Black-Scholes Mathematical Analysis

Partial Differential Equations and Option Pricing

Lecturer

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Outline

Proving the Black-Scholes Equation is Parabolic

The Black-Scholes Partial Differential Equation

The Black-Scholes equation for option pricing is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Where:

- $V(S, t) =$ option value
- $S =$ underlying asset price
- $t =$ time
- $\sigma =$ volatility
- $r =$ risk-free interest rate

General Form of Second-Order PDEs

A general second-order PDE in two variables has the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F \cdot u + G = 0$$

Classification of PDEs
PDEs are classified based on the discriminant $\Delta = B^2 - 4AC$:

- **Elliptic**: $\Delta < 0$
- **Parabolic**: $\Delta = 0$
- **Hyperbolic**: $\Delta > 0$

Identifying Coefficients in Black-Scholes

Rewriting the Black-Scholes equation in standard form with $x = S$ and $y = t$:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$$

Coefficients

- $A = \frac{1}{2}\sigma^2 S^2$
- $B = 0$
- $C = 0$
- $D = rS$
- $E = 1$
- $F = -r$

Computing the Discriminant

$$\Delta = B^2 - 4AC = 0^2 - 4 \cdot \left(\frac{1}{2}\sigma^2 S^2\right) \cdot 0 = 0$$

Conclusion
Since the discriminant $\Delta = 0$, the Black-Scholes equation is **parabolic**.

Physical Interpretation

The parabolic nature reflects:

1. **Diffusion Process**: The $\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ term represents diffusion in the asset price, similar to heat diffusion
2. **Time Evolution**: Information propagates through the system over time, characteristic of parabolic PDEs
3. **No Second Time Derivative**: Unlike wave equations (hyperbolic), there's no "acceleration" term in time

Important Implications

- Parabolic PDEs have unique solutions under appropriate boundary conditions
- Numerical methods for parabolic PDEs are well-established
- The solution exhibits smoothing properties typical of diffusion processes

Itô's Formula and Stochastic Calculus

The Need for Stochastic Calculus

Problem with Ordinary Calculus
When dealing with random processes like stock prices, ordinary calculus fails because:

- Brownian motion isn't differentiable
- The quadratic term $(dx)^2$ doesn't vanish
- We need special rules for stochastic differentiation

Brownian Motion Properties
For Brownian motion $W(t)$:

- $W(0) = 0$ (starts at zero)
- Independent, normally distributed increments
- Continuous paths but nowhere differentiable
- **Key property**: $(dW)^2 = dt$

Itô's Formula: The Stochastic Chain Rule

For a function $f(W, t)$ where W is Brownian motion:

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right) dt + \frac{\partial f}{\partial W} dW$$

Key Insight

The second derivative term appears because $(dW)^2 = dt \neq 0$ in stochastic calculus!

In ordinary calculus: $df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx$

In stochastic calculus: We get an extra $\frac{1}{2}\frac{\partial^2 f}{\partial W^2} dt$ term.

Geometric Brownian Motion

Stock prices follow geometric Brownian motion (GBM):
$dS = \mu S dt + \sigma S dW$

Where:

- μ = expected return (drift)
- σ = volatility
- dW = Brownian motion increment

Why GBM?

- Stock prices can't go negative
- Percentage changes are more natural than absolute changes
- Leads to lognormal distribution of future prices

Itô's Formula for Functions of GBM

For a function $V(S, t)$ where S follows GBM, Itô's formula gives:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

The Derivation Logic

1. Start with Taylor expansion around (S, t)
2. Use $dS = \mu S dt + \sigma S dW$
3. Compute $(dS)^2 = \sigma^2 S^2 dt$ (key step!)
4. Keep terms up to order dt

The Crucial Stochastic Rules

Fundamental Rules

- $(dt)^2 = 0$ - second-order infinitesimals vanish
- $(dW)^2 = dt$ - **this is the key stochastic rule**
- $dt \times dW = 0$ - deterministic and random parts are orthogonal

Computing $(dS)^2$

$(dS)^2 = (\mu S dt + \sigma S dW)^2$
$= \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 (dt)(dW) + \sigma^2 S^2 (dW)^2$
$= 0 + 0 + \sigma^2 S^2 dt = \sigma^2 S^2 dt$

Why Itô's Formula Matters for Options

The formula tells us how option value $V(S, t)$ changes:

$$dV = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dW$$

Key Observations

- **Same randomness**: Both stock and option driven by the same dW
- **The drift term**: Contains the volatility effect $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$
- **The diffusion term**: Proportional to $\frac{\partial V}{\partial S}$ (the delta!)
- **Foundation for hedging**: This shared randomness enables delta hedging

Derivation of the Black-Scholes Equation

Key Assumptions for Black-Scholes

1. **Geometric Brownian Motion**: Stock price follows

$$dS = \mu S dt + \sigma S dW$$

   where dW is a Wiener process (Brownian motion)
2. **Constant Parameters**: Risk-free rate r, volatility σ are constant
3. **No Dividends**: The stock pays no dividends
4. **Perfect Market**: No transaction costs, continuous trading, unlimited borrowing/lending at rate r
5. **European Exercise**: Option can only be exercised at expiration

Step 1: Stock Price Model

The stock price follows geometric Brownian motion:

$$dS = \mu S dt + \sigma S dW$$

Where:

- μ is the expected return (drift)
- σ is the volatility
- dW is a Wiener process (random walk)

Interpretation
The stock has a deterministic trend μS and random fluctuations σS.

Step 2: Option Value Changes (Itô's Lemma)

For option value $V(S, t)$, applying Itô's lemma:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

Key Insight
Both dS and dV contain the same random term dW - this is crucial for hedging.

Step 3: Constructing the Hedged Portfolio

Create a portfolio Π consisting of:

- **Long 1 option** (value $= V$)
- **Short Δ shares** (value $= -\Delta S$)

So: $\Pi = V - \Delta S$

The change in portfolio value is:

$$d\Pi = dV - \Delta\, dS$$

Step 4: The Critical Substitution

Substitute the expressions for dV and dS:

$$d\Pi = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta \mu S\right] dt \tag{1}$$

$$+ \left[\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S\right] dW \tag{2}$$

Factoring the dW coefficient:

$$\text{dW coefficient} = \sigma S \left[\frac{\partial V}{\partial S} - \Delta\right]$$

Step 4: Eliminating Risk - The Magic Choice

To eliminate the random dW term, we set:

$$\Delta = \frac{\partial V}{\partial S}$$

Why This Works

$\frac{\partial V}{\partial S}$ is the option's sensitivity to stock price changes. By shorting exactly this many shares, we create perfect hedge:

- Stock goes up $\rightarrow$ option gains, short stock loses
- Stock goes down $\rightarrow$ option loses, short stock gains
- **Net effect: zero risk!**

With this choice: $d\Pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt$

Step 5: No-Arbitrage Condition

Since the portfolio is now risk-free, it must earn the risk-free rate r:

$$d\Pi = r\Pi \, dt$$

But $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S} S$

Therefore:

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt = r \left[V - \frac{\partial V}{\partial S} S\right] dt$$

Step 6: The Black-Scholes Equation Emerges

Dividing by dt and rearranging:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Rearranging to standard form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Remarkable Results

- The expected return μ disappeared completely!
- Option prices depend only on volatility σ, not expected return
- Perfect hedging creates risk-free portfolio

The Deep Intuition

What We Accomplished
By setting $\Delta = \frac{\partial V}{\partial S}$, we matched the sensitivities:

- Option position sensitivity = Stock position sensitivity
- Random fluctuations cancel out perfectly
- Only deterministic terms remain

Economic Interpretation
In an efficient market with no arbitrage opportunities:

- Risk-free portfolios must earn the risk-free rate
- This constraint determines option prices uniquely
- The mathematics captures this economic principle