fail to be a parameter on X if and only if $\dim X = \dim(X \cap \mathbb{V}(f_0))$. This happens if and only if f_0 is a zero divisor on a top-dimensional component of X. Let V be the reduced subscheme of some top-dimensional irreducible component of X and let $\mathcal{I}_V$ be the defining ideal sheaf of V. Then the set of zero divisors of degree d on V will form a linear subspace in $\mathscr{A}_{0,d}$ corresponding to the elements of the vector subspace $H^0(\mathcal{I}_V(d))$. The codimension of $H^0(\mathcal{I}_V(d)) \subseteq S_d$ is precisely given by the Hilbert function of the homogeneous coordinate ring of V in degree d. By applying Lemma 3.1(1), we conclude that for all d this linear space has codimension at least $\binom{n+d}{d}$. Since $\mathscr{D}_{0,d}(X)$ is the union of these linear spaces over all top-dimensional components of X, this proves that $\operatorname{codim} \mathscr{D}_{0,d}(X) \geq \binom{n+d}{d}$.

Take the induction hypothesis that we have proven the statement for $\mathscr{D}_{j,d}(X')$ for all $X' \subseteq \mathbb{P}^r$ and all $j \le k-1$. We separate $\mathscr{D}_{k,d}(X) = \mathscr{D}_{k,d}^{\mathrm{bad}}(X) \sqcup \mathscr{D}_{k,d}^{\mathrm{good}}(X)$ and will show that each locus has sufficiently large codimension. We begin with $\mathscr{D}_{k,d}^{\mathrm{bad}}(X)$. By using the factorization from Remark 4.3, we can realize $\mathscr{D}_{k,d}^{\mathrm{bad}}(X) \subseteq \mathscr{A}_{k,d} \cong \mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d}$. By definition of $\mathscr{D}_{k,d}^{\mathrm{bad}}(X)$, the image of $\mathscr{D}_{k,d}^{\mathrm{bad}}(X)$ in $\mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d}$ is $\mathscr{D}_{k-1,d}(X) \times \mathscr{A}_{0,d}$. It follows that

$$\operatorname{codim}(\mathscr{D}_{k,d}^{\mathrm{bad}}(X), \mathscr{A}_{k,d}) = \operatorname{codim}(\mathscr{D}_{k-1,d}(X), \mathscr{A}_{k-1,d}) \ge \binom{n-k+1+d}{n-k+1} \ge \binom{n-k+d}{n-k},$$

where the middle inequality follows by induction.

Now consider an arbitrary point $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ in $\mathscr{D}_{k,d}^{\mathrm{good}}(X)$. By definition, $f_0, f_1, \ldots, f_{k-1}$ are parameters on X, and thus $\pi(\boldsymbol{f}) \in \mathscr{A}_{k-1,d} \setminus \mathscr{D}_{k-1,d}(X)$. Using the splitting of Remark 4.3, the fiber of $\mathscr{D}_{k,d}^{\mathrm{good}}(X)$ over $\boldsymbol{f}$ can be identified with $\mathscr{D}_{0,d}(X')$ where $X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{k-1})$. Since $(f_0, f_1, \ldots, f_{k-1}) \notin \mathscr{D}_{k-1,d}(X)$, we have that $\dim X' = n-k$. The inductive hypothesis thus guarantees that $\operatorname{codim} \mathscr{D}_{0,d}(X') \ge \binom{\dim X' + d}{d} = \binom{n-k+d}{d}$. □

5. Probabilistic analysis, I: Proof of Theorem 1.3

The main result of this section is Proposition 5.1 which provides an effective bound for finding parameters, and which we will use to prove Theorem 1.5. We also use this to give a new proof of Theorem 1.3 for $k < n$. Throughout this section, we let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a projective scheme of dimension n over a finite field $\mathbb{F}_q$. Recall that $S_d = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$. We define

$$\mathrm{Par}_{d,k} = \{f_0, f_1, \ldots, f_k \text{ that are parameters on } X\} \subset S_d^{k+1}.$$

In Theorem 1.3, we compute the following limit (which a priori might not exist):

$$\lim_{d\to\infty} \mathrm{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are parameters on } X) := \lim_{d\to\infty} \frac{\#\mathrm{Par}_{d,k}}{\# S_d^{k+1}}.$$

Proposition 5.1. *If $k < n$ then*

$$\mathrm{Prob}(f_0, f_1, \ldots, f_k \textit{ of degree } d \textit{ are parameters on } X) \ge 1 - \widehat{\deg}(X)(1 + d + d^2 + \cdots + d^k) q^{-\binom{n-k+d}{n-k}}.$$

Proof. We induct on k and largely follow the structure of the proof of Theorem 1.2. First, let $k = 0$. A polynomial f_0 will fail to be a parameter on X if and only if it is a zero divisor on a top-dimensional component V of X. There are at most $\widehat{\deg}(X)$ many such components. As argued in the proof of Theorem 1.2, the set of zero divisors on V corresponds to the elements of $H^0(\mathbb{P}^r, \mathcal{I}_V(d))$ which has codimension at least $\binom{n+d}{d}$ in S_d. It follows that

$$\mathrm{Prob}(f_0 \text{ of degree } d \text{ is not a parameter on } X) \le \widehat{\deg}(X) q^{-\binom{n+d}{d}}.$$

Now consider the induction step. We will separately compute the probability that $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ lies in $\mathscr{D}_{k,d}^{\mathrm{bad}}(X)$ and the probability that $\boldsymbol{f}$ lies in $\mathscr{D}_{k,d}^{\mathrm{good}}(X)$. By definition, the projection π maps $\mathscr{D}_{k,d}^{\mathrm{bad}}(X)$

onto $\mathscr{D}_{k-1,d}(X)$, and by induction

$$\begin{aligned}\operatorname{Prob}(\pi(\boldsymbol{f}) \in \mathscr{P}_{k-1,d}(X)(\mathbb{F}_q)) &\leq \widehat{\deg}(X)(1+d+d^2+\cdots+d^{k-1})q^{-\binom{n-k+1+d}{n-k+1}} \\ &\leq \widehat{\deg}(X)(1+d+d^2+\cdots+d^{k-1})q^{-\binom{n-k+d}{n-k}}.\end{aligned}$$

We now assume $\boldsymbol{f} \notin \mathscr{D}^{\text{bad}}_{k,d}(X)$. We thus have that $f_0, f_1, \ldots, f_{k-1}$ are parameters on X. As in the proof of Theorem 1.2, the fiber $\pi^{-1}(\boldsymbol{f})$ can be identified with $\mathscr{D}_{0,d}(X')$ where $X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{k-1})$. By construction $\dim X' = n-k$ and by Lemma 2.3, $\widehat{\deg}(X') \leq \widehat{\deg}(X) \cdot d^k$. Our inductive hypothesis thus implies that

$$\operatorname{Prob}\left(\begin{array}{l}(f_0, f_1, \ldots, f_k) \in \mathscr{D}_{k,d}(X)(\mathbb{F}_q) \text{ given that} \\ (f_0, f_1, \ldots, f_{k-1}) \notin \mathscr{D}_{k-1,d}(X)(\mathbb{F}_q)\end{array}\right) \leq \widehat{\deg}(X')q^{-\binom{n-k+d}{n-k}} \leq \widehat{\deg}(X) \cdot d^k q^{-\binom{n-k+d}{n-k}}.$$

Combining the estimates for $\mathscr{D}^{\text{bad}}_{k,d}(X)$ and $\mathscr{D}^{\text{good}}_{k,d}(X)$ yields the proposition. □

Proof of Theorem 1.3. If $k < n$, then we apply Proposition 5.1 to obtain

$$\lim_{d\to\infty} \operatorname{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are parameters on } X\end{array}\right) \geq \lim_{d\to\infty} 1 - \widehat{\deg}(X)(d^0 + d^1 + \cdots + d^k)q^{-\binom{n-k+d}{n-k}} = 1.$$

Now let $k = n$. For completeness, we summarize the proof of [Bucur and Kedlaya 2012, Theorem 1.2]. We fix e, which will go to ∞, and separate the argument into low, medium, and high degree cases.

Low degree argument. For a zero dimensional subscheme Y, we have that S_d surjects on $H^0(Y, \mathcal{O}_Y(d))$ when $d \geq \deg Y - 1$ [Poonen 2004, Lemma 2.1]. So if $d > \deg P - 1$, the probability that $f_0, f_1, \ldots, f_n$ all vanish at a closed point $P \in X$ is $1 - q^{-(n+1)\deg P}$. If $Y \subseteq X$ is the union of all points of degree $\leq e$, and if $d \geq \deg Y - 1$, then the surjection onto $H^0(Y, \mathcal{O}_Y(d))$ implies that the probabilities at the points $P \in Y$ behave independently. This yields:

$$\operatorname{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ are parameters on } X \\ \text{at all points } P \in X \text{ where } \deg(P) \leq e\end{array}\right) = \prod_{\substack{P \in X \\ \deg(P) \leq e}} 1 - q^{-(n+1)\deg P}.$$

Medium degree argument. Our argument is nearly identical to [Poonen 2004, Lemma 2.4], and covers all points whose degree lies in the range $\left[e+1, \frac{d}{n+1}\right]$. For any such point $P \in X$, S_d surjects onto $H^0(P, \mathcal{O}_P(d))$ and thus the probability that $f_0, f_1, \ldots, f_n$ all vanish at P is $q^{-\ell(n+1)}$. By [Lang and Weil 1954], $\#X(\mathbb{F}_{q^\ell}) \leq Kq^{\ell n}$ for some constant K independent of ℓ. We have

$$\begin{aligned}\operatorname{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ all vanish} \\ \text{at some } P \in X \text{ where } e < \deg(P) \leq \left\lfloor \frac{d}{n+1} \right\rfloor\end{array}\right) &\leq \sum_{\ell=e+1}^{\lfloor \frac{d}{n+1} \rfloor} \#X(\mathbb{F}_{q^\ell})q^{-\ell(n+1)} \\ &\leq \sum_{\ell=e+1}^{\infty} Kq^{\ell n}q^{-(n+1)\ell} \\ &= \frac{Kq^{-e-1}}{1-q^{-1}}.\end{aligned}$$

This tends to 0 as $e \to \infty$, and therefore does not contribute to the asymptotic limit.

High degree argument. By the case when $k = n - 1$, we may assume that $f_0, f_1, \dots, f_{n-1}$ form a system of parameters with probability $1 - o(1)$. So we let V be one of the irreducible components of this intersection (over $\mathbb{F}_q$) and we let R be its homogeneous coordinate ring. If $\deg V \le \frac{d}{n+1}$, then it can be ignored as we considered such points in the low and medium degree cases. Hence, we can assume $\deg V > \frac{d}{n+1}$. Since $\dim R_\ell \ge \min\{\ell + 1, \deg R\}$ for all ℓ, the probability that f_n vanishes along V is at most $q^{-\lfloor d/(n+1)\rfloor - 1}$. Hence the probability of vanishing on some high degree point is bounded by $O(d^n q^{-\lfloor d/(n+1)\rfloor - 1})$ which is $o(1)$ as $d \to \infty$.

Combining the various parts as $e \to \infty$, we see that the low degree argument converges to $\zeta_X(n+1)^{-1}$ and the contributions from the medium and high degree points go to 0. □

Remark 5.2. It might be interesting to consider variants of Theorem 1.3 that allow imposing conditions along closed subschemes, similar to Poonen's Bertini with Taylor coefficients [Poonen 2004, Theorem 1.2]. For instance, [Kedlaya 2005, Theorem 1] might be provable by such an approach, though this would be more complicated than the original proof.

Proposition 5.1 yields an effective bound on the degree of a full system of parameters over a finite field. Sharper bounds can be obtained if one allows the f_i to have different degrees.

Corollary 5.3. (1) *If d_1 satisfies $d_1^{n-1} q^{-d_1 - 1} < (n \cdot \widehat{\deg}(X))^{-1}$, then there exist $g_0, g_1, \dots, g_{n-1}$ of degree d_1 that are parameters on X.*

(2) *Let X' be 0-dimensional. If $\max\{d_2 + 1, q\} \ge \widehat{\deg}(X')$ then there exists a degree d_2 parameter on X'.*

Proof. Applying Proposition 5.1 in the case $k = n - 1$ yields (1). For (2), let f be a random degree d polynomial and let $P \in X'$ be a closed point. Since the dimension of the image of S_d in $H^0(P, \mathcal{O}_P(d))$ is at least $\min\{d + 1, \deg P\}$, the probability that f vanishes at P is at worst $q^{-\min\{d+1, \deg P\}}$ which is at least q^{-1}. It follows that the probability that a degree d function vanishes on some point of X' is at worst $\sum_{P \in X'} q^{-1} \le \widehat{\deg}(X') q^{-1}$. Thus if $q > \widehat{\deg}(X')$, this happens with probability strictly less than 1. On the other hand, if $d + 1 \ge \widehat{\deg}(X')$ then polynomials of degree d surject onto $H^0(X', \mathcal{O}_{X'}(d))$ and hence we can find a parameter on X' by choosing a polynomial that restricts to a unit on X'. □

Proof of Theorem 1.5. If $\dim X = 0$, then we can directly apply Corollary 5.3(2) to find a parameter of degree d. So we assume $n := \dim X > 0$. Since $d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$ it follows that $(n \cdot \widehat{\deg}(X))^{-1} > q^{-d} d^n > q^{-d-1} d^{n-1}$. Applying Corollary 5.3(1), we find $g_0, g_1, \dots, g_{n-1}$ in degree d that are parameters on X. Let $X' = X \cap V(g_0, g_1, \dots, g_{n-1})$. Since $\max\{d, \frac{q}{d^n}\} \ge \widehat{\deg}(X)$ it follows that $\max\{d^{n+1}, q\} \ge d^n \widehat{\deg}(X) \ge \widehat{\deg}(X')$, and Corollary 5.3(2) yields a parameter g_n of degree d^{n+1} on X'. Thus $g_0^{d^n}, g_1^{d^n}, \dots, g_{n-1}^{d^n}, g_n$ are parameters of degree d^{n+1} on X. □

6. Probabilistic analysis, II: The error term and proof of Theorem 1.4

In this section, we let $k < n$ and we analyze the error terms in Theorem 1.3 more precisely. In particular, we prove Theorem 1.4, which shows that the probabilities are controlled by the probability of vanishing along an $(n-k)$-dimensional subvariety, with varieties of lowest degree contributing the most.

Our proof of Theorem 1.4 adapts Poonen's sieve in a couple of key ways. The first big difference is that instead of sieving over closed points, we will sieve over $(n-k)$-dimensional subvarieties of X; this is because polynomials $f_0, f_1, \dots, f_k$ will fail to be parameters on X only if they all vanish along some $(n-k)$-dimensional subvariety.

The second difference is that the resulting probability formula will not be a product of local factors. This is because the values of a function can never be totally independent along two higher dimensional varieties with a nontrivial intersection. For instance, Lemma 6.1 shows that the probability that a degree d polynomial vanishes along a line is $q^{-(d+1)}$, but the probability of vanishing along two lines that intersect in a point is $q^{-(2d+1)} > (q^{-(d+1)})^2$.

The following result characterizes the individual probabilities arising in our sieve.

Lemma 6.1. *If $Z \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ is a reduced, projective scheme over a finite field $\mathbb{F}_q$ with homogeneous coordinate ring R then*

$$\operatorname{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ all vanish along } Z) = \left(\frac{1}{\#R_d}\right)^{k+1}.$$

If d is at least the Castelnuovo–Mumford regularity of the ideal sheaf of Z, then

$$\operatorname{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ all vanish along } Z) = q^{-(k+1)h^0(Z,\mathcal{O}_Z(d))}.$$

Proof. Let $I \subseteq S$ be the homogeneous ideal defining Z, so that $R = S/I$. An element $h \in S_d$ vanishes along Z if and only if it restricts to 0 in R_d i.e., if and only if it lies in I_d. Since we have an exact sequence of $\mathbb{F}_q$-vector spaces:

$$0 \to I_d \to S_d \to R_d \to 0$$

we obtain

$$\operatorname{Prob}(h \text{ vanishes on } Z) = \frac{\#I_d}{\#S_d} = \frac{1}{\#R_d}.$$

For $k+1$ elements of S_d, the probabilities of vanishing along Z are independent and this yields the first statement of the lemma.

We write $\tilde{I}$ for the ideal sheaf of Z. If d is at least the regularity of $\tilde{I}$, then $H^1(\mathbb{P}^r_{\mathbb{F}_q}, \tilde{I}(d)) = 0$. Hence there is a natural isomorphism between R_d and $H^0(Z, \mathcal{O}_Z(d))$. Thus, we have

$$\frac{1}{\#R_d} = q^{-h^0(Z,\mathcal{O}_Z(d))},$$

yielding the second statement. □

Proof of Theorem 1.4. Throughout the proof, we set $\epsilon_{e,k}$ to be the error term for a given e and k, namely $\epsilon_{e,k} := q^{-e(k+1)\binom{n-k+d}{n-k}}$. We also set:

$$\begin{aligned}
\mathrm{Par}_{d,k} &:= \{f_0, f_1, \ldots, f_k \text{ are parameters on } X\} \\
\mathrm{Low}_{d,k,e} &:= \left\{\begin{array}{l} f_0, f_1, \ldots, f_k \text{ all vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } \deg(Z) \le e \end{array}\right\} \\
\mathrm{Med}_{d,k,e} &:= \left\{\begin{array}{l} (f_0, f_1, \ldots, f_k) \notin \mathrm{Low}_{d,k,e} \text{ which all vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } e < \deg(Z) \le e(k+1) \end{array}\right\} \\
\mathrm{High}_{d,k,e} &:= \left\{\begin{array}{l} (f_0, f_1, \ldots, f_k) \notin \mathrm{Low}_{d,ke} \cup \mathrm{Med}_{d,k,e} \text{ which all vanish along} \\ \text{a variety } Z \text{ where } \dim Z = (n-k) \text{ and } e(k+1) < \deg(Z) \end{array}\right\}.
\end{aligned}$$

Note that if $f_0, f_1, \ldots, f_k$ all vanish along a variety of dimension $> n-k$ then they will also all vanish along a high degree variety, and hence we do not need to count this case separately. For $\boldsymbol{f} = f_0, f_1, \ldots, f_k \in S_d^{k+1}$, we thus have

$$\begin{aligned}
\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Par}_{d,k}) &= 1 - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e} \cup \mathrm{Med}_{d,k,e} \cup \mathrm{High}_{d,k,e}) \\
&= 1 - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e}) - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Med}_{d,k,e}) - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{High}_{d,k,e}).
\end{aligned}$$

It thus suffices to show that

$$\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e}) = \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \le e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o(\epsilon_{e,k})$$

and that $\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Med}_{d,k,e})$ and $\mathrm{Prob}(\boldsymbol{f} \in \mathrm{High}_{d,k,e})$ are each in $o(\epsilon_{e,k})$.

We proceed by induction on k. When $k = 0$ the condition that f_0 is a parameter on X is equivalent to f_0 not vanishing along a top-dimensional component of X. Thus, combining Lemma 6.1 with an inclusion/exclusion argument implies the exact result:

$$\mathrm{Prob}(f_0 \in \mathrm{Par}_{d,0}) = 1 - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))}.$$

By basic properties of the Hilbert polynomial, as $d \to \infty$ we have

$$h^0(Z, \mathcal{O}_Z(d)) = \frac{\deg(Z)}{n!} d^n + o(d^n) = \deg(Z) \binom{n+d}{d} + o(d^n).$$

Hence for the fixed degree bound e, we obtain

$$\begin{aligned}\operatorname{Prob}(f \in \operatorname{Par}_{d,0}) &= 1 - \sum_{\substack{Z\subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \le e}} (-1)^{|Z|-1} q^{-h^0(Z,\mathcal{O}_Z(d))} - \sum_{\substack{Z\subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z > e}} (-1)^{|Z|-1} q^{-h^0(Z,\mathcal{O}_Z(d))} \\ &= 1 - \sum_{\substack{Z\subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \le e}} (-1)^{|Z|-1} q^{-h^0(Z,\mathcal{O}_Z(d))} + o(\epsilon_{e,0}).\end{aligned}$$

We now consider the induction step. Let $\boldsymbol{f} = (f_0, f_1, \dots, f_k)$ drawn randomly from S_d^{k+1}. Here we separate into low, medium, and high degree cases.

Low degree argument. Let $\boldsymbol{V}_{k,e}$ denote the set of integral projective varieties $V \subseteq X$ of dimension $n-k$ and degree $\le e$. We have $\boldsymbol{f} \in \operatorname{Low}_{d,k,e}$ if and only if $\boldsymbol{f}$ vanishes on some $V \in \boldsymbol{V}_{k,e}$. Since $\boldsymbol{V}_{k,e}$ is a finite set, we may use an inclusion-exclusion argument to get

$$\operatorname{Prob}(\boldsymbol{f} \in \operatorname{Low}_{d,k,e}) = \sum_{\substack{Z\subseteq X \text{ a union of} \\ V \in \boldsymbol{V}_{k,e}}} (-1)^{|Z|-1} \operatorname{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ all vanish along } Z).$$

If $\deg Z > e$ then Lemma 6.1 implies that those terms can be absorbed into the error term $o(\epsilon_{e,k})$. Moreover, assuming that Z is a union of $V \in \boldsymbol{V}_{k,e}$ satisfying $\deg(Z) \le e$ is equivalent to assuming Z is reduced and equidimensional of dimensional $n-k$. We thus have

$$= \sum_{\substack{Z\subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \le e}} (-1)^{|Z|-1} \operatorname{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ all vanish along } Z) + o(\epsilon_{e,k}).$$

Medium degree argument. We know that $\operatorname{Prob}(\boldsymbol{f} \in \operatorname{Med}_{d,k,e})$ is bounded by the sum of the probabilities that f vanishes along some irreducible variety V in $\boldsymbol{V}_{k,e(k+1)} \setminus \boldsymbol{V}_{k,e}$.

$$\operatorname{Prob}(\boldsymbol{f} \in \operatorname{Med}_{d,k,e}) \le \sum_{Z \in \boldsymbol{V}_{k,e(k+1)} \setminus \boldsymbol{V}_{k,e}} \operatorname{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ all vanish along } Z).$$

Lemma 6.1 implies that each summand on the right-hand side lies in $o(\epsilon_{e,k})$. This sum is finite and thus $\operatorname{Prob}(\boldsymbol{f} \in \operatorname{Med}_{d,k,e})$ is in $o(\epsilon_{e,k})$.

High degree argument. Proposition 5.1 implies that $f_0, f_1, \dots, f_{k-1}$ are parameters on X with probability $1 - o(q^{-\binom{n-k+1+d}{d}}) \ge 1 - o(\epsilon_{e,k})$ for any e. Hence we may restrict our attention to the case where $f_0, f_1, \dots, f_{k-1}$ are parameters on X.

Let $V_1, V_2, \dots, V_s$ be the irreducible components of $X' := X \cap \mathbb{V}(f_0, f_1, \dots, f_{k-1})$ that have dimension $n-k$. We have that $f_0, f_1 \dots, f_k$ fail to be parameters on X if and only if f_k vanishes on some V_i. We can assume that f_k does not vanish on any V_i where $\deg V_i \le e(k+1)$ as we have already accounted for this possibility in the low and medium degree cases. After possibly relabeling the components, we let $V_1, V_2, \dots, V_t$ be the components of degree $> e(k+1)$ and $X'' = V_1 \cup V_2 \cup \cdots \cup V_t$. Using Lemma 2.3, we compute $\widehat{\deg}(X'') \leq \widehat{\deg}(X') = \widehat{\deg}(X) \cdot d^k$. It follows that X'' has at most $\widehat{\deg}(X)d^k/(e(k+1))$ irreducible components.

Now for the key point: since the value of d is not necessarily larger than the Castelnuovo–Mumford regularity of V_i, we cannot use a Hilbert polynomial computation to bound the probability that f_k vanishes along V_i. Instead, we use the lower bound for Hilbert functions obtained in Lemma 3.1. Let $\epsilon = \frac{1}{2}$, though any choice of ϵ would work. We write $R(V_i)$ for the homogeneous coordinate ring of V_i. For any $1 \leq i \leq t$, Lemmas 3.1 and 6.1 yield

$$\text{Prob}(f_k \text{ of degree } d \text{ vanishes along } V_i) = q^{-\dim R(V_i)_d} \leq q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$$

whenever $d \geq Ce^{k+1}$. Combining this with our bound on the number of irreducible components of X'' gives $\text{Prob}(\boldsymbol{f} \in \text{High}_{d,k,e}) \leq \frac{1}{e(k+1)} \widehat{\deg} X d^k q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$ which is in $o(\epsilon_{e,k})$. □

Corollary 6.2. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme and let $k < n$. Then*

$$\lim_{d\to\infty} q^{(k+1)\binom{n-k+d}{n-k}} \text{Prob}\begin{pmatrix} f_0, f_1, \dots, f_k \textit{ of degree } d \\ \textit{are } \underline{\textit{not}} \textit{ parameters on } X \end{pmatrix} = \#\{(n-k)\textit{-planes } L \subseteq \mathbb{P}^r_{\mathbb{F}_q} \textit{ such that } L \subseteq X\}.$$

Proof. Let N denote the number of $(n-k)$-planes $L \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ such that $L \subseteq X$. Choosing $e = 1$ in Theorem 1.4, we compute that

$$\text{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ are parameters on } X) = 1 - Nq^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}}).$$

It follows that

$$\text{Prob}(f_0, f_1, \dots, f_k \text{ of degree } d \text{ are } \underline{\text{not}} \text{ parameters on } X) = Nq^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}}).$$

Dividing both sides by $q^{-(k+1)\binom{n-k+d}{n-k}}$ and taking the limit as $d \to \infty$ yields the corollary. □

7. Passing to $\mathbb{Z}$ and $\mathbb{F}_q[t]$

In this section we prove Corollaries 1.6 and 1.7.

Definition 7.1. Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and fix a finitely generated, free B-module B^s and a subset $\mathcal{S} \subseteq B^s$. Given $a \in B^s$ we write $a = (a_1, a_2, \dots, a_s)$. The *density* of $\mathcal{S} \subseteq B^s$ is

$$\text{Density}(\mathcal{S}) := \begin{cases} \lim_{N\to\infty} \frac{\#\{a\in\mathcal{S} \mid \max\{|a_i|\}\leq N\}}{\#\{a\in\mathbb{Z}^s \mid \max\{|a_i|\}\leq N\}} & \text{if } B = \mathbb{Z}, \\ \lim_{N\to\infty} \frac{\#\{a\in\mathcal{S} \mid \max\{\deg a_i\}\leq N\}}{\#\{a\in\mathbb{F}_q[t]^s \mid \max\{\deg a_i\}\leq N\}} & \text{if } B = \mathbb{F}_q[t]. \end{cases}$$

Proof of Corollary 1.6. For clarity, we will prove the result over $\mathbb{Z}$ in detail and at the end, mention the necessary adaptations for $\mathbb{F}_q[t]$.

We first let $k < n$. Given degree d polynomials $f_0, f_1, \dots, f_k$ with integer coefficients and a prime p, let $\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k$ be the reduction of these polynomials mod p. Then $\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k$ will be parameters on X_p if and only if the point $\overline{\boldsymbol{f}} = (\overline{f}_0, \overline{f}_1, \dots, \overline{f}_k)$ lies $\mathcal{D}_{d,k}(X_{\mathbb{F}_p})$. As noted in Remark 4.1, this is equivalent to asking that $\overline{f}$ is an $\mathbb{F}_p$-point of $\mathscr{D}_{k,d}(X_{\mathbb{Z}})$. Thus, we may apply [Ekedahl 1991, Theorem 1.2] to $\mathscr{D}_{d,k}(X_{\mathbb{Z}}) \subseteq \mathscr{A}_{k,d}$ (using $M = 1$) to conclude that

$$\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{that restrict to parameters on } X_p \text{ for all } p \end{array}\right\} = \prod_p \text{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array}\right).$$

Applying Proposition 5.1 to estimate the individual factors; we have:

$$\begin{aligned}\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \text{ that} \\ \text{restrict to parameters on } X_p \text{ for all } p \end{array}\right\} &= \lim_{d\to\infty} \prod_p \text{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array}\right) \\ &\geq \lim_{d\to\infty} \prod_p (1 - \widehat{\deg}(X_p)(1 + d + \cdots + d^k) p^{-\binom{n-k+d}{n-k}}).\end{aligned}$$

Lemma 7.2 shows that there is an integer D where $D \geq \widehat{\deg}(X_p)$ for all p. Moreover, $1+d+\cdots+d^k \leq kd^k$ for all d, and hence:

$$\geq \lim_{d\to\infty} \prod_p (1 - Dkd^k p^{-\binom{n-k+d}{n-k}}).$$

For $d \gg 0$ we can make $Dkd^k p^{-\binom{n-k+d}{n-k}} \leq p^{-d/2}$ for all p simultaneously. Using $\zeta(n)$ for the Riemann zeta function, we get:

$$\geq \lim_{d\to\infty} \prod_p (1 - p^{-d/2}) \geq \lim_{d\to\infty} \zeta(d/2)^{-1} = 1.$$

We now consider the case $k = n$. This follows by a "low degree argument" exactly analogous to [Poonen 2004, Theorem 5.13]. Fix a large integer N and let Y be the union of all closed points $P \in X$ whose residue field $\kappa(P)$ has cardinality at most N. Since Y is a finite union of closed points, we see that for $d \gg 0$, there is a surjection

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \to H^0(Y, \mathcal{O}_Y(d)) \cong \bigoplus_{\substack{P\in X \\ \#\kappa(P)\leq N}} H^0(P, \mathcal{O}_P(d)) \to 0.$$

It follows that we have a product formula

$$\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ do not all} \\ \text{vanish on a point } P \text{ with } \#\kappa(P) \leq N \end{array}\right\} = \prod_{P\in X, \#\kappa(P)\leq N} \left(1 - \frac{1}{\#\kappa(P)^{n+1}}\right)$$

This is certainly an upper bound on the density of $f_0, f_1, \ldots, f_n$ that are parameters on X_p for all p. As $N \to \infty$ the right-hand side approaches $\zeta_X(n+1)^{-1}$. However, since the dimension of X is $n+1$, this zeta function has a pole at $s = n+1$ [Serre 1965, Theorems 1 and 3(a)]. Hence this asymptotic density equals 0. This completes the proof over $\mathbb{Z}$.

Over $\mathbb{F}_q[t]$, the key adaptation is to use [Poonen 2003, Theorem 3.1] in place of Ekedahl's result. Poonen's result is stated for a pair of polynomials, but it applies equally well to n-tuples of polynomials such as the n-tuples defining $\mathscr{D}_{k,d}(X)$. In particular, one immediately reduces to proving an analogue of

[Poonen 2003, Lemma 5.1], for n-tuples of polynomials which are irreducible over $\mathbb{F}_q(t)$ and which have gcd equal to 1; but the $n = 2$ version of the lemma then implies the $n \geq 2$ versions of the lemma.[2] The rest of our argument over $\mathbb{Z}$ works over $\mathbb{F}_q[t]$. □

Lemma 7.2. *Let $X \subseteq \mathbb{P}^r_B$ be any closed subscheme. There is an integer D where $D \geq \widehat{\deg}(X_s)$ for all $s \in \operatorname{Spec} B$.*

Proof. First we take a flattening stratification for X over B [EGA IV$_4$ 1967, Corollaire 6.9.3]. Within each stratum, the maximal degree of a minimal generator is semicontinuous, and we can thus find a degree e where X_s is generated in degree e for all $s \in \operatorname{Spec} B$. By [Bayer and Mumford 1993, Proposition 3.5], we then obtain that $\widehat{\deg}(X) \leq \sum_{j=0}^{n} e^{r-j}$. In particular defining $D := re^r$ will suffice. □

To prove Corollary 1.7, we use Corollary 1.6 to find a submaximal collection $f_0, f_1, \ldots, f_{n-1}$ which restrict to parameters on X_s for all $s \in \operatorname{Spec} B$. This cuts X down to a scheme $X' = X \cap \mathbb{V}(f_0, f_1, \ldots, f_{n-1})$ with 0-dimensional fibers over each point s. When $B = \mathbb{Z}$, such a scheme is essentially a union of orders in number fields, and we find the last element f_n by applying classical arithmetic results about the Picard groups of rings of integers of number fields. When $B = \mathbb{F}_q[t]$, we use similar facts about Picard groups of affine curves over $\mathbb{F}_q$.

An example illustrates this approach. Let $X = \mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x, y])$. A polynomial of degree d will be a parameter on X as long as the $d+1$ coefficients are relatively prime. Thus as $d \to \infty$, the density of these choices will go to 1. However, once we have fixed one such parameter, say $5x - 3y$, it is much harder to find an element that will restrict to a parameter on $\mathbb{Z}[x, y]/(5x - 3y)$ modulo p for all p. In fact, the only possible choices are the elements which restrict to units on $\operatorname{Proj}(\mathbb{Z}[x, y]/(5x - 3y))$. Among the linear forms, these are

$$\pm(7x - 4y) + c(5x - 3y) \text{ for any } c \in \mathbb{Z}.$$

Hence, these elements arise with density zero, and yet they form a nonempty subset.

Lemmas 7.3 and 7.4 below are well-known to experts, but we sketch the proofs for clarity.

Lemma 7.3. *If $X' \subseteq \mathbb{P}^r_{\mathbb{Z}}$ is closed and finite over* $\operatorname{Spec}(\mathbb{Z})$, *then* $\operatorname{Pic}(X')$ *is finite.*

Proof. We first reduce to the case where X' is reduced. Let $\mathcal{N} \subseteq \mathcal{O}_{X'}$ be the nilradical ideal. If X' is nonreduced then there is some integer $m > 1$ for which $\mathcal{N}^m = 0$. Let X'' be the closed subscheme defined by $\mathcal{N}^{m-1}$. We have a short exact sequence $0 \to \mathcal{N}^{m-1} \to \mathcal{O}^*_{X'} \to \mathcal{O}^*_{X''} \to 1$ where the first map sends $f \mapsto 1 + f$. Since X' is affine and noetherian and $\mathcal{N}^{m-1}$ is a coherent ideal sheaf, we have that $H^1(X', \mathcal{N}^{m-1}) = H^2(X', \mathcal{N}^{m-1}) = 0$ [Hartshorne 1977, Theorem III.3.7]. Taking cohomology of the above sequence thus yields an isomorphism $\operatorname{Pic}(X') \cong \operatorname{Pic}(X'')$. Iterating this argument, we may assume X' is reduced.

We now have $X' = \operatorname{Spec}(B)$ where B is a finite, reduced $\mathbb{Z}$-algebra. If Q is a minimal prime of B, then B/Q is either zero dimensional or an order in a number field, and hence has a finite Picard group [Neukirch 1999, Theorem I.12.12]. If B has more than one minimal prime, then we let Q' be the

[2] We thank Bjorn Poonen for pointing out this reduction.

intersection of all of the minimal primes of B except for Q, and we again have an exact sequence in cohomology

$$\cdots \to (B/(Q+Q'))^* \to \mathrm{Pic}(X') \to \mathrm{Pic}(B/Q) \oplus \mathrm{Pic}(B/Q') \to \cdots$$

Since $(B/(Q+Q'))^*$ is a finite set, and since B/Q and B/Q' have fewer minimal primes than B, we may use induction to conclude that $\mathrm{Pic}(X')$ is finite. □

Lemma 7.4. *If C is an affine curve over $\mathbb{F}_q$, then* $\mathrm{Pic}(C)$ *is finite.*

Proof. If C fails to be integral, then an argument entirely analogous to the proof of Lemma 7.3 reduces us to the case C is integral. We next assume that C is nonsingular and integral, and that $\overline{C}$ is the corresponding nonsingular projective curve. Since C is affine we have $\mathrm{Pic}(C) = \mathrm{Pic}^0(C) \subseteq \mathrm{Pic}^0(\overline{C}) \cong \mathrm{Jac}(\overline{C})(\mathbb{F}_q)$, the last of which is a finite group. If C is singular, then the finiteness of $\mathrm{Pic}(C)$ follows from the nonsingular case by a minor adaptation of the proof of [Neukirch 1999, Proposition I.12.9]. □

Proof of Corollary 1.7. By Corollary 1.6, for $d \gg 0$ we can find polynomials $f_0, f_1, \dots, f_{n-1}$ of degree d that restrict to parameters on X_s for all $s \in \mathrm{Spec}\, B$. Let $X' := \mathbb{V}(f_0, f_1, \dots, f_{n-1}) \cap X$, which is finite over B by construction. Let A be the finite B-algebra where $\mathrm{Spec}\, A = X'$. Lemma 7.3 or 7.4 implies that $H^0(X', \mathcal{O}_{X'}(e)) = A$ for some e. We can thus find a polynomial f_n of degree e mapping onto a unit in the B-algebra A. It follows that $\mathbb{V}(f_n) \cap X' = \varnothing$. Replace f_i by f_i^e for $i = 0, \dots, n-1$ and replace f_n by f_n^d. Then we have $f_0, f_1, \dots, f_n$ of degree $d' := de$ and restricting to parameters on X_s for all $s \in \mathrm{Spec}(B)$ simultaneously.

We thus obtain a proper morphism $\pi : X \to \mathbb{P}^n_B$ where $X_s \to \mathbb{P}^n_{\kappa(s)}$ is finite for all s. Since π is quasifinite and proper, it is finite by [EGA IV$_3$ 1966, Théorème 8.11.1]. □

The following generalizes Corollary 1.7 to other graded rings.

Corollary 7.5. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and let R be a graded, finite type B-algebra where* $\dim R \otimes_{\mathbb{Z}} \mathbb{F}_p = n+1$ *for all p. Then there exist $f_0, f_1, \dots, f_n$ of degree d for some d such that $B[f_0, f_1, \dots, f_n] \subseteq R$ is a finite extension.*

Proof. After replacing R by a high degree Veronese subring R', we may assume that R' is generated in degree one and contains no R'_+-torsion submodule, where $R'_+ \subseteq R'$ is the homogeneous ideal of strictly positive degree elements. Let $r+1$ be the number of generators of R'_1. Then there is a surjection $\phi\colon B[x_0, x_1, \dots, x_r] \to R'$ inducing an embedding of $X := \mathrm{Proj}(R') \subseteq \mathbb{P}^r_B$. Since R' contains no R'_+-torsion submodule, the kernel of ϕ will be saturated with respect to $(x_0, x_1, \dots, x_r)$ and hence R' will equal the homogeneous coordinate ring of X. Choosing $f_0, f_1, \dots, f_n$ as in Corollary 1.7, it follows that $B[f_0, f_1, \dots, f_n] \subseteq R'$ is a finite extension, and thus so is $B[f_0, f_1, \dots, f_n] \subseteq R$. □

8. Examples

Example 8.1. By Corollary 6.2, it is more difficult to randomly find parameters on surfaces that contain lots of lines. Consider $\mathbb{V}(xyz) \subset \mathbb{P}^3$ which contains substantially more lines than $\mathbb{V}(x^2+y^2+z^2) \subset \mathbb{P}^3$.

Using [Macaulay2] to select 1,000,000 random pairs (f_0, f_1) of polynomials of degree two, the proportion that failed to be systems of parameters were

	$\mathbb{V}(xyz)$	$\mathbb{V}(x^2+y^2+z^2)$
$\mathbb{F}_2$	.2638	.1179
$\mathbb{F}_3$	.0552	.0059
$\mathbb{F}_5$	.0063	.0004

Example 8.2. Let $X \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ be a smooth cubic surface. Over the algebraic closure X has 27 lines, but it has between 0 and 27 lines defined over $\mathbb{F}_q$. For example, working over $\mathbb{F}_4$, the Fermat cubic surface X' defined by $x^3+y^3+z^3+w^3$ has 27 lines, while the cubic surface X defined by $x^3+y^3+z^3+aw^3$ where $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ has no lines defined over $\mathbb{F}_4$ [Debarre et al. 2017, Section 3]. It will thus be more difficult to find parameters on X than on X'. Using [Macaulay2] to select 100,000 random pairs (f_0, f_1) of polynomials of degree two, 0.62% failed to be parameters on X whereas no choices whatsoever failed to be parameters on X'. This is in line with the predictions from Corollary 6.2; for instance, in the case of X, we have $27 \cdot 4^{-2\cdot 3} \approx 0.66\%$.

Example 8.3. Let $X = [1:4] \cup [3:5] \cup [4:5] = \mathbb{V}((4x-y)(5x-3y)(5x-4y)) \subseteq \mathbb{P}^1_{\mathbb{Z}}$ and let R be the homogeneous coordinate ring of X. The fibers are 0-dimensional so finding a Noether normalization $X \to \mathbb{P}^0_{\mathbb{Z}}$ is equivalent to finding a single polynomial f_0 that restricts to a unit on each of the points simultaneously. We can find such an f_0 of degree d if and only if the induced map of free $\mathbb{Z}$-modules $\mathbb{Z}[x, y]_d \to R_d$ is surjective. A computation in [Macaulay2] shows that this happens if and only if d is divisible by 60.

Example 8.4. Let $R = \mathbb{Z}[x]/(3x^2-5x) \cong \mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{3}\right]$. This is a flat, finite type $\mathbb{Z}$-algebra where every fiber has dimension 0, yet it is not a finite extension of $\mathbb{Z}$. However, if we take the projective closure of $\operatorname{Spec}(R)$ in $\mathbb{P}^1_{\mathbb{Z}}$, then we get $\operatorname{Proj}(\bar{R})$ where $\bar{R} = \mathbb{Z}[x, y]/(3x^2-5xy)$. If we then choose $f_0 := 4x-7y$, we see that $\mathbb{Z}[f_0] \subseteq \bar{R}$ is a finite extension of graded rings.

Example 8.5. Let $\boldsymbol{k}$ be a field and let $X = [1:1+t] \cup [1-t:1] = \mathbb{V}((y-(1+t)x)(x-(1-t)y)) \subseteq \mathbb{P}^1_{\boldsymbol{k}[t]}$. Let R be the homogeneous coordinate ring of X. In degree d, we have the map $\phi_d : \boldsymbol{k}[t][x, y]_d \cong \boldsymbol{k}[t]^{d+1} \to R_d \cong \boldsymbol{k}[t]^2$. Choosing the standard basis $x^d, x^{d-1}y, \dots, y^d$ for the source of ϕ_d, and the two points of X for the target, we can represent ϕ_d by the matrix

$$\begin{pmatrix} 1 & 1+t & (1+t)^2 & \cdots & (1+t)^d \\ (1-t)^d & (1-t)^{d-1} & (1-t)^{d-2} & \cdots & 1 \end{pmatrix}.$$

It follows that $\operatorname{im}\phi_d = \operatorname{im}\left(\begin{smallmatrix} t^2 & (1+t)^d \\ 0 & 1 \end{smallmatrix}\right) = \operatorname{im}\left(\begin{smallmatrix} t^2 & 1+dt \\ 0 & 1 \end{smallmatrix}\right)$. The image of ϕ_d thus contains a unit if and only if the characteristic of $\boldsymbol{k}$ is p and $p \mid d$. In particular, if $\boldsymbol{k} = \mathbb{Q}$, then we cannot find a polynomial f_0 inducing a finite map $X \to \mathbb{P}^0_{\mathbb{Q}[t]}$.

Example 8.6. Let $\boldsymbol{k}$ be any field, let $B = \boldsymbol{k}[s, t]$, and let $X = [s:1] \cup [1:t] = \mathbb{V}((x-sy)(y-tx)) \subseteq \mathbb{P}^1_B$. We claim that for any $d > 0$, there does not exist a polynomial that restricts to a parameter on X_b for each point $b \in B$. Assume for contradiction that we had such an $f = \sum_{i=0}^{d} c_i s^i t^{d-i}$ with $c_i \in B$. After scaling, we obtain

$$f([s:1]) = c_0 s^d + c_1 s^{d-1} + \cdots + c_d = 1 \quad \text{and} \quad f([1:t]) = c_0 + c_1 t + \cdots + c_d t^d = \lambda$$

where $\lambda \in B^* = \boldsymbol{k}^*$. Substituting for c_d we obtain

$$f([1:t]) = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} + (1 - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s)) t^d = \lambda,$$

which implies that

$$\begin{aligned} \lambda - t^d &= c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s) t^d \\ &= (c_0 - c_0 s^d t^d) + (c_1 t - c_1 s^{d-1} t^d) + \cdots + (c_{d-1} t^{d-1} - c_{d-1} s t^d) \\ &= (1 - st) h(s, t) \end{aligned}$$

where $h(s, t) \in \boldsymbol{k}[s, t]$. This implies that $\lambda - t^d$ is divisible by $(1 - st)$, which is a contradiction.

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juliette.bruce@math.wisc.edu
Department of Mathematics, University of Wisconsin, Madison, WI, United States

derman@math.wisc.edu
Department of Mathematics, University of Wisconsin, Madison, WI, United States

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ON THE TOP-WEIGHT RATIONAL COHOMOLOGY OF $\mathcal{A}_g$

MADELINE BRANDT, JULIETTE BRUCE, MELODY CHAN, MARGARIDA MELO, GWYNETH MORELAND, AND COREY WOLFE

Abstract. We compute the top-weight rational cohomology of $\mathcal{A}_g$ for $g = 5$, 6, and 7, and we give some vanishing results for the top-weight rational cohomology of $\mathcal{A}_8, \mathcal{A}_9$, and $\mathcal{A}_{10}$. When $g = 5$ and $g = 7$, we exhibit nonzero cohomology groups of $\mathcal{A}_g$ in odd degree, thus answering a question highlighted by Grushevsky. Our methods develop the relationship between the top-weight cohomology of $\mathcal{A}_g$ and the homology of the link of the moduli space of principally polarized tropical abelian varieties of rank g. To compute the latter we use the Voronoi complexes used by Elbaz-Vincent-Gangl-Soulé. In this way, our results make a precise connection between the rational cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$. Our computations also give natural candidates for compactly supported cohomology classes of $\mathcal{A}_g$ in weight 0 that produce the stable cohomology classes of the Satake compactification of $\mathcal{A}_g$ in weight 0, under the Gysin spectral sequence for the latter space.

1. Introduction

Let $\mathcal{A}_g$ be the moduli stack of principally polarized complex abelian varieties of dimension g. It is well-known that $\mathcal{A}_g$ is a separated Deligne-Mumford stack, isomorphic to the quotient of the Siegel upper half plane $\mathbb{H}_g$ under the action of the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Therefore $\mathcal{A}_g$ is smooth of dimension $d = \binom{g+1}{2}$, but it is not proper for $g > 0$. Since $\mathcal{A}_g$ is a complex algebraic variety, the rational cohomology groups of $\mathcal{A}_g$ admit a weight filtration in the sense of mixed Hodge theory, with graded pieces $\mathrm{Gr}_j^W H^\bullet(\mathcal{A}_g;\mathbb{Q})$ which may appear for j from 0 to $2d$. We refer to the piece of weight $2d$ as the *top-weight rational cohomology* of $\mathcal{A}_g$.

The orbifold Euler characteristic and the stable cohomology of $\mathcal{A}_g$ are classically understood [Har71, Bor74]. However, the full cohomology ring $H^\bullet(\mathcal{A}_g;\mathbb{Q})$ is a mystery even for small g. The cases when $g \leq 2$ are classically known, and the case when $g = 3$ is the work of Hain [Hai02]. The full cohomology ring for $g \geq 4$ is already unknown, though when $g = 4$, much information can be determined from [HT12, HT18], where the complete Betti tables for both the Voronoi and the perfect cone compactifications of $\mathcal{A}_4$ are computed. In particular, the top-weight cohomology of $\mathcal{A}_4$ vanishes; see Remark 6.7.

We compute the top-weight rational cohomology of $\mathcal{A}_g$ for $2 \leq g \leq 7$. For $g = 3$ and 4, our computations agree with the above-mentioned results of Hain and Hulek-Tommasi, respectively. Our first main result is then the following Theorem A.

Theorem A. *The top-weight rational cohomology of $\mathcal{A}_g$ for $g = 5$, 6, and 7, is*

$$\mathrm{Gr}_{30}^W H^k(\mathcal{A}_5;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 15, 20, \\ 0 & \text{else,} \end{cases}$$

$$\mathrm{Gr}_{42}^W H^k(\mathcal{A}_6;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 30, \\ 0 & \text{else,} \end{cases}$$

$$\mathrm{Gr}_{56}^W H^k(\mathcal{A}_7;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 28, 33, 37, 42 \\ 0 & \text{else.} \end{cases}$$

This answers an open question of Grushevsky, who asked whether $\mathcal{A}_g$ ever has nonzero odd cohomology [Gru09, Open Problem 7].

For broader context, recall that from the description of $\mathcal{A}_g$ as the quotient $[\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})]$, it is a rational classifying space for the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. Thus, $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$. The situation is analogous to that of the moduli space of curves $\mathcal{M}_g$, which is a rational classifying space for the mapping class group Mod_g via its action on Teichmüller space. Moreover, in both cases, we find ourselves in the advantageous situation that $\mathcal{M}_g$ and $\mathcal{A}_g$ are smooth and separated Deligne Mumford stacks with coarse moduli spaces which are algebraic varieties, permitting Deligne's mixed Hodge theory to be applied to study the rational cohomology of these groups. The results of this paper use this algebro-geometric perspective to find new nonzero classes in a canonical quotient of $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$: the *top-weight* quotient, in the sense of mixed Hodge theory. (Recall that in general, the rational cohomology of a complex algebraic variety X of dimension d admits a weight filtration with graded pieces $\mathrm{Gr}_j^W H^k(X;\mathbb{Q})$. As $\mathrm{Gr}_j^W H^k(X;\mathbb{Q})$ vanishes whenever $j > 2d$, $\mathrm{Gr}_{2d}^W H^k(X;\mathbb{Q})$ is referred to as the *top weight* part of $H^k(X;\mathbb{Q})$.)

Indeed, in this paper, we develop methods for studying $\mathcal{A}_g$ that are analogous to those employed in [CGP21] for $\mathcal{M}_g$. The moduli spaces $\mathcal{A}_g$ admit toroidal compactifications $\overline{\mathcal{A}_g}^{\Sigma}$, which are proper Deligne-Mumford stacks (see [FC90, Theorem 5.7]). The compactifications $\overline{\mathcal{A}_g}^{\Sigma}$ are associated to admissible decompositions Σ of Ω_g^{rt}, the rational closure of the cone of positive definite quadratic forms in g variables (see Section 2.3). The same data is also used to construct the moduli space $A_g^{\mathrm{trop},\Sigma}$ of tropical abelian varieties of dimension g in the category of generalized cone complexes (see Section 2.6).

Then for any admissible decomposition Σ of Ω_g^{rt} and for each $i \geq 0$, and writing $LA_g^{\mathrm{trop},\Sigma}$ for the link of the cone point of $A_g^{\mathrm{trop},\Sigma}$, the following canonical identification holds:

$$\widetilde{H}_{i-1}(LA_g^{\mathrm{trop},\Sigma};\mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g;\mathbb{Q}).$$

This statement can be deduced from [OO18, Corollary 2.9] (see pp. 24–25 of op. cit.); because the language is different and for self-containedness, we give a short proof in Theorem 3.1. Briefly, there exist admissible decompositions Σ for which $\overline{\mathcal{A}}_g^{\Sigma}$ is a smooth simple normal crossings compactification of $\mathcal{A}_g$ whose boundary complex is identified with $LA_g^{\mathrm{trop},\Sigma}$. However, the homeomorphism type of $LA_g^{\mathrm{trop},\Sigma}$ is independent of Σ: see Section 3 or [Oda19, A.14]. The conclusion follows by applying the generalization to Deligne-Mumford stacks, spelled out in [CGP21], of Deligne's comparison theorems in mixed Hodge theory (see Theorem 3.1).

We then compute the topology of $A_g^{\mathrm{trop},\Sigma}$ by considering the *perfect* or *first Voronoi* toroidal compactification $\overline{\mathcal{A}_g}^{\mathrm{P}}$ and its tropical version $A_g^{\mathrm{trop},\mathrm{P}}$, associated to the *perfect cone decomposition* (Fact 2.6). This decomposition is very well studied and enjoys interesting combinatorial properties, which are well-suited for our computations. We identify the homology of the link of $A_g^{\mathrm{trop},\mathrm{P}}$ with the homology of the *perfect chain complex* $P_\bullet^{(g)}$ (Definition 4.1, Proposition 4.4), using the framework of cellular chain complexes of symmetric CW-complexes due to Allcock-Corey-Payne [ACP22].

To compute the homology of the complex $P_\bullet^{(g)}$ we use a related complex $V_\bullet^{(g)}$, called the Voronoi complex. This was introduced in [EVGS13, LS78] to compute the cohomology of the modular groups $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$. They use the perfect form cell decomposition of Ω_g^{rt}, which is invariant under the action of each of these groups, and then relate the equivariant homology of Ω_g^{rt} modulo its boundary with the cohomology of $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$, respectively. For this purpose, the homology of $V_\bullet^{(4)}$ was computed by Lee and Szczarba in [LS78] for $\mathrm{SL}_4(\mathbb{Z})$ (and we adapt this computation to the case of $\mathrm{GL}_4(\mathbb{Z})$ in this paper), while for $g = 5, 6, 7$ the complex $V_\bullet^{(g)}$ was computed in [EVGS13] with the help of a computer program using lists of perfect forms for $g \leq 7$ by Jaquet [JC93]. In

Theorem 4.13 we show that the complexes $P_\bullet^{(g)}$ and $V_\bullet^{(g)}$ sit in an exact sequence

(1) $$0 \longrightarrow P_\bullet^{(g-1)} \longrightarrow P_\bullet^{(g)} \xrightarrow{\pi} V_\bullet^{(g)} \longrightarrow 0.$$

This sequence together with the results in [EVGS13] are then crucial to get our main result.

In Section 5 we consider a subcomplex of $P_\bullet^{(g)}$ called the inflation complex and prove that it is acyclic. Using this result, we show that $\mathrm{Gr}^W_{g^2+g} H^i(\mathcal{A}_g;\mathbb{Q}) = 0$ for $i > g^2$, which recovers the vanishing in top weight of the rational cohomology of $\mathcal{A}_g$ in degree above the virtual cohomological dimension (which for $\mathcal{A}_g$ is equal to g^2).

For $g = 8, 9$, and 10, full calculations of the top-weight cohomology of $\mathcal{A}_g$ are beyond the scope of our computations. However, our computations for $g = 7$ together with a vanishing result of [SEVKM19] allow us to deduce, in Section 6.5, the vanishing of $\mathrm{Gr}^W_{(g+1)g} H^\bullet(\mathcal{A}_g;\mathbb{Q})$ in a range slightly larger than what is implied by virtual cohomological dimension bounds.

Theorem B. *The top-weight rational cohomology of $\mathcal{A}_8$, $\mathcal{A}_9$, and $\mathcal{A}_{10}$ vanish in the following ranges:*

$$\mathrm{Gr}^W_{72} H^i(\mathcal{A}_8;\mathbb{Q}) = 0 \quad \textit{for } i \geq 60,$$
$$\mathrm{Gr}^W_{90} H^i(\mathcal{A}_9;\mathbb{Q}) = 0 \quad \textit{for } i \geq 79,$$
$$\mathrm{Gr}^W_{110} H^i(\mathcal{A}_{10};\mathbb{Q}) = 0 \quad \textit{for } i \geq 99.$$

To provide some broader context for our main results on $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$, we now highlight two interesting connections: first, to the stable cohomology of Satake compactifications, and second, to the cohomology of general linear groups $\mathrm{GL}_g(\mathbb{Z})$. More details appear in Section 7.

Relationship with the stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$. By Poincaré duality, the top-weight cohomology of $\mathcal{A}_g$ studied in this paper admits a perfect pairing with weight 0 compactly supported cohomology of $\mathcal{A}_g$. These weight 0 classes, in turn, have an interesting, not yet fully understood relationship with the stable cohomology ring of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$, whose structure was first understood by Charney-Lee [CL83].

Indeed, the stable cohomology ring of $\mathcal{A}_g^{\mathrm{Sat}}$ is freely generated by extensions of the well-known odd λ-classes and by less understood classes y_6, y_{10}, $y_{14}, \ldots$ which were proven to be of weight 0 by Chen-Looijenga in [CL17]. This predicts the existence of infinitely many top-weight cohomology classes of $\mathcal{A}_g$ as g grows. More precisely, the classes found in the present paper, with Poincaré duality applied, give natural candidates for the "sources" of the y_j's in the sense of persisting in a Gysin spectral sequence relating the compactly supported cohomology groups of the space $\mathcal{A}_g^{\mathrm{Sat}}$ and those of the spaces $\mathcal{A}_{g'}$ for $g' \leq g$. See Table 4 at the end of the paper for a summary of everything that is known on the E_1 page of this spectral sequence in weight 0.

This connection was explained to us by O. Tommasi and provides significant additional interest in our main results; we discuss it in detail in Section 7.

Relationship with the rational cohomology of $\mathrm{GL}_g(\mathbb{Z})$. Second, we would like to emphasize the connection between $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ and $H^*(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ provided by our main results. The possibility of such a connection is essentially present in [AMRT75], but the precise connection employed in this paper, which is a key step in proving our main Theorems A and B, has been underutilized in the literature.

Indeed, Theorem 4.13 of this paper shows the exactness of the sequence (1) relating the perfect complexes $P^{(g-1)}$ and $P^{(g)}$ on the one hand, and the Voronoi complexes $V^{(g)}$. Again, these complexes are related to $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ and $H^*(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ respectively: precisely, for all k,

$$H^{\binom{g}{2}-k}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \cong H_{k+g-1}(V^{(g)})$$

and

$$H_{k-1}(P^{(g)}) \cong \mathrm{Gr}^W_{g^2+g} H^{g^2+g-k}(\mathcal{A}_g;\mathbb{Q}) \twoheadleftarrow H^{g^2+g-k}(\mathcal{A}_g;\mathbb{Q}).$$

(See [Sou00], [EVGS13, §3.4] and Proposition 4.4, respectively). For example, in view of exactness of (1), it is immediately possible to pass vanishing results on the top weight quotient of $H^*(\mathcal{A}_g;\mathbb{Q})$ and vanishing results on $H^*(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$ back and forth. For instance, recall that Church-Farb-Putman conjectured [CFP14, Conjecture 2] that

$$H^{\binom{g}{2}-i}(\mathrm{SL}_g(\mathbb{Z});\mathbb{Q}) = 0 \text{ for all } i < g-1,$$

which implies the analogous statement for $\mathrm{GL}_g(\mathbb{Z})$. The conjecture is true for $i=0$ by [LS76], for $i=1$ by [CP17], and for $i=2$ by the recent preprint [BMP+22], which appeared after the original version of this paper. As corollaries of these results and the results of this paper, we thus have, for all $g>0$,

Corollary 1.1.

$$\mathrm{Gr}^W_{g^2+g} H^{g^2-k}(\mathcal{A}_g;\mathbb{Q}) = 0 \text{ when } k \leq 2,$$

which agrees with the $g=9$ and $g=10$ vanishing results in Theorem B. More generally, the Church-Farb-Putman conjecture would imply that

$$\mathrm{Gr}^W_{g^2+g} H^{g^2-i}(\mathcal{A}_g;\mathbb{Q}) = 0 \text{ whenever } i < g-1.$$

That is, it would imply vanishing of $E_1^{p,q}$ in the spectral sequence in Table 4 for all $q < p-1$; see Section 7.

It would be very interesting to find connections to the cohomology of $\mathrm{GL}_g(\mathbb{Z})$ that go deeper in the weight filtration on $H^*(\mathcal{A}_g;\mathbb{Q})$.

The paper is organized as follows. In Section 2, we give the necessary preliminaries. This includes a discussion of generalized cone complexes, their links, and their homology. We then discuss admissible decompositions of the rational closure of the set of positive definite quadratic forms, and focus in particular on the perfect cone decomposition. We also give a brief introduction to matroids and to perfect cones associated to matroids. Then, we give some background on the tropical moduli space $A_g^{\mathrm{trop},\Sigma}$, and on the construction of toroidal compactifications of $\mathcal{A}_g$ out of admissible decompositions.

In Section 3, we prove Theorem 3.1), which relates the top-weight cohomology of $\mathcal{A}_g$ to the reduced rational homology of the link of $A_g^{\mathrm{trop},\Sigma}$. In Section 4, we show that the perfect chain complex $P_\bullet^{(g)}$ computes the top-weight cohomology of $\mathcal{A}_g$ (Proposition 4.4). We also relate this chain complex to the Voronoi complex $V_\bullet^{(g)}$ (Theorem 4.13). In Section 5, we introduce the inflation subcomplex, which we show is acyclic in Theorem 5.15. We prove an analogous result for the coloop subcomplex $C_\bullet^{(g)}$ of the regular matroid complex $R_\bullet^{(g)}$, which may be useful for future results.

In Section 6, we put together the results obtained in Section 4 with the computations of [LS78] for $g=4$ and [EVGS13] in $g=5,6$, and 7 to describe the top-weight cohomology of $\mathcal{A}_g$ for $g=4,5,6$, and 7 and to give the above mentioned bound for the vanishing of the cohomology of $\mathcal{A}_g$ in top weight for $g=8,9$, and 10. This proves Theorems A and B. In Section 7, we discuss the relationship with the stable cohomology of the Satake compactification, including some open questions which are partially addressed by our main results and which deserve further attention.

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2. Preliminaries

In this section we give preliminaries and introduce notation.

2.1. **Cones and generalized cone complexes.** A *rational polyhedral cone* σ in $\mathbb{R}^g$ (or just a *cone*, for simplicity) is the non-negative real span of a finite set of integer vectors $v_1, v_2, \ldots, v_n \in \mathbb{Z}^g$,

$$\sigma \coloneqq \mathbb{R}_{\geq 0}\langle v_1, v_2, \ldots, v_n\rangle \coloneqq \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

We assume all cones $\sigma \subset \mathbb{R}^g$ are *strongly convex*, meaning σ contains no nonzero linear subspaces of $\mathbb{R}^g$. The *dimension* of σ is the dimension of its linear span. The cone σ is said to be *smooth* if it is possible to choose the generating vectors $v_1, \ldots, v_n$ so that they are a subset of a $\mathbb{Z}$-basis of $\mathbb{Z}^g$. Note that some sources refer to what we call smooth cones as *basic* cones. A d dimensional cone σ is said to be *simplicial* if it is generated by d vectors, which are linearly independent over $\mathbb{R}$. A *face* of σ is any nonempty subset of σ that minimizes a linear functional on $\mathbb{R}^g$. Faces of σ are themselves rational polyhedral cones. A *facet* is a face of codimension one.

Given cones $\sigma \in \mathbb{R}^g$ and $\sigma' \in \mathbb{R}^{g'}$, a *morphism* $\sigma \to \sigma'$ is a continuous map from σ to σ' obtained as the restriction of a linear map $\mathbb{R}^g \to \mathbb{R}^{g'}$ sending $\mathbb{Z}^g$ to $\mathbb{Z}^{g'}$. A *face morphism* is a morphism of cones $\sigma \to \sigma'$ sending σ isomorphically to a face of σ'. Notice that isomorphisms of cones are examples of face morphisms. Denote with Cones the category of cones with face morphisms.

The one-dimensional faces of σ are called the *extremal rays* of σ, and there are only finitely many of these. Given an extremal ray ρ of σ, the semigroup $\rho \cap \mathbb{Z}^g$ is generated by a unique element u_ρ called the *ray generator* of ρ. An automorphism of a strongly convex cone permutes its finitely many ray generators, and is uniquely determined by this permutation. So, $\mathrm{Aut}(\sigma)$ is finite.

A *generalized cone complex* (see [ACP15]) is a topological space with a presentation as a colimit $X \coloneqq \varinjlim_{i\in\mathcal{I}} \sigma_i$ of an arbitrary diagram of cones $\sigma\colon \mathcal{I} \to$ Cones, in which all morphisms of cones are face morphisms. A morphism $(X = \varinjlim_{i\in\mathcal{I}} \sigma_i) \to (X' = \varinjlim_{i\in\mathcal{I}} \sigma_i')$ is a continuous map $f\colon X \to X'$ such that for each cone σ_i in the presentation of X, there exists a cone σ_j' in the presentation of

X' and a morphism of cones $f_i \colon \sigma_i \to \sigma'_j$ such that the following diagram commutes.

$$\begin{tikzcd} \sigma_i \arrow[r, "f_i"] \arrow[d] & \sigma'_j \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

We remark that the category of generalized cone complexes is equivalent to the one of stacky fans as defined in [CMV13, Def. 2.1.7].

2.2. **Links of generalized cone complexes.** For any cone $\sigma \subset \mathbb{R}^g$, let us define the *link* of σ at the origin to be the topological space $L\sigma = (\sigma - \{0\})/\mathbb{R}_{>0}$, where the action of $\mathbb{R}_{>0}$ is by scalar multiplication. Thus $L\sigma$ is homeomorphic to a closed ball of dimension $\dim \sigma - 1$. A face morphism of cones $\sigma \to \sigma'$ induces a morphism of links $L\sigma \to L\sigma'$, making L a functor from Cones to topological spaces.

Let $X = \varinjlim_{i \in \mathcal{I}} \sigma_i$ be a generalized cone complex, where $\sigma \colon \mathcal{I} \to$ Cones is a diagram of cones. We define the *link* of X as the colimit

$$LX = \varinjlim (L \circ \sigma).$$

Thus LX is a topological space, equipped with a colimit presentation as above. In fact, LX is a *symmetric CW-complex*, by [ACP22, Example 2.4]. The definition of symmetric CW-complex generalizes the *symmetric Δ-complexes* of [CGP21]. Roughly, a symmetric CW-complex is like a CW-complex, except with closed n-balls replaced by quotients thereof by finite subgroups of the orthogonal group $O(n)$.

Let X be a finite generalized cone complex, meaning that the indexing category $\mathcal{I}$ is equivalent to one with a finite number of objects and morphisms. We now write down a chain complex isomorphic to the *cellular chain complex* of LX, in the sense of [ACP22, §4], [CGP21, §3]), whose homology is identified with the singular homology of LX.

For each $p \geq -1$, let $\mathrm{Cones}_p(X)$ denote the finite groupoid whose objects are all $(p+1)$-dimensional faces of σ_i, for all $i \in \mathcal{I}$, with a morphism $\tau \to \tau'$ for each isomorphism of cones $\phi \colon \tau \xrightarrow{\cong} \tau'$ such that the following diagram commutes.

$$\begin{tikzcd} \tau \arrow[rr, "\phi"] \arrow[dr] & & \tau' \arrow[dl] \\ & X & \end{tikzcd}$$

Let τ be a cone in $\mathrm{Cones}_p(X)$. We make use of three compatible notions of orientation found in the literature: (i) an orientation of τ is an orientation of $L\tau$ [LS78], (ii) an orientation of τ is an orientation of the suspension of $L\tau$ [ACP22], and (iii) an orientation of τ is an orientation of $\mathbb{R}\tau$, the $\mathbb{R}$-linear span of τ; i.e., it is a choice of ordered basis for $\mathbb{R}\tau$, up to a change of basis with positive determinant [EVGS13]. For the first two definitions, it is clear that an orientation on τ induces an orientation on the faces of τ as well. For the third definition, given a facet τ' of τ, the induced orientation on τ' is any one such that the quantity $\epsilon(\tau', \tau)$, defined as follows, is 1.

Let $B = (v_1, \ldots, v_n, v)$ where $B' = (v_1, \ldots, v_n)$ is an orientation of τ' and v is a ray generator of a ray of τ not contained in τ'. Set $\epsilon(\tau', \tau)$ to be the sign of the orientation of B in the oriented vector space $\mathbb{R}\tau$. Note that this sign does not depend on the choice of v. These definitions are compatible, in that a choice of orientation under one definition yields a choice of orientation under the other two, and under this correspondence a cone morphism $\tau \to \sigma$ is orientation-preserving under one definition if it is orientation-preserving under all three. Say that $\tau \in \mathrm{Cones}_p(X)$ is *alternating* if all automorphisms $\tau \to \tau$ in $\mathrm{Cones}_p(X)$ are orientation-preserving on τ.

Choose a set Γ_p of representatives of isomorphism classes of alternating cones in $\mathrm{Cones}_p(X)$, and for each $\tau \in \Gamma_p$ fix an orientation ω_τ of τ. If ρ' is a facet of τ, then ω_τ induces an orientation of ρ', which we denote $\omega_\tau|_{\rho'}$. Let $C_p(LX)$ be the $\mathbb{Q}$-vector space with basis Γ_p. We define a differential

$$\partial\colon C_p(LX) \to C_{p-1}(LX)$$

by extending linearly on $C_p(LX)$ the following definition: given $\tau \in \Gamma_p$ and $\rho \in \Gamma_{p-1}$, set

$$\partial(\tau)_\rho = \sum_{\rho'} \eta(\rho', \rho)$$

where ρ' ranges over the facets of τ that are isomorphic in $\mathrm{Cones}_{p-1}(X)$ to ρ, and where $\eta(\rho', \rho) = \pm 1$ according to whether an isomorphism $\phi\colon \rho' \to \rho$ in $\mathrm{Cones}_p(X)$ takes the orientation $\omega_\tau|_{\rho'}$ to ω_ρ or $-\omega_\rho$. Note that $\eta(\rho', \rho)$ is well defined, i.e., independent of choice of ϕ, precisely because ρ is alternating.

Let $C_\bullet(LX)$ denote the complex

$$\cdots \xrightarrow{\partial} C_p(LX) \xrightarrow{\partial} C_{p-1}(LX) \xrightarrow{\partial} \cdots C_{-1}(LX) \to 0.$$

The main proposition in this subsection is the following.

Proposition 2.1. Let X be a finite generalized cone complex. We have that $C_\bullet(LX)$ is a complex, i.e., $\partial^2 = 0$.

(1) If X is connected, we have, for each $p \geq 0$,

$$H_p(C_\bullet(LX)) \cong \widetilde{H}_p(LX; \mathbb{Q}).$$

(2) More generally, for each $p > 0$, we have canonical isomorphisms

$$H_p(C_\bullet(LX)) \cong H_p(LX; \mathbb{Q}),$$

and for $p = 0$ we have

$$H_0(C_\bullet(LX)) \cong \ker(H_0(LX; \mathbb{Q}) \to \mathbb{Q}\Gamma_{-1}).$$

Proposition 2.1 follows from [ACP22, Theorem 4.2], by tracing through their definition of the cellular chain complex of LX. We give a self-contained proof sketch below.

Proof sketch. Write $LX^{(p)}$ for the p-skeleton of LX, i.e., the union of the images of $L\sigma$ in X, for σ ranging over cones of dimension at most $p+1$ in X. By a standard argument analogous to [Hat02, Theorem 2.2.27], the complex

$$\cdots H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \xrightarrow{\delta_p} H_{p-1}(LX^{(p-1)}, LX^{(p-2)}; \mathbb{Q}) \cdots \tag{2}$$

has homology canonically identified with the singular homology of LX. Moreover,

$$H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \cong \bigoplus_\tau H_p((L\tau)/\operatorname{Aut}(\tau), (\partial L\tau)/\operatorname{Aut}(\tau); \mathbb{Q}),$$

where τ ranges over a set of representatives of isomorphism classes in $\mathrm{Cones}_p(X)$. Here $\operatorname{Aut}(\tau) = \mathrm{Iso}_{\mathrm{Cones}_p(X)}(\tau, \tau)$ is the automorphism group of τ in $\mathrm{Cones}_p(X)$. Since $|\operatorname{Aut}(\tau)|$ is invertible in $\mathbb{Q}$, it follows that

$$H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q}) \quad\cong\quad \bigoplus_\tau H_p(L\tau, \partial L\tau; \mathbb{Q})_{\operatorname{Aut}(\tau)},$$

and for each $\tau \in \mathrm{Cones}_p(X)$, we have

$$H_p(L\tau, \partial L\tau; \mathbb{Q})_{\operatorname{Aut}(\tau)} \cong \begin{cases} \mathbb{Q} & \text{if } \tau \text{ is alternating,} \\ 0 & \text{else.} \end{cases},$$

which identifies $H_p(LX^{(p)}, LX^{(p-1)}; \mathbb{Q})$ with $C_p(LX)$. $\square$

Remark 2.2. A statement analogous to Proposition 2.1 holds with $\mathbb{Q}$ replaced by a commutative ring R, if the order of $\mathrm{Aut}(\tau)$ is invertible in R for each τ.

Remark 2.3. See also the proofs in [LS78, §3], as well as [EVGS13, §3.3], which are written in the special cases of the *Voronoi complexes* for $\mathrm{SL}_g(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$, but apply essentially verbatim to prove Proposition 2.1. The Voronoi complex of $\mathrm{GL}_g(\mathbb{Z})$ plays an important role in this paper.

2.3. **Admissible decompositions.** We now introduce admissible decompositions of the rational closure of the set of positive definite quadratic forms, which are used in the construction of toroidal compactifications of the moduli space of abelian varieties, as well as in the construction of the moduli space of tropical abelian varieties.

We denote by $\mathbb{R}^{\binom{g+1}{2}}$ the vector space of quadratic forms in $\mathbb{R}^g$, which we identify with $g \times g$ symmetric matrices with coefficients in $\mathbb{R}$. We denote by Ω_g the cone in $\mathbb{R}^{\binom{g+1}{2}}$ of positive definite quadratic forms. We define the rational closure of Ω_g to be the set Ω_g^{rt} of positive semidefinite quadratic forms whose kernel is defined over $\mathbb{Q}$. The group $\mathrm{GL}_g(\mathbb{Z})$ acts on the vector space $\mathbb{R}^{\binom{g+1}{2}}$ of quadratic forms by $h \cdot Q \coloneqq hQh^t$, where $h \in \mathrm{GL}_g(\mathbb{Z})$ and h^t is its transpose. The cones Ω_g and Ω_g^{rt} are preserved by this action of $\mathrm{GL}_g(\mathbb{Z})$.

Remark 2.4. A positive semidefinite quadratic form Q in $\mathbb{R}^g$ belongs to Ω_g^{rt} if and only if there exists $h \in \mathrm{GL}_g(\mathbb{Z})$ such that

$$hQh^t = \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}$$

for some positive definite quadratic form Q' in $\mathbb{R}^{g'}$, with $0 \leq g' \leq g$ (see [Nam80, Sec. 8]).

The cones Ω_g and Ω_g^{rt} are not polyhedral cones. However, one can consider decompositions of these spaces into rational polyhedral cones, as in the following definition.

Definition 2.5. [[Nam80, Lemma 8.3], [FC90, Chap. IV.2]] An *admissible decomposition* of Ω_g^{rt} is a collection $\Sigma = \{\sigma_\mu\}$ of rational polyhedral cones of Ω_g^{rt} such that:

(i) if σ is a face of $\sigma_\mu \in \Sigma$ then $\sigma \in \Sigma$,
(ii) the intersection of two cones σ_μ and σ_ν of Σ is a face of both cones,
(iii) if $\sigma_\mu \in \Sigma$ and $h \in \mathrm{GL}_g(\mathbb{Z})$ then $h\sigma_\mu h^t \in \Sigma$,
(iv) the set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of cones is finite, and
(v) $\cup_{\sigma_\mu \in \Sigma} \sigma_\mu = \Omega_g^{\mathrm{rt}}$.

We say that two cones $\sigma_\mu, \sigma_\nu \in \Sigma$ are equivalent if they are in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit.

There are three known families of admissible decompositions of Ω_g^{rt} described for all g: the perfect cone decomposition, the second Voronoi decomposition, and the central cone decomposition (see [Nam80, Chap. 8] and the references there). In this paper, we work with the perfect cone decomposition, which we now describe.

2.4. **The perfect cone decomposition.** Given a positive definite quadratic form Q, consider the set of nonzero integral vectors where Q attains its minimum,

$$M(Q) \coloneqq \{\xi \in \mathbb{Z}^g \setminus \{0\} \,:\, Q(\xi) \leq Q(\zeta), \forall \zeta \in \mathbb{Z}^g \setminus \{0\}\}.$$

The elements of $M(Q)$ are called the *minimal vectors* of Q. Let $\sigma[Q]$ denote the rational polyhedral subcone of Ω_g^{rt} given by the non-negative linear span of the rank one forms $\xi \cdot \xi^t \in \Omega_g^{\mathrm{rt}}$ for elements ξ of $M(Q)$, i.e.

$$\sigma[Q] \coloneqq \mathbb{R}_{\geq 0}\langle \xi \cdot \xi^t \rangle_{\xi \in M(Q)}. \tag{3}$$

The *rank* of the cone $\sigma[Q]$ is defined to be the maximum rank of an element of $\sigma[Q]$; in fact the rank of $\sigma[Q]$ is exactly the dimension of the span of $M(Q)$, see Lemma 4.8.

Fact 2.6. [Vor08] The set of cones

$$\Sigma_g^{\mathrm{P}} := \{\sigma[Q] \,:\, Q \text{ a positive definite form on } \mathbb{R}^g\}$$

is an admissible decomposition of Ω_g^{rt}, known as the *perfect cone decomposition*.

The quadratic forms Q such that $\sigma[Q]$ has maximal dimension $\binom{g+1}{2}$ are called *perfect*, hence the name of this admissible decomposition.

Example 2.7. Let us compute Σ_2^{P}. In this case, there is a unique perfect form up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, namely

$$Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

One can compute that $M(Q) = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1)\}$. Thus, up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, there is a unique perfect cone $\sigma[Q]$ of maximal dimension 3, with ray generators

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

One may check that for $i \in \{0, 1, 2\}$, all i-dimensional faces of $\sigma[Q]$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent; hence there is a unique perfect cone of each dimension up to the action of $\mathrm{GL}_2(\mathbb{Z})$.

Remark 2.8. The cones $\sigma[Q] \in \Sigma_g^{\mathrm{P}}$ need not be simplicial for $g \geq 4$ (see [Nam80, p. 93]).

2.5. **Matroidal Perfect Cones.** We now give a brief introduction to matroids and their associated orbits of perfect cones. Further background on matroids can be found in [Oxl92].

Definition 2.9. A *matroid* $M = (E, \mathcal{C})$ on a finite set E is a subset $\mathcal{C} \subset \mathcal{P}(E) \setminus \{\emptyset\}$, called the set of *circuits* of M, satisfying the following axioms:

- (C1) No proper subset of a circuit is a circuit.
- (C2) If $C_1, C_2 \in \mathcal{C}$ are distinct and $c \in C_1 \cap C_2$ then $(C_1 \cup C_2) - \{c\}$ contains a circuit.

A matroid $M = (E, \mathcal{C})$ is said to be *simple* if it has no circuits of length 1 or 2. A matroid $M = (E, \mathcal{C})$ is called *representable* over a field $\mathbb{F}$ if there is a matrix A over $\mathbb{F}$ such that E bijects to the columns of A with the circuits $\mathcal{C}$ of M indexing the minimal linearly dependent sets of columns of A. The matrix A is known as an $\mathbb{F}$-representation of M. An *automorphism* of a matroid is a bijection $\phi : E \to E$ such that for any subset $C \subset E$, C is a circuit of M if and only of $\phi(C)$ is a circuit of M.

Definition 2.10. A matroid is *regular* if and only if it is representable over every field.

A matroid M being regular is equivalent to M being representable over $\mathbb{R}$ by a totally unimodular matrix (i.e., a matrix such that every minor is either $-1, 0$, or 1). The *rank* of a regular matroid M is the smallest number r such that M is representable over $\mathbb{R}$ by a $r \times n$ totally unimodular matrix for some n [Oxl92, Lemma 2.2.21, page 85].

Definition 2.11. Let G be a graph. The *graphic matroid* $M(G)$ is the matroid with ground set $E(G)$ whose circuits are subsets of $E(G)$ forming a simple cycle of G.

Since graphic matroids are regular, they are representable over fields of any characteristic. This can be seen directly by constructing the following matrix representing $M(G)$. Fix an orientation of the edges of G. Let $A(G)$ be the $|V(G)| \times |E(G)|$ matrix with entries

$$A(G)_{ij} = \begin{cases} 0 \text{ if } v_i \notin e_j, \\ -1 \text{ if } v_i \text{ is the head of } e_j, \\ 1 \text{ if } v_i \text{ is the tail of } e_j. \end{cases}$$

This matrix represents the matroid $M(G)$ over any field.

Construction 2.12. Given a simple, regular matroid M of rank $\leq g$ choose a $g \times n$ totally unimodular matrix A that represents M over $\mathbb{R}$. Denoting the columns of A by $v_1, v_2, \ldots, v_n$ we let $\sigma_A(M) \subset \Omega_g^{\mathrm{rt}}$ be the rational polyhedral cone:

$$\sigma_A(M) := \mathbb{R}_{\geq 0} \left\langle v_1 v_1^t, v_2 v_2^t, \ldots, v_n v_n^t \right\rangle .$$

By [MV12, Theorem 4.2.1], the cone $\sigma_A(M)$ is a perfect cone in Σ_g^{P}.

The cone $\sigma_A(M)$ is uniquely determined by M up to the action of $\mathrm{GL}_g(\mathbb{Z})$. In particular, if A and A' are two different totally unimodular matrices representing M over $\mathbb{R}$ then there exists an element $h \in \mathrm{GL}_g(\mathbb{Z})$ such that $h\sigma_A(M)h^t = \sigma_{A'}(M)$ (see [MV12, Lemma 4.0.5(ii)]). We therefore denote the $\mathrm{GL}_g(\mathbb{Z})$-orbit of $\sigma_A(M)$ by $\sigma(M)$.

In the case of graphic matroids, Construction 2.12 can be made very explicit. As this is useful in Section 6, we take the time to explain it here. Fix $g > 0$. We now construct cones of Σ_g^{P} from graphs on $g+1$ vertices. The rows of the $(g+1) \times |E(G)|$ matrix $A(G)$ as constructed above are linearly dependent. Let $A^*(G)$ be the matrix obtained from $A(G)$ by deleting the last row. The matrices $A(G)$ and $A^*(G)$ are both representations of $M(G)$. Let $v_1, \ldots, v_d$ be the columns of $A^*(G)$. Then $\sigma(M(G)) := \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_d v_d^t\rangle \in \Sigma_g^{\mathrm{P}}$ is a perfect cone (see [MV12, Theorem 4.2.1]).

Definition 2.13. The *principal cone* is $\sigma_g^{\mathrm{prin}} := \sigma(M(K_{g+1}))$, the cone corresponding to the complete graph K_{g+1}.

When $g = 2$, this is the cone discussed in Example 2.7. More generally, for arbitrary g the principal cone can be defined as the cone corresponding to the quadratic form

$$\begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{bmatrix}.$$

These two definitions agree by [BMV11, Lemma 6.1.3].

The faces of σ_g^{prin} may be understood as follows. Since $M(K_{g+1})$ is a simple matroid, the principal cone in Σ_g^{P} is simplicial by [BMV11, Theorem 4.4.4(iii)]. Therefore, a codimension i face of the principal cone comes from a graph obtained by deleting i edges from K_{g+1}.

Remark 2.14. Automorphisms of the graph G give automorphisms of the matroid $M(G)$, but not all automorphisms of $M(G)$ arise in this way. However, if G is 3-connected, then $\mathrm{Aut}(G) = \mathrm{Aut}(M(G))$ (this is proved by Whitney in [Whi32], see [HPW72, Lemmas 1 and 2]). The group $\mathrm{Aut}(M(G))$ is isomorphic to the group of permutations of the rays of $\sigma(M(G))$ induced by elements of $GL_g(\mathbb{Z})$ stabilizing $\sigma(M(G))$ [Cha12, Theorem 5.10].

2.6. **The tropical moduli space A_g^{trop}.** We now introduce the moduli space of tropical abelian varieties, which is a generalized cone complex constructed in [BMV11] and later worked out in [CMV13]. Our aim is to compute the homology of the link of A_g^{trop}, as this is canonically isomorphic to the top-weight rational cohomology of $\mathcal{A}_g$ (see Theorem 3.1).

Definition 2.15. A *principally polarized tropical abelian variety* (or, for simplicity, just *tropical abelian variety*) of dimension g is a pair $A = (\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is a positive semidefinite symmetric bilinear form on $\mathbb{R}^g$ with rational null space. We say that $A = (\mathbb{R}^g/\mathbb{Z}^g, Q)$ is *pure* if Q is positive definite.

Two tropical abelian varieties $(\mathbb{R}^g/\mathbb{Z}^g, Q)$ and $(\mathbb{R}^g/\mathbb{Z}^g, Q')$ are *isomorphic* if there is $h \in \mathrm{GL}_g(\mathbb{Z})$ such that $Q' = hQh^t$. The set of isomorphism classes of tropical abelian varieties of dimension g is in bijective correspondence with the orbits in $\Omega_g^{\mathrm{rt}}/\mathrm{GL}_g(\mathbb{Z})$.

Given an admissible decomposition Σ of Ω_g^{rt}, one can define a generalized cone complex $A_g^{\mathrm{trop},\Sigma}$ by considering the stratified quotient of Ω_g^{rt} with respect to Σ (see [CMV13, Def. 2.2.2]). Precisely, $A_g^{\mathrm{trop},\Sigma}$ is the generalized cone complex obtained as the colimit

$$A_g^{\mathrm{trop},\Sigma} := \varinjlim \{\sigma\}_{\sigma\in\Sigma}$$

with arrows given by inclusion of faces composed with the action of the group $\mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}: given two cones σ_i and $\sigma_j \in \Sigma$ and $h \in \mathrm{GL}_g(\mathbb{Z})$ with $h\sigma_i h^t$ a face of σ_j, we consider its associated lattice-preserving linear map $L_{i,j,g} : \sigma_i \hookrightarrow \sigma_j$ in the diagram. The space $A_g^{\mathrm{trop},\Sigma}$ is the *moduli space of tropical abelian varieties* of dimension g with respect to Σ.

2.7. **Toroidal compactifications of the moduli space $\mathcal{A}_g$.** In this paper, $\mathcal{A}_g$ denotes the moduli stack of principally polarized abelian varieties of dimension g. It is a smooth Deligne-Mumford algebraic stack of dimension $d = \binom{g+1}{2}$, and the coarse moduli space of principally polarized abelian varieties, denoted A_g, is a quasiprojective variety.

The moduli stack $\mathcal{A}_g$ is not proper for $g > 0$, and there are different constructions of compactifications of $\mathcal{A}_g$. In particular, it is possible to construct normal crossings compactifications of $\mathcal{A}_g$ via the theory of toroidal compactifications. Both the constructions of $\mathcal{A}_g$ and of its toroidal compactifications as algebraic stacks were achieved in [AMRT75] over the complex numbers and in [FC90] over an arbitrary base. Even though we work over the complex numbers, we often refer to the constructions in [FC90] as these are more conveniently stated within the algebraic category and specifically for moduli of abelian varieties (rather than quotients of bounded symmetric domains as in [AMRT75]).

Let Σ be an admissible decomposition of Ω_g^{rt} (in the sense of Definition 2.5). Then one may associate to Σ a *toroidal compactification* $\overline{\mathcal{A}_g}^{\Sigma}$ of $\mathcal{A}_g$, which is a proper Deligne-Mumford stack, although in general it is not smooth. The fact that $\mathcal{A}_g \subset \overline{\mathcal{A}_g}^{\Sigma}$ is toroidal means that $(\mathcal{A}_g, \overline{\mathcal{A}_g}^{\Sigma})$ is étale-locally isomorphic to a torus inside a toric variety.

By construction, the toroidal compactification $\overline{\mathcal{A}_g}^{\Sigma}$ comes with a stratification into locally closed subsets. These are in order-reversing bijection, with respect to the order relation given by the closure, with the $\mathrm{GL}_g(\mathbb{Z})$-equivalence classes of the relative interiors of the cones in Σ. For example, the origin of Ω_g^{rt}, which is the unique zero-dimensional cone in every admissible decomposition Σ, corresponds to the open substack $\mathcal{A}_g$, which is the unique stratum of $\overline{\mathcal{A}_g}^{\Sigma}$ of maximal dimension d. At the other extreme, the maximal dimensional cones in Σ correspond to the zero-dimensional strata of $\overline{\mathcal{A}_g}^{\Sigma}$.

We study the *perfect* toroidal compactification $\mathcal{A}_g^{\mathrm{perf}} = \overline{\mathcal{A}_g}^{\Sigma_g^{\mathrm{P}}}$ of $\mathcal{A}_g$, i.e., the toroidal compactification of $\mathcal{A}_g$ associated to the perfect cone decomposition Σ_g^{P}. The geometric significance of the perfect cone compactification was highlighted in work of Shepherd-Barron [SB06], who shows that $\mathcal{A}_g^{\mathrm{perf}}$ is the canonical model of $\mathcal{A}_g$ for $g \geq 12$. For our purposes it is particularly nice because the number of strata of codimension l in the boundary of $\mathcal{A}_g^{\mathrm{perf}} \setminus \mathcal{A}_g$ is independent of g if $l \leq g$ (see [GHT18b, Prop. 7.1]).

3. A comparison theorem for $\mathcal{A}_g$ and A_g^{trop}

Let Σ be any admissible decomposition of Ω_g^{rt}. As mentioned above, we can use Σ to construct both a toroidal compactification of $\mathcal{A}_g$, and the generalized cone complex $A_g^{\mathrm{trop},\Sigma}$, the moduli space of tropical abelian varieties associated to Σ. In this section, we record the relationship between the homology of $A_g^{\mathrm{trop},\Sigma}$ with the top-weight cohomology of $\mathcal{A}_g$, as deduced from Deligne's comparison theorems and the framework in [CGP21]. This precise relationship was already remarked by Odaka-Oshima [OO18, Corollary 2.9], as we explain further in Remark 3.2, but it is useful to have a self-contained proof, below.

Theorem 3.1. For each $i \geq 0$ and admissible decomposition Σ, we have a canonical isomorphism

$$\widetilde{H}_{i-1}(LA_g^{\mathrm{trop},\Sigma};\mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g;\mathbb{Q}),$$

where $d = \binom{g+1}{2}$ is the complex dimension of $\mathcal{A}_g$.

Proof. First, by replacing Σ with another admissible decomposition of Ω_g^{rt} that refines it, we may assume that every cone of Σ is smooth and that it enjoys the following additional property: for any $h \in \mathrm{GL}_g(\mathbb{Z})$ and $\sigma \in \Sigma$, we have that h fixes, pointwise, the cone $h\sigma h^t \cap \sigma$. Such a refinement is well known to exist [FC90, IV.2, p. 98]. For example, one may be obtained by taking the barycentric refinement, which is simplicial, and then taking an appropriate *smooth* refinement which can be constructed as in [CLS11, Theorem 11.1.9]. The homeomorphism type of $LA_g^{\mathrm{trop},\Sigma}$ is unchanged when passing to a refinement.

Then, by [FC90, Theorem 5.7], it follows that $\overline{\mathcal{A}_g}^{\Sigma}$ is a smooth, separated Deligne-Mumford stack which is a simple normal crossings compactification of $\mathcal{A}_g$ and whose boundary complex is $LA_g^{\mathrm{trop},\Sigma}$. Now the desired result follows from the following comparison theorem: we have a canonical isomorphism

$$\widetilde{H}_{i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}});\mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{X};\mathbb{Q}),$$

for any normal crossings compactification $\mathcal{X} \subset \overline{\mathcal{X}}$ of smooth, separated Deligne-Mumford stacks over $\mathbb{C}$, where $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$ denotes the boundary complex of the pair $(\mathcal{X}, \overline{\mathcal{X}})$ and $d = \dim \mathcal{X}$ is the complex dimension of $\mathcal{X}$. This comparison theorem follows from Deligne's mixed Hodge theory [Del71, Del74] in the case of complex varieties; we refer to [CGP21] for the generalization to Deligne-Mumford stacks.

□

Remark 3.2. Let us briefly explain how Theorem 3.1 appears in [Oda19] and [OO18], since the language of those papers is somewhat different. Odaka and Odaka-Oshima study certain "hybrid" compactifications of arithmetic quotients $\Gamma\backslash D$ of Hermitian symmetric domains. The case of $\mathcal{A}_g$ is the case $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$ and D is the "unit disc" of complex symmetric matrices Z with $Z^t\overline{Z} < \mathrm{Id}_g$. The point is that the boundary of these compactifications is homeomorphic to $L\mathcal{A}_g^{\mathrm{trop},\Sigma}$, so the comparison statement in [OO18, Corollary 2.9], which relies on [CGP21], combined with Theorem 2.1 of op. cit., reduces to Theorem 3.1 in this case.

It is worth emphasizing the independence of choice of the admissible decomposition of Σ, as remarked in [Oda19, A.14], that was implicit in the discussion above. More precisely, for any two admissible decompositions Σ_1 and Σ_2 of Ω_g^{rt}, we have a homeomorphism of links

$$LA_g^{\mathrm{trop},\Sigma_1} \cong LA_g^{\mathrm{trop},\Sigma_2}.$$

Indeed, it is well known that any two admissible decompositions Σ_1 and Σ_2 admit a common refinement $\widetilde{\Sigma}$ which is an admissible decomposition [FC90, IV.2, p. 97], and by the construction of §2.2, we have canonical homeomorphisms $LA_g^{\mathrm{trop},\Sigma_1} \cong LA_g^{\mathrm{trop},\tilde{\Sigma}} \cong LA_g^{\mathrm{trop},\Sigma_2}$.

4. The Perfect and Voronoi Chain Complexes

In computing the top-weight cohomology of $\mathcal{A}_g$ there are two chain complexes that play central roles: the perfect chain complex $P_\bullet^{(g)}$ and the Voronoi chain complex $V_\bullet^{(g)}$. In this section we define both of these complexes, and show that the homology of the perfect chain complex $P_\bullet^{(g)}$ computes the top-weight cohomology of $\mathcal{A}_g$. Further, we show that the perfect and Voronoi complexes are related via a short exact sequence of chain complexes, which is useful as the Voronoi complex has seen more extensive study [EVGS13, SEVKM19]. We make use of this short exact sequence to prove our main results in Section 6.

4.1. **The perfect chain complex.** We first fix some notation, most of which we adapt from Section 2.2. For $n \in \mathbb{Z}$, let $\Sigma_g^{\mathrm{P}}[n]$ be the set of perfect cones in Σ_g^{P} of dimension $n+1$, and denote the finite set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of such cones by $\Sigma_g^{\mathrm{P}}[n]/\mathrm{GL}_g(\mathbb{Z})$. We write $\sigma \sim \sigma'$ if and only if σ and σ' lie in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit. Recall that a cone $\sigma \in \Sigma_g^{\mathrm{P}}$ is *alternating* if and only if every element of $\mathrm{GL}_g(\mathbb{Z})$ stabilizing σ induces an orientation-preserving cone morphism of σ. If σ is an alternating cone then every cone in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit as σ is alternating. We call such $\mathrm{GL}_g(\mathbb{Z})$-orbits alternating. Let $\Gamma_n^{(g)} = \Gamma_n$ be a set of representatives for the alternating elements of $\Sigma_g^{\mathrm{P}}[n]/\mathrm{GL}_g(\mathbb{Z})$.

For each n and each $\sigma \in \Gamma_n$, choose an orientation ω_σ on σ; the $\mathrm{GL}_g(\mathbb{Z})$-action extends this choice to a choice of orientation on every alternating cone in Σ_g^{P}. If $\rho \subset \sigma$ is an alternating facet of σ, denote the orientation induced on ρ by $\omega_\sigma|_\rho$. Now let $\eta(\rho,\sigma)$ be 1 if the orientation on ρ agrees with the orientation induced by σ (i.e., $\omega_\rho = \omega_\sigma|_\rho$) and -1 otherwise. Finally, given $\sigma \in \Gamma_n$ and $\sigma' \in \Gamma_{n-1}$ define

$$\delta(\sigma',\sigma) \coloneqq \sum_{\substack{\rho\subset\sigma \\ \rho\sim\sigma'}} \eta(\rho,\sigma) \tag{4}$$

where the sum is over all facets ρ of σ in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit as σ'. With this notation in hand we can now define the perfect chain complex.

Definition 4.1. The *perfect chain complex* $(P_\bullet^{(g)}, \partial_\bullet)$ is the rational complex defined as follows. For each n, $P_n^{(g)}$ is the $\mathbb{Q}$-vector space with basis indexed by Γ_n. The differential $\partial_n : P_n^{(g)} \to P_{n-1}^{(g)}$ is given by

$$\partial(e_\sigma) \coloneqq \sum_{\sigma' \in \Gamma_{n-1}} \delta(\sigma',\sigma) e_{\sigma'}.$$

Notice that $P_n^{(g)}$ is only possibly nonzero in the range $-1 \le n \le \binom{g+1}{2} - 1$, but even within this range $P_n^{(g)}$ may be zero since alternating perfect cones do not necessarily exist in every dimension (see Example 4.2). While in many cases $\delta(\sigma',\sigma)$ is equal to $-1, 0$, or 1, this need not always be the case since a cone may have two or more facets that are $GL_g(\mathbb{Z})$-equivalent.

Example 4.2. When $g = 2$, recall from Example 2.7 that up to the action of $\mathrm{GL}_2(\mathbb{Z})$ there is precisely one cone of maximal dimension,

$$\sigma_3 \coloneqq \mathbb{R}_{\ge 0}\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle.$$

Since σ_3 is simplicial, its faces correspond to all subsets of the above ray generators. One can show that up to the action of $\mathrm{GL}_2(\mathbb{Z})$ there is at most one cone in each dimension:

$$\sigma_2 \coloneqq \mathbb{R}_{\ge 0}\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle, \quad \sigma_1 \coloneqq \mathbb{R}_{\ge 0}\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle, \quad \sigma_0 \coloneqq \mathbb{R}_{\ge 0}\left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle.$$

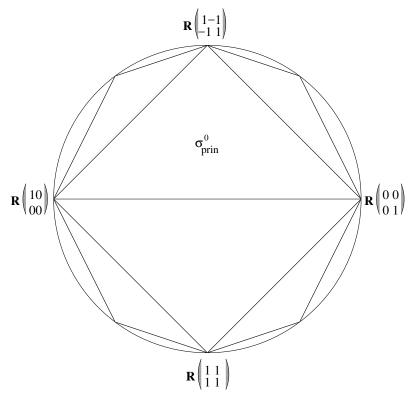


FIGURE 1. A section of Ω_2^{rt} and its perfect cone decomposition.

Thus, to determine Γ_{-1}, Γ_0, Γ_1, and Γ_2, it is enough to see which of σ_0, σ_1, σ_2, and σ_3 are alternating. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can show that A stabilizes both σ_2 and σ_3 and that the induced cone morphism is orientation-reversing. Thus, Γ_1 and Γ_2 are empty. On the other hand, since the action of $\mathrm{GL}_2(\mathbb{Z})$ fixes the cone point σ_0, both σ_0 and σ_1 are alternating. So, $\Gamma_{-1} = \{\sigma_0\}$ and $\Gamma_0 = \{\sigma_1\}$. From this we see that the complex $P_\bullet^{(2)}$ is:

$$\begin{array}{ccccccccc} P_2^{(2)} & & P_1^{(2)} & & P_0^{(2)} & & P_{-1}^{(2)} & & P_{-2}^{(2)} \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q}\langle e_{\sigma_1}\rangle & \xrightarrow{\partial_0} & \mathbb{Q}\langle e_{\sigma_0}\rangle & \longrightarrow & 0 \end{array}$$

where ∂_0 sends e_{σ_1} to either e_{σ_0} or $-e_{\sigma_0}$ depending on the chosen orientations.

Remark 4.3. To be precise, the perfect complex $P_\bullet^{(g)}$ as constructed in Definition 4.1 is only unique up to isomorphism. In particular, the choice of representatives for Γ_n or reference orientations may result in different but isomorphic chain complexes. For instance, in Example 4.2, the differential ∂_0 is only determined up to sign.

The next proposition shows that the perfect complex $P_\bullet^{(g)}$ is isomorphic to the cellular chain complex associated to the symmetric CW complex $LA_g^{\mathrm{trop,P}}$. Thus, by Theorem 3.1, the homology of $P_\bullet^{(g)}$ computes the top-weight rational cohomology of $\mathcal{A}_g$.

Proposition 4.4. For each $i \geq 0$, there exist canonical isomorphisms

$$H_{i-1}(P_\bullet^{(g)}) \cong \widetilde{H}_{i-1}(LA_g^{\mathrm{trop,P}};\mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-i}(\mathcal{A}_g;\mathbb{Q}).$$

where $A_g^{\mathrm{trop,P}} = A_g^{\mathrm{trop},\Sigma_g^{\mathrm{P}}}$ is the tropical moduli space constructed in Section 2.6.

Proof. By construction, $P_\bullet^{(g)}$ is naturally isomorphic to the cellular chain complex of $LA_g^{\mathrm{trop,P}}$ as defined in Section 2.2. Observe that the space $A_g^{\mathrm{trop,P}}$ is connected since it deformation retracts to the cone point. Thus, the first isomorphism then follows from part (1) of Proposition 2.1 and the second isomorphism follows from Theorem 3.1. □

Example 4.5. By Example 4.2, we see that $P_\bullet^{(2)}$ has trivial homology in all degrees. Thus, Proposition 4.4 recovers the fact that $\mathcal{A}_2$ has trivial top-weight cohomology [Igu62].

4.2. **The Voronoi Complex.** Now we introduce a closely related complex, called the Voronoi complex $V_\bullet^{(g)}$, as considered in [EVGS13, SEVKM19][1]. We shall soon see that $V_\bullet^{(g)}$ is a quotient of $P_\bullet^{(g)}$, obtained by setting to zero the generators corresponding to cones contained in the boundary of Ω_g^{rt}.

For each $n \in \mathbb{Z}$, let $\overline{\Gamma}_n^{(g)} = \overline{\Gamma}_n$ be the subset of Γ_n consisting of those cones σ such that $\sigma \cap \Omega_g \neq \emptyset$. For each $\sigma \in \overline{\Gamma}_n$, let ω_σ be an orientation of σ, and for each $\sigma \in \overline{\Gamma}_{n-1}$ define $\delta(\sigma', \sigma)$ as before in (4). With this notation, we can now define the Voronoi complex.

Definition 4.6. The *Voronoi chain complex* $(V_\bullet^{(g)}, d_\bullet)$ is the complex where $V_n^{(g)}$ is the $\mathbb{Q}$-vector space with basis indexed by $\overline{\Gamma}_n$ and the differential $d_n : V_n^{(g)} \to V_{n-1}^{(g)}$ is given by

$$d(e_\sigma) \coloneqq \sum_{\sigma' \in \overline{\Gamma}_{n-1}} \delta(\sigma', \sigma) e_{\sigma'}.$$

Example 4.7. The $\mathrm{GL}_2(\mathbb{Z})$-orbits of alternating cones in Σ_2^{P} are all contained in $\Omega_2^{\mathrm{rt}} \setminus \Omega_2$. Hence the Voronoi complex $V_\bullet^{(2)}$ is zero in all degrees.

There is a natural surjection of chain complexes $P_\bullet^{(g)} \twoheadrightarrow V_\bullet^{(g)}$ given by quotienting $P_n^{(g)}$ by the subcomplex spanned by those cones contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$. Our next goal, achieved in Theorem 4.13, is to show that the kernel of the map above can naturally be identified with $P_\bullet^{(g-1)}$. We begin by noting the following two lemmas studying those cones lying in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$.

Lemma 4.8. Let $\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_n v_n^t \rangle$ with $v_1, \ldots, v_n \in \mathbb{Z}^g$ be a perfect cone. Then σ is contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ if and only if $\dim \operatorname{span}_{\mathbb{R}}\langle v_1, v_2, \ldots, v_n \rangle < g$.

Lemma 4.9. If $\sigma \in \Sigma_g^{\mathrm{P}}$ is a perfect cone and $\sigma \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ then there is a matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ and a cone $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$, where $g' < g$ and $\sigma' \cap \Omega_{g'} \neq \emptyset$, with

$$A \sigma A^t = \left\{ \left(\begin{array}{c|c} Q' & 0 \\ \hline 0 & 0 \end{array} \right) \middle| Q' \in \sigma' \right\}. \tag{5}$$

In this situation, say σ' is a *reduction* of σ.

We now show that, in a sense that we shall make precise, the action of $\mathrm{GL}_g(\mathbb{Z})$ on σ does not depend on the ambient matrix size g. For example, given a cone $\sigma \in \Sigma_g^{\mathrm{P}}$ and a reduction $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$ of σ, we will see that σ is alternating if and only if σ' is alternating. We begin with the following definition.

Definition 4.10. Given perfect cones $\sigma_1, \sigma_2 \in \Sigma_g^{\mathrm{P}}$, let $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ denote the set of morphisms $\rho : \sigma_1 \to \sigma_2$ which are restrictions from the action of $\mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}:

$$\begin{tikzcd} \Omega_g^{\mathrm{rt}} \arrow[r, two heads, "X \mapsto AXA^t"] & \Omega_g^{\mathrm{rt}} \\ \sigma_1 \arrow[u, hook] \arrow[r, "\rho"] & \sigma_2 \arrow[u, hook] \end{tikzcd}.$$

The following two results concerning homomorphisms of cones contained in the boundary of Ω_g^{rt} are standard and possibly well-known to experts. We include proofs here, however, as we are unaware of suitable references.

[1] In [EVGS13, SEVKM19] the Voronoi complex is defined as a complex of free $\mathbb{Z}$-modules, while our definition of Voronoi complex is as a complex of $\mathbb{Q}$-vector spaces.

Proposition 4.11. If $\sigma_1, \sigma_2 \in \Sigma_g^{\mathrm{P}}$ are perfect cones contained in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ and $\sigma_1', \sigma_2' \in \Sigma_{g'}^{\mathrm{P}}$ are reductions of σ_1 and σ_2 respectively, then there exists a bijection

$$\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2) \xleftrightarrow{\sim} \mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2').$$

Proof. By Lemma 4.9, we may assume σ_i are in the form of (5). Then if $\rho' \in \mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2')$ arises from the action of a matrix $A' \in \mathrm{GL}_{g'}(\mathbb{Z})$ on $\Omega_{g'}^{\mathrm{rt}}$, then extending it by a $(g-g') \times (g-g')$ identity matrix gives a matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ that induces a cone morphism $\rho : \sigma_1 \to \sigma_2$.

In the other direction, suppose that $\rho \in \mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ comes from the action of a matrix $A \in \mathrm{GL}_g(\mathbb{Z})$ on Ω_g^{rt}. Write $\mathbb{R}^{g'}$ for the coordinate subspace of $\mathbb{R}^g$ of vectors in which the last $g - g'$ coordinates are zero. Let $\sigma_1 = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_n v_n^t \rangle$. By Lemma 4.8, the vectors $v_1, \ldots, v_n$ span $\mathbb{R}^{g'}$. Since $A\sigma_1 A^t = \sigma_2$, it follows again from Lemma 4.8 that $Av_1, \ldots, Av_n$ also span $\mathbb{R}^{g'}$; thus A restricts to a map $A' \colon \mathbb{R}^{g'} \to \mathbb{R}^{g'}$, with $A(\mathbb{Z}^{g'}) \subseteq \mathbb{Z}^{g'}$. Similarly, A^{-1} restricts to $(A')^{-1} \colon \mathbb{R}^{g'} \to \mathbb{R}^{g'}$, and $(A')^{-1}(\mathbb{Z}^{g'}) \subseteq \mathbb{Z}^{g'}$. Therefore $A' \in \mathrm{GL}_{g'}(\mathbb{Z})$ is an invertible integer matrix, with $A'\sigma_1'(A')^t = \sigma_2'$.

Finally, a direct computation shows that these constructions are mutual inverses. □

As a corollary of Proposition 4.11, the properties of being in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit and being alternating do not depend on g—that is, they are preserved by taking reductions.

Corollary 4.12. Two perfect cones $\sigma_1, \sigma_2 \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ are in the same $\mathrm{GL}_g(\mathbb{Z})$-orbit if and only if there exists a $g' < g$ and reductions $\sigma_1', \sigma_2' \in \Sigma_{g'}^{\mathrm{P}}$ that are in the same $\mathrm{GL}_{g'}(\mathbb{Z})$-orbit. A perfect cone $\sigma \subset \Omega_g^{\mathrm{rt}} \setminus \Omega_g$ is alternating if and only if there exists a reduction $\sigma' \in \Sigma_{g'}^{\mathrm{P}}$ which is alternating.

Proof. Two perfect cones σ_1 and σ_2 are in the same orbit if and only if $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ is nonempty. Then the claim follows from Proposition 4.11, since $\mathrm{Hom}_{\Omega_g^{\mathrm{rt}}}(\sigma_1, \sigma_2)$ is nonempty if and only if $\mathrm{Hom}_{\Omega_{g'}^{\mathrm{rt}}}(\sigma_1', \sigma_2')$ is nonempty. Similarly, the proof of Proposition 4.11, applied to $\sigma = \sigma_1 = \sigma_2$, shows that σ has an orientation-reversing automorphism if and only if its reduction σ' does. □

Corollary 4.12 allows us to naturally identify the set of $\mathrm{GL}_g(\mathbb{Z})$-orbits of alternating perfect cones in $\Omega_g^{\mathrm{rt}} \setminus \Omega_g$ with the set of $\mathrm{GL}_{g-1}(\mathbb{Z})$-orbits of alternating perfect cones in $\Omega_{g-1}^{\mathrm{rt}}$. Thus, we have the following theorem.

Theorem 4.13. We have a short exact sequence of chain complexes

$$0 \longrightarrow P_\bullet^{(g-1)} \longrightarrow P_\bullet^{(g)} \xrightarrow{\pi} V_\bullet^{(g)} \longrightarrow 0.$$

Proof. By construction, the kernel of $\pi : P_n^{(g)} \to V_n^{(g)}$ is generated by those basis vectors e_σ where $\sigma \in \Gamma_n^{(g)} \setminus \overline{\Gamma}_n^{(g)}$. (Recall that $\Gamma_n^{(g)}$ denotes a set of representatives of alternating $\mathrm{GL}_g(\mathbb{Z})$-orbits of cones in Σ_g^P, and $\overline{\Gamma}_n^{(g)}$ denotes the subset of those that meet Ω_g.) By Corollary 4.12, such cones are in bijection with elements of $\Gamma_n^{(g-1)}$. The differentials on $P_\bullet^{(g)}$ and $P_\bullet^{(g-1)}$ are defined in the same fashion, so the result follows. □

Theorem 4.13 reflects the stratification of LA_g^{trop} by the spaces $L\Omega_{g'}/\mathrm{GL}_{g'}(\mathbb{Z})$, for $g' = 1, \ldots, g$, which are rational classifying spaces for $\mathrm{GL}_{g'}(\mathbb{Z})$; this is the underlying geometric reason that it is possible to relate the cohomology of $\mathcal{A}_g$ to that of $\mathrm{GL}_{g'}(\mathbb{Z})$, as we do here. This possible relationship was suggested in the more general setting of arithmetic quotients of Hermitian symmetric domains in [OO18, §2.4].

5. The inflation complex and the coloop complex

In this section, we define a subcomplex of $P_\bullet^{(g)}$, called the inflation complex $I_\bullet^{(g)}$. We shall show in Theorem 5.15 that $I_\bullet^{(g)}$ is acyclic. This acyclicity result implies a vanishing result for $H_k(P_\bullet^{(g)})$ in low degrees, obtained in Corollary 5.16, and it is invoked in the computations in the next section for $g = 6$ and $g = 7$. In Section 5.2, we define an analogous subcomplex, the *coloop* complex, of the regular matroid complex, and prove an analogous acyclicity result. The acyclicity of the coloop complex will not be used in this paper, but should likely be useful for future study of the regular matroid complex.

5.1. **The inflation complex.**

Definition 5.1.

(1) Let $S \subset \mathbb{Z}^g$ be a finite set. Say $v \in S$ is a $\mathbb{Z}^g$*-coloop* of S if v is part of a $\mathbb{Z}$-basis $v, w_2, \ldots, w_g$ for $\mathbb{Z}^g$ such that any $w \in S \setminus \{v\}$ is in the $\mathbb{Z}$-linear span of $w_2, \ldots, w_g$. Equivalently, v is a $\mathbb{Z}^g$-coloop if, up to the action of $\mathrm{GL}_g(\mathbb{Z})$, we may write $v = (0, \ldots, 0, 1)$ and $w = (*, \ldots, *, 0)$ for all $w \in S \setminus \{v\}$.

(2) Now let $\sigma = \sigma[Q]$ be a perfect cone in Σ_g^{P}. Recall that the set $M(Q)$ of minimal vectors has the property that $v \in M(Q)$ if and only if $-v \in M(Q)$; let $M'(Q) = \{v_1, \ldots, v_n\}$ be a choice of one of $\{v, -v\}$ for each $v \in M(Q)$. So

$$\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_n v_n^t\rangle.$$

We say $v \in M'(Q)$ is a *coloop* of σ if v is a $\mathbb{Z}^g$-coloop in $M'(Q)$.

Remark 5.2. The definition of a $\mathbb{Z}^g$-coloop is inspired by the notion of a coloop of a matroid (i.e., an element not belonging to any circuit). Indeed, if $S \subset \mathbb{Z}^g$ is any finite set and $v \in S$, then v being a $\mathbb{Z}^g$-coloop of S implies that v is a coloop of S, considered as vectors in $\mathbb{R}^g$. The converse does not hold: for example, let $(v_1, v_2) = ((0, 1), (3, 2))$. Then v_1 and v_2 are coloops of the matroid $M(v_1, v_2)$ over $\mathbb{R}$, but neither is a $\mathbb{Z}^2$-coloop.

On the other hand, we prove in Lemma 5.22 that if M is a regular matroid, then M has a coloop if and only if a totally unimodular matrix A representing M has column vectors with a $\mathbb{Z}^g$-coloop, if and only if $\sigma(M)$ has a coloop in the sense of Definition 5.1(2).

Example 5.3. Consider the quadratic form defined by the positive definite matrix

$$Q = \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{pmatrix}.$$

The minimum of Q on $\mathbb{Z}^3 - \{0\}$ is 1 and

$$M(Q) = \{(0, \pm 1, 0), (0, 0, \pm 1), \pm(1, 0, -1), \pm(0, 1, -1)\} \subset \mathbb{R}^3.$$

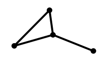
The corresponding perfect cone $\sigma[Q]$ has a coloop, in particular, letting

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

we see that

$$A \begin{pmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{pmatrix} A^t = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so $M(AQA^t)$ is $\{(\pm 1, 0, 0), (0, \pm 1, 0), \pm(1, 1, 0), (0, 0, \pm 1)\}$.



The cone $\sigma[Q]$ can also be realized as the matroidal cone $\sigma[M(G)]$ where G is the graph below. The coloop corresponds to the bridge edge of G.

Lemma 5.4. Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{Z}^g$ and suppose that $v_1 \neq v_2$ are both $\mathbb{Z}^g$-coloops for S. Then there is a $\mathbb{Z}$-basis $v_1, v_2, w_3, \ldots, w_g$ for $\mathbb{Z}^g$ such that

$$v_3, \ldots, v_n \in \mathbb{Z}\langle w_3, \ldots, w_g \rangle.$$

Proof. By restricting to $\mathbb{R}\langle v_1, \ldots, v_n \rangle$, we may assume that $\mathbb{R}\langle v_1, \ldots, v_n \rangle = \mathbb{R}^g$. Now the fact that both v_1 and v_2, being coloops, are in every basis for $\mathbb{R}^g$ chosen from $\{v_1, \ldots, v_n\}$, implies that $v_3, \ldots, v_n$ span a $(g-2)$-dimensional subspace V of $\mathbb{R}^g$. Let $w_3, \ldots, w_g$ be a $\mathbb{Z}$-basis for $V \cap \mathbb{Z}^g$. We need only verify that $v_1, v_2, w_3, \ldots, w_g$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^g$. Let $x \in \mathbb{Z}^g$. Since $v_1, v_2, w_3, \ldots, w_g$ form a $\mathbb{Q}$-basis, we have

$$x = a_1 v_1 + a_2 v_2 + a_3 w_3 + \cdots + a_g w_g, \qquad \text{for some } a_i \in \mathbb{Q},$$

and it suffices to show that $a_i \in \mathbb{Z}$ for all $i = 1, 2, 3, \ldots, g$. First, we show $a_1 \in \mathbb{Z}$. Since v_1 is a $\mathbb{Z}^g$-coloop, after a change of $\mathbb{Z}$-basis, we may assume that $v_1 = (0, \ldots, 0, 1)$ and that $v_2, \ldots, v_n$ have last coordinate zero. Since $w_3, \ldots, w_g \in \mathrm{span}_{\mathbb{R}}\langle v_3, \ldots, v_n \rangle$, each w_i also has last coordinate zero. Therefore $a_1 \in \mathbb{Z}$. By a similar argument, $a_2 \in \mathbb{Z}$. Then $a_3 w_3 + \cdots + a_g w_g \in V \cap \mathbb{Z}^g$ for $a_i \in \mathbb{Q}$. But $w_3, \ldots, w_g$ is a $\mathbb{Z}$-basis for $V \cap \mathbb{Z}^g$. Therefore it must be that $a_3, \ldots, a_g \in \mathbb{Z}$, as desired.

□

Corollary 5.5. Let $S \subset \mathbb{Z}^{g-1}$ be a finite set, and identify $\mathbb{Z}^{g-1}$ with its image in $\mathbb{Z}^g$ under $w \mapsto (w, 0)$. Then v is a coloop of S if and only if v is a coloop of $S \cup \{e_g\} \subset \mathbb{Z}^g$.

Proof. The forward direction is direct from the definitions. For the backward direction, suppose v is a coloop of $S \cup \{e_g\}$. Then *both* v and e_g are coloops, so by Lemma 5.4, up to the action of $\mathrm{GL}_g(\mathbb{Z})$, we may assume that $v = e_{g-1}$ and $w = (*, \ldots, *, 0, 0)$ for all $w \in S \setminus \{v\}$. Therefore v was a coloop of $S \subset \mathbb{Z}^{g-1}$. □

Corollary 5.6. A cone with two or more coloops is not alternating. That is, if $\sigma = \sigma[Q]$ where $v \neq v' \in M'(Q)$ are two distinct coloops, then σ is not alternating.

Proof. The cone σ has an orientation-reversing automorphism induced by an element of $GL_g(\mathbb{Z})$ swapping the two coloops. □

We now describe two operations on cones, inflation and deflation, that add or remove coloops respectively. Inflation is described in [EVGS13, Section 6.1], and can be performed for any cone, but we shall consider it for the cones in the set $\Sigma^P_{g,\mathrm{nco}}[n]$ defined below.

Recall that $\Sigma^{\mathrm{P}}_g[n]$ denotes the set of $(n+1)$-dimensional perfect cones, and $\Sigma^{\mathrm{P}}_g[n]/GL_g(\mathbb{Z})$ denotes the collection of $GL_g(\mathbb{Z})$-orbits of $(n+1)$-dimensional perfect cones.

Definition 5.7. We define two subsets of $\Sigma^{\mathrm{P}}_g[n]$ as follows:

$$\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n] \coloneqq \{\sigma \in \Sigma^{\mathrm{P}}_g[n] : \mathrm{rank}(\sigma) \leq g-1 \text{ and } \sigma \text{ has no coloop}\},$$
$$\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n] \coloneqq \{\sigma \in \Sigma^{\mathrm{P}}_g[n] : \sigma \text{ has exactly one coloop}\}.$$

We then define $\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$ and $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n]/GL_g(\mathbb{Z})$ to be the collection of $GL_g(\mathbb{Z})$-orbits of the respective sets.

We now define inflation and deflation as operations on $\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$ and $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n]/GL_g(\mathbb{Z})$, and we show these operations are well defined in Lemma 5.12.

Definition 5.8. *Inflation* is the map

$$\mathrm{ifl} : \Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z}) \longrightarrow \Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z})$$

defined as follows. Given an element of $\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$, choose a representative

$$\sigma = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t\rangle,$$

so that the g-th entry of the w_i are zero (see Lemma 4.9). Let

$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t, e_g e_g^t\rangle.$$

Then set $\mathrm{ifl}([\sigma]) = [\tilde{\sigma}]$.

Remark 5.9. We check that inflation is well defined in Lemma 5.12. However, we pause to point out that $\tilde{\sigma}$ is indeed a perfect cone, as noted in [EVGS13, Section 6.1]: if $Q \in \Omega_{g-1}$ is a positive definite quadratic form such that $\sigma[Q] = \mathbb{R}_{\geq 0}\langle \tilde{w}_1 \tilde{w}_1^t, \ldots, \tilde{w}_k \tilde{w}_k^t\rangle$, where $\tilde{w}_i$ denotes the truncation of w_i by the last entry, then the inflation of σ is the cone associated to the quadratic form

$$\tilde{Q} = \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & m(Q) \end{array}\right)$$

where $m(Q)$ is the minimum value of Q on $\mathbb{Z}^{g-1} \setminus \{0\}$. Moreover, $\tilde{\sigma}$ has exactly one coloop by Corollary 5.5 and the fact that σ had no coloops.

Example 5.10. Continuing Example 5.3, we see that if Q' is the positive definite quadratic form

$$Q' = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix},$$

then $M(Q') = \{\pm(1,0,0), \pm(0,1,0), \pm(1,-1,0)\}$. Thus, the cone $\sigma[Q]$ is the inflation of $\sigma[Q']$. We may describe the cone $\sigma[Q']$ as

$$\sigma[Q'] = \left\{ \left(\begin{array}{c|c} Q'' & 0 \\ \hline 0 & 0 \end{array}\right) \;\middle|\; Q'' \in \sigma\left[\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right] \right\},$$

from which we see that $\sigma[Q']$ does not meet Ω_3, but its inflation $\sigma[Q]$ does. In general, the inflation of a perfect cone corresponding to a quadratic form of rank r will itself be a perfect cone corresponding to a quadratic form of rank $r+1$.

Definition 5.11. We define the *deflation* operation as a map

$$\mathrm{dfl} : \Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z}) \longrightarrow \Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$$

given as follows. Given an element of $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z})$, pick a $GL_g(\mathbb{Z})$-representative

$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t, e_g e_g^t\rangle,$$

where each w_i is zero in the last coordinate. Let $\sigma = \mathbb{R}_{\geq 0}\langle w_1 w_1^t, \ldots, w_k w_k^t\rangle$. It is routine to check that σ really is a perfect cone, and moreover, it has no coloops by Corollary 5.5. Then we set $\mathrm{dfl}([\tilde{\sigma}]) = [\sigma]$. We now show that inflation and deflation are well defined.

Lemma 5.12. For each $n \in \mathbb{N}$, inflation is a well-defined operation on $\Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/GL_g(\mathbb{Z})$, and deflation is a well-defined operation on $\Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/GL_g(\mathbb{Z})$. Furthermore, these operations are inverses of each other.

Proof. We start with inflation. Given $[\sigma] \in \Sigma^{\mathrm{P}}_{g,\mathrm{nco}}[n]/\mathrm{GL}_g(\mathbb{Z})$, let $\sigma_1 = \mathbb{R}_{\geq 0}\langle v_1v_1^t, \ldots, v_kv_k^t\rangle$ and $\sigma_2 = \mathbb{R}_{\geq 0}\langle w_1w_1^t, \ldots, w_kw_k^t\rangle$ be two $\mathrm{GL}_g(\mathbb{Z})$-representatives of $[\sigma]$ such that the g-th entry of each of the v_i, w_j is zero. By Proposition 4.11, there exist reductions σ_1' of σ_1 and σ_2' of σ_2 as well as an $A \in \mathrm{GL}_{g-1}(\mathbb{Z})$ sending σ_1' to σ_2' in $\Omega^{\mathrm{rt}}_{g-1}$. Then

$$A' = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$$

yields an equivalence between the two inflations.

Now let $[\sigma] \in \Sigma^{\mathrm{P}}_{g,\mathrm{co}}[n+1]/\mathrm{GL}_g(\mathbb{Z})$. Let

$$\sigma_1 = \mathbb{R}_{\geq 0}\langle v_1v_1^t, \ldots, v_kv_k^t, e_ge_g^t\rangle, \ \ \sigma_2 = \mathbb{R}_{\geq 0}\langle w_1w_1^t, \ldots, w_kw_k^t, e_ge_g^t\rangle$$

be two $\mathrm{GL}_g(\mathbb{Z})$ representatives of $[\sigma]$ such that the v_i, w_j have g-th coordinate zero. Then by the proof of Proposition 4.11, there exists an $A \in \mathrm{GL}_g(\mathbb{Z})$ such that

$$Av_i = \pm w_i \text{ for } i = 1, \ldots, k, \quad Ae_g = \pm e_g,$$

possibly after reordering the v_i. Indeed, A must take the coloop $\pm e_g$ to $\pm e_g$. Then A gives an equivalence between the deflations $\mathbb{R}_{\geq 0}\langle v_1v_1^t, \ldots, v_kv_k^t\rangle \sim \mathbb{R}_{\geq 0}\langle w_1w_1^t, \ldots, w_kw_k^t\rangle$.

We now have that inflation and deflation are well defined, and it is clear from the definitions that these two operations are inverses. $\square$

Lemma 5.13. Let $\sigma = \mathbb{R}_{\geq 0}\langle v_1v_1^t, \ldots, v_kv_k^t\rangle$ be a perfect cone in Σ^{P}_g of rank $< g$ with no coloop. Then $[\sigma]$ is alternating if and only if $\mathrm{ifl}([\sigma])$ is alternating.

Proof. We may assume that $v_1, \ldots, v_n$ have last coordinate 0, so that, letting

$$\tilde{\sigma} = \mathbb{R}_{\geq 0}\langle v_1v_1^t, \ldots, v_kv_k^t, e_ge_g^t\rangle,$$

we have $\mathrm{ifl}(\sigma) = \tilde{\sigma}$. We claim there is a natural bijection

$$\mathrm{Aut}(\sigma) \longleftrightarrow \mathrm{Aut}(\tilde{\sigma}), \tag{6}$$

where $\mathrm{Aut}(\sigma) = \mathrm{Hom}_{\Omega^{\mathrm{rt}}_g}(\sigma, \sigma)$ (see Definition 4.10) and similarly for $\mathrm{Aut}(\tilde{\sigma})$. Moreover, we claim that (6) takes orientation-preserving/reversing automorphisms of σ to orientation-preserving/reversing automorphisms of $\tilde{\sigma}$, respectively.

Given $\rho \in \mathrm{Aut}(\sigma)$ arising from a matrix $A \in \mathrm{GL}_{g-1}(\mathbb{Z})$, the matrix

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

yields an automorphism $\tilde{\rho}$ of $\tilde{\sigma}$. The linear span of $\tilde{\sigma}$ is the sum of the linear span of σ and that of $e_ge_g^t$. Moreover, $\tilde{\rho}$ fixes the ray $e_ge_g^t$ of $\tilde{\sigma}$ and acts on the linear span of σ according to A; in particular, $\tilde{\rho}$ is orientation-preserving if and only if ρ was.

Next, suppose $\tilde{A} \in \mathrm{GL}_g(\mathbb{Z})$ induces $\tilde{\rho} \in \mathrm{Aut}(\tilde{\sigma})$. Recall that e_g is the only coloop of $\tilde{\sigma}$, by Corollary 5.5. Therefore $Ae_g = \pm e_g$, and hence A induces an automorphism of σ. Finally, it is routine to check that the maps constructed between $\mathrm{Aut}(\sigma)$ and $\mathrm{Aut}(\tilde{\sigma})$ are two-sided inverses.

$\square$

Definition 5.14. Let $I^{(g)}_\bullet$ be the subcomplex of $P^{(g)}_\bullet$ generated in degree n by cones $\sigma \in \Gamma_n$ of rank $\leq g-1$ and cones of rank g with a coloop.

Theorem 5.15. The chain complex $I^{(g)}_\bullet$ is acyclic.

Proof. By Lemmas 5.12 and 5.13 there is a matching of cones generating $I^{(g)}_\bullet$, given by

$$\sigma \to \begin{cases} \mathrm{ifl}(\sigma) & \text{if } \sigma \text{ has no coloop,} \\ \mathrm{dfl}(\sigma) & \text{if } \sigma \text{ has a coloop.} \end{cases}$$

Here, we have abused notation slightly, since ifl is an operation on orbits rather than orbit representatives. Thus, when we write $\mathrm{ifl}(\sigma) = \sigma'$ for $\sigma \in \Gamma_n$ we mean that σ' is the unique orbit representative in Γ_{n+1} such that $\mathrm{ifl}([\sigma]) = [\sigma']$. Similarly for deflation.

Now let σ be a generator in $I_\bullet^{(g)}$ of maximal degree; then σ must have a coloop v. We claim that $\sigma' = \mathrm{dfl}(\sigma)$ is not a facet of any other generator $\tau \neq \sigma$ of $I_\bullet^{(g)}$. Indeed, suppose

$$\tau = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_k v_k^t\rangle$$

is a generator of $I_\bullet^{(g)}$ containing σ' as a facet. If τ had a coloop, say v_n, then since σ' has no coloop, σ' must not contain the ray $v_n v_n^t$. But

$$\mathbb{R}_{\geq 0}\langle v_1 v_1^t, \ldots, v_{k-1} v_{k-1}^t\rangle$$

is already a facet of τ, so it must be $\sigma' = \mathrm{dfl}(\sigma) = \mathrm{dfl}(\tau)$, implying $\tau = \sigma$. So τ has no coloop. But then $\mathrm{ifl}(\tau)$ is a generator of $I_\bullet^{(g)}$ and it would have higher rank than σ.

Thus, the complex $I_\bullet^{(g)'}$ spanned by all cones except σ and $\mathrm{dfl}(\sigma)$ is a subcomplex. Then we have a short exact sequence

$$0 \longrightarrow I_\bullet^{(g)'} \longrightarrow I_\bullet^{(g)} \longrightarrow I_\bullet^{(g)}/I_\bullet^{(g)'} \longrightarrow 0,$$

where $I_\bullet^{(g)}/I_\bullet^{(g)'}$ is isomorphic to $0 \to \sigma \to \mathrm{dfl}(\sigma) \to 0$. Hence $I_\bullet^{(g)'} \to I_\bullet^{(g)}$ is a quasi-isomorphism. Repeating this, we deduce inductively that I is quasi-isomorphic to 0. $\square$

As a corollary of Theorem 5.15, we are able to prove the following vanishing result for the cohomology of $P_\bullet^{(g)}$ in low degrees.

Corollary 5.16. If $k \leq g-2$ then $H_k\left(P_\bullet^{(g)}\right) = 0$.

Proof. Since the inflation complex $I_\bullet^{(g)}$ is acyclic by Theorem 5.15, it is enough to show that $I_k^{(g)} = P_k^{(g)}$ for all $k \leq g-2$. For this, we simply need the well-known fact that the rank of a perfect cone is at most its dimension. Indeed, let $\sigma = \mathbb{R}_{\geq 0}\langle v_1 v_1^t, v_2 v_2^t, \ldots, v_n v_n^t\rangle \in \Sigma_g^{\mathrm{P}}$ be an alternating cone of dimension $k+1$. If $Q \in \sigma$ then

$$Q = \lambda_1 v_{i_1} v_{i_1}^t + \lambda_2 v_{i_2} v_{i_2}^t + \ldots + \lambda_{k+1} v_{i_{k+1}} v_{i_{k+1}}^t$$

for some $\{i_1, \ldots, i_{k+1}\} \subset \{1, \ldots, n\}$ and some $\lambda_i \in \mathbb{R}_{\geq 0}$. In particular, since $v_j v_j^t$ is a rank one quadratic form, this implies that the rank of Q is at most $k+1$. Thus, if $k+1 \leq g-1$ then $\mathrm{rank}(\sigma) < g$, implying that the orbit of σ represents an element of $I_k^{(g)}$. $\square$

Remark 5.17. The inflation operation is, of course, a special case of taking the block sum of two perfect cones. In this way one obtains a product map on chain complexes $P^{(g_1)} \otimes P^{(g_2)} \to P^{(g_1+g_2)}$, and a corresponding product on homology. This is reminiscent of the result of [GHT18a] describing the stable cohomology of the matroidal partial compactifications $\overline{\mathcal{A}}_g^{\mathrm{matr}}$ via 1-sums of irreducible regular matroids. Perhaps if one had nonvanishing statements for the latter product, then the cohomology classes detected in this paper could be used to construct infinite families of top-weight classes in $\mathcal{A}_g$.

Remark 5.18. The proof of Corollary 5.16 shows that any cone of dimension less than or equal to $g-1$ does not intersect Ω_g. This implies that $H_k(V_\bullet^{(g)}) = 0$ for all $k \leq g-2$.

Remark 5.19. The virtual cohomological dimension of $\mathcal{A}_g$ is

$$\mathrm{vcd}(\mathcal{A}_g) = \mathrm{vcd}(\mathrm{Sp}(2g, \mathbb{Z})) = g^2,$$

by [BS73], see [CL17]. In particular,

$$\mathrm{Gr}^W_{g^2+g} H^i(\mathcal{A}_g;\mathbb{Q}) = 0 \quad \text{for all } i > g^2,$$

which is equivalent to, setting $i = 2\dim(\mathcal{A}_g) - j - 1 = g^2 + g - j - 1$, that

$$H_j(P^{(g)}) = 0 \quad \text{for all } j < g - 1.$$

Corollary 5.16 thus reproves, in a completely different way, the vanishing in top weight of rational cohomology of $\mathcal{A}_g$ in degree above the virtual cohomological dimension.

5.2. **The Regular Matroid Complex and Inflation.** In this section, we introduce two combinatorially defined subcomplexes, $R_\bullet^{(g)}$ and $C_\bullet^{(g)}$, of $P_\bullet^{(g)}$ coming from regular matroids and regular matroids with coloops respectively. These are not used further in this paper. Nevertheless, the matroidal cones in Σ_g^P have geometric significance: Alexeev and Brunyate, in proving the existence of a compactified Torelli map $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{perf}}$, conjectured an open locus on which $\overline{\mathcal{A}_g}^{\mathrm{perf}}$ and $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ are isomorphic and on which the two Torelli maps $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{perf}}$ and $\overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^{\mathrm{Vor}}$ agree [AB12]. The fourth author and Viviani verified their conjecture, showing that the *matroidal partial compactification* $\mathcal{A}_g^{\mathrm{matr}}$, whose strata correspond to cones arising from regular matroids, is the largest such open subset [MV12]. For possible future use in studying $R^{(g)}$, we establish in this section that the complex $C_\bullet^{(g)}$, which is a matroid analogue of $I_\bullet^{(g)}$, is acyclic.

Given a cone $\sigma \in \Sigma_g^{\mathrm{P}}[n]$, we say that σ is a matroidal cone if and only if there exists a simple, regular matroid M of rank at most g such that $[\sigma] = \sigma(M)$ where $\sigma(M)$ is as described in Construction 2.12. Matroidal cones are simplicial. Since the faces of a matroidal cone are themselves matroidal cones, the set of representatives of alternating cones arising from simple, regular matroids forms a subcomplex of $P^{(g)}$.

Definition 5.20. The *regular matroid complex* $R_\bullet^{(g)}$ is the subcomplex of $P_\bullet^{(g)}$ generated in degree n by cones $\sigma \in \Gamma_n$ such that σ is a matroidal cone.

Remark 5.21. When $g = 2$ and $g = 3$, the complexes $R_\bullet^{(g)}$ and $P_\bullet^{(g)}$ are in fact equal. It would be interesting to understand in general how much larger $P_\bullet^{(g)}$ is compared to $R_\bullet^{(g)}$.

Recall that an element e of a matroid M is a *coloop* if it does not belong to any of the circuits of M; equivalently, e is a coloop if it belongs to every base of M. When M is a regular matroid, this is equivalent to the existence of a totally unimodular matrix $A = [v_1, v_2, \ldots, v_n]$ representing M such that $v_i = (*, *, \ldots, *, 0)$ for $i = 1, 2, \ldots, n-1$ and $e = v_n = (0, 0, \ldots, 0, 1)$. It is worth establishing that the notions of a matroid coloop and a $\mathbb{Z}^g$-coloop agree for matroidal cones, as we show in the next lemma.

Lemma 5.22. Let M be a simple, regular matroid of rank $\leq g$. The cone $\sigma(M)$ has a $\mathbb{Z}^g$-coloop if and only if the matroid M has a coloop.

Proof. Suppose that $\sigma(M)$ has a $\mathbb{Z}^g$-coloop. By definition, there exists a quadratic form $Q \in \Omega_g$ such $[\sigma(Q)] = \sigma(M)$ and $M'(Q) = \{v_1, v_2, \ldots, v_n\}$ where $v_i = (*, *, \ldots, *, 0)$ for $i = 1, \ldots, n-1$ and $v_n = (0, 0, \ldots, 0, 1)$. Then by the construction of $\sigma(M)$, the matrix $A = [v_1, v_2, \ldots, v_n]$ is a totally unimodular matrix representing M over $\mathbb{R}$. Therefore v_n is a coloop of the matroid M.

For the other direction, suppose that the regular matroid M on the ground set $\{1, \ldots, n\}$ is represented by a full-rank totally unimodular $g' \times n$ matrix $A = [v_1, v_2, \ldots, v_n]$, for some $g' \leq g$, and that n is a coloop of M. Then n is in every base of M, so v_n is in every full-rank $g' \times g'$ submatrix of A. Reorder so that the rightmost $g' \times g'$ submatrix is full rank; call it B. Then $B \in \mathrm{GL}_{g'}(\mathbb{Z})$ by total unimodularity of A. Consider the matrix $B^{-1}A$, which still represents M. The rightmost $g' \times g'$ submatrix of $B^{-1}A$ is the identity. Moreover, each of the first $n - g'$ columns is of the form $(*, \ldots, *, 0)$, for otherwise it could replace the last column in the rightmost square submatrix to form a full rank square matrix, contradicting that n was a coloop. This shows that v_n is a $\mathbb{Z}^{g'}$-coloop of $v_1, \ldots, v_n \in \mathbb{Z}^{g'}$, and after padding by zeroes, v_n is a $\mathbb{Z}^g$-coloop of the of $v_1, \ldots, v_n$. $\square$

Definition 5.23. The *coloop complex* $C_\bullet^{(g)}$ is the subcomplex of $P_\bullet^{(g)}$ generated in degree n by cones $\sigma \in \Gamma_n$ such that σ is a matroidal cone and either:

(1) the rank of σ is $< g$,
(2) the rank of σ is equal to g and σ has one coloop.

By Lemma 5.22, the generators for $C_\bullet^{(g)}$ are the generators of $R_\bullet^{(g)}$ that are also generators of $I_\bullet^{(g)}$; in summary, we have inclusions of complexes

$$\begin{tikzcd} C_\bullet^{(g)} \arrow[r, hook] \arrow[d, hook] & R_\bullet^{(g)} \arrow[d, hook] \\ I_\bullet^{(g)} \arrow[r, hook] & P_\bullet^{(g)} \end{tikzcd}.$$

Similar to the inflation complex, the coloop complex is acyclic.

Theorem 5.24. The chain complex $C_\bullet^{(g)}$ is acyclic.

We omit the details of the proof of Theorem 5.24. It is closely analogous to the proof of Theorem 5.15, the key step being the following lemma.

Lemma 5.25. There is a bijection of sets between

$$\left\{ \begin{array}{c} \text{alternating, regular matroids} \\ \text{of rank} < g \text{ with } 0 \text{ coloops} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{alternating, regular matroids} \\ \text{of rank} \leq g \text{ with } 1 \text{ coloop} \end{array} \right\}.$$

6. Computations on the cohomology of $\mathcal{A}_g$

In this section, we compute the top-weight cohomology of $\mathcal{A}_g$ for $3 \leq g \leq 7$, proving Theorem A. When $g = 3, 4$, and 5, we do this by studying the cones of Σ_g^{P} arising from matroids, from which we explicitly compute the chain complex $P_\bullet^{(g)}$. We handle the cases when $g = 6$ and $g = 7$ by utilizing the long exact sequence in homology arising from Theorem 4.13, as well as the fact that the inflation subcomplex $I_\bullet^{(g)}$ is acyclic (see Theorem 5.15). Additionally, we prove a vanishing result for the top-weight cohomology of $\mathcal{A}_g$ for $g = 8, 9$, and 10 in Theorem 6.16.

6.1. The complex $P_\bullet^{(3)}$.

For $g = 3$, the fact that every perfect cone is matroidal allows us to compute the complex $P_\bullet^{(3)}$ directly. Using this description of $P_\bullet^{(3)}$, we then compute the top-weight cohomology of $\mathcal{A}_3$.

Proposition 6.1. The chain complex $P_\bullet^{(3)}$ is

$$\begin{array}{ccccccccccccccccc} & & P_5^{(3)} & & P_4^{(3)} & & P_3^{(3)} & & P_2^{(3)} & & P_1^{(3)} & & P_0^{(3)} & & P_{-1}^{(3)} & & \\ 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} & \longrightarrow & 0. \end{array}$$

Proof. The only top-dimensional perfect cone of $\Sigma_3^{\mathrm{P}}/GL_3(\mathbb{Z})$ is the principal cone σ_3^{prin} coming from the complete graph K_4 [Vor08, p. 151]. The principal cone σ_3^{prin} is alternating because the automorphisms of K_4 are all alternating permutations of its edges, and every automorphism of σ_3^{prin} arises from $\mathrm{Aut}(K_4)$ by Remark 2.14. Thus, we have $P_5^{(3)} \cong \mathbb{Q}$.

The automorphisms on the codimension i faces of σ_3^{prin} arise from matroids of graphs obtained from K_4 by deleting i edges, see Figure 2. For $i = 1, \ldots, 4$, each of the matroids associated to graphs with i edges removed from K_4 has an automorphism given by an odd permutation of the edges; see Figure 3 for an example. So we have $P_j^{(3)} = 0$ for $1 \leq j \leq 4$. The single ray and vertex of $\Sigma_3^{\mathrm{P}}/GL_3(\mathbb{Z})$ are alternating, so $P_j^{(3)} \cong \mathbb{Q}$ for $j = 0$ and $j = -1$.

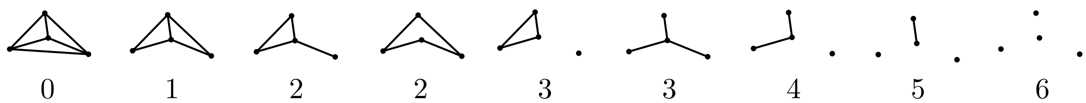


FIGURE 2. Graphs obtained by deleting the indicated number of edges from K_4, giving isomorphism classes of graphic matroids.

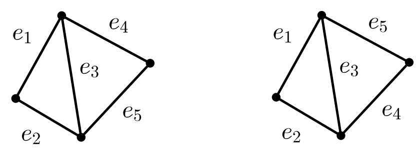


FIGURE 3. An automorphism of $M[K_4 \backslash \{e_6\}]$ interchanging e_4 and e_5.

$\square$

Theorem 6.2. The top-weight cohomology of $\mathcal{A}_3$ is

$$\mathrm{Gr}_{12}^W H^i(\mathcal{A}_3; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 6 \\ 0 \text{ else.} \end{cases}$$

Proof. The top-weight cohomology of $\mathcal{A}_3$ is the homology of $P_\bullet^{(3)}$ by Theorem 3.1. $\square$

Remark 6.3. Theorem 6.2 agrees with the work of Hain [Hai02], who computes the full cohomology ring of $\mathcal{A}_3$. Hain deduces in particular $H^6(\mathcal{A}_3; \mathbb{Q}) = E$ where E is a mixed Hodge structure that is an extension $0 \to \mathbb{Q}(-3) \to E \to \mathbb{Q}(-6) \to 0$, where $\mathbb{Q}(n)$ denotes the Tate Hodge structure of dimension 1 and weight $-2n$.

Example 6.4. While we do not need it here, we note that using the fact that all of the perfect cones in Σ_3^{P} arise from graphic matroids, one can check that the inflation complex $I_\bullet^{(3)}$ is the following:

$$\begin{array}{ccccccccccccccccc} & & I_5^{(3)} & & I_4^{(3)} & & I_3^{(3)} & & I_2^{(3)} & & I_1^{(3)} & & I_0^{(3)} & & I_{-1}^{(3)} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} & \longrightarrow & 0. \end{array}$$

6.2. **The complex $P_\bullet^{(4)}$.** In this section, we explicitly compute the complex $P_\bullet^{(4)}$ by using the matroidal description of the principal cone given in Section 2.5 together with the description of a similar complex for $\mathrm{SL}_g(\mathbb{Z})$-alternating cones described in [LS78]. We then use $P_\bullet^{(4)}$ to compute the top-weight cohomology of $\mathcal{A}_4$.

Proposition 6.5. The chain complex $P_\bullet^{(4)}$ is

$$\begin{array}{ccccccccccccccccccccc}
P_9^{(4)} & & P_8^{(4)} & & P_7^{(4)} & & P_6^{(4)} & & P_5^{(4)} & & P_4^{(4)} & & P_3^{(4)} & & P_2^{(4)} & & P_1^{(4)} & & P_0^{(4)} & & P_{-1}^{(4)} \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} \to 0.
\end{array}$$

Proof. By Theorem 4.13, we have, in any degree ℓ, that $\dim P_\ell^{(4)} = \dim P_\ell^{(3)} + \dim V_\ell^{(4)}$. We have already computed $P_\bullet^{(3)}$, so we now compute $V_\bullet^{(4)}$. In [LS78], the authors compute a complex $C_\bullet$ which is generated in degree i by the $(i+1)$-dimensional $\mathrm{SL}_4(\mathbb{Z})$-alternating perfect cones meeting Ω_g up to $\mathrm{SL}_4(\mathbb{Z})$-equivalence. Their results [LS78, Proposition 3.1] are summarized in the first three columns of Table 1. The cone $\sigma(D_4)$ is the cone corresponding to the quadratic form

$$D_4 = \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

For $i \notin \{4,5,6,8,9\}$, they show that $C_i = 0$.

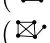
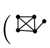
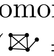
i	C_i	$\mathrm{SL}_4(\mathbb{Z})$-alternating cones of dim $i{+}1$	$\mathrm{GL}_4(\mathbb{Z})$-alternating?
4	$\mathbb{Q}$	$\sigma($$)$	no
5	$\mathbb{Q}$	$\sigma($$)$	no
6	$\mathbb{Q}$	$\sigma($$)$	yes
8	$\mathbb{Q}$	$\sigma($$)$	no
9	$\mathbb{Q}^2$	σ_4^{prin}, $\sigma(D_4)$	no

TABLE 1. $SL_4(\mathbb{Z})$-alternating cones of $\Sigma_4^P/SL_4(\mathbb{Z})$.

We now compute the Voronoi complex $V_\bullet^{(4)}$. As far as we know, this computation—for $\mathrm{GL}_4(\mathbb{Z})$, as opposed to $\mathrm{SL}_4(\mathbb{Z})$—constitutes a small gap in the literature, which we fill here. To obtain the complex $V_\bullet^{(4)}$, we must pass from $\mathrm{SL}_4(\mathbb{Z})$ to $\mathrm{GL}_4(\mathbb{Z})$. In doing so, two things may happen. First, two $\mathrm{SL}_4(\mathbb{Z})$-inequivalent cones may be $\mathrm{GL}_4(\mathbb{Z})$-equivalent. This does not occur by the corollary following Lemma 4.4 in [LS78]: the $\mathrm{GL}_4(\mathbb{Z})$-orbits of cones in Σ_4^{P} are equal to the $\mathrm{SL}_4(\mathbb{Z})$-orbits. Second, a cone which is $\mathrm{SL}_4(\mathbb{Z})$-alternating may no longer be $\mathrm{GL}_4(\mathbb{Z})$-alternating. We now check whether this occurs for the cones in Table 1.

In degree 9, neither cone is alternating since transposition matrices stabilize these cones but reverse orientation, as is observed in [LS78, p. 107]. In degrees 4, 5, and 8 the graphic matroids giving rise to each of the cones in Table 1 have an automorphism coming from an odd permutation of the ground set elements, so these cones are not alternating. Therefore $V_9^{(4)} = V_8^{(4)} = V_5^{(4)} = V_4^{(4)} = 0$. In degree 6, the cone $\sigma($$)$ is alternating because any automorphism of $M($ $= K_4 \cup \{e\})$ fixes e and $\sigma(K_4)$ is alternating, so $V_6^{(4)} = \mathbb{Q}$.

We now explain the nonzero morphisms. Since $V_\bullet^{(4)}$ is 0 in degree less than 2, the map $P_0^{(4)} \to P_{-1}^{(4)}$ is an isomorphism. We now compute the map $P_6^{(4)} \to P_5^{(4)}$. We have that $P_6^{(4)}$ is generated by one cone $\sigma($ $= K_4 \cup \{e\})$. Its faces are the cones obtained from $K_4 \cup \{e\}$ by deleting one edge. Only deleting the edge e yields a graph which gives an alternating perfect cone, and this cone generates $P_5^{(4)}$. So, this map is an isomorphism. $\square$

Theorem 6.6. The top-weight cohomology $\mathrm{Gr}_{20}^W H^i(\mathcal{A}_4;\mathbb{Q})$ of $\mathcal{A}_4$ is 0 for all i.

Proof. The top-weight cohomology of $\mathcal{A}_4$ is given by the homology of the chain complex $P_\bullet^{(4)}$ by Theorem 3.1. This chain complex has no homology. $\square$

In fact, our explicit description of $P_\bullet^{(4)}$ shows that $P_\bullet^{(4)} = I_\bullet^{(4)}$, since every nonzero generator is either of rank < 4 or has a coloop. The acyclicity of $P^{(4)}$ is then consistent with Theorem 5.15.

Remark 6.7. Theorem 6.6 can be deduced from the results in [HT12]. In particular, the weight 0 compactly supported cohomology of $\mathcal{A}_4$ is encoded in the last two columns of Table 1 in loc. cit., which describes the first page of a spectral sequence converging to the cohomology of the second Voronoi compactification $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ of $\mathcal{A}_4$. Here, these two columns contain the compactly supported cohomology of two strata of $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ whose union is exactly $\mathcal{A}_4$: the fifth column corresponds to the Torelli locus, while the sixth column corresponds to its complement in $\mathcal{A}_4$. By Poincaré duality (9), as described in Section 7, if $\mathcal{A}_4$ had top-weight cohomology it would also have compactly supported cohomology in weight 0. However, even though there are some undetermined entries in the sixth column of the aforementioned table, a close look at the table shows that the weight 0 part must vanish. Indeed, there are no weight 0 classes in the table northwest of the undetermined entries, so any weight 0 classes in the sixth column would persist in the E_∞ page of the spectral sequence and yield weight 0 classes of $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$. But this is impossible as $\overline{\mathcal{A}}_4^{\mathrm{Vor}}$ is a smooth compactification of $\mathcal{A}_4$.

6.3. The complex $P_\bullet^{(5)}$.

By using the short exact sequence given in Theorem 4.13, we now compute the complex $P_\bullet^{(5)}$. From this we compute the top-weight cohomology of $\mathcal{A}_5$.

Proposition 6.8. The chain complex $P_\bullet^{(5)}$ is

$$\begin{array}{ccccccccccccccccc}
 & & P_{14}^{(5)} & & P_{13}^{(5)} & & P_{12}^{(5)} & & P_{11}^{(5)} & & P_{10}^{(5)} & & P_{9}^{(5)} & & P_{8}^{(5)} & & P_{7}^{(5)} \\
0 & \longrightarrow & \mathbb{Q}^3 & \xrightarrow{\partial_{14}} & \mathbb{Q}^2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\partial_{11}} & \mathbb{Q}^6 & \xrightarrow{\partial_{10}} & \mathbb{Q}^7 & \xrightarrow{\partial_9} & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow \\
 & & P_{6}^{(5)} & & P_{5}^{(5)} & & P_{4}^{(5)} & & P_{3}^{(5)} & & P_{2}^{(5)} & & P_{1}^{(5)} & & P_{0}^{(5)} & & P_{-1}^{(5)} \\
 & & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\sim} & \mathbb{Q} & \longrightarrow 0.
\end{array}$$

Proof. By Theorem 4.13, we have in any degree ℓ that $\dim P_\ell^{(5)} = \dim P_\ell^{(4)} + \dim V_\ell^{(5)}$. We have already computed $P_\bullet^{(4)}$, so we now study $V_\bullet^{(5)}$, which was computed in [EVGS13]. Recall from Section 4.1 that Γ_n denotes the set of representatives of alternating perfect cones of dimension $n+1$. In [EVGS13, Table 1], the cardinality of Γ_n, is given by

n	4	5	6	7	8	9	10	11	12	13	14
$\lvert\Gamma_n\rvert$	0	0	0	0	1	7	6	1	0	2	3

In [EVGS13, Section 6.2], there is an explicit description of the differential maps.

Since $V_\bullet^{(5)}$ is supported in degrees > 7, while $P_\bullet^{(4)}$ is supported in degrees < 7, the differential maps $P_j^{(5)} \to P_{j-1}^{(5)}$ for $j < 7$ are inherited from $P_\bullet^{(4)}$, and likewise the differential maps $P_j^{(5)} \to P_{j-1}^{(5)}$ for $j > 7$ are inherited from $V_\bullet^{(5)}$. $\square$

Theorem 6.9. The top-weight cohomology of $\mathcal{A}_5$ is

$$\mathrm{Gr}_{30}^W H^i(\mathcal{A}_5;\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 15 \text{ or } 20, \\ 0 \text{ else.} \end{cases}$$

Proof. By Proposition 6.8 and [EVGS13, Theorem 4.3] we have that $H_9(P_\bullet^{(5)}) = \mathbb{Q}$ and $H_{14}(P_\bullet^{(5)}) = \mathbb{Q}$. Then by Theorem 3.1, we obtain the desired result. □

Remark 6.10. Grushevsky asks if $\mathcal{A}_g$ ever has nonzero odd cohomology [Gru09, Open Problem 7]. Theorem 6.9 confirms that $\mathcal{A}_5$ does in degree 15. Furthermore, we will see in Theorem 6.12 that Grushevsky's question is also answered affirmatively for $\mathcal{A}_7$, where

$$\dim \mathrm{Gr}^W_{56} H^{33}(\mathcal{A}_7;\mathbb{Q}) = \dim \mathrm{Gr}^W_{56} H^{37}(\mathcal{A}_7;\mathbb{Q}) = 1.$$

6.4. **The top-weight cohomology of $\mathcal{A}_6$ and $\mathcal{A}_7$.** In [EVGS13, Theorem 4.3][2] Elbaz-Vincent, Gangl, and Soulé computed the homology of the Voronoi complex $V_\bullet^{(g)}$ for $g = 5, 6$, and 7. Combining this, together with Proposition 6.8, we are able to compute the top-weight cohomology of $\mathcal{A}_6$ and $\mathcal{A}_7$.

Theorem 6.11. The top-weight cohomology of $\mathcal{A}_6$ is

$$\mathrm{Gr}^W_{42} H^i(\mathcal{A}_6;\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 30, \\ 0 \text{ else.} \end{cases}$$

Proof. By Proposition 4.4, we need to show that $H_{11}(P_\bullet^{(6)}) \cong \mathbb{Q}$ and $H_i(P_\bullet^{(6)}) = 0$ for $i \neq 11$. Consider the long exact sequence in homology arising from the short exact sequence of chain complexes given in Theorem 4.13. Combining this with the computation of the homology of $V_\bullet^{(6)}$ [EVGS13, Theorem 4.3] and the homology of $P_\bullet^{(5)}$ given in Proposition 6.8, our computation of $H_k(P_\bullet^{(6)})$ reduces to the four cases in Table 2.

i	$H_i(P_\bullet^{(5)})$	$H_i(P_\bullet^{(6)})$	$H_i(V_\bullet^{(6)})$
≥ 16	0	0	0
15	0	0	$\mathbb{Q}$
14	$\mathbb{Q}$	0	0
13	0	0	0
12	0	0	0
11	0	$\mathbb{Q}$	$\mathbb{Q}$
10	0	0	$\mathbb{Q}$
9	$\mathbb{Q}$	0	0
≤ 8	0	0	0

TABLE 2. The long exact sequence in homology for $g = 6$.

- **Case 1:** $(i \leq 8, i = 12, 13, i \geq 16)$: For these values of i, both $H_i(P_\bullet^{(5)})$ and $H_i(V_\bullet^{(6)})$ are equal to zero, so $H_i(P_\bullet^{(6)}) = 0$.
- **Case 2:** $(i = 14, 15)$: The long exact sequence in homology gives the exact sequence

$$0 \longrightarrow H_{15}(P_\bullet^{(6)}) \longrightarrow \mathbb{Q} \xrightarrow{\delta^6_{15}} \mathbb{Q} \longrightarrow H_{14}(P_\bullet^{(6)}) \longrightarrow 0.$$

[2] Elbaz-Vincent, Gangl, and Soulé define the Voronoi complex as a complex of free $\mathbb{Z}$-modules, and in [EVGS13, Theorem 4.3] they compute the integral homology of this complex. Our definition of the Voronoi complex $V_\bullet^{(g)}$ is a complex of $\mathbb{Q}$-vector spaces, but this causes no problems as we are only interested in the rational homology of $V_\bullet^{(g)}$.

Exactness implies that the connecting homomorphism δ^6_{15} is either an isomorphism or the zero map. By [EVGS13, Theorem 6.1] we know that inflating the cones in $V^{(5)}_\bullet$ gives an isomorphism of chain complexes $V^{(6)}_\bullet \cong V^{(5)}_\bullet[1] \oplus F_\bullet$ for some complex $F_\bullet$. Combining this with [EVGS13, Theorem 4.3] shows the nontrivial homology class in $H_{15}(V^{(6)}_\bullet)$ is the inflation of a nontrivial homology class in $H_{14}(V^{(5)}_\bullet)$. By Proposition 6.8, the nontrivial homology class in $H_{14}(P^{(5)}_\bullet)$ is the nontrivial homology class in $H_{14}(V^{(5)}_\bullet)$, so $H_{15}(V^{(6)}_\bullet)$ is generated by the inflation of the nontrivial class $H_{14}(P^{(5)}_\bullet)$. By the proof of the acyclicity of the inflation complex $I^{(g)}_\bullet$ (Theorem 5.15), this implies the connecting map δ^6_{15} is an isomorphism. The exact sequence above then implies that $H_k(P^{(6)}_\bullet) = 0$ for both $k = 14$ and $k = 15$.

- **Case 3:** ($i = 11$): Since $H_{11}(P^{(5)}_\bullet)$ and $H_{10}(P^{(5)}_\bullet)$ vanish, the long exact sequence in homology gives us

$$0 \longrightarrow H_{11}(P^{(6)}_\bullet) \longrightarrow \mathbb{Q} \longrightarrow 0.$$

This exactness implies that $H_{11}(P^{(6)}_\bullet)$ is isomorphic to $\mathbb{Q}$.

- **Case 4:** ($i = 9, 10$): By considering the long exact sequence in homology in the range $i = 10$ to $i = 9$ we have the following exact sequence:

$$0 \longrightarrow H_{10}(P^{(6)}_\bullet) \longrightarrow \mathbb{Q} \xrightarrow{\delta^6_{10}} \mathbb{Q} \longrightarrow H_9(P^{(6)}_\bullet) \longrightarrow 0.$$

An analysis similar to that in Case 2 shows that connecting map δ^6_{11} is an isomorphism, implying by exactness that $H_i(P^{(6)}_\bullet) = 0$ for both $i = 9$ and $i = 10$.

□

We now compute the top-weight rational cohomology of $\mathcal{A}_7$.

Theorem 6.12. The top-weight cohomology of $\mathcal{A}_7$ is

$$\mathrm{Gr}^W_{56} H^i(\mathcal{A}_7; \mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ if } i = 28, 33, 37, 42 \\ 0 \text{ else.} \end{cases}$$

Proof. We compute the homology of $P^{(7)}_\bullet$ in a similar fashion to the proof of Theorem 6.11, by considering the long exact sequence in homology arising from the short exact sequence of chain complexes given in Theorem 4.13. Table 3 records the homology of $P^{(6)}_\bullet$ and $V^{(7)}_\bullet$, which are given in Table 2 and [EVGS13, Theorem 4.3] respectively.

Both **Case 1** ($i \neq 11, 12, 13, 18, 22, 27$) and **Case 2** ($i = 13, 18, 22, 27$) follow from the exactness of the long exact sequence on homology in a manner analogous to Cases 1 and 3 in the proof of Theorem 6.11.

For **Case 3** ($i = 11, 12$), the long exact sequence in homology gives the exact sequence

$$0 \longrightarrow H_{12}(P^{(7)}_\bullet) \longrightarrow \mathbb{Q} \xrightarrow{\delta^7_{12}} \mathbb{Q} \longrightarrow H_{11}(P^{(7)}_\bullet) \longrightarrow 0.$$

Now δ^7_{12} is either an isomorphism or it is the zero map. As discussed in [EVGS13, §6.3], the nontrivial homology class in $H_{12}(V^{(7)}_\bullet)$ is the inflation of a nontrivial homology class in $H_{11}(V^{(6)}_\bullet)$. However, since by the proof of Theorem 6.11, the nontrivial homology class in $H_{11}(P^{(6)}_\bullet)$ is the nontrivial homology class in $H_{11}(V^{(6)}_\bullet)$, this implies that $H_{12}(V^{(7)}_\bullet)$ is generated by the inflation of the nontrivial class $H_{11}(P^{(6)}_\bullet)$. By the proof of the acyclicity of the inflation complex $I^{(g)}$

i	$H_i(P_\bullet^{(6)})$	$H_i(P_\bullet^{(7)})$	$H_i(V_\bullet^{(7)})$
≥ 28	0	0	0
27	0	$\mathbb{Q}$	$\mathbb{Q}$
26	0	0	0
25	0	0	0
24	0	0	0
23	0	0	0
22	0	$\mathbb{Q}$	$\mathbb{Q}$
21	0	0	0
20	0	0	0
19	0	0	0
18	0	$\mathbb{Q}$	$\mathbb{Q}$
17	0	0	0
16	0	0	0
15	0	0	0
14	0	0	0
13	0	$\mathbb{Q}$	$\mathbb{Q}$
12	0	0	$\mathbb{Q}$
11	$\mathbb{Q}$	0	0
10	0	0	0
9	0	0	0
≤ 8	0	0	0

TABLE 3. The long exact sequence in homology for $g = 7$.

(Theorem 6.11), this implies the connecting map δ^7_{12} is an isomorphism. The exact sequence above then implies $H_k(P_\bullet^{(7)}) = 0$ for $i = 11$ and $i = 12$.

□

Theorem A now follows directly from Theorems 6.6, 6.9, 6.11, and 6.12. As a corollary of this we are able to deduce the top-weight Euler characteristic of $\mathcal{A}_g$ for $2 \leq g \leq 7$.

Corollary 6.13. The top-weight Euler characteristic of $\mathcal{A}_g$ for $2 \leq g \leq 7$ is

$$\chi^{\mathrm{top}}(\mathcal{A}_g) = \begin{cases} 1 & \text{if } g = 3, 6 \\ 0 & \text{if } g = 2, 4, 5, 7. \end{cases}$$

Remark 6.14. One can also deduce the top-weight Euler characteristic of $\mathcal{A}_g$ for $5 \leq g \leq 7$ directly from the numbers listed in [EVGS13, Figures 1 and 2]. It would be interesting to know whether a closed formula for the top-weight Euler characteristic of $\mathcal{A}_g$ exists in general.

Remark 6.15. We have established

$$\mathrm{Gr}^W_{(g+1)g} H^{g(g-1)}(\mathcal{A}_g; \mathbb{Q}) \neq 0 \tag{7}$$

for $g = 3, 5, 6$, and 7 ($g = 3$ also follows from [Hai02]). We ask whether (7) holds for all $g \geq 5$. Equivalently, the question is whether $H_{2g-1}(P^{(g)}) \neq 0$ for all $g \geq 5$. The connection to the stable cohomology of the Satake compactification, as summarized in Table 4, gives evidence for this question, as explained in Section 7; see Question 7.1. We also note the possible relationship with the main theorems of [CGP21] on the rational cohomology of $\mathcal{M}_g$, which use the fact that

$H_{2g-1}(G^{(g)}) \neq 0$ for $g = 3$ and $g \geq 5$ ([Bro12], [Wil15], see [CGP21, Theorem 2.7]). We leave this interesting investigation as an open question.

6.5. **Results for $g \geq 8$.** While full calculations for the top-weight cohomology of $\mathcal{A}_g$ in the range $g \geq 8$ are beyond the scope of current computations, we can nevertheless use our previous computation of the top-weight cohomology of $\mathcal{A}_7$ together with a vanishing result of [SEVKM19] to show that the top-weight cohomology of $\mathcal{A}_8, \mathcal{A}_9$, and $\mathcal{A}_{10}$ vanishes in a certain range slightly larger than what is given by the virtual cohomological dimension.

Theorem 6.16. The top-weight rational cohomology of $\mathcal{A}_8$, $\mathcal{A}_9$, and $\mathcal{A}_{10}$ vanishes in the following ranges:

$$\begin{aligned} \operatorname{Gr}^W_{72} H^i(\mathcal{A}_8; \mathbb{Q}) &= 0 \quad \text{for } i \geq 60 \\ \operatorname{Gr}^W_{90} H^i(\mathcal{A}_9; \mathbb{Q}) &= 0 \quad \text{for } i \geq 79 \\ \operatorname{Gr}^W_{110} H^i(\mathcal{A}_{10}; \mathbb{Q}) &= 0 \quad \text{for } i \geq 99. \end{aligned}$$

Proof. By Theorem 4.5 of [SEVKM19] for $g = 8, 9$, and 10 the homology $H_i(V_\bullet^{(g)}) = 0$ for $i \leq 11$, and further, $H_{12}(V_\bullet^{(8)}) = 0$. Considering the long exact sequence in homology:

$$\cdots \longrightarrow H_{i+1}\left(V_\bullet^{(g)}\right) \xrightarrow{\delta} H_i\left(P_\bullet^{(g-1)}\right) \longrightarrow H_i\left(P_\bullet^{(g)}\right) \longrightarrow H_i\left(V_\bullet^{(g)}\right) \longrightarrow \cdots$$

coming from the short exact sequence of chain complexes given in Theorem 4.13, we see that this vanishing implies that

$$H_i(P_\bullet^{(7)}) \cong H_i(P_\bullet^{(8)}) \cong H_i(P_\bullet^{(9)}) \cong H_i(P_\bullet^{(10)})$$

for $i \leq 10$ and $H_{11}(P_\bullet^{(7)}) \cong H_{11}(P_\bullet^{(8)})$. By our computation of the homology of $P_\bullet^{(7)}$ in the proof of Theorem 6.12 we know that $H_i(P_\bullet^{(7)}) = 0$ for all $i \leq 12$, implying that for $g = 8, 9$, and 10, the homology $H_i(P_\bullet^{(g)}) = 0$ for $i \leq 10$, and further, $H_{11}(P_\bullet^{(8)}) = 0$. The result now follows from Proposition 4.4. $\square$

Remark 6.17. These vanishing bounds for $g = 8, 9, 10$ are slightly larger than the bounds provided by Corollary 5.16, equivalently, the fact that $\operatorname{vcd} \mathcal{A}_g = g^2$ (see Remark 5.19), which imply that

$$\begin{aligned} \operatorname{Gr}^W_{72} H^i(\mathcal{A}_8; \mathbb{Q}) &= 0 \quad \text{for } i \geq 65 \\ \operatorname{Gr}^W_{90} H^i(\mathcal{A}_9; \mathbb{Q}) &= 0 \quad \text{for } i \geq 82 \\ \operatorname{Gr}^W_{110} H^i(\mathcal{A}_{10}; \mathbb{Q}) &= 0 \quad \text{for } i \geq 101. \end{aligned}$$

The result for $g = 10$, however, is subsumed by the more general fact that the top-weight cohomology of $\mathcal{A}_g$ vanishes in degrees 0 and 1 below the vcd, as we shall note in §7 below.

7. Relationship with the stable cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$.

Our results on the existence of certain top-weight cohomology classes of $\mathcal{A}_g$ can be related to results of Chen-Looijenga [CL17] and Charney-Lee [CL83], which predict that, as g grows, there should be infinitely many of these classes. This connection was brought to our attention by O. Tommasi, and we thank her for explaining her ideas to us in detail.

Recall that $\mathcal{A}_g$ admits a compactification $\mathcal{A}_g^{\mathrm{Sat}}$, called the Satake or Baily-Borel compactification, first constructed as a projective variety by Baily and Borel in [BB66]. This compactification can be seen as a minimal compactification in the sense that it admits a morphism from all toroidal compactifications of $\mathcal{A}_g$. The reader interested in learning more about the vast literature on $\mathcal{A}_g$ and its compactifications can look at the very nice surveys [Gru09] and [HT18]. There are natural maps

$\mathcal{A}_g^{\mathrm{Sat}} \to \mathcal{A}_{g+1}^{\mathrm{Sat}}$, and the groups $H^k(\mathcal{A}_g^{\mathrm{Sat}};\mathbb{Q})$ stabilize for $k < g$ [CL83]. Moreover, as Charney-Lee prove, the stable cohomology ring $H^\bullet(\mathcal{A}_\infty^{\mathrm{Sat}};\mathbb{Q})$ of the Satake compactifications is freely generated by the classes λ_i for i odd, and the classes y_{4j+2} for $j = 1, 2, 3, \ldots$, where y_{4j+2} is in degree $4j+2$. Here, the λ-classes extend the i^{th} Chern class of the Hodge bundle on $\mathcal{A}_g$; in particular they are algebraic, and hence never have weight 0. But the classes y_j have weight 0, as proven recently by Chen-Looijenga [CL17]. This result is very important in the discussion that follows.

Recall also that $\mathcal{A}_g^{\mathrm{Sat}}$ admits a stratification by locally closed substacks

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0.$$

Thus the spectral sequence on compactly supported cohomology associated to this stratification is

$$E_1^{p,q} = H_c^{p+q}(\mathcal{A}_p;\mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{A}_g^{\mathrm{Sat}};\mathbb{Q}), \tag{8}$$

where $p = 0, \ldots, g$. This spectral sequence may be interpreted in the category of mixed Hodge structures. Passing to the weight 0 subspace, we see that the existence of the products of the y_j classes in the stable cohomology ring of the Satake compactification implies the existence of infinitely many cohomology classes in $\mathrm{Gr}_0^W H_c^j(\mathcal{A}_g;\mathbb{Q})$ for all g, and hence by the perfect pairing

$$\mathrm{Gr}_0^W H_c^j(\mathcal{A}_g;\mathbb{Q}) \times \mathrm{Gr}_{(g+1)g}^W H^{(g+1)g-j}(\mathcal{A}_g;\mathbb{Q}) \to \mathbb{Q} \tag{9}$$

provided by Poincaré duality, infinitely many classes $\mathrm{Gr}_{(g+1)g}^W H^*(\mathcal{A}_g;\mathbb{Q})$ in top weight.

With Poincaré duality applied, all of the known results on the top-weight cohomology of $\mathcal{A}_g$, including our Theorems 6.9, 6.11, and 6.12, can thus be summarized in Table 4, which shows the weight 0 part of the E_1 page of the spectral sequence (8).

Implicit in Table 4 is the fact that all terms below the p-axis are zero. This follows from the fact that $\mathrm{vcd}(\mathcal{A}_g) = \mathrm{vcd}(\mathrm{Sp}(2g,\mathbb{Z})) = g^2$, or, just as well, from the fact that $\mathrm{vcd}(\mathrm{GL}_g(\mathbb{Z})) = \binom{g}{2}$ [BS73]. In fact, the vanishing below the p-axis as well as in the rows $q = 0$, 1, and 2, apart from $(p,q) = (0,0)$, can be deduced from the fact that the rational cohomology of $\mathrm{GL}_g(\mathbb{Z})$ vanishes in degrees 0, 1, and 2 below the vcd. Indeed, we have, for all k,

$$H^{\binom{g}{2}-k}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \cong H_k(\mathrm{GL}_g(\mathbb{Z});\mathrm{St}\otimes\mathbb{Q}) \cong H_{k+g-1}(V^{(g)})$$

where St denotes the Steinberg module [Sou00]; these are all zero when $g > 1$ for $k = 0$ [LS76], $k = 1$ [CP17], and $k = 2$ [BMP+22]. Then Theorem 4.13 implies that also $H_{k+g-1}(P^{(g)}) = 0$ and $k = 0, 1$, and 2 so also $\mathrm{Gr}_{g^2+g}^W H^{g^2-k}(\mathcal{A}_g;\mathbb{Q}) = (\mathrm{Gr}_0^W H_c^{g+k}(\mathcal{A}_g;\mathbb{Q}))^\vee = 0$ for $g > 0$ and $k \le 2$ by Proposition 4.4.

As explained to us by Tommasi, the classes in Theorems 6.9, 6.11, and 6.12, as well as the already-known class in $\mathrm{Gr}_{12}^W H^6(\mathcal{A}_3;\mathbb{Q})$ [Hai02] give natural candidates for classes in $\mathrm{Gr}_0^W H_c^{p+q}(\mathcal{A}_p;\mathbb{Q})$ that produce the classes y_{4j+2} in the spectral sequence (8), in the sense that they persist in the E_∞ page in the Gysin spectral sequence for g sufficiently large. Indeed, looking at the $p = q$ diagonal on the E_1 page of the spectral sequence in Table 4, we are led to ask:

Question 7.1.

(1) Is $\mathrm{Gr}_0^W H_c^{2g}(\mathcal{A}_g;\mathbb{Q}) \neq 0$ for $g = 3$ and all $g \geq 5$?
(2) Moreover, do these cohomology classes produce the stable cohomology classes in $\mathrm{Gr}_0^W H^\bullet(\mathcal{A}_\infty^{\mathrm{Sat}};\mathbb{Q})$?
(3) Is $\mathrm{Gr}_0^W H_c^k(\mathcal{A}_g;\mathbb{Q}) = 0$ for $k < 2g$?

As discussed in the introduction, an affirmative answer to the third question in the range $k < 2g-1$ would be implied by [CFP14, Conjecture 2]. Our Theorems 6.2, 6.9, 6.11 and 6.12 verify the first and third questions for $g \leq 7$. They also verify the second question for $g = 3$ and for $g = 5$. Indeed, $\mathrm{Gr}_0^W H_c^6(\mathcal{A}_3;\mathbb{Q})$ and $\mathrm{Gr}_0^W H_c^{10}(\mathcal{A}_5;\mathbb{Q})$ are the only nonzero terms in the antidiagonals $p+q = 6$ and $p+q = 10$, respectively; so they produce the classes $y_6 \in \mathrm{Gr}_0^W H^6(\mathcal{A}_\infty^{\mathrm{Sat}};\mathbb{Q})$ and

21	0	0	0	0	0	0	0	$\mathbb{Q}$				
20	0	0	0	0	0	0	0	0				
19	0	0	0	0	0	0	0	0				
18	0	0	0	0	0	0	0	0				
17	0	0	0	0	0	0	0	0				
16	0	0	0	0	0	0	0	$\mathbb{Q}$				
15	0	0	0	0	0	0	0	0				
14	0	0	0	0	0	0	0	0				
13	0	0	0	0	0	0	0	0				
12	0	0	0	0	0	0	0	$\mathbb{Q}$				
11	0	0	0	0	0	0	0	0				
10	0	0	0	0	0	$\mathbb{Q}$	0	0				
9	0	0	0	0	0	0	0	0				
8	0	0	0	0	0	0	0	0				
7	0	0	0	0	0	0	0	$\mathbb{Q}$				
6	0	0	0	0	0	0	$\mathbb{Q}$	0				
5	0	0	0	0	0	$\mathbb{Q}$	0	0				
4	0	0	0	0	0	0	0	0	0			
3	0	0	0	$\mathbb{Q}$	0	0	0	0	0			
2	0	0	0	0	0	0	0	0	0	0		
1	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
0	$\mathbb{Q}$	0	0	0	0	0	0	0	0	0	0	$\cdots$
	0	1	2	3	4	5	6	7	8	9	10	$\cdots$

TABLE 4. The page $E_1^{p,q} = \mathrm{Gr}_0^W H_c^{p+q}(\mathcal{A}_p;\mathbb{Q}) \Rightarrow \mathrm{Gr}_0^W H^{p+q}(\mathcal{A}_g^{\mathrm{Sat}};\mathbb{Q})$ of the Gysin spectral sequence, for g sufficiently large. The blank entries for $p \geq 8$ are currently unknown.

$y_{10} \in \mathrm{Gr}_0^W H^{10}(\mathcal{A}_\infty^{\mathrm{Sat}};\mathbb{Q})$, respectively. It is natural to guess that the other terms in Table 4 similarly produce products of y_j's: for example, that $\mathrm{Gr}_0^W H_c^{12}(\mathcal{A}_6;\mathbb{Q})$ produces y_6^2, and that $\mathrm{Gr}_0^W H_c^{14}(\mathcal{A}_7;\mathbb{Q})$ produces y_{14}, and so on.

Finally, Tommasi also remarks that the odd degree classes in weight 0 compactly supported cohomology of $\mathcal{A}_g$ detected so far, namely

$$\mathrm{Gr}_0^W H_c^{15}(\mathcal{A}_5;\mathbb{Q}),\ \mathrm{Gr}_0^W H_c^{19}(\mathcal{A}_7;\mathbb{Q}),\ \text{and}\ \mathrm{Gr}_0^W H_c^{23}(\mathcal{A}_7;\mathbb{Q}),$$

must of course be killed by a differential on some page of the spectral sequence, since $\mathcal{A}_g^{\mathrm{Sat}}$ has no weight 0 stable cohomology in odd degrees. This implies the existence of some even degree classes in $\mathrm{Gr}_0^W H_c^\bullet(\mathcal{A}_g;\mathbb{Q})$ which kill the odd degree classes and which are not related by this spectral sequence to the products of y_js. It would be very interesting to explicitly identify such classes.

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Department of Mathematics, Brown University, Box 1917, Providence, RI 02912
Email address: madeline_brandt@brown.edu

Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720
Email address: juliette.bruce@berkeley.edu

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912
Email address: melody_chan@brown.edu

Department of Mathematics and Physics, Università Roma Tre, Largo San Leonardo Murialdo, 00146 Roma, Italy
Email address: melo@mat.uniroma3.it

Department of Mathematics, Harvard University, 1 Oxford St, Cambridge, MA 02138
Email address: gwynm@math.harvard.edu

Department of Mathematics, Tulane University, New Orleans, LA
Email address: cwolfe@tulane.edu

CHARACTERIZING MULTIGRADED REGULARITY ON PRODUCTS OF PROJECTIVE SPACES

JULIETTE BRUCE, LAUREN CRANTON HELLER, AND MAHRUD SAYRAFI

Abstract. We explore the relationship between multigraded Castelnuovo–Mumford regularity, truncations, Betti numbers, and virtual resolutions. We prove that on a product of projective spaces X, the multigraded regularity region of a module M is determined by the minimal graded free resolutions of the truncations $M_{\geq \mathbf{d}}$ for $\mathbf{d} \in \operatorname{Pic} X$. Further, by relating the minimal graded free resolutions of M and $M_{\geq \mathbf{d}}$ we provide a new bound on multigraded regularity of M in terms of its Betti numbers. Using this characterization of regularity and this bound we also compute the multigraded Castelnuovo–Mumford regularity for a wide class of complete intersections.

1. Introduction

Let S be the polynomial ring on $n+1$ variables over an algebraically closed field $\Bbbk$ and $\mathfrak{m}$ its maximal homogeneous ideal. A coherent sheaf $\mathscr{F}$ on the projective space $\mathbb{P}^n = \operatorname{Proj} S$ is d-regular for $d \in \mathbb{Z}$ if

(1) $H^i(\mathbb{P}^n, \mathscr{F}(b)) = 0$ for all $i > 0$ and all $b \geq d - i$.

The Castelnuovo–Mumford regularity of $\mathscr{F}$ is then the minimum d such that $\mathscr{F}$ is d-regular. In [EG84], Eisenbud and Goto considered the analogous condition on the local cohomology of a finitely generated graded S-module M, proving the equivalence of the following:

(2) $H^i_{\mathfrak{m}}(M)_b = 0$ for all $i \geq 0$ and all $b > d - i$;
(3) the truncation $M_{\geq d}$ has a linear free resolution;
(4) $\operatorname{Tor}_i(M, \Bbbk)_b = 0$ for all $i \geq 0$ and all $b > d + i$.

In particular, if $M = \bigoplus_p H^0(\mathbb{P}^n, \mathscr{F}(p))$ is the graded S-module corresponding to $\mathscr{F}$ (so that $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$) then conditions (1) through (4) are equivalent (c.f. [Eis05, Prop. 4.16]).

In [MS04], Maclagan and Smith introduced the notion of multigraded Castelnuovo–Mumford regularity for finitely generated $\operatorname{Pic}(X)$-graded modules over the Cox ring of a smooth projective toric variety X. In essence their definition is a generalization of condition (2). In this setting the multigraded regularity of a module is a semigroup inside $\operatorname{Pic} X$ rather than a single integer.

When $X = \mathbb{P}^n$ the minimum element of the multigraded regularity recovers the classical Castelnuovo–Mumford regularity. However, when X has higher Picard rank, translating the geometric definition of Maclagan and Smith into algebraic conditions like (3) and (4) above is an open problem. In this article we focus on the case when X is a product of projective spaces and explore the relationship between multigraded regularity, truncations, Betti numbers, and virtual resolutions.

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The obvious way one might hope to generalize Eisenbud and Goto's result to products of projective spaces is false: the truncation $M_{\geq \mathbf{d}}$ of a $\mathbf{d}$-regular $\mathrm{Pic}(X)$-graded module M can have nonlinear maps in its minimal free resolution (see Example 4.2). We show that under a mild saturation hypothesis, multigraded Castelnuovo–Mumford regularity is determined by a slightly weaker linearity condition, which we call *quasilinearity* (see Definition 4.3).

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and B the corresponding irrelevant ideal. The following complex contains all allowed twists for a quasilinear resolution generated in degree zero on a product of 2 projective spaces:

$$0 \longleftarrow S \longleftarrow \begin{array}{c} S(-1,\ \ 0) \\ \oplus \\ S(\ \ 0,-1) \end{array} \oplus\ S(-1,-1) \longleftarrow \begin{array}{c} S(-2,\ \ 0) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(\ \ 0,-2) \end{array} \oplus \begin{array}{c} S(-2,-1) \\ \oplus \\ S(-1,-2) \end{array} \longleftarrow \cdots.$$

Within each term, the summands in the left column (green) are linear syzygies while those in the right column (pink) are nonlinear syzygies. In general, for twists $-\mathbf{b}$ appearing in the i-th step of a quasilinear resolution, the sum of the positive components of $\mathbf{b} - \mathbf{d} - \mathbf{1}$ is at most $i-1$, where $\mathbf{d}$ is the degree of all generators.

Our main theorem characterizes multigraded regularity of modules on products of projective spaces in terms of the Betti numbers of their truncations.

Theorem A. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$.*

In [BES20, Thm. 2.9] Berkesch, Erman, and Smith established a similar result characterizing multigraded regularity of modules on products of projective spaces in terms of the existence of short virtual resolutions of a certain shape. In contrast, Theorem A constructs specific, efficiently computable virtual resolutions that determine multigraded regularity (see Section 4.2). These resolutions still have length at most the dimension of the space.

The proof of Theorem A is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel. Precisely, if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$ we prove the equality

$$\dim_{\Bbbk} \mathrm{Tor}^S_j(M_{\geq \mathbf{d}}, \Bbbk)_{\mathbf{a}} = h^{|\mathbf{a}|-j}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) \quad \text{for } |\mathbf{a}| \geq j \geq 0, \tag{1.1}$$

where the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The regularity of M implies certain cohomological vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, which, using (1.1), implies quasilinearity of the resolution of $M_{\geq \mathbf{d}}$. Conversely, building on [BES20, Thm. 2.9], a computation of $H^i_B(S)$ shows that the cokernel of a quasilinear resolution generated in degree $\mathbf{d}$ is $\mathbf{d}$-regular.

Since a linear resolution is necessarily quasilinear, Theorem A implies that having a linear truncation at $\mathbf{d}$ is strictly stronger than being $\mathbf{d}$-regular. That is to say, when $H^0_B(M) = 0$:

$$\begin{array}{c} M_{\geq \mathbf{d}} \text{ has a linear resolution} \\ \text{generated in degree } \mathbf{d} \end{array} \Longrightarrow \begin{array}{c} M_{\geq \mathbf{d}} \text{ has a quasilinear resolution} \\ \text{generated in degree } \mathbf{d} \end{array} \Longleftrightarrow M \text{ is } \mathbf{d}\text{-regular.}$$

Using (1.1), we also get a cohomological characterization of when $M_{\geq \mathbf{d}}$ has a linear resolution. In addition, Corollary 6.5 implies that when a $\mathbf{d}$-regular module M has a linear presentation, then the virtual resolution used in the proof of [BES20, Thm. 2.9] is in fact isomorphic to the minimal free resolution of $M_{\geq \mathbf{d}}$.

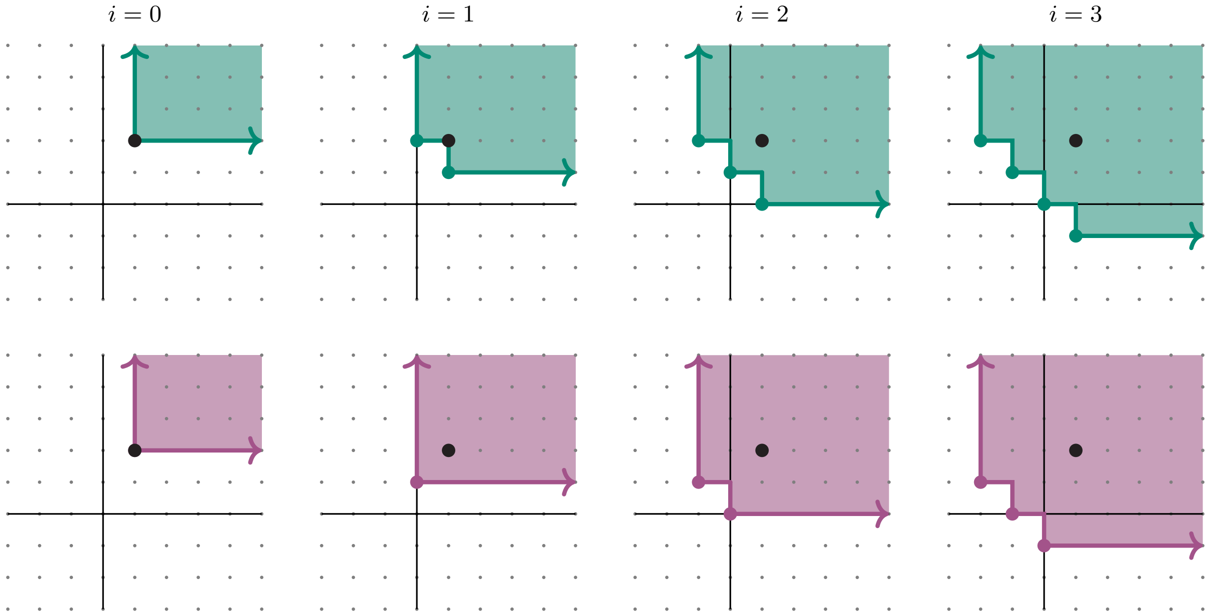

FIGURE 1. The top row shows the regions $L_i(1,2)$ in green, and the bottom row $Q_i(1,2)$ in pink for $i = 0,1,2,3$, from left to right, as defined in Section 2.1.

Unlike in the case of a single projective space, the multigraded Betti numbers of a module M do not determine its multigraded regularity. For instance, in Example 5.1 we construct two modules with the same multigraded Betti numbers but different multigraded regularities. Hence the Betti numbers of M also do not determine the Betti numbers of $M_{\geq \mathbf{d}}$. Still, we can intersect combinatorially defined regions $L_i(\mathbf{b})$ and $Q_i(\mathbf{b})$ (see Figure 1) to specify a subset of the degrees $\mathbf{d} \in \mathbb{Z}^r$ where $M_{\geq \mathbf{d}}$ has a linear or quasilinear resolution generated in degree $\mathbf{d}$.

Theorem B. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module, and define the set $\beta_i(M) \coloneqq \{\mathbf{b} \in \mathbb{Z}^r \mid \operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{b}} \neq 0\}$.*

(1) If $\mathbf{d} \in \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{b} \in \beta_i(M)} Q_i(\mathbf{b})$ then $M_{\geq \mathbf{d}}$ has a quasilinear resolution generated in degree $\mathbf{d}$.

(2) If $\mathbf{d} \in \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{b} \in \beta_i(M)} L_i(\mathbf{b})$ then $M_{\geq \mathbf{d}}$ has a linear resolution generated in degree $\mathbf{d}$.

On a single projective space we recover condition (4) of Eisenbud–Goto. Our proof of Theorem B is based on the observation that we can construct a possibly nonminimal free resolution of $M_{\geq \mathbf{d}}$ from the truncations of the terms in the minimal free resolution of M.

A number of inner[1] bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, [MS04, Cor. 7.3] used a local cohomology long exact sequence argument to deduce such a bound. These methods were extended in

[1]We use the terms inner and outer bound since in general there is no total ordering on $\operatorname{reg} X$ when $\operatorname{Pic} X \neq \mathbb{Z}$. For a single projective space an inner bound corresponds to an upper bound and an outer bound to a lower bound.

[BC17, Thm. 4.14] using a local cohomology spectral sequence. Our bound in Theorem B is generally larger and thus closer to the actual regularity than these results.

Moreover, Theorem B is sharp in a number of examples. For instance, we use Theorem A to show that the containment in (1) is equal to the regularity for all saturated *ample* complete intersections, meaning those determined by ample hypersurfaces.

Theorem C. *Suppose $\langle f_1, \dots, f_c\rangle \subset B$ is a saturated complete intersection of codimension c in S, so the affine subvariety defined by it contains the irrelevant locus $V(B)$. Then*

$$\operatorname{reg} \frac{S}{\langle f_1, \dots, f_c\rangle} = Q_c\left(\sum_{i=1}^{c} \deg f_i\right).$$

Note that on a product of projective spaces the intermediate cohomology of a complete intersection does not necessarily vanish. Even the local cohomology of a hypersurface in a product of projective spaces is not determined by its degree [BC17, Sec. 4.5]. Thus computing the multigraded regularity of complete intersections on products of projective spaces is more complicated than in the case of a single projective space.

This highlights a theme from the literature on multigraded regularity [MS04; HW04; SV04; SVW06; HÒ7; CMR07; BC17; CN20], Tate resolutions [EES15; BE21], virtual resolutions [BES20; BKLY21; HNT21; Lop21; Yan21], and syzygies [HSS06; HS07; Her10; Bru19; Bru20], that algebraic and homological properties of modules on toric varieties are more nuanced than in the standard graded setting.

Outline. The organization of the paper is as follows: Section 2 gathers background results and fixes our notation. Section 3 defines minimal virtual resolutions and constructs one from the Beilinson spectral sequence. Readers not familiar with derived categories can skip Sections 3.2 and 3.3 except the statement of Proposition 3.7. Section 4 proves Theorem A, describing the relationship between multigraded regularity and quasilinear truncations. Section 5 proves Theorem B, describing the relationship between multigraded Betti numbers and resolutions of truncations, and Theorem C, computing regularity for a class of complete intersections. Section 6 sharpens our theorems in the case of linear truncations. Finally, Section 7 summarizes our results about the regions defined by truncations, Betti numbers, and multigraded regularity.

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2. Notation and Background

Throughout we denote the natural numbers by $\mathbb{N} = \{0, 1, 2, \dots\}$. When referring to vectors in $\mathbb{Z}^r$ we use a bold font. Given a vector $\mathbf{v} = (v_1, \dots, v_r) \in \mathbb{Z}^r$ we denote the sum $v_1 + \cdots + v_r$ by $|\mathbf{v}|$. For $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^r$ we write $\mathbf{v} \leq \mathbf{w}$ when $v_i \leq w_i$ for all i, and use $\max\{\mathbf{v}, \mathbf{w}\}$ to denote the vector whose i-th component is $\max\{v_i, w_i\}$. We reserve $\mathbf{e}_1, \dots, \mathbf{e}_r$ for the standard basis of $\mathbb{Z}^r$ and for brevity we write $\mathbf{1}$ for $(1, 1, \dots, 1) \in \mathbb{Z}^r$ and $\mathbf{0}$ for $(0, 0, \dots, 0) \in \mathbb{Z}^r$.

Fix a Picard rank $r \in \mathbb{N}$ and dimension vector $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$. We denote by $\mathbb{P}^{\mathbf{n}}$ the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of r projective spaces over a field $\Bbbk$. Given $\mathbf{b} \in \mathbb{Z}^r$ we let

$$\mathscr{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{b}) \coloneqq \pi_1^* \mathscr{O}_{\mathbb{P}^{n_1}}(b_1) \otimes \cdots \otimes \pi_r^* \mathscr{O}_{\mathbb{P}^{n_r}}(b_r)$$

where π_i is the projection of $\mathbb{P}^{\mathbf{n}}$ to $\mathbb{P}^{n_i}$. This gives an isomorphism $\operatorname{Pic} \mathbb{P}^{\mathbf{n}} \cong \mathbb{Z}^r$, which we use implicitly throughout.

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}}$, which is isomorphic to the polynomial ring $\Bbbk[x_{i,j} \mid 1 \leq i \leq r,\ 0 \leq j \leq n_i]$ with $\deg(x_{i,j}) = \mathbf{e}_i$. Further, let $B = \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \dots, x_{i,n_i} \rangle \subset S$ be the irrelevant ideal. For a description of the Cox ring and the relationship between coherent $\mathscr{O}_{\mathbb{P}^{\mathbf{n}}}$-modules and $\mathbb{Z}^r$-graded S-modules, see [Cox95; CLS11]. In particular, the twisted global sections functor Γ_* given by $\mathscr{F} \mapsto \bigoplus_{\mathbf{p} \in \mathbb{Z}^r} H^0(\mathbb{P}^{\mathbf{n}}, \mathscr{F}(\mathbf{p}))$ takes coherent sheaves on $\mathbb{P}^{\mathbf{n}}$ to S-modules. Given a $\mathbb{Z}^r$-graded S-module M, let $\beta_i(M) \coloneqq \{\mathbf{b} \in \mathbb{Z}^r \mid \operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{b}} \neq 0\}$ denote the set of multidegrees of i-th syzygies of M.

2.1. Multigraded Regularity.

In order to streamline our definitions of regions inside the Picard group of $\mathbb{P}^{\mathbf{n}}$, we introduce the following subsets of $\mathbb{Z}^r$: for $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{N}$ let

$$L_i(\mathbf{d}) \coloneqq \bigcup_{|\lambda| = i} (\mathbf{d} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r + \mathbb{N}^r) \quad \text{for } \lambda_1, \dots, \lambda_r \in \mathbb{N}$$
$$Q_i(\mathbf{d}) \coloneqq L_{i-1}(\mathbf{d} - \mathbf{1}) \quad \text{for } i > 0 \quad \text{and} \quad Q_0(\mathbf{d}) = \mathbf{d} + \mathbb{N}^r.$$

Note that for fixed $\mathbf{d} \in \mathbb{Z}^r$ we have $L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d})$ for all i.

Example 2.1. When $r = 2$ the regions $L_i(\mathbf{d})$ and $Q_i(\mathbf{d})$ can be visualized as in Figure 1. For $i > 1$ they are shaped like staircases with $i + 1$ and i "corners," respectively; in other words, the semigroup $L_i(\mathbf{d})$ is generated by $i + 1$ elements and $Q_i(\mathbf{d})$ by i elements.

Remark 2.2. An alternate description of $L_i(\mathbf{d})$ will also be useful: it is the set of $\mathbf{b} \in \mathbb{Z}^r$ so that the sum of the positive components of $\mathbf{d} - \mathbf{b}$ is at most i. (This ensures that we can distribute the λ_j so that $\mathbf{b} + \sum_j \lambda_j \mathbf{e}_j \geq \mathbf{d}$.)

With this notation in hand we can recall the definition of multigraded regularity.

Definition 2.3. [MS04, Def. 1.1] Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. We say M is $\mathbf{d}$-*regular* for $\mathbf{d} \in \mathbb{Z}^r$ if the following hold:

1. $H_B^0(M)_{\mathbf{p}} = 0$ for all $\mathbf{p} \in \bigcup_{1 \leq j \leq r} (\mathbf{d} + \mathbf{e}_j + \mathbb{N}^r)$,
2. $H_B^i(M)_{\mathbf{p}} = 0$ for all $i > 0$ and $\mathbf{p} \in L_{i-1}(\mathbf{d})$.

The *multigraded Castelnuovo–Mumford regularity* of M is then the set

$$\operatorname{reg}(M) \coloneqq \left\{ \mathbf{d} \in \mathbb{Z}^r \;\middle|\; M \text{ is } \mathbf{d}\text{-regular} \right\} \subset \operatorname{Pic} \mathbb{P}^{\mathbf{n}} \cong \mathbb{Z}^r.$$

It follows directly from the definition that if M is $\mathbf{d}$-regular, then M is $\mathbf{d}'$-regular for all $\mathbf{d}' \geq \mathbf{d}$. For other properties of multigraded regularity, such as $\mathbf{0}$-regularity of S, see [MS04].

Remark 2.4. Several alternate notions of Castelnuovo–Mumford regularity for the multigraded setting exist in the literature. The initial extension was introduced by Hoffman and Wang for a product of two projective spaces [HW04]. Following Maclagan and Smith's definition, Botbol and Chardin gave a more general definition working over an arbitrary base ring [BC17]. Recently, in their work on Tate resolutions on toric varieties, Brown and Erman introduced a modified notion of multigraded regularity for a weighted projective space, which they then extended to other toric varieties [BE21, §6.1].

2.2. **Truncations and Local Cohomology.** In this section we collect facts about truncations and local cohomology that will be used repeatedly. As in the case of a single projective space, the truncation of a graded module on a product of projective spaces at multidegree $\mathbf{d}$ contains all elements of degree at least $\mathbf{d}$.

Definition 2.5. For $\mathbf{d} \in \mathbb{Z}^r$ and M a $\mathbb{Z}^r$-graded S-module, the *truncation* of M at $\mathbf{d}$ is the $\mathbb{Z}^r$-graded S-submodule $M_{\geq \mathbf{d}} \coloneqq \bigoplus_{\mathbf{d}' \geq \mathbf{d}} M_{\mathbf{d}'}$.

Immediate from the definition is the following lemma.

Lemma 2.6. *The truncation map $M \mapsto M_{\geq \mathbf{d}}$ is an exact functor of $\mathbb{Z}^r$-graded S-modules.*

Remark 2.7. Since truncation is exact, if $F_\bullet$ is graded free resolution of a module M then the term by term truncation $(F_\bullet)_{\geq \mathbf{d}}$ is a resolution of $M_{\geq \mathbf{d}}$. However, in general the truncation of a free module is not free, so $(F_\bullet)_{\geq \mathbf{d}}$ is generally not a free resolution of $M_{\geq \mathbf{d}}$.

We denote by $H^p_B(M)$ the p-th local cohomology of M supported at the irrelevant ideal B. For $p > 0$ and $\mathbf{a} \in \mathbb{Z}^r$ there exist natural isomorphisms

$$H^p\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})\big) \cong H^{p+1}_B(M)_{\mathbf{b}},$$

and for $p = 0$ there is a $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow H^0_B(M) \longrightarrow M \longrightarrow \Gamma_*(\widetilde{M}) \longrightarrow H^1_B(M) \longrightarrow 0. \tag{2.1}$$

An important tool for computing local cohomology is the local Čech complex

$$\check{C}^\bullet(B, M)\colon 0 \longrightarrow M \longrightarrow \bigoplus M[g_i^{-1}] \longrightarrow \bigoplus M[g_i^{-1}, g_j^{-1}] \longrightarrow \cdots$$

where the g_i range over the generators of B. We index the local Čech complex so that the summands of $\check{C}^p(B, M)$ are localizations of M at p distinct generators of B. Then we have

$$H^p_B(M) \cong H^p(\check{C}^\bullet(B, M)).$$

See [Iye+07] and [CLS11, §9] for more details.

Note that inverting a generator of B inverts a variable from each factor of $\mathbb{P}^{\mathbf{n}}$, so the distinguished open sets corresponding to the generators of B form an affine cover $\mathfrak{U}_B$ of $\mathbb{P}^{\mathbf{n}}$. Denote by $\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})$ the Čech complex of a sheaf $\mathscr{F}$ with respect to $\mathfrak{U}_B$:

$$\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})\colon 0 \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i} \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i \cap \mathfrak{U}_j} \longrightarrow \cdots .$$

Lemma 2.8. *Given a complex of graded S-modules $L \to M \to N$ such that $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact, the complex $\check{C}^p(B, L) \to \check{C}^p(B, M) \to \check{C}^p(B, N)$ is exact for each $p \geq 0$.*

Proof. Fix p. Then $\check{C}^p(B,L) \to \check{C}^p(B,M) \to \check{C}^p(B,N)$ splits as a direct sum of complexes

$$L[g_1^{-1},\dots,g_p^{-1}] \to M[g_1^{-1},\dots,g_p^{-1}] \to N[g_1^{-1},\dots,g_p^{-1}]$$

each of which can be obtained by applying $\Gamma(U,-)$ to $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$, where U is the complement of $V(g_1,\dots,g_p)$. Since U is affine they are exact. $\square$

Since $M/M_{\geq \mathbf{d}}$ is annihilated by a power of B, a module M and its truncation define the same sheaf on $\mathbb{P}^{\mathbf{n}}$. In particular $H_B^p(M) = H_B^p(M_{\geq \mathbf{d}})$ for $p \geq 2$. The long exact sequence of local cohomology applied to $0 \to M_{\geq \mathbf{d}} \to M \to M/M_{\geq \mathbf{d}} \to 0$ gives

$$0 \longrightarrow H_B^0(M_{\geq \mathbf{d}}) \longrightarrow H_B^0(M) \longrightarrow M/M_{\geq \mathbf{d}} \longrightarrow H_B^1(M_{\geq \mathbf{d}}) \longrightarrow H_B^1(M) \longrightarrow 0.$$

Hence $H_B^0(M) = 0$ implies $H_B^0(M_{\geq \mathbf{d}}) = 0$. Since $M/M_{\geq \mathbf{d}}$ is zero in degrees larger than $\mathbf{d}$ we also have $H_B^1(M_{\geq \mathbf{d}})_{\geq \mathbf{d}} = H_B^1(M)_{\geq \mathbf{d}}$. An immediate consequence is the following lemma, which we will use repeatedly to reduce to the case when $\mathbf{d} = \mathbf{0}$.

Lemma 2.9. *A $\mathbb{Z}^r$-graded S-module M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ is $\mathbf{d}$-regular.*

2.3. **Koszul Complexes and Cotangent Sheaves.** For each factor $\mathbb{P}^{n_i}$ of $\mathbb{P}^{\mathbf{n}}$, the Koszul complex on the variables of $S_i = \operatorname{Cox} \mathbb{P}^{n_i}$ is a resolution of $\Bbbk$:

$$K^i_\bullet \colon 0 \leftarrow S_i \leftarrow S_i^{n_i+1}(-1) \leftarrow \bigwedge^2 S_i^{n_i+1}(-1) \leftarrow \cdots \leftarrow \bigwedge^{n_i+1} S_i^{n_i+1}(-1) \leftarrow 0. \tag{2.2}$$

The Koszul complex $K_\bullet$ on the variables of S is the tensor product of the complexes $\pi_i^* K^i_\bullet$.

For $1 \leq a \leq n$ let $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ be the kernel of $\bigwedge^{a-1} S_i^{n_i+1} \leftarrow \bigwedge^a S_i^{n_i+1}$ and let $\Omega^a_{\mathbb{P}^{n_i}}$ denote its sheafification. The minimal free resolution of $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ then consists of the terms of $K^i_\bullet$ with homological index greater than a. Write $\hat{\Omega}^0_{\mathbb{P}^{n_i}}$ for the kernel of $\Bbbk \leftarrow S_i$ (so that $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$) and take $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ to be 0 otherwise. For $\mathbf{a} \in \mathbb{Z}^r$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$ define

$$\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} \coloneqq \pi_1^* \Omega^{a_1}_{\mathbb{P}^{n_1}} \otimes \cdots \otimes \pi_r^* \Omega^{a_r}_{\mathbb{P}^{n_r}}$$

and write $\hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ for the analogous tensor product of the modules $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$.

Given a free complex $F_\bullet$ and a multidegree $\mathbf{a} \in \mathbb{Z}^r$, denote by $F_\bullet^{\leq \mathbf{a}}$ the subcomplex of $F_\bullet$ consisting of free summands generated in degrees at most $\mathbf{a}$.

Lemma 2.10. *Fix $\mathbf{a} \in \mathbb{Z}^r$ and let $K_\bullet$ be the Koszul complex on the variables of S. The subcomplex $K_\bullet^{\leq \mathbf{a}}$ is equal to $K_\bullet$ in degrees $\leq \mathbf{a}$, and its sheafification is exact except at homological index $|\mathbf{a}|$, where it has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.*

Proof. The first statement follows from the fact that the terms appearing in $K_\bullet$ but not $K_\bullet^{\leq \mathbf{a}}$ have no elements in degrees $\leq \mathbf{a}$.

Note that $K_\bullet^{\leq \mathbf{a}}$ is a tensor product of pullbacks of subcomplexes of the $K^i_\bullet$ in (2.2):

$$K_\bullet^{\leq \mathbf{a}} = \pi_i^*(K^1_\bullet)^{\leq a_1} \otimes \cdots \otimes \pi_r^*(K^r_\bullet)^{\leq a_r}.$$

After sheafification, each complex $\pi_i^*(K^i_\bullet)^{\leq \mathbf{a}}$ is exact away from its kernel $\pi_i^* \Omega^{a_i}_{\mathbb{P}^{n_i}}$, which appears at homological index a_i. Thus $\widetilde{K}_\bullet^{\leq \mathbf{a}}$ has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$, appearing in index $|\mathbf{a}|$. $\square$

3. Structure of Virtual Resolutions

Let X be a smooth projective toric variety with $\mathrm{Pic}(X)$-graded Cox ring S and irrelevant ideal B. Consider a graded S-module M. While a minimal free resolution $F_\bullet$ of M can be easily computed using Gröbner methods, it does not always provide a faithful reflection of the geometry of X. For example, when the Picard rank of X is greater than one, the length of $F_\bullet$ may exceed $\dim X$. To bridge this gap, Berkesch, Erman, and Smith introduced *virtual resolutions* in [BES20].

Definition 3.1. A $\mathrm{Pic}(X)$-graded complex of free S-modules $F_\bullet$ is a *virtual resolution* of M if the complex $\widetilde{F_\bullet}$ of locally free sheaves on X is a resolution of the sheaf $\widetilde{M}$.

Despite more faithfully capturing the geometry of X, virtual resolutions are often less rigid than minimal free resolutions. For example, a module M generally has many non-isomorphic virtual resolutions. In this section we consider virtual resolutions containing no degree 0 maps, which we show in certain situations are subcomplexes of minimal free resolutions. This additional structure is necessary to our work in Section 6.

Virtual resolutions will also appear in our proof of Theorem A. Inspired by the work of Berkesch, Erman, and Smith, we use a Fourier-Mukai construction to give a virtual resolution of M whose Betti numbers are computable in terms of certain cohomology groups. In Section 4.1 we then relate this virtual resolution to the minimal free resolution of $M_{\geq \mathbf{d}}$.

Remark 3.2. When $X = \mathbb{P}^{\mathbf{n}}$, Berkesch, Erman, and Smith constructed virtual resolutions of length at most $\dim \mathbb{P}^{\mathbf{n}}$. Note that since $\widetilde{M} = \widetilde{M_{\geq \mathbf{d}}}$, a free resolution of $M_{\geq \mathbf{d}}$ is automatically a virtual resolution of M. In Section 4.1 we use this fact to give an alternative construction of short virtual resolutions on $\mathbb{P}^{\mathbf{n}}$.

3.1. Subcomplexes of Minimal Free Resolutions.

A complex of S-modules is *trivial* if it is a direct sum of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow S \xrightarrow{\;1\;} S \longrightarrow 0 \longrightarrow \cdots .$$

A free resolution of a finitely generated $\mathrm{Pic}(X)$-graded S-module M is isomorphic to the direct sum of the $\mathrm{Pic}(X)$-graded minimal free resolution of M and a trivial complex. With this in mind we introduce the following notion of a minimal virtual resolution.

Definition 3.3. A virtual resolution $F_\bullet$ is *minimal* if it is not isomorphic to a $\mathrm{Pic}(X)$-graded chain complex of the form $F'_\bullet \oplus F''_\bullet$ where $F''_\bullet$ is a trivial complex.

Note that, unlike in the case of ordinary free resolutions, minimal virtual resolutions are not unique, even up to isomorphism. Further, minimal virtual resolutions need not have the same length. That said, analogous to the case of minimal free resolutions, minimal virtual resolutions are characterized by having no constant entries in their differentials.

Lemma 3.4. *A virtual resolution of M is minimal if and only if its differentials have no degree 0 components.*

Proof. Since S is positively graded, a graded version of Nakayama's Lemma holds (see [MS05, pp. 155-156]). The statement follows immediately from an argument similar to those in [Eis95, Thm. 20.2, Exc. 20.1]. $\square$

The following structure result shows that a minimal virtual resolution F of a module M satisfying certain conditions on the Betti numbers arises as a subcomplex of the minimal free resolution of $H_0(F_\bullet)$. We use this in the proof of Corollary 6.5 to show that the virtual resolution which we construct in Section 3.3 is exact. Here we denote by $\operatorname{Eff}(X)$ the cone generated by the degrees of the variables of S in $\operatorname{Pic} X$.

Proposition 3.5. *Let $(F_\bullet, \varphi_\bullet)$ be a finite minimal virtual resolution and let $N = H_0(F_\bullet)$. Suppose that*

(1) $\dim_{\Bbbk} \operatorname{Tor}_i(F_\bullet, \Bbbk)_d \leq \dim_{\Bbbk} \operatorname{Tor}_i(N, \Bbbk)_d$ for all d and all i;
(2) whenever $c - d \in \operatorname{Eff}(X)$ and $\operatorname{Tor}_i(F_\bullet, \Bbbk)_c \neq 0$ equality holds in (1).

Then $F_\bullet$ is a subcomplex of the minimal free resolution of N.

Proof. First, we will inductively construct a resolution $(G_\bullet, \psi_\bullet)$ of N which contains $(F_\bullet, \varphi_\bullet)$ as a subcomplex. Let $G_0 = F_0$, $G_1 = F_1$, and $\psi_1 = \varphi_1$, so that $H_0(G_\bullet) = N$.

Suppose G_i has been defined for $0 \leq i \leq n-1$ so that $F_\bullet$ is a summand and $G_\bullet$ is exact for $0 < i < n-1$. Consider φ_n as a map $F_n \to G_{n-1}$ by composing with the inclusion $F_{n-1} \hookrightarrow G_{n-1}$. Choose $z_1, \ldots, z_s \in \ker \psi_{n-1}$ such that their images generate $\ker \psi_{n-1} / \operatorname{im} \varphi_n$. Let $G_n = F_n \oplus S(-\mathbf{a}_1) \oplus \cdots \oplus S(-\mathbf{a}_s)$ where $\deg z_j = \mathbf{a}_j$. Define ψ_n by $\psi_n|_{F_n} = \varphi_n$ and $\psi_n(g_j) = z_j$, where g_j is the generator of $S(-\mathbf{a}_j)$. Then $\operatorname{im} \psi_n = \ker \psi_{n-1}$, so that $G_\bullet$ is a complex and exact at $n-1$.

We will now show by induction that it is possible to prune $G_\bullet$ to a minimal free resolution of N that contains $F_\bullet$ as a subcomplex. At each step, take a nonminimal homogeneous relation among the images of generators of some G_i. Write it as

$$\psi_i\Big(\sum a_j f_j + \sum b_j g_j\Big) = 0,$$

where $f_j \in F_i$, $g_j \in G_i \setminus F_i$, and $a_j, b_j \neq 0$ for all j. As $F_\bullet$ is minimal, at least one g_j does appear. Since each G_i has only finitely many generators, it is possible to choose a relation whose degree c satisfies $c - d \notin \operatorname{Eff}(X)$ for all degrees $d \neq c$ of other available relations.

Assume by induction that no generator of $F_\bullet$ has been removed in a previous step. Since the chosen relation is nonminimal, at least one of its coefficients is a unit. If some b_j is a unit then we may remove the corresponding g_j and continue pruning.

Suppose instead that all unit coefficients appear among the a_j. In this case we must prune some f_k in order to remove the relation. Note that by homogeneity

$$\deg f_k = \deg a_k f_k = c = \deg b_j g_j = \deg b_j + \deg g_j$$

for all j. Thus $c - \deg g_j = \deg b_j \in \operatorname{Eff}(X)$, so equality holds in (1) for $d = \deg g_j$ by hypothesis. By choice of c we cannot remove anything of degree $\deg g_j$ in a subsequent step. Hence g_j appears in the minimal free resolution of N, so by the equality in (1) some generator f of F_i with degree d must be removed. However, it cannot have been removed before f_k by the induction hypothesis, and it cannot be removed after f_k by choice of c. This is a contradiction, so we are never required to prune a generator of $F_\bullet$, completing the proof. □

In the language of [BES20], this proposition implies that a virtual resolution that appears to be a *virtual resolution of a pair* based only on its Betti numbers can indeed be produced by that construction. Note that the proposition is not true without conditions on the Betti numbers. For instance, [BPC21, Ex. 1.2] gives a minimal virtual resolution which is not a subcomplex of the minimal free resolution of its cokernel.

3.2. **Fourier–Mukai Transforms.** The sheafification of a virtual resolution of M is a resolution of $\widetilde{M}$ by direct sums of line bundles. More generally, following [EES15, §8], we define a *free monad* of a coherent sheaf $\mathscr{F}$ to be a finite complex

$$\mathcal{L}\colon 0 \leftarrow \mathcal{L}_{-s} \leftarrow \cdots \leftarrow \mathcal{L}_{-1} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \mathcal{L}_t \leftarrow 0$$

whose terms are direct sums of line bundles and whose homology is $H_\bullet(\mathcal{L}) = H_0(\mathcal{L}) \simeq \mathscr{F}$.

In this section we introduce a type of geometric functor between derived categories known as a Fourier–Mukai transform. We will use a particular instance in Section 3.3 to prove that a complex constructed from the Beilinson spectral sequence is a free monad. See [Huy06, §5] for background and further details.

Let X and Y be smooth projective varieties and consider the two projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y. \end{tikzcd}$$

A *Fourier–Mukai transform* is a functor

$$\Phi_{\mathcal{K}}\colon\ \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$$

between the derived categories of bounded complexes of coherent sheaves. It is represented by an object $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times Y)$ and constructed as a composition of derived functors

$$\mathscr{F} \mapsto \mathbf{R}p_*\big(\mathbf{L}q^*\mathscr{F} \otimes^{\mathbf{L}} \mathcal{K}\big).$$

Here $\mathbf{L}q^*$, $\mathbf{R}p_*$, and $-\otimes^{\mathbf{L}}\mathcal{K}$ are the derived functors induced by q^*, p_*, and $-\otimes\mathcal{K}$, respectively. Moreover, since q is flat $\mathbf{L}q^*$ is the usual pull-back, and if $\mathcal{K}$ is a complex of locally free sheaves $-\otimes^{\mathbf{L}}\mathcal{K}$ is the usual tensor product. In fact, all equivalences between $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ arise in this way.

A special case of the Fourier–Mukai transform occurs when $Y = X$ and $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times X)$ is a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal subscheme $\iota\colon \Delta \to X \times X$. Such $\mathcal{K}$ is referred to as a *resolution of the diagonal*.

Using the projection formula, one can see that the Fourier–Mukai transform $\Phi_{\mathcal{O}_\Delta}$ is simply the identity in the derived category; that is to say, it produces quasi-isomorphisms. We will use this fact in the proof of Proposition 3.7.

3.3. **The Beilinson Spectral Sequence.** Returning to the case of products of projective spaces, we consider coherent sheaves on $X = \mathbb{P}^{\mathbf{n}}$. We construct a free monad for $\widetilde{M}$ from the Beilinson spectral sequence on $\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ and describe its Betti numbers. When M is $\mathbf{0}$-regular it is a minimal virtual resolution, which we will use in Sections 4 and 6. See [OSS80, §3.1] for a geometric exposition and [Huy06, §8.3] or [AO89, §3] for an algebraic exposition on a single projective space.

For sheaves $\mathscr{F}$ and $\mathscr{G}$ on $\mathbb{P}^{\mathbf{n}}$, denote $p^*\mathscr{F} \otimes q^*\mathscr{G}$ by $\mathscr{F} \boxtimes \mathscr{G}$. Consider the vector bundle

$$\mathcal{W} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{T}^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i),$$

where $\mathcal{T}^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}$ is the pullback of the tangent bundle, as in the Euler sequence on the factor $\mathbb{P}^{n_i}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}} \longrightarrow \mathcal{O}^{n_i+1}_{\mathbb{P}^{n_i}}(\mathbf{e}_i) \longrightarrow \mathcal{T}_{\mathbb{P}^{n_i}} \longrightarrow 0. \qquad (3.1)$$

There is a canonical section $s \in H^0(\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}, \mathcal{W})$ whose vanishing cuts out the diagonal subscheme $\Delta \subset \mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ (see [BES20, Lem. 2.1]), giving a Koszul resolution of $\mathcal{O}_\Delta$:

$$\mathcal{K}\colon 0 \longleftarrow \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \longleftarrow \mathcal{W}^\vee \longleftarrow \bigwedge^2 \mathcal{W}^\vee \longleftarrow \cdots \longleftarrow \bigwedge^n \mathcal{W}^\vee \longleftarrow 0. \tag{3.2}$$

The terms of $\mathcal{K}$ can be written as

$$\mathcal{K}_j = \bigwedge^j \left(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i) \boxtimes \Omega^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \right) = \bigoplus_{|\mathbf{a}|=j} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}), \qquad \text{for } 0 \le j \le |\mathbf{n}|. \tag{3.3}$$

As in Section 3.2, we are interested in the derived pushforward of $q^*\widetilde{M} \otimes \mathcal{K}$, which we will compute by resolving the second term of each box product with a Čech complex to obtain a spectral sequence. Since $\mathcal{K}$ is a resolution of the diagonal, the pushforward will be quasi-isomorphic to $\widetilde{M}$.

Consider the double complex

$$C^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big),$$

with vertical maps from the Čech complexes and horizontal maps from $\mathcal{K}$. Since taking Čech complexes is functorial and exact we have $\operatorname{Tot}(C) \sim q^*\widetilde{M} \otimes \mathcal{K}$, which is a resolution of $q^*\widetilde{M} \otimes \mathcal{O}_\Delta$ because $\mathcal{K}$ is locally free. Moreover, since the first term of each box product in $q^*\widetilde{M} \otimes \mathcal{K}$ is locally free, the columns of C are p_*-acyclic (c.f. [Har66, Prop. 3.2], [AO89, Lem. 3.2]). Hence the pushforward

$$E_0^{-s,t} = p_*(C^{-s,t}) = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes \Gamma\Big(\mathbb{P}^{\mathbf{n}}, \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big)\Big) \tag{3.4}$$

satisfies $\operatorname{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M}$. With this notation, the *Beilinson spectral sequence* is the spectral sequence of the double complex E_0, whose (vertical) first page has terms

$$E_1^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes H^t\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big) = \mathbf{R}^t p_*(q^*\widetilde{M} \otimes \mathcal{K}_s). \tag{3.5}$$

Beilinson's resolution of the diagonal and the associated spectral sequence are crucial ingredients in constructions of Beilinson monads, Tate resolutions, and virtual resolutions [EFS03; EES15; BES20]. Recently, Brown and Erman [BE21] expanded these constructions to toric varieties using a noncommutative analogue of a Fourier–Mukai transform. More generally, Costa and Miró-Roig [CMR07] have introduced a Beilinson type spectral sequence for a smooth projective variety under certain conditions on its derived category.

The main result of this section is the next proposition, which describes the Betti numbers of a free monad constructed from the Beilinson spectral sequence (c.f. [BES20, Thm. 2.9]). A key component of the construction is the following lemma.

Lemma 3.6. *Let $(C_\bullet, d_\bullet)$ be a bounded above complex of free S-modules and let $B_i = \operatorname{im} d_{i-1}$ and $Z_i = \ker d_i$. If every homology module Z_i/B_i of $C_\bullet$ is free then there is a splitting $f_i\colon C_i \to B_i \oplus Z_i/B_i \oplus C_i/Z_i$ such that f and d commute on each summand.*

Proof. Since $C_\bullet$ is bounded above, there is some k such that $B_i = 0$ for all $i > k$, so in particular $B_{k+1} \cong C_k/Z_k$ is free. Since C_k is free, the exact sequence $0 \to Z_k \to C_k \to$

$C_k/Z_k \to 0$ implies that Z_k is free and $C_k \cong Z_k \oplus C_k/Z_k$. Since Z_k/B_k is free by assumption, the exact sequence $0 \to B_k \to Z_k \to Z_k/B_k \to 0$ implies that B_k is free and $Z_k \cong B_k \oplus Z_k/B_k$. Together, we get $C_k \cong B_k \oplus Z_k/B_k \oplus C_k/Z_k$, and the freeness of B_k means that we can induct backwards on the whole complex. ☐

Proposition 3.7. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There is a free monad $\mathcal{L}$ for $\widetilde{M}$ with terms*

$$\mathcal{L}_k = \bigoplus_{|\mathbf{a}|=k} \mathcal{O}_{\mathbb{P}^\mathbf{n}}(-\mathbf{a}) \otimes H^{|\mathbf{a}|-k}(\mathbb{P}^\mathbf{n}, \widetilde{M} \otimes \Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a}))$$

so that

(1) *the free complex $F_\bullet = \Gamma_*(\mathcal{L})$ has Betti numbers $\beta_{k,\mathbf{a}}(F_\bullet) = h^{|\mathbf{a}|-k}\big(\mathbb{P}^\mathbf{n}, \widetilde{M} \otimes \Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a})\big)$;*

(2) *if $H^i\big(\mathbb{P}^\mathbf{n}, \widetilde{M} \otimes \Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a})\big) = 0$ for $i > |\mathbf{a}|$ then $F_\bullet$ is a minimal virtual resolution for M.*

Proof. Let $\mathcal{K}$ be the resolution of the diagonal from (3.3) and let $\Phi_\mathcal{K}$ be the corresponding Fourier–Mukai transform. The Beilinson spectral sequence has (vertical) first page $E_1^{-s,t}$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
\mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
k=0 & & k=1 & & k=2 & &
\end{array} \tag{3.6}$$

Since both (3.4) and (3.5) have locally free terms, by Lemma 3.6 the vertical differential of E_0 satisfies the splitting hypotheses of [EFS03, Lem. 3.5], which implies that the total complex of E_0 is homotopy equivalent to a complex $\mathcal{L}$ with terms $\mathcal{L}_k = \bigoplus_{s-t=k} E_1^{-s,t}$. Hence

$$\mathcal{L} \sim \mathrm{Tot}(E_0) = \Phi_\mathcal{K}(\widetilde{M}) \sim \widetilde{M}.$$

Since the terms of E_1 are direct sums of line bundles, the complex $\mathcal{L}$ is a free monad for $\widetilde{M}$.

Observe that the only terms with twist $\mathbf{a}$ appear in $\mathcal{K}_s$ for $s = |\mathbf{a}|$ and that the Betti numbers in homological index k come from the higher direct images $E_1^{-s,t}$ on diagonals with $s - t = k$. Hence $\beta_{k,\mathbf{a}}(F_\bullet)$ is the rank of $\mathcal{O}_{\mathbb{P}^\mathbf{n}}(-\mathbf{a})$ in $E_1^{-|\mathbf{a}|,|\mathbf{a}|-k}$ which is $h^{|\mathbf{a}|-k}(\mathbb{P}^\mathbf{n}, \widetilde{M} \otimes \Omega^\mathbf{a}(\mathbf{a}))$.

Lastly, note that the hypothesis of part (2) implies that the terms of (3.6) on diagonals with $k < 0$ vanish; hence the free monad $\mathcal{L}$ is a locally free resolution. Since each map in the construction from [EFS03, Lem. 3.5] increases the index $-s$, the differentials in $F_\bullet$ are minimal, so $F_\bullet$ is a minimal virtual resolution. ☐

Remark 3.8. In the proof of [BES20, Prop. 1.2], Berkesch, Erman, and Smith show that if M is sufficiently twisted so that all higher direct images of $\widetilde{M} \otimes \Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a})$ vanish, then the E_1 page will be concentrated in one row, which results in a linear virtual resolution. Similarly in [EES15, Prop. 1.7], Eisenbud, Erman, and Schreyer prove that for sufficiently positive twists, the truncation of M has a linear free resolution. However, in both cases the positivity condition is stronger than $\mathbf{0}$-regularity for M, as illustrated by the following example.

Example 3.9. Write $S = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$ and consider the ideal $I = (y_0 + y_1 + y_2, x_0y_0 + x_0y_1 + x_0y_2 + x_1y_0 + x_1y_1)$. Then $M = S/I$ is a bigraded, $(0,0)$-regular S-module. The global sections of the Beilinson spectral sequence for $\widetilde{M}$ has first page

$$\begin{tikzcd}
0 & 0 \arrow[l] & S(-1,-1) \arrow[l] \arrow[dll, dotted, "x_1y_2"'] & S(-1,-2) \arrow[l, "y_0+y_1+y_2"'] \arrow[dll, dotted, "-x_1y_2"'] & 0 \arrow[l] \\
S & S(0,-1) \arrow[l, "y_0+y_1+y_2"] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the dotted diagonal maps are lifts of maps from the second page of the spectral sequence, which agree with the maps from [EFS03, Lem. 3.5].

In the next section we state and prove Theorem A by illustrating the restrictions on the virtual resolution above that follow from the regularity of $\widetilde{M}$ and using them to bound the shape of the minimal free resolution of a truncation of M. Later, in Corollary 6.5, we examine the maps in the first row in order to compute the Beilinson spectral sequence when M has a linear presentation.

4. A Criterion for Multigraded Regularity

To investigate the relationship between multigraded regularity and resolutions of truncations we first need to establish a definition of linearity for a multigraded resolution. We would like the differentials to be given by matrices with entries of total degree at most 1. However, we will examine only the twists in the resolution, requiring that they lie in the L regions from Section 2.1. In particular, we will identify a complex with a map of degree > 1 as nonlinear even if that map is zero.

Definition 4.1. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *linear* if F_0 is generated in a single multidegree $\mathbf{d}$ and the twists appearing in F_j lie in $L_j(-\mathbf{d})$.

We require F_0 to be generated in a single degree so that the truncation of a module with a linear resolution also has a linear resolution (see Proposition 4.5). Otherwise, for instance, the minimal resolution of M in the following example would be considered linear, yet the resolution of its truncation $M_{\geq(1,0)}$ would not.

Example 4.2. Write $S = \Bbbk[x_0, x_1, y_0, y_1]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ and let M be the module with resolution $S(-1,0)^2 \oplus S(0,-1)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$ given by the presentation matrix

$$\begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_0 \\ -y_0 & 0 & -y_0 & 0 \\ 0 & -y_1 & 0 & -y_1 \end{bmatrix}.$$

A *Macaulay2* computation shows that M is $(1,0)$-regular. However, the minimal graded free resolution of the truncation $M_{\geq(1,0)}$ is

$$0 \longleftarrow S(-1,0)^2 \longleftarrow S(-2,-1)^2 \longleftarrow 0$$

which is not linear because $(-2,-1) \notin L_1(-1,0)$.

This example shows that a module can be $\mathbf{d}$-regular yet have a nonlinear resolution for $M_{\geq \mathbf{d}}$. Thus in order to characterize regularity in terms of truncations we need to weaken the definition of linear. We will use the larger Q regions from Section 2.1 in order to allow some maps of higher degree.

Definition 4.3. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *quasilinear* if F_0 is generated in a single multidegree $\mathbf{d}$ and for each j the twists appearing in F_j lie in $Q_j(-\mathbf{d})$.

Example 4.4. Unlike on a single projective space, the resolution of S/B for the irrelevant ideal B on a product of projective spaces is not linear. However it is quasilinear. On $\mathbb{P}^1 \times \mathbb{P}^2$, for instance, S/B has resolution

$$0 \longleftarrow S \longleftarrow S(-1,-1)^6 \longleftarrow \begin{matrix} S(-1,-2)^6 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^3 \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0,$$

which has generators in degree $(0,0)$ and relations in degree $(1,1)$. Thus the resolution is not linear, since $(-1,-1) \notin L_1(0,0)$. However $(-1,-1) \in Q_1(0,0)$ is compatible with quasilinearity.

This condition is inspired by [BES20, Thm. 2.9], which characterized regularity in terms of the existence of virtual resolutions with Betti numbers similar to those of S/B—see Section 4.2 for a more complete discussion. Note that both linear and quasilinear reduce to the standard definition of linear on a single projective space. As one might expect from that setting, they satisfy the property below, which will follow from Theorems 5.4 and 5.5.

Proposition 4.5. *Let M be a $\mathbb{Z}^r$-graded S-module. If $M_{\geq \mathbf{d}}$ has a linear (respectively quasilinear) resolution and $\mathbf{d}' \geq \mathbf{d}$ then $M_{\geq \mathbf{d}'}$ has a linear (respectively quasilinear) resolution.*

A linear resolution for $M_{\geq \mathbf{d}}$ implies that M is $\mathbf{d}$-regular when $H^0_B(M) = 0$. To obtain a converse that generalizes Eisenbud–Goto's result one should instead check that the resolution is quasilinear. This gives a criterion for regularity that does not require computing cohomology.

Theorem 4.6. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ such that F_0 is generated in degree $\mathbf{d}$.*

We prove one direction of Theorem 4.6 in Section 4.1 (Theorem 4.8) and the other in Section 4.2 (Theorem 4.14).

4.1. **Regularity Implies Quasilinearity.** In Proposition 3.7 we constructed a virtual resolution with Betti numbers determined by the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By resolving the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$ in terms of line bundles and tensoring with $\widetilde{M}$, we can relate the cohomological vanishing in the definition of multigraded regularity to the shape of this virtual resolution. The following lemma implies that when M is $\mathbf{d}$-regular the virtual resolution is quasilinear, i.e., the coefficients of twists outside of $Q_i(-\mathbf{d})$ are zero. The lemma is a variant of [BES20, Lem. 2.13] (see Section 4.2).

Lemma 4.7. *If a $\mathbb{Z}^r$-graded S-module M is $\mathbf{0}$-regular then $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) = 0$ for all $-\mathbf{a} \notin Q_i(\mathbf{0})$ and all $i > 0$.*

Proof. Fix i and $\mathbf{a} \in \mathbb{Z}^r$ with $-\mathbf{a} \notin Q_i(\mathbf{0})$, and suppose that $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \neq 0$. We will show that M is not $\mathbf{0}$-regular. We must have $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$, else $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}) = 0$. Let ℓ be the number of nonzero coordinates in $\mathbf{a}$.

A tensor product of locally free resolutions for the factors $\pi_i^*(\Omega^{a_i}_{\mathbb{P}^{n_i}})$ gives a locally free resolution for $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. Since $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$ we can use $r - \ell$ copies of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ and ℓ linear resolutions, each generated in total degree 1, to obtain such a resolution $\mathcal{F}_\bullet$ (see Section 2.3). Thus the twists in $\mathcal{F}_j$ have nonpositive coordinates and total degree $-j-\ell$, so they are in $L_{j+\ell}(\mathbf{0})$.

Since $\mathcal{F}$ is locally free the cokernel of $\widetilde{M} \otimes \mathcal{F}$ is isomorphic to $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By a standard spectral sequence argument, explained in the proof of Theorem 4.14, the nonvanishing of $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$ implies the existence of some j such that $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$.

If $i = 0$ then

$$|\mathbf{a}| - i + j \geq \ell - i + j = j + \ell.$$

If $i > 0$ then $\mathbf{a} - \mathbf{1}$ has ℓ nonnegative coordinates that sum to $|\mathbf{a}| - \ell$. Thus $|\mathbf{a}| - \ell > i - 1$, since $-\mathbf{a} \notin Q_i(\mathbf{0}) = L_{i-1}(-\mathbf{1})$ (see Remark 2.2). This also gives

$$|\mathbf{a}| - i + j \geq (\ell + i) - i + j = j + \ell.$$

so in either case $L_{j+\ell}(\mathbf{0}) \subseteq L_{|\mathbf{a}|-i+j}(\mathbf{0})$. Therefore $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$ for F_j with twists in $L_{j+\ell}(\mathbf{0})$ implies that M is not $\mathbf{0}$-regular. □

See [CMR07, Thm. 5.5] for a similar result relating Hoffman and Wang's definition of regularity [HW04] to a different cohomology vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Motivated by the quasilinearity of the virtual resolution in Proposition 3.7, we will prove that the $\mathbf{d}$-regularity of M implies that the minimal free resolution of $M_{\geq \mathbf{d}}$ is quasilinear. Let K be the Koszul complex from Section 2.3 and $\check{C}^p(B, \cdot)$ the Čech complex as in Section 2.2. We will use the spectral sequence of a double complex with rows from subcomplexes of K and columns given by Čech complexes in order to relate the Betti numbers of $M_{\geq \mathbf{d}}$ to the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Theorem 4.8. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M)_{\mathbf{d}} = 0$. If M is $\mathbf{d}$-regular then $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$.*

Proof. Without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 2.9).

By Proposition 3.7 there exists a free monad $G_\bullet$ of M with j-th Betti number given by $h^{|\mathbf{a}|-j}(\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$. Since M is $\mathbf{0}$-regular the vanishing of these cohomology groups results in a quasilinear virtual resolution by Lemma 4.7 and (2) from Proposition 3.7. Let $F_\bullet$ be the minimal free resolution of M. We will show that the Betti numbers of $F_\bullet$ are equal to those of $G_\bullet$, so that $F_\bullet$ is also quasilinear and $F_0 = G_0$ is generated in degree $\mathbf{d}$.

Fix a degree $\mathbf{a} \in \mathbb{Z}^r$. Construct a double complex $E^{\bullet,\bullet}$ by taking the Čech complex of each term in $M \otimes K^{\leq \mathbf{a}}_\bullet$ and including the Čech complex of $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ as an additional column. Index $E^{\bullet,\bullet}$ so that

$$E^{s,t} = \begin{cases} \check{C}^t\Big(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|+1-s}\Big) & \text{if } s > 0, \\ \check{C}^t\Big(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\Big) & \text{if } s = 0. \end{cases}$$

We will compare the vertical and horizontal spectral sequences of $E^{\bullet,\bullet}$ in degree $\mathbf{a}$. By Lemma 2.10 and the fact that $K_\bullet^{\le \mathbf{a}}$ is locally free, the sheafification of the 0-th row $E^{\bullet,0}$ is exact. Thus by Lemma 2.8 the rows of $E^{\bullet,\bullet}$ are exact for $t \neq 0$.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots \\
\check{C}^2(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) \arrow[u] \arrow[r] & \check{C}^2(B, M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|}) \arrow[u] \arrow[r] & \check{C}^2(B, M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \check{C}^2(B, M \otimes K^{\le \mathbf{a}}_{0}) \arrow[u] \\
\check{C}^1(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) \arrow[u] \arrow[r] & \check{C}^1(B, M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|}) \arrow[u] \arrow[r] & \check{C}^1(B, M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \check{C}^1(B, M \otimes K^{\le \mathbf{a}}_{0}) \arrow[u] \\
M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} \arrow[u] \arrow[r] & M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|} \arrow[u] \arrow[r] & M \otimes K^{\le \mathbf{a}}_{|\mathbf{a}|-1} \arrow[u] \arrow[r] & \cdots \arrow[r] & M \otimes K^{\le \mathbf{a}}_{0} \arrow[u]
\end{tikzcd}
$$

Since the elements of M have degrees $\ge \mathbf{0}$, the elements of degree $\mathbf{a}$ in $M \otimes K_\bullet$ come from elements of degree $\le \mathbf{a}$ in $K_\bullet$. Thus by Lemma 2.10 the homology of $M \otimes K_\bullet^{\le \mathbf{a}}$ in degree $\mathbf{a}$ is the same as that of $M \otimes K_\bullet$. Hence the cohomology of the 0-th row $E^{\bullet,0}$ in degree $\mathbf{a}$ computes the degree $\mathbf{a}$ Betti numbers of F_j for $0 \le j \le |\mathbf{a}|$, i.e., for $s > 0$,

$$H^s(E^{\bullet,0})_{\mathbf{a}} = \operatorname{Tor}_{|\mathbf{a}|+1-s}(M, \Bbbk)_{\mathbf{a}}. \tag{4.1}$$

The vertical cohomology of $E^{\bullet,\bullet}$ gives the local cohomology of the terms of $M \otimes K_\bullet^{\le \mathbf{a}}$ along with $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Consider the degree $\mathbf{a}$ part of this double complex. The cohomology coming from $M \otimes K_\bullet^{\le \mathbf{a}}$ has summands of the form $H^i_B(M(-\mathbf{b}))_{\mathbf{a}} = H^i_B(M)_{\mathbf{a}-\mathbf{b}}$ where $\mathbf{b} \le \mathbf{a}$. These vanish because M is $\mathbf{0}$-regular, except possibly $H^0_B(M)_{\mathbf{0}}$ which vanishes by hypothesis, so the only nonzero terms come from $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.

Since $K_\bullet^{\le \mathbf{a}}$ is a resolution of $\Bbbk$ in degrees $\le \mathbf{a}$, there are no elements of degree $\mathbf{a}$ in $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Hence, using (2.1),

$$H^1_B\Big(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\Big)_{\mathbf{a}} = H^0\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big).$$

Therefore the cohomology of the 0-th column $E^{0,\bullet}$ in degree $\mathbf{a}$ is

$$H^t(E^{0,\bullet})_{\mathbf{a}} = H^t_B(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}})_{\mathbf{a}} = H^{t-1}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{4.2}$$

for $t > 0$, i.e., the Betti numbers of $G_\bullet$ indexed differently.

Since both spectral sequences of the double complex $E^{\bullet,\bullet}$ converge after the first page, their total complexes agree in degree $\mathbf{a}$, so by equating the dimensions of (4.1) and (4.2) in total degree $|\mathbf{a}| + 1 - j$ we get

$$\dim_{\Bbbk} \operatorname{Tor}_j(M, \Bbbk)_{\mathbf{a}} = \dim_{\Bbbk} H^{|\mathbf{a}|-j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{4.3}$$

for $|\mathbf{a}| \ge j \ge 0$. When $j > |\mathbf{a}|$, neither $F_\bullet$ nor $G_\bullet$ has a nonzero Betti number for degree reasons, and when $\mathbf{a}$ has $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} = 0$ the argument above still holds. Hence the Betti numbers of $G_\bullet$ and $F_\bullet$ are equal in degree $\mathbf{a}$. $\square$

Example 4.9. A smooth hyperelliptic curve of genus 4 can be embedded into $\mathbb{P}^1 \times \mathbb{P}^2$ as a curve of degree $(2, 8)$. An example of such a curve is given explicitly in [BES20, Ex. 1.4] as the B-saturation I of the ideal

$$\langle x_0^2y_0^2 + x_1^2y_1^2 + x_0x_1y_2^2, x_0^3y_2 + x_1^3(y_0 + y_1)\rangle.$$

Using Theorem 4.8 it is relatively easy to check that S/I is not $(2,1)$-regular: the minimal, graded, free resolution of $(S/I)_{\geq(2,1)}$ is

$$0 \longleftarrow S(-2,-1)^9 \longleftarrow \begin{matrix} S(-3,-1)^7 \\ \oplus \\ S(-2,-2)^{10} \\ \oplus \\ S(-2,-3)^2 \end{matrix} \longleftarrow \begin{matrix} S(-3,-2)^6 \\ \oplus \\ S(-2,-3)^3 \\ \oplus \\ S(-3,-3)^3 \end{matrix} \longleftarrow S(-3,-3)^2 \longleftarrow 0$$

which is not quasilinear because $(-2,-3) \notin Q_1(-2,-1)$.

To check that a module M is $\mathbf{d}$-regular directly from Definition 2.3, condition (2) requires one to show that $H^i_B(M)_\mathbf{p}$ vanishes for all $i > 0$ and all $\mathbf{p} \in \bigcup_{|\lambda|=i}(\mathbf{d} - \lambda_1\mathbf{e}_1 - \cdots - \lambda_r\mathbf{e}_r + \mathbb{N}^r)$ with $\lambda \in \mathbb{N}^r$. The proof of Theorem 4.8, when combined with Theorem 4.6 and Lemma 4.7, shows that on a product of projective spaces the full strength of this condition is unnecessary. In particular, one only needs to consider λ_j with $\lambda_j \leq n_j + 1$.

Proposition 4.10. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If*

(1) $H^0_B(M)_\mathbf{p} = 0$ for all $\mathbf{p} \geq \mathbf{d}$

(2) $H^i_B(M)_\mathbf{p} = 0$ for all $i > 0$ and all $\mathbf{p} \in \bigcup_{|\lambda|=i}(\mathbf{d} - \sum_1^r \lambda_j\mathbf{e}_j + \mathbb{N}^r)$ where $0 \leq \lambda_j \leq n_j + 1$

then M is $\mathbf{d}$-regular.

Proof. The only difference between (2) above and condition (2) in Definition 2.3 is the restriction to $\lambda_j \leq n_j + 1$. By the proof of Theorem 4.8, if $H^0_B(M)_\mathbf{b} = 0$ and M satisfies the hypotheses of Proposition 3.7 and Lemma 4.7 then M has a quasilinear resolution generated in degree $\mathbf{d}$ and is thus $\mathbf{d}$-regular by Theorem 4.6. In the proof of Lemma 4.7 it is sufficient for the cohomology of $M(\mathbf{d})$ to vanish in degrees appearing in the resolution of some $\Omega^\mathbf{a}_{\mathbb{P}^\mathbf{n}}(\mathbf{a})$, which excludes those with coordinates not $\leq \mathbf{n} + \mathbf{1}$. ☐

Example 4.11. On $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, to show that a module M is $\mathbf{0}$-regular using Definition 2.3 one must check that $H^3_B(M)_\mathbf{p} = 0$ for $\mathbf{p}$ in the region with minimal elements

$$(-3,0,0), (-2,-1,0), (-2,0,-1), \ldots, (0,-3,0), \ldots, (0,0,-3).$$

However, Proposition 4.10 implies that a smaller region is sufficient. For instance, we need not check that $H^3_B(M)_\mathbf{p} = 0$ for $\mathbf{p}$ equal to each of $(-3,0,0)$, $(0,-3,0)$, and $(0,0,-3)$.

Remark 4.12. One may also deduce Proposition 4.10 from the proofs in [BES20] without the hypothesis that $H^0_B(M)_\mathbf{d} = 0$. See Section 4.2 for further discussion.

4.2. **Quasilinearity Implies Regularity.** We will now prove the reverse implication of Theorem 4.6, namely that a quasilinear resolution generated in degree $\mathbf{d}$ for $M_{\geq\mathbf{d}}$ implies that M is $\mathbf{d}$-regular. We use a hypercohomology spectral sequence argument, which relates the local cohomology of M to that of the terms in a resolution for $M_{\geq\mathbf{d}}$.

The following lemma will show that entire diagonals in our spectral sequence vanish when the resolution is quasilinear. Thus the local cohomology modules $H^i_B(M)$ to which the diagonals converge also vanish in the same degrees.

Lemma 4.13. *If $i, j \in \mathbb{N}$ then $H_B^{i+j+1}(S)_{\mathbf{a}+\mathbf{b}} = 0$ for all $\mathbf{a} \in L_i(\mathbf{0})$ and all $\mathbf{b} \in Q_j(\mathbf{0})$.*

Proof. Note that $L_i(\mathbf{0}) + Q_j(\mathbf{0}) = L_i(\mathbf{0}) + L_{j-1}(-\mathbf{1}) = L_{i+j-1}(-\mathbf{1})$ as sets. We also have $H_B^0(S) = H_B^1(S) = 0$, so it suffices to show that $H_B^{k+1}(S)_{\mathbf{c}} = H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c})) = 0$ for $k \geq 1$ and $\mathbf{c} \in L_{k-1}(-\mathbf{1})$.

The cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ is given by the Künneth formula. Fix a nonempty set of indices $J \subseteq \{1, \ldots, r\}$ and consider the term

$$\left[\bigotimes_{j\in J} H^{n_j}(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right] \otimes \left[\bigotimes_{j\notin J} H^{0}(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right],$$

which contributes to $H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c}))$ for $k = \sum_{j\in J} n_j$. It will be nonzero if and only if $d_j \leq -n_j - 1$ for $j \in J$ and $d_j \geq 0$ for $j \notin J$. If $\mathbf{c} \in L_{k-1}(-\mathbf{1})$ then

$$\mathbf{c} \geq -\mathbf{1} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r$$

for some λ_i with $\sum \lambda_i = k - 1 = -1 + \sum_{j\in J} n_j$. It is not possible for the right side to have components $\leq -n_j - 1$ for all $j \in J$. Since all cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ arises in this way, the lemma follows. □

In [BES20, Thm. 2.9] Berkesch, Erman, and Smith show for M with $H_B^0(M) = H_B^1(M) = 0$ that M is $\mathbf{d}$-regular if and only if M has a virtual resolution $F_\bullet$ so that the degrees of the generators of $F(\mathbf{d})_\bullet$ are at most those appearing in the minimal free resolution of S/B. This Betti number condition is stronger than quasilinearity, but the additional strength is not used in their proof, so the existence of such a virtual resolution is equivalent to the existence of a quasilinear one.

Since a resolution of $M_{\geq \mathbf{d}}$ is a type of virtual resolution, the reverse implication of Theorem 4.6 mostly reduces to this result. We present a modified proof for completeness. In particular, we do not need to require $H_B^1(M) = 0$ because we have more information about the cokernel of our resolution.

From this perspective Theorem 4.6 says that the regularity of M is determined not only by the Betti numbers of its virtual resolutions, but by the Betti numbers of only those virtual resolutions that are actually minimal free resolutions of truncations of M. Thus we provide an explicit method for checking whether M is $\mathbf{d}$-regular.

Theorem 4.14. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H_B^0(M) = 0$. If $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$, then M is $\mathbf{d}$-regular.*

Proof. Without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 2.9).

Let $F_\bullet$ be a quasilinear resolution of M, so that the twists of F_j are in $Q_j(\mathbf{0})$. Then the spectral sequence of the double complex $E^{\bullet,\bullet}$ with terms

$$E^{s,t} = \check{C}^t(B, F_{-s})$$

converges to the cohomology $H_B^i(M)$ of M in total degree i. The first page of the vertical spectral sequence has terms $H_B^t(F_{-s})$, so $H_B^{i+j}(F_j)_{\mathbf{a}} = 0$ for all j (i.e., for all $(s,t) = (-j, i+j)$) implies $H_B^i(M)_{\mathbf{a}} = 0$.

Therefore it suffices to show that $H_B^{i+j}(S(\mathbf{b}))_{\mathbf{a}} = 0$ for $i \geq 1$ and all $\mathbf{a} \in L_{i-1}(\mathbf{0})$ and $\mathbf{b} \in Q_j(\mathbf{0})$, as is done in Lemma 4.13. □

5. Multigraded Regularity and Betti Numbers

Unlike in the single graded setting, it is possible for two modules on a product of projective spaces to have the same multigraded Betti numbers but different multigraded regularities.

Example 5.1. Let M be the module on $\mathbb{P}^1 \times \mathbb{P}^1$ with resolution

$$S(-1,0)^2 \oplus S(0,-1)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$$

given in Example 4.2. Computation shows that M is $(1,0)$-regular but not $(0,1)$-regular. Notice that all of the twists appearing in the minimal resolution of M are symmetric with respect to the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the cokernel N given by exchanging x and y in the presentation matrix has the same multigraded Betti numbers as M. However N is not $(1,0)$-regular because M was not $(0,1)$-regular.

Remark 5.2. Example 5.1 answers a question of Botbol and Chardin [BC17, Ques. 1.2].

5.1. **Inner Bound from Betti Numbers.** While the multigraded Betti numbers of a module do not determine its regularity, in this section we show that they do determine a subset of the regularity. In particular, the following lemma restricts the possible Betti numbers of a truncation of M given the Betti numbers of M. Intuitively, it states that the degrees of Betti numbers of $M_{\geq \mathbf{d}}$ come from the maximum of $\mathbf{d}$ and the degrees of Betti numbers of M, possibly after adding some linear terms.

Lemma 5.3. *Let M be a $\mathbb{Z}^r$-graded S-module. If $M_{\geq \mathbf{d}}$ has $\operatorname{Tor}^S_{m'}(M_{\geq \mathbf{d}}, \Bbbk)_{\mathbf{b}'} \neq 0$ for some $\mathbf{b}' \in \mathbb{Z}^k$ then there exist $\mathbf{b} \leq \mathbf{b}'$ and $m \leq m'$ such that $\operatorname{Tor}^S_m(M, \Bbbk)_{\mathbf{b}} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ where $\mathbf{c} = \max\{\mathbf{b}, \mathbf{d}\}$.*

Proof. Let $0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ be the minimal free resolution of M. Then the termwise truncation $0 \leftarrow M_{\geq \mathbf{d}} \leftarrow (F_0)_{\geq \mathbf{d}} \leftarrow (F_1)_{\geq \mathbf{d}} \leftarrow \cdots$ is also exact by Lemma 2.6. For each i, let $G^i_\bullet$ be a minimal free resolution of $(F_i)_{\geq \mathbf{d}}$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G^0_1 & & G^1_1 & & G^2_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G^0_0 & & G^1_0 & & G^2_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & (F_0)_{\geq \mathbf{d}} & \longleftarrow & (F_1)_{\geq \mathbf{d}} & \longleftarrow & (F_2)_{\geq \mathbf{d}} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We will see in Corollary 6.3 that $S(-\mathbf{b})_{\geq \mathbf{d}}$ has a linear resolution for all $\mathbf{b} \in \mathbb{Z}^k$. Thus the $G^i_\bullet$ are linear. By taking iterated mapping cones we can construct a free resolution of $M_{\geq \mathbf{d}}$ with terms

$$0 \leftarrow G^0_0 \leftarrow G^0_1 \oplus G^1_0 \leftarrow G^0_2 \oplus G^1_1 \oplus G^2_0 \leftarrow \cdots . \tag{5.1}$$

Then $\mathbf{b}'$ corresponds to the degree of a generator of some G^i_j with $i + j = m'$. Since $G^i_\bullet$ is linear, there is a minimal generator of $(F_i)_{\geq \mathbf{d}}$ with degree $\mathbf{c}$ such that $|\mathbf{b}' - \mathbf{c}| = j$.

However the generators of $(F_i)_{\geq \mathbf{d}}$ have degrees equal to $\max\{\mathbf{b}, \mathbf{d}\}$ for degrees $\mathbf{b}$ of generators of F_i. These correspond to $\mathbf{b} \in \mathbb{Z}^k$ such that $\operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{b}} \neq 0$. Thus the lemma holds for $m = i$, so that $m' - m = j = |\mathbf{b}' - \mathbf{c}|$ as desired. $\square$

Lemma 5.3 shows that each Betti number of $M_{\geq \mathbf{d}}$ comes from a Betti number of M in a predictable way. Note that the process cannot be reversed—not all Betti numbers of M produce minimal Betti numbers of $M_{\geq \mathbf{d}}$. However, the Betti numbers of M limit the degrees where a nonlinear truncation could exist. The following theorem identifies such degrees.

Theorem 5.4. *Let M be a $\mathbb{Z}^r$-graded S-module. For all $\mathbf{d} \in \bigcap L_m(\mathbf{b})$, the truncation $M_{\geq \mathbf{d}}$ has a linear resolution generated in degree $\mathbf{d}$, where the intersection is over all m and all $\mathbf{b}$ with $\operatorname{Tor}_m^S(M, \Bbbk)_{\mathbf{b}} \neq 0$.*

Proof. We may assume that $\mathbf{d} = \mathbf{0}$. Suppose instead that $M_{\geq \mathbf{0}}$ does not have a linear resolution generated in degree $\mathbf{0}$. Then there exist $\mathbf{b}' \in \mathbb{N}^k$ and $m' \in \mathbb{Z}$ such that $\operatorname{Tor}_{m'}^S(M_{\geq \mathbf{0}}, \Bbbk)_{\mathbf{b}'} \neq 0$ and $|\mathbf{b}'| > m'$.

By Lemma 5.3 there exist $\mathbf{b}$ and m so that $\operatorname{Tor}_m^S(M, \Bbbk)_{\mathbf{b}} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ where $\mathbf{c} = \max\{\mathbf{b}, \mathbf{0}\}$. The sum of the positive components of $\mathbf{b}$ is

$$|\mathbf{c}| = |\mathbf{b}'| - |\mathbf{b}' - \mathbf{c}| > m' - (m' - m) = m$$

so $\mathbf{0} \notin L_m(\mathbf{b})$ (see Remark 2.2). $\square$

An analogous statement to Theorem 5.4 exists for truncations with quasilinear resolutions. By Theorem 4.6 it also gives a subset of the multigraded regularity. We will see in Section 5.2 that this inner bound is sharp.

Theorem 5.5. *Let M be a $\mathbb{Z}^r$-graded S-module. For all $\mathbf{d} \in \bigcap Q_m(\mathbf{b})$, the truncation $M_{\geq \mathbf{d}}$ has a quasilinear resolution generated in degree $\mathbf{d}$, where the intersection is over all m and all $\mathbf{b}$ with $\operatorname{Tor}_m^S(M, \Bbbk)_{\mathbf{b}} \neq 0$.*

Proof. Assume $\mathbf{d} = \mathbf{0}$ and suppose instead that $M_{\geq \mathbf{0}}$ does not have a quasilinear resolution generated in degree $\mathbf{0}$. If $M_{\geq \mathbf{0}}$ is not generated in degree $\mathbf{0}$ then some generator of M has a degree $\mathbf{b}$ with a positive coordinate, so that $\mathbf{0} \notin \mathbf{b} + \mathbb{N}^r = Q_0(\mathbf{b})$.

Otherwise there exist $\mathbf{b}' \in \mathbb{N}^k$ and $m' \in \mathbb{Z}$ such that $\operatorname{Tor}_{m'}^S(M_{\geq \mathbf{0}}, \Bbbk)_{\mathbf{b}'} \neq 0$ and $|\mathbf{b}'| > m' + \ell' - 1$ where ℓ' is the number of nonzero coordinates in $\mathbf{b}'$. Thus by Lemma 5.3 there exist $\mathbf{b}$ and m so that $\operatorname{Tor}_m^S(M, \Bbbk)_{\mathbf{b}} \neq 0$ and $|\mathbf{b}' - \mathbf{c}| \leq m' - m$ for $\mathbf{c} = \max\{\mathbf{b}, \mathbf{0}\}$.

Let ℓ be the number of coordinates for which $\mathbf{c}$ differs from $\mathbf{c}' = \max\{\mathbf{b}, \mathbf{1}\}$. Then $|\mathbf{c}'| = |\mathbf{c}| + \ell$, so the sum of the positive components of $\mathbf{b} - \mathbf{1}$ is

$$\begin{aligned}
|\mathbf{c}' - \mathbf{1}| &= |\mathbf{c}| + \ell - r \\
&= |\mathbf{b}'| - |\mathbf{b}' - \mathbf{c}| - r + \ell \\
&> (m' + \ell' - 1) - (m' - m) - r + \ell \\
&= m - 1 + \ell' - (r - \ell).
\end{aligned}$$

Note that $r - \ell$ is the number of nonzero coordinates in $\mathbf{c}$. Since $\mathbf{b}' \geq \mathbf{0}$ and $\mathbf{b}' \geq \mathbf{b}$ we have $\mathbf{b}' \geq \mathbf{c} \geq \mathbf{0}$, so $\ell' \geq r - \ell$. Hence the right side of the inequality is $\geq m - 1$, so $\mathbf{0} \notin L_{m-1}(\mathbf{b} - \mathbf{1}) = Q_m(\mathbf{b})$ (see Remark 2.2). $\square$

Corollary 5.6. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If $H^0_B(M) = 0$, then*

$$\bigcap_{i\in\mathbb{N}} \bigcap_{\mathbf{b}\in\beta_i(M)} Q_i(\mathbf{b}) \subseteq \operatorname{reg}(M).$$

We can now prove Proposition 4.5.

Proof of Proposition 4.5. Suppose that $M_{\geq \mathbf{d}}$ has a linear resolution. We will apply Theorem 5.4 to $M_{\geq \mathbf{d}}$ to show that $M_{\geq \mathbf{d}'}$ has a linear resolution for $\mathbf{d}' \geq \mathbf{d}$ as desired. We may assume that the intersection contains all possible terms that could arise from a linear resolution:

$$\bigcap_{i\in\mathbb{N}} \bigcap_{-\mathbf{b}\in L_i(-\mathbf{d})} L_i(\mathbf{b})$$

Note that $-\mathbf{b} \in L_i(-\mathbf{d})$ if and only if $\mathbf{d} \in L_i(\mathbf{b})$. Thus $\mathbf{d} \in L_i(\mathbf{b})$ for all $\mathbf{b}$, so $\mathbf{d}'$ is in the intersection as well. For quasilinear resolutions replace L with Q. □

Other bounds on the multigraded regularity of a module in terms of its Betti numbers exist in the literature. For example, Maclagan and Smith use a long exact sequence argument to bound regularity in [MS04, Thm. 1.5, Cor 7.2]. While our theorem has the added hypothesis that $H^0_B(M) = 0$, it is often sharper than Maclagan and Smith's.

Example 5.7. In [MS04, Ex. 7.6] Maclagan and Smith consider the B-saturated ideal $I = \langle x_{1,0} - x_{1,1}, x_{2,0} - x_{2,1}, x_{3,0} - x_{3,1}\rangle \cap \langle x_{1,0} - 2x_{1,1}, x_{2,0} - 2x_{2,1}, x_{3,0} - 2x_{3,1}\rangle$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. They show that the regularity of S/I is

$$\operatorname{reg}(S/I) = \big((1,0,0) + \mathbb{N}^3\big) \cup \big((0,1,0) + \mathbb{N}^3\big) \cup \big((0,0,1) + \mathbb{N}^3\big)$$

and their bound from the Betti numbers of S/I is

$$\big((2,2,1) + \mathbb{N}^3\big) \cup \big((2,1,2) + \mathbb{N}^3\big) \cup \big((1,2,2) + \mathbb{N}^3\big) \subset \operatorname{reg}(S/I).$$

However, Corollary 5.6 implies that $(1,1,1) + \mathbb{N}^r \subseteq \operatorname{reg}(S/I)$, giving a larger inner bound.

5.2. **Regularity of Complete Intersections.** As an application of Theorems A and B, in this section we compute the multigraded regularity of a saturated complete intersection satisfying minor hypotheses on its generators. To do this we make the bound from Corollary 5.6 explicit in the case of complete intersections. We then use our characterization of regularity to prove that the resulting bound is sharp by explicitly constructing truncations outside this region that do not have quasilinear resolutions.

Lemma 5.8. *If $\mathbf{b}, \mathbf{c} \in \mathbb{N}^r$ with $b_j, c_j > 0$ for all j then $Q_{i+1}(\mathbf{b}+\mathbf{c}) \subseteq Q_i(\mathbf{b})$ for all $i > 0$.*

Proof. By definition the semigroup generators of $Q_{i+1}(\mathbf{b}+\mathbf{c})$ are of the form $\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v}$ where $\mathbf{v} \in \mathbb{N}^r$ and $|\mathbf{v}| = i$. Thus it is enough to show that each $\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v}$ is in $Q_i(\mathbf{b})$. Since $|\mathbf{v}| = i$ it has at least one nonzero coordinate, say v_j. From this we have

$$\mathbf{b}+\mathbf{c}-\mathbf{1}-\mathbf{v} = (\mathbf{b}-\mathbf{1}-(\mathbf{v}-\mathbf{e}_j)) + (\mathbf{c}-\mathbf{e}_j).$$

The desired containment follows from the above equality given that $|\mathbf{v}-\mathbf{e}_j| = i-1$ and that by assumption $\mathbf{c}-\mathbf{e}_j$ is in $\mathbb{N}^r$. □

Theorem 5.9. *Let $I = \langle f_1, \ldots, f_c \rangle \subset B$ be a saturated complete intersection of codimension c in S, meaning that the f_i form a regular sequence of elements from B and $H^0_B(S/I) = 0$. Then*

$$\operatorname{reg}(S/I) = Q_c\left(\sum_{i=1}^{c} \deg f_i\right).$$

Proof. Write $\mathbf{a} = \sum_{i=1}^{c} \deg f_i$. By Theorem 4.6 it suffices to show that $(S/I)_{\geq \mathbf{d}}$ has a quasilinear resolution generated in degree $\mathbf{d}$ if and only if $\mathbf{d} \in Q_c(\mathbf{a})$. We will prove one direction by showing that $Q_c(\mathbf{a})$ is the bound from Corollary 5.6, i.e., that

$$\bigcap_{j\in\mathbb{N}} \bigcap_{\mathbf{b}\in\beta_j(S/I)} Q_j(\mathbf{b}) = Q_c(\mathbf{a})$$

By hypothesis the minimal free resolution $F_\bullet$ of S/I is a Koszul complex, so the elements of $\beta_j(S/I)$ are sums of j choices of $\deg f_i$. In particular $\beta_0(S/I) = \{\mathbf{0}\}$ and $\beta_c(S/I) = \{\mathbf{a}\}$. We have $Q_c(\mathbf{a}) \subset \mathbb{N}^r = Q_0(\mathbf{0})$, so it suffices to show that

$$Q_{j+1}(\deg f_{i_1} + \cdots + \deg f_{i_j} + \deg f_{i_{j+1}}) \subseteq Q_j(\deg f_{i_1} + \cdots + \deg f_{i_j})$$

for all $0 < j < c$ and all $1 \leq i_1 < \cdots < i_{j+1} \leq c$, since each of the other sets in the intersection can be obtained from $Q_c(\mathbf{a})$ in this way. Note that since $I \subset B$, all coordinates of each $\deg f_i$ are positive; therefore the inclusion follows from Lemma 5.8.

Now we need that $(S/I)_{\geq \mathbf{d}}$ does not have a quasilinear resolution if $\mathbf{d} \notin Q_c(\mathbf{a})$. Specifically, we will show that the resolution of $(S/I)_{\geq \mathbf{d}}$ has a c-th syzygy in degree $\mathbf{a}' = \max\{\mathbf{d}, \mathbf{a}\}$. If $\mathbf{d} \notin Q_c(\mathbf{a})$ then $\mathbf{d} \notin Q_c(\mathbf{a}')$ and thus $-\mathbf{a}' \notin Q_c(-\mathbf{d})$, so this will complete our argument.

The proof of Lemma 5.3 constructs a possibly nonminimal free resolution (5.1) of $(S/I)_{\geq \mathbf{d}}$ from resolutions of truncations of the F_j. Since $(F_c)_{\geq \mathbf{d}}$ has a generator of degree $\mathbf{a}'$, the minimal resolution of $(S/I)_{\geq \mathbf{d}}$ will contain a c-th syzygy of degree $\mathbf{a}'$ unless there is a nonminimal map from the generators G^c_0 of $(F_c)_{\geq \mathbf{d}}$ to $G^{c-1}_0 \oplus \cdots \oplus G^0_{c-1}$. Suppose for contradiction that this is true.

The degrees of the summands in G^{c-1-i}_i have the form $\max\{\mathbf{d}, \mathbf{b}\} + \mathbf{v}$ where $\mathbf{b}$ is the sum of the degrees of $c - 1 - i$ choices of the generators f_j and some $\mathbf{v} \in \mathbb{N}^r$ with $|\mathbf{v}| = i$. In order to have a degree 0 map we need $\max\{\mathbf{d}, \mathbf{b}\} + \mathbf{v} = \mathbf{a}' = \max\{\mathbf{d}, \mathbf{a}\}$ for some $\mathbf{b}$ and $\mathbf{v}$. Since all coordinates of each $\deg f_j$ are positive $b_j + i + 1 \leq a_j$ for each j, so $b_j + v_j \neq a_j$. Thus $\mathbf{d} \geq \mathbf{b}$, so $\mathbf{d} + \mathbf{v} = \mathbf{a}'$, contradicting the fact that $\mathbf{d} \notin Q_c(\mathbf{a}')$. □

Note the assumption that $H^0_B(S/I) = 0$ is automatically satisfied if $\operatorname{codim}(P) \neq \operatorname{codim}(I)$ for all minimal primes P over B. However, based on a number of examples it seems that a weaker saturation hypothesis may be sufficient.

Example 5.10. Write $S = \Bbbk[x_0, x_1, x_2, y_0, y_1, y_2]$ and consider the saturated complete intersection ideal $I = (x_0y_0, x_1y_1^2)$ that defines a surface in $\mathbb{P}^2 \times \mathbb{P}^2$. Then Theorem 5.9 implies

$$\operatorname{reg}(S/I) = Q_2\big((2,3)\big) = \big((0,2) + \mathbb{N}^2\big) \cup \big((1,1) + \mathbb{N}^2\big).$$

6. Linear Truncations

As demonstrated by Example 4.2, in general $\mathbf{d}$-regularity is a stronger condition than having a linear resolution for $M_{\geq \mathbf{d}}$. Still, linear truncations have been independently studied in the literature [EES15; BES20].

Our main result in this section is a cohomological vanishing condition that specifies when $M_{\geq \mathbf{d}}$ has a linear resolution. Our arguments largely mimic those for the analogous statements about quasilinear resolutions by switching the roles of L and Q. However, in this case we can identify not only the terms but also the maps in the first page of the Beilinson spectral sequence with those in the resolution of $M_{\geq \mathbf{d}}$.

Lemma 6.1. *Let M be a $\mathbb{Z}^r$-graded S-module. If $H^i(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})) = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_i(\mathbf{0})$, then $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) = 0$ for all $i \geq 0$ and all $-\mathbf{a} \notin L_i(\mathbf{0})$.*

Proof. We will modify the argument from Lemma 4.7.

Suppose that $-\mathbf{a} \notin L_i(\mathbf{0})$ and $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \neq 0$. Since $\mathbf{a} \geq 0$ we have $|\mathbf{a}| > i$. There must exist j such that $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$, where the twists $\mathbf{b}$ in F_j have total degree $-j - \ell$ for ℓ the number of nonzero coordinates in $\mathbf{a}$. Each twist has ℓ negative coordinates, so that the positive coordinates of $-\mathbf{1} - \mathbf{b}$ sum to $j + \ell - \ell = j$. Hence $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})) \neq 0$ for some $\mathbf{b} \in L_j(-\mathbf{1}) = Q_{j+1}(\mathbf{0}) \subseteq Q_{|\mathbf{a}|-i+j}(\mathbf{0})$ with $|\mathbf{a}|-i+j > 0$. ☐

As in our main theorem, the conclusion of this lemma ensures the vanishing of certain Betti numbers of $M_{\geq \mathbf{d}}$.

Theorem 6.2. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$. Then $M_{\geq \mathbf{d}}$ has a linear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$ if and only if $H^i_B(M)_{\mathbf{b}} = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_{i-1}(\mathbf{d})$.*

Proof. The proof of the forward implication is analogous to the proof of Theorem 4.14, switching the roles of L and Q. For the reverse, notice that the proof of Theorem 4.8 shows that the virtual resolution of M from Proposition 3.7 has the same Betti numbers as the minimal free resolution of $M_{\geq \mathbf{d}}$, i.e.,

$$\dim_{\Bbbk} \operatorname{Tor}_j(M_{\geq \mathbf{d}}, \Bbbk)_{\mathbf{a}} = \dim_{\Bbbk} H^{|\mathbf{a}|-j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$$

for $|\mathbf{a}| \geq j \geq 0$ and both are 0 otherwise. The vanishing of the right hand side for $-\mathbf{a} \notin L_j(\mathbf{0})$, given by Lemma 6.1, then implies that the minimal free resolution of $M_{\geq \mathbf{d}}$ is linear. ☐

Corollary 6.3. *The minimal free resolution of $S(-\mathbf{b})_{\geq \mathbf{d}}$ is linear for all $\mathbf{b}, \mathbf{d} \in \mathbb{Z}^r$.*

Proof. By adjusting $\mathbf{d}$ we may assume that $\mathbf{b} = \mathbf{0}$. Note that $S_{\geq \mathbf{d}} = S_{\geq \mathbf{d}'}$ for $\mathbf{d}' = \max\{\mathbf{d}, \mathbf{0}\} \in \mathbf{0} + \mathbb{N}^r$. Thus by Theorem 6.2 and Proposition 4.5 it suffices to show that $H^i_B(S)_{\mathbf{b}} = 0$ for all $i > 0$ and all $\mathbf{b} \in Q_{i-1}(\mathbf{0})$, which follows from Lemma 4.13. ☐

A key part in the proofs of Theorems 4.8 and 6.2 is that if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$, then the Betti numbers of $M_{\geq \mathbf{d}}(\mathbf{d})$ and the virtual resolution $F_\bullet$ of $M(\mathbf{d})$ constructed in Proposition 3.7 are equal. In the case when $H^0_B(M) = H^1_B(M) = 0$ and $F_\bullet$ has a linear presentation, however, we can prove an even stronger result: in fact $F_\bullet$ is the minimal free resolution of $M_{\geq \mathbf{d}}(\mathbf{d})$. We do this by using the following lemma to show that the cokernel of $F_\bullet$ is $M_{\geq \mathbf{d}}(\mathbf{d})$ and then using Proposition 3.5.

Lemma 6.4. *If $H^0_B(M)_0 = H^1_B(M)_0 = 0$ then the first map in the virtual resolution from Proposition 3.7:*

$$S \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}) \leftarrow \bigoplus_{i=1}^{r} S(-\mathbf{e}_i) \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{e}_i}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i))$$

is the linear part of a presentation of $M_{\geq \mathbf{0}}$.

Proof. Recall from Section 3.3 that the first map in the virtual resolution comes from the first map in the resolution of the diagonal (3.2), which is the dual of a section

$$s\colon \mathcal{O}_{\mathbb{P}^{\mathbf{n}}} \boxtimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}} \to \mathcal{W} = \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{T}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(-\mathbf{e}_i).$$

We can describe s on the i-th summand of $\mathcal{W}$ by a map to its presentation, which we obtain by pulling back the Euler sequence (3.1) along π_i and taking the box product with $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i)$.

$$\begin{tikzcd}
 & & & \mathcal{O}_{\mathbb{P}^{\mathbf{n}}} \boxtimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}} \arrow[dl, "\begin{bmatrix} x_{i,0}\boxtimes 1 \\ \vdots \\ x_{i,n_i}\boxtimes 1 \end{bmatrix}"'] \arrow[d, "s_i"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i) \arrow[r, "\begin{bmatrix} 1\boxtimes x_{i,0} \\ \vdots \\ 1\boxtimes x_{i,n_i} \end{bmatrix}"] & \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}^{n_i+1} \arrow[r] & \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{T}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(-\mathbf{e}_i) \arrow[r] & 0
\end{tikzcd}$$

Summing over i, dualizing, and tensoring with $q^*\widetilde{M}$ gives a diagram of sheaves on the product $\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$. In particular, the map s gives the presentation

$$0 \longleftarrow q^*\widetilde{M} \otimes \mathcal{O}_{\Delta} \longleftarrow \mathcal{O}_{\mathbb{P}^{\mathbf{n}}} \boxtimes \widetilde{M} \longleftarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i) \boxtimes (\widetilde{M} \otimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(\mathbf{e}_i))$$

Pushing forward along p_* and taking global sections, we get the commutative diagram

$$\begin{tikzcd}
 & & \bigoplus_{i=1}^{r} S(-\mathbf{e}_i) \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{e}_i)) \\
 & & \bigoplus_{i=1}^{r} S(-\mathbf{e}_i) \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M})^{n_i+1} \arrow[u, "\bigoplus_{i=1}^{r} 1\otimes\begin{bmatrix} x_{i,0} \cdots x_{i,n_i} \end{bmatrix}"] \arrow[dl, "\begin{bmatrix} x_{1,0} \cdots x_{r,n_r} \end{bmatrix}\otimes 1"'] \\
M & S \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}) \arrow[l, "g"] & \bigoplus_{i=1}^{r} S(-\mathbf{e}_i) \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(\mathbf{e}_i)) \arrow[l, "f"] \arrow[u] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

where the vertical maps come from the Euler sequence and the horizontal maps from the resolution of the diagonal. Note that $g\colon S \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}) \to M$ is surjective onto $M_{\mathbf{0}}$ by hypothesis, so it suffices to show that in total degree 1 the kernel of g is the image of f.

Take an element $v \in S \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M})$ of total degree 1 such that $g(v) = 0$. Since the diagonal map is surjective in total degree 1, we get a lift w of v. Then w maps to 0 vertically because M and $\bigoplus S(-\mathbf{e}_i) \otimes H^0(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{e}_i))$ are isomorphic in total degree 1 and this isomorphism commutes with the diagram above. Since the vertical sequence is exact w is in the image of the map below and v is in the image of f. □

Corollary 6.5. *If M is $\mathbf{d}$-regular with $H_B^0(M) = H_B^1(M) = 0$ and the virtual resolution $G_\bullet$ of $M(\mathbf{d})$ from Proposition 3.7 has a linear presentation, then $G_\bullet$ is the minimal free resolution of $M_{\geq \mathbf{d}}(\mathbf{d})$.*

Proof. Since M is $\mathbf{d}$-regular, by the proof of Theorem 4.8 the minimal free resolution $F_\bullet$ of $M_{\geq \mathbf{d}}(\mathbf{d})$ and the virtual resolution $G_\bullet$ have the same Betti numbers. Hence $M_{\geq \mathbf{d}}(\mathbf{d})$ has a linear presentation, so by Lemma 6.4 it is the cokernel of $G_\bullet$. Since $F_\bullet$ and $G_\bullet$ have the same Betti numbers they must be isomorphic by Proposition 3.5. $\square$

The hypotheses of Corollary 6.5 do not require the truncation to have a linear resolution.

Example 6.6. Consider the ideal $I = (x_0y_0 + x_0y_1 + x_0y_2, x_1y_0y_1 + x_1y_0y_2 + x_1y_1y_2)$ inside $S = \operatorname{Cox} \mathbb{P}^1 \times \mathbb{P}^2$. Then $N = S/I$ is a bigraded $(1,1)$-regular S-module. The virtual resolution $G_\bullet$ of $M = N_{\geq (1,1)}(1,1)$ comes from the spectral sequence with first page

$$\begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] & S(-1,-2) \arrow[l] \arrow[dl, dotted] & 0 \arrow[l] \\
S^5 & S(-1,0)^3 \oplus S(0,-1)^7 \arrow[l] & S(-1,-1)^3 \oplus S(0,-2)^2 \arrow[l] & S(-1,-2) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the diagonal map is from the virtual resolution. Observe that $G_\bullet$ has a linear presentation; hence it is a resolution by Corollary 6.5.

In light of the isomorphism in Corollary 6.5 and the more general equality of Betti numbers in the proof of Theorem 4.8, it seems reasonable to guess that the hypothesis of the corollary on the presentation of $M_{\geq \mathbf{d}}$ may be unnecessary.

Conjecture 6.7. *If M is $\mathbf{d}$-regular with $H^0_B(M) = H^1_B(M) = 0$, then the virtual resolution of $M(\mathbf{d})$ from Proposition 3.7 is the minimal free resolution of $M_{\geq \mathbf{d}}(\mathbf{d})$.*

7. Generalizing Eisenbud–Goto

Recall Eisenbud–Goto's conditions (2) through (4) from the introduction. As we have seen, these conditions diverge substantially for products of projective spaces. However, they can each be generalized to give interesting, albeit different, regions inside $\operatorname{Pic} \mathbb{P}^{\mathbf{n}}$.

If M is a finitely generated $\mathbb{Z}^r$-graded S-module, then (2) defines the multigraded regularity region $\operatorname{reg}(M) \subset \operatorname{Pic} \mathbb{P}^{\mathbf{n}}$ of Maclagan and Smith. On the other hand condition (3) naturally generalizes to two truncation regions. First, the obvious generalization gives the linear truncation region:

$$\operatorname{trunc}^L(M) \coloneqq \{\mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a linear resolution generated in degree } \mathbf{d}\}.$$

Second, our characterization of regularity gives the quasilinear truncation region:

$$\operatorname{trunc}^Q(M) \coloneqq \{\mathbf{d} \in \mathbb{Z}^r \mid M|_{\geq \mathbf{d}} \text{ has a quasilinear resolution generated in degree } \mathbf{d}\}.$$

Finally, condition (4) on the Betti numbers of M also naturally generalizes to two Betti regions; the L-Betti region as in Theorem 5.4 and the Q-Betti region as in Theorem 5.5:

$$\operatorname{betti}^L(M) \coloneqq \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{d} \in \beta_i(M)} L_i(\mathbf{d}), \qquad \operatorname{betti}^Q(M) \coloneqq \bigcap_{i \in \mathbb{N}} \bigcap_{\mathbf{d} \in \beta_i(M)} Q_i(\mathbf{d}).$$

Theorem 4.6 now states that $\operatorname{reg}(M) = \operatorname{trunc}^Q(M)$ when $H^0_B(M) = 0$. Moreover, since all linear resolutions are quasilinear we get $\operatorname{trunc}^L(M) \subseteq \operatorname{trunc}^Q(M)$. Similarly, since $L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d})$, by definition $\operatorname{betti}^L(M) \subseteq \operatorname{betti}^Q(M)$.

Theorem 5.4 shows that the L-Betti region $\operatorname{betti}^L(M)$ is a subset of the linear truncation region $\operatorname{trunc}^L(M)$. Similarly, Theorem 5.5 shows that the Q-Betti region $\operatorname{betti}^Q(M)$ is a subset of the quasilinear truncation region $\mathrm{trunc}^Q(M)$. We can summarize all of the above relations in the following highly non-commutative diagram:

$$\begin{tikzcd}
\mathrm{betti}^L(M) \arrow[r, hook, "5.4"] \arrow[d, hook] & \mathrm{trunc}^L(M) \arrow[d, hook] & \\
\mathrm{betti}^Q(M) \arrow[r, hook, "5.5"] & \mathrm{trunc}^Q(M) \arrow[r, equal, "4.6"] & \mathrm{reg}(M)
\end{tikzcd}$$

We saw in Section 6 that we can switch the roles of Q and L in the proof of Theorem 4.6 to complete the upper right corner of this diagram. The resulting cohomological characterization of $\mathrm{trunc}^L(M)$ in Theorem 6.2 is related to the positivity conditions described in Remark 3.8. We suspect that the reversal of Q and L between the Betti number and cohomological conditions has a deeper explanation in terms of the BGG correspondence.

We illustrate the four regions above in the following example.

Example 7.1. Let I be the B-saturated ideal in Example 4.9, defining a smooth hyperelliptic curve of genus 4 embedded into $\mathbb{P}^1 \times \mathbb{P}^2$ as a curve of degree $(2, 8)$. As noted in [BES20, Ex. 1.4], using *Macaulay2* one finds that the minimal graded free resolution of I is:

$$S \longleftarrow \begin{matrix} S(-3,-1) \\ \oplus \\ S(-2,-2) \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8) \end{matrix} \longleftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7) \\ \oplus \\ S(-1,-8)^2 \end{matrix} \longleftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8) \end{matrix} \longleftarrow S(-3,-7) \longleftarrow 0.$$

From this we can calculate that $\mathrm{betti}^L(S/I)$ and $\mathrm{betti}^Q(S/I)$ are both equal to $(2,7) + \mathbb{N}^2$. These regions, depicted in Figure 2, can also be computed using `linearTruncationsBound` and `regularityBound` from the *Macaulay2* package `LinearTruncations`, which implement Theorems 5.4 and 5.5, respectively [CHN21].

Further, using the functions `linearTruncations` and `multigradedRegularity` from the package `VirtualResolutions` [ABLS20], we can compute where S/I has a linear or quasilinear truncation inside the box $[0,9]^2$. We see that the minimal elements of $\mathrm{trunc}^L(S/I)$ are $(1,5), (2,2)$, and $(5,1)$. On the other hand the minimal elements of $\mathrm{trunc}^Q(S/I)$—which equals $\mathrm{reg}(S/I)$ as I is saturated—are $(1,5), (2,2)$, and $(4,1)$.

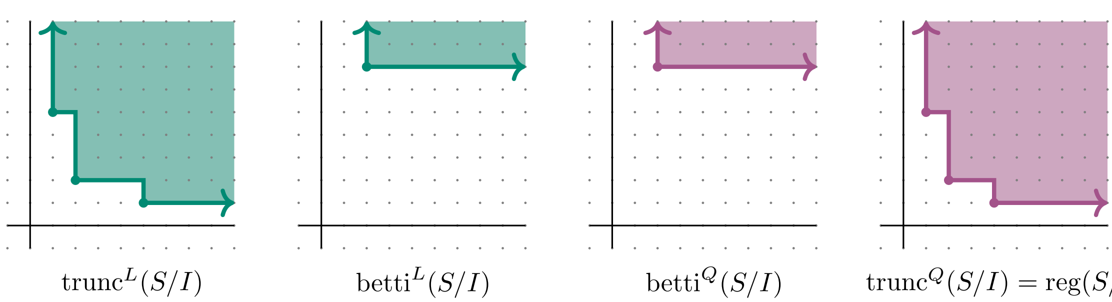

FIGURE 2. The four regions for Example 4.9 inside $\mathrm{Pic}\,\mathbb{P}^1 \times \mathbb{P}^2$.

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Department of Mathematics, University of California, Berkeley, CA
Email address: juliette.bruce@berkeley.edu
URL: https://juliettebruce.github.io

Department of Mathematics, University of California, Berkeley, CA
Email address: lch@math.berkeley.edu
URL: https://math.berkeley.edu/~lch/

School of Mathematics, University of Minnesota, Minneapolis, MN
Email address: mahrud@umn.edu
URL: https://math.umn.edu/~mahrud/