Asymptotic Syzygies in the Semi-Ample Setting

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Background

• Given a smooth projective variety $X \subset \mathbb{P}^r$, the **homogeneous coordinate ring** of X is $S_X := S/I_X$, and captures geometric information about X as an embedded variety in $\mathbb{P}^r$.

• Here S is the polynomial ring $\mathbb{C}[x_0, x_1 \ldots, x_r]$ and I_X is the ideal of polynomials vanishing on X.

• The **minimal graded free resolution** of S_X as a graded S-module has the form:

$$0 \longleftarrow S_X \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_r \longleftarrow 0$$

where F_i is a finitely generated graded S-module, and so can be written as:

$$F_p = \bigoplus_{q \in \mathbb{Z}} S(-q)^{\oplus \beta_{p,q}(S_X)}.$$

• Here $\beta_{p,q}(S_X)$ is the number of pth-syzygies of degree q, which are the **graded Betti numbers** of X.

• We form the **graded Betti table** of X by placing $\beta_{p,p+q}(S_X)$ in the (p,q)-th spot.

Example - Seven Points in $\mathbb{P}^3$

• Suppose $X \subset \mathbb{P}^3$ is 7 points in general linear position. There are two possible minimal free resolutions of S_X:

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-2)^3 \\ \oplus \\ S(-3) \end{matrix} \longleftarrow S(-4)^6 \longleftarrow S(-5)^3$$

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-2)^3 \\ \oplus \\ S(-3)^3 \end{matrix} \longleftarrow \begin{matrix} S(-3)^2 \\ \oplus \\ S(-4)^6 \end{matrix} \longleftarrow S(-5)^3$$

• These resolutions correspond to the following Betti tables:

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	1	6	3

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

• The second resolution occurs if and only if there is an irreducible cubic curve passing through the points in X.

The Ample Setting

• Our goal is to understand the relationship between the geometry of the embedding of X and its graded Betti numbers.

• For example, one might wonder how the syzygies of X behave after re-embeding by a d'uple Veronese embedding:

$$\begin{tikzcd} X \arrow[r, hook, "|A|"] \arrow[dr, "|dA|"'] & \mathbb{P}^r \arrow[d, hook, "\iota_d"] \\ & \mathbb{P}^{r_d - 1}. \end{tikzcd}$$

Theorem 1 (Ein-Lazarsfeld). *Let X be a smooth projective variety with $\dim X = n$, and let A (ample) and B be line bundles on X. Fixing a row $q \in [1, n]$ if $d \gg 0$ then*

$$\beta_{p,p+q}(X; B + dA) \neq 0 \quad \textit{for all} \quad p \in [P_-, P_+]$$

where:

• $P_- = O(d^q)$ *is determined by* q*th row of* $\beta(\mathbb{P}^q, \mathcal{O}(d))$,

• $P_+ = r_d - O(d^{n-1})$ *is determined by the* $(n-q)$*th row of* $\beta(\mathbb{P}^{n-q}, \mathcal{O}(d))$,

• *and*

$$\lim_{d \to \infty} \frac{\textit{the \% of non-zero entries}}{\textit{in the } q\textit{th row of } \beta(X, B + dA)} = \lim_{d \to \infty} \frac{P_+ - P_-}{r_d} = 1.$$

• Projective space determines P_- and P_+ in that the asymptotic is what arrises for those cases of projective space.

• This gives an inductive structure to the Betti table as most rows are controlled by varieties of smaller dimension.

The Semi-Ample Setting

• One may wonder how this story changes if ample is replaced by other weaker notions of positivity.

• A line bundle D is **semi-ample** if there exists a $k \gg 0$ such that $|kD|$ defines a regular map to $\mathbb{P}^N$ for some N.

• The quintessential example of semi-ample line bundles are $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ on $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, which give the coordinate projections.

Theorem 2 (Bruce). *Let $n_1, n_2, d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Fixing a row $q \in [1, n_1 + n_2]$ if $d_1, d_2 \gg 0$ then:*

$$\beta_{p,p+q}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, \mathcal{O}(d_1, d_2)) \neq 0 \quad \textit{for all} \quad p \in [P_-, P_+]$$

where:

• $P_- = \min\left\{ O\left(d_1^a d_2^b\right) \;\middle|\; \begin{matrix} a + b = q - 1, \\ a \leq n_1, b \leq n_2 \end{matrix} \right\}$,

• $P_+ = r_d - \min\left\{ O\left(d_1^{n_1 - a} d_2^{n_2 - b}\right) \;\middle|\; \begin{matrix} a + b = q, \\ a \leq n_1, b \leq n_2 \end{matrix} \right\}$.

• These bounds seem to have a geometric origin giving an inductive structure similar to Ein and Lazarsfeld's.

Heuristic 1 (Bruce). *Let $n_1, n_2, d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Fixing a row $q \in [1, n_1 + n_2]$ if $d_1, d_2 \gg 0$ then:*

$$\beta_{p,p+q}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, \mathcal{O}(d_1, d_2)) \neq 0 \quad \textit{for all} \quad p \in [P_-, P_+]$$

where:

• P_- *is determined by subvarieties of the form* $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ *where* $a + b = q$, *and*

• P_+ *is determined by subvarieties of the form* $\mathbb{P}^{n_1 - a} \times \mathbb{P}^{n_2 - b} \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ *where* $a + b = q$.

Example - $q = 2$

• When $q = 2$ it is possible to give explicit formulas for P_-, and as predicted by the heuristic there are three possibilities:

$$P_- = \min\{\underbrace{3d_1 - 2}_{\mathbb{P}^1 \times *}, \underbrace{2(d_1 + d_2) - 2}_{\mathbb{P}^1 \times \mathbb{P}^1}, \underbrace{3d_2 - 2}_{* \times \mathbb{P}^1}\}.$$

• From this we see that the third part of Ein and Lazarsfeld's results no longer holds:

$$\lim_{d_1 \to \infty} \frac{\text{\% of non-zero entries}}{\begin{matrix}\text{in the 2nd row of} \\ \beta\left(\mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2)\right)\end{matrix}} = \lim_{d_1 \to \infty} \frac{r_d - 3 - [2(d_1 + d_2) - 2]}{r_d} = 1 - \frac{2}{d_2 + 1}.$$

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