Numerical Analysis for Non-Linear Optimization

The notes for this week have been taken from > Kelley, W. G. (2010). The theory of differential equations. Springer.

First-Order Linear Equations

In the next week, we will study equation of the form:

$$x'(t) = f(t, x(t))$$

where $f : (a, b) \times (c, d) \to \mathbb{B}$ is continuous, $-\infty \leq a < b \leq \infty$ and $-\infty \leq c, d \leq \infty$.

Variation of Constants Formula

Consider

$$\begin{cases} x' = p(t)x + q(t) \\ x(t_0) = x_0 \end{cases} \tag{1.5}$$

where $p, q : (a, b) \to \mathbb{R}$ are continuous functions, $-\infty \leq a < b \leq \infty$, $t_0 \in (a, b)$ and $x_0 \in \mathbb{R}$.

Then for $t \in (a, b)$, the solution of the above equation is given by

$$x(t) = e^{\int_{t_0}^{t} p(\tau)\, d\tau} x_0 + e^{\int_{t_0}^{t} p(\tau)\, d\tau} \int_{t_0}^{t} e^{-\int_{t_0}^{s} p(\tau)\, d\tau} q(s)\, ds, \tag{1.6}$$

Example

Consider $p(\tau) = \tau$ and $q(\tau) = \sin(\tau)$. First, let's write a function to compute the solution of the above equation.

```
def phi(p, q, x0, t0, t):
    tau, s = var('tau s')
    # Compute the exponential integrating factor
    exp_int_p = exp(integrate(p(tau), (tau, t0, t)))

    # First term
    term1 = exp_int_p * x0

    # Inner integral inside the second term
    inner_exp = exp(-integrate(p(tau), (tau, t0, s)))
    inner_integrand = inner_exp * q(s)
    inner_integral = integrate(inner_integrand, (s, t0, t))
```

```
    # Second term
    term2 = exp_int_p * inner_integral

    return term1 + term2
```

Then, let's define the adeuacte parameters:

```
t, t0 = var('t t0')

assume( t0 > 0)
assume(t > t0)

p(t) = -2
q(t) = 140 + 20 * exp(-2*t)
x0 = 40
x = phi(p, q, x0, 0, t)
print(x.expand())
```

Finally, let's verify that the given function is a solution of the initial value problem.

```
diff(x,t).expand()
```

```
(p(t)*x + q(t)).expand()
```

```
x(t=0)
```

Autonomous Equations

The equation

$$x' = f(x) \tag{1.6}$$

is called autonomous because f doesn't depends explicitly on t. We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function.

Theorem 1.8
If x is a solution of the autonomous differential equation (1.6) on an interval (a, b), where $-\infty \leq a < b \leq \infty$, then for any constant c, the function y defined by $y(t) := x(t - c)$, for $t \in (a + c, b + c)$, is a solution of (1.6) on $(a + c, b + c)$.

Definition 1.9
If $f(x_0) = 0$ we say that x_0 is an**equilibrium point**for the differential equation (1.6). If, in addition, there is a $\delta > 0$ such that

$$f(x) \neq 0 \quad \text{for } |x - x_0| < \delta,\ x \neq x_0,$$

then we say x_0 is an**isolated equilibrium point.

Example (1.10)

Consider the equation $x' = -2(x - 70)$.

- $x = 70$ is the only equilibrium point.
- Any solution is in the form $x(t) = De^{-2t} + 70$.
- We cna verify that $x(t - c)$ is a solution.

Definition 1.11
Let ϕ be a solution of (1.6) with maximal interval of existence (α, ω). Then the set

$$\{\phi(t) : t \in (\alpha, \omega)\}$$

is called an orbit for the differential equation (1.6).

Note that the orbits for

$$x' = -2(x - 70)$$

are the images of the sets

$$(-\infty, 70), \quad \{70\}, \quad (70, \infty).$$

under the solution $x(t)$.

```
var('x t x0')
f(x) = -2 * (x - 70)

def G(x0):
    return 70 + (x0 - 70) * exp(-2 * t)

phi = G(x0)
```

```
phi.diff(t).expand()
```

```
f(phi).expand()
```

```
phi(t = 0)
```

```
# Initial conditions
x0_values = [40, 50, 60, 70, 80, 90, 100]
solutions = [ phi(x0=x0) for x0 in x0_values]

# Slope field in the (t, x) plane
slope_field = plot_slope_field(
    f, (t, 0, 5), (x, 40, 100), color='lightgray')
```

```
# Plot solution curves
solution_plots = sum(
    [plot(sol, (t, 0, 3),) for sol, x0 in zip(solutions, x0_values)])

# Equilibrium line
equilibrium_line = plot(
    70, (t, 0, 5),
    color='black', linestyle='--', legend_label='Equilibrium x = 70')

# Show the complete plot
show(slope_field + solution_plots + equilibrium_line, figsize=6)
```

Theorem 1.12
Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then two orbits of (1.6) are either disjoint sets or are the same set.

Example 1.13 *(Logistic Growth)*
The logistic law of population growth (Verhulst [52], 1838) is

$$N' = rN\left(1 - \frac{N}{K}\right),$$

where N is the number of individuals in the population, $r(1 - N/K)$ is the *per capita growth rate* that declines with increasing population, and $K > 0$ is the *carrying capacity* of the environment.
What are the orbits of the differential equation in this case?

```
var('t r K')
N = function('N')(t)
Np = r*N*(1-N/K)
factor(diff(Np, t).subs({diff(N,t):Np}))
```

```
# Define variables
var('t N r K N0')

f = r * N * (1-N/K)

# phi = (K*N0*exp(r*t))/(K + N0*(exp(r*t) - 1))
def G(N0, K, r):
    return (K*N0*exp(r*t))/(K + N0*(exp(r*t) - 1))

phi = G(N0, K, r)
```

```
phi.diff(t).factor()
```

```
f(N = phi).factor()
```

```
phi(t=0)
```

```
# Initial conditions
import numpy as np

K = 100
r = 0.1
N0_values = np.arange(1, 2*K, 10)

solutions = [ G(N0, K, r) for N0 in N0_values]

T = 110

# Slope field in the (t, x) plane
slope_field = plot_slope_field(f(r=r, K =K), (t, 0, T), (N, 1, 2*K), color='lightgray')

# Plot solution curves
solution_plots = sum([plot(sol, (t, 0, T),) for sol, N0 in zip(solutions, N0_values)])

# Equilibrium line
equilibrium_line = plot(K, (t, 0, T), color='black', linestyle='--', legend_label=f'N = {K}'

# Show the complete plot
show(slope_field + solution_plots + equilibrium_line, figsize=6)
```

Definition 1.15 We say that an equilibrium point x_0 of the differential equation (1.6) is *stable* provided given any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x_1 - x_0| < \delta$ it follows that the solution $\phi(\cdot, x_1)$ exists on $[0, \infty)$ and $|\phi(t, x_1) - x_0| < \epsilon$, for $t \geq 0$.

If, in addition, there is a $\delta_0 > 0$ such that $|x_1 - x_0| < \delta_0$ implies that $\lim_{t\to\infty} \phi(t, x_1) = x_0$, then we say that the equilibrium point x_0 is *asymptotically stable*. If an equilibrium point is not stable, then we say that it is *unstable*.

For the differential equation $N' = rN(1 - N/K)$ the equilibrium point $N_1 = 0$ is unstable and the equilibrium point $N_2 = K$ is asymptotically stable.

Potential Energy Function

Definition 1.16 We say that F is a *potential energy function* for the differential equation (1.6) provided $f(x) = -F'(x)$.

Theorem 1.17 If F is a potential energy function for (1.6), then $F(x(t))$ is strictly decreasing along any nonconstant solution x. Also, x_0 is an equilibrium point of (1.6) iff $F'(x_0) = 0$. If x_0 is an isolated equilibrium point of (1.6) such that F has a local minimum at x_0, then x_0 is asymptotically stable.

Example 1.18

Find the potential function for $x' = -2(x - 70)$.

```
var('u')

f = -2*(u-70)

F = -integrate(f, u, 0, x)
F

tc = 70
f(u=tc)

X = function('X')(t)
ode = diff(X, t) == f(u=X)
ode

var('X0')
phi = desolve(ode, X, [0, X0])

F(x=phi).diff(t).expand()

(-f(u=phi)^2).expand()
```

Generalized Logistic Equation

Suppose p, q are continuous, and x is the solution of

$$x' = -p(t)x + q(t) \tag{1.8}$$

with $x(t) \neq 0$ on I.

Then $y(t) = \frac{1}{x(t)}, t \in I$ is a solution of the *generalized linear equation*:

$$y' = (p(t) - q(t)y)\, y \tag{1.9}$$

Theorem 1.19: If $y_0 \neq 0$ and

$$\frac{1}{y_0} + \int_{t_0}^{t} q(s) e^{\int_{t_0}^{s} p(\tau)\, d\tau}\, ds \neq 0, \quad t \in I,$$

then the solution of the IVP

$$y' = (p(t) - q(t)y)\, y, \quad y(t_0) = y_0, \quad t_0 \in I \tag{1.10}$$

is given by

$$y(t) = \frac{e^{\int_{t_0}^{t} p(\tau)\, d\tau}}{\frac{1}{y_0} + \int_{t_0}^{t} q(s) e^{\int_{t_0}^{s} p(\tau)\, d\tau}\, ds}. \tag{1.11}$$

Corollary 1.20 If $y_0 \neq 0$ and

$$\frac{1}{y_0} - \frac{1}{K} + \frac{1}{K} e^{\int_{t_0}^{t} p(s)\,ds} \neq 0, \quad t \in I,$$

then the solution of the IVP

$$y' = p(t)\left[1 - \frac{y}{K}\right] y, \quad y(t_0) = y_0 \tag{1.13}$$

is given by

$$y(t) = \frac{e^{\int_{t_0}^{t} p(s)\,ds}}{\frac{1}{y_0} - \frac{1}{K} + \frac{1}{K} e^{\int_{t_0}^{t} p(s)\,ds}}. \tag{1.14}$$

Theorem 1.21 Assume $p : [t_0, \infty) \to [0, \infty)$ is continuous and $\int_{t_0}^{\infty} p(t)\,dt = \infty$. Let $y(t)$ be the solution of the IVP (1.13) with $y_0 > 0$, then $y(t)$ exists on $[t_0, \infty)$. Also if $0 < y_0 < K$, then $y(t)$ is nondecreasing with $\lim_{t\to\infty} y(t) = K$. If $y_0 > K$, then $y(t)$ is nonincreasing with $\lim_{t\to\infty} y(t) = K$.

Bifurcation

In the context of differential equations, a bifurcation refers to a qualitative change in the behavior of a system as a parameter is varied. More specifically, it occurs when a small smooth change made to the value of a parameter causes a sudden 'bifurcation' or splitting in the structure of the system's solutions — such as the number or stability of equilibrium points.

Example 1.22 $x' = \lambda(x-1)$

```
var("t x")

for l in [-1,0,1]:

    f = l * (x-1)

    g = Graphics()

    g += plot_slope_field(f, (t, -1, 1), (x, -1,  3), color='blue')

    g += plot( 1 , (t, -1, 1), color = 'red')

    g.show()

var("t x")

for l in [-1,0,1]:
```

```
a = sqrt(abs(l))

f = l + x^2

g = Graphics()

g += plot_slope_field(f, (t, -1, 1), (x, -a-1,  a+1), color='blue')

if l == 0 :
    g += plot(0, (t,-1,1), color = "red")

if l  < 0:
    g += plot(-a, (t,-1,1), color = "red")
    g += plot(a, (t,-1,1), color = "red")

g.show()
```