The top weight cohomology of A_g ~ Juliette Bruce, Berkeley / MSRI

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- A_g = moduli space of principally polarized abelian varieties of dim. g

  $\dim A_g = \binom{g+1}{2} = d$

- Question: (Grushevsky '09): Does A_g have non-trivial odd cohomology for some g?

- Previous results...

  A) Igusa '62: $H^k(A_2;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=0,2 \\ 0 & \text{else} \end{cases}$

  B) Hain '02: $H^k(A_3;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=0,2,4 \\ \mathbb{Q}^2 & k=6 \\ 0 & \text{else} \end{cases}$ ← Also computed Hodge structure

  C) Tommasi '05: $H^5(\mathcal{M}_4;\mathbb{Q}) \cong \mathbb{Q}$

- Theorem: (Bruce, Brandt, Chan, Melo, Moreland, Wolfe): For $2 \leq g \leq 7$ the top weight cohomology of A_g is zero except in the following cases:

  1) $Gr^W_{12} H^k(A_3;\mathbb{Q}) \cong \mathbb{Q}$ when $k=6$

  2) $Gr^W_{30} H^k(A_5;\mathbb{Q}) \cong \mathbb{Q}$ when $k=15, 20$

  3) $Gr^W_{42} H^k(A_6;\mathbb{Q}) \cong \mathbb{Q}$ when $k=30$

  4) $Gr^W_{56} H^k(A_7;\mathbb{Q}) \cong \mathbb{Q}$ when $k=28, 33, 37, 42$

  Answer's Grushevsky's question affirmatively.

- The weight filtration is supported in degrees $[0,2d] \longrightarrow H^k(A_g;\mathbb{Q}) \twoheadrightarrow Gr^W_{2d} H^k(A_g;\mathbb{Q})$.

- Theorem: (BBCMMW): There exists a chain complex (the perfect complex) $P_\bullet^g$ such that there is a canonical isomorphism:

  ← Kontsevich's Graph complex.

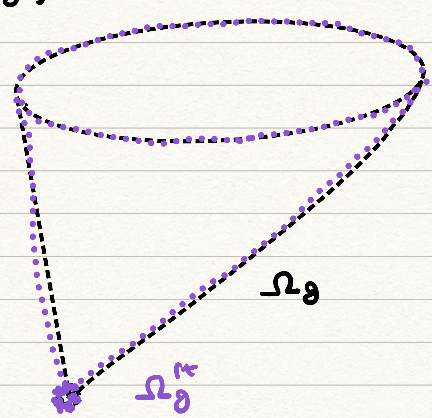
$$H_{k-1}(P_\bullet^g) \cong Gr^W_{2d} H^{2d-k}(A_g;\mathbb{Q}).$$

$$\Omega_g = \left\{ \begin{array}{c} \text{positive definite} \\ g\times g \text{ Quadratic} \\ \text{Forms}/\mathbb{R} \end{array} \right\} \subseteq \mathbb{R}^{\binom{g+1}{2}}$$

$$\Omega_g^{rt} = \left\{ \begin{array}{c} \text{Positive semi-definite} \\ g\times g \text{ Quadratic Forms} \\ \text{w/ rational kernel} \end{array} \right\}$$

the rational closure of Ω_g

$GL_g(\mathbb{Z})$ ↻ $A \mapsto A^T$



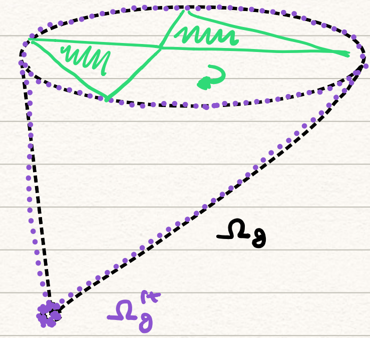
- **Def:** An admisible decomposition of Ω_g^{rt} is a set of rational polyhedral cones $\Sigma = \{\sigma\}$ such that

  1) Σ covers Ω_g^{rt}

  2) Σ is closed under intersections and faces

  3) If $A \in GL_g(\mathbb{Z})$ and $\sigma \in \Sigma$ then $A\sigma A^T \in \Sigma$

  4) $\# \Sigma / GL_g(\mathbb{Z}) < \infty$



- **The Perfect Cone Decomposition:** Let $Q \in \Omega_g$
  - $M(Q) = \{ \bar{x} \in \mathbb{Z}^g - \{0\} \mid \bar{x} \text{ minimizes } Q|_{\mathbb{Z}^g - \{0\}} \}$
  - $\sigma[Q] = \mathbb{R}_{\geq 0} \langle \bar{x}\bar{x}^T \mid \bar{x} \in M(Q) \rangle$
  - $\Sigma_g^P = \{ \sigma[Q] \mid Q \in \Omega_g \}$

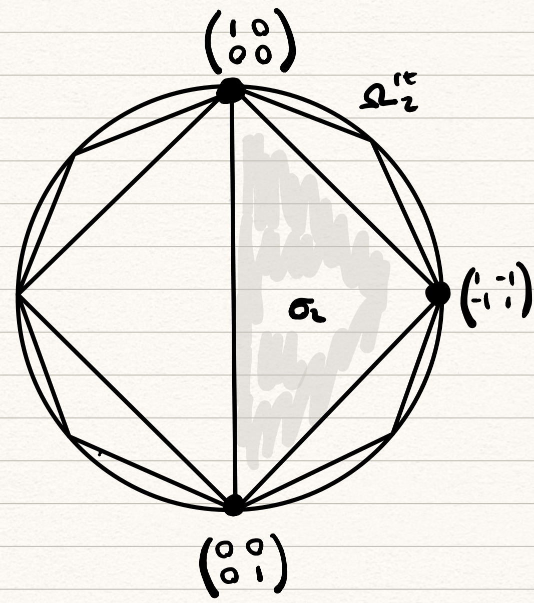
- **Theorem:** (Voronoi 1908): Σ_g^P is an admisible decomposition of Ω_g^{rt}.

- **Ex:** $(g=2)$:

$$Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \rightsquigarrow x^2 + xy + y^2$$

$$M(Q) = \{ (\pm 1, 0), (0, \pm 1), \pm(1,-1) \}$$

$$\sigma[Q] = \mathbb{R}_{\geq 0} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$



- **Def:** A cone σ is alternating $\iff$ given $A \in GL_g(\mathbb{Z})$ such that $A\sigma A^T = \sigma$ then A is orientation preserving on σ.

- **Def:** The perfect complex $P_\bullet^g$ is the complex where

$$P_k^g = \mathbb{Q} \left\langle \begin{array}{c} GL_g(\mathbb{Z})\text{-orbits of alternating} \\ \text{cones } \sigma \in \Sigma_g^P \text{ w/ } \dim \sigma = k+1 \end{array} \right\rangle$$

$$\partial_k([\sigma]) = \sum_{\substack{\tau \subseteq \sigma \\ \text{codim}=1}} (-1)^{\bullet} [\tau]$$

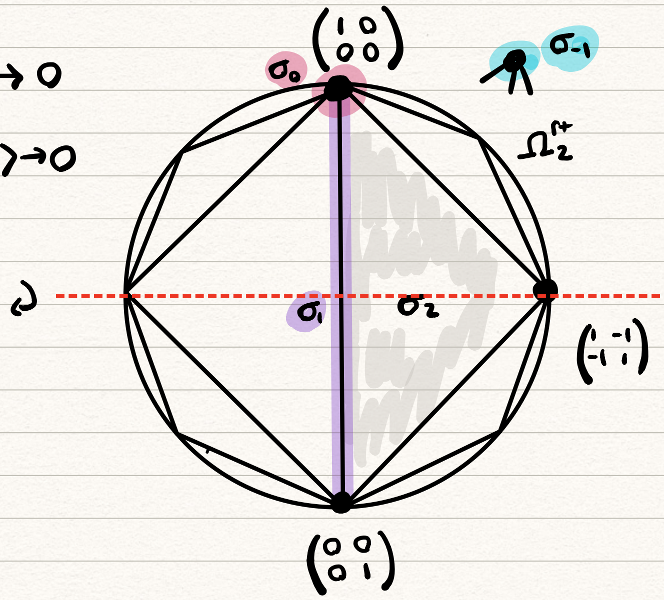
• **Ex:** $(g=0)$

$$0 \longrightarrow P_2^2 \longrightarrow P_1^2 \longrightarrow P_0^2 \longrightarrow P_{-1}^2 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Q}\langle\sigma_0\rangle \xrightarrow{\sim} \mathbb{Q}\langle\sigma_{-1}\rangle \to 0$$

$\Rightarrow H_k(P_\bullet^2) = 0$ for all k

$\Rightarrow \mathrm{Gr}_6^W H^k(A_2;\mathbb{Q}) = 0$ for all k.



• **Fact:** The data of an admisible decomposition Σ of Ω_g^{rt} gives rise to

1) A torodial compactification $A_g \subseteq \overline{A}_g^\Sigma$ (Ash, Mumford, Rapoport, Tai / Faltings, Choi)

2) The moduli space of tropical abelian varieties (Brannetti, Melo, Viviani / Chan, Melo, Viviani)

Proof of Theorem:

$$\begin{tikzcd}
H_{k-1}(P_\bullet^g) \arrow[rr, leftrightarrow, "\sim"] \arrow[d, leftrightarrow, "s"] & & \mathrm{Gr}_{2d}^W H^{2d-k}(A_g;\mathbb{Q}) \\
\widetilde{H}_{k-1}(LA_g^{\mathrm{Tr},P}) \arrow[r, phantom, "\cong"] & \widetilde{H}_{k-1}(\Delta(A_g \subseteq \overline{A}_g^P)) \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

* This is a bit of a lie because of smoothness *

Delinge / Chan, Galatius, Payne

■

• **Why stop at $g=7$?**

1) We are making use of work of Elbaz-Vincent, Gangl, Soule and Sikirić, Elbaz-Vincent, Kupers, Martinet

2) Well...

g	1	2	3	4	5	6	7	8	9
# top dim cones	1	1	1	2	3	7	33	10916	>500,000