Contents

Chapter 1

Some PL topology

In this chapter we will collect some basic facts from the piecewise-linear category. Most of the proofs are omitted and can be found in standard references like [1], [2]. If the reader is already familiar with basics from PL topology, the chapter may be skipped without loss of continuity.

Definition 1. *Let $v_0, ..., v_k \in \mathbb{R}^n$ be points in some affine space such that $\{v_1 - v_0, ..., v_k - v_0\}$ is a set of linearly independent vectors. We call*

$$[v_0, ..., v_k] := \left\{ \lambda_0 v_0 + ... + \lambda_k v_k \,\middle|\, \sum_{i=0}^{k} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\}$$

*the simplex spanned by $\{v_0, ..., v_k\}$. Its dimension is k and we call it a k-**simplex** for short. The points that span a simplex are called vertices. For a simplex σ we say that τ is a **face** of σ if τ is a simplex spanned by a nonempty subset of the vertices of σ and we abbreviate this by writing $\tau < \sigma$.*

Definition 2. *A **simplicial complex** K is a set of simplices that satisfies the following conditions:*

- *Every face of a simplex in K is also contained in K.*
- *The intersection of any two simplices $\sigma, \tau \in K$ is either empty or a face of both σ and τ.*

*We define the **plyhedron** of K by $|K| := \bigcup\{\sigma|\sigma \in K\}$. A **subcomplex** of K is a subset $L \subset K$ such that L itself is a simplicial complex. The **p-skeleton** of K is the subcomplex of K, given by $K_{(p)} = \{\sigma \in K | dim(\sigma) \leq p\}$.*

We denote the set of vertices of K by $V(K)$. The ***dimension*** *of K is the largest dimension of any simplex contained in K. If no such maximum exists, we say that K is of dimension ∞. K is said to be of* ***pure dimension*** *n if every simplex of K is a face of some n-simplex in K.*

Definition 3. *Let K be a simplicial complex of pure dimension n. The* ***boundary*** *$Bd(K)$ of K is defined to be the (possibly empty) subcomplex of K of pure dimension $n-1$, whose $(n-1)$-simplices are those $(n-1)$-simplices of K which are incident to precisely one n-simplex in K.*

Definition 4. *For a simplicial complex K and a simplex $\sigma \in K$, we call*

$$St(\sigma, K) = \{\rho \in K | \rho < \tau, \sigma < \tau \text{ for some } \tau \in K\}$$

the ***star*** *of σ in K. The* ***link*** *of σ in K is given by*

$$Lk(\sigma, K) = \{\rho \in St(\sigma, K) | \sigma \cap \rho = \emptyset\}.$$

Definition 5. *Suppose that there are two simplicial complexes K and K' such that $|K| = |K'|$. If every simplex of K' is contained in some simplex of K, we say that K' is a* ***subdivision*** *of K and write $K' \lhd K$.*

Example 1. *Given a simplicial complex K there is always an inductive process that produces a subdivision of K. Assume that $K_{(p-1)}$ has already been subdivided and let $\sigma = [v_0, ..., v_p]$ be a p-simplex in K. The point $\hat{\sigma} = \frac{1}{p+1} \sum_{i=0}^{p} v_i$ lies in the interior of σ and is called its* ***barycenter****. The* ***barycentric subdivision*** *of σ is the decomposition of σ into the p-simplices $[\hat{\sigma}, w_0, ..., w_{p-1}]$ where, inductively, $[w_o, ..., w_{p-1}]$ is a $(p-1)$-simplex in the barycentric subdivision of a face $[v_0, ..., \overline{v_i}, ..., v_p]$. (In this notation, the vertex v_i is omitted.) Continuing this procedure for every p-simplex σ leads to a decomposition of all simplices in $K_{(p)}$. The induction starts at $p = 0$ when the barycentric subdivision of a 0-simplex $[v_0]$ is just $[v_0]$ itself. It is guaranteed that this process delivers a subdivision $K^1 \lhd K$, called the* ***first barycentric subdivision*** *of K. For details, see [2]. More generally, the r-th barycentric subdivision is inductively given by $K^r = (K^{r-1})^1$.*

Definition 6. *A topological space X is said to be* **triangulable** *if there exists a simplicial complex T and a homeomorphism $\phi : |T| \to X$. The triple (T, X, ϕ) is called a* ***triangulation*** *of X. In this situation we will simply say that T is a triangulation of X, by abuse of notation.*

Definition 7. *A* ***PL space*** *is a pair $(X, \mathcal{T})$ consisting of a topological space X and a class $\mathcal{T}$ of locally finite triangulations of X which satisfies the following conditions:*

- *If $T \in \mathcal{T}$ then $T' \in \mathcal{T}$ for any subdivision $T' \lhd T$.*
- *If $T, T' \in \mathcal{T}$ then there exists $T'' \in \mathcal{T}$ such that both $T'' \lhd T$ and $T'' \lhd T'$.*

*We will simply write X for a PL space $(X, \mathcal{T})$ if there is no danger of confusion. A **closed PL subspace** of X is a subcomplex of a suitable triangulation of X.*

Definition 8. *Given simplicial complexes K and L we call a map $f : |K| \to |L|$ **simplicial** if f maps each simplex of K linearly onto some simplex of L. A map $g : |K| \to |L|$ between PL spaces is said to be a **PL map** if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $g : K' \to L'$ is simplicial.*

Note that a simplicial map $f : K \to L$ is given by linear extension of a (set-theoretic) function $V(K) \to V(L)$.

Definition 9. *A **simplicial isomorphism** between two simplicial complexes K and L is given by a bijection $f : V(K) \to V(L)$ such that $[v_0, ..., v_k]$ is a k-simplex in K if and only if $[f(v_0), ..., f(v_k)]$ is a k-simplex in L. In particular, extending f linearly yields a homeomorphism between the underlying polyhedra $|K|$ and $|L|$. A map $g : |K| \to |L|$ between PL spaces is a **PL isomorphism** if there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $g : |K'| \to |L'|$ is a simplicial isomorphism.*

Theorem 1. *(Simplicial Approximation Theorem). Let $f : |K| \to |L|$ be a map between polyhedra. Then there exist subdivisions $K' \lhd K$ and $L' \lhd L$ and a simplicial map $g : |K'| \to |L'|$ which is ϵ-homotopic to f, i.e. if $\epsilon : |L| \to (0, \infty)$ is a map, then there is a map $H : |K| \times [0,1] \to |L|$ with $H(|K| \times \{0\}) = f$, $H(|K| \times \{1\}) = g$ and diam$(H(\{x\} \times [0,1])) < \epsilon(f(x))$.*

Proof. See [3], e.g. □

JOIN
DIMENSION
PL MANIFOLDS

Chapter 2

A bordism approach to homology theories

The purpose of this chapter is to give a geometric treatment of homology theories, as described by S. Buoncristiano, C. Rourke and B. Sanderson in [5]. First we use local link properties to determine a class of polyhedra. Then, in an analagous manner to ordinary bordism theory, we define groups of bordism classes of maps, whose domains lie in the polyhedral class defined before. We observe that interesting examples of generalized homology theories can be interpreted in this way, including ordinary homology, $(\mathbb{Z}/2)$-homology and PL bordism theory.
For the rest of this chapter we make the convention that every polyhedron is PL and of pure dimension.

Definition 10. *Suppose we are given a class $\mathcal{L}_n$ of $(n-1)$-polyhedra which is closed under PL isomorphism. Then a* ***closed $\mathcal{L}_n$-manifold*** *is a polyhedron M such that the link of each vertex of M lies in $\mathcal{L}_n$.*

Definition 11. *A* ***theory with singularities*** *$\mathcal{L}$ consists of a class $\mathcal{L}_n$ of $(n-1)$-polyhedra for every $n = 0, 1, ...$ which satisfy the following compatibility conditions:*

1. *each member of $\mathcal{L}_n$ is a closed $\mathcal{L}_{n-1}$-manifold.*
2. *$S\mathcal{L}_{n-1} \subset \mathcal{L}_n$ (i.e. the suspension of an $(n-1)$-link is always an n-link).*
3. *$c\mathcal{L}_{n-1} \cap \mathcal{L}_n = \emptyset$ (i.e. the cone of an $(n-1)$-link is never an n-link).*
4. *$S(c\mathcal{L}_{n-2}), c(c\mathcal{L}_{n-2}) \subset c\mathcal{L}_{n-1}$.*

*Then an $\mathcal{L}_n$**-manifold with boundary** consists of a polyhedron whose links of vertices lie either in $\mathcal{L}_n$ or in $c\mathcal{L}_{n-1}$. The **boundary** consists of the subpolyhedron spanned by vertices whose links lie in the latter class.*

Proposition 1. *Let W be an $\mathcal{L}_n$-manifold with boundary in a fixed theory with singularities $\mathcal{L}$. Then the boundary of W, denoted by ∂W, is well-defined and is itself a closed $\mathcal{L}_{n-1}$-manifold.*

Proof. By the third requirement of Def. 11, the link of a vertex in W is contained in $c\mathcal{L}_{n-1}$ if and only if it is not contained in $\mathcal{L}_n$. This shows that ∂W is well-defined. For the second statement, note that ∂W is a subpolyhedron by definition and that if v is a vertex in ∂W with $Lk(v, W) = cL$ for some $L \in \mathcal{L}_{n-1}$, then $Lk(v, \partial W) = L$. $\square$

DEFINITION BERARBEITEN:
LINKS STABIL BZGL SUBDIVISION?
KOMPAKTE POLYEDER
ORIENTIERUNGEN
EVTL ORIENTIERUNGSERHALTENDE ISOMORPHISMEN

Definition 12. *Let $\mathcal{L}$ be a theory with singularities and let (X, A) be a pair of topological spaces (i.e. $A \subset X$ is a subspace). For two compact, oriented $\mathcal{L}_i$-manifolds (possibly with boundary) P, Q assume that we are given continuous maps $f : (P, \partial P) \to (X, A)$ and $g : (Q, \partial Q) \to (X, A)$. An **oriented bordism** between f and g is a triple (F, W, Z), where W is a compact, oriented $\mathcal{L}_{i+1}$-manifold with boundary, s.t. $\partial W \cong P \sqcup -Q \cup Z$, Z is a compact, oriented $\mathcal{L}_i$-manifold (possibly with boundary), s.t. $\partial Z \cong \partial P \sqcup -\partial Q$ and $F : (W, Z) \to (X, A)$ is a continuous map with $F_{|P} = f$ and $F_{|Q} = g$. If there exists a bordism between them, f and g are said to be **bordant** and we abbreviate this by writing $f \sim_{bord} g$. We call F a **null-bordism** for f if $Q = \emptyset$ and we say that f is **null-bordant** if there exists a null-bordism for f.*

Definition 13. *With the notation as in the previous definition, $f : (P, \partial P) \to (X, A)$ and $g : (Q, \partial Q) \to (X, A)$ are called **isomorphic** if there exists a PL isomorphism $h : P \to Q$ s.t. the diagram*

$$\begin{tikzcd}
(P, \partial P) \arrow[rr, "h"] \arrow[dr, "f"'] & & (Q, \partial Q) \arrow[dl, "g"] \\
& (X, A) &
\end{tikzcd}$$

commutes. Let $Isom_i^{\mathcal{L}}(X, A)$ be the set of isomorphism classes of maps

$f : (P, \partial P) \to (X, A)$, where P varies over all compact $\mathcal{L}_i$-manifolds of a fixed theory $\mathcal{L}$.

Proposition 2. *The relation $\sim_{bord}$ is an equivalence relation on $Isom_i^{\mathcal{L}}(X, A)$.*

Proof. Let $[f : (P, \partial P) \to (X, A)] \in Isom_i^{\mathcal{L}}(X, A)$ and let $I = [0, 1]$ denote the closed interval. The only links in I are S^0 and $pt.$ and so there are four types of links in $P \times I$, namely $S^0 * L \cong SL$, $pt. * L \cong cL$, $S^0 * cL' \cong S(cL')$ and $pt. * cL' \cong c(cL')$, where $L \in \mathcal{L}_i$ and $L' \in \mathcal{L}_{i-1}$ are some links of P. The second statement of Def. 11 ensures that the first type of links is contained in $\mathcal{L}_{i+1}$ and by the fourth statement all other types of links lie in $c\mathcal{L}_i$. There exists a canonical orientation of $P \times I$ s.t. $\partial(P \times I) = P \sqcup -P \cup \partial P \times I$. So, $P \times I$ is an oriented $\mathcal{L}_{i+1}$-manifold and we define $F : (P \times I, \partial(P \times I)) \to (X, A)$ by $F(x, s) = f(x)$ for every $s \in I$. Then $f \sim_{bord} f$ via F and this bordism is well-defined on the isomorphism class of f, which shows reflexivity. For symmetry, we only need to obersve that disjoint union commutes, up to isomorphism. Now suppose $f \sim_{bord} g$ and $g \sim_{bord} h$ via bordisms F and G, respectively, where $g : (Q, \partial Q) \to (X, A)$, $F : (W, Z) \to (X, A)$ and $G : (W', Z') \to (X, A)$. If $W \cup_Q W'$ denotes the space obtained by glueing W and W' along Q, we let $H : W \cup_Q W' \to X$ be the unique map with $H_{|W} = F$ and $H_{|W'} = G$. If v is any vertex in Q, we have $Lk(v, W) = cLk(v, Q) = Lk(v, W')$, and therefore $Lk(v, W \cup_Q W') = SLk(v, Q)$, which lies in $\mathcal{L}_{i+1}$, by definition. Since the links of all other vertices remain unaltered, we see that $W \cup_Q W'$ is in fact an $\mathcal{L}_{i+1}$-manifold, which clearly is compact, as W and W' are. Moreover, we have $\partial(W \cup_Q W') = dom(f) \sqcup -dom(h) \cup (Z \cup_{\partial Q} Z')$ with $\partial(Z \cup_{\partial Q} Z') = \partial(dom(f)) \sqcup -\partial(dom(h))$, and the same argument as before shows that $Z \cup_{\partial Q} Z'$ is an $\mathcal{L}_i$-manifold. We conclude $f \sim_{bord} h$ via $H : (W \cup_Q W', Z \cup_{\partial Q} Z') \to (X, A)$, and again H is a well-defined bordism on isomorphism classes of f and h. This shows transitivity and completes the proof. □

Definition 14. *For a theory with singularities $\mathcal{L}$, we let*

$$\Omega_i^{\mathcal{L}}(X, A) = Isom_i^{\mathcal{L}}(X, A)/ \sim_{bord}$$

*and call it the i**-th (relative) $\mathcal{L}$-bordism set** with base (X, A). Moreover, we define*

$$\Omega_i^{\mathcal{L}}(X) = \Omega_i^{\mathcal{L}}(X, \emptyset)$$

*and call it the i**-th (absolute) $\mathcal{L}$-bordism set** with base X.*

Remark 1. *1. There is an obvious notion of unoriented $\mathcal{L}$-bordism by removing all references to orientations. We denote the associated bordism sets by $\underline{\Omega}_i^{\mathcal{L}}(X, A)$.*

2. *For the absolute case, note that $\Omega_i^{\mathcal{L}}(X)$ consists of bordism classes of maps $f : P \to X$, where P is a, necessarily closed $\mathcal{L}_i$-manifold. For some $g : Q \to X$, the bordism relation between f and g reduces to the existence of some compact, oriented $\mathcal{L}_{i+1}$-manifold W with $\partial W \cong P \sqcup -Q$ and a map $F : W \to X$ s.t. $F_{|P} = f$ and $F_{|Q} = g$.*

So far, we have not discussed interactions of two $\mathcal{L}$-manifolds with each other. For example, the product of two $\mathcal{L}_i$-manifolds is not an $\mathcal{L}_i$-manifold, in general. However, the following is true:

Proposition 3. *$\Omega_i^{\mathcal{L}}(X, A)$ is an abelian group with respect to disjoint union.*

Proof. Let $[f : (P, \partial P) \to (X, A)], [g : (Q,\ \partial Q) \to (X, A)] \in \Omega_i^{\mathcal{L}}(X, A)$. Then $P \sqcup Q$ is an $\mathcal{L}_i$-manifold, as the links are unaltered by disjoint union, and so $f \sqcup g : (P \sqcup Q, \partial P \sqcup \partial Q) \to (X, A)$ represents a class in $\Omega_i^{\mathcal{L}}(X, A)$. Consequently, we let $[f] \sqcup [g] := [f \sqcup g]$. Suppose that f' and g' are representatives of $[f]$ and $[g]$, respectively, via bordisms F and G. Then $F \sqcup G$ is a bordism between $f \sqcup g$ and $f' \sqcup g'$, which shows that the above definition is a well-defined operation. Associativity and commutativity clearly hold. The identity element is given by the class of the empty map (and so by the class of any null-bordant map). If $[-f]$ denotes the class of f with reversed orientation of its domain, we observe once again that $F : (P \times I, \partial(P \times I)) \to (X, A)$ with $F(x, s) = f(x)$ is a bordism between f and $-f$. We conclude

$$[f] \sqcup [-f] = [f \sqcup -f] = 0$$

and so the inverse of $[f]$ is given by $[-f]$. In the unoriented setting, we can remark that every bordism class is self-inverse. □

The previous Proposition allows us to speak of bordism groups, rather than of bordism sets and we adapt this terminology for the rest of this discussion.
So far, we did not explain connections of $\mathcal{L}$-bordism groups for varying base spaces. This will be done in the following.

Definition 15. *Let $\boldsymbol{CW^2}$ denote the category of pairs of finite CW spaces with continuous maps between them and let **AbGrp** be the category of abelian groups with group homomorphisms between them. Moreover, let T denote the covariant functor on $\boldsymbol{CW^2}$, given by*

$T(X, A) = (A, \emptyset)$, for any $(X, A) \in \boldsymbol{CW^2}$,
$T(f) = f_{|(A,\emptyset)} : (A, \emptyset) \to (B, \emptyset)$, for any $f : (X, A) \to (Y, B) \in \boldsymbol{CW^2}$.

*A **generalized homology theory** is a pair (H_*, ∂_*), consisting of a sequence of functors $H_i : \boldsymbol{CW^2} \to \boldsymbol{AbGrp}$, together with a sequence of natural transformations $\partial_i : H_i \to H_{i-1} \circ T$, s.t. the following three conditions are satisfied:*

1. ***Homotopy-invariance:*** *If $f, g \in \boldsymbol{CW^2}$ are homotopic maps, then $H_i(f) = H_i(g)$ for all i.*

2. ***Excision:*** *Let $(X, A) \in \boldsymbol{CW^2}$ and suppose U is a subspace of X with $\overline{U} \subset int(A)$, then the inclusion $j : (X - U, A - U) \to (X, A)$ induces isomorphisms*

$$j_* : H_i(X - U, A - U) \to H_i(X, A),$$

for all i.

3. ***Long exact sequence:*** *If $(X, A) \in \boldsymbol{CW^2}$ and if $j : (A, \emptyset) \to (X, \emptyset)$ and $k : (X, \emptyset) \to (X, A)$ denote the inclusions, the sequence*

$$\cdots \longrightarrow H_{i+1}(X, A) \xrightarrow{\partial_{i+1}} H_i(A) \xrightarrow{H_i(j)} H_i(X)$$

$$\xrightarrow{H_i(k)} H_i(X, A) \xrightarrow{\partial_i} H_{i-1}(A) \longrightarrow \cdots$$

is exact, for all i.

*A generalized homology theory (H_*, ∂_*) is called **ordinary** if the following additional requirement holds:*

4. ***Dimension:*** *$H_i(pt.) = 0$ for $i \neq 0$. Then $H_0(pt.)$ is called the **coefficient group** of H_*.*

Theorem 2. *For a theory with singularities $\mathcal{L}$, the pair $(\Omega_*^{\mathcal{L}}, \partial_*)$ is a generalized homology theory, where*

$$\begin{aligned} \partial_i :\ \Omega_i^{\mathcal{L}}(X, A) &\to \Omega_{i-1}^{\mathcal{L}}(A) \\ [f : (P, \partial P) \to (X, A)] &\mapsto [f_{|\partial P} : \partial P \to A] \end{aligned}$$

for any pair $(X, A) \in \boldsymbol{CW^2}$.

Proof. First, if $\phi : (X, A) \to (Y, B) \in \boldsymbol{CW^2}$, then the induced map of ϕ is given by

$$\begin{aligned} \phi_* : \Omega_i^{\mathcal{L}}(X, A) &\to \Omega_i^{\mathcal{L}}(Y, B) \\ [f] &\mapsto [\phi \circ f], \end{aligned}$$

which is well-defined since if F is a bordism over (X, A), then $\phi \circ F$ is a bordism over (Y, B). Moreover, this construction respects the corresponding group structures, behaves functorial and we have $(id)_* = id$. Similarly, ∂_i is well-defined, as if $F : (W, Z) \to (X, A)$ is a bordism between f and g over (X, A), then $F_{|Z}$ is a bordism between $f_{|\partial dom(f)}$ and $g_{|\partial dom(g)}$ over A. The maps ∂_i are natural, as composition of maps commutes with restriction to the boundary.

1. Homotopy-invariance: Let $\phi, \psi : (X, A) \to (Y, B)$ be two homotopic maps via a homotopy H. For $[f] \in \Omega_i^{\mathcal{L}}(X, A)$, the map

$$F : (dom(f) \times I) \to (X, A)$$

   with $F(x, s) = H(f(x), s)$ is a bordism between $\phi \circ f$ and $\psi \circ f$ (with reversed orientation of its domain). So, we have

$$\phi_*([f]) = [\phi \circ f] = [\psi \circ f] = \psi_*([f])$$

   and since $[f]$ was chosen arbitrarily, we conclude $\phi_* = \psi_*$.

2. Excision: For $(X, A) \in \boldsymbol{CW^2}$ and $U \subset A$ with $\overline{U} \subset int(A)$, let

$$j_i : \Omega_i^{\mathcal{L}}(X - U, A - U) \to \Omega_i^{\mathcal{L}}(X, A)$$

   denote the map induced by inclusion. We will show that j_i is an isomorphism. To see surjectivity, consider $[f : (P, \partial P) \to (X, A)] \in \Omega_i^{\mathcal{L}}(X, A)$ and let $U_1 = f^{-1}(U)$ and $A_1 = f^{-1}(A)$. We choose a triangulation T of P, fine enough such that the smallest subcomplex of T which contains every simplex that meets $M - A_1$, is contained in $M - U_1$. This is possible, since $d(M - int(A_1), \overline{U_1}) > 0$ for any metric d on P. If we denote this subcomplex by K, note that for any vertex $v \in K$ either $Lk(v, K) = Lk(v, T)$ or, by the first condition of Def. 11, there exists a vertex $w \in L := Lk(v, T)$ such that $Lk(w, L) = Lk(v, Bd(K))$. It follows that

$$Lk(v, K) = w * Lk(v, Bd(K)) = w * Lk(w, L) \cong c(Lk(w, L))$$

   and since $Lk(w, L) \in \mathcal{L}_{i-1}$, $|K|$ is in fact an $\mathcal{L}_i$-manifold. By construction, $f_1 := f|_{|K|}$ defines a class in $\Omega_i^{\mathcal{L}}(X - U, A - U)$ and

$$\begin{aligned} F : \frac{P \times I}{(P - |K|) \times \{1\}} &\to (X, A) \\ (x, s) &\mapsto f(x) \end{aligned}$$

defines a bordism between f and f_1 over (X, A). Consequently, we have $j_i([f_1]) = [f]$. For injectivity, suppose $j_i([f]) = 0$ and let F be the corresponding null-bordism for f over (X, A). Then, the same construction as before applied to F provides a null-bordism F_1 for f over $(X - U, A - U)$, which shows $[f] = 0$.

3. Long exact sequence: For some $i \geq 0$, let

$$\cdots \longrightarrow \Omega_i^{\mathcal{L}}(A) \xrightarrow{j_i} \Omega_i^{\mathcal{L}}(X) \xrightarrow{k_i} \Omega_i^{\mathcal{L}}(X, A)$$

$$\xrightarrow{\partial_i} \Omega_{i-1}^{\mathcal{L}}(A) \xrightarrow{j_{i-1}} \Omega_{i-1}^{\mathcal{L}}(X) \longrightarrow \cdots$$

be the sequence of the induced maps of inclusions j, k and restriction map ∂_i.
$im(j_i) = ker(k_i)$: If $[f : P \to A] \in \Omega_i^{\mathcal{L}}(A)$, then $F : (P \times I, -P) \to (X, A)$ with $F(x, s) = f(x)$ is a null-bordism for f over (X, A), which shows $k_i \circ j_i = 0$. Now, suppose that $[g : Q \to X] \in \Omega_i^{\mathcal{L}}(X)$ with $k_i([g]) = 0$. This means that there exists a null-bordism $G : (W, Z) \to (X, A)$ for g over (X, A). We then have

$$j_i([G_{|Z} : -Z \to A]) = [g],$$

as $G : W \to X$ is a bordism between $G_{|Z}$ and g over X.
$im(k_i) = ker(\partial_i)$: Since $\partial_i \circ k_i$ is given by restriction to an empty boundary, $\partial_i \circ k_i = 0$ is obvious. On the other hand, if

$$\partial_i([f : (P, \partial P) \to (X, A)]) = [f_{|\partial P} : \partial P \to A] = 0,$$

there exists a null-bordism $F : W \to A$ for $f_{|\partial P}$ over A. In particular, we have $\partial W = \partial P$. Let Q denote the cylinder $P \times I$, in which we add W by glueing it along the boundary of $P \times \{0\}$. Then Q is an $\mathcal{L}_{i+1}$-manifold with P and $P \cup_{\partial P} W$ sitting inside its boundary. We define the map $G : Q \to X$ to be f on every level of the cylinder and to be F on W. The condition $F_{|\partial W} = f_{|\partial P}$ ensures that G is well-defined and continuous. Moreover, G is a bordism between $G_{|P \cup_{\partial P} W}$ and f over (X, A) and since $P \cup_{\partial P} W$ has no boundary, we conclude

$$k_i([G_{|P \cup_{\partial P} W}]) = [f].$$

$im(\partial_i) = ker(j_{i-1})$: If $[f : (P, \partial P) \to (X, A)] \in \Omega_i^{\mathcal{L}}(X, A)$, we have

$$j_{i-1} \circ \partial_i([f]) = [f_{|\partial P} : \partial P \to X] = 0,$$

as a null-bordism over X is given by f. For the other implication, consider $[g : Q \to A] \in \Omega^{\mathcal{L}}_{i-1}(A)$ with $j_{i-1}([g]) = 0$. If $G : W \to X$ is a corresponding null-bordism over X, it follows that

$$\partial_i([G : (W, Q) \to (X, A)]) = [g],$$

which shows exactness of the sequence and completes the proof. $\square$

Next, we will give examples of classes $\mathcal{L}$ that determine the corresponding bordism theory $\Omega^{\mathcal{L}}_*$.

Example 2. *Consider the class of links $\mathcal{L}$, given by $\mathcal{L}_0 := \{\emptyset\}$, $\mathcal{L}_n := \{X | X \cong S^{n-1}\}$ for $n \geq 1$. Then a closed $\mathcal{L}_n$-manifold is a polyhedron M of dimension n, in which every vertex v has a PL sphere as its link. Therefore M is a closed PL manifold, as a small open neighborhood of v is PL isomorphic to an open PL ball. Similarly, an $\mathcal{L}_n$-manifold with boundary is simply an n-dimensional PL manifold with boundary, and so the associated theory $\Omega^{\mathcal{L}}_*$ is "ordinary" (oriented) PL bordism theory, denoted by Ω^{PL}_*.*

Example 3. *We define a class of links $\mathcal{L}$ as follows: $\mathcal{L}_0 := \{\emptyset\}$, $\mathcal{L}_1 := \{X | X \cong S^0\}$, and for $n \geq 2$ we let $\mathcal{L}_n$ be the class of **all** closed $\mathcal{L}_{n-1}$-manifolds. We would like to compute the coefficient group $\Omega^{\mathcal{L}}_*(pt.)$. For this, note that an $\mathcal{L}_0$-manifold is a disjoint union of points and an $\mathcal{L}_1$-manifold with boundary is a disjoint union of (closed) intervals. This means that a point does not bound in $\mathcal{L}$, and so generates $\Omega^{\mathcal{L}}_0(pt.) \cong \mathbb{Z}$. Moreover, if P is a closed $\mathcal{L}_n$-manifold for $n \geq 1$, then cP is an $\mathcal{L}_{n+1}$-manifold with boundary. Indeed, let c denote the cone point. Then $Lk(c, cP) = P \in \mathcal{L}_{n+1}$ and if v denotes some other vertex of cP, then v lies in P and if $L := Lk(v, P)$, we have $Lk(v, cP) = cL$, which is the cone on a link in $\mathcal{L}_n$. This also shows, that the boundary of cP consists of P, and so P bounds in $\mathcal{L}$. We conclude $\Omega^{\mathcal{L}}_n(pt.) = 0$ and denote the corresponding theory by Ω^{ord}_*. Note that in the unoriented setting of the same class, the higher coefficient groups still vanish. But in difference to the oriented case, we have $\Omega^{\mathcal{L}}_0(pt.) = \mathbb{Z}/2$, since the disjoint union of two points is then the boundary of an interval. We denote the unoriented theory associated to $\mathcal{L}$ by $\Omega^{ord}_*(-; \mathbb{Z}/2)$. As the previous two generalized homology theories additionally satisfy the dimension axiom, these represent ordinary homology theory with $\mathbb{Z}$- and $\mathbb{Z}/2$-coefficients, respectively.*

Chapter 3

The basic sets $Q_i^{\overline{p}}$

In this chapter we will define and study the so called "basic sets", originally introduced by M. Goresky and R. McPherson in [4]. Given a stratified PL pseudomanifold X and a perversity $\overline{p}$, we construct subpolyhedra $Q_i^{\overline{p}}$ of X for every $i \geq 0$. They are designed to give a connection between ordinary homology groups of these basic sets and the intersection homology groups of the whole space X.
We begin with a brief introduction to PL intersection homology theory.

3.1 Intersection Homology

Definition 16. *A* ***0-dimensional PL stratified pseudomanifold*** *is a countable set of points with the discrete topology. An* ***n-dimensional PL stratified pseudomanifold*** *X is a PL space together with a filtration of closed PL subspaces*

$$X = X_n \supset X_{n-1} = X_{n-2} \supset ... \supset X_0 \supset X_{-1} = \emptyset$$

such that the following conditions are satisfied:

- *Every $X_{n-k} - X_{n-k-1}$ is a (possibly empty) PL manifold of dimension $n-k$.*
- *$X - X_{n-2}$ is dense in X.*
- ***Local normal triviality:*** *For every point $x \in X_{n-k} - X_{n-k-1}$ there exists an open neighborhood U of x in X and a compact PL stratified pseudomanifold L of dimension $k-1$ with filtration*

$$L = L_{k-1} \supset L_{k-3} \supset ... \supset L_0 \supset L_{-1} = \emptyset$$

and a PL isomorphism

$$\phi : U \to \mathbb{R}^{n-k} \times c^\circ L$$

(where c° denotes the open cone) which restricts to PL isomorphism $\phi_| : U \cap X_{n-l} \to \mathbb{R}^{n-k} \times c^\circ L_{k-l-1}$. We say that ϕ is ***stratum-preserving****.*

A closed subset X_{n-k} occuring in the filtration of X is called ***stratum*** *of codimension k. We call $X_{n-k} - X_{n-k-1}$ the* ***pure stratum*** *of* codimension *k.*

Definition 17. *Let X be a PL space and T be an admissible triangulation for X. A* ***simplicial i-chain*** *with respect to T is a function*

$$\xi : \{\sigma \in T | \sigma \text{ is an oriented } i\text{-simplex}\} \to \mathbb{Z}$$

with $\xi(-\sigma) = -\xi(\sigma)$, where $-\sigma$ denotes the simplex σ with opposite orientation. Let $C_i^T(X)$ denote the abelian group of all simplicial i-chains. Suppose $T' \lhd T$ is a subdivision. For $\xi \in C_i^T(X)$, we can assign a canonical element $\xi' \in C_i^{T'}(X)$ by

$$\xi'(\sigma') = \begin{cases} 0, & \text{if } \sigma' \text{ is not contained in an } i\text{- simplex of } T, \\ \xi(\sigma), & \text{if } \sigma' \text{ is contained in the } i\text{-simplex } \sigma \in T. \end{cases}$$

This yields a map

$$C_i^T(X) \to C_i^{T'}(X),$$

which we call the canonical map. Now, define

$$C_i(X) := colim\ C_i^T(X),$$

where the colimit ranges over all triangulations of the PL structure of X, with respect to the canonical maps. In other words, $C_i(X)$ consists of equivalence classes, represented by elements $\xi \in C_i^T(X)$, where ξ and $\xi' \in C_i^{T'}(X)$ are equivalent if there is a common admissible subdivision T'' of T and T', such that the images of ξ and ξ' under the canonical maps coincide in $C_i^{T''}(X)$. For any i, the simplicial boundary maps (see. [2], ch.2)

$$\partial_i^T : C_i^T(X) \to C_{i-1}^T(X)$$

give rise to boundary maps

$$\partial_i : C_i(X) \to C_{i-1}(X).$$

satisfying $\partial_i \circ \partial_{i-1} = 0$. *The associated homology groups*

$$H_i^{BM}(X) := H_i(C_*(X))$$

are called ***Borel-Moore homology*** *groups of* X.

Remark 2. *Note that we do not impose finiteness assumptions on the chains of the Borel-Moore complex, as it is done for the ordinary chain complex.*

Definition 18. *A* ***perversity*** *is a function* $\overline{p} : \mathbb{N}_{\geq 2} \to \mathbb{N}$ *such that:*

- $\overline{p}(2) = 0$,
- $\overline{p}(k) \leq \overline{p}(k+1) \leq \overline{p}(k) + 1$.

Given two perversities $\overline{p}, \overline{q}$, *we write* $\overline{p} \leq \overline{q}$ *if* $\overline{p}(k) \leq \overline{q}(k)$ *for every* k.

Example 4. *There are at least four important perversities:*

- *The* ***zero perversity*** $\overline{0}$, *defined as* $\overline{0}(k) = 0$ *for all* k.
- *The* ***lower-middle perversity*** $\overline{m} = \{0, 0, 1, 1, 2, 2, 3, 3, ...\}$
- *The* ***upper-middle perversity*** $\overline{n} = \{0, 1, 1, 2, 2, 3, 3, ...\}$
- *The* ***top perversity*** $\overline{t}$, *given by* $\overline{t}(k) = k - 2$ *for all* k.

Two perversities $\overline{p}, \overline{q}$ *are said to be* ***complementary*** *if* $\overline{p} + \overline{q} = \overline{t}$.

For the rest of this chapter, let X be a fixed PL stratified pseudomanifold of dimension n with strata X_{n-k} and let $\overline{p}$ be a perversity.

Definition 19. *A subspace* $Y \subset X$ *is said to be* $(\overline{p}, i)$***-allowable*** *if* $dim(Y) \leq i$ *and* $dim(Y \cap X_{n-k}) \leq i + (n-k) - n + \overline{p}(k) = i - k + \overline{p}(k)$, *for all* k.

Definition 20. *For a triangulation* T *of* X *and* $\xi \in C_i^T(X)$ *let* $|\xi|$ *denote the* ***support*** *of* ξ, *i.e. the union of those simplices* $\sigma \in T$ *with* $\xi(\sigma) \neq 0$. *Suppose* $T' \lhd T$ *is an admissible subdivision of* T *and* $\xi' \in C_i^T(X)$ *is the image of* ξ *under the canonical map. Then* $|\xi| = |\xi'|$, *and so any element* $\alpha \in C_i(X)$ *has a well-defined support* $|\alpha|$.

Definition 21. *Let $IC_i^{\overline{p}}(X)$ be the subgroup of $C_i(X)$ consisting of those chains ξ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable. Then the boundary maps of the complex $C_*(X)$ restrict to boundary maps*

$$\partial_i : IC_i^{\overline{p}}(X) \to IC_{i-1}^{\overline{p}}(X)$$

by the constraint on the boundaries of elements in $IC_i^{\overline{p}}(X)$. Therefore, we have a chain complex $(IC_*^{\overline{p}}(X), \partial_*)$, and we define

$$IH_i^{\overline{p}}(X) := H_i(IC_*^{\overline{p}}(X))$$

to be the i-th ***intersection homology*** *group of X, for perversity $\overline{p}$.*

3.2 Basic sets

For the PL stratified pseudomanifold X^n, we fix an admissible triangulation T if not otherwise stated. By T^1, we denote the first barycentric subdivision of T and the p-skeleton of T is denoted by $T_{(p)}$, as usual.

Definition 22. *For each $i \geq 0$ and fixed perversity $\overline{p}$ we define a function $L_i^{\overline{p}} : \{0, ..., n+1\} \to \mathbb{N}$ as follows:*

$$L_i^{\overline{p}}(0) = i, \quad L_i^{\overline{p}}(1) = i-1, \quad L_i^{\overline{p}}(n+1) = -1,$$

and for $2 \leq c \leq n$ we let

$$L_i^{\overline{p}}(c) = \begin{cases} -1 & \textit{if } i - c + p(c) \leq -1 \\ n - c & \textit{if } i - c + p(c) \geq n - c \\ i - c + p(c) & \textit{otherwise.} \end{cases}$$

Furthermore, let $\Delta_i^{\overline{p}}(c) = L_i^{\overline{p}}(c) - L_i^{\overline{p}}(c+1)$. Then the ***i-th basic set*** *$Q_i^{\overline{p}}$ of X with respect to T is the subcomplex of T^1, which is spanned by the following set of barycenters of simplices in T:*

$$\{\hat{\sigma} | \sigma \in T, \; \Delta_i^{\overline{p}}(n - dim(\sigma)) = 1\}$$

From now on we will consider basic sets $Q_i^{\overline{p}}$ with respect to a fixed triangulation T of X without further mention.

Remark 3. *1. Note that $L_i^{\overline{p}}(c)$ represents the largest possible dimension of intersection of any $(\overline{p}, i)$-allowable set with X_{n-c}.*

2. *Using the perversity restriction $\overline{p}(c) \leq \overline{p}(c+1) \leq \overline{p}(c)+1$, a simple case distinction shows that $\Delta_i^{\overline{p}}(c)$ is either 0 or 1.*

3. *A similar distinction shows that if $\Delta_i^{\overline{p}}(c) = 1$, then $\Delta_{i+1}^{\overline{p}}(c) = 1$. We conclude that $Q_i^{\overline{p}}$ is a subcomplex of $Q_{i+1}^{\overline{p}}$.*

Proposition 4. *For any k-simplex $\sigma \in T$, we have*

$$dim(Q_i^{\overline{p}} \cap \sigma) = L_i^{\overline{p}}(n-k).$$

Proof. If σ^1 denotes the first barycentric subdivision of σ, then $Q_i^{\overline{p}} \cap \sigma$ is a subcomplex of σ^1, spanned by barycenters of faces $\tau < \sigma$ with $\Delta_i^{\overline{p}}(n - dim(\tau)) = 1$. If $\tau_0 < \tau_1 < \ldots < \tau_{k-1} < \tau_k := \sigma$ is as sequence of faces, where each τ_j is a j-simplex, then under consideration of Rem.3.2. a top-dimensional simplex in $Q_i^{\overline{p}} \cap \sigma$ is spanned by

$$\begin{aligned}\sum_{j=0}^{k} \Delta_i^{\overline{p}}(n - dim(\tau_j)) = \sum_{j=0}^{k} \Delta_i^{\overline{p}}(n-j) &= L_i^{\overline{p}}(n-k) - L_i^{\overline{p}}(n+1) \\ &= L_i^{\overline{p}}(n-k) + 1\end{aligned}$$

vertices, and so $dim(Q_i^{\overline{p}} \cap \sigma) = L_i^{\overline{p}}(n-k)$. □

Corollary 1. *$Q_i^{\overline{p}}$ is of dimension i.*

Proof. For any n-simplex $\sigma \in T$, we have $dim(Q_i^{\overline{p}} \cap \sigma) = L_i^{\overline{p}}(0) = i$. □

Lemma 1. *For complementary perversities $\overline{p}$ and $\overline{q}$, the equation*

$$L_i^{\overline{p}}(c) + L_{n-i+1}^{\overline{q}}(c) = n - c - 1$$

holds for $2 \leq c \leq n+1$.

Proof. □

Proposition 5. *For $i \geq 1$ and complementary perversities $\overline{p}$ and $\overline{q}$ there are simplex-preserving deformation retractions*

$$\begin{aligned}X - (Q_{n-i+1}^{\overline{q}} \cap |T_{(n-2)}|) &\to Q_i^{\overline{p}}, \\ X - Q_{n-i+1}^{\overline{q}} &\to Q_i^{\overline{p}} \cap |T_{(n-2)}|.\end{aligned}$$

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