

# MA2008B - Numerical Analysis for Non-Linear Optimization

Lecture Notes

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# 1 Control Theory

In this module, we introduce the fundamental concepts of control theory, contrasting classical methods with modern state-space approaches. We also explore the solution of linear time-invariant differential equations.

## 1.1 Conventional and State-Space Methods

### 1.1.1 Mathematical Review

Control theory serves as the foundation for analyzing and designing systems that maintain desired behaviors. We distinguish between two primary frameworks:

**Definition 1.1.1.1** (Conventional Control (Classical)): A frequency-domain approach using **Transfer Functions** ( $G(s)$ ). It is primarily used for Single-Input Single-Output (SISO), Linear, Time-Invariant (LTI) systems. Key metrics include overshoot, settling time, and steady-state error.

**Definition 1.1.1.2** (State-Space Methods (Modern)): A time-domain approach using **differential equations**. It models systems via state variables:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

This method handles Multiple-Input Multiple-Output (MIMO), non-linear, and time-varying systems.

The relationship between the two representations for LTI systems is given by:

$$G(s) = C(sI - A)^{-1}B + D$$

### 1.1.2 Solved Problems

*Example 1.1.2.1 (Transfer Function to State-Space):* Given the transfer function  $G(s) = \frac{1}{s^2+3s+2}$ , find a state-space representation.

**Solution:** The differential equation corresponds to  $\ddot{y} + 3\dot{y} + 2y = u$ . Let  $x_1 = y$  and  $x_2 = \dot{y}$ . Then:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u\end{aligned}$$

In matrix form:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \ 0), \quad D = 0$$

*Example 1.1.2.2 (State-Space to Transfer Function):* Given  $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $C = (1 \ 1)$ ,  $D = 0$ . Find  $G(s)$ .

**Solution:** Calculate  $(sI - A)^{-1}$ :

$$sI - A = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix} \Rightarrow (sI - A)^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix}$$

Then  $G(s) = C(sI - A)^{-1}B$ :

$$G(s) = (1 \ 1) \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+3}{(s+1)(s+2)}$$

### 1.1.3 Supplementary Problems

**Exercise 1.1.3.1** (System Classification): Classify the system  $\dot{x} = -x + x^3$  as linear or non-linear, and time-invariant or time-varying.

**Exercise 1.1.3.2** (Dimensionality): For a MIMO system with 2 inputs and 3 outputs, what are the dimensions of the  $D$  matrix?

## 1.2 Solving Linear Time-Invariant Differential Equations

### 1.2.1 Mathematical Review

A Linear Time-Invariant (LTI) system  $\dot{x} = Ax + Bu$  has solutions described by the matrix exponential.

**Theorem 1.2.1.1** (Solution to Homogenous System): For a homogenous system  $\dot{x} = Ax$  (where  $u(t) = 0$ ), the solution is:

$$x(t) = e^{At}x(0)$$

where  $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$  is the **Matrix Exponential**.

**Theorem 1.2.1.2** (General Solution): For the non-homogenous system  $\dot{x} = Ax + Bu$ , the general solution is:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

### 1.2.2 Solved Problems

*Example 1.2.2.1 (Scalar Decay):* Solve the scalar system  $\dot{x} = -2x$  with initial condition  $x(0) = 5$ .

**Solution:** Here  $A = -2$  (a  $1 \times 1$  matrix). The solution is:

$$x(t) = e^{-2t}x(0) = 5e^{-2t}$$

This represents an exponential decay.

*Example 1.2.2.2 (Diagonal Matrix Exponential):* Find  $e^{At}$  for  $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ .

**Solution:** Since  $A$  is diagonal, the matrix exponential is simply the exponential of diagonal elements:

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}$$

### 1.2.3 Supplementary Problems

**Exercise 1.2.3.1** (Nilpotent Matrix): Find  $e^{At}$  for the nilpotent matrix  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

**Exercise 1.2.3.2** (Forced Response): Write the integral expression for the solution of  $\dot{x} = -x + 1$  with  $x(0) = 0$ .

## 1.3 Transfer Function

### 1.3.1 Mathematical Review

The **Transfer Function**  $G(s)$  of a Linear Time-Invariant (LTI) system is defined as the ratio of the Laplace transform of the output  $Y(s)$  to the Laplace transform of the input  $U(s)$ , assuming all initial conditions are zero.

**Definition 1.3.1.1** (Transfer Function):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}$$

where  $n \geq m$  for physical realizability. The roots of the denominator are the **poles**, and the roots of the numerator are the **zeros**.

**Impulse Response:** The system output when the input is a Dirac delta function  $\delta(t)$ . Its Laplace transform is  $G(s)$ .

### 1.3.2 Solved Problems

*Example 1.3.2.1 (RC Circuit):* Consider a series RC circuit where the input is voltage  $v_{\{e\}}(t)$  and output is capacitor voltage  $v_{c(t)}$ . Find  $G(s)$ .

**Solution:** Kirchhoff's voltage law:

$$v_{\{\epsilon\}}(t) = Ri(t) + v_{c(t)}$$

Current relation:  $i(t) = C\dot{v}_{c(t)}$ . Substituting:

$$v_{\{\epsilon\}}(t) = RC\dot{v}_{c(t)} + v_{c(t)}$$

Taking Laplace transform (zero initial conditions):

$$V_{\{\epsilon\}}(s) = (RCs + 1)V_{c(s)}$$

Transfer function:

$$G(s) = \frac{V_{c(s)}}{V_{\{\epsilon\}}}(s) = \frac{1}{RCs + 1}$$

*Example 1.3.2.2 (Poles and Stability):* Given  $G(s) = \frac{10}{s^2 + 2s + 5}$ , find the poles and determine stability.

**Solution:** The poles are roots of  $s^2 + 2s + 5 = 0$ . Using quadratic formula:

$$s = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm -2j$$

Since the real part ( $-1$ ) is negative, the system is **stable**.

### 1.3.3 Supplementary Problems

**Exercise 1.3.3.1** (Mechanical System): Find transfer function  $\frac{X(s)}{F(s)}$  for a mass-spring-damper system:  $M\ddot{x} + B\dot{x} + Kx = f(t)$ .

**Exercise 1.3.3.2** (Zero Locations): How do zeros in the right-half plane (non-minimum phase) affect the step response?

## 1.4 Block Diagrams and Signal Flow Graphs

### 1.4.1 Mathematical Review

**Block Diagrams** graphically represent systems where functional blocks are connected by signals.

**Definition 1.4.1.1** (Block Diagram Algebra):

- **Series:**  $G_{\{=\}}(s) = G_1(s)G_2(s)$
- **Parallel:**  $G_{\{=\}}(s) = G_1(s) + G_2(s)$
- **Feedback Loop:** For negative feedback  $H(s)$ , the closed-loop transfer function is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

**Signal Flow Graphs (SFG)** are directed graphs where nodes represent variables and branches represent gains. **Mason's Gain Formula** computes the transfer function of an SFG.

### 1.4.2 Solved Problems

*Example 1.4.2.1 (Feedback Reduction):* A system has forward path  $G(s) = \frac{K}{s+1}$  and unity negative feedback ( $H(s) = 1$ ). Find the closed loop transfer function  $T(s)$ .

**Solution:** Using the feedback formula:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{s+1}}{1 + \frac{K}{s+1}}$$

Multiply numerator and denominator by  $(s + 1)$ :

$$T(s) = \frac{K}{s + 1 + K}$$

*Example 1.4.2.2 (Mason's Rule Application):* Find the transfer function  $\frac{C}{R}$  for a simple loop with forward gain  $G$  and feedback  $H$ .

**Solution:**

- Forward paths:  $P_1 = G$ .
- Loops:  $L_1 = -GH$ .
- Determinant  $\Delta = 1 - (\sum L_i) = 1 - (-GH) = 1 + GH$ .
- Path cofactor  $\Delta_1 = 1$  (loop touches the path).
- Result:  $T = \frac{P_1 \Delta_1}{\Delta} = \frac{G}{1+GH}$ .

**1.4.3 Supplementary Problems**

**Exercise 1.4.3.1** (Multiple Loops): Simplify a block diagram with two nested feedback loops.

**Exercise 1.4.3.2** (Signal Flow Construction): Draw the Signal Flow Graph corresponding to the equations:  $x_2 = ax_1 + bx_3$ ,  $x_3 = cx_2$ .

**1.5 State-Space Analysis****1.5.1 Mathematical Review**

State-space analysis uses the vector-matrix representation of systems.

**Definition 1.5.1.1** (State Controllability): A system is **controllable** if the state vector can be transferred from any initial state to any final state in finite time. The Controllability Matrix is:

$$\mathcal{C} = [B, AB, A^2B, \dots, A^{n-1}B]$$

The system is fully controllable if  $\text{rank}(\mathcal{C}) = n$ .



**Definition 1.5.1.2** (State Observability): A system is **observable** if the initial state can be determined from the output history. The Observability Matrix is:

$$\mathcal{O} = \left[ C^T, A^T C^T, (A^T)^2 C^T, \dots, (A^T)^{n-1} C^T \right]^T$$

The system is fully observable if  $\text{rank}(\mathcal{O}) = n$ .

### 1.5.2 Solved Problems

*Example 1.5.2.1 (Controllability Check):* Check controllability for  $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

**Solution:** Compute  $\mathcal{C} = [B, AB]$ .

$$AB = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$$

Determinant is  $0(-3) - 1(1) = -1 \neq 0$ . Rank is 2 ( $n = 2$ ). Therefore, the system is **controllable**.

*Example 1.5.2.2 (Observability Check):* Check observability for same  $A$  with  $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ .

**Solution:** Compute  $\mathcal{O} = [C, CA]^T$ .

$$CA = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is the identity matrix. Rank is 2. Therefore, the system is **observable**.

### 1.5.3 Supplementary Problems

**Exercise 1.5.3.1** (Uncontrollable Mode): Identify the uncontrollable mode in a system with diagonal  $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

**Exercise 1.5.3.2** (Duality): Prove that the pair  $(A, B)$  is controllable if and only if  $(A^T, B^T)$  is observable? (Note: Check duality property accurately).

## 1.6 Basic Design Principles of Control Systems

### 1.6.1 Mathematical Review

Control system design involves selecting components and parameters to satisfy performance, stability, and robustness specifications.

**Definition 1.6.1.1** (Design Specifications): Common metrics include:

- **Rise Time ( $t_r$ )**: Time to reach the vicinity of the final value.
- **Settling Time ( $t_s$ )**: Time to stay within a tolerance band (e.g., 2%).
- **Percent Overshoot ( $M_p$ )**: Peak value relative to steady state.
- **Phase Margin / Gain Margin**: Measures of relative stability.

**Theorem 1.6.1.1** (PID Control): A Proportional-Integral-Derivative controller has the law:

$$u(t) = K_p e(t) + K_i \int e(\tau) d\tau + K_d \dot{e}(t)$$

- **P**: Improves speed.
- **I**: Eliminates steady-state error.
- **D**: Improves damping (reduces overshoot).

### 1.6.2 Solved Problems

*Example 1.6.2.1 (Parameter Selection for Damping)*: A second order system has characteristic equation  $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ . Select  $\zeta$  for  $M_p \approx 5\%$ .

**Solution:** Percentage overshoot is given by  $M_p = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}}$ . For  $M_p = 0.05$ :

$$\ln(0.05) = -\pi \frac{\zeta}{\sqrt{1-\zeta^2}} \approx -3$$

$$\pi^2 \zeta^2 \approx 9(1 - \zeta^2) \Rightarrow (\pi^2 + 9)\zeta^2 \approx 9$$

$$\zeta \approx \sqrt{\frac{9}{19}} \approx 0.69$$

A damping ratio of  $\zeta \approx 0.7$  typically gives  $\approx 5\%$  overshoot.

*Example 1.6.2.2 (Steady State Error):* Find steady-state error for a unit step input with forward gain  $G(s) = \frac{10}{s+1}$  in unity feedback.

**Solution:** System Type is 0 (no integrators in  $G(s)$ ). Static position error constant  $K_p = \lim_{s \rightarrow 0} G(s) = \frac{10}{1} = 10$ . Steady state error  $e_{ss} = \frac{1}{1+K_p} = \frac{1}{11}$ .

### 1.6.3 Supplementary Problems

**Exercise 1.6.3.1** (PID Tuning): Describe the Ziegler-Nichols tuning method for a PID controller.

**Exercise 1.6.3.2** (Root Locus Design): How does adding a pole at the origin affect the root locus and stability?