MA2008B: Assignment 03

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Instructions

- Show all your work for full credit.
- Write your answers clearly and neatly.
- Submit your assignment by the due date.

1 Partial Differential Equations

Problem 1.1. *Show that the solution to the initial value problem is unique provided that it is sufficiently smooth and decays sufficiently fast at infinity, as follows:*

Suppose that $u_1(x,t)$ *and* $u_2(x,t)$ *are both solutions to the initial value problem (4.5)–(4.9):*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \infty < x < \infty, \quad t > 0, \tag{4.5}$$

with

$$u(x,0) = u_0(x), \tag{4.6}$$

where
(i) $u_0(x)$ *is sufficiently well behaved,* *(4.7)*
(ii)

$$\lim_{|x|\to\infty} u_0(x)e^{-ax^2} = 0$$

for any $a > 0$, *(4.8)*
and lastly where

$$\lim_{|x|\to\infty} u(x,t)e^{-ax^2} = 0$$

for any $a > 0$, $t > 0$, *(4.9)*
Show that $v(x,t) = u_1 - u_2$ *is also a solution of (4.5) with* $v(x,0) = 0$.

Show that if

$$E(t) = \int_{-\infty}^{\infty} v^2 \, dx,$$

then

$$E(t) \geq 0, \quad E(0) = 0,$$

and, by integrating by parts, that

$$\frac{dE}{dt} \leq 0;$$

thus $E(t) \equiv 0$*, hence* $v(x,t) \equiv 0$*.*

Note, though, that as yet we have no guarantee that $v(x,t)$ *exists, nor that the above manipulations can be justified.*

Problem 1.2. *Show that* $\sin nx \, e^{-n^2 t}$ *is a solution of the forward diffusion equation, and that* $\sin nx \, e^{n^2 t}$ *is a solution of the backward diffusion equation. Now try to solve the initial value problem for the forward and backward equations in the interval* $-\pi < x < \pi$ *by expanding the solution in a Fourier series in* x *with coefficients depending on* t*. What difference do you see between the two problems? Which is well-posed?*

References

[1] P. Wilmott, J. Dewynne, S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, 1993.