Spacefilling Curves and the Planar Travelling Salesman Problem

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Abstract. To construct a short tour through points in the plane, the points are sequenced as they appear along a spacefilling curve. This heuristic consists essentially of sorting, so it is easily coded and requires only $O(N)$ memory and $O(N \log N)$ operations. Its performance is competitive with that of other fast methods.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*routing and layout*; *sequencing and scheduling*

General Terms: Algorithms, Performance, Verification

Additional Key Words and Phrases: Fractal geometry, spacefilling curves

1. *Introduction*

The travelling salesman problem (TSP) is to construct a circuit of minimum total length that visits each of N given points. Even in the plane, this problem is NP-complete [10], so that instances of practical interest cannot be solved *exactly* in reasonable time. Accordingly, attention has focused on fast algorithms that generate good, but not necessarily optimal, tours.

The authors have introduced an appealingly simple and practical solution to this problem. Its primary virtues are easy implementation and fast execution with acceptable accuracy [1], and it has been used successfully in a variety of practical applications, including a commercial routing system that requires no computer [2, 3]. Based on the spacefilling curve ψ, a continuous mapping from the unit interval $C = [0, 1]$ *onto* the unit square $S = [0, 1]^2$, it is performed as follows:

SPACEFILLING HEURISTIC.
(1) For each point $p \in S$ to be visited, compute a $\theta \in C$ such that $p = \psi(\theta)$.
(2) Sort the points by their corresponding θ's.

In other words, this heuristic visits points in the order of their appearance along the spacefilling curve.

The Spacefilling Heuristic is appealing because it is so simple. We show here that it also performs well, and is competitive with other methods. We establish

The authors have been supported by the Office of Naval Research under Contract No. N00014-80-k-0709, and by the National Science Foundation under grant ECS 83-51313.

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worst-case bounds on the heuristic tour length and the ratio of heuristic to optimal tour lengths (Theorems 4.1 and 4.2), and almost sure bounds (for increasingly large random point sets) on the heuristic tour length (Theorem 5.3) and the distribution of points along the path of the heuristic tour (Theorem 5.4).

The Spacefilling Heuristic is of special interest because it is based on spacefilling curves. Originally devised as topological counterexamples nearly a century ago, these were long regarded as "mathematical monstrosities." It is only recently that their usefulness has been recognized [9]. Our work represents the first application of fractal geometry to combinatorial optimization. Standard combinatorial arguments are inappropriate because they rely on the combinatorial structure of the problem; spacefilling curves ignore this structure. Our analysis utilizes techniques not often encountered in combinatorics, including measure theory and a nonlinear metric.

To streamline this presentation, we consider only a single curve in the plane. Following [2], our methods can be generalized to the TSP in d-space, and to more general combinatorial problems, such as matching and clustering.

The paper is organized as follows: The spacefilling curve is defined in Section 2. The computational effort required by the algorithm is determined in Section 3. The worst-case and probabilistic analyses are given in Sections 4 and 5, respectively. Section 6 discusses the Spacefilling Heuristic's performance relative to that of competing methods.

Table I summarizes our performance analysis of the Spacefilling Heuristic. For comparison, the performances of comparable methods, cited by Bentley [5] as particularly simple, are also included. These are:

Nearest Neighbor (NN). Start at an arbitrary point and successively visit the nearest unvisited point. After all points have been visited, return to the start.

Minimum Spanning Tree (MST). Construct the minimum spanning tree of the point set and duplicate all the links of the tree. Sequence the points as they would appear in a traversal of the doubled tree. Pass through the sequence and remove all representations after the first of each point.

Strip. Partition the square into $\sqrt{N/3}$ vertical strips. Sequence the points in each strip by vertical position, alternately top-to-bottom and bottom-to-top, and visit the strips from left to right. Then return to the starting point.

Using parameters given in Section 6, the length of Spacefilling Heuristic tours over large statistically independent point sets can be accurately predicted. For points uniformly distributed in the square, our informal tests indicate similar (within 15%) average tour lengths obtained by all four methods. For an extensive computational comparison of these (and other) heuristics, see Johnson et al. [7].

2. *The Spacefilling Curve*

The particular "curve" to be considered here is constructed by successively partitioning S as shown in Figure 1. The kth partition of S consists of 2^k identical triangles, each labeled with the binary representation of an integer in the range $0, \ldots, 2^k - 1$. One vertex of each triangle is marked to distinguish it from the others. The interval C is similarly partitioned, as shown in Figure 2, into subintervals, each labeled by a string of binary digits, and marked at one endpoint.

Each successive partition is obtained by bisecting every triangle [subinterval] in the previous partition. A new label is created by right-appending an additional

TABLE I. COMPARISON OF SIMPLE TSP HEURISTICS[a]

	NN	MST	Strip	Spacefilling
Ease of coding	Good	Poor	Good	Good
Memory	$O(N)$	$O(N)$	$O(N)$	$O(N)$
Parallelizability	Poor	Poor	Good	Good
Worst-case effort				
To solve	$O(N^2)$*	$O(N^2 \log N)$*	$O(N \log N)$	$O(N \log N)$
To modify	Re-solve	Re-solve*	$O(\log N)$	$O(\log N)$
Longest tour	$2.15\sqrt{N}$	$3.04\sqrt{N}$	$2.31\sqrt{N}$	$2\sqrt{N}$
Worst-case ratio				
Bound	$O(\log N)$	2	$O(N)$	$O(\log N)$
Known	$O\left(\frac{\log N}{\log\log N}\right)$	2	$O(N)$	$O(\log N)$
Performance on nonuniform data	Good	Good	Poor	Good

[a] To make as clean a comparison as possible, we consider only the most straightforward implementations and ignore enhancements such as sophisticated data structures or subroutines designed to mitigate pathological behavior. An asterisk indicates that the entry may be reduced, but only at considerable programming expense.

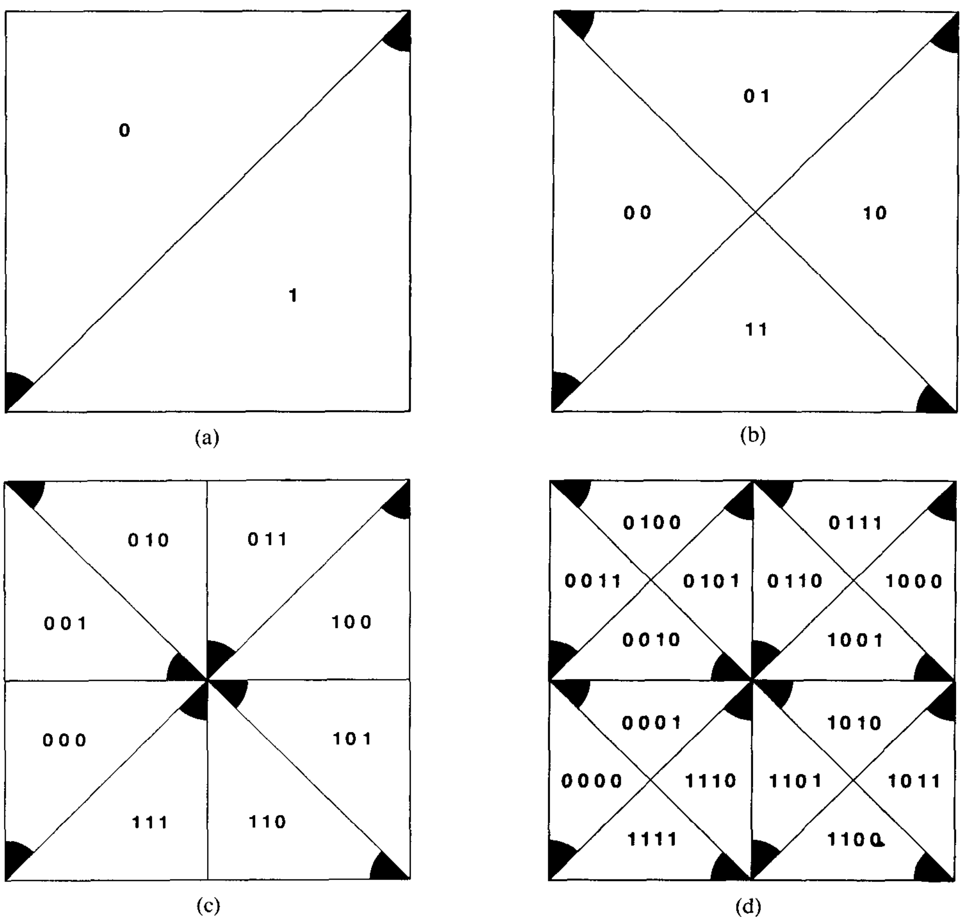

FIG. 1. Successive partitions of the unit square S into 2^k identical triangles, each containing a k-digit binary label and a marked vertex. (a) $k = 1$. (b) $k = 2$. (c) $k = 3$. (d) $k = 4$.

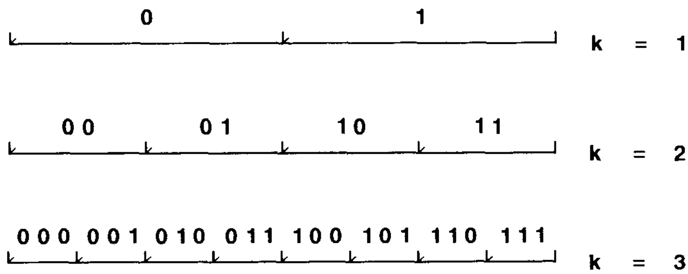

FIG. 2. Successive partitions of the unit interval C into 2^k identical subintervals. The label contains the first k binary digits of numbers in the subinterval's interior. The marked endpoint is always to the left.

digit to the parent's label; the half containing the parent's marked vertex [endpoint] receives a zero, and the other half receives a one.

The spacefilling curve ψ maps each subinterval of C *onto* the subtriangle of S bearing an identical label, and it maps the marked endpoint of the subinterval to the marked vertex of the triangle. The marked vertices are therefore visited in a sequence determined by their labels, as shown in Figure 3.

If θ has a finite k-digit binary expansion, then it is a marked endpoint in the kth partition of C, and so is mapped to its associated vertex, whose coordinates each have $k/2$-digit expansions. Thus, the spacefilling curve can be thought of as *encoding* the two coordinates' digits into a single number. This idea is pursued in Section 3, where we show that the effort required to invert ψ is linear in the size of the input data.

Since $\psi(\frac{1}{16}) = \psi(\frac{3}{16}) = (0, \frac{1}{2})$ (see Figure 3, $k = 4$), ψ is not one-to-one. Therefore, there may exist more than one acceptable θ to select in Step 1 of the Spacefilling Heuristic. For clarity of presentation, we assume that this θ is selected according to some mapping $\lambda: S \to C$, and require only that $\psi(\lambda(p)) = p$ for all $p \in S$. The nonuniqueness of λ will turn out to be of no theoretical or practical importance. It occurs because any point p lying on a boundary of the partition shown in Figure 1 is contained by more than one triangle; each triangle contains the initial binary digits of a possible value for $\lambda(p)$. There may be up to eight such triangles, corresponding to two alternative binary encodings of each of four actual values. For example, $(\frac{1}{2}, \frac{1}{2})$ is contained in eight triangles (see Figure 1, $k = 3$); $\lambda(\frac{1}{2}, \frac{1}{2})$ may be $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, or $\frac{7}{8}$, for which there are eight possible encodings: 0.001, 0.011, 0.101, 0.111, 0.000111$\cdots$, 0.010111$\cdots$, 0.100111$\cdots$, and 0.110111$\cdots$.

Our performance analysis of the Spacefilling Heuristic is based on two fundamental properties of ψ and λ, which are stated in the following lemmas. Their proofs, along with a formal definition of ψ, are given in Appendix A.

The first Lemma formalizes an observation of [9, p. 65] that a spacefilling curve preserves nearness: points close together in C map (via ψ) onto points close together in S. We take the measure of nearness on the square S to be Euclidean distance, denoted by $D[\cdot, \cdot]$. Since we consider C to be a circuit with $\psi(0) = \psi(1)$, the natural metric on C is

$$\Delta[\theta, \theta'] = \min\{|\theta - \theta'|, 1 - |\theta - \theta'|\}.$$

Using these metrics, we can now state

LEMMA 2.1. *$D[\psi(\theta), \psi(\theta')] \leq 2\sqrt{\Delta[\theta, \theta']}$ for all $\theta, \theta' \in C$.*

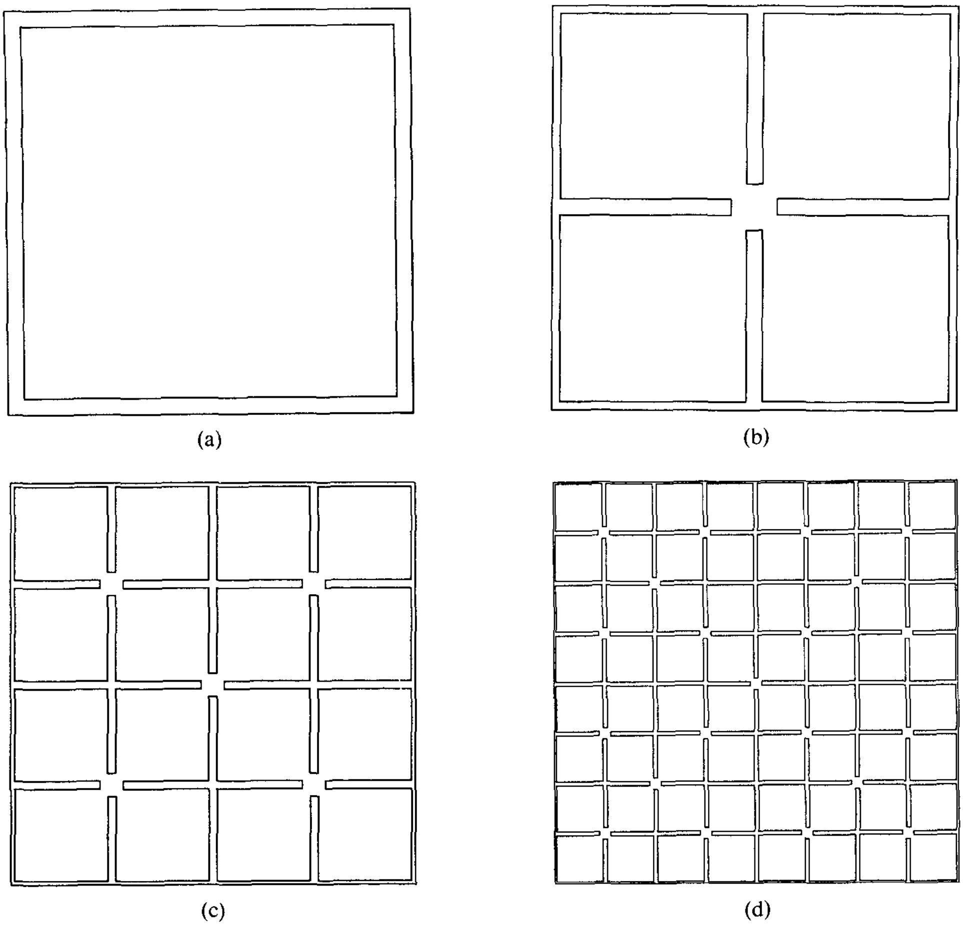

FIG. 3. The conventional representation of an approximation to a spacefilling curve shows a path through the marked vertices of the triangles of Figure 1, sequenced according to their labels. (a) $k = 2$. (b) $k = 4$. (c) $k = 6$. (d) $k = 8$.

The second property is that a spacefilling curve is homogeneous: subcurves of equal length "fill" regions of equal area. That is, for any interval I in C area$\{\psi(I)\}$ = length$\{I\}$. This is required in the worst-case analysis, and will permit us to determine the distribution of $\lambda(P)$ when P is a random variable. It is formally stated as:

LEMMA 2.2. *The spacefilling curve ψ and its "inverse" λ are Borel-measurable, Lebesgue measure-preserving transformations, and $\lambda(\psi(\theta)) = \theta$ almost everywhere in C.*

3. *Computational Effort*

Suppose at first that the basic unit of computation is an arithmetic operation upon a real number of fixed precision.

To perform the first step of the Spacefilling Heuristic, one must compute $\lambda(p)$ for each given point p. This can be accomplished iteratively by identifying, at the kth iteration, a triangle of the kth partition containing p. This triangle's label contains the first k digits of the binary expansion of the result, $\lambda(p)$. To perform the subsequent iteration, bisect the triangle and decide which half contains p. Each iteration requires a fixed number of arithmetic operations; the total number of iterations is the number of digits to which $\lambda(p)$ is calculated; and λ must be evaluated once for each of N given points. The first step of the Spacefilling Heuristic, therefore, requires $O(N)$ arithmetic operations, and is dominated by the second step, sorting. Consequently, we have

PROPOSITION 3.1. *The spacefilling heuristic requires $O(N \log N)$ arithmetic operations.*

Remark. If a sorting procedure such as BINSORT [8] is used, then the heuristic can be performed in linear expected time.

Since the heuristic tour is simply a sorted list, every subsequence of the heuristic tour is itself a heuristic tour. Thus, if the set of points to be visited changes slightly, the current solution need be modified only locally to produce a new heuristic tour. This observation has important practical consequences in applications in which routes must be updated frequently [3]. It is formally stated as

PROPOSITION 3.2. *Inserting a point into or deleting a point from an N-point heuristic tour requires only $O(\log N)$ arithmetic operations.*

Finally, since a value θ is required for each of N points, we have

PROPOSITION 3.3. *The Spacefilling Heuristic requires $O(N)$ memory registers.*

We now consider bit operations and bits of input data rather than arithmetic operations and real registers. If the coordinates of p are specified to n binary digits (i.e., if each coordinate is an integer multiple of 2^{-n}), then p is the marked vertex of some triangle of partition $k = 2n + 2$ (see Figure 1), and so $\lambda(p)$ can be expressed *exactly* using k bits (i.e., it is an integer multiple of 2^{-k}). The memory requirement, therefore, remains linear in the input data size. However, the procedure for Step 1 described at the start of this section no longer runs in linear time, since each arithmetic operation requires at least $O(n)$ bit operations, and the number of arithmetic operations is linear in the input data size. To overcome this difficulty, we must streamline Step 1.

Note that all triangles in the various partitions decompose similarly into subtriangles. It follows that the digits of $\lambda(p)$ yet to be computed are entirely determined by the position of p relative to the triangle whose label is the string of digits already determined. Thus, one can replace p and the triangle containing it by an alternative point occupying a similar position within an alternative triangle, without affecting the remaining digits. The streamlined procedure exploits this idea by repositioning p to the appropriate point in the upper-left half triangle of S on odd-numbered iterations, or the left quarter triangle of S on even-numbered iterations. These two triangles will be bisected by examining one bit only.

More specifically, the first digit of $\lambda((x, y))$ is 0 if $x \leq y$, or 1 if $x \geq y$. (If $x = y$, either digit may be selected.) Moreover, if the first digit is 0, then (x, y) is already located in the upper-left half triangle of S. If the first digit is 1, perform the substitution $(x, y) \leftarrow (1 - x, 1 - y)$. Now, (x, y) occupies a position in the upper-left half triangle corresponding to the one it previously occupied in the lower right half, and the remaining digits of λ are unaffected. When bisecting the upper-left half triangle in S, if (x, y) lies in the upper-right half, perform the substitution $(x, y) \leftarrow (1 - y, x)$ to move to the lower-left half, the left quarter triangle in S.

Finally, when bisecting the left quarter triangle in S, specify digit 0 and substitute $(x, y) \leftarrow (2x, 2y)$ if $y \le \frac{1}{2}$, or specify digit 1 and substitute $(x, y) \leftarrow (2y - 1, 2x)$ if $y \ge \frac{1}{2}$, to transform the position in either subtriangle to a relative position in the upper left half of S. Repeat until a sufficient number of digits have been obtained.

A simple realization of this idea is outlined in Table II. The procedure arguments are integers X and Y in the range 0 to M, inclusive, representing the given coordinates of a point in a square of size M. The result is constructed in an integer variable I, whose binary expansion is the K-digit label of a triangle containing the original point. At each iteration, K is incremented, and an additional digit is appended to the label by doubling, and possibly incrementing, I. Note that no multiplications or divisions are required.

If M is a power of 2, and one maintains variables $U = X + Y$ and $V = M + X - Y$ in addition to X and Y, each bisection can be performed by testing a single bit of Y or V. Each pair of substitutions removes the most-significant bit from each of X, Y, U, and V. When all bits have been processed, the procedure is complete. Its computational effort (in bit operations) is therefore linear in the data size. Step 2 can be performed in a linear number of bit operations using RADIXSORT.

4. *Deterministic Analysis*

We now consider how long the heuristic tour can be in the worst-case. Throughout this section, L and L^* will denote the heuristic and the optimal tour lengths, respectively, corresponding to an arbitrary set Π of N points in S.

THEOREM 4.1. $L \le 2\sqrt{N}$

PROOF. Let $\theta_{[1]}, \ldots, \theta_{[N]}$ be a sorted list generated by the heuristic, and set $\Delta_i = \theta_{[i+1]} - \theta_{[i]}$, $i = 1, \ldots, N-1$, and $\Delta_N = 1 + \theta_{[1]} - \theta_{[N]}$. By Lemma 2.1,

$$L \le \sum_{i=1}^{N} 2\sqrt{\Delta_i} \equiv G(\Delta_1 \cdots \Delta_N).$$

But $\Delta_i \ge 0$ and $\sum_{i=1}^{N} \Delta_i = 1$, so the symmetric concave function G achieves its minimum at $\Delta_i = 1/N$. □

Although Theorem 4.1 guarantees that the heuristic tour cannot be very long, the optimal tour might be considerably shorter. Theorem 4.2 establishes an asymptotic upper bound of $O(\log N)$ on their ratio. Although we had long conjectured that this bound could be further reduced to a constant, we recently learned that Bertsimas and Grigni [6] have constructed a counterexample for which our bound is tight.

THEOREM 4.2. $L/L^* = O(\log N)$.

PROOF. Let $\#\{\cdot\}$ denote cardinality, and $1\{\cdot\}$ the indicator function. If w_k is the length of the kth link along the tour, and

$$H(t) = \#\{k: w_k \ge t\},$$

then

$$L = \sum w_k = \sum \int_0^\infty 1\{w_k > t\}\, dt = \int_0^\infty H(t)\, dt. \tag{4.1}$$

TABLE II. OUTLINE OF AN ALGORITHM TO COMPUTE POSITIONS ALONG THE SPACEFILLING CURVE

CONSTANT M, KMAX	M is the size of the square. KMAX is the number of binary digits to compute.
Function $L(X, Y)$	X and Y are integers in the range 0 to M, the coordinates of a point in the $M \times M$ square.
Integer I, K, Z	I is where we build the result to be returned.
$I = 0$	K is the number of bits computed so far.
$K = 1$	(Z is a temporary variable, used for swapping X and Y.)
if $X > Y$ **then**	Adjust if in lower right half of S.
$I = 1$	
$X = M - X$	
$Y = M - Y$	
end if	
while $K <$ KMAX **do**	
$I = I + I$	Bisect the upper left half triangle in S and determine one more bit of I.
$K = K + 1$	
if $X + Y > M$ **then**	Adjust if in upper right half of the triangle.
$I = I + 1$	
$Z = M - Y$	
$Y = X$	
$X = Z$	
end if	
if $K <$ KMAX **then**	
$I = I + I$	Bisect the left quarter triangle in S and determine one more bit of I.
$K = K + 1$	
$X = X + X$	Expand the triangle so its lower half overlaps the upper left half triangle in S.
$Y = Y + Y$	
if $Y > M$ **then**	Adjust if in upper half of the triangle.
$I = I + 1$	
$Z = Y - M$	
$Y = M - X$	
$X = Z$	
end if	
end if	
end while	
return(I)	I contains the position of the original X, Y along the spacefilling curve, expressed as a KMAX-bit integer.
end function	

We show that there is a number Ω (independent of Π) such that

$$H(t) \le \frac{\Omega L^*}{t}, \qquad \text{for all} \quad t \in [0, L^*]. \tag{4.2}$$

Since $H(t) \le N$ for all t and $H(t) = 0$ for $t > L^*$, (4.1) and (4.2) combine to give the desired result,

$$\frac{L}{L^*} \le \int_0^{L^*} \min\left(N, \frac{\Omega L^*}{t}\right) \frac{dt}{L^*} \le \Omega \int_0^1 \min\left(N, \frac{1}{a}\right) da = O(\log N).$$

To obtain (4.2), consider the set $\Pi(t)$ of points in S that lie within distance t of some point in Π. Also, let Π' denote a collection of $\lceil L^*/t \rceil$ points placed at equal intervals along the path of the optimal tour, so that each point in Π lies within distance t of some point in Π'. Now each point in $\Pi(t)$ must lie within distance

$2t$ of a point in Π', so

$$\text{area}\{\Pi(t)\} \leq \left\lceil \frac{L^*}{t} \right\rceil \pi (2t)^2 \leq 8\pi t L^*, \; t \in [0, L^*]. \tag{4.3}$$

Finally, let $[\theta_k = \lambda(p_k), k = 1, \ldots, N]$ be the sorted list from which the heuristic tour is constructed, and consider the intervals

$$C_k(t) = \left\{ \theta \bmod 1 : \theta_k < \theta < \theta_k + \frac{t^2}{4} \right\}.$$

By Lemma 2.1, if $w_k \geq t$, then $C_k(t) \cap C_{k'} = \varnothing$, $k \neq k'$. Thus, there are at least $H(t)$ mutually disjoint intervals $C_k(t)$. Moreover, by Lemma 2.1, $\psi(C_k(t)) \subseteq \Pi(t)$, and by Lemma 2.2, $\psi(C_k(t))$ has area $t^2/4$. It follows from Lemma 2.2 that

$$\text{area}\{\Pi(t)\} \geq H(t) \frac{t^2}{4}. \tag{4.4}$$

The claim (4.2) follows immediately from (4.3) and (4.4). □

5. *Probabilistic Analysis*

Let $\{P_i\}$ be an infinite sequence of statistically independent, identically distributed points in S having absolutely continuous distribution $\Pr\{p \in A\} = \int\int_A f(x, y)\, dy\, dx$.

(We have excluded singular point distributions because the distributions of their corresponding θ's cannot be easily described. Consequently, we assume throughout this section that point coordinates are infinite-precision real numbers, and that their corresponding θ's are computed *exactly*. The outcome of the probabilistic analysis is a statement about ideal computation that, nonetheless, has useful implications for actual computation.) From Lemma 2.2, we obtain

PROPOSITION 5.1. *The random variables $\Theta_i = \lambda(P_i)$ are statistically independent with density $g(\theta) = f(\psi(\theta))$.*

Now let L_N $[L_N^*]$ denote the (random) length of the heuristic [optimal] tour over $\{P_1, \ldots, P_N\}$, respectively. In a classic work, Beardwood et al. [4] showed that $L_N^*/K(f)\sqrt{N} \to \beta^*$ (almost surely), where β^* is a constant and

$$K(f) = \int\int_S \sqrt{f(x, y)}\, dy\, dx. \tag{5.1}$$

We seek similar asymptotic bounds on the heuristic tour lengths L_N. Unfortunately, the most natural statement of convergence, $L_N/K(f)\sqrt{N} \to \beta$, does not hold, even for uniformly distributed points. We prove instead a result whose practical implications are the same: that $L_N/K(f)\sqrt{N}$ "converges" to a narrow interval $[\beta^-, \beta^+]$, where

$$\beta^- = \liminf_{N\to\infty} E\left\{\frac{L_N}{\sqrt{N}}\right\} \quad \text{and} \quad \beta^+ = \limsup_{N\to\infty} E\left\{\frac{L_N}{\sqrt{N}}\right\}, \tag{5.2}$$

the expectations in (5.2) being taken for uniformly distributed points.

More specifically, Steele's analysis of optimal tours [11, 12] can be adapted, according to the outline in Appendix B, to establish

PROPOSITION 5.2

(*a*) *For uniformly distributed points*

$$\liminf_{N\to\infty} \frac{L_N}{\sqrt{N}} = \beta^- \text{ and } \limsup_{N\to\infty} \frac{L_N}{\sqrt{N}} = \beta^+ \qquad (\text{almost surely}).$$

(*b*) *For points distributed according to any density* f

$$\liminf_{N\to\infty} \frac{L_N}{K(f)\sqrt{N}} \geq \beta^- \text{ and } \limsup_{N\to\infty} \frac{L_N}{K(f)\sqrt{N}} \leq \beta^+ \qquad (\text{almost surely}).$$

Proposition 5.2 states that the limit points of the random sequence $L_N/K(f)\sqrt{N}$ may span the interval $[\beta^-, \beta^+]$. We show that this interval is narrow, but of positive width.

THEOREM 5.3. $\beta^+ - \beta^- < 2 \times 10^{-4}$ *and* $\beta^- \neq \beta^+$.

PROOF. Since β^- and β^+ are defined by the *deterministic* sequence (5.2), the analysis can be restricted to expectations over uniformly distributed point sets. Let $m(t)$ denote the expected Euclidean distance between two uniformly distributed points in S whose corresponding θ's lie exactly distance t apart, that is,

$$m(t) = \int_0^1 D[\psi(\theta), \psi(\theta + t \bmod 1)]\, d\theta,$$

and define

$$r(t) = \lim_{k\to\infty} \frac{m(2^{-k}t)}{\sqrt{2^{-k}t}}, \qquad \rho(z) = r(2^{-z}). \tag{5.3}$$

By Lemma 2.1,

$$\rho(z) \leq 2. \tag{5.4}$$

The recursive structure of ψ suggests that $m(t/2) \sim m(t)/\sqrt{2}$. This is used in Lemma C1 to demonstrate the existence of the limit (5.3). Clearly, $\rho(z) = \rho(z \bmod 1)$. Although $\rho(z)$ is periodic, it need not be *constant*.

Further, define $g(s) = s^{3/2}e^{-s}$ and $\gamma(z) = \sum_{k=-\infty}^{\infty} \ln(2) g(2^{k+z})$. Clearly, $\gamma(z) = \gamma(z \bmod 1)$. Furthermore, γ displays very little variation:

$$\gamma^- = \min_z \gamma(z) \simeq 0.88620, \qquad \gamma^+ = \max_z \gamma(z) \simeq 0.88626. \tag{5.5}$$

Denote by $a * b$ the cyclic convolution of functions on $[0, 1]$:

$$[a * b](z) = \int_0^1 a((z - z')\bmod 1) b(z')\, dz'. \tag{5.6}$$

We show that $E\{L_N\}/\sqrt{N}$ is asymptotic to $[\rho * \gamma](\log_2 N)$. It then follows that

$$\beta^- = \min_z \{[\rho * \gamma](z)\}, \qquad \beta^+ = \max_z \{[\rho * \gamma](z)\}. \tag{5.7}$$

By (5.3)–(5.7)

$$\beta^+ - \beta^- \le \max_z |\rho(z)|[\gamma^+ - \gamma^-] < 2 \times 10^{-4}.$$

Moreover, if $\rho^*_m = \int_0^1 \exp(2\pi mjz)\rho(z)\,dz$ and $\gamma^*_m = \int_0^1 \exp(2\pi mjz)\gamma(z)\,dz$ are the Fourier series coefficients for ρ and γ, respectively, then $\beta^- = \beta^+$ iff $\rho^*_m\gamma^*_m = 0$ for all $m > 0$. Direct numerical evaluation yields $\rho^*_1 = (0.00007, -0.0017) \neq 0$ and $\gamma^*_1 = (0.000013, -0.0000073) \neq 0$, so $[\rho * \gamma]$ contains at least one harmonic term, and thus $\beta^- \neq \beta^+$.

It remains to show that $E\{L_N\}/\sqrt{N}$ is asymptotic to $[\rho * \gamma](\log_2 N)$. Let $\Theta_i = \lambda(P_i)$, $i = 1, \ldots, N$, be the (unsorted) list obtained in Step 1 of the algorithm, and let $\Delta_i = \min_{j\neq i}\{(\theta_j - \theta_i) \bmod 1\}$ be random variables representing the distance between the ith (unsorted) θ and its successor along the heuristic tour. By symmetry, the pairs (Θ_i, Δ_i) are identically distributed. Moreover, each Θ_i is uniformly distributed (by Proposition 5.1), and for each i, Θ_i and Δ_i are pairwise independent (since the Θ_i are mutually independent), so

$$\begin{aligned}\frac{E\{L_N\}}{\sqrt{N}} &= \sum_{i=1}^{N} \frac{E\{D[\psi(\Theta_i), \psi(\Theta_i + \Delta_i \bmod 1)]\}}{\sqrt{N}} \\ &= \sqrt{N}E\{D[\psi(\Theta_1), \psi((\Theta_1 + \Delta_1) \bmod 1)]\} \\ &= \sqrt{N}E\{m(\Delta_1)\}. \end{aligned} \tag{5.8}$$

For large N, $\{\Theta_i\}$ will approximate a Poisson process on C, and the distribution of Δ_1 will approach a negative exponential of mean $1/N$. Moreover, since Δ_1 is small when N is large, the limiting form (5.3) is substituted into (5.8), to obtain

$$\lim_{N\to\infty} \sqrt{N}\left\{E\{m(\Delta_1)\} - \int_0^\infty r(t)\sqrt{t}N e^{-Nt}\,dt\right\} = 0. \tag{5.9}$$

This is formally demonstrated in Lemma C2. Finally, for $a = 1/N$, $z = \log_2(Nt)$,

$$\begin{aligned}\sqrt{N}\int_0^\infty r(t)\sqrt{t}N e^{-Nt}\,dt &= \int_0^\infty r(t)g(Nt)\,\frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} \int_{2^k a}^{2^{k+1}a} r(t)g(Nt)\,\frac{dt}{t} \\ &= \int_a^{2a} r(t) \sum_{k=-\infty}^{\infty} g(2^k Nt)\,\frac{dt}{t} \\ &= \int_0^1 \rho((\log_2 N) - z)\gamma(z)\,dz = [\rho * \gamma](\log_2 N). \end{aligned} \tag{5.10}$$

Combining (5.8)–(5.10) yields

$$\lim_{N\to\infty}\left\{\frac{E\{L_N\}}{\sqrt{N}} - [\rho * \gamma](\log_2 N)\right\} = 0. \qquad \square$$

We now show that points are smoothly distributed along the path of the heuristic tour. This makes the tours obtained by the Spacefilling Heuristic particularly well suited to applications such as multivehicle routing, where the grand tour must be divided into sections of nearly equal length that contain nearly equal numbers of points [3]. Let $p^* \in S$ be a fixed reference point (the "central depot"), set $\theta^* = \lambda(p^*)$, and define

$$F(z) = \int_{\theta^*}^{\theta^*+z} f(\psi(\theta \bmod 1))\, d\theta,$$

$$G(z) = K(f)^{-1} \int_{\theta^*}^{\theta^*+z} \sqrt{f(\psi(\theta \bmod 1))}\, d\theta,$$

$$H(\alpha) = G(F^{-1}(\alpha)),$$

$$H^-(\alpha) = \left(\frac{\beta^-}{\beta^+}\right) H(\alpha), \qquad H^+(\alpha) = \left(\frac{\beta^+}{\beta^-}\right) H(\alpha).$$

H is well defined because F and G are nondecreasing and $F(z) = F(z')$ whenever $G(z) = G(z')$. Note that $H(\alpha) = \alpha$ when $f \equiv 1$, that is, when the points P_i are uniformly distributed on S. Since $\beta^- \simeq \beta^+$, it follows that $H^- \simeq H \simeq H^+$.

THEOREM 5.4. *Let $L_N(\alpha)$ be the length of a path that follows the heuristic tour of $\{p^*, P_1, \ldots, P_n\}$ from p^*, in the direction of increasing θ, until $\lfloor \alpha N \rfloor$ points have been visited. Then, for any $0 \le \alpha \le 1$,*

$$\liminf_{N\to\infty} \frac{L_N(\alpha)}{L_N} \ge H^-(\alpha) \quad \text{and} \quad \limsup_{N\to\infty} \frac{L_N(\alpha)}{L_N} \le H^+(\alpha) \quad (\text{almost surely}).$$

PROOF. Define

$$\lambda^*(p) = ((\lambda(p) - \theta^*) \bmod 1), \qquad S(z) = \{p : \lambda^*(p) \le z\},$$
$$\Pi_N(z) = \{p^*, P_1, \ldots, P_N\} \cap S(z), \qquad z_N = \min\{z : \#\{\Pi_N(z)\} \ge \lfloor \alpha N \rfloor\},$$

and let $\tilde{L}_N(z)$ denote the length of the heuristic tour through $\Pi_N(z)$. The first $\lfloor \alpha N \rfloor$ points in the tour of $\{p^*, P_1, \ldots, P_N\}$ form a set that coincides with $\Pi_N(z_N)$. So $\tilde{L}_N(z_N)$ and $L_N(\alpha)$ differ by the length of a single arc:

$$|\tilde{L}_N(z_N) - L_N(\alpha)| \le \operatorname{diam}\{S\} = \sqrt{2}. \tag{5.11}$$

Further define $\Phi(p) = \Pr\{\lambda^*(P_i) \le \lambda^*(p)\}$. By Proposition 5.1, $\Phi(p) = F(\lambda^*(p))$. Since F is nondecreasing, a heuristic tour starting at p^* can be constructed by sorting points from smallest to largest $\Phi(p)$. Moreover, each $\Phi(P_i)$ is uniformly distributed on $[0, 1)$. Since $F(z_N) = \max\{\Phi(p) : p \in \Pi_N(z_N)\}$ is the $\lfloor \alpha N \rfloor$th largest of N such uniform random variables,

$$\lim_{N\to\infty} F(z_N) = \alpha \qquad (\text{almost surely}). \tag{5.12}$$

Finally, define

$$K_N(z) = \min\{i : \#\{\Pi_i(z)\} \ge \lfloor \alpha N \rfloor\}, \qquad f_z(p) = \begin{cases} \dfrac{f(p)}{F(z)}, & \text{if } p \in S(z), \\ 0, & \text{otherwise,} \end{cases}$$

and let z^* be any solution of $F(z^*) = \alpha$. Now

$$\Pi_{K_N(z_N)}(z_N) = \Pi_N(z_N), \tag{5.13}$$

and, in addition to p^*, $\Pi_{K_N(z)}(z)$ contains $\lfloor \alpha N \rfloor - 1$ independent points of density f_z. By Proposition 5.2,

$$\liminf_{N\to\infty} \frac{\tilde{L}_{K_N(z)}(z)}{\sqrt{N}} \ge \beta^- K(f_z)\sqrt{\alpha},$$

and

$$\limsup_{N\to\infty} \frac{\tilde{L}_{K_N(z)}(z)}{\sqrt{N}} \le \beta^+ K(f_z)\sqrt{\alpha} \qquad \text{(almost surely)}.$$

But

$$K(f_z) = K(f)\,\frac{G(z)}{\sqrt{F(z)}} = K(f)\,\frac{H(F(z))}{\sqrt{F(z)}},$$

so,

$$\liminf_{N\to\infty} \frac{\tilde{L}_{K_N(z)}(z)}{\sqrt{N}} \ge \beta^- K(f) H(F(z))\sqrt{\alpha/F(z)},$$

and

$$\limsup_{N\to\infty} \frac{\tilde{L}_{K_N(z)}(z)}{\sqrt{N}} \le \beta^+ K(f) H(F(z))\sqrt{\alpha/F(z)}, \qquad \text{(almost surely)}.$$

By (5.12) and the monotonicity of $\tilde{L}_{\cdot}(\cdot)$ and $K_N(\cdot)$,

$$\liminf_{N\to\infty} \frac{\tilde{L}_{K_N(z_N)}(z_N)}{\sqrt{N}} \ge \liminf_{N\to\infty} \frac{\tilde{L}_{K_N(z^*)}(z^*)}{\sqrt{N}} \ge \beta^- K(f) H(\alpha),$$

and

$$\limsup_{N\to\infty} \frac{\tilde{L}_{K_N(z_N)}(z_N)}{\sqrt{N}} \le \limsup_{N\to\infty} \frac{\tilde{L}_{K_N(z^*)}(z^*)}{\sqrt{N}} \le \beta^+ K(f) H(\alpha) \qquad \text{(almost surely)}.$$

By (5.11) and (5.13), then

$$\liminf_{N\to\infty} \frac{L_N(\alpha)}{\sqrt{N}} \ge \beta^- K(f) H(\alpha),$$

and

$$\limsup_{N\to\infty} \frac{L_N(\alpha)}{\sqrt{N}} \le \beta^+ K(f) H(\alpha) \qquad \text{(almost surely)}.$$

Proposition 5.2 supplies corresponding bounds for $L_N/\sqrt{N}$. □

6. *Comparison with Competing Heuristics*

We have estimated that β^- and β^+ lie in the range 0.956 ± 10^{-3}. Thus, for uniformly distributed point sets, the heuristic tour is asymptotic to $0.956\sqrt{N}$. But as the distribution becomes clustered, both the heuristic and optimal tours will decrease in length. Indeed, the ratio of heuristic to optimal tour lengths approaches the narrow range $[\beta^-/\beta^*, \beta^+/\beta^*]$ for statistically independent point sets following *any* distribution. Using the familiar estimate $\beta^* \cong 0.765$, the Spacefilling Heuristic would achieve tours 25% longer than optimal. More recent calculations [7] suggest, however, that β^* is somewhat lower. The heuristic tours could therefore be as much as 35% longer than optimal, asymptotically.

When the distribution is known, we can use β^- and β^+ (according to Proposition 5.2) to predict the heuristic tour length without actually constructing the tour. To illustrate this, consider the family of truncated normal densities $\nu_\sigma(x, y) = c\exp\{-((x - 0.5)^2 + (y - 0.5)^2)/2\sigma^2\}$ on S, where c is chosen so that $\int\int_S \nu_\sigma\, dy\, dx = 1$. When σ is large, this distribution is nearly uniform over S. As σ decreases, the points become more concentrated at the center of the square, and the optimal and heuristic tour lengths decrease (Table III).

TABLE III. TOUR LENGTHS FOR A TRUNCATED NORMAL DISTRIBUTION IN S[a]

σ	Radius	$K(f)$	Heuristic tour ($\sqrt{N}$)
$\gg 1$	0.382	1.000	0.956
1	0.379	1.000	0.956
0.5	0.368	0.999	0.955
0.2	0.301	0.956	0.914
0.1	0.177	0.692	0.662
$\ll 0.1$	1.78σ	7.09σ	6.78σ

[a] The radius (mean distance from a point to the center of S) shows the points becoming more concentrated as σ decreases.

Of the heuristics listed in Table I, the most familiar (and the easiest to code) is NN. But NN and MST require $O(N^2)$ effort. This is an order of magnitude slower than Strip and Spacefilling, which are essentially sorting routines and can be performed in linear expected time.

The longest tours produced by Strip and Spacefilling are about the same. For large, independent, uniformly distributed point sets, they produce tours of similar average length: Our experiments show $0.922\sqrt{N}$ for Strip vs. $0.956\sqrt{N}$ for Spacefilling. (These numbers correspond to the basic heuristics described in Section 1; either can be implemented with additional steps to improve performance.)

The worst-case ratio of heuristic to optimal tour lengths is considerably worse for Strip than it is for Spacefilling. Indeed, Strip performs poorly on clustered data [5]. To formalize this claim, consider the rectilinear length of a Strip tour of N points generated from a continuous, strictly positive density f. As N increases, the highest point in each strip will approach the top of the strip, and the lowest point in each strip will approach the bottom, a distance of one unit away. There are $\sqrt{N/3}$ strips, so the total vertical distance approaches $\sqrt{N/3}$. Moreover, each point point is nearly uniformly distributed horizontally over the width of its strip. So the average horizontal interpoint distance is (strip width)/3 = $\sqrt{1/3N}$. There are N points, so the total horizontal distance also approaches $\sqrt{N/3}$. Rectilinear distance is at most $\sqrt{2}\,\cdot$ (Euclidean distance), so

$$\liminf_{N\to\infty} \frac{E\{\text{Strip tour length}\}}{\sqrt{N}} \geq \sqrt{2/3} \simeq 0.816.$$

This compares unfavorably with the corresponding limits for Spacefilling Heuristic tours (Table III), which become arbitrarily small as the point distribution becomes more clustered, and which never exceed 0.956, for *any* distribution.

Thus, the Spacefilling Heuristic performs at least comparably with other "fast" TSP heuristics, for each of the measures listed in Table I, and it outperforms each of its competitors in some important way. It deserves serious consideration when fast, moderately accurate TSP solutions are required.

Appendix A Formal Development of the Spacefilling Curve ψ

To begin, we formalize the construction illustrated in Figure 1. The marked vertex of the triangle labeled by the k-digit binary expansion of i will be denoted by $p_k(i)$.

From Figure 1, we have

$$\begin{aligned} &p_1(0) = p_1(2) = p_2(0) = p_2(4) = (0, 0),\\ &p_1(1) = p_2(2) = (1, 1),\\ &p_2(1) = (0, 1), \qquad p_2(3) = (1, 0), \end{aligned}$$

and we recursively construct the remaining marked vertices according to

$$p_k(i) = \left\{ \begin{array}{ll} p_{k-1}\left(\dfrac{i}{2}\right), & \text{if } \ i \text{ is even,} \\ \dfrac{p_{k-2}(\lfloor i/4 \rfloor + p_{k-2}(\lceil i/4 \rceil)}{2}, & \text{if } \ i \text{ is odd,} \end{array} \right\} \quad k = 3, 4, \ldots.$$

Note that since $p_k(i) = p_{k+1}(2i)$,

$$p_k(i) = p_{k+k'}(2^{k'} i). \tag{A1}$$

The ith triangle of the kth partition is now defined by

$$T_k(i) = \operatorname{conv}\{p_k(i), p_{k+1}(2i + 1), p_k(i + 1)\}. \tag{A2}$$

Likewise, the marked endpoints in Figure 2 are given by $\theta_k(i) = i2^{-k}$, so

$$\theta_k(i) = \theta_{k+k'}(2^{k'} i). \tag{A3}$$

The subintervals take the form $I_k(i) = [\theta_k(i), \theta_k(i + 1)]$.

Proposition A1. $D^2[p_k(i), p_k(i')] \le 4 \,|\, \theta_k(i) - \theta_k(i')\,|$.

Proof. Assume without loss of generality that $i' > i$. If $i' - i = 1$, then, by (A2), $p_k(i)$ and $p_k(i')$ are acute vertices of the isosceles right triangle $T_k(i)$, and since this triangle has area 2^{-k},

$$D^2[p_k(i), p_k(i')] = 4 \cdot 2^{-k} = 4(\theta_k(i') - \theta_k(i)).$$

We now proceed by induction on $i' - i$. Select k' so that i and $(i' - 1)$ correspond to k-digit labels whose initial k' digits match, but whose $(k' + 1)$th digits differ. Since $i' - i > 1$, it follows that $k' < k$. Let j' be an index of the k'th partition whose corresponding label contains the initial k' digits of i and $(i' - 1)$, and let j be an index of the kth partition whose initial k' digits match those in i, $(i' - 1)$ or j, followed by a single one and filled out by zeroes. Now $A = p_k(i)$ and $B = p_k(i')$ are both contained in $T_{k'}(j')$, but $p_k(i)$ is in $T_{k'+1}(2j')$, whereas $p_k(i')$ is in $T_{k'+1}(2j' + 1)$. Since $T_{k'}(j')$ was bisected at its right vertex $C = p_{k'+1}(2j' + 1) = p_k(j)$ to form these subtriangles, it follows that the angle ACB cannot exceed 90°. Therefore

$$\overline{AB}^2 \le \overline{AC}^2 + \overline{CB}^2.$$

But $\overline{AB}^2$ is $D^2[p_k(i), p_k(i')]$, the left-hand side of the inequality we seek to establish. The construction of j guarantees that $i < j < i'$, so $j - i$ and $i' - j$ are both less than $i' - i$, and, by induction,

$$\begin{aligned} AC^2 + CB^2 &= D^2[\theta_k(i), \theta_k(j)] + D^2[\theta_k(j), \theta_k(i')]\\ &\le 4[\theta_k(j) - \theta_k(i)] + 4[\theta_k(i') - \theta_k(j)]\\ &= 4[\theta_k(i') - \theta_k(i)]. \end{aligned}$$

□

COROLLARY A2. $D^2[p_k(i), p_{k'}(i')] \le 4\,|\theta_k(i) - \theta_{k'}(i')|$.

PROOF. The substitution $(k, i, k', i') \to (k + k', i2^{k'}, k + k', i'2^k)$, justified by (A1) and (A3), converts this claim to the form required by Proposition A1. □

Now define countable sets $\hat{S} = \{p_k(i)\}$ and $\hat{C} = \{\theta_k(i)\}$. There is a $\hat{\psi}: \hat{C} \to \hat{S}$ mapping $\theta_k(i)$ to $p_k(i)$ since, by (A1) and (A3),

$$\theta_k(i) = \theta_{k'}(i') \Rightarrow i2^{k'} = i'2^k \Rightarrow p_k(i) = p_{k'}(i').$$

Note that $\hat{S}$ is dense in S and $\hat{C}$ is dense in C. As a consequence of Corollary A2, $\hat{\psi}$ can be extended in the usual way to form $\psi: C \to S$. Since $\hat{\psi}$ is onto $\hat{S}$, ψ is onto S. Moreover,

$$\psi(I_k(i)) = T_k(i), \tag{A4}$$

since, by construction, $\theta_{k'}(i') \in I_k(i) \Rightarrow p_{k'}(i') \in T_k(i)$, and so $\hat{\psi}(I_k(i) \cap \hat{C}) = T_k(i) \cap \hat{S}$, from which (A4) follows directly.

Since ψ is continuous and onto, it is a spacefilling curve. Lemma 2.1 is a direct consequence of Corollary A2. We now prove Lemma 2.2.

Since ψ is continuous (Lemma 2.1), it is measurable. Define measurable sets

$$S^1 = S - \bigcup_k \bigcup_{i \ne j} (T_k(i) \cap T_k(j)),$$
$$C^1 = \psi^{-1}(S^1),$$

and let ψ^1 $[\lambda^1]$ denote the restriction of ψ $[\lambda]$ to domain C^1 $[S^1]$. Note that $S - S^1$ is a set of measure zero. By (A4)

$$T_k(i) \cap S^1 \subseteq \lambda^{-1}(I_k(i)) \subseteq T_k(i),$$

so $\lambda^{-1}(I_k(i))$ is a measurable set, and $\text{area}\{\lambda^{-1}(I_k(i))\} = 2^{-k} = \text{length}\{I_k(i)\}$. Thus, λ is both measurable and measure-preserving. But ψ^1 is one-to-one, since $\psi(\theta) = \psi(\theta')$ with $\theta \ne \theta'$ implies the existence of k, $i \ne i'$ such that $\theta \in I_k(i)$ and $\theta' \in I_k(i')$, and, by (A4), $\psi(\theta) = \psi(\theta') \in T_k(i) \cap T_k(i') \subset S - S^1$. Consequently, λ^1 is the inverse of ψ^1, and $\lambda^{-1}(C^1) = S^1$. Since λ is measure-preserving, $C - C^1$ is also a set of measure zero. Thus, λ is almost everywhere the (uniquely determined) inverse of ψ.

Appendix B. Outline of the Proof of Lemma 5.2

For $H = [0, \frac{1}{2}]$ and $\theta_1, \ldots, \theta_N \in H$, let $W^-(\theta_1, \ldots, \theta_N)$ be the length of the open path that visits the points $\psi(\theta_i)$ from smallest θ to largest θ, and $W^+(\theta_1, \ldots, \theta_N) = W^-(0, \theta_1, \ldots, \theta_N, \frac{1}{2})$. For independent uniformly distributed random variables $\{\hat{C}_i\}$ in H, let $\Omega_N^- = W^-(\hat{C}_1, \ldots, \hat{C}_N)$, $\Omega_N^+ = W^+(\hat{C}_1, \ldots, \hat{C}_N)$, and

$$B^- = \liminf_{N\to\infty} E\left\{\frac{\Omega_N^-}{\sqrt{N}}\right\}, \qquad B^+ = \limsup_{N\to\infty} E\left\{\frac{\Omega_N^+}{\sqrt{N}}\right\}.$$

The analysis in [11] is easily adapted to show that $(\Omega_N^- - E\{\Omega_N^-\})/\sqrt{N} \to 0$ and $(\Omega_N^+ - E\{\Omega_N^+\})/\sqrt{N} \to 0$ (almost surely). Thus

$$\liminf_{N\to\infty} \frac{\Omega_N^-}{\sqrt{N}} = B^- \quad \text{and} \quad \limsup_{N\to\infty} \frac{\Omega_N^+}{\sqrt{N}} = B^+ \quad \text{(almost surely)}.$$

From this, it is easy to demonstrate part (a).

Next, consider subtours of points uniformly distributed in any triangle $T_k(i)$. The lengths of these tours, scaled by the triangle size $2^{-(k-1)/2}$, do not depend on i, k. In this way, the proof of Lemma 3.2 of [12] may be adapted to show that (b) holds whenever there is a $k \geq 0$ such that f is piecewise constant over the interior of each triangle $T_k(i)$, $i = 0, \ldots, 2^k - 1$.

Finally, the proof of Theorem 2 of [12] may be adapted to show that (b) holds for any f.

Appendix C. Two Lemmas Required in the Proof of Theorem 5.3

The existence of r and ρ in (5.3) is justified by

LEMMA C1. *There is a continuous function r on* $(0, \infty)$ *such that*

(*a*) $$\left|\frac{m(t)}{\sqrt{t}} - r(t)\right| \leq 4t, \quad t \leq \frac{1}{4}.$$

(*b*) $$r\left(\frac{t}{2}\right) = r(t).$$

PROOF. By Lemma 2.2,

$$m(t) \leq 2\sqrt{t},$$

and by the triangle inequality

$$m(t + \epsilon) \leq m(t) + m(\epsilon) \leq m(t) + 2\sqrt{\epsilon},$$

so $m(\cdot)$ is continuous.

S consists of two identical triangles, so

$$m(t) = 2\int_0^{1/2} D[\psi(\theta), \psi(\theta + t)]\, d\theta.$$

Define

$$m^-(t) = 2\int_0^{1/2-t} D[\psi(\theta), \psi(\theta + t)]\, d\theta$$

and

$$m^+(t) = m^-(t) + 4t^{3/2}.$$

Clearly

$$m^-(t) \leq m(t),$$

and by Lemma 2.2,

$$m(t) \leq m^-(t) + 2\int_{1/2-t}^{1/2} 2\sqrt{t}\, d\theta = m^+(t).$$

By construction of ψ,

$$\begin{aligned}\frac{D[\psi(\theta), \psi(\theta')]}{\sqrt{2}} &= D\left[\psi\left(\frac{\theta}{2}\right), \psi\left(\frac{\theta'}{2}\right)\right] \\ &= D\left[\psi\left(\frac{1}{4} + \frac{\theta}{2}\right), \psi\left(\frac{1}{4} + \frac{\theta'}{2}\right)\right], \qquad \theta \leq \theta' \leq \frac{1}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned} m^-(t) &\geq 2\int_0^{1/4-t} D[\psi(\theta), \psi(\theta+t)]\,d\theta + 2\int_{1/4}^{1/2-t} D[\psi(\theta), \psi(\theta+t)]\,d\theta \\ &= 4\int_0^{1/4-t} D[\psi(\theta), \psi(\theta+t)]\,d\theta \\ &= 2\int_0^{1/2-2t} D\left[\psi\left(\frac{\theta}{2}\right), \psi\left(\frac{\theta+t}{2}\right)\right] d\theta \\ &= 2\int_0^{1/2-2t} \frac{D[\psi(\theta), \psi(\theta+t)]}{\sqrt{2}}\,d\theta \\ &= \frac{m^-(2t)}{\sqrt{2}} \end{aligned}$$

or equivalently,

$$m^-(t) \leq \sqrt{2}\, m^-\left(\frac{t}{2}\right).$$

Similarly,

$$m^+(t) \geq \sqrt{2}\, m^+\left(\frac{t}{2}\right).$$

Let $m_k^-(t) = 2^{k/2} m^-(t/2^k)$ and $m_k^+(t) = 2^{k/2} m^+(t/2^k)$. Now

$$m_0^-(t) \leq m_1^-(t) \leq \cdots \leq m_1^+(t) \leq m_0^+(t).$$

Since $m_k^+(t) - m_k^-(t) = 4t^{3/2}/2^k$, it follows that $m_k^-(t)$ and $m_k^+(t)$ converge uniformly to a continuous function $m^*(t)$ satisfying

$$m^*(t) = \sqrt{2}\, m^*\left(\frac{t}{2}\right) \tag{C1}$$

and

$$|\, m(t) - m^*(t)\,| \leq 4t^{3/2}. \tag{C2}$$

Let $r(t) = m^*(t)/\sqrt{t}$. Part (a) follows from (C2) and (b) follows from (C1). Continuity of r follows from that of m^*. □

The limit (5.8) is justified by

LEMMA C2. $lim_{N\to\infty} \sqrt{N}\{E[m(\Delta_1)\} - \int_0^\infty r(t)\sqrt{t}Ne^{-Nt}\,dt\} = 0.$

PROOF. The exact density of Δ_1 is $f_N(t) = N(1-t)^{N-1}$, $0 \leq t \leq 1$, so

$$\begin{aligned} &\sqrt{N}\left| E\{m(\Delta_1)\} - \int_0^1 r(t)\sqrt{t}f_N(t)\,dt \right| \\ &\leq \sqrt{N}\int_0^1 |\, m(t) - r(t)\sqrt{t}\,|\, f_N(t)\,dt \\ &\leq \int_0^1 4(Nt)^{3/2}(1-t)^{N-1}\,dt \qquad \text{(by Lemma C1a)} \end{aligned}$$

$$\leq \int_0^1 4(Nt)^{3/2} e^{-(N-1)t}\, dt \qquad (\text{since } 1 - t \leq e^{-t})$$

$$\leq \int_0^\infty 4(Nt)^{3/2} e^{-(N-1)t}\, dt$$

$$= O\left(\frac{1}{N}\right).$$

It remains to show that the following sequence vanishes as $N \to \infty$,

$$\left| \left[\sqrt{N} \int_0^1 r(t)\sqrt{t} f_N(t)\, dt \right] - \left[\sqrt{N} \int_0^1 r(t)\sqrt{t} N e^{-Nt}\, dt \right] \right|$$

$$\leq \int_0^1 2N^{3/2}\sqrt{t}\, |(1-t)^{(N-1)} - e^{-Nt}|\, dt \qquad (\text{by (5.3)–(5.4)})$$

$$\leq \int_0^\infty 2\sqrt{\tau} \left\{ \left| \max\left(0, 1 - \frac{\tau}{N}\right)^{(N-1)} - e^{-\tau} \right| \right\} d\tau. \qquad (\tau = Nt)$$

Since $(1 - \tau/N)^N$ converges upward to $e^{-\tau}$, the integrand of this upper bound converges pointwise to zero as $N \to \infty$, and by Lebesgue's dominated convergence theorem, the limit integral vanishes as well. □

ACKNOWLEDGMENTS. We thank Bob Foley, Arnie Rosenthal, Dick Serfozo, Craig Tovey, John Vande Vate, and especially Steve Hackman, for their many helpful comments.

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RECEIVED NOVEMBER 1984; REVISED OCTOBER 1985, JUNE 1987, FEBRUARY 1988, AND JANUARY 1989; ACCEPTED JANUARY 1989