Syzygies of Surfaces via Distributed Computing

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- Given homogenous polynomials $f_1, \ldots, f_t \in S = \mathbb{C}[x_1, \ldots, x_r]$ we associate to it a space (variety):

$$X = \left\{ \vec{p} \in \mathbb{C}^r \;\; : \;\; f_1(\vec{p}) = f_2(\vec{p}) = \cdots = f_t(\vec{p}) = 0 \right\} \subset \mathbb{C}^r,$$

  and a ring (coordinate ring):

$$S_X = \frac{S}{\langle f_1, \ldots, f_t \rangle}.$$

- This talk is focused on how the syzygies of S_X relate to the geometry of X.

Definition

Let $(f_1, f_2, \dots, f_n)$ and $(g_1, g_2, \dots, g_n)$ be (tuples) of polynomials. We say that $(g_1, g_2, \dots, g_n)$ is a **syzygy** of $(f_1, f_2, \dots, f_n)$ if:

$$f_1 g_1 + f_2 g_2 + \cdots + f_n g_n = 0.$$

Introduction

- **Example** Let $f_1 = y^2 - xz$, $f_2 = yz - xw$, and $f_3 = z^2 - yw$ then

$$-z \cdot f_1 + y \cdot f_2 - x \cdot f_2 = -z \cdot (y^2 - xz) + y \cdot (yz - xw) - x \cdot (z^2 - yw) = 0$$
$$w \cdot f_1 - z \cdot f_2 + y \cdot f_2 = w \cdot (y^2 - xz) - z \cdot (yz - xw) + y \cdot (z^2 - yw) = 0$$

  so $(g_1, g_2, g_3) = (-z, y, -x)$ and $(g_1, g_2, g_3) = (w, -z, y)$ are syzygies of $(f_1, f_2, f_3) = (y^2 - xz, yz - xw, z^2 - yw)$.

- In fact, *every* syzygy is a multiple of these two!!
- But then we could take the syzygies this first set of syzygies we found....

- Let $S = \mathbb{C}[x_0, x_1, \ldots, x_r]$ be the graded polynomial ring.
- If M is a finitely generated graded S-module then there exists a minimal graded free resolution of M:

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_0} F_1 \xleftarrow{\phi_1} \cdots$$

- The $\ker \phi_0$ correspond to the (first) syzygies of M.

Theorem (Hilbert Syzygy Theorem)

If M is a finitely generated graded S-module then the minimal graded free resolution of M has length at most $r+1$

- **Notation:** We write $S(-q)$ for the free S-module of rank one generated in degree q.

Introduction

- Given the minimal graded free resolution of M:

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_0} F_1 \xleftarrow{\phi_1} \cdots,$$

since F_p is a graded free module there is an isomorphism:

$$F_p \cong \bigoplus_{q\in\mathbb{Z}} S(-q)^{\oplus \beta_{p,q}(M)}.$$

- The $\beta_{p,q}(M)$ are called the graded Betti numbers of M.
- We often put these into a table called the Betti table of M:

$$\beta(M) := \begin{array}{c|ccccc} & 0 & 1 & 2 & \cdots & p \\ \hline 0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots & \\ 1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \\ 2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ q & & & & & \beta_{p,p+q} \end{array}$$

Example - Twisted Cubic

- The twisted cubic curve is the image of the map:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3 \qquad\qquad [s:t] \longmapsto \left[s^3 : s^2t : st^2 : t^3\right],$$

and in equations it is defined by $y^2 - xz$, $yz - xw$, and $z^2 - yw$.

- The minimal graded free resolution of its coordinate ring is:

$$0 \longleftarrow S_X \longleftarrow S \longleftarrow S(-2)^3 \xleftarrow{\;1\;} S(-3)^2 \longleftarrow 0.$$

- The Betti table for the twisted cubic is then:

$$\beta(S_X) := \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & 2 & - \\ 2 & - & - & - & - \end{array}$$

What Is Know? (dim $= 1$)

- Beginning with the work of M. Green in the 1980′s there has been substantial work on the case when X is a curve.
  - Green′s Conjecture (Voisin),
  - Gonality Conjecture (Ein-Lazarsfeld),
  - Prym-Green conjecture (Farkas-Kemeny),
  - and many more...
- **Example:** Consider the d'uple rational curve:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d \qquad\qquad [s:t] \longmapsto \left[s^d : s^{d-1}t : \cdots : st^{d-1} : t^d\right].$$

- The minimal free resolution for this is given by the Eagon-Northcott complex.

What Is Know? (dim > 1)

- Surprisingly little is know!
- The last few years has seen a number of fascinating conjectures:
  - which entries of the Betti table are non-zero, and
  - what the relative size of each entry in each row.
- These conjectures suggest things are complicated, and very different from the case of curves.
- But there are only a handful of specific examples know.
- **Example:** Consider the d'uple Veronese surface:

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^{\binom{2+d}{d}-1}$$

until recently the Betti tables were only know for $d = 2, 3, 4$.

Example - Veronese Surface

- The dimension of S is growing quadratically in d, which means the computational complexity ratchets up quite dramatically.
- **Example:** When $d = 3$, V_d is defined by the 27 quadric polynomials in 10 variables shown below:

$$
\begin{array}{llll}
x_7x_8 - x_6x_9, & x_5x_8 - x_4x_9, & x_4x_8 - x_3x_9, & x_1x_4 - x_0x_7, \\
x_2x_8 - x_1x_9, & x_5x_7 - x_3x_9, & x_4x_7 - x_3x_8, & x_2x_3 - x_0x_7, \\
x_2x_7 - x_1x_8, & x_5x_6 - x_3x_8, & x_4x_6 - x_3x_7, & x_2x_6 - x_1x_7, \\
x_4x_5 - x_1x_9, & x_3x_5 - x_1x_8, & x_2x_5 - x_0x_9, & x_1x_2 - x_0x_4 \\
x_1x_5 - x_0x_8, & x_3x_4 - x_1x_7, & x_1x_3 - x_0x_6, & x_2x_4 - x_0x_8,
\end{array}
$$

$$x_8^2 - x_7x_9,\ x_7^2 - x_6x_8,\ x_2^2 - x_0x_5,\ x_1^2 - x_0x_3,$$

$$x_5^2 - x_2x_9,\ x_4^2 - x_1x_8,\ x_3^2 - x_1x_6$$

- Things only get more complicated... 315 polynomials in 28 variables...

Our Goal

Goal

Systematically gather new examples of Betti tables of Veronese embeddings of $\mathbb{P}^2$.

- **Notation:** For the remaining part of the talk we will use the following notation:
  - $V_d =$ d'uple Veronese embedding of $\mathbb{P}^2$,
  - $S = \mathbb{C}[x, y, z]$,
  - $S_d = \mathbb{C}$-vector space spanned by the monomials of degree d in S,

Computational Approach

- Looking at the Koszul complex:

$$\bigwedge^{p+1} S_d \otimes S_{(q-1)d} \xrightarrow{\partial_{p+1}} \bigwedge^{p} S_d \otimes S_{qd} \xrightarrow{\partial_{p}} \bigwedge^{p-1} S_d \otimes S_{(q+1)d}$$

  where ∂_p is defined by:

$$\partial_p\Big(m_1 \wedge \cdots \wedge m_p \otimes f\Big) = \sum_{k=1}^{p} (-1)^k m_1 \wedge \cdots \wedge \widehat{m}_k \wedge \cdots \wedge m_p \otimes (m_k f)$$

  one can show that

$$\beta_{p,p+q}(V_d) = \dim\ker\partial_p - \dim\operatorname{img}\partial_{p+1} = \operatorname{corank}\partial_p - \operatorname{rank}\partial_{p+1}$$

- Wsriting $K_{p,q}(2;d)$ for the cohomology of the above sequence:

$$\beta_{p,p+q}(V_d) = \dim K_{p,q}(2,d).$$

Computational Approach

- So computing syzygies of V_d reduces to computing the ranks of certain matrices.
- But....
  - these matrices are gigantically huge,
  - are generally high rank,
  - and there are lots of them.
- **Example:** One of the matrices we would need to compute the rank of has dimensions:

$$254{,}103{,}788{,}400 \times 902{,}737{,}143{,}000.$$

- Size and scale makes naive approaches impossible.
- Thus, we take advantage of
  - symmetries and patterns within our examples,
  - sparse numerical linear algebra methods (LU-factorizations),
  - and high-throughput and high performance computing.

Computational Approach - Relevant Range

- We don't actually have to compute every entry via differentials.
- Given the minimal graded free resolution of M:

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{\phi_0} F_1 \xleftarrow{\phi_1} \cdots,$$

  since this is graded and dimension of vector spaces is additive:

$$\dim M_d = \sum_i (-1)^i \dim(F_i)_d = \sum_i (-1)^i \beta_{i,d}(M)$$

- So when there is only one entry on the diagonal it is determined by the Hilbert function.
- It is much easier to compute the Hilbert function.

Computational Approach - Relevant Range

- **Example:** Here is the Betti table for V_5:

$\beta(2;5) =$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	165	1830	10710	41616	117300	250920	417690	548080	568854	464100	291720	134640	39780	4858	375	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	2002	4200	2160	595	90	6

- There are only four values not determined by the Hilbert function.
  - The relevant range is $\{(14,1),(15,1),(13,2),(14,2)\}$.

Computational Approach - Symmetries

- Our matrices in question turn out to have natural symmetries.
- In particular, $(\mathbb{C}^*)^3$ acts on the Koszul complex, which gives a decomposition of its cohomology.
- More concretely our resolution breaks into multigraded strands:

$$\left(\bigwedge^{p+1} S_d \otimes S_{(q-1)d}\right)_{\mathbf{a}} \xrightarrow{\partial_{p+1,\mathbf{a}}} \left(\bigwedge^{p} S_d \otimes S_{qd}\right)_{\mathbf{a}} \xrightarrow{\partial_{p,\mathbf{a}}} \left(\bigwedge^{p-1} S_d \otimes S_{(q+1)d}\right)_{\mathbf{a}}.$$

- We define the multigraded Betti numbers of V_d by:

$$\beta_{p,\mathbf{a}}(V_d) = \dim\ker\partial_{p,\mathbf{a}} - \dim\operatorname{img}\partial_{p+1,\mathbf{a}}$$

- These are related to the graded Betti numbers via the following:

$$\beta_{p,p+q}(V_d) = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^3 \\ |\mathbf{a}|=(p+q)d}} \beta_{p,\mathbf{a}}(V_d).$$

Computational Approach - Symmetries

- **Example:** The Betti table for V_3 is:

$$\beta(V_3) = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1 & - & - & - & - & - & - & - \\ 1 & - & \color{red}{27} & 105 & 189 & 189 & 105 & 27 & - \\ 2 & - & - & 0 & - & - & - & - & 1 \end{array}.$$

- Focusing on the red 27, we have

$$K_{1,1}(2;3) \cong \mathbb{C}^{27} \quad \text{and} \quad \beta_{1,2}(V_3) = 27.$$

- As a $\mathbb{Z}^3$-graded vector space, $K_{1,1}(2;3)$ has 19 distinct multidegrees, which we encode via a generating series

$$t_0^4 t_1^2 + t_0^3 t_1^3 + t_0^2 t_1^4 + t_0^4 t_1 t_2 + 2t_0^3 t_1^2 t_2 + 2t_0^2 t_1^3 t_2 + t_0 t_1^4 t_2$$

$$+ t_0^4 t_2^2 + 2t_0^3 t_1 t_2^2 + 3t_0^2 t_1^2 t_2^2 + 2t_0 t_1^3 t_2^2 + t_1^4 t_2^2 + t_0^3 t_2^3 + 2t_0^2 t_1 t_2^3$$

$$+ 2t_0 t_1^2 t_2^3 + t_1^3 t_2^3 + t_0^2 t_2^4 + t_0 t_1 t_2^4 + t_1^2 t_2^4.$$

- Thus for instance $K_{1,1}(0;3)_{(4,2,0)} = \mathbb{C}$ and $K_{1,1}(0;3)_{(2,2,2)} = \mathbb{C}^3$.

Computational Approach - Symmetries

- **Example:** Now let us focus on the blue entry in Betti table for V_3:

$$\beta(V_3) = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1 & - & - & - & - & - & - & - \\ 1 & - & 27 & 105 & 189 & 189 & 105 & 27 & - \\ 2 & - & - & {\color{blue}0} & - & - & - & - & 1 \end{array}.$$

- Consider $K_{2,2}(0;3)_{(7,3,2)}$, which is computed by

$$\left(\bigwedge^3 S_3 \otimes S_{1\cdot 3}\right)_{(7,3,2)} \xrightarrow{\partial_{3,(7,3,2)}} \left(\bigwedge^2 S_3 \otimes S_{2\cdot 3}\right)_{(7,3,2)} \xrightarrow{\partial_{2,(7,3,2)}} \left(\bigwedge^1 S_3 \otimes S_{3\cdot 3}\right)_{(7,3,2)}.$$

- We use products of monomials for our bases.
- For example, the following is a basis vector in the source:

$$x^3 \wedge x^2 y \otimes x^2 y^2 z^2 \in \left(\bigwedge^2 S_3 \otimes S_{2\cdot 3}\right)_{(7,3,2)}.$$

- We then have:

$$\partial_{2,(7,3,2)}(x^3 \wedge x^2 y \otimes x^2 y^2 z^2) = x^3 \otimes x^4 y^3 z^2 - x^2 y \otimes x^5 y^2 z^2.$$

Computational Approach - Symmetries

- Working over all monomials, we represent $\partial_{2,(7,3,2)}$ by a matrix:

$$
\begin{array}{c|c}
 & \begin{array}{cccc} x^3 \wedge x^2y \otimes x^2y^2z^2 & x^3 \wedge xy^2 \otimes x^3yz^2 & x^3 \wedge x^2z \otimes x^2y^3z & \cdots \end{array} \\
\begin{array}{c} x^3 \otimes x^4y^3z^2 \\ x^2y \otimes x^5y^2z^2 \\ x^2z \otimes x^5y^3z \\ xy^2 \otimes x^6y^2z^2 \\ \\ \end{array} & \begin{pmatrix} 1 & 1 & 1 & \cdots \\ -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & \cdots \\ 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{array}
$$

- The dimension of $K_{2,2}(0;3)_{(7,3,2)}$ is determined by the ranks and sizes of these matrices. Since $\partial_{2,(7,3,2)}$ has 23 columns, we have

$$
\begin{aligned}
\dim K_{2,2}(0;3)_{(7,3,2)} &= \dim\ker \partial_{2,(7,3,2)} - \dim \operatorname{img} \partial_{3,(7,3,2)} \\
&= \left(23 - \operatorname{rank} \partial_{2,(7,3,2)}\right) - \operatorname{rank} \partial_{3,(7,3,2)} \\
&= 23 - 8 - 15 = 0.
\end{aligned}
$$

Computational Approach - Symmetries

- This allows us to break our lots of gigantically huge matrices into tons of *huge* matrices.
- **Example:** The number of matrices and the size of the largest matrix we need to compute the ranks of for a few examples is shown below:

d	b	# of Matrices	Largest Matrix
6	0	1,028	$596,898 \times 1,246,254$
	1	148	$7,345 \times 9,890$
	2	148	$7,345 \times 9,890$
	3	1,028	$596,898 \times 1,246,254$
	4	1,753	$4,175,947 \times 12,168,528$
	5	1,753	$4,175,947 \times 12,168,528$

Computational Approach - Sparsity

- Despite being huge our matrices have very few non-zero entries.
- **Example:** For the $4,175,947 \times 12,168,528$ matrix from the previous slide less than .000164% of the entries are non-zero.
- This **sparsity** allows us to use sparse numerical linear algebra algorithms like LU and QR factorizations.

$$QAP = LU = L \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

- These algorithms are substantially faster and use astronomically less memory.
- But... these methods are numerical, and so errors may creep in.

Computational Approach - HTC

- We then use **high throughput computing** to handle the thousands of jobs we have.
- Instead of running all of our jobs sequentially on one super computer we distribute our jobs all over campus.
  - We do this via HTCondor, the Center for High Throughput Computing, and the Open Science Grid.
- **Example:** One part of a medium sized example required the follow computing resources

# Matrices	Max Run Time	Ram (GB)
151	18 min.	< 1
16	1 hr.	$1 - 10$
17	16 hr.	$20 - 80$
2	3 **days**	> 450

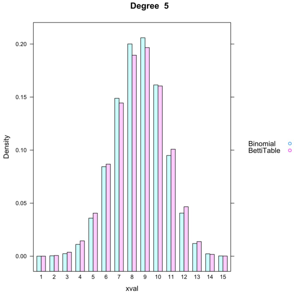
Results

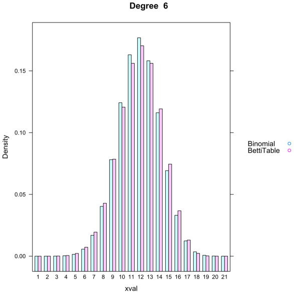
- It worked!!
- We computed the graded Betti numbers, multigraded Betti numbers and more for V_d when $d = 5, 6$.
  - We also compute this data for all auxiliary line bundles on $\mathbb{P}^2$.
- We have made all of this data available at:

  syzygydata.com.
- A Macaulay2 package with all of our data will also be out soon.
- These new examples have allowed us to see previously unseen patterns and structures within these syzygies.

Conjecture - Normally Distributed

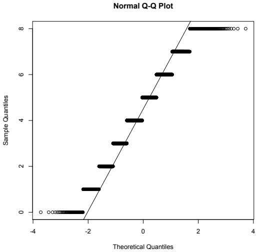
- Ein, Erman, and Lazarsfeld conjectured that any row of a Betti table should converge, after rescaling, to a normal distribution.
- This is true for curves and "randomly" chosen Betti tables.
- Our data provides the first computational evidence in support of this conjecture for surfaces.

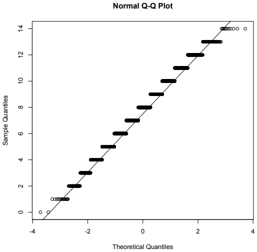




Conjecture - Normally Distributed

- In addition to the qualitative visual evidence we give statistical evidence for the rows being approximately normally distributed.
- Here we have the Q-Q plots compare the first row for $d = 5$ and $d = 6$ to a normal distribution of best fit.





- If these two distributions were approximately the same we would expect the points to be roughly distributed along the line $y = x$.

Conjectures - Schur Functors

- Our computations produced a lot of data, which consists of really big numbers.
- The size of the individual Betti numbers – in the hundreds of millions – obscures many of the underlying structures.
- For this reason it is useful to use another symmetry to package this data in a more condensed format.

Conjectures - Schur Functors

- The action of $\mathrm{GL}_3(\mathbb{C})$ on $S = \mathbb{C}[x, y, z]$ by linear change of coordinates descends to the Koszul complex.
- This makes the cohomology of the Koszul complex a representation of $\mathrm{GL}_3(\mathbb{C})$.
  - Every representation of $\mathrm{GL}_3(\mathbb{C})$ decomposes into irreducible.
  - Irreducible representations of $\mathrm{GL}_3(\mathbb{C})$ are indexed by partitions $\vec{\lambda} = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$.
- So the cohomology of Koszul complex decomposes:

$$K_{p,q}(2,d) = \bigoplus_{\substack{\vec{\lambda} \in \mathbb{Z}^3 \\ |\vec{\lambda}| = d(p+q)}} \mathbb{S}_{\lambda}(\mathbb{C}^3)^{\oplus m_{p,\vec{\lambda}}(2,d)}.$$

Computational Approach - Symmetries

- **Example:** Again focusing on the red 27 in Betti table for V_3 is:

$$\beta(V_3) = \begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 1 & - & - & - & - & - & - & - \\ 1 & - & \color{red}{27} & 105 & 189 & 189 & 105 & 27 & - \\ 2 & - & - & - & - & - & - & - & 1 \end{array}.$$

- As a $\mathbb{Z}^3$-graded vector space, we saw $K_{1,1}(2;3)$ has 19 distinct multidegrees, which we encoded via a generating series

$$t_0^4t_1^2 + t_0^3t_1^3 + t_0^2t_1^4 + t_0^4t_1t_2 + 2t_0^3t_1^2t_2 + 2t_0^2t_1^3t_2 + t_0t_1^4t_2$$

$$+t_0^4t_2^2 + 2t_0^3t_1t_2^2 + 3t_0^2t_1^2t_2^2 + 2t_0t_1^3t_2^2 + t_1^4t_2^2 + t_0^3t_2^3 + 2t_0^2t_1t_2^3$$

$$+2t_0t_1^2t_2^3 + t_1^3t_2^3 + t_0^2t_2^4 + t_0t_1t_2^4 + t_1^2t_2^4.$$

- As a Schur module, $K_{1,1}(0;3)$ is isomorphic to the irreducible representation $\mathbb{S}_{(4,2,0)}(\mathbb{C}^3)$.

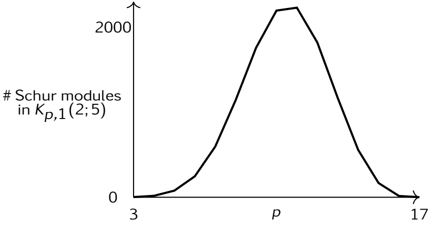
Question - Unimodality

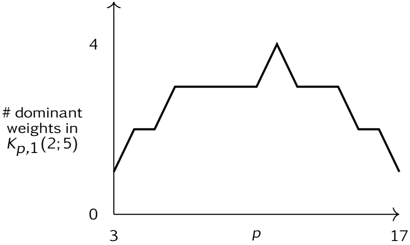
Question (Bruce, et. al)

When is each of the following a unimodal function of p?

1. *The rank of $\beta_{p,p+q}(V_d)$;*
2. *The number of distinct irred. Schur modules in $K_{p,q}(\mathbb{P}^2; d)$;*
3. *The total number of irred. Schur modules in $K_{p,q}(\mathbb{P}^2; d)$;*
4. *The largest multiplicity of a Schur module in $K_{p,q}(\mathbb{P}^2; d)$;*

- **Example:** For $d = 5$, $(b = 2)$, we see the predicted unimodal behavior in the first row:





Conjecture - Dominate Weights

- We say a partition $\vec{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a)$ dominates another partition $\vec{\lambda'} = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_b)$ if and only if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \lambda'_k$$

for all $k \leq \min\{a, b\}$.

- This places a partial ordering on all the space of weights.
- **Example:** The Schur functor decomposition of $K_{14,1}(2;5)$ is:

$$K_{14,1}(5;0) \cong \mathbb{S}_{(34,21,20)} \oplus \mathbb{S}_{(33,25,17)} \oplus \mathbb{S}_{(33,24,18)} \oplus \cdots$$

The weight $(33, 24, 18)$ is dominated by $(33, 25, 17)$ but is not dominated by $(34, 21, 20)$.

Conjecture - Dominate Weights

- Ein, Erman, and Lazarsfeld constructed a subset

$$E_{p,q}(2;d) \subset K_{p,q}(2;d)$$

of special monomial syzygies, and conjectured that:

$$E_{p,q}(2;d) \neq 0 \iff K_{p,q}(2;d) \neq 0.$$

- Our data suggests there is a much deeper connection here!!

Conjecture (Bruce, et. al)

For all p, q, and d:

$$\operatorname{domWeights} E_{p,q}(2;d) = \operatorname{domWeights} K_{p,q}(2;d)$$

- The monomial syzygies in Ep,q(Pn,b;d) represent only a small fraction of the total syzygies

Conjecture - Dominate Weights

- This conjecture raises a number of other interesting questions.

Question (Bruce, et. al)

Find a compelling combinatorial description of $\mathrm{domWeights}\, K_{p,q}(2; d)$.

Question (Bruce, et. al)

Let $\vec{\lambda} \in \mathrm{domWeights}\, K_{p,q}(2; d)$. *Does the representation* $\mathbb{S}_{\lambda}(\mathbb{C}^3)$ *appear in* $K_{p,q}(2; d)$ *with multiplicity one?*

Question (Bruce, et. al)

When is the number of dominant weights in $K_{p,q}(2; d)$ *a unimodal function of* p*?*

- Note when $d = 3$ the dominate weights are non-unimodal, but this is the only such example.

Conjecture - Dominate Weights

- The conjecture also suggests a mysterious uniformity among all of the $K_{p,q}(2;d)$ lying in a single row of a Betti table.
- If we vary only p, then the monomial syzygies constructed in $E_{p,q}(2;d)$ naturally form a graded lattice, with a unique maximal and minimal element.
- In other words, it may be natural to think of the entire qth row as a single object

$$K_{\bullet,q}(2;d) := \bigoplus_{p} K_{p,q}(2;d).$$

Question (Bruce, et. al)

Is this vector space naturally a representation (or even an irreducible representation) over a larger group.

- Precisely such a phenomenon occurs when $d = 2$ (Sam).

- **Example:** Consider $K_{\bullet,1}(2;3)$, which corresponds to the first row of the Betti table of V_3.
- The dominant weights of $K_{p,1}(2;3)$ are in bijection with the weights of the monomial syzygies in the pth row of this lattice:

$$\begin{tikzcd}
 & x^2y \otimes x^2y \arrow[dl, dash] \arrow[dr, dash] & \\
x^2y \wedge xy^2 \otimes x^2y \arrow[dr, dash] & & x^2y \wedge x^2z \otimes x^2y \arrow[dl, dash] \\
 & x^2y \wedge xy^2 \wedge x^2z \otimes x^2y \arrow[d, dash] & \\
 & x^2y \wedge xy^2 \wedge x^2z \wedge xyz \otimes x^2y \arrow[dl, dash] \arrow[dr, dash] & \\
x^2y \wedge xy^2 \wedge x^2z \wedge xyz \wedge y^2z \otimes x^2y \arrow[dr, dash] & & x^2y \wedge xy^2 \wedge x^2z \wedge xyz \wedge xz^2 \otimes x^2y \arrow[dl, dash] \\
 & x^2y \wedge xy^2 \wedge x^2z \wedge xyz \wedge y^2z \wedge xz^2 \otimes x^2y &
\end{tikzcd}$$

What Next?

- Beyond the questions and conjectures we raise there are likely many more lurking in our data.
- Even using distributed computing and spare linear algebra we are bumping into the limits of our computing power.
- An even bigger problem is that the errors from the numerical nature of our algorithms are too large.
- The Betti table for V_7 may be within reach using our current methods, but beyond this...
- That said our methods could be adopted to study other families:
    - $\mathbb{P}^3$,
    - $\mathbb{P}^1 \times \mathbb{P}^1$, and
    - many more!