June 2025

Empirical Risk Minimization and Regularization

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IE University (demo class)

Outline

1. Empirical Risk Minimization

2. L_2 regularization

3. L_1 regularization

Empirical Risk Minimization

A tale of two terms

$$\min_{\mathbf{w},b} \quad \underbrace{\sum_{n=1}^{N} \ell(h_{\mathbf{w},b}(\mathbf{x}_n), y_n)}_{\text{Loss}}$$

$$\text{s.t.} \quad \underbrace{r(\mathbf{w}) \leq C}_{\text{Regularization}}$$

The regulzarization generates a **constraint** that limits learned parameter values in some way, in order to favour simpler solutions

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Empirical Risk Minimization

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The regularization term can take many different forms. The most popular ones are:

- L_2 regulatization
- L_1 regularization

L_2 regularization

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Where

$$\|\mathbf{w}\|_2^2 = \left(\|\mathbf{w}\|_2\right)^2 = \left(\sqrt{w_1^2 + w_2^2 + \cdots + w_d^2}\right)^2 = \sum_{i=1}^{d} w_i^2$$

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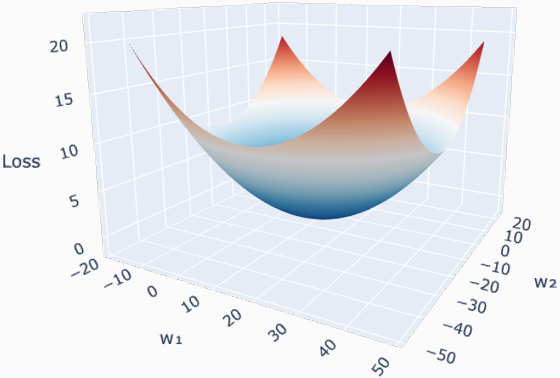
λ is the *regularization strength* → controls the tradeoff between loss and regularization

L_2 regularization

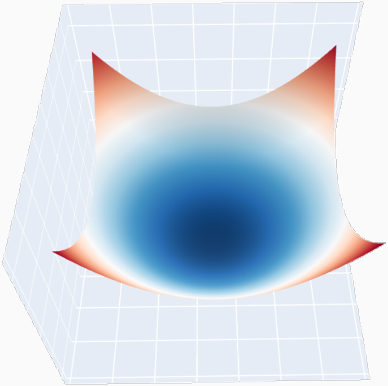
L_2 regularization has the effect of **shrinking** the estimated parameters $\mathbf{w}$ to smaller values

Smaller values for $\mathbf{w}$ generate more *conservative* predictions, potentially preventing overfitting

L_2 regularization



L_2 regularization



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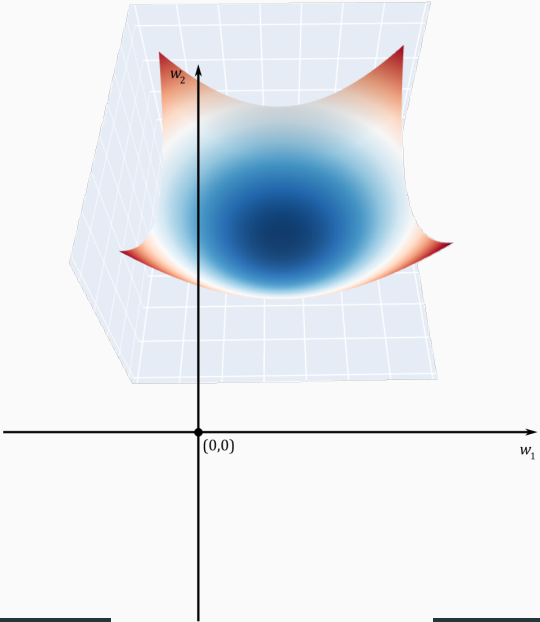

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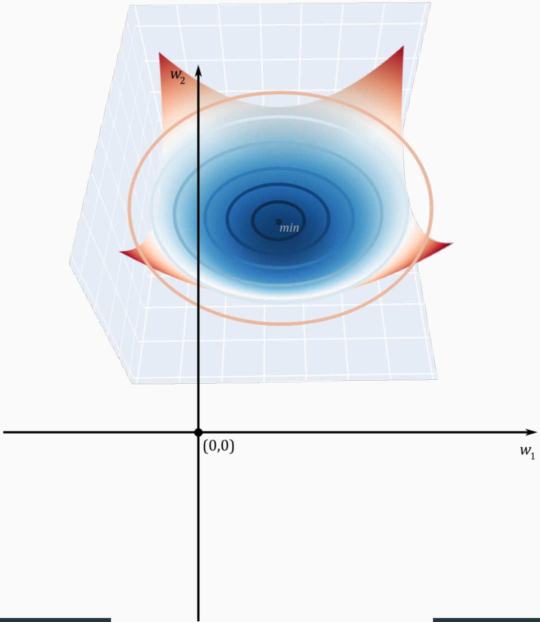

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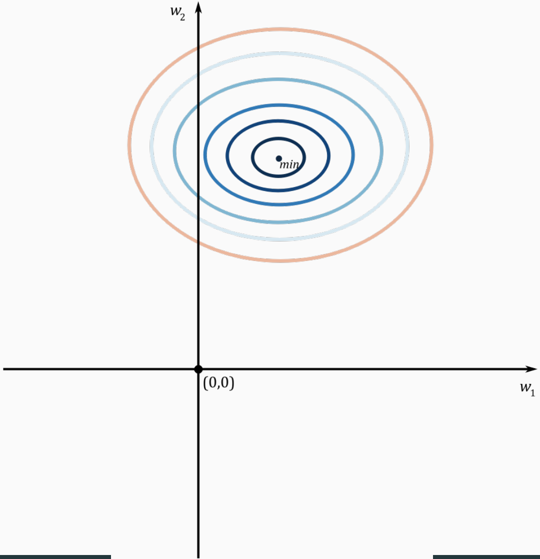

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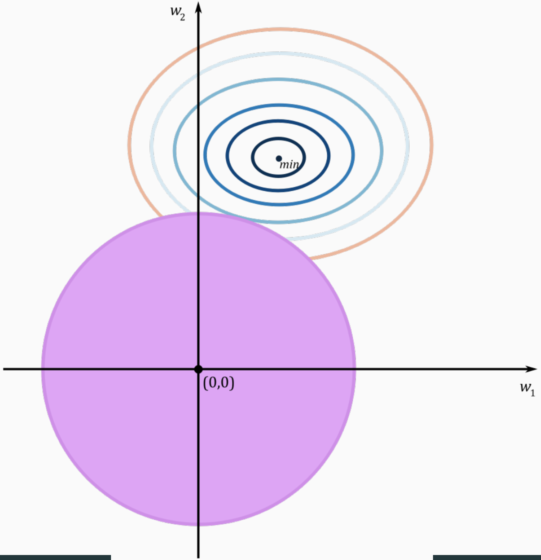

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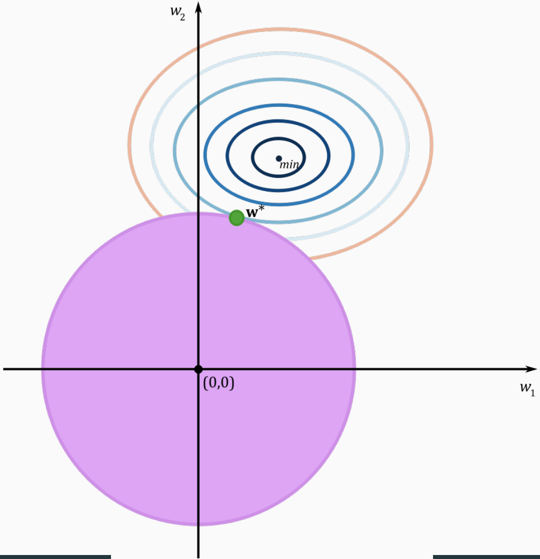
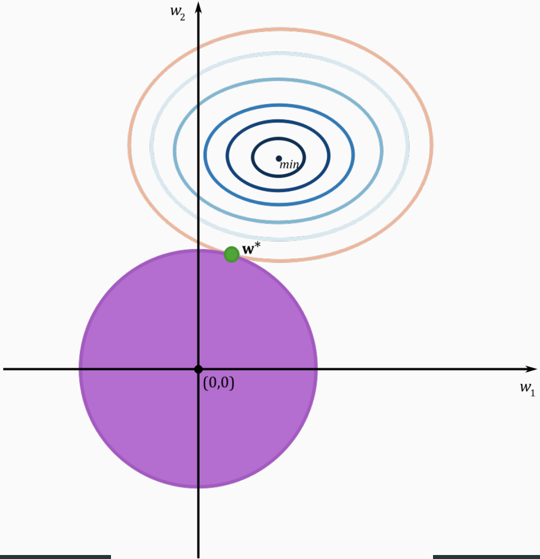
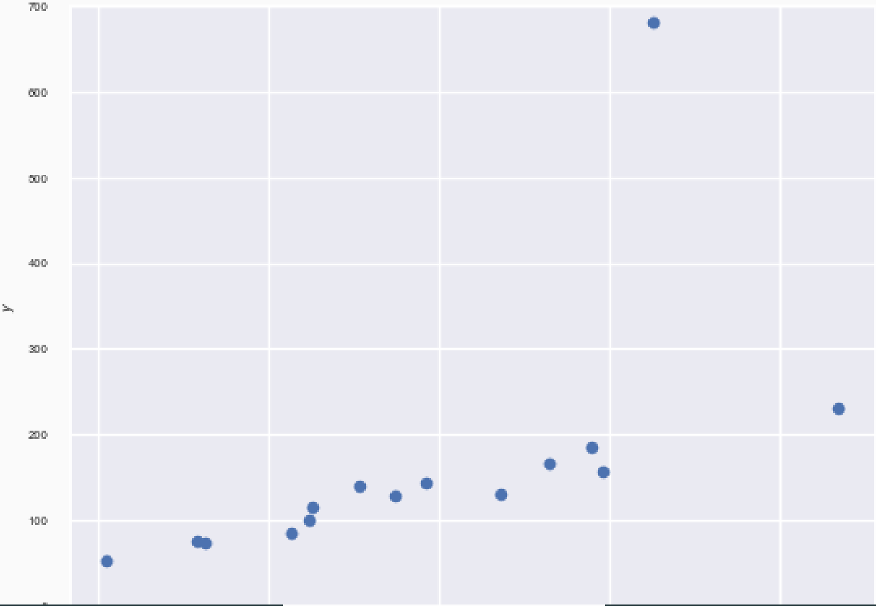


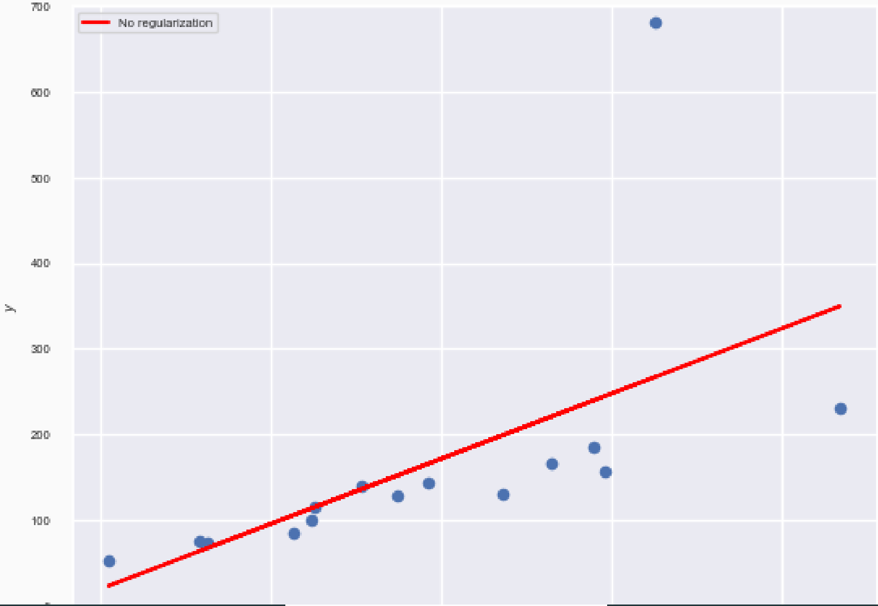
What happens if we increase λ?



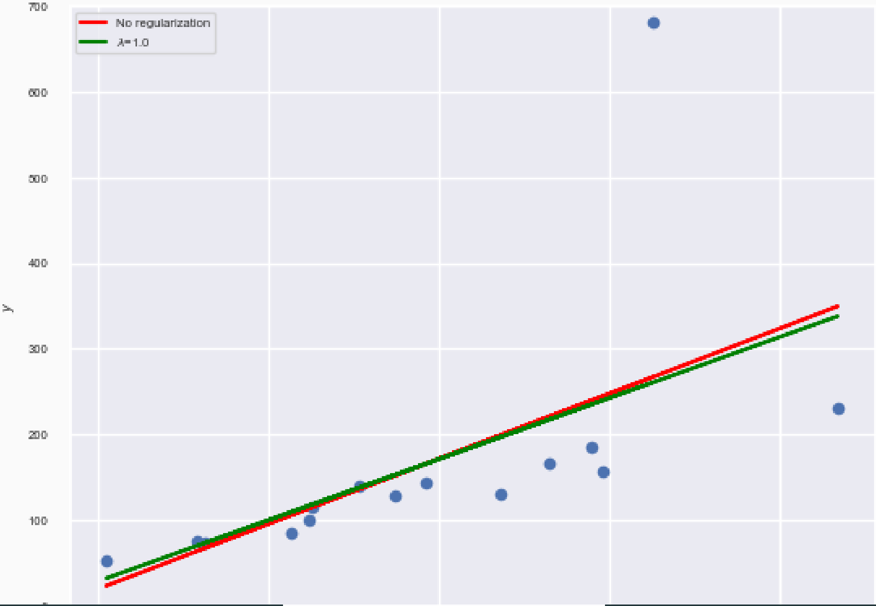
Sample dataset



Linear regression



L_2-regularized linear regression



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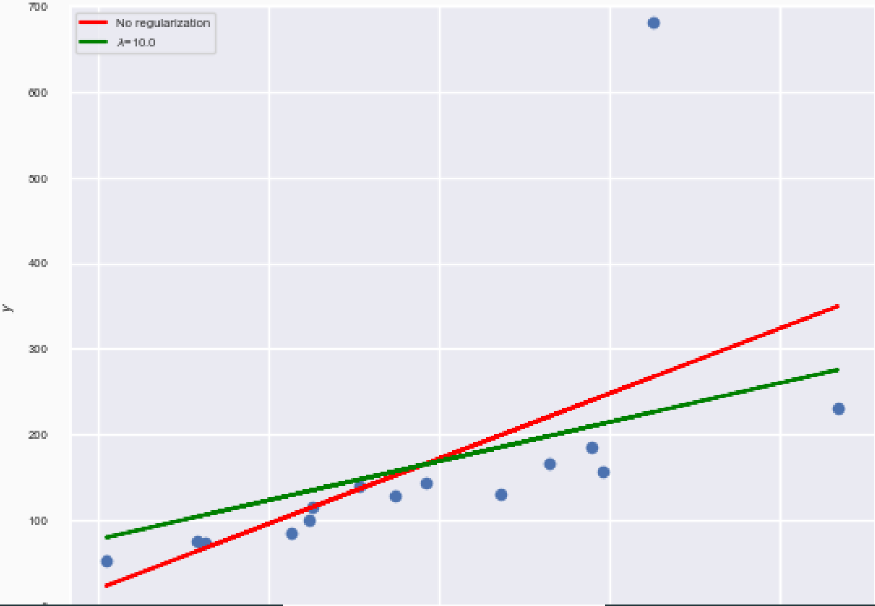

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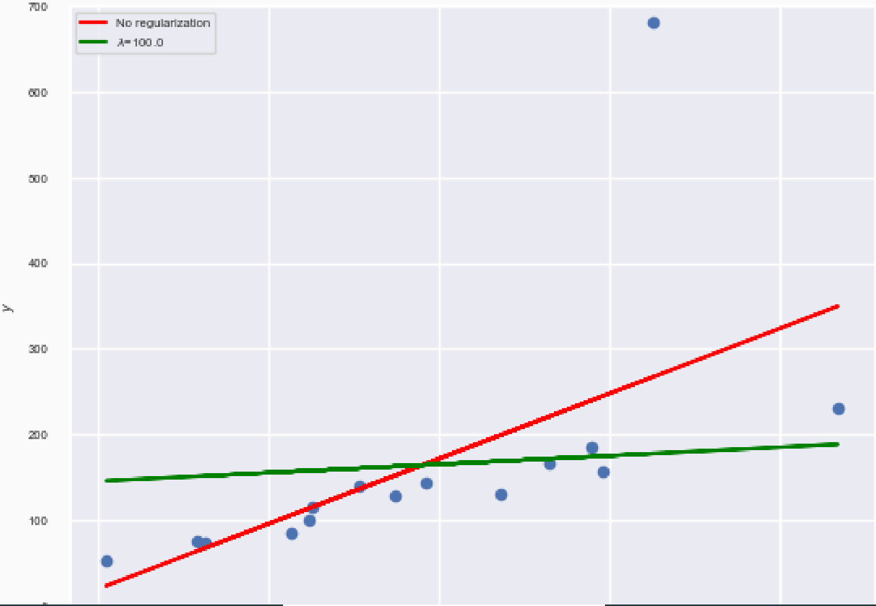

L_1 regularization

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Where

$$\|\mathbf{w}\|_1 = \sum_{i=1}^{d} |w_i|$$

L_1 regularization has the effect of turning some of the parameters in $\mathbf{w}$ to exactly 0

This induces **model sparsity**, where some of the features/dimensions are completely disregarded—a w_i of 0 makes the feature it multiplies to have no effect on the final prediction

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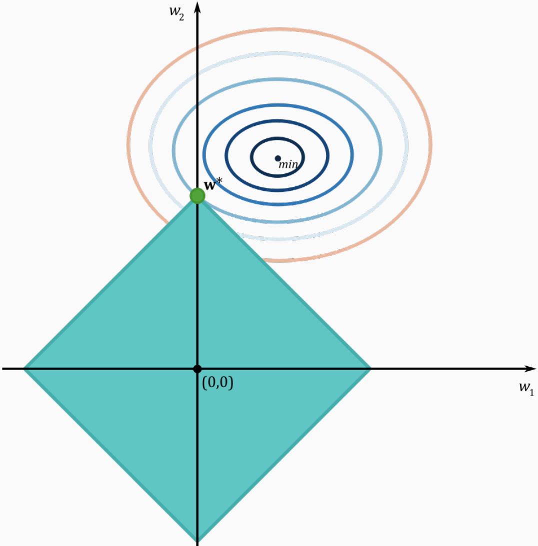
Therefore, L_1 regularization performs **feature selection** implicitly

L_1 regularization



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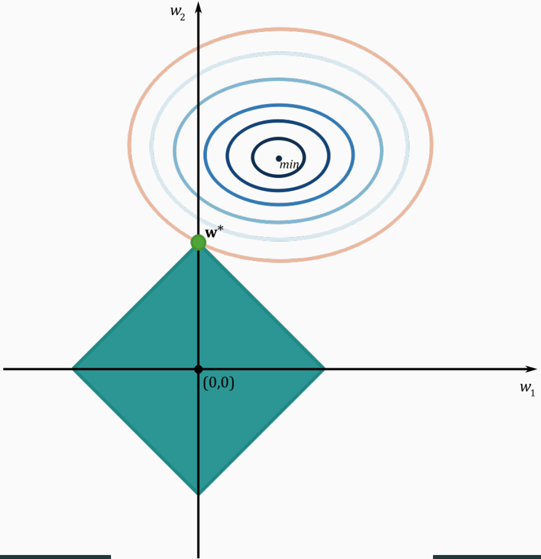

L_2 and L_1 regularization

For the specific case of linear regression:

- Linear regression with L_2 regularization → *Ridge*
- Linear regression with L_1 regularization → *LASSO*
- Linear regression with both L_1 and L_2 regularization → *Elastic Net*

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$$\text{Elastic Net} \;=\; \min_{\mathbf{w},b} \; \frac{1}{N}\sum_{n=1}^{N}(\mathbf{w}^\top\mathbf{x}_n + b - y_n)^2 + \lambda(\underbrace{\alpha\,\|\mathbf{w}\|_1}_{\substack{L_1\\ \text{regularization}}} + \underbrace{\left(\frac{1-\alpha}{2}\right)\|\mathbf{w}\|_2^2}_{\substack{L_2\\ \text{regularization}}})$$

$$\alpha \in [0,1)$$

Questions?