Numerical Methods For Ordinary Differential Equations

References:

1. Chasnov, J. R. (2012). Numerical methods. Hong Kong University of Science and Technology. https://www.math.hkust.edu.hk/~machas/numerical-methods.pdf
2. Liu, J., Spiegel, M. R. (1999). Mathematical Handbook of Formulas and Tables. United Kingdom: McGraw-Hill.

Here we give methods to solve the following initial-value problem of an ordinary differential equation:

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

The methods will use a computational grid:

$$t_n = t_0 + nh \tag{2}$$

where h is the grid size.

Example

Consider $f(t,x) = rx(K-x)$, with the following parameters:

```
r = 0.15
K = 100
t0 = 0
tf = 1
x0 = 1
n = 10
step_size =  (tf-t0)/n
step_size
```

```
1/10
```

```
t = var('t')

x = function('x')(t)

ode = x.diff(t) == r*x*(K-x)

solution = desolve(ode, dvar=x, ivar=t, ics=[t0,x0])

solution
```

```
-1/15*log(x(t) - 100) + 1/15*log(x(t)) == -1/15*I*pi + t - 1/15*log(99)
```

```
u = var('u')
solution = solution.subs({x: u})
solution
```

```
-1/15*log(u - 100) + 1/15*log(u) == -1/15*I*pi + t - 1/15*log(99)
```

```
solution.solve(u, to_poly_solve=True)
```

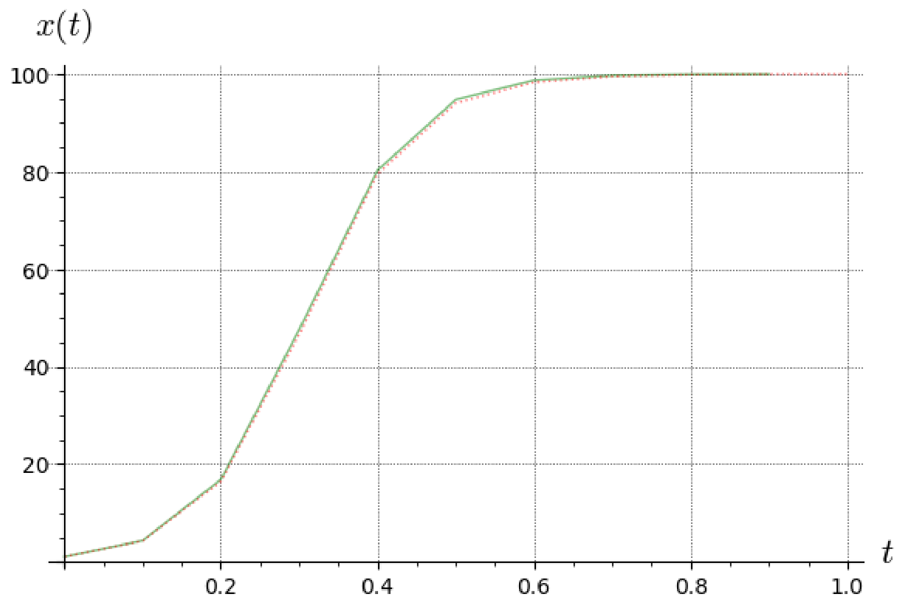
```
[u == 100*e^(15*t)/(e^(15*t) + 99)]
```

```
x_exact = solution.solve(u, to_poly_solve=True)[0].rhs()
x_exact
```

```
100*e^(15*t)/(e^(15*t) + 99)
```

```
custom_grid = [t0 + i*step_size for i in range(n)]
exact_solution = [[_t.n(), x_exact(t=_t).n()] for _t in custom_grid]
```

```
t,x = var('t x')
numerical_solution = desolve_rk4( r*x*(K-x), x, ivar = t, ics=[t0,x0], end_points=tf, step=s
sage_solution = list_plot(exact_solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gr
sage_solution += list_plot(numerical_solution, plotjoined=True, alpha=0.5, color="red", line
sage_solution.show()
```



Euler Method: Manual Reproduction of the Code

We consider the ODE:

$$\frac{dx}{dt} = 0.15\,x(100 - x), \quad x(0) = 1$$

with parameters:

- $r = 0.15$
- $K = 100$
- $t_0 = 0$, $t_f = 1$
- $x_0 = 1$
- $n = 10$, $h = \frac{1-0}{10} = 0.1$

Euler's method iterates:

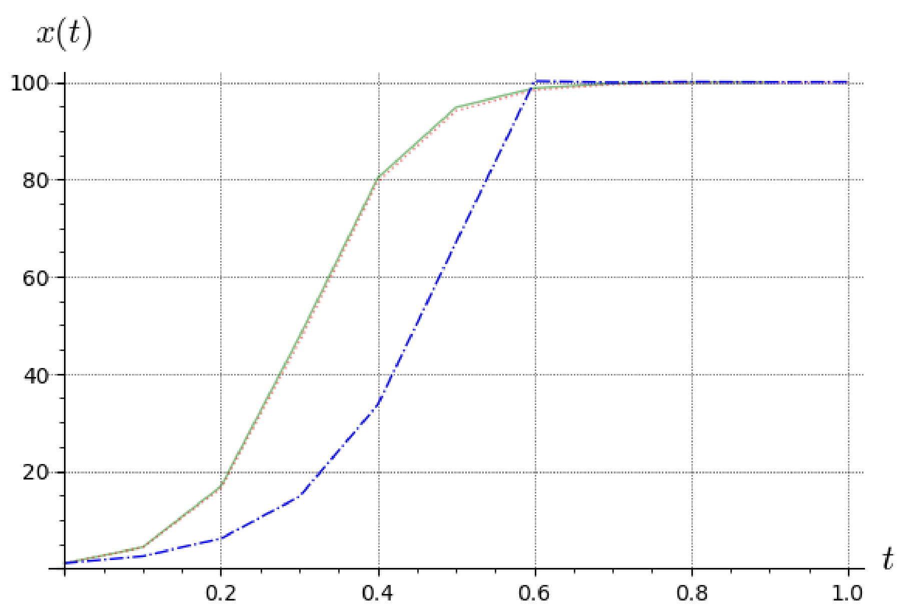
$$x_{n+1} = x_n + h \cdot f(t_n, x_n) \quad \text{where} \quad f(t, x) = 0.15\,x(100 - x)$$

Iteration Table

Step n	t_n	x_n
0	0.0	1.000000
1	0.1	2.485000
2	0.2	6.119872
3	0.3	14.737887
4	0.4	33.586637
5	0.5	67.045660
6	0.6	100.187342
7	0.7	99.905803
8	0.8	100.046966
9	0.9	99.976484
10	1.0	100.011750

```
def euler_method(f, x0, t0, t_end, h):
    """
    Implements the Euler method for solving x' = f(t, x)

    Parameters:
        f: a function of (t, x)
        x0: initial value x(t0)
        t0: initial time
        t_end: final time
        h: step size

    Returns:
        A list of (t, x) points
    """
    steps = int((t_end - t0) / h)
```

```
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        x = x + h * f(t, x)
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))
t, x = var('t x')
f(t, x) = r * x * (K - x)  # logistic equation

solution = euler_method(f, x0=x0, t0=t0, t_end=tf, h=step_size)

# Plot
euler_solution = list_plot(solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlin
compare_euler = sage_solution + euler_solution
compare_euler.plot()
```



Modified Euler Method (Heun's Method)

Update rule:

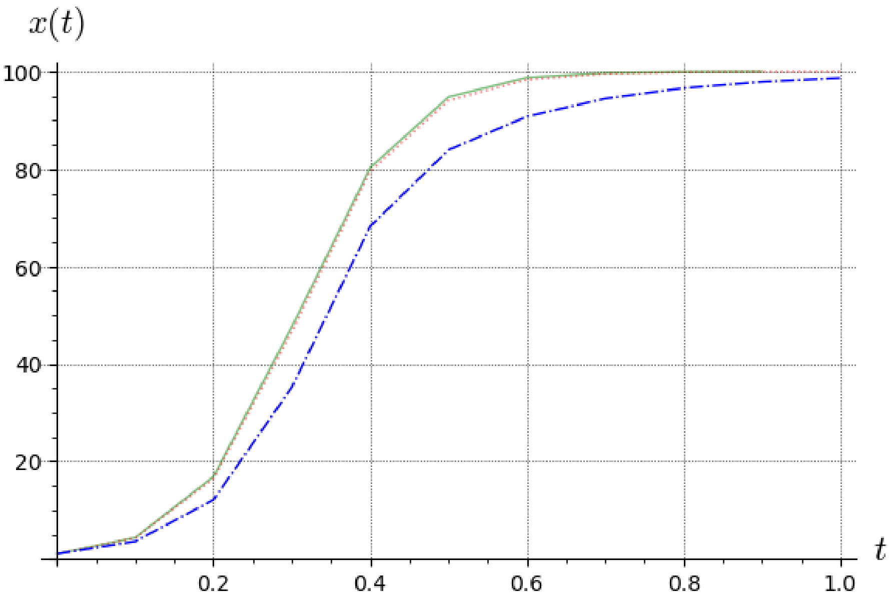
$$k_1 = h \cdot f(t_n, x_n)$$
$$k_2 = h \cdot f(t_n + h, x_n + k_1)$$
$$x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2)$$

Iteration Table

Step n	t_n	x_n
0	0.0	1.000000
1	0.1	3.559936
2	0.2	12.098199
3	0.3	35.210581
4	0.4	68.238787
5	0.5	83.927939
6	0.6	90.793752
7	0.7	94.480669
8	0.8	96.624886
9	0.9	97.916019
10	1.0	98.706641

```
def modified_euler_method(f, x0, t0, t_end, h):
    """
    Implements the Modified Euler method (Heun's Method) for solving x' = f(t, x)

    Parameters:
        f: a function of (t, x)
        x0: initial value x(t0)
        t0: initial time
        t_end: final time
        h: step size

    Returns:
        A list of (t, x) points
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0
```

```
    for _ in range(steps):
        k1 = h*f(t, x)
        k2 = h*f(t + h, x + k1)
        x = x + 1/2 * (k1 + k2)
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))

solution = modified_euler_method(f, x0=x0, t0=t0, t_end=tf, h=step_size)

# Plot the numerical approximation
heun_solution = list_plot(solution, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridline
compare_heun = sage_solution + heun_solution
compare_heun.plot()
```



Runge-Kutta 2nd Order Method (General Form)

With parameters:

- $\alpha = \frac{1}{2}$
- $\beta = \frac{1}{2}$
- $a = 0$, $b = 1$

The method iterates:

$$k_1 = h \cdot f(t_n, x_n)$$
$$k_2 = h \cdot f(t_n + \alpha h, x_n + \beta k_1)$$
$$x_{n+1} = x_n + a \cdot k_1 + b \cdot k_2$$

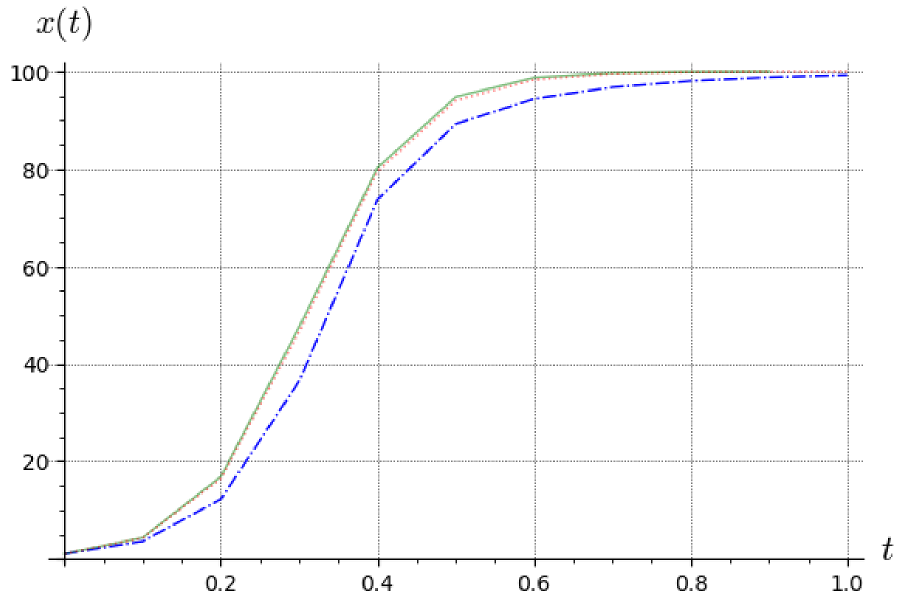
Iteration Table

Step n	t_n	x_n
0	0.0	1.000000
1	0.1	3.568205
2	0.2	12.224380
3	0.3	36.467980
4	0.4	73.746264
5	0.5	89.280670
6	0.6	94.404944
7	0.7	96.815240
8	0.8	98.112218
9	0.9	98.856498
10	1.0	99.298710

```
def rk2_general_form(f, x0, t0, t_end, h=0.01,
                       alpha=1/2, beta=1/2, a=0, b=1):
    """
    General second-order Runge-Kutta method using parameters alpha, beta, a, b.

    Solves x' = f(t, x) over [t0, t_end] with initial condition x(t0) = x0.

    Parameters:
        f: right-hand side function f(t, x)
        x0: initial condition x(t0)
        t0: initial time
        t_end: final time
        h: time step size
        alpha: time increment coefficient for k2
        beta: slope coefficient for k2
        a, b: weights for k1 and k2 in the update

    Returns:
        List of (t, x) points approximating the solution
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
```

```
    x = x0

    for _ in range(steps):
        k1 = h * f(t, x)
        k2 = h * f(t + alpha * h, x + beta * k1)
        x = x + a * k1 + b * k2
        t = t + h
        t_vals.append(t)
        x_vals.append(x)

    return list(zip(t_vals, x_vals))
```

```
sol = rk2_general_form(f, x0=x0, t0=t0, t_end=tf, h=step_size)

heun_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines=Tr
compare_heun = heun_solution  + sage_solution
compare_heun.plot()
```



Runge-Kutta 2nd Order Method — Ralston's Method

Parameters:

- $\alpha = \frac{3}{4}$
- $\beta = \frac{3}{4}$
- $a = \frac{1}{3}$, $b = \frac{2}{3}$

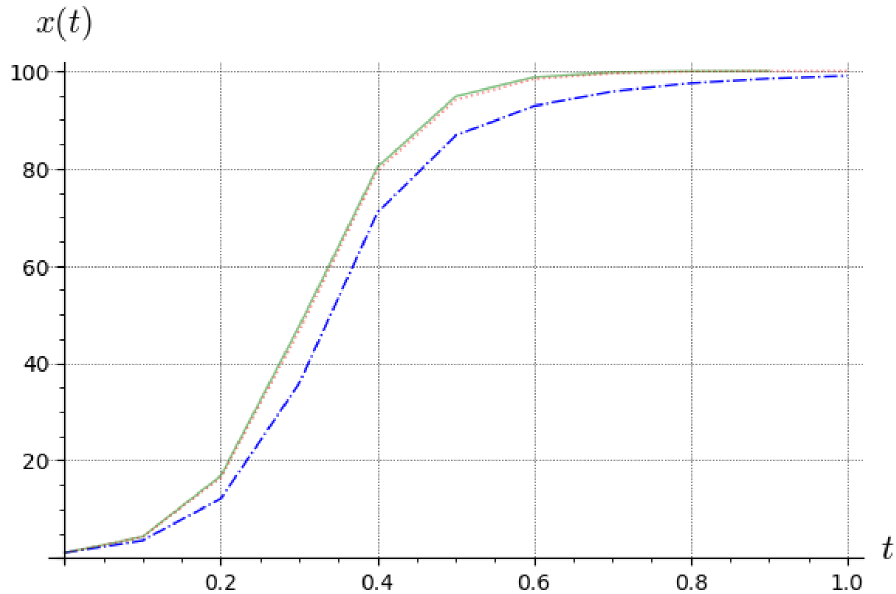
The method iterates:

$$
\begin{aligned}
k_1 &= h \cdot f(t_n, x_n) \\
k_2 &= h \cdot f\left(t_n + \alpha h, x_n + \beta k_1\right) \\
x_{n+1} &= x_n + a \cdot k_1 + b \cdot k_2
\end{aligned}
$$

Iteration Table

Step n	t_n	x_n
0	0.0	1.000000
1	0.1	3.564071
2	0.2	12.161179
3	0.3	35.834899
4	0.4	70.962143
5	0.5	86.778367
6	0.6	92.828066
7	0.7	95.838010
8	0.8	97.505961
9	0.9	98.479518
10	1.0	99.063879

```
sol = rk2_general_form(f, x0=x0, t0=t0, t_end=tf, h=step_size, alpha=3/4, beta=3/4, a=1/3, b

ralston_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines
compare_heun = ralston_solution  + sage_solution
compare_heun.plot()
```



Runge-Kutta 4th Order Method (RK4)

The method iterates:

$$
\begin{aligned}
k_1 &= h \cdot f(t_n, x_n) \\
k_2 &= h \cdot f\left(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}\right) \\
k_3 &= h \cdot f\left(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}\right) \\
k_4 &= h \cdot f(t_n + h, x_n + k_3) \\
x_{n+1} &= x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

Iteration Table

Step n	t_n	x_n
0	0.0	1.000000
1	0.1	4.259248
2	0.2	16.428180
3	0.3	46.613716
4	0.4	79.536875
5	0.5	94.077402
6	0.6	98.359221

Step n	t_n	x_n
7	0.7	99.549650
8	0.8	99.876726
9	0.9	99.966283
10	1.0	99.990780

```
def rk4(f, x0, t0, t_end, h=0.01):
    """
    Fourth-order Runge-Kutta method as specified in the image.

    Solves x' = f(t, x) over [t0, t_end] with initial condition x(t0) = x0.

    Parameters:
        f: right-hand side function f(t, x)
        x0: initial condition x(t0)
        t0: initial time
        t_end: final time
        h: time step size

    Returns:
        List of (t, x) points approximating the solution
    """
    steps = int((t_end - t0) / h)
    t_vals = [t0]
    x_vals = [x0]

    t = t0
    x = x0

    for _ in range(steps):
        # Calculate the four k values according to the formulas
        k1 = h * f(t, x)
        k2 = h * f(t + 0.5 * h, x + 0.5 * k1)
        k3 = h * f(t + 0.5 * h, x + 0.5 * k2)
        k4 = h * f(t + h, x + k3)

        # Update x using the weighted average
        x = x + (1/6) * (k1 + 2*k2 + 2*k3 + k4)

        # Increment time
        t = t + h

        # Store results
        t_vals.append(t)
        x_vals.append(x)
```

```
    return list(zip(t_vals, x_vals))

sol = rk4(f, x0=x0, t0=t0, t_end=tf, h=step_size)

rk4_solution  = list_plot(sol, plotjoined=True, axes_labels=['$t$', '$x(t)$'], gridlines=Tru
compare_rk4 = rk4_solution  + sage_solution
compare_rk4.plot()
```