- **$\mathbb{Q}$**: Let X be a smooth, projective, geomectrically integral curve with genus $g \geq 2$ defined ove $\mathbb{Q}$ desoribe $X(\mathbb{Q})$?
  - ↳ This has many answers.......

- **Theorem**: (Faltings) $\# X(\mathbb{Q}) < \infty$.
  - ↳ This proof does not give a "workable" bounded nor does it actully compute $X(\mathbb{Q})$.....

- The gool of Chobouty and colemon is to do bette than this....

- If we are going to succeed we probably need something to put some additional structure on $X(\mathbb{Q})$......

- **Def**: Let X be a nice curve s.t. $X(K) \neq \emptyset$ then the Jacobian of X is an algebraic variety J s.t. there are functorial isomorisms

$$J(K') \xrightarrow{\simeq} \mathrm{Pic}^{\circ}(X(K'))$$

for all extensions K'/K.

- **Recall** + $\mathrm{Pic}(X)$ "=" $\mathrm{Div}(X)$ / linearly equivalence

"=" $\mathbb{Z}\{\text{closed points}\}$ / linearly equivalence

+ $\deg(D) = \sum_i n_i$

+ $\mathrm{Pic}^{\circ}(X)$ = degree 0 divisors....

- Whatever this J is all we need is that.....

- <u>Fact</u>: + J is a group, that is finitely generated, and abelian.
  - + There is a natural embedding:

$$\begin{aligned} X &\overset{i}{\hookrightarrow} J \\ P &\longmapsto [P - O] \end{aligned}$$

where $O \in X(\mathbb{Q})$.

- Strategy to answer our question:
  1) Understand $J(\mathbb{Q})$.
  2) Determine which points of $J(\mathbb{Q}) \in X$.

- Doing #1 is hard, but "maybe" possible.

- The goal now becomes to get to a place where $J(\mathbb{Q})$ is finite....

  If $J(\mathbb{Q})$ is finite and $D \in J(\mathbb{Q})$ then if $i(P) = D$ then

$$P = D + O + (f)$$

For some $f \in \underbrace{L(D+O)}_{\text{This we can understand.}}$.

- <u>KEY IDEA</u>: Find some finite subset of $J(\mathbb{Q})$ containing $X(\mathbb{Q})$.
  - ↳ Then we could try and count things.

- Now $J(\mathbb{R})$ is a g-dimensional real Lie group. Now $J(\mathbb{Q}) \subseteq J(\mathbb{R})$ So we could take its closure making $\overline{J(\mathbb{Q})}$ a real lie group Now is

$$\overline{J(\mathbb{Q})} \cap X(\mathbb{R}) \subseteq J(\mathbb{R}) \quad \text{Finite?}$$

well....

$$\dim(\overline{J(\mathbb{Q})} \cap X(\mathbb{R})) \text{ "=" } \underset{\uparrow\, ??}{\dim(J(\mathbb{Q}))} + \underset{\uparrow\, 1}{\dim X(\mathbb{R})} - \underset{\uparrow\, g}{\dim J(\mathbb{R})}$$

- So if $\dim \overline{J(\mathbb{Q})} < g$ we are in buisness.....
  - ↳ Sadly this is not the case and $\dim \overline{J(\mathbb{Q})} = g$ "most of the time"
    - ↳ When $J(\mathbb{Q})$ is dense in J we expect $\overline{J(\mathbb{Q})}$ to be open in $J(\mathbb{R})$.

- Game change..... P-adic manifolds & p-adic Lie group.
  - ↳ Why? Who knows......

- Def: A function $f : U \longrightarrow \mathbb{Q}_p^m$ where $U \subseteq \mathbb{Q}_p^n$ is open is locally analytic iff $\forall p \in U$ there is a neighborhood $p \in V_p \subseteq U$ s.t. $\pi_i \circ f|_{V_p}$ is analytic i.e. is a convergent power series.

- Def: A p-adic manifold is a topological space X together with an atlas $\{(\phi_i, U_i)\}$ where $\phi_i : U_i \longrightarrow \mathbb{Q}_p^n$ is a homeomorphism s.t.

$$\phi_i^{-1} \circ \phi_j : U_j \longrightarrow U_i$$

is locally analytic.

- Def: A p-adic lie group is a p-adic manifold together with a group structure s.t. the multiplication is locally analytic.

- Facts: 1) $X(\mathbb{Q}_p)$ is a p-adic manifold (assuming X is smooth)

  2) $J(\mathbb{Q}_p)$ is a p-adic lie group...

- Now we play the same game $\overline{J(\mathbb{Q})} \subseteq J(\mathbb{Q}_p)$ be the p-adic closure

$$\underbrace{\dim\left[X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}\right]}_{=0 \text{ hopefully.}} = \underbrace{\dim X(\mathbb{Q}_p)}_{1} + \dim \overline{J(\mathbb{Q})} - \underbrace{\dim J(\mathbb{Q}_p)}_{g}$$

- So if $\dim \overline{J(\mathbb{Q})} < g$ we are in game.... (maybe)

  ↳ same place as before.

- Lemma: $\dim \overline{J(\mathbb{Q})} \leq \operatorname{rank}(J(\mathbb{Q}))$

- Thus if $\operatorname{rank}(J(\mathbb{Q})) < g$ we are good to go.

- Coleman-Chabauty-Hypothesis: $\dim \overline{J(\mathbb{Q})} < g$.

- We now suspect $X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$ is finite, but how could we count any of this.... USE THE LIE GROUP STRUCTURE.

- Since $\overline{J(\mathbb{Q})} \subseteq J(\mathbb{Q}_p)$ and $\dim \overline{J(\mathbb{Q})} \lneq \dim(J(\mathbb{Q}_p))$ we know

$$T\overline{J(\mathbb{Q})} \subsetneq TJ(\mathbb{Q}_p)$$

moreover there is a map

$$\lambda : TJ(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$

That vanishes on. $T\overline{J(\mathbb{Q})}$.

- Moreover since ~~...~~ $J(\mathbb{Q}_p)$ is a lie group we have the log map

$$\log : J(\mathbb{Q}_p) \longrightarrow TJ(\mathbb{Q}_p)$$

So by composing these we get an locally analytic map

$$\eta_J : J(\mathbb{Q}_p) \xrightarrow{\log} TJ(\mathbb{Q}_p) \xrightarrow{\lambda} \mathbb{Q}_p$$

That vanishes on $\overline{J(\mathbb{Q})}$!! So we get an analytic map

$$\eta : X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$

that vanishes exactly on $X(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$!!

- So to bound $X(\mathbb{Q})$ we just need to bound the # of zeros of η !!
  - So what this seems untrackable....
  - Again harness the power of Lie groups?

- In particular, we use the fact that η is related to integration.

- <u>Fact</u>: The functional $\lambda: TJ(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ corresponds to a non-zero differential $\omega_J \in H^0(J_{\mathbb{Q}_p}, \Omega^1)$. Then the map η is given by

$$\eta_J^*: \underset{J}{\cancel{}}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$

$$Q \longmapsto \underbrace{\int_0^Q \omega_J}_{\text{what exactly this is ..... is somewhat mysterious.}}$$

- <u>Prop</u>: η_J is characterized as
  1) Being a group homomorphism.
  2) There is an open neighborhood $U \subseteq J(\mathbb{Q}_p)$ s.t $\forall Q \in U$, we may compute $\eta_J(Q)$ be formally expanding ω_J in local coordinates & formally anti differentiating.

- So now we may think about η as an integral, and it seems like we might be able to actually count zeros.

• Doing exactly this - in a clever way - we get the following theorem of coleman.

• <u>Theorem</u>: (coleman '85): Let X be a smooth nice curve of genus $g \geq 2$ and p be a prime of good reduction for X s.t.

$$\dim \overline{J(\mathbb{Q})} < g \qquad (CCH).$$

If $p > 2g$ then

$$\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + 2g - 2. \blacksquare$$

• <u>Ex</u>: $X: y^2 = x(x-1)(x-2)(x-5)(x-6)$

$$\left.\begin{array}{l} \bullet \text{ smooth} \\ \bullet \text{ genus 2 curve} \\ \bullet\ J(\mathbb{Q}) \text{ has rank one} \end{array}\right\} \dim \overline{J(\mathbb{Q})} \leq \operatorname{rank} J(\mathbb{Q}) < g \qquad (CCH)$$

Moreover this curve has good reduction at $p = 7$.

$$\# X(\mathbb{F}_7) = 8$$

So coleman implies

$$\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_7) + 2g - 2 = 8 + 2 \cdot 2 - 2 = 10$$

$$X(\mathbb{Q}) = \{(0,0), (1,0), (2,0), (5,0), (6,0), (3, \pm 6). (10, \pm 120), \infty\}$$

Sketch of Proof : 3 Steps:

1) Properties of η:

A) If $Q_i, Q_i' \in X(\mathbb{Q}_p)$ st $div(f) = \sum [Q_i' - Q_i]$ then

$$\sum_i \int_{Q_i}^{Q_i'} \omega \equiv 0$$

B) If $Q, Q' \in X(\mathbb{Q}_p)$ have the same image under the map $X(\mathbb{Q}_p) \twoheadrightarrow \mathfrak{X}(\mathbb{F}_p)$ then

$$\int_Q^{Q'} \omega$$

may be computed by expanding ω as a power series.

C) If $Q_i, Q_i' \in X(\mathbb{Q}_p)$ st $\sum [Q_i - Q_i'] \in \overline{J(\mathbb{Q})}$ then

$$\sum \int_{Q_i'}^{Q_i} \omega \equiv 0.$$

i.e. $\omega \in \mathbb{Q}_p[[t]]$ for some neighborhood.

↳ Here $\displaystyle\int_{Q_i}^{Q_i'} \omega = \int_0^{[Q_i - Q_i]} \omega_J$ where $\omega_J \longmapsto \omega$ under the isomorphism $H^0(J_{\mathbb{Q}_p}) \longrightarrow H^0(X_{\mathbb{Q}_p})$.

2) p-adic zero count: If $f(t) \in \mathbb{Q}_p[[t]]$ st $f'(t) \in \mathbb{Z}_p[[t]]$

& $\operatorname{ord}_{t=0} \left[f'(t) \bmod p \right] < p-2 \Rightarrow f(t)$ has at most $m+1$ zeros in $p\mathbb{Z}_p$

↳ Newton polygon

• 3 : Put it togethe....