Not for distribution

COVER SHEET FOR PROPOSAL TO THE NATIONAL SCIENCE FOUNDATION

PROGRAM ANNOUNCEMENT/SOLICITATION NO./DUE DATE		☐ Special Exception to Deadline Date Policy	FOR NSF USE ONLY
PD 20-1264	10/14/2022		NSF PROPOSAL NUMBER
FOR CONSIDERATION BY NSF ORGANIZATION UNIT(S) (Indicate the most specific unit known, i.e. program, division, etc.)			
DMS - ALGEBRA,NUMBER THEORY,AND COM			

DATE RECEIVED	NUMBER OF COPIES	DIVISION ASSIGNED	FUND CODE	UEI (Unique Entity Identifier)	FILE LOCATION
	1	03040000 DMS	1264	E3FDXZ6TBHW3	

EMPLOYER IDENTIFICATION NUMBER (EIN) OR TAXPAYER IDENTIFICATION NUMBER (TIN)	SHOW PREVIOUS AWARD NO. IF THIS IS	IS THIS PROPOSAL BEING SUBMITTED TO ANOTHER FEDERAL AGENCY? YES ☐ NO ☒ IF YES, LIST ACRONYM(S)
050258809	☐ A RENEWAL ☐ AN ACCOMPLISHMENT-BASED RENEWAL	

NAME OF ORGANIZATION TO WHICH AWARD SHOULD BE MADE	ADDRESS OF AWARDEE ORGANIZATION, INCLUDING 9 DIGIT ZIP CODE
BROWN UNIVERSITY	1 PROSPECT ST PROVIDENCE,RI 02912-9127 US
AWARDEE ORGANIZATION CODE (IF KNOWN) 0034017000	
NAME OF PRIMARY PLACE OF PERF Brown University	ADDRESS OF PRIMARY PLACE OF PERF, INCLUDING 9 DIGIT ZIP CODE 151 Thayer Street PROVIDENCE,RI 02912-9127 US

IS AWARDEE ORGANIZATION (Check All That Apply) ☐ SMALL BUSINESS ☐ FOR-PROFIT ORGANIZATION ☐ MINORITY BUSINESS ☐ WOMAN-OWNED BUSINESS ☐ IF THIS IS A PRELIMINARY PROPOSAL THEN CHECK HERE

TITLE OF PROPOSED PROJECT	SHOW LETTER OF INTENT ID IF APPLICABLE
Multigraded Homological Algebra and Geometry	

REQUESTED AMOUNT	PROPOSED DURATION (1-60 MONTHS)	REQUESTED STARTING DATE	SHOW RELATED PRELIMINARY PROPOSAL NO. IF APPLICABLE
$ 174,142	36 months	07/01/2023	

THIS PROPOSAL INCLUDES ANY OF THE ITEMS LISTED BELOW

- ☐ BEGINNING INVESTIGATOR
- ☐ DISCLOSURE OF LOBBYING ACTIVITIES
- ☐ PROPRIETARY & PRIVILEGED INFORMATION
- ☐ HISTORIC PLACES
- ☐ VERTEBRATE ANIMALS IACUC App. Date ______
  PHS Animal Welfare Assurance Number ______
- ☒ TYPE OF PROPOSAL Research
- ☐ HUMAN SUBJECTS Human Subjects Assurance Number ______
  Exemption Subsection ______ or IRB App. Date ______
- ☐ FUNDING OF INT'L BRANCH CAMPUS OF U.S IHE
- ☐ FUNDING OF FOREIGN ORGANIZATION OR FOREIGN INDIVIDUAL
- ☒ INTERNATIONAL ACTIVITIES: COUNTRY/COUNTRIES INVOLVED
  GM UK CA
- ☒ COLLABORATIVE STATUS
  Non-Collaborative

PI/PD DEPARTMENT	PI/PD POSTAL ADDRESS
Department of Mathematics	151 THAYER ST Providence,RI 02912 US
PI/PD FAX NUMBER	

NAMES (TYPED)	High Degree	Yr of Degree	Telephone Number	Email Address
PI/PD NAME Juliette Bruce	PhD	2020	401-863-1867	juliette_bruce1@brown.edu
CO-PI/PD				
CO-PI/PD				
CO-PI/PD				
CO-PI/PD				

CERTIFICATION PAGE

Certification for Authorized Organizational Representative (or Equivalent)

By electronically signing and submitting this proposal, the Authorized Organizational Representative (AOR) is: (1) certifying that statements made herein are true and complete to the best of his/her knowledge; and (2) agreeing to accept the obligation to comply with NSF award terms and conditions if an award is made as a result of this application. Further, the applicant is hereby providing certifications regarding conflict of interest (when applicable), flood hazard insurance (when applicable), responsible conduct of research, and organizational support as set forth in the NSF Proposal & Award Policies & Procedures Guide (PAPPG). Willful provision of false information in this application and its supporting documents or in reports required under an ensuing award is a criminal offense (U. S. Code, Title 18, §1001).

Certification Regarding Conflict of Interest

The AOR is required to complete certifications stating that the organization has implemented and is enforcing a written policy on conflicts of interest (COI), consistent with the provisions of PAPPG Chapter IXA; and that, to the best of his/her knowledge, all financial disclosures required by the conflict of interest policy were made; and that conflicts of interest, if any, were, or prior to the organizations expenditure of any funds under the award, will be, satisfactorily managed, reduced or eliminated in accordance with the organizations conflict of interest policy. Conflicts that cannot be satisfactorily managed, reduced or eliminated and research that proceeds without the imposition of conditions or restrictions when a conflict of interest exists, must be disclosed to NSF via use of the Notifications and Requests Module in FastLane.

Certification Regarding Flood Hazard Insurance

Two sections of the National Flood Insurance Act of 1968 (42 USC §4012a and §4106) bar Federal agencies from giving financial assistance for acquisition or construction purposes in any area identified by the Federal Emergency Management Agency (FEMA) as having special flood hazards unless the:

(1) community in which that area is located participates in the national flood insurance program; and
(2) building (and any related equipment) is covered by adequate flood insurance.

By electronically signing the Certification Pages, the Authorized Organizational Representative (or equivalent) located in FEMA-designated special flood hazard areas is certifying that adequate flood insurance has been or will be obtained in the following situations:
(1) for NSF grants for the construction of a building or facility, regardless of the dollar amount of the grant; and
(2) for other NSF grants when more than $25,000 has been budgeted in the proposal for repair, alteration or improvement (construction) of a building or facility.

Certification Regarding Responsible Conduct of Research (RCR)

(This certification is not applicable to conference proposals.)
By electronically signing the Certification Pages, the Authorized Organizational Representative is certifying that, in accordance with the NSF Proposal & Award Policies & Procedures Guide, Chapter IX.B., the institution has a plan in place to provide appropriate training and oversight in the responsible and ethical conduct of research to undergraduates, graduate students and postdoctoral researchers who will be supported by NSF to conduct research.
The AOR shall require that the language of this certification be included in any award documents for all subawards at all tiers.

Certification Regarding Organizational Support

By electronically signing the Certification Pages, the Authorized Organizational Representative (or equivalent) is certifying that there is organizational support for the proposal as required by Section 526 of the America COMPETES Reauthorization Act of 2010. This support extends to the portion of the proposal developed to satisfy the Broader Impacts Review Criterion as well as the Intellectual Merit Review Criterion, and any additional review criteria specified in the solicitation. Organizational support will be made available, as described in the proposal, in order to address the broader impacts and intellectual merit activities to be undertaken.

Certification Regarding Dual Use Research of Concern

By electronically signing the certification pages, the Authorized Organizational Representative is certifying that the organization will be or is in compliance with all aspects of the United States Government Policy for Institutional Oversight of Life Sciences Dual Use Research of Concern.

Certification Regarding the Meeting Organizer's Written Policy or Code-of-Conduct that Addresses Sexual Harassment, Other Forms of Harassment, and Sexual Assault

(This certification is only applicable to travel proposals)
By electronically signing the Cover Sheet, the AOR is certifying that prior to the proposer's participation in the meeting, the proposer will assure that the meeting organizer has a written policy or code-of-conduct that addresses sexual harassment, other forms of harassment, and sexual assault, and that includes clear and accessible means of reporting violations of the policy or code-of-conduct. The policy or code-of-conduct must address the method for making a complaint as well as how any complaints received during the meeting will be resolved. The proposer is not required to submit the meeting organizer's policy or code-of-conduct for review by NSF.

Certification Regarding Family Leave Status (or equivalent)

(This certification is only applicable to career-life balance supplemental funding requests)
By electronically signing the certification pages, the Authorized Organizational Representative hereby certifies that the request for a technician (or equivalent) is because the (PI/co-PI/senior personnel/ NSF Graduate Research Fellow/postdoctoral researcher/graduate student) is, or will be, on family leave status (or equivalent) from the organization in accordance with the organization's policies. The Authorized Organizational Representative also affirms that the organization is able to fill the position for which funding is being requested, in an appropriate timeframe.

AUTHORIZED ORGANIZATIONAL REPRESENTATIVE	SIGNATURE	DATE
NAME		

TELEPHONE NUMBER	EMAIL ADDRESS	FAX NUMBER

Project Summary: Multigraded Homological Algebra and Geometry

Overview.
This proposal develops new connections between multigraded algebra and geometry. The first project expands our understanding of the relationship between minimal graded free resolutions and geometry by generalizing Green's conjecture on the syzygies of canonical curves. The second project is devoted to developing new tools in multigraded commutative algebra to gain a better understanding of the geometry of multigraded Hilbert schemes.

Intellectual Merit.
The PI's first project expands our understanding of the connections between minimal graded free resolutions and the geometry of algebraic varieties. More specifically, this project generalizes Green's conjecture concerning the syzygies of canonical curves to the canonical rings of stacky curves. Towards this goal, the PI proposes developing algebraic and geometric tools to study the properties of the canonical rings of stacky curves. This includes a substantial program to extend classical results characterizing subvarieties of projective space of minimal degree (and their minimal graded free resolutions) to other toric varieties. This project includes several applications like a large-scale computational exploration of syzygies and homological algebra on toric varieties.

A second project proposed by the PI deepens our understanding of the geometry of multigraded Hilbert schemes. Despite their increasing usefulness throughout algebraic geometry and commutative algebra, the geometry of multigraded Hilbert schemes remains quite mysterious. In particular, unlike their classical counterparts, very little is known about when multigraded Hilbert schemes are non-empty, connected, or smooth. This project develops new tools in multigraded commutative algebra to approach answering such geometric questions. In particular, the project looks to generalize a number of classical results and tools in commutative algebra – like Macaulay's theorem, Gotzmann's theorem, and lex ideals – to the multigraded setting. This project contains significant connections to computation and combinatorics.

Broader Impacts.
As an LGBTQ+ woman, the PI has worked hard to promote diversity, inclusivity, and justice in the mathematical community. This proposal will further the PI's work in these directions as she continues to mentor students, to work to support women and LGBTQ+ people in mathematics, and organize conferences. The PI plans to continue mentoring one undergraduate and two graduate students (a majority of whom identify with generally underrepresented groups) she is working with on research projects. In Summer 2024 the PI will organize an undergraduate research program exploring questions in this proposal for students from underrepresented groups.

The PI is the inaugural president of *Spectra, the Association for LBGTQ+ Mathematicians*, and has served as a board member for several years. Since 2020 the PI has organized an annual conference for transgender and non-binary mathematicians and she plans to continue organizing this conference, with the first in-person iteration scheduled for 2024. In Fall 2023 PI will organize a mentoring program supporting LGBTQ+ undergraduate students applying to graduate school.

The PI has organized many conferences including Spec($\overline{\mathbb{Q}}$) a conference for LGBTQ+ mathematicians in algebra, geometry, and number theory (2022), *Western Algebraic Geometry Symposium* (2021), *Gender Equity in the Mathematical Study of Combinatorics* (2021), the *Graduate Workshop in Commutative Algebra for Women and Mathematicians of Minority Genders* (2019), *Geometry & Arithmetic of Surfaces* (2019), and a five-day conference dedicated to developing open-source computer software for algebra and geometry *M2@UW* (2018). The PI plans to organize a follow-up to *Graduate Workshop in Commutative Algebra for Women and Mathematicians of Minority Genders* planned for Summer 2023, as well as a conference in multigraded algebra and geometry.

TABLE OF CONTENTS

For font size and page formatting specifications, see PAPPG section II.B.2.

*Proposers may select any numbering mechanism for the proposal. The entire proposal however, must be paginated. Complete both columns only if the proposal is numbered consecutively.

Project Description: Multigraded Homological Algebra and Geometry

This proposal involves research in commutative algebra and algebraic geometry with several connections to computation and combinatorics, as well as a wide array of broader impacts.

- § 1 **Results of Prior NSF Support.**
- § 2 **Syzygies Beyond Curves.** We will extend Green's conjecture describing the syzygies of canonical curves to the canonical rings of stacky curves. This includes developing a classification of varieties of minimal degree (and their minimal graded free resolutions) for toric varieties, and the development of a Bertini theorem for weighted projective spaces.
- § 3 **Multigraded Hilbert Functions and Schemes.** By developing tools in multigraded commutative algebra this project seeks to improve our understanding of the geometry of multigraded Hilbert schemes. In particular, this project looks to characterize when multigraded Hilbert schemes are non-empty, connected, or smooth.
- § 4 **Broader Impacts.**

1. Results of Prior NSF Support

During 2020-2022 the PI was supported by an NSF Postdoctoral Research Fellowship (NSF Grant No. MSPRF DMS-2002239, $150,000) titled *Asymptotic Syzygies in Algebraic Geometry.* In this period the PI made major contributions to extending our understanding of homological algebra on toric varieties (see Section 1.2) and the cohomology of moduli spaces [9]. These works resulted in 4 new papers being posted to the arXiv [9, 18–20].

During 2015-2020 the PI was supported by an NSF Graduate Research Fellowship (NSF Grant No. DGE-1256259, $150,000) title *Syzygies in Algebraic Geometry.* In this period the PI made substantial contributions to commutative algebra and algebraic geometry, including furthering our understanding of syzygies on higher dimensional varieties (see Section 1.1) and extending classical results in algebraic geometry and commutative algebra to finite fields (see Section 1.3). This resulted in the PI posting 8 new papers to the arXiv [2, 10–16].

Additionally, of the 12 new papers the PI posted to the arXiv during these periods of support 9 of these above mentioned papers were accepted for publication, including in such journals as *Algebra & Number Theory* [11], *Geometry & Topology* [9], *Journal of Algebra* [18], and *Experimental Mathematics* [13]. The PI contributed to the release of 4 open-source software packages, one public-facing database, and two articles for the *Notices of the AMS* [8, 17]. All of these are available via the algebra software *Macaulay2* or the PI's website and Github pages. A more detailed summary of these results and their intellectual merits continues in the following sections.

1.1 Syzygies in Algebraic Geometry Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$0 \longleftarrow M \xleftarrow{\epsilon} F_0 \xleftarrow{d_1} \cdots \qquad\qquad \cdots \xleftarrow{d_{k-1}} F_{k-1} \xleftarrow{d_k} F_k \longleftarrow \cdots$$

where each F_p is a graded free R-module, and hence can be written as $\bigoplus_q R(-p)^{\beta_{p,q}}$. The module $R(-q)$ is the ring R with a twisted grading, so that $R(-q)_d$ is equal to R_{d-q} where R_{d-q} is the graded piece of degree $d-q$. The $\beta_{p,q}$'s are the *Betti numbers* of M, and they count the number of p-syzygies of M of degree q. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^n$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_n]$ and I_X is the ideal of homogeneous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{p,q}$ for its canonical embedding [3,87,88]. See also [35, Conjecture 9.6] and [4,80].

1.1.1 Asymptotic Syzygies Much of my work has focused on studying the asymptotic properties of syzygies of projective varieties. Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [68,69]. However, the modern viewpoint arose from the pioneering work of Green [43,44] and later Ein and Lazarsfeld [34].

To give a flavor of the results of asymptotic syzygies we will focus on the question: in what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{n_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [38] we set,

$$\rho_q(X, L_d) \;\;:=\;\; \frac{\#\left\{p \in \mathbb{N} \mid \big|\, \beta_{p,p+q}(X, L_d) \neq 0\right\}}{n_d},$$

which is the percentage of degrees in which non-zero syzygies appear [35, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [43] *Let $X \subset \mathbb{P}^n$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that $\deg L_d = d$ then*

$$\lim_{d\to\infty} \rho_2(X; L_d) = 0.$$

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [33,58,70,72–74].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [34, Theorem C] *Let $X \subset \mathbb{P}^n$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then*

$$\lim_{d\to\infty} \rho_q(X; L_d) = 1.$$

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [15, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q(X; \mathcal{O}(d_1, d_2)) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\; j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.} \\ \textit{terms}\end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [59] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q\left(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)\right)$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [34, Conjectures 7.1, 7.5].

1.1.2 Syzygies via Highly Distributed Computing It is quite difficult to compute examples of syzygies. For example, until recently the syzygies of the projective plane embedded by the d-uple Veronese embedding were only known for $d \leq 5$. My co-authors and I exploited recent advances in numerical linear algebra and high-throughput high-performance computing to generate a number of new examples of Veronese syzygies. A follow-up project used similar computational approaches to compute the syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$ in over 200 new examples. This data provided support for several existing conjectures, as well as led us to make a number of new conjectures [13, 18]. The resulting data is publicly available via the website SyzygyData and a package for Macaualy2 [14, 60].

1.2 Multigraded Castelnuovo–Mumford Regularity Introduced by Mumford, the Castelnuovo–Mumford Regularity of a projective variety $X \subset \mathbb{P}^n$ is a measure of the complexity of X given in terms of the vanishing of certain cohomology groups of X. Roughly speaking one should think about Castelnuovo–Mumford regularity as being a measure of geometric complexity. Such a measure can be easily extended to modules over a standard graded polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$ by requiring the analogous vanishing conditions for local cohomology.

Mumford was interested in such a measure as it plays a key role in constructing Hilbert and Quot schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. However, Eisenbud and Goto showed that regularity is also closely connected to interesting homological properties.

Theorem 1.4. [31] *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $M = \bigoplus_{e\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent:*

(1) M is d-regular;
(2) $\beta_{p,q}(M) = 0$ for all $p \geq 0$ and $q > d + i$;
(3) $M_{\geq d}$ has a linear resolution.

My collaborators and I have worked to generalize this result to the multigraded setting, i.e. from coherent sheaves on a single projective space to sheaves on a product of projective spaces. In particular, fixing a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ we let $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ and $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\mathrm{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$.

Maclagan and Smith generalized Castelnuovo–Mumford regularity to this setting in terms of certain cohomology vanishing. Fixing some notation given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let:

$$L_i(\mathbf{d}) \coloneqq \bigcup_{\substack{\mathbf{v}\in\mathbb{N} \\ |\mathbf{v}|=i}} (\mathbf{d} - \mathbf{v}) + \mathbb{N}^r.$$

Note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i+1)$-corners. Roughly speaking we define regularity by requiring the i-th cohomology of certain twists of $\mathcal{F}$ to vanish on L_i.

Definition 1.5. [64, Definition 6.1] *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if*

$$H^i\left(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})\right) = 0 \qquad \textit{for all } \mathbf{e} \in L_i(\mathbf{d}).$$

The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:

$$\mathrm{reg}(\mathcal{F}) \coloneqq \left\{ \mathbf{d} \in \mathbb{Z}^r \;\middle|\; \mathcal{F} \textit{ is } \mathbf{d}\textit{-regular} \right\} \subset \mathbb{Z}^r.$$

The obvious approaches to generalize Theorem 1.4 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity [19, Example 5.1] Despite this we show that part (3) of Theorem 1.4 can be generalized. To do so we introduce the following generalization of linear resolutions.

Definition 1.6. *A complex $F_\bullet$ of $\mathbb{Z}^r$-graded free S-modules is* $\mathbf{d}$*-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d}-\mathbf{1})$.*

Theorem 1.7. [19, Theorem A] *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H_B^0(M)=0$:*

$$M \text{ is } \mathbf{d}\text{-regular} \iff M_{\geq \mathbf{d}} \text{ has a } \mathbf{d}\text{-quasilinear resolution.}$$

The proof of Theorem 1.7 is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel. Precisely, if M is $\mathbf{d}$-regular and $H_B^0(M)=0$ we prove the that

$$\dim_{\mathbb{C}} \operatorname{Tor}_j^S(M_{\geq \mathbf{d}}, \mathbb{C})_{\mathbf{a}} = h^{|\mathbf{a}|-j}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{a}}(\mathbf{a})\big) \quad \text{for } |\mathbf{a}| \geq j \geq 0,$$

where the $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{a}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The result then follows from showing that M being $\mathbf{d}$-regular is equivalent to certain vanishings of the right-hand side above.

1.2.1 Multigraded Regularity of Powers of Ideals Building on the work of many people [5, 24, 83, 84], Cutkosky, Herzog, Trung [27] and independently Kodiyalam [55] showed the Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^n$ has surprisingly predictable asymptotic behavior. In particular, given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$, there exist constants $d, e \in \mathbb{Z}$ such that $\operatorname{reg}(I^t) = dt + e$ for $t \gg 0$.

Building upon our work discussed above, my collaborators and I generalized this result to arbitrary toric varieties. In particular, Definition 1.5 can be extended to all toric varieties by letting S be Cox ring of the toric variety X, replacing $\mathbb{Z}^r$ with the Picard group of X, and replacing $\mathbb{N}^r$ with the nef cone of X. My collaborators and I show that the multigraded regularity of powers of ideals is bounded and translates in a predictable way. In particular, the regularity of I^t essentially translates within $\operatorname{Nef} X$ in fixed directions at a linear rate.

Theorem 1.8. [20, Theorem 4.1] *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on I, such that for each integer $t > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators f_i of I, we have*

$$t\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^t) \subseteq t\mathbf{q}_2 + \operatorname{Nef} X.$$

1.3 Varieties over Finite Fields Over a finite field, many classical statements from algebraic geometry no longer hold. For example, if $X \subset \mathbb{P}^n$ is a smooth projective variety of dimension r over $\mathbb{C}$, Bertini's theorem states that, if $H \subset \mathbb{P}^n$ is a generic hyperplane, then $X \cap H$ is smooth of dimension $r-1$. Famously, however, this fails if $\mathbb{C}$ is replaced by a finite field $\mathbf{F}_q$. Using an ingenious probabilistic sieving argument, Poonen showed that if one is willing to replace the role of hyperplanes by hypersurfaces of arbitrarily large degree, then a version of Bertini's theorem is true [76]. Specifically Poonen showed that as, $d \to \infty$, the percentage of hypersurfaces $H \subset \mathbb{P}^n_{\mathbf{F}_q}$ of degree d such that $X \cap H$ is smooth is determined by the Hasse-Weil zeta function of X.

1.3.1 A Probabilistic Study of Systems of Parameters Given an n dimensional projective variety $X \subset \mathbb{P}^n$, a collection of homogeneous polynomials $f_0, f_1, \ldots, f_k$ of degree d is a (partial) system of parameters if $\dim X \cap \mathbb{V}(f_0, f_1, \ldots, f_k) = \dim X - (k+1)$. Systems of parameters are closely tied to Noether normalization, as the existence of a finite (i.e. surjective with finite fibers) map $X \to \mathbb{P}^r$ is equivalent to the existence of a system of parameters of length $r+1$.

Inspired by work of Poonen [76] and Bucur and Kedlaya [21], Daniel Erman and I computed the asymptotic probability that randomly chosen homogeneous polynomials $f_0, f_1, \ldots, f_k$ over $\mathbf{F}_q$ form a system of parameters. By adapting Poonen's closed point sieve to sieve over higher dimensional varieties, we showed that, when $k < n$, the probability that randomly chosen $f_0, f_1, \ldots, f_k$ form a partial system of parameters is controlled by a zeta-function-like power series that enumerates higher dimensional varieties instead of closed points. In the following, $|Z|$ is the number of irreducible components of Z, and $\dim Z \equiv k$ if Z is equidimensional of dimension k.

Theorem 1.9. [11, Theorem 1.4] *Let $X \subseteq \mathbb{P}^n_{\mathbf{F}_q}$ be a projective scheme of dimension r. Fix e and let $k < r$. The probability that random polynomials $f_0, \dots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\begin{pmatrix} f_0, \dots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{pmatrix} = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv r-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)\binom{r-k+d}{r-k}}\right).$$

From this we proved the first explicit bound for Noether normalization over $\mathbf{F}_q$ and gave a new proof of recent results on Noether normalizations of families over $\mathbb{Z}$ and $\mathbf{F}_q[t]$ [25, 42].

1.4 Broader Impacts from Prior NSF Support

1.4.1 Organizing I have organized 9 conferences: *Math Careers Beyond Academia* (50 participants), *M2@UW* (45 participants), *Geometry and Arithmetic of Surfaces* (40 participants), *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* (35 participants), *CAZoom* (70 participants), *Western Algebraic Geometry Symposium* (100 participants), *GEMS of Combinatorics* (40 participants), Spec($\overline{\mathbb{Q}}$) (50 participants), and *BATMOBILE* (November 2022). Additionally, I have organized three special sessions at AMS Sectional Meetings and the Joint Math Meetings. When organizing these conferences I have given paid special attention to promoting women and other underrepresented groups in mathematics. For example, *Graduate Workshop in Commutative Algebra for Women & Mathematicians of Other Minority Genders* and *GEMS in Combinatorics* focused on forming communities of women and non-binary researchers in commutative algebra and combinatorics respectively. Further, Spec($\overline{\mathbb{Q}}$) highlighted the research of LGBTQ+ mathematicians in algebra, geometry, and number theory.

1.4.2 Math Circles From 2015 through 2018 I was heavily involved in the Madison Math Circle, including 2 years as the lead organizer. As an organizer, I worked to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations focused on underrepresented groups. This led to weekly attendance to more than double. I also created a new outreach arm of the circle, which visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This dramatically expanded the reach of the Madison Math Circle, and during my final year as an organizer, the circle reached over 300 students. While a post-doc at UC Berkeley I volunteered as a speaker with the Berkeley Math Circle.

1.4.3 Mentoring I have actively sought out ways to mentor undergraduate and graduate students, especially those from generally underrepresented groups. While a graduate student, I led reading courses with three undergraduates. One of these students, an undergraduate woman, worked with me for over a year, during which time I helped her apply for summer research projects. Working with *Girls' Math Night Out* I lead two girls in high school through a project exploring cryptography. During 2018-2019, I mentored 6 first-year graduate students (all women or non-binary students). I also mentored two undergraduate women via the AWM's Mentoring Network.

I advised two summer research projects for undergraduate students. The first of these projects ran virtually during Summer 2021 when 6 undergraduates worked on combinatorial questions related to my work in [9]. In Summer 2022 I advised an undergraduate student on a research project related to my work on syzygies discussed in Section 1.1. This work is ongoing and will hopefully result in a paper to be posted later this year. This student is now applying to graduate schools in math.

As a postdoc, I began research projects with three graduate students (a majority of whom identify with a generally underrepresented group). These projects have resulted in two pre-prints, with additional projects still ongoing. Throughout the Spring and Summer of 2022, I did a reading course with a first-year graduate woman on algebraic geometry.

1.4.4 Virtual Mathematics In response to the COVID-19 pandemic and the shift of many mathematical activities to virtual formats, I worked to find ways for these online activities to reach those often at the periphery. During Summer and Fall 2020, I helped with Ravi Vakil's *Algebraic Geometry in the Time of Covid* project. This massive online open-access course in algebraic geometry brought together $\sim 2,000$ participants from around the world. In Spring 2021, I organized an 8-week virtual reading course for undergraduates in algebraic geometry and commutative algebra.

1.4.5 A More Inclusive Community. In Fall of 2016, I led the creation of a committee on inclusivity and diversity within the Mathematics Department, as a member of this committee, I drafted the department's commitment to inclusivity and non-discrimination and created template syllabi statements that let students know about these department policies.

While a graduate student I co-founded oSTEM@UW as a resource for LGBTQ+ students in STEM, which eventually grew to over fifty active members. As one member said, "It made me very happy to see other friendly LGBTQ+ faces around... Thanks so much for organizing this stuff – it's really helpful". From 2017-2020 I lead the campus social organization for LGBTQ+ graduate and post-graduate students, which had over 350 members.

Since Fall 2020 I have organized *Trans Math Day*, an annual one-day virtual conference promoting the work of transgender and non-binary mathematicians. Highlighting the importance of such conferences one student participant said, "I've been really considering leaving mathematics. [Trans Math Day 2020] reminded me why I'm here and why I want to stay. ... If a conference like this had been around for me five years ago, my life would have been a lot better." Trans Math Day regularly has 50 participants. The next Trans Math Day is in December 2022.

Since Fall 2020 I have been a board member for *Spectra: The Association for LGBTQ+ Mathematicians.* As a board member I have overseen the growth and formalization of the organization, including the creation and adoption of bylaws, the creation of an invited lecture at the Joint Mathematics Meetings, and a fundraising campaign that has raised over $20,000. I am currently the inaugural president of Spectra, and as of right now we have over 500 people on our mailing/membership lists.

2. Syzygies Beyond Curves

Let X be a projective variety and L a very ample line bundle. Considering the embedding of X into projective space $X \hookrightarrow \mathbb{P}H^0(X, L) \cong \mathbb{P}^n$ defined by L, let $S_X = S/I_X$ where $S = \mathbb{C}[x_0, \ldots, x_n]$ is the standard graded polynomial ring and I_X is the homogeneous defining ideal of X. The minimal graded free resolution of S_X as an S-module is an exact sequence:

$$0 \longleftarrow S_X \xleftarrow{\epsilon} F_0 \xleftarrow{d_1} \cdots \qquad\qquad \cdots \xleftarrow{d_{k-1}} F_{k-1} \xleftarrow{d_k} F_k \longleftarrow \cdots$$

where F_p is a graded free S-module, and the differentials satisfy some minimality conditions. Since F_p is a graded free S-module it can be written as $\bigoplus_q S(-q)^{\beta_{p,q}(S_X)}$. The module $S(-q)$ is the ring S with a twisted grading, so that $S(-q)_d$ is equal to S_{d-q} where S_{d-q} is the graded piece of degree $d-q$. The $\beta_{p,q}(S_X)$ are known as the graded Betti numbers (or syzygies) of the pair (X, L), and we often denote them as $\beta_{p,q}(X; L)$ to indicate their dependence not just on X but on the line bundle L as well. That is to say the syzygies depend on the extrinsic embedding of X into $\mathbb{P}^n$. An overarching theme in the study of syzygies in algebraic geometry is understanding ways that minimal graded free resolutions and grade Betti numbers also capture the intrinsic geometry of X.

The connection between the geometry of X and its minimal graded free resolutions has been made most precise when X is a curve. For example, consider the following classical theorem of Noether and Petri concerning canonical curves.

Theorem 2.1 (Noether-Petri Theorem)**.** *Let X be a smooth projective curve of genus g. If X does not have a degree 2 cover of $\mathbb{P}^1$ then the canonical bundle K_X defines a projectively normal* *embedding of X into $\mathbb{P}^{g-1}$. Further, if X does not admit a degree 3 cover of $\mathbb{P}^1$ then the image of $X \subset \mathbb{P}^{g-1}$ is defined by quadrics.*

In the language of minimal graded free resolutions this theorem can be translated as follows:

- If X does not admit a cover of $\mathbb{P}^1$ of degree 2 then $\beta_{0,0}(X; K_X) = 1$ and $\beta_{0,q}(X; K_X) = 0$ for all $q \neq 0$.
- If X does not admit a cover of $\mathbb{P}^1$ of degree 3 then $\beta_{1,q}(X; K_X) = 0$ for all $q \neq 2$.

From this we see that syzygies of a smooth projective curve X seem to be closely related to the degrees for which X admits a cover of $\mathbb{P}^1$. This heuristic turns out to be true in several ways, but to understand the higher syzygies of canonical curves one needs a slightly more subtle invariant. The Clifford index of a smooth curve X is defined to be:

$$\operatorname{Cliff}(X) := \min \left\{ \deg(L) - 2\dim H^0(X, L) + 2 \;\middle|\; \begin{matrix} & L \in \operatorname{Pic}(X) & \\ \deg(L) \leq g-1 & & \dim H^0(X, L) \geq 2 \end{matrix} \right\}.$$

As a vast generalization of the Noether-Petri theorem, the following conjecture of Green says that the syzygies of canonical curves (i.e. the syzygies of a (non-hyperelliptic) curve embedded by the complete linear series of the canonical divisor) are controlled by the Clifford index.

Conjecture 2.2 (Green's Conjecture [43])**.** *If X is a smooth projective curve then:*

$$\beta_{p,p+2}(X; K_X) = 0 \qquad \textit{for all} \qquad p < \operatorname{Cliff}(X).$$

Stated somewhat differently Green's conjecture say that one can read the Clifford index of a smooth curve (which is an intrinsic invariant of the curve, not dependent on the embedding) from the vanishing of the syzygies of the canonical embedding of the curve (which are extrinsic to the curve, dependent on the choice of the embedding).

Breakthrough work of Voisin proved this conjecture for general curves of even genus [87,88], and in recent years several new proofs of this result have been found [3,54]. Additionally, substantial work has gone into finding refinements and extensions of Green's Conjecture [26,28,39–41].

Remark 2.3. Green's conjecture can be formulated entirely algebraically: Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and $I \subset S$ be a prime ideal containing no linear forms and whose degree two part is spanned by quadrics of rank ≤ 4. Further, assume that S/I is normal, Gorenstein, has dimension two, and $\deg(S/I) = 2n$. If $\beta_{p,p+2}(S/I) = 0$ for all $p \leq C$ then I contains the 2×2 minors of a 1-generic $i \times j$ matrix where $i + j - n = C$ [36, Conjecture 1.3].

Despite these results for curves, much remains unknown about how syzygies encode geometry for other types of varieties. The overarching goal of this first project is to deepen our understanding of the ways minimal graded free resolutions encode geometry beyond the case of curves. In particular, this project seeks to extend Green's conjecture to stacky curves.

2.1 Syzygies of Stacky Curves Formally a *stacky curve* (over a field $\mathbb{K}$) is a smooth proper geometrically connected Deligne-Mumford stack of dimension 1 over $\mathbb{K}$ which contains a dense open subscheme. However, intuitively one may think about a stacky curve as being a curve with a finite number of special points – called stacky points – which have "fractional degrees". Note this intuition can be made precise in the sense that any stacky curve is (Zariski) locally the quotient of a smooth affine curve by a finite group [86, Lemma 5.3.10]. For simplicity for the remainder of this section we will assume we $\mathbb{K} = \mathbb{C}$, however, everything remains true over an arbitrary field if one places minor conditions on the stacky curve.

In [86] Voight and Zureick-Brown develop a theory of divisors, including canonical divisors, for stacky curves. Roughly this theory proceeds by noting that every stacky curve $\mathcal{X}$ admits a coarse space $\pi : \mathcal{X} \to X$ where X is a smooth (non-stacky) curve. The theory of divisors on $\mathcal{X}$ can be developed by carefully transferring properties of divisors on X to $\mathcal{X}$ via the map π. Two key features in this theory differ from that of divisors on non-stacky curves. First, since the degree of a stacky point need not be an integers the degree of a divisor on a stacky curve maybe be a rational number. Second, when working with a stacky curve $\mathcal{X}$ and divisor D instead of thinking of D as defining an embedding of $\mathcal{X}$ into projective space we work directly with the section ring $R(\mathcal{X}; D) := \bigoplus_{k\in\mathbb{Z}} H^0(\mathcal{X}, kD)$. When D is the canonical divisor $K_{\mathcal{X}}$ we refer to $R(\mathcal{X}; K_{\mathcal{X}})$ as the *canonical ring* of $\mathcal{X}$.

In this setup Voight and Zureick-Brown generalize the Noether-Petri Theorem (see Theorem 2.1) to stacky curves by bounding the degrees of the generators and relations of the canonical ring.

Theorem 2.4. [86, Theorem 8.4.1] *Let $\mathcal{X}$ be stacky curve whose coarse space has genus ≥ 1 and let e be the maximum order of the stabilizer groups of the stacky points; then the canonical ring $R(\mathcal{X}; K_{\mathcal{X}})$ is generated by elements in degree at most $3e$ with relations in degree at most $6e$.*

As a consequence of this result, we know that under minor hypotheses the canonical ring of a stacky curve is a finitely generated graded $\mathbb{C}$-algebra with a non-standard $\mathbb{Z}$-grading. That is there exists a $\mathbb{Z}$-graded polynomial ring $S = \mathbb{C}[x_1, x_2, \ldots, x_n]$ where $\deg(x_i) = w_i \in \mathbb{Z}$ such that $R(\mathcal{X}; K_{\mathcal{X}}) \cong S/I$ for some homogeneous ideal $I \subset S$.

Example 2.5. Suppose $\mathcal{X}$ is a stacky curve with two stacky points each having $\mathbb{Z}/2\mathbb{Z}$ as its stabilizer and whose coarse space has genus 1. The canonical ring of $\mathcal{X}$ can be minimally presented as

$$R(\mathcal{X}; K_{\mathcal{X}}) \cong \mathbb{C}[x_1, x_2, x_3]/\langle x_3^2 x_1 - b x_3 x_2^6 - x_1^2 - a x_2^8 \rangle$$

where $a, b \in \mathbb{C}$ and $\deg(x_1) = 4$, $\deg(x_2) = 1$, and $\deg(x_3) = 2$ [86, Remark 8.3.6].

We may translate Theorem 2.4 into a statement about the graded Betti numbers of $R(\mathcal{X}; K_{\mathcal{X}})$: Thinking of $R(\mathcal{X}; K_{\mathcal{X}})$ as a graded S-module, $\beta_{0,q} = 1$ if $q = 0$ and is zero otherwise and $\beta_{1,q} = 0$ for all $q > 6e$. A natural next question is to consider the minimal graded free resolution of $R(\mathcal{X}; K_{\mathcal{X}})$ as a module over this weighted polynomial ring. Given its centrality in the classical study of syzygies of curves an overarching goal of this project is to generalize Green's conjecture to stacky curves.

Research Problem 2.6. *Find and prove an appropriate generalization of Green's conjecture describing the vanishing of the syzygies of the canonical rings of stacky curves.*

As far as I am aware, Problem 2.6 remains widely unconsidered and nothing is known about the higher syzygies of canonical rings of stacky curves. However, Theorem 2.4 and the fact that the geometry of stacky curves, while often complex and nuanced, is closely related to the geometry of classical curves (i.e. via the coarse space) make Problem 2.6 a natural approach to furthering our understanding of the relationship between syzygies and geometry.

Example 2.7. Continuing Example 2.5 let $\mathcal{X}$ be a stacky curve with two stacky points each having $\mathbb{Z}/2\mathbb{Z}$ as its stabilizer and whose coarse space has genus 1. Letting $S = \mathbb{C}[x_1, x_2, x_3]$ where $\deg(x_1) = 4$, $\deg(x_2) = 1$, and $\deg(x_3) = 2$ the minimal graded free resolution of $R(\mathcal{X}; K_{\mathcal{X}})$ is:

$$0 \longleftarrow R(\mathcal{X}; K_{\mathcal{X}}) \xleftarrow{\epsilon} S \xleftarrow{d_0} S(-8) \longleftarrow 0$$

with the map d_0 being multiplication by $x_3^2 x_1 - b x_3 x_2^6 - x_1^2 - a x_2^8$. Thus, the non-zero graded Betti numbers of $R(\mathcal{X}; K_{\mathcal{X}})$ are: $\beta_{0,0}(R(\mathcal{X}; K_{\mathcal{X}})) = 1$ and $\beta_{1,8}(R(\mathcal{X}; K_{\mathcal{X}})) = 1$.

Remark 2.8. Voight and Zureick-Brown's interest in canonical rings of stacky curves arose primarily out of the fact that such rings are connected to modular forms. Much of the subsequent work on stacky curves has followed similar arithmetic motivations [6, 57]. If an answer to Problem 2.6 is found it would be interesting to understand what implications it might have in arithmetic geometry.

Remark 2.9. One could generalize the definition of the Clifford index to stacky curves verbatim and hope that it controls the syzygies of the canonical rings of stacky curves just like in Green's conjecture. However, since the degree of a divisor on a stacky curve is generally not an integer it is not obvious that the Clifford index of a stacky curve would be an integer. However, one can check that a stacky version of Clifford's theorem for effective divisors holds, and so at the very least this definition of the Clifford index would be a non-negative rational number.

In attempting to generalize Green's conjecture to stacky curves, it would be helpful to know whether the canonical rings of stacky curves share the same nice algebraic properties as canonical rings of classical curves. In particular, the canonical rings of (non-stacky) curves are Gorenstein [35, Proposition 9.5], which implies their syzygies have symmetries coming from a duality statement. Based on a number of examples explored as part of Problem 2.11 it seems like the canonical rings of stacky curves are also Gorenstein. As such, I am working to prove the following goal theorem.

Goal Theorem 2.10. *If $\mathcal{X}$ is a stacky curve whose coarse space has genus ≥ 3 then the canonical ring $R(\mathcal{X}; K_{\mathcal{X}})$ is Gorenstein.*

I am approaching proving Goal Theorem 2.10 by first showing that $R(\mathcal{X}; K_{\mathcal{X}})$ is Cohen-Macaulay and then establishing generalizations of results in local cohomology, namely multigraded localy duality and the relationship between local cohomology and sheaf cohomology on $\mathcal{X}$.

2.2 Computing Syzygies of Canonical Stacky Curves The proof of Theorem 2.4 builds upon the ideas of Schreyer [81] to show the existence of a Gröbner basis for the canonical ring of a stacky curve whose elements have certain degrees. However, Voight and Zureick-Brown do not explicitly construct such Gröbner bases. Instead, they prove the existence of such a Gröbner basis via a delicate induction argument, which allows them to reduce to cases when $\mathcal{X}$ has relatively few stacky points, the stacky points of $\mathcal{X}$ have stabilizers of small order, or the coarse space of $\mathcal{X}$ has small genus. It should be possible to make their arguments explicit for simple stacky curves.

With this in mind, one approach toward Problem 2.6 is to make Voight and Zureick-Brown's arguments explicit for a large number of examples, and then use a computer algebra system like Macaulay2 to compute the minimal graded free resolutions for these examples.

Research Problem 2.11. *Compute the minimal graded free resolution of the canonical ring $R(\mathcal{X}, K_{\mathcal{X}})$ for stacky curves $\mathcal{X}$ where:*

- *the coarse space of $\mathcal{X}$ has genus ≤ 4,*
- *there are at most 4 stacky points, and*
- *the stabilizers of the stacky points have order at most 5.*

Building upon code generously shared by Voight and Zureick-Brown, I have carried out Problem 2.11 for most of the genus zero examples. In many of these computed cases, and as noted in [86, Appendix], the canonical ring is a (weighted) complete intersection. However, there are examples showing that even for small examples the canonical ring of a stacky curve is quite complicated.

Example 2.12. Let $\mathcal{X}$ be a stacky curve with three stacky points whose stabilizers are $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/5\mathbb{Z}$ whose coarse space has genus zero. The canonical ring $R(\mathcal{X}; K_{\mathcal{X}})$ can be presented as S/I where $S = \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ where $\deg(x_1) = 3$, $\deg(x_2) = \deg(x_3) = 4$, and $\deg(x_4) = \deg(x_5) = 5$ and I is an ideal generated by 7 homogenous polynomials of degrees $8, 8, 9, 9, 10, 13, 13$. The minimal graded free resolution of $R(\mathcal{X}; K_{\mathcal{X}})$ is shown below (with the degrees suppressed for brevity):

$$0 \longleftarrow R(\mathcal{X}; K_{\mathcal{X}}) \longleftarrow S \longleftarrow S^{\oplus 7} \longleftarrow S^{\oplus 20} \longleftarrow S^{\oplus 30} \longleftarrow S^{\oplus 22} \longleftarrow S^{\oplus 6} \longleftarrow 0.$$

The approach of using large-scale computations to find conjectures concerning syzygies is very much in the same spirit as my prior work on the syzygies of surfaces [13, 18]. While the precise techniques need to address Problem 2.11 will likely be different much of the general framework of organizing, distributing, and publicizing such computations will likely be similar.

2.3 Varieties of Minimal Degree in Toric Varieties One approach to proving Green's conjecture, originally pursued by Schreyer [80], is to note that given a cover $X \to \mathbb{P}^1$ of minimal degree then the image of $X \subset \mathbb{P}^{g-1}$ under the canonical embedding lies on a rational normal scroll. The minimal graded free resolution of a rational normal scroll is known to be an Eagon-Northcott complex, and via general considerations one can show that the minimal graded free resolution of a rational normal scroll containing X injects into the minimal graded free resolution of X.

From this perspective, one way to develop and prove a version of Green's conjecture for stacky curves is to find the correct analog of rational normal scrolls. More precisely, let $\mathcal{X}$ be a stacky curve and $R(\mathcal{X}; K_{\mathcal{X}}) \cong S/I$ a presentation for its canonical ring, where S is a weighted polynomial ring. If one could show there exists a homogeneous ideal $J \subset I \subset S$ such that we know the minimal graded free resolution of S/J then it would be possible to deduce information about the syzygies of $R(\mathcal{X}; K_{\mathcal{X}})$. Note such an ideal J defines a subscheme of weighted projective space.

One characterization of rational normal scrolls is that they have the smallest possible degree among all non-degenerate varieties. More precisely, a variety $X \subset \mathbb{P}^n$ is *non-degenerate* if it is not contain in a hyperplane. A straightforward argument using projection shows that if $X \subset \mathbb{P}^n$ is a non-degenerate subvariety then $\deg(X) \geq \operatorname{codim}(X) + 1$. We say that a subvariety $X \subset \mathbb{P}^n$ is a *variety of minimal degree* if X is non-degenerate and $\deg(X) = \operatorname{codim}(X) + 1$.

Theorem 2.13. [32] *If $X \subset \mathbb{P}^n$ is a variety of minimal degree then X is the cone over a variety of minimal degree. Moreover, if X is smooth and $\operatorname{codim}(X) > 1$ then X is either a rational normal scroll or a the Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$.*

With this in mind, one approach I will pursue to try to find the correct analog of rational normal scrolls for a stacky Green's conjecture is to classify subvarieties of minimal degree when $\mathbb{P}^n$ is replaced by other toric varieties, namely weighted projective space.

Research Problem 2.14. *Let Y be a smooth projective toric variety. Classify all subvarieties of Y of minimal degree, and compute their minimal graded free resolutions.*

Outside of when $Y = \mathbb{P}^n$ Problem 2.14 remains open. In fact, a central hurdle to answering Problem 2.14 is that one first needs to define what it means to be a "variety of minimal degree" when Y is not $\mathbb{P}^n$. Note there are even subtleties in discussing the degree of a subvariety $X \subset Y$ as the usual approaches used for subvarieties of $\mathbb{P}^n$ (i.e. via Hilbert polynomials, intersecting with linear spaces, and intersection theory in the Chow ring) do not necessarily generalize.

Given these subtleties, instead of approaching Problem 2.14 in full generality, I have focused on the case when Y is a weighted projective space. That is to say $Y = \mathbb{P}(w_1, w_2, \ldots, w_n)$ is projective variety associated to the non-standard $\mathbb{Z}$-graded polynomial ring $S = \mathbb{C}[x_1, x_2, \ldots, x_n]$ where $\deg(x_i) = w_i \in \mathbb{Z}$. This is a particularly interesting case to consider in light of Problem 2.6 since the canonical ring of a stacky curve is naturally a quotient of a weighted polynomial ring.

In this setting, there are two classes of varieties with obvious notions of degree, namely curves and hypersurfaces. The degree of a hypersurface is the degree of its defining equation. The degree of a curve $X \subset \mathbb{P}(w_1, \ldots, w_n)$ is the degree of $\mathcal{O}_X(1)$ as a line bundle on X. Further, we say that a subvariety $X \subset \mathbb{P}(w_1, \ldots, w_n)$ is *non-degenerate* if its defining ideal I_X is contained in $\langle x_1, x_2, \ldots, x_n \rangle^2$. We say that X is a *cone* if I_X is contained in a sub-polynomial ring of S. Finally, if X is a curve or hypersurface we say it is a *variety of minimal degree* if it is non-degenerate, not a cone, and has the smallest degree amongst all such subvarieties.

Going forward we further restrict ourselves, letting $S = \mathbb{C}[x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b]$ with $\deg(x_i) = 1$ and $\deg(y_i) = 2$ and write $\mathbb{P}(1^a, 2^b)$ for the corresponding weighted projective space.

Example 2.15. Let $X \subset \mathbb{P}(1^a, 2^b)$ be a hypersurface. If X has degree one then by definition it is degenerate. Supposing X has degree 2 then if its defining equation involves one of the y_i's it is degenerate, and if its defining equation only involves the x_i's it is a cone. Thus, unlike in $\mathbb{P}^n$ hypersurfaces of minimal degree in $\mathbb{P}(1^a, 2^b)$ have degree three.

In this setting, my co-authors and I have made progress understanding curves of minimal degree.

Proposition 2.16. *If $C \subset \mathbb{P}(1^a, 2^2)$ is a smooth curve of minimal degree then C has degree a and is isomorphic to the image of the map:*

$$\phi : \mathbb{P}^1 \to \mathbb{P}(1^a, 2^2) \qquad \textit{given by} \qquad [s : t] \mapsto [s^a : s^{a-1}t : \cdots : st^{a-1} : st^{2a-1} : t^{2a}].$$

We are currently in the process of generalizing Proposition 2.16 to $\mathbb{P}(1^a, 2^b)$ for any a and b satisfying some minor conditions. Additionally, motivated by Problem 2.6 and the fact that varieties of minimal degree in $\mathbb{P}^n$ have particular nice minimal graded free resolutions another natural next step is to compute the defining equations and syzygies for the curves in Proposition 2.16.

Research Problem 2.17. *Compute the defining equations and minimal graded free resolutions for the curves defined in Proposition 2.16.*

More generally, one approach that I am currently pursuing to classify varieties of minimal degree in $\mathbb{P}(1^a, 2^b)$ is to prove a Bertini-like theorem for weighted projective spaces. In particular, the goal would be to show that taking a general hyperplane section – i.e. intersecting with a hypersurface defined by a polynomial of degree one – would preserve the degree, the codimension, the nondegeneracy, and the smoothness/irreducibility of a subvariety of weighted projective space.

Goal Theorem 2.18. *Fix integers $1 \leq b < a$. Let $X \subset \mathbb{P}(1^a, 2^b)$ be an irreducible, smooth, non-degenerate subvariety, which is not a cone. For a general hyperplane $H \subset \mathbb{P}(1^a, 2^b)$ the hyperplane section $X \cap H$ is irreducible, smooth, non-degenerate, not a cone, and* $\deg(X) = \deg(X \cap H)$.

From such a theorem, varieties of minimal degree could then be classified by building off of a classification of curves of minimal degree. For example, combining Goal Theorem 2.18 and Proposition 2.16 would imply that $X \subset \mathbb{P}(1^a, 2^2)$ is a variety of minimal degree if and only if $X \cap H$ is isomorphic to the curve described in Proposition 2.16 for an generic linear subspace $H \subset \mathbb{P}(1^a, 2^2)$.

3. Multigraded Hilbert Functions and Schemes

A central object of study in both commutative algebra and algebraic geometry has been Hilbert schemes parameterizing subvarieties of projective space. Given a polynomial $p \in \mathbb{Q}[t]$ the Hilbert scheme $\mathrm{Hilb}^p(\mathbb{P}^n)$ parameterizes subschemes of $\mathbb{P}^n$ whose Hilbert polynomial is equal to p. Stated differently, if $S = \mathbb{K}[x_0, x_1, \ldots, x_n]$ is a standard graded polynomial ring then $\mathrm{Hilb}^p(\mathbb{P}^n)$ parameterizes (saturated) homogeneous ideals $I \subset S$ whose Hilbert polynomial is equal to p. Hilbert schemes have proven to be both ubiquitous and extremely valuable throughout algebraic geometry. Further, much of our understanding of the geometry of Hilbert schemes (e.g. when they are non-empty, connected, smooth, etc.) comes from deep results combining combinatorics and commutative algebra. The goal of this project is to develop similar tools in multigraded commutative algebra to provide a better understanding of the geometry of multirgraded Hilbert schemes.

Roughly speaking, just as classical Hilbert schemes parametrize (saturated) homogeneous ideals in a standard graded polynomial ring, multigraded Hilbert schemes parametrize certain ideals in a polynomial ring with a different grading. We now introduce some multigraded commutative algebra.

Fix a field $\mathbb{K}$ and let R be a $\mathbb{K}$-algebra. There is a correspondence between the monomials in $T = R[x_1, x_2, \ldots, x_n]$ and vectors in $\mathbb{N}^n$ given by identifying the monomial $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ with the vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. Given a semi-group homomorphism $\deg : \mathbb{N}^n \to A$ there is an induced A-grading on T by setting $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{u}) \in A$. Given $\mathbf{a} \in A$ let $T_{\mathbf{a}}$ be the free R-module generated by the monomials of T of degree $\mathbf{a}$. There is a decomposition of T as $T \cong \oplus_{\mathbf{a}\in A} T_{\mathbf{a}}$. We say that a T-module M is A-graded if there is a direct sum decomposition $M = \oplus_{\mathbf{a}\in A} M_{\mathbf{a}}$ such that $T_{\mathbf{a}} \cdot M_{\mathbf{b}} \subset M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in A$. An ideal $I \subset T$ is homogeneous, with respect to the given A-grading, if it is A-graded when considered as a T-module.

Given a homogeneous ideal $I \subset T$ in order to discuss the Hilbert function of I we would like for $(T/I)_{\mathbf{a}}$ to be a locally free R-module for all $\mathbf{a} \in A$. However, this need not always be the case [66, Section 18.5]. With this in mind, we restrict our attention to a certain class of homogeneous ideals, which avoids these pathologies. A homogeneous ideal $I \subset S$ is admissible if and only if $(S/I)_{\mathbf{a}}$ is a locally free R-module of finite rank for all $\mathbf{a} \in A$. The Hilbert function of a homogeneous admissible ideal $I \subset T$ is the function:

$$h_I : A \to \mathbb{N} \qquad \text{given by} \qquad \mathbf{a} \mapsto \operatorname{rank}_R(T/I)_{\mathbf{a}}.$$

With this definitions in hand, having fixed an A-graded polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ and a function $h : A \to \mathbb{N}$ the multigraded Hilbert scheme is a scheme which parameterizes homogeneous admissible ideals in S with Hilbert polynomial h. The existence of such a scheme, and the framework discussed in this section, was established by work of Haiman and Sturmfels.

Theorem 3.1. [47] *Fix an abelian group A, and let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an A-graded polynomial ring. Given a function $h : A \to \mathbb{N}$, consider the functor $\mathcal{H}^h_S : \mathbb{K}$-algebras $\to$ Sets given by:*

$$\mathcal{H}^h_S \coloneqq \left\{ I \subset R \otimes_{\mathbb{K}} S \;\middle|\; \begin{matrix} I \text{ is an admissible homogeneous ideal} \\ h_I(\mathbf{a}) = h(\mathbf{a}) \text{ for all } \mathbf{a} \in A \end{matrix} \right\}.$$

There exists a quasi-projective $\mathbb{K}$-scheme Hilb^h_S, called the multigraded Hilbert scheme of S and h, which represents $\mathcal{H}^h_S$. Moreover, if the grading on S is positive, then Hilb^h_S is projective.

Remark 3.2. Multigraded Hilbert schemes are quite general and a number of important families of schemes in algebraic geometry can be realized as multigraded Hilbert schemes, this includes: classical Hilbert schemes $\mathrm{Hilb}^p(\mathbb{P}^n)$ and toric Hilbert schemes [75]. Moreover, if X is a smooth toric variety there exists a closed subscheme of a certain multigraded Hilbert scheme which parameterizes all toric subvarieties of X with a given multigraded Hilbert polynomial [63, Theorem 6.2].

Since their introduction, multigraded Hilbert schemes have proven both interesting in their own right [49, 62, 77, 79] and useful tools for approach other problems in algebraic geometry [1, 29, 37, 56]. For example, multigraded Hilbert schemes have played a crucial role in recent results analyzing when the classical Hilbert scheme of points is irreducible [23, 30, 53], as well as, in recent work studying border rank [22, 50]. Despite this, much of the geometry of multigraded Hilbert schemes remains a mystery. The overall goal of this project is to develop tools in multigraded commutative algebra to approach understanding the geometry of multigraded Hilbert schemes.

3.1 Non-Emptyness of Multigraded Hilbert Schemes Having fixed an A-graded polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ and a function $h : A \to \mathbb{N}$ a natural first question concerning multigraded Hilbert schemes is to ask when is Hilb^h_S non-empty. That is to say, what conditions on the function h ensure that there exists an admissible ideal $I \subset S$ with Hilbert function h? In most cases, this question remains open, and I plan to pursue it as part of this research proposal.

Research Problem 3.3. *Give a combinatorial characterization of when a function $h : A \to \mathbb{N}$ is the Hilbert function of an admissible homogeneous ideal.*

Note that for classical Hilbert schemes on $\mathbb{P}^n$ (i.e. when S is $\mathbb{Z}$-standard graded) this question has a beautiful answer touching on deep results in commutative algebra, which we briefly recall here. Given a positive integer a for any fixed positive integer i there is a unique way to express a as:

$$a = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$ are integers. We call the above representation the i-Macaulay expansion of a. (Note in some of the literature this is referred to as the i-binomial expansion of a.) Further, if the i-Macaulay expansion of a is as above we define:

$$a^{\langle i \rangle} = \binom{m_i + 1}{i+1} + \binom{m_{i-1}+1}{i-1+1} + \cdots + \binom{m_j+1}{j+1}.$$

Using these expansions Macaulay was able to characterize the Hilbert functions of standard graded $\mathbb{C}$-algebras as being exactly those functions whose growth is bounded in a precise way.

Theorem 3.4 (Macaulay's Theomrem [61]). *If $h : \mathbb{N} \to \mathbb{N}$ is a function the following are equivalent:*

(1) h is the Hilbert function of a standard graded $\mathbb{C}$-algebra, and

(2) $h(0) = 1$ and $h(i+1) \leq h(i)^{\langle i \rangle}$ for all $i \geq 1$.

In proving this theorem Macaulay also showed that the bound $h(i+1) \leq h(i)^{\langle i \rangle}$ is sharp and is achieved for all i by certain special monomial ideals called lexicographic (lex) ideals (also called lexsegment ideals by some in the literature). Geometrically, we can interpret Macaulay's result as saying that all non-empty Hilbert schemes have a special canonical point, often called the lexicographic (lex) point, corresponding to the lexicographic (lex) ideal. Moreover, the failure of such a point to exist arises because a Hilbert function fails to satisfy certain growth conditions.

For multigraded polynomial rings, no generalization of Macaulay's theorem is known in wide generality. The difficulty in answering Problem 3.3 lies in the fact that when working with multigraded rings lex ideals need not exist [62, Example 3.13]. In particular, to answer Problem 3.3 one needs to find an appropriate replacement for lex ideals.

In recent years this approach has seen some attention. For example, when $S = \mathbb{Z}[x_1, x_2]$ Maclagan and Smith showed there are distinguished ideals, which they call the lex-most ideals, with many of the same properties of lex ideals. In particular, these lex-most ideals are maximal, with respect to some partial order, amongst all monomial ideals with a given Hilbert function [62, Proposition 3.12]. It would be interesting to see if this partial order can be generalized to polynomial rings with more variables in such a way so that the resulting poset of monomial ideals has a unique maximal element.

In particular, given an ideal $I \subset S$ and a monomial $m \in S$ let $L_m(I)$ denote the set of standard monomials of I with degree equal to $\deg(m)$ that are lexicographically less than or equal to m. Using this notation, we define a partial order as follows, if $I, J \subset S$ are monomial ideals with Hilbert function h then we say that $J \preceq_{\text{Lex}} I$ if $\#L_m(J) \leq \#L_m(I)$ for every monomial $m \in S$. Let $\mathcal{L}_h$ denote the poset of monomial ideals with Hilbert function h with respect to the $\preceq_{\text{Lex}}$ partial ordering. A goal of this project would be to prove the following generalization of [62, Proposition 3.12].

Goal Theorem 3.5. *Let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an A-graded polynomial ring. If $h : A \to \mathbb{N}$ is a Hilbert function then the poset $\mathcal{L}_h$ has a unique maximal element.*

I hope to prove Goal Theorem 3.5 by first reducing to the case when the function h has finite support, and then generalizing the combinatorial arguments in [62]. The hope is that identifying such extremal ideals will provide a stepping stone, similar to lex ideals, in answering Problem 3.3.

A different approach to generalizing lex ideals, is to characterize monomial ideals with the largest A-graded Betti numbers. That is, under minor assumptions on the grading (namely that it is positive), if $I \subset S$ is homogenous we may consider the minimal A-graded free resolution of S/I:

$$0 \longleftarrow S/I \xleftarrow{\;\epsilon\;} F_0 \xleftarrow{\;d_1\;} \cdots \qquad \cdots \xleftarrow{\;d_{k-1}\;} F_{k-1} \xleftarrow{\;d_k\;} F_k \longleftarrow \cdots$$

where F_p is free and has a direct sum decomposition as $F_p \cong \oplus_{\mathbf{a} \in A} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}(S/I)}$. Define a partial order on the set of monomial ideals with Hilbert function h by saying that $J \preceq_{\text{betti}} I$ if and only if $\beta_{p,\mathbf{a}}(J) \leq \beta_{p,\mathbf{a}}(I)$ for all $p \in \mathbb{Z}$ and all $\mathbf{a} \in A$. Let $\mathcal{B}_h$ denote the poset of monomial ideals with Hilbert function h with respect to the $\preceq_{\text{betti}}$ partial ordering.

Research Problem 3.6. *Let $S = \mathbb{K}[x_1, x_2, \dots, x_n]$ be a positively A-graded polynomial ring. Fixing a Hilbert function $h : A \to \mathbb{N}$ characterize the maximal elements of the poset $\mathcal{B}_h$.*

When S has the standard grading results of Bigatti, Hulett, and Pardue [7, 51, 71] imply that the maximal element of $\mathcal{B}_h$ is unique and is a lex ideal. In particular, the maximal elements of $\mathcal{L}_h$ and $\mathcal{B}_h$ are equal. When S is not standard graded, even if $S = \mathbb{K}[x_1, x_2]$, this need not be the case [62, Example 3.14]. Thus, the natural place where I will begin approaching Problem 3.6 is when $S = \mathbb{K}[x_1, x_2]$ with a relatively simple grading, for example $\mathbb{Z}^2$-grading where $\deg(x_i) = \mathbf{w}_i \in \mathbb{Z}^2$.

Remark 3.7. In the standard graded case the classification of Hilbert functions due to Macaulay, and subsequent refinements proven by Gotzmann and Green are closely related to Castelnuovo-Mumford regularity [45, 46]. For multigraded rings the analogous connections between multigraded Hilbert functions and multigraded Castelnuovo-Mumford regularity is an active area of research [19, 20, 63].

3.2 Connectedness of Multigraded Hilbert Schemes One of the first results concerning the geometry of classical Hilbert schemes, originally due to Hartshorne, is that $\mathrm{Hilb}^p(\mathbb{P}^n)$ is always connected [48]. In comparison, multigraded Hilbert schemes can be disconnected.

Theorem 3.8. [79] *There exists a $\mathbb{Z}^6$-grading on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_{26}]$ and a function $h : \mathbb{Z}^6 \to \mathbb{N}$ such that Hilb^h_S is disconnect with at least 17 connected components.*

The above example is the smallest known disconnected multigraded Hilbert scheme. It would be interesting to understand whether such disconnected examples are possible with fewer variables.

Research Problem 3.9. *Find the smallest positive integer n such that there exists an A-graded polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ (for some abelian group A) and a function $h : A \to \mathbb{N}$ where the corresponding multigraded Hilbert scheme Hilb^h_S is disconnected.*

I will approach Problem 3.9 by searching for configurations of vectors in $\mathbb{Z}^r$ whose graphs of (unimodular) triangulations are disconnected. Building upon [49, 65, 79] such examples can be used to construct disconnected multigraded Hilbert schemes. Finding such a collection of vectors can be approached computationally, as well as, theoretically via the combinatorics of oriented matroids.

In the opposite direction, Maclagan and Smith have shown that when $S = \mathbb{K}[x_1, x_2]$ then for any grading and function the corresponding multigraded Hilbert scheme is not only connected, but is in fact smooth and irreducible [62, Theorem 1.1]. As part of this proposal, I would like to see if there are other large classes of multigraded Hilbert schemes which are connected.

Research Problem 3.10. *Fixing an A-graded polynomial ring $S = \mathbb{K}[x_1, x_2, \dots, x_n]$, find a characterization, depending on S and A, of all functions $h : A \to \mathbb{N}$ for which the multigraded Hilbert scheme Hilb^h_S is connected.*

In full generality, this question is quite ambitious, however, even progress in special cases would be new and interesting. For example, if we restrict to the case when $S = \mathbb{C}[x_1, x_2, x_3]$ and $A = \mathbb{Z}^r$ this question is completely open, but is especially interesting since in this case reducible multigraded Hilbert schemes are known to exist [52]. Further, one can likely build upon Goal Theorem 3.5 via degeneration technique to provide at least a partial answer to Problem 3.10.

3.3 Smoothness of Multigraded Hilbert Schemes In general, classical Hilbert schemes exhibit a number of pathologies. For example, Mumford showed that there exist Hilbert schemes with irreducible components that are generically non-reduced [67]. Further, Vakil's so-called "Murphy's Law" for Hilbert schemes shows that every singularity type appears on some Hilbert scheme of points in $\mathbb{P}^4$ [85]. Recent work of Skjelnes and Smith has shown that despite these pathologies it is possible to characterize smooth Hilbert schemes on $\mathbb{P}^n$ [82]. A natural follow-up to this work, which I plan to pursue, is to provide a similar characterization of smooth multigraded Hilbert schemes.

Research Problem 3.11. *Characterize for which A-graded polynomial rings S and which functions $h : A \to \mathbb{N}$ the multigraded Hilbert scheme Hilb_S^h is smooth.*

Outside of classical Hilbert schemes relatively little is known about the singularities of multigraded Hilbert schemes [62, 77]. In particular, it is unclear what an answer to Problem 3.11 might look like. With this in mind, it is worth noting that the results of Skjelnes and Smith were discovered via large exploratory computations with the computer algebra system Macaulay2. While much more involved, Problem 3.11 is similarly amenable to computational exploration.

Research Problem 3.12. *Using Macaulay2, conduct a search for smooth multigraded Hilbert schemes Hilb_S^h when S has ≤ 6 variables.*

Problem 3.12 presents a number of subtleties. For example, how should one choose the grading on S and how should one choose which function h. A reasonable answer to this first challenge is to consider the case when $A = \mathbb{Z}^r$ with $\deg(x_i) \in \mathbb{Z}^r$ chosen uniformly within some box. The challenge of knowing which Hilbert functions to explore is best approached by answering Problems 3.3.

4. Broader Impacts

4.1 Organization: I am in the early stages of organizing a week-long conference, *Gender Equity in the Mathematical Study (GEMS) of Commutative Algebra* to support young women and non-binary researchers in commutative algebra and related fields. I also plan to organize a research conference in multigraded commutative algebra and algebraic geometry.

4.2 Mentorship: In Fall 2023 I plan to organize a mentorship program helping guide LGBTQ+ undergraduates through the process of applying to graduate programs in mathematics and helping young LGBTQ+ graduate students establish themselves. The plan for this program is to break participants into groups with each group having LGBTQ+ mathematicians at various career stages, thus allowing participants to exchange advice, find support, and build mentoring networks.

Currently, I have ongoing research projects with two graduate students and one undergraduate which build upon the broader impacts and intellectual merits of this project

- *Multigraded Regularity:* Two graduate students, Lauren Cranton Heller and Mahrud Sayrafi, and I are working on a broad project to understand the homological properties of multigraded Castelnuovo–Mumford regularity on toric varieties (see Section 1.2).
- *Varieties of Minimal Degree:* Lauren Cranton Heller, Ritvik Ramkumar (who was a graduate student when this began), and I are working to understand varieties of minimal degree in weighted projective space (see Section 2.3).
- *Explicit Non-Vanishing Syzygies:* In ongoing work with Daniel Rostamloo, an undergraduate at UC, Berkeley, I am seeking to refine Theorem 1.3. In particular, we are looking to give more precise non-vanishing statements for certain products of projective spaces.

In the summer of 2024, I will organize a summer undergraduate research program for 1-3 students to computational explore the combinatorics and geometry of multigraded Hilbert schemes in the spirit of those problems laid out in Section 3. When organizing this program I will prioritize women, LGBTQ+ mathematicians, and mathematicians from other underrepresented groups.

4.3 A More Inclusive Community: Continuing my efforts to create a more welcoming and inclusive mathematical environment for LGBTQ+ mathematicians in 2024 I will organize the fourth annual *Trans Math Day* as an in-person conference highlighting the research contributions of transgender and non-binary mathematicians. Further, I will remain a board member of *Spectra: The Association for LGBTQ+ Mathematicians* through at least 2023. During this time I will continue to help Spectra grow, as well as oversee the creation of a manual of best practices for hosting mathematical events that are inclusive and welcoming of LGBTQ+ mathematicians.

References

[1] Valery Alexeev and Michel Brion, *Stable reductive varieties. I. Affine varieties*, Invent. Math. **157** (2004), no. 2, 227–274.

[2] Ayah Almousa, Juliette Bruce, Michael Loper, and Mahrud Sayrafi, *The virtual resolutions package for Macaulay2*, J. Softw. Algebra Geom. **10** (2020), no. 1, 51–60.

[3] Marian Aprodu, Gavrill Farkas, Ştefan Papadima, Claudiu Raicu, and Jerzy Weyman, *Koszul modules and Green's conjecture*, Inventiones mathematicae (2019).

[4] Dave Bayer and David Eisenbud, *Graph curves*, Adv. Math. **86** (1991), no. 1, 1–40. With an appendix by Sung Won Park.

[5] Aaron Bertram, Lawrence Ein, and Robert Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, J. Amer. Math. Soc. **4** (1991), no. 3, 587–602.

[6] Manjul Bhargava and Bjorn Poonen, *The local-global principle for integral points on stacky curves*, Journal of Algebraic Geometry (2022). to appear.

[7] Anna Maria Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2317–2334.

[8] Anthony Bonato, Juliette Bruce, and Ron Buckmire, *Spaces for all: the rise of LGBTQ+ mathematics conferences*, Notices Amer. Math. Soc. **68** (2021), no. 6, 998–1003.

[9] Madeline Brandt, Juliette Bruce, Melody Chan, Margarida Melo, Gwyneth Moreland, and Corey Wolfe, *On the top-weight rational cohomology of $\mathcal{A}_g$*, Geometry & Topology (2022). to appear.

[10] Madeline Brandt, Juliette Bruce, Taylor Brysiewicz, Robert Krone, and Elina Robeva, *The degree of* $\mathrm{SO}(n, \mathbb{C})$, Combinatorial algebraic geometry, Fields Inst. Commun., vol. 80, Fields Inst. Res. Math. Sci., Toronto, ON, 2017, pp. 229–246.

[11] Juliette Bruce and Daniel Erman, *A probabilistic approach to systems of parameters and Noether normalization*, Algebra Number Theory **13** (2019), no. 9, 2081–2102.

[12] Juliette Bruce and Wanlin Li, *Effective bounds on the dimensions of Jacobians covering abelian varieties*, Proc. Amer. Math. Soc. **148** (2020), no. 2, 535–551.

[13] Juliette Bruce, Daniel Erman, Steve Goldstein, and Jay Yang, *Conjectures and computations about Veronese syzygies*, Exp. Math. **29** (2020), no. 4, 398–413.

[14] ———, *The Schur-Veronese package in Macaulay2*, J. Softw. Algebra Geom. **11** (2021), no. 1, 83–87.

[15] Juliette Bruce, *Asymptotic syzygies in the setting of semi-ample growth* (2019). Pre-print: arxiv:1904.04944.

[16] ———, *The quantitative behavior of asymptotic syzygies for Hirzebruch surfaces*, J. Commut. Algebra **14** (2022), no. 1, 19–26.

[17] ———, *A word from... Juliette Bruce, Inaugural President of Spectra*, Notices Amer. Math. Soc. **69** (2022), no. 6, 898–899.

[18] Juliette Bruce, Daniel Corey, Daniel Erman, Steve Goldstein, Robert P. Laudone, and Jay Yang, *Syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$: data and conjectures*, J. Algebra **593** (2022), 589–621.

[19] Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi, *Characterizing Multigraded Regularity on Products of Projective Spaces* (2021). Pre-print: arxiv:2110.10705.

[20] ———, *Bounds on Multigraded Regularity* (2022). Pre-print: arxiv:2208.11115.

[21] Alina Bucur and Kiran S. Kedlaya, *The probability that a complete intersection is smooth*, J. Théor. Nombres Bordeaux **24** (2012), no. 3, 541–556 (English, with English and French summaries).

[22] Weronika Buczyńska and Jarosław Buczyński, *Apolarity, border rank, and multigraded Hilbert scheme*, Duke Math. J. **170** (2021), no. 16, 3659–3702.

[23] Dustin A. Cartwright, Daniel Erman, Mauricio Velasco, and Bianca Viray, *Hilbert schemes of 8 points*, Algebra Number Theory **3** (2009), no. 7, 763–795.
[24] Karen A. Chandler, *Regularity of the powers of an ideal*, Comm. Algebra **25** (1997), no. 12, 3773–3776.
[25] Ted Chinburg, Laurent Moret-Bailly, Georgios Pappas, and Martin J. Taylor, *Finite morphisms to projective space and capacity theory*, J. Reine Angew. Math. **727** (2017), 69–84.
[26] Alessandro Chiodo, David Eisenbud, Gavril Farkas, and Frank-Olaf Schreyer, *Syzygies of torsion bundles and the geometry of the level ℓ modular variety over $\overline{M}_g$*, Invent. Math. **194** (2013), no. 1, 73–118.
[27] S. Dale Cutkosky, Jürgen Herzog, and Ngô Viêt Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Mathematica **118** (1999), no. 3, 243–261.
[28] Anand Deopurkar, *The canonical syzygy conjecture for ribbons*, Math. Z. **288** (2018), no. 3-4, 1157–1164.
[29] Simon Donaldson and Song Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II*, J. Differential Geom. **107** (2017), no. 2, 327–371.
[30] Theodosios Douvropoulos, Joachim Jelisiejew, Bernt Ivar Utstøl Nødland, and Zach Teitler, *The Hilbert scheme of 11 points in $\mathbb{A}^3$ is irreducible*, Combinatorial algebraic geometry, Fields Inst. Commun., vol. 80, Fields Inst. Res. Math. Sci., Toronto, ON, 2017.
[31] David Eisenbud and Shiro Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), no. 1, 89–133.
[32] David Eisenbud and Joe Harris, *On varieties of minimal degree (a centennial account)*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 3–13.
[33] Lawrence Ein and Robert Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), no. 1, 51–67.
[34] ———, *Asymptotic syzygies of algebraic varieties*, Invent. Math. **190** (2012), no. 3, 603–646.
[35] David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.
[36] ———, *Green's conjecture: an orientation for algebraists*, Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990), Res. Notes Math., vol. 2, Jones and Bartlett, Boston, MA, 1992, pp. 51–78.
[37] Jordan S. Ellenberg and Daniel Erman, *Furstenberg sets and Furstenberg schemes over finite fields*, Algebra Number Theory **10** (2016), no. 7, 1415–1436.
[38] Daniel Erman and Jay Yang, *Random flag complexes and asymptotic syzygies*, Algebra Number Theory **12** (2018), no. 9, 2151–2166.
[39] Gavril Farkas, Mircea Mustaţă, and Mihnea Popa, *Divisors on $M_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 4, 553–581 (English, with English and French summaries).
[40] Gavril Farkas and Michael Kemeny, *The generic Green-Lazarsfeld secant conjecture*, Invent. Math. **203** (2016), no. 1, 265–301.
[41] ———, *The Prym-Green conjecture for torsion line bundles of high order*, Duke Math. J. **166** (2017), no. 6, 1103–1124.
[42] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270.
[43] Mark L. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), no. 1, 125–171.
[44] ———, *Koszul cohomology and the geometry of projective varieties. II*, J. Differential Geom. **20** (1984), no. 1, 279–289.

[45] Mark Green, *Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann*, Algebraic curves and projective geometry (Trento, 1988), Lecture Notes in Math., vol. 1389, Springer, Berlin, 1989, pp. 76–86.

[46] Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70 (German).

[47] Mark Haiman and Bernd Sturmfels, *Multigraded Hilbert schemes*, J. Algebraic Geom. **13** (2004), no. 4, 725–769.

[48] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. **29** (1966), 5–48.

[49] Milena Hering and Diane Maclagan, *The T-graph of a multigraded Hilbert scheme*, Exp. Math. **21** (2012), no. 3, 280–297.

[50] Roser Homs, Joachim Jelisiejew, Mateusz Michałek, and Tim Seynnaeve, *Bounds on complexity of matrix multiplication away from Coppersmith-Winograd tensors*, J. Pure Appl. Algebra **226** (2022), no. 12, Paper No. 107142, 16.

[51] Heather A. Hulett, *Maximum Betti numbers of homogeneous ideals with a given Hilbert function*, Comm. Algebra **21** (1993), no. 7, 2335–2350.

[52] A. Iarrobino, *Reducibility of the families of* 0*-dimensional schemes on a variety*, Invent. Math. **15** (1972), 72–77.

[53] Joachim Jelisiejew, *Pathologies on the Hilbert scheme of points*, Invent. Math. **220** (2020), no. 2, 581–610.

[54] Michael Kemeny, *Universal secant bundles and syzygies of canonical curves*, Invent. Math. **223** (2021), no. 3, 995–1026.

[55] Vijay Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128** (2000), no. 2, 407–411.

[56] Alex Küronya, Victor Lozovanu, and Catriona Maclean, *Convex bodies appearing as Okounkov bodies of divisors*, Adv. Math. **229** (2012), no. 5, 2622–2639.

[57] Aaron Landesman, Peter Ruhm, and Robin Zhang, *Spin canonical rings of log stacky curves*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 6 (English, with English and French summaries).

[58] Robert Lazarsfeld, Giuseppe Pareschi, and Mihnea Popa, *Local positivity, multiplier ideals, and syzygies of abelian varieties*, Algebra Number Theory **5** (2011), no. 2, 185–196.

[59] Alexander Lemmens, *On the n-th row of the graded Betti table of an n-dimensional toric variety*, J. Algebraic Combin. **47** (2018), no. 4, 561–584.

[60] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[61] F. S. MacAulay, *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc. (2) **26** (1927), 531–555.

[62] Diane Maclagan and Gregory G. Smith, *Smooth and irreducible multigraded Hilbert schemes*, Adv. Math. **223** (2010), no. 5, 1608–1631.

[63] ———, *Uniform bounds on multigraded regularity*, J. Algebraic Geom. **14** (2005), no. 1, 137–164.

[64] ———, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.

[65] D. Maclagan and R. R. Thomas, *Combinatorics of the toric Hilbert scheme*, Discrete Comput. Geom. **27** (2002), no. 2, 249–272.

[66] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[67] David Mumford, *Further pathologies in algebraic geometry*, Amer. J. Math. **84** (1962), 642–648.

[68] ———, *Varieties defined by quadratic equations*, Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), Edizioni Cremonese, Rome, 1970, pp. 29–100.

[69] D. Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354.
[70] Giorgio Ottaviani and Raffaella Paoletti, *Syzygies of Veronese embeddings*, Compositio Math. **125** (2001), no. 1, 31–37.
[71] Keith Pardue, *Deformation classes of graded modules and maximal Betti numbers*, Illinois J. Math. **40** (1996), no. 4, 564–585.
[72] Giuseppe Pareschi, *Syzygies of abelian varieties*, J. Amer. Math. Soc. **13** (2000), no. 3, 651–664.
[73] Giuseppe Pareschi and Mihnea Popa, *Regularity on abelian varieties. I*, J. Amer. Math. Soc. **16** (2003), no. 2, 285–302.
[74] ———, *Regularity on abelian varieties. II. Basic results on linear series and defining equations*, J. Algebraic Geom. **13** (2004), no. 1, 167–193.
[75] Irena Peeva and Mike Stillman, *Toric Hilbert schemes*, Duke Math. J. **111** (2002), no. 3, 419–449.
[76] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.
[77] Ritvik Ramkumar and Alessio Sammartano, *On the smoothness of lexicographic points on Hilbert schemes*, J. Pure Appl. Algebra **226** (2022), no. 3, Paper No. 106872, 12.
[78] ———, *On the smoothness of lexicographic points on Hilbert schemes*, J. Pure Appl. Algebra **226** (2022), no. 3, Paper No. 106872, 12.
[79] Francisco Santos, *Non-connected toric Hilbert schemes*, Math. Ann. **332** (2005), no. 3, 645–665.
[80] Frank-Olaf Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. **275** (1986), no. 1, 105–137.
[81] ———, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math. **421** (1991), 83–123.
[82] Roy Skjelnes and Gregory G. Smith, *Smooth Hilbert schemes: their classification and geometry*, Journal für die reine und angewandte Mathematik (Crelle's Journal),. to appear.
[83] Karen E. Smith and Irena Swanson, *Linear bounds on growth of associated primes*, Comm. Algebra **25** (1997), no. 10, 3071–3079.
[84] Irena Swanson, *Powers of ideals: primary decompositions, Artin-Rees lemma and regularity*, Math. Ann. **307** (1997), no. 2, 299–313.
[85] Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590.
[86] John Voight and David Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), no. 1362, v+144.
[87] Claire Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 4, 363–404.
[88] ———, *Green's canonical syzygy conjecture for generic curves of odd genus*, Compos. Math. **141** (2005), no. 5, 1163–1190.

NSF BIOGRAPHICAL SKETCH

NAME: Bruce, Juliette

ORCID: 0000-0001-9766-7896

POSITION TITLE & INSTITUTION: Postdoctoral Research Associate in Mathematics, Brown University

(a) PROFESSIONAL PREPARATION -(see PAPPG Chapter II.C.2.f.(a))

INSTITUTION	LOCATION	MAJOR / AREA OF STUDY	DEGREE (if applicable)	YEAR YYYY
University of Michigan	Ann Arbor, MI	Mathematics / Political Science	BS	2014
University of Wisconsin	Madison, WI	Mathematics	MA	2016
University of Wisconsin	Madison, WI	Mathematics	PHD	2020

(b) APPOINTMENTS -(see PAPPG Chapter II.C.2.f.(b))

2020 - present Postdoctoral Research Associate in Mathematics, Brown University, Department of Mathematics, Providence, RI

2020 - 2022 NSF Postdoctoral Research Fellow, University of California, Berkeley, Department of Mathematics, Berkeley, CA

2020 - 2021 Postdoctoral Fellow, Mathematical Sciences Research institute (MSRI), Berkeley, CA

2015 - 2018 NSF Graduate Research Fellow, University of Wisconsin, Department of Mathematics, Madison, WI

2014 - 2020 Graduate Student Teaching/Research Assistant, University of Wisconsin, Department of Mathematics, Madison, WI

(c) PRODUCTS -(see PAPPG Chapter II.C.2.f.(c))

Products Most Closely Related to the Proposed Project

1. Brandt M, Bruce J, Chan M, Melo M, Moreland G, Wolfre C. On the top weight cohomology of $\mathcal{A}_{g}$. Geometry & Topology. Forthcoming. Available from: https://arxiv.org/abs/2012.02892
2. Juliette Bruce. The quantitative behavior of asymptotic syzygies for Hirzebruch surfaces. Journal of Commutative Algebra. 2022; 14(1):19-26. Available from: https://doi.org/10.1216/jca.2022.14.19 DOI: 10.1216/jca.2022.14.19
3. Bruce J, Corey D, Erman D, Goldstein S, Laudone R, Yang J. Syzygies of $\mathbb{P}^{1}\times\mathbb{P}^{1}$: Data and conjectures. Journal of Algebra. 2022 March; 593:589-621. Available from: https://linkinghub.elsevier.com/retrieve/pii/S0021869321005160 DOI: 10.1016/j.jalgebra.2021.10.023
4. Bruce J, Erman D, Goldstein S, Yang J. Conjectures and Computations about Veronese Syzygies. Experimental Mathematics. 2018 June 21; 29(4):398-413. Available from: https://www.tandfonline.com/doi/full/10.1080/10586458.2018.1474506 DOI: 10.1080/10586458.2018.1474506

5. Bruce J, Erman D. A probabilistic approach to systems of parameters and Noether normalization. Algebra & Number Theory. 2019 December 7; 13(9):2081-2102. Available from: https://msp.org/ant/2019/13-9/p05.xhtml DOI: 10.2140/ant.2019.13.2081

Other Significant Products, Whether or Not Related to the Proposed Project

1. Bruce J, Cranton Heller L, Sayrafi M. Characterizing Multigraded Regularity on Products of Projective Spaces. arXiv [Preprint]. 2021 October 20 [revised 2022 January 03]. Available from: https://arxiv.org/abs/2110.10705 DOI: https://doi.org/10.48550/arXiv.2110.10705
2. Bruce J, Cranton Heller L, Sayrafi M. Bounds on Multigraded Regularity. arXiv [Preprint]. 2022 August . Available from: https://arxiv.org/abs/2208.11115 DOI: https://doi.org/10.48550/arXiv.2208.11115
3. Bruce J. Asymptotic Syzygies in the Setting of Semi-Ample Growth. arXiv [Preprint]. 2019 April 9. Available from: https://arxiv.org/abs/1904.04944 DOI: https://doi.org/10.48550/arXiv.1904.04944
4. Almousa A, Bruce J, Loper M, Sayrafi M. The virtual resolutions package for Macaulay2. The Journal of Software in Algebra and Geometry. 2020; 10:51-60. Available from: https://msp.org/jsag/2020/10-1/jsag-v10-n1-p06-p.pdf DOI: https://msp.org/jsag/2020/10-1/jsag-v10-n1-p06-p.pdf
5. Bruce J, Li W. Effective bounds on the dimensions of Jacobians covering abelian varieties. Proceedings of the American Mathematical Society. 2019; 148(2):535-551. Available from: https://www.ams.org/proc/2020-148-02/S0002-9939-2019-14756-0/ DOI: 10.1090/proc/14756

(d) SYNERGISTIC ACTIVITIES -(see PAPPG Chapter II.C.2.f.(d))

1. Conference Organizer M2@UW: I organized a 4-day conference, titled M2@UW, in April 2018 focused on creating new software packages for Macaulay2 — an open source computer algebra system — by bringing together developers and users of all skill levels and experiences. The conference had over 45 participants from all career stages (undergraduate students through tenured professors) from around the country. As an organizer, I created conference activities that promoted the development of collaborative relationships that cut across standard topic collaborations and involved a diverse group of researchers. Multiple published software packages and papers arose from work that began or continued at this conference.
2. Undergraduate Research Advisor: Since the Summer of 2021, I have mentored and advised Daniel Rostamloo -- an undergraduate at the University of California, Berkeley -- through two research projects. The first research project was part of an online summer undergrad research program where, together with five other undergraduate students, Daniel explored combinatorial questions related to the graphical matroid locus within the moduli space of tropical abelian varieties. Following the completion of this project, I began advising Daniel on a second research project proving sharp non-vanishing results for the syzygies of certain products of projective spaces. This project is ongoing, and I will hopefully result in a pre-print to be posted later this year. Daniel is now applying to graduate programs in mathematics.
3. Board Member (President) of Spectra, The Association for LGBTQ+ Mathematicians: Since the Fall of 2022 I have served as a board member for Spectra, The Association for LGBTQ+ Mathematicians through which I worked to make the mathematical community a more welcoming, inclusive, and supportive place for LGBTQ+ people. As a board member I oversaw

the growth and formalization of the organization including: our mailing/member lists reaching over 500 people, the creation and adoption of the groups first bylaws, the establishment of an invited lecture held annually at the Joint Mathematics Meetings, and the launch fundraising campaign that has raised over $20,000. Further, as a board member I led Spectra's efforts to work with publishers and other mathematically societies to adopt more inclusive publishing practices. In 2022 I am serving as the inaugural president of Spectra.

4. Organizer, Madison Math Circle: Between 2015 and 2018 I served as a volunteer, and eventually lead organizer, with the Madison Math Circle. When I began volunteering with the Madison Math Circle the circle's main programming was a weekly on-campus lecture given by a member of the math department. After roughly a year I stepped into the role of lead organizer overseeing the administrative needs of the circle. As an organizer, I worked to build stronger connections between the Madison Math Circle, local schools and teachers, and other outreach organizations focused on underrepresented groups. These ties helped the weekly attendance of the circle to more than double. I also led the creation of a new outreach arm of the circle, which visits high schools around the state of Wisconsin to better serve students from underrepresented groups. This dramatically expanded the reach of the Madison Math Circle, and during my final year as an organizer the circle reached over 300 students.
5. co-Organizer of Algebraic Geometry in the Time of COVID: In response to the COVID-19 pandemic causing many mathematical activities to ship to virtual formats I worked create online events and activities support those most likely to be harmed by the lose of in-person activities as well as those often at the periphery. During the Summer and Fall of 2020 I helped shepherd Ravi Vakil's Algebraic Geometry in the Time of Covid (AGITOC) project. This massive online open access course in algebraic geometry brought together over 2,000 registered participants at various career stages -- ranging high school and undergraduate students to tenured professors and people working in industry -- from around the world. This project consisted break the participants in to small discussion groups hosted on an online discussion platform together with 17 2-hour online lectures delivered by Ravi Vakil, with myself and other organizers helping add additional content to the lectures. As a co-organizer I helped process participants registration information, assigning them into their discussion groups. I also mentored a number of discuss groups totally roughly 200 students.

Other Personnel Biographical Information

Data Not Available

SUMMARY PROPOSAL BUDGET

YEAR 1

ORGANIZATION Brown University				FOR NSF USE ONLY PROPOSAL NO.	DURATION (months) Proposed	Granted
PRINCIPAL INVESTIGATOR / PROJECT DIRECTOR Juliette Bruce				AWARD NO.		

A. SENIOR PERSONNEL: PI/PD, Co-PI's, Faculty and Other Senior Associates (List each separately with title, A.7. show number in brackets)	NSF Funded Person-months CAL	ACAD	SUMR	Funds Requested By proposer	Funds granted by NSF (if different)
1. Juliette Bruce - Principal Inv	0.0			0	
2.					
3.					
4.					
5.					
6. ( ) OTHERS (LIST INDIVIDUALLY ON BUDGET JUSTIFICATION PAGE)	0.0			0	
7. (1) TOTAL SENIOR PERSONNEL (1 - 6)	0.0			0	
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)					
1. (0) POST DOCTORAL SCHOLARS	0.0			0	
2. (0) OTHER PROFESSIONALS (TECHNICIAN, PROGRAMMER, ETC.)	0.0			0	
3. (0) GRADUATE STUDENTS				0	
4. (3) UNDERGRADUATE STUDENTS				7,600	
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)				0	
6. (0) OTHER				0	
TOTAL SALARIES AND WAGES (A + B)				7,600	
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				450	
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A + B + C)				8,050	
D. EQUIPMENT (LIST ITEM AND DOLLAR AMOUNT FOR EACH ITEM EXCEEDING $5,000.) Computer $ 5000.0					
TOTAL EQUIPMENT				5,000	
E. TRAVEL 1. DOMESTIC (INCL. U.S. POSSESSIONS)				5,000	
2. INTERNATIONAL				7,000	
F. PARTICIPANT SUPPORT COSTS 1. STIPENDS $ 0 2. TRAVEL 0 3. SUBSISTENCE 0 4. OTHER 0					
TOTAL NUMBER OF PARTICIPANTS (0) TOTAL PARTICIPANT COSTS				0	
G. OTHER DIRECT COSTS					
1. MATERIALS AND SUPPLIES				0	
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION				0	
3. CONSULTANT SERVICES				0	
4. COMPUTER SERVICES				1,600	
5. SUBAWARDS				0	
6. OTHER				0	
TOTAL OTHER DIRECT COSTS				1,600	
H. TOTAL DIRECT COSTS (A THROUGH G)				26,650	
I. INDIRECT COSTS (F&A)(SPECIFY RATE AND BASE) 59.5% MTDC (Rate: 59.5, Base:21650.0)					
TOTAL INDIRECT COSTS (F&A)				12,882	
J. TOTAL DIRECT AND INDIRECT COSTS (H + I)				39,532	
K. FEE				0	
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				39,532	
M. COST SHARING PROPOSED LEVEL $ 0	AGREED LEVEL IF DIFFERENT $				

PI/PD NAME Juliette Bruce	FOR NSF USE ONLY INDIRECT COST RATE VERIFICATION		
ORG. REP. NAME*	Date Checked	Date Of Rate Sheet	Initials - ORG

***ELECTRONIC SIGNATURES REQUIRED FOR REVISED BUDGET**

SUMMARY PROPOSAL BUDGET

YEAR 2

ORGANIZATION Brown University				FOR NSF USE ONLY PROPOSAL NO.	DURATION (months) Proposed	Granted
PRINCIPAL INVESTIGATOR / PROJECT DIRECTOR Juliette Bruce				AWARD NO.		
A. SENIOR PERSONNEL: PI/PD, Co-PI's, Faculty and Other Senior Associates (List each separately with title, A.7. show number in brackets)	NSF Funded Person-months CAL	ACAD	SUMR	Funds Requested By proposer	Funds granted by NSF (if different)	
1. Juliette Bruce - Principal Inv	2.0			15,555		
2.						
3.						
4.						
5.						
6. ( ) OTHERS (LIST INDIVIDUALLY ON BUDGET JUSTIFICATION PAGE)	0.0			0		
7. (1) TOTAL SENIOR PERSONNEL (1 - 6)	2.0			15,555		
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)						
1. (0) POST DOCTORAL SCHOLARS	0.0			0		
2. (0) OTHER PROFESSIONALS (TECHNICIAN, PROGRAMMER, ETC.)	0.0			0		
3. (0) GRADUATE STUDENTS				0		
4. (3) UNDERGRADUATE STUDENTS				7,600		
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)				0		
6. (0) OTHER				0		
TOTAL SALARIES AND WAGES (A + B)				23,155		
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				5,039		
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A + B + C)				28,194		
D. EQUIPMENT (LIST ITEM AND DOLLAR AMOUNT FOR EACH ITEM EXCEEDING $5,000.) Computer $ 0.0						
TOTAL EQUIPMENT				0		
E. TRAVEL 1. DOMESTIC (INCL. U.S. POSSESSIONS)				5,000		
2. INTERNATIONAL				7,000		
F. PARTICIPANT SUPPORT COSTS 1. STIPENDS $ 0 2. TRAVEL 0 3. SUBSISTENCE 0 4. OTHER 0						
TOTAL NUMBER OF PARTICIPANTS (0) TOTAL PARTICIPANT COSTS				0		
G. OTHER DIRECT COSTS						
1. MATERIALS AND SUPPLIES				0		
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION				0		
3. CONSULTANT SERVICES				0		
4. COMPUTER SERVICES				1,600		
5. SUBAWARDS				0		
6. OTHER				0		
TOTAL OTHER DIRECT COSTS				1,600		
H. TOTAL DIRECT COSTS (A THROUGH G)				41,794		
I. INDIRECT COSTS (F&A)(SPECIFY RATE AND BASE) 59.5% MTDC (Rate: 59.5, Base:41794.0)						
TOTAL INDIRECT COSTS (F&A)				24,867		
J. TOTAL DIRECT AND INDIRECT COSTS (H + I)				66,661		
K. FEE				0		
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				66,661		
M. COST SHARING PROPOSED LEVEL $ 0	AGREED LEVEL IF DIFFERENT $					
PI/PD NAME Juliette Bruce				FOR NSF USE ONLY INDIRECT COST RATE VERIFICATION		
ORG. REP. NAME*				Date Checked	Date Of Rate Sheet	Initials - ORG

*ELECTRONIC SIGNATURES REQUIRED FOR REVISED BUDGET

SUMMARY PROPOSAL BUDGET

YEAR 3

ORGANIZATION Brown University	FOR NSF USE ONLY PROPOSAL NO.	DURATION (months) Proposed	Granted
PRINCIPAL INVESTIGATOR / PROJECT DIRECTOR Juliette Bruce	AWARD NO.		

A. SENIOR PERSONNEL: PI/PD, Co-PI's, Faculty and Other Senior Associates (List each separately with title, A.7. show number in brackets)	NSF Funded Person-months CAL	ACAD	SUMR	Funds Requested By proposer	Funds granted by NSF (if different)
1. Juliette Bruce - Principal Inv	2.0			16,178	
2.					
3.					
4.					
5.					
6. ( ) OTHERS (LIST INDIVIDUALLY ON BUDGET JUSTIFICATION PAGE)	0.0			0	
7. (1) TOTAL SENIOR PERSONNEL (1 - 6)	2.0			16,178	
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)					
1. (0) POST DOCTORAL SCHOLARS	0.0			0	
2. (0) OTHER PROFESSIONALS (TECHNICIAN, PROGRAMMER, ETC.)	0.0			0	
3. (0) GRADUATE STUDENTS				0	
4. (3) UNDERGRADUATE STUDENTS				7,600	
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)				0	
6. (0) OTHER				0	
TOTAL SALARIES AND WAGES (A + B)				23,778	
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				5,223	
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A + B + C)				29,001	
D. EQUIPMENT (LIST ITEM AND DOLLAR AMOUNT FOR EACH ITEM EXCEEDING $5,000.) Computer $ 0.0					
TOTAL EQUIPMENT				0	
E. TRAVEL 1. DOMESTIC (INCL. U.S. POSSESSIONS)				5,000	
2. INTERNATIONAL				7,000	
F. PARTICIPANT SUPPORT COSTS 1. STIPENDS $ 0 2. TRAVEL 0 3. SUBSISTENCE 0 4. OTHER 0					
TOTAL NUMBER OF PARTICIPANTS (0) TOTAL PARTICIPANT COSTS				0	
G. OTHER DIRECT COSTS					
1. MATERIALS AND SUPPLIES				0	
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION				0	
3. CONSULTANT SERVICES				0	
4. COMPUTER SERVICES				1,600	
5. SUBAWARDS				0	
6. OTHER				0	
TOTAL OTHER DIRECT COSTS				1,600	
H. TOTAL DIRECT COSTS (A THROUGH G)				42,601	
I. INDIRECT COSTS (F&A)(SPECIFY RATE AND BASE) 59.5% MTDC (Rate: 59.5, Base:42601.0)					
TOTAL INDIRECT COSTS (F&A)				25,348	
J. TOTAL DIRECT AND INDIRECT COSTS (H + I)				67,949	
K. FEE				0	
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				67,949	
M. COST SHARING PROPOSED LEVEL $ 0	AGREED LEVEL IF DIFFERENT $				

PI/PD NAME Juliette Bruce	FOR NSF USE ONLY INDIRECT COST RATE VERIFICATION		
ORG. REP. NAME*	Date Checked	Date Of Rate Sheet	Initials - ORG

***ELECTRONIC SIGNATURES REQUIRED FOR REVISED BUDGET**

SUMMARY PROPOSAL BUDGET

Cumulative

ORGANIZATION Brown University	FOR NSF USE ONLY PROPOSAL NO.	DURATION (months) Proposed	Granted	
PRINCIPAL INVESTIGATOR / PROJECT DIRECTOR Juliette Bruce	AWARD NO.			

A. SENIOR PERSONNEL: PI/PD, Co-PI's, Faculty and Other Senior Associates (List each separately with title, A.7. show number in brackets)	NSF Funded Person-months CAL	ACAD	SUMR	Funds Requested By proposer	Funds granted by NSF (if different)
1. Juliette Bruce - Principal Inv	4.0			31,733	
2.					
3.					
4.					
5.					
6. ( ) OTHERS (LIST INDIVIDUALLY ON BUDGET JUSTIFICATION PAGE)					
7. (1) TOTAL SENIOR PERSONNEL (1 - 6)	4.0			31,733	
B. OTHER PERSONNEL (SHOW NUMBERS IN BRACKETS)					
1. (0) POST DOCTORAL SCHOLARS	0.0			0	
2. (0) OTHER PROFESSIONALS (TECHNICIAN, PROGRAMMER, ETC.)	0.0			0	
3. (0) GRADUATE STUDENTS				0	
4. (9) UNDERGRADUATE STUDENTS				22,800	
5. (0) SECRETARIAL - CLERICAL (IF CHARGED DIRECTLY)				0	
6. (0) OTHER				0	
TOTAL SALARIES AND WAGES (A + B)				54,533	
C. FRINGE BENEFITS (IF CHARGED AS DIRECT COSTS)				10,712	
TOTAL SALARIES, WAGES AND FRINGE BENEFITS (A + B + C)				65,245	
D. EQUIPMENT (LIST ITEM AND DOLLAR AMOUNT FOR EACH ITEM EXCEEDING $5,000.)					
TOTAL EQUIPMENT				5,000	
E. TRAVEL 1. DOMESTIC (INCL. U.S. POSSESSIONS)				15,000	
2. INTERNATIONAL				21,000	
F. PARTICIPANT SUPPORT COSTS 1. STIPENDS $ 0 2. TRAVEL 0 3. SUBSISTENCE 0 4. OTHER 0					
TOTAL NUMBER OF PARTICIPANTS (0) TOTAL PARTICIPANT COSTS				0	
G. OTHER DIRECT COSTS					
1. MATERIALS AND SUPPLIES				0	
2. PUBLICATION COSTS/DOCUMENTATION/DISSEMINATION				0	
3. CONSULTANT SERVICES				0	
4. COMPUTER SERVICES				4,800	
5. SUBAWARDS				0	
6. OTHER				0	
TOTAL OTHER DIRECT COSTS				4,800	
H. TOTAL DIRECT COSTS (A THROUGH G)				111,045	
I. INDIRECT COSTS (F&A)(SPECIFY RATE AND BASE)					
TOTAL INDIRECT COSTS (F&A)				63,097	
J. TOTAL DIRECT AND INDIRECT COSTS (H + I)				174,142	
K. FEE				0	
L. AMOUNT OF THIS REQUEST (J) OR (J MINUS K)				174,142	

M. COST SHARING PROPOSED LEVEL $ 0	AGREED LEVEL IF DIFFERENT $		
PI/PD NAME Juliette Bruce	FOR NSF USE ONLY INDIRECT COST RATE VERIFICATION		
ORG. REP. NAME*	Date Checked	Date Of Rate Sheet	Initials - ORG

*ELECTRONIC SIGNATURES REQUIRED FOR REVISED BUDGET

Budget Justification

Senior Personnel:

The PI will devote 100% effort for 2 Summer Months per year in years 2 and 3 on the research projects proposed. This proposed level of commitment is appropriate to the scope of work and in order to fulfill the research objectives.

The PI is aware of the NSF policy limiting NSF support for senior personnel to two months in any year. For the purpose of this restriction, Brown University defines a year as a 12-month period spanning from July 1 to June 30.

Other Personnel:

2 Undergraduate Student Research Assistants per year (summer)

1 Undergraduate Student (academic year) to work on website development (approximately $1600 per year). The PI will hire a student to create and maintain a website that will be dedicated to fostering an inclusive research network for underrepresented groups and members of the LBGTQ community in mathematics.

Fringe Benefits:

PI (faculty summer) @ 29.5%
Undergraduate Student (summer) @7.5%

Travel:

Domestic:

The PI requests $4,000 per year for domestic travel. These funds are requested for the PI to attend research-related conferences, seminars, or workshops that may arise during the course of the grant research. She also plans to take 2 to 3 trips per year to meet with her collaborators at the U. of Minnesota (C. Berkesch and M. Sayrafi), U. of San Diego (A. Boocher), UC Berkeley (L. Cranton Heller, D. Eisenbud), U. of Wisconsin (D. Erman), Texas A&M (P. Klein), Washington U., St. Louis (W. Li), Cornell (R. Ramkumar) and San Francisco State (D. Ross).

Foreign:

The PI requests $5,000 per year for foreign travel. Funds are requested for the PI to attend international research-related conferences, seminars, and workshops that may arise during the course of the grant research. She also plans to take 2 trips per year to Europe to meet with collaborators at Technische Univeristat, Berlin (D. Corey), Edinburgh U. (M. Hering), and U. of

Warwick (D. Maclagan). In addition to these collaborators, she might also visit J. Levinson (Simon Fraser University, Canada) and B. Sturmfels (Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany).

Travel for Collaborators:

The PI requests $2,000 per year to invite her colleagues (such as mentioned in the domestic and foreign travel justification) to Brown to conduct joint mathematical research related to this grant. These colleagues and collaborators may also present their relevant work at seminars held in the Mathematics Department. The PI will pay a portion of their travel expenses.

Equipment:

Desktop Computer: The PI requests 5k purchase a high end desktop computer and peripherals in year one of the grant. The computer will be used to run dedicated computations on the work proposed.

Other Direct Costs:

Computing:

User fees to the Computation and Visualization at Brown University (CCV) for VM hosting of high speed multi-parallel computations. User fees are estimated at $400 per quarter annually.

Indirect Costs:

59.5% MTDC (years 1, 2 and 3)

*PI/co-PI/Senior Personnel Name: Juliette Bruce

***Required fields**

Note: NSF has provided 15 project/proposal and 10 in-kind contribution entries for users to populate. Please leave any unused entries blank.

Project/Proposal Section:

Current and Pending Support includes all resources made available to an individual in support of and/or related to all of his/her research efforts, regardless of whether or not they have monetary value.[1] Information must be provided about all current and pending support, including this project, for ongoing projects, and for any proposals currently under consideration from whatever source, irrespective of whether such support is provided through the proposing organization or is provided directly to the individual. This includes, for example, Federal, State, local, foreign, public or private foundations, non-profit organizations, industrial or other commercial organizations, or internal funds allocated toward specific projects. Concurrent submission of a proposal to other organizations will not prejudice its review by NSF, if disclosed.[2]

[1] If the time commitment or dollar value is not readily ascertainable, reasonable estimates should be provided.
[2] The Biological Sciences Directorate exception to this policy is delineated in PAPPG Chapter II.D.2.

Projects/Proposals

1.*Project/Proposal Title : FRG: Collaborative Research:Matroids, graphs, and algebraic geometry

*Status of Support : (•) Current ( ) Pending ( ) Submission Planned ( ) Transfer of Support

Proposal/Award Number (if available): DMS-2053221

*Source of Support: NSF

*Primary Place of Performance : Brown University

Project/Proposal Start Date (MM/YYYY) (if available) : 07/2021

Project/Proposal End Date (MM/YYYY) (if available) : 06/2024

*Total Award Amount (including Indirect Costs): $ 427,722

*Person-Month(s) (or Partial Person-Months) Per Year Committed to the Project

	*Year (YYYY)	*Person Months (##.##)
1.	2023	12.00
2.	2024	12.00
3.		

	Year (YYYY)	Person Months (##.##)
4.		
5.		

*Overall Objectives : The work of PIs Chan, Huh, Payne and Proudfoot is to build on recent breakthroughs to address fundamental questions at the interface between matroid theory, graph theory, and algebraic geometry.

*Statement of Potential Overlap : J. Bruce's role as Postdoctoral Research Associate on this project is related to combinatorial algebraic geometry, with a particular focus on Matroid complexes and the top-weight cohomology of A_g.

There is no overlap with the current proposal.

Projects/Proposals

2.*Project/Proposal Title : Multigraded Homological Algebra and Geometry (This Proposal)

*Status of Support : ○ Current ◉ Pending ○ Submission Planned ○ Transfer of Support

Proposal/Award Number (if available):

*Source of Support: NSF

*Primary Place of Performance : Brown University

Project/Proposal Start Date (MM/YYYY) (if available) : 07/2023

Project/Proposal End Date (MM/YYYY) (if available) : 06/2026

*Total Award Amount (including Indirect Costs): $ 174,142

*Person-Month(s) (or Partial Person-Months) Per Year Committed to the Project

	*Year (YYYY)	*Person Months (##.##)
1.	2023	0.00
2.	2024	2.00
3.	2025	2.00

	Year (YYYY)	Person Months (##.##)
4.		
5.		

*Overall Objectives : This proposal develops new connections between multigraded algebra and geometry. The first project expands our understanding of the relationship between minimal graded free resolutions and geometry by generalizing Green's conjecture on the syzygies of canonical curves. The second project is devoted to developing new tools in multigraded commutative algebra to gain a better understanding of the geometry of multigraded Hilbert schemes.

*Statement of Potential Overlap : The projects proposed do not overlap with the FRG Grant. These projects fall under the same broad area of mathematics, however the tools, goals and results are all entirely distinct.

Facilities, Equipment and Other Resources

Laboratory:

N/A

Clinical:

N/A

Animal:

N/A

Computer:

Brown University has made significant investments to promote analysis and handling of massive data sets. A key investment is the Center for Computation and Visualization (CCV) which supports the generation, storage, backup, analysis and visualization of large datasets as well as the transfer and sharing of data at high bandwidths. CCV's HPC cluster of more than 300 multi-core nodes, 1.7 PB of high-speed data storage connected with 100 Gigabits per second networks, and 1.5 PB of archival storage supports many big data researchers at Brown and elsewhere and is continually growing and being updated. CCV also operates a visualization lab which includes a state of the art 100 megapixel virtual reality system to display appropriate data sets.

Brown University also provides software licenses and access to Maple, Mathematica, Matlab, R, Python, CUDA, Perl, Julia, TensorFlow, Blender, Paraview, Tecplot, Intel compilers, PGI compilers, and other applications.

Space:

All faculty, graduate students, and visitors are provided with sufficient office space to conduct their teaching and research.

Data Management Plan for “Multigraded Homological Algebra and Geometry”

I envision the only data generated from this proposal to be

(1) mathematical articles and pre-prints, and
(2) computer code written for use in my research.

All mathematical articles will be made publicly available via the arXiv pre-print server, before eventually being submitted to peer-reviewed journals. Any computer code generated as part of this project, which might be interesting to others, will be made publicly accessible from the PI’s website or GitHub repository.

Data Not Available

Table 1

1	Your Name:	Your Organizational Affiliation(s), last 12 mo	Last Active Date
	Bruce, Juliette E	Brown University	
		University of California, Berkeley	07/31/22

Table 2

2	Name:	Type of Relationship	Optional (email, Department)	Last Active Date
R	Newton, Kit	Family	knewton@dvc.edu	

Table 3

3	Advisor/Advisee Name:	Organizational Affiliation	Optional (email, Department)
G	Erman, Daniel	University of Wisconsin, Madison	derman@math.wisc.edu
T			

Table 4

4	Name:	Organizational Affiliation	Optional (email, Department)	Last Active Date
A	Almousa, Ayah	University of Minnesota - Twin Cities	almou007@umn.edu	05/19/20
C	Berkesch, Christine	University of Minnesota - Twin Cities	cberkesc@umn.edu	
C	Bonato, Anthony	Ryerson University	abonato@ryerson.ca	
C	Boocher, Adam	University of San Diego	aboocher@sandiego.edu	
A	Brandt, Madeline	University of California, Berkeley	madeline_brandt@brown.edu	
C	Buckmire, Ron	Ocidental College	ron@oxy.edu	
C	Cavalieri, Renzo	Colorado State University	renzo@math.colostate.edu	08/01/22
A	Chan, Melody	Brown University	melody_chan@brown.edu	
A	Corey, Daniel	Technische Universität Berlin	corey@math.tu-berlin.de	
A	Cranton Heller, Lauren	University of California - Berkeley	lch@math.berkeley.edu	
A	Eisenbud, David	University of California - Berkeley	de@math.berkeley.edu	07/31/22
A	Erman, Daniel	University of Wisconsin - Madison	derman@math.wisc.edu	
A	Goldstein, Steve	University of Wisconsin - Madison	sgoldstein@wisc.edu	04/29/21
C	Hering, Milena	Edinburgh University	m.hering@ed.ac.uk	
C	Kelly, Tyler, L	University of Birmingham	t.kelly.1@bham.ac.uk	08/01/22
C	Klein, Patricia	Texas A&M University	pjklein@tamu.edu	
A	Laudone, Robert, P	University of Michigan	Robert.laudone@gmail.com	04/29/21
A	Li, Wanlin	Washington University in St. Louis	wanlin@wustl.edu	07/06/19
A	Loper, Michael, C	University of Wisconsin - River Falls	michael.loper@uwrf.edu	05/19/20

C	Maclagan, Diane	University of Warwick	D.Maclagan@warwick.ac.uk	
A	Melo, Margarida	Roma Tre University	melo@mat.uniroma3.it	07/01/22
A	Moreland, Gwyneth	Harvard University	gwynm@math.harvard.edu	07/01/22
C	Ramkumar, Ritvik	Cornell University	ritvikr@cornell.edu	
C	Ross, Dustin	San Francisco State University	rossd@sfsu.edu	
A	Sayrafi, Mahrud	University of Minnesota - Twin Cities	mahrud@umn.edu	
A	Wolfe, Corey	University of Tulane	cwolfe@tulane.edu	07/01/22
C	Voight, John	Dartmouth College	jvoight@gmail.com	08/01/22
A	Yang, Jay	McMaster University	yangj306@mcmaster.ca	04/29/21

Table 5

5	Name:	Organizational Affiliation	Journal/Collection	Last Active Date
B	Chapman, Scott	Sam Houston State University	Communications in Algebra	06/01/22

Data Not Available

List of Reviewers Not to Include

Data Not Available