A Distributed Numerical Approach to Syzygies of $\mathbb{P}^2$

Juliette Bruce (joint with Daniel Erman, Steve Goldstein, & Jay Yang)
University of Wisconsin - Madison

Definitions

- Letting $S = \mathbb{C}[x_0, \dots, x_n]$ be the polynomial ring we set:

$$S(n, b; d) := \bigoplus_{k \in \mathbb{Z}} S_{dk+b} \subset S,$$

which we think of as an $R = \mathrm{Sym}(S_d)$ module. This corresponds to the section ring of $\iota^* \mathcal{O}_{\mathbb{P}^n}(b)$ where $\iota : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ is the d'uple embedding.

- The **Koszul cohomology groups** of $S(n, b; d)$ are

$$K_{p,q}(n, b; d) := \mathrm{Tor}^R_p(S(n, b; d), \mathbb{C})_{p+q},$$

and we set

$$k_{p,q}(n, b; d) := \dim_{\mathbb{C}} K_{p,q}(n, b; d).$$

Example - $\mathbb{P}^2 \subseteq \mathbb{P}^9$ embedded by $\mathcal{O}_{\mathbb{P}^2}(3)$

- The Betti table of $S(2, 0; 3)$ is

	0	1	2	3	4	5	6	7
0	1	.	.	.	.	.	.	.
1	.	**27**	105	189	189	105	27	.
2	.	.	.	.	.	.	.	1

- So $K_{1,1}(2, 0; 3) = \mathbb{C}^{27}$. As a $\mathbb{Z}^3$-graded vector space, $K_{1,1}(2, 0; 3)$ has 19 distinct multidegrees, encoded via the multigraded Hilbert series:

$$\begin{aligned} HS_{K_{1,1}(2,0;3)}(t_0, t_1, t_2) = {} & t_0^4t_1^2 + t_0^3t_1^3 + t_0^2t_1^4 + t_0^4t_1t_2 + 2t_0^3t_1^2t_2 + 2t_0^2t_1^3t_2 + t_0t_1^4t_2 + t_0^4t_2^2 + 2t_0^3t_1t_2^2 \\ & + 3t_0^2t_1^2t_2^2 + 2t_0t_1^3t_2^2 + t_1^4t_2^2 + t_0^3t_2^3 + 2t_0^2t_1t_2^3 + 2t_0t_1^2t_2^3 + t_1^3t_2^3 + t_0^2t_2^4 + t_0t_1t_2^4 + t_1^2t_2^4. \end{aligned}$$

- As a Schur module, $K_{1,1}(2, 0; 3)$ is isomorphic to the irreducible representation $\mathbb{S}_{(4,2,0)}$.

Example - $\mathbb{P}^2 \subseteq \mathbb{P}^{27}$ embedded by $\mathcal{O}_{\mathbb{P}^2}(6)$

- The Betti table of $S(2, 2; 6)$ is:

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	6	123	1128	5775	16170	11628	.	.	.	.	.	.	.
1	.	.	.	1470	27498	333960	1738110	5958150	15502575	32303040	55383195	79341720	95834100
2	.	.	.	.	.	.	.	.	.	.	.	.	.

	13	14	15	16	17	18	19	20	21	22	23	24	25
0	.	.	.	.	.	.	.	.	.	.	.	.	.
1	98062800	85136340	62626470	38864595	20189400	8671575	3020820	827310	168360	22350	1050	.	.
2	.	.	.	.	.	.	.	.	.	.	231	48	3

Goal 1. *Use numerical linear algebra and high speed massively distributed computing to systematically gather new examples of Betti tables of d'uple embeddings of $\mathbb{P}^2$.*

Methodology

- We work one multidegree at a time, computing the cohomology of the multigraded Koszul complex by representing the differentials as matrices and then computing their ranks.

Result 1. *A computation of $K_{p,q}(2, b; d)$ for all $0 \leq b, d \leq 6$ and all p, q together with the Schur and multigraded decompositions.*

- **Step #1 - Pre-Computation:** We use known vanishing and duality results together with facts about Hilbert series to reduce the number of necessary rank computations.
- **Step #2 - Main Computation:** We construct the matrices, and use sparse LU factorization together with distributed high throughput computing to find the ranks.
- **Step #3 - Post-Processing:** We process the rank data to produce the multigraded decomposition. From these we compute the Betti numbers and find the Schur decomposition.

Example - $\mathbb{P}^2 \subseteq \mathbb{P}^{27}$ embedded by $\mathcal{O}_{\mathbb{P}^2}(6)$

- For $S(2, 0; 6)$ there are 600 matrices whose rank must be computed. The largest of these is $600,000 \times 600,000$.
- The computation of $K_{16,2}(2, 0; 6)$, has 178 relevant matrices:

# Matrices	Max Run Time	Ram (GB)
151	18 min.	< 1
16	1 hr.	$1 - 10$
17	16 hr.	$20 - 80$
2	3 **days**	> 450

Conjecture: Dominant and EEL Weights

- Ein, Erman, & Lazarsfeld constructed a set $E_{p,q}(n, b; d)$ of monomial syzygies, and conjectured:

$$K_{p,q}(n, b; d) = 0 \iff E_{p,q}(n, b; d) = 0.$$

- Counterintuitively our examples suggest $E_{p,q}(n, b; d)$ determines substantially more than just non-vanishing.

Conjecture 1. *For all n, d, b, p and q:*

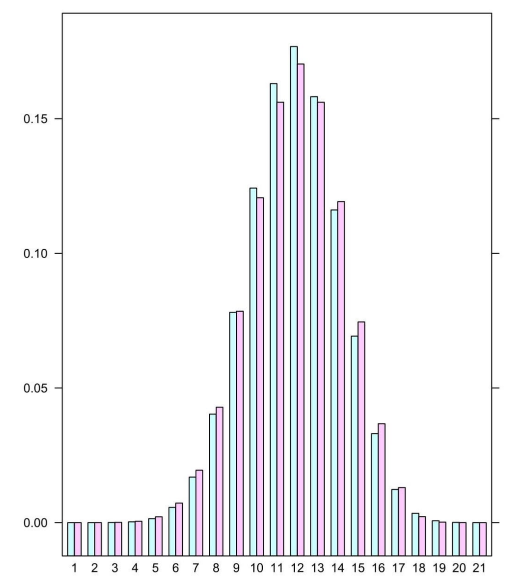
$$\mathrm{domWeights}\, K_{p,q}(n, b; d) = \mathrm{domWeights}\, E_{p,q}(n, b; d).$$

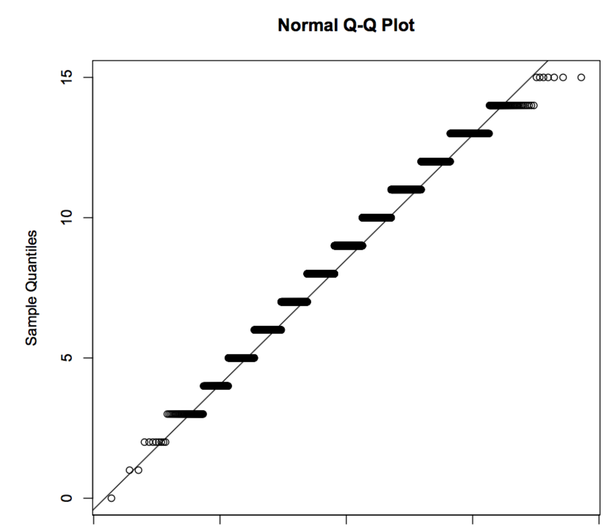
Conjecture: Normality of Betti Numbers

- Our data provides evidence for the Normality Conjecture of Ein, Erman, & Lazarsfeld.

Conjecture 2 (EEL)**.** *Fix n, b, and q. There exists a function $F_{n,b,q}(d)$ such that as $d \to \infty$:*

$$F_{n,b,q}(d) \cdot k_{p_d,q}(n, b; d) \to e^{-a^2/2}.$$





- Plotting the normalized $k_{p,1}(2, 0; 6)$'s (pink) versus the binomial distribution of best fit (blue), as well as the normal QQ-plot show support for this case of the conjecture.

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