• **Q**: How can we measure the "complexity" of a variety $X \subseteq \mathbb{P}^r_{\mathbb{C}}$

↑ throughout I will work over $\mathbb{C}$ to make life easier.

• Reasons such a measure would be useful:

1) We know when nice behavior begins

↳ e.g. $h_X(t) = \chi(X, \mathcal{O}_X(t))$ becomes polynomial

2) We can make seemingly infinite problems finite
3) We can use it to perform induction.

} e.g. Hilb & Quot schemes

4) We can get a sense for how long computations take.

↳ e.g. Bayer-Stillman

• The way we will go about constructing such a notion of complexity, in my mind follows the script of a more general theme in mathematics:

"Study wiggly, curvy, complex things by approximating them by linear things"

↳ eg. + calculus
+ manifolds
+ group reps.

• **Set Up**: Given $X \subseteq \mathbb{P}^r_{\mathbb{C}}$:

+ $S = \mathbb{C}[x_0, \ldots\ldots, x_r]$

+ $S_d = \mathbb{C}\langle \text{homog. degree } d \text{ forms}\rangle = \mathbb{C}\cdot\langle \text{degree } d \text{ monomials}\rangle \cong \mathbb{C}^{\binom{r+d}{d}}$

+ I_X = homog. ideal defining X.

+ $S_X = S/I_X$ = homogeneous cordinate ring.

• **Def**: A graded S-module is an S-module M together with a decomposition as abelian groups

$$M \cong \bigoplus_{d\in\mathbb{Z}} M_d$$

s.t. $S_e \cdot M_d \subseteq M_{d+e}$.

• <u>Two Examples</u>: 1) If $I \leq S$ is a homog. ideal (i.e. generated by homog. poly) then

$$I_d = I \cap S_d = \{\text{degree } d \text{ homog. polys in } I\}$$

is a natural grading.

2) Given $a \in \mathbb{Z}$, we let $S(a)$ to be S as an S-module, and shift the grading so that

$$S(a)_d = S_{a+d}.$$

i.e. since 1 generates S as an S-module $S(a)$ is a free S-module with generator in degree $-a$.

↳ e.g. $S\langle x^2\rangle \cong S(-2)$.

• <u>Def</u>: A minimal free resolution of a f.g.g. S-module M is a chain complex

$$0 \leftarrow F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2 \leftarrow \cdots\cdots \quad \cdots F_{n-1} \xleftarrow{\delta_n} F_n \leftarrow \cdots$$

of graded S-modules s.t.

1) each F_i is free meaning

$$F_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$$

2)
$$H^i(F_\cdot) = \begin{cases} M & i = 0 \\ 0 & \text{else} \end{cases}$$

3) the matrices representing δ_i contain no units.

• <u>Ex</u>: Let $S = \mathbb{C}[x_0, x_1]$, $I = \langle x_0, x_1\rangle$

$$\begin{tikzcd}
 & & & 0 & & \\
0 & S/I \arrow[l] & S \arrow[l, two heads] & I \arrow[l] \arrow[u] & 0 \arrow[l] & \\
 & & & S(-1)^2 \arrow[u, two heads] \arrow[ul, "\delta_1"] & & \\
 & & 0 \arrow[r, leftrightarrow] & (x_0 : x_1) \arrow[u] & S(-2) \arrow[l, two heads, "\cong"] \arrow[ul, "\delta_2"'] & 0 \arrow[l] \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

<u>Note</u>: If
$Ax_0 = Bx_1$
$\Rightarrow A'x_0x_1 = B'x_0x_1$
$\Rightarrow (A' - B')x_0x_1 = 0$
So $(x_0 : x_1) \cong S\langle x_0x_1\rangle$.

• Eg (cont): So the resolution of S/I is:

$$0 \longleftarrow S \xleftarrow{(x,y)} S(-1)\oplus S(-1) \xleftarrow{\binom{-y}{x}} S(-2) \longleftarrow 0$$

since the differentials have no units this is minimal.

• Thm: (Hilbert): Minimal free resolutions exist and are unique up to chain isomorphism.

• From here there are two "obvious" notions of complexity staring us in the face:

1) $pd(M)$ = how long is $F_\bullet$

↳ Hilbert $\leq r+1$

↳ Stillman's conjecture: $pd(I)$ can be bounded independent of r.

2) How big these generators - i.e. the $-j$'s - are getting.

• We would be tempted to say:

$$reg(M) = \max_{\beta_{i,j}\neq 0}\{\beta_{i,j} \neq 0\},$$

but for various reasons it is actually most useful to normalize this.

• Note the fact we require our δ_i's to not contain units means if $\beta_{i,j} \neq 0$ then $\beta_{i+1,k} \neq 0$ for some $k > j$. i.e. we cannot have

$$\cdots \longleftarrow S(-j) \xleftarrow{\delta_{i+1}} S(-j) \longleftarrow \cdots,$$

and so we normalize regularity as follows.

• Def: The regularity of a f.g.g S-module is

$$reg(M) = \max\{\beta_{i,j} - i \mid \beta_{i,j} \neq 0\}.$$

• Ex: 1) $reg(S) = 0$ since $0 \longleftarrow S \longleftarrow S \longleftarrow 0$ is a minimal resolution.

2) If F is a free S-module then the minimal resolution is

$$0 \longleftarrow F \longleftarrow F \longleftarrow 0$$

and so $reg(F)$ = max degree of a minimal generator of F.

3) $S = \mathbb{C}[x_0,x_1]$, $I = \langle x_0,x_1\rangle$ then

$$reg(S/I) = 0.$$

This is true for all Koszul resolutions.

Ex: 4) $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ via $[s:t] \longmapsto [s^3 : s^2t : t^2s : s^3]$ has a resolution

$$S \xleftarrow{(x_1x_3 \;\; x_0x_3 \;\; x_0x_2)} S(-2)^3 \xleftarrow{\begin{pmatrix} x_0 & 0 \\ -x_1 & x_1 \\ 0 & -x_3 \end{pmatrix}} S(-3)^2 \longleftarrow 0$$

And so we see that

$$\operatorname{reg}(S_X) = 1.$$

• **How this is useful:**

1) Recall the Hilbert function is defined to be

$$h_M(d) = \dim_{\mathbb{C}} (S_X)_d,$$

Since this is additive is LES we see that

$$h_M(d) = \sum_i h_{F_i}(d)(-1)^i = \sum_{i,j} (-1)^i \beta_{i,j} \binom{d-j+r}{r}$$

$$= \sum_{i,j} (-1)^i \beta_{i,j} \left[\frac{(d-j+r)\cdots\cdots(d-j+1)}{r!} \right]$$

Each of these is a polynomial in d so long as

$$d \geq j - r$$

But for $d \geq \operatorname{reg}(M) \geq j - i \Rightarrow$ the desired inequality.

• Prop: $h_M(d)$ is a polynomial for $d > \operatorname{reg}(M)$.

2) Suppose we wanted to build a space parameterizing $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ s.t. X has Hiblert polynomial $\Phi(t) \in \mathbb{Q}[t]$. One way to do this would be as follows:

1) For any d, $(I_X)_d \subseteq S_d$ is a linear subspace, and so defines a point
$$X_d \in \mathrm{Gr}\left(\binom{r+d}{d} - h_X(d), \binom{r+d}{d} \right).$$

2) For $d \gg 0$ large X_d determines X i.e I_{X_d} determines X.

So to construct such a thing we must just hope there is a uniform d for all X we might be able to pick. This leads to the question:

Q: Can we bound $\operatorname{reg}(X)$?

• **Thm**: (Gotzmann): Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a variety with $\mathrm{reg}(X) = \mathrm{reg}(I_X) = R$ then there exists integers $a_1 \geq a_2 \geq \cdots \geq a_R \geq 0$:

$$P_X(t) = \sum_{k=1}^{R} \binom{t + a_k - k + 1}{a_k}.$$

↑ A sort of converse - i.e. what polynomials are Hilbert polynomials is true.

• In general, regularity of arbitrary ideals is quite poorly behaved.

• **Thm**: (Mayr-Meyer, Bayer-Stillman, Ullery): There exists families of ideals $I \subseteq \mathbb{C}[x_0, \ldots, x_{10n}]$

$$\mathrm{reg}(I) \geq \deg(I)^{2^{n-1}}$$

↑ work of others shows this is about as worse as it can get.

• In the geometric setting, this are often somewhat more well behaved, we hope....

• **Thm**: (Gruson-Lazarsfeld-Peskine): Let $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ be a reduced, non-degenerate, irreducible curve then:

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 1.$$

• **Conj**: (Eisenbud-Goto Regularity Conj): Let $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ be non-degenerate, and connected in codimension one, then

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 1.$$

• Work of Peeva-McCullough shows this to be false in a major way.

• In a slightly different direction; is the following bound

• <u>Thm</u>: Let $X \subseteq \mathbb{P}^r_{\mathbb{C}}$ be a smooth irreducible variety. Setting $e = \operatorname{codim} X$; if X is cut out scheme theoretically by hypersurfaces of degree

$$d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$$

then X is $(d_1 + \cdots + d_e) - e + 1$ regular.

• This is a corollary of the following vanishing result.

• <u>Thm</u>: With X as above:

$$H^i(\mathbb{P}^r, \mathcal{I}_X^\wedge(k)) = 0 \qquad \forall i \geqslant 1 \text{ with } k \geqslant \wedge d_1 + \cdots + d_e - r.$$

• This is because we can also phrase regularity for sheaves as vanishing of sufficient twists of cohomology. Notice in some ways this itself is a measure of complexity.

• <u>Def</u>: Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. We say $\mathcal{F}$ is d-regular if

$$H^i(\mathbb{P}^r, \mathcal{F}(d-i)) = 0 \qquad \forall i \geqslant 1.$$

• Note we could have seen some of this hint of vanishing of cohomology by the relation to the Hilbert polynomial since H_X becomes a polynomial as soon as higher cohomology vanishes i.e. as soon as Serre vanishing kicks in.

• In many settings i.e. under fairly mild hypotheses the two definitions of regularity are related.

• <u>Thm</u>: • Let $\mathcal{F}$ be a sheaf (coherent) on $\mathbb{P}^r_{\mathbb{C}}$ then

$$\operatorname{reg} \Gamma_*(\mathcal{F}) = \operatorname{reg}(\mathcal{F}).$$

• Let M be a graded S-module and $\widetilde{M}$ the associated sheaf on $\mathbb{P}^n_{a}$, then.

$$\operatorname{reg}(M) \geqslant \operatorname{reg}(\widetilde{M}).$$

↑ Failure of equality is due to the failure of the natural map

$$M \longrightarrow \bigoplus_e H^0(\mathbb{P}^n_a, \widetilde{M}(e))$$

to be an isomorphism.

• The obstruction to this isomorphism is captured by local cohomology in the following LES

$$0 \longrightarrow H^0_m(M) \longrightarrow M \longrightarrow \bigoplus_e H^0(\mathbb{P}^n, \tilde{M}(e)) \longrightarrow H^1_m(M) \longrightarrow 0 .$$

• Note we (rarely) need to worry about this in the case of I_X since

$$I_X \cong \bigoplus_e H^0(\mathbb{P}^r, \mathcal{I}_X(e))$$

However issues arise when, we consider S_X if X is not projective normal. That is if the map $(*)$ below

$$0 \longrightarrow H^0(\mathbb{P}^r, \mathcal{I}_X(e)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(e)) \xrightarrow{(*)} H^0(\mathbb{P}^r, \mathcal{O}_X(e)) \longrightarrow H^1(X, \mathcal{I}_X(e)) \longrightarrow \cdots$$

is not a surjection, for example $H^1(X, \mathcal{I}_X(e))$ does not vanish for some e we will not be in this case.

• Ex: Consider $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ embeded by the non-complete linear system corresponding to the map:

$$[s,t] \longrightarrow [s^3 : s^2t, st^2].$$