A Comparative Study of L^1-Regularized and Nearly-Isotonic Regression

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Isotonic regression consists of finding a monotone function g that best approaches a function for a certain norm $\|\cdot\|$. In the discrete and non-decreasing setting, given observations $((x_i, y_i))_{1\leq i\leq n}$, where the $(x_i)_{1\leq i\leq n}$ are non-decreasingly ordered, this reduces to solving the following optimization problem:

$$\min_{g\in\mathcal{M}} \|g-y\|^2 \tag{1}$$

where $\mathcal{M}$ is the subset of $\mathbb{R}^n$ of non-decreasingly ordered vectors. This problem can for example be solved using the Frank-Wolfe method which is written as follows:

Algorithm 1 Frank-Wolfe method to minimize $f(g) = \|g-y\|^2$ over $\mathcal{M}$ with line-search

1: Initialize $g^0 \in \mathcal{M}$ and $k \leftarrow 0$
2: **for** $k = 1, 2, \ldots, K$ **do**
3: Compute $\nabla f(g^k) = 2(g^k - y)$
4: Solve linear optimization problem $\tilde{g}^k \leftarrow \arg\min_{g\in\mathcal{M}}\{f(g^k) + \nabla f(g^k)^\top (g - g^k)\}$
5: Perform line-search $\bar{\alpha}^k \leftarrow \arg\min_{\alpha\in[0,1]} f(g^k + \alpha(\tilde{g}^k - g^k))$
6: Update $g^{k+1} \leftarrow g^k + \bar{\alpha}^k(\tilde{g}^k - g^k)$
7: **end for**

This project concerns two relaxations of isotonic regression: L^1-regularized isotonic regression and nearly-isotonic regression. We plan to apply this to a dataset that is monotone, but yet have to find one. For instance, college admissions (ranking as a function of test scores) might be an idea.

While isotonic regression enforces strict monotonicity, real-world datasets often include features whose relationship with the outcome is weak, noisy, or only approximately monotonic. Instead of manually selecting features to include in a monotonic model, we introduce L^1-regularization to automatically promote sparsity – allowing the model to retain only those features that contribute meaningfully to a monotonic trend. L^1-regularization thus induces sparsity and selects the most relevant features while maintaining strict monotonicity. The associated problem writes:

$$\min_{g\in\mathcal{M}} \frac{1}{2}\|g-y\|^2 + \lambda\|Dg\|_1 \tag{2}$$

where $D\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the difference operator.

Nearly-isotonic regression allows controlled monotonicity violations to improve the fit (as discussed in the work of Tibshirani et al. [4]). The associated problem writes:

$$\min_{g\in\mathbb{R}^n} \frac{1}{2}\|g-y\|^2 + \lambda\sum_{i=1}^{n-1}(g_i - g_{i+1})^+. \tag{3}$$

Our project will include a brief literature review of the theoretical properties of both L^1-regularized and nearly-isotonic regression, focusing on their formulation, optimization, and convergence behavior i.e. we will look into the theoretical properties. We will then conduct numerical simulations on (for example) college admissions data to evaluate how the two methods differ in terms of predictive performance, interpretability (sparsity), and monotonicity adherence, and how they compare to classical isotonic regression (also from a theoretical point of view). This analysis will be complemented by a comparison of how various algorithms covered in class – such as projected gradient descent, proximal methods, Frank-Wolfe, and interior point methods – can be applied to each formulation.[3][2][1]

1 Introduction

2 Related Work

3 Problem Formulation and Preliminaries

4 Algorithmic Framework

5 Theoretical Analysis

6 Numerical Experiments

7 Discussion

8 Conclusions and Future Work

References

[1] Mohan S Acharya, Asfia Armaan, and Aneeta S Antony. A Comparison of Regression Models for Prediction of Graduate Admissions. In *2019 International Conference on Computational Intelligence in Data Science (ICCIDS)*, pages 1–5, 2019.

[2] Vladimir Pastukhov. Fused Lasso Nearly-Isotonic Signal Approximation in General Dimensions. *Statistics and Computing*, 34(4):120, 2024.

[3] Joseph Salmon. Isotonic Regression. https://josephsalmon.eu/blog/isotonic/, November 10 2024. Accessed April 20, 2025.

[4] Ryan J Tibshirani, Holger Hoefling, and Robert Tibshirani. Nearly-Isotonic Regression. *Technometrics*, 53(1):54–61, 2011.