Teoriforelesning 13
INTEGRASJON II

Nøkkelbegreper
- Substitusjon
- Delvis integresjon
- Delbrøksoppspalting
- Uegentlige integreler
- Andvendelser av integralet (buelengde og tverrsnittmetoden)

Substitusjon (7.4)

Kjerneregelen: $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$

Dvs: $\int f'(g(x)) \underbrace{g'(x)dx} = f(g(x)) + C$

$\int f'(u)du = f(u) + C$

$= f(g(x)) + C$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$\Downarrow$

"$du = g'(x)dx$"

Eks

i) $\int 2(1+2x)^5 dx$

ii) $\int_0^1 2(1+2x)^5 dx$

i) $u = 1+2x$

$\frac{du}{dx} = 2 \rightarrow du = 2dx$

$\int 2(1+2x)^5 dx = \int u^5 du$

$= \frac{1}{6}u^6 + C = \underline{\frac{1}{6}(1+2x)^6 + C}$

ii) $\int_0^1 2(1+2x)^5 dx$

$= \int_1^3 u^5 du = \left[\frac{1}{6}u^6\right]_1^3$

$= \underline{\frac{1}{6}(3^6 - 1)}$

Eks

$\int 2x^3 (\underbrace{x^2+1}_{u})^{1/3} dx = \int \underbrace{x^2}_{u-1} (\underbrace{x^2+1}_{u})^{\frac{1}{3}} \underbrace{2x\,dx}_{du}$

$u = x^2+1$

$du = 2x\,dx$

$u - 1 = x^2$

$= \int (u-1)u^{1/3} du = \int u^{4/3} - u^{1/3} du$

$= \frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} + C$

$= \underline{\frac{3}{7}(x^2+1)^{7/3} - \frac{3}{4}(x^2+1)^{4/3} + C}$

Eks

$$\int \sin^3 x \cos^5 x \, dx = \int \sin x (1-\cos^2 x) \cos^5 x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2) u^5 \, du = -\frac{1}{6}u^6 + \frac{1}{8}u^8 + C$$

$$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C$$

$$\int a^x dx = \int (e^{\ln a})^x dx$$

(fra regel 3)

$$= \int e^{\overbrace{\ln a \cdot x}^{u}} dx = \ldots = \frac{1}{\ln a} e^{\ln a \cdot x} + C$$

$$= \frac{1}{\ln a} a^x + C$$

Delvis integrasjon (7.5)

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

$$\underbrace{\int [u(x)v(x)]' dx}_{= u(x)v(x)} = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

Eks

$$\int_1^e \underbrace{x^2}_{u'} \underbrace{\ln x}_{v} dx = \left[\frac{1}{3}x^3 \ln x\right]_1^e - \int_1^e \frac{1}{3}x^3 \cdot \frac{1}{x} dx$$

$u' = x^2$, $u = \frac{1}{3}x^3$, $v = \ln x$, $v' = \frac{1}{x}$

$$= \frac{1}{3}e^3 - \frac{1}{3}\int_1^e x^2 dx$$

$$= \frac{1}{3}e^3 - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^e$$

$$= \frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1)$$

$$= \frac{2}{9}e^3 + \frac{1}{9}$$

Eks

$$I = \int \underbrace{e^x}_{u'} \underbrace{\sin x}_{v} \, dx = e^x \sin x - \int \underbrace{e^x}_{u'} \underbrace{\cos x}_{v} \, dx$$

$u = e^x \quad v' = \cos x$

$u' = e^x, \; u = e^x$
$v = \cos x, \; v' = \sin x$

$$= e^x \sin x - \left(e^x \cos x + \underbrace{\int e^x \sin x \, dx}_{I \, (+C)}\right)$$

$$I = e^x \sin x - e^x \cos x - I + C$$

$$2I = e^x \sin x - e^x \cos x + C$$

$$I = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C$$

Delbrøksoppspalting (7.10)

$$\int \frac{P(x)}{Q(x)} \, dx$$ Polynomer

Eks

$$\int \frac{x^4+1}{x^3+x^2+x+1} \, dx$$

Steg 1: Bruk polynomdivisjon hvis grad$(P) \geq$ grad(Q).

Eks

$$\begin{array}{l} x^4+1 : x^3+x^2+x+1 = x-1 + \dfrac{2}{x^3+x^2+x+1} \\ -\,(x^4+x^3+x) \\ \hline \quad -x^3-x^2-x+1 \\ -\,(-x^3-x^2-x-1) = 2 \end{array}$$

$$\int \frac{x^4+1}{x^3+x^2+x+1} \, dx = \int x-1+\frac{2}{x^3+x^2+x+1} \, dx$$

Steg 2: Faktoriser $Q(x)$ i et produkt av enklere polynomer

(dvs. i et produkt med faktorer av typen $(x-r)$, (ax^2+bx+c))

Eks

$$x^3+x^2+x+1 = (x+1)(x^2+1)$$

Steg 3: Bruk delbrøksoppspalting

$$\frac{2}{x^3+x^2+x+1} = \frac{2}{(x+1)(x^2+1)} = \frac{\overset{1}{A}}{(x+1)} + \frac{\overset{-1}{B}x+\overset{4}{C}}{(x^2+1)}$$

$$= \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$= \frac{x^2\overset{0}{\widetilde{(A+B)}}+x\overset{0}{\widetilde{(B+C)}}+\overset{2}{\widetilde{(A+C)}}}{(x+1)(x^2+1)} = \frac{2}{(x+1)(x^2+1)}$$

$A+B=0 \to A=-B$
$B+C=0 \to B=-C$
$A+C=2 \to A+C = A-(-C)$
$= A-B = 2A = 2$

$\Rightarrow A = C = 1,\ B = -1$

$$I = \int \frac{x^4+1}{x^3+x^2+x+1}\,dx = \int (x-1) + \frac{1}{x+1} + \frac{(-x+1)}{x+1}\,dx$$

$$= \int (x-1) + \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= \int (x-1)\,dx = \frac{1}{2}x^2 - x + C$$

$$= \frac{1}{x+1}\,dx = \ln|x+1| + C$$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

$$\underline{I = \frac{1}{2}x^2 - x + \ln|x+1| - \frac{1}{2}\ln|x^2+1| + \arctan x + C}$$

Merk

i) $\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}\,dx$

$= \int -\frac{1}{2(x+1)} + \frac{1}{2(x-1)}\,dx$

$= \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{x+1}\,dx$

Kan stoppe her

$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= \frac{1}{2}\ln\frac{|x-1|}{|x+1|} + C = \ln\left(\frac{|x-1|}{|x+1|}\right)^{\frac{1}{2}} + C$

$= \ln\sqrt{\frac{|x-1|}{|x+1|}} + C$

$\frac{1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$

-1

$A+B=0 \quad A=-B$

$-A+B=1$

$-2A=1 \Rightarrow A=-\frac{1}{2} \quad B=\frac{1}{2}$

ii) $\frac{x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$ Kommer ikke på eksamen

$(x-3)$

$(x^2+1)^3$

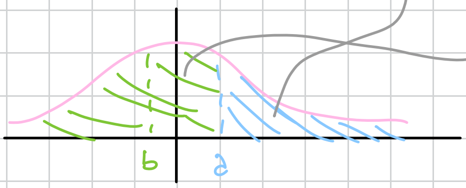
$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(\_\_)}$

Temaforelesning 13
INTEGRASJON II

Uegentlige integraler (7.11)

To typer

i) Ubegrenset område: $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$



$$\int_a^{\infty} f(x)dx = \lim_{M\to\infty} \int_a^{M} f(x)dx$$

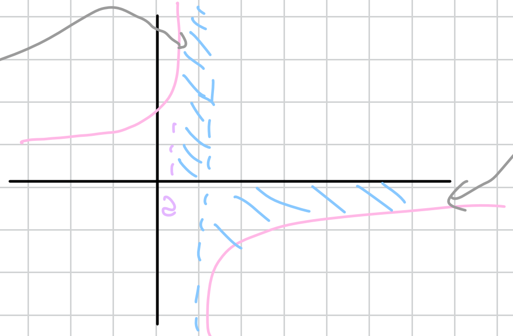
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

ii) Ubegrenset funksjon: $\int_a^{a_0} f(x)dx$, $\int_{a_0}^{b} f(x)dx$,

der vi antar at f har en vertikal asymptote i $x = a$

$$\int_a^{a_0} f(x)dx = \lim_{t\to a_0^-} \int_a^{t} f(x)dx$$

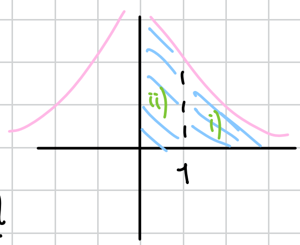
$$\int_{a_0}^{b} f(x)dx = \lim_{t\to a_0^+} \int_t^{b} f(x)dx$$



Eks

i) $\int_1^{\infty} \frac{1}{x^2}dx$ konv/div?

ii) $\int_0^{1} \frac{1}{x^2}dx$ konv/div?



i) $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{x^2}dx = \lim_{M\to\infty} \left[-\frac{1}{x}\right]_1^M = \lim_{M\to\infty} \left(-\frac{1}{M}+1\right) = \underline{1} \rightarrow \underline{\underline{\text{konv}}}$

ii) $\int_0^{1} \frac{1}{x^2}dx = \lim_{t\to 0^+} \int_0^{1} \frac{1}{x^2}dx = \lim_{t\to 0^+} \left[-\frac{1}{x}\right]_t^1 = \lim_{t\to 0^+} \left(-1+\frac{1}{t}\right) \rightarrow \infty$

$\underline{\underline{\text{div}}}$

Eks (p-integralet)

Vi ser på $\int_1^\infty \frac{1}{x^p}dx$ og $\int_0^1 \frac{1}{x^p}dx$

for generell p.

p = 1:

$$\int \frac{1}{x}dx = \ln x + C$$

$$\int_1^\infty \frac{1}{x}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x}dx = \lim_{M\to\infty}\left[\ln x\right]_1^M = \lim_{M\to\infty}\ln M \to \infty \quad \underline{\text{div}}$$

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \lim_{t\to 0^+}(-\ln t) \to \infty \quad \underline{\text{div}}$$

For $p \neq 1$:

$$\int \frac{1}{x^p}dx = \frac{1}{(-p+1)}x^{-p+1} + C$$

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{M\to\infty}\int_1^M \frac{1}{x^p}dx = \lim_{M\to\infty}\left[\frac{1}{-p+1}x^{-p+1}\right]_1^M = \lim_{M\to\infty}\frac{1}{-p+1}\left(M^{-p+1}-1\right)$$

$$= \lim_{M\to\infty}\frac{1}{1-p}\left[\frac{1}{M^{p-1}}-1\right] = \begin{cases}\frac{1}{p-1} & \text{for } p>1 \to \text{konv}\\ \infty & \text{for } p<1 \to \text{div}\end{cases}$$

$$\int_0^1 \frac{1}{x^p}dx \begin{cases}\to \text{div for } p>1\\ \to \text{konv for } p<1\end{cases}$$

Eks

$I = \int_1^\infty \frac{1}{\sqrt{x^4+x^2}}dx$ konv/div?

$$0 \leq \frac{1}{\sqrt{x^4+x^2}} \leq \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$$

Dermed:

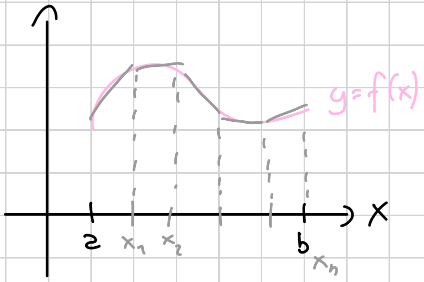
$$\int_1^\infty \frac{1}{\sqrt{x^4+x^2}} \leq \int_1^\infty \frac{1}{x^2}dx \quad \text{konv,}$$

Så I konvergerer også

Mindre enn konv → konv
Større enn div → div
Mindre enn div → sier ingenting

Anvendelser av integralet

Buelengde



Partisjon: $\{x_0, x_1, x_2, \dots, x_n\}$, $x_0 = a$, $x_n = b$

Lengden av linjestykket fra $(x_{i-1}, f(x_{i-1}))$ til $(x_i, f(x_i))$ er

$$L_i = \sqrt{\underbrace{(x_i - x_{i-1})}_{\Delta x_i}{}^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \Delta x_i \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}$$

$$\overset{?}{=} \Delta x_i \sqrt{1 + f'(x_i^*)^2}$$

Totalt: $L \approx \sum_{i=1}^{n} L_i$

Ved MVS finnes en $x_i^* \in [x_{i-1}, x_i]$ slik at

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Dermed:

$$L = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \Delta x_i \sqrt{1 + [f'(x_i^*)]^2}$$

Lar nå $n \to \infty$ og får

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

Eks Finn buelengden til $f(x) = 1 + 2x^{3/2}$ fra $x = 0$ til $x = 1$

$$f(x) = 1 + 2x^{3/2}$$

$$[f'(x)]^2 = \left[2 \cdot \frac{3}{2} x^{1/2}\right]^2$$

$$= 9x$$

$$L = \int_0^1 \sqrt{1 + 9x}\, dx = \frac{1}{9}\left[\frac{2}{3}(1 + 9x)^{3/2}\right]_0^1$$

$$= \frac{2}{27}(10\sqrt{10} - 1) \approx 2{,}27$$

Volum ved tverrsnittmetoden

Deler legemet i Δx tykke skiver.

Volumet av skiven i intervallet $[x_{i-1}, x_i]$ er $V_i \approx A(x_i^*)\Delta x_i$

Totalt volum:

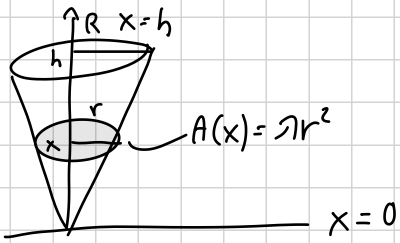
Riemannsum

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} A(x_i^*)\Delta x_i$$

Hvis A er gitt som funksjon av x får vi:

$$\boxed{V = \int_a^b A(x)\,dx}$$

Eks (7.14.1)



Skive ved høyden x: Hva er r?

$$\frac{x}{h} = \frac{r}{R} \rightarrow r = \frac{Rx}{h}$$

$$A(x) = \pi r^2 = \pi\left(\frac{Rx}{h}\right)^2$$

$$V = \int_0^h A(x)\,dx = \frac{\pi R^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi R^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \frac{\pi R^2}{h^2}\frac{1}{3}h^3 = \underline{\frac{1}{3}\pi R^2 h}$$