Doodling Daydreams

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The Madison Math Circle is an outreach organization seeking to show middle and high schoolers the fun and excitement of math! For more info about the Madison Math Circle please visit:

`math.wisc.edu/outreach/mathcircle`.

Take Home Questions

Question 1. Suppose $X \subset \mathbb{R}^2$ is a region in the plane, and let $N_r(X)$ be the set of points a distance at most r from X. As these are figures in the plane there are various features we can measure about them, in particular, their perimeter and area. How do $\mathrm{Perm}(X)$, $\mathrm{Perm}(N_r(X))$, $\mathrm{Area}(X)$, and $\mathrm{Area}(N_r(X))$ relate?

Exercise 1. Compute $\mathrm{Perm}(X)$, $\mathrm{Perm}(N_r(X))$, $\mathrm{Area}(X)$, and $\mathrm{Area}(N_r(X))$ assuming X is the following shapes (you should get a formula in terms of r):

(1) a line segment of length 1,

(2) a line segment of length 19,

(3) a circle of radius 1,

(4) a circle of radius 19,

(5) a square of side length 1,

(6) a square of side length 19,

(7) and a regular hexagon of side length 1.

Question 2. ($\star\star\star$) Suppose you wrap a string tight around the equator of the Earth. (For simplicity let's assume the Earth is a perfect sphere.) You then cut this string and add an additional 1m segment before reconnecting the ends. If you were able to place this new loop of string at a contestant distance from the Earth how far would it be from the surface of the Earth?

Exercise 2. We can generalize our definition of N_r from Exercise 1 and the talk to regions in space. If $X \subset \mathbb{R}^3$ is a region in space then $N_r(X)$ is the set of all points in space $\mathbb{R}^3$ such that are a distance at most r from X. Describe, as precisely as possible, $N_r(X)$ for the following regions:

(1) a point,

(2) a line segment of length 1, =

(3) a sphere of radius 1,

(4) a cube of side length 1,

(5) a rectangular box of length ℓ, width w, and height h,

(6) and circle of radius radius R [1]

Question 3. Suppose $X \subset \mathbb{R}^3$ is a region in space. How do Perm(X), Perm($N_r(X)$), Area(X), Area($N_r(X)$), Vol(X), and Vol($N_r(X)$) relate?

Exercise 3. Compute Perm(X), Perm($N_r(X)$), Area(X), Area($N_r(X)$), Vol(X), and Vol($N_r(X)$) assuming X is the following shapes (you should get a formula in terms of r):

(1) a point,

(2) a line segment of length 1,

(3) a line segment of length 17,

(4) a sphere of radius 1,

(5) a sphere of radius 17

(6) a cube of side length 1,

(7) a cube of side length 17,

(8) and a rectangular box of length ℓ, width w, and height h.

Question 4. ($\star\star\star$) When you check your luggage with Delta they require they length, width, and height of the bag to sum to less than or equal to 62 inches. That is, if your bag has length ℓ, width w, and height h you need that $\ell + w + h \leq 62$. Is it possible to cheat Delta's policy by somehow putting an oversized bag inside of a legal bag. Can you prove your answer? (You might want to assume your bag is a rectangular box.)

Question 5. Can we generalize N_r to higher dimensions? More complicated shapes?

Question 6. ($\star\star\star$) Let $\mathcal{M}_{g,n}$ be the moduli space of hyperbolic Riemann surfaces of genus g with n punctures. Let Vol($\mathcal{M}_{g,n}$) denote the Weil-Peterson volume of $\mathcal{M}_{g,n}$. Describe the asymptotic behavior of Vol($\mathcal{M}_{g,n}$) as $g \to \infty$.

Question 7. ($\star\star\star$) How could you have won a Fields Medal (i.e. math's Nobel Prize)?[2]

[1] Beware! This one is trickier than it first might appear, and a complete answer may end up depending on the relationship between both R and r.

[2] One good route would have been to answer the previous question (c.f. Maryam Mirzakhani).