Falling liquid films: Kuramoto-Sivashinsky equation

Falling liquid films are an example of unstable fluid flow which generates waves and show transition to turbulence. See [Ruyer-Quil et al, “Dynamics of falling liquid films”, Eur. Phys. J. E, v. 37, 30 (2014)] for a review. Many qualitative features can be described by the one-dimensional weakly nonlinear Kuramoto-Sivashinsky (KS) equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

Here u describes velocity on the surface and the deviation of the height (in this approximation these quantities are proportional to each other).

(1) Show that in peresence of coefficients at all terms of the equation, it can be reduced to the free-of-parameters form above by rescaling t, u, x.

KS equation has no parameters, but its solutions depend on boundary conditions. Below periodic domain is considered $u(x,t) = u(x+L,t)$, so that spatial period L is the only parameter of the problem.

It is convenient to solve KS equation in the Fourier space:

$$u(x,t) = \sum_{n=-\infty}^{\infty} C_n(t)\exp[in\frac{2\pi}{L}x] \qquad C_n(t) = \frac{1}{L}\int_0^L dx\; u(x,t)\exp[-in\frac{2\pi}{L}x] \qquad C_{-n} = C_n^*$$

The basic wavevector is $K = 2\pi/L$.

(2) Derive infinite system of equations for $\frac{dC_n}{dt}$ by differentiating in time the expression for C_n and substituting KS equation for $\frac{\partial u}{\partial t}$, use partial integration.

(3) Write down a finite Galerkin approximation by using a finite number of Fourier modes $|n| < N$.

(4) Show that in linear approximation the problem reduces to a system

$$\frac{dC_n}{dt} = (n^2K^2 - n^4K^4)C_n$$

(4.1) What is the condition for mode C_n to grow in time?
(4.2) At which L there is no growing in time modes?
(4.3) How many modes grow at given value of L?

For numerical solution of the system of ordinary differential equations for C_n use methods described in [Cox and Matthews, J. Comp. Phys. v 176, 430-455 (2002)] for systems of type

$$\frac{dC}{dt} = aC + F(C,t)$$

(here C, a, F are vectors). In particular, method ETD2RK ($O(h^2)$, h is the time step) reads:

$$\tilde{C}_n = C_n e^{ah} + F_n\frac{e^{ah}-1}{a}$$

$$C_{n+1} = \tilde{C}_n + \frac{(F(\tilde{C}_n, t_n + h) - F_n)(e^{ah} - 1 - ah)}{a^2h}$$

(5) Implement the ETD2RK method and solve KS equation for different values of parameter L (starting from the minimal value at which waves first become unstable). Use for N the triple number of unstable modes. Represent the field $u(x,t)$ as a color 3D plot (use, e.g., gnuplot drawing program). Show that for $L \gtrsim 20$ solutions of KS equation are turbulent. Check how the solutions depend on number of modes N.

(6) Suppose there are particles (e.g. dust) on top of the film. These particles are advected according to

$$\frac{dX}{dt} = u(X,t)$$

where u is solution of KS equation. Spread many particles on top of a turbulent film, and find their evolution by solving simultaneously the KS equation (in turbulent regime) and the equations for the particle's advection. What is the final stage of the evolution?