Thesis

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Chapter 1

Maxwell's Equations

1.1 Minkowski space

The Monkowski space is $\mathbb{R}^4$, like a vector space, with the *Lorentz inner product*, which is the bilinear form defined by

$$\Lambda(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3,$$

for $x = (x^0,\dots,x^3), y = (y^0,\dots,y^3) \in \mathbb{R}^4$. Physically, x^0 denotes the time coordinate, and $x^1,\dots,x^3$ denote the space ones. It is common in the physics literature to denote x by x^μ, and it is called *a contravariant vector.*

Remark. We shall use the *Heaviside Lorentz units* where

$$\epsilon_0 = \mu_0 = c = 1,$$

which are the electric constant, magnetic constant and speed light, respectively.

The Lorentz form is defined by the matrix

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1). \tag{1.1}$$

We employ the *Einsteins summation convention*: In any product of vector and tensor in which an index appears once as a subscript and once as a superscript, that index is to be summed from 0 to 3. For example

$$\Lambda(x,y) = \eta_{\alpha\beta}x^\alpha y^\beta. \tag{1.2}$$

Another convention we shall adopt is using the matrix g to raise or lower indices:

$$x_\mu = g^{\mu\nu}x^\mu, p^\mu = g_{\mu\nu}p_\nu.$$

We said x_μ is *a covariant vector.*

Strictly speking, vector whose component are denoted by subscripts should be constructed as elements of the dual space $(\mathbb{R}^4)^*$, and the map $x^\alpha \mapsto x_\alpha$ is the isomorphism of $\mathbb{R}^4$ with $(\mathbb{R}^4)^*$ induced by the Lorentz form. Practically, the effect is to change the sign of the space components. Then, formula (1.2) can be written as

$$\Lambda(x,y) = x^\alpha y_\alpha = x_\alpha y^\alpha.$$

Remark. The Lorentz inner product of a vector x with itself is denoted by x^2 and its Euclidean norm by $|x|$.

Vectors x such tha $x^2 > 0, x^2 = 0$ or $x^2 < 0$ are called *timelike, lightlike or spacelike,* respectively.

The notations for derivatives on $\mathbb{R}^4$ is

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha},$$

so that, with respect to space-time coordinates $\mathbf{x}, t$

$$\partial_\alpha = (\partial_0, \ldots, \partial_3) = (\partial_t, \nabla_\mathbf{x}), \partial^\beta = (\partial^0, \ldots, \partial^3) = (\partial_t, -\nabla_\mathbf{x}), \tag{1.3}$$

and ∂^2 is the wave operator or d'Alembertian:

$$\partial^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = \partial_t^2 - \nabla_\mathbf{x}^2.$$

1.2 Covariant form of Maxwell's equations

Maxwell electromagnetic theory provides an important example of a gauge theory. The Maxwell equations in a classical form are given by:

$$\nabla \cdot B = 0 \tag{1.4}$$
$$\nabla \times E + \partial B/\partial t = 0 \tag{1.5}$$
$$\nabla \cdot E = \rho \tag{1.6}$$
$$\nabla \times B - \partial E/\partial t = J \tag{1.7}$$

where the electric field E and the magnetic field B are time dependent vector fields on some subset of $\mathbb{R}^3$ and ρ and J are the charges and current densities respectively.

From (1.6) and (1.7), we obtain *the continuty equation*

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0. \tag{1.8}$$

From (1.4), there is a vector function A, called the vector potential, such that

$$B = \nabla \times A. \tag{1.9}$$

Subtituting into (1.5), we obtain

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0, \tag{1.10}$$

and therefore, there is a scalar potential Φ, such that

$$E = -\nabla\Phi - \frac{\partial A}{\partial t}. \tag{1.11}$$

Maxwell's equations can be written in *covariant form* by introducing

$$A^\alpha = (\Phi, A) = (A^0, \dots, A^3) \tag{1.12}$$

$$J^\alpha = (\rho, J) = (J^0, \dots, J^3). \tag{1.13}$$

Using this notation, we define

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \tag{1.14}$$

and from (1.9) and (1.10), we obtain the *contravariant field tensor*

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}. \tag{1.15}$$

The components of the fields in equations (1.9) (1.11) can be identified as

$$E_i = F^{0i} \tag{1.16}$$

$$B_i = \frac{1}{2}\epsilon^{ijk} F^{jk}, \; i, j, k = 1, 2, 3, \tag{1.17}$$

where

$$\epsilon^{ijk} = \begin{cases} 1 & (i,j,k) = (1,2,3), (3,1,2), (2,3,1), \\ -1 & (i,j,k) = (1,3,2), (3,2,1), (2,1,3), \\ 0 & otherwise \end{cases}$$

is the *Levi-Civita symbol.*

For the *covariant field tensor* defined by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{1.18}$$

we obtain

$$F_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\lambda}F^{\gamma\lambda} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}. \tag{1.19}$$

The homogeneous Maxwell's equations (1.4) and (1.5) correspond to the Jacobi identities:

$$\partial^\gamma F^{\alpha\beta} + \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} = 0, \tag{1.20}$$

where $\alpha, \beta, \gamma \in \{0, \dots, 3\}$.

For example, if $\gamma = 1, \alpha = 2, \beta = 3$ we have from (1.15) and (1.20),

$$\partial^1 F^{32} + \partial^2 F^{13} + \partial^3 F^{21} = -(\partial^1 B_1 + \dots + \partial^3 B_3) = -(\partial_1 B_1 + \dots + \partial_3 B_3) = 0, \tag{1.21}$$

which indeed corresponds to (1.4).

The inhomogeneous Maxwell's equations (1.7) and (1.6) can be written as

$$\partial_\beta F^{\alpha\beta} = J^\alpha. \tag{1.22}$$

For example, if $\alpha = 0$, we have from (1.15) and (1.22)

$$\partial_0 F^{00} + \dots + \partial_3 F^{03} = \partial_1 E_1 + \dots + \partial_3 E_3 = \rho, \tag{1.23}$$

which indeed corresponds to (1.7).

Remark. Maxwell's equations have been reduced to (1.20) and (1.22). The continuity equation in covariant form can be obtained from (1.22) by operating ∂_μ on both sides. Then

$$\partial_\alpha J^\alpha = \partial_\alpha \partial_\beta F^{\alpha\beta} = 0, \tag{1.24}$$

since $\partial_\alpha \partial_\beta$ is symmetric in α and β while $F^{\alpha\beta}$ is antisimmetric in α and β. The expression (1.24) is *the conservation of electric charge* whose underlying symmetry is gauge invariance.

1.3 Gauge transformation

Equations (1.9) and (1.11) show that A determines B, as well as part of E. Notice that B eis left invariant by the transformation

$$A \mapsto A' = A + \nabla\chi, \tag{1.25}$$

for any scalar function χ. The invariance of E es accomplished by the transformation

$$\Phi \mapsto \Phi' = \Phi - \frac{\partial\chi}{\partial t}. \tag{1.26}$$

The transformations (1.25) and (1.26) are called *gauge transformations,* and the invariance of the fields under such transformations is called *gauge invariance.*

In the lenguage of covariance, we see that $A^\alpha = (\Phi, A)$ is not unique: the same electromagnetic field tensor $F^{\alpha\beta}$ can be obtained from potential

$$A^\alpha = \left(\Phi - \frac{\partial\chi}{\partial t}, A + \nabla\chi\right). \tag{1.27}$$

Substituting (1.27) into (1.14) we obtain

$$\begin{aligned} F^{\alpha\beta} &= \partial^\alpha A^\beta + \partial^\alpha\partial^\beta\chi - \partial^\beta A^\alpha - \partial^\beta\partial^\alpha\chi \\ &= \partial^\alpha A^\beta - \partial^\beta A^\alpha + \left[\partial^\alpha, \partial^\beta\right]\chi \\ &= \partial^\alpha A^\beta - \partial^\beta A^\alpha, \end{aligned}$$

using the fact that $\left[\partial^\alpha, \partial^\beta\right] = 0$.

Remark. The transformation $A^\alpha \mapsto A'^\alpha = A^\alpha + \partial^\alpha\chi$ is a gauge transformation.

1.4 The metric

Definition 1.1. A real vector space V is called a metric vector space if on V, a scalar product is defined as a map

$$g : V \times V \to \mathbb{R},$$

such that for all $u, v \in V$ and $\Lambda \in \mathbb{R}$ the following propieties are satisfied:

(Bilinear) $g(\lambda u + v, w) = \lambda g(u, v) + g(u, w)$

(Symmetric) $g(u, v) = g(v, u)$

(Non-degenerate) $\forall u \in V, g(u, v) = 0 \iff v = 0$

If e_α is the orthonormal basis in V, then

$$g(e_\alpha, e_\beta) = g_{\alpha\beta} = \pm\delta_{\alpha\beta} = \begin{cases} \pm 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \tag{1.28}$$

The signature of the metric is defined by the number of $+1's$ and $-1's$, usually denoted by (p, q). The idea if a metric can be extended to the space $\mathfrak{F}(M)$ of all vector fields and the space $\Omega^1(M)$ of all 1-forms on a manifold M. On a smooth manifold M, a metric g assigns to each point $p \in M$ a metric g_p on the tangent space T_pM in a smooth varying way

$$g_p : T_pM \times T_pM \to \mathbb{R},$$

which satisfies the above propieties with λ replaced by $f \in C^\infty(M)$, and $g(u, v)$ is a funtion on M whose value at p is $g_p(u_p, v_p)$, where $u, v \in \mathfrak{F}(M)$ and $u_p, v_p \in T_pM$.

If the signature of g is $(n, 0)$, n being the dimension of M, we say that g is a Riemmanian metric, while if g has the signature of $(n-1, 1)$, we say that g is Lorentzian. A manifold equipped with a metric will be called a semi-Riemmannian manifold denoted by (M, g).

Setting $\tilde{g}(u)(v) = g(u, v)$, we obtain an isomorphism

$$\tilde{g} : T_pM \to T_p^*M, \tag{1.29}$$

which can be proved by using the non-degeneracy propierty and the fact that $\dim T_pM = \dim T_p^*M$.

Let $\{\partial_\alpha\}$ be a basis of a vector field on an open neighbourhood U of a point in M, then, the components of the metric are given by

$$g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta). \tag{1.30}$$

If the dimension of M es n, then $g_{\alpha\beta}$ is an $n \times n$ matrix. The non-degeneracy property shows that $g_{\alpha\beta}$ is invertible and we shall denote the inverse by $g^{\alpha\beta}$, also $g_{\alpha\gamma}g^{\alpha\beta} = \delta_\gamma^\beta$. This leads to the raising and lowering of indices which is a process of converting vector fields to 1-forms. With the help of (1.29), one can easily convert between tangent vectors and contangent vectors.

Example 1.1. If $u = u^\alpha\partial_\alpha$ is a vector field on a chart, then the corresponding 1-form can be calculated can be calculated as follows:

$$\begin{aligned}
\tilde{g}(u)(v) &= g_{\alpha\beta}u^\alpha v^\beta \\
&= g_{\alpha\gamma}\delta_\beta^\gamma u^\alpha v^\beta \\
&= g_{\alpha\gamma}u^\alpha v^\beta dx^\gamma(\partial^\beta) \\
&= (g_{\alpha\gamma}u^\alpha dx^\gamma)v^\beta\partial_\beta = (u_\gamma dx^\gamma)v
\end{aligned}$$

Therefore $\tilde{(u)} = u_\gamma dx^\gamma$, where $u_\gamma = g_{\alpha\gamma}u^\alpha$.

Conversely, if $\xi = \xi_\beta dx^\beta$ is a 1-form on a chart, then the corresponding vector field can be calculated as follows:

$$\begin{aligned}
\tilde{g}^{-1}(\chi)(\eta) &= g^{\beta\gamma}\xi_\beta\eta_\gamma \\
&= g^{\beta\alpha}\xi_\beta\eta_\gamma dx^\gamma(\partial_\alpha) \\
&= (\xi^\alpha\partial_\alpha)\eta,
\end{aligned}$$

Therefore $\tilde{g}^{-1}(\xi) = \xi^\alpha\partial_\alpha$, where $\xi^\alpha = g^{\beta\alpha}\xi_\beta$.

Using the fact that we can switch from 1-form to vector fields and vice versa with the help of a metric, we define the inner product of the two 1-form ξ, η as

$$\langle \xi, \eta \rangle = g^{\beta\gamma} \xi_\beta \eta_\gamma. \tag{1.31}$$

if $\theta^1 \wedge \cdots \wedge \theta^p$ and $\sigma^1 \wedge \cdots \wedge \sigma^p$ are orthonormal basis of $p-$forms, then

$$\langle \theta^1 \wedge \cdots \wedge \theta^p, \sigma^1 \wedge \cdots \wedge \sigma^p \rangle = \det \left[g(\theta^\alpha, \sigma^\beta) \right], \tag{1.32}$$

we define

$$\langle \theta^\alpha, \theta^\beta \rangle = \begin{cases} \varepsilon_\beta = \pm 1, & \alpha = \beta \\ 0, & \alpha \neq \beta. \end{cases} \tag{1.33}$$

Definition 1.2. Let M be an $n-$dimensional manifold, we define the volume form on a chart (U_α, ϕ_α) as

$$\mathfrak{Y} = dx^1 \wedge \cdots \wedge dx^n;$$

if the manifold is a semi-Riemannian manifold we have,

$$\mathfrak{Y} = \sqrt{|\det g_{\alpha\beta}|} dx^1 \wedge \cdots \wedge dx^n.$$

1.5 The Hodge Star Operator

Definition 1.3. The Hodge star operator is the unique linear map on a semi-Riemannian manifold from $p-$forms to $(n-p)-$forms defined by

$$\star : \Omega^p(M) \to \Omega^{(n-p)(M)},$$

such that for all $\xi, \eta \in \Omega^p(M)$,

$$\xi \wedge \star\eta = \langle \xi, \eta \rangle \mathfrak{Y}.$$

This is an isomorphism between $p-$forms and $(n-p)-$forms, and $\star\eta$ is called the dual of η. Suppose that $dx^1, \cdots, dx^n$ are positively oriented orthonormal basis of 1-forms on some chart (U_α, ϕ_α) on a manifold M.

Let $1 \leq i_1 < \cdots < i_p \leq n$ be an ordered distinct increasing indices and let $j_1 < \cdots < j_{n-p}$ be their complement in the set $\{1, ..., n\}$. Then

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} = \text{sign}(I) dx^1 \wedge \cdots \wedge dx^n, \tag{1.34}$$

where, $\text{sign}(I)$ es the sign of the permutation

$$\begin{pmatrix} i_1 & ... & i_p & j_1 & ... & j_{n-p} \\ 1 & ... & p & p+1 & ... & n \end{pmatrix}.$$

In other words, the wedge products of a $p-$forms and $(n-p)-$forms yield the volume form up to a sign.

Remark. $\star 1 \in \Omega^n(M)$ is the volume form $\mathfrak{Y}$. For a domain $D \subset M$, $\int_D \mathfrak{Y}$ is the volume of D. In particular, if M is compact, $\int_M \mathfrak{Y}$ is called the volume of M.

We state the following proposition without proof:

Proposition 1.1 ([4], 4.7)**.** *The $\star-$ operator of Hodge has the following properties. For any $f, g \in C^\infty(M)$ and $\xi, \eta \in \Omega^p(M)$ we have*

(i) $\star(f\xi + g\eta) = f \star \xi + g \star \eta$

(ii) $\star \star \xi = (-1)^{p(n-p)}\xi$

(iii) $\xi \wedge \star\eta = \eta \wedge \star\xi$

(iv) $\star(\xi \wedge \star\eta) = \langle \xi, \eta \rangle$

(v) $\langle \star\xi, \star\eta \rangle = \langle \xi, \eta \rangle$

Now let M be an oriented Riemannian manifold. For any $X \in \mathfrak{F}(M)$, let ξ_X be the 1-form corresponding to X by the isomorphism $\mathfrak{F}(M) \cong \Omega^1(M)$. We set

$$\nabla \cdot X = \star d \star \xi_X.$$

It is called the *divergence* of X. If M comes with opposite orientation, $\star$ changes to $-\star$. Since $\star$ appears twice in the definition of $\nabla\cdot$, it follows that $\nabla \cdot X$ is defined independently of the coise of orientation.

Let I_n denote the set $\{1, ..., n\}$.

Example 1.2. On the Euclidean space $\mathbb{R}^n$, to $X = \sum_{k \in I_n} u^k \partial_k$, there correspond $\xi_X = \sum_{k \in I_n} u^k dx^k$. Hence we get

$$\nabla \cdot X = \sum_{k \in I_n} \partial_k u^k \tag{1.35}$$

by direct computation. This is the original definition of divergence.

We state the following theorem without proof:

Theorem 1.2 ([4], 4.9)**.** *Let M be an oriented compact Riemannian manifold. If X is a vector field on M, then we have the equality*

$$\int_M \nabla \cdot X \ \mathfrak{Y}_M = \int_{\partial M} \langle X, n \rangle \ \mathfrak{Y}_{\partial M},$$

where n is the outward unit normal vector field on ∂M. In particular, if M is a closed manifold, then

$$\int_M \nabla \cdot X \ \mathfrak{Y}_M = 0.$$

Remark. Let $d : \Omega^\star(\mathbb{R}^3) \to \Omega^\star(\mathbb{R}^3)$ be the exterior derivative, $f \in C^\infty(M)$ and $A \in \Omega^1(\mathbb{R}^3)$. Then

$$df = \sum_{k \in I_3} \partial_k f dx^k. \tag{1.36}$$

Thus, for $dx = (dx^1, dx^2, dx^3)$,

$$df = \langle \nabla f, dx \rangle. \tag{1.37}$$

Let $A = \sum_{k \in I_3} A_k dx^k$ be a 1-form in $\mathbb{R}^3$. Then

$$\begin{aligned} dA &= \sum_{n \in I_3} (\partial_{\sigma^n(1)} A_{\sigma^n(2)} - \partial_{\sigma^n(2)} A_{\sigma^n(1)}) dx^{\sigma^n(1)} \wedge dx^{\sigma^n(2)}, \\ \star dA &= \sum_{n \in I_3} (\partial_{\sigma^n(2)} A_{\sigma^n(3)} - \partial_{\sigma^n(3)} A_{\sigma^n(2)}) dx^{\sigma^n(1)}, \end{aligned} \tag{1.38}$$

where $\sigma^n(\cdot) = (1\ 2\ 3)^n \in S_3$. Therefore,

$$\star\, dA = \langle (\nabla \times A), dx \rangle. \tag{1.39}$$

In this way, $d(df) = \nabla \times (\nabla f) = 0$ and $d(dA) = \nabla \cdot (\nabla \times A) = 0$.

1.6 The Homogeneous Maxwell's Equations

Instead of treating the magnetic field as a vector $B = (B_1, B_2, B_3)$ we will treat it as a 2-form

$$B = \sum_{n \in I_3} B_{\sigma^n(1)} dx^{\sigma^n(2)} \wedge dx^{\sigma^n(3)}. \tag{1.40}$$

Similary, instead of treating the electric field as a vector $E = (E_1, E_2, E_3)$, we will teat it as a 1-form

$$E = \sum_{k \in I_3} E_k dx^k. \tag{1.41}$$

Assume that M is a semi-Riemmanian Manifold equipped with Minkowski, i.e., as a 4-dimensional Lorentzian manifold or *spacetime*. Furthermore, we shall assume that the spacetime M can be split into 3-dimensional manifold S, "space", with a Riemmanian metric and another space $\mathbb{R}$ for time. Then,

$$M = \mathbb{R} \times S.$$

Let $x^i,\ i \in I_3$ denote local coordinates on an open subset $U \subset S$, and let x^0 denote the coordinate on $\mathbb{R}$, then the local coordinates on $\mathbb{R} \times U \subset M$ will be those given by $x^\alpha = (x^0, ..., x_3) = (t, x)$. with the metric defined $\eta_{\alpha\beta}$ by (1.1).

We can then combine the electric and magnetic fields into a unified electromagnetic field F, which is a 2-form on $\mathbb{R} \times U \subset M$ defined by

$$F = B + E \wedge dx^0. \tag{1.42}$$

In comoponent form we have

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{1.43}$$

where $F_{\alpha\beta}$ is given by (1.19).

Explicity, we have

$$F = \sum_{k \in I_3} E_k dx^k \wedge dx^0 + B = \sum_{n \in I_3} B_{\sigma^n(1)} dx^{\sigma^n(2)} \wedge dx^{\sigma^n(3)}. \tag{1.44}$$

Taking the exterior derivative if (1.42) we obtain

$$dF = d(B + E \wedge dx^0) = dB + dE \wedge dx^0. \tag{1.45}$$

In general, for any differential form η on spacetime, we have

$$\eta = \eta_I dx^I, \tag{1.46}$$

where I range over $I_n^p := \{A \subset I_n | \# A = p\}$, and η_I is a function of spacetime.

Taking the exterior derivative of (1.46), we obtain

$$\begin{aligned} d\eta &= \partial_\alpha \eta_I dx^\alpha \wedge dx^I \\ &= \sum_{k \in I_3} \partial_k \eta_I dx^k \wedge dx^I + \partial_0 \eta_I dx^0 \wedge dx^I \\ &= d_s \eta\eta + dx^0 \wedge \partial_0 \eta, \end{aligned}$$

Then, $d = d_s + dx^0 \wedge \partial_0$, where d_s is the exterior derivative of space and $x^0 = t$.

Since B, E are differential forms on a spacetime, we shall split the exterior derivative into spacelike part and timelike part. Using the identity above, we obtain the following form (1.45)

$$\begin{aligned} dF &= d_s B + dx^0 \wedge \partial_0 B + (d_s E + dx^0 \wedge \partial_0 E) \wedge dx^0 \\ &= d_s B + (d_s E + \partial_0 B) \wedge dx^0 + dx^0 \wedge dx^0 \wedge \partial_0 E \\ &= d_s B + (d_s E + \partial_0 B) \wedge dx^0. \end{aligned}$$

Now, $dF = 0$ is the same as

$$d_s B = 0 \tag{1.47}$$

$$d_s E + \partial_0 B = 0. \tag{1.48}$$

The equations (1.47) and (1.48) are exactly the same as (1.4) and (1.5). Hence, the homogeneous Maxwell's equations correspond to the closed form $dF = 0$ which is similar to the Jacobi identities (1.20).

1.7 Inhomogeneous Mawxwell's Equations

Starting from (1.44) and using the relations:

$$\star \left(dx^{\sigma^n(1)} \wedge dx^0\right) = dx^{\sigma^n(2)} \wedge dx^{\sigma^n(3)}, \; n \in I_3. \tag{1.49}$$

we obtain:

$$\star F = \sum_{n \in I_3} E_{\sigma^n(1)} dx^{\sigma^n(2)} \wedge dx^{\sigma^n(3)} - \sum_{k \in I_3} B_k dx^k \wedge dx^0, \tag{1.50}$$

or

$$\star F = \frac{1}{2} (\star F)_{\alpha\beta} dx^\alpha \wedge dx^\beta \tag{1.51}$$

where

$$(\star F)_{\alpha\beta} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}. \tag{1.52}$$

Thus, the effect in (1.19) of the dual operator on F amounts to the exchange

$$E_i \mapsto -B_i, \; B_i \mapsto E_i, \; i \in I_3.$$

Combining the charge density ρ and the current density J into a unified vector field on Minkowski spacetime, we obtain

$$\mathbf{J} = J^\alpha \partial_\alpha = \rho\partial_0 + J^1\partial_1 + J^2\partial_2 + J^3\partial_3. \tag{1.53}$$

Using the result of example 1.1, with Minkowski metric (1.1), we obtain the 1-form

$$J = J_\beta dx^\beta = \sum_{k\in I_3} J^k dx^k - \rho dx^0, \tag{1.54}$$

where

$$J_\beta = \eta_{\alpha\beta} J^\alpha. \tag{1.55}$$

Let $\star_s$ denote the Hodge star operator on space, using relations

$$dx^{\sigma^n(1)} = \star(dx^{\sigma^n(2)} \wedge dx^{\sigma^n(3)}),\ n \in I_3, \tag{1.56}$$

we can see that (1.50) is the same as

$$\star F = \star_s E - \star_s B \wedge dx^0 \tag{1.57}$$

which amounts to the exchange

$$E \mapsto - \star_s B,\ B \mapsto \star_s E,$$

in (1.42), taking the exterior derivative of (1.57) and applying the Hodge star operator, we obtain

$$\star d \star F = - \star_s d_s \star_s E \wedge dx^0 - \partial_0 E + \star_s d_s \star_s B. \tag{1.58}$$

If we set $\star d \star F = J$ and equate components, we obtain

$$\star_s d_s \star_s E = \rho \tag{1.59}$$

$$-\partial_0 E + \star_s d_s \star_s B = J^i dx^i,\ i \in I_3, \tag{1.60}$$

which is exactly the inhomogeneous Maxwell's equations (1.6) and (1.7).

Thus, the Maxwell's equations have been rewritten as the following ones

$$dF = 0 \tag{1.61}$$

$$\star d \star F = J \tag{1.62}$$

The continuity equation in differential form will be derived from (1.62).

From prop. 1.1,we obtain $\star^2 = 1$. Applying $d\star$, to (1.62), we obtain

$$d \star J = dd \star F = 0. \tag{1.63}$$

Using the relations

$$\star\, dx^{\tilde\sigma^n(0)} + \sum_{n\in I_3} dx^{\tilde\sigma^n(1)} \wedge dx^{\tilde\sigma^n(2)} \wedge dx^{\tilde\sigma^n(3)} = 0, \tag{1.64}$$

where $\tilde\sigma^n(\cdot) = (1\ 2\ 3\ 0)^n \in S_4$, we obtain, with $J^0 = \rho$,

$$\star J = \sum_{n\in I_4} (-1)^n J^{\tilde\sigma^n(0)} dx^{\tilde\sigma^n(1)} \wedge dx^{\tilde\sigma^n(2)} \wedge dx^{\tilde\sigma^n(3)}. \tag{1.65}$$

Operating the exterior derivative on (1.65), we obtain

$$d \star J = \sum_{k \in I_4} \partial_k J^k dx^0 \wedge ... \wedge dx^3. \tag{1.66}$$

Therefore $d \star J = 0$ correspond to

$$\sum_{k \in I_4} \partial_k J^k = 0, \tag{1.67}$$

which is exactly the continuity equation (1.8).

1.8 The Vacuum Maxwell's Equations

In free space or *vacuum*, Maxwell's equations correspond to $\rho, \mathbf{J} = 0$, i.e., $J = 0$, which amounts to the exchange

$$F \mapsto \star F.$$

We say that $F \in \Omega^2(M)$ is *self-dual* if $\star F = F$ and *anti-self-dual* if $\star F = -F$. In 3-dimensional Riemannian manifold, it was shown that $\star^2 = 1$. This implies that the Hodge star operator has eigenvalues ± 1. Therefore, if we take $F_\pm = \frac{1}{2}(F \pm \star F)$, we can consider any $F \in \Omega^2(M)$ as a sum of a self-dual and anti-self-dual:

$$F = F_+ + F_-,$$

where $\star F = \pm F_\pm$.

However, in the Lorentzian case $\star^2 = -1$, which implies that the eigenvalues are $\pm i$. If we consider complex-valued differential forms on M, it follows that, for any $F \in \Omega^2(M)$, we have

$$F = F_+ + F_-,$$

where $\star F = \pm i F_\pm$.

In both cases, if F is a self-dual or anti-self-dual 2-form satisfying

$$dF = 0, \tag{1.68}$$

automatically it satisfies

$$\star\, d \star F = 0. \tag{1.69}$$

Certainly, F is a complex-valued in the Lorentzian case, but we can always split the real and the imaginary parts and obtain a real solution using the fact that (vacuum) Maxwell's equations are linear, which correspond to either (1.68) or (1.69).

Chapter 2

Theory of Gauge Fields

2.1 Principal Fiber Bundles

Classical gauge theories are mathematically described using principal fiber bundles.

Definition 2.1 ([5], 1.1.1). A principal fiber bundle $\pi : P \to M$ with structure group G consists of smooth manifolds P, M and a Lie Group G together with a smooth surjective projection map $\pi : P \to M$, where the Lie group G has a free smooth right action on P and $\pi^{-1}(\pi(p)) = \{pg : g \in G\}$. If $x \in M$, then $\pi^{-1}(x)$ is called the fiber above x.

Furthermore, we require that for each $x \in M$ there exists an open set U containing x and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times G$ of the form $\phi_U(p) = (\pi(p), s_U(p))$, where $s_U : P \to G$ has the property $s_U(pg) = s_U(p)g$ for all $g \in G, p \in \pi^{-1}(U)$. The map T_U is called local trivialization (or, in physics language, a choice of gauge).

Let $T_U : \pi^{-1}(U) \to U \times G$ and $T_V : \pi^{-1}(V) \to V \times G$ be two local trivializations of a principal fiber bundle $\pi : P \to M$. The *transition function* from U to V is the map $g_{UV} : U \cap V \to G$ defined, for $x = \pi^{-1}(p) \in U \cap V$, by $g_{UV}(x) = s_V(p)s_U(p)^{-1}$. Thus is independent of the choice of $p \in \pi^{-1}(x)$.

Theorem 2.1 ([5], 1.1.5). *A principal bundle $\pi : P \to M$, with structure group G is a trivial bundle if and only if it admits a global section $M \to P$.*

Proof. Let $\sigma : U \to P$ be a given local section. Then define the smooth map $T_U : \pi^{-1}(U) \to U \times G$ by $T_U(\sigma(x)g) = (\pi(p), g)$. We verify that indeed $T_U(\sigma(x)gh) = (\pi(p), gh)$ so that T_U is a local trivialization. Conversely, let $T_U : \pi^{-1}(()U) \to U \times G$ be a given local trivialization, then define a section $\sigma : U \to \pi^{-1}(U)$ by $\sigma(x) = \phi_u^{-1}(x, e)$. This, we see that local sections correspond to a local trivializations. In particular, this implies that a global sections exists if ans only if the bundle is trivial. □

Example 2.1 (Square root). The map $z \mapsto z^2$ in S^1 induces a principal bundle $\pi : S^1 \to S^1$ with structure group $\mathbb{Z}^2$. It is locally trivial since, locally, on the circle there always exists a smooth square root function. Since the total space is S^1 and not $S^1 \times \mathbb{Z}_2$ this bundle is not trivial. Hence there does not exist a continuous square root function on S^1.

Example 2.2 (Hopf fibration)**.** Identify $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$ by $(x_1,x_2,x_3)\mapsto(z=x_1+ix_2, x=_{x_3})$ and $\mathbb{R}^4$ with $\mathbb{C}^2$ by identifying $(x_1,...,x_4)\mapsto(z_1=x_1+ix_2, z_2=x_3+ix_4)$. Then the unit sphere S^2 in $\mathbb{R}^3$ is identified with $\{(z,x)||z|^2+|x|^2\}$ and S^3 is identified with $\{(z_1,z_2||z_1|^2+|z_2|^2)=1\}$.

The Hopf fibration is defined by

$$p(z_1,p_2)=(2z_1z_2^*,|z_1|^2|z_2|^2).$$

Then p maps S^3 onto S^2 as can be checked:

$$4|z_1|^2|z_2|^2+(|z_1|^2-|z_2|^2)^2=(|z_1|^2+|z_1|^2)^2=1.$$

It can be shown that p maps elements of S^3 to the same point in S^2 if and only if these points are the same up to a factor $\lambda\in U(1)$. The bundle is not global trivial, but for Hopf fibration it is enough to remove a single point $m\in S^2$, thus one can take $U=S^2-m$ as trivializing neighbourhoods, and any point in S^2 has a neighbourhood of this form. Hence $p:S^3\to S^2$ is a principal $U(1)-$ bundle.

2.2 Connections and gauge potential

Definition 2.2 ([5], 1.2.1)**.** A *connection* assigns to each $p\in P$ a subspace $H_p\subset T_pP$ such that

(i) $T_pP=V_p\bigoplus H_p$, where $V_p=\{X\in T_pP|\pi_*(X)=0\}$.

(ii) $R_{g*}H_p=H_p$, where R_g is multiplication by the right

(iii) For each $p\in P$, there exist a neighbourhood U and vector fields $X_1,...,X_n$ on U such that H_p is spanned by $X_1(p),...,X_n(p)$ for all $p\in U$.

We will also use the following equivalent definition:

Definition 2.3 ([5], 1.2.2)**.** A *connection* is a $\mathfrak{g}-$valued 1-form ω on P satisfying thw following conditions:

(i) For the fundamental vector fields $A^*(p):=\frac{d}{dt}(p\exp(tA))|_{t=0}$,

$$\forall p\in P,\omega_p(A^*)=A,$$

(ii) For any $g\in G$,

$$R_g^*\omega=\mathrm{ad}_{g^{-1}}\omega,$$

where $\mathrm{ad}_{g^{-1}}\in Aut(\mathfrak{g})$ is the adjoint action of g on $\mathfrak{g}$. To be more precise, we require $(R_{g*}\omega)(p)=\mathrm{ad}_{g^{-1}}\omega_p$, i. e., $\omega_{pg}(R_{g*}X_p)=ad_{g^{-1}}(\omega_p(X_p))$, where $ad:G\to GL(\mathfrak{g})$ denotes the adjoint action of G on $\mathfrak{g}$.

The equivalence of definitions 2.2 and 2.3 can be seen as follows: given a connection in the form of definition 2.2, one defines a $\mathfrak{g}-$valued 1-form ω by $\omega(A^*)=A$ and $\omega_p(X_p)=0$ for all $X_p\in H_p$.

Conversely, given a connection 1-form ω as in definition 2.3 define $H_p=\{X\in T_pP|\omega_p(X_p)=0\}$. Since the action of G is free, the map $A\mapsto A_p^*$ is injective.

In order to see this, let A be such that $A^*_p = 0$, then

$$\begin{aligned}\frac{d}{dt}(p\exp(tA)) &= \frac{d}{ds}(p\exp((s+t)A))|_{s=0}\\ &= \frac{d}{ds}(p\exp(sA)\exp(tA))|_{s=0}\\ &= R_{\exp(tA)*}\frac{d}{ds}(p\exp(sA))|_{s=0}\\ &= R_{\exp(tA)*}A^*_p = 0.\end{aligned}$$

So one obtains a vector space isomorphism $\mathfrak{g} = V_p$. From part 1) of definition 2.3, it follows that $H_p \bigoplus V_p = T_pP$.

It can been shown that $[A^*, B^*]_p = [A, B]^*_p$ for all $A, B \in \mathfrak{g}$, where $[\cdot,\cdot]$ on vector fields is the usual Lie-bracket, i.e., $[X, Y] = \frac{d}{dt}(\phi^{-1}_{t*}Y_{\phi_t(p)})|_{t=0}$ where $\frac{d}{dt}\phi_t = X$ in a neighbourhood of p.

Proposition 2.2. *The vector fields on M can be identified with the $G-$invariant horizontal fields on P.*

Proof. The map π_* establish an isomorphism between H_p and $T_{\pi p}M$, so it indeeds identifies a subspace of T_pP with the tangent space of M. This allows us to lift vector fields on M to unique horizontal vector fields on P. This horizontal lift is $G-$invariant, i.e., $R_{g*}\tilde{X} = \tilde{X}$, since $R_{g*}H_p = H_{pg}$. Conversely, every $G-$invariant vector field $\tilde{X}$ is a lift of a vector field $X = \pi_*\tilde{X}$ on M, which is well-defined since $\tilde{X}$ is $G-$invariant and π is surjective. □

Lemma 2.3. *For a horizontal lift $\tilde{X}$ of X, we have $[A^*, \tilde{X}] = 0$, for all $A \in \mathfrak{g}$.*

Proof. Use the formula $[A^*, \tilde{X}] = \frac{d}{dt}\left(\phi^{-1}_{t*}\tilde{X}_{\phi_t(p)}\right)|_{t=0}$, where $\phi_t(p) = p\exp(tA)$, and the $G-$invariance of $\tilde{X}$, which implies $\phi_{t*}\tilde{X} = \tilde{X}$, for all t. □

Remark. By choosing a local trivialisation, i.e, a local section $\sigma : U \to P$, one can pull back ω to a 1-form $\omega_U = \sigma^*\omega$ on $U \subset M$. In the particular case that G is a matrix Lie group and two local sections $\sigma : U \to P$ and $\sigma : V \to P$ are given such that $U \cap V \neq \emptyset$, then the transformation rule between ω_U and ω_V is given by

$$\omega_V = g^{-1}_{uv}dg_{UV} + g^{-1}_{UV}\omega_U g_{UV}. \tag{2.1}$$

(C.f. [[5]], definition 1.2.3 and theorem 1.2.5.)

Conversely, a collection of local $\mathfrak{g}-$valued 1-forms $\{\omega_U\}$, where $\{U_i\}$ is a cover of M such that P is locally trivial on U_i, for all i, satisfying the transformation rule (2.1) glue together into a global $\mathfrak{g}-$valued 1-form ω on P satisfying the conditions of definition 2.3.

The pullback ω_U on M are known as gauge potentials in physics. The transformation property (2.1) is what physicists may recognize as the transformation rule of a *gauge potential* in gauge theories. A connection on P is therefore also known as a gauge potential. The choice of the local sections is called a choice of gauge.

Definition 2.4. A gauge theory on M with a gauge group G consists of a principal $G-$bundle $\pi : P \to M$ endowed with a connection ω.

2.3 Associated bundles and particles

Let P be a principal $G-$bundle and F a smooth manifold on which G acts smoothly from the left. Let $P \times F$ be the direct product and identify two elements $(p_1, f_1) \sim (p_2, f_2)$ if and only if $(p_1 g, g^{-1} f_1) = (p_2, f_2)$ for some $g \in G$.

This action of G is free and transitive since the action of G on P is thus. Therefore, the quotient $P \times_G F$ is a manifold and even a fiber bundle with fiber F under the projection map $\pi(p, f) = \pi(p)$.

Definition 2.5. The bundle $P \times_G F$ is called an associated bundle of P.

It can be shown that local sections of P induce local trivialisations of $P \times_G F$. If $F = V$ is a finite-dimensional real or complex vector space, then the fibers of $P \times_G F$ inherit the same structure if we define

$$[(p, v_1)] + [(p, v_2)] = [p, v_1 + v_2], \ \lambda[(p, v)] = [p, \lambda v],$$

with $p \in P, v_1, v_2, v \in V, \lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$.

We can introduce particle fields to a gauge theory by introducing an associated vector bundle of P. Sections of this associated bundles are interpreted as the particle fields.

Example 2.3. Let $G = SU(2)$ and consider the fundamental representation $\rho : SU(2) \to GL_2(\mathbb{C})$. Then $\Gamma(P \times_{SU(2)} \mathbb{C}^2)$ form a $C^\infty(M)-$module of a particle fields that are $\mathbb{C}^2-$valued locally.

If E is an associated vector bundle of P with fiber V, the $C^\infty(M)-$module of sections $\Gamma^\infty(E)$ can be identified with $V-$valued $G-$equivariant functions on P.

Lemma 2.4. *In general, if P is a $G-$principal fiber bundle and $E = P \times_G V$ is an associated vector bundle, there is a natural isomorphism of*

$$C^\infty(M)\text{-}modules \cong C^\infty(P)^G\text{-}modules,$$

between $\Gamma(M, E)$ and $C^\infty(P, V)^G$, where the last space is the space of all G-equivariant V-valued functions on P. A function $f : P \to V$ is called G-equivariant if for all $g \in G$ and $p \in P$, $f(pg) = g^{-1} \cdot f(p)$.

Proof. Let f be such an equivariant function. Then $s \in \Gamma(M, E)$ is defined by $s(x) = [p, f(p)]$, where $\pi(p) = x$. Since f is G-invariant this is well-defined:

$$[pg, f(pg)] = [pg, g^{-1} \cdot f(p)] = [p.f(p)],$$

so that $s(x)$ is independent of the choice of p in the fiber of x.

Conversely, given $s \in \Gamma(M, E)$, a G-equivariant function $f : P \to V$ is defined by $s(x) = [p, f(p)]$. □

Remark.

- The isomorphism $C^\infty(M) \cong C^\infty(P)^G$ is just a spacial case of this lemma, where G acts trivially on $\mathbb{C}$. Note that $f \in C^\infty(M)$ can be identified with a G-invariant function $\tilde{f}$ on P through $\tilde{f} = f \circ \pi$.

- If we want to check that the constructed f or s are smooth, we need the fact that the action of G on $P \times V$ is proper and free. This implies that $E = P \times_G V$ has a natural smooth structure with respect to the quotient topology (note that in the text we already used the fact that E is smooth).

  Moreover, with this smooth structure the quotient map $P \times V \to E$ is smooth and $P \times V$ is a principal fiber bundle over E. Using the fact that locally the quotient map $P \times V \to E$ has a smooth inverse (namely, a smooth local section from E to $P \times V$) we can show that the defined s or f in the proof of the lemma are smooth if and only if the other is smooth.

This identification is very useful: to study sections of associated bundles, it is enough to consider equivariant functions on P. From now on, we will implicitly make this identification. Using this identification, we can carry over structures on P to similar structures on the associated bundles. One of these structures is the connection. It induces a covariant derivative on associated vector bundles.

Definition 2.6 ([5], 3.1.2)**.** Let $\rho : G \to GL(V)$ be a finite-dimensional representation of G. Given a connection ω, we can introduce the space $\overline{\Omega}^k(P, V)$ of differential forms on P such that $\phi(X_1, ..., X_k) = 0$ if one of the X_i is a vertical vector field, and $R_g^*\phi = g^{-1}\phi$.

The *vertical condition* implies that we are actually considering forms on the manifold M and the equivariance condition means that we are considering sections of the associated bundle $\Gamma(P \times_G V)$.

Proposition 2.5. *Two connections ω, ω' differ by an element in $\overline{\Omega}^1(P, \mathfrak{g})$.*

Proof. It can be checked that $(\omega - \omega')(X) = 0$ on vertical fields $X = A^*$ by the first condition in def. 2.3. The property $R_g^*(\omega - \omega') = \mathrm{ad}_{g^{-1}}(\omega - \omega')$ follows from the second condition in the same definition. □

Corolary 2.6. *For a given ω the space $\overline{\Lambda^1}(P, \mathfrak{g})$ is in one-to-one correspondence with the space of all connection 1-forms $\mathcal{C}$ on P, through the assignment $\tau \mapsto \tau + \omega$.*

A connection on a principal bundle P naturally induces a connection on any associated bundle $E = P \times_G V$.

Definition 2.7 ([5], 3.1.3)**.** For a connection ω, the covariant derivative on $P \times_G V$ is defined by

$$D_\omega \phi = (d\phi)^H,$$

This is indeed a map $\overline{\Omega}^\bullet(P, V) \to \overline{\Omega}^{\bullet+1}(P, V)$, since

$$\begin{aligned} R_g^* D_\omega \phi &= R_g^*(d\phi)^H \\ &= (R_g^* d\phi)^H \\ &= (dR_g^*\phi)^H \\ &= (d(\mathrm{ad}_{g^{-1}})\phi)^H \\ &= \mathrm{ad}_{g^{-1}}(d\phi)^H. \end{aligned}$$

If $\mathfrak{g}$ acts on V, then we have an action of $\tilde{\Omega}(P,\mathfrak{g})$ on $\tilde{\Omega}(P,V)$ given by

$$(\phi\dot{\wedge}\tau)(X_1,...,X_{k+l}) = \sum_{\sigma\in S_{k+l}} (-1)^\sigma \phi(X_{\sigma(1)},...,X_{\sigma(k)}) \cdot \tau(X_{\sigma(k+1)},...,X_{\sigma(k+l)}),$$

where $\phi \in \tilde{\Omega}(P,\mathfrak{g})$ and $\tau \in \tilde{\Omega}(P,V)$.

In this way, the set $\Omega(P,\mathfrak{g}) = \Omega(P)\otimes\mathfrak{g}$ is a differential graded Lie-algebra, where $\mathfrak{g}$ acts on itself in the adjoint representation. We will denote $[\phi_1,\phi_2] = \phi_1\dot{\wedge}\phi_2$.

With a differential graded Lie-algebra we mean that the bracket $[\cdot,\cdot]$ satisfies

(i) $[\phi_1,\phi_2] = -(-1)^{kl}[\phi_2,\phi_1]$

(ii) $(-1)^{mk}\left[[\phi_1,\phi_2],\phi_3\right] + (-1)^{lk}\left[[\phi_2,\phi_3],\phi_1\right] + (-1)^{lm}\left[[\phi_3,\phi_1],\phi_2\right] = 0$

(iii) $d\,[\phi_1,\phi_2] = [d\phi_1,\phi_2] + (-1)^k\,[\phi_1,d\phi_2]\,.$

Theorem 2.7 ([5], 3.1.5)**.** *For $\tau \in \overline{\Omega}^k(P,V)$, $\nabla_\omega\tau = d\tau + \omega\dot{\wedge}\tau$.*

Proof. This can be proved point-wise, so if $v_p \in T_pP$ is horizontal, we can assume that $v_p = X_p$, where X is a G-invariant horizontal vector field. Similarly, if $v_p \in T_p$ is vertical, we can assume that $v_p = A^*_p$ for some $A \in \mathfrak{g}$.

If all the fields $X_1,...,X_k$ are horizontal, then $(\omega\dot{\wedge}\tau)(X_1,...,X_k) = 0$ and $X_i^H = X_i$, so that both sides coincide.

If at least two vector fields of the $X_1,...,X_{k+1}$ are vertical at the point p and are extended to fundamental vector fields, and the other vector are extended to G-invariant horizontal fields, the $\nabla_\omega\tau = 0 = \omega\dot{\wedge}\tau$, so we must show that $d\tau = 0$. We have

$$\begin{aligned} d\tau(X_1,...,X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}X_i(\tau(X_1,...,\hat{X}_i,...,X_{k+1})) \\ &+\sum_{i<j}(-1)^{i+j}\tau([X_i,X_j],X_2,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}), \end{aligned}$$

which is zero when at least two of the X_i are vertical, because $[A^*,B^*] = [A,B]^*$.

If precisely one of the X_i is vertical, say X_1, then $[X_1,X_i] = 0$ for all i and one needs to show that

$$X_1(\tau(X_2,...,X_k)) + \omega(X_1)\cdot\tau(X_2,...,X_k) = 0,$$

which follows from ($g_t = \exp(tA)$)

$$\begin{aligned} X_1(\tau(X_2,...,X_k)) &= \frac{d}{dt}[\tau(R_{g_t*}X_2,...,R_{g_t*}X_{k+1})] \\ &= \frac{d}{dt}[g_t^{-1}\cdot\tau(X_2,...,X_k)] \\ &= -A\cdot\tau(X_2,...,X_{k+1}) \\ &= -\omega(X_1)\cdot\tau(X_2,...,X_{k+1}). \end{aligned}$$

The theorem now follows by linearity. $\square$

Corolary 2.8. *If $\tau \in \overline{\Omega}^\bullet(P,\mathfrak{g})$, then $\nabla_\omega\tau = d\tau + [\omega,\tau]\,.$*

Proposition 2.9. *For* $\tau \in \overline{\Omega}^{\bullet}(P,V)$, $\nabla^2_\omega \tau = F_\omega \tau$, *where* $F_\omega = d\omega + \frac{1}{2}\omega \dot{\wedge} \omega \in \overline{\Omega}^2(P,\mathfrak{g})$ *is the curvature of* ∇_ω.

Proof. One checks that

$$\begin{aligned}\nabla_\omega(d\tau + \omega\dot{\wedge}\tau) &= d^2\tau + d\omega\dot{\wedge}\tau - \omega\dot{\wedge}d\tau + \omega\dot{\wedge}d\tau + \omega\dot{\wedge}(\omega\dot{\wedge}\tau)\\ &= d\omega\dot{\wedge}\tau + \frac{1}{2}(\omega\dot{\wedge}\omega)\dot{\wedge}\tau.\end{aligned}$$

It remains to check that $F_\omega(X_1,X_2) = (d\omega + \frac{1}{2}[\omega,\omega])[X_1,X_2]$ vanishes when either X_1 or X_2 is vertical. This is again a point-wise calculation, so we assume that $X_1 = A^*$. Note that $[\omega,\omega](X_1,X_2) = 2[\omega(X_1),X_2] = 2[\omega(X_1),\omega(X_2)]$ so that

$$\begin{aligned}F_\omega(X_1,X_2) &= d\omega(X_1,X_2) + [\omega(X_1),\omega(X_2)]\\ &= X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1,X_2]) + [\omega(X_1),\omega(X_2)].\end{aligned}$$

Since $\omega(X_1) = A$ is constant, the second term vanishes. It remains to show that $X_1(\omega(X_2)) - \omega([X_1,X_2]) + [\omega(X_1),\omega(X_2)] = 0$. If X_2 is G-invariant horizontal, all terms are zero. If $X_2 = B^*$ for some $B \in \mathfrak{g}$, then $X_1(\omega(X_2)) = 0$ and $\omega([X_1,X_2]) = [\omega(X_1),\omega(X_2)]$ because $[A^*,B^*] = [A,B]^*$. □

Proposition 2.10 (Bianchi identity)**.** $\nabla_\omega F_\omega = 0$.

Proof. Note that

$$\begin{aligned}\nabla_\omega F_\omega &= dF_\omega + [F_\omega,\omega]\\ &= d^2\omega + \frac{1}{2}d([\omega,\omega]) + [\omega,d\omega] + [\omega,[\omega,\omega]]\\ &= \frac{1}{2}d([\omega,\omega]) + [d\omega,\omega] = 0,\end{aligned}$$

since $d([\omega,\omega]) = [d\omega,\omega] - [\omega,d\omega] = -2[\omega,d\omega]$. We also used the Jacobi identity to show that $[\omega,[\omega,\omega]] = 0$. □

2.4 Relation with physics

Let P be a principal G-bundle and assume for simplicity that G is a matrix Lie group, like $SU(n)$. Let $\rho : G \to GL(V)$ be a finite-dimensional representation of G. Construct the associated bundle $P \times_G V$. Then a local section of P will induce a local trivialisation of $P \times_G V$.

Let s be a local sections of the associated bundle $P \times_G V$. On some local trivialisation (U,σ) of P, induced by a local section $\sigma : U \to P$, the section s can be considered as a V-valued function. This is the point of view most physicists take.

If we consider s as a G-equivariant function $\bar{s} : P \to V$, then on $U \subset M$ the section s is realized as a V-valued function as $\bar{s} \circ \sigma$. In these local coordinates, the covariant derivative $s \mapsto (ds)^H$ takes the form

$$\begin{aligned}(ds)^H \circ \sigma &= \sigma^*(ds + \omega\dot{\wedge}s)\\ &= d\sigma^*s + \sigma^*\omega\dot{\wedge}\sigma^*s\\ &= (d + \omega_U)\sigma^*s.\end{aligned}$$

That is, on local trivialisation, a connection ω takes the form $d+\omega_U$, where ω_U transforms according to the rule (2.1). This is the usual form of a connection one encounters in physics.

The statement that two connections differ by an element of $\overline{\Omega}^1(P,\mathfrak{g})$ translates, on the level of vector bundles, to the one that two connection ∇, ∇' differ by an element of $\Gamma(M, T^*M \otimes E)$. The curvature $F_\omega = d\omega + \frac{1}{2}\omega \wedge \omega$ lies in $\overline{\Omega}^2(P,\mathfrak{g})$ and therefore corresponds to an element $F \in \Gamma(T^*M \otimes T^*M \otimes E)$, which is also known as the curvature connection ∇ on E.

The local form on M of the curvature $F_\omega = d\omega + \frac{1}{2}[\omega,\omega]$ is $F_u = d\omega_U + \frac{1}{2}[\omega_U, \omega_U]$, the usual field strength tensor in field theory.

2.5 Gauge group and gauge algebra

The gauge group as the group of G-equivariant diffeomorphisms preserving the fibers $GA(P) = \{f \in Diff(P) | f \text{ equivariant and } f \circ \pi = \pi\}$. This group is isomorphic to $C^\infty(P,G)^G$, where G acts on itself in the adjoint representation, and $f \in GA(P)$ is related to an element $\tau \in C^\infty(M,G)^G$ by $f(p) = p\tau(p)$. We have isomorphisms $GA(P) \cong C^\infty(M,G) \cong \Gamma(P \times_{Ad} G)$.

The bundle $\operatorname{Ad} P = P \times_{Ad} G$ is called the adjoint bundle of P and it is obtained as an associated bundle of P by the adjoint action of G on itself. The fibers have the structure of a gauge since $\operatorname{Ad} g \in AutG$, for all $g \in G$. The group structure of the fibers naturally induces a group structure on the section $\Gamma(M, \operatorname{Ad} P.)$

From now on we will work with the group $C^\infty(M,G)^G$.

The group $C^\infty(M,G)^G$ acts on $\Gamma(P \times_G V)$ as $(f\phi)(p) = f(p)\phi(p)$ or more generally, on $\overline{\Omega}^k(P,V)$ as $(f \cdot \phi) = f(p) \cdot (X_1, ..., X_k)$. The covariant derivative transforms as $\nabla_\omega \mapsto f\nabla_\omega f^{-1} := \nabla^f_\omega$, so that $f \cdot \nabla_\omega \phi = \nabla^f_\omega(f\phi)$. This implies that the curvature transforms as $F \mapsto fFf^{-1}$ and ω as $\omega \mapsto fdf^{-1} + f\omega f^{-1}$. Locally, f can be considered as a change of local trivialization $\sigma \mapsto f \circ \sigma$.

Note that for any gauge transformation f, the 1-form $f^*\omega$ is again a connection. Two connections on P are called equivalent if they are related by a gauge transformation as above.

Definition 2.8. The gauge algebra is the Lie algebra of infinitesimal gauge transformation $\Gamma(M, \operatorname{ad} P) \cong C^\infty(P,\mathfrak{g})^G$, where G acts in the adjoint representation on $\mathfrak{g}$. Here $\operatorname{ad} P = P \times_{ad} \mathfrak{g}$.

The Lie algebra structure on $\Gamma(M, \operatorname{ad} P)$ is given by

$$[H_1, H_2](p) = [H_1(p), H_2(p)] \,.$$

Moreover, there is a map $Exp : \Gamma(M, \operatorname{ad} P) \to \Gamma(M, \operatorname{Ad} P)$ given by

$$\operatorname{Exp}(H)(p) = \exp(H(p)),$$

which is well-defined since

$$\begin{aligned}
\operatorname{Exp}(H)(pg) &= \exp H(pg) \\
&= \exp(\operatorname{ad}_{g^{-1}} H(p)) \\
&= \operatorname{Ad}_{g^{-1}} \exp(H(p)) \\
&= \operatorname{Ad}_{g^{-1}}(\operatorname{Exp}(H)(p)).
\end{aligned}$$

The gauge algebra acts on $\overline{\Omega}^k(P, V)$ as

$$(H \cdot \phi)(X_1, ..., X_k) = H(p) \cdot \phi(X_1, ..., X_k).$$

Equivalently, it can be defined by $H \cdot \phi = \frac{d}{dt}(\mathrm{Exp}(tH) \cdot \phi)|_{t=0}$.

Definition 2.9. The action functional for the matrix Lie groups is given by

$$S(\omega) = \int_M \mathrm{Tr}\, F_\omega \wedge \star F_\omega,$$

where $\star$ denotes the Hodge star operator.

If one looks for a local extremum, one finds the Yang-Mills equation:

$$\nabla_\omega(\star F) = 0,$$

which is similar to the Bianchi identity $\nabla_\omega F = 0$, which is always satisfied. We will discuss the Yang-Mills equations in more detail in our next chapter.

Chapter 3

Yang-Mills Fields

3.1 Electromagnetic fields

A source-free electromagnetic field is the prototype of Yang-Mills fields. We will shoe that a source-free electromagnetic field is a gauge field with gauge group $U(1)$.

Let $P(M^4, U(1))$ be a principal $U(1)$-principal bundle over the Minkowski space M^4. Any principal bundle over M^4 is trivializable. We choose a fixed trivialization of $P(M^4, U(1))$ and use it to write $P(M^4, U(1))$. The Lie algebra

$$\mathfrak{u}(1) = \{z \in \mathbb{C} | z = -\bar{z}\}$$

of $U(1)$ may identified with $i\mathbb{R}$.

Thus a connection form on P may be written as $i\omega$, $\omega \in \Lambda^1(P)$, by choosing i as the basis of the Lie algebra $i\mathbb{R}$. The gauge field can be written as $i\Omega$, where $\Omega = d\omega \in \Lambda^2(P)$. The Bianchi identity $d\Omega = 0$ is an immediate consequence of this result.

The bundle $\mathrm{ad}(P)$ is also trivial and we have $\mathrm{ad}(P) = M^4 \times u(1)$. Thus the gauge field $F_\omega \in \Lambda^2\left(M^4, \mathrm{ad}(P)\right)$, on the base M^4, can be written as iF, $F \in \Lambda^2\left(M^4\right)$. Using the global gauge $s : M^4 \to P$ defined by $s(x) = (x, 1), \forall x \in M^4$, we can pull the connection form $i\omega$ on P to M^4 to obtain the gauge potential $iA = is^*\omega$. Thus in this case, we have a global potential $A \in \Lambda^1\left(M^4\right)$ and the corresponding gauge field $F = dA$. The Bianchi identity $dF = 0$ for F follows from the exactness of the 2-form F.

The field equations $\delta F = 0$, for $\delta = \star d \star$ are obtained as the Euler-Lagrange equations minimizing the action $\int |F|^2$, where $|F|$ is the pseudo-norm induced by the Lorentz metric on M^4 and the trivial inner product on the Lie algebra $\mathfrak{u}(1)$.

We note that the action represents the total energy of the electromagnetic field. The two equations

$$dF = 0, \ \delta F = 0 \tag{3.1}$$

are the Maxwell's equations for a source-free electromagnetic field.

A gauge transformation f is a section of $\mathrm{Ad}(P) = M^4 \times U(1)$. It is completely determined by the function $\psi \in \mathcal{F}(M^4)$ such that

$$f(x) = \left(x, e^{i\psi(x)}\right) \in \mathrm{Ad}(P), \forall x \in M^4.$$

If iB denotes the potential obtained by the action of the gauge transformation f on iA, then we have

$$iB = e^{-i\psi}(iA)e^{i\psi} + e^{-i\psi}de^{i\psi}, \text{ or } B = A + d\psi,$$

which is the classical formulation of the gauge transformation f.

Chapter 4

Appendices

4.1 Lie groups and Lie algebras

Definition 4.1. Let G be an n-manifold and a group such that the groups operation $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2$ and the function $G \to G$ given by $g \mapsto g^{-1}$ are C^∞ maps. Then G is called a *Lie group*.

Definition 4.2. Let $L_g : G \to G$ be defined by $L_g(g') = gg'$; L_g is a diffeomorphism. Let e be the identity element of G, and let $A \in T_eG$. Define $\bar{A} \in \Gamma(TG)$ by $\bar{A}_g = L_{g*}(A)$; $\bar{A}$ is called the *left-invariant vector field* determined by A.

Definition 4.3. Let $\mathfrak{G} = T_eG$ and ,for $A, B \in \mathfrak{G}$, define $[A, B] \in \mathfrak{G}$ by $[A, B] = \left[\bar{A}, \bar{B}\right]_e$. Note that it is *anti-symmetric* and it satisfies the *Jacobi identity*. Then $\mathfrak{G}$, together with the bracket operation $[,]$, is called the *Lie algebra* of G.

For $A \in \mathfrak{G}$, we can prove that $\bar{A}$ is a complete vector field. Let $\{\varphi_t\}$ be the one-parameter group of diffeomorphism generated by $\bar{A} \in \mathfrak{G}$. Let $\gamma : \mathbb{R} \to G$ be the curve through e defined by $\gamma(t) = \varphi_t(e)$.

We prove that $\gamma(s+t) = \gamma(s)\gamma(t)$ (group multiplication). Let $s \in \mathbb{R}$ be fixed and let $\gamma_1(t) = \gamma(s+t)$, while $\gamma_2(t) = \gamma(s)\gamma(t)$. Then $\gamma_1'(t) = \gamma'(s+t) = \bar{A}_{\gamma(s+t)}$ and

$$
\begin{aligned}
\gamma_2'(t) &= L_{\gamma(s)}*(\gamma'(t)) \\
&= L_{\gamma(s)}*(\bar{A}_{\gamma(t)}) \\
&= L_{\gamma(s)} * (L_{\gamma(t)} * A) = \bar{A}_{\gamma(s)\gamma(t)}.
\end{aligned}
$$

Thus $\gamma : \mathbb{R} \to G$ is homomorphism. Conversely, given a curve and homomorphism $\sigma : \mathbb{R} \to G$, then $\psi_t : G \to G$, defined by $\psi_t(g) = g\sigma(t)$ is a one-parameter group of diffeomorphism of G such that

$$
\bar{B}_g \equiv \frac{d}{dt}\psi_t(g)|_{t=0}
$$

defines the left-invariant vector field $\bar{B}$ determined by $B \equiv \bar{B}_e$. Thus, there is a one-to-one correspondence $A \leftrightarrow \gamma$.

Definition 4.4. We define the *exponential map* $\exp : \mathfrak{G} \to G$ by $\exp(A) = \gamma(1)$. Note that $\gamma(t) = \exp(tA)$, and $\gamma_t(g) = g\gamma(t) = g\exp(tA)$.

Example 4.1. Let V be a vector space with $\dim V = m < \infty$, and let $GL(V)$ be the group of invertible linear functions $F : V \to V$. By regarding $GL(V)$ as a group of matrices, it is simple to see that $GL(V)$, which is a open subset of $\mathbb{R}^{m^2}$, is a Lie group.

Let $I \in GL(V)$ be the identity, and denote $T_I(GL(V))$ by $\mathfrak{Gl}(V)$. Note that $\mathfrak{Gl}(V)$ can be identified whit the vector space of *all* linear functions $A : V \to V$, the correspondence being

$$A \leftrightarrow \left.\frac{d}{dt}(I + tA)\right|_{t=0}.$$

For $A \in \mathfrak{Gl}(V)$, let

$$\operatorname{Exp}(A) = I + A + \frac{1}{2!}A^2 + ...$$

It can be proved that the sum converges, and that

$$\operatorname{Exp}((t+s)A) = \operatorname{Exp}(tA)\operatorname{Exp}(sA).$$

Thus, $\operatorname{Exp}(A)\operatorname{Exp}(-A) = I$ and so $\operatorname{Exp}(A) \in \mathfrak{Gl}(V)$. Note that $t \mapsto \operatorname{Exp}(tA)$ is a curve and a homomorphism with

$$\left.\frac{d}{dt}\operatorname{Exp}(tA)\right|_{t=0} = A.$$

It follows from the discussion previous to definition 4.4 that Exp is the exponential map for $GL(V)$. In 4.3, we will prove that, for $A, B \in \mathfrak{Gl}(V)$, $[A, B] = AB - BA$.

Definition 4.5. A *Lie subgroup* of a Lie group G is a submanifold (of G) that is also a subgroup. A Lie subgroup H of G is itself a Lie group. Since the homomorphisms $\gamma : \mathbb{R} \to H$ are also homomorphisms into G, we have that $\exp : \mathfrak{H} \to H$ is just $\exp : \mathfrak{G} \to G$ restricted to $\mathfrak{H}$.

Theorem 4.1. *Let G and G' be Lie groups, and let $F : G \to G'$ be a C^∞ homomorphism. The $F_{*e} : \mathfrak{G} \to \mathfrak{G}'$ is a linear function such that $F_{*e}([A, B]) = [F_{*e}A, F_{*e}]$, i.e., F_{*e} is a homomorphism of Lie algebras.*

Proof. Note that

$$F \circ L_g(g') = F(gg') = F(g)F(g') = (L_{F(g)} \circ F)(g').$$

Thus

$$F_{*g}(\bar{A}_g) = F_{*g}(L_{*g}A) = L_{F(g)*e'}(F_{*e}A) = (\overline{F_{*e}A})_{F(g)},$$

and so $F_*(\bar{A}) = (\overline{F_{*e}}A)$. Thus

$$F_{*e}([A, B]) = \left[F_*(\bar{A}), F_*(\bar{B})\right]_{e'} = \left[(\overline{F_{*e}A}), (\overline{F_{*e}B})\right]_{e'} = [F_{*e}A, F_{*e}B].$$

□

Definition 4.6. For $g \in G$, let $\mathrm{Ad}_g : G \to G$ be the C^∞ *adjoint isomorphism* given by $\mathrm{Ad}_g(g') = gg'g^{-1}$. We let $\mathfrak{Ad}_g : \mathfrak{G} \to \mathfrak{G}$ be the induced isomorphism of $\mathfrak{G}$ provided by theorem 4.1, i.e., $\mathfrak{Ad}_g = \mathfrak{Ad}_{d*e}$. Let $\mathfrak{ad} : G \to GL(\mathfrak{G})$ be the homomorphism $g \mapsto \mathfrak{ad}_g$. Then theorem 4.1 gives us an induced homomorphism $\mathfrak{ad} : \mathfrak{G} \to \mathfrak{Gl}(\mathfrak{G})$, i.e., $\mathfrak{ad} = \mathfrak{ad}_{*e}$.

Theorem 4.2. *For $A, B \in \mathfrak{G}$, we have*

$$\mathfrak{ad}(A)(B) = \frac{\partial^2}{\partial s \partial t} (\exp(tA) \exp(sB) \exp(-tA)) \Big|_{s,t=0} = [A, B].$$

Proof. Let $\{\varphi_t\}$ be the one-parameter group generated by $\bar{A}$. By the end of definition 4.4, we have $\varphi_t(g) = g \exp tA$. Using $L_{\bar{A}} \bar{B} = [\bar{A}, \bar{B}]$, we have (at $s = t = 0$)

$$\begin{aligned}
[A, B] = [\bar{A}, \bar{B}]_e &= \frac{d}{dt} \varphi_{-t*} \left(\bar{B}_{\varphi_t(e)} \right) \\
&= \frac{d}{dt} \varphi_{-t*} \left(\frac{d}{dt} \varphi_t(e) \exp(sB) \right) \\
&= \frac{d}{dt} \frac{d}{ds} \varphi_{-t} \left(\varphi_t(e) \exp(sB) \right) \\
&= \frac{\partial^2}{\partial t \partial s} \left(\exp(tA) \exp(sB) \exp(-tA) \right) \\
&= \frac{d}{dt} \mathfrak{ad} \left(\exp(tA) \right) (B) = \mathfrak{ad}_{*e}(A)(B) \\
&= \mathfrak{ad}(A)(B)
\end{aligned}$$

□

Corolary 4.3. *If G is any Lie subgroup of $GL(V)$, then the bracket operation on $\mathfrak{G} \subset \mathfrak{Gl}(V)$ is given by $[A, B] = AB - BA$.*

Proof. By 4.5 it suffices to consider the case in which $G = GL(V)$. Using theorem 4.2 with exp = Exp (see example 4.1), we have

$$[A, B] = \frac{\partial^2}{\partial s \partial t} (\operatorname{Exp}(tA) \operatorname{Exp}(sB) \operatorname{Exp}(-tA)) \Big|_{s,t=0} = AB - BA.$$

□

Definition 4.7. Let $e_1, ..., e_n$ be a basis for the Lie algebra $\mathfrak{G}$ of G. The *structure constants* $c_{ij}^k \in \mathbb{R}$ are defined by $[e_i, e_j] = \sum c_{ij}^k e_k.$ Note that $[e_i, e_j] = -[e_j, e_i]$, which implies $c_{ij}^k = -c_{ij}^k$. Similarly, the Jacobi identity implies

$$\sum_m c_{im}^h c_{jk}^m + c_{km}^h c_{ij}^m + c_{jm}^h c_{ki}^m = 0,$$

for all h, i, j, k.

Example 4.2 ($SU(n)$, the Special Unitary Group)**.** The computation of the Lie algebra of a Lie group of matrices is illustrated here for the group $SU(n)$, which is frequently used in elementary particle physics.

Let $\mathfrak{Gl}(n, \mathbb{C})$ be the space of all $n \times n$ matrices with complex entries. For $A \in \mathfrak{Gl}(n, \mathbb{C})$, let A^* denote the conjugate of the transpose of A. Recall that the unitary group is

$$U(n) = \{A \in \mathfrak{Gl}(n, \mathbb{C}) | AA^* = I\}$$

and

$$SU(n) = \{A \in U(n) | \det A = 1\}.$$

If $t \mapsto A(t)$ is a curve in $U(n)$ with $A(0) = I$, then at $t = 0$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}(I) = \frac{d}{dt}\left\{A(t)A(t)^*\right\} \\ &= A'(0)A(0)^* + A(0)A'(0)^* = A'(0) + A'(0)^*. \end{aligned}$$

Thus, for $\mathfrak{S} = \left\{B \in \mathfrak{Gl}(n, \mathbb{C}) | B + B^* = 0\right\}$, we have $\mathfrak{S} \supset \mathfrak{U}(n)$, the Lie algebra of $U(n)$.

Conversely, if $B \in \mathfrak{S}$, then $(\operatorname{Exp} B)(\operatorname{Exp} B)^* = (\operatorname{Exp} B)(\operatorname{Exp} B^*) = I$, and so $\operatorname{Exp} B \in U(n)$. At $t = 0$,

$$B = \frac{d}{dt} \operatorname{Exp} tB \in \mathfrak{U}(n),$$

whence $\mathfrak{U}(n) = \mathfrak{S}$.

The Lie subalgebra $\mathfrak{SU}(n)$ of $SU(n)$ is the subalgebra of $\mathfrak{U}(n)$ consisting of matrices with trace 0, i.e.,

$$\mathfrak{SU}(n) 0 \left\{B \in \mathfrak{U}(n) | \mathfrak{tr} B = 0\right\}.$$

This follows fro the formula $\det(\operatorname{Exp} B) = e^{\operatorname{tr} B}$, which is valid for any $n \times n$ matrix. We can prove this formula as follows, Let $f(t) = \det(\operatorname{Exp} tB)$. at $h = 0$, we have

$$\begin{aligned} f' &= \frac{d}{dt} f(t+h) = \frac{d}{dt}\left[\det(\operatorname{Exp} tB) \det(\operatorname{Exp} hB)\right] \\ &= \det(\operatorname{Exp} tB) \frac{d}{dh} \det(I + hB) \\ &= \det(\operatorname{Exp} tB) \mathfrak{tr} B = (\mathfrak{tr} B) f(t). \end{aligned}$$

Thus, $f(t) = f(0)e^{(\operatorname{tr} B)t} = e^{(\operatorname{tr} B)t}$, and setting $t = 1$ yields the result.

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