Contents

Chapter 1

Some PL topology

In this chapter we will collect some basic facts from the piecewise-linear category. Most of the proofs are omitted and can be found in standard references like [1], [2]. If the reader is already familiar with basics from PL topology, the chapter may be skipped without loss of continuity.

Definition 1. *Let $v_0, ..., v_k \in \mathbb{R}^n$ be points in some affine space such that $\{v_1 - v_0, ..., v_k - v_0\}$ is a set of linearly independent vectors. We call*

$$[v_0, ..., v_k] := \left\{ \lambda_0 v_0 + ... + \lambda_k v_k \,\middle|\, \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}$$

*the simplex spanned by $\{v_0, ..., v_k\}$. Its dimension is k and we call it a k-**simplex** for short. The points that span a simplex are called vertices. For a simplex σ we say that τ is a **face** of σ if τ is a simplex spanned by a nonempty subset of the vertices of σ and we abbreviate this by writing $\tau < \sigma$.*

Definition 2. *A **simplicial complex** K is a set of simplices that satisfies the following conditions:*

- *Every face of a simplex in K is also contained in K.*
- *The intersection of any two simplices $\sigma, \tau \in K$ is either empty or a face of both σ and τ.*

*We define the **plyhedron** of K by $|K| := \bigcup\{\sigma | \sigma \in K\}$. The **p-skeleton** of K is given by $K^{(p)} = \{\sigma \in K | dim(\sigma) \leq p\}$. A **subcomplex** of K is a subset $L \subset K$ such that L itself is a simplicial complex. We denote the set of vertices of K by $V(K)$.*

Definition 3. *Suppose that there are two simplicial complexes K and K' such that $|K| = |K'|$. If every simplex of K' is contained in some simplex of K, we say that K' is a **subdivision** of K and write $K' \lhd K$.*

Example 1. *Given a simplicial complex K there is always an inductive process that produces a subdivision of K. Assume that $K^{(p-1)}$ has already been subdivided and let $\sigma = [v_0, ..., v_p]$ be a p-simplex in K. The point $\hat{\sigma} = \frac{1}{p+1}\sum_{i=0}^{p} v_i$ lies in the interior of σ and is called its **barycenter**. The **barycentric subdivision** of σ is the decomposition of σ into the p-simplices $[\hat{\sigma}, w_0, ..., w_{p-1}]$ where, inductively, $[w_o, ..., w_{p-1}]$ is a $(p-1)$-simplex in the barycentric subdivision of a face $[v_0, ..., \overline{v_i}, ..., v_p]$. (In this notation, the vertex v_i is omitted.) Continuing this procedure for every p-simplex σ leads to a decomposition of all simplices in $K^{(p)}$. The induction starts at $p = 0$ when the barycentric subdivision of a 0-simplex $[v_0]$ is just $[v_0]$ itself. It is guaranteed that this process delivers a subdivision $K^1 \lhd K$, called the **first barycentric subdivision** of K. For details, see [2]. More generally, the r-th barycentric subdivision is inductively given by $K^r = (K^{r-1})^1$.*

Definition 4. *A topological space X is said to be* **triangulable** *if there exists a simplicial complex T and a homeomorphism $\phi : |T| \to X$. The triple (T, X, ϕ) is called a **triangulation** of X. In this situation we will simply say that T is a triangulation of X, by abuse of notation.*

Definition 5. *A **PL space** is a pair $(X, \mathcal{T})$ consisting of a topological space X and a class $\mathcal{T}$ of locally finite triangulations of X which satisfies the following conditions:*

- *If $T \in \mathcal{T}$ then $T' \in \mathcal{T}$ for any subdivision $T' \lhd T$.*
- *If $T, T' \in \mathcal{T}$ then there exists $T'' \in \mathcal{T}$ such that both $T'' \lhd T$ and $T'' \lhd T'$.*

*We will simply write X for a PL space $(X, \mathcal{T})$ if there is no danger of confusion. A **closed PL subspace** of X is a subcomplex of a suitable triangulation of X.*

Definition 6. *Given simplicial complexes K and L we call a map $f : |K| \to |L|$ **simplicial** if f maps each simplex of K linearly onto some simplex of L. A map $g : |K| \to |L|$ between PL spaces is said to be a **PL map** if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $g : K' \to L'$ is simplicial.*

Note that a simplicial map $f : K \to L$ is given by linear extension of a (set-theoretic) function $V(K) \to V(L)$.

Definition 7. *A **simplicial isomorphism** between two simplicial complexes K and L is given by a bijection $f : V(K) \to V(L)$ such that $[v_0, ..., v_k]$ is a k-simplex in K if and only if $[f(v_0), ..., f(v_k)]$ is a k-simplex in L. In particular, extending f linearly yields a homeomorphism between the underlying polyhedra $|K|$ and $|L|$. A map $g : |K| \to |L|$ between PL spaces is a **PL isomorphism** if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $g : |K'| \to |L'|$ is a simplicial isomorphism.*

Theorem 1.

STARS AND LINKS SIMPLICIAL APPROXIMATION DIMENSION PL MANIFOLDS

Definition 8. *A **0-dimensional PL stratified pseudomanifold** is a countable set of points with the discrete topology. An **n-dimensional PL stratified pseudomanifold** X is a PL space together with a filtration of closed PL subspaces*

$$X = X_n \supset X_{n-1} = X_{n-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

such that the following conditions are satisfied:

- *Every $X_{n-k} - X_{n-k-1}$ is a (possibly empty) PL manifold of dimension $n - k$.*
- *$X - X_{n-2}$ is dense in X.*
- ***Local normal triviality:** For every point $x \in X_{n-k} - X_{n-k-1}$ there exists an open neighborhood U of x in X and a compact PL stratified pseudomanifold L of dimension $k - 1$ with filtration*

  $$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

  and a PL isomorphism

  $$\phi : U \to \mathbb{R}^{n-k} \times c^\circ L$$

  *(where c° denotes the open cone) which restricts to PL isomorphism $\phi_| : U \cap X_{n-l} \to \mathbb{R}^{n-k} \times c^\circ L_{k-l-1}$. We say that ϕ is **stratum-preserving**.*

Bibliography

[1] Colin Patrick Rourke, Brian Joseph Sanderson, *Introduction to piecewise-linear topology.*

[2] Allen Hatcher, *Algebraic Topology.* p.120.