MA2008B - Numerical Analysis for Non-Linear Optimization

Lecture Notes

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Contents

1 Control Theory

In this module, we introduce the fundamental concepts of control theory, contrasting classical methods with modern state-space approaches. We also explore the solution of linear time-invariant differential equations.

1.1 Conventional and State-Space Methods

1.1.1 Mathematical Review

Control theory serves as the foundation for analyzing and designing systems that maintain desired behaviors. We distinguish between two primary frameworks:

Definition 1.1.1.1 (Conventional Control (Classical)): A frequency-domain approach using **Transfer Functions** ($G(s)$). It is primarily used for Single-Input Single-Output (SISO), Linear, Time-Invariant (LTI) systems. Key metrics include overshoot, settling time, and steady-state error.

Example 1.1.1.1 (Simple Transfer Function): A cruise control system might be modeled as $G(s) = \frac{K}{ms+b}$, relating force input to velocity output.

Definition 1.1.1.2 (State-Space Methods (Modern)): A time-domain approach using **differential equations**. It models systems via state variables:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

This method handles Multiple-Input Multiple-Output (MIMO), non-linear, and time-varying systems.

Example 1.1.1.2 (Simple State-Space): For the same cruise control system, let v be the state x. Then $m\dot{v} + bv = u$ becomes:

$$\dot{v} = -\frac{b}{m}v + \frac{1}{m}u$$

Here $A = \left[-\frac{b}{m}\right]$, $B = \left[\frac{1}{m}\right]$.

The relationship between the two representations for LTI systems is given by:

$$G(s) = C(sI - A)^{-1}B + D$$

1.1.2 Solved Problems

Example 1.1.2.1 (Transfer Function to State-Space): Given the transfer function $G(s) = \frac{1}{s^2+3s+2}$, find a state-space representation.

Solution: The differential equation corresponds to $\ddot{y} + 3\dot{y} + 2y = u$. Let $x_1 = y$ and $x_2 = \dot{y}$. Then:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

In matrix form:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \;\; 0), \quad D = 0$$

Example 1.1.2.2 (State-Space to Transfer Function): Given $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = (1 \;\; 1)$, $D = 0$. Find $G(s)$.

Solution: Calculate $(sI - A)^{-1}$:

$$sI - A = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix} \Rightarrow (sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix}$$

Then $G(s) = C(sI - A)^{-1}B$:

$$G(s) = (1 \;\; 1) \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+3}{(s+1)(s+2)}$$

1.1.3 Supplementary Problems

Exercise 1.1.3.1 (System Classification): Classify the system $\dot{x} = -x + x^3$ as linear or non-linear, and time-invariant or time-varying.

Exercise 1.1.3.2 (Dimensionality): For a MIMO system with 2 inputs and 3 outputs, what are the dimensions of the D matrix?

1.2 Solving Linear Time-Invariant Differential Equations

1.2.1 Mathematical Review

A Linear Time-Invariant (LTI) system $\dot{x} = Ax + Bu$ has solutions described by the matrix exponential.

Theorem 1.2.1.1 (Solution to Homogenous System): For a homogenous system $\dot{x} = Ax$ (where $u(t) = 0$), the solution is:

$$x(t) = e^{At}x(0)$$

where $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ is the **Matrix Exponential**.

Example 1.2.1.1 (1D Homogenous): If $\dot{x} = 3x$, then $A = 3$, and $x(t) = e^{3t}x(0)$.

Theorem 1.2.1.2 (General Solution): For the non-homogenous system $\dot{x} = Ax + Bu$, the general solution is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Example 1.2.1.2 (Constant Input): If $\dot{x} = -x + 1$ with $x(0) = 0$, then $x(t) = \int_0^t e^{-(t-\tau)}d\tau = 1 - e^{-t}$.

1.2.2 Solved Problems

Example 1.2.2.1 (Scalar Decay): Solve the scalar system $\dot{x} = -2x$ with initial condition $x(0) = 5$.

Solution: Here $A = -2$ (a 1×1 matrix). The solution is:

$$x(t) = e^{-2t}x(0) = 5e^{-2t}$$

This represents an exponential decay.

Example 1.2.2.2 (Diagonal Matrix Exponential): Find e^{At} for $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Solution: Since A is diagonal, the matrix exponential is simply the exponential of diagonal elements:

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

1.2.3 Supplementary Problems

Exercise 1.2.3.1 (Nilpotent Matrix): Find e^{At} for the nilpotent matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Exercise 1.2.3.2 (Forced Response): Write the integral expression for the solution of $\dot{x} = -x + 1$ with $x(0) = 0$.

1.3 Transfer Function

1.3.1 Mathematical Review

The **Transfer Function** $G(s)$ of a Linear Time-Invariant (LTI) system is defined as the ratio of the Laplace transform of the output $Y(s)$ to the Laplace transform of the input $U(s)$, assuming all initial conditions are zero.

Definition 1.3.1.1 (Transfer Function):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$$

where $n \geq m$ for physical realizability. The roots of the denominator are the **poles**, and the roots of the numerator are the **zeros**.

Example 1.3.1.1 (Zero-Pole Example): $G(s) = \frac{s+2}{s(s+3)}$. Zero at $s = -2$. Poles at $s = 0, s = -3$.

Impulse Response: The system output when the input is a Dirac delta function $\delta(t)$. Its Laplace transform is $G(s)$.

1.3.2 Solved Problems

Example 1.3.2.1 (RC Circuit): Consider a series RC circuit where the input is voltage $v_{\{\in\}}(t)$ and output is capacitor voltage $v_{c(t)}$. Find $G(s)$.

Solution: Kirchhoff's voltage law:

$$v_{\{\in\}}(t) = Ri(t) + v_{c(t)}$$

Current relation: $i(t) = C\dot{v}_{c(t)}$. Substituting:

$$v_{\{\in\}}(t) = RC\dot{v}_{c(t)} + v_{c(t)}$$

Taking Laplace transform (zero initial conditions):

$$V_{\{\in\}}(s) = (RCs + 1)V_{c(s)}$$

Transfer function:

$$G(s) = \frac{V_{c(s)}}{V_{\{\in\}}}(s) = \frac{1}{RCs + 1}$$

Example 1.3.2.2 (Poles and Stability): Given $G(s) = \frac{10}{s^2+2s+5}$, find the poles and determine stability.

Solution: The poles are roots of $s^2 + 2s + 5 = 0$. Using quadratic formula:

$$s = \frac{-2 + -\sqrt{4 - 20}}{2} = -1 + -2j$$

Since the real part (-1) is negative, the system is **stable**.

1.3.3 Supplementary Problems

Exercise 1.3.3.1 (Mechanical System): Find transfer function $\frac{X(s)}{F(s)}$ for a mass-spring-damper system: $M\ddot{x} + B\dot{x} + Kx = f(t)$.

Exercise 1.3.3.2 (Zero Locations): How do zeros in the right-half plane (non-minimum phase) affect the step response?

1.4 Block Diagrams and Signal Flow Graphs

1.4.1 Mathematical Review

Block Diagrams graphically represent systems where functional blocks are connected by signals.

Definition 1.4.1.1 (Block Diagram Algebra):
- **Series:** $G_{\{=\}}(s) = G_1(s)G_2(s)$
- **Parallel:** $G_{\{=\}}(s) = G_1(s) + G_2(s)$
- **Feedback Loop:** For negative feedback $H(s)$, the closed-loop transfer function is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Example 1.4.1.1 (System in Series): Two blocks $G_1(s) = s$ and $G_2(s) = \frac{1}{s}$ in series give $G_{\{=\}}(s) = s \cdot \left(\frac{1}{s}\right) = 1$.

Signal Flow Graphs (SFG) are directed graphs where nodes represent variables and branches represent gains. **Mason's Gain Formula** computes the transfer function of an SFG.

1.4.2 Solved Problems

Example 1.4.2.1 (Feedback Reduction): A system has forward path $G(s) = \frac{K}{s+1}$ and unity negative feedback ($H(s) = 1$). Find the closed loop transfer function $T(s)$.

Solution: Using the feedback formula:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{s+1}}{1 + \frac{K}{s+1}}$$

Multiply numerator and denominator by $(s + 1)$:

$$T(s) = \frac{K}{s + 1 + K}$$

Example 1.4.2.2 (Mason's Rule Application): Find the transfer function $\frac{C}{R}$ for a simple loop with forward gain G and feedback H.

Solution:

- Forward paths: $P_1 = G$.
- Loops: $L_1 = -GH$.
- Determinant $\Delta = 1 - (\sum L_i) = 1 - (-GH) = 1 + GH$.
- Path cofactor $\Delta_1 = 1$ (loop touches the path).
- Result: $T = \frac{P_1 \Delta_1}{\Delta} = \frac{G}{1+GH}$.

1.4.3 Supplementary Problems

Exercise 1.4.3.1 (Multiple Loops): Simplify a block diagram with two nested feedback loops.

Exercise 1.4.3.2 (Signal Flow Construction): Draw the Signal Flow Graph corresponding to the equations: $x_2 = ax_1 + bx_3$, $x_3 = cx_2$.

1.5 State-Space Analysis

1.5.1 Mathematical Review

State-space analysis uses the vector-matrix representation of systems.

Definition 1.5.1.1 (State Controllability): A system is **controllable** if the state vector can be transferred from any initial state to any final state in finite time. The Controllability Matrix is:

$$\mathcal{C} = [B, AB, A^2 B, ..., A^{n-1} B]$$

The system is fully controllable if $\operatorname{rank}(\mathcal{C}) = n$.

Example 1.5.1.1 (Uncontrollable System): A system with states decoupled from input u: $\dot{x}_1 = -x_1, \dot{x}_2 = -x_2 + u$, is not fully controllable (cannot influence x_1).

Definition 1.5.1.2 (State Observability): A system is **observable** if the initial state can be determined from the output history. The Observability Matrix is:

$$\mathcal{O} = \left[C^T, A^T C^T, \left(A^T\right)^2 C^T, ..., \left(A^T\right)^{n-1} C^T\right]^T$$

The system is fully observable if $\mathrm{rank}(\mathcal{O}) = n$.

Example 1.5.1.2 (Unobservable System): If $y = x_2$ but $\dot{x}_1 = -x_1$ (independent of x_2), we cannot deduce x_1 from y.

1.5.2 Solved Problems

Example 1.5.2.1 (Controllability Check): Check controllability for $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution: Compute $\mathcal{C} = [B, AB]$.

$$AB = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$$

Determinant is $0(-3) - 1(1) = -1 \neq 0$. Rank is 2 ($n = 2$). Therefore, the system is **controllable**.

Example 1.5.2.2 (Observability Check): Check observability for same A with $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Solution: Compute $\mathcal{O} = [C, CA]^T$.

$$CA = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is the identity matrix. Rank is 2. Therefore, the system is **observable**.

1.5.3 Supplementary Problems

Exercise 1.5.3.1 (Uncontrollable Mode): Identify the uncontrollable mode in a system with diagonal $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Exercise 1.5.3.2 (Duality): Prove that the pair (A, B) is controllable if and only if (A^T, B^T) is observable? (Note: Check duality property accurately).

1.6 Basic Design Principles of Control Systems

1.6.1 Mathematical Review

Control system design involves selecting components and parameters to satisfy performance, stability, and robustness specifications.

Definition 1.6.1.1 (Design Specifications): Common metrics include:

- **Rise Time (t_r):** Time to reach the vicinity of the final value.
- **Settling Time (t_s):** Time to stay within a tolerance band (e.g., 2%).
- **Percent Overshoot (M_p):** Peak value relative to steady state.
- **Phase Margin / Gain Margin:** Measures of relative stability.

Example 1.6.1.1 (Fast vs Stable): A requirement of $t_s < 1s$ requires fast decay, while $M_p < 5\%$ limits oscillation.

Theorem 1.6.1.1 (PID Control): A Proportional-Integral-Derivative controller has the law:

$$u(t) = K_p e(t) + K_i \int e(\tau) d\tau + K_d \dot{e}(t)$$

- **P:** Improves speed.
- **I:** Eliminates steady-state error.
- **D:** Improves damping (reduces overshoot).

Example 1.6.1.2 (PI Controller): Ideally used when D is sensitive to noise. Law: $u(t) = K_p e(t) + K_i \int e(\tau) d\tau$.

1.6.2 Solved Problems

Example 1.6.2.1 (Parameter Selection for Damping): A second order system has characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. Select ζ for $M_p \approx 5\%$.

Solution: Percentage overshoot is given by $M_p = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}$. For $M_p = 0.05$:

$$\ln(0.05) = -\pi\frac{\zeta}{\sqrt{1-\zeta^2}} \approx -3$$

$$\pi^2\zeta^2 \approx 9(1-\zeta^2) \Rightarrow (\pi^2+9)\zeta^2 \approx 9$$

$$\zeta \approx \sqrt{\frac{9}{19}} \approx 0.69$$

A damping ratio of $\zeta \approx 0.7$ typically gives $\approx 5\%$ overshoot.

Example 1.6.2.2 (Steady State Error): Find steady-state error for a unit step input with forward gain $G(s) = \frac{10}{s+1}$ in unity feedback.

Solution: System Type is 0 (no integrators in $G(s)$). Static position error constant $K_p = \lim_{s\to 0} G(s) = \frac{10}{1} = 10$. Steady state error $e_{\text{ss}} = \frac{1}{1+K_p} = \frac{1}{11}$.

1.6.3 Supplementary Problems

Exercise 1.6.3.1 (PID Tuning): Describe the Ziegler-Nichols tuning method for a PID controller.

Exercise 1.6.3.2 (Root Locus Design): How does adding a pole at the origin affect the root locus and stability?