MA2008B: Assignment 03

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Period 03

Instructions

- Show all your work for full credit.
- Write your answers clearly and neatly.
- Submit your assignment by the due date.

1 Partial Differential Equations

1. Show that the solution to the initial value problem is unique provided that it is sufficiently smooth and decays sufficiently fast at infinity, as follows:

   Suppose that $u_1(x,t)$ and $u_2(x,t)$ are both solutions to the initial value problem (4.5)–(4.9):

   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \infty < x < \infty, \quad t > 0, \tag{4.5}$$

   with

   $$u(x,0) = u_0(x), \tag{4.6}$$

   where

   (i) $u_0(x)$ is sufficiently well behaved, (4.7)

   (ii)

   $$\lim_{|x|\to\infty} u_0(x)e^{-ax^2} = 0$$

   for any $a > 0$, (4.8)

   and lastly where

   $$\lim_{|x|\to\infty} u(x,t)e^{-ax^2} = 0$$

   for any $a > 0$, $t > 0$, (4.9)

Show that $v(x,t) = u_1 - u_2$ is also a solution of (4.5) with $v(x,0) = 0$.

Show that if

$$E(t) = \int_{-\infty}^{\infty} v^2 \, dx,$$

then

$$E(t) \geq 0, \quad E(0) = 0,$$

and, by integrating by parts, that

$$\frac{dE}{dt} \leq 0;$$

thus $E(t) \equiv 0$, hence $v(x,t) \equiv 0$.

Note, though, that as yet we have no guarantee that $v(x,t)$ exists, nor that the above manipulations can be justified.

References

[1] James Stewart, *Calculus: Early Transcendentals*, 8th Edition, Cengage Learning, 2015.

[2] Walter Rudin, *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill, 1976.

[3] P. Wilmott, J. Dewynne, S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, 1993.