Syzygies, Homological Methods, Finite Fields

Overview. The applicant has studied questions in commutative algebra, algebraic geometry, and arithmetic geometry. Many of her projects involved studying the geometry of algebraic varieties via homological commutative algebra. Additionally, the applicant has done work in the intersection of algebraic geometry and arithmetic geometry, extending classical results in algebraic geometry to finite fields. The applicant will receive her Ph.D in the Spring of 2020.

Asymptotic Syzygies. Much of the applicant's work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work [Gre84a, Gre84b, EL12]. The applicant's thesis focuses on the behavior of asymptotic syzygies in the setting of semi-ample growth, which is a weakening of the previously considered positivity conditions. Her exploration of asymptotic syzygies in the setting of semi-ample growth began by proving nonvanishing results for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$. Subsequently, she has studied the quantitative properties of the syzygies of Hirzebruch surfaces.

Syzygies via Highly Distributed Computing It. It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. The applicant and her co-authors exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. This data provided support for several existing conjectures, as well as led to a number of new conjectures. The resulting data has been made publicly available via SyzygyData.com as well as, a package for Macaualy2 [BEGSY19, M2].

Bertini Theorems Over Finite Fields. Over a finite field, a number of classical statements from algebraic geometry no longer hold. For example, if $X \subset \mathbb{P}^r$ is a smooth projective variety of dimension n over $\mathbb{C}$, Bertini's theorem states that, if $H \subset \mathbb{P}^r$ is a generic hyperplane, then $X \cap H$ is smooth of dimension $n-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Using an ingenious probabilistic sieving argument, it was shown that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree, then a version of Bertini's theorem is true [Poo04] The applicant has extended these probabilistic sieving techniques in a number of new directions. In one project building upon work in [BK12, Poo04] the applicant and her co-author computed the asymptotic probability that randomly chosen homogenous polynomials $f_0, f_1, \ldots, f_k$ over $\mathbf{F}_q$ form a system of parameters. In another project the applicant and a different co-author proved an effective Bertini theorem over finite fields, and used this to generalize a classical statement about Jacobians covering abelian varieties to finite fields.