

A Probabilistic Approach to Noether Normalization

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Definitions

- If $X \subset \mathbb{P}^r$ is a closed subscheme, $f_0, \dots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ are a **(partial) system of parameters** on X if

$$\dim \mathbb{V}(f_0, \dots, f_k) \cap X = \dim X - (k + 1).$$

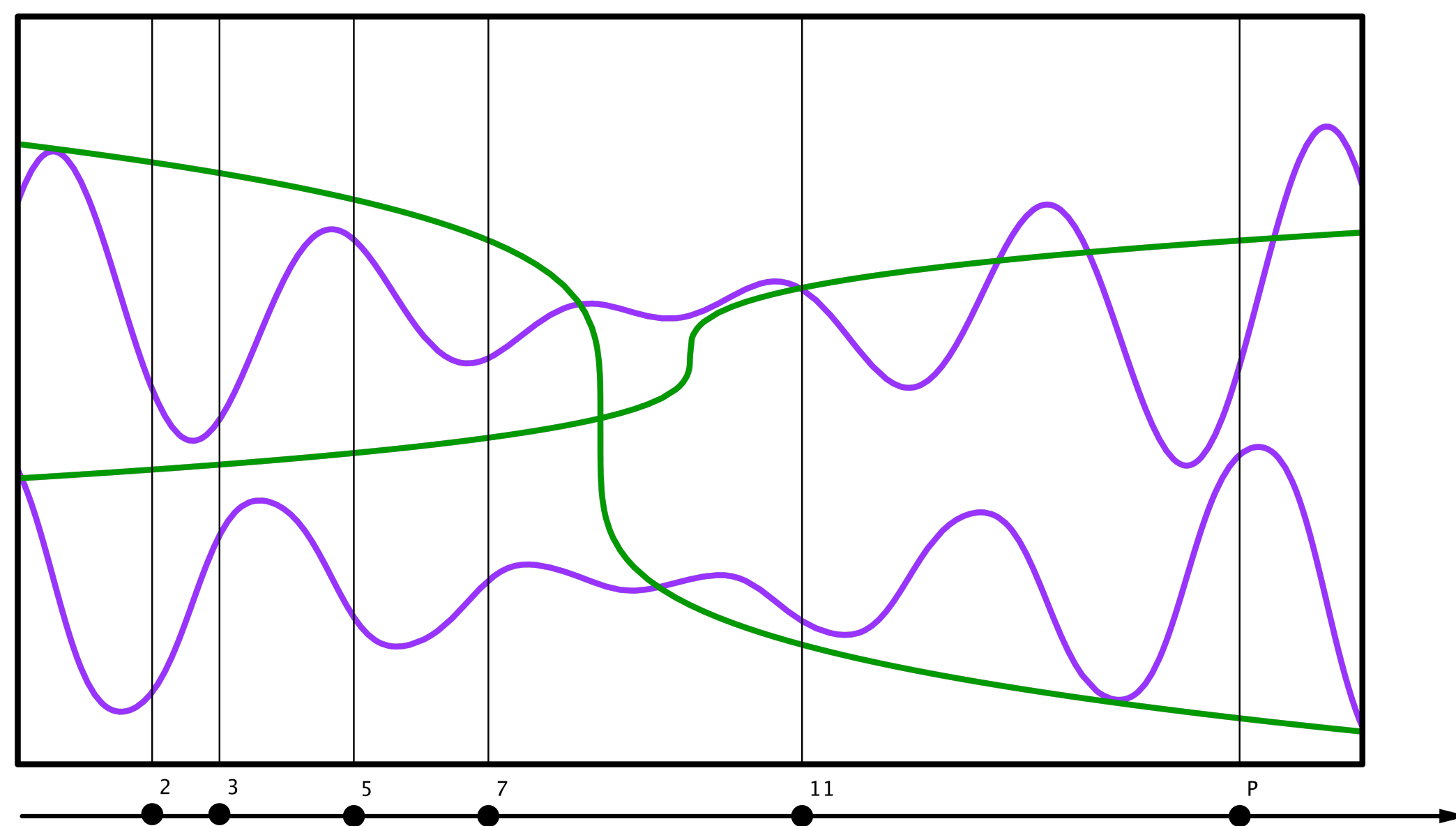


Figure 1: A non-example of fiberwise parameters for $\mathbb{P}^1_{\mathbb{Z}}$.

- If $X \subset \mathbb{P}^r_B$ is a closed subscheme a **fiberwise system of parameters** on X is a collection $f_0, \dots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$, which restrict to parameters on each fiber of X over B .

Example - Points in $\mathbb{P}^1_{\mathbb{Z}}$

- A fiberwise parameter for a set of points in $\mathbb{P}^1_{\mathbb{Z}}$ is a homogeneous polynomial f such that $f = \pm 1$ at every point.
- For one point the **Euclidean algorithm** shows a linear parameter always exists.
- The points $[4 : 5]$ and $[1 : 8]$ have no fiberwise parameter f with $\deg f < 9$. If $\deg(f) = 9$ one can choose:

$$\begin{aligned} f(x, y) = & 590316869666239369788336486577332627591x^9 - y^9 \\ & - 54805820786420002024577742687553134041x^8y \\ & + 60895356429355557805086415854172570449x^7y^2 \\ & - 201231722866854045440755923284664255x^6y^3 - 99438623xy^8 \\ & - 30126091505240316687953753225193x^5y^4 - 1977608122820402x^2y^7 \\ & + 620958323405203261138096698x^4y^5 - 1230273968422817725004x^3y^6. \end{aligned}$$

The Main Result

Theorem 1. Let $X \subset \mathbb{P}^r_{\mathbb{Z}}$ be a closed subscheme. If every fiber of X over \mathbb{Z} has dimension n then there exists a linear series in $\mathcal{O}_X(d)$, for some $d > 0$, inducing a finite morphism:

$$\phi : X \rightarrow \mathbb{P}^n_{\mathbb{Z}}.$$

- The existence of such ϕ is subtle, even potentially unexpected, as the sections $f_0, \dots, f_n \in H^0(X, \mathcal{O}_X(d))$ defining such a finite map have **density zero**; even if we let $d \rightarrow \infty$.
- Despite not being able to construct ϕ by randomly choosing f_0, \dots, f_n , our proof of Theorem 1 relies on being able to generate partial systems of parameters probabilistically.

Theorem 2. Let $X \subseteq \mathbb{P}^r_B$ be a closed subscheme of dimension n . The “probability” that random chosen forms f_0, \dots, f_k of degree d are a (fiberwise) system of parameters as $d \rightarrow \infty$ is:

	$B = \mathbb{F}_q$	$B = \mathbb{Z}$
$k < n$	1	1
$k = n$	$\zeta_X(n+1)^{-1}$	0

- The case when $B = \mathbb{F}_q$ and $k = n$ is originally due to Bucur and Kedlaya.
- **Step #1 - (Probabilistic):** By Theorem 2 we can find fiberwise parameters f_0, \dots, f_{n-1} on X simply by picking high degree forms at random, and so reduce Theorem 1 to

$$X' = X \cap \mathbb{V}(f_0, \dots, f_{n-1}).$$

- **Step #2 - (Arithmetic):** Such X' is essentially a union of orders in number fields. Using the fact an order has a finite Picard group we show that $\text{Pic}(X')$ is finite. This finiteness allows us to construct the last parameter.
- Similar arguments hold if one replaces \mathbb{Z} with $\mathbb{F}_q[t]$.

Distribution of Parameters

- The probabilities over \mathbb{F}_q appearing in Theorem 2 are computed via an adaptation of Poonen’s sieving argument.
- When $k < n$, this results in a description of the distribution of parameters in terms of a **higher dimensional analogue** of the Hasse-Weil zeta function.

Proposition 1. Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a closed subscheme of dimension n . Then there is an explicit function $f(k, n, d)$ such that

$$\text{Prob} \left(\begin{array}{c} (f_0, \dots, f_k) \\ \text{of degree } d \text{ are } \textbf{not} \\ \text{parameters on } X \end{array} \right) \sim \# \left\{ \begin{array}{c} (n-k)\text{-planes} \\ L \subset X \end{array} \right\} \cdot f.$$

Example - Parameters on Surfaces

- In \mathbb{P}^3 the surface $\mathbb{V}(xyz)$ contains substantially more lines than $\mathbb{V}(x^2 + y^2 + z^2)$. Selecting 10^6 random pairs (f_0, f_1) of degree two, the proportion that *failed* to be parameters were:

	$\mathbb{V}(xyz)$	$\mathbb{V}(x^2 + y^2 + z^2)$
\mathbb{F}_2	.2638	.1179
\mathbb{F}_3	.0552	.0059
\mathbb{F}_5	.0063	.0004

- Over \mathbb{F}_4 the Fermat cubic surface

$$X = \mathbb{V}(x^3 + y^3 + z^3 + w^3) \subset \mathbb{P}^3_{\mathbb{F}_4},$$

has 27 \mathbb{F}_4 -lines. Proposition 1 implies $\approx 0.66\%$ of pairs (f_0, f_1) of degree two should not be parameters. Simulating 10^5 such pairs we found 0.62% failed to be parameters.

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