# Partial Differential Equations in Finance

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# Overview of Topics

- Charting theory and nature of boundary and initial conditions
- Explicit solutions, including original Black-Scholes formula
- Special problems arising when there are free boundaries

# Key Questions in Financial PDEs

- Physical interpretation of the equations?
- Mathematical properties of the solution?
- Techniques for obtaining explicit solutions?

PDEs in finance:

- Fundamental equations (e.g., Black-Scholes)
- Linear vs. nonlinear problems

# Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
   - Appropriate boundary or initial/final conditions?
   - Nature of the mathematical problem?
   - Smooth or discontinuous solutions?
2. Can we develop analytical tools to solve the equation?
3. How should we solve the equation numerically if necessary?

# First Order Linear PDE

Consider the equation:

$$\alpha(s,t)\frac{\partial u}{\partial s} + \beta(s,t)\frac{\partial u}{\partial t} = \gamma(s,t)u(t)$$

## Linearity Property

If $u_1, u_2$ are solutions, then $c_1 u_1 + c_2 u_2$ is also a solution for constants $c_1, c_2$.

## Proof.

Substitute $u = c_1 u_1 + c_2 u_2$ into the PDE and use linearity of derivatives. $\square$

# Constant Coefficient Case

When $\alpha, \beta$ are constants and $\gamma \equiv 0$:

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector $\vec{v} = (\alpha_0, \beta_0)$ with $\beta_0 \alpha_0 \neq 0$.
The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \rangle$$

Thus, (4.2) $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

# Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 s + \alpha_0 t, \quad \zeta = \beta_0 s - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Computing $\frac{\partial u}{\partial \xi}$:

$$\frac{\partial u}{\partial \xi} = \beta_0 \frac{\partial u}{\partial s} + \alpha_0 \frac{\partial u}{\partial t} \equiv 0$$

Therefore, $u$ depends only on $\zeta$: $u = F(\zeta)$

# Characteristics and Information Propagation

- $\xi = \beta_0 s - \alpha_0 t$ is called a **characteristic**
- Characteristics represent directions in which information propagates
- Analogous to constants of integration in ODEs

Important Note

Without boundary conditions, the solution remains arbitrary (like the constant $C$ in ODEs)

# General Second-Order Linear PDE

Consider the general form in two variables:

$a(x,t)u_{xx}+2b(x,t)u_{xt}+c(x,t)u_{tt}+d(x,t)u_x+e(x,t)u_t+f(x,t)u=g(x,t$

where $a, b, c, d, e, f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are coefficient functions.

## Principal Part

The highest-order terms form the principal part:

$$L_p(u) = au_{xx} + 2bu_{xt} + cu_{tt}$$

# Classification of PDEs

Define the discriminant:

$$\Delta(L) = b^2 - ac : (x, t) \mapsto b(x, t)^2 - a(x, t)c(x, t)$$

The PDE is classified as:

- **Hyperbolic** if $\Delta(L) > 0$ (e.g., wave equation)
- **Parabolic** if $\Delta(L) = 0$ (e.g., heat equation)
- **Elliptic** if $\Delta(L) < 0$ (e.g., Laplace equation)

# Coordinate Transformations

## Definition

A transformation $(\xi, \eta) = (\xi(x,t), \eta(x,t))$ is called a change of coordinates if the Jacobian is non-zero:

$$J := \xi_x \eta_t - \xi_t \eta_x \neq 0$$

## Theorem

*The type of a linear second-order PDE in two variables is invariant under a change of coordinates.*

# Proof of Invariance

Under transformation $(\xi, \eta)$, the principal part becomes:

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta}$$

where:

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2 \\ B &= a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t \\ C &= a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2 \end{aligned}$$

The new discriminant satisfies:

$$\Delta_{new} = AC - B^2 = J^2(b^2 - ac) = J^2\Delta_{original}$$

# Canonical Forms

- **Hyperbolic**: $u_{\xi\xi} - u_{\eta\eta} =$ lower order terms (wave equation)
- **Parabolic**: $u_{\xi\xi} =$ lower order terms (heat equation)
- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} =$ lower order terms (Laplace equation)

## Examples

- Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$
- Heat equation: $u_t = k(u_{xx} + u_{yy})$
- Laplace equation: $u_{xx} + u_{yy} = 0$

# Simplification Through Coordinates

A proper linear change of coordinates can transform:

- Hyperbolic equations to $u_{\xi\eta} = 0$
- Parabolic equations to $u_{\xi\xi} = 0$
- Elliptic equations to $u_{\xi\xi} + u_{\eta\eta} = 0$

Important Note

The classification determines:

- Nature of characteristics
- Appropriate boundary conditions
- Solution techniques

# Heat Equation

The fundamental heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Physical interpretation:

- $u(x, t)$ represents temperature distribution along a thin insulated rod
- Models heat diffusion over time

# Mathematical Properties

Key characteristics:

- Linear equation (superposition principle applies)
- Second order in space ($x$)
- First order in time ($t$)
- Parabolic type (discriminant $\Delta = 0$)

## Analytic Solutions

Solutions are analytic functions of $x$:

- For any $t^* > 0$ and $a < x_0 < b$
- $u(x, t^*)$ can be expressed as a convergent power series in $(x - x_0)$

# Solution Behavior

Important features:

- **Smoothing property**: Solutions become instantly smooth for $t > 0$
- **Infinite propagation speed**: Initial disturbances spread infinitely fast
- **Maximum principle**: Extreme values occur only at initial/boundary conditions

## Financial Interpretation

Analogous to:

- Diffusion of volatility in option pricing
- Smoothing of price shocks over time