Partial Differential Equations in Finance

Dr. Juliho David Castillo Colmenares

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Overview of Topics

- Charting theory and nature of boundary and initial conditions
- Explicit solutions, including original Black-Scholes formula
- Special problems arising when there are free boundaries

Key Questions in Financial PDEs

- Physical interpretation of the equations?
- Mathematical properties of the solution?
- Techniques for obtaining explicit solutions?

PDEs in finance:

- Fundamental equations (e.g., Black-Scholes)
- Linear vs. nonlinear problems

Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
   - Appropriate boundary or initial/final conditions?
   - Nature of the mathematical problem?
   - Smooth or discontinuous solutions?
2. Can we develop analytical tools to solve the equation?
3. How should we solve the equation numerically if necessary?

First Order Linear PDE

Consider the equation:

$$\alpha(s,t)\frac{\partial u}{\partial s} + \beta(s,t)\frac{\partial u}{\partial t} = \gamma(s,t)u(t)$$

Linearity Property

If u_1, u_2 are solutions, then $c_1 u_1 + c_2 u_2$ is also a solution for constants c_1, c_2.

Proof.

Substitute $u = c_1 u_1 + c_2 u_2$ into the PDE and use linearity of derivatives. $\square$

Constant Coefficient Case

When α, β are constants and $\gamma \equiv 0$:

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector $\vec{v} = (\alpha_0, \beta_0)$ with $\beta_0 \alpha_0 \neq 0$.
The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \rangle$$

Thus, (4.2) $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 s + \alpha_0 t, \quad \zeta = \beta_0 s - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Computing $\frac{\partial u}{\partial \xi}$:

$$\frac{\partial u}{\partial \xi} = \beta_0 \frac{\partial u}{\partial s} + \alpha_0 \frac{\partial u}{\partial t} \equiv 0$$

Therefore, u depends only on ζ: $u = F(\zeta)$

Characteristics and Information Propagation

- $\xi = \beta_0 s - \alpha_0 t$ is called a **characteristic**
- Characteristics represent directions in which information propagates
- Analogous to constants of integration in ODEs

Important Note

Without boundary conditions, the solution remains arbitrary (like the constant C in ODEs)

General Second-Order Linear PDE

Consider the general form in two variables:

$$a(x,t)u_{xx} + 2b(x,t)u_{xt} + c(x,t)u_{tt} + d(x,t)u_x + e(x,t)u_t + f(x,t)u = g(x,t$$

where $a, b, c, d, e, f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are coefficient functions.

Principal Part

The highest-order terms form the principal part:

$$L_p(u) = au_{xx} + 2bu_{xt} + cu_{tt}$$

Classification of PDEs

Define the discriminant:

$$\Delta(L) = b^2 - ac : (x, t) \mapsto b(x, t)^2 - a(x, t)c(x, t)$$

The PDE is classified as:

- **Hyperbolic** if $\Delta(L) > 0$ (e.g., wave equation)
- **Parabolic** if $\Delta(L) = 0$ (e.g., heat equation)
- **Elliptic** if $\Delta(L) < 0$ (e.g., Laplace equation)

Coordinate Transformations

Definition

A transformation $(\xi, \eta) = (\xi(x,t), \eta(x,t))$ is called a change of coordinates if the Jacobian is non-zero:

$$J := \xi_x \eta_t - \xi_t \eta_x \neq 0$$

Theorem

The type of a linear second-order PDE in two variables is invariant under a change of coordinates.

Proof of Invariance

Under transformation (ξ, η), the principal part becomes:

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta}$$

where:

$$\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2 \\
B &= a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t \\
C &= a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2
\end{aligned}$$

The new discriminant satisfies:

$$\Delta_{new} = AC - B^2 = J^2(b^2 - ac) = J^2\Delta_{original}$$

Canonical Forms

- **Hyperbolic**: $u_{\xi\xi} - u_{\eta\eta} =$ lower order terms (wave equation)
- **Parabolic**: $u_{\xi\xi} =$ lower order terms (heat equation)
- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} =$ lower order terms (Laplace equation)

Examples

- Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$
- Heat equation: $u_t = k(u_{xx} + u_{yy})$
- Laplace equation: $u_{xx} + u_{yy} = 0$

Simplification Through Coordinates

A proper linear change of coordinates can transform:

- Hyperbolic equations to $u_{\xi\eta} = 0$
- Parabolic equations to $u_{\xi\xi} = 0$
- Elliptic equations to $u_{\xi\xi} + u_{\eta\eta} = 0$

Important Note

The classification determines:

- Nature of characteristics
- Appropriate boundary conditions
- Solution techniques

Heat Equation

The fundamental heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Physical interpretation:

- $u(x, t)$ represents temperature distribution along a thin insulated rod
- Models heat diffusion over time

Mathematical Properties

Key characteristics:

- Linear equation (superposition principle applies)
- Second order in space (x)
- First order in time (t)
- Parabolic type (discriminant $\Delta = 0$)

Analytic Solutions

Solutions are analytic functions of x:

- For any $t^* > 0$ and $a < x_0 < b$
- $u(x, t^*)$ can be expressed as a convergent power series in $(x - x_0)$

Solution Behavior

Important features:

- **Smoothing property**: Solutions become instantly smooth for $t > 0$
- **Infinite propagation speed**: Initial disturbances spread infinitely fast
- **Maximum principle**: Extreme values occur only at initial/boundary conditions

Financial Interpretation

Analogous to:

- Diffusion of volatility in option pricing
- Smoothing of price shocks over time

Dirichlet Boundary Value Problem

Heat equation on finite interval:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L, \ t > 0$$

With conditions:

- Initial condition: $u(x, 0) = u_0(x)$
- Boundary conditions:
  - $u(-L, t) = g_1(t)$ (left end temperature)
  - $u(L, t) = g_2(t)$ (right end temperature)

Physical Interpretation

Models heat diffusion in a finite rod with controlled end temperatures

Neumann Boundary Value Problem

Same heat equation with flux conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L, \; t > 0$$

With conditions:

- Initial condition: $u(x,0) = u_0(x)$
- Flux boundary conditions:
  - $-\frac{\partial u}{\partial x}(-L,t) = h_1(t)$ (left end flux)
  - $\frac{\partial u}{\partial x}(L,t) = h_2(t)$ (right end flux)

Physical Interpretation

Models heat diffusion with controlled heat flow at boundaries

Infinite Domain Problem

Taking $L \to \infty$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \ t > 0$$

With initial condition:

$$u(x, 0) = u_0(x)$$

Growth Conditions

For well-posedness:

- $u_0(x)$ piecewise continuous (finite jumps allowed)
- $\lim_{|x|\to\infty} \frac{u_0(x)}{e^{ax^2}} = 0 \quad \forall a > 0$
- $\lim_{|x|\to\infty} \frac{u(x,t)}{e^{ax^2}} = 0 \quad \forall a, t > 0$

Semi-Infinite Domain with Mixed Conditions

Example problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \; t > 0$$

With conditions:

- Initial condition: $u(x, 0) = u_0(x)$ (sufficiently smooth)
- Boundary condition: $u(0, t) = g(t)$
- Growth conditions at infinity

Well-Posedness

The problem is well-posed when:

- Initial/boundary conditions are compatible
- Growth conditions are satisfied
- Solution exists, is unique, and depends continuously on data

Time Reversal Transformation

Consider the time transformation:

$$t = T - \tau, \quad T \text{ constant}$$

The time derivative transforms as:

$$\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{\partial u}{\partial t}$$

Applying to the heat equation:

$$\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial x^2} \quad \text{(Backward Heat Equation)}$$

Important Property

The backward heat equation is generally **ill-posed** for most initial conditions

Fundamental Solution

The fundamental solution to the (forward) heat equation:

$$u_g(x,t) = \begin{cases} \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)} & t > 0 \\ \delta_0(x) & t = 0 \end{cases}$$

Consider the variant:

$$u(x,t) = \frac{1}{2\sqrt{\pi(\tau - t)}} e^{-\frac{x^2}{4(\tau - t)}}$$

- Solves the backward heat equation
- Initial condition: $u_0(x) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/(4\tau)}$
- **Blows up** at $t = \tau$ (singularity)

Final Value Problem

The backward heat equation with final condition:

$$u_T(x) = u_g(x, \tau)$$

Well-Posed Case

This specific final value problem is well-posed when:

- Properly transformed to a forward problem via $t \mapsto T - t$
- The final condition decays sufficiently fast
- Solution exists in a limited time domain $t < \tau$

Example

In finance, appears in:

- Option pricing with terminal payoff conditions
- Implicit finite difference methods

Mathematical Implications

Key observations about the backward heat equation:

- **Ill-posedness**: Small changes in final conditions can lead to large solution changes
- **Regularization needed** for practical applications
- **Time reversal** requires special treatment

Numerical Challenges

Requires:

- Special discretization schemes
- Stability analysis
- Often needs additional constraints