The Degree of SO(r) and Low-rank SDP

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The Main Question

Question (B. Sturmfels)

What is the degree of the special orthogonal group?

Set Up - SO(r)

- The **special orthogonal group** is the group:

$$\mathrm{SO}(r) := \left\{ M \in \mathrm{Mat}(r,r) \quad \middle| \det M = 1 \quad MM^t = \mathrm{Id} \right\}.$$

- The conditions for $M \in \mathrm{SO}(r)$ are polynomial in the entries of M.
- **Example:** If $r = 2$ then:

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \middle| \begin{array}{c} x_{1,1}x_{2,2} - x_{1,2}x_{2,1} = 1, \\ x_{1,1}^2 + x_{1,2}^2 = 1, x_{2,1}^2 + x_{2,2}^2 = 1, \\ x_{1,1}x_{2,1} + x_{1,2}x_{2,2} = 0 \end{array} \right\} \subset \mathbb{C}^4$$

- This makes SO(r) an algebraic subvariety of $\mathbb{C}^{r^2}$.

Set Up - Degree

- The degree of a variety is a geometric invariant that intuitively measures the complexity of the variety.

Definition

If $X \subset \mathbb{C}^r$ is a subvariety of dimension k then the **degree** *of X is:*

$$\deg X := \#X \cap H,$$

where $H \subset \mathbb{C}^r$ is a generic plane of dimension $r - k$.

- Generic (essentially) means that the plane is transverse to X.
- More formally we are interested in the degree of the projective closure of $\mathrm{SO}(r)$ under the embedding given by the equations:

$$\det M = \mathrm{Id} \quad \text{and} \quad MM^t = \mathrm{Id}\,.$$

Some Answers

- Using symbolic and numerical techniques Macaulay2 / Bertini can answer this for small r.

r	Symbolic	Numerical
2	2	2
3	8	8
4	40	40
5	384	384
6	-	4768
7	-	111616
8	-	-

Table: The degree of SO(r) computed various ways.

Some Answers

- These computations quickly become difficult as the dimension (and degree) of $\mathrm{SO}(r)$ grow rapidly in r.
- Standard techniques were only effective for $r \leq 5$.
- A more efficient approach is to use monodromy methods.
  1. Pick random hyperplanes H and H' s.t. $H \cap \mathrm{SO}(r)$ contains Id.
  2. Pick a homotopy γ from $H \cap \mathrm{SO}(r)$ to $H' \cap \mathrm{SO}(r)$.
  3. Pick another homotopy γ' from $H' \cap \mathrm{SO}(r)$ to $H \cap \mathrm{SO}(r)$.
  4. Look at the point $\gamma'(\gamma(\mathrm{Id})) \in H \cap \mathrm{SO}(r)$.
- We were able to compute the degree of $\mathrm{SO}(7)$ in ≈ 12 hours, compared to the $30+$ days we expected with other methods.

The Answer

Theorem (B^3KR)

$$\deg \mathrm{SO}(r) = 2^{r-1} \det \left(\binom{2r-2i-2j}{r-2i} \right)_{1 \le i,j \le \lfloor \frac{r}{2} \rfloor}.$$

- We prove this by making explicit an earlier result of Kazarnovskij.

The Answer

Theorem (B^3KR)

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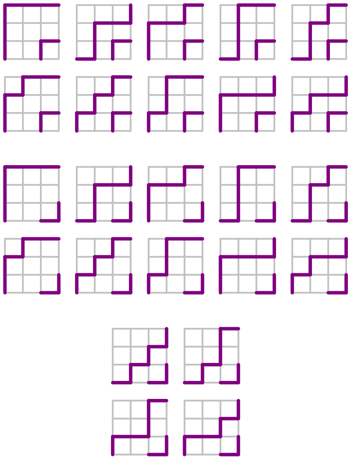
- We prove this by making explicit an earlier result of Kazarnovskij.
- The appearance of the binomial coefficents allows us to get by following corollary via the Gessel-Viennot Lemma.

Corollary (B^3KR)

$$\deg \mathrm{SO}(r) = 2^{r-1} \cdot \# \left\{ \begin{matrix} \textit{Certain Non-intersecting} \\ \textit{Planar Lattice Paths} \end{matrix} \right\}.$$

The Answer

- **Ex:** The degree of $SO(5)$ is $384 = 2^4 \cdot 24$, and here are 24 non-intersecting planar lattice paths:



The Real Main Question

Question (B. Recht)

Why does the augmented Lagrangian algorithm for solving low-rank semidefinite programing problems work?

Linear Programing

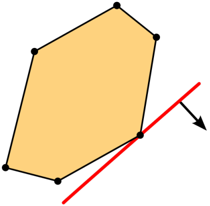
Linear Programing

Given a cost function $\vec{c} \in \mathbb{R}^n$ *and constraints* $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$ *the linear programing question* $(c; a_1, \ldots, a_m; b)$ *is to*

$$\min_{x \in \mathbb{R}^n} \langle \vec{c}, x \rangle$$
$$\textit{subject to}\ \langle \vec{a}_i, x \rangle = b_i \quad \textit{and}\ x \geq 0.$$

Linear Programing

- **Idea:** Optimize some function $\langle \vec{c}, x \rangle$ over a convex polytope given by the intersection of the positive orthant with the region defined by the hyperplanes $\langle \vec{a}_i, x \rangle = b_i$.



- There are many algorithms (ellipsoid, simplex, etc.) for efficiently solving linear programing problems.

Theorem (Khachiyan)

LP problems can be solved in polynomial time in n, the # of unknowns.

Semidefinite Programing

- Let $\mathbb{S}^n$ be the vector space of $n \times n$ real symmetric matrices.
- We can equip $\mathbb{S}^n$ with an inner product by setting:

$$\langle A, B \rangle := \operatorname{tr}\left(A^t B\right).$$

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- We can equip $\mathbb{S}^n$ with an inner product by setting:

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- A matrix M is **positive semidefinite** if and only if for every $\vec{x} \in \mathbb{R}^n$:

$$\vec{x}^t M \vec{x} \geq 0.$$

Lemma

The set of positive semidefinite matrices forms a convex cone inside of $\mathbb{S}^n$.

Semidefinite Programing

- **Idea:** Semidefinite programing generalizes linear programing by replacing $\mathbb{R}^n$ with $\mathbb{S}^n$ and the positive orthant with the p.s.d cone.

Semidefinite Programing

Given a cost function $C \in \mathbb{S}^n$ and constraints $A_1, A_2, \ldots, A_m \in \mathbb{S}^n$, $\vec{b} \in \mathbb{R}^m$ the semidefinite programing question $(C; A_1, \ldots, A_m; \vec{b})$ is to

$$\min_{X \in \mathbb{S}^n} \langle C, X \rangle$$

$$\textit{subject to} \langle A_i, X \rangle = b_i$$

$$\textit{and } X \textit{ is p.s.d.}$$

Solving SDP Problems

- Notice the region we are optimizing over is still convex, and so methods of convex optimization can be applied.

Theorem

A SDP problem can be solved in polynomial time in n, the # of unknowns.

- In practice, for many applications n is huge, which makes many of the standard algorithms costly, especially when the problem is sparse.

Solving SDP Problems

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Theorem

A SDP problem can be solved in polynomial time in n, the # of unknowns.

- In practice, for many applications n is huge, which makes many of the standard algorithms costly, especially when the problem is sparse.
- **However**, the rank of the solution is often quite small.

Lemma

Let $M \in \mathbb{S}^n$ then:

$$M \text{ is p.s.d, } \operatorname{rank} M \leq r \iff M = RR^t, \text{ for some } R \in \operatorname{Mat}(n, r).$$

Solving SDP Problems

non-Linear Semidefinite Programing (Burer and Monteiro)

Given a cost function $C \in \mathbb{S}^n$ *and constraints* $A_1, A_2, \ldots, A_m \in \mathbb{S}^n$, $\vec{b} \in \mathbb{R}^m$ *the non-linear semidefinite programing question* $(C; A_1, \ldots, A_m; \vec{b})$ *is to*

$$\min_{R \in \mathrm{Mat}_{n,r}} \langle C, RR^t \rangle$$

$$\textit{subject to} \langle A_i, RR^t \rangle = b_i.$$

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$$\textit{subject to} \left\langle A_i, RR^t \right\rangle = b_i.$$

- Notice the condition that the solution RR^t be p.s.d is now implicit.

Advantages	**Disadvantages**
Fewer Unknowns	non-Linear in Constraints
	non-Linear in Cost Function
	non-Convex

non-Linear SDP Problems

- Burer and Monteiro proposed an algorithm that experimentally outperforms existing methods of solving SDP problems.
- Their method is essentially Lagrange multipliers.
- Despite its appealing experimental behavior it is not known to converge, and there are generally numerous local minima.

Question

How many critical points does the Lagrange equation arising from non-linear semidefinite programming (nLSDP) problem have?

Number of Critical Points

Theorem (B^3KR)

There is an explicit constant $\delta(n, m, r)$ such that:

$$\#\left\{\begin{matrix}\textit{critical points}\\ \textit{of}\\ \textit{nLSDP Algorithm}\end{matrix}\right\} = \delta(n, m, r)\deg \mathrm{SO}(r)$$

- Combining this with our other main result given an explicit formula.

Corollary

$$\#\left\{\begin{matrix}\textit{critical points}\\ \textit{of}\\ \textit{nLSDP Algorithm}\end{matrix}\right\} = \delta(n, m, r)2^{r-1}\det\left(\binom{2r-2i-2j}{r-2i}\right)_{1\le i,j\le\lfloor\frac{r}{2}\rfloor}.$$

More On Convergence

- Notice that that the number of critical points grows rapidly in r.
- The appealing behavior of this algorithm is unexplained by our result and remains mysterious.
- **But...** some of the critical points we counted are extraneous. In particular, when solving an SDP problem we are only interested in those critical points that:
  - are real (rational) valued points,
  - and satisfies additional linear constraints.
- It would be interesting to refine our theorem so that it only counts such critical points.

Real Locus of SO(r)

Question

Can the degree of SO(r) *always be realized over* $\mathbb{R}$*?*

Question

Does there exist a plane H *such that:*

$$\deg \mathrm{SO}(r) = \#(H \cap \mathrm{SO}(r) \cap \mathbb{R}^{r^2})?$$

- This is a question of real algebraic geometry, and seems quite difficult.
- **Example:** Both of these smooth surfaces have degree two:

$$X_1 := \left\{x^2 + y^2 = -1\right\}$$
$$X_2 := \left\{x^2 + y^2 = 1\right\},$$

but X_1 has no real points. The degree of X_2 can be realized over $\mathbb{R}$.

Real Locus of SO(r)

- We explored this question computationally using Macaulay2 / Bertini.
- We randomly generated generic hyperplanes and recorded the number of real points witnessed in the resulting intersection.

	Degree	Most $\mathbb{R}$-Points	# of Trials
SO(3)	8	8	1398000
SO(4)	40	30	1004100
SO(5)	384	76	48200

- We relied on polynomial homotopy continuation and monodromy techniques for these computations.

Real Locus of SO(r)

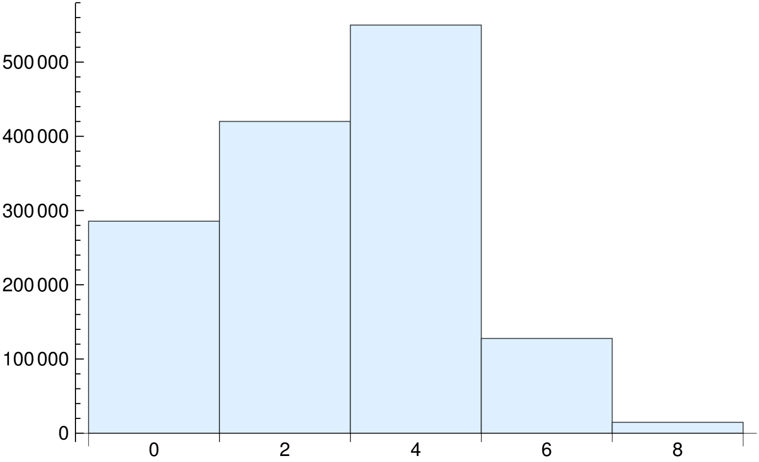

Figure: Number of Real Points in Witness Sets for SO(3)

Real Locus of SO(r)

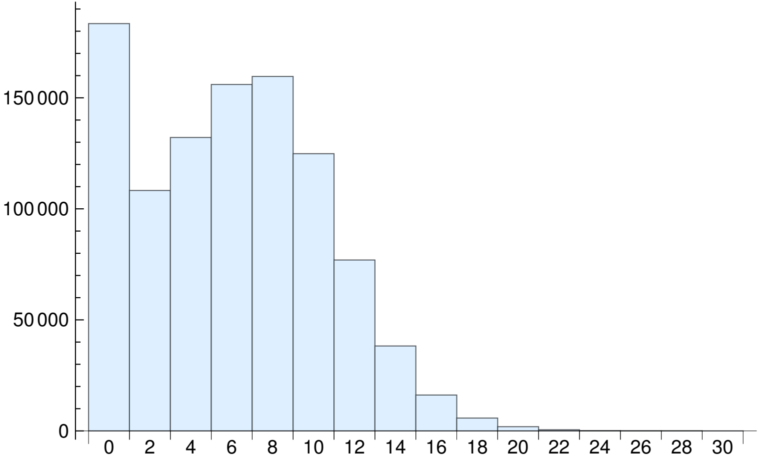

Figure: Number of Real Points in Witness Sets for SO(4)

Real Locus of SO(r)

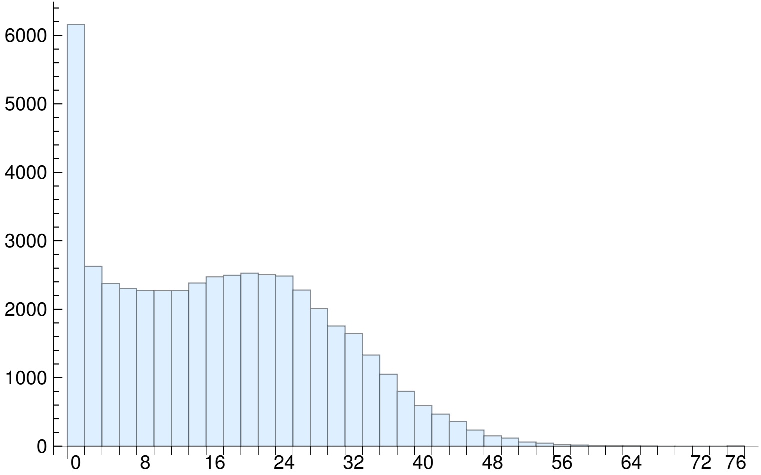

Figure: Number of Real Points in Witness Sets for SO(5)

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