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ABSTRACT. Classically, Noether normalization says that any projective (resp. affine) variety of dimension n over a field admits a finite surjective morphism to $\mathbb{P}^n$ (resp. $\mathbb{A}^n$). I will discuss whether we can generalize such theorems to other bases like $\mathbb{Z}$, $\mathbb{C}[t]$, etc. This is based on joint work with Daniel Erman appearing in http://arxiv.org/abs/1604.01704.

1. INTRODUCTION

Goal for this talk is to understand the varying situations in which Noether normalization works and does not work; filling in this table as we go:

	Affine (Linear)	**Proj. (Linear)**	**Affine**	**Proj.**
$\lvert\mathbf{k}\rvert = \infty$				
$\lvert\mathbf{k}\rvert < \infty$				
$\mathbb{Z}$				
$\mathbb{C}[t]$				
$\mathbb{Q}[t]$				
$\mathbb{F}_q[t]$				
$\mathbb{P}^1_{\mathbb{C}}$				

With this in mind I would like to explain what each of the four columns means by reviewing Noether normalization in the classical setting. Let's beginning with the most classical, which is originally due to Emmy Noether; although versions of this result were certainly used by others sans proof before this.

Theorem A (Affine Linear). *Let $\mathbf{k}$ be an infinite field and R be finitely generated $\mathbf{k}$-algebra of dimension n. There exists **linear** polynomials $f_1, \dots, f_n \in R$ such that*

(i) R is a finitely generated $\mathbf{k}[f_1, \dots, f_n]$ module;

(ii) $\mathbf{k}[f_1, \dots, f_n] \cong \mathbf{k}[x_1, \dots, x_n]$.

Translating this result with our handy-dandy algebra-geometry dictionary we see that this says if $X \subset \mathbb{A}^N_{\mathbf{k}}$ is an n dimensional affine variety then there is a morphism

$$\phi : X \to \mathbb{A}^n_{\mathbf{k}}$$

which is surjective and finite. (Recall that a morphism is finite if and only if it is proper with finite fibres, so in most of our cases we should just think finite fibres.) In fact in this case constructing such a ϕ tends to be relatively easy: pick a point $p_1 \in \mathbb{A}^N_{\mathbf{k}} - X$ and consider the projection

$$\phi_1 : \mathbb{A}^N_{\mathbf{k}} - \{p_1\} \to \mathbb{A}^{N-1}_{\mathbf{k}},$$

and proceed inductively by picking a point $p_1 \in \mathbb{A}^{N-1}_{\mathbf{k}} - \phi_1(X)$ noting that projection from a point is a finite morphism.

That said when one wants to pass from the case of an infinite field to a finite field there is some subtlety. For example, we can no longer take $f_1, \dots, f_n$ to necessarily be linear, or even of any fixed given degree, because over a finite field their are $\mathbf{k}$-algebras in which all linear forms are zero divisors.

Example 1.1. For example, consider the ring $R = \mathbf{k}[x,y]/\langle xy(x-1)(y-1)(x-y)\rangle$ where $\mathbf{k} \cong \mathbb{F}_2$. In this ring by construction every linear form is a zero divisor. Moreover one can check that if ℓ is a linear form $\mathbf{k}[\ell] \subset R$ is not a integral extension, and so is not module finite.

That said if we are willing to drop the condition that our elements $f_1, \dots, f_n$ be of any particular degree we can get the same result over finite fields.

Theorem B (Affine). *Let $\mathbf{k}$ be an field and R be finitely generated $\mathbf{k}$-algebra of dimension n. There exists elements $f_1, \dots, f_n \in R$ such that*

(i) R is a finitely generated $\mathbf{k}[f_1, \dots, f_n]$ module;

(ii) $\mathbf{k}[f_1, \dots, f_n] \cong \mathbf{k}[x_1, \dots, x_n]$.

The remaining two versions of Noether normalization I would like to discuss are similar to the two above expect that they are in the graded i.e. projective setting as opposed to the affine.

Theorem C (Projective Linear). *Let $\mathbf{k}$ be an infinite field and R be finitely generated graded $\mathbf{k}$-algebra of dimension $n+1$. There exists **linear** homogenous polynomials $f_0, \dots, f_n \in R$ such that* *(i) R is a finitely generated $\mathbf{k}[f_0, \ldots, f_n]$ module;*

(ii) $\mathbf{k}[f_0, \ldots, f_n] \cong \mathbf{k}[x_0, \ldots, x_n]$.

Geometrically this statement corresponds to essentially the same statement as before, expect in the projective situation: if $X \subset \mathbb{P}^N_{\mathbf{k}}$ is an n dimensional projective variety over an infinite field $\mathbf{k}$ then there is a finite surjective morphism:

$$\phi : X \to \mathbb{P}^n_{\mathbf{k}}$$

which is given by linear projections. Once again if $\mathbf{k}$ is finite there still exists such a map, but we can no longer guarantee that it is given by linear projections.

Theorem D (Projective)**.** *Let $\mathbf{k}$ be a field and R be finitely generated graded $\mathbf{k}$-algebra of dimension $n+1$. There exists homogenous polynomials $f_0, \ldots, f_n \in R$ such that*

(i) R is a finitely generated $\mathbf{k}[f_0, \ldots, f_n]$ module;

(ii) $\mathbf{k}[f_0, \ldots, f_n] \cong \mathbf{k}[x_0, \ldots, x_n]$.

Returning to our table we should think that the columns correspond to these four versions of Noether normalization as follows:

- Theorem A $\iff$ Affine Linear $\iff$ deg. one finite surj. morphism $\phi : X \to \mathbb{A}^n_B$.
- Theorem B $\iff$ Affine $\iff$ finite surj. morphism $\phi : X \to \mathbb{A}^n_B$.
- Theorem C $\iff$ Proj. Linear $\iff$ deg. one finite surj. morphism $\phi : X \to \mathbb{P}^n_B$.
- Theorem D $\iff$ Proj. $\iff$ finite surj. morphism $\phi : X \to \mathbb{P}^n_B$.

Thus, based on what we have said so far we can fill the table in for the case of fields.

	A	**C**	**B**	**D**
	Affine (Linear)	**Proj. (Linear)**	**Affine**	**Proj.**
$\lvert\mathbf{k}\rvert = \infty$	☺	☺	☺	☺
$\lvert\mathbf{k}\rvert < \infty$	☹	☹	☺	☺
$\mathbb{Z}$				
$\mathbb{C}[t]$				
$\mathbb{Q}[t]$				
$\mathbb{F}_q[t]$				
$\mathbb{P}^1_{\mathbb{C}}$				

The goal for the remain part of the talk is to try and fill in as much of this table as possible. That is put somewhat differently we would like to try and answer the following question.

Question 1.2. *Are similar versions of Noether normalization true if we replace* $\mathbf{k}$ *by a one dimensional ring i.e.* $\mathbb{Z}$, $\mathbb{C}[t]$, $\mathbb{F}_q[t]$, *etc.?*

2. A Storm Brews

Thinking about it for a moment there are a few entires in the table, which we can quickly fill in...

Counterexample 2.1. Consider the extremely "nice" $\mathbb{Z}$-algebra $\mathbb{Z}[x]/\langle 3x^2 - x\rangle$, which is finitely generated a $\mathbb{Z}$-algebra, flat, with zero dimensional fibres over $\mathbb{Z}$ i.e. "extremely nice". Despite all this niceness we know that as $\mathbb{Z}$-modules:

$$\frac{\mathbb{Z}[x]}{\langle 3x^2 - x\rangle} \cong \mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{3}\right],$$

which of course is not finitely generated as a $\mathbb{Z}$-module. (The element $\frac{1}{3}$ is not integral over $\mathbb{Z}$.)

Similarly affine Noether normalization does not in general hold over $\mathbf{k}[t]$ for any field $\mathbf{k}$. In particular, consider the following example.

Counterexample 2.2. Letting $B = \mathbf{k}[t]$ and considering the finitely generated B-algebra $B[x]/\langle tx^2 - x\rangle$ we see that

$$\frac{B[x]}{\langle tx^2 - x\rangle} \cong B \oplus B\left[\frac{1}{t}\right].$$

Now the element t^{-1} clearly does not satisfy a monic polynomial over B implying that this is not a finitely generated B-module. Hence Theorems A and B do not hold over $\mathbf{k}[t]$.

	A	**C**	**B**	**D**
╲	**Affine (Linear)**	**Proj. (Linear)**	**Affine**	**Proj.**
$\lvert\mathbf{k}\rvert = \infty$	☺	☺	☺	☺
$\lvert\mathbf{k}\rvert < \infty$	☹	☹	☺	☺
$\mathbb{Z}$	☹		☹	
$\mathbb{C}[t]$	☹		☹	
$\mathbb{Q}[t]$	☹		☹	
$\mathbb{F}_q[t]$	☹		☹	
$\mathbb{P}^1_{\mathbb{C}}$				

Moreover we cannot hope for linear projective Noether normalizations in any of these cases as the following examples show.

Counterexample 2.3. Let $\mathbf{k}$ be any field and consider the family $X \subseteq \mathbb{P}^2_{\mathbf{k}[t]}$ defined as $\mathbb{V}((x-ty)(tx-y))$. We observe that there do not exist sections of $\mathcal{O}_X(1)$ which yield a finite map to $\mathbb{P}^1_{\mathbf{k}[t]}$. Assume for contradiction that there were such linear forms $f_0 = a_1(t)x+a_2(t)y+a_3(t)z$ and $f_1 = b_1(t)x+b_2(t)y+b_3(t)z$. Then these must restrict to parameters on each component of X. Write $X = X_1 \cup X_2$ where $X_1 = \mathbb{V}(x-ty)$ and $X_2 = \mathbb{V}(tx-y)$.

Three linear forms on $\mathbb{P}^2$ will have no base locus if and only if the corresponding 3×3 determinant of their coefficients is a unit. It follows that f_0, f_1 restrict to parameters on each fiber of X_1 if and only if

$$\det\begin{pmatrix} 1 & -t & 0 \\ a_1(t) & a_2(t) & a_3(t) \\ b_1(t) & b_2(t) & b_3(t) \end{pmatrix} \in \mathbf{k}[t]^* = \mathbf{k}^*.$$

Namely, if this determinant were not a unit, then it would lie inside some maximal ideal of $\mathbf{k}[t]$, and f_0, f_1 would fail to be parameters over that point of $\mathbf{k}[t]$.

Let $\alpha := a_2b_3 - a_3b_2 \in \mathbf{k}[t]$ and $\beta := a_1b_3 - a_3b_1 \in \mathbf{k}[t]$ so that the determinant of the above matrix is $\alpha + t\beta$. Since this is a unit, we can, after rescaling the a_i, assume that $\alpha + t\beta = 1$. If f_0, f_1 restrict to parameters on each fiber of X_2, then a similar computation shows that $t\alpha + \beta = u$ from some $u \in \mathbf{k}[t]^* = \mathbf{k}^*$. Substituting $\alpha = 1 - t\beta$ we get

$$t(1-t\beta)+\beta = u \quad \Rightarrow \quad (1-t^2)\beta = u - t \quad \Rightarrow \quad \beta = \frac{u-t}{1-t^2} \notin \mathbf{k}[t].$$

But this contradicts the fact that $\beta \in \mathbf{k}[t]$, and hence there exist no such linear forms f_0 and f_1. However, in degree 2, one can check that $z^2, x^2 - txy + y^2$ restrict to parameters on all fibers of X.

Notice that if we let $\mathbf{k}[t] = \mathbb{Z}$ and $t = 3$ essentially the same argument shows that we also cannot have linear projective Normalizations to hold over $\mathbb{Z}$. Additionally, if we take our base B to be $\mathbb{P}^1_{\mathbb{C}}$ one cannot obtain a Noether normalization type result; as the following proposition shows.

Proposition 2.4. [BE16] *Over $\mathbb{C}$, let $X \to \mathbb{P}^1$ be a very general hypersurface of bi-degree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Then X admits no finite map to a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*

Proof. Theorem 1.1 of [Ott15] gives an explicit description of the Nef and psuedo-effective cones of such hypersurfaces. From this one can check X admits no such morphism. $\square$

	A	C	B	D
	Affine (Linear)	Proj. (Linear)	Affine	Proj.
$\lvert\mathbf{k}\rvert = \infty$	☺	☺	☺	☺
$\lvert\mathbf{k}\rvert < \infty$	☹	☹	☺	☺
$\mathbb{Z}$	☹	☹	☹	
$\mathbb{C}[t]$	☹	☹	☹	
$\mathbb{Q}[t]$	☹	☹	☹	
$\mathbb{F}_q[t]$	☹	☹	☹	
$\mathbb{P}^1_{\mathbb{C}}$	☹	☹	☹	☹

3. The Eye of the Storm

At this point things are not looking very promising for Noether normalization in other settings. (We are left with our only hope being (non-linear) projective Noether normalization over $\mathbb{Z}$ or $\mathbf{k}[t]$.) That said the following lemma, whose proof I will punt on, suggests that maybe we should have a bit more hope for the projective setting as opposed to the affine version.

Lemma 3.1. [BH93, Theorem 1.5.17] *Let $\mathbf{k}$ be a field and let R be a $(k+1)$-dimensional graded $\mathbf{k}$-algebra where $R_0 = \mathbf{k}$. If $f_0, \dots, f_k$ are homogeneous elements of degree d and $R/(f_0, \dots, f_k)$ has finite length, then the extension $\mathbf{k}[z_0, \dots, z_k] \to R$ given by $z_i \mapsto f_i$ is a finite extension.*

Lemma 3.1 is clearly false if we drop the graded and homogenous conditions.

Example 3.2. For example, if we let $f_0 = x$ and $f_1 = x + 1$ be elements of $\mathbb{C}[x, y]$ then their quotient $\mathbb{C}[x, y]/\langle x, x + 1\rangle$ is finite length. (It is the zero ring since $\langle f_0, f_1\rangle = \langle 1\rangle$.) However, $\mathbb{C}[x, y]$ is clearly not module finite over $\mathbb{C}[f_0, f_1] \cong \mathbb{C}[x]$.

Hence in some ways this suggests that finding a Noether normalization in projective case may be easier than in the affine case. (Note this is opposite of how Noether normalization is often presented: the affine case is done first and the projective follows as sort of an afterthought.) That is if $X \subset \mathbb{P}^N_{\mathbf{k}}$ is a projective variety over a field $\mathbf{k}$ then to given a finite morphism:

$$\phi : X \to \mathbb{P}^n_{\mathbf{k}}$$

all we have to do is find functions $f_0, \dots, f_n \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ such that $\mathbb{V}(f_0, \dots, f_n) \cap X = \varnothing$. That is we wish to find a system of parameters on X. More generally;

Definition 3.3. *If $X \subset \mathbb{P}^N$ is a projective variety of dimension n then $(f_0, \dots, f_k) \in \mathbb{H}^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^{\oplus(k+1)}$ are a **(partial) systems of parameters** on X if and only if:*

$$\dim \mathbb{V}(f_0, \dots, f_k) \cap X = \dim X - (k+1) = n - (k+1).$$

Note we take the convention that a variety has dimension -1 if and only if it is empty. This way giving a finite morphism $\phi : X \to \mathbb{P}^n_{\mathbf{k}}$ is equivalent to giving (full) system of parameters on X.

Everything I have said so far in this section – i.e. Lemma 3.1 – only applies to fields, and the only things left in our chart are $\mathbb{Z}$, $\mathbb{C}[t]$, $\mathbb{Q}[t]$, $\mathbb{F}_q[t]$, which are decidedly not fields. That said we can bootstrap Lemma 3.1 up to these cases if we work with "fibrewise" systems of parameters. For example, if $X \subset \mathbb{P}^N_{\mathbb{Z}}$ is a "nice" projective variety of dimension n then in order to show the existence of a morphism $\phi : X \to \mathbb{P}^n_{\mathbb{Z}}$ it suffices to find $(f_0, \dots, f_n) \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^{\oplus(n+1)}$ such that $(f_0, \dots, f_n)_s$ are systems of parameters on X_s for all $s \in \operatorname{Spec} \mathbb{Z}$.

4. A Cleansing Storm

With this in mind we might ask: How "hard" it is to find such a system of "fibrewise" parameters (say for $X \subset \mathbb{P}^N_{\mathbb{Z}}$)? In order to get some level of familiarity, with this question let us briefly return to where our base is a field. In this setting the collection of points in $\mathbb{P}(\mathbb{H}^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))^{\oplus(k+1)})$ corresponding to tuples $(f_0, \dots, f_k)$, which *fail* to be systems of parameters on X forms a (non-empty) proper Zariski closed subset. Thus, when $\mathbf{k}$ is infinite systems of parameters of a given degree not only exist, but are generic in the space of tuples of homogenous of some degree.

As we previously saw the question of Noether normalization over a finite field is a more nuanced, and this nuance carries over to the ubiquity of systems of parameters. For example, as we saw earlier it can be the case that for specific d their are no systems of parameters. That said using sieve methods adapted from the work of Poonen on Bertini theorems over finite fields we are able to show that (asymptotically) systems of parameters are still quite common.

Theorem 4.1 ([BE16]). *Let $X \subseteq \mathbb{P}^N_{\mathbb{F}_q}$ be an n-dimensional closed subscheme. Then we have*

$$\lim_{d \to \infty} \operatorname{Prob} \begin{pmatrix} (f_0, \dots, f_k) \text{ of degree } d \\ \text{are parameters on } X \end{pmatrix} = \begin{cases} 1 & \text{if } k < n \\ \zeta_X(n+1)^{-1} & \text{if } k = n \end{cases}$$

where $\zeta_X(s)$ is the arithmetic zeta function of X.

Notice that since $\zeta_X(n+1)^{-1} > 0$ this means that even over a finite field we can construct a finite map probabilistically by picking functions of sufficiently high degree at random. (We can of course do this in the infinite field case.) With these examples in mind we might hope that over $\mathbb{Z}$ we could construct "fibrewise" systems of parameters probabilistically.This is *not* the case...

Theorem 4.2 ([BE16]). *Let $X \subseteq \mathbb{P}^N_{\mathbb{Z}}$ be a closed subscheme with an n-dimensional fiber whose general fiber over $\mathbb{Z}$ has dimension n. Then*

$$\lim_{d\to\infty} \text{Density}\left\{\begin{matrix} (f_0,\dots,f_k) \textit{ of degree } d \textit{ that restrict} \\ \textit{to parameters on } X_p \textit{ for all } p \end{matrix}\right\} = \begin{cases} 1 & \textit{if } k < n \\ 0 & \textit{if } k = n \textit{ and all } d. \end{cases}$$

Notice that this not only means that over $\mathbb{Z}$ we cannot construct a finite morphism $\phi : X \to \mathbb{P}^n_{\mathbb{Z}}$ probabilistically, but that even finding such a morphism is quite difficult. This taken into account with all the additional evidence suggesting Noether normalizations over non-fields is prone to failure makes the following result all the more surprising. (Well at least to me.)

Theorem 4.3 ([BE16]). *Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and $X \subseteq \mathbb{P}^N_B$ be a closed subscheme. If each fiber of X over B has dimension n, then for some d, there exists a linear series in $\mathcal{O}_X(d)$ inducing a finite morphism $\pi : X \to \mathbb{P}^n_B$.*

I will briefly sketch the argument for this theorem in the case of $B = \mathbb{Z}$; to give some sense of the tools we use. That said the tl;dr version is: Use Theorem 4.2 to construct a partial system of parameters. This cuts X down to something finite over $\operatorname{Spec}\mathbb{Z}$, which has a finite Picard group.

Proof. Applying Theorem 4.2 for sufficiently large $e > 0$ we can construct find $(f_0,\dots,f_{n-1})$, which are parameters on X. This means that $X' = X \cap \mathbb{V}(f_0,\dots,f_{n-1})$ is finite over $\operatorname{Spec}\mathbb{Z}$, which in turns implies that $\operatorname{Pic}(X')$ is finite. Hence for some $k \gg 0$ we know that

$$H^0(X',\mathcal{O}_{X'}(k)) \cong \mathbb{Z},$$

and so some element f_n of $H^0(X',\mathcal{O}_{X'}(k))$ maps onto a unit implying that $\mathbb{V}(f_n) \cap X' = \emptyset$. Taking sufficient powers of $f_0,\dots,f_n$ – so that they all have degree d – gives the result. $\square$

Example 4.4. This example illustrates how taking the projective closure can "fix" the failure of finiteness. Consider the $\mathbb{Z}$-algebra $R := \mathbb{Z}[x]/(3x^2 - 5x) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{3}]$ mentioned in the introduction. This is a flat, finite type $\mathbb{Z}$-algebra where every fiber has dimension 0, yet it is not a finite extension of $\mathbb{Z}$. However, if we take the projective closure of $\operatorname{Spec}(R)$ in $\mathbb{P}^1_{\mathbb{Z}}$, then we get $\operatorname{Proj}(\overline{R})$ where $\overline{R} = \mathbb{Z}[x,y]/(3x^2 - 5xy)$. If we then choose $f_0 := 4x - 7y$, we see that $\mathbb{Z}[f_0] \subseteq \overline{R}$ is then a finite extension of graded rings.

	A	C	B	D
╲	Affine (Linear)	Proj. (Linear)	Affine	Proj.
$\lvert\mathbf{k}\rvert = \infty$	☺	☺	☺	☺
$\lvert\mathbf{k}\rvert < \infty$	☹	☹	☺	☺
$\mathbb{Z}$	☹	☹	☹	☺
$\mathbb{C}[t]$	☹	☹	☹	???
$\mathbb{Q}[t]$	☹	☹	☹	???
$\mathbb{F}_q[t]$	☹	☹	☹	☺
$\mathbb{P}^1_{\mathbb{C}}$	☹	☹	☹	☹

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