

Asymptotic Syzygies in the Semi-Ample Setting

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Background

- Given a smooth projective variety $X \subset \mathbb{P}^r$, the **homogeneous coordinate ring** of X is $S_X := S/I_X$, and captures geometric information about X as an embedded variety in \mathbb{P}^r .
- Here S is the polynomial ring $\mathbb{C}[x_0, x_1, \dots, x_r]$ and I_X is the ideal of polynomials vanishing on X .
- The **minimal graded free resolution** of S_X as a graded S -module has the form:

$$0 \longleftarrow S_X \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_r \longleftarrow 0$$

where F_i is a finitely generated graded S -module, and so can be written as:

$$F_p = \bigoplus_{q \in \mathbb{Z}} S(-q)^{\oplus \beta_{p,q}(S_X)}.$$

- Here $\beta_{p,q}(S_X)$ is the number of p th-syzygies of degree q , which are the **graded Betti numbers** of X .
- We form the **graded Betti table** of X by placing $\beta_{p,p+q}(S_X)$ in the (p, q) -th spot.

Example - Seven Points in \mathbb{P}^3

- Suppose $X \subset \mathbb{P}^3$ is 7 points in general linear position. There are two possible minimal free resolutions of S_X :

$$0 \longleftarrow S \longleftarrow \begin{array}{c} S(-2)^3 \\ \oplus \\ S(-3) \end{array} \longleftarrow S(-4)^6 \longleftarrow S(-5)^3$$

$$0 \longleftarrow S \longleftarrow \begin{array}{c} S(-2)^3 \\ \oplus \\ S(-3)^3 \end{array} \longleftarrow \begin{array}{c} S(-3)^2 \\ \oplus \\ S(-4)^6 \end{array} \longleftarrow S(-5)^3$$

- These resolutions correspond to the following Betti tables:

| | 0 | 1 | 2 | 3 | | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | - | - | - | 0 | 1 | - | - | - |
| 1 | - | 3 | - | - | 1 | - | 3 | 2 | - |
| 2 | - | 1 | 6 | 3 | 2 | - | 3 | 6 | 3 |

- The second resolution occurs if and only if there is an irreducible cubic curve passing through the points in X .

The Ample Setting

- Our goal is to understand the relationship between the geometry of the embedding of X and its graded Betti numbers.
- For example, one might wonder how the syzygies of X behave after re-embedding by a d -uple Veronese embedding:

$$\begin{array}{ccc} X & \xleftarrow{|A|} & \mathbb{P}^r \\ & \searrow |dA| & \downarrow \iota_d \\ & & \mathbb{P}^{rd-1} \end{array}$$

Theorem 1 (Ein-Lazarsfeld). *Let X be a smooth projective variety with $\dim X = n$, and let A (ample) and B be line bundles on X . Fixing a row $q \in [1, n]$ if $d \gg 0$ then*

$$\beta_{p,p+q}(X; B + dA) \neq 0 \quad \text{for all } p \in [P_-, P_+]$$

where:

- $P_- = O(d^q)$ is determined by q th row of $\beta(\mathbb{P}^q, \mathcal{O}(d))$,
- $P_+ = r_d - O(d^{n-1})$ is determined by the $(n - q)$ th row of $\beta(\mathbb{P}^{n-q}, \mathcal{O}(d))$,
- and

$$\lim_{d \rightarrow \infty} \frac{\text{the \% of non-zero entries in the } q\text{th row of } \beta(X, B + dA)}{r_d} = \lim_{d \rightarrow \infty} \frac{P_+ - P_-}{r_d} = 1.$$

- Projective space determines P_- and P_+ in that the asymptotic is what arises for those cases of projective space.
- This gives an inductive structure to the Betti table as most rows are controlled by varieties of smaller dimension.

The Semi-Ample Setting

- One may wonder how this story changes if ample is replaced by other weaker notions of positivity.
- A line bundle D is **semi-ample** if there exists a $k \gg 0$ such that $|kD|$ defines a regular map to \mathbb{P}^N for some N .
- The quintessential example of semi-ample line bundles are $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, which give the coordinate projections.

Theorem 2 (Bruce). *Let $n_1, n_2, d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Fixing a row $q \in [1, n_1 + n_2]$ if $d_1, d_2 \gg 0$ then:*

$$\beta_{p,p+q}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, \mathcal{O}(d_1, d_2)) \neq 0 \quad \text{for all } p \in [P_-, P_+]$$

where:

- $P_- = \min \left\{ O(d_1^a d_2^b) \mid \begin{array}{l} a + b = q - 1, \\ a \leq n_1, b \leq n_2 \end{array} \right\},$
- $P_+ = r_d - \min \left\{ O(d_1^{n_1-a} d_2^{n_2-b}) \mid \begin{array}{l} a + b = q, \\ a \leq n_1, b \leq n_2 \end{array} \right\}.$

- These bounds seem to have a geometric origin giving an inductive structure similar to Ein and Lazarsfeld's.

Heuristic 1 (Bruce). *Let $n_1, n_2, d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Fixing a row $q \in [1, n_1 + n_2]$ if $d_1, d_2 \gg 0$ then:*

$$\beta_{p,p+q}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, \mathcal{O}(d_1, d_2)) \neq 0 \quad \text{for all } p \in [P_-, P_+]$$

where:

- P_- is determined by subvarieties of the form $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ where $a + b = q$, and
- P_+ is determined by subvarieties of the form $\mathbb{P}^{n_1-a} \times \mathbb{P}^{n_2-b} \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ where $a + b = q$.

Example - $q = 2$

- When $q = 2$ it is possible to give explicit formulas for P_- , and as predicted by the heuristic there are three possibilities:

$$P_- = \min \left\{ \underbrace{3d_1 - 2}_{\mathbb{P}^1 \times *}, \underbrace{2(d_1 + d_2) - 2}_{\mathbb{P}^1 \times \mathbb{P}^1}, \underbrace{3d_2 - 2}_{* \times \mathbb{P}^1} \right\}.$$

- From this we see that the third part of Ein and Lazarsfeld's results no longer holds:

$$\lim_{d_1 \rightarrow \infty} \frac{\text{\% of non-zero entries in the 2nd row of } \beta(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2))}{r_d} = \lim_{d_1 \rightarrow \infty} \frac{r_d - 3 - [2(d_1 + d_2) - 2]}{r_d} = 1 - \frac{2}{d_2 + 1}.$$

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