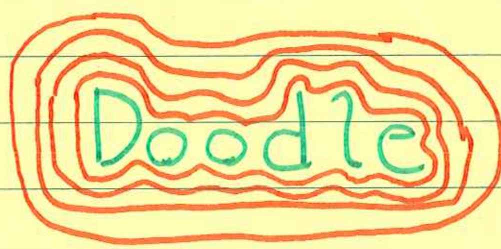
§0. INTRO

- This is a sort of meta-lecture, it is more about the process of how mathematicians DO math, not about math itself.

- As a result this will have more questions than answers, but that's okay most mathematicians have more Q's than A's.

§1. A Question



- **Q0:** Do my doodles get more circular?

- Q: (30s): Talk to your neighbor about this.

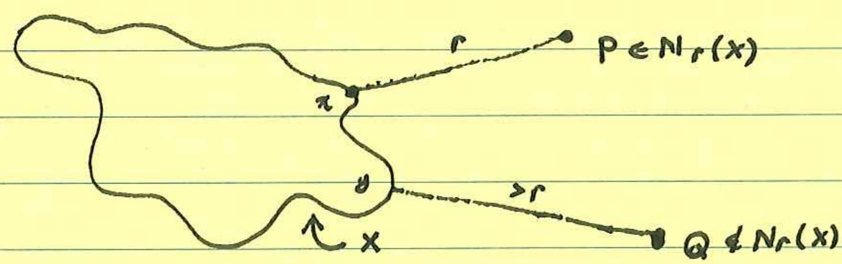
- We have an issue; this question is too vague, Flabby, etc. to have a reasonable mathematical answer.

- We need to make this more precise:

$$\text{"Do my } \underbrace{\text{doodles}}_{\#0} \ \underbrace{\text{get}}_{\#1} \ \underbrace{\text{more circular}}_{\#3} \text{?"}$$

- **Def:** If X is a figure in the plane:

$$N_r(X) := \left\{ \begin{array}{c} \text{Pts a distant} \\ \leq r \\ \text{from some pt of } X \end{array} \right\} = \left\{ \begin{array}{c} \text{Points } p \text{ s.t there is a point} \\ x \in X \text{ with } |p - x| \leq r \end{array} \right\}$$

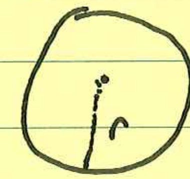


• Ex: 1) What if X is the point $(0,0)$?

↳ $N_r(X) = \{$pts a distance $\le r$ from $(0,0)\}$

$= \{$pts p s.t. $|p-(0,0)| \le r\}$

$= \{$pts p s.t. $\sqrt{(p_1-0)^2+(p_2-0)^2} \le r\}$



2) What if X is a circle of radius R?

↳ $N_{Rr}(X)$ = circle of radius $R+r$.

$* \; N_r(X) = N_{Rr}(N_R((0,0)) = N_{r+R}(0,0)$.

• Q1: Does $\overbrace{N_r(N_r(N_r(\cdots N_r(X)\cdots)))}^{n\text{-times}}$ get more circular if we iterate N_r a bunch?

Q: (30s): Why is this question the "same" as our original?

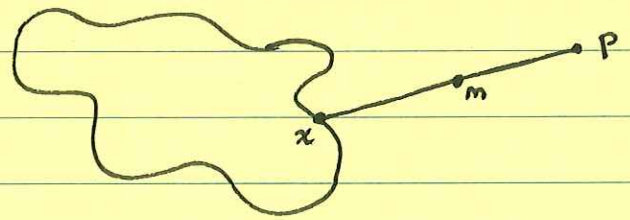
• This question is still pretty flabby, for example

+ "~~flabby~~ more circular"?

+ "iterate"?

• There is a bigger issue... we previously didn't have an "r". How does this change with r?

• Q1.5: How do $N_1(N_1(X))$ & $N_2(X)$ compare?

• 

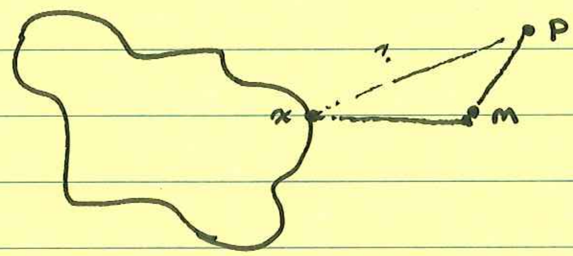
+ Suppose $p \in N_2(X)$.
+ so there exists a point $x \in X$ s.t.

$$|x - p| \leq 2.$$

+ Let ℓ be the line segment between x and p.
+ Let m be the midpoint of ℓ.

• Q: (1 min): What is the distance between x and m?
" " " " " m and p?

+ Hence $m \in N_1(X)$ and since $|m - p| \leq 1 \Rightarrow p \in N_1(N_1(X))$, and so $N_2(X)$ is contained in $N_1(N_1(X))$.

• 

+ suppose $p \in N_1(N_1(X))$
+ So there exists $m \in N_1(X)$ with $|\cancel{x}p - m| \leq 1$
+ But if $m \in N_1(X)$ this means there exist $x \in X$ with $|x - m| \leq 1$.

• Q: What can we say about $|x - p|$?

• The Δ-inequality exactly says $|x - p| \leq 2$! This means $p \in N_2(X)$, and so $N_1(N_1(X)$ is contained in $N_2(X)$ meaning

$$N_1(N_1(X)) = N_2(X)!!$$

• Q: Was 1, 1, and 2 special?

• **Thm:** Let $a, b > 0$ then

$$N_a(N_b(X)) = N_{a+b}(X).$$ ■

• **Q2:** Does $N_{nr}(X)$ get more circular as n gets large?

• This is pretty mathematical! If you've seen limits letting n get large makes some sense - if not we can imagine.

• But more importanty Notice as n gets big $nr > R$ for any fixed R, and so

$$N_{nR}(X) \supseteq N_R(X). \qquad (*)$$

And so r plays ~~very~~ no role in our question!

• **Q3:** Does $N_R(X)$ get more circular as R gets large?

we should be able to squeeze it between to circleles C_1 & C_2 s.t. C_1 & C_2 are close!

• Pick a point $p \in X$: and let $D_t(p)$ be a circle around p s.t.

$$X \subset D_t(p).$$

Now we have that

$$D_R(p) = N_R(\{p\}) \subseteq N_R(X) \subseteq N_R(D_t(p)) = D_{R+t}(p) \qquad (**)$$

A circle of radius R

A circle of radius of $R+t$.

So $N_R(X)$ is squeezed between two circles!!

• **Answer:** The answer to Q3 is Yes!!

• **But** ... why are (*) and (**) true?

• This often happens, you make a question precise, see the answer to your question, just to realize you were a bit blazea about some of your argument..... sometimes we get lucky & this does not matter.

• **Lemma**: If $Y \subseteq X$ then

$$N_R(Y) \text{ is contained in } N_R(X).$$

• You should think about why this is true.

§2. Another Question

• **QO**: How does the perimeter of $N_r(X)$ relate to the peremeter of X?

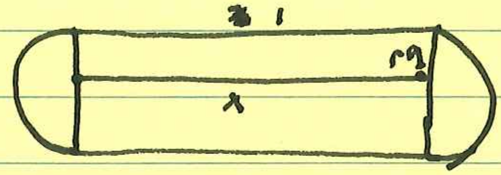
• **Ex**: 1) If X is a point then $N_r(X)$ is a circle of radius r so

$$\text{Perm}(N_r(X)) = \text{Prem}(\text{Circle radius } r) = 2\pi r$$

2) If X is a circle of radius R then $N_r(X)$ is a circle of radius $R+r$.

$$\text{Perm}(N_r(X)) = \text{Perm}(\text{circle radius } R+r) = 2\pi(R+r).$$

3) If X is the line segment $[0,1]$ then $N_r(X)$ looks something like



So we have $\text{Perm}(N_r(X)) = 2 + 2\pi r$

• Q (30s): Does anyone notice a patern?

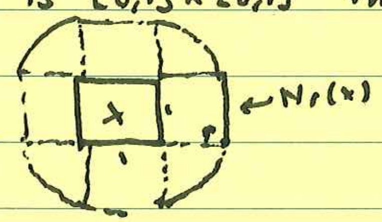
• Notice that

1) $\text{Perm}(N_r(X)) = 2\pi r + 0 = 2\pi r + \text{Perm}(X)$

2) $\text{Perm}(N_r(X)) = 2\pi(r+R) = 2\pi r + 2\pi R = 2\pi r + \text{Perm}(X)$

3) $\text{Perm}(N_r(X)) = 2\pi r + 2 \neq 2\pi r + \text{Perm}(X)$ ....

• But if X is $[0,1] \times [0,1]$ then $N_r(X)$



And so

$$\text{Perm}(N_r(X)) = 4 + 2\pi r$$
$$= \text{Perm}(X) + 2\pi r$$

So it seems like sometimes we have the following relation.

$$\text{Perm}(N_r(X)) = \text{Perm}(X) + 2\pi r.$$

• Q: If X is a ____________ then for any $r > 0$

$$\text{Perm}(N_r(X)) = \text{Perm}(X) + 2\pi r$$

• I will leave it to you to try and fill in the blank.