Partial Differential Equations in Finance

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Overview of Topics

- Charting theory and nature of boundary and initial conditions
- Explicit solutions, including original Black-Scholes formula
- Special problems arising when there are free boundaries

Key Questions in Financial PDEs

- ▶ Physical interpretation of the equations?
- ▶ Mathematical properties of the solution?
- ▶ Techniques for obtaining explicit solutions?

PDEs in finance:
- ▶ Fundamental equations (e.g., Black-Scholes)
- ▶ Linear vs. nonlinear problems

Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
   - Appropriate boundary or initial/final conditions?
   - Nature of the mathematical problem?
   - Smooth or discontinuous solutions?

2. Can we develop analytical tools to solve the equation?

3. How should we solve the equation numerically if necessary?

First Order Linear PDE

Consider the equation:

$$\alpha(s,t)\frac{\partial u}{\partial s} + \beta(s,t)\frac{\partial u}{\partial t} = \gamma(s,t)u(t)$$

Linearity Property

If u_1, u_2 are solutions, then $c_1 u_1 + c_2 u_2$ is also a solution for constants c_1, c_2.

Proof.

Substitute $u = c_1 u_1 + c_2 u_2$ into the PDE and use linearity of derivatives. $\qquad\square$

Constant Coefficient Case

When α, β are constants and $\gamma \equiv 0$:

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector $\vec{v} = (\alpha_0, \beta_0)$ with $\beta_0 \alpha_0 \neq 0$.
The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \rangle$$

Thus, (4.2) $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 s + \alpha_0 t, \quad \zeta = \beta_0 s - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Computing $\frac{\partial u}{\partial \xi}$:

$$\frac{\partial u}{\partial \xi} = \beta_0 \frac{\partial u}{\partial s} + \alpha_0 \frac{\partial u}{\partial t} \equiv 0$$

Therefore, u depends only on ζ: $u = F(\zeta)$

Characteristics and Information Propagation

- $\xi = \beta_0 s - \alpha_0 t$ is called a **characteristic**
- Characteristics represent directions in which information propagates
- Analogous to constants of integration in ODEs

Important Note

Without boundary conditions, the solution remains arbitrary (like the constant C in ODEs)

General Second-Order Linear PDE

Consider the general form in two variables:

$$a(x,t)u_{xx}+2b(x,t)u_{xt}+c(x,t)u_{tt}+d(x,t)u_x+e(x,t)u_t+f(x,t)u = g(x,t$$

where $a, b, c, d, e, f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are coefficient functions.

Principal Part

The highest-order terms form the principal part:

$$L_p(u) = au_{xx} + 2bu_{xt} + cu_{tt}$$

Classification of PDEs

Define the discriminant:

$$\Delta(L) = b^2 - ac : (x, t) \mapsto b(x, t)^2 - a(x, t)c(x, t)$$

The PDE is classified as:

- **Hyperbolic** if $\Delta(L) > 0$ (e.g., wave equation)
- **Parabolic** if $\Delta(L) = 0$ (e.g., heat equation)
- **Elliptic** if $\Delta(L) < 0$ (e.g., Laplace equation)

Coordinate Transformations

Definition
A transformation $(\xi, \eta) = (\xi(x, t), \eta(x, t))$ is called a change of coordinates if the Jacobian is non-zero:

$$J := \xi_x \eta_t - \xi_t \eta_x \neq 0$$

Theorem
The type of a linear second-order PDE in two variables is invariant under a change of coordinates.

Proof of Invariance

Under transformation (ξ, η), the principal part becomes:

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta}$$

where:

$$A = a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2$$
$$B = a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t$$
$$C = a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2$$

The new discriminant satisfies:

$$\Delta_{new} = AC - B^2 = J^2(b^2 - ac) = J^2\Delta_{original}$$

Canonical Forms

- **Hyperbolic**: $u_{\xi\xi} - u_{\eta\eta} =$ lower order terms (wave equation)
- **Parabolic**: $u_{\xi\xi} =$ lower order terms (heat equation)
- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} =$ lower order terms (Laplace equation)

Examples

- Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$
- Heat equation: $u_t = k(u_{xx} + u_{yy})$
- Laplace equation: $u_{xx} + u_{yy} = 0$

Simplification Through Coordinates

A proper linear change of coordinates can transform:

- ▶ Hyperbolic equations to $u_{\xi\eta} = 0$
- ▶ Parabolic equations to $u_{\xi\xi} = 0$
- ▶ Elliptic equations to $u_{\xi\xi} + u_{\eta\eta} = 0$

Important Note

The classification determines:

- ▶ Nature of characteristics
- ▶ Appropriate boundary conditions
- ▶ Solution techniques