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The Euclidean Traveling Salesman Problem and a Space-Filling Curve

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Abstract - We elucidate the relationship between space-fillling curves and the Euclidean Traveling[2] Salesman Problem (TSP) by reference to a particular space-filling curve whose scaling behaviour is strongly related to the conjectured scaling behaviour of the optimal TSP tour. We suggest that space-filling curves can be used to generate testbed TSPs: sets of points which in the limit cover a planar surface and for which tours of minimum length are known.

INTRODUCTION

In this paper we discuss a way to generate large problem instances of TSP, with *unique* global optima. This approach is complementary to TSPLIB—a public domain database, accessible via ftp[3], which contains many large TSP instances solved to optimality through the use of exact methods like branch-and-cut. We suggest that space-filling curves can be used to generate large TSP instances which, as well as being cleanly defined, can provide significant insight on the characteristics of the TSP and the nature of heuristic approximation methods.

It has been noted elsewhere [1] that there is a strong relationship between the well defined notion of *computational complexity*, see for example [2], and the less well defined notion of complexity which is used in the literature describing fractals, chaos and emergent behaviour. Whilst, this paper does not make a formal connection between the two notions, we take the opportunity of pointing out specific connections which apertain to our particular problem.

NPEANO

In the widely-disseminated freeware software package called FRACTINT (Version 17.2) there is a program developed by A. Mariano which can be used to generate fractals by interpretation of L-systems [3,

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[2]US spelling is used by convention.

[3]TSPLIB is compiled and maintained by Gerhard Reinelt of the "Institut fuer Mathematik, Universitaet Augsburg", E-Mail `reinelt@augsopt.uni-augsburg.de`.

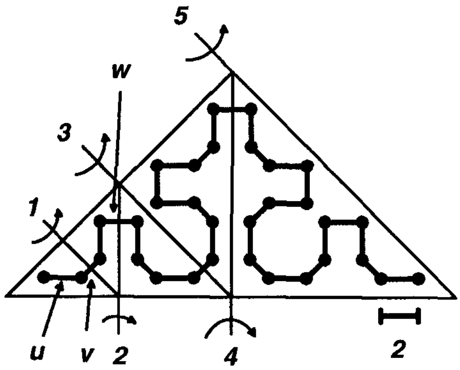

Figure 1: Construction of the NPeano/2 curve

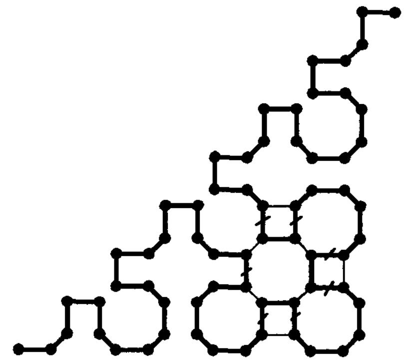

Figure 2: A NPeano/2 Curve where the NPeano Curve is a Suboptimal TSP tour

4]. The inspiration and starting point of our work was a fractal referred to by Mariano as Peano2. In this section we define variants of that curve which we refer to as NPeano and NPeano/2. Although these curves can be generated by L-systems, we choose to describe our curves with reference to the Multiple Reduction Copier Machine (MRCM) metaphor (see Peitgen *et al* [5]), in a way which is strongly analogous to Hilbert's famous definition [6].

The basic unit of the NPeano/2 Curve consists of a line of length 2, which is considered to be inside an isosceles triangle of base 6 and height 3 as shown in Figure 1. We refer to the base of the triangle as a *long segment* and the non-base sides as *short segments.* We replicate this basic unit and rotate the replicant around the apex of the triangle by 90°. We connect the two adjacent terminals of the curve at the ends closest to the apex of the resulting triangle as shown. We then iterate, applying the same replication, rotation and connection to the resulting triangle. We refer to the figure generated after m iterations as the MPeano/2 curve of order m.

At this point it is useful to consider properties of the lines that make up the curve. If we consider replicants of a line to have the same property as a line itself, lines can have one of three properties.

Firstly, they were defined in the basic triangle. These we refer to as type u; they have length 2. Secondly, they connect lines of type u and have length $\sqrt{2}$. These we refer to as type v. Thirdly, they connect lines of type u and have length 2. These we refer to as type w.

At any given order we refer to the number of lines of type u as U_m, the number of type v as V_m, and the number of type w as W_m. By construction we have the following.

$$U_0 = 1, V_0 = 0 \text{ and } W_0 = 0. \tag{1}$$

For odd values of m

$$U_m = 2U_{m-1}, V_m = 2V_{m-1} + 1 \text{ and } W_m = 2W_{m-1}, \tag{2}$$

and for even values of m

$$U_m = 2U_{m-1}, V_m = 2V_{m-1} \text{ and } W_m = 2W_{m-1} + 1. \tag{3}$$

The values of U, V and W are found in Table 1. We note that $U_m = 2^m$, that $V_m = \lceil 2^{m+1}/3 \rceil$ and

order	0	1	2	3	4	5	6	7	8
U	1	2	4	8	16	32	64	128	256
V	0	0	1	2	5	10	21	42	85
W	0	1	2	5	10	21	42	85	170

Table 1: Numbers of lines of various types, for different orders of NPeano/2

$W_m = \lceil 2^m/3 \rceil$.

Like Hilbert's curve, NPeano/2 is not strictly self-similar, but note that the box-counting (or rather triangle-counting) dimension of the curve is 2. In practice we will be interested in a pair of linked NPeano/2 curves, as shown in later figures, which we shall refer to as the NPeano curve. The NPeano curve of a given order is constructed from two copies of the NPeano/2 curve of the same order which are abbutted along the bases of their enclosing triangles. The two curves are connected by a pair of edges which in the case of odd orders are of length $\sqrt{2}$, and of even orders are of length 2. MPeano curves bear resemblance to Peano2 curves of Mariano.

Theorem 1 *The NPeano/2 curve is self-avoiding.*

Proof By induction on the order. □

In later sections we will need the following lemma.

Lemma 1 *In the construction of the NPeano/2 curve, the enclosing triangle either has a base which is composed of a sequence of long segments and non-base sides composed of short segments, or a base composed of short segments and non-base sides composed of long segments*

Proof By induction on the order.
Base The theorem holds for the basic triangle.
Induction By construction, the base of the triangle at order m is two copies of one of the non-base sides at order $m-1$, and the non-base sides at order m are copies of the base at order $m-1$. □

TRAVELING SALESMAN

The TSP is well-studied [7] and known to be NP complete [2]. For the purposes of this paper we will be interested in the following non-standard wording of the optimisation problem corresponding to the Euclidean version.

> Find the shortest *Hamiltonian cycle* through a complete graph on a set of cities, with edges weighted by Euclidean inter-city distance.

In what follows we could have worked with the shortest *Hamiltonian path* problem and the NPeano/2 curve rather than the NPeano curve.

Uniform Points Sets in a Unit Square

In order to test approximation algorithms, workers in the TSP have, on occasion, used random instances since there exist asymptotic expected length formulae for the optimal tour (L_{opt}) in some specific cases. In 1959 Beardwood, Halton and Hammersley gave the following formula

$$L_{opt}(N, A) = K\sqrt{NA} \tag{4}$$

for the optimal tour length of N cities randomly, uniformly distributed over a rectangular area of A units [8]. Later computer experiments by D. Stein in 1977 gave empirical bounds on the constant K

$$0.765 \leq K \leq 0.765 + \frac{4}{N} \tag{5}$$

but more recently, E. Bonomi and J.L. Lutton gave an experimentally derived value of $K = 0.749$ for Beardwood's formula in the limit $N \longrightarrow \infty$ [9]. For uniform point sets of size 10^4 inside a unit square D.S. Johnson has observed a lower bound on K of 0.715 and conjectures that for a set of that cardinality K has the value 0.725 [10]. The divergence of results may be connected with finite size effects associated with the different calculations.

The Relationship Between the TSP and Fractals

We can consider a TSP tour as a fractal curve. In the case of the optimal tour of an infinite number of cities uniformly distributed in the unit square, we note that the uniformity of the distribution of cities ensures that the tour is space-filling. That is, the tour has box-counting dimension 2. We also note that the optimal tour is 2-opt [11], thus it has no crosses: it is self-avoiding.

In order to map between a curve and a set of cities which constitute a TSP instance we introduce cities into the basic unit during the construction of the curve. In the simplest case we introduce a city at each end of lines of type u, and our construction now defines the set of cities which are depicted at corners of the curve in Fig. 1. These cities form a TSP instance, but unfortunately the Peano/2 curve is not the optimal solution. See Fig 2 for a counter-example.

MPEANO

We can modify the set of cities created by our curve in the following way.

Lines of type u contain 3 cities: one at each end and one at their mid-points. Lines of type v still contain no city. Lines of type w contain a city at their mid-points.

We shall refer to the newly defined curve as MPeano/2. MPeano is created by joining two MPeano/2 curves as for NPeano. In the case of even orders, the connecting lines are of length 2, and we bisect them with a city. In the case of odd orders, the connecting lines are of length $\sqrt{2}$ and do not contain a city.

Theorem 2 *In the curve defined by MPeano/2 every non-terminal city is adjacent to its two unique nearest neighbour cities.*

Proof By induction on the order.

Base The theorem holds for the basic triangle, and we note the following properties also hold.

1. All edges are of length 1 or $\sqrt{2}$.

2. No city is closer than 1 unit to a long segment that forms part of the enclosing triangle.

Length (L)	Area (A)	MPeano		MNPeano	
		Cities (N)	$K = \frac{L}{\sqrt{NA}}$	Cities (N)	$K = \frac{L}{\sqrt{NA}}$
$24 + 4\sqrt{2}$	50	28	0.7926127	24	0.8561196
$88 + 20\sqrt{2}$	242	108	0.7192856	92	0.7793263
$344 + 84\sqrt{2}$	1058	428	0.6877378	364	0.7457515
$1368 + 340\sqrt{2}$	4418	1708	0.6730398	1452	0.7299639
$5464 + 1364\sqrt{2}$	18050	6828	0.6659392	5804	0.7223
$21848 + 5460\sqrt{2}$	72962	27308	0.6624486	23212	0.7185233
$87384 + 21844\sqrt{2}$	293378	109228	0.660718	92844	0.7166485
$349528 + 87380\sqrt{2}$	1176578	436908	0.6598563	371372	0.7157145
$1398104 + 349524\sqrt{2}$	4712450	1747628	0.6594264	1485484	0.7152453
$5592408 + 1398100\sqrt{2}$	18862082	6990508	0.6592117	5941932	0.7150124
$22369624 + 5592404\sqrt{2}$	75472898	27962028	0.6591044	23767724	0.714899

Table 2: Parameter scaling of fractal TSPs

3. No city is closer than $\sqrt{2}/2$ to a short segment that forms part of the enclosing triangle, and any non-terminal city that achieves that bound is connected to two cities at distance 1.

Induction When a triangle is replicated, and after rotation, by Lemma 1 the two adjoining sides of the triangles are composed uniquely of long segments or of short segments. Properties 1–3 above ensure that edges between non-terminal cities in different triangles would always be longer than the edges that form part of the curve. After adjoinment, the curve is connected by new edges which are made between nearest neighbours; all new connections are of length 1 or $\sqrt{2}$; and a connection of length 1 is made between any non-terminal cities of distance $\sqrt{2}/2$ from a short segment forming part of the boundary. Thus the three properties continue to hold. □

Corollary 1 *The curve defined by MPeano is an optimal TSP tour among the corresponding set of cities.*

The set of cities defined by MPeano is thus a TSP with a known optimal solution, and might be used as a test-bed for TSP algorithms, but we find it unsatisfactory for a number of reasons.

Scaling

We can investigate how the length of the optimal tour scales for MPeano. The results are summarized in table 2. Here we have defined A, the area of the MPeano TSP as the area of the smallest square that inscribes the curve. In later sections we shall define area slightly differently. The value of K is lower than that which is expected for uniform point sets inside a unit square, implying there is something "special" about sets of cities defined by MPeano.

Frustration

We proved the optimality of the MPeano tour by showing that all cities are connected to both of their two nearest neighbours. This implies that the MPeano TSP can be solved exactly by the NNStart heuristic [10]. In general instances of the TSP, a tour can be optimal without this property. Given a set C of cities, it is useful to define a quantity we shall refer to as *frustration*, f, such that given a city $c \in C$

whose nearest neighbours are n_c and m_c and a tour T, with edges $\{c, a_c\}$ and $\{c, b_c\}$,

$$f(c,T) = dist(c,a_c) + dist(c,b_c) - (dist(c,n_c) + dist(c,m_c)). \tag{6}$$

For any $B \subseteq C$ we also define $f(B,T) = \sum_{c \in B} f(c,T)$, and note that since the nearest neighbour distances are tour independent, the optimisation version of the TSP consists of minimising $f(C,T)$.

Justification

Now, let us consider another version of MPeano where the curve is identical but the cardinality of the set of cities is doubled by bisecting each edge with a new city. In that case we have a new value of scaling factor, $K' = K/\sqrt{2}$ and following that argument we can see that the value of K' can be made arbitrarily small.

This leads us to an interesting problem. Let C be the set of cities defined by MPeano. What is the cardinality of the minimum subset of C for which the curve defined by MPeano is superimposable upon the unique optimal tour? It is also interesting to consider the general case which, in a sense, is an inverse problem to the TSP.

> Given an arbitrary space-filling curve, T, find a set cities, C, of minimal cardinality such that T is an optimal tour for C.

It is interesting to consider what would be the value of K for C and T. In practice we may also wish to put some constraint of "uniformity" on the set C.

MNPEANO

In order to address some of the above issues, we now define a curve with a new set of cities which we refer to as MNPeano/2. Its construction is identical to MPeano except that in the construction of the basic unit we "forget" to include the city that bisects the edge of type u. We "remember" to include it in the first copy that we make of the basic unit. Thereafter we merely copy units. MNPeano is shown in Fig 3. Fig 4 shows a magnified portion which is relevant to the proof of Theorem 3.

Theorem 3 *The curve defined by MNPeano is the shortest tour through the set of cities defined by MNPeano.*

Proof We note that, in contrast to the case with MPeano, there exist cities that are not connected to both their nearest neighbours. These cities are always located at one of the 90° corners of the curve. By the rotation in the construction, these frustrated cities are always at least $\sqrt{10}$ units apart.

We shall refer to the set of all frustrated cities as B, and note that $\forall b \in B, f(b,T) = 2 - \sqrt{2}$, whereas $\forall c \in C - B, f(c,T) = 0$. Let us assume as a hypothesis to be proven contradictory that there exists a tour T' which is shorter than T. That is $f(C,T') < f(C,T)$. This implies that for some set of $n > 0$ cities, $B' \subset B$, $f(B',T') = 0$. That is they are connected to their nearest neighbour as shown on Fig 4. We shall refer to the set of such neighbours as N and, since $\sqrt{10} > 2\sqrt{2}$, each element of B has a distinct neighbour in N. Note that each city in N is no longer connected to both its nearest neighbours, thus $f(N,T') \geq n(\sqrt{2}-1)$. Furthermore this implies that there exists a set, M of n cities to which the cities of

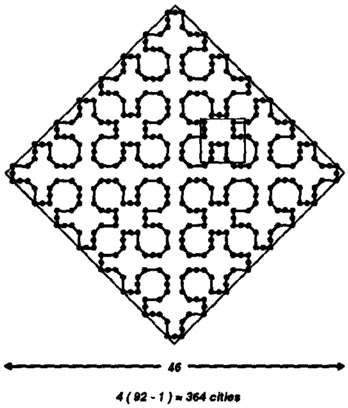

Figure 3: MNPeano curve, order 6

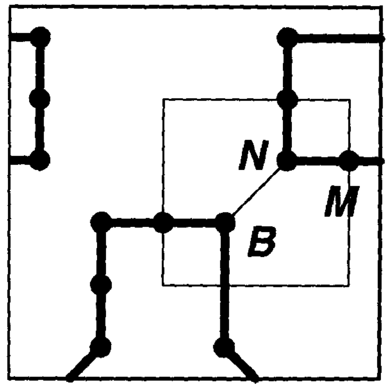

Figure 4: Magnified Section of MNPeano Curve

N were connected which are no longer connected to their nearest neighbour, that is $f(M,T') \geq n(\sqrt{2}-1)$. Now since

$$f(C,T') \;\geq\; f(N,T') + f(M,T') + f(B-B',T') \tag{7}$$

$$f(C,T') \;\geq\; 2n(\sqrt{2}-1) + (|B|-n)(2-\sqrt{2}) > |B|(2-\sqrt{2}) = f(C,T), \tag{8}$$

which leads to a contradiction. □

Table 2 contains the scaling relationship for MNPeano, and the relationship between K for MNPeano and Johnson's bound is striking.

ASYMPTOTIC VALUES FOR K

We now derive a value for K for MNPeano which is exact in the limit of increasing order. We work with MNPeano/2, but our result trivially applies to MNPeano. In this section we choose to consider the area of curve as the area of the isosceles triangle of the construction, that is the area at an order m is given by $A_m = 9 \times 2^m$. Our value of A differs slightly from that used in Table 2, and so the value of $K_m = L_m/\sqrt{N_m, A_m}$ we derive here differs slighly from K shown in Table 2. Although they are asymptotic to the same value, K_m converges much faster.

We consider lines of type u as contributing 5/2 cities to the tour, counting the city at each end and the city that bisects every other line. We consider lines of type v as contributing 0 cities, and those of w as contributing 1 city, thus the total number of cities at order m is given by

$$N_m = 5U_m/2 + W_m = 5(2^m)/2 + \lceil 2^m/3 \rceil \tag{9}$$

Lines of type u are of length 2. Lines of type v are of length $\sqrt{2}$ and those of type w are of length 2. The length L_m of the tour at order m is thus

$$L_m = 2(U_m + W_m) + \sqrt{2}V_m = 2(2^m + \lceil 2^m/3 \rceil) + \sqrt{2}\lceil 2^{m+1}/3 \rceil. \tag{10}$$

Asymptotically we can remove the ceiling operators and

$$
\begin{aligned}
K &= L/\sqrt{NA} && (11)\\
&= \frac{(4+\sqrt{2})/3}{\sqrt{17 \times 3/8}} && (12)\\
&= 0.71478270 && (13)
\end{aligned}
$$

CONCLUSION

The Generator as an Algorithm for Solving Fractal TSPs

The computational complexity of problems such as the TSP is defined for the worst case, and it should not be assumed that restricted versions of the TSP (such as those generated from MNPeano curves) remain NP-Complete. Indeed, solving MNPeano TSPs is, in one sense, polynomial, since the generator function of the MNPeano curve is an algorithm which solves the MNPeano TSP in time which is linear in the number of cities. It should be noted however that the curve can be specified without enumerating the position of every city.

It is interesting to remark an analogy between the inverse problem described above, and another research direction. Given an observed sequence of structures, the problem of finding L-Systems which generate them is called the *syntactic inference problem* (also known as the *realization* problem) (see [4] and references therein). Among the issues treated is the *decidability* of questions such as the *membership* problem. A particular type of L-Systems called *table L-Systems* or *TOL-Systems* was developed for simulating changes of states due to environmental changes (for example, in plants going from a vegetative to flowering state). This systems have been formalized in [4, 12]. While for others the time required is bounded by a polynomial, for the TOL-Systems the decision is NP-Complete [13, 14, 15].

General-Purpose Algorithms for solving Fractal TSPs

We noted earlier that the general-purpose NNstart heuristic is capable of exactly solving TSPs generated by MPeano. Intriguingly, the general-purpose Multiple Fragment (MF) Algorithm [10] generates optimal solutions to both MPeano TSPs and MNPeano TSPs. In the general-purpose implementation of the MF algorithm by Bentley, the complexity of their solution is $O(N \log N)$. Interestingly, MF performs particularly well on uniform points sets in the unit square, but is not robust, that is it does not perform well on non-uniform points sets where the optimal tour has fractal dimension less than 2. It might, therefore, be appropriate to use a fractal-like measures of dimension to determine whether a particular TSP instance would be amenable to the MF heuristic.

Significance

A word of caution is needed in comparing bounds for our space-filling curve and those for the TSP. First, we are not yet aware of how Johnson's bound was derived; in [10], which is its source, it is referenced as a personal communication. Second, we haven't yet proved that a there exists no *smaller* set of cities which mantains the optimality of the path of the space-filling curve for all orders. Third, the relationship between the asymptotics of the TSP and of space-filling curves deserves a more serious treatment, so we will limit ourselves to expressing the hope that our pointing out of this *"numerical coincidence"* may lead to new algorithmic insights and to a better estimate of the asymptotic TSP tour length.

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