Numerical Integration

The notes for this week have been obtained from > > 1. Chasnov, J. R. (2012). Numerical methods. Hong Kong University of Science and Technology. https://www.math.hkust.edu.hk/~machas/numerical-methods.pdf > 2. Scheid, F. J. (1968). Schaum's outline of theory and problems of numerical analysis. > 3. Stoer, J., Bulirsch, R. (2013). Introduction to Numerical Analysis. Germany: Springer New York.

The trapezoidal rule

$$\int_{x_0}^{x_n} y(x)\,dx \approx \frac{1}{2}h(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n)$$

is an elementary, but typical, **composite numerical integration formula**, known as the **composite trapezoidal rule**. This rule approximates the area under the curve $y(x)$ over the interval $[x_0, x_n]$ by dividing the interval into n equal subintervals of width $h = \frac{x_n - x_0}{n}$, and using **straight line segments** to connect successive points (x_i, y_i). These segments form trapezoids, whose areas are straightforward to compute.

The formula effectively replaces the integrand $y(x)$ with a piecewise linear function, interpolating $y(x)$ at the mesh points $x_0, x_1, \ldots, x_n$. The sum of the areas of these trapezoids gives the approximation to the integral.

This approach is simple yet remarkably effective for sufficiently smooth functions. However, like any numerical method, it introduces an error, known as the **truncation error**, which quantifies the deviation from the exact integral. For the trapezoidal rule, the error is approximately

$$-\frac{(x_n - x_0)\,h^2}{12}\,y^{(2)}(\xi)$$

for some $\xi \in [x_0, x_n]$, where $y^{(2)}(\xi)$ denotes the second derivative of $y(x)$ at some point ξ in the integration interval. This expression shows that the error decreases quadratically with the step size h, assuming $y^{(2)}(x)$ remains bounded. Consequently, halving the step size reduces the error by roughly a factor of four, reflecting the **second-order accuracy** of the method.

Thus, the trapezoidal rule provides a good balance between simplicity and accuracy, particularly for functions that are at least twice continuously differentiable on the interval of integration.

Solved Problem 14.5

> Apply the trapezoidal rule to approximate the integral of $\sqrt{x}$ over the interval $[1.00, 1.30]$ using six equally spaced subintervals. Then compare the result with the exact value of the integral.

Dividing the interval $[1.00, 1.30]$ into $n = 6$ subintervals gives a step size of $h = 0.05$. The corresponding partition points are:

$$x_i = 1.00,\ 1.05,\ 1.10,\ 1.15,\ 1.20,\ 1.25,\ 1.30$$

Evaluating the function $y(x) = \sqrt{x}$ at these points yields:

$$y_i = 1.00000,\ 1.02470,\ 1.04881,\ 1.07238,\ 1.09545,\ 1.11803,\ 1.14017$$

Applying the trapezoidal rule:

$$\int_{1.00}^{1.30} \sqrt{x}\, dx \approx \frac{0.05}{2} \left[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6\right]$$

$$= \frac{0.05}{2} \left[1.00000 + 2(1.02470 + 1.04881 + 1.07238 + 1.09545 + 1.11803) + 1.14017\right]$$

$$= \frac{0.05}{2} \left[1.00000 + 2(5.35937) + 1.14017\right] = \frac{0.05}{2} \cdot 12.85891 = 0.32147$$

The exact value of the integral is:

$$\int_{1.00}^{1.30} \sqrt{x}\, dx = \frac{2}{3} \left(1.3^{3/2} - 1\right) \approx 0.32149$$

Thus, the trapezoidal approximation is accurate to five decimal places, with an absolute error of approximately 0.00002.

Solved Problem 14.6

Derive an estimate of the truncation error of the trapezoidal rule.

To estimate the truncation error of the composite trapezoidal rule, consider approximating the definite integral

$$\int_a^b y(x)\, dx$$

by dividing the interval $[a, b]$ into n equal subintervals of width $h = \frac{b-a}{n}$. On each subinterval $[x_i, x_{i+1}]$, the trapezoidal rule approximates the area under $y(x)$ by using a linear segment between $y(x_i)$ and $y(x_{i+1})$, forming a trapezoid. The total error in the approximation comes from the accumulation of local errors over each subinterval.

Local Truncation Error via Taylor Expansion Consider a single subinterval $[x_i, x_{i+1}]$, and denote $x = x_i + t$, where $0 \le t \le h$. Let's expand $y(x)$ about x_i using Taylor's theorem:

$$y(x_i + t) = y(x_i) + ty'(x_i) + \frac{t^2}{2}y''(\xi_t),$$

for some $\xi_t \in [x_i, x_i + t] \subseteq [x_i, x_{i+1}]$.

Now integrate $y(x)$ over the interval:

$$\int_{x_i}^{x_{i+1}} y(x)\,dx = \int_0^h \left[y(x_i) + ty'(x_i) + \frac{t^2}{2}y''(\xi_t) \right] dt.$$

Compute each term separately:

- $\int_0^h y(x_i)\,dt = hy(x_i)$
- $\int_0^h ty'(x_i)\,dt = \frac{h^2}{2}y'(x_i)$
- $\int_0^h \frac{t^2}{2}y''(\xi_t)\,dt = \frac{1}{2}\int_0^h t^2 y''(\xi_t)\,dt$

Since $y''(\xi_t)$ varies with t, we can't integrate it exactly, but by the **mean value theorem for integrals**, there exists some point $\xi_i \in [x_i, x_{i+1}]$ such that:

$$\int_0^h t^2 y''(\xi_t)\,dt = y''(\xi_i)\int_0^h t^2\,dt = y''(\xi_i)\cdot\frac{h^3}{3}.$$

Thus, the full integral becomes:

$$\int_{x_i}^{x_{i+1}} y(x)\,dx = hy(x_i) + \frac{h^2}{2}y'(x_i) + \frac{1}{2}\cdot\frac{h^3}{3}y''(\xi_i) = hy(x_i) + \frac{h^2}{2}y'(x_i) + \frac{h^3}{6}y''(\xi_i).$$

Now consider the **trapezoidal approximation** over the same interval:

$$\frac{h}{2}\left[y(x_i) + y(x_{i+1})\right].$$

We expand $y(x_{i+1})$ using Taylor's theorem again:

$$y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{\xi}_i)$$

for some $\tilde{\xi}_i \in [x_i, x_{i+1}]$. Substituting into the trapezoidal approximation:

$$\frac{h}{2}\left[y(x_i)+y(x_i)+hy'(x_i)+\frac{h^2}{2}y''(\tilde{\xi}_i)\right]=hy(x_i)+\frac{h^2}{2}y'(x_i)+\frac{h^3}{4}y''(\tilde{\xi}_i).$$

Now subtract the trapezoidal approximation from the exact integral:

$$E_i=\int_{x_i}^{x_{i+1}} y(x)\,dx-\frac{h}{2}\left[y(x_i)+y(x_{i+1})\right]$$

$$=\left(hy(x_i)+\frac{h^2}{2}y'(x_i)+\frac{h^3}{6}y''(\xi_i)\right)-\left(hy(x_i)+\frac{h^2}{2}y'(x_i)+\frac{h^3}{4}y''(\tilde{\xi}_i)\right)$$

$$=\frac{h^3}{6}y''(\xi_i)-\frac{h^3}{4}y''(\tilde{\xi}_i)=-\frac{h^3}{12}y''(\bar{\xi}_i)$$

for some $\bar{\xi}_i \in [x_i, x_{i+1}]$, assuming y'' is smooth enough to interpolate between ξ_i and $\tilde{\xi}_i$.

Global Truncation Error Summing the local errors over all subintervals gives the total truncation error:

$$E=\sum_{i=0}^{n-1} E_i=-\frac{h^3}{12}\sum_{i=0}^{n-1} y''(\bar{\xi}_i).$$

Assuming $y''(x)$ is continuous, the sum can be approximated as $ny''(\xi)$ for some $\xi \in [a,b]$. Substituting $nh=b-a$, we obtain the classical global error estimate:

$$\text{Truncation error}=-\frac{(b-a)h^2}{12}y''(\xi),$$

where $\xi \in [a,b]$. This result shows that the global error of the trapezoidal rule is proportional to h^2 and depends on the second derivative of the integrand.

Solved Problem 14.7

Apply the estimate of Problem 14.6 to our square root integral.

To estimate the error in the trapezoidal approximation of the integral

$$\int_{1.00}^{1.30} \sqrt{x}\,dx,$$

we apply the global truncation error formula:

$$\text{Error} \approx -\frac{(b-a)\,h^2}{12}\,y^{(2)}(\xi),$$

where $h = 0.05$, $b - a = 0.30$, and the second derivative of $y(x) = \sqrt{x}$ is

$$y^{(2)}(x) = -\frac{1}{4}x^{-3/2}.$$

Using $\xi = 1.0$ to obtain a conservative estimate, we compute:

$$\text{Error} \approx -\frac{0.30 \cdot (0.05)^2}{12} \cdot \left(-\frac{1}{4}\right) = 0.000015625.$$

This estimated error is slightly less than the actual error of 0.00002, as previously observed. However, when the trapezoidal approximation 0.32147 is corrected by adding this estimated error, the result becomes

$$0.32147 + 0.000015625 = 0.32149,$$

which matches the exact value of the integral to five decimal places. Thus, the error estimate is consistent and provides a reliable correction to the numerical approximation.

Solved Problem 14.8

> Estimate the effect of inaccuracies in the y_k values on results obtained by the trapezoidal rule.

With Y_k denoting the true values, as before, we find

$$\frac{h}{2}(e_0 + 2e_1 + \cdots + 2e_{n-1} + e_n)$$

as the error introduced into the trapezoidal approximation due to inaccuracies in the y_k values, where $e_k = Y_k - y_k$ represents the deviation of each computed or measured value y_k from the true value Y_k.

This expression arises because the trapezoidal rule weights the endpoint values y_0 and y_n once, and all interior values $y_1, \ldots, y_{n-1}$ twice. Consequently, the total contribution of the errors e_k to the integral follows the same weighting structure.

Now, suppose the magnitude of each error is bounded by some constant E, so that

$$|e_k| \leq E \quad \text{for all } k = 0, 1, \ldots, n.$$

In the worst case, all errors reach this upper bound and contribute maximally to the output error. Therefore, we can bound the total error as follows:

$$\left|\frac{h}{2}(e_0 + 2e_1 + \cdots + 2e_{n-1} + e_n)\right| \leq \frac{h}{2}\left(E + 2(n-1)E + E\right).$$

Simplifying:

$$= \frac{h}{2} \cdot 2nE = nhE.$$

Since $nh = b - a$, this gives the final bound:

$$\text{Maximum output error} \leq (b - a)E.$$

This result shows that the overall impact of uniformly bounded errors in the y_k values is linearly proportional to both the length of the integration interval and the maximum pointwise error. It offers a useful and intuitive measure of how sensitive the trapezoidal rule is to inaccuracies in the input data.

Solved Problem 14.9

Apply the above to the square root integral of Problem 14.5.

We have $(b - a)E = (0.30)(0.00005) = 0.000015$, so that this source of error is negligible.

Simpson's rule

$$\int_{x_0}^{x_n} y(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

is also a composite formula, and comes from using connected parabolic segments as the approximation to $y(x)$. It is one of the most heavily used formulas for approximate integration. The truncation error is about

$$-(x_n - x_0)\frac{h^4}{180}y^{(4)}(\xi)$$

for some $\xi \in [x_0, x_n]$.

Solved Problem 14.10

Derive the composite Simpson's Rule by approximating the integrand with quadratic interpolants over subintervals and assembling a general integration formula.

Step 1: The Setup We begin with the task of approximating the definite integral:

$$\int_a^b f(x)\,dx$$

Suppose the interval $[a, b]$ is subdivided into n equal subintervals, where n is **even**. Define:

- Step size:
$$h = \frac{b-a}{n}$$
- Nodes:
$$x_k = a + kh, \quad \text{for } k = 0, 1, \ldots, n$$
- Function values:
$$y_k = f(x_k)$$

We'll derive Simpson's Rule by approximating the integrand $f(x)$ over **pairs of subintervals** using a **quadratic polynomial** through three consecutive points.

Step 2: Approximating over a Single Segment Consider three consecutive points:

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h$$

Using **Lagrange interpolation**, we construct a quadratic polynomial $P(x)$ that satisfies:

$$P(x_k) = y_k = f(x_k), \quad \text{for } k = 0, 1, 2$$

The interpolating polynomial is:

$$P(x) = y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Integrating $P(x)$ from x_0 to x_2, we find:

$$\int_{x_0}^{x_2} P(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

This is **Simpson's Rule over a single pair of intervals**.

Step 3: Generalizing to the Full Interval Now, we apply this idea repeatedly across the entire interval $[a, b]$, grouping points in sets of three:

- From x_0 to x_2:
$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$
- From x_2 to x_4:
$$\frac{h}{3}(y_2 + 4y_3 + y_4)$$
- From x_4 to x_6:
$$\frac{h}{3}(y_4 + 4y_5 + y_6)$$
- ...
- Up to x_{n-2} to x_n:
$$\frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding these expressions, the middle terms appear multiple times and must be combined. The full expression becomes:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

Solved Problem 14.11

Apply Simpson's rule to the integral of Problem 14.5.

We apply Simpson's Rule to approximate the integral:

$$\int_{1.00}^{1.30} \sqrt{x}\,dx$$

using 6 equally spaced nodes with step size $h = 0.05$. The Simpson's Rule formula gives:

$$\int_{1.00}^{1.30} \sqrt{x}\,dx \approx \frac{0.05}{3}\left[1.0000 + 4(1.02470 + 1.07238 + 1.11803) + 2(1.04881 + 1.09544) + 1.14017\right]$$

Evaluating the expression:

$$= \frac{0.05}{3}\left[1.0000 + 4(3.21511) + 2(2.14425) + 1.14017\right] = \frac{0.05}{3}(19.28911) = 0.32149$$

which is correct to five places.

Solved Problem 14.12

Estimate the truncation error of Simpson's Rule.

To understand the truncation error of Simpson's Rule, we begin by examining what happens over a single application of the rule — that is, over just two subintervals.

Suppose we have three points: x_0, $x_1 = x_0 + h$, and $x_2 = x_0 + 2h$. On the interval $[x_0, x_2]$, we construct a quadratic interpolating polynomial $P(x)$ that matches the function $f(x)$ at those three points. Simpson's Rule then approximates the integral $\int_{x_0}^{x_2} f(x)\,dx$ by integrating the polynomial $P(x)$ instead:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} P(x)\,dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)).$$

This approximation is exact if $f(x)$ is a polynomial of degree at most three. However, for a general smooth function, there is a difference between the actual function and its quadratic interpolant. This difference is captured by an error term from interpolation theory, which says that

$$f(x) - P(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{3!} f^{(3)}(\xi_x)$$

for some $\xi_x \in [x_0, x_2]$. When this expression is integrated over $[x_0, x_2]$, the result is a precise formula for the **local truncation error** of Simpson's Rule:

$$E_{\text{local}} = \int_{x_0}^{x_2} f(x)\,dx - \int_{x_0}^{x_2} P(x)\,dx = -\frac{h^5}{90} f^{(4)}(\xi),$$

for some $\xi \in [x_0, x_2]$. This tells us that the approximation deviates from the true integral by a term involving the fourth derivative of the function and the fifth power of the step size. The power of h^5 reflects how rapidly the local error shrinks as the intervals get finer.

Now, if the full interval $[a, b]$ is divided into n equal subintervals with even n, then Simpson's Rule is applied $n/2$ times — once per pair of subintervals. Each application contributes its own local error term of the form above, and the total error is the sum of these:

$$E = -\frac{h^5}{90} \left(f^{(4)}(\xi_1) + f^{(4)}(\xi_2) + \cdots + f^{(4)}(\xi_{n/2}) \right).$$

Assuming $f^{(4)}$ is continuous, this sum can be approximated as

$$E \approx -\frac{h^5}{90} \cdot \frac{n}{2} f^{(4)}(\xi)$$

for some $\xi \in [a, b]$. Substituting $nh = b - a$, the total truncation error becomes

$$\boxed{\text{Truncation error} = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \quad \xi \in [a, b]}.$$

This final result shows that Simpson's Rule has fourth-order accuracy, with the error shrinking proportionally to h^4 and depending on how curved the function is, as measured by the fourth derivative.

Solved Problem 14.13

Apply the estimate of Problem 14.12 to our square root integral.

We apply the Simpson's Rule truncation error formula to the integral

$$\int_{1.00}^{1.30} \sqrt{x}\, dx,$$

which was evaluated using 6 subintervals of width $h = 0.05$. The truncation error formula from Problem 14.12 is

$$\text{Error} = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \quad \text{for some } \xi \in [a, b].$$

For $f(x) = \sqrt{x} = x^{1/2}$, the fourth derivative is

$$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}.$$

This matches the expression shown in the problem. Since the function $f^{(4)}(x)$ is decreasing on the interval $[1.00, 1.30]$, the maximum absolute value occurs at the left endpoint:

$$|f^{(4)}(1.00)| = \frac{15}{16} = 0.9375.$$

Substituting into the truncation error formula:

- $b - a = 0.30$
- $h = 0.05$
- $h^4 = (0.05)^4 = 6.25 \times 10^{-8}$

The error is bounded by

$$|\text{Error}| \leq \frac{(0.30)(6.25 \times 10^{-8})}{180} \cdot \frac{15}{16} = 1.0417 \times 10^{-10}.$$

This result shows the truncation error is less than 0.00000000011, which is far smaller than the approximation 0.00000001 suggested in the book. The estimate given is conservative but valid, and the conclusion that the error is **minute** is accurate.

The Integration Formulas of Newton and Cotes (Stoer, section 3.1)

The integration formulas of Newton and Cotes are obtained if the integrand is replaced by a suitable interpolating polynomial $P(x)$ and if then

$$\int_a^b P(x)dx$$

is taken as an approximate value for

$$\int_a^b f(x)dx.$$

Consider a uniform partition of the closed interval $[a, b]$ given by

$$x_i = a + ih, \quad i = 0, 1, \ldots, n,$$

of step length $h := (b - a)/n$, $n > 0$ integer, and let P_n be the interpolating polynomial of degree n or less with

$$P_n(x_i) = f_i := f(x_i) \quad \text{for} \quad i = 0, 1, \ldots, n.$$

By Lagrange's interpolation formula (2.1.1.4),

$$P_n(x) = \sum_{i=0}^{n} f_i L_i(x), \quad L_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k},$$

or, introducing the new variable t such that $x = a + ht$,

$$L_i(x) = \varphi_i(t) := \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{t - k}{i - k}.$$

Integration gives

$$\int_a^b P_n(x)\,dx = \sum_{i=0}^{n} f_i \int_a^b L_i(x)\,dx = h \sum_{i=0}^{n} f_i \int_0^1 \varphi_i(t)\,dt = h \sum_{i=0}^{n} f_i \alpha_i.$$

Note that the coefficients or *weights*

$$\alpha_i := \int_0^n \varphi_i(t)\,dt.$$

depend solely on n; in particular, they do not depend on the function f to be integrated, or on the boundaries a, b of the integral.

If $n = 2$, for instance, then

$$\begin{aligned}
\alpha_0 &= \int_0^1 \frac{t(t-1)}{2-0} dt = \frac{1}{2}\int_0^1 (t^2 - t)\,dt = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{2}\left(-\frac{1}{6}\right) = -\frac{1}{12}, \\
\alpha_1 &= \int_0^1 \frac{-(t)(t-2)}{1} dt = \int_0^1 (2t - t^2) dt = \left(t^2 - \frac{t^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}, \\
\alpha_2 &= \int_0^1 \frac{t(t-1)}{2-0} dt = \frac{1}{2}\int_0^1 (t^2 - t)\,dt = -\frac{1}{12},
\end{aligned}$$

and we obtain the following approximate value:

$$\int_a^b P_2(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2)$$

for the integral $\int_a^b f(x)\,dx$. This is **Simpson's rule**.

For any natural number n, the **Newton-Cotes formulas**

$$\int_a^b P_n(x)\,dx = h\sum_{i=0}^n f_i\alpha_i, \quad f_i := f(a + ih), \quad h := \frac{b-a}{n}, \tag{3.1.1}$$

provide approximate values for $\int_a^b f(x)\,dx$. The weights α_i, $i = 0, 1, \ldots, n$, have been tabulated. They are rational numbers with the property

$$\sum_{i=0}^n \alpha_i = n. \tag{3.1.2}$$

This follows from (3.1.1) when applied to $f(x) \equiv 1$, for which $P_n(x) \equiv 1$. If s is a common denominator for the fractional weights α_i so that the numbers

$$\sigma_i := s\alpha_i, \quad i = 0, 1, \ldots, n,$$

are integers, then (3.1.1) becomes

$$\int_a^b P_n(x)\,dx = h\sum_{i=0}^{n} f_i\alpha_i = \frac{b-a}{ns}\sum_{i=0}^{n}\sigma_i f_i. \tag{3.1.3}$$

For sufficiently smooth functions $f(x)$ on the closed interval $[a, b]$, it can be shown [see Steffensen (1950)] that the approximation error may be expressed as follows:

$$\int_a^b P_n(x)\,dx - \int_a^b f(x)\,dx = h^{p+1}\cdot K\cdot f^{(p)}(\xi), \quad \xi \in (a, b). \tag{3.1.4}$$

Here (a, b) denotes the open interval from a to b. The values of p and K depend only on n but not on the integrand f.

For $n = 1, 2, \ldots, 6$, we find the Newton–Cotes formulas given in the following table. For larger n, some of the values σ_i become negative and the corresponding formulas are unsuitable for numerical purposes, as cancellations tend to occur in computing the sum (3.1.3).

n	σ_i	ns	Error Term	Name
1	1 1	2	$h^3\,\frac{1}{12}f^{(2)}(\xi)$	Trapezoidal rule
2	1 4 1	6	$h^5\,\frac{1}{90}f^{(4)}(\xi)$	Simpson's rule
3	1 3 3 1	8	$h^5\,\frac{3}{80}f^{(4)}(\xi)$	3/8-rule
4	7 32 12 32 7	90	$h^7\,\frac{14}{945}f^{(6)}(\xi)$	Milne's rule
5	19 75 50 50 75 19	288	$h^7\,\frac{275}{12096}f^{(6)}(\xi)$	—
6	41 216 27 272 27 216 41	840	$h^9\,\frac{9}{1400}f^{(8)}(\xi)$	Weddle's rule

Solved Problem

Approximate $\int_{1.0}^{1.3}\sqrt{x}\,dx$ using the 3/8 rule:

Use the formula:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right]$$

with $a = 1.0$, $b = 1.3$, so $h = \frac{1.3-1.0}{3} = 0.1$

Nodes:

$$x_0 = 1.0, \quad x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3$$

Function values:

$$f_0 = \sqrt{1.0} = 1.0000$$

$$f_1 = \sqrt{1.1} \approx 1.0488$$

$$f_2 = \sqrt{1.2} \approx 1.0954$$

$$f_3 = \sqrt{1.3} \approx 1.1402$$

Plug in:

$$\int_{1.0}^{1.3} \sqrt{x}\, dx \approx \frac{0.3}{8}(1.0000 + 3(1.0488) + 3(1.0954) + 1.1402)$$

$$= 0.0375 \cdot 8.573 \approx 0.3215$$