

Computing Syzygies

joint w/ Daniel Erman, Steve Goldstein, Joy Yang

Where I work and live occupies the ancestral and current home land of the Muwukma Ohlone Tribe and other familial descendants of the sovereign Verano Band of Alameda County.

Black and Indigenous Lives Matter.

Where I work and live is only a few miles away from the site of Compton's Cafeteria, where, 51 years ago, in the face of transphobia and police violence, & state violence one of the first actions in the modern LGBTQ movement occurred lead by Black, Brown, and Indigenous trans women.



Hoppy Trans Week of Visibility.

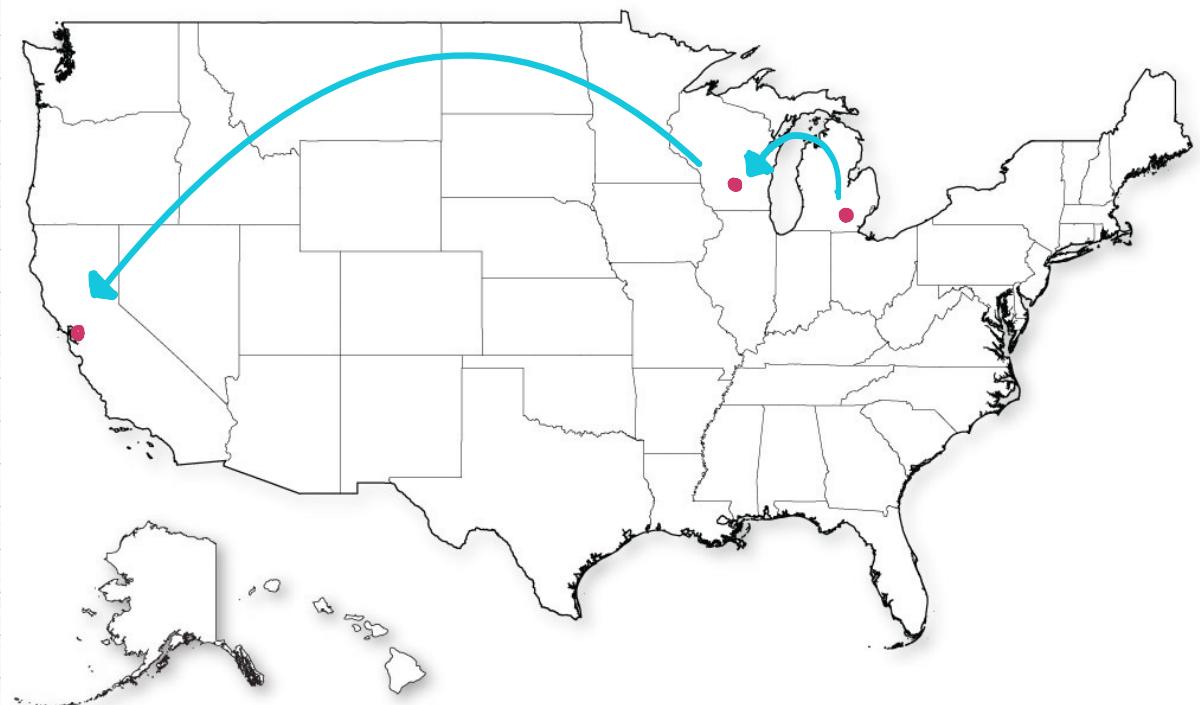
Black and Indigenous LGBTQ+ Lives Matter.



Who am I ?



Juliette Bruce
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Some Quotes

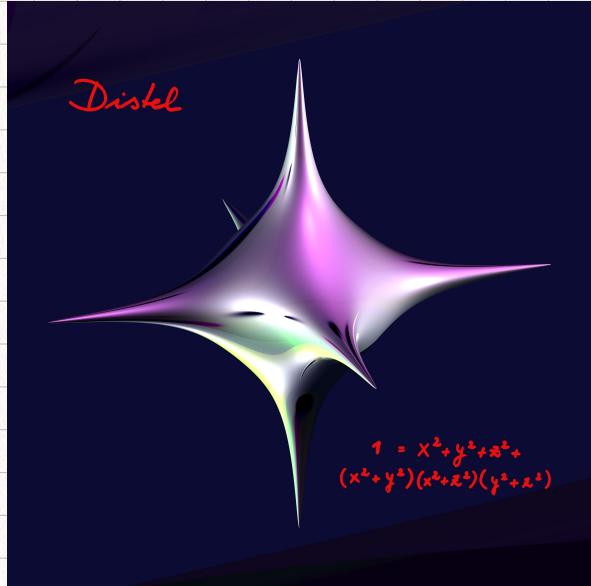
"Eventually you can't help but figure out that, while gender is a construct, so is a traffic light, and if ignore either of them, you get hit by cars. Which, also, are constructs."

~Imogen Binnie, Nevada~

"Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil & paper."

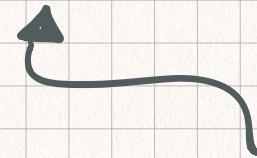
~George Polya '85~

Syzygies & Algebroic Varieties



algebraic variety

= "Solution set to a system
of polynomial equations"



astronomical syzygy

= when 3⁺ astronomical
bodies are roughly collinear.

Syzygies & Algebraic Varieties

- A syzygy is a polynomial relation among a collection of polynomials.
- Ex: Consider the monomials of degree 3 on s and t : s^3, s^2t, st^2, t^3 .

- 0th Syzygies

$$(s^3)(st^2) - (s^2t)^2 = 0$$

$$(s^2t)(t^3) - (st^2)^2 = 0$$

$$(s^3)(t^3) - (s^2t)(st^2) = 0$$

$$\begin{aligned}x &= s^3, \quad y = s^2t \\z &= st^2, \quad w = t^3\end{aligned}$$

$$F_1: xz - y^2 = 0$$

$$F_2: yw - z^2 = 0$$

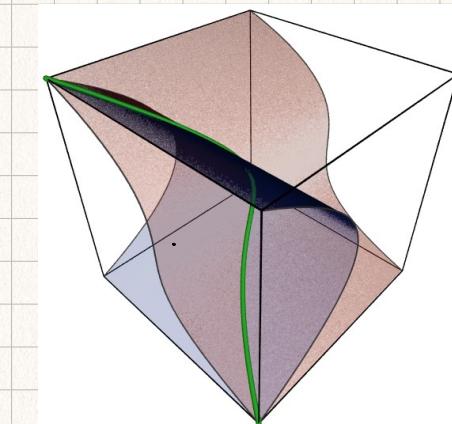
$$F_3: xw - yz = 0$$

- 1st Syzygies

$$z \cdot F_1 + x \cdot F_2 - y \cdot F_3 = 0$$

$$\omega \cdot F_1 + y \cdot F_2 - z \cdot F_3 = 0$$

there are no relations among these



Veronese Syzygies

- We can generalize to studying the syzygies of the monomials of degree d on n variables.

		# of variable						
		2	3	4	5	•	•	•
degree	3	✓	✓	✓	✗	✗	✗	✗
	4	✓	✓	✗	✗	✗	✗	✗
	5	✓	✓	✗	✗	✗	✗	✗
	6	✓	✗	✗	✗	✗	✗	✗
	7	✓	✗	✗	✗	✗	✗	✗
	7 ⁺	✓	~	✗	✗	✗	✗	✗

Studying syzygies
of the def. eqs.
 $\mathbb{P}^{n-1} \xrightarrow{\nu_d} \mathbb{P}^N$
 veronese map

Veronese Syzygies ($n=3$)

- The standard approach using Gröbner bases bogs down when $n=3$ and $d=5$, but we can convert the question of computing syzygies into linear algebra.

Via
Mössel
cohomology
techniques

- Namely we need compute the ranks of a number of really really really large matrices...

$$254,103,788,400 \times 902,737,143,000$$

- Using symmetries (multigradings) these huge matrices can be broken into a large number of very big matrices.



Part of the $d=6, n=3$ computation breaks down to 1,753 matrices w/ the largest being

$$4,175,947 \times 12,168,528$$

Veronese Syzygies ($n=3$)

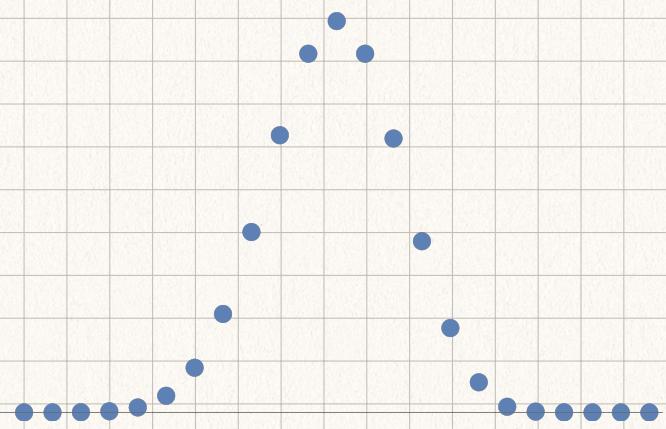
- We leverage advances in sparse numerical linear algebra.
 - we correct minor errors using the underlying representation theory
- We distribute our computations over a network of hundreds of computers (high throughput) including a number of high performance nodes.
- One small part of the $d=6$ computation required the following resources:

# of matrices	Avg. Run time	Memory (GB)
151	20 mins.	< 1
16	1 hr.	1 - 10
17	16 hr.	20 - 80
2	3 days	> 500

Veronese Syzygies ($n=3$)

Syzygy Data . com

- This data has provided further evidence for a number of existing conjectures:



- It also suggests that deeper symmetries are controlled by special "monomial syzygies"!
- But our understanding of the syzygies of surfaces is still very limited !!

What about :

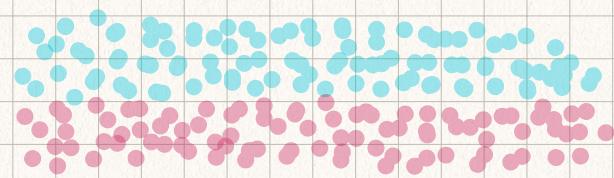
+ $\mathbb{P}^1 \times \mathbb{P}^1$

+ Hirzebruch surfaces

+ Elliptic Surfaces



Look forward to...



Spec(\bar{Q})

