A Probabilistic Approach to Noether Normalization

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Definitions

• If $X \subset \mathbb{P}^r$ is a closed subscheme, $f_0, \ldots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ are a **(partial) system of parameters** on X if

$$\dim \mathbb{V}(f_0, \ldots, f_k) \cap X = \dim X - (k+1).$$

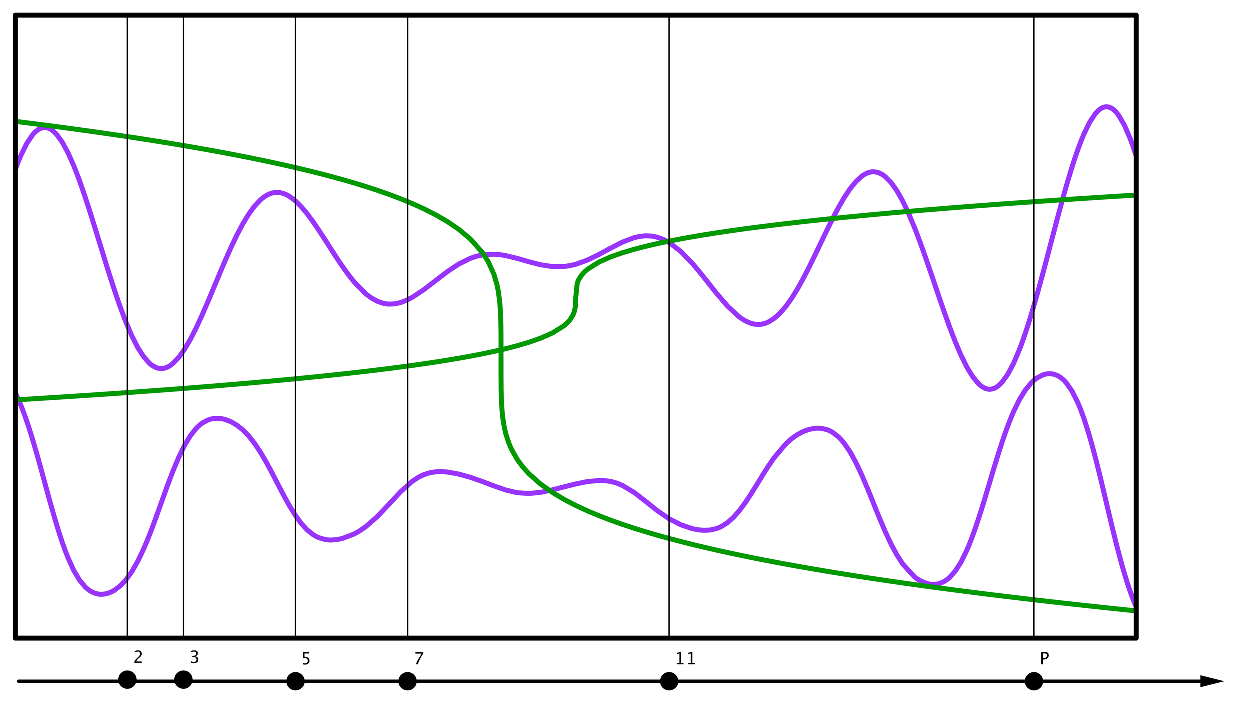

Figure 1: A non-example of fiberwise parameters for $\mathbb{P}^1_{\mathbb{Z}}$.

• If $X \subset \mathbb{P}^r_B$ is a closed subscheme a **fiberwise system of parameters** on X is a collection $f_0, \ldots, f_k \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$, which restrict to parameters on each fiber of X over B.

Example - Points in $\mathbb{P}^1_{\mathbb{Z}}$

• A fiberwise parameter for a set of points in $\mathbb{P}^1_{\mathbb{Z}}$ is a homogenous polynomial f such that $f = \pm 1$ at every point .

• For one point the **Euclidean algorithm** shows a linear parameter always exists.

• The points $[4:5]$ and $[1:8]$ have no fiberwise parameter f with $\deg f < 9$. If $\deg(f) = 9$ one can choose:

$$\begin{aligned} f(x,y) = {} & 5903168696662393697883364865773326275 91x^9 - y^9 \\ & - 5480582078642000202457777426875531340 41x^8y \\ & + 6089535642935555780508641585417257044 9x^7y^2 \\ & - 2012317228668540454407559232846642 55x^6y^3 - 99438623xy^8 \\ & - 30126091505240316687953753225193x^5y^4 - 1977608122820402x^2y^7 \\ & + 620958323405203261138096698x^4y^5 - 123027396842281772 5004x^3y^6. \end{aligned}$$

The Main Result

Theorem 1. *Let $X \subset \mathbb{P}^r_{\mathbb{Z}}$ be a closed subscheme. If every fiber of X over $\mathbb{Z}$ has dimension n then there exists a linear series in $\mathcal{O}_X(d)$, for some $d > 0$, inducing a finite morphism:*

$$\phi : X \to \mathbb{P}^n_{\mathbb{Z}}.$$

• The existence of such ϕ is subtle, even potentially unexpected, as the sections $f_0, \ldots, f_n \in H^0(X, \mathcal{O}_X(d))$ defining such a finite map have **density zero**; even if we let $d \to \infty$.

• Despite not being able to construct ϕ by randomly choosing $f_0, \ldots, f_n$, our proof of Theorem 1 relies on being able to generate partial systems of parameters probabilistically.

Theorem 2. *Let $X \subseteq \mathbb{P}^r_B$ be a closed subscheme of dimension n. The "probability" that random chosen forms $f_0, \ldots, f_k$ of degree d are a (fiberwise) system of parameters as $d \to \infty$ is:*

	$B = \mathbb{F}_q$	$B = \mathbb{Z}$
$k < n$	1	1
$k = n$	$\zeta_X(n+1)^{-1}$	0

• The case when $B = \mathbb{F}_q$ and $k = n$ is originally due to Bucur and Kedlaya.

• **Step #1 - (Probabilistic):** By Theorem 2 we can find fiberwise parameters $f_0, \ldots, f_{n-1}$ on X simply by picking high degree forms at random, and so reduce Theorem 1 to

$$X' = X \cap \mathbb{V}(f_0, \ldots, f_{n-1}).$$

•**Step #2 - (Arithmetic):** Such X' is essentially a union of orders in number fields. Using the fact an order has a finite Picard group we show that $\mathrm{Pic}(X')$ is finite. This finiteness allows us to construct the last parameter.

• Similar arguments hold if one replaces $\mathbb{Z}$ with $\mathbb{F}_q[t]$.

Distribution of Parameters

• The probabilities over $\mathbb{F}_q$ appearing in Theorem 2 are computed via an adaptation of Poonen's sieving argument.

• When $k < n$, this results in a description of the distribution of parameters in terms of a **higher dimensional analogue** of the Hasse-Weil zeta function.

Proposition 1. *Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a closed subscheme of dimension n. Then there is an explicit function $f(k,n,d)$ such that*

$$\mathrm{Prob}\begin{pmatrix} (f_0, \ldots, f_k) \\ \textit{of degree } d \textit{ are } \textbf{\textit{not}} \\ \textit{parameters on } X \end{pmatrix} \sim \#\left\{ \begin{matrix} (n-k)\textit{-planes} \\ L \subset X \end{matrix} \right\} \cdot f.$$

Example - Parameters on Surfaces

• In $\mathbb{P}^3$ the surface $\mathbb{V}(xyz)$ contains substantially more lines than $\mathbb{V}(x^2+y^2+z^2)$. Selecting 10^6 random pairs (f_0, f_1) of degree two, the proportion that *failed* to be parameters were:

	$\mathbb{V}(xyz)$	$\mathbb{V}(x^2+y^2+z^2)$
$\mathbb{F}_2$	.2638	.1179
$\mathbb{F}_3$	.0552	.0059
$\mathbb{F}_5$	.0063	.0004

• Over $\mathbb{F}_4$ the Fermat cubic surface

$$X = \mathbb{V}(x^3+y^3+z^3+w^3) \subset \mathbb{P}^3_{\mathbb{F}_4},$$

has 27 $\mathbb{F}_4$-lines. Proposition 1 implies $\approx$ 0.66% of pairs (f_0, f_1) of degree two should not be parameters. Simulating 10^5 such pairs we found 0.62% failed to be parameters.

Acknowledgments

The author was partially supported by the NSF GRFP under grant No. DGE-1256259; as well as The Graduate School and the Office of the Vice Chancellor for Research and Graduate Education at the University of Wisconsin with funding from the Wisconsin Alumni Research Foundation.