Juliette Bruce's Research Statement

My research interests lie in algebraic geometry, commutative algebra, and arithmetic geometry. In particular, I am interested in using homological and combinatorial methods to study the geometry of zero loci of systems of polynomials (i.e. algebraic varieties). I am also interested in studying the arithmetic properties of varieties over finite fields. Further, I am passionate about promoting inclusivity, diversity, and justice in the mathematics community. Broadly speaking my current research follows these ideas in two directions.

- **Homological Algebra on Toric Varieties:** A classical story in algebraic geometry is that homological methods and tools like minimal free resolutions and Castelnuovo–Mumford regularity capture the geometry of subvarieties of projective space in nuanced ways. My work has sought to generalize this story by developing ways homological algebra can be used to study the geometry of toric varieties (i.e., "nice' compactifications of the torus $(\mathbb{C}^\times)^n$).
- **Cohomology of Moduli Spaces and Arithmetic Groups:** Despite its importance in algebraic geometry and number theory much remains unknown about the topology of $\mathcal{A}_g$, the moduli space of abelian varieties of dimension g. I have been working to study a canonical "part" of the cohomology of $\mathcal{A}_g$, called the top-weight cohomology. This turns out to be closely connected to the study of cohomology of various arithmetic groups like $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$, as well as the study of automorphic forms.

1. Homological Algebra on Toric Varieties

Given a graded module M over a graded ring R, a helpful tool for understanding the structure of M is its minimal graded free resolution. In essence, a minimal graded free resolution is a way of approximating M by a sequence of free R-modules. More formally, a *graded free resolution* of a module M is an exact sequence

$$\cdots \to F_k \xrightarrow{d_k} F_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

where each F_i is a graded free R-module, and hence can be written as $\bigoplus_j R(-j)^{\beta_{i,j}}$. The module $R(-j)$ is the ring R with a twisted grading, so that $R(-j)_d$ is equal to R_{d-j} where R_{d-j} is the graded piece of degree $d-j$. The $\beta_{i,j}$'s are the *Betti numbers* of M, and they count the number of i-syzygies of M of degree j. We will use syzygy and Betti number interchangeably throughout.

Given a projective variety X embedded in $\mathbb{P}^r$, we associate to X the ring $S_X = S/I_X$, where $S = \mathbb{C}[x_0, \ldots, x_r]$ and I_X is the ideal of homogenous polynomials vanishing on X. As S_X is naturally a graded S-module we may consider its minimal graded free resolution, which is often closely related to both the extrinsic and intrinsic geometry of X. An example of this phenomenon is Green's Conjecture, which relates the Clifford index of a curve with the vanishing of certain $\beta_{i,j}$ for its canonical embedding [Voi02, Voi05, AFP+19]. See also [Eis05, Conjecture 9.6] and [Sch86, BE91, FP05, Far06, AF11, FK16, FK17].

Much of my work can be viewed as understanding how minimal graded free resolutions capture the geometry when the role of $\mathbb{P}^r$ is replaced by another variety Y. In particular, I have focused on the case when Y is a toric variety, i.e., a compactification of the torus $(\mathbb{C}^\times)^r$ where the action of the torus extends to the boundary. Examples of toric varieties include projective space, products of projective spaces, and Hirzebruch surfaces. Work of Cox shows there is a correspondence between (toric) subvarieties of a fixed toric variety and quotients of a polynomial ring similar to the story discussed above for $\mathbb{P}^r$ [Cox95]. As such recent years have seen substantial work looking to use homological algebra and to better understand the geometry of toric varieties [ABLS20, BES20, BE22, BE23a, BB21, CEVV09, EES15, GVT15, MS04, MS05].

1.1 Asymptotic Syzygies Broadly speaking, asymptotic syzygies is the study of the graded Betti numbers (i.e. the syzygies) of a projective variety as the positivity of the embedding grows. In many ways, this perspective dates back to classical work on the defining equations of curves of high degree and projective normality [Mum66, Mum70]. However, the modern viewpoint arose from the pioneering work of Green [Gre84a, Gre84b] and later Ein and Lazarsfeld [EL12].

To give a flavor of the results of asymptotic syzygies we will focus on the question: In what degrees do non-zero syzygies occur? Going forward we will let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety embedded by a very ample line bundle L_d. Following [EY18] we set,

$$\rho_q(X, L_d) \quad := \quad \frac{\#\left\{p \in \mathbb{N}| \mid \beta_{p,p+q}(X, L_d) \neq 0\right\}}{r_d},$$

which is the percentage of degrees in which non-zero syzygies appear [Eis05, Theorem 1.1]. The asymptotic perspective asks how $\rho_q(X; L_d)$ behaves along the sequence of line bundles $(L_d)_{d\in\mathbb{N}}$.

With this notation in hand, we may phrase Green's work on the vanishing of syzygies for curves of high degree as computing the asymptotic percentage of non-zero quadratic syzygies.

Theorem 1.1. [Gre84a] *Let $X \subset \mathbb{P}^r$ be a smooth projective curve. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles on X such that $\deg L_d = d$ then $\lim_{d\to\infty} \rho_2(X; L_d) = 0$.*

Put differently, asymptotically the syzygies of curves are as simple as possible, occurring in the lowest possible degree. This inspired substantial work, with the intuition being that syzygies become simpler as the positivity of the embedding increases [OP01, EL93, LPP11, Par00, PP03, PP04].

In a groundbreaking paper, Ein and Lazarsfeld showed that for higher dimensional varieties this intuition is often misleading. Contrary to the case of curves, they show that for higher dimensional varieties, asymptotically syzygies appear in every possible degree.

Theorem 1.2. [EL12, Theorem C] *Let $X \subset \mathbb{P}^r$ be a smooth projective variety, $\dim X \geq 2$, and fix an index $1 \leq q \leq \dim X$. If $(L_d)_{d\in\mathbb{N}}$ is a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and ample then $\lim_{d\to\infty} \rho_q(X; L_d) = 1$.*

My work has focused on the behavior of asymptotic syzygies when the condition that $L_{d+1} - L_d$ is constant and ample is weakened to assuming $L_{d+1} - L_d$ is semi-ample. Recall a line bundle L is *semi-ample* if $|kL|$ is base point free for $k \gg 0$. The prototypical example of a semi-ample line bundle is $\mathcal{O}(1,0)$ on $\mathbb{P}^n \times \mathbb{P}^m$. My exploration of asymptotic syzygies in the setting of semi-ample growth thus began by proving the following nonvanishing result for $\mathbb{P}^n \times \mathbb{P}^m$ embedded by $\mathcal{O}(d_1, d_2)$.

Theorem 1.3. [Bru19, Corollary B] *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exist constants $C_{i,j}$ and $D_{i,j}$ such that*

$$\rho_q(X; \mathcal{O}(d_1, d_2)) \geq 1 - \sum_{\substack{i+j=q \\ i\leq n,\, j\leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\begin{pmatrix}\textit{lower ord.} \\ \textit{terms}\end{pmatrix}.$$

Notice if both $d_1 \to \infty$ and $d_2 \to \infty$ then $\rho_q(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2)) \to 1$, recovering the results of Ein and Lazarsfeld for $\mathbb{P}^n \times \mathbb{P}^m$. However, if d_1 is fixed and $d_2 \to \infty$ (i.e. semi-ample growth) my results bound the asymptotic percentage of non-zero syzygies away from zero. This together with work of Lemmens [Lem18] has led me to conjecture that, unlike in previously studied cases, in the semi-ample setting $\rho_q(\mathbb{P}^n \times \mathbb{P}^m; \mathcal{O}(d_1, d_2))$ does not approach 1. Proving this would require a vanishing result for asymptotic syzygies, which is open even in the ample case [EL12, Conjectures 7.1, 7.5].

The proof of Theorem 1.3 is based upon generalizing the monomial methods of Ein, Erman, and Lazarsfeld. Such a generalization is complicated by the difference between the Cox ring and homogenous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$. A central theme in this work is to exploit the fact that a key regular sequence I use has a number of non-trivial symmetries.

This work suggests that the theory of asymptotic syzygies in the setting of semi-ample growth is rich and substantially different from the other previously studied cases. Going forward I plan to use this work as a jumping-off point for the following question.

Question 1.4. *Let $X \subset \mathbb{P}^{r_d}$ be a smooth projective variety and fix an index $1 \leq q \leq \dim X$. Let $(L_d)_{d\in\mathbb{N}}$ be a sequence of very ample line bundles such that $L_{d+1} - L_d$ is constant and semi-ample, can one compute $\lim_{d\to\infty} \rho_q(X; L_d)$?*

A natural next case in which to consider Question 1.4 is that of Hirzebruch surfaces. I addressed a different, but related question for a narrow class of Hirzebruch surfaces in [Bru22].

1.2 Multigraded Castelnuovo–Mumford Regularity Introduced by Mumford, the Castelnuovo–Mumford Regularity of a projective variety $X \subset \mathbb{P}^r$ is a measure of the complexity of X given in terms of the vanishing of cohomology groups of X. Roughly, one should think about Castelnuovo–Mumford regularity as being a numerical measure of geometric complexity. Mumford was interested in such a measure as it plays a key role in constructing Hilbert schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. Eisenbud and Goto showed that regularity is also closely connected to interesting homological properties.

Theorem 1.5. [EG84] *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$ and $M = \bigoplus_{e\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent: (1) M is d-regular; (2) $\beta_{p,q}(M) = 0$ for all $p \geq 0$ and $q > d + i$; (3) $M_{\geq d}$ has a linear resolution.*

Maclagan and Smith introduced multigraded Castelnuovo–Mumford regularity, where $\mathbb{P}^r$ can be replaced by any toric variety. Similarly to the definition in the classical setting multigraded Castelnuovo–Mumford regularity is defined in terms of the vanishing of cohomology groups, however, the multigraded regularity of a subvariety or module is not a single number, but instead an infinite subset of $\mathbb{Z}^r$.

As an example, let us consider the case of products of projective spaces. Fixing a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ we let $\mathbb{P}^{\mathbf{n}} \coloneqq \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ and $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\mathrm{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$. Fixing some notation given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let $L_i(\mathbf{d}) \coloneqq \bigcup_{\mathbf{v}\in\mathbb{N},|\mathbf{v}|=i} (\mathbf{d} - \mathbf{v}) + \mathbb{N}^r$. Note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i+1)$-corners. Roughly speaking we define regularity by requiring the i-th cohomology of certain twists of $\mathcal{F}$ to vanish on L_i.

Definition 1.6. [MS04, Definition 6.1] *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if $H^i(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})) = 0$ for all $\mathbf{e} \in L_i(\mathbf{d})$. The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:*

$$\mathrm{reg}(\mathcal{F}) \coloneqq \left\{\mathbf{d} \in \mathbb{Z}^r \;\middle|\; \mathcal{F} \textit{ is } \mathbf{d}\textit{-regular}\right\} \subset \mathbb{Z}^r.$$

The obvious approaches to generalize Theorem 1.5 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity [BCHS21, Example 5.1] Despite this we show that part (3) of Theorem 1.5 can be generalized. To do so we introduce the following generalization of linear resolutions.

Definition 1.7. *A complex $F_\bullet$ of $\mathbb{Z}^r$-graded free S-modules is $\mathbf{d}$-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d} - \mathbf{1})$.*

Theorem 1.8. [BCHS21, Theorem A] *Let M be a (saturated) finitely generated $\mathbb{Z}^r$-graded S-module:*

$$M \textit{ is } \mathbf{d}\textit{-regular} \iff M_{\geq \mathbf{d}} \textit{ has a } \mathbf{d}\textit{-quasilinear resolution.}$$

The proof of Theorem 1.8 is based in part on a spectral sequence argument that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the Fourier–Mukai transform of $\widetilde{M}$ with Beilinson's resolution of the diagonal as the kernel. Recent breakthroughs [HHL23, BE23b] understanding resolutions of the diagonal on arbitrary toric varieties mean that there is hope one may be able to generalize the above argument to arbitrary toric varieties. With this in mind, I am interested in pursuing the following question

Question 1.9. *How can Theorem 1.8 be generalized to arbitrary smooth projective toric varieties? in particular, what is the correct definition of quasilinear resolutions?*

2. Cohomology of Moduli Spaces and Arithmetic Groups

Some of the most classical objects in algebraic geometry are moduli spaces, i.e., spaces that parameterize a given collection of geometric objects. The quintessential example of a moduli space is $\mathcal{M}_g$, the moduli space of (smooth) genus g curves, also known as the moduli space of compact Riemann surfaces of genus g. Despite their classical nature, much remains unknown about the geometry of many moduli spaces. For example, the rational cohomology of $\mathcal{M}_g$ is only known for $g \leq 4$. However, classical results suggest that $\mathcal{M}_g$ should have a lot of cohomology because its Euler characteristic grows super exponentially. Recent groundbreaking work of Chan, Galatius, and Payne has shed the first direct light on this phenomenon by constructing new non-trivial cohomology classes, and showing that the dimension of certain cohomology groups of $\mathcal{M}_g$ grow at least exponentially.

Theorem 2.1. [CGP21, Theorem 1.1] *For $g \geq 2$ the dimension of $H^{4g-6}(\mathcal{M}_g;\mathbb{Q})$ grows at least exponentially. In particular* $\dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) > \beta^g$ *for any real number $\beta < \beta_0$ where $\beta_0 \approx 1.3247\ldots$ is the real solution of $t^3 - t - 1 = 0$.*

Much of my recent work has sought to build up the groundwork laid by Chan, Galatius, and Payne to study the rational cohomology of other moduli spaces. Of particular, interest to me has been the moduli space of abelian varieties and various generalizations. This work has deep connections to the cohomology of various arithmetic groups like $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{GL}_g(\mathbb{Z})$.

2.1 Cohomology of $\mathcal{A}_g$ The moduli space of (principally polarized) abelian varieties of dimension g, is a smooth variety $\mathcal{A}_g$ (truthfully a smooth Delinge–Mumford stack) whose points are in one to one correspondence with isomorphism classes of principally polarized abelian varieties of dimension g. Concretely, we may view it as the quotient $[\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})]$ where H_g is the Siegel upper half-space. Notice this means that $\mathcal{A}_g$ is a rational classifying space for the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Similar to the moduli space of curves $\mathcal{A}_g$ has long been studied, but much remains unknown about its geometry. For example, the (singular) cohomology is $\mathcal{A}_g$ is only fully known for $g \leq 3$, with $g = 0, 1$ being relatively easy, $g = 2$ which is a classical result of Igusa, and $g = 3$ due to work of Hain. In fact the cohomology of $\mathcal{A}_g$ is so mysterious until recently work by myself and co-authors it was unknown whether $H^{2i+1}(\mathcal{A}_g;\mathbb{Q}) \neq 0$ for some g and i. This was a question originally posed by Grushevsky.

Building upon the work of Chan, Galatius, and Payne, my co-authors and I developed new methods for understanding a certain canonical quotient of the cohomology of $\mathcal{A}_g$. In particular, our results construct non-trivial cohomology classes in $H^k(\mathcal{A}_g;\mathbb{Q})$ in a number of new cases.

Theorem 2.2. [BBC+22, Theorem A] *The rational cohomology $H^k(\mathcal{A}_g;\mathbb{Q}) \neq 0$ for:*

$$(g,k) = (5,15), (5,20), (6,30), (7,28), (7,33), (7,37), \text{ and } (7,42).$$

For broader context, since $\mathcal{A}_g$ is a rational classifying space for $\mathrm{Sp}_{2g}(\mathbb{Z})$ there is natural isomorphism $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$. In particular, the above results provide new non-vanishing results for $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$. However, my work takes advantage of the fact that since $\mathcal{A}_g$ is a smooth and separated Deligne Mumford stack with a coarse moduli space which is an algebraic variety, permitting Deligne's mixed Hodge theory to be applied to study the rational cohomology of these groups. In particular, the rational cohomology of a complex algebraic variety X of dimension d admits a weight filtration with graded pieces $\mathrm{Gr}^W_j H^k(X;\mathbb{Q})$. As $\mathrm{Gr}^W_j H^k(X;\mathbb{Q})$ vanishes whenever $j > 2d$, $\mathrm{Gr}^W_{2d} H^k(X;\mathbb{Q})$ is referred to as the *top-weight* part of $H^k(X;\mathbb{Q})$.) In this way we deduce Theorem 2.2 above as a corollary to computing the top-weight cohomology of $\mathcal{A}_g$ for all $g \leq 7$.

2.2 Cohomology of $\mathcal{A}_g(m)$ The moduli space $\mathcal{A}_g$ actually a special instance of the moduli space of (principally polarized) abelian varieties of dimension g with level m-structure. Denoted by $\mathcal{A}_g(m)$, we may view it as the quotient $[\mathbb{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})](m)$ where $\mathrm{Sp}_{2g}(\mathbb{Z})(m)$ is the principal congruence subgroup $\ker(\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$. Note that when $m = 1$, we have that $\mathcal{A}_g(m)$ is isomorphic to $\mathcal{A}_g$. From this perspective, one may hope to generalize Theorem 2.2 and underlying methods my co-authors and I developed in [BBC+22] to studying the rational cohomology of $\mathcal{A}_g(m)$ and $\mathrm{Sp}_{2g}(\mathbb{Z})(m)$. In ongoing work, Melody Chan and I are developing such generalizations.

Goal Theorem 2.3. *Let $d = \binom{g+1}{2}$ be the dimension of $\mathcal{A}_g(m)$. For any integers $m \geq 1$ and $g \geq 0$ there exists a cellular complex $LA_g(m)^{\mathrm{trop}}$ such that for all $i \geq 0$ there is a natural isomorphism*

$$\tilde{H}_{i-1}\left(LA_g(m)^{\mathrm{trop}};\mathbb{Q}\right) \cong \mathrm{Gr}^W_{2d} H^i\left(\mathcal{A}_g(m);\mathbb{Q}\right),$$

The methods behind Goal Theorem 2.3 show new connections between the cohomology of $\mathcal{A}_g(m)$ and the cohomology of $\mathrm{GL}_g(\mathbb{Z})(m)$. The cohomology of $\mathrm{Sp}(2g,\mathbb{Z})(m)$ – and hence $\mathcal{A}_g(m)$ – and $\mathrm{GL}_g(\mathbb{Z})(m)$ are closely connected to automorphic forms. It is natural to wonder whether our methods for computing the top-weight cohomology of $\mathcal{A}_g(m)$ shed new light on automorphic forms. In an ongoing conversation with, Melody Chan, and Peter Sarnak I hope to address this question.

Question 2.4. *What is the relationship between the top-weight cohomology of $\mathcal{A}_g(m)$ and Siegel–Eisenstein series?*