Black-Scholes Mathematical Analysis

Partial Differential Equations and Option Pricing

Lecturer

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Outline

The Black-Scholes Partial Differential Equation

The Black-Scholes equation for option pricing is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Where:

- $V(S, t) =$ option value
- $S =$ underlying asset price
- $t =$ time
- $\sigma =$ volatility
- $r =$ risk-free interest rate

General Form of Second-Order PDEs

A general second-order PDE in two variables has the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F \cdot u + G = 0$$

Classification of PDEs

PDEs are classified based on the discriminant $\Delta = B^2 - 4AC$:

- **Elliptic**: $\Delta < 0$
- **Parabolic**: $\Delta = 0$
- **Hyperbolic**: $\Delta > 0$

Identifying Coefficients in Black-Scholes

Rewriting the Black-Scholes equation in standard form with $x = S$ and $y = t$:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$$

Coefficients

- $A = \frac{1}{2}\sigma^2 S^2$
- $B = 0$
- $C = 0$
- $D = rS$
- $E = 1$
- $F = -r$

Computing the Discriminant

$$\Delta = B^2 - 4AC = 0^2 - 4 \cdot \left(\frac{1}{2}\sigma^2 S^2\right) \cdot 0 = 0 - 0 = 0$$

Conclusion

Since the discriminant $\Delta = 0$, the Black-Scholes equation is **parabolic**.

Physical Interpretation

The parabolic nature reflects:

1. **Diffusion Process**: The $\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ term represents diffusion in the asset price, similar to heat diffusion
2. **Time Evolution**: Information propagates through the system over time, characteristic of parabolic PDEs
3. **No Second Time Derivative**: Unlike wave equations (hyperbolic), there's no "acceleration" term in time

Important Implications

- Parabolic PDEs have unique solutions under appropriate boundary conditions
- Numerical methods for parabolic PDEs are well-established
- The solution exhibits smoothing properties typical of diffusion processes

The Need for Stochastic Calculus

Problem with Ordinary Calculus

When dealing with random processes like stock prices, ordinary calculus fails because:

- Brownian motion isn't differentiable
- The quadratic term $(dx)^2$ doesn't vanish
- We need special rules for stochastic differentiation

Brownian Motion Properties

For Brownian motion $W(t)$:

- $W(0) = 0$ (starts at zero)
- Independent, normally distributed increments
- Continuous paths but nowhere differentiable
- **Key property**: $(dW)^2 = dt$

Itô's Formula: The Stochastic Chain Rule

For a function $f(W, t)$ where W is Brownian motion:

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right) dt + \frac{\partial f}{\partial W} dW$$

Key Insight

The second derivative term appears because $(dW)^2 = dt \neq 0$ in stochastic calculus!

In ordinary calculus: $df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx$

In stochastic calculus: We get an extra $\frac{1}{2}\frac{\partial^2 f}{\partial W^2} dt$ term.

Geometric Brownian Motion

Stock prices follow geometric Brownian motion (GBM):
$dS = \mu S dt + \sigma S dW$
Where:

- μ = expected return (drift)
- σ = volatility
- dW = Brownian motion increment

Why GBM?

- Stock prices can't go negative
- Percentage changes are more natural than absolute changes
- Leads to lognormal distribution of future prices

Itô's Formula for Functions of GBM

For a function $V(S, t)$ where S follows GBM, Itô's formula gives:
$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW$

The Derivation Logic

1. Start with Taylor expansion around (S, t)
2. Use $dS = \mu S dt + \sigma S dW$
3. Compute $(dS)^2 = \sigma^2 S^2 dt$ (key step!)
4. Keep terms up to order dt

The Crucial Stochastic Rules

Fundamental Rules

- $(dt)^2 = 0$ - second-order infinitesimals vanish
- $(dW)^2 = dt$ - **this is the key stochastic rule**
- $dt \times dW = 0$ - deterministic and random parts are orthogonal

Computing $(dS)^2$

$$(dS)^2 = (\mu S dt + \sigma S dW)^2 = \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 (dt)(dW) + \sigma^2 S^2 (dW)^2$$
$$= 0 + 0 + \sigma^2 S^2 dt = \sigma^2 S^2 dt$$

Why Itô's Formula Matters for Options

The formula tells us how option value $V(S, t)$ changes:

$$dV = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dW$$

Key Observations

- **Same randomness**: Both stock and option driven by the same dW
- **The drift term**: Contains the volatility effect $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$
- **The diffusion term**: Proportional to $\frac{\partial V}{\partial S}$ (the delta!)
- **Foundation for hedging**: This shared randomness enables delta hedging

Key Assumptions for Black-Scholes

1. **Geometric Brownian Motion**: Stock price follows

$$dS = \mu S dt + \sigma S dW$$

   where dW is a Wiener process (Brownian motion)
2. **Constant Parameters**: Risk-free rate r, volatility σ are constant
3. **No Dividends**: The stock pays no dividends
4. **Perfect Market**: No transaction costs, continuous trading, unlimited borrowing/lending at rate r
5. **European Exercise**: Option can only be exercised at expiration

Step 1: Stock Price Model

The stock price follows geometric Brownian motion:

$$dS = \mu S dt + \sigma S dW$$

Where:

- μ is the expected return (drift)
- σ is the volatility
- dW is a Wiener process (random walk)

Interpretation

The stock has a deterministic trend μS and random fluctuations σS.

Step 2: Option Value Changes (Itô's Lemma)

For option value $V(S, t)$, applying Itô's lemma:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

Key Insight

Both dS and dV contain the same random term dW - this is crucial for hedging.

Step 3: Constructing the Hedged Portfolio

Create a portfolio Π consisting of:

- **Long 1 option** (value $= V$)
- **Short Δ shares** (value $= -\Delta S$)

So: $\Pi = V - \Delta S$
The change in portfolio value is:

$$d\Pi = dV - \Delta\, dS$$

Step 4: The Critical Substitution

Substitute the expressions for dV and dS:

$$d\Pi = \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \Delta\mu S\right] dt \tag{1}$$

$$+ \left[\sigma S\frac{\partial V}{\partial S} - \Delta\sigma S\right] dW \tag{2}$$

Factoring the dW coefficient:

$$\text{dW coefficient} = \sigma S\left[\frac{\partial V}{\partial S} - \Delta\right]$$

Step 4: Eliminating Risk - The Magic Choice

To eliminate the random dW term, we set:

$$\Delta = \frac{\partial V}{\partial S}$$

Why This Works

$\frac{\partial V}{\partial S}$ is the option's sensitivity to stock price changes. By shorting exactly this many shares, we create perfect hedge:

- Stock goes up $\rightarrow$ option gains, short stock loses
- Stock goes down $\rightarrow$ option loses, short stock gains
- **Net effect: zero risk!**

With this choice: $d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt$

Step 5: No-Arbitrage Condition

Since the portfolio is now risk-free, it must earn the risk-free rate r:

$$d\Pi = r\Pi \, dt$$

But $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S} S$
Therefore:

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt = r \left[V - \frac{\partial V}{\partial S} S \right] dt$$

Step 6: The Black-Scholes Equation Emerges

Dividing by dt and rearranging:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Rearranging to standard form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Remarkable Results

- The expected return μ disappeared completely!
- Option prices depend only on volatility σ, not expected return
- Perfect hedging creates risk-free portfolio

The Deep Intuition

What We Accomplished

By setting $\Delta = \frac{\partial V}{\partial S}$, we matched the sensitivities:

- Option position sensitivity $=$ Stock position sensitivity
- Random fluctuations cancel out perfectly
- Only deterministic terms remain

Economic Interpretation

In an efficient market with no arbitrage opportunities:

- Risk-free portfolios must earn the risk-free rate
- This constraint determines option prices uniquely
- The mathematics captures this economic principle