MA2008B - Numerical Analysis for Non-Linear Optimization

Lecture Notes

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Contents

1 Module 1: Control Theory

In this module, we introduce the fundamental concepts of control theory, contrasting classical methods with modern state-space approaches. We also explore the solution of linear time-invariant differential equations.

1.1 1.1 Conventional and State-Space Methods

1.1.1 Mathematical Review

Control theory serves as the foundation for analyzing and designing systems that maintain desired behaviors. We distinguish between two primary frameworks:

Definition 1.1.1.1 (Conventional Control (Classical)): A frequency-domain approach using **Transfer Functions** ($G(s)$). It is primarily used for Single-Input Single-Output (SISO), Linear, Time-Invariant (LTI) systems. Key metrics include overshoot, settling time, and steady-state error.

Definition 1.1.1.2 (State-Space Methods (Modern)): A time-domain approach using **differential equations**. It models systems via state variables:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

This method handles Multiple-Input Multiple-Output (MIMO), non-linear, and time-varying systems.

The relationship between the two representations for LTI systems is given by:

$$G(s) = C(sI - A)^{-1}B + D$$

1.1.2 Solved Problems

Example 1.1.2.1 (Transfer Function to State-Space): Given the transfer function $G(s) = \frac{1}{s^2+3s+2}$, find a state-space representation.

Solution: The differential equation corresponds to $\ddot{y} + 3\dot{y} + 2y = u$. Let $x_1 = y$ and $x_2 = \dot{y}$. Then:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

In matrix form:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \;\; 0), \quad D = 0$$

Example 1.1.2.2 (State-Space to Transfer Function): Given $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $C = (1 \;\; 1)$, $D = 0$. Find $G(s)$.

Solution: Calculate $(sI - A)^{-1}$:

$$sI - A = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix} \Rightarrow (sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix}$$

Then $G(s) = C(sI - A)^{-1}B$:

$$G(s) = (1 \;\; 1)\begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+3}{(s+1)(s+2)}$$

1.1.3 Supplementary Problems

Exercise 1.1.3.1 (System Classification): Classify the system $\dot{x} = -x + x^3$ as linear or non-linear, and time-invariant or time-varying.

Exercise 1.1.3.2 (Dimensionality): For a MIMO system with 2 inputs and 3 outputs, what are the dimensions of the D matrix?

1.2 1.2 Solving Linear Time-Invariant Differential Equations

1.2.1 Mathematical Review

A Linear Time-Invariant (LTI) system $\dot{x} = Ax + Bu$ has solutions described by the matrix exponential.

Theorem 1.2.1.1 (Solution to Homogenous System): For a homogenous system $\dot{x} = Ax$ (where $u(t) = 0$), the solution is:

$$x(t) = e^{At}x(0)$$

where $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ is the **Matrix Exponential**.

Theorem 1.2.1.2 (General Solution): For the non-homogenous system $\dot{x} = Ax + Bu$, the general solution is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

1.2.2 Solved Problems

Example 1.2.2.1 (Scalar Decay): Solve the scalar system $\dot{x} = -2x$ with initial condition $x(0) = 5$.

Solution: Here $A = -2$ (a 1×1 matrix). The solution is:

$$x(t) = e^{-2t}x(0) = 5e^{-2t}$$

This represents an exponential decay.

Example 1.2.2.2 (Diagonal Matrix Exponential): Find e^{At} for $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Solution: Since A is diagonal, the matrix exponential is simply the exponential of diagonal elements:

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

1.2.3 Supplementary Problems

Exercise 1.2.3.1 (Nilpotent Matrix): Find e^{At} for the nilpotent matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Exercise 1.2.3.2 (Forced Response): Write the integral expression for the solution of $\dot{x} = -x + 1$ with $x(0) = 0$.