Betti Tables of Graph Curve

- Given a free resolution of M (a graded R-module)

$$\cdots \longrightarrow F_n \xrightarrow{\delta_n} F_{n-1} \longrightarrow \cdots \xrightarrow{\delta_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

(So this is an exact sequence) each F_i may be written as

$$F_i \cong \bigoplus_j R(-j)^{b_{ij}}$$ ← these are called the betti numbers

where $(R(-j))_d = (R)_{d-j}$.

	0	1	2	3	
0	1	•	•	•	- - -
1	•	b_{12}	b_{23}	b_{34}	- - - ← quadratic strand
2	•	b_{13}	b_{24}	b_{35}	- - - ← cubic strand.
⋮	⋮	⋮	⋮	⋮	⋱

- We have wierd indexing because many of the betti #'s must be zero.

- We are interested in modules over $\mathbb{C}[x_0, \ldots, x_n]$.

- If $X \subseteq \mathbb{P}^n$ then $I_X \subset S = \mathbb{C}[x_0, \_\_ x_n]$ is the homogenous ideal (of functions vanishing on X) then $S_X = S/I_X$ is a graded S-module.

- We are actually interested in one particular type of free resolution "minimal free resolution".

• **Idea**: A free resolution is minimal if $F_0, F_1, F_2, \ldots$ have the least number of generators possible.

* **Def**: If $m = (x_0, \ldots, x_n) \leq S$ then a free resolution

$$\cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots \longrightarrow M \longrightarrow 0$$

is minimal $\Longleftrightarrow$ $\mathrm{Img}(\delta_i) \subseteq mF_i \quad \forall i.$

• This connects to the intuitive idea via Nakayama's lemma.

+ **Lemma**: If M is a f.g. graded S-module and $m_1, \ldots, m_s \in M$ are such that $\overline{m_1}, \ldots, \overline{m_s} \in M/mM$ generate then $m_1, \ldots, m_s$ generate M. ∎

+ **COR**: A free resolution

$$\cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots = \mathbb{F}$$

is minimal iff δ_i takes a basis of F_i to a minimal generating set for $\mathrm{Img}(\delta_i)$, $\forall i$.

Proof: Consider the RES $F_{i+1} \xrightarrow{\delta_{i+1}} F_i \longrightarrow \mathrm{Img}(\delta_i) \longrightarrow 0$.

Now $\mathbb{F}$ is minimal $\Longleftrightarrow$

$$\overline{\delta}_{i+1} : F_{i+1}/mF_{i+1} \longrightarrow F_i/mF_i \equiv 0$$

$\Longleftrightarrow$ $F_i/mF_i \longrightarrow \mathrm{Img}(\delta_i)/m\mathrm{Img}(\delta_i)$ is an isom.

$\Longleftrightarrow$ claim. ∎

- **Theorem**: If M is a f.g. S-module then any 2 minimal free resolutions are isomorphic and
$$b_{i,j} = \dim_{\mathbb{C}} \operatorname{Tor}_i^S(M, \mathbb{C})_j .$$

Proof: See Eisenbud "Geometry of Syzygies" I.1.7. $\blacksquare$

- When we refer to the betti table of M we are refeing to the betti table of the minimal free resolution.

- **Def**: If $G = (E, V)$ is a simple, connected, strictly subtrivalent graph then the graph curve $\widetilde{G}$ associated to G is a collection of lines $\{L_v \mid v \in V\}$ s.t. $L_v \cong \mathbb{P}^1$ and L_v intesects $L_{v'}$ iff there is an edge from v to v'.

- Note: This is defined abstractly and is not embedded.
- $\widetilde{G}$ is determined only by the combinatorial data of G since $\operatorname{Aut}(\mathbb{P}^1)$ acts 3-transitively and so we may take the intersections to be whatever we like.
- Strictly subtrivalent = every vertice has degree ≤ 3 and there is one vertex of degree < 3.

• **Notation**: Throughout the rest of the talk if $G = (E,V)$ is a graph

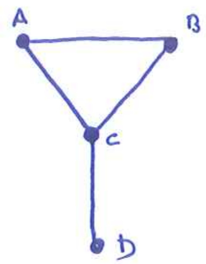
$d = |V| = $ degree of $\widetilde{G}$

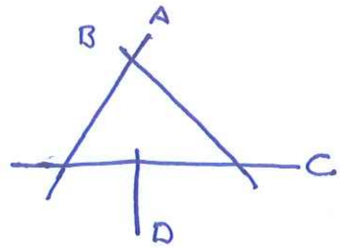
$m = |E|$

$g = h^1(\widetilde{G}, \mathbb{C}) = $ arithmetic genus $= m - d + 1$

(See Burnham, Rosen, SIDman, Vermeire)

• **Ex**

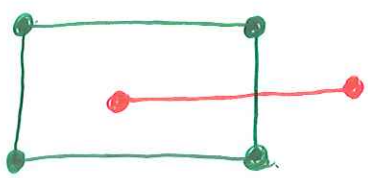




$d = 4$
$m = 4$
$g = 1$

• This is sort of like the ~~dual~~ line/derived/conjugate graph.

• It is not the dual grap as dual graph is faces to vetices



Assumption: (For the remainder): G is a

1) G is planar
2) G is simple
3) G is connected
4) G is strictly subtrivalent
5) If $H \subset G$ is connected subgraph of degree d' and genus g' then $d' \geq 2g' + 1$

- Strictly Subtrivolent = All vertices have degree ≤ 3 with at least one vertice having degree < 3.

- Condition #5 is sort of an ~~ala~~ analog of the result (Esienbud 8A.8.1) (Castelunvo 1893, Mattuck 1961, Mumford 1970, Green & Lazarsfeld 1985) that a smooth irreduclbe curve $C \subset \mathbb{P}^n$ with $d \geq 2g+1$ is arithmetic cohen Macauly.

- Specifically, a ~~ ~~ graph with the above assumptions has the following result.

~~Theorem: If~~

- Theorem: (Burnham, Rosen, Sidman, Vermeire, 2012): With $\widetilde{G}$ as above $\widetilde{G}$ can be embedded into $\mathbb{P}^{d-g}$ s.t. $\widetilde{G}$ is arithmetically cohen macaulay.

- Recall $X \subset \mathbb{P}^n$ is arithmetically cohen-macaulay if S_X is Cohen-Macaulay.

  ↳ A local ring R is cohen-macaulay if its Krull dimension equals its depth

$$\mathrm{depth}_I(M) = \min\{i \mid \mathrm{Ext}^i(R/I, M) \neq 0\} \leq \dim(M)$$

A non-local ring R is cohen-macaulay iff it is locally cohen-macaulay.

- If this means nothing read ACM as nice.

- **Ex**: 1) Regular local rings $k[\![t]\!]$
  2) Gorenstien Rings (things arising from complete intersections)
  3) R^G where R is CM and G reductive algebraic (finite) group (Hochster & Roberts)

- **non-Ex**: $k[\![x,y]\!]/(x^2, xy)$ depth $=0$, dim $=1$.

- **Ex**: Let C be a smooth rational curve of degree $2a$ lying on a smooth quadric ~~hyper~~surface $Q \subseteq \mathbb{P}^3$, $Q = X_0X_3 - X_1X_2$

C is parameterized by

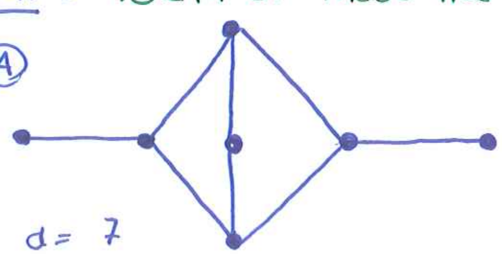
$$X_0 = s^{2a}, \quad X_1 = s^{2a-1}t, \quad X_2 = t^{2a-1}s, \quad X_3 = t^{2a}$$

C is non-ACM as it is genus zero.

↳ the only smooth rational ACM curve in $\mathbb{P}^3$ is the twisted cubic.

- **Ex**: Both of these has $d=7$, $m=8$, $g=2$

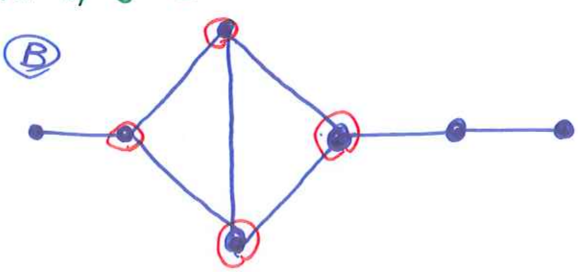
Ⓐ



$d = 7$
$m = 8$
$g = 2$

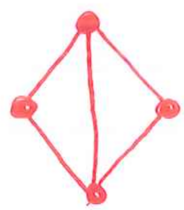
ok

Ⓑ



Not ok

B does not meet assumption #5



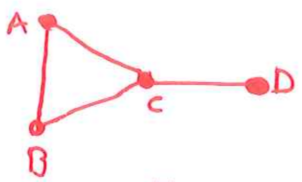
$d = 4$
$m = 5$
$g = 2$

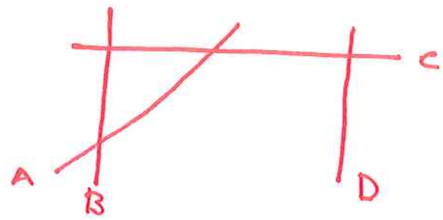
$2g+ = 2(2)+1 = 5$

$d = 4 \not\geq 5 = 2g+1$

- The theorem of (Burnham, et al.) is actually essentially just the embedding you would think, and is very natural. $I_{\widetilde{G}}$ is generated by ^elements of the form $x_i x_j$, $(x_j - x_k)x_i$, and $(x_j - x_k)(x_l - x_m)$ where the indices are distinct.

- <u>Ex</u>:





$\subseteq \mathbb{P}^3$

$d = 4$
$g = 1$

$A = (z_0, z_2)$ $\quad$ $C = (z_2, z_3)$

$B = (z_1, z_2)$ $\quad$ $D = (z_1, z_3)$

Notice $A \cap B = (z_0, z_1, z_2) = 1$ point!

- Why care about graph curves?

~~1) There has been~~

- 1) Studying singular things arise naturally from studying smooth things.

  ↳ <u>Eg</u>. the moduli space of smooth genus g curves M_g has a compactification $\overline{M_g}$, where not every curve is smooth.

  - Sometimes to prove ~~something~~ something regarding smooth curves we deform to the singular case.

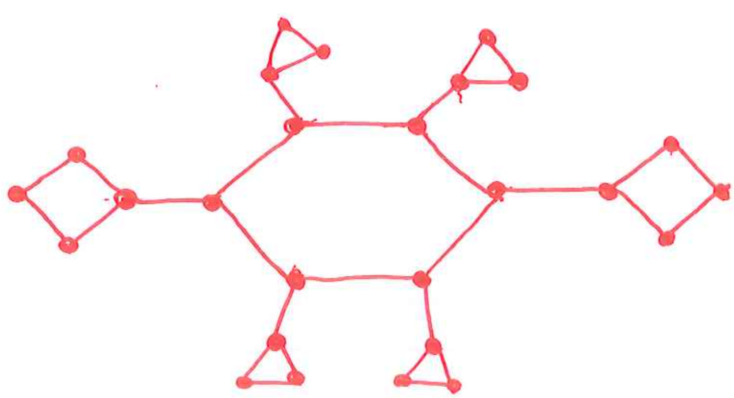
- This was the original motivation for graph curves. Eisenbud & Bayer 1991 wanted to use them to prove ~~gen~~ Green's Conjecture.

- <u>Theorem</u>: (Voisin 2002, 2005)(Green '80's): Let C be a generic smooth projective curve over $\mathbb{C}$ embedded in $\mathbb{P}^{g-1}$ by the complete canonical series. The length of the first strand of the minimal free resolution of I_C is $g-3-\text{cliff}$

  ↳ $\text{Cliff}(X) = \min\{d - 2r(D) \mid \text{all special divisors } D\}$ where D is special if $\ell(K-D) > 0$ where K is the canonical divisor and $r(D) = \ell(D) - 1$.

2) <u>Computation</u>: You can get your hands on pretty wierd curves and compute their betti table easily.

<u>Ex</u>: Could you find a curve (or its betti table) of a genus 7 degree 26 curve?



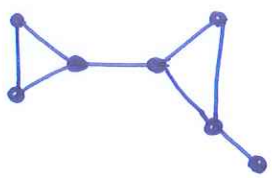
$g = 7$
$d = 26$
$\in \mathbb{P}^{19}$

You can compute in Macaulay II easily (relatively).

• 3) The singular case has more variation.

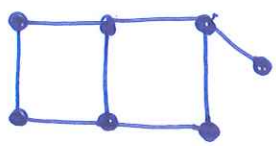
Thm: (Green): If C is a smooth curve of genus g and degree d where $d \geq 2g+1+k$ for $k \geq 1$ then the first $p+1$ entries of the quadratic strand are determined by g & d.

• **Ex** $d=7$, $m=8$, $g=2$, $\subseteq \mathbb{P}^5$

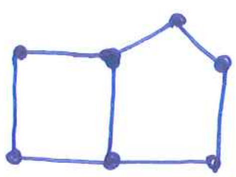


1	-	-	-	-
-	8	14	9	2
-	2	6	6	2

Notice the variation in the quadratic strand.



1	-	-	-	-
-	8	12	6	1
-	-	3	5	2



1	-	-	-	-
-	8	12	4	-
-	-	1	4	2

//

• **Prop** (D–): Let $\tilde{G} \subseteq \mathbb{P}^n$ have genus g then

1) $\tilde{G}$ is 3 regular, $b_{i,k}(\tilde{G}) = 0 \quad \forall k \geq i+3$

2) $b_{i,i+2} = h^1(\tilde{G}, M_L(1))$

3) $b_{i,i+1} = h^1(\tilde{G}, \Lambda^{i+1} M_L) - g\binom{n+1}{i+1} + b_{i-1,i+1}(\tilde{G})$

4) $b_{i,j} = 0 \quad \forall\, i \geq n+1$

<u>Proof</u>: 4) follows from Hilbert's Syzygy Theorem:

- <u>Theorem</u>: A finitely generated module M over $\mathbb{R}[x_1, \dots, x_n]$ has a ~~minimal~~ free resolution of length at most n.

For 1) - 3) use the fact the following is exact (Eisenbud 5.8)

$$0 \longrightarrow \operatorname{Tor}_i(S_C, \mathbb{C})_k \longrightarrow H^1(C, \Lambda^{i+1} M_L(k-i-1)) \longrightarrow H^1(C, \Lambda^{i+1} \Gamma(k-i-1))$$

$$\longrightarrow H^1(C, \Lambda^i(M_L(k-i))) \longrightarrow 0$$

where Γ is the trivial vector bundle of rank $n+1$ and M_L is the kernel

$$\Gamma(C, \mathcal{O}_C(1)) \twoheadrightarrow \mathcal{O}_C(1)$$

where $C \subset \mathbb{P}^n$ is an ACM curve.

$\blacksquare$

- So to compute these betti numbers we need to find cohomology groups.

- This prop says our betti tables look like

							n		
	1	-	-	-	-	-	-		
	-	$b_{i,i+1}$	-	-	-	-	$b_{n,n+1}$	0	0
2	-	$b_{2,3}$	-	-	-	-	$b_{n,n+2}$	0	0
	0	0	0	0	0	0	0	0	

- How we compute these cohomology dimensions is not very enlightening so I will avoid the proofs, and instead focus on the results.

Theorem: (D–): Let P_n be the path on n-vertices then

$$b_{i,i+1}(\widetilde{P_n}) = n\binom{n-i}{i} - \binom{n}{i+1}$$ (quadratic)

$$b_{i,i+2}(\widetilde{P_n}) = 0$$ (cubic)

∎

Ex:

$d=5,\ g=1$

 $\subseteq \mathbb{P}^4$

1	–	–	–	–
–	10	20	15	4
0	0	0	0	0

$d=6,\ g=1$

 $\subseteq \mathbb{P}^5$

1	–	–	–	–	–
–	15	40	45	24	4
0	0	0	0	0	0

question: Distribution of these?

$d=7,\ g=1$

 $\subseteq \mathbb{P}^6$

1	–	–	–	–	–	–
–	21	70	105	84	35	6
0	0	0	0	0	0	0

Theorem: (E. Bollico, 2003): Let C_{n+1} be the cyclic graph on $n+1$ vertices

$$b_{i,i+1} = n\binom{n-i}{i} - \binom{n-i}{i} - \binom{n}{i+1} \qquad i<n$$

$$b_{n,n+1} = 0 \qquad i=n$$

quadratic

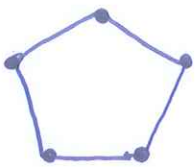
$$b_{i,i+2} = 0 \qquad i<n$$

$$b_{i,i+2} = 1 \qquad i=n$$

cubic

∎

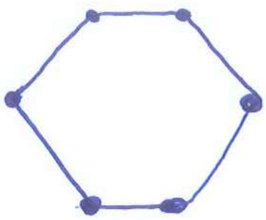
• Ex :



$d = 5,\ g = 1$

$\in \mathbb{P}^4$

1	-	-	-
-	5	5	-
-	-	-	1



$d = 6,\ g = 1$

$\in \mathbb{P}^5$

1	-	-	-	-
-	9	16	9	-
-	-	-	-	1

C_8 $\quad d = 8,\ g = 1$ $\quad \in \mathbb{P}^7$

1	-	-	-	-	-	-
-	20	64	90	64	20	-
-	-	-	-	-	-	1

• We now know know all paths and cycles and so to finish of all genus zero and genus one graphes we need

trees  & hairy cycles 

• Towards this we have the following theorem on the quadratic strand (recall we needed 2 things (a cohomology rank & the cubic strand)

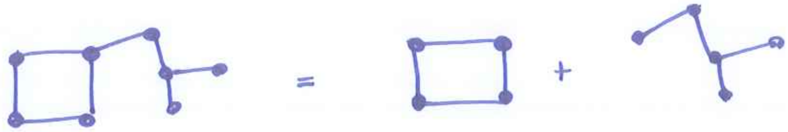
• Theorem: (D-) : If $G \subseteq \mathbb{P}^n$ ~~then~~ is of genus g then

$$b_{i,i+1}(\tilde{G}) = n\binom{n-1}{i} - g\binom{n+1}{i-1} - \binom{n}{i+1} + b_{i-1,i+1}(\tilde{G}).$$

$\blacksquare$

• Thus, to compute the Betti table of any graph curve we are left to find the cubic strand.

- We are not able to compute the cubic strand in general.
- Notice however all trees and hairy cycles break up to paths and cycles



and the intersection is a point. It turns out in this case we can say something.

- **Theorem** : (D–): Let $\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2 \subseteq \mathbb{P}^n$ be st. $\widetilde{G}_1 \cap \widetilde{G}_2 = \{p\}$ is reduced. If $\widetilde{G}_i \subseteq \mathbb{P}^{n_i}$ then

$$b_{i,i+2}(\widetilde{G}) = \sum_{s=0}^{n-n_1} \binom{n-n_1}{s} b_{i-s,i-s+2}(\widetilde{G}_1) + \sum_{t=0}^{n-n_2} \binom{n-n_2}{t} b_{i-t,i-t+2}(\widetilde{G}_2).$$

Prop : Let X, Y be projective varieties in $\mathbb{P}^n$ and $\ell \subseteq \mathbb{P}^n$ a line st $X = Y \cup \ell$ and ℓ intersects Y transversely at a point then

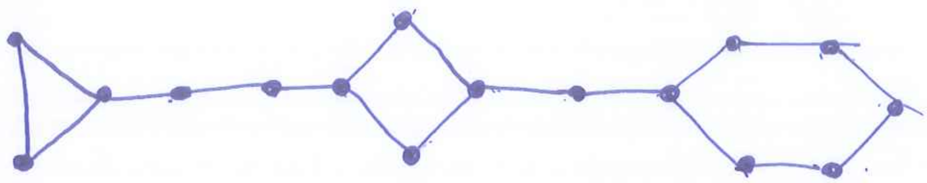
$$H^1(X, \Lambda^{i+1} M_L(1)) \cong H^1(Y, \Lambda^{i+1} M_L(1)) \qquad \blacksquare$$

Prop : Suppose $X \subseteq \mathbb{P}^r \subseteq \mathbb{P}^n$ where X spans $\mathbb{P}^r$. Letting $M_L = \Omega_{\mathbb{P}^n}(1) \otimes \mathcal{O}_X$ and $\widetilde{M}_L = \Omega_{\mathbb{P}^r}(1) \otimes \mathcal{O}_X$ we have

$$h^j(X, \Lambda^{i+1} M_L(k-i-1)) = \sum_{t=0}^{n-r} \binom{n-r}{t} h^j(X, \Lambda^{i+1-t} \widetilde{M}_L(k-i-1)). \qquad \blacksquare$$

Proof : Combine these. $\blacksquare$

Ex:



$$d = 16 \qquad g = 3 \qquad \subsetneq \mathbb{P}^{13}$$

$$b_{i,i+2} = \binom{11}{i-1} + \binom{10}{i-2} + \binom{8}{i-4}$$

1	–	–	–	–	–	–	–	–
–	75	537	1959	4553	7306	8378	6937	4114
–	1	12	65	211	458	700	770	610

–	–	–	–
1699	461	73	5
341	128	29	3

- This is called a tree of cycles.

- A tree of cycles is a tree with replacing non-adjacent vertices with cycles.

• Def: The girth γ of a graph G is length of shortest cycle in G.

• Prop: If G is a tree of cycles with girth γ then

$b_{\gamma-2,\gamma}(\bar{G}) = \#$ of cycles of length γ.

$\blacksquare$

• Prop: If G is a tree of cycles $\bar{G} \subseteq \mathbb{P}^n$ then

$b_{n-1,n}(\bar{G})$ is the $\#$ of bridges in G.

$\square$

• Bridge = edge which when removed is disconnected

$\square$

• Conj: Let $G = C_4^k$ be a bunch of C_4's glued along one edge

$$b_{2,4}(\bar{G}) = b_{1,2}(\bar{G}) - 1$$

$$b_{i,i+1}(\bar{G}) = b_{i+2,i+4}(\bar{G}) \quad i \geq 2$$

$$b_{k,k+2}(\bar{G}) = (b_{k+1,k+3}(\bar{G}))^2 - 1$$

$\square$

• Proof : (Quadratic Strand)

$$\chi(\bar{G}, \Lambda^{i+1} M_L) = -h^1(\bar{G}, \Lambda^{i+1} M_L)$$

$$0 \longrightarrow H^0(\bar{G}, \Lambda^{i+1} M_L) \longrightarrow \bigoplus_{i}^{d} H^0(\ell_i, \Lambda^{i+1} M_L|_{\ell_i})$$

$$\longrightarrow H^0(\{P_1, \dots P_m\}, \Lambda^{i+1} M_L) \quad \text{exact}$$

$\bar{G} = \bigcup \ell_i$