- $\mathrm{Hilb}(\mathbb{P}^n)$ is the scheme "parameterizing" closed subschemes of $\mathbb{P}^n$.

- Last time there was a tizzy over "parameterize", and for good reason.... we need more than just a bijection

$$\mathrm{Hilb}(\mathbb{P}^n) \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{closed subschemes} \\ \text{of } \mathbb{P}^n \end{array} \right\}$$

we want this bijection to be "geometric" + "natural"....

- **Def**: Define a functor:

$$\mathcal{H}^n : \mathrm{Sch} \longrightarrow \mathrm{Set}$$

$$\mathcal{H}^n(T) := \left\{ (\mathcal{F}, q) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ is a sheaf on } \mathbb{P}^n_T \text{ s.t.} \\ \quad + \mathcal{F} \text{ is coherent} \\ \quad + \mathcal{F} \text{ is flat} \\ \quad + \mathcal{F} \text{ is proper support} \\ q : \mathcal{O}_{\mathbb{P}^n_T} \twoheadrightarrow \mathcal{F} \end{array} \right\} \Big/ \sim$$

where $(\mathcal{F}, q_1) \simeq (\mathcal{G}, q_2)$ if and only if there exists there exists an isomorphism $i : \mathcal{F} \longrightarrow \mathcal{G}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^n_T} \arrow[r, "q_1"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F} \arrow[d, "i", "\wr"'] \\ \mathcal{O}_{\mathbb{P}^n_T} \arrow[r, "q_2"] & \mathcal{G} \end{tikzcd}$$

- **Lemma/Exr**: $(\mathcal{F}, q_1) \simeq (\mathcal{G}, q_2)$ if and only if

$$\mathrm{Ker}(q_1) = \mathrm{Ker}(q_2).$$

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- $\mathcal{H}^n(T) = \left\{ \mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_T} \;\middle|\; \begin{array}{l} \mathcal{I} \text{ is coherent} \\ \mathcal{O}_{\mathbb{P}^n_T}/\mathcal{I} \text{ is flat \& proper} \end{array} \right\}$

$$= \left\{ X \subseteq \mathbb{P}^n_T \;\middle|\; \begin{array}{l} X \text{ is + closed} \\ \quad + \text{proper} \\ \quad + \text{flat} \\ \text{subscheme of } \mathbb{P}^n_T \end{array} \right\}.$$

- Note I introduced $\mathcal{H}^n$ in this - more complicated - way for two reasons: 1) because it generalizes to the Quot-functor by replacing $\mathcal{O}_{\mathbb{P}^n}$ with a different coherent sheaf, 2) it highlights one of the differences bantered about "moduli" v.s. "parameter" that being equivalence.....

- $\mathrm{Hilb}(\mathbb{P}^n)$ is the scheme s.t.

$$\mathcal{H}^n \cong \mathrm{Hom}_{sch}(-, \mathrm{Hilb}(\mathbb{P}^n)).$$

- this is the general frame work of a moduli problem....
  1) define a moduli functor
  2) pray to the heavens.
     ↳ often your functor is not representable....

- <u>Thm</u>: (Grothendieck '60): The functor $\mathcal{H}^n$ is representable, (i.e. $\mathrm{Hilb}(\mathbb{P}^n)$ is a thing.).

- As a general warning I am being sloppy about my Noetherianity conditions as well as restrictions on the base scheme, as well as ...

~~[illegible]~~ :

• **Thm:** Let $X \subseteq \mathbb{P}^n_T$ be a projective scheme then:
(18.6.1):

1) $\chi(X, \mathcal{O}_X(m)) = \sum (-1)^i \dim H^i(X, \mathcal{O}_X(m))$

$$= \dim H^0(X, \mathcal{O}_X(m)) \quad \text{for } m \gg 0.$$

2) $\chi(X, \mathcal{O}_X(m))$ · is a polynomial P_X of degree $\dim X$ for $m \gg 0$.

~~Thm: Euler characteristic [illegible] families.~~

• **Ex:** Let $X \subseteq \mathbb{P}^3$ be the twisted cubic, i.e. the image of the map

$$\begin{aligned} \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3 \\ [s:t] &\longmapsto [s^3 : s^2t : t^2s : s^3]. \end{aligned}$$

Then we need to compute $H^0(X, \mathcal{O}_X(m))$ for $m \gg 0$. But R-R says

$$\begin{aligned} \dim H^0(X, \mathcal{O}_X(m)) &= \deg(X) \cdot m + 1 - 0 \\ &= 3m + 1. \end{aligned}$$

• **Def:** Let $\Phi \in \mathbb{Q}[m]$, then $\mathrm{Hilb}(\mathbb{P}^n, \Phi)$ "parameterizes" closed subschemes of $\mathbb{P}^n$, with Hilbert polynomial Φ.

• **Thm:** The functor $\mathcal{H}^n_\Phi$ is representable by a projective scheme $\mathrm{Hilb}(\mathbb{P}^n, \Phi)$, and moreover,

$$\mathrm{Hilb}(\mathbb{P}^n) = \coprod_{\Phi} \mathrm{Hilb}(\mathbb{P}^n, \Phi).$$

- I want to spend the remainder of my talk focusing the following questions:

  1) How can we "construct" $\mathrm{Hilb}(\mathbb{P}^n, \Phi)$?

  2) When is $\mathrm{Hilb}(\mathbb{P}^n, \Phi) \neq \emptyset$?

the key - or a key - for both is the following result.

- Thm: (Gotzmann '73): Suppose $\Phi(m) \in \mathbb{Q}[m]$ s.t.

$$\Phi(m) = \sum_{k=1}^{r} \binom{m + a_k - k + 1}{a_k}$$

where $a_1 \geq a_2 \geq \cdots\cdots \geq a_r \geq 0$. There exists a closed subscheme $X \subseteq \mathbb{P}^n$ s.t. $P_X(m) = \Phi(m)$. Moreover, ~~in~~ for any such subscheme:

  1) $\mathcal{I}_X(r)$ is globally generated,

  2) $H^i(X, \mathcal{O}_X(j)) = 0 \quad \forall j \geq r$ and $i > 0$,

  3) r is the Castelnuovo-Mumford regularity of X.

- This is quite a mouthfull, so let's digest it bit by bit..

- Part of this was proven - in a slightly different context - by Macaulay '26.

- Ex: Notice if we let $r = 4$ and take $1 \geq 1 \geq 1 \geq 0$ then

$$\sum_{k=1}^{4} \binom{m + a_k - k + 1}{a_k} = \binom{m+1}{1} + \binom{m}{1} + \binom{m-1}{1} + \binom{m-2}{0} = 3m+1.$$

  Hence, the Theorem claims $\mathrm{Hilb}(\mathbb{P}^n, 3m+1) \neq \emptyset$, which makes sense for $n \geq 3$ since the twisted cubic is in here.

- The first clause in the theorem exactly answers question #2.

- <u>Cor</u>: Let $\Phi \in \mathbb{Q}[m]$. The Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n, \Phi) \neq \emptyset \iff$ there exists natural numbers $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ s.t.

$$\Phi(m) = \sum_{k=1}^{r} \binom{m + a_k - k + 1}{a_k}.$$

$\blacksquare$

- Part #1 tells us that X can be cut out by equations of degree r.

- Part #2 says the Hilbert function of X is polynomial for $m \geq r$.

- Part #3 is... we'll come back to it... first let us see how parts 1 + 2 are useful to use.

- To show $\mathcal{H}^n_\Phi$ is representable we want to work in two steps
  1) Build something that "parameterizes" $\mathcal{H}^n_\Phi$.
  2) Show the thing we built represents $\mathcal{H}^n_\Phi$.

- For step #1 we want to put this somewhere...

- $$\left\{ \begin{array}{c} X \subseteq \mathbb{P}^n \text{ closed subscheme} \\ P_X(m) = \Phi(m) \end{array} \right\} \hookrightarrow ??$$

- By part 1 of the previous theorem there is a canonical vector space associated to such an X... $H^0(X, \mathcal{O}_X(r))$.

  Because this uniquely determines $\mathcal{I}_X$.

• Part 2 tells us that for any such X:

$$\dim H^0(X, \mathcal{O}_X(r)) = \Phi(r)!$$

So every such X can be recorded as a vector space of the same dimension.

•
$$\left\{ \begin{array}{c} X \subseteq \mathbb{P}^n \\ \text{closed subscheme} \\ P_X(m) = \Phi(m) \end{array} \right\} \hookrightarrow Gr(\Phi(r), P_{\mathbb{P}^n}(r)).$$

$$X \longmapsto H^0(X, \mathcal{O}_X(r)).$$

• Ex: For $\Phi(m) = 3m+1$ we see

$$\mathrm{Hilb}(\mathbb{P}^3, 3m+1) \subseteq Gr(13, 35)$$

15 dim

286 dimensional

- I now want to come back to this Theorem of Gotzmann and use the remaining time to sketch its proof.

- There are sort of 2 steps
  1) Show 3) ⟹ 1) & 2)
  2) Show the relation between B) and the expression of P_X in its Gotzmann expansion.

- <u>Def</u>: Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. The Castelnuovo-Mumford is the smallest r s.t.

$$H^i(\mathbb{P}^n, \mathcal{F}(r-i)) = 0.$$

- We can somewhat think about this as being the smallest r for which Serre Vanishing kicks in.

- <u>Thm</u>: (Mumford): Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, with $r = reg(\mathcal{F})$. Then $\forall k \geq 0$:

  1) $\mathcal{F}(r+k)$ is generated by global sections.

  2) The map

$$H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}(r+k))$$

  is surjective.

• Rem : "Constructing $\mathcal{M}_g$"

• Let C be a smooth curve of genus g.

↳ K_C has degree $2g-2$.

• Now if $g \geq 2$ then

$$\deg(2gK_C) = 2g(2g-2) = 4g^2 - 4g > 2g+1$$

$\Rightarrow$ $2gK_C$ is very ample.

$$\Rightarrow C \hookrightarrow \mathbb{P}^{2g(2g-2)-g}.$$

• Prop: two curves C_1 & C_2 embedded into $\mathbb{P}^{2g(2g-2)-g}$ by $2gK_{C_1}$ & $2gK_{C_2}$ are isomorphic $\Leftrightarrow$ $\exists$ an automorphism of $\mathbb{P}^{2g(2g-2)-g}$ realizing this.

• Every such C has hilbert polynomial

$$(4gm-1)(g-1)$$

...