Numerical Methods For Ordinary Differential Equations

References:

1. Chasnov, J. R. (2012). Numerical methods. Hong Kong University of Science and Technology. https://www.math.hkust.edu.hk/~machas/numerical-methods.pdf
2. Liu, J., Spiegel, M. R. (1999). Mathematical Handbook of Formulas and Tables. United Kingdom: McGraw-Hill.

Here we give methods to solve the following initial-value problem of an ordinary differential equation:

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

The methods will use a computational grid:

$$t_n = t_0 + nh \tag{2}$$

where h is the grid size.

Example

Consider $f(t,x) = rx(K-x)$, with the following parameters:

```
r = 0.15
K = 100
t0 = 0
tf = 1
x0 = 1
n = 10
step_size =  (tf-t0)/n
step_size
```

```
1/10
```

```
t = var('t')

x = function('x')(t)

ode = x.diff(t) == r*x*(K-x)

solution = desolve(ode, dvar=x, ivar=t, ics=[t0,x0])

solution
```

```
-1/15*log(x(t) - 100) + 1/15*log(x(t)) == -1/15*I*pi + t - 1/15*log(99)
```

```
u = var('u')
solution = solution.subs({x: u})
solution
```

```
-1/15*log(u - 100) + 1/15*log(u) == -1/15*I*pi + t - 1/15*log(99)
```

```
solution.solve(u, to_poly_solve=True)
```

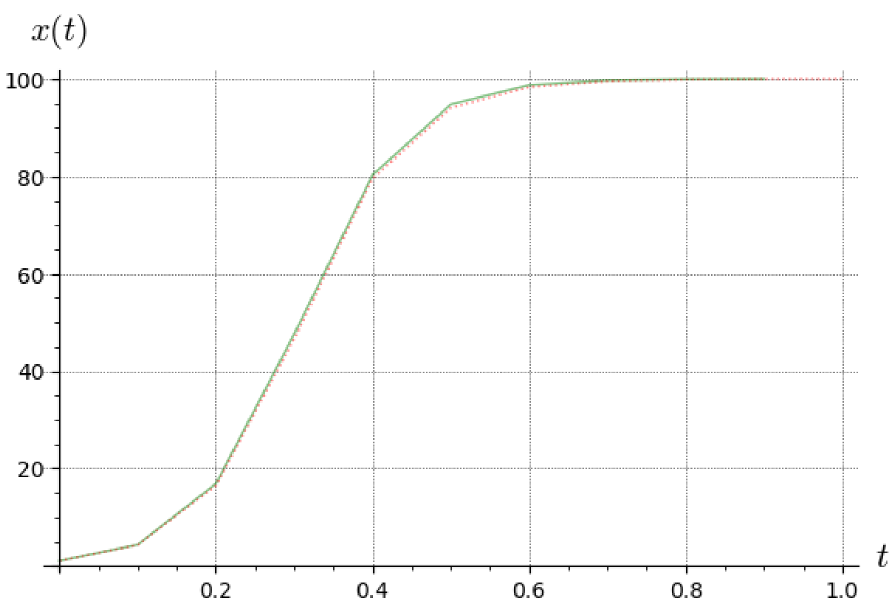
```
[u == 100*e^(15*t)/(e^(15*t) + 99)]
```

```
x_exact = solution.solve(u, to_poly_solve=True)[0].rhs()
x_exact
```

```
100*e^(15*t)/(e^(15*t) + 99)
```

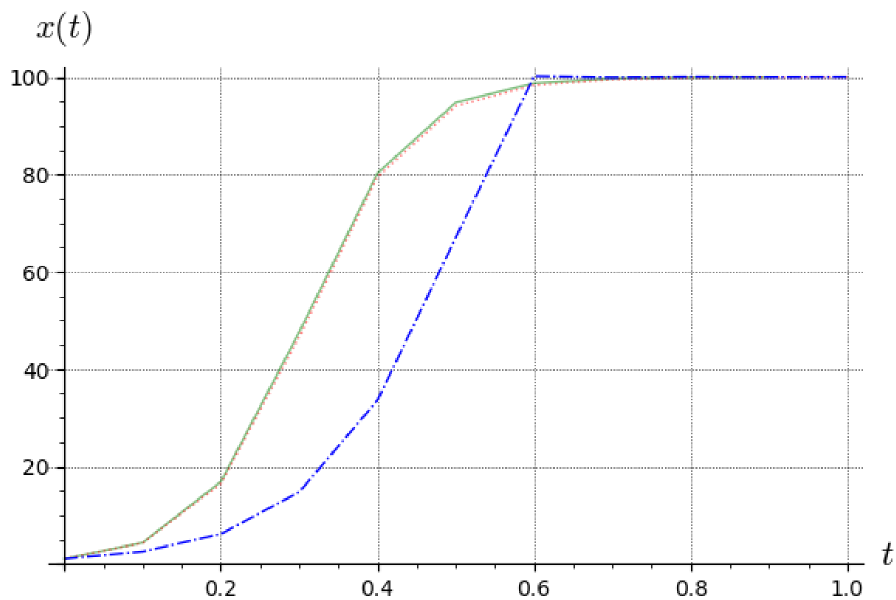
```
custom_grid = [t0 + i*step_size for i in range(n)]
exact_solution = [[_t.n(), x_exact(t=_t).n()] for _t in custom_grid]
```

```
t,x = var('t x')
numerical_solution = desolve_rk4( r*x*(K-x), x, ivar = t, ics=[t0,x0], end_points=tf, step=s
sage_solution = list_plot(exact_solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gr
sage_solution += list_plot(numerical_solution, plotjoined=True, alpha=0.5, color="red", line
sage_solution.show()
```




```
def euler_method(f, x0, t0, t_end, h):
    """
    Implements the Euler method for solving x' = f(t, x)
```

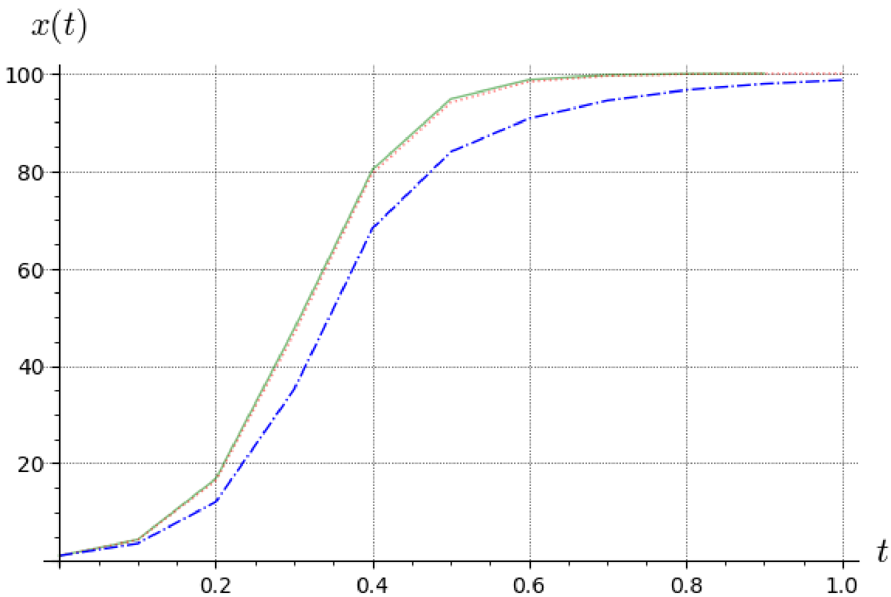
```
    Parameters:
        f: a function of (t, x)
        x0: initial value x(t0)
        t0: initial time
        t_end: final time
        h: step size

    Returns:
        A list of (t, x) points
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        x = x + h * f(t, x)
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))
t, x = var('t x')
f(t, x) = r * x * (K - x)  # logistic equation

solution = euler_method(f, x0=x0, t0=t0, t_end=tf, h=step_size)

# Plot
euler_solution = list_plot(solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlin
compare_euler = sage_solution + euler_solution
compare_euler.plot()
```




```
def modified_euler_method(f, x0, t0, t_end, h):
    """
    Implements the Modified Euler method (Heun's Method) for solving x' = f(t, x)

    Parameters:
        f: a function of (t, x)
        x0: initial value x(t0)
        t0: initial time
        t_end: final time
        h: step size

    Returns:
        A list of (t, x) points
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        k1 = h*f(t, x)
        k2 = h*f(t + h, x + k1)
        x = x + 1/2 * (k1 + k2)
```

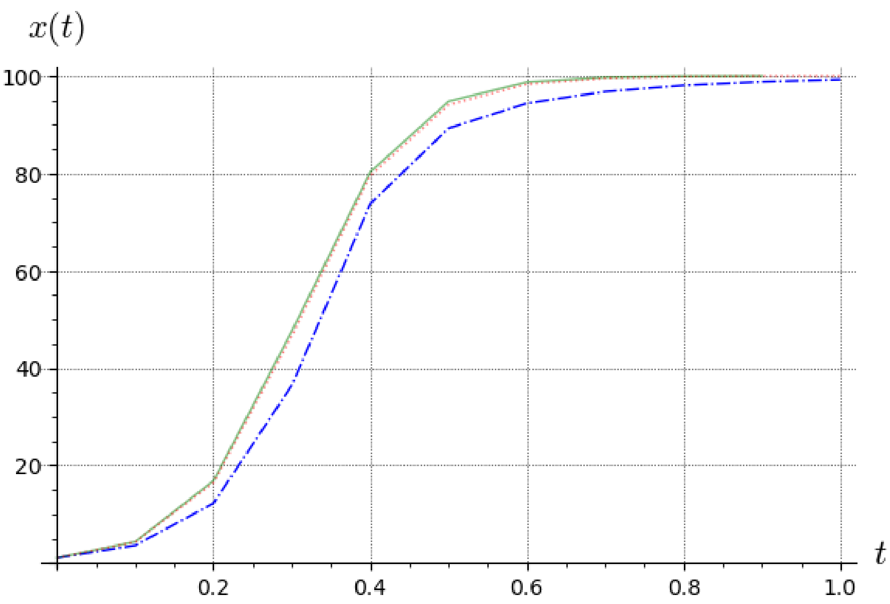
```
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))
```

```
solution = modified_euler_method(f, x0=x0, t0=t0, t_end=tf, h=step_size)

# Plot the numerical approximation
heun_solution = list_plot(solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridline
compare_heun = sage_solution + heun_solution
compare_heun.plot()
```

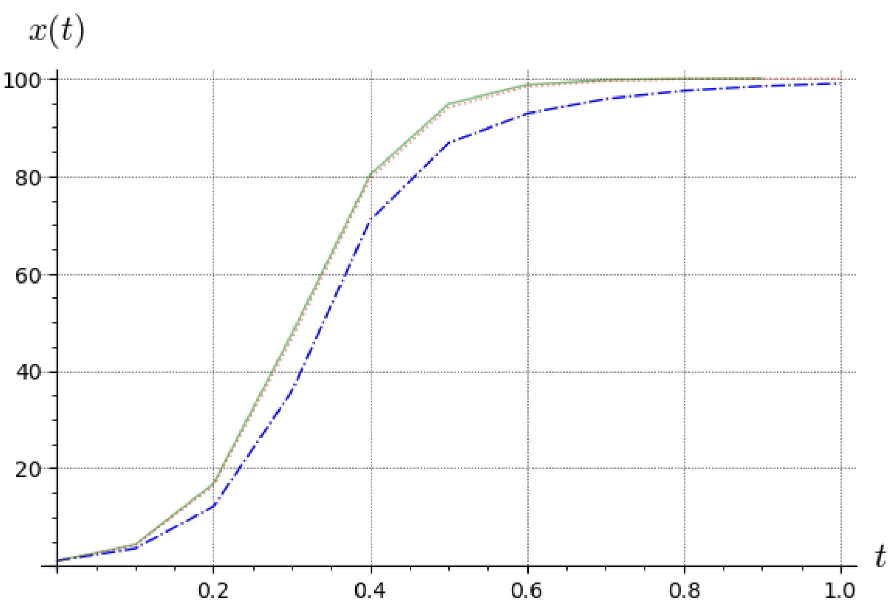



```
def rk2_general_form(f, x0, t0, t_end, h=0.01,
                     alpha=1/2, beta=1/2, a=0, b=1):
    """
    General second-order Runge-Kutta method using parameters alpha, beta, a, b.

    Solves x' = f(t, x) over [t0, t_end] with initial condition x(t0) = x0.

    Parameters:
        f: right-hand side function f(t, x)
        x0: initial condition x(t0)
        t0: initial time
        t_end: final time
        h: time step size
```

```
        alpha: time increment coefficient for k2
        beta: slope coefficient for k2
        a, b: weights for k1 and k2 in the update

    Returns:
        List of (t, x) points approximating the solution
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        k1 = h * f(t, x)
        k2 = h * f(t + alpha * h, x + beta * k1)
        x = x + a * k1 + b * k2
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))
sol = rk2_general_form(f, x0=x0, t0=t0, t_end=tf, h=step_size)

heun_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines=Tr
compare_heun = heun_solution  + sage_solution
compare_heun.plot()
```




```
sol = rk2_general_form(f, x0=x0, t0=t0, t_end=tf, h=step_size, alpha=3/4, beta=3/4, a=1/3, b

ralston_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines
compare_heun = ralston_solution  + sage_solution
compare_heun.plot()
```



```
def rk4(f, x0, t0, t_end, h=0.01):
    """
    Fourth-order Runge-Kutta method as specified in the image.

    Solves x' = f(t, x) over [t0, t_end] with initial condition x(t0) = x0.

    Parameters:
        f: right-hand side function f(t, x)
        x0: initial condition x(t0)
        t0: initial time
        t_end: final time
        h: time step size

    Returns:
        List of (t, x) points approximating the solution
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        # Calculate the four k values according to the formulas
        k1 = h * f(t, x)
        k2 = h * f(t + 0.5 * h, x + 0.5 * k1)
        k3 = h * f(t + 0.5 * h, x + 0.5 * k2)
        k4 = h * f(t + h, x + k3)

        # Update x using the weighted average
        x = x + (1/6) * (k1 + 2*k2 + 2*k3 + k4)

        # Increment time
        t = t + h

        # Store results
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))

sol = rk4(f, x0=x0, t0=t0, t_end=tf, h=step_size)

ralston_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines
compare_heun = ralston_solution  + sage_solution
```

```
compare_heun.plot()
```