Partial Differential Equations in Finance

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Overview of Topics

- Charting theory and nature of boundary and initial conditions
- Explicit solutions, including original Black-Scholes formula
- Special problems arising when there are free boundaries

Key Questions in Financial PDEs

- Physical interpretation of the equations?
- Mathematical properties of the solution?
- Techniques for obtaining explicit solutions?

PDEs in finance:

- Fundamental equations (e.g., Black-Scholes)
- Linear vs. nonlinear problems

Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
   - Appropriate boundary or initial/final conditions?
   - Nature of the mathematical problem?
   - Smooth or discontinuous solutions?
2. Can we develop analytical tools to solve the equation?
3. How should we solve the equation numerically if necessary?

First Order Linear PDE

Consider the equation:

$$\alpha(s,t)\frac{\partial u}{\partial s} + \beta(s,t)\frac{\partial u}{\partial t} = \gamma(s,t)u(t)$$

Linearity Property

If u_1, u_2 are solutions, then $c_1 u_1 + c_2 u_2$ is also a solution for constants c_1, c_2.

Proof.

Substitute $u = c_1 u_1 + c_2 u_2$ into the PDE and use linearity of derivatives. □

Constant Coefficient Case

When α, β are constants and $\gamma \equiv 0$:

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector $\vec{v} = (\alpha_0, \beta_0)$ with $\beta_0 \alpha_0 \neq 0$.
The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \rangle$$

Thus, (4.2) $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 s + \alpha_0 t, \quad \zeta = \beta_0 s - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Computing $\frac{\partial u}{\partial \xi}$:

$$\frac{\partial u}{\partial \xi} = \beta_0 \frac{\partial u}{\partial s} + \alpha_0 \frac{\partial u}{\partial t} \equiv 0$$

Therefore, u depends only on ζ: $u = F(\zeta)$

Characteristics and Information Propagation

- $\xi = \beta_0 s - \alpha_0 t$ is called a **characteristic**
- Characteristics represent directions in which information propagates
- Analogous to constants of integration in ODEs

Important Note

Without boundary conditions, the solution remains arbitrary (like the constant C in ODEs)