sheet1

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1 Sheet 1 - McCulloch–Pitts Neurons and Perceptron Learning

No AI tool has been used to produce this solution

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[2]:
```
import numpy as np

from numpy.typing import NDArray
from typing import Final, Any, TypeAlias

import itertools

import matplotlib.pyplot as plt
from matplotlib.axes import Axes
from collections.abc import Sequence
```

[3]:
```
plt.style.use("ggplot")
plt.rcParams["figure.dpi"] = 300

RNG: Final[np.random.Generator] = np.random.default_rng(42)
```

1.1 Exercise 1 - McCulloch–Pitts Neuron

[4]:
```
class Perceptron:

    def __init__(self, input_size: int, rng: np.random.Generator = RNG) -> None:
        k: np.float64 = np.sqrt(1 / input_size)
        self.w: NDArray[np.float64] = rng.uniform(-k, k, size=input_size)

    def __call__(self, x: NDArray[np.float64]) -> np.float64:
        return np.sign(self.w @ x)

    def fit(
        self, xs: NDArray[np.float64], ys: NDArray[np.float64], learning_rate:␣
↪float, max_epochs: int = 1000
    ) -> int:
        for epoch in range(1, max_epochs + 1):
```

```
            previous_weights: NDArray[np.float64] = self.w.copy()

            # Update weights
            for x, y in zip(xs, ys, strict=True):
                self.w += learning_rate * (y - self(x)) * x

            # Check for convergence
            if np.allclose(previous_weights, self.w):
                return epoch

        raise RuntimeError(f"Exceeded maximum number of epochs ({max_epochs!r})␣
 ↪without convergence.")
```

[5]:
```
perceptron: Perceptron = Perceptron(input_size=3)
perceptron.w = np.array([3, 2, 2], dtype=np.float64)

print("x1  x2   y")
for x1, x2 in itertools.product([-1, 1], repeat=2):
    prediction: np.float64 = perceptron(
        np.array([-1, x1, x2], dtype=np.float64)
    ).astype(np.int64)
    print(f"{x1:>2}  {x2:>2}  {prediction:>2}")
```

```
x1  x2   y
-1  -1  -1
-1   1  -1
 1  -1  -1
 1   1   1
```

The truth table above is equalivalent to an AND operation.

2 Exercise 2 - Activation Functions

[6]:
```
def sigmoid(x: NDArray[np.float64], a: np.float64 | float) -> NDArray[np.
 ↪float64]:
    return 2 / (1 + np.exp(-a * x))
```

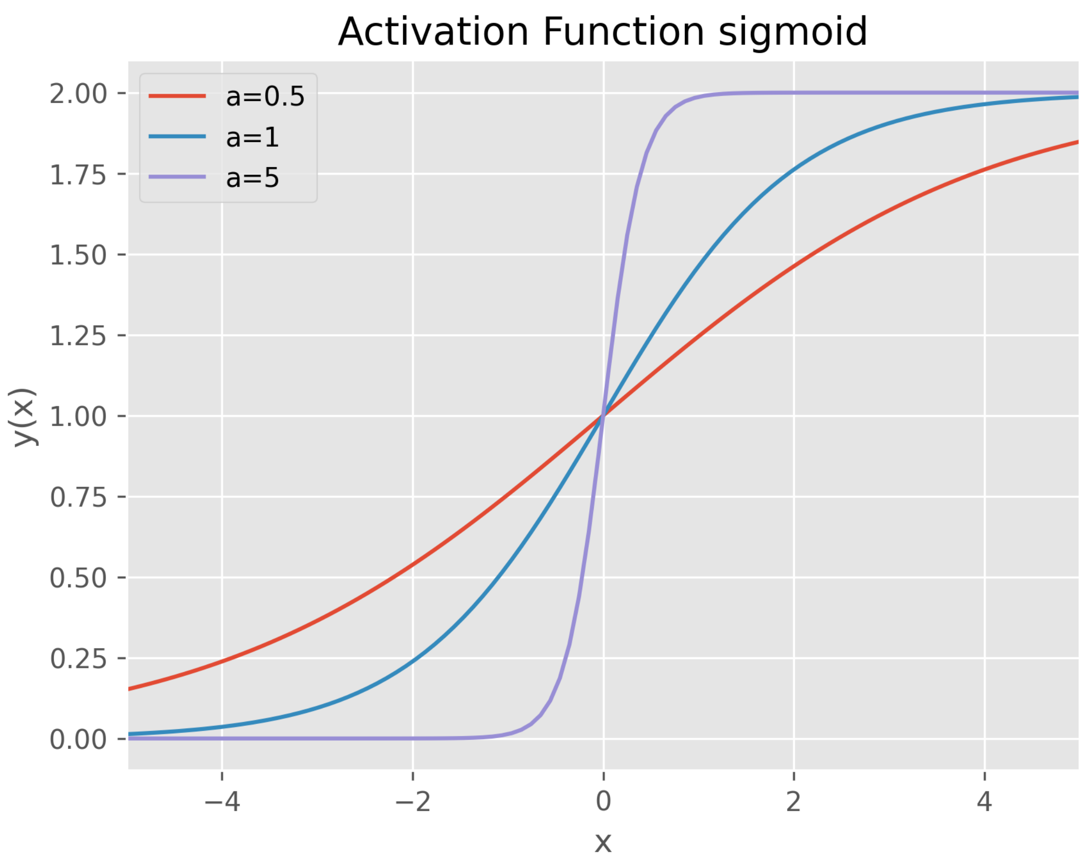
[7]:
```
def tanh(x: NDArray[np.float64], a: np.float64 | float) -> NDArray[np.float64]:
    return np.tanh(a * x)
```

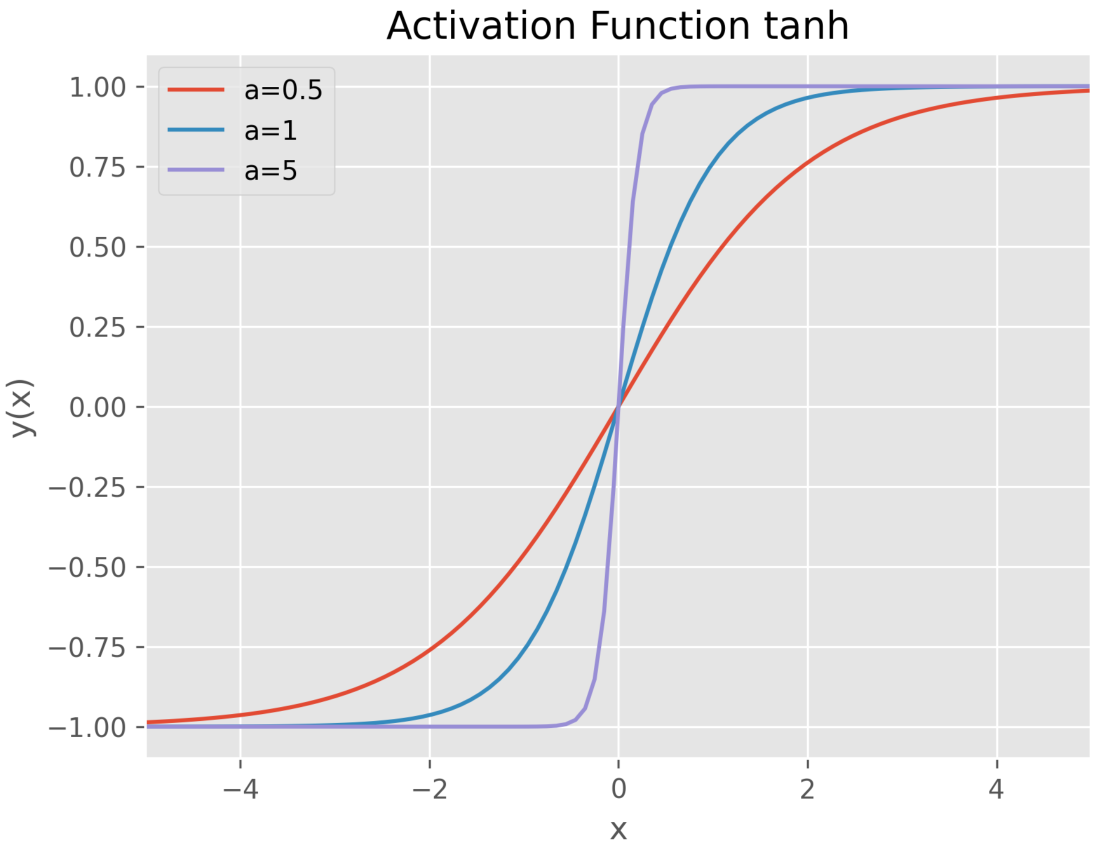
[8]:
```
def piecewise_linear(
    x: NDArray[np.float64], a: np.float64 | float
) -> NDArray[np.float64]:
    return np.piecewise(
        x,
        [x >= 1 / a, np.logical_and(-1 / a < x, x < 1 / a), x <= -1 / a],
        [1, lambda x: a * x, -1],
```

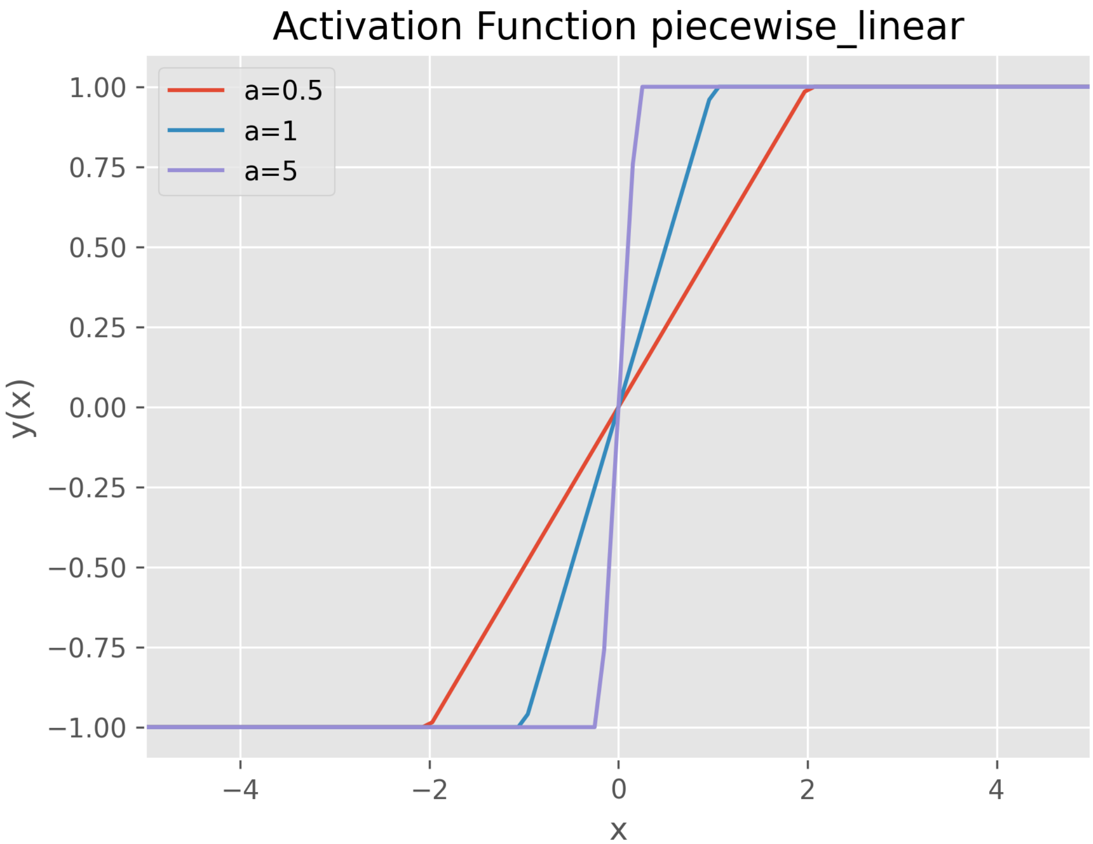
```
    )
```

[9]:
```
def plot_activation(activation: Any, a_choices: tuple[int, ...] = (0.5, 1, 5))␣
 ↪-> None:
    x: NDArray[np.float64] = np.linspace(-5, 5, 100, dtype=np.float64)

    _, ax = plt.subplots()
    for a in a_choices:
        ax.plot(x, activation(x, a), label=f"{a=}")

    ax.legend(loc="upper left")
    ax.set_xlabel("x")
    ax.set_ylabel("y(x)")

    ax.set_title(f"Activation Function {activation.__name__}")

    ax.margins(x=0)
```

[10]:
```
plot_activation(sigmoid)
plot_activation(tanh)
plot_activation(piecewise_linear)
```

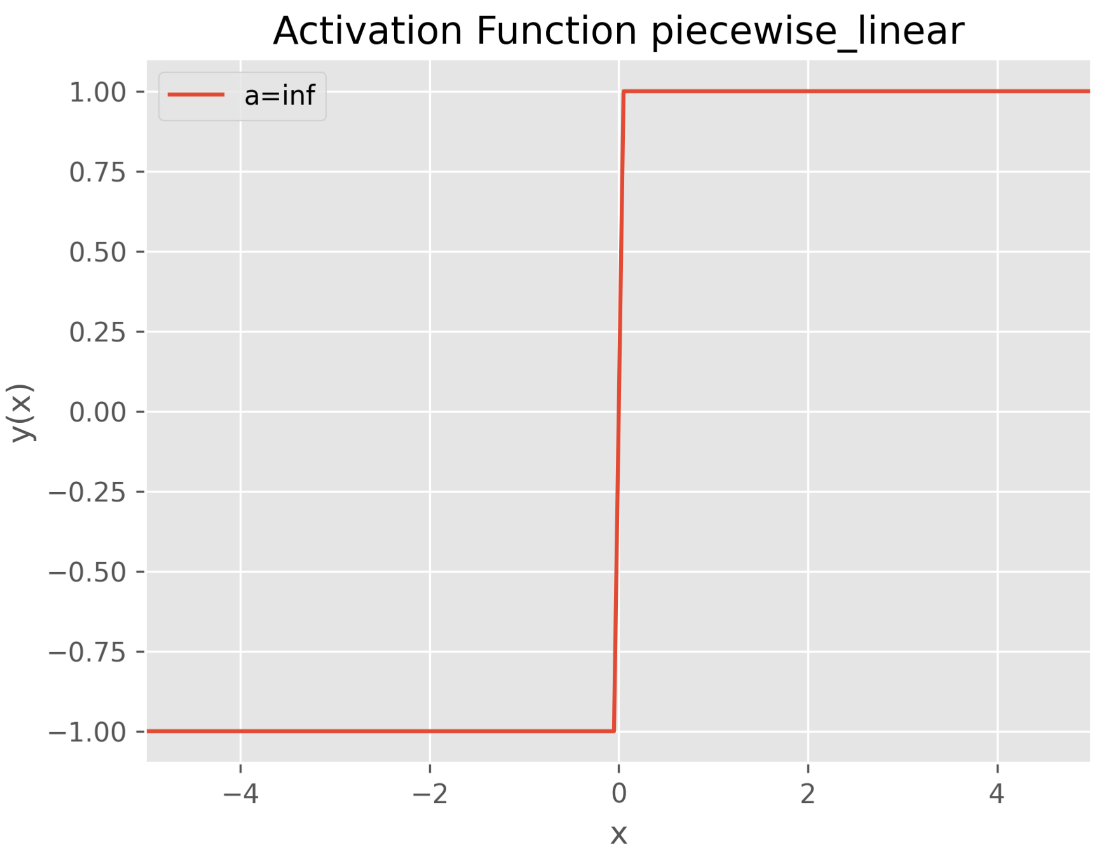



Activation Function tanh
a=0.5
a=1
a=5
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y(x)
−4
−2
0
2
4
x



To approach the Heaviside function we choose any of the activation functions above and let a approach inifinity.

[11]:
```
plot_activation(piecewise_linear, a_choices=(np.inf,))
```



3 Exercise 3 - Rosenblatt's Perceptron

[12]:
```python
Dataset: TypeAlias = tuple[NDArray[np.float64], NDArray[np.float64]]

def generate_dataset(n: int, rng: np.random.Generator = RNG) -> Dataset:
    xs: NDArray[np.float64] = np.concatenate(
        (
            np.full(shape=(n, 1), fill_value=-1.0, dtype=np.float64),
            rng.normal(size=(n, 2)),
        ),
        axis=1,
    )

    difference: NDArray[np.float64] = xs[:, 2] - xs[:, 1]
    ys: NDArray[np.float64] = np.piecewise(
        difference,
        [difference >= 0.5, difference < 0.5],
        [1, -1],
    )
```

```python
    return xs, ys
```

[13]:
```python
training_dataset: Dataset = generate_dataset(n=1000)
perceptron = Perceptron(input_size=3)
convergence_epoch: int = perceptron.fit(*training_dataset, learning_rate=0.1)
print(f"Convergerd after {convergence_epoch} epochs")
```

```
Convergerd after 60 epochs
```

We use a learning rate $0 < \eta \ll 1$ to ensure a stable learning process until convergence. A smaller (but appropriate) learning rate ensures the update rule does not overshoot an optimum. Nevertheless, in this simple task, a larger learning rate, e.g. 5, leads to convergence as well. The learning rate still needs to be positive since a negative learning rate would reverse the direction of the weight updates, thus increasing the training error rather than decreasing it.

It is not biologically plausible.

[14]:
```python
validation_dataset = generate_dataset(n=1000)

all_correct: bool = True
for x, y_true in zip(*validation_dataset):
    prediction: np.float64 = perceptron(x)

    if prediction != y_true:
        all_correct = False
        print(
            f"Prediction mismatch: x={x}, y_true={y_true.item()}, 
 ↪prediction={prediction.item()}"
        )

if all_correct:
    print("All inputs have been classified correctly.")
```

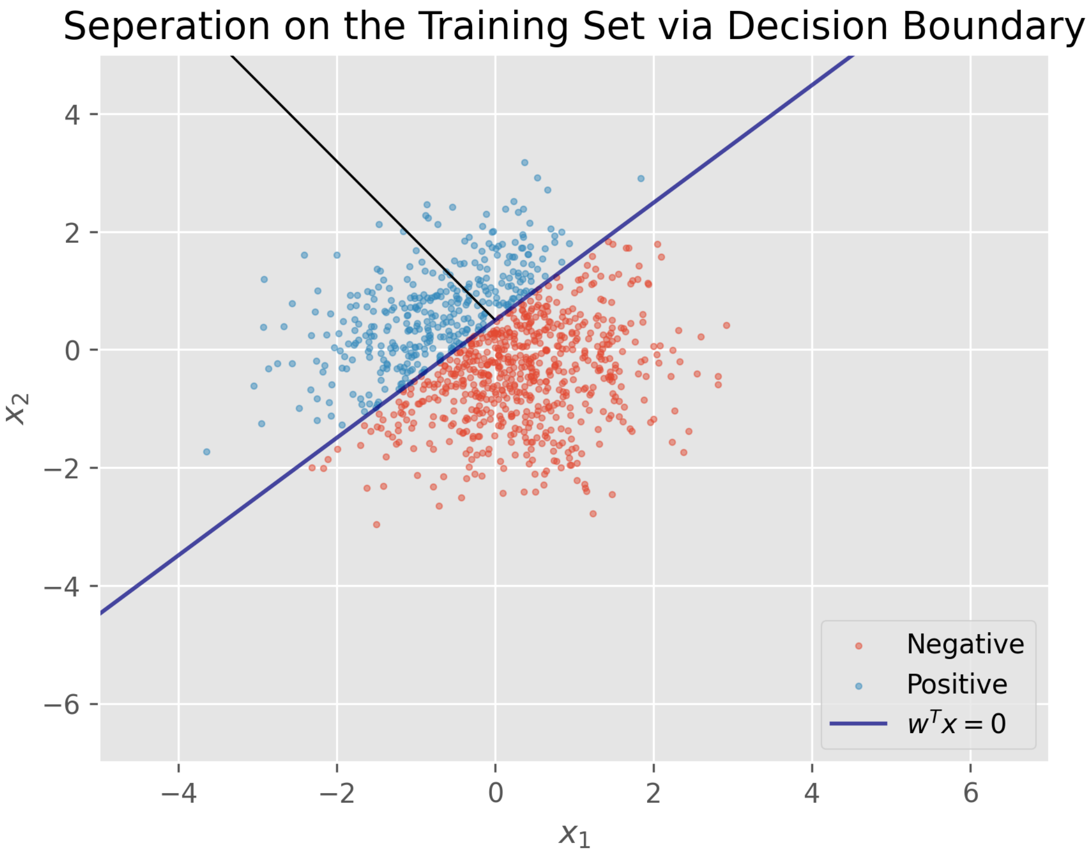
```
Prediction mismatch: x=[-1.         -0.07347866  0.42612395], y_true=-1.0,
prediction=1.0
```

[15]:
```python
def plot_data_points_2d(ax: Axes, xs: NDArray[np.float64], ys: NDArray[np.
 ↪float64], **kwargs: Any) -> None:
    assert xs.shape[1] == 2, "xs must be 2D data"

    x1s: NDArray[np.float64] = xs[:, 0]
    x2s: NDArray[np.float64] = xs[:, 1]

    ax.scatter(x1s[ys == -1], x2s[ys == -1], label="Negative", **kwargs)
    ax.scatter(x1s[ys ==1], x2s[ys == 1], label="Positive", **kwargs)

    ax.set_xlabel("$x_{1}$")
    ax.set_ylabel("$x_{2}$")
```

[16]:
```
def plot_decision_boundary_2d(ax: Axes, w: NDArray[np.float64]) -> None:
    assert w.shape == (3, )

    intercept = (0, w[0] / w[2])
    ax.axline(
        xy1=intercept,
        slope=-w[1] / w[2],
        label="$w^{T}x=0$",
        color="navy",
        alpha=0.7,
    )

    ax.quiver(
        *intercept,
        *w[1:],
        # angles="xy",
        scale_units="xy",
        scale=1,
        width=0.0025,
    )
```

[17]:
```
_, ax = plt.subplots()

plot_data_points_2d(ax, xs=training_dataset[0][:,1:], ys=training_dataset[1],␣
 ↪s=5, alpha=0.5)
plot_decision_boundary_2d(ax, w=perceptron.w)

ax.set_xlim(-5, 7)
ax.set_ylim(-7, 5)

ax.set_title("Seperation on the Training Set via Decision Boundary")

ax.legend(loc="lower right")
```

[17]:
```
<matplotlib.legend.Legend at 0x7f12acb2bac0>
```



The weight vector w is near-optimal in the sense of its direction. Nevertheless, it is suboptimal in the sense of its magnitude since there is no normalization constraint enforced during or after the training. Therefore, the weight vector's magnitude gets increasingly large.

The function is a generalization of the logical OR operation.

3.1 Exercise 4 - Linear Separability

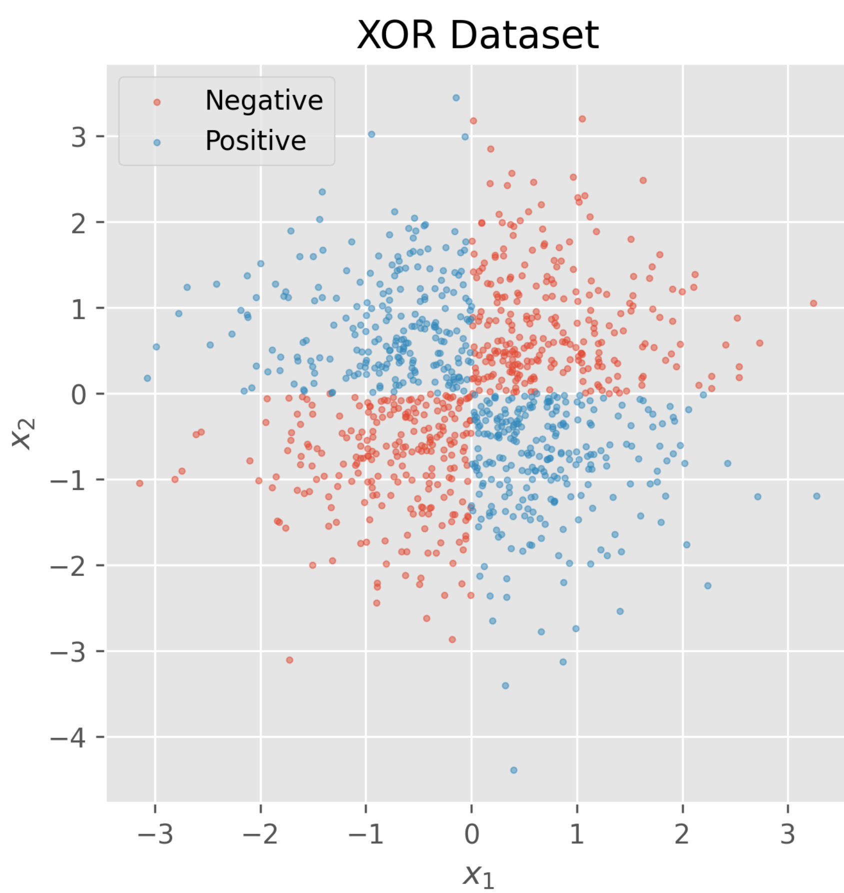
[18]:
```
def generate_xor_dataset(n: int, rng: np.random.Generator = RNG) -> Dataset:
    xs: NDArray[np.float64] = np.concatenate(
        (
            np.full(shape=(n, 1), fill_value=-1.0, dtype=np.float64),
            rng.normal(size=(n, 2)),
        ),
        axis=1,
    )

    ys: NDArray[np.float64] = ((xs[:, 1] > 0) != (xs[:, 2] > 0)).astype(np.
↪float64)
    ys[ys == 0] = -1
```

```
    return xs, ys
```

[19]:
```
soft_xor_xs, soft_xor_ys = generate_xor_dataset(n=1000)
```

[20]:
```
_, ax = plt.subplots(figsize=(5,5))

plot_data_points_2d(ax, xs=soft_xor_xs[:, 1:], ys=soft_xor_ys, s=5, alpha=0.5)
ax.set_title("XOR Dataset")
ax.legend(loc="upper left")
```

[20]: <matplotlib.legend.Legend at 0x7f12b018bdf0>



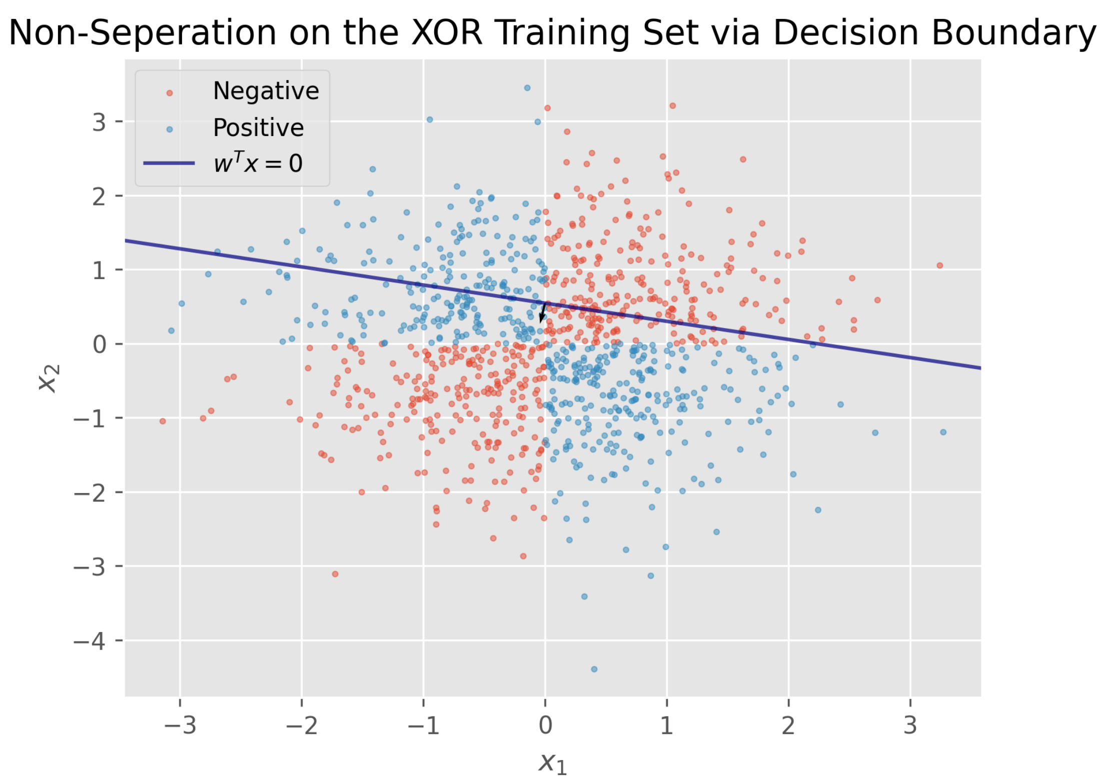
[21]:
```
xor_perceptron = Perceptron(input_size=3)

max_epochs: int = 1000
try:
    convergence_epoch: int = xor_perceptron.fit(xs=soft_xor_xs, ys=soft_xor_ys,␣
 ↪learning_rate=0.1, max_epochs=max_epochs)
```

```
    print(f"Convergerd after {convergence_epoch} epochs")
except RuntimeError:
    print(f"No convergence after {max_epochs} epochs")
```

```
No convergence after 1000 epochs
```

[22]:
```
_, ax = plt.subplots()

plot_data_points_2d(ax, xs=soft_xor_xs[:,1:], ys=soft_xor_ys, s=5, alpha=0.5)
plot_decision_boundary_2d(ax, w=xor_perceptron.w)

ax.set_title("Non-Seperation on the XOR Training Set via Decision Boundary")
ax.legend(loc="upper left")
```

[22]:
```
<matplotlib.legend.Legend at 0x7f12acabecb0>
```



Since XOR is not separable by a linear function, a perceptron that can only fit linear decision boundaries can not learn the XOR function. The coordinates of the weights keep changing the sign. Thus, they can't converge. (Printing the entire weight sequence would lead to too much output.)

BONUS 14 of 16 operations could be learned by perceptron, since they form (piece-wise) linearly separable data, i.e. could be transformed to linearly separate data. Only two operations that do not form linearly separable data are XOR and equivalence (XNOR).

```
[ ]:
```