Test ideals, Cohen-Macaulay modules, and singularities of commutative rings

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CAZoom April 2020

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2. Describe new perspectives on how to do this
3. Give some theoretical extensions of this perspective to closure operations, interior operations, test ideals, cores, and hulls
4. Give examples of computations of test ideals of maximal Cohen-Macaulay modules.

Big Cohen-Macaulay modules and algebras, a brief history

Definition (Hochster 1975)

Let (R, m) be a Noetherian local ring. An R-module B (not necessarily finitely-generated, is a (balanced) big Cohen-Macaulay module if $mB \neq B$ and every system of parameters on R is a regular sequence on B.

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They were first studied by Hochster in the context of proving the homological conjectures, a family of related conjectures many of which were proved in the last case in the last year or two, following André's proof of the existence of big Cohen-Macaulay algebras in the mixed characteristic case.

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Ma-Schwede 2018: Used big Cohen-Macaulay algebras to define analogues of F-regular and F-rational singularities, using a big Cohen-Macaulay version of a test ideal for pairs.

Big Cohen-Macaulay module and algebra closures

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If B is a big Cohen-Macaulay algebra, it is sufficient to show that $1 \otimes x \in im(B \otimes_R N \to B \otimes_R M)$.

This is based on ideas used by Hochster and Huneke. What's new is that in my thesis I studied this as an example of a closure operation, a more general object that includes examples like tight closure, integral closure, Frobenius closure, plus closure, solid closure, and others.

Closure operations

Definition

A closure operation on a class of R-modules $\mathcal{M}$ is a map that takes each pair $N \subseteq M$ of modules in $\mathcal{M}$ to a new submodule N_M^{cl} of M such that:

1. $N \subseteq N_M^{\mathrm{cl}}$,
2. $\left(N_M^{\mathrm{cl}}\right)_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}$,
3. *and if* $N \subseteq L \subseteq M$, *then* $N_M^{\mathrm{cl}} \subseteq L_M^{\mathrm{cl}}$.

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Basic example:
Trivial closure: $N_M^{\mathrm{cl}} = N$ for all $N \subseteq M \in \mathcal{M}$

Module closures

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Plus closure is cl_{R^+}, where R^+ is the absolute integral closure of R.
Tight closure, integral closure, Frobenius closure, and solid closure are all variants on module closures. They all use a family of modules or algebras instead of one, or an extra multiplier.

Properties of big Cohen-Macaulay module closures

While module closures in general are useful, big Cohen-Macaulay module closures have a number of special properties.

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Theorem (RG 2016, 2019)

Let R be a complete local domain.

1. *R is regular if and only if cl_B is trivial for all big Cohen-Macaulay modules B.*
2. *If R has characteristic $p > 0$, R is weakly F-regular if and only if cl_B is trivial for all big Cohen-Macaulay algebras B.*

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These results establish the connection between big Cohen-Macaulay modules and algebras and singularities of the ring. But closure operations are only one of the tools used to study singularities.

Test ideals

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This is technically a single invariant of the ring, but is definitely still too complicated to compute. For tight closure, we have a number of different ways to represent the test ideal, some of which make it more possible to compute.

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Let R be a Noetherian local ring B an R-module. If R is complete or B is finitely-generated, then the cl_B-test ideal of R agrees with the trace ideal $tr_B(R)$, which is defined as:

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$$tr_B(R) := \sum_{f \in Hom_R(B,R)} f(B).$$

When B is finitely-generated, we can compute $\mathrm{tr}_B(R)$ using Macaulay2 (for specific examples). In particular, we can do this for finitely-generated Cohen-Macaulay modules, known as maximal Cohen-Macaulay modules.

Trace ideals

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Faber 2019: Studies trace ideals of finitely-generated Cohen-Macaulay modules over rings of finite Cohen-Macaulay type.

Lindo-Pande 2018: Every ideal is a trace ideal if and only if the ring is Artinian Gorenstein.

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3. *If $\tau_{\mathrm{cl}_B}(R) = R$ for some big Cohen-Macaulay module B, then R is Cohen-Macaulay.*
4. *If B is a finitely-generated Cohen-Macaulay module, then $V(\tau_{\mathrm{cl}_B}(R))$ is contained in the singular locus of R.*

Main result continued:

Theorem (Part 2)

1. *If R is a Cohen-Macaulay ring with canonical module ω, then R is Gorenstein if and only if $\tau_{\mathrm{cl}_\omega}(R) = R$.*

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3. *If R has countably many indecomposable finitely-generated Cohen-Macaulay modules, then $\tau_{\mathrm{cl}_B}(R)$ may not be m-primary for all finitely-generated Cohen-Macaulay modules B.*

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2. Compute examples of finitely-generated Cohen-Macaulay test ideals–now doable in Macaulay2–to study their connection to singularities

Closure-interior Duality: Interior Operations

Definition

An interior operation int on $\mathcal{M}$ is a map that takes each R-module $M \in \mathcal{M}$ to a submodule $\mathrm{int}(M) \subseteq M$ *such that*

1. $\mathrm{int}(\mathrm{int}(M)) = \mathrm{int}(M)$
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Examples:

1. Trivial interior: $\text{int}(M) = M$ for all R-modules M.
2. Tight interior (Epstein-Schwede 2014)
3. Module trace:
$$\text{tr}_B(M) = \sum_{f \in \text{Hom}_R(B,M)} f(B).$$

Dual of a closure operation

Let R be a complete local Noetherian ring.
Let cl be a closure operation on $\mathcal{M}$. Let $\alpha(M)$ denote 0_M^{cl}. We define a dual to α on all modules $M \in \mathcal{M}^\vee$ as follows:

$$\alpha^\smile(M) = \left(\frac{M^\vee}{\alpha(M^\vee)}\right)^\vee.$$

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Given an interior operation int on $\mathcal{M}^\vee$, we define the dual of int on all modules $M \in \mathcal{M}$ to be

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Proposition

Let R *be a complete Noetherian local ring. If cl is a residual closure operation on* $\mathcal{M}$ *and* $\alpha(M) := 0_M^{\mathrm{cl}}$, *then* $\alpha^{\smile}(M)$ *is an interior operation on* $\mathcal{M}^{\vee}$.
If int *is an interior operation on* $\mathcal{M}^{\vee}$, *then* $\mathrm{int}^{\smile}(M) = 0_M^{\mathrm{cl}}$ *for a unique residual closure operation cl on* $\mathcal{M}$.

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If int *is an interior operation on $\mathcal{M}^{\vee}$, then* $\mathrm{int}^{\smile}(M) = 0_M^{\mathrm{cl}}$ *for a unique residual closure operation cl on $\mathcal{M}$.*
Further, $\alpha^{\smile\smile}(M) = \alpha(M)$ for all $M \in \mathcal{M}$, and $\mathrm{int}^{\smile\smile}(M) = \mathrm{int}(M)$ *for all $M \in \mathcal{M}^{\vee}$.*

Closures are dual to their test ideals

Theorem (Epstein–RG 2019)

Let R be a complete Noetherian local ring, let cl be a closure operation on the set of Artinian R-modules $\mathcal{A}$, and let $\alpha(M) = 0_M^{\text{cl}}$ for all $M \in \mathcal{A}$. Then

$$\alpha^{\smile}(R) = Ann_R(\alpha(E)) = \bigcap_{M \in \mathcal{A}^\vee} Ann(\alpha(M)).$$

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In the case that cl is residual, this implies the dual of cl, when applied to R, is actually the cl-test ideal of R.

Module closures are dual to trace modules

Another consequence of the theorem is the following:

Corollary (Epstein–RG 2019)

Let R be a complete Noetherian local ring. Let B be an R-module and set $\alpha(M) := 0_M^{\mathrm{cl}_B}$. Then $\alpha^{\smile}(M) = tr_B(M)$.

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Let R be a complete Noetherian local ring. Let B be an R-module and set $\alpha(M) := 0_M^{\mathrm{cl}_B}$. Then $\alpha^\smile(M) = tr_B(M)$.

This is similar to the result stated earlier that the test ideal of a module closure equals the trace ideal. Here, we require the ring to be complete, but in Epstein–RG 2019 this result is a product of the general closure-interior duality, rather than something specific to module closures.

Other directions for closure-interior duality

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6. Dual of core

Core-Hull Duality: Definition of core

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Definition

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This was originally defined for integral closure, then for tight closure.

Nakayama closures

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Definition

Let $(R,)$ *be a Noetherian local ring and* cl *a closure operation on the class of finitely-generated* R*-modules . We say that* cl *is a Nakayama closure if for* $L \subseteq M \subseteq N \in$ *satisfying* $L \subseteq M \subseteq (L+M)_N^{\mathrm{cl}}$ *implies that* $L_N^{\mathrm{cl}} = M_N^{\mathrm{cl}}$.

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If cl is any closure operation satisfying the Nakayama property, then the cl core will exist.

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Examples of Nakayama closures: tight closure, integral closure, Frobenius closure

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Definition (Epstein-RG-Vassilev 2020)

Let int *be an interior operation on the class of Artinian R-modules $\mathcal{M}$, where (R, m) is a Noetherian local ring. We say that* int *is Nakayama if whenever $A \subseteq B$ are Artinian modules such that*

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Let int *be the dual of* cl*. Then* cl *is a Nakayama closure if and only if* int *is a Nakayama interior.*

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More examples: Duals of all Nakayama closures, such as tight interior, integral interior, Frobenius interior.

Core-Hull duality

Definition (Vassilev 2020)

Let int *be an interior operation defined on the class of artinian R-modules $\mathcal{M}$. If $M \supseteq N$ are elements of $\mathcal{M}$, the* int*-hull of a submodule M with respect to N is*

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Theorem (Epstein-RG-Vassilev 2020)

Let R be a complete Noetherian local ring, $A \subseteq B$ Artinian R-modules, and int *a Nakayama interior defined on Artinian R-modules. Then the* int*-hull of A in B exists, and is dual to the* cl*-core of $(B/A)^\vee$ in $B^\vee$, where* cl *is the closure operation dual to* int.

And now for something completely different: Examples

Recall the following result from earlier:

Theorem (Pérez-RG 2019)

Let R be a Noetherian local ring.

1. *If R is a Cohen-Macaulay ring with canonical module ω, then R is Gorenstein if and only if $\tau_{\mathrm{cl}_\omega}(R) = R$.*
2. *If R has only finitely many indecomposable finitely-generated Cohen-Macaulay modules, then τ_{cl_B} is m-primary for all finitely-generated Cohen-Macaulay modules B.*
3. *If R has countably many indecomposable finitely-generated Cohen-Macaulay modules, then $\tau_{\mathrm{cl}_B}(R)$ may not be m-primary for all finitely-generated Cohen-Macaulay modules B.*
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MCM-test ideal

Our goal is to compute the following test ideal:

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Let $MCM(R)$ denote the set of all maximal Cohen-Macaulay module R-modules, i.e. the finitely-generated Cohen-Macaulay R-modules. The maximal Cohen-Macaulay module module test ideal of R is

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Plus, for nice enough rings, it turns out that every big Cohen-Macaulay module has a direct summand that is a finitely-generated Cohen-Macaulay module, and that is enough to imply that $\tau_{sing}(R) = \tau_{MCM}(R)$.

Veronese subrings

Example (Benali-Pothagoni-RG 2020)

If $R = k[[x^d, x^{d-1}y, ..., xy^{d-1}, y^d]]$ where k is a field, then $\tau_{MCM}(R) = m$. Furthermore, $\tau_M(R) = m$ for all non-free indecomposable maximal Cohen-Macaulay modules M.

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Maximal Cohen-Macaulay modules (viewed as submodules of $S = k[[x, y]]$):

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In fact, setting $M_i = x^iR + x^{i-1}yR + \ldots + y^iR$, $\mathrm{Hom}_R(M_i, R)$ is generated by the following maps:

$$f_j(u) = x^{d-i-j}y^ju, \ 0 \le j \le d - i.$$

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5. *(E_8) Let $R = [[x, y, z]]/(z^2 + x^3 + y^5)$. Then $\tau_{MCM}(R) = (x, y^2, z)$.*

Countable Cohen-Macaulay type

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Let k be a field of characteristic not equal to 2, 3, or 5.

1. *(Whitney umbrella) Let $R = k[[x, y, z]]/(x^2y + z^2)$, where k is a field of some arbitrary characteristic. Then $\tau_{MCM}(R) = (x^2, z)$.*

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Notice in some of the non-domain examples, $\tau_{MCM}(R) = 0$, confirming that the domain hypothesis is needed for the results from Pérez-RG.

Main computational tool

Theorem (Benali-Pothagoni-RG 2020)

Suppose $R = k[[x,y,z]]/(z^2 + g(x,y))$ *and* φ *is a matrix over* $k[[x,y]]$ *such that* $(zid_n - \varphi, zid_n + \varphi)$ *is a matrix factorization of* $z^2 + g(x,y)$*. If* $M = coker(\overline{zid_n - \varphi})$*, then the trace ideal of* M *is generated by the entries of* $\overline{zid_n + \varphi}$*.*