The Degree of SO($n, \mathbb{C}$)

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Abstract We provide a closed formula for the degree of $\mathrm{SO}(n, \mathbb{C})$. In addition, we test symbolic and numerical techniques for computing the degree of $\mathrm{SO}(n, \mathbb{C})$. As an application of our results, we give a formula for the number of critical points of a low-rank semidefinite programming problem. Finally, we provide evidence for a conjecture regarding the real locus of $\mathrm{SO}(n, \mathbb{C})$.

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1 Introduction

The *special orthogonal group* $\mathrm{SO}(n, \mathbb{R})$ is the group of automorphisms of $\mathbb{R}^n$ which preserve the standard inner product and have determinant equal to one. The complex special orthogonal group is the complexification of $\mathrm{SO}(n, \mathbb{R})$ or, more explicitly, the

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group of matrices $\mathrm{SO}(n,\mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : \det(M) = 1 \text{ and } M^{\mathsf{T}}M = I\}$. Since the defining conditions are polynomials in the entries of the matrix M, the group $\mathrm{SO}(n,\mathbb{C})$ is also a complex variety.

The degree of a complex variety $X \subset \mathbb{C}^n$ is the generic number of points in the intersection of X with a linear space of complementary dimension. Problem 4 on Grassmannians in [19] seeks a formula for the degree of $\mathrm{SO}(n,\mathbb{C})$. Our main result provides this.

Theorem 1.1 *The degree of* $\mathrm{SO}(n,\mathbb{C})$ *equals* $2^{n-1}\det\left[\binom{2n-2i-2j}{n-2i}\right]_{1\le i,j\le\lfloor n/2\rfloor}$.

Our proof of Theorem 1.1 uses a formula of Kazarnovskij [14] for the degree of the image of a representation of a connected reductive algebraic group over an algebraically closed field; see Theorem 2.4 for more information. By applying this formula to the case of the standard representation of $\mathrm{SO}(n,\mathbb{C})$, we are able to express the degree in terms of its root data and other invariants. As an added feature, Theorem 4.2 provides a combinatorial interpretation of this degree in terms of non-intersecting lattice paths. In contrast with Theorem 1.1, the combinatorial statement has the benefit of being obviously non-negative.

In order to verify Theorem 1.1, as well as explore the structure of $\mathrm{SO}(n,\mathbb{C})$ in further depth, it is useful to compute this degree explicitly. We were able to do this, for small n, using symbolic and numerical computations. A comparison of the success of these approaches is illustrated in Table 1.

Remark 1.2 Let $\Bbbk$ be a field of characteristic zero. We can define $\mathrm{SO}(n,\Bbbk)$ using the same system of equations because they are defined over the prime field $\mathbb{Q}$. For a field $\Bbbk$ that is not algebraically closed, the degree of a variety can be defined in terms of the Hilbert series of its coordinate ring. Since the Hilbert series does not depend on the choice of $\Bbbk$, the degree does not either. We choose to work over $\mathbb{C}$ not only for simplicity, but also so that we may use the above definition of degree.

Remark 1.3 Our methods are not restricted to $\mathrm{SO}(n,\mathbb{C})$ and can be used to compute the degree of other algebraic groups. For example, we provide a similar closed formula for the degree of the symplectic group in Sect. 3 and a combinatorial reinterpretation in Sect. 4.

Table 1 Degree of $\mathrm{SO}(n,\mathbb{C})$ computed in various ways

n	Symbolic	Numerical	Formula
2	2	2	2
3	8	8	8
4	40	40	40
5	384	384	384
6	-	4768	4768
7	-	111616	111616
8	-	-	3433600
9	-	-	196968448

This project started in the spring of 2014, when Benjamin Recht asked the fifth author to describe the geometry of a low-rank optimization problem; see Sect. 5. In particular, Benjamin asked why the augmented Lagrangian algorithm for solving this problem [5] almost always recovers the correct optimum despite the existence of multiple local minima. It quickly became clear that to even compute the number of local extrema, one needs to know the degree of the orthogonal group. In Sect. 5, we find a formula for the number of critical points of the low-rank semidefinite programming problem; see Theorem 5.3.

The rest of this article is organized as follows. In Sect. 2, we give the reader a brief introduction to algebraic groups and state the Kazarnovskij Theorem. Section 3 proves Theorem 1.1 by applying the Kazarnovskij Theorem and simplifying the resulting expressions. After simplification, we are left with a determinant of binomial coefficients that can be interpreted combinatorially using the celebrated Gessel–Viennot Lemma; see Sect. 4. The relationship between the degree of SO($n, \mathbb{C}$) and the degree of the low-rank optimization programming problem is elaborated upon in Sect. 5. Section 6 contains descriptions of the symbolic and numerical techniques involved in the explicit computation of $\deg \mathrm{SO}(n, \mathbb{C})$. Finally, in Sect. 7, we explore questions involving the real points on SO($n, \mathbb{C}$).

2 Background

In this section, we provide the reader with the language to understand the Kazarnovskij Theorem, our main tool for determining the degree of SO($n, \mathbb{C}$). We invite those who already are familiar with Lie theory to skip to the statement of Theorem 2.4. Aside from applying Theorem 2.4, no understanding of the material in this section is necessary for understanding the remainder of the proof of Theorem 1.1. A more thorough treatment of the theory of algebraic groups can be found in [6, 8, 13].

An *algebraic group* G is a variety equipped with a group structure such that multiplication and inversion are both regular maps on G. When the unipotent radical of G is trivial and G is over an algebraically closed field, we say that G is a *reductive group*. Throughout this section, we let G denote a connected reductive algebraic group over an algebraically closed field $\Bbbk$. Let $\mathbb{G}_{\mathrm{m}}$ denote the multiplicative group of $\Bbbk$, so as a set $\mathbb{G}_{\mathrm{m}} = \Bbbk \setminus \{0\}$. Let T denote a fixed maximal torus of G, that is a subgroup of G isomorphic to $\mathbb{G}_{\mathrm{m}}^r$ and which is maximal with respect to inclusion. The number $r \in \mathbb{N}$ is well-defined and is called the *rank* of G. After fixing T, we define the *Weyl group* of G, denoted $W(G)$, to be the quotient of the normalizer of T by its centralizer: $W(G) := N_G(T)/Z_G(T)$. Like the rank, the group $W(G)$ does not depend on the choice of T up to isomorphism.

Example 2.1 The map $\mathrm{R}\colon \mathbb{G}_{\mathrm{m}} \to \mathrm{SO}(2,\Bbbk)$, given by $\mathrm{R}(t) := \frac{1}{2}\begin{bmatrix} t+t^{-1} & -i(t-t^{-1}) \\ i(t-t^{-1}) & t+t^{-1} \end{bmatrix}$, parametrizes $\mathrm{SO}(2,\Bbbk)$ and is a group isomorphism. If $\Bbbk = \mathbb{C}$, then the rotation by an angle θ corresponds to the matrix $\mathrm{R}(e^{i\theta})$. Therefore, the algebraic group $\mathrm{SO}(2,\Bbbk)$ has rank 1.

If $r \geq 1$, then the maximal tori of rank r in their respective algebraic groups are

$$T_{2r} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) \end{bmatrix} : t_i \in \mathbb{G}_{\mathrm{m}} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r,\Bbbk)\,,$$

$$T_{2r+1} := \left\{ \begin{bmatrix} \mathrm{R}(t_1) & 0 & 0 \cdots & 0 & 0 \\ 0 & \mathrm{R}(t_2) & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \mathrm{R}(t_r) & 0 \\ 0 & 0 & 0 \cdots & 0 & 1 \end{bmatrix} : t_i \in \mathbb{G}_{\mathrm{m}} \right\} \cong \mathrm{SO}(2,\Bbbk)^r \subset \mathrm{SO}(2r+1,\Bbbk)\,.$$

Therefore, we have $\operatorname{rank}\mathrm{SO}(2r,\Bbbk) = \operatorname{rank}\mathrm{SO}(2r+1,\Bbbk) = r$ and see that the rank of $\mathrm{SO}(n,\Bbbk)$ depends on the parity of n.

The *character group* $M(T)$ is the set of algebraic group homomorphisms from T to $\mathbb{G}_{\mathrm{m}}$. In other words, $M(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(T, \mathbb{G}_{\mathrm{m}})$ consists of the group homomorphisms defined by polynomial maps. Since T is isomorphic to $\mathbb{G}_{\mathrm{m}}^r$, all such homomorphisms must be of the form $(t_1, t_2, \ldots, t_r) \mapsto t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r}$ for some integers $a_1, a_2, \ldots, a_r$. Hence, the character group $M(T)$ is isomorphic to $\mathbb{Z}^r$ and, for this reason, it is often called the character lattice. The *group of 1-parameter subgroups* $N(T) := \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_{\mathrm{m}}, T)$ is dual to $M(T)$ and is also isomorphic to $\mathbb{Z}^r$. Indeed, each 1-parameter subgroup is of the form $t \mapsto (t^{b_1}, t^{b_2}, \ldots, t^{b_r})$ for some integers $b_1, b_2, \ldots, b_r$. Moreover, there exists a natural bilinear pairing $M(T) \times N(T) \to \mathrm{Hom}_{\mathrm{AlgGrp}}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$ given by $\langle \chi, \sigma \rangle \mapsto \chi \circ \sigma$.

Now, if $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then we attach to it special characters called weights. A *weight* of the representation ρ is a character $\chi \in M(T)$ such that the set

$$V_\chi := \bigcap_{s \in T} \ker\bigl(\rho(s) - \chi(s)\,\mathrm{Id}_V\bigr)$$

is non-trivial. This condition is equivalent to saying that all of the matrices in $\{\rho(s) : s \in T\}$ have a simultaneous eigenvector $v \in V$ such that the associated eigenvalue for $\rho(s)$ is $\chi(s)$. We write C_V for the convex hull of the weights of the representation ρ.

Example 2.2 An important example for us comes from the defining representation $\rho\colon \mathrm{SO}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$. Let $e_1, e_2, \ldots, e_n$ denote the standard basis for $\mathbb{C}^n$. For any $t \in \mathbb{G}_{\mathrm{m}}$, the matrix $\mathrm{R}(t) \in \mathrm{SO}(2, \mathbb{C})$ has eigenvectors $e_1 + ie_2$ and $e_1 - ie_2$ with eigenvalues t and t^{-1} respectively. From the explicit description of the maximal torus T in Example 2.1, it follows that the eigenvectors of ρ are all vectors of the form $e_{2j-1} \pm ie_{2j}$ with $1 \leq j \leq r$ and the corresponding eigenvalues are $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}$. These eigenvalues, viewed as characters, are the weights of ρ. Additionally, when $n = 2r + 1$, we see that e_{2r+1} is an eigenvector with eigenvalue 1, corresponding to the trivial character.

Another representation of a matrix group $G \subseteq \mathrm{End}(V)$ is the *adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{GL}\big(\mathrm{End}(V)\big)$, where $\mathrm{Ad}(g)$ the linear map defined by $A \mapsto gAg^{-1}$. The *roots* of G are the nonzero weights of the adjoint representation. Given a linear functional ℓ on $M(T)$, we define the *positive roots* of G with respect to ℓ to be the roots χ such that $\ell(\chi) > 0$. We denote the positive roots of G by $\alpha_1, \alpha_2, \ldots, \alpha_l$. For the algebraic groups in this paper, we can choose ℓ to be the inner product with the vector $(r, r-1, \ldots, 1)$, so that a root of the form $e_j - e_k$ is positive if and only if $j < k$. To each root α, we associate a *coroot* $\check{\alpha}$, defined to be the linear function $\check{\alpha}(\mathbf{x}) := 2\langle \mathbf{x}, \alpha\rangle / \langle \alpha, \alpha\rangle$ where the pairing is $W(G)$-invariant. Throughout this paper, we fix the pairing to be the standard inner product.

Example 2.3 We now describe the roots of $\mathrm{SO}(n, \mathbb{C})$, starting with $n = 2r$. The simultaneous eigenvectors of $\mathrm{Ad}(s)$ over all $s \in T$ are matrices A with the following structure. These matrices are zero outside a (2×2)-block B in rows $2j-1, 2j$ and columns $2k-1, 2k$ for some $1 \leq j, k \leq r$. Furthermore, $B = v_1 v_2^{\mathsf{T}}$ with each vector v_k, for $1 \leq k \leq 2$, equals one of the eigenvectors of $\mathrm{R}(t)$, namely $e_1 \pm ie_2$. If $s \in T$ has blocks along the diagonal $\mathrm{R}(t_j)$ with $t_1, t_2, \ldots, t_r \in \mathbb{G}_{\mathrm{m}}$, then the matrix $\mathrm{Ad}(s)(A)$ will also be zero except in the same (2×2)-block, and that block will be $\mathrm{R}(t_j) B\, \mathrm{R}(t_k)^{\mathsf{T}} = t_j^{\pm 1} t_k^{\pm 1} B$, where the signs depend on the choices of v_1 and v_2. Taking the exponent vectors of these eigenvalues, we see that the roots of $\mathrm{SO}(2r, \mathbb{C})$ are the characters of the form $\pm(e_j \pm e_k)$ for $1 \leq j, k \leq r$.

When $n = 2r + 1$, the matrix A has an extra row and column. If the matrix A has support only in the last column, then we have $\mathrm{Ad}(s)(A) = sAs^{-1}$. But s^{-1} acts trivially on the left, while s acts on the last column as an element of $\mathrm{GL}(n, \mathbb{C})$ as in the standard representation. As in Example 2.2, the eigenvalues are $t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}, 1$. The same weights appear for A with support in the last row. Hence, the roots of $\mathrm{SO}(2r + 1, \mathbb{C})$ are $\pm(e_j \pm e_k)$ for $1 \leq j, k \leq r$ and $\pm e_i$ for $1 \leq i \leq r$.

Associated to the algebraic group G is a Lie algebra $\mathfrak{g}$ that comes equipped with a Lie bracket $[\cdot, \cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing; if $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$, then we have $y \in \mathfrak{h}$. Let $S(\mathfrak{h}^*)$ be the ring of polynomial functions on $\mathfrak{h}$. The Weyl group $W(G)$ acts on $\mathfrak{h}$, and this extends to an action of $W(G)$ on $S(\mathfrak{h}^*)$. The space $S(\mathfrak{h}^*)^{W(G)}$ of polynomials which are invariant up to the action of $W(G)$ is generated by r homogeneous polynomials whose degrees, $c_1 + 1, c_2 + 1, \ldots, c_r + 1$, are uniquely determined. The values $c_1, c_1, \ldots, c_r$ are called *Coxeter exponents*.

Table 2 Data required to apply the Kazarnovskij Theorem

Group	Dimension	Rank	Positive roots	Weights	$\lvert W(G)\rvert$	Coxeter exponents
$\mathrm{SO}(2r+1,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1,3,5,\ldots,2r-1$
$\mathrm{Sp}(2r,\mathbb{C})$	$\binom{2r+1}{2}$	r	$\{e_i \pm e_j\}_{i<j} \cup \{2e_i\}$	$\{\pm e_i\}$	$r!2^r$	$1,3,5,\ldots,2r-1$
$\mathrm{SO}(2r,\mathbb{C})$	$\binom{2r}{2}$	r	$\{e_i \pm e_j\}_{i<j}$	$\{\pm e_i\}$	$r!2^{r-1}$	$1,3,5,\ldots,2r-3,r-1$

We are now prepared to state the Kazarnovskij Theorem.

Theorem 2.4 (Kazarnovskij Theorem, [6, Proposition 4.7.18]) *Let G be a connected reductive algebraic group of dimension m and rank r over an algebraically closed field. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation with finite kernel, then we have*

$$\deg\left(\overline{\rho(G)}\right) = \frac{m!}{|W(G)|\,(c_1!c_2!\cdots c_r!)^2\,|\ker(\rho)|}\int_{C_V}(\check{\alpha}_1\check{\alpha}_2\cdots\check{\alpha}_r)^2\,dv\,,$$

where $W(G)$ is the Weyl group, the c_i are Coxeter exponents, C_V is the convex hull of the weights, and the $\check{\alpha}_i$ are the coroots.

If ρ is the standard representation for an algebraic group G, then it follows that $\deg\overline{\rho(G)} = \deg G$. Thus, in order to compute $\deg \mathrm{SO}(n,\mathbb{C})$, all we must do is apply this theorem for the standard representation of $\mathrm{SO}(n,\mathbb{C})$. The relevant data for this theorem is given in Table 2 for $\mathrm{SO}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$.

3 Main Result: The Degree of $\mathrm{SO}(n,\mathbb{C})$

We now prove our main result, Theorem 1.1. At the end of this section, we also use the same method to obtain a formula for the degree of the symplectic group.

We begin by applying Theorem 2.4 to $\mathrm{SO}(2r,\mathbb{C})$ and $\mathrm{SO}(2r+1,\mathbb{C})$ to obtain

$$\deg \mathrm{SO}(2r,\mathbb{C}) = \frac{\binom{2r}{2}!}{r!2^{r-1}\left((r-1)!\right)^2\prod_{k=1}^{r-1}\left((2k-1)!\right)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\,dv\,,$$

$$\deg \mathrm{SO}(2r+1,\mathbb{C}) = \frac{\binom{2r+1}{2}!}{r!2^{r}\prod_{k=1}^{r}\left((2k-1)!\right)^2}\int_{C_V}\prod_{1\le i<j\le r}(x_i^2-x_j^2)^2\prod_{i=1}^{r}(2x_i)^2\,dv\,.$$

To compute the degree of $\mathrm{SO}(n,\mathbb{C})$, it suffices to find formulas for these integrals. We do this by expanding the integrand into monomials and integrating the result. We first use the well-known expression for the determinant of the Vandermonde matrix:

$$\prod_{1\le i<j\le r}(y_j-y_i) = \sum_{\sigma\in\mathfrak{S}_r}\mathrm{sgn}(\sigma)\prod_{i=1}^{r}y_i^{\sigma(i)-1}\,,$$

where $\mathfrak{S}_r$ denotes the symmetric group on $\{1, 2, \ldots, r\}$. Substituting $y_i = x_i^2$ and squaring the entire expression yields

$$\prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 = \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-4}. \tag{1}$$

Every variable in the integrand is being raised to an even power, and C_V is the convex hull of weights $\{\pm e_i\}$. Because of this symmetry, the integrals over C_V are 2^r times the same integrals over the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \ldots, e_r) \subset \mathbb{R}^r$. Hence, we have reduced the computation to understanding the integral of any monomial over the simplex Δ_r. The following lemma provides the required formula.

Lemma 3.1 ([15, Lemma 4.23]) *Consider the r-simplex $\Delta_r := \operatorname{conv}(0, e_1, e_2, \ldots, e_r)$ in $\mathbb{R}^r$. If $\mathbf{a} = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r_{>0}$, then we have*

$$\int_{\Delta_r} \mathbf{x}^{\mathbf{a}}\, d\mathbf{x} = \int_{\Delta_r} x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}\, dx_1 dx_2 \cdots dx_r = \frac{1}{(r + \sum_i a_i)!} \prod_i a_i!\,.$$

With these preliminaries, we can now prove the key technical result in this section.

Proposition 3.2 *We have*

$$\int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2\, dv = \frac{r!2^r}{\binom{2r}{2}!} \det\left[(2i + 2j - 4)!\right]_{1 \le i,j \le r},$$

$$\int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv = \frac{r!2^{3r}}{\binom{2r+1}{2}!} \det\left[(2i + 2j - 2)!\right]_{1 \le i,j \le r}.$$

Proof Exploiting the symmetry of C_v along with equation (1) gives

$$I_{\text{odd}}(r) := \int_{C_V} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv = 2^r \int_{\Delta_r} \prod_{1 \le i < j \le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2\, dv$$

$$= 2^{3r} \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \int_{\Delta_r} \prod_{i=1}^{r} x_i^{2\sigma(i)+2\tau(i)-2}\, dv\,.$$

As the integrand is homogeneous of degree $4\binom{r}{2} + 2r$, Lemma 3.1 yields

$$I_{\text{odd}}(r) = \frac{2^{3r}}{\left(4\binom{r}{2} + 3r\right)!} \sum_{\sigma, \tau \in \mathfrak{S}_r} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{r} \big(2\sigma(i) + 2\tau(i) - 2\big)!\,.$$

Since $\sigma \in \mathfrak{S}_r$, we may reindex the product by $\sigma^{-1}(i)$ rather than i to obtain $\prod_{i=1}^{r}\big(2\sigma(i)+2\tau(i)-2\big)! = \prod_{i=1}^{r}\big(2i+2\tau\sigma^{-1}(i)-2\big)!$. Ranging over all $\sigma, \tau \in \mathfrak{S}_r$, each permutation in $\mathfrak{S}_r$ appears exactly $r!$ times as the composition $\upsilon := \tau\sigma^{-1}$ and $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\upsilon)$. Therefore, we have

$$\begin{aligned} I_{\text{odd}}(r) &= \frac{r!2^{3r}}{\big(4\binom{r}{2}+3r\big)!} \sum_{\upsilon\in\mathfrak{S}_r} \operatorname{sgn}(\upsilon) \prod_{i=1}^{r} (2i+2\upsilon(i)-2)! \\ &= \frac{r!2^{3r}}{\binom{2r+1}{2}!} \det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

The calculation for $I_{\text{even}}(r) := \int_{C_V} \prod_{1\le i<j\le r}(x_i^2 - x_j^2)^2 \, dv$ follows the same steps. □

Proof of Theorem 1.1 Combining Theorem 2.4, the data in Table 2, and Proposition 3.2, we have

$$\begin{aligned} \deg \mathrm{SO}(2r+1,\mathbb{C}) &= \frac{\binom{2r+1}{2}!}{r!2^r \prod_{k=1}^{r}\big((2k-1)!\big)^2} \int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r} (2x_i)^2 \, dv \\ &= \frac{2^{2r}}{\prod_{k=1}^{r}\big((2k-1)!\big)^2} \det\big[(2i+2j-2)!\big]_{1\le i,j\le r}. \end{aligned}$$

Since the determinant is linear in each row and column, we obtain

$$\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r} \det\left[\frac{(2i+2j-2)!}{(2i-1)!(2j-1)!}\right] = 2^{2r} \det\left[\binom{2i+2j-2}{2i-1}\right]_{1\le i,j\le r}.$$

Reversing the order of the rows and columns of the final matrix and reindexing produces the required formula. Similarly, for the even case, we have

$$\begin{aligned} \deg \mathrm{SO}(2r,\mathbb{C}) &= \frac{\binom{2r}{2}!}{r!2^{r-1}\big((r-1)!\big)^2 \prod_{k=1}^{r-1}\big((2k-1)!\big)^2} \int_{C_V} \prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \, dv \\ &= \frac{2}{\big((r-1)!\big)^2 \prod_{k=1}^{r-1}\big((2k-1)!\big)^2} \det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \\ &= \frac{2(2^{r-1})^2}{\prod_{k=1}^{r-1}(2k)^2 \prod_{k=1}^{r-1}\big((2k-1)!\big)^2} \det\big[(2i+2j-4)!\big]_{1\le i,j\le r} \end{aligned}$$

$$= \frac{2^{2r-1}}{\prod_{k=1}^{r}\big((2k-2)!\big)^2}\det\big[(2i+2j-4)!\big]_{1\le i,j\le r}$$

$$= 2^{2r-1}\det\left[\frac{(2i+2j-4)!}{(2i-2)!(2j-2)!}\right]_{1\le i,j\le r}$$

$$= 2^{2r-1}\det\left[\binom{2i+2j-4}{2i-2}\right] = 2^{2r-1}\det\left[\binom{4r-2i-2j}{2r-2i}\right]_{1\le i,j\le r}.$$

□

Since the orthogonal group $\mathrm{O}(n,\mathbb{C}) := \{M \in \mathbb{C}^{n\times n} : M^{\mathsf{T}}M = MM^{\mathsf{T}} = I\}$ has two connected components that are isomorphic to $\mathrm{SO}(n,\mathbb{C})$, we immediately get a formula for the degree of $\mathrm{O}(n,\mathbb{C})$.

Corollary 3.3 *The degree of* $\mathrm{O}(n,\mathbb{C})$ *equals* $2^n\det\big[\binom{2n-2i-2j}{n-2i}\big]_{1\le i,j\le\lfloor\frac{n}{2}\rfloor}$.

We also easily obtain the degree of the symplectic group $\mathrm{Sp}(2r,\mathbb{C})$. By definition, we have $\mathrm{Sp}(2r,\mathbb{C}) := \{M \in \mathbb{C}^{2r\times 2r} : M^{\mathsf{T}}\Omega M = \Omega\}$ where

$$\Omega := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix} \in \mathbb{C}^{2r\times 2r}.$$

Corollary 3.4 *We have* $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r}\deg \mathrm{Sp}(2r,\mathbb{C})$ *and*

$$\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\left[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\right]_{1\le i,j\le r}.$$

Proof Comparing the first two rows in Table 2, we see that the Weyl groups for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$ have the same cardinality, the Coxeter exponents are equal, the convex hull of the weights are equal, and there is a natural bijection between the coroots. In fact, among the r^2 coroots for $\mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{C})$, $r(r-1)$ are equal and r differ by a factor of 2 with the coroots for $\mathrm{Sp}(2r,\mathbb{C})$ being larger. Hence, Theorem 2.4 implies that $\deg \mathrm{SO}(2r+1,\mathbb{C}) = 2^{2r}\deg \mathrm{Sp}(2r,\mathbb{C})$ and Theorem 1.1 shows that $\deg \mathrm{Sp}(2r,\mathbb{C}) = \det\big[\binom{2(2r+1)-2i-2j}{(2r+1)-2i-1}\big]_{1\le i,j\le r}$. □

4 Non-intersecting Lattice Paths

This section gives a combinatorial interpretation for the determinant appearing in our formulas for the degree of $\mathrm{SO}(n,\mathbb{C})$. In particular, we show that this determinant counts appropriate collections of non-intersecting lattice paths by using the celebrated Lindström–Gessel–Viennot Lemma; see [1, Chap. 29] or [10, Theorem 1].

To sketch this approach, let Q be a locally-finite directed acyclic graph. Since there are no directed cycles in Q and every vertex in Q is the tail of only finitely many arrows, it follows that there are only finitely many directed paths (connected sequences of distinct arrows all oriented in the same direction) between any two vertices. For pair a, b of vertices in Q, let $m_{a,b} \in \mathbb{N}$ be number of directed paths from a to b. Given two finite lists $A := \{a_1, a_2, \ldots, a_r\}$ and $B := \{b_1, b_2, \ldots, b_r\}$ of vertices, the associated *path matrix* is $M := [m_{a_i,b_j}]_{1\le i,j\le r} \in \mathbb{N}^{r\times r}$. A *path system* P from A to B consists of a permutation $\sigma \in \mathfrak{S}_r$ together with r directed paths from a_i to $b_{\sigma(i)}$. For $\sigma \in \mathfrak{S}_r$, set $\operatorname{sgn}(\sigma) := (-1)^k$ where k is the number of inversions in σ. If the paths in P are pairwise vertex-disjoint, then P is a *non-intersecting* path system. The following "lemma" relates $\det M$ with non-intersecting path systems.

Lemma 4.1 (Lindström–Gessel–Viennot) *If A and B are finite lists, having the same cardinality and consisting of vertices from a locally-finite directed acyclic graph, then the determinant of the associated path matrix M equals the signed sum of the non-intersecting path systems from A to B:* $\det M = \sum_P \operatorname{sgn}(\sigma)$. □

For our application, consider the directed grid graph whose vertices are the lattice points in $\mathbb{Z}^2$ and whose arrows are unit steps in either the north or east direction. In other words, the vertex $(i, j) \in \mathbb{Z}^2$ is the tail of exactly two arrows: one with head $(i, j+1)$ and the other with head $(i+1, j)$. The next result provides our combinatorial reinterpretation for the degree of $\mathrm{SO}(n, \mathbb{C})$.

Proposition 4.2 *Let $n \in \mathbb{N}$. If $N(n)$ is the number of non-intersecting path systems in the directed grid graph from $A := \{(2-n, 0), (4-n, 0), \ldots, (2\lfloor n/2\rfloor - n, 0)\}$ to $B := \{(0, n-2), (0, n-4), \ldots, (0, n-2\lfloor n/2\rfloor)\}$, then we have*

$$\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1}N(n)\,.$$

Proof By construction, the only non-intersecting path systems in our directed grid graph have direct paths from $(2i-n, 0)$ to $(0, n-2i)$ for $0 \le i \le \lfloor n/2\rfloor$. Hence, the associated element in $\mathfrak{S}_{\lceil n/2\rceil}$ is the identity permutation and the determinant of the associated path matrix counts the total number of non-intersecting path systems.

The number of directed paths from $(0, 0)$ to (i, j) in our directed grid graph is $\binom{i+j}{i}$; simply choose which i arrows in the connected sequence are oriented east. Since the grid graph is invariant under translation, it follows that the number of direct paths from the vertex $(2i-n, 0)$ to $(0, n-2j)$ equals $\binom{2n-2i-2j}{n-2i}$. Therefore, the path matrix associated to A and B is $M = \left[\binom{2n-2i-2j}{n-2i}\right]_{1\le i,j\le\lfloor n/2\rfloor}$. Combining Theorem 1.1 and Lemma 4.1, we conclude that $\deg \mathrm{SO}(n, \mathbb{C}) = 2^{n-1}N(n)$. □

Remark 4.3 From Corollaries 3.3–3.4, we also see that $\deg \mathrm{O}(n, \mathbb{C}) = 2^n N(n)$ and $\deg \mathrm{Sp}(2r, \mathbb{C}) = N(2r+1)$.

Example 4.4 For $n = 5$, the 24 non-intersecting path systems are illustrated in Fig. 1. It follows that $\deg \mathrm{SO}(5, \mathbb{C}) = 2^4(24) = 384$.

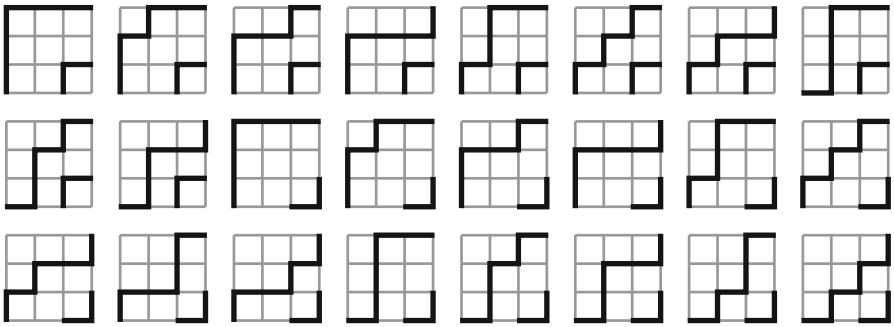

Fig. 1 The non-intersecting path systems from $\{(-3, 0), (-1, 0)\}$ to $\{(0, 1), (0, 3)\}$

Theorem 4.2 suggests that there might be a deeper relationship between the degree of SO($n, \mathbb{C}$) and lattice paths. It would be interesting to find a direct connection. Since the degree of Sp($2r, \mathbb{C}$) does not have a coefficient involving a power of 2, it may be the natural place to look for a combinatorial proof.

5 The Degree of a Low-Rank Optimization Problem

In this section, we show how the degree of SO($n, \mathbb{C}$) arises in counting the number of critical points for a particular optimization problem.

To motivate our particular problem, we first consider a more general framework. The *trace* $\mathrm{tr}(A)$ of a square matrix C is the sum of the entries on the main diagonal, and a real symmetric matrix X is *positive semidefinite*, written $X \succeq 0$, if all of its eigenvalues are nonnegative. A semidefinite programming problem has the form:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\mathrm{tr}(CX)$, for all real symmetric matrices $X \in \mathbb{R}^{n\times n}$, subject to the constraints that $X \succeq 0$ and $\mathrm{tr}(A_i X) = b_i$ for all $1 \le i \le m$. (SDP)

Many practical problems can be modeled as, and many NP-hard problems can be approximated by, semidefinite programming problems; see [3, 11]. Although semidefinite programming problems can often be efficiently solved by interior point methods, this invariably becomes computationally prohibitive for large n. Since the rank of an optimal solution is often much smaller than n, Burer and Monteiro [5] study the hierarchy of relaxations in which X is replaced by the low-rank positive semidefinite matrix RR^{T}. Specifically, the new optimization problem is:

For real symmetric matrices $C, A_1, A_2, \ldots, A_m \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^m$, minimize $\mathrm{tr}(CRR^{\mathsf{T}})$, for all $R \in \mathbb{R}^{n\times r}$, subject to the constraints that $\mathrm{tr}(A_i RR^{\mathsf{T}}) = b_i$ for all $1 \le i \le m$. (NOP)

When $r < (n+1)/2$, this alternative formulation has the advantage of reducing the number of unknowns from $\binom{n+1}{2}$ to nr. However, the objective function and the contraints are no longer linear—they are quadratic and the feasible set is non-convex.

Burer and Monteiro [5] propose a fast algorithm for solving (NOP). Despite the existence of multiple local minima, this algorithm quickly finds the global minimum in practice. To help understand this phenomenon, we examine the *critical points*, those points where the partial derivatives of the associated Lagrangian function vanish, of (NOP). Before giving our formula for the number of critical points of the new optimization problem, we need the following notation.

Definition 5.1 For positive integers i and j, let $\psi_i := 2^{i-1}$, let $\psi_{0,j} := \psi_j$, and let $\psi_{i,j} := \sum_{k=i}^{j-1} \binom{i+j-2}{k}$. For $r > 2$, set

$$\psi_{i_1,i_2,\ldots,i_r} := \begin{cases} \operatorname{pf}\left[\psi_{i_k,i_\ell}\right]_{1\le k<\ell\le r} & \text{if } r \text{ is even} \\ \operatorname{pf}\left[\psi_{i_k,i_\ell}\right]_{0\le k<\ell\le r} & \text{if } r \text{ is odd,} \end{cases}$$

where pf denotes the Pfaffian of a skew-symmetric matrix. For positive integer m and n, we define $\delta(m,n,r) := \sum_I \psi_I \psi_{I'}$, where the sum runs over all strictly increasing subsequences $I := \{i_1, i_2, \ldots, i_{n-r}\}$ of $\{1,2,\ldots,n\}$ with $i_1 + i_2 + \cdots + i_{n-r} = m$ and $I' := \{1,2,\ldots,n\} \setminus I$ denotes the complement.

Remark 5.2 Originally defined in [16] as the number of critical points for (SDP) in which the matrix X has rank r, the number $\delta(m,n,r)$ is called the *algebraic degree* of the semidefinite programming problem. Our defining formula for $\delta(m,n,r)$ was subsequently computed in [2].

Theorem 5.3 *The number of critical points for* (NOP) *is* $2\delta(m,n,r)\deg \mathrm{SO}(r,\mathbb{C})$.

Proof Given new variables $y_1, y_2, \ldots, y_m$, the Lagrangian function associated to (NOP) is $L(R,y) := \operatorname{tr}(CRR^{\mathsf{T}}) - \sum_{i=1}^m y_i\big(\operatorname{tr}(A_i RR^{\mathsf{T}}) - b_i\big)$. Taking the partial derivatives of $L(R,y)$ yields the equations

$$\left(C - \sum_{i=1}^m y_i A_i\right) RR^{\mathsf{T}} = 0 \qquad \text{and} \qquad \operatorname{tr}(A_i RR^{\mathsf{T}}) = b_i\,, \quad \text{for } 1 \le i \le m,$$

which define the set of critical points. Analogously, the critical points for (SDP) are determined by the equations

$$\left(C - \sum_{i=1}^m y_i A_i\right) X = 0 \qquad \text{and} \qquad \operatorname{tr}(A_i X) = b_i\,, \quad \text{for } 1 \le i \le m.$$

Nie, Ranestad, and Sturmfels [16] show that the number of critical points for (SDP), for which the rank of X equals r, is $\delta(m,n,r)$. Comparing the defining systems of equations for the critical points of (NOP) and (SDP), we see that the fibre of the map $(R,y) \mapsto (RR^{\mathsf{T}}, y)$ over each point (X,y) consists of all points (R,y') for which

$X = RR^{\mathsf{T}}$ and $y' = y$. Given X and R such that $X = RR^{\mathsf{T}}$, all other matrices S such that (S, y) lies in the fibre over (X, y) have the form $S = RU$ where U is an orthogonal $(r \times r)$-matrix. In other words, the fibre is isomorphic to a copy of the orthogonal group. Therefore, the number of critical points for (NOP) equals $2\delta(m, n, r) \deg \mathrm{SO}(r, \mathbb{C})$. □

Since the number of critical points for (NOP) grows rapidly with the rank r, the appealing behaviour of the algorithm in [5] still needs to be explained.

Remark 5.4 For applications, the most important critical points for (NOP) are real and satisfy the equation $\left(C - \sum_{i=1}^{m} y_i A_i\right) \succeq 0$.

6 Computational Methods

Since Theorem 1.1 provides a formula for the degree of SO$(n, \mathbb{C})$, this family of examples becomes an interesting testing ground for various symbolic and numerical methods for computing degrees. In this section, we outline three algorithmic techniques for calculating the degree of a variety. The first is based on Gröbner bases, the second uses polynomial homotopy continuation, and the third involves numerical monodromy. Table 1 summarizes the results of our computations, and the related *Macaulay2* code appears in the Appendix. Beyond contrasting these algorithms, we hope that the different routines and auxiliary data, such as Gröbner bases or witness sets, will lead to new insights into the degrees of varieties.

The standard symbolic algorithm for determining the degree of a variety first finds a Gröbner basis of the defining ideal and then uses combinatorial properties of the initial ideal to return the Hilbert polynomial; the degree can be easily extracted from the highest degree term of the Hilbert polynomial. As this method is independent of the ground field, one can speed up the calculation by working over a small finite field. With this algorithm, we were able to compute the degree of SO$(n, \mathbb{C})$ for all $2 \leq n \leq 5$, but it was the slowest among the methods we compared.

The basic numerical strategy for computing the degree of SO$(n, \mathbb{C})$ randomly chooses a linear subspace L of complementary dimension and counts the number of complex solutions S to the zero-dimensional system of polynomial equations corresponding to SO$(n, \mathbb{C}) \cap L$. The triple $\left(\mathrm{SO}(n, \mathbb{C}), L, S\right)$ is called a *witness set* for SO$(n, \mathbb{C})$. This triple is a fundamental data type in numerical algebraic geometry: the computation of a witness set is often a necessary input to other numerical algorithms, including sampling points on the variety, studying its asymptotic behaviour, computing its monodromy group, or even studying its real locus; see Sect. 7. Both numerical algorithms presented below produce a witness set for SO$(n, \mathbb{C})$.

Polynomial homotopy continuation computes a witness set by finding numerical approximations for the complex solutions S. First, one constructs a polynomial system that has a similar structure to the target system and has a simple solution set. This start system is embedded in a homotopy relating it to the target system and the numerical solutions of the start system are traced towards solutions of the target

system. Start systems correspond to root counts. For dense systems, one typically uses the Bézout bound whereas, for sparse systems, one uses the mixed volume of the appropriate Newton polytopes. However, for $\mathrm{SO}(n, \mathbb{C})$, both of these bounds are equal $2^{n(n+1)/2}$, which grows quickly (for $n = 6$, it is already 2 097 152). Because of the number of paths that must be tracked, we were only able to compute the degree of $\mathrm{SO}(n, \mathbb{C})$ for all $2 \leq n \leq 5$ using this method.

Our third technique takes advantage of monodromy; see [7]. Suppose L and L' are two linear subspaces of complementary dimension to $\mathrm{SO}(n, \mathbb{C})$. Given a point on the linear slice $W := \mathrm{SO}(n, \mathbb{C}) \cap L$, we can numerically track this solution along some path γ to a point in another slice $W' := \mathrm{SO}(n, \mathbb{C}) \cap L'$. Tracking the second point along a different path γ' back to W yields another point in W and induces a permutation $\sigma_{\gamma,\gamma'}$ on the points in W. Iterating this process, one expects to populate the witness set associated to W. Although there are algorithms [17] which certify that a witness set is complete, one frequently uses heuristic stopping criteria because they are much faster. This monodromy method is implemented in the *MonodromySolver* package for *Macaulay2* [9]. With the naive stopping criterion that no new points were found after ten consecutive iterations, we were able to calculate with this method the degree $\mathrm{SO}(n, \mathbb{C})$ for all $6 \leq n \leq 7$.

7 Real Points on $\mathrm{SO}(n, \mathbb{C})$

Motivated by the applications to optimization, this section investigates the structure of the real points in $\mathrm{SO}(n, \mathbb{C})$. Taking advantage of the numerical monodromy algorithm, we collect experimental data counting the number of real points in witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$.

More precisely, we use the `random` function in *Macaulay2* [9] to generate a sample of linear slices of $\mathrm{SO}(n, \mathbb{C})$. Homotopy continuation allows us to track solutions from a precomputed witness set to those lying on each randomly chosen linear slice. We determine how many solutions in the random slice are real by checking whether each coordinate is within a 0.001 numerical tolerance of being real. One can actually certify reality using *alphaCertify* [12], which implements Smale's α-theory. However, for the sake of speed, we limited these formal checks to at least one witness set achieving the maximum observed number of real points. The results of computing 1,398,000, 1,004,100, and 48,200 witness sets for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ are displayed in Figs. 2 and 3.

The raw data and actually code can be found at [4]. In rare examples, the process failed to return a witness set on the randomly chosen linear slice, because the homotopy continuation was ill-conditioned. In particular, we observed 2, 51, and 81 such failures for $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SO}(4, \mathbb{C})$, and $\mathrm{SO}(5, \mathbb{C})$ respectively. Despite the fact that all witness sets computed for $\mathrm{SO}(4, \mathbb{C})$ and $\mathrm{SO}(5, \mathbb{C})$ had fewer than 40 and 384 solutions, we are not convinced that there exists a non-trivial upper bound for the number of real solutions on a witness set of $\mathrm{SO}(n, \mathbb{C})$ exists. In fact, we conjecture that, for all $n \geq 2$, $\mathrm{SO}(n, \mathbb{C})$ admits a real witness set.

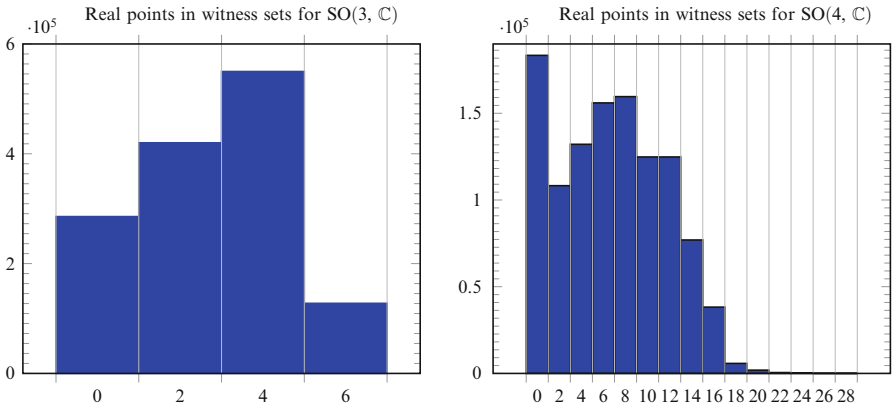

Fig. 2 Some histograms for the number of real solutions found in each witness set

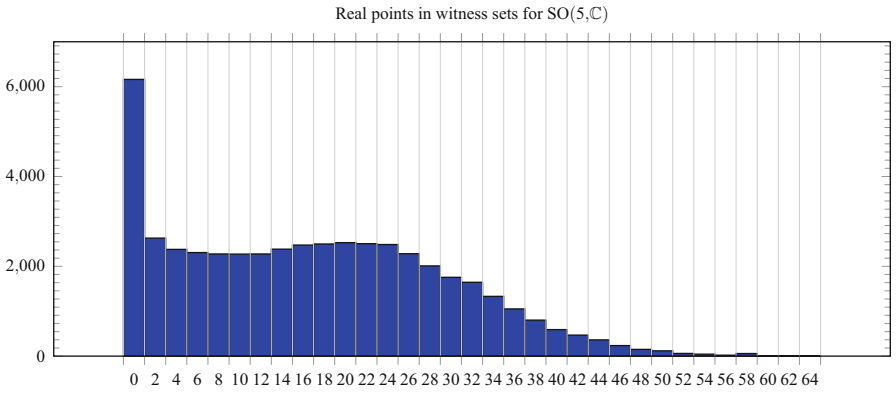

Fig. 3 Another histogram for the number of real solutions found in each witness set

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Appendix: *Macaulay2* Code

This section contains *Macaulay2* [9] code for computing the degree of $\mathrm{SO}(n,\mathbb{C})$. We typically compute the degree of $\mathrm{O}(n,\mathbb{C})$, and divide by to 2 to obtain the degree of $\mathrm{SO}(n,\mathbb{C})$, because this approach eliminates the polynomial of highest degree, the condition that the determinant equal 1.

First, we compute the degree of SO(5) using Gröbner bases. The computation is done over the finite field $\mathbb{Z}/2\mathbb{Z}$ for $\mathrm{O}(5,\mathbb{C})$ and the result is halved to give the degree of $\mathrm{SO}(5,\mathbb{C})$.

```
deg1SO = n -> (
    R := ZZ/2[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    J := minors(1, M * transpose(M) - id_(R^n));
    (degree J) // 2)
```

Our second function uses the package *NumericalAlgebraicGeometry* to solve the zero-dimensional system arising from a linear slice of the variety $\mathrm{O}(3,\mathbb{C})$. The command `solveSystem` employs the standard method of polynomial homotopy continuation.

```
needsPackage ``NumericalAlgebraicGeometry'';
deg2SO = n -> (
    R := CC[x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := apply(binomial(n,2),
        i -> random(1,R) - random(CC));
    S := solveSystem(polys | linearSlice);
    #S // 2)
```

We next provide code that computes the degree of $\mathrm{SO}(n,\mathbb{C})$ using the package *MonodromySolver*. Again we do not include the determinant condition, but this time we do *not* need to halve the result. This is because our starting point, the identity matrix, lies on $\mathrm{SO}(n,\mathbb{C})$ and this method only discovers points on the irreducible component corresponding to our starting point. The linear slices are parametrized by the t and c variables which are varied within the function `monodromySolve` to create monodromy loops. The method stops when ten consecutive loops provide no new points. Although it is possible that this stopping criterion is satisfied prematurely, in our case the program stopped at the correct number.

```
needsPackage ``MonodromySolver'';
deg3SO = n -> (
    d := binomial(n,2);
    R := CC[c_1..c_d,
        t_(1,1,1)..t_(d,n,n)][x_(1,1)..x_(n,n)];
    M := genericMatrix(R,n,n);
```

```
    B := M * transpose(M) - id_(R^n);
    polys := unique flatten entries B;
    linearSlice := for i from 1 to d list (
        c_i + sum flatten for j from 1 to n list (
            for k from 1 to N list t_(i,j,k)*x_(j,k)));
    G := polySystem( polys | linearSlice);
    setRandomSeed 0;
    (p0, x0) := createSeedPair(G,
        flatten entries id_(CC^n));
    (V, npaths) = monodromySolve(G, p0, {x_0},
        NumberOfNodes => 2, NumberOfEdges => 4);
    # flatten points V.PartialSols)
```

Finally, we may use Theorem 1.1 to compute the degree of SO($n, \mathbb{C}$).

```
deg4SO = n -> (
    r := n // 2;
    M := matrix table(toList(1..r), toList(1..r),
        (i,j) -> binomial(2*n-2*i-2*j, n-2*i));
    2^(n-1) * det(M))
```

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A probabilistic approach to systems of parameters and Noether normalization

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A probabilistic approach to systems of parameters and Noether normalization

Juliette Bruce and Daniel Erman

We study systems of parameters over finite fields from a probabilistic perspective and use this to give the first effective Noether normalization result over a finite field. Our central technique is an adaptation of Poonen's closed point sieve, where we sieve over higher dimensional subvarieties, and we express the desired probabilities via a zeta function-like power series that enumerates higher dimensional varieties instead of closed points. This also yields a new proof of a recent result of Gabber, Liu and Lorenzini (2015) and Chinburg, Moret-Bailly, Pappas and Taylor (2017) on Noether normalizations of projective families over the integers.

Given an n-dimensional projective scheme $X \subseteq \mathbb{P}^r$ over a field, Noether normalization says that we can find homogeneous polynomials that induce a finite morphism $X \to \mathbb{P}^n$. Such a morphism is determined by a system of parameters, namely by choosing homogeneous polynomials $f_0, f_1, \ldots, f_n$ of degree d where $X \cap V(f_0, f_1, \ldots, f_n) = \varnothing$. Such a system of polynomials $f_0, f_1, \ldots, f_n$ is a system of parameters on the homogeneous coordinate ring of X. More generally, for $k \leq n$ we say that $f_0, f_1, \ldots, f_k$ *are parameters on* X if

$$\dim \mathbb{V}(f_0, f_1, \ldots, f_k) \cap X = \dim X - (k+1).$$

By convention, the empty set has dimension -1.

Over an infinite field any generic choice of $\leq n+1$ linear polynomials will automatically be parameters on X. Over a finite field we can ask:

Questions 1.1. Let $\mathbb{F}_q$ be a finite field and $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme:

(1) What is the probability that a random choice $f_0, f_1, \ldots, f_k$ of polynomials of degree d will be parameters on X?

(2) Can one effectively bound the degrees d for which such a finite morphism exists?

We will provide new insight into these questions by studying the distribution of systems of parameters from both a geometric and probabilistic viewpoint.

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For the geometric side, we fix a field $\boldsymbol{k}$ and let $S = \boldsymbol{k}[x_0, x_1, \ldots, x_r]$ be the coordinate ring of $\mathbb{P}^r_{\boldsymbol{k}}$. We write S_d for the vector space of degree d polynomials in S. In Section 4, we define a scheme $\mathscr{D}_{k,d}(X)$ parametrizing collections that do not form parameters. The $\boldsymbol{k}$-points of $\mathscr{D}_{k,d}(X)$ are

$$\mathscr{D}_{k,d}(X)(\boldsymbol{k}) = \{(f_0, f_1, \ldots, f_k) \text{ that are } \textit{not} \text{ parameters on } X\} \subset \underbrace{S_d \times \cdots \times S_d}_{k+1 \text{ copies}}.$$

We prove an elementary bound on the codimension of these closed subschemes of the affine space $S_d^{\oplus k+1}$.

Theorem 1.2. *Let $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ be an n-dimensional closed subscheme. We have*:

$$\operatorname{codim} \mathscr{D}_{k,d}(X) = \begin{cases} \geq \binom{n-k+d}{n-k} & \text{if } k < n, \\ = 1 & \text{if } k = n. \end{cases}$$

This generalizes several results from the literature: the case $k = n$ is a classical result about Chow forms [Gelfand et al. 1994, 3.2.B]. For $d = 1$ and $k < n$, the bound is sharp, by a classical result about determinantal varieties.[1] The bound for the case $k = 0$ appears in [Benoist 2011, Lemme 3.3]. If $k < n$, then the codimension grows as $d \to \infty$ and this factors into our asymptotic analysis over finite fields. It also leads to a uniform convergence result that allows us to go from a finite field to $\mathbb{Z}$.

For the probabilistic side, we work over a finite field $\mathbb{F}_q$ and compute the asymptotic probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X. The following result, which follows from known results in the literature, shows that there is a bifurcation between the $k = n$ and $k < n$ cases, reflecting Theorem 1.2.

Theorem 1.3 [Bucur and Kedlaya 2012; Poonen 2013]. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme. The asymptotic probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\lim_{d \to \infty} \text{Prob}(f_0, f_1, \ldots, f_k \textit{ of degree } d \textit{ are parameters on } X) = \begin{cases} 1 & \text{if } k < n, \\ \zeta_X(n+1)^{-1} & \text{if } k = n, \end{cases}$$

where $\zeta_X(s)$ is the arithmetic zeta function of X.

The maximal case $k = n$ follows from the $k = m + 1$ case of Bucur and Kedlaya [2012, Theorem 1.2] (though they assume that X is smooth, their proof does not need that assumption when $k = m + 1$) and is proven using Poonen's closed point sieve. Moreover, the result in both cases could be derived from a slight modification of [Poonen 2013, Proof of Theorem 2.1]. See also [Charles and Poonen 2016, Corollary 1.4] for a similar result.

The main results in our paper stem from a deeper investigation of the cases where $k < n$, as the limiting value of 1 is only the beginning of the story. In the following theorem, we use $|Z|$ to denote the number of irreducible components of a scheme Z, and we write $\dim Z \equiv k$ if Z is equidimensional of dimension k.

[1] See [Bruns and Vetter 1988, Theorem 2.5] for a modern statement and proof. That result has a complicated history, discussed in [Bruns and Vetter 1988, Section 2.E], with some cases dating as far back as [Macaulay 1916, Section 53].

Theorem 1.4. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a projective scheme of dimension n. Fix e and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree d are parameters on X is*

$$\operatorname{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are parameters on } X \end{array}\right) = 1 - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z,\mathcal{O}_Z(d))} + o(q^{-e(k+1)\binom{n-k+d}{n-k}}).$$

Theorem 1.4 illustrates that the probability of finding a sequence $f_0, f_1, \ldots, f_k$ of parameters on X is intimately tied to the codimension k geometry of X. Note that, by basic properties of the Hilbert polynomial, as $d \to \infty$ we have

$$h^0(Z, \mathcal{O}_Z(d)) = \frac{\deg(Z)}{(n-k)!} d^{n-k} + o(d^{n-k}) = \deg(Z)\binom{n-k+d}{n-k} + o(d^{n-k}).$$

It follows that the term $q^{-(k+1)h^0(Z,\mathcal{O}_Z(d))}$ lies in $o(q^{-e(k+1)\binom{n-k+d}{n-k}})$ if and only if $\deg(Z) > e$.

For instance, setting $e = 1$, the sum simplifies to $1 - N \cdot q^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}})$, where N is the number of $(n-k)$-dimensional linear subspaces lying in X. It would thus be more difficult to find parameters on a variety X containing lots of linear spaces, as illustrated in Example 8.1. More generally, the probability of finding parameters for $k < n$ depends on a power series that counts the number of $(n-k)$-dimensional subvarieties of varying degrees, in analogue with the appearance of the zeta function in the $k = n$ case.

Our approach to Theorem 1.4 is motivated by a simple observation: $f_0, f_1, \ldots, f_k$ fail to be parameters if and only if they all vanish along some $(n-k)$-dimensional subvariety of X. We thus develop an analogue of Poonen's sieve where closed points are replaced by $(n-k)$-dimensional varieties. Sieving over higher dimensional varieties presents new challenges, especially bounding the error. This error depends on the Hilbert function of these varieties, and one key innovation is a uniform lower bound for Hilbert functions given in Lemma 3.1.

This perspective also leads to our second main result: an answer to Questions 1.1.(2) where the bound is in terms of the sum of the degrees of the irreducible components. If $X \subseteq \mathbb{P}^r$ has minimal irreducible components $V_1, V_2, \ldots, V_s$ (considered with the reduced scheme structure), then we define $\widehat{\deg}(X) := \sum_{i=1}^s \deg(V_i)$ (see Definition 2.2). We set $\log_q 0 = -\infty$.

Theorem 1.5. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ where $\dim X = n$. If $\max\{d, \frac{q}{d^n}\} \geq \widehat{\deg}(X)$ and*

$$d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$$

then there exist $f_0, f_1, \ldots, f_n$ of degree d^{n+1} inducing a finite morphism $\pi : X \to \mathbb{P}^n_{\mathbb{F}_q}$.

The bound is asymptotically optimal in q. Namely, if we fix $\widehat{\deg}(X)$, then as $q \to \infty$, the bound becomes $d = 1$. Thus, a linear Noether normalization exists if $q \gg \widehat{\deg}(X)$. For a fixed q, we expect the bound could be significantly improved. (Even the case $\dim X = 0$ would be interesting, as it is related to Kakeya type problems over finite fields [Ellenberg and Erman 2016; Ellenberg et al. 2010].)

Theorem 1.5 provides the first explicit bound for Noether normalization over a finite field. (One could potentially derive an explicit bound from Nagata's argument [1962, Chapter I.14], though the inductive nature of that construction would at best yield a bound that is multiply exponential in the largest degree of a defining equation of X.)

After computing the probabilities over finite fields, we combine these analyses and characterize the distribution of parameters on projective B-schemes where $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. We use standard notions of density for a subset of a free B-module; see Definition 7.1.

Corollary 1.6. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. If $X \subseteq \mathbb{P}^r_B$ is a closed subscheme whose general fiber over B has dimension n, then*

$$\lim_{d\to\infty} \text{Density}\left\{ \begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \text{ that} \\ \text{restrict to parameters on } X_p \text{ for all } p \end{array} \right\} = \begin{cases} 1 & \text{if } k < n, \\ 0 & \text{if } k = n \text{ and all } d. \end{cases}$$

The density over B thus equals the product over all the fibers of the asymptotic probabilities over $\mathbb{F}_q$. In the case $B = \mathbb{Z}$, our proof relies on Ekedahl's infinite Chinese remainder theorem [Ekedahl 1991, Theorem 1.2] combined with Proposition 5.1, which illustrates uniform convergence in p for the asymptotic probabilities in Theorem 1.3. In the case $B = \mathbb{F}_q[t]$, we use Poonen's analogue of Ekedahl's result [Poonen 2003, Theorem 3.1].

When $k = n$, an analogue of Corollary 1.6 for smoothness is given by Poonen [2004, Theorem 5.13]. Moreover, while it is unknown if there are any smooth hypersurfaces of degree > 2 over $\mathbb{Z}$ (see for example the discussion in [Poonen 2009]), the density zero subset from Corollary 1.6 turns out to be nonempty for large d. This leads to a new proof of a recent result about uniform Noether normalizations.

Corollary 1.7. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$. Let $X \subseteq \mathbb{P}^r_B$ be a closed subscheme. If each fiber of X over B has dimension n, then for some d, there exist homogeneous polynomials $f_0, f_1, \ldots, f_n \in B[x_0, x_1, \ldots, x_r]$ of degree d inducing a finite morphism $\pi : X \to \mathbb{P}^n_B$.*

Corollary 1.7 is a special case of a recent result of Chinburg, Moret-Bailly, Pappas and Taylor [2017, Theorem 1.2] and of Gabber, Liu and Lorenzini [2015, Theorem 8.1]. This corollary can fail when B is any of $\mathbb{Q}[t]$ or $\mathbb{Z}[t]$ or $\mathbb{F}_q[s, t]$, as in those cases, the Picard group of a finite cover of Spec B can fail to be torsion. See Section 8 for explicit examples and counterexamples and see [Chinburg et al. 2017; Gabber et al. 2015] for generalizations and applications.

There are a few earlier results related to Noether normalization over the integers. For instance [Moh 1979] shows that Noether normalizations of semigroup rings always exist over $\mathbb{Z}$; and [Nagata 1962, Theorem 14.4] implies that given a family over any base, one can find a Noether normalization over an open subset of the base. Relative Noether normalizations play a key role in [Achinger 2015, Section 5]. There is also the incorrect claim in [Zariski and Samuel 1960, page 124] that Noether normalizations exist over any infinite domain (see [Abhyankar and Kravitz 2007]). Brennan and Epstein [2011] analyze the distribution of systems of parameters from a different perspective, introducing the notion of a generic matroid to relate various different systems of parameters. In addition, after our paper was posted, work of

Charles [2017] on arithmetic Bertini theorems appeared which, under the additional hypothesis that X is integral and flat, implies a stronger version of Corollary 1.6 where one also obtains bounds on the norms of the functions.

This paper is organized as follows. Section 2 gathers background results and Section 3 involves a key lower bound on Hilbert functions. Section 4 contains our geometric analysis of parameters including a proof of Theorem 1.2. Sections 5 and 6 contain the probabilistic analysis of parameters over finite fields: Section 5 proves Theorem 1.3 and Theorem 1.5 while Section 6 gives the more detailed description via an analogue of the zeta function enumerating $(n-k)$-dimensional subvarieties, including the proof of Theorem 1.4. Section 7 contains our analysis over $\mathbb{Z}$ including proofs of Corollaries 1.6 and 1.7 and related corollaries. Section 8 contains examples.

2. Background

In this section, we gather some algebraic and geometric facts that we will cite throughout.

Lemma 2.1. *Let $\boldsymbol{k}$ be a field and let R be a $(k+1)$-dimensional graded $\boldsymbol{k}$-algebra where $R_0 = \boldsymbol{k}$. If $f_0, f_1, \ldots, f_k$ are homogeneous elements of degree d and $R/\langle f_0, f_1, \ldots, f_k\rangle$ has finite length, then the extension $\boldsymbol{k}[z_0, z_1, \ldots, z_k] \to R$ given by $z_i \mapsto f_i$ is a finite extension.*

Proof. See [Bruns and Herzog 1993, Theorem 1.5.17]. □

This lemma implies that if $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ has dimension n, and if $f_0, f_1, \ldots, f_n$ are parameters on X, then the map $\phi\colon X \to \mathbb{P}^n_{\boldsymbol{k}}$ given by sending $x \mapsto [f_0(x) : f_1(x) : \cdots : f_n(x)]$ is a finite morphism. In particular, if R is the homogeneous coordinate of X, then the ideal $\langle f_0, f_1, \ldots, f_n\rangle \subseteq R$ has finite colength, and thus the base locus of ϕ is the empty set. In other words, ϕ defines a genuine morphism. Moreover, the lemma shows that the corresponding map of coordinate rings $\phi^\sharp : R \to \boldsymbol{k}[z_0, z_1, \ldots, z_n]$ is finite, and this implies that ϕ is finite.

Definition 2.2. Let $X \subseteq \mathbb{P}^r$ be a projective scheme with minimal irreducible components $V_1, \ldots, V_s$ (considered with the reduced scheme structure). We define $\widehat{\deg}(X) := \sum_{i=1}^s \deg(V_i)$. For a subscheme $X' \subseteq \mathbb{A}^r$ with projective closure $\overline{X}' \subseteq \mathbb{P}^r$ we define $\widehat{\deg}(X') := \widehat{\deg}(\overline{X}')$.

This provides a notion of degree which ignores nonreduced structure but takes into account components of lower dimension. Similar definitions have appeared in the literature: for instance, in the language of [Bayer and Mumford 1993, Section 3], we would have $\widehat{\deg}(X) = \sum_{j=0}^{\dim X} \text{geom-deg}_j(X)$.

Lemma 2.3. *Let $\boldsymbol{k}$ be any field and let $X \subseteq \mathbb{A}^r_{\boldsymbol{k}}$. Let $f_0, f_1, \ldots, f_t$ be polynomials in $\boldsymbol{k}[x_1, \ldots, x_r]$. If $X' = X \cap \mathbb{V}(f_0, f_1, \ldots, f_t)$, then $\widehat{\deg}(X') \le \widehat{\deg}(X) \cdot \prod_{i=0}^t \deg(f_i)$.*

Proof. This follows from the refined version of Bezout's theorem [Fulton 1984, Example 12.3.1]. □

3. A uniform lower bound on Hilbert functions

For a subscheme of $\mathbb{P}^r$, the Hilbert function in degree d is controlled by the Hilbert polynomial, at least if d is very large related to some invariants of the subscheme. We analyze the Hilbert function at the

other extreme, where the degree of the subscheme may be much larger than d. The following lemma, which applies to subschemes of arbitrarily high degree, provides uniform lower bounds that are crucial to bounding the error in our sieves.

Lemma 3.1. *Let $\boldsymbol{k}$ be an arbitrary field and fix some $e \geq 0$. Let $V \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ be any closed, m-dimensional subscheme of degree $> e$ with homogeneous coordinate ring R:*

(1) *We have* $\dim R_d \geq h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$ *for all d.*

(2) *For any $0 < \epsilon < 1$, there exists a constant C depending only on m and ϵ (but not on d or $\boldsymbol{k}$ or R) such that*

$$\dim R_d > (e+\epsilon) \cdot h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$$

for all $d \geq Ce^{m+1}$.

Proof. If $\boldsymbol{k}'$ is a field extension of $\boldsymbol{k}$, then the Hilbert series of R is the same as the Hilbert series of $R \otimes_{\boldsymbol{k}} \boldsymbol{k}'$. We can thus assume that $\boldsymbol{k}$ is an infinite field. For part (1), we simply take a linear Noether normalization $\boldsymbol{k}[t_0, t_1, \ldots, t_m] \subseteq R$ of the ring R [Eisenbud 1995, Theorem 13.3]. This yields $\boldsymbol{k}[t_0, t_1, \ldots, t_m]_d \subseteq R_d$, giving the statement about Hilbert functions.

We prove part (2) of the lemma by induction on m. Let $S = \boldsymbol{k}[x_0, x_1, \ldots, x_r]$ and let $I_V \subseteq S$ be the saturated, homogeneous ideal defining V. Thus $R = S/I_V$. If $m = 0$, then we have $\dim R_d \geq \min\{d+1, \deg V\} \geq \min\{d+1, e+1\}$ which is at least $e+\epsilon$ for all $d \geq e$. This proves the case $m = 0$, where the constant C can be chosen to be 1.

Now assume the claim holds for all closed subschemes of dimension less than m. Let $V \subset \mathbb{P}^r_{\boldsymbol{k}}$ be a closed subscheme with $\dim V = m \geq 1$. Fix $0 < \epsilon < 1$. Since we are working over an infinite field, [Eisenbud 1995, Lemma 13.2(c)] allows us to choose a linear form ℓ that is a nonzero divisor in R. This yields a short exact sequence $0 \to R(-1) \xrightarrow{\cdot \ell} R \to R/\ell \to 0$. Since $R/\ell = S/(I_V + \langle \ell \rangle)$, this yields the equality

$$\dim R_i = \dim R_{i-1} + \dim(S/(I_V + \langle \ell \rangle))_i. \tag{1}$$

Letting $W = V \cap V(\ell)$ we know that $\dim W = m-1$ and $\deg W = \deg V$. Moreover, if I_V is the saturated ideal defining V and if I_W is the saturated ideal defining W, then since I_W contains $I_V + \langle \ell \rangle$, we have $\dim(S/(I_V + \langle \ell \rangle))_i \geq \dim(S/I_W)_i$. Combining with (1) yields

$$\dim R_i \geq \dim R_{i-1} + \dim(S/I_W)_i. \tag{2}$$

Now, by induction, in the case $m-1$ and $\epsilon' := \frac{1+\epsilon}{2}$, there exists C' depending on ϵ' and $m-1$ (or equivalently depending on ϵ and m) where

$$\dim(S/I_W)_i \geq (e+\epsilon') \binom{m-1+i}{m-1} \tag{3}$$

for all $i \geq C'e^m$. Now let $d \geq C'e^m$. Iteratively applying (2) for $i = d, d-1, d-2, \ldots, \lceil C'e^m \rceil$, we obtain:

$$\dim R_d \geq \dim R_{\lceil C'e^m \rceil - 1} + \sum_{i=\lceil C'e^m \rceil}^{d} \dim(S/I_W)_i.$$

By dropping the $\dim R_{\lceil C'e^m \rceil - 1}$ term and applying (3), we conclude that

$$\dim R_d \geq \sum_{i=\lceil C'e^m \rceil}^{d} (e+\epsilon')\binom{m-1+i}{m-1}.$$

The identity $\sum_{i=a}^{b} \binom{i+k}{k} = \binom{b+k+1}{k+1} - \binom{a+k}{k+1}$ implies that $\sum_{i=\lceil C'e^m \rceil}^{d} (e+\epsilon')\binom{m-1+i}{m-1}$ can be rewritten as $(e+\epsilon')\left(\binom{m+d}{m} - \binom{m-1+\lceil C'e^m \rceil}{m}\right)$. There exists a constant C depending on ϵ and m so that $(\epsilon' - \epsilon)\binom{m+d}{m} = \left(\frac{1}{2} - \frac{\epsilon}{2}\right)\binom{m+d}{m} \geq (e+\epsilon')\binom{m-1+\lceil C'e^m \rceil}{m}$ for all $d \geq \lceil Ce^{m+1} \rceil$. Thus, for all $d \geq \lceil Ce^{m+1} \rceil$ we have

$$\dim R_d \geq (e+\epsilon')\binom{m+d}{m} - (\epsilon' - \epsilon)\binom{m+d}{d} = (e+\epsilon)\binom{m+d}{m}. \qquad \square$$

Remark 3.2. Asymptotically in e, the bound of Ce^2 is the best possible for curves. For instance, let $C \subseteq \mathbb{P}^r$ be a curve of degree $(e+1)$ lying inside some plane $\mathbb{P}^2 \subseteq \mathbb{P}^r$. Let R be the homogeneous coordinate ring of C. If $d \geq e$ then the Hilbert function is given by

$$\dim R_d = (e+1)d - \tfrac{e^2-e}{2}.$$

Thus, if we want $\dim R_d \geq (e+\epsilon)(d+1)$, we will need to let $d \geq (e^2 + e + 2\epsilon)/(2(1-\epsilon)) \approx \frac{1}{2}e^2$. It would be interesting to know if the bound Ce^{m+1} is the best possible for higher dimensional varieties.

4. Geometric analysis

In this section we analyze the geometric picture for the distribution of parameters on X. The basic idea behind the proof of Theorem 1.2 is that $f_0, f_1, \ldots, f_k$ fail to be parameters on X if and only if they all vanish along some $(n-k)$-dimensional subvariety of X. Since the Hilbert polynomial of a $(n-k)$-dimensional variety grows like d^{n-k}, when we restrict a degree d polynomial f_j to such a subvariety, it can be written in terms of $\approx d^{n-k}$ distinct monomials. The polynomial f_j will all vanish along the subvariety if and only if all of the $\approx d^{n-k}$ coefficients vanish. This rough estimate explains the growth of the codimension of $\mathscr{D}_{k,d}(X)$ as $d \to \infty$.

We begin by constructing the schemes $\mathscr{D}_{k,d}(X)$. Fix $X \subseteq \mathbb{P}^r_{\boldsymbol{k}}$ a closed subscheme of dimension n over a field $\boldsymbol{k}$. Given $k < n$ and $d > 0$, let $\mathscr{A}_{k,d}$ be the affine space $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))^{\oplus k+1}$ and $\boldsymbol{k}[c_{0,1}, \ldots, c_{k,\binom{r+d}{d}}]$ be the corresponding polynomial ring. We enumerate the monomials in $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ as $m_1, \ldots, m_{\binom{r+d}{d}}$, and then define the universal polynomial

$$F_i := \sum_{j=1}^{\binom{r+d}{d}} c_{i,j} m_j \in \boldsymbol{k}[c_{0,1}, \ldots, c_{k,\binom{r+d}{d}}] \otimes_{\boldsymbol{k}} \boldsymbol{k}[x_0, x_1, \ldots, x_r].$$

Given a closed point $c \in \mathscr{A}_{k,d}$ we can specialize $F_0, F_1, \ldots, F_k$ and obtain polynomials $f_0, f_1, \ldots, f_k \in \kappa(c)[x_0, x_1, \ldots, x_r]$, where $\kappa(c)$ is the residue field of c. We will thus identify each element of $\mathscr{A}_{k,d}(\boldsymbol{k})$ with a collection of polynomials $\boldsymbol{f} = (f_0, f_1, \ldots, f_k) \in \boldsymbol{k}[x_0, x_1, \ldots, x_r]$.

Now define $\Sigma_{k,d}(X) \subseteq X \times \mathscr{A}_{k,d}$ via the equations $F_0, F_1, \ldots, F_k$. Consider the second projection $p_2 \colon \Sigma_{k,d}(X) \to \mathscr{A}_{k,d}$. Given a point $\boldsymbol{f} = (f_0, f_1, \ldots, f_k) \in \mathscr{A}_{k,d}$, the fiber $p_2^{-1}(\boldsymbol{f}) \subseteq X$ can be identified with the points lying in $X \cap \mathbb{V}(f_0, f_1, \ldots, f_k)$. For generic choices of $\boldsymbol{f}$ (after passing to an infinite field if necessary) the polynomials $f_0, f_1, \ldots, f_k$ will define an ideal of codimension $k+1$, and thus the fiber $p_2^{-1}(\boldsymbol{f})$ will have dimension $n-k-1$.

There is a closed sublocus $\mathscr{D}_{k,d}(X) \subsetneq \mathscr{A}_{k,d}$ where the dimension of the fiber is at least $n-k$, and we give $\mathscr{D}_{k,d}(X)$ the reduced scheme structure. It follows that $\mathscr{D}_{k,d}(X)$ parametrizes collections $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ of degree d polynomials which fail to be parameters on X.

Remark 4.1. If we fix $X_{\mathbb{Z}} \subseteq \mathbb{P}^r_{\mathbb{Z}}$, then we can follow the same construction to obtain a scheme $\mathscr{D}_{k,d}(X_{\mathbb{Z}}) \subseteq \mathscr{A}_{k,d}$. Writing $X_{\boldsymbol{k}}$ as the pullback $X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \boldsymbol{k}$, we observe that the equations defining $\Sigma_{k,d}(X_{\boldsymbol{k}})$ are obtained by pulling back the equations defining $\Sigma_{k,d}(X_{\mathbb{Z}})$. It follows that $\mathscr{D}_{k,d}(X_{\mathbb{Z}}) \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec}(\boldsymbol{k})$ has the same set-theoretic support as $\mathscr{D}_{k,d}(X_{\boldsymbol{k}})$.

Definition 4.2. We let $\mathscr{D}^{\mathrm{bad}}_{k,d}(X)$ be the locus of points in $\mathscr{D}_{k,d}(X)$ where $f_0, f_1, \ldots, f_{k-1}$ already fail to be parameters on X and let $\mathscr{D}^{\mathrm{good}}_{k,d}(X) := \mathscr{D}_{k,d}(X) \setminus \mathscr{D}^{\mathrm{bad}}_{k,d}(X)$. We set $\mathscr{D}^{\mathrm{bad}}_{0,d}(X) = \varnothing$.

Remark 4.3. We have a factorization:

$$\mathscr{A}_{k,d} \to \mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d}$$
$$(f_0, f_1, \ldots, f_k) \mapsto ((f_0, f_1, \ldots, f_{k-1}), f_k).$$

We let $\pi : \mathscr{D}_{k,d}(X) \to \mathscr{A}_{k-1,d}$ be the induced projection, which will we use to work inductively.

Proof of Theorem 1.2. First consider the case $k = n$. There is a natural rational map from $\mathscr{A}_{n,d}$ to the Grassmanian $\mathrm{Gr}(n+1, S_d)$ given by sending the point $(f_0, f_1, \ldots, f_n) \in \mathscr{A}_{n,d}$ to the linear space that those polynomials span. Inside of the Grassmanian, the locus of choices of $(f_0, f_1, \ldots, f_n)$ that all vanish on a point of X is a divisor in the Grassmanian defined by the Chow form; see [Gelfand et al. 1994, 3.2.B]. The preimage of this hypersurface in $\mathscr{A}_{n,d}$ is a hypersurface contained in $\mathscr{D}_{n,d}(X)$, and thus $\mathscr{D}_{n,d}(X)$ has codimension 1.

For $k < n$, we will induct on k. Let $k = 0$. A polynomial f_0 will fail to be a parameter on X if and only if $\dim X = \dim(X \cap \mathbb{V}(f_0))$. This happens if and only if f_0 is a zero divisor on a top-dimensional component of X. Let V be the reduced subscheme of some top-dimensional irreducible component of X and let $\mathcal{I}_V$ be the defining ideal sheaf of V. Then the set of zero divisors of degree d on V will form a linear subspace in $\mathscr{A}_{0,d}$ corresponding to the elements of the vector subspace $H^0(\mathcal{I}_V(d))$. The codimension of $H^0(\mathcal{I}_V(d)) \subseteq S_d$ is precisely given by the Hilbert function of the homogeneous coordinate ring of V in degree d. By applying Lemma 3.1(1), we conclude that for all d this linear space has codimension at least $\binom{n+d}{d}$. Since $\mathscr{D}_{0,d}(X)$ is the union of these linear spaces over all top-dimensional components of X, this proves that $\operatorname{codim} \mathscr{D}_{0,d}(X) \geq \binom{n+d}{d}$.

Take the induction hypothesis that we have proven the statement for $\mathscr{D}_{j,d}(X')$ for all $X' \subseteq \mathbb{P}^r$ and all $j \leq k-1$. We separate $\mathscr{D}_{k,d}(X) = \mathscr{D}^{\text{bad}}_{k,d}(X) \sqcup \mathscr{D}^{\text{good}}_{k,d}(X)$ and will show that each locus has sufficiently large codimension. We begin with $\mathscr{D}^{\text{bad}}_{k,d}(X)$. By using the factorization from Remark 4.3, we can realize $\mathscr{D}^{\text{bad}}_{k,d}(X) \subseteq \mathscr{A}_{k,d} \cong \mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d}$. By definition of $\mathscr{D}^{\text{bad}}_{k,d}(X)$, the image of $\mathscr{D}^{\text{bad}}_{k,d}(X)$ in $\mathscr{A}_{k-1,d} \times \mathscr{A}_{0,d}$ is $\mathscr{D}_{k-1,d}(X) \times \mathscr{A}_{0,d}$. It follows that

$$\operatorname{codim}(\mathscr{D}^{\text{bad}}_{k,d}(X), \mathscr{A}_{k,d}) = \operatorname{codim}(\mathscr{D}_{k-1,d}(X), \mathscr{A}_{k-1,d}) \geq \binom{n-k+1+d}{n-k+1} \geq \binom{n-k+d}{n-k},$$

where the middle inequality follows by induction.

Now consider an arbitrary point $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ in $\mathscr{D}^{\text{good}}_{k,d}(X)$. By definition, $f_0, f_1, \ldots, f_{k-1}$ are parameters on X, and thus $\pi(\boldsymbol{f}) \in \mathscr{A}_{k-1,d} \setminus \mathscr{D}_{k-1,d}(X)$. Using the splitting of Remark 4.3, the fiber of $\mathscr{D}^{\text{good}}_{k,d}(X)$ over $\boldsymbol{f}$ can be identified with $\mathscr{D}_{0,d}(X')$ where $X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{k-1})$. Since $(f_0, f_1, \ldots, f_{k-1}) \notin \mathscr{D}_{k-1,d}(X)$, we have that $\dim X' = n-k$. The inductive hypothesis thus guarantees that $\operatorname{codim} \mathscr{D}_{0,d}(X') \geq \binom{\dim X'+d}{d} = \binom{n-k+d}{d}$. □

5. Probabilistic analysis, I: Proof of Theorem 1.3

The main result of this section is Proposition 5.1 which provides an effective bound for finding parameters, and which we will use to prove Theorem 1.5. We also use this to give a new proof of Theorem 1.3 for $k < n$. Throughout this section, we let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a projective scheme of dimension n over a finite field $\mathbb{F}_q$. Recall that $S_d = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$. We define

$$\operatorname{Par}_{d,k} = \{f_0, f_1, \ldots, f_k \text{ that are parameters on } X\} \subset S_d^{k+1}.$$

In Theorem 1.3, we compute the following limit (which a priori might not exist):

$$\lim_{d\to\infty} \operatorname{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are parameters on } X) := \lim_{d\to\infty} \frac{\#\operatorname{Par}_{d,k}}{\#S_d^{k+1}}.$$

Proposition 5.1. *If $k < n$ then*

$$\operatorname{Prob}(f_0, f_1, \ldots, f_k \textit{ of degree } d \textit{ are parameters on } X) \geq 1 - \widehat{\deg}(X)(1 + d + d^2 + \cdots + d^k)q^{-\binom{n-k+d}{n-k}}.$$

Proof. We induct on k and largely follow the structure of the proof of Theorem 1.2. First, let $k = 0$. A polynomial f_0 will fail to be a parameter on X if and only if it is a zero divisor on a top-dimensional component V of X. There are at most $\widehat{\deg}(X)$ many such components. As argued in the proof of Theorem 1.2, the set of zero divisors on V corresponds to the elements of $H^0(\mathbb{P}^r, \mathcal{I}_V(d))$ which has codimension at least $\binom{n+d}{d}$ in S_d. It follows that

$$\operatorname{Prob}(f_0 \text{ of degree } d \text{ is not a parameter on } X) \leq \widehat{\deg}(X)q^{-\binom{n+d}{d}}.$$

Now consider the induction step. We will separately compute the probability that $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ lies in $\mathscr{D}^{\text{bad}}_{k,d}(X)$ and the probability that $\boldsymbol{f}$ lies in $\mathscr{D}^{\text{good}}_{k,d}(X)$. By definition, the projection π maps $\mathscr{D}^{\text{bad}}_{k,d}(X)$

onto $\mathscr{D}_{k-1,d}(X)$, and by induction

$$\begin{aligned}\operatorname{Prob}(\pi(\boldsymbol{f}) \in \mathscr{P}_{k-1,d}(X)(\mathbb{F}_q)) &\leq \widehat{\deg}(X)(1+d+d^2+\cdots+d^{k-1})q^{-\binom{n-k+1+d}{n-k+1}} \\ &\leq \widehat{\deg}(X)(1+d+d^2+\cdots+d^{k-1})q^{-\binom{n-k+d}{n-k}}.\end{aligned}$$

We now assume $\boldsymbol{f} \notin \mathscr{D}^{\text{bad}}_{k,d}(X)$. We thus have that $f_0, f_1, \ldots, f_{k-1}$ are parameters on X. As in the proof of Theorem 1.2, the fiber $\pi^{-1}(\boldsymbol{f})$ can be identified with $\mathscr{D}_{0,d}(X')$ where $X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{k-1})$. By construction $\dim X' = n-k$ and by Lemma 2.3, $\widehat{\deg}(X') \leq \widehat{\deg}(X) \cdot d^k$. Our inductive hypothesis thus implies that

$$\operatorname{Prob}\left(\begin{array}{l}(f_0, f_1, \ldots, f_k) \in \mathscr{D}_{k,d}(X)(\mathbb{F}_q) \text{ given that} \\ (f_0, f_1, \ldots, f_{k-1}) \notin \mathscr{D}_{k-1,d}(X)(\mathbb{F}_q)\end{array}\right) \leq \widehat{\deg}(X')q^{-\binom{n-k+d}{n-k}} \leq \widehat{\deg}(X) \cdot d^k q^{-\binom{n-k+d}{n-k}}.$$

Combining the estimates for $\mathscr{D}^{\text{bad}}_{k,d}(X)$ and $\mathscr{D}^{\text{good}}_{k,d}(X)$ yields the proposition. □

Proof of Theorem 1.3. If $k < n$, then we apply Proposition 5.1 to obtain

$$\lim_{d\to\infty} \operatorname{Prob}\left(\begin{array}{l}f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are parameters on } X\end{array}\right) \geq \lim_{d\to\infty} 1 - \widehat{\deg}(X)(d^0+d^1+\cdots+d^k)q^{-\binom{n-k+d}{n-k}} = 1.$$

Now let $k = n$. For completeness, we summarize the proof of [Bucur and Kedlaya 2012, Theorem 1.2]. We fix e, which will go to ∞, and separate the argument into low, medium, and high degree cases.

Low degree argument. For a zero dimensional subscheme Y, we have that S_d surjects on $H^0(Y, \mathcal{O}_Y(d))$ when $d \geq \deg Y - 1$ [Poonen 2004, Lemma 2.1]. So if $d > \deg P - 1$, the probability that $f_0, f_1, \ldots, f_n$ all vanish at a closed point $P \in X$ is $1 - q^{-(n+1)\deg P}$. If $Y \subseteq X$ is the union of all points of degree $\leq e$, and if $d \geq \deg Y - 1$, then the surjection onto $H^0(Y, \mathcal{O}_Y(d))$ implies that the probabilities at the points $P \in Y$ behave independently. This yields:

$$\operatorname{Prob}\left(\begin{array}{l}f_0, f_1, \ldots, f_n \text{ of degree } d \text{ are parameters on } X \\ \text{at all points } P \in X \text{ where } \deg(P) \leq e\end{array}\right) = \prod_{\substack{P\in X \\ \deg(P)\leq e}} 1 - q^{-(n+1)\deg P}.$$

Medium degree argument. Our argument is nearly identical to [Poonen 2004, Lemma 2.4], and covers all points whose degree lies in the range $\left[e+1, \frac{d}{n+1}\right]$. For any such point $P \in X$, S_d surjects onto $H^0(P, \mathcal{O}_P(d))$ and thus the probability that $f_0, f_1, \ldots, f_n$ all vanish at P is $q^{-\ell(n+1)}$. By [Lang and Weil 1954], $\#X(\mathbb{F}_{q^\ell}) \leq Kq^{\ell n}$ for some constant K independent of ℓ. We have

$$\begin{aligned}\operatorname{Prob}\left(\begin{array}{l}f_0, f_1, \ldots, f_n \text{ of degree } d \text{ all vanish} \\ \text{at some } P \in X \text{ where } e < \deg(P) \leq \left\lfloor \frac{d}{n+1} \right\rfloor\end{array}\right) &\leq \sum_{\ell=e+1}^{\left\lfloor \frac{d}{n+1} \right\rfloor} \#X(\mathbb{F}_{q^\ell})q^{-\ell(n+1)} \\ &\leq \sum_{\ell=e+1}^{\infty} Kq^{\ell n}q^{-(n+1)\ell} \\ &= \frac{Kq^{-e-1}}{1-q^{-1}}.\end{aligned}$$

This tends to 0 as $e \to \infty$, and therefore does not contribute to the asymptotic limit.

High degree argument. By the case when $k = n-1$, we may assume that $f_0, f_1, \ldots, f_{n-1}$ form a system of parameters with probability $1 - o(1)$. So we let V be one of the irreducible components of this intersection (over $\mathbb{F}_q$) and we let R be its homogeneous coordinate ring. If $\deg V \le \frac{d}{n+1}$, then it can be ignored as we considered such points in the low and medium degree cases. Hence, we can assume $\deg V > \frac{d}{n+1}$. Since $\dim R_\ell \ge \min\{\ell+1, \deg R\}$ for all ℓ, the probability that f_n vanishes along V is at most $q^{-\lfloor d/(n+1)\rfloor - 1}$. Hence the probability of vanishing on some high degree point is bounded by $O(d^n q^{-\lfloor d/(n+1)\rfloor - 1})$ which is $o(1)$ as $d \to \infty$.

Combining the various parts as $e \to \infty$, we see that the low degree argument converges to $\zeta_X(n+1)^{-1}$ and the contributions from the medium and high degree points go to 0. □

Remark 5.2. It might be interesting to consider variants of Theorem 1.3 that allow imposing conditions along closed subschemes, similar to Poonen's Bertini with Taylor coefficients [Poonen 2004, Theorem 1.2]. For instance, [Kedlaya 2005, Theorem 1] might be provable by such an approach, though this would be more complicated than the original proof.

Proposition 5.1 yields an effective bound on the degree of a full system of parameters over a finite field. Sharper bounds can be obtained if one allows the f_i to have different degrees.

Corollary 5.3. (1) *If d_1 satisfies $d_1^{n-1} q^{-d_1-1} < (n \cdot \widehat{\deg}(X))^{-1}$, then there exist $g_0, g_1, \ldots, g_{n-1}$ of degree d_1 that are parameters on X.*

(2) *Let X' be 0-dimensional. If $\max\{d_2+1, q\} \ge \widehat{\deg}(X')$ then there exists a degree d_2 parameter on X'.*

Proof. Applying Proposition 5.1 in the case $k = n-1$ yields (1). For (2), let f be a random degree d polynomial and let $P \in X'$ be a closed point. Since the dimension of the image of S_d in $H^0(P, \mathcal{O}_P(d))$ is at least $\min\{d+1, \deg P\}$, the probability that f vanishes at P is at worst $q^{-\min\{d+1, \deg P\}}$ which is at least q^{-1}. It follows that the probability that a degree d function vanishes on some point of X' is at worst $\sum_{P \in X'} q^{-1} \le \widehat{\deg}(X') q^{-1}$. Thus if $q > \widehat{\deg}(X')$, this happens with probability strictly less than 1. On the other hand, if $d+1 \ge \widehat{\deg}(X')$ then polynomials of degree d surject onto $H^0(X', \mathcal{O}_{X'}(d))$ and hence we can find a parameter on X' by choosing a polynomial that restricts to a unit on X'. □

Proof of Theorem 1.5. If $\dim X = 0$, then we can directly apply Corollary 5.3(2) to find a parameter of degree d. So we assume $n := \dim X > 0$. Since $d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$ it follows that $(n \cdot \widehat{\deg}(X))^{-1} > q^{-d} d^n > q^{-d-1} d^{n-1}$. Applying Corollary 5.3(1), we find $g_0, g_1, \ldots, g_{n-1}$ in degree d that are parameters on X. Let $X' = X \cap V(g_0, g_1, \ldots, g_{n-1})$. Since $\max\{d, \frac{q}{d^n}\} \ge \widehat{\deg}(X)$ it follows that $\max\{d^{n+1}, q\} \ge d^n \widehat{\deg}(X) \ge \widehat{\deg}(X')$, and Corollary 5.3(2) yields a parameter g_n of degree d^{n+1} on X'. Thus $g_0^{d^n}, g_1^{d^n}, \ldots, g_{n-1}^{d^n}, g_n$ are parameters of degree d^{n+1} on X. □

6. Probabilistic analysis, II: The error term and proof of Theorem 1.4

In this section, we let $k < n$ and we analyze the error terms in Theorem 1.3 more precisely. In particular, we prove Theorem 1.4, which shows that the probabilities are controlled by the probability of vanishing along an $(n-k)$-dimensional subvariety, with varieties of lowest degree contributing the most.

Our proof of Theorem 1.4 adapts Poonen's sieve in a couple of key ways. The first big difference is that instead of sieving over closed points, we will sieve over $(n-k)$-dimensional subvarieties of X; this is because polynomials $f_0, f_1, \ldots, f_k$ will fail to be parameters on X only if they all vanish along some $(n-k)$-dimensional subvariety.

The second difference is that the resulting probability formula will not be a product of local factors. This is because the values of a function can never be totally independent along two higher dimensional varieties with a nontrivial intersection. For instance, Lemma 6.1 shows that the probability that a degree d polynomial vanishes along a line is $q^{-(d+1)}$, but the probability of vanishing along two lines that intersect in a point is $q^{-(2d+1)} > (q^{-(d+1)})^2$.

The following result characterizes the individual probabilities arising in our sieve.

Lemma 6.1. *If $Z \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ is a reduced, projective scheme over a finite field $\mathbb{F}_q$ with homogeneous coordinate ring R then*

$$\text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ all vanish along } Z) = \left(\frac{1}{\#R_d}\right)^{k+1}.$$

If d is at least the Castelnuovo–Mumford regularity of the ideal sheaf of Z, then

$$\text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ all vanish along } Z) = q^{-(k+1)h^0(Z,\mathcal{O}_Z(d))}.$$

Proof. Let $I \subseteq S$ be the homogeneous ideal defining Z, so that $R = S/I$. An element $h \in S_d$ vanishes along Z if and only if it restricts to 0 in R_d i.e., if and only if it lies in I_d. Since we have an exact sequence of $\mathbb{F}_q$-vector spaces:

$$0 \to I_d \to S_d \to R_d \to 0$$

we obtain

$$\text{Prob}(h \text{ vanishes on } Z) = \frac{\#I_d}{\#S_d} = \frac{1}{\#R_d}.$$

For $k+1$ elements of S_d, the probabilities of vanishing along Z are independent and this yields the first statement of the lemma.

We write $\tilde{I}$ for the ideal sheaf of Z. If d is at least the regularity of $\tilde{I}$, then $H^1(\mathbb{P}^r_{\mathbb{F}_q}, \tilde{I}(d)) = 0$. Hence there is a natural isomorphism between R_d and $H^0(Z, \mathcal{O}_Z(d))$. Thus, we have

$$\frac{1}{\#R_d} = q^{-h^0(Z,\mathcal{O}_Z(d))},$$

yielding the second statement. □

Proof of Theorem 1.4. Throughout the proof, we set $\epsilon_{e,k}$ to be the error term for a given e and k, namely $\epsilon_{e,k} := q^{-e(k+1)\binom{n-k+d}{n-k}}$. We also set:

$$\begin{aligned}
\mathrm{Par}_{d,k} &:= \{f_0, f_1, \ldots, f_k \text{ are parameters on } X\} \\
\mathrm{Low}_{d,k,e} &:= \left\{ \begin{array}{l} f_0, f_1, \ldots, f_k \text{ all vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } \deg(Z) \le e \end{array} \right\} \\
\mathrm{Med}_{d,k,e} &:= \left\{ \begin{array}{l} (f_0, f_1, \ldots, f_k) \notin \mathrm{Low}_{d,k,e} \text{ which all vanish along a variety } Z \\ \text{where } \dim Z = (n-k) \text{ and } e < \deg(Z) \le e(k+1) \end{array} \right\} \\
\mathrm{High}_{d,k,e} &:= \left\{ \begin{array}{l} (f_0, f_1, \ldots, f_k) \notin \mathrm{Low}_{d,ke} \cup \mathrm{Med}_{d,k,e} \text{ which all vanish along} \\ \text{a variety } Z \text{ where } \dim Z = (n-k) \text{ and } e(k+1) < \deg(Z) \end{array} \right\}.
\end{aligned}$$

Note that if $f_0, f_1, \ldots, f_k$ all vanish along a variety of dimension $> n-k$ then they will also all vanish along a high degree variety, and hence we do not need to count this case separately. For $\boldsymbol{f} = f_0, f_1, \ldots, f_k \in S_d^{k+1}$, we thus have

$$\begin{aligned}
\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Par}_{d,k}) &= 1 - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e} \cup \mathrm{Med}_{d,k,e} \cup \mathrm{High}_{d,k,e}) \\
&= 1 - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e}) - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{Med}_{d,k,e}) - \mathrm{Prob}(\boldsymbol{f} \in \mathrm{High}_{d,k,e}).
\end{aligned}$$

It thus suffices to show that

$$\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Low}_{d,k,e}) = \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \le e}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))} + o(\epsilon_{e,k})$$

and that $\mathrm{Prob}(\boldsymbol{f} \in \mathrm{Med}_{d,k,e})$ and $\mathrm{Prob}(\boldsymbol{f} \in \mathrm{High}_{d,k,e})$ are each in $o(\epsilon_{e,k})$.

We proceed by induction on k. When $k = 0$ the condition that f_0 is a parameter on X is equivalent to f_0 not vanishing along a top-dimensional component of X. Thus, combining Lemma 6.1 with an inclusion/exclusion argument implies the exact result:

$$\mathrm{Prob}(f_0 \in \mathrm{Par}_{d,0}) = 1 - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))}.$$

By basic properties of the Hilbert polynomial, as $d \to \infty$ we have

$$h^0(Z, \mathcal{O}_Z(d)) = \frac{\deg(Z)}{n!} d^n + o(d^n) = \deg(Z)\binom{n+d}{d} + o(d^n).$$

Hence for the fixed degree bound e, we obtain

$$\begin{aligned}\text{Prob}(f \in \text{Par}_{d,0}) &= 1 - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z > e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} \\ &= 1 - \sum_{\substack{Z \subseteq X \text{reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-h^0(Z, \mathcal{O}_Z(d))} + o(\epsilon_{e,0}).\end{aligned}$$

We now consider the induction step. Let $\boldsymbol{f} = (f_0, f_1, \ldots, f_k)$ drawn randomly from S_d^{k+1}. Here we separate into low, medium, and high degree cases.

Low degree argument. Let $\boldsymbol{V}_{k,e}$ denote the set of integral projective varieties $V \subseteq X$ of dimension $n-k$ and degree $\leq e$. We have $\boldsymbol{f} \in \text{Low}_{d,k,e}$ if and only if $\boldsymbol{f}$ vanishes on some $V \in \boldsymbol{V}_{k,e}$. Since $\boldsymbol{V}_{k,e}$ is a finite set, we may use an inclusion-exclusion argument to get

$$\text{Prob}(\boldsymbol{f} \in \text{Low}_{d,k,e}) = \sum_{\substack{Z \subseteq X \text{ a union of} \\ V \in \boldsymbol{V}_{k,e}}} (-1)^{|Z|-1} \text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ all vanish along } Z).$$

If $\deg Z > e$ then Lemma 6.1 implies that those terms can be absorbed into the error term $o(\epsilon_{e,k})$. Moreover, assuming that Z is a union of $V \in \boldsymbol{V}_{k,e}$ satisfying $\deg(Z) \leq e$ is equivalent to assuming Z is reduced and equidimensional of dimensional $n-k$. We thus have

$$= \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} \text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ all vanish along } Z) + o(\epsilon_{e,k}).$$

Medium degree argument. We know that $\text{Prob}(\boldsymbol{f} \in \text{Med}_{d,k,e})$ is bounded by the sum of the probabilities that f vanishes along some irreducible variety V in $\boldsymbol{V}_{k,e(k+1)} \setminus \boldsymbol{V}_{k,e}$.

$$\text{Prob}(\boldsymbol{f} \in \text{Med}_{d,k,e}) \leq \sum_{Z \in \boldsymbol{V}_{k,e(k+1)} \setminus \boldsymbol{V}_{k,e}} \text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ all vanish along } Z).$$

Lemma 6.1 implies that each summand on the right-hand side lies in $o(\epsilon_{e,k})$. This sum is finite and thus $\text{Prob}(\boldsymbol{f} \in \text{Med}_{d,k,e})$ is in $o(\epsilon_{e,k})$.

High degree argument. Proposition 5.1 implies that $f_0, f_1, \ldots, f_{k-1}$ are parameters on X with probability $1 - o(q^{-\binom{n-k+1+d}{d}}) \geq 1 - o(\epsilon_{e,k})$ for any e. Hence we may restrict our attention to the case where $f_0, f_1, \ldots, f_{k-1}$ are parameters on X.

Let $V_1, V_2, \ldots, V_s$ be the irreducible components of $X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{k-1})$ that have dimension $n-k$. We have that $f_0, f_1 \ldots, f_k$ fail to be parameters on X if and only if f_k vanishes on some V_i. We can assume that f_k does not vanish on any V_i where $\deg V_i \leq e(k+1)$ as we have already accounted for this possibility in the low and medium degree cases. After possibly relabeling the components, we let $V_1, V_2, \ldots, V_t$ be the components of degree $> e(k+1)$ and $X'' = V_1 \cup V_2 \cup \cdots \cup V_t$. Using Lemma 2.3,

we compute $\widehat{\deg}(X'') \leq \widehat{\deg}(X') = \widehat{\deg}(X) \cdot d^k$. It follows that X'' has at most $\widehat{\deg}(X)d^k/(e(k+1))$ irreducible components.

Now for the key point: since the value of d is not necessarily larger than the Castelnuovo–Mumford regularity of V_i, we cannot use a Hilbert polynomial computation to bound the probability that f_k vanishes along V_i. Instead, we use the lower bound for Hilbert functions obtained in Lemma 3.1. Let $\epsilon = \frac{1}{2}$, though any choice of ϵ would work. We write $R(V_i)$ for the homogeneous coordinate ring of V_i. For any $1 \leq i \leq t$, Lemmas 3.1 and 6.1 yield

$$\text{Prob}(f_k \text{ of degree } d \text{ vanishes along } V_i) = q^{-\dim R(V_i)_d} \leq q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$$

whenever $d \geq Ce^{k+1}$. Combining this with our bound on the number of irreducible components of X'' gives $\text{Prob}(\boldsymbol{f} \in \text{High}_{d,k,e}) \leq \frac{1}{e(k+1)}\widehat{\deg} X d^k q^{-(e(k+1)+\epsilon)\binom{n-k+d}{n-k}}$ which is in $o(\epsilon_{e,k})$. □

Corollary 6.2. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme and let $k < n$. Then*

$$\lim_{d\to\infty} q^{(k+1)\binom{n-k+d}{n-k}} \text{Prob}\left(\begin{array}{c} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{are } \underline{\text{not}} \text{ parameters on } X \end{array}\right) = \#\{(n-k)\text{-planes } L \subseteq \mathbb{P}^r_{\mathbb{F}_q} \text{such that } L \subseteq X\}.$$

Proof. Let N denote the number of $(n-k)$-planes $L \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ such that $L \subseteq X$. Choosing $e = 1$ in Theorem 1.4, we compute that

$$\text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are parameters on } X) = 1 - Nq^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}}).$$

It follows that

$$\text{Prob}(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are } \underline{\text{not}} \text{ parameters on } X) = Nq^{-(k+1)\binom{n-k+d}{n-k}} + o(q^{-(k+1)\binom{n-k+d}{n-k}}).$$

Dividing both sides by $q^{-(k+1)\binom{n-k+d}{n-k}}$ and taking the limit as $d \to \infty$ yields the corollary. □

7. Passing to $\mathbb{Z}$ and $\mathbb{F}_q[t]$

In this section we prove Corollaries 1.6 and 1.7.

Definition 7.1. Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and fix a finitely generated, free B-module B^s and a subset $\mathcal{S} \subseteq B^s$. Given $a \in B^s$ we write $a = (a_1, a_2, \ldots, a_s)$. The *density* of $\mathcal{S} \subseteq B^s$ is

$$\text{Density}(\mathcal{S}) := \begin{cases} \lim_{N\to\infty} \frac{\#\{a\in\mathcal{S} | \max\{|a_i|\}\leq N\}}{\#\{a\in\mathbb{Z}^s | \max\{|a_i|\}\leq N\}} & \text{if } B = \mathbb{Z}, \\ \lim_{N\to\infty} \frac{\#\{a\in\mathcal{S} | \max\{\deg a_i\}\leq N\}}{\#\{a\in\mathbb{F}_q[t]^s | \max\{\deg a_i\}\leq N\}} & \text{if } B = \mathbb{F}_q[t]. \end{cases}$$

Proof of Corollary 1.6. For clarity, we will prove the result over $\mathbb{Z}$ in detail and at the end, mention the necessary adaptations for $\mathbb{F}_q[t]$.

We first let $k < n$. Given degree d polynomials $f_0, f_1, \ldots, f_k$ with integer coefficients and a prime p, let $\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k$ be the reduction of these polynomials mod p. Then $\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k$ will be parameters on X_p if and only if the point $\overline{\boldsymbol{f}} = (\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k)$ lies $\mathscr{D}_{d,k}(X_{\mathbb{F}_p})$. As noted in Remark 4.1, this is

equivalent to asking that $\overline{\boldsymbol{f}}$ is an $\mathbb{F}_p$-point of $\mathscr{D}_{k,d}(X_{\mathbb{Z}})$. Thus, we may apply [Ekedahl 1991, Theorem 1.2] to $\mathscr{D}_{d,k}(X_{\mathbb{Z}}) \subseteq \mathscr{A}_{k,d}$ (using $M = 1$) to conclude that

$$\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{that restrict to parameters on } X_p \text{ for all } p \end{array}\right\} = \prod_p \text{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array}\right).$$

Applying Proposition 5.1 to estimate the individual factors; we have:

$$\begin{aligned}\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \text{ that} \\ \text{restrict to parameters on } X_p \text{ for all } p \end{array}\right\} &= \lim_{d\to\infty} \prod_p \text{Prob}\left(\begin{array}{l} f_0, f_1, \ldots, f_k \text{ of degree } d \\ \text{restrict to parameters on } X_p \end{array}\right) \\ &\geq \lim_{d\to\infty} \prod_p (1 - \widehat{\deg}(X_p)(1 + d + \cdots + d^k) p^{-\binom{n-k+d}{n-k}}).\end{aligned}$$

Lemma 7.2 shows that there is an integer D where $D \geq \widehat{\deg}(X_p)$ for all p. Moreover, $1+d+\cdots+d^k \leq kd^k$ for all d, and hence:

$$\geq \lim_{d\to\infty} \prod_p (1 - Dkd^k p^{-\binom{n-k+d}{n-k}}).$$

For $d \gg 0$ we can make $Dkd^k p^{-\binom{n-k+d}{n-k}} \leq p^{-d/2}$ for all p simultaneously. Using $\zeta(n)$ for the Riemann zeta function, we get:

$$\geq \lim_{d\to\infty} \prod_p (1 - p^{-d/2}) \geq \lim_{d\to\infty} \zeta(d/2)^{-1} = 1.$$

We now consider the case $k = n$. This follows by a "low degree argument" exactly analogous to [Poonen 2004, Theorem 5.13]. Fix a large integer N and let Y be the union of all closed points $P \in X$ whose residue field $\kappa(P)$ has cardinality at most N. Since Y is a finite union of closed points, we see that for $d \gg 0$, there is a surjection

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \to H^0(Y, \mathcal{O}_Y(d)) \cong \bigoplus_{\substack{P\in X \\ \#\kappa(P)\leq N}} H^0(P, \mathcal{O}_P(d)) \to 0.$$

It follows that we have a product formula

$$\text{Density}\left\{\begin{array}{l} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ do not all} \\ \text{vanish on a point } P \text{ with } \#\kappa(P) \leq N \end{array}\right\} = \prod_{P\in X, \#\kappa(P)\leq N} \left(1 - \frac{1}{\#\kappa(P)^{n+1}}\right)$$

This is certainly an upper bound on the density of $f_0, f_1, \ldots, f_n$ that are parameters on X_p for all p. As $N \to \infty$ the right-hand side approaches $\zeta_X(n+1)^{-1}$. However, since the dimension of X is $n+1$, this zeta function has a pole at $s = n+1$ [Serre 1965, Theorems 1 and 3(a)]. Hence this asymptotic density equals 0. This completes the proof over $\mathbb{Z}$.

Over $\mathbb{F}_q[t]$, the key adaptation is to use [Poonen 2003, Theorem 3.1] in place of Ekedahl's result. Poonen's result is stated for a pair of polynomials, but it applies equally well to n-tuples of polynomials such as the n-tuples defining $\mathscr{D}_{k,d}(X)$. In particular, one immediately reduces to proving an analogue of

[Poonen 2003, Lemma 5.1], for n-tuples of polynomials which are irreducible over $\mathbb{F}_q(t)$ and which have gcd equal to 1; but the $n = 2$ version of the lemma then implies the $n \geq 2$ versions of the lemma.[2] The rest of our argument over $\mathbb{Z}$ works over $\mathbb{F}_q[t]$. □

Lemma 7.2. *Let $X \subseteq \mathbb{P}^r_B$ be any closed subscheme. There is an integer D where $D \geq \widehat{\deg}(X_s)$ for all $s \in \operatorname{Spec} B$.*

Proof. First we take a flattening stratification for X over B [EGA IV$_4$ 1967, Corollaire 6.9.3]. Within each stratum, the maximal degree of a minimal generator is semicontinuous, and we can thus find a degree e where X_s is generated in degree e for all $s \in \operatorname{Spec} B$. By [Bayer and Mumford 1993, Proposition 3.5], we then obtain that $\widehat{\deg}(X) \leq \sum_{j=0}^{n} e^{r-j}$. In particular defining $D := re^r$ will suffice. □

To prove Corollary 1.7, we use Corollary 1.6 to find a submaximal collection $f_0, f_1, \ldots, f_{n-1}$ which restrict to parameters on X_s for all $s \in \operatorname{Spec} B$. This cuts X down to a scheme $X' = X \cap \mathbb{V}(f_0, f_1, \ldots, f_{n-1})$ with 0-dimensional fibers over each point s. When $B = \mathbb{Z}$, such a scheme is essentially a union of orders in number fields, and we find the last element f_n by applying classical arithmetic results about the Picard groups of rings of integers of number fields. When $B = \mathbb{F}_q[t]$, we use similar facts about Picard groups of affine curves over $\mathbb{F}_q$.

An example illustrates this approach. Let $X = \mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x, y])$. A polynomial of degree d will be a parameter on X as long as the $d + 1$ coefficients are relatively prime. Thus as $d \to \infty$, the density of these choices will go to 1. However, once we have fixed one such parameter, say $5x - 3y$, it is much harder to find an element that will restrict to a parameter on $\mathbb{Z}[x, y]/(5x - 3y)$ modulo p for all p. In fact, the only possible choices are the elements which restrict to units on $\operatorname{Proj}(\mathbb{Z}[x, y]/(5x - 3y))$. Among the linear forms, these are

$$\pm(7x - 4y) + c(5x - 3y) \text{ for any } c \in \mathbb{Z}.$$

Hence, these elements arise with density zero, and yet they form a nonempty subset.

Lemmas 7.3 and 7.4 below are well-known to experts, but we sketch the proofs for clarity.

Lemma 7.3. *If $X' \subseteq \mathbb{P}^r_{\mathbb{Z}}$ is closed and finite over* $\operatorname{Spec}(\mathbb{Z})$, *then* $\operatorname{Pic}(X')$ *is finite.*

Proof. We first reduce to the case where X' is reduced. Let $\mathcal{N} \subseteq \mathcal{O}_{X'}$ be the nilradical ideal. If X' is nonreduced then there is some integer $m > 1$ for which $\mathcal{N}^m = 0$. Let X'' be the closed subscheme defined by $\mathcal{N}^{m-1}$. We have a short exact sequence $0 \to \mathcal{N}^{m-1} \to \mathcal{O}^*_{X'} \to \mathcal{O}^*_{X''} \to 1$ where the first map sends $f \mapsto 1 + f$. Since X' is affine and noetherian and $\mathcal{N}^{m-1}$ is a coherent ideal sheaf, we have that $H^1(X', \mathcal{N}^{m-1}) = H^2(X', \mathcal{N}^{m-1}) = 0$ [Hartshorne 1977, Theorem III.3.7]. Taking cohomology of the above sequence thus yields an isomorphism $\operatorname{Pic}(X') \cong \operatorname{Pic}(X'')$. Iterating this argument, we may assume X' is reduced.

We now have $X' = \operatorname{Spec}(B)$ where B is a finite, reduced $\mathbb{Z}$-algebra. If Q is a minimal prime of B, then B/Q is either zero dimensional or an order in a number field, and hence has a finite Picard group [Neukirch 1999, Theorem I.12.12]. If B has more than one minimal prime, then we let Q' be the

[2] We thank Bjorn Poonen for pointing out this reduction.

intersection of all of the minimal primes of B except for Q, and we again have an exact sequence in cohomology

$$\cdots \to (B/(Q+Q'))^* \to \operatorname{Pic}(X') \to \operatorname{Pic}(B/Q) \oplus \operatorname{Pic}(B/Q') \to \cdots$$

Since $(B/(Q+Q'))^*$ is a finite set, and since B/Q and B/Q' have fewer minimal primes than B, we may use induction to conclude that $\operatorname{Pic}(X')$ is finite. □

Lemma 7.4. *If C is an affine curve over $\mathbb{F}_q$, then* $\operatorname{Pic}(C)$ *is finite.*

Proof. If C fails to be integral, then an argument entirely analogous to the proof of Lemma 7.3 reduces us to the case C is integral. We next assume that C is nonsingular and integral, and that $\overline{C}$ is the corresponding nonsingular projective curve. Since C is affine we have $\operatorname{Pic}(C) = \operatorname{Pic}^0(C) \subseteq \operatorname{Pic}^0(\overline{C}) \cong \operatorname{Jac}(\overline{C})(\mathbb{F}_q)$, the last of which is a finite group. If C is singular, then the finiteness of $\operatorname{Pic}(C)$ follows from the nonsingular case by a minor adaptation of the proof of [Neukirch 1999, Proposition I.12.9]. □

Proof of Corollary 1.7. By Corollary 1.6, for $d \gg 0$ we can find polynomials $f_0, f_1, \ldots, f_{n-1}$ of degree d that restrict to parameters on X_s for all $s \in \operatorname{Spec} B$. Let $X' := \mathbb{V}(f_0, f_1, \ldots, f_{n-1}) \cap X$, which is finite over B by construction. Let A be the finite B-algebra where $\operatorname{Spec} A = X'$. Lemma 7.3 or 7.4 implies that $H^0(X', \mathcal{O}_{X'}(e)) = A$ for some e. We can thus find a polynomial f_n of degree e mapping onto a unit in the B-algebra A. It follows that $\mathbb{V}(f_n) \cap X' = \varnothing$. Replace f_i by f_i^e for $i = 0, \ldots, n-1$ and replace f_n by f_n^d. Then we have $f_0, f_1, \ldots, f_n$ of degree $d' := de$ and restricting to parameters on X_s for all $s \in \operatorname{Spec}(B)$ simultaneously.

We thus obtain a proper morphism $\pi : X \to \mathbb{P}^n_B$ where $X_s \to \mathbb{P}^n_{\kappa(s)}$ is finite for all s. Since π is quasifinite and proper, it is finite by [EGA IV$_3$ 1966, Théorème 8.11.1]. □

The following generalizes Corollary 1.7 to other graded rings.

Corollary 7.5. *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and let R be a graded, finite type B-algebra where* $\dim R \otimes_{\mathbb{Z}} \mathbb{F}_p = n+1$ *for all p. Then there exist $f_0, f_1, \ldots, f_n$ of degree d for some d such that $B[f_0, f_1, \ldots, f_n] \subseteq R$ is a finite extension.*

Proof. After replacing R by a high degree Veronese subring R', we may assume that R' is generated in degree one and contains no R'_+-torsion submodule, where $R'_+ \subseteq R'$ is the homogeneous ideal of strictly positive degree elements. Let $r+1$ be the number of generators of R'_1. Then there is a surjection $\phi\colon B[x_0, x_1, \ldots, x_r] \to R'$ inducing an embedding of $X := \operatorname{Proj}(R') \subseteq \mathbb{P}^r_B$. Since R' contains no R'_+-torsion submodule, the kernel of ϕ will be saturated with respect to $(x_0, x_1, \ldots, x_r)$ and hence R' will equal the homogeneous coordinate ring of X. Choosing $f_0, f_1, \ldots, f_n$ as in Corollary 1.7, it follows that $B[f_0, f_1, \ldots, f_n] \subseteq R'$ is a finite extension, and thus so is $B[f_0, f_1, \ldots, f_n] \subseteq R$. □

8. Examples

Example 8.1. By Corollary 6.2, it is more difficult to randomly find parameters on surfaces that contain lots of lines. Consider $\mathbb{V}(xyz) \subset \mathbb{P}^3$ which contains substantially more lines than $\mathbb{V}(x^2+y^2+z^2) \subset \mathbb{P}^3$.

Using [Macaulay2] to select 1,000,000 random pairs (f_0, f_1) of polynomials of degree two, the proportion that failed to be systems of parameters were

	$\mathbb{V}(xyz)$	$\mathbb{V}(x^2+y^2+z^2)$
$\mathbb{F}_2$	.2638	.1179
$\mathbb{F}_3$	.0552	.0059
$\mathbb{F}_5$	.0063	.0004

Example 8.2. Let $X \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ be a smooth cubic surface. Over the algebraic closure X has 27 lines, but it has between 0 and 27 lines defined over $\mathbb{F}_q$. For example, working over $\mathbb{F}_4$, the Fermat cubic surface X' defined by $x^3+y^3+z^3+w^3$ has 27 lines, while the cubic surface X defined by $x^3+y^3+z^3+aw^3$ where $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ has no lines defined over $\mathbb{F}_4$ [Debarre et al. 2017, Section 3]. It will thus be more difficult to find parameters on X than on X'. Using [Macaulay2] to select 100,000 random pairs (f_0, f_1) of polynomials of degree two, 0.62% failed to be parameters on X whereas no choices whatsoever failed to be parameters on X'. This is in line with the predictions from Corollary 6.2; for instance, in the case of X, we have $27 \cdot 4^{-2\cdot 3} \approx 0.66\%$.

Example 8.3. Let $X = [1:4] \cup [3:5] \cup [4:5] = \mathbb{V}((4x-y)(5x-3y)(5x-4y)) \subseteq \mathbb{P}^1_{\mathbb{Z}}$ and let R be the homogeneous coordinate ring of X. The fibers are 0-dimensional so finding a Noether normalization $X \to \mathbb{P}^0_{\mathbb{Z}}$ is equivalent to finding a single polynomial f_0 that restricts to a unit on each of the points simultaneously. We can find such an f_0 of degree d if and only if the induced map of free $\mathbb{Z}$-modules $\mathbb{Z}[x, y]_d \to R_d$ is surjective. A computation in [Macaulay2] shows that this happens if and only if d is divisible by 60.

Example 8.4. Let $R = \mathbb{Z}[x]/(3x^2-5x) \cong \mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{3}\right]$. This is a flat, finite type $\mathbb{Z}$-algebra where every fiber has dimension 0, yet it is not a finite extension of $\mathbb{Z}$. However, if we take the projective closure of $\operatorname{Spec}(R)$ in $\mathbb{P}^1_{\mathbb{Z}}$, then we get $\operatorname{Proj}(\bar{R})$ where $\bar{R} = \mathbb{Z}[x, y]/(3x^2-5xy)$. If we then choose $f_0 := 4x-7y$, we see that $\mathbb{Z}[f_0] \subseteq \bar{R}$ is a finite extension of graded rings.

Example 8.5. Let $\boldsymbol{k}$ be a field and let $X = [1:1+t] \cup [1-t:1] = \mathbb{V}((y-(1+t)x)(x-(1-t)y)) \subseteq \mathbb{P}^1_{\boldsymbol{k}[t]}$. Let R be the homogeneous coordinate ring of X. In degree d, we have the map $\phi_d : \boldsymbol{k}[t][x, y]_d \cong \boldsymbol{k}[t]^{d+1} \to R_d \cong \boldsymbol{k}[t]^2$. Choosing the standard basis $x^d, x^{d-1}y, \ldots, y^d$ for the source of ϕ_d, and the two points of X for the target, we can represent ϕ_d by the matrix

$$\begin{pmatrix} 1 & 1+t & (1+t)^2 & \cdots & (1+t)^d \\ (1-t)^d & (1-t)^{d-1} & (1-t)^{d-2} & \cdots & 1 \end{pmatrix}.$$

It follows that $\operatorname{im}\phi_d = \operatorname{im}\left(\begin{smallmatrix} t^2 & (1+t)^d \\ 0 & 1 \end{smallmatrix}\right) = \operatorname{im}\left(\begin{smallmatrix} t^2 & 1+dt \\ 0 & 1 \end{smallmatrix}\right)$. The image of ϕ_d thus contains a unit if and only if the characteristic of $\boldsymbol{k}$ is p and $p \mid d$. In particular, if $\boldsymbol{k} = \mathbb{Q}$, then we cannot find a polynomial f_0 inducing a finite map $X \to \mathbb{P}^0_{\mathbb{Q}[t]}$.

Example 8.6. Let $\boldsymbol{k}$ be any field, let $B = \boldsymbol{k}[s, t]$, and let $X = [s:1] \cup [1:t] = \mathbb{V}((x-sy)(y-tx)) \subseteq \mathbb{P}^1_B$. We claim that for any $d > 0$, there does not exist a polynomial that restricts to a parameter on X_b for each

point $b \in B$. Assume for contradiction that we had such an $f = \sum_{i=0}^{d} c_i s^i t^{d-i}$ with $c_i \in B$. After scaling, we obtain

$$f([s:1]) = c_0 s^d + c_1 s^{d-1} + \cdots + c_d = 1 \quad \text{and} \quad f([1:t]) = c_0 + c_1 t + \cdots + c_d t^d = \lambda$$

where $\lambda \in B^* = \boldsymbol{k}^*$. Substituting for c_d we obtain

$$f([1:t]) = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} + (1 - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s))t^d = \lambda,$$

which implies that

$$\begin{aligned} \lambda - t^d &= c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s) t^d \\ &= (c_0 - c_0 s^d t^d) + (c_1 t - c_1 s^{d-1} t^d) + \cdots + (c_{d-1} t^{d-1} - c_{d-1} s t^d) \\ &= (1 - st) h(s,t) \end{aligned}$$

where $h(s,t) \in \boldsymbol{k}[s,t]$. This implies that $\lambda - t^d$ is divisible by $(1-st)$, which is a contradiction.

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