Contents

Chapter 1

Some PL topology

In this chapter we will collect some basic facts from the piecewise-linear category. Most of the proofs are omitted and can be found in standard references like [1]. If the reader is already familiar with basics from PL topology, the chapter may be skipped without loss of continuity.

Definition 1. *Let $v_0, ..., v_k \in \mathbb{R}^n$ be points in some affine space such that $\{v_1 - v_0, ..., v_k - v_0\}$ is a set of linearly independent vectors. We call*

$$[v_0, ..., v_k] := \left\{ \lambda_0 v_0 + ... + \lambda_k v_k \middle| \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\} \tag{1.1}$$

*the simplex spanned by $\{v_0, ..., v_k\}$. Its dimension is k and we call it a **k-simplex** for short. The points that span a simplex are called vertices. For a simplex σ we say that τ is a **face** of σ if τ is a simplex spanned by a nonempty subset of the vertices of σ and we abbreviate this by writing $\tau < \sigma$.*

Definition 2. *A **simplicial complex** K is a set of simplices that satisfies the following conditions:*

- *Every face of a simplex in K is also contained in K.*
- *The intersection of any two simplices $\sigma, \tau \in K$ is either empty or a face of both σ and τ.*

*We define the **plyhedron** of K by $|K| := \bigcup\{\sigma | \sigma \in K\}$. The **p-skeleton** of K is given by $K^{(p)} = \{\sigma \in K | dim(\sigma) \leq p\}$. A **subcomplex** of K is a subset $L \subset K$ such that L itself is a simplicial complex.*

Definition 3. *Suppose that there are two simplicial complexes K and K' such that $|K| = |K'|$. If every simplex of K' is contained in some simplex of K, we say that K' is a* ***subdivision*** *of K and write $K' \lhd K$.*

Definition 4. *A topological space X is said to be* **triangulable** *if there exists a simplicial complex T and a homeomorphism $\phi : |T| \to X$. The triple (T, X, ϕ) is called a* ***triangulation*** *of X. In this situation we will simply say that T is a triangulation of X, by abuse of notation.*

Definition 5. *A* ***PL space*** *is a pair $(X, \mathcal{T})$ consisting of a topological space X and a class $\mathcal{T}$ of locally finite triangulations of X which satisfies the following conditions:*

- *If $T \in \mathcal{T}$ then $T' \in \mathcal{T}$ for any subdivision $T' \lhd T$.*
- *If $T, T' \in \mathcal{T}$ then there exists $T'' \in \mathcal{T}$ such that both $T'' \lhd T$ and $T'' \lhd T'$.*

We will simply write X for a PL space $(X, \mathcal{T})$ if there is no danger of confusion. A ***closed PL subspace*** *of X is a subcomplex of a suitable triangulation of X.*

Definition 6. *Given simplicial complexes K and L we call a map $f : |K| \to |L|$* ***simplicial*** *if f maps each simplex of K linearly onto some simplex of L. A map $g : |K| \to |L|$ between polyhedra is said to be a* ***PL map*** *if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $g : K' \to L'$ is simplicial.*

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Definition 7. *A* ***0-dimensional PL stratified pseudomanifold*** *is a countable set of points with the discrete topology. An* ***n-dimensional PL stratified pseudomanifold*** *X is a PL space together with a filtration of closed PL subspaces*

$$X = X_n \supset X_{n-1} = X_{n-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

such that the following conditions are satisfied:

- *Every $X_{n-k} - X_{n-k-1}$ is a (possibly empty) PL manifold of dimension $n - k$.*
- *$X - X_{n-2}$ is dense in X.*

- ***Local normal triviality:*** *For every point* $x \in X_{n-k} - X_{n-k-1}$ *there exists an open neighborhood* U *of* x *in* X *and a compact PL stratified pseudomanifold* L *of dimension* $k-1$ *with filtration*

  $$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

  and a PL homeomorphism

  $$\phi : U \to \mathbb{R}^{n-k} \times c^\circ L$$

  (where c° *denotes the open cone) which restricts to PL homeomorphisms* $\phi_| : U \cap X_{n-l} \to \mathbb{R}^{n-k} \times c^\circ L_{k-l-1}$*. We say that* ϕ *is* ***stratum-preserving****.*

Bibliography

[1] Colin Patrick Rourke, Brian Joseph Sanderson, *Introduction to piecewise-linear topology.*