# Frequency-based Analysis of Financial Time Series

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## Abstract

We perform an analysis of spectral density with the magnitude of Fourier transform (FFT) for stock prices, logarithm of stock prices and daily returns using many sets of historical data which had been selected to be representative of a wide range of stocks. The results show that the stock prices and their logarithms both have the spectral density with $1/f^2$ form which is the prediction for the spectral density of a random walk and illustrates the long term memory and predictability of the financial time series, while the spectral density of returns is approximately flat like white noise which means the samples are highly uncorrelated and are not predictable. The scale invariance of stock prices can also be seen with this phenomenon. The random walk model is compliant with the most of these results.

**Keywords**: Stock price, Fourier transform, FFT, linear regression, Random-walk

# Motivation

Research on financial time series and price fluctuations is of prime importance in finance. It has many applications including in the field of control and risk assessment of new financial instruments (derivatives, structured products, etc.). Current research mainly uses statistical methods and approaches based on the signal processing have been largely minority. The purpose of this project is to investigate financial time series through their frequency behavior. The properties of scale invariance (fractal) observed in the structure of the magnitude of the Fourier transform of these signals should be verified. That might be of interest to predictive applications.

One of the other side objectives of doing this analysis project was producing some supporting results for the article [1] which is now under revision for submitting in the "*IEEE Transactions* on Signal Processing".

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## CHAPTER I:

# An Introduction to Analysis of Financial Time Series

## Introduction

For many years economists, statisticians, teachers of finance and also some other scientists like physicists, engineers… have been interested in developing and testing models of financial time series behavior. It has been such interesting that some other fields of science have opened new branches related to finanace [18] and some new fields like Econophysics, Finance engineering,… were born.

Research on financial time series and price fluctuations is of prime importance in finance. It has many applications for example in the field of control and risk assessment of new financial instruments [1].

*Financial time series* are continually brought to our attention. Daily news reports in newspapers and other medias inform us for instance of the latest stock market index values, stock prices, currency exchange rates, electricity prices, and interest rates. It is usually desirable to monitor price behavior frequently and to try to understand the probable development of the prices in the future. Private and corporate investors, businessmen, anyone involved in the international trade and the brokers and analysts who advice these people can all benefit from a deeper understanding of price behavior. Many traders deal with the risks associated with changes in prices.

There are two main objectives for investigating financial time series. First, it is important to understand how prices behave and the second is to use our knowledge of price behavior to reduce risk or take better decisions for future. The variance of the time series is particularly relevant. Tomorrow's price is uncertain and it must therefore be described by a probability distribution.

Time series models may for example be used for prediction or forecasting, option pricing and risk management.

In this article, first we try to briefly give a basic knowledge about financial concepts related to our project then we explain more about our investigating problem and review some previous works in that area, consequently we present our work and analysis with interpreting the results and at the end, we will give some suggestions for possible further researches in such a field.

## 1.1. What are financial time series?

Time series are sequences of measurement values usually recorded at regular time intervals, more precisely a time series can be mathematically represented as a discrete time, continuous state stochastic process $X = \{x_t\ , t = 1,2, \dots, N\}$; where $\boldsymbol{t}$ is the time index and $N$ is the total number of observations.

A financial time series is a time series related to the value of a financial instrument, e.g. stock prices, exchange rates, number of trades, shares and so on. The time increment can be everything from seconds to years.

Figure 1 shows an example of financial time series which is the daily adjusted stock price of DELL Inc. during about 3000 working days from 30.01.1997 until 30.12.2008:

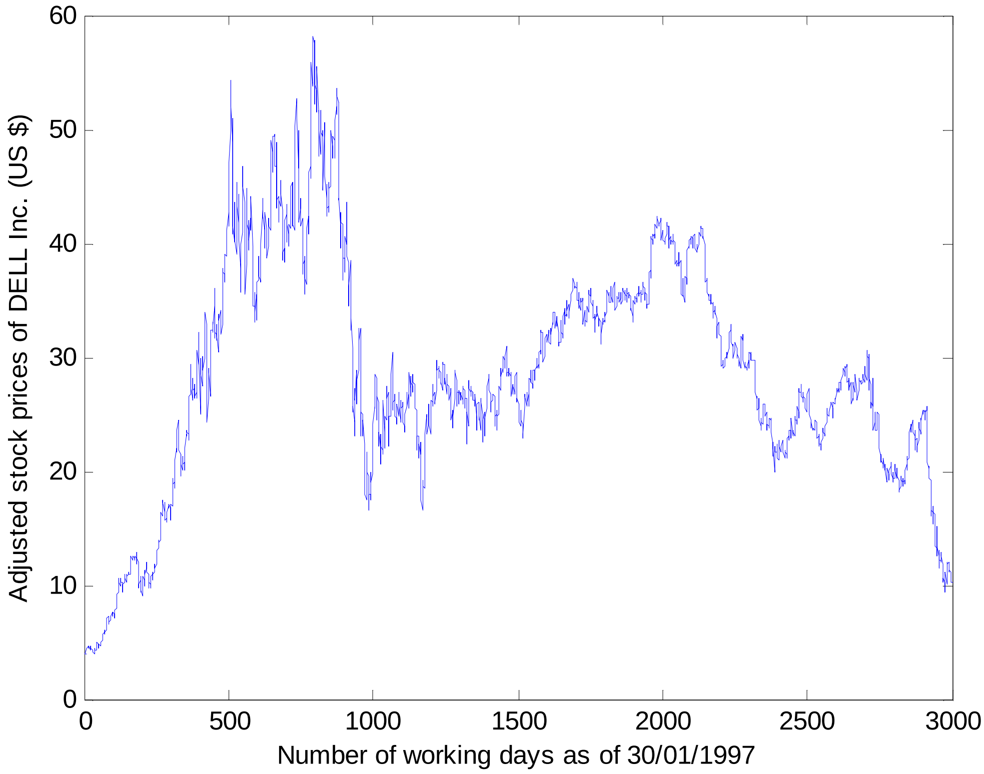


Figure 1

Any time series comprises a whole range of information which may consist of cycles of repeated duration, sudden changes, long-term and short-term decline or rise, and / or correlation amongst the numbers in the sequence.

In this project, we focus on stock prices rather than other financial instruments and try to analyze their Fourier magnitude spectrum.

### 1.1.1. Some properties of financial time series

In this part we describe some important properties of financial time series related to our work as follow:

**1- Regularity:** As discussed above, the measurement values of a time series are usually recorded at *regular* time intervals, this *regularity* is so important because without *regularity*, some parameters like moving averages, autocorrelations, and so on would not make sense. *Non-regular* time series are also of interest, but we say less about them. In this project we mostly work on daily recorded stock prices which are completely *regular*.

**2- Stationarity:** The stationarity of time series briefly means that the statistical characteristics of them do not vary over time! More precisely, a process is stationary when its joint probability distribution does not change when shifted in time and as a result, parameters such as the mean and variance, if exist, also do not change over time.

A weaker form of stationarity is known as wide-sense stationarity (WSS) or covariance stationarity and only requires that 1st and 2nd moment**s** do not vary with respect to time.

The financial time series are originally non-stationary but the magnitude spectrum analysis of stock prices shows that we can find some harmonically stationary objects in feature space of them [2]!

The stationarity is also important because the spectral analysis and Fourier transform basically assume that the signal is stationary [3] and therefore for getting reliable and reasonable results, the signal must be almost stationary at least in some of its frequency domain statistics.

**3- Historicity:** Some time series exhibit *historicity* which means the past is an indicator of the future, but some others don't show any especial *historicity*. For financial time series, evidence of the *historicity* is so interesting because if it is the case, it will give us the ability of prediction (predictability) which is the main motivation of analysis and modeling of financial time series.

## 1.2. Review of previous studies:

As mentioned several times before, the analysis and modeling of financial time series is too important for predictive applications like risk management and so on. The efforts for such a modeling have a long history which started with the Brownian motion model by Bachelard in 1900 [4].

In general, there are two main approaches for analyzing the behavior of time series, one is in "time domain" and other is in "frequency domain". Based on these two approaches, we can classify all analysis studies in 3 groups as follow:

1. **Studies based on Statistical approach**
2. **Studies based on Spectral analysis**
3. **Studies based on Time-frequency analysis and Wavelets**

The first group is more related to time domain analysis like correlation analysis, the second one is more in frequency domain and the last one is a kind of time-frequency analysis.

We must note that there have also been some studies in which there are combined analysis i.e. they belongs to more than one group e.g. the random walk model can be analyzed in both time and frequency domain together.

In the following, a brief introduction to each group will be given:

### 1.2.1. Studies based on Statistical approach

The first and also more studies for analysis and modeling of financial time series have been based on statistical approach. In this approach, it is tried to model the behavior of financial time series using statistical methods and models, for example the Brownian motion model which as mentioned above was the first describing model for stock markets is a kind of these statistical models. This kind of models mostly considers the randomness properties of financial time series and takes them as non-stationary processes.

Direct statistical analysis of financial prices is difficult and doesn't give any good sense, because consecutive prices are highly correlated, and the variances of prices often increase with time. That is why it is usually more convenient to analyze changes in prices. For such a consideration in the most statistical analysis, *Returns* are used instead of prices itself. Daily *returns* of prices are defined by:

$$r_t = \frac{x_t - x_{t-1}}{x_{t-1}} \qquad (1)$$

Where $x_t$ is the price of the asset in day $t$.

The *return* value is independent of the price value in each time $t$ and gives a sense for relative changes which can easily be used to give appropriate results for prices. The important fact that the *returns* are not highly correlated as prices and their statistical properties don't usually change over time makes them as a good alternative in statistical analysis.

In figure 2, the return values of DELL are shown beside the consecutive prices during 3000 working days.

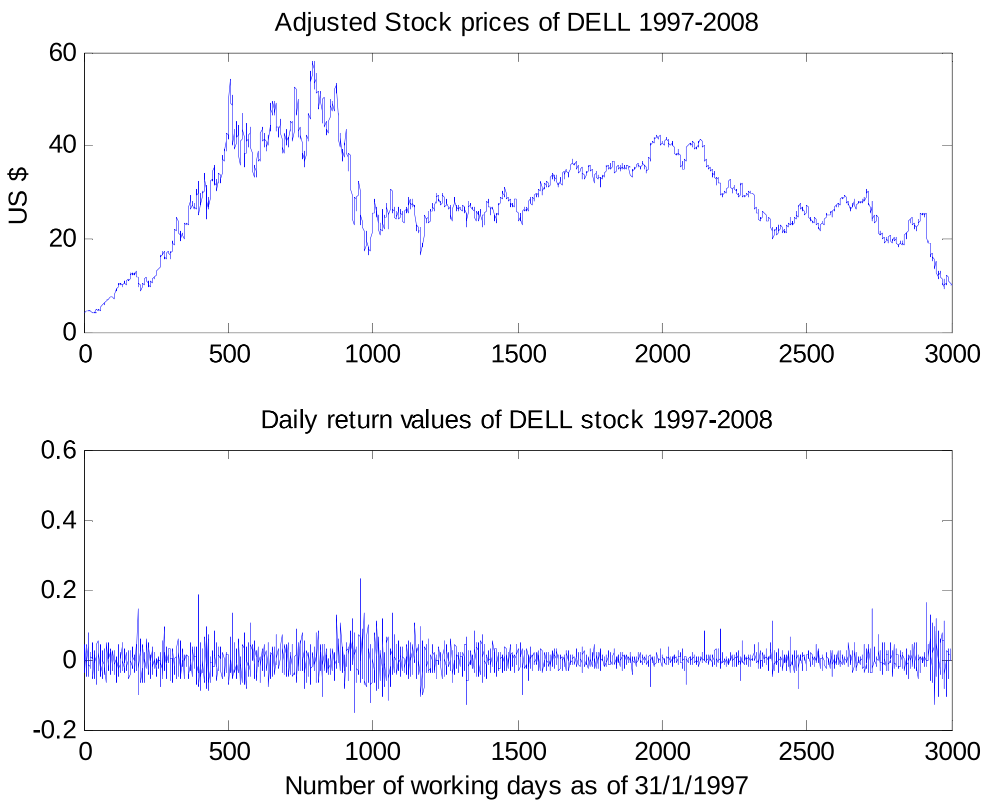


Figure 2

Many statistical models for financial time series and especially stock prices have been given till now (see [5]). Random-walk model, Auto regressive (AR) models such as ARMA and Econometric models like ARCH, GARCH, EGARCH … [5,6] are the examples of such studies, among which we will explain the random walk model later which is somehow related to our results.

### 1.2.2. Studies based on spectral analysis

The Spectral and Fourier analysis of time series are among frequency domain analyzing approaches. The seed of the analysis in frequency domain was originally in physical sciences and astronomy. The fact that light passing through a prism is decomposed into its various colour components as per corresponding frequencies, led to the idea that a time series could also be similarly broken into components with different frequencies.

Spectral analysis decomposes a stationary time series into a set of frequency bands in terms of its contribution to the termed power of the series. If a band of frequencies is important, the spectrum will exhibit a relative peak in this band. On the other hand, if the spectrum is flat indicating that every component is present in equal amount, the interpretation is that the series is merely a sequence of uncorrelated readings which is purely random or technically a white noise series.[7]

The two approaches, "time domain" and "frequency domain" are mathematically related in that the autocovariance function of the time domain approach is the Fourier transform of the spectrum of the frequency domain formulation and vice versa. Very often it is not clear as to why one approach should be preferred over other. The choice between the two procedures depends on both technical and substantive use.

As it can be inferred from the title of this project, our analyses completely lie in this group of studies; therefore in the next chapter we will explain it in more detail.

### 1.2.3. Studies based on Time-frequency analysis and Wavelets

The usefulness of wavelets is its ability to localize data in time-scale space. At high scales (shorter time intervals), the wavelet has a small time support and is thus, better able to focus on short lived, strong transients like discontinuities, ruptures and singularities. At low scales, the wavelet's time support is large, making it suited for identifying long periodic features. [7]

Fourier and spectral analysis are very appealing when working with time series. However, restricting the researches to stationary time series is not very appealing since most financial time series exhibit quite complicated patterns over time (e.g. trends, abrupt changes, transient events etc.). The spectral tools cannot efficiently capture these events. In fact, if the frequency components are not stationary such that they may appear, disappear and then reappear over time, spectral tools may miss such frequency components.

The spectral density does not provide any information on the time localization of different frequency components. That is, on the basis of spectral density, it is not possible to identify the exact time period when the frequency components are active. By considering the frequency representation of time series, one may know which frequency components are active but not when they were active. The reverse is true in time domain - one knows when things happened but has no information about corresponding frequency.

To overcome the problem of simultaneous analysis of time and frequency, a new set of basis functions are needed. The wavelet transform uses a basis function (called wavelets) that is stretched and shifted to capture features that are local in time and local in frequency. The wavelet filter is long in time when capturing low- frequency events and hence has good frequency resolution. Conversely, the wavelet is short in time when capturing high-frequency events and therefore has good time resolution for these events. By combining several combinations of shifting and stretching of the wavelets, the wavelet transform is able to capture all the information in a time series and associate it with specific time horizon and locations in time.

The wavelet transform adapts itself to capture features across a wide range of frequencies and has the ability to capture events that are local in time. This makes the wavelet transform an ideal tool for studying non-stationary or transient time series. The following points demonstrate the convenient usage of wavelet based methods.

Wavelet filter provides insight into the dynamics of financial time series beyond that of current methodology. A number of concepts such as non-stationarity, multi-resolution and approximate decorrelations emerge from wavelet filters. Wavelet analysis provides a natural platform to deal with the time varying characteristics found in most real world time series and thus the assumption of stationarity may not be invoked. Wavelets provide an easy way to the study the multiresoution properties of a process. It is important to realize that financial time series may not need to follow the same relationship as a function of time horizon (scale). Hence, a transform that decomposes a process into different time horizons is appealing as it reveals structural breaks and identifies local and global dynamic properties of a process at these time scales. Further, wavelets provide a convenient way of dissolving the correlation structure of a process across time scales. This would indicate that the wavelet coefficients at one level are not associated with coefficients at different scales or within their scale.

CHAPTER II:

# Spectral Analysis of Financial Time Series: A review on select works

# Introduction

The behavior of financial time series, in particular stock prices, has always remained as an intriguing phenomenon for the general public as well as academics and policy makers. An important reason may be that these markets are characterized by complex dynamics, which cannot be captured by traditional approaches developed in various spheres of analyzing these phenomena. To be specific, the quest for retrieving the hidden information in a financial data has caught the imagination of researchers in recent years. Barring standard statistical and econometric treatments of financial data, researchers in present times have venture into quite alien areas to explain the observed dynamics in financial time series. In this regard, two prominent approaches namely spectral methods and wavelets may be mentioned. Though these methods are not of recent origin, their application in financial markets is relatively new and at its infancy. Therefore, it is useful as well as interesting to take cognizance of these methods.

There have been several kinds of spectral analysis in the literature such as study on power spectrum, cross spectrum, bi spectrum,… but in this project we focus on power spectrum as a kind of spectral analysis.

The first financial application of spectral methods was in 1959 and the first paper (Granger) and the first book (Granger and Hatanaka) were published in 1961, 1964 respectively. After that, the use of spectral methods spread to different areas of economics. Here, we present a select review of studies pertaining to stock price behavior with a passing reference to few other macroeconomic applications.

## 2.1. A review on previous studies based on power spectrum [7]

An apparent feature of a power spectrum that can be easily noted are the peaks, such as at the seasonal frequencies and any shape that is complicated compared to the simple shapes that arise from a white noise or first order autoregressive and moving average models. Economies have been seen to follow swings with alternating periods of prosperity and depression, known as the business cycle. An early application of spectral techniques was to investigate these swings. It should be emphasized that the business cycle has never been at all regular, or deterministic, and so corresponds to one, or several, frequency bands rather than to particular frequency points. The apparent problem with this topic is that the business cycle corresponds to rather low frequencies and so estimation of this component is difficult unless very long series are available. The situation is a little improved by considering a number of different series from the same economy, as this provides little extra information; most parts of the economy are inclined to move together at low frequencies. Although some evidence was found for certain low frequency components being especially important (see, for instance, Howrey, 1968 and Harkness, 1968), in general all low frequencies were usually observed to be important for the levels of major economic variables. The relative importance of low frequency components compared to all higher components was found so frequently that a spectrum that steadily declined from low to higher frequencies, except possibly at seasonal frequencies, was called the 'typical spectral shape' in Granger (1966). The resulting spectrum of the New York commercial paper rate for the period 1876-1914, is not a typical one for an economic series as the low frequencies are considerably more imposing and a peak has been found for a frequency other than that correspondingly to the annual component (Granger and Hatanaka, 1964).

Granger and Hatanaka (1964) estimated the spectrum taking the monthly mid range of Woolworth stock prices quoted on the New York stock exchange for the period January 1946 to December 1960. The spectrum was seen to be very smooth and with the low frequencies predominating a shape frequently found for the spectrum of an economic series. There were no important peaks except that centered on period 2.8 months, which is the 'alias' of a weekly cycle. In another study in the same year for the period 1879-1914, they found that the estimated power spectrum of the call money rate was similar in general appearance to that of the commercial paper rate but with less prominent peaks corresponding to the annual and 40-months components and their harmonics.

There are a number of studies relating to testing of random walk hypothesis of stock price behavior. If the spectrum is flat indicating that every component is present in an equal amount, the interpretation is that the series is nearly a sequence of uncorrelated readings, which is purely random or technically a white noise process. If the spectrum shows any clear peak(s) and spike(s) at a particular frequency, then the conclusion that there is one frequency or frequencies which are of particular importance, resulting in a periodic cycle appearing in the series. Granger and Morgenstern (1963) applying spectral technique to New York stock market prices found that short run movements of series obey the simple random walk hypothesis but that long run components are of greater importance than suggested by this hypothesis. The seasonal variation and the business cycle components are shown to be of little or no importance and a surprisingly small connection was found between the amount of stocks sold and the stock price series.

In another attempt to test for random walk hypothesis of share prices on the New York and London stock exchanges, Godfrey et al (1964) found that most of the points of the spectra were within the 95% confidence limits. However, the strong long run (two or more years) components were larger than expected. The annual component was not apparent in any of the series studied, but several series showed very faint evidence at the harmonics of the seasonal components. In a similar fashion Granger and Rees (1968) found that a random walk model with no seasonal component appeared to fit the shorter run fluctuations of the series rather well for rate of return on a bond of specific maturity. Granger and Morgenstern (1970) studied the spectrum of the Sydney ordinary share index over 160 weeks from 1961-64. They observed that the spectrum seemed flat, which was not surprising, as the series was not long enough to reveal other components if they were present.

Sharma and Kennedy (1977) made a comparative analysis of stock price behavior on the Bombay, London, and New York stock exchanges. Their results indicated that the spectral densities estimated for the first differences series (raw and log transformed) of each index, confirmed the randomness of series, and no systematic cyclical component or periodicity was present. Based on these tests, their view was that stocks on the Bombay Stock Exchange obeyed a random walk and were equivalent in this sense to the behavior of stock prices in the markets of advanced industrialized countries.

In contrast with the above studies, Praetz (1973) found departures from the random walk hypothesis in case of Australian share prices and share price indices and moreover there was a clearly defined seasonal pattern in share price indices. Kulkarni (1978) presented auto-spectral test of the random walk hypothesis about share price movements on the Indian stock exchanges. All the weekly series (all India, Bombay, Calcutta, Madras, Ahmedabad, and Delhi) seemed to behave alike except for Calcutta and Delhi. The presence of 4 week lags and hence auto- covariance function of 4 lags was an indication of non-random walk among weekly series. All other weekly series except these two seemed to be free from any seasonal or other cycles. Of the six monthly series analyzed, four of them show a lag structure of four months in their spectral representation. Thus, this result indicated the presence of non-random walk behavior. All the monthly series were found to be influenced by one or two seasonal and other harmonics (cycles). In another study, Ranganatham and Subramaniam (1993) made an attempt to test weak form of efficient market hypothesis (EMH) and failing to find any support for this, concluded that there was no random walk.

Poterba and Summers (1987) and Lo and MacKinlay (1988) challenged the conventional view that stock price returns were unpredictable i.e., do not form a martingale difference sequence. They suggested that the spectral shape tests may be interpreted as searching over all frequencies of spectral density for martingale difference violations, whereas the variance bounds tests may be interpreted as examining the zero frequency in isolation. Durlauf (1991) extended this literature on using spectral shape to test various hypotheses, which has concentrated on a single statistic, to more general question of analyzing spectral distribution deviations from the straight line as a problem of weak convergence in a random function space. A general asymptotic theory for spectral distribution function permits the construction of many test statistics of the martingale hypothesis. Using the data sets of Lo - MacKinlay (1988) and Poterba and Summers (1987), he found that weekly and monthly stock returns revealed some evidence against the null hypothesis that holding returns are martingale differences. His result confirmed that stock price exhibited long run mean reversion. Violations of the random walk theory appear to be robust to a relatively diffuse formulation of a researcher's beliefs concerning the class of alternatives. Another study by Fond and Ouliaris (1995) supported the view against the martingale hypothesis for exchange rates data and they viewed this rejection is due to long memory influences.

Apart form the studies relating to random walk and martingale behavior, a few studies have been made to analyze some other aspects of stock price behavior. On the basis of eleven original descriptive spectral characteristics obtained from log spectrum of the return on Helsinki stock exchange, Knif and Luoma (1992) developed three main principal characteristics of the spectrum, i.e., size, shape, and variability. The empirical results indicated that the spectral approach could be used for the descriptive as well as the analytical analysis of stock market behavior. Knif, Pynnonen and Luoma (1995) studied the differences in the spectral characteristics between the two stock markets - the Finnish and Swedish. Their results indicated differences between the return spectra of two markets and more volatile Swedish market exhibited a two-day periodicity and autoregressive dependence of about two weeks. In a recent study, Barkoulas and Baum (2000) used the spectral regression test for fractional dynamic behavior in a number of Japanese financial time series, viz. spot exchange rates, forward exchange rates, stock prices, and currency forward premia. Long memory is indicated by the fact that the spectral density becomes unbounded as the frequency approaches zero; the series has power at low frequencies.

Bonanno, Lillo and Mantegna (2000) have confirmed the $1/f^2$ behavior for power spectrum of logarithm of the stock prices in [3] as well as [6], [8].

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CHAPTER III:

# Analysis and Simulation Results

## Introduction

In this main chapter we are interesting to analyze the magnitude spectrum of historical stock prices in several aspects for some various data bases. These analyses are among the “spectral analysis” methods of time series and as discussed before, spectral analysis is usually used for stationary time series which is somehow a good assumption for adjusted stock prices in frequency domain statistics.

In each case we have a time series $\{x_t;\ t = 0, 1, \dots, N-1\}$ where $N$ is the total nuber of observations (days) for which we do the analysis. Our main instrument for such an analysis is Fourier transform or more precisely FFT of time series which is defined by:

$$x_t \xrightarrow{FFT} X_k = \sum_{t=0}^{N-1} x_t e^{\frac{-2\pi j}{N} tk} \quad k = 0, 1, \dots, N-1 \qquad (2)$$

For converting $k$ to a realized and usable frequency $f_k$, we must note that since we have used daily prices as the input signal, the sampling frequency ($F_s = 1/T_s$) is equal to 1 $[\frac{1}{day}]$ and so we can reallocate the frequencies as below:

$$f = \left\{ f_k = \frac{k}{N-1} \cdot F_s = \frac{k}{N-1}\ ;\ \ k = 0, 1, \dots, N-1 \right\} \qquad (3)$$

The unit of this new set of discrete frequencies is $[\frac{1}{day}]$ and can make sense of real frequencies that we need in our analyses! Also we know that by sampling theorem just those components of the signal which have frequency lower than or equal to $F_s/2 = 0.5\ day^{-1}$, will be measured without aliasing. Regarding this point and the fact that we are interesting in the magnitude of Fourier coefficients where $X_{N-k} = X_k^*$, we limit our frequency axis between 0 and 0.5 in all simulations.

The magnitude of Fourier transform $|X(f)|$ gives a good sense about the power spectrum of time series $S_x(f)$ as below:

$$S_x(f) = |X(f)|^2 \qquad (4)$$

As said before, at first we study the $1/f$ behavior of the magnitude spectrum of the historical stock prices. To do so, we use the FFT as a good instrument and compute the magnitude of that, then for better seeing the shape of the spectrum, we will go to log-log scale, i.e. the logarithm of the magnitude of FFT values versus the logarithm of frequency index. If we want to fit the magnitude spectrum $|X(f)|$ with a power-law function with respect to frequency $f$ i.e. $|X(f)| \propto 1/f^{\gamma}$, we can use linear fitting for the logarithm of that with respect to $\log(f)$ according to the following relation:

$$|X(f)| = \frac{C}{f^{\gamma}} \quad \xrightarrow{yields} \quad \log|X(f)| = \log C - \gamma \cdot \log f \qquad (5)$$

The slope of such a regression line is $-\gamma$ by which we will be able to estimate the exponent $\gamma$ or see whether how much the magnitude spectrum is close to an especial power-law shape e.g. $1/f$.

Using a logarithmic binning can also be a good way in order to avoid overestimating the high-frequency components.

For evaluation of fitting, we can use some goodness-of-fit statistics like MSE (Mean Square Error), R-square (Coefficient of determination), Norm of residuals and so on. In our case in which we are doing the linear fitting for log-log scale, R-square (Coefficient of determination) can be a very useful goodness-of-fit statistic especially for comparison.

R-square is a coefficient with the value between 0 and 1 and will be 1 when we have a complete fitting (best case) and will be 0 when we have the worst case of fitting.

## 3.1. Nature and Sources of the Data

For our analysis, we need the historical stock prices; fortunately there exists a good source of such a database which is Yahoo! finance [15] in which we could find many historical stock prices mostly in NYSE (New York Stock Exchange) as our database. Another important point is that we have used adjusted prices in all simulations. The stock prices in the past are adjusted to reflect subsequent splits, dividing,… allowing the data to be compared with current prices

## 3.2. Study the magnitude spectrum of stock prices

The analysis was done as discussed in previous section for several different types of stock prices. All used stock prices had been adjusted by yahoo finance.

In the figures 3 to 7, the analysis results for some set of historical adjusted stock prices of IBM, UBS AG, Dell Inc. and General Motors over the time period of 1007 working days will be shown.

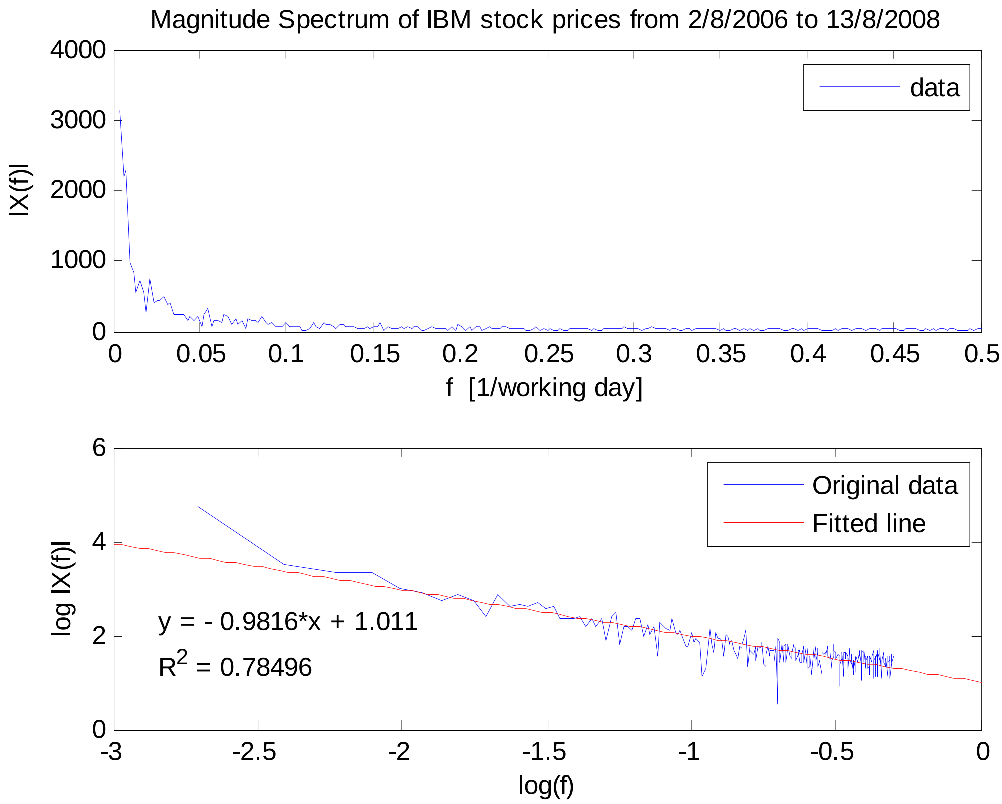


Figure 3

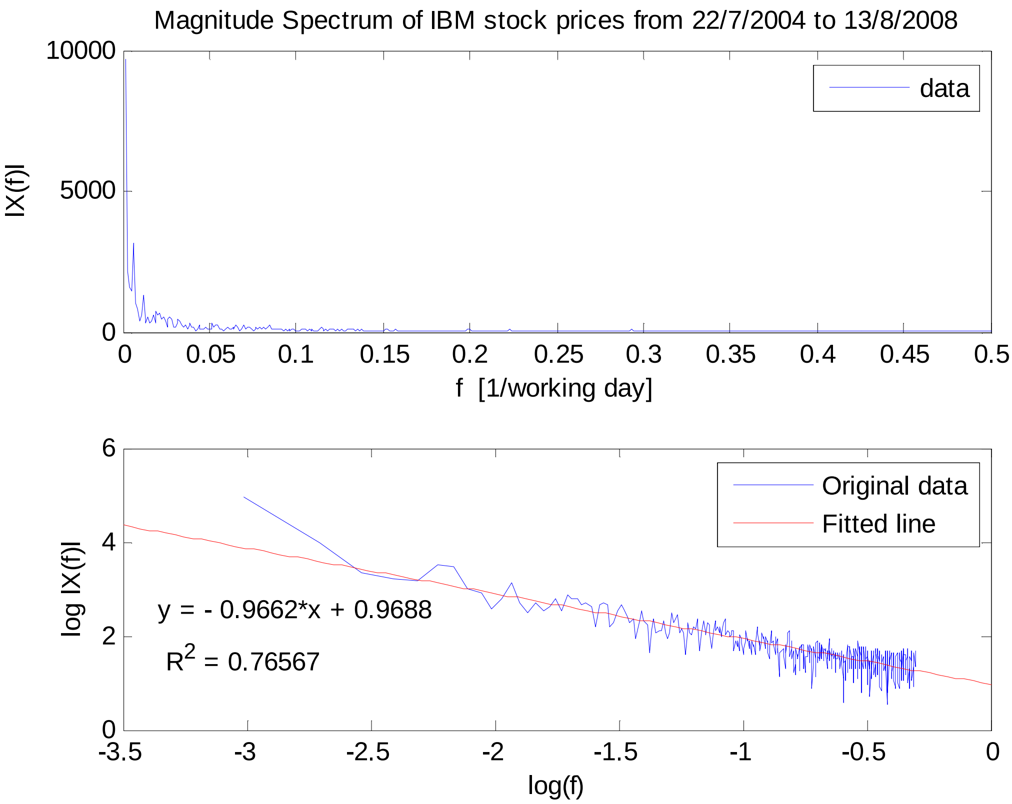


Figure 4

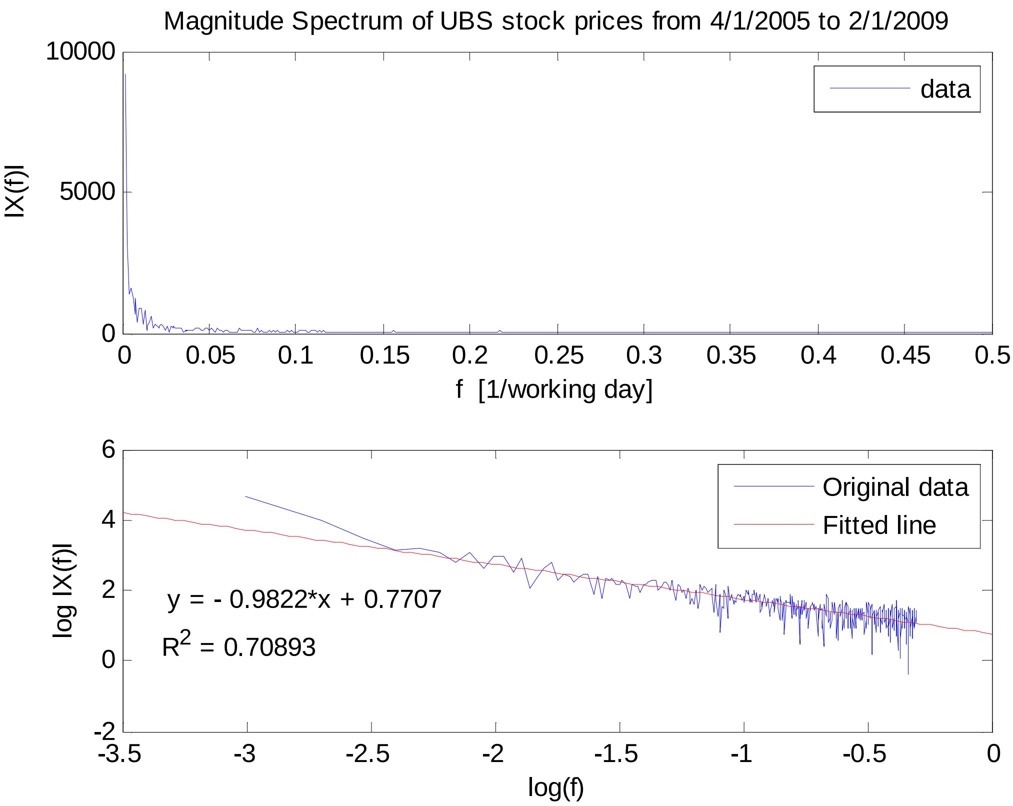


Figure 5

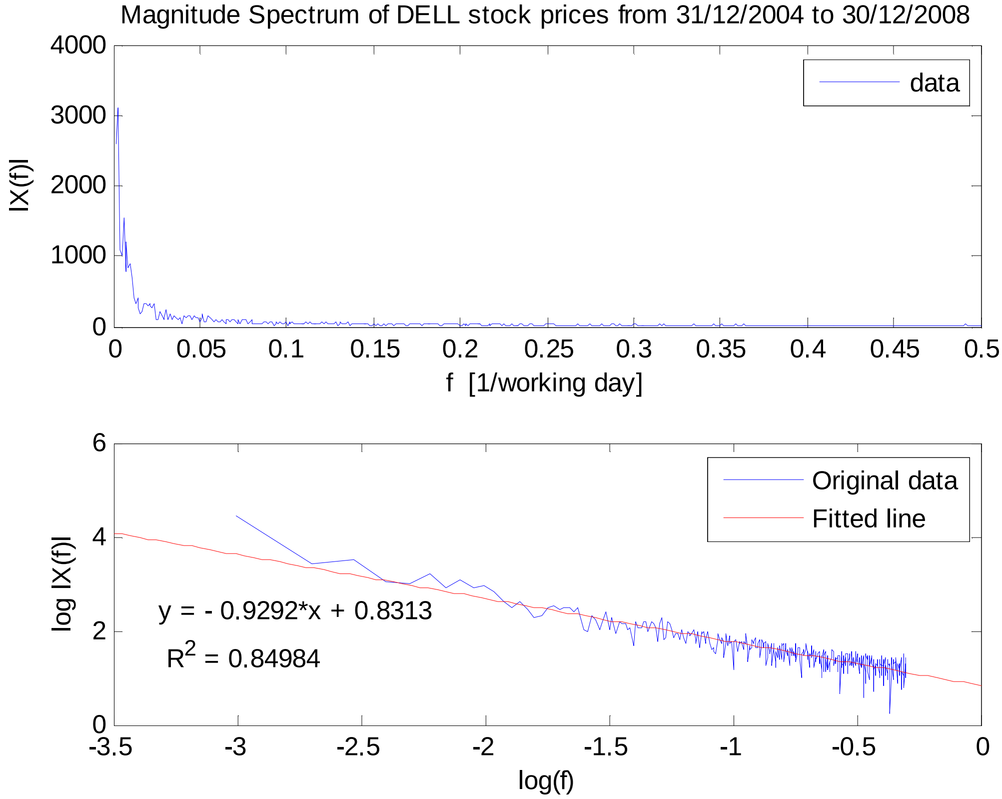


Figure 6

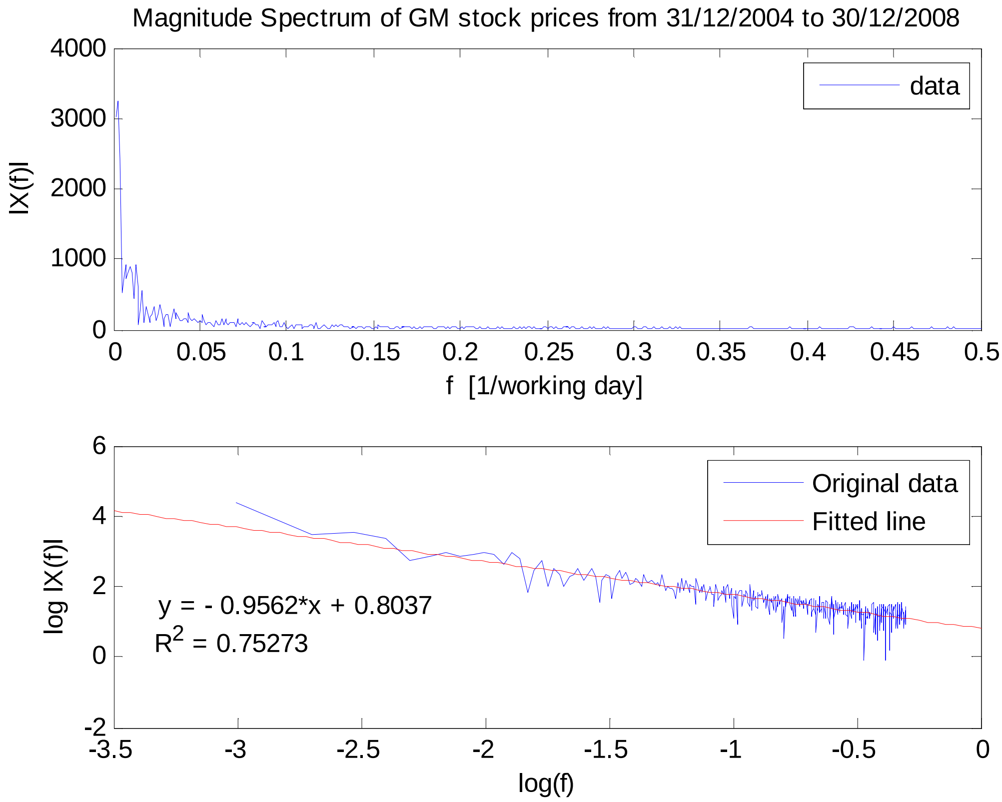


Figure 7

We have done the same analysis on many other sets of data and obtained the same results with the slope of too close to -1 which means $\gamma \approx -1$ but the R-square as the goodness-of-fit parameter is in the range of $0.7 \sim 0.8$ for the most cases but it is almost stable and persistent!

In the figures 8, 9, the 1/f shape of magnitude spectrum of stock prices can be seen in the smaller time period of 106 working days from 31/7/2008 to 30/12/2008. The results become more interesting if we keep in mind that this period has been at the heart of ***global financial crisis of 2008*** and also General Motors has been one of the worst cases in the sense of losing stock prices. This robustness can illustrate the invariance of the scale of the process (Fractal property).

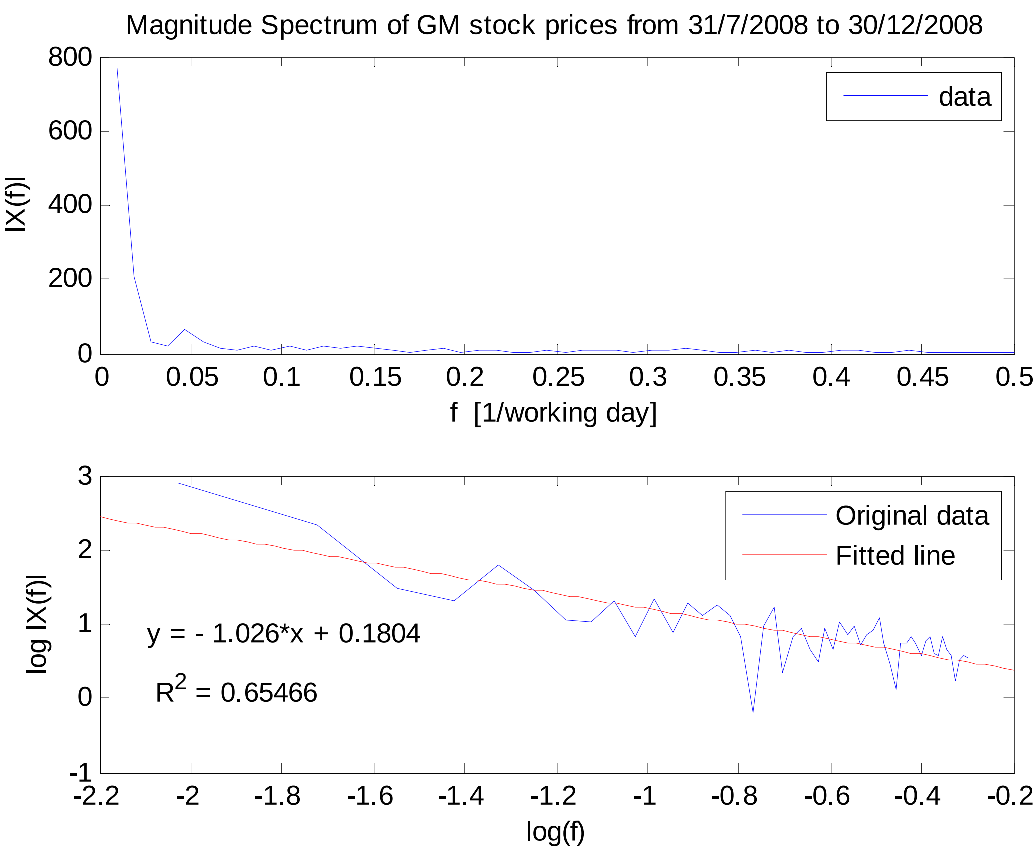

Figure 8

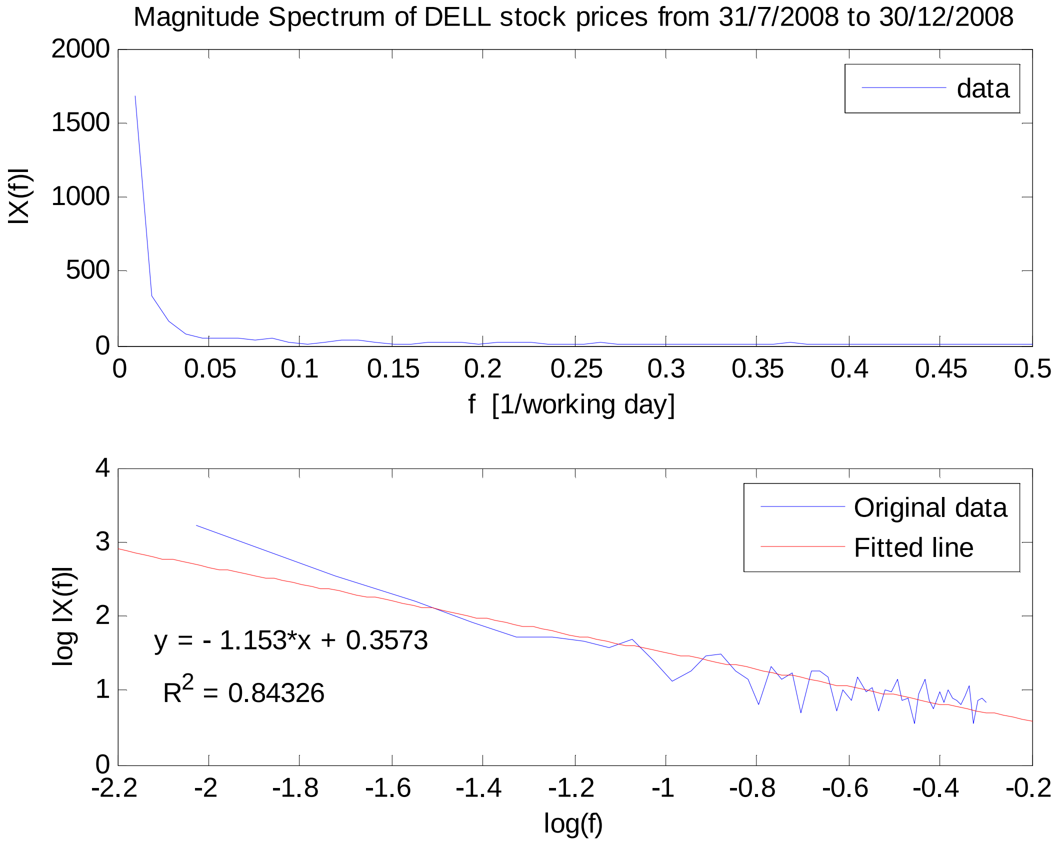

Figure 9

## 3.2. More precise analysis using moving windows:

In a more precise manner, we used a moving window and each time shifted it with an increment of 100 days over the time period 1985-2008 or about 6000 working days for the General Electric Co. (GE) stock and calculated regression slope and R-square. The figures 10, 11 show the results for two different moving window sizes of 1024 and 512 days respectively.

Furthermore we calculated the arithmetic average of slops and R-square values over all windows which were as following table:

| Window size (days) | Mean slope | Mean $R^2$ |
|---|---|---|
| 1024 | $-0.9341$ | 0.7559 |
| 512 | $-0.9822$ | 0.7355 |

Table 1: mean slope and $R^2$ for GE

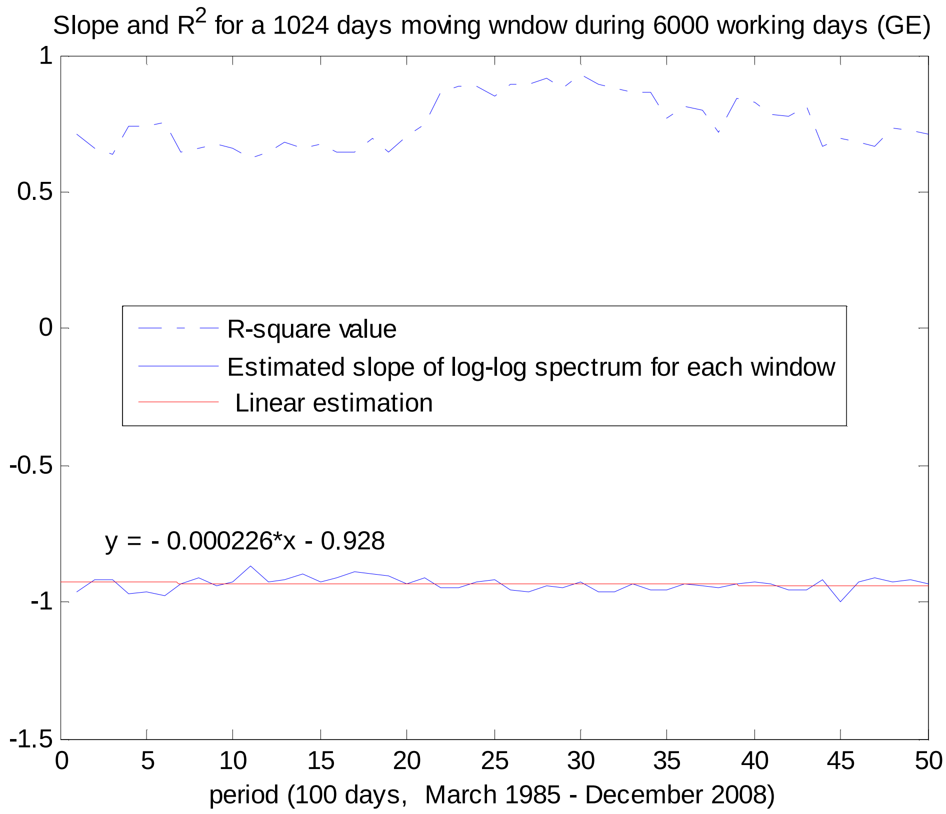


Figure 10

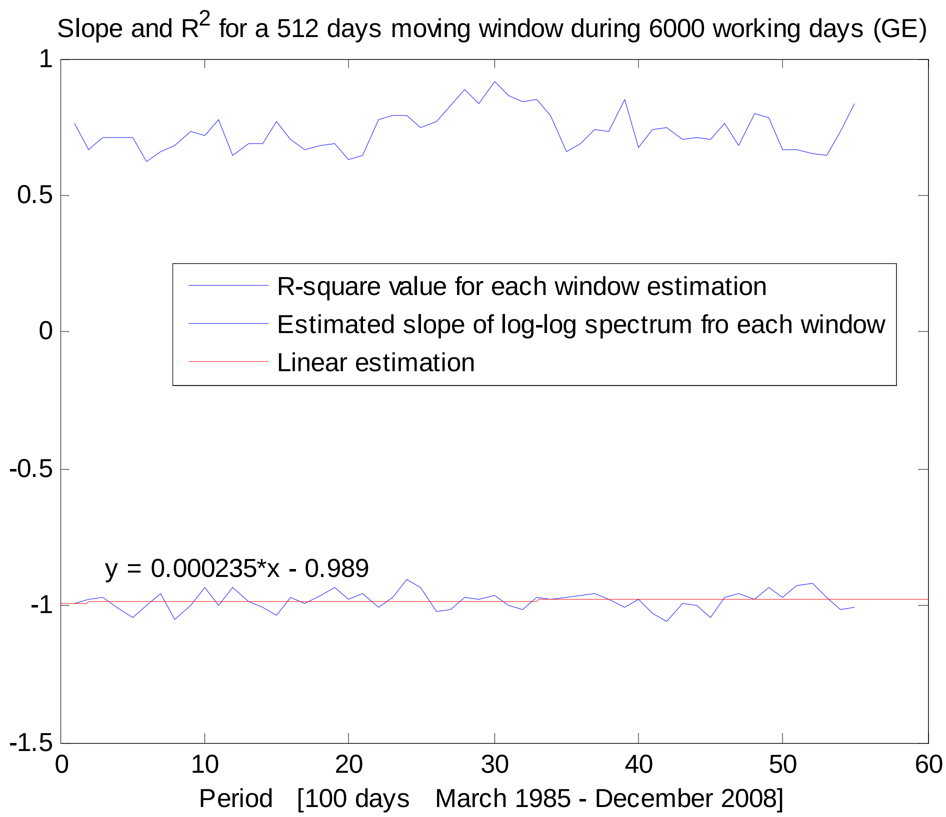


Figure 11

Above Figures (10, 11) show that the R-square is significant and stable over the period and the slope is also very close to -1 in all windows over the period.

## 3.3. Study the logarithm of prices and returns

As said before, especially in the statistical analysis of financial prices, it is more convenient to analyze the changes in price e.g. using *returns* rather than prices as discussed in section 1.2.1.

Also in many studies related to the financial time series, the logarithm of price is used instead of price itself, the seed of such a change is that if we suppose $l_t = \log x_t$ is our new value, then we have:

$$d_t = l_t - l_{t-1} = \log x_t - \log x_{t-1} = \log \frac{x_t}{x_{t-1}} = \log(1 + \frac{x_t - x_{t-1}}{x_{t-1}}) \quad (6)$$

But usually the daily changes of prices are much less with respect to the prices i.e. $x_t - x_{t-1} \ll x_{t-1}$ and so we can approximate the last logarithm as follow:

$$d_t = l_t - l_{t-1} = \log(1 + \frac{x_t - x_{t-1}}{x_{t-1}}) \approx \frac{x_t - x_{t-1}}{x_{t-1}} = r_t \quad (7)$$

It means that if we use the logarithm of prices instead of them, just the daily difference between values is approximately equal to the return values. This result is so interesting because in the case of logarithm of price, unlike the price itself, the daily change of values ($d_t$) which is called "*geometric return*" (Jorion, 1997) makes a good sense for relative changes and is independent of the magnitude values.

In this section, we consider the logarithm of stock prices $l_t = \log x_t$ and do the analysis like section 3.2. for their magnitude spectrum$|L(f)|$.

In the following figure we can see the 1/f-like behavior is also seen in the magnitude spectrum of logarithm of stock prices:

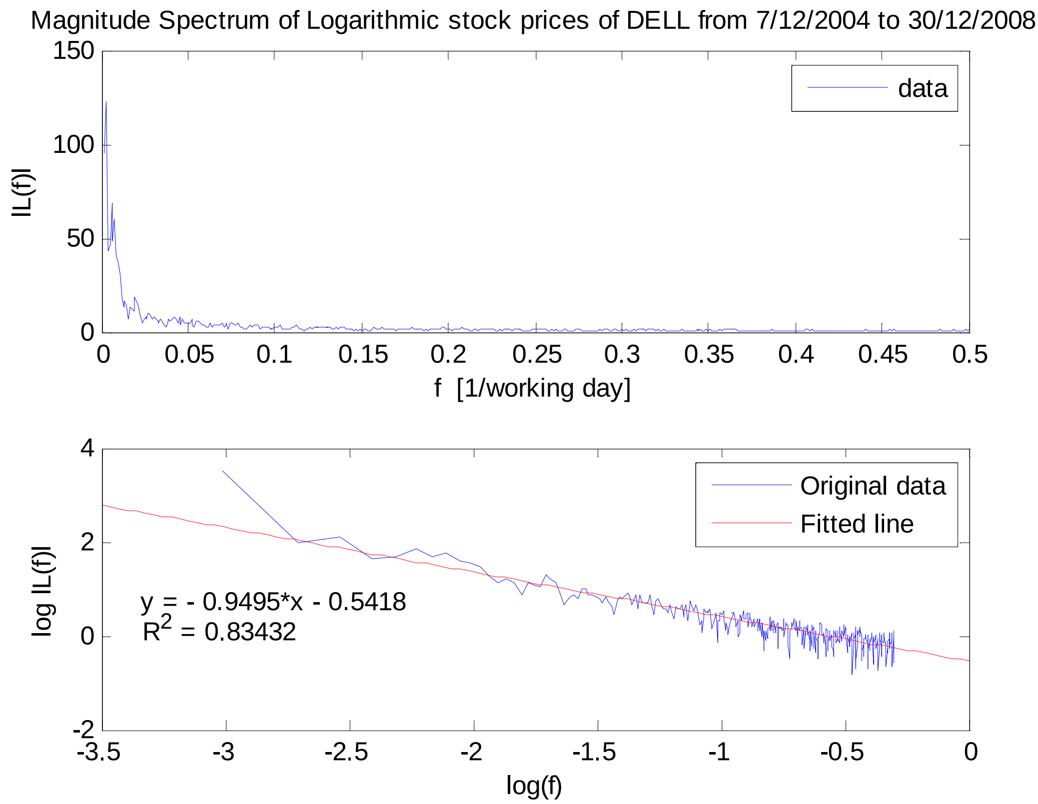


Figure 12

A more precise analysis on General Electric Co. (GE) logarithmic stock prices has been done using a moving window like section 3.2. with the increment of 100 days in each step and the result can be seen in figure 13:

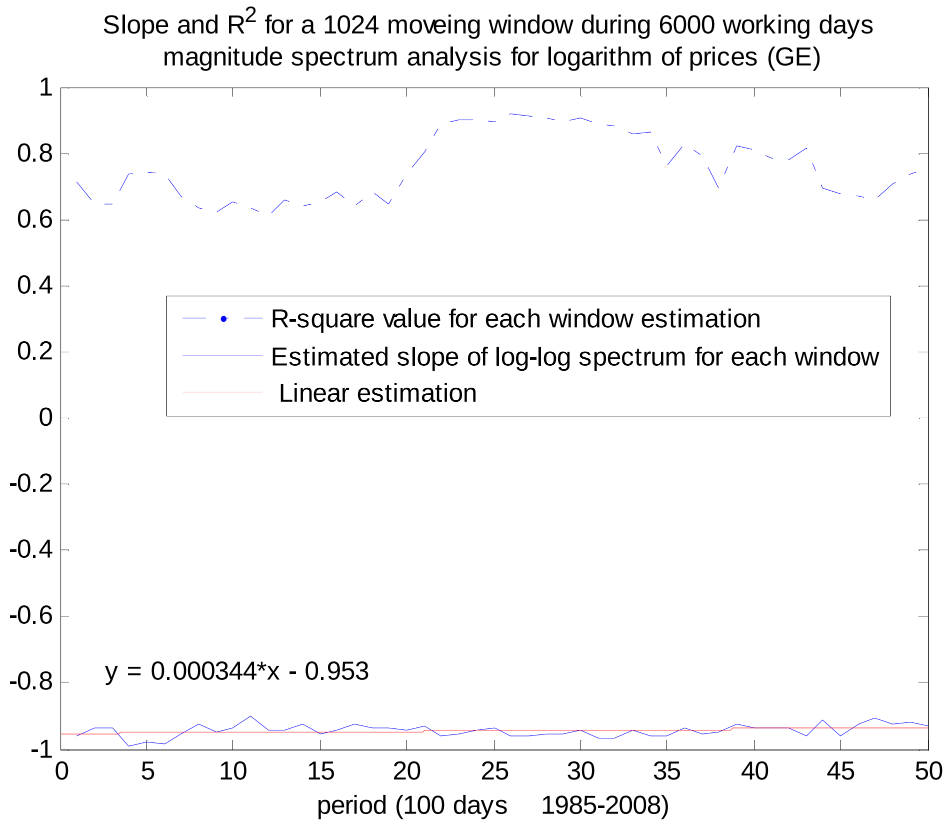


Figure 13

The average of slope and R-square over all windows has been given in the tables 2, 3, for stock prices of Microsoft Corporation (MSFT) and General Electric Co. (GE) respectively:

| Window size (days) | Mean slope for price | Mean $R^2$ for the price | Mean slope for the logarithmic price | Mean $R^2$ for the logarithmic price |
|---|---|---|---|---|
| 1024 | $-0.9328$ | 0.7492 | $-0.9350$ | 0.7635 |
| 512 | $-0.9667$ | 0.7357 | $-0.9794$ | 0.7363 |
| 256 | $-1.0354$ | 0.7137 | $-1.0526$ | 0.7094 |

Table 2: MSFT - November 1986 to August 2008

| Window size (days) | Mean slope for price | Mean $R^2$ for the price | Mean slope for the logarithmic price | Mean $R^2$ for the logarithmic price |
|---|---|---|---|---|
| 1024 | $-0.9341$ | 0.7559 | $-0.9441$ | 0.7559 |
| 512 | $-0.9822$ | 0.7355 | $-0.9960$ | 0.7270 |
| 256 | $-1.0555$ | 0.7135 | $-1.0790$ | 0.7026 |

Table 4: GE - March 1985 to December 2008

All results show that the power spectral of logarithm of the stock prices behave like $1/f^2$ the same as prices itself or even better in many cases.

We have also considered the returns and do the same analysis for their magnitude spectrum. In the figure 14 you can see the result of such an analysis for DELL Inc. over the period 2004-2008. The spectrum analysis of daily returns shows that their power spectrum is somehow flat i.e. they are acting like white noise.

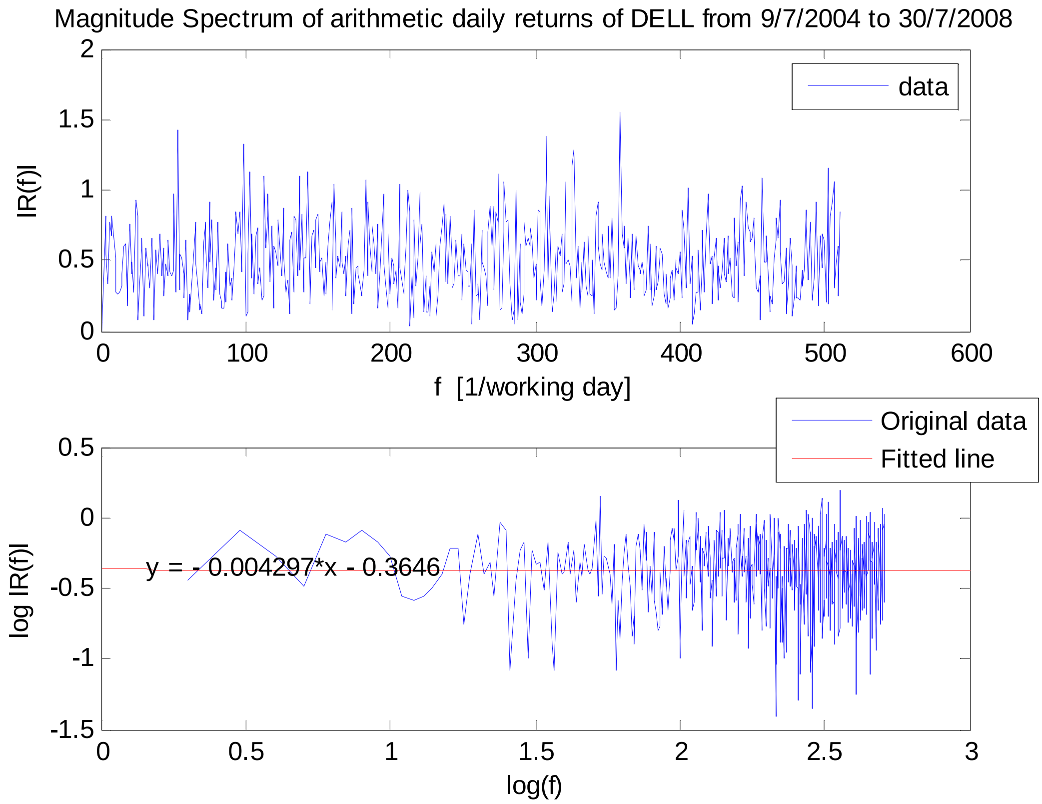


Figure 14

## 3.4. Interpretation of the results:

All above simulations and analyses for several set of historical adjusted stock prices ( $x_t$ ) shows that the Fourier magnitude spectrum of them is well approximated by $|X(f)| \propto 1/f$ or equivalently their spectral power density is approximated by $S_X(f) \propto 1/f^2$. We have also seen the same results (or even better in some cases) for the logarithm of prices. The spectral density of returns is also approximated by a white sequence. This form of results is the prediction for the spectral density of a *random- walk* and so our results are totally compliant with [3],[14],[15].

### 3.4.1. More about Random-walk

For better interpretation of the results first we need to know about some basic types of random processes especially *random-walk* noise, so in the following we will give a brief introduction to them:

Basically, there are three basic types of random processes that are often present in the physical systems and can be distinguished by the forms of the dependence of the power spectral density $S(f)$ on the frequency $f$: white ($1/f^0$), flicker ($1/f$), and *random-walk* ($1/f^2$) noises.

*Random-walk* is simply defined as $X_t = \mu + X_{t-1} + \varepsilon_t$ over a process $X_t$; where μ is the drift of the process and increments $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t, \ldots$ are serially independent random variables. Usually the sequence $\{\varepsilon_t\}$ is identically distributed with mean zero and variance $\sigma^2$ (this is not a necessary assumption). The variance of the process at time $t$ is given by $Var(X_t) = t\sigma^2$ i.e. in this model, the variance of process varies linearly with time.

Since in this model, the changes per time unit i.e. $n_t = X_t - X_{t-1}$ is modeled with serially independent identically distributed variables, the power spectrum of such a series in frequency domain can be interpreted as a white noise which has a constant power in all frequencies, and so we will have:

$$n_t = X_t - X_{t-1} \xrightarrow{TD \overset{FT}{\rightarrow} FD} |N(f)| = \sqrt{2 - 2\cos(2\pi f)} \cdot |X(f)| \;\cong\; 2\pi f . |X(f)| \text{ for low } f \qquad (8)$$

$$\textit{or equivalently} \quad S_n(f) \cong 4\pi^2 f^2 \cdot S_x(f) \qquad (9)$$

Therefore if the power spectral density $S_n(f)$ wants to be white, the power spectral density $S_x(f)$ has to have the $1/f^2$ shape with respect to frequency $f$ .

AS discussed earlier, the results of our "frequency domain" simulations confirm the random-walk model for stock prices and also for the logarithm of stock prices with a good approximation!

As seen in section 3.3., the spectral analysis of daily returns (especially geometric returns) earns an approximately flat power spectrum which means the daily returns are somehow acting like white noise. This result can also be explained by *random-walk* model which has been discussed above: If we accept the $1/f^2$ shape for the power spectrum of logarithm of stock prices, Since the returns are the daily changes of these values as shown in (7), they must have a white spectrum according to (9).

Also using a totally different approach, a good theoretical model has been proposed to explain the behavior of magnitude spectrum in under publishing article [1]. The results of this proposed model which considers an oscillatory chain behavior for the stocks are totally compliant with the random-walk model discussed above even in variance analysis.

### 3.4.2. Predictability

One way to describe mathematically the memory hidden in a sequence of random numbers is by the spectral density or correlation between distant points.

In a white sequence (flat power spectrum), points are absolutely uncorrelated, while in both $1/f$ and $1/f^2$ sequences, points are correlated over any time scale. It means that the white sequences are pure random and have nor historicity neither predictability while some other like $1/f^2$ sequences can be predicted because of correlation between their points. This fact can be the most important result of such a project which gives us hope for prediction applications like risk management in financial time series.

In a more precise and mathematical definition, "Predictability" means the convergence of the integral $\int S(f)df$ at low frequencies and so the time series with $1/f^2$-like power spectrum (random-walk) are well predictable in this sense. This kind of sequences also has a good originality or strong correlation between points at the high frequency part of the spectrum. The "Predictability" can also be interpreted as a long time memory for sequences with $1/f^2$ behavior like stock prices or logarithm of them and short time temporal memory for sequences like returns.

But we must note that the random-walk model assumes that at a given moment, it is not possible to estimate where in the business cycle the economy is. [5]

### 3.4.3. High frequency behavior:

The effectiveness of the model decreases in higher frequencies. As seen in almost all results, there are more variations and distortions in higher frequency components of our results. This can be reasonably explained by the random-walk model and the fact that the daily changes (returns) are white sequences, therefore in higher frequencies where the power spectrum of stock prices decreases with $1/f^2$ the effect of returns becomes more and more exactly like a white noise!

## 3.5. Suggestions for further research

As discussed earlier, since the main goal of analysis and modeling of financial time series is predictive applications, the proposed models must be as precise as possible and cover more properties and details of them. Therefore we can suggest 2 main ways for the further works; one is more precise modeling and one is using current results and models in predictive applications. For example in our simulations, the goodness-of-fit parameter i.e. R-square was in average around 70-80% which can be considered as a good range especially in the sense of persistency and stability of the model but probably with some modifications we can get better values and so more precise models.

Regarding these main ways, the following 4 suggestions for future works can be made:

### 3.5.1. Time-frequency analysis and wavelets

Restricting the researches to stationary time series is not very appealing since the most financial time series exhibit quite complicated patterns over time. The wavelet transform intelligently adapts itself to capture features across a wide range of frequencies and thus has the ability to capture events that are local in time (non-stationarity and transient time series). The most useful applications of wavelet analysis involve segregating observed data into time-scales, identification of structural breaks, denoising, scaling and finding the relationship among different time series. Though there have been many number of studies recently in such an area like [16],[17],[18],[19],[20].

### 3.5.2. Phase analysis

In this project we just did our analysis on magnitude spectrum but one can go for phase analysis of financial time series such as finding the phase distribution or analysis of phase patterns in Fourier transform and so on. The phase analysis can have good predictive applications. There has also been published a good work in this area in [9].

### 3.5.3. Similarity search between time series

Using the models have been still proposed or using some spectral analysis like cross-spectrum or bispectrum one can investigate how similar two time series are. This is the concepet of similarity search between time series which is becoming an interesting issue in finance research recently. There are some good works in this area even with using the fourier components [10,11].

### 3.5.4. Work on other financial instrument

In this project we just worked on the stock prices as financial instrument but working on other financial instruments such as , exchange rates, daily number of trades, shares and so on can be appealing for example in [3] it is seen that the spectral analysis for the number of daily trades shows the $1/f$ behavior unlike $1/f^2$ behavior of stock prices spectrum or in [16] it has been mentioned that the spectral density of stock volatility also shows $1/f$ behavior.

## 3.6. Conclusion

We confirm that the time series of the stock prices are characterized by $1/f^2$ spectral density as well as the logarithm of stock prices while the daily returns show a more white behavior in frequency domain. This result is somehow compliant with the random-walk model even in time domain and gives us the hope of prediction for stock prices. The $1/f^2$ behavior observed in the spectral density of stock prices (and also logarithm of stock prices) reflects the long-time memory for logarithm of prices while gives a short time temporal memory for price returns.

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## Bibliography:


[1] E. Sorouchyari, *Magnitude spectrum of stock prices: an oscillatory model,* under review (May 2008) to be submitted in IEEE Transactions on Signal Processing.

[2] David Ceballos Hornero and Mª Teresa Sorrosal i Forradellas, *Time aggregation problems in Financial Time Series*. Available at: **http://www.chronos.msu.ru/EREPORTS/hornero_time_aggreg.pdf**

[3] Bonanno G., Lillo F., Mantegna R.N., *Dynamics of the number of trades of financial securities*, (2000) Physica A: Statistical Mechanics and its Applications, 280 (1), pp. 136-141.

[4] Bachelier, L., *Theory of Speculation*, PhD thesis, 1900.

[5] Kjersti Aas, Xeni K. Dimakos, *Statistical modelling of financial time series: An introduction,* March 2004

[6] P.H. Cootner (Ed.), *The Random Character of Stock Market Prices*, MIT Press, Cambridge, MA, 1964.

[7] P. C. Biswal*, An Analysis of Stock Prices in India: Wavelets and Spectral Applications*, PhD thesis, Department of Economics, University of Hyderabad, India, july 2002.

[8] P.A. Samuelson, Ind. Manage. Rev. 6 (1965) 41.

[9] Ming-Chya Wu, Ming-Chang Huang, Hai-Chin Yu, and Thomas C. Chiang, *Phase distribution and phase correlation of financial time series.*

[10] R. Agrawal, C. Faloutsos, A. Swami, *Efficient Similary Search In Sequence Databases.*

[11] Davood Rafiei and Alberto Mendelzon*, Similarity-Based Queries for Time Series Data.*

[12] Ian Kaplan, *DFT of a non-stationary time series*, Sep. 2001. Available at: http://www.bearcave.com/misl/misl_tech/signal/nonstat/index.html

[13] Dennis Shasha , Time Series in Finance: the array database approach, Department of Computer Science, New York University. Available at: http://cs.nyu.edu/shasha/papers/jagtalk.html

[14] 1/f (One-Over-F) Noise (or Related Topics) in Financial Data. Available at: http://linkage.rockefeller.edu/wli/moved.8.04/1fnoise/1fnoise_finance.html

[15] Yahoo! Finance available at: http://finance.yahoo.com

[16] Kin-pong Chan, Ada Wai-chee Fu, *Efficient Time SeriesMatching byWavelets,*

[17] Gonul Turhan-Sayan, Serdar Sayan, *Use of Time-Frequency Representations in the Analysis of Stock Market Data*, Jan. 2001

[18] J. R. Norsworthy, D. Li, R. Gorener, *Wavelet-Based Analysis of Time Series: An Export from Engineering to Finance*

[19] Ping Chen, *A Random-Walk or Color-Chaos on the Stock Market? -Time-Frequency Analysis of S&P Indexes*, Studies in Nonlinear Dynamics & Econometrics, 1996

[20] T. Vuorenmaa, The Discrete Wavelet Transform with Financial Time Series Applications