How To Not Prove Things

- Introduction ?:

- Fix your favorite field k.

- Def: A k-variety is a "space" that locally looks like

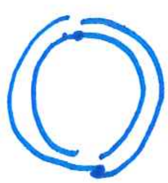
$$\mathbb{V}(f_1, \dots, f_s) \in k^n$$

For $f_1, \dots, f_s \in k[x_0, \dots, x_n]$.

- Now we want to think about the model structure of the category or sheaves of topologicalspaces or....

- Ex: 1) $\mathbb{A}^1_k = \mathbb{V}(x_1) = k$

2) $\mathbb{A}^n_k = \mathbb{V}(x_1, x_2) = k^2$

3) $\mathbb{P}^1_k$ 

- Def: If k'/k is a field extension a k'-point is......

- Def: If $x \in X$ - is a closed point - then

$$\mathcal{O}_{X,x} \text{ ':=' } \{ f \in k(V) \mid f \text{ is regular at } x \}$$

is the regular local ring of X at x:

$$m_x \text{ ':=' } \{ f \in \mathcal{O}_{X,x} \mid f \text{ vanishes at } x \}$$

• **Def**: The residue of a point $x \in X$ is

$$k(x) = \mathcal{O}_{X,x} / m_x$$

• **Ex**: If k is algebraically closed then $k(x) = k$.

• **Ex**: 1) $X = \mathbb{A}^1_{\mathbb{F}_q}$

$$\left\{ \text{Points } \mathbb{A}^1_{\mathbb{F}_q} \right\} \overset{\text{" "}}{\longleftrightarrow} \left\{ \text{Primes in } \mathbb{F}_q[x] \right\} \overset{\text{" "}}{\longleftrightarrow} \left\{ \begin{matrix} \text{monic irreducible} \\ \text{Polynomials } \mathbb{F}_q[x] \end{matrix} \right\}$$

2) Local ring corresponding to P is

$$\mathcal{O}_{X,x} = \mathbb{F}_q[x]_P$$

$$m_x = P\,\mathbb{F}_q[x]_P$$

3) The residue field of $x \in X$

$$\mathcal{O}_{X,x}/m_x = \mathbb{F}_q[x]_P / P\,\mathbb{F}_q[x]_P = \left(\mathbb{F}_q[x]/P\right)_{\bar{P}}$$

but $P = \langle f \rangle$ for some irreducible polynomial $f \in \mathbb{F}_q[x]$ and so

$$k(x) = \mathcal{O}_{X,x}/m_x = \left(\mathbb{F}_q[x]/P\right)_{\bar{P}} \cong \mathbb{F}_q[x]/P \cong \mathbb{F}_{q^{\deg(f)}}.$$

• **Q**: $\#\{\text{Pts. with residue field } \mathbb{F}_{p^e}\} = \; ? \; = \frac{1}{e} \sum_{d \mid e} \mu\left(\frac{e}{d}\right) p^d.$

A: $\mathbb{F}_{p^e}$ is the splitting field of $X^{p^e} - X$ and

$$x^{p^e} - x = \prod_{d \mid e} \prod_{\substack{\text{monic} \\ \deg(f)=d}} f$$

$$p^e = \deg(p^e) = \sum_{d \mid e} d \, \# \left\{ \begin{matrix} \text{Pts with residue} \\ \text{field } \mathbb{F}_{p^e} \end{matrix} \right\}$$

• **Def**: If $x \in X$ then

$$\deg(x) = [k(x) : k].$$

• **Prop**: Let k be a finite field and K/k be a degree r field extension.

$$|X(K)| = \sum_{d|r} d \cdot \{x \in X_{cl} \mid \deg(x) = d\}$$

Proof: (For those who don't know schemes close your eyes):

$$X(K) = \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spec}\, K, X) = \coprod_{x \in X^{cl}} \mathrm{Hom}_{k\text{-alg}}(k(x), K)$$

Now $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q) \cong \mathbb{Z}/r$ acts on $\mathrm{Hom}(k(x), K)$ transitively and if $\deg(x) = d$ then the stabilizer is $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_{q^d})$. ∎

• **Idea**: ~~[illegible]~~ Think about the $\mathbb{A}^1_{\mathbb{F}_p}$ case. A degree d point is a irreducible polynomial of $\mathbb{F}_p[x]$. If we move to a sufficiently large field i.e. a degree d extension these poly should factor giving us d points,

• Weil Conjectures:

• Let X be a variety of $\mathbb{F}_q$.

• **Def**: The Hasse-Weil Zeta function of X is

$$Z_X(t) = \exp\left(\sum_{m \geq 0} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m\right) \in \mathbb{Q}[[t]]$$

- Basic Properties:

1) $Z_X(t) = \prod_{x \in X^{cl}} (1 - t^{\deg(x)})^{-1} \in \mathbb{Z}[[t]]$

2) If $Y \subseteq X$ is closed then

$$Z_X(t) = Z_Y(t) Z_{X \setminus Y}(t).$$

- Ex: 1) $Z_{\mathbb{A}^n}(t) = \dfrac{1}{(1 - q^n t)}$

2) $Z_{\mathbb{P}^n}(t) = \dfrac{1}{(1-t)(1-qt) \cdots\cdots (1-q^n t)}$

- Weil Conjectures: Let X be a smooth, projective, (geometrically connected) variety of dimension n over $\mathbb{F}_q$:

1) (Rationality): $Z_X(t) \in \mathbb{Q}(t)$

2) (Functional Equation): $Z_X\left(\dfrac{1}{q^n t}\right) = \pm q^c t^c Z_X(t)$

3) (Reimann-Hypothesis):

$$Z_X(t) = \frac{P_1(t) P_3(t) \cdots\cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots\cdots P_{2n}(t)}$$

where $P_i \in \mathbb{Z}[t]$ and

$$\deg(P_i) \overset{!}{=} \dim\left(H^i(X, \mathbb{Q}) \right)$$

Easy — Dwork (1960), Grothendieck (1965)

Grothendieck (1965)

Deligne (1974).

hardest — Grothendieck 1965

- Goal #1: Fail to prove the rationality of $Z_X(t)$.

- Goal #2: See the relation between the geometry and arithmetic.

§ The Grothendieck Ring:

- Let k be your favorite field.

- Def: We let
$$\mathrm{Var}_k = \{ k\text{-varieties} \} / \text{Isomorphism}$$

- Def: The Grothendieck group of k-varieties is the abelian group
$$K_0(\mathrm{Var}_k) = \mathbb{Z}\langle \mathrm{Var}_k \rangle / \sim$$
where $\sim$ is the equivalence relation generated by
$$[X] = [Y] + [X \setminus Y]$$
where $Y \hookrightarrow X$ is a closed embedding.

- Q: What is the zero in $K_0(\mathrm{Var}_k)$?

A: For any k-variety k we have that
$$[X] = [X] + [X \setminus X] = [X] + [\emptyset].$$

- We place a ring structure on $K_0[\mathrm{Var}_k]$ by
$$[X] \cdot [Y] = [X \times Y] = [X \times_{\operatorname{spec} k} Y].$$

- Since $X \times Y \cong Y \times X$ this is commutative.

- Moreover since $X \times \{*\} \cong X$ we see that $[pt]$ is the multiplicative unit.

- **Lemma**: $K_0(Var_k)$ is a commutative ring with unit. ∎

- **Q**: What is this ring?

  ↳ This is really a serious question...

- **Ex**: $K_0(BCW)$ =: do the same thing with CW-complexes that have cells in bounded dimension.

  $[\mathbb{R}] = [\mathbb{R}^{<0}] + [0] + [\mathbb{R}^{<0}] = [2\mathbb{R}] + [0] \Rightarrow [\mathbb{R}] = -1$

  In fact we can do this for every thing .... In fact, compactly supported

  $\chi_c : K_0(bCW) \xrightarrow{\cong} \mathbb{Z}$ gives an isomorphism.

- **A**: The only way - I know of - to study $K_0(Var_k)$ is by studying maps to and from it.

- **Def**: A <u>motivic measure</u> is a ring homomorphism

$$\mu : K_0[Var_k] \longrightarrow R.$$

- **Ex**: 1) Euler-Characteristics: If H is a "nice" cohomology theory - compactly supported cohomology, compactly supported etale cohomology, ... - then we normally get a map

$$\chi_H : K_0[Var_k] \longrightarrow \mathbb{Z}.$$

+ If $k = \mathbb{C}$ and we consider our varieties with the analytic topology then

$$\chi(\mathbb{P}^2) = 3 \quad \text{and} \quad \chi(\mathbb{A}^1) = 1$$

So $K_0(Var_k)$ has at least 2 non-equivalent elements.

2) Point Counting: If $k = \mathbb{F}_q$ then we get a map

$$\# : \mathbb{K}Var_{\mathbb{F}_q} \longrightarrow \mathbb{N}$$

$$[X] \longmapsto \# X$$

Any easy check shows this descends to a motivic measure....

$$\# \, K_0(Var_k) \longrightarrow \mathbb{N}.$$

3) ... well this is all we got so far......
There is still a lot to know about $K_0(Var_k)$.......

Q: What are the primes of $K_0[Var_k]$?

- What is $\operatorname{Spec} K_0[Var_k]$?
- Is $K_0[Var_k]$ a domain?

- So the whole Weil Conjecture thing......

- § connecting weil to grothendieck:

- Def: If X is a variety then

$$\operatorname{Sym}^n X = \underbrace{X \times X \times \cdots \times X}_{n\text{-times}} / S_n$$

↳ If X is not quasi-projective then $\operatorname{Sym}^n X$ need not be a variety, but it is an algebraic space so we knot bother with this......

• **Prop**: If X is a variety over $\mathbb{F}_q$ then

$$Z_X(t) = \exp\left(\sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{k} t^k\right) = \sum_{n\geq 0} |\mathrm{Sym}^n(X)|(\mathbb{F}_q)\, t^n.$$

• Idea......

• **Def**: If $[X] \in K_0(\mathrm{Var}_k)$ then the motivic zeta function of $[X]$ is

$$\mathbb{Z}_{[X]}(t) = \sum_{n\geq 0} [\mathrm{Sym}^n X]\, t^n.$$

• **Prop**: $\mathbb{Z}_{-}(t) : K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_k)[[t]]^{\times}$ is a group homomorphism.

Proof: 1) If $X \simeq Y$ then $\mathrm{Sym}^n X \simeq \mathrm{Sym}^n Y$

2) If $Y \xhookrightarrow{cl} X$ is closed then

$$[\mathrm{Sym}^n(X)] = \sum_{p+q=n} [\mathrm{Sym}^p(Y)][\mathrm{Sym}^q(X\setminus Y)]. \quad \blacksquare$$

• Notice if $\mu : K_0(\mathrm{Var}_k) \longrightarrow R$ is a motivic measure then

$$\mu\left(\mathbb{Z}_{[X]}(t)\right) = \sum_{n\geq 0} \mu([\mathrm{Sym}^n X])\, t^n \in R[[t]].$$

So.....

$$\#\left(\mathbb{Z}_{[X]}(t)\right) = \sum_{n\geq 0} \#\mathrm{Sym}^n X\, t^n = Z_X(t). \quad !!!$$

• To prove $Z_X(t)$ is rational all we must do is prove

$\mathbb{Z}_X(t)$ is rational !!

• <u>Ex</u>: Returning to $K_0(b^cW)$ not we may also discuss symetric powers, defined in the same way as before.

$$\mathrm{Sym}^n([pt]) = [pt]$$

so

$$Z_{[pt]}(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}.$$

That said we saw above that $[X] = \chi_c(X)[pt]$.

$$\mathbb{Z}_{[X]}(t) = \left(\frac{1}{1-t}\right)^{\chi_c(X)} \quad !!$$

• <u>Theorem</u>: (Kapranov): If X is a smooth, geometrically connected, projective genus g curve over a perfect field k with a k-point then

$$\mathbb{Z}_{[X]}(t) = \frac{f(t)}{(1-t)(1-[\mathbb{A}^1]t)}$$

where $f(t) \in K_0(\mathrm{Var}_k)[t]$ of degree $\leq 2g$.

<u>Proof</u>: There is a morphism

$$\pi_n : \mathrm{Sym}^n(X) \longrightarrow \mathrm{Pic}_n(X)$$

which is a fiber bundle for $n \geq 2g-1$ with fibre $\mathbb{P}^{n-g}$.

Thus, for $n \geq 2g-1$ we have that

$$[\mathrm{Sym}^n X] = (1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^{n-g})[\mathrm{Jac}\, X].$$

Then we get that

$$\mathbb{Z}_{[X]}(t) = \sum_{n=0}^{2g-2} [\mathrm{Sym}^n X] t^n + \sum_{n=2g-1}^{\infty} (1 + \mathbb{L} + \cdots \mathbb{L}^{n-g})[\mathrm{Jac}\, X] = \boxed{} + \frac{[\mathrm{Jac}\, X]}{1-\mathbb{L}}\left(\frac{1}{1-t} - \frac{\mathbb{L}^g}{1-\mathbb{L}t}\right)$$

■

- The case when X is a curve without a rational point was done by D. Litt.

- **Thm**: (Larsen & Lunts): A complex surface X has rational zeta function
  $\Leftrightarrow$ the Kodaira dimension of X $= -\infty$.

- We will not prove this theorem, but instead prove a similar result fundamental to the above proof.

- **Thm**: (Larsen & Lunts): If $k = \mathbb{C}$ there exists a motivic measure
$$\mu : K_0(Var_k) \longrightarrow K$$
such that if X is a smooth surface with $h^{2,0}(X) \geq 2$ then
$$\mu(Z_{[X]}(t))$$
is not rational.

- § Stable Birational Geometry:

- **Def**: Two varieties X and Y are birational iff $\exists$
  $f : X \longrightarrow Y$ s.t. f is an isomorphism on a dense set.

- **Def**: Two varieties X and Y are stably birational iff $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^m$ for some $m, n \geq 0$.

- **Thm**: Let G be a abelian commutative monoid and $\mathbb{Z}[G]$ the associated ring. Let $\mathcal{M}$ be the multiplicative monoid of isomorphism classes of smooth complete irreducible varieties. If
$$\psi : \mathcal{M} \longrightarrow G$$

is a homomorphism of monoid such that

A) $\psi([X]) = \psi(Y)$ if X is birational to Y

B) $\psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.

then there exists a unique ring homomorphism

$$\phi : K_0(Var_k) \longrightarrow \mathbb{Z}[G]$$

where $\Phi([X]) = \psi([X])$ for all $[X] \in \mathcal{M}$.

↳ Need to be working over an algebraically closed field char = 0.

• <u>Thm</u>: (Weak Factorization): (Abramovich, Karu, Matsuki, Wlodarczyk): If

$$X_1 \overset{\emptyset}{- - - \to} X_2$$

is a birational equivalence between complete non-singular varieties over an algebraically closed field of characteristic zero where $U \subseteq X_1$ is the open dense set where $\emptyset$ is an isomorphism then there exists a sequence of birational maps

$$X_1 = V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \cdots \longrightarrow V_i \xrightarrow{\varphi_{i+1}} V_{i+1} \longrightarrow \cdots\cdots \xrightarrow{\varphi_n} V_n = X_2$$

such that

1) $\emptyset = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$

2) φ_i is an isomorphism on U

3) either $\varphi_i : V_i \longrightarrow V_{i+1}$ or φ_i^{-1} is a blow up at a center away from U.

↳ This is a souped up version of resolution of singularities.

↳ The same proof works over a perfect field of char = p for varieties of dimension d assuming embed resolution of singularities in dim d+1.

↳ we currently only have this in surfaces.

- **Thm**: (Hironaka): We have embedded resolution of singularities in characteristic zero. over an algebraic closed field.

- There are two independent proofs of the above theorem. However, the both utilize these theorems in crucial ways.

- **Thm**: (Bittner): As a group $K_0\langle Var_k\rangle$ is isomorphic to

$$\mathbb{Z}\left\langle \begin{matrix}\text{isomorphism classes} \\ \text{Smooth complete projective} \\ k\text{- Varieties}\end{matrix} \right\rangle \Big/ \sim$$

where 1) $[\emptyset] = 0$

2) If $Y \subseteq X$ is a closed smooth subvariety and $Bl_Y X$ is the blow-up of X along Y and E is the exceptional divisor

$$[Bl_Y X] - [E] = [X] - [Y].$$

- **Proof**: Obvious.

- **Def**:

$$SB = \text{multiplicative monoid of Stable birational classes}$$

- Note birational $\Rightarrow$ stably birational.
- We have a map

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Psi_{SB}} & SB \\ \| & & \| \\ \text{iso classes of smooth} & & \text{stable birational classes} \end{array}$$

• Note this morphism is surjective, by Hironaka's theorem.

1) Moreover, if $X \simeq Y$ then $[X]_{SB} = [Y]_{SB}$ so Ψ_{SB} satisfies #1.

2) $\mathbb{P}^n \times \{*\} \cong \mathbb{P}^0 \times \mathbb{P}^n \Rightarrow [\mathbb{P}^n]_{SB} = [*] = 1 \quad \forall n.$

Thus, we get an induced map (by the theorem)

$$K_0(\mathrm{Var}_k) \xrightarrow{\Phi_{SB}} \mathbb{Z}[SB]$$

Any morphism as in the theorem factors through this

This is surjective.

• <u>Psuedo</u> - A: (Q 2): Let $X_1, \ldots\ldots X_k, Y_1 \ldots \ldots Y_r$ be smooth complete varities and $m_i, n_i \in \mathbb{Z}$ s.t.

$$\sum_i m_i [X_i] = \sum_j n_j [Y_j] \quad (\text{in } K_0(\mathrm{Var}_k))$$

then 1) $k = r$

2) After relabing $X \simeq_{SB} Y$ & $m_i = n_i$

<u>Proof:</u> Apply Φ_{SB} to Both sides

$$\Phi_{SB}\left(\sum m_i [X_i]\right) = \sum m_i \Phi_{SB}([X_i])$$

$$\underbrace{= \sum_{m_i} m_i \Psi_{SB}[X_i] = \sum_i n_i \Psi_{SB}(Y_i)}_{\text{In } \mathbb{Z}[SB].} = \Phi_{SB}\left(\sum m_i [X_i]\right)$$

∎

• **Thm:** Any variety X may be written ~~uniquely~~ in $K_0(Var_k)$ as

$$[X] = [Y_1] + \cdots\cdots + [Y_n]$$

where the Y_i are complete and smooth, and unique up to stable birationality. ■

• This is like cut and paste - but insted we resolve and complete.

• **Thm:** The morphism Φ_{SB} induces an isomorphism

$$\frac{K_0(Var_k)}{[\mathbb{L}]} \cong \mathbb{Z}[SB]$$

• **Pf:** 1) Note Φ_{SB} is surjective

2) WTS $Ker(\Phi_{SB}) = \langle \mathbb{L} \rangle$.

• Note $[\mathbb{P}^1] = [\mathbb{A}^1] + [*]$ & $\Phi_{SB}(\mathbb{P}^1) = 1$

$\Rightarrow \Phi_{SB}(\mathbb{A}^1) = 0$.

• Suppose $T \in Ker(\Phi_{SB})$. By the previous theorme we may write

$T = X_1 + \cdots\cdots X_n - Y_1 \cdots\cdots - Y_m$ ← smooth & complete.

$$\Rightarrow \Phi_{SB}(T) = \sum_i \Psi_{SB}(X_i) - \sum_i \Psi_{SB}(Y_i) = 0$$

$\Rightarrow n = m$ and After relabeling $X_i \simeq_{SB} Y_i$.

So it suffices to show If X & Y are smooth, complete, and stably birational

$[X] - [Y] \in \langle \mathbb{L} \rangle$.

Notice

$$[X \times \mathbb{P}^n] - [X] = [X][\mathbb{P}^n] - [X]$$
$$= [X](1 + \cdots\cdots \mathbb{L}^n) - [X]$$
$$= [X](\mathbb{L} + \cdots\cdots \mathbb{L}^n).$$

Now if $X \simeq_{SB} Y$ so $X \times \mathbb{P}^n \simeq Y \times \mathbb{P}^m$ thus we have

$$[X] - [X \times \mathbb{P}^n] - (Y - [Y \times \mathbb{P}^m])$$
$$= [X] - [Y] - ([X \times \mathbb{P}^n] + [Y \times \mathbb{P}^m])$$

$$= [X](\mathbb{L} + \cdots\cdots \mathbb{L}^n) - [Y](\mathbb{L} + \cdots \mathbb{L}^m) \in \langle \mathbb{L} \rangle$$

So if $[X \times \mathbb{P}^n] - [Y \times \mathbb{P}^m] \in \langle \mathbb{A}^1 \rangle$ we are good. Thus, we may reduce to the case $X \simeq Y$. By weak factorization we may assume X is the blow-up of Y at a smooth center Z, with exceptional divisor E. Then we have $[E] = [\mathbb{P}^t] \times [Z]$ &

$$[X] - [Y] = [E] - [Z] = [\mathbb{P}^t][Z] - [Z]$$ ∎

- <u>Cor</u>: If $\mu : K_0[\mathrm{Var}_k] \longrightarrow R$ is a motivic measure the following are equivalent
  1) $\mu(\mathbb{A}^1) = 0$
  2) If X and Y are smooth and complete with $X \simeq Y \Rightarrow \mu[X] = \mu[Y]$.

∎

- $\mu(\mathbb{Z}_X(t))$ is not always rational....

  - $C \subseteq \{$ Polys with positive leading coeff $\} \subseteq \mathbb{Z}[t]$

  - $\mathcal{H} = \mathrm{Frac}(\mathbb{Z}[C])$

  - $\Psi_h(X) = 1 + h^{1,0}(X)t + \cdots\cdots + h^{d,0}(X)t^d \in C$

    $\uparrow X$ smooth-complete, dim $= d$

  - Kunneth + other stuff $\Rightarrow \Psi_h : \mathcal{M} \longrightarrow C$ is a monoid map.

  $\Rightarrow \mu_h : K_0(\mathrm{Var}_k) \longrightarrow \mathcal{H}.$

- <u>Thm</u>: With μ_h as above let X be a smooth complete ~~variety~~ proj. surface.

  if $h^0(X, \omega_X) \geq 2 \Rightarrow \mu_h(\mathbb{Z}_X(t))$ is not rational.

- <u>Def</u>: A power series $f(t) \in A[\![t]\!]$ is <u>globally rational</u> iff $g(t), h(t) \in A[t]$ with $f(t)$ the unique solution to $g(t)x = h(t)$.

- <u>Def</u>: A power series $f(t) \in A[\![t]\!]$ is <u>pointwise rational</u> $\Leftrightarrow \forall\ \phi : A \longrightarrow k \leftarrow$ A field

  $\Phi(f)$ is globally rational.

- <u>Def</u>: A powerseries $A[\![t]\!]$, $f(t) = \sum a_i t^i$ is <u>determinantally rational</u> $\Leftrightarrow$

  $\exists$ integers m, n s.t

$$\det \begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+m} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+m+1} \\ \vdots & & & \\ a_{i+m} & \cdots & \cdots & a_{i+2m} \end{vmatrix} = 0$$

  $\forall\ i > n.$

- **Thm:** If A is an domain these are equivalent.

- **Thm:** A ~~rational~~ complex surface X has rational motivic zeta function $\iff \kappa(X) = -\infty$.

- **Pseudo-Answer** (Q#1): No.

- **Thm** (Borisov): Over $\mathbb{C}$ there are 2 smooth Calabi-Yau 3-folds X_W & Y_W with

$$([X_W]-[Y_W])(\mathbb{L}^2-1)(\mathbb{L}-1)\mathbb{L}^7 = 0.$$

- **Thm:** (Borisov): $\mathbb{L}$ is a zero-divisor over $\mathbb{C}$.

  Pf: NTS

$$([X_W]-[Y_W])(\mathbb{L}^2-\mathbb{L})(\mathbb{L}-1) \neq 0 \qquad (*)$$

$\Rightarrow$ Enough to show not zero mod $\mathbb{L}$. By Larsen and Lunts

$(*) = 0 \mod \mathbb{L} \Rightarrow X_W \simeq_{SB} Y_W$. But for these $X_W \simeq Y_W$ SB $\Rightarrow$

$X_W \simeq Y_W$ birational, but this is not true. ∎

- **Thm:** The cut and paste conjecture fail,

  Pf: cut and paste $\Rightarrow$

$$X_W \times GL(2,\mathbb{C}) \times \mathbb{C}^6 \simeq Y_W \times GL(2,\mathbb{C}) \times \mathbb{C}^6$$

$\Rightarrow X_W \simeq_{SB} Y_W \Rightarrow X_W \simeq Y_W.$

∎

- Thm: (Zakharevich): Every element in the kernel of multiplication by $\mathbb{L}$ may be represented as $[X]-[Y]$. where
  1) $[X] \neq [Y]$
  2) $X \times \mathbb{A}^1$. is not piecewise isomorphic to $Y \times \mathbb{A}^1$
  3) $[X \times \mathbb{A}^1] = [Y \times \mathbb{A}^1]$.

  $\leadsto$ Gets the larsen-Lunts results & stuff.

- So why care
  - Means we don't "get" an "easy" "geometric" proof of Weil.
  - Has consequenseces for geometry....
    + Nick Addington
    + (Galkin & Shinder): If $\mathbb{L}$ is not a zero or zero divisor then a rational smooth cubic fourfold in $\mathbb{P}^5$ must have its Fano variety of lines be birational to the symmetric square of a K3 surface.