Asymptotic Syzygies in the Semi-Ample Setting

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Background

- Asymptotic syzygies is the study of the graded Betti numbers of a variety as the positivity of the embedding grows.
- Let X be a smooth projective variety and $\{L_d\}$ be a sequence of (very ample) line bundles

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}.$$

- To this we associate the homogenous coordinate ring of X:

$$S(X, L_d) = \bigoplus_{k \in \mathbb{Z}} H^0(X, kL_d).$$

- The **minimal graded free resolution** of $S(X, L_d)$ as a graded S-module has the form:

$$S(X, L_d) \twoheadleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_{r_d} \longleftarrow 0,$$

and the **graded Betti numbers** of X are

$$\beta_{p,q}(X, d) = \#\left\{\begin{matrix}\text{minimal generators} \\ \text{of } F_p \text{ of degree } q\end{matrix}\right\} = \begin{matrix}\text{number of syzygies} \\ \text{of degree } q \text{ at step } p\end{matrix}.$$

- We form the **graded Betti table** of X, denoted $\beta(X, L_d)$, by placing $\beta_{p,p+q}(X, L_d)$ in the (p,q)-th spot.

Example - Seven Points in $\mathbb{P}^3$

- If $X \subset \mathbb{P}^3$ is 7 points in general linear position. There are two possible minimal free resolutions of:

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-2)^3 \\ \oplus \\ S(-3) \end{matrix} \longleftarrow S(-4)^6 \longleftarrow S(-5)^3$$

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-2)^3 \\ \oplus \\ S(-3)^3 \end{matrix} \longleftarrow \begin{matrix} S(-3)^2 \\ \oplus \\ S(-4)^6 \end{matrix} \longleftarrow S(-5)^3$$

- These resolutions correspond to the following Betti tables:

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	1	6	3

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

Asymptotic Non-vanishing

- We are interested in how many of the graded Betti numbers in each row are non-zero, and so define

$$\rho_q(X, L_d) := \frac{\#\left\{p \in \mathbb{N} \,\middle|\, \beta_{p,p+q}(X, L_d) \neq 0\right\}}{r_d} = \begin{matrix}\text{percent of non-zero entries} \\ \text{in the } q\text{-row of } \beta(X, L_d)\end{matrix}.$$

- Notice that $\rho_q(X, L_d)$ is between 0 and 1.

Goal. Understand the asymptotic behavior of $\rho_q(X, L_d)$ under varying positivity conditions on the line bundles $\{L_d\}$.

- For curves asymptotically the syzygies occur in the simplest possible way.

Theorem (Green)**.** If $\dim X = 1$ and $\deg L_d = d$ then

$$\lim_{d \to \infty} \rho_2(X, L_d) = 0.$$

- For higher dimensional varieties the syzygies behave in a more complicated fashion.

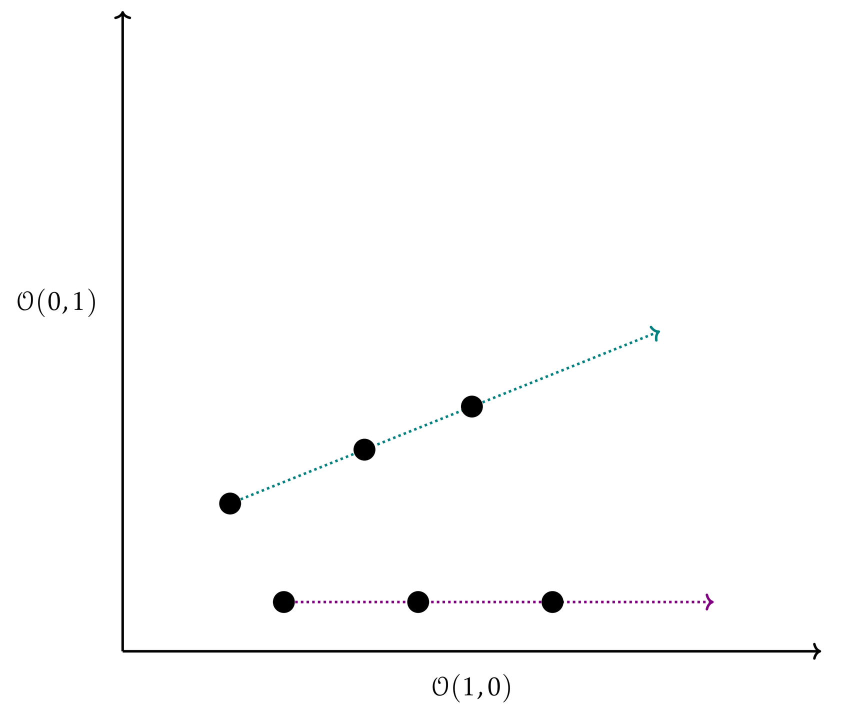
Theorem (Ein & Lazarsfeld)**.** Let $\dim X \geq 2$ and fix an index $1 \leq q \leq n$. If $L_{d+1} - L_d$ is constant and ample then

$$\lim_{d \to \infty} \rho_q(X, L_d) = 1.$$

Theorem (Juliette Bruce)**.** Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. There exists constants $C_{i,j}$ and $D_{i,j}$ such that

$$\rho_q(X, \mathcal{O}(d_1, d_2)) \geq 1 - \sum_{\substack{i+j=q \\ 0 \leq i \leq n \\ 0 \leq j \leq m}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n-i} d_2^{m-j}}\right) - O\left(\begin{matrix}\text{lower ord.} \\ \text{terms}\end{matrix}\right).$$

- My results generalizes Ein & Lazarsfeld's by weakening the positivity condition to include semi-ample bundles.
- A line bundle L is **semi-ample** if there exists a $k \gg 0$ such that $|kL|$ defines a regular map to $\mathbb{P}^N$ for some N.



- The asymptotic behavior in my theorem is dependent, in a nuanced way, on the relationship between d_1 and d_2.

Corollary (Juliette Bruce)**.** Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and fix an index $1 \leq q \leq n+m$. If d_2 is fixed then

$$\lim_{d_1 \to \infty} \rho_q(X, \mathcal{O}(d_1, d_2)) \geq 1 - \frac{m!}{(m-q)! d_2^q} - \frac{m!}{(n-q)! d_2^{n+m-q}}.$$

Example - $\mathbb{P}^1 \times \mathbb{P}^5$

- When $q = 2$ my theorem says that the percent of non-zero syzygies is given by:

$$\rho_2\left(\mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(d_1, d_2)\right) \geq 1 - \frac{20}{d_2^2} - \frac{60}{d_1 d_2^3} - \frac{5}{d_1 d_2} - \frac{120}{d_2^4} - O\left(\begin{matrix}\text{lower} \\ \text{ord.}\end{matrix}\right).$$

- In particular, if d_2 is fixed then we get

$$\lim_{d_1 \to \infty} \rho_2\left(\mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(d_1, d_2)\right) \geq 1 - \frac{20}{d_2^2} - \frac{120}{d_2^4}.$$

- We do not believe this limit approaches one.

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