A Probabilistic Approach to Noether Normalization

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Definitions

• If $X \subset \mathbb{P}^r$ is a closed subscheme, $f_0, \dots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ are a **(partial) system of parameters** on X if

$$\dim \mathbb{V}(f_0, \dots, f_k) \cap X = \dim X - (k+1).$$

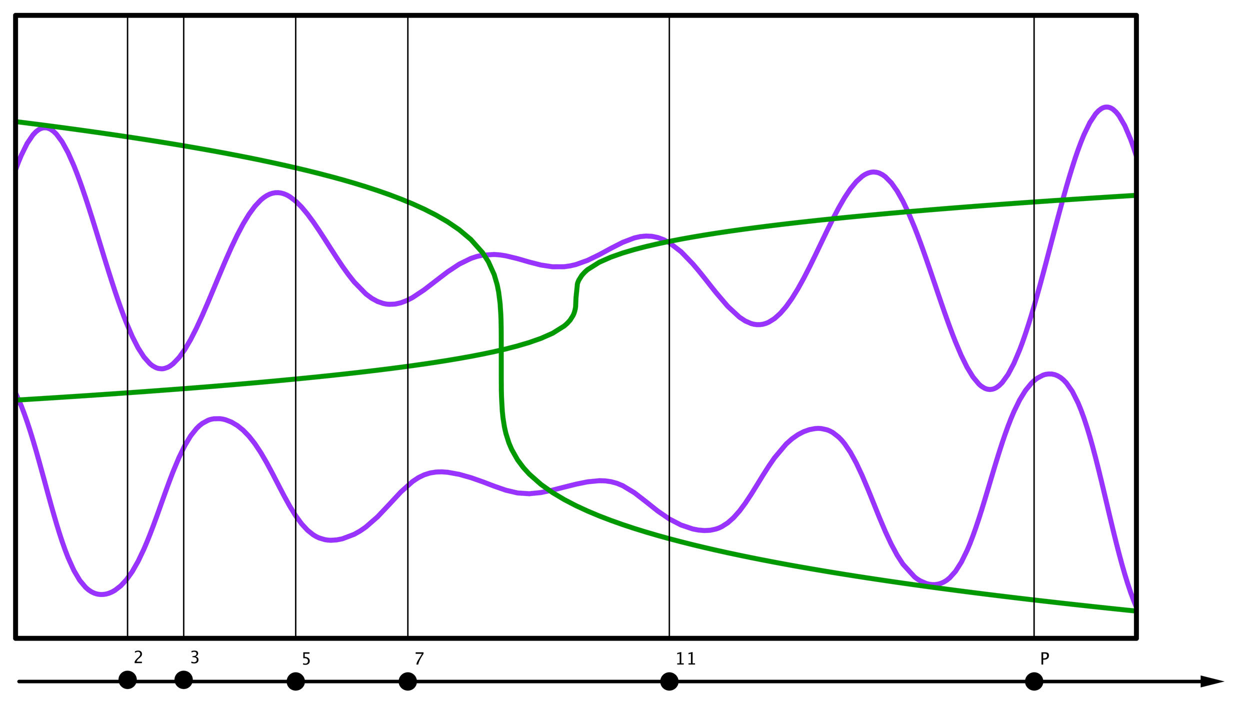

Figure 1: A non-example of fiberwise parameters for $\mathbb{P}^1_{\mathbb{Z}}$.

• If $X \subset \mathbb{P}^r_B$ is a closed subscheme a **fiberwise system of parameters** on X is a collection $f_0, \dots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$, which restrict to parameters on each fiber of X over B.

Example - Points in $\mathbb{P}^1_{\mathbb{Z}}$

• A fiberwise parameter for a set of points in $\mathbb{P}^1_{\mathbb{Z}}$ is a homogenous polynomial f such that $f = \pm 1$ at every point .

• For one point the **Euclidean algorithm** shows a linear parameter always exists.

• The points $[4:5]$ and $[1:8]$ have no fiberwise parameter f with $\deg f < 9$. If $\deg(f) = 9$ one can choose:

$$\begin{aligned} f(x,y) = {} & 590316869666239369788336486577332627591x^9 - y^9 \\ & - 548058207864200020245777742687553134041x^8y \\ & + 608953564293555578050864158541725704 49x^7y^2 \\ & - 201231722866854045440755923284664255x^6y^3 - 99438623xy^8 \\ & - 30126091505240316687953753225193x^5y^4 \\ & - 1977608122820402x^2y^7 + 62095832340520326113809669 8x^4y^5 \\ & - 1230273968422817725004x^3y^6. \end{aligned}$$

Distribution of Parameters

Theorem 1. *Let $X \subseteq \mathbb{P}^r_B$ be a closed subscheme of dimension n. The "probability" that random chosen forms $f_0, \dots, f_k$ of degree d are a (fiberwise) system of parameters as $d \to \infty$ is:*

	$B = \mathbb{F}_q$	$B = \mathbb{Z}$
$k < n$	1	1
$k = n$	$\zeta_X(n+1)^{-1}$	0

• The case $B = \mathbb{F}_q$ and $k = n$ is due to Bucur and Kedlaya.

• We use an adaptation of Poonen's sieving argument, which when $k < n$, results in a description of the distribution of parameters in terms of a **higher dimensional analogue** of the Hasse-Weil zeta function.

Proposition 1. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a closed subscheme of dimension n. Then there is an explicit function $f(k,n,d)$ such that*

$$\operatorname{Prob}\left(\begin{array}{c} (f_0, \dots, f_k) \\ \textit{of degree } d \textit{ are } \textbf{\textit{not}} \\ \textit{parameters on } X \end{array} \right) \sim \# \left\{ \begin{array}{c} (n-k)\textit{-planes} \\ L \subset X \end{array} \right\} \cdot f.$$

Example - Parameters on Surfaces

• Over $\mathbb{F}_4$ the Fermat cubic surface

$$X = \mathbb{V}(x^3 + y^3 + z^3 + w^3) \subset \mathbb{P}^3_{\mathbb{F}_4},$$

has 27 $\mathbb{F}_4$-lines. Proposition 1 implies $\approx 0.66\%$ of pairs (f_0, f_1) of degree two should not be parameters. Simulating 10^5 such pairs we found 0.62% failed to be parameters.

• In $\mathbb{P}^3$ the surface $\mathbb{V}(xyz)$ contains substantially more lines than $\mathbb{V}(x^2+y^2+z^2)$. Selecting 10^6 random pairs (f_0, f_1) of degree two, the proportion that *failed* to be parameters were:

	$\mathbb{V}(xyz)$	$\mathbb{V}(x^2+y^2+z^2)$
$\mathbb{F}_2$	.2638	.1179
$\mathbb{F}_3$	.0552	.0059
$\mathbb{F}_5$	.0063	.0004

Application: Uniform Noether Normalization

• As an application we recover a recent result of Gabber-Liu-Lorenzini and Chinburg-More-Bailly-Pappas-Taylor.

Theorem 2. *Let $X \subset \mathbb{P}^r_{\mathbb{Z}}$ be a closed subscheme. If every fiber of X over $\mathbb{Z}$ has dimension n then there exists a linear series in $\mathcal{O}_X(d)$, for some $d > 0$, inducing a finite morphism:*

$$\phi : X \to \mathbb{P}^n_{\mathbb{Z}}.$$

• The existence of such ϕ is subtle, even potentially unexpected, as the sections $f_0, \dots, f_n \in H^0(X, \mathcal{O}_X(d))$ defining such a finite map have **density zero**; even if we let $d \to \infty$.

• **Step #1 - (Probabilistic):** By Theorem 1 we can find fiberwise parameters $f_0, \dots, f_{n-1}$ on X simply by picking high degree forms at random, and so reduce Theorem 2 to

$$X' = X \cap \mathbb{V}(f_0, \dots, f_{n-1}).$$

•**Step #2 - (Arithmetic):** Such X' is essentially a union of orders in number fields. Using the fact an order has a finite Picard group we show that $\operatorname{Pic}(X')$ is finite. This finiteness allows us to construct the last parameter.

Application: Effective Noether Normalization

• Applying Proposition 1 we also obtain an effective version of Noether normalization over $\mathbb{F}_q$.

Proposition 2. *Let $X \subset \mathbb{P}^r_{\mathbb{F}_q}$ be a closed irreducible subscheme of dimension $n > 0$. There exist forms of degree d^n defining a finite morphism $\pi : X \to \mathbb{P}^n_{\mathbb{F}_q}$ so long as:*

$$d > \log_q \deg X + n \log_q d + \log_q n$$
$$d > \sqrt[n]{q \cdot \deg X}.$$

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