

# Partial Differential Equations in Finance

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# Overview of Topics

- ▶ Charting theory and nature of boundary and initial conditions
- ▶ Explicit solutions, including original Black-Scholes formula
- ▶ Special problems arising when there are free boundaries

# Key Questions in Financial PDEs

- ▶ Physical interpretation of the equations?
- ▶ Mathematical properties of the solution?
- ▶ Techniques for obtaining explicit solutions?

PDEs in finance:

- ▶ Fundamental equations (e.g., Black-Scholes)
- ▶ Linear vs. nonlinear problems

# Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
  - ▶ Appropriate boundary or initial/final conditions?
  - ▶ Nature of the mathematical problem?
  - ▶ Smooth or discontinuous solutions?
2. Can we develop analytical tools to solve the equation?
3. How should we solve the equation numerically if necessary?

# First Order Linear PDE

Consider the equation:

$$\alpha(s, t) \frac{\partial u}{\partial s} + \beta(s, t) \frac{\partial u}{\partial t} = \gamma(s, t) u(t)$$

## Linearity Property

If  $u_1, u_2$  are solutions, then  $c_1 u_1 + c_2 u_2$  is also a solution for constants  $c_1, c_2$ .

## Proof.

Substitute  $u = c_1 u_1 + c_2 u_2$  into the PDE and use linearity of derivatives.



# Constant Coefficient Case

When  $\alpha, \beta$  are constants and  $\gamma \equiv 0$ :

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector  $\vec{v} = (\alpha_0, \beta_0)$  with  $\beta_0 \alpha_0 \neq 0$ .

The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \left\langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right\rangle$$

Thus, (4.2)  $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

# Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 s + \alpha_0 t, \quad \zeta = \beta_0 s - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Computing  $\frac{\partial u}{\partial \xi}$ :

$$\frac{\partial u}{\partial \xi} = \beta_0 \frac{\partial u}{\partial s} + \alpha_0 \frac{\partial u}{\partial t} \equiv 0$$

Therefore,  $u$  depends only on  $\zeta$ :  $u = F(\zeta)$

# Characteristics and Information Propagation

- ▶  $\xi = \beta_0 s - \alpha_0 t$  is called a **characteristic**
- ▶ Characteristics represent directions in which information propagates
- ▶ Analogous to constants of integration in ODEs

## Important Note

Without boundary conditions, the solution remains arbitrary (like the constant  $C$  in ODEs)



# General Second-Order Linear PDE

Consider the general form in two variables:

$$a(x, t)u_{xx} + 2b(x, t)u_{xt} + c(x, t)u_{tt} + d(x, t)u_x + e(x, t)u_t + f(x, t)u = g(x, t)$$

where  $a, b, c, d, e, f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are coefficient functions.

## Principal Part

The highest-order terms form the principal part:

$$L_p(u) = au_{xx} + 2bu_{xt} + cu_{tt}$$

# Classification of PDEs

Define the discriminant:

$$\Delta(L) = b^2 - ac : (x, t) \mapsto b(x, t)^2 - a(x, t)c(x, t)$$

The PDE is classified as:

- ▶ **Hyperbolic** if  $\Delta(L) > 0$  (e.g., wave equation)
- ▶ **Parabolic** if  $\Delta(L) = 0$  (e.g., heat equation)
- ▶ **Elliptic** if  $\Delta(L) < 0$  (e.g., Laplace equation)

# Coordinate Transformations

## Definition

A transformation  $(\xi, \eta) = (\xi(x, t), \eta(x, t))$  is called a change of coordinates if the Jacobian is non-zero:

$$J := \xi_x \eta_t - \xi_t \eta_x \neq 0$$

## Theorem

*The type of a linear second-order PDE in two variables is invariant under a change of coordinates.*

# Proof of Invariance

Under transformation  $(\xi, \eta)$ , the principal part becomes:

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta}$$

where:

$$A = a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t$$

$$C = a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2$$

The new discriminant satisfies:

$$\Delta_{new} = AC - B^2 = J^2(b^2 - ac) = J^2\Delta_{original}$$

# Canonical Forms

- ▶ **Hyperbolic:**  $u_{\xi\xi} - u_{\eta\eta} = \text{lower order terms}$  (wave equation)
- ▶ **Parabolic:**  $u_{\xi\xi} = \text{lower order terms}$  (heat equation)
- ▶ **Elliptic:**  $u_{\xi\xi} + u_{\eta\eta} = \text{lower order terms}$  (Laplace equation)

## Examples

- ▶ Wave equation:  $u_{tt} = c^2(u_{xx} + u_{yy})$
- ▶ Heat equation:  $u_t = k(u_{xx} + u_{yy})$
- ▶ Laplace equation:  $u_{xx} + u_{yy} = 0$

# Simplification Through Coordinates

A proper linear change of coordinates can transform:

- ▶ Hyperbolic equations to  $u_{\xi\eta} = 0$
- ▶ Parabolic equations to  $u_{\xi\xi} = 0$
- ▶ Elliptic equations to  $u_{\xi\xi} + u_{\eta\eta} = 0$

## Important Note

The classification determines:

- ▶ Nature of characteristics
- ▶ Appropriate boundary conditions
- ▶ Solution techniques

# Heat Equation

The fundamental heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Physical interpretation:

- ▶  $u(x, t)$  represents temperature distribution along a thin insulated rod
- ▶ Models heat diffusion over time

# Mathematical Properties

Key characteristics:

- ▶ Linear equation (superposition principle applies)
- ▶ Second order in space ( $x$ )
- ▶ First order in time ( $t$ )
- ▶ Parabolic type (discriminant  $\Delta = 0$ )

## Analytic Solutions

Solutions are analytic functions of  $x$ :

- ▶ For any  $t^* > 0$  and  $a < x_0 < b$
- ▶  $u(x, t^*)$  can be expressed as a convergent power series in  $(x - x_0)$



# Solution Behavior

Important features:

- ▶ **Smoothing property:** Solutions become instantly smooth for  $t > 0$
- ▶ **Infinite propagation speed:** Initial disturbances spread infinitely fast
- ▶ **Maximum principle:** Extreme values occur only at initial/boundary conditions

## Financial Interpretation

Analogous to:

- ▶ Diffusion of volatility in option pricing
- ▶ Smoothing of price shocks over time

# Dirichlet Boundary Value Problem

Heat equation on finite interval:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L, \quad t > 0$$

With conditions:

- ▶ Initial condition:  $u(x, 0) = u_0(x)$
- ▶ Boundary conditions:
  - ▶  $u(-L, t) = g_1(t)$  (left end temperature)
  - ▶  $u(L, t) = g_2(t)$  (right end temperature)

## Physical Interpretation

Models heat diffusion in a finite rod with controlled end temperatures

# Neumann Boundary Value Problem

Same heat equation with flux conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L, \quad t > 0$$

With conditions:

- ▶ Initial condition:  $u(x, 0) = u_0(x)$
- ▶ Flux boundary conditions:
  - ▶  $-\frac{\partial u}{\partial x}(-L, t) = h_1(t)$  (left end flux)
  - ▶  $\frac{\partial u}{\partial x}(L, t) = h_2(t)$  (right end flux)

## Physical Interpretation

Models heat diffusion with controlled heat flow at boundaries

# Infinite Domain Problem

Taking  $L \rightarrow \infty$ :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0$$

With initial condition:

$$u(x, 0) = u_0(x)$$

## Growth Conditions

For well-posedness:

- ▶  $u_0(x)$  piecewise continuous (finite jumps allowed)
- ▶  $\lim_{|x| \rightarrow \infty} \frac{u_0(x)}{e^{ax^2}} = 0 \quad \forall a > 0$
- ▶  $\lim_{|x| \rightarrow \infty} \frac{u(x, t)}{e^{ax^2}} = 0 \quad \forall a, t > 0$

# Semi-Infinite Domain with Mixed Conditions

Example problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

With conditions:

- ▶ Initial condition:  $u(x, 0) = u_0(x)$  (sufficiently smooth)
- ▶ Boundary condition:  $u(0, t) = g(t)$
- ▶ Growth conditions at infinity

## Well-Posedness

The problem is well-posed when:

- ▶ Initial/boundary conditions are compatible
- ▶ Growth conditions are satisfied
- ▶ Solution exists, is unique, and depends continuously on data