# Partial Differential Equations in Finance

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# Introduction to PDEs in Financial Applications

# Overview of Topics

- Charting theory and nature of boundary and initial conditions
- Explicit solutions, including original Black-Scholes formula
- Special problems arising when there are free boundaries

# Key Questions in Financial PDEs

- Physical interpretation of the equations?
- Mathematical properties of the solution?
- Techniques for obtaining explicit solutions?

PDEs in finance:

- Fundamental equations (e.g., Black-Scholes)
- Linear vs. nonlinear problems

# Considerations for PDEs in Finance

1. Does the equation make sense as a well-posed problem?
    - Appropriate boundary or initial/final conditions?
    - Nature of the mathematical problem?
    - Smooth or discontinuous solutions?
2. Can we develop analytical tools to solve the equation?
3. How should we solve the equation numerically if necessary?

# First Order Linear PDEs

# First Order Linear PDE

Consider the equation:

$$\alpha(s,t)\frac{\partial u}{\partial s} + \beta(s,t)\frac{\partial u}{\partial t} = \gamma(s,t)u(t)$$

**Linearity Property**
If $u_1, u_2$ are solutions, then $c_1 u_1 + c_2 u_2$ is also a solution for constants $c_1, c_2$.

**Proof.**
Substitute $u = c_1 u_1 + c_2 u_2$ into the PDE and use linearity of derivatives. $\square$

# Constant Coefficient Case

When $\alpha, \beta$ are constants and $\gamma \equiv 0$:

$$\alpha_0 \frac{\partial u}{\partial s} + \beta_0 \frac{\partial u}{\partial t} = 0 \quad (4.2)$$

Define the vector $\vec{v} = (\alpha_0, \beta_0)$ with $\beta_0 \alpha_0 \neq 0$.

The directional derivative:

$$\nabla_{\vec{v}} u = \langle \alpha_0, \beta_0 \rangle \cdot \langle \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \rangle$$

Thus, (4.2) $\Leftrightarrow \nabla_{\vec{v}} u \equiv 0$

# Characteristic Coordinates

Introduce new coordinates:

$$\xi = \beta_0 S + \alpha_0 t, \quad \zeta = \beta_0 S - \alpha_0 t$$

The Jacobian determinant:

$$\begin{vmatrix} \beta_0 & \alpha_0 \\ \beta_0 & -\alpha_0 \end{vmatrix} = -2\alpha_0\beta_0 \neq 0$$

Let's solve the system for $S, t$:

$$S = \frac{1}{2\beta_0}(\xi + \zeta) \tag{1}$$

$$t = \frac{1}{2\alpha_0}(\xi - \zeta) \tag{2}$$

Then

$$\frac{\partial S}{\partial \xi} = \frac{1}{2\beta_0} \tag{3}$$

$$\frac{\partial t}{\partial \xi} = \frac{1}{2\alpha_0} \tag{4}$$

Computing $\frac{\partial u}{\partial \xi}$:

$$
\begin{aligned}
\frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial S}\frac{\partial S}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} && (5)\\
&= \frac{1}{2\beta_0}\frac{\partial u}{\partial S} + \frac{1}{2\alpha_0}\frac{\partial u}{\partial t} && (6)\\
&= \frac{1}{2\beta_0\alpha_0}\left(\alpha_0\frac{\partial u}{\partial S} + \beta_0\frac{\partial u}{\partial t}\right) && (7)\\
&= 0 && (8)
\end{aligned}
$$

Therefore, $u$ depends only on $\zeta$: $u = F(\zeta)$

# Characteristics and Information Propagation

- $\xi = \beta_0 s - \alpha_0 t$ is called a **characteristic**
- Characteristics represent directions in which information propagates
- Analogous to constants of integration in ODEs

**Important Note**
Without boundary conditions, the solution remains arbitrary (like the constant $C$ in ODEs)

# Second-Order Linear PDEs

# General Second-Order Linear PDE

Consider the general form in two variables:

$$a(x,t)u_{xx} + 2b(x,t)u_{xt} + c(x,t)u_{tt} \\ + d(x,t)u_x + e(x,t)u_t + f(x,t)u = g(x,t) \quad (9)$$

where $a, b, c, d, e, f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are coefficient functions.

**Principal Part**
The highest-order terms form the principal part:

$$L_p(u) = au_{xx} + 2bu_{xt} + cu_{tt}$$

# Classification of PDEs

Define the discriminant:

$$\Delta(L) = b^2 - ac : (x,t) \mapsto b(x,t)^2 - a(x,t)c(x,t)$$

The PDE is classified as:

- **Hyperbolic** if $\Delta(L) > 0$ (e.g., wave equation)
- **Parabolic** if $\Delta(L) = 0$ (e.g., heat equation)
- **Elliptic** if $\Delta(L) < 0$ (e.g., Laplace equation)

# Coordinate Transformations

**Definition**
A transformation $(\xi, \eta) = (\xi(x,t), \eta(x,t))$ is called a change of coordinates if the Jacobian is non-zero:

$$J := \xi_x \eta_t - \xi_t \eta_x \neq 0$$

**Theorem**
*The type of a linear second-order PDE in two variables is invariant under a change of coordinates.*

# Proof of Invariance

Under transformation $(\xi, \eta)$, the principal part becomes:

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta}$$

where:

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2 \\ B &= a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t \\ C &= a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2 \end{aligned}$$

The new discriminant satisfies:

$$\Delta_{new} = AC - B^2 = J^2(b^2 - ac) = J^2\Delta_{original}$$

# Canonical Forms

- **Hyperbolic**: $u_{\xi\xi} - u_{\eta\eta} =$ lower order terms (wave equation)
- **Parabolic**: $u_{\xi\xi} =$ lower order terms (heat equation)
- **Elliptic**: $u_{\xi\xi} + u_{\eta\eta} =$ lower order terms (Laplace equation)

**Examples**

- Wave equation: $u_{tt} = c^2(u_{xx} + u_{yy})$
- Heat equation: $u_t = k(u_{xx} + u_{yy})$
- Laplace equation: $u_{xx} + u_{yy} = 0$

# Simplification Through Coordinates

A proper linear change of coordinates can transform:

- Hyperbolic equations to $u_{\xi\eta} = 0$
- Parabolic equations to $u_{\xi\xi} = 0$
- Elliptic equations to $u_{\xi\xi} + u_{\eta\eta} = 0$

**Important Note**
The classification determines:

- Nature of characteristics
- Appropriate boundary conditions
- Solution techniques

# Heat Equation Analysis

# Heat Equation

The fundamental heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Physical interpretation:

- $u(x, t)$ represents temperature distribution along a thin insulated rod
- Models heat diffusion over time

# Mathematical Properties

Key characteristics:

- Linear equation (superposition principle applies)
- Second order in space ($x$)
- First order in time ($t$)
- Parabolic type (discriminant $\Delta = 0$)

**Analytic Solutions**
Solutions are analytic functions of $x$:

- For any $t^* > 0$ and $a < x_0 < b$
- $u(x, t^*)$ can be expressed as a convergent power series in $(x - x_0)$

# Solution Behavior

Important features:

- **Smoothing property**: Solutions become instantly smooth for $t > 0$
- **Infinite propagation speed**: Initial disturbances spread infinitely fast
- **Maximum principle**: Extreme values occur only at initial/boundary conditions

**Financial Interpretation**
Analogous to:

- Diffusion of volatility in option pricing
- Smoothing of price shocks over time

# Initial Value Problems with Boundary Conditions

# Dirichlet Boundary Value Problem

Heat equation on finite interval:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L,\ t > 0$$

With conditions:

- Initial condition: $u(x, 0) = u_0(x)$
- Boundary conditions:
  - $u(-L, t) = g_1(t)$ (left end temperature)
  - $u(L, t) = g_2(t)$ (right end temperature)

**Physical Interpretation**
Models heat diffusion in a finite rod with controlled end temperatures

# Neumann Boundary Value Problem

Same heat equation with flux conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -L < x < L,\ t > 0$$

With conditions:

- Initial condition: $u(x,0) = u_0(x)$
- Flux boundary conditions:
  - $-\frac{\partial u}{\partial x}(-L,t) = h_1(t)$ (left end flux)
  - $\frac{\partial u}{\partial x}(L,t) = h_2(t)$ (right end flux)

**Physical Interpretation**
Models heat diffusion with controlled heat flow at boundaries

# Infinite Domain Problem

Taking $L \to \infty$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0$$

With initial condition:

$$u(x,0) = u_0(x)$$

**Growth Conditions**
For well-posedness:

- $u_0(x)$ piecewise continuous (finite jumps allowed)
- $\lim_{|x|\to\infty} \frac{u_0(x)}{e^{ax^2}} = 0 \quad \forall a > 0$
- $\lim_{|x|\to\infty} \frac{u(x,t)}{e^{ax^2}} = 0 \quad \forall a, t > 0$

# Semi-Infinite Domain with Mixed Conditions

Example problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \ t > 0$$

With conditions:

- Initial condition: $u(x, 0) = u_0(x)$ (sufficiently smooth)
- Boundary condition: $u(0, t) = g(t)$
- Growth conditions at infinity

**Well-Posedness**

The problem is well-posed when:

- Initial/boundary conditions are compatible
- Growth conditions are satisfied
- Solution exists, is unique, and depends continuously on data

# Backward Heat Equation

# Time Reversal Transformation

Consider the time transformation:

$$t = T - \tau, \quad T \text{ constant}$$

The time derivative transforms as:

$$\frac{\partial u}{\partial \tau} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} = -\frac{\partial u}{\partial t}$$

Applying to the heat equation:

$$\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial x^2} \quad \text{(Backward Heat Equation)}$$

**Important Property**

The backward heat equation is generally **ill-posed** for most initial conditions

# Fundamental Solution

The fundamental solution to the (forward) heat equation:

$$u_g(x,t) = \begin{cases} \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)} & t > 0 \\ \delta_0(x) & t = 0 \end{cases}$$

Consider the variant:

$$u(x,t) = \frac{1}{2\sqrt{\pi(\tau - t)}} e^{-\frac{x^2}{4(\tau - t)}}$$

- Solves the backward heat equation
- Initial condition: $u_0(x) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/(4\tau)}$
- **Blows up** at $t = \tau$ (singularity)

# Final Value Problem

The backward heat equation with final condition:

$$u_T(x) = u_g(x, \tau)$$

**Well-Posed Case**
This specific final value problem is well-posed when:

- Properly transformed to a forward problem via $t \mapsto T - t$
- The final condition decays sufficiently fast
- Solution exists in a limited time domain $t < \tau$

**Example**
In finance, appears in:

- Option pricing with terminal payoff conditions
- Implicit finite difference methods

# Mathematical Implications

Key observations about the backward heat equation:

- **Ill-posedness**: Small changes in final conditions can lead to large solution changes
- **Regularization needed** for practical applications
- **Time reversal** requires special treatment

**Numerical Challenges**
Requires:

- Special discretization schemes
- Stability analysis
- Often needs additional constraints