

MA2008B: Assignment 03

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Period 03

Instructions

- Show all your work for full credit.
- Write your answers clearly and neatly.
- Submit your assignment by the due date.

1 Partial Differential Equations

1. Show that the solution to the initial value problem is unique provided that it is sufficiently smooth and decays sufficiently fast at infinity, as follows:

Suppose that $u_1(x, t)$ and $u_2(x, t)$ are both solutions to the initial value problem (4.5)–(4.9):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \infty < x < \infty, \quad t > 0, \quad (4.5)$$

with

$$u(x, 0) = u_0(x), \quad (4.6)$$

where

(i) $u_0(x)$ is sufficiently well behaved, (4.7)

(ii)

$$\lim_{|x| \rightarrow \infty} u_0(x) e^{-ax^2} = 0$$

for any $a > 0$, (4.8)

and lastly where

$$\lim_{|x| \rightarrow \infty} u(x, t) e^{-ax^2} = 0$$

for any $a > 0$, $t > 0$, (4.9)

Show that $v(x, t) = u_1 - u_2$ is also a solution of (4.5) with $v(x, 0) = 0$.

Show that if

$$E(t) = \int_{-\infty}^{\infty} v^2 dx,$$

then

$$E(t) \geq 0, \quad E(0) = 0,$$

and, by integrating by parts, that

$$\frac{dE}{dt} \leq 0;$$

thus $E(t) \equiv 0$, hence $v(x, t) \equiv 0$.

Note, though, that as yet we have no guarantee that $v(x, t)$ exists, nor that the above manipulations can be justified.

2. Show that $\sin nx e^{-n^2 t}$ is a solution of the forward diffusion equation, and that $\sin nx e^{n^2 t}$ is a solution of the backward diffusion equation. Now try to solve the initial value problem for the forward and backward equations in the interval $-\pi < x < \pi$ by expanding the solution in a Fourier series in x with coefficients depending on t . What difference do you see between the two problems? Which is well-posed?

References

- [1] P. Wilmott, J. Dewynne, S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, 1993.