Black-Scholes Mathematical Analysis

Partial Differential Equations and Option Pricing

Lecturer

May 25, 2025

Outline

The Black-Scholes Partial Differential Equation

The Black-Scholes equation for option pricing is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Where:

- $V(S, t) =$ option value
- $S =$ underlying asset price
- $t =$ time
- $\sigma =$ volatility
- $r =$ risk-free interest rate

General Form of Second-Order PDEs

A general second-order PDE in two variables has the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F \cdot u + G = 0$$

Classification of PDEs

PDEs are classified based on the discriminant $\Delta = B^2 - 4AC$:

- **Elliptic**: $\Delta < 0$
- **Parabolic**: $\Delta = 0$
- **Hyperbolic**: $\Delta > 0$

Identifying Coefficients in Black-Scholes

Rewriting the Black-Scholes equation in standard form with $x = S$ and $y = t$:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$$

Coefficients

- $A = \frac{1}{2}\sigma^2 S^2$
- $B = 0$
- $C = 0$
- $D = rS$
- $E = 1$
- $F = -r$

Computing the Discriminant

$$\Delta = B^2 - 4AC = 0^2 - 4 \cdot \left(\frac{1}{2}\sigma^2 S^2\right) \cdot 0 = 0 - 0 = 0$$

Conclusion

Since the discriminant $\Delta = 0$, the Black-Scholes equation is **parabolic**.

Physical Interpretation

The parabolic nature reflects:

1. **Diffusion Process**: The $\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ term represents diffusion in the asset price, similar to heat diffusion
2. **Time Evolution**: Information propagates through the system over time, characteristic of parabolic PDEs
3. **No Second Time Derivative**: Unlike wave equations (hyperbolic), there's no "acceleration" term in time

Important Implications

- Parabolic PDEs have unique solutions under appropriate boundary conditions
- Numerical methods for parabolic PDEs are well-established
- The solution exhibits smoothing properties typical of diffusion processes