Korteweg – de Vries equation (1.5 LP)

The one dimensional Korteweg-de-Vries (KdV) equation is a universal equation describing weakly nonlinear and weakly dispersive waves. There are many application fields, onee of the most important are shallow water waves. It is famous as one of the paradigmatic examples of integrable nonlinear partial differential equations bearing **solitons** (see http://www.scholarpedia.org/article/Soliton; article "Korteweg-deVries equation" in Encyclopedia of Nonlinear Science; book "Physics of Solitons" by Dauxois and Peyard; articles in wikipedia on Solitons and KdV equation).

If d is the water depth and g is the gravity acceleration, the KdV equation for the elevation η can be written as:

$$\eta_t + \sqrt{gd}\left(\eta_x + \frac{d^2}{6}\eta_{xxx} + \frac{3}{2d}\eta\eta_x\right) = 0$$

(1) Rescale η, t, x and perform a Galilean transformation into a reference frame moving with linear velocity, to get a dimensionless KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

(2) Check that a soliton wave

$$u(x,t) = \frac{A}{\cosh^2(kx - \omega t - \eta_0)}$$

is a solution if $A = 2k^2$, $\omega = 4k^3$. (3) Check that the following expressions are integrals (i.e. $dP_m/dt = 0$):

$$P_0 = \int_{-\infty}^{\infty} u(x,t)\, dx$$

$$P_1 = \int_{-\infty}^{\infty} u^2(x,t)\, dx$$

$$P_2 = \int_{-\infty}^{\infty} \left[2u^3(x,t) - \left(\frac{\partial u}{\partial x}\right)^2\right] dx$$

Problem 1: Implement the following explicit numerical scheme for KdV (Zabusky-Kruskal scheme):

$$\begin{aligned}
u &= (1/3)(u_{i-1}^n + u_i^n + u_{i+1}^n) \\
u_t &= \frac{1}{2\Delta t}(u_i^{n+1} - u_i^{n-1}) \\
u_x &= \frac{1}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) \\
u_{xxx} &= \frac{1}{2\Delta x^3}(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n) \\
u_t + 6uu_x + u_{xxx} &= \frac{1}{2\Delta t}(u_i^{n+1} - u_i^{n-1}) + 2(u_{i-1}^n + u_i^n + u_{i+1}^n)\frac{1}{2\Delta t}(u_{i+1}^n - u_{i-1}^n) \\
&\quad + \frac{1}{2\Delta x^3}(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n)
\end{aligned}$$

Use an explicit first-order in time scheme for the first step. Stability of this scheme requires

$$\frac{\Delta t}{\Delta x}\left|-2u_0 + \frac{1}{(\Delta x)^2}\right| < \frac{2}{3\sqrt{3}}$$

where u_0 is the maximum value of $u(x,t)$ [Can you derive this formula for the linear equation, where $u_0 = 0$?]

With the initial condition

$$u(x,0) = \frac{N(N+1)}{\cosh^2(x)}$$

the dimensionless KdV equation generates a N-soliton solution ($N = 1, 2, 3, ...$).

(4) With periodic boundary conditions, generate 3-soliton solution and observe collision of solitons. Check how exact are the integrals $P_{0,1,2}$ conserved.

(5) Set $N = 2.5$ and observe creation of 2 solitons and of a wave train.

Problem 2: Tsunami wave can be considered as a soliton, which propagates over a variable dimensionless depth $h(x)$. In this case the KdV equation can be written as [see Johnson,Some numerical solutions of a variable-coefficient Korteweg-de Vries equation (with applications to solitary wave development on a shelf), J. Fluid Mech. (1972), vol. 54, part 1, pp. 81-91]

$$u_t + \frac{1}{h^{7/4}(x)} 6uu_x + h^{1/2}(x)u_{xxx} = 0$$

(6) Consider a profile

$$h(x) = \begin{cases} 1 & x < 0 \\ \frac{1+h_0+(1-h_0)\cos(\pi x/L)}{2} & 0 < x < L \\ h_0 & x > L \end{cases}$$

and study how one soliton moving with positive velocity from $x < 0$ is transformed on this step.

(7) Describe the change of the amplitude in dependence on h_0, and possible created small-amplitude waves.

(*8) Describe transformation of a soliton over an underwater hill of width L and height h_0: $h(x) = 1 - h_0 \exp(-x^2/L^2)$

Problem 3 (Bonus 0.5 LP): Compare accuracy of the Zabusky-Kruskal with two schemes presented in Paper [Wang, Wang, Hu, An Explicit Scheme for the KdV Equation, CHIN.PHYS.LETT., Vol. 25, No. 7 (2008) 2335] for the tasks 1(4) and 1(5).