On "Optimal auction duration: A price formation viewpoint" by Jusselin, Mastrolia, and Rosenbaum

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1 Introduction

Modern electronic markets for equities, foreign exchange and listed derivatives overwhelmingly rely on the continuous double auction implemented through a centralized limit-order book (CLOB). Investors or traders are individuals and corporations asking for liquidity, and are therefore called *liquidity consumers*. On the other hand, *market makers*, a firm or an individual, are actively quoting two-sided markets (both buyers and sellers) in a security, providing bids and offer along with the market size of each. A market maker is called a *liquidity provider*. Market makers provide liquidity and depth to markets and profit from the difference in the bid-ask spread. They do so because they may see a decline in the value of a security after it has been purchased from a seller and before it is sold to a buyer. This induces two major drawbacks: (i) investors systematically have to pay the bid-ask spread as transaction costs, and, quoting Farmer and Skouras (2012), (ii) this *trading mechanism [. . .] endows a huge advantage to being faster than other traders, creating evolutionary pressures that drive an arms race for ever-more speed* [3]. There is no evidence that the LOB is the optimal market structure for all the market participants having different preferences.[5].

This motivates the search for alternative designs that preserve liquidity while de-monetizing speed. A leading proposal is the frequent batch auction: orders arriving during a short interval h are pooled and cleared simultaneously at a single uniform clearing price. By eliminating price-time priority within the batch, such auctions dismantle the mechanical component of latency arbitrage. Indeed, Budish, Cramton and Shim (2015) indicate that an interval as small as h $= 100$ ms can extinguish speed-based rents without materially widening spreads or degrading welfare for liquidity demanders [1]. Yet batch auctions are not free of drawbacks. Strategic traders have a strong incentive to wait until the very end of the interval before submitting or canceling orders – they want to see as much incoming information as possible while still participating in the same clearing. This late-in-cycle "pile-up" reduces displayed depth early in the batch and mechanically widens the instantaneous spread, especially when h is longer than a few hundred milliseconds. Consequently *the suggested optimal duration is a trade-off between averaging effect (a long duration allows a large number of agents to take part in the auction, hence reducing uncertainty about the efficient price) and volatility risk (a short duration leads to small volatility risk)* [4].

This tradeoff is exactly treated in "Optimal auction duration: A price formation viewpoint" by Jusselin, Mastrolia and Rosenbaum. They *provide a sound and operational quantitative analysis of the optimal auction duration on a financial market, and compare the efficiency of this mechanism with that of a CLOB* [4]. Thus, the first part of the paper consists in evaluating the quality-of-price-formation process metric $E(h)$ to determine the optimal auction duration. The second part of the paper then examines the case of strategic market takers. By considering that market takers aim at minimizing their trading costs by adapting their trading intensities to the market state, the authors formalize a game and show the existence of a Nash equilibrium to this game, as well as the expression of the associated optimal auction duration.

This report aims at elaborating on the work of Jusselin, Mastrolia and Rosenbaum under the lens of the IEOR 222 course at the University of California at Berkeley. The report was written by Yilin Chen, Julius Graf, and Konstantin Zhivotov. We will first detail the mathematical and model setup before formalizing the optimization problem we seek to solve. We will then provide a summary of the main theoretical results and mathematical ideas leading up to these results. More precisely, we will solve the control problem for a strategic market seller, when market buyers have an increasing arrival rate to the market. Finally, we aim at reproducing the main numerical results. Specifically, we want to deliver a graphical representation of $h \mapsto E(h)$ and perform a sensitivity analysis of the auction clearing price with respect to various model parameters, in particular the supply function nominal K. The report closes with a brief discussion of tractable model extensions that could serve as avenues for future research.

2 Mathematical and Model Setup

2.1 Auction Market Design

In this section, we introduce our stochastic auction-market framework, strongly inspired by Jusselin et al. (2020) [4] and Mastrolia and Xu (2025) [5]. An auction market is organized in independent sequential auctions. Each auction is triggered by the arrival of a market order and has a finite duration h. Let $(\tau_i^{\mathrm{op}})_{i\in\mathbb{N}^*}$ (resp. $(\tau_i^{\mathrm{cl}})_{i\in\mathbb{N}^*}$) be the sequence of auction opening (resp. clearing) times. We additionally set $\tau_0^{\mathrm{cl}} = 0$ and observe that $\tau_i^{\mathrm{cl}} = \tau_i^{\mathrm{op}} + h$. We will denote $\Delta_i = \tau_i^{\mathrm{cl}} - \tau_{i-1}^{\mathrm{cl}}$. At time τ_i^{cl}, the i-th auction is cleared according to a clearing rule that will be defined later on. This clearing rule determines a clearing price $P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}}$, which in essence maximizes the matched number of buyers and sellers.

We introduce the formal probability space setting as done in [5]. Let $\Omega = \mathcal{S}^3 \times \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where $\mathcal{S}$ is the set of left limit and right continuous non-decreasing functions from $\mathbb{R}_+$ into $\mathbb{N}$, and $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ is the set of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel algebra of Ω. Let $X = (X^1, X^2, X^3, X^4) := (N^a, N^b, W, N^{\mathrm{mm}})$ be the canonical process on Ω, defined as $X_t(\omega) = (\omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t))$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Consider then $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ the natural filtration associated to X, and $\mathbb{P}$ the probability measure under which W is a standard Brownian motion, and N^a, N^b and N^{mm} are independent Poisson processes with some fixed non-random intensities ν, ν, and μ, for $\nu, \mu > 0$. We will see that and why $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ defines an auction market.

2.2 Market Makers and Market Takers

The market has to support three actors: market makers, selling market takers, and buying market takers. Market makers arrive in the market throughout the day and provide liquidity by sending their limit orders. More precisely, during the i-th market period, the arrival times are given by $(\tau_{i-1}^{\mathrm{cl}} + \tau_k^{i,\mathrm{mm}})_{k\in\mathbb{N}}$, where $\tau_k^{i,\mathrm{mm}}$ denotes the k-th arrival time of an underlying counting process $N^{i,\mathrm{mm}}$. In particular, market makers can arrive at the market between two auctions, as the latter are triggered by arrivals of market takers only. The supply function of the k-market maker (the k-th arrival of market maker to the market) associates to a price p the number of shares the k-th market maker is willing to sell at price p or above (if positive) or buy at price p or below (if negative), depending on the view the k-th market maker has on the price of the asset. The "true" price of the asset is given through a process P defined by $P_t = P_0 + \sigma_f W_t$ for all $t \in \mathbb{R}_+$, with $\sigma_f > 0$ and $P_0 \in \mathbb{R}_+$. The supply functions are then given by the following assumption.

Assumption 1. *The supply function of the k-th market maker is given by*

$$S_k(p) = K(p - \tilde{P}_k) \quad \text{with} \quad \tilde{P}_k = \mathbb{E}\Big[P_{\tau_i^{\mathrm{cl}}} \;\Big|\; \mathcal{F}_{\tau_{i-1}^{\mathrm{cl}} + \tau_k^{i,\mathrm{mm}}}\Big] + g_k$$

where $K > 0$ is a nominal and $(g_k)_{k\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables with variance $\sigma^2 > 0$, independent of all other processes.

We interpret $\tilde{P}_k$ as the view on the price of the asset by the k-th market maker (i.e. at time $\tau_{i-1}^{\mathrm{cl}} + \tau_k^{i,\mathrm{mm}}$), and g_k as a noise in the determination of the efficient price by the k-th market maker. We want our auction market to be regenerative, which is why the consider that unmatched market orders (at time τ_i^{cl}) in the i-th market period get canceled and thus not considered for the $(i+1)$-th period. We will later go more into detail on the regenerative property. The key idea is that the market resets itself across auction periods, each auction period having the same mathematical properties (i.e., order flows, inter-clearing times). In particular, at time τ_i^{cl}, there are $N^{i,\mathrm{mm}}_{\Delta_i}$ market makers to be matched.

Next, as said, the auction market should similarly support market takers on both the buy and sell side. During the i-th market period, the buying (resp. selling) market order arrival times are given by $(\tau^{\mathrm{cl}}_{i-1} + \tau^{i,a}_k)_{k\in\mathbb{N}}$ (resp. $(\tau^{\mathrm{cl}}_{i-1} + \tau^{i,b}_k)_{k\in\mathbb{N}}$), where $\tau^{i,a}_k$ (resp. $\tau^{i,b}_k$) denotes the k-th arrival time of an underlying counting process $N^{i,a}$ (resp. $N^{i,b}$).

Assumption 2. $(N^{i,\mathrm{mm}}, N^{i,a}, N^{i,b})$ *is independent of the efficient price process* P.

This independence assumption can also be viewed as the independence between the processes $(N^{i,\mathrm{mm}}, N^{i,a}, N^{i,b})$ and the Brownian motion W. We are now able to define properly the auction market.

Definition 2.1 (Auction market)**.** *The space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is called an auction market if it supports:*

1. *Three independent Poisson processes* N^a, N^b, N^{mm} *modeling buy orders, sell orders, and market-maker arrivals in each auction,*

2. *A Brownian motion* W*, independent of* $(N^a, N^b, N^{\mathrm{mm}})$*, and an "efficient-price" process* P *defined by* $P_t = P_0 + \sigma_f W_t$ *for* $t \in \mathbb{R}_+$ *and* $\sigma_f > 0$.

Let $I^i = v(N^{i,a} - N^{i,b})$ for a nominal $v \in \mathbb{R}$ be the cumulated imbalance process for market takers. The demand of market takers at the clearing time τ^{cl}_i of the i-th auction is thus given by $I^i_{\Delta_i}$. Furthermore, as we said that an auction is triggered by the arrival of a market taker, the opening time of the i-th auction is given by $\tau^{\mathrm{op}}_i = \tau^{\mathrm{cl}}_{i-1} + \tau^{i,a}_1 \wedge \tau^{i,b}_1$.

Assumption 3 (Regenerative market)**.** *The processes* $(N^{i,\mathrm{mm}}, N^{i,a}, N^{i,b})_{i\in\mathbb{N}}$ *are i.i.d. distributed as* $(N^{\mathrm{mm}}, N^a, N^b)$ *and the random variables* $(\tau^{i,a}_1 \wedge \tau^{i,b}_1)_{i\in\mathbb{N}}$ *are i.i.d. following a* ν*-exponential distribution, for* $\nu > 0$*. These two assumptions mean that "the market order flow is a Poisson process". Furthermore, we assume that* $N^{1,a}_{\tau^{\mathrm{cl}}_1}, N^{1,b}_{\tau^{\mathrm{cl}}_1} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

The regenerative assumption means that for each auction period, the stochastic processes governing the arrivals of market makers, buyers, and sellers are drawn from the same underlying probability distributions $(N^{\mathrm{mm}}, N^a, N^b)$ and are independent of the processes in any other auction period. This aligns with assuming that the time until the next auction opens, given by $\tau^{i,a}_1 \wedge \tau^{i,b}_1$ (the first arrival time of either a buyer or a seller in the i-th auction period, measured from τ^{cl}_{i-1}), is i.i.d. following an exponential distribution with parameter $\nu > 0$. The market can be split into i.i.d. auction intervals, each auction having the same properties.

Furthermore, in practice, a market taker will not arrive at the market if no liquidity is provided. Therefore, we will assume that for each market period, market takers arrive to a non-empty limit order book. The assumption we made earlier − unmatched market orders at the clearing time τ^{cl}_i are canceled and not carried over to the $(i+1)$-th period − reinforces the regenerative aspect of the market.

Assumption 4. *The first market maker always arrives before the first market order is executed. In other words,* $\tau^{i,\mathrm{mm}}_1 < \left(\tau^{i,a}_1 \wedge \tau^{i,b}_1\right) + h$ *a.s. for all* $i \in \mathbb{N}^*$.

2.3 Clearing Price Rule

Let us now define how the clearing price $P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i}$ is defined at the end of the i-th auction. The goal is to "maximize the exchanged volume". Let $F^+(p)$ (resp. $F^-(p)$) be the total number of shares that buyers (resp. sellers) are willing to buy (resp. sell) at price p. The exchanged volume then is $F^+(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i}) \wedge F^-(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i})$. To maximize this exchanged volume, the clearing price must satisfy $F^+(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i}) = F^-(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i})$ (otherwise liquidity remains on one side or another). In terms of the previous notation, $P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i}$ is a solution to

$$\sum_{k=1}^{N^{i,\mathrm{mm}}_{\Delta_i}} S_k(p) = I^i_{\Delta_i}.$$

Since $S_k(p) = K(p - \tilde{P}_k)$, this is a linear equation in p, yielding

$$P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}} = \frac{1}{N^{i,\mathrm{mm}}_{\Delta_i}} \sum_{k=1}^{N^{i,\mathrm{mm}}_{\Delta_i}} \tilde{P}_k + \frac{1}{K} \frac{I^i_{\Delta_i}}{N^{i,\mathrm{mm}}_{\Delta_i}}. \tag{1}$$

Example 1. *Let there be 3 market participants: market maker 1 with supply function* $S_1(p) = 10(p - \tilde{P}_1)$*, market maker 2 with supply function* $S_1(p) = 10(p - \tilde{P}_2)$*, and a buyer with volume* $v = 10000$*. Here* $\tilde{P}_k$ *represent the views of the price of the asset by k-th market maker when they send their orders. Now, denoting* $F^-(p)/F^+(p)$ *as total number of shares that buyers/sellers are willing to buy/sell, we arrive at the optimal clearing price equation* P^{cl} *where algebraic supply-demand function of all market participants together is netted to 0:*

$$F^-(P^{\mathrm{cl}}) - F^+(P^{\mathrm{cl}}) = 0$$

In our case we have:

$$F^-(p) - F^+(p) = S_1(p) + S_2(p) - v = 20p - 10(\tilde{P}_1 + \tilde{P}_2) - v$$

such that the clearing price is given by

$$P^{\mathrm{cl}} = \frac{10(\tilde{P}_1 + \tilde{P}_2) + v}{20}.$$

We confirm this equilibrium graphically.

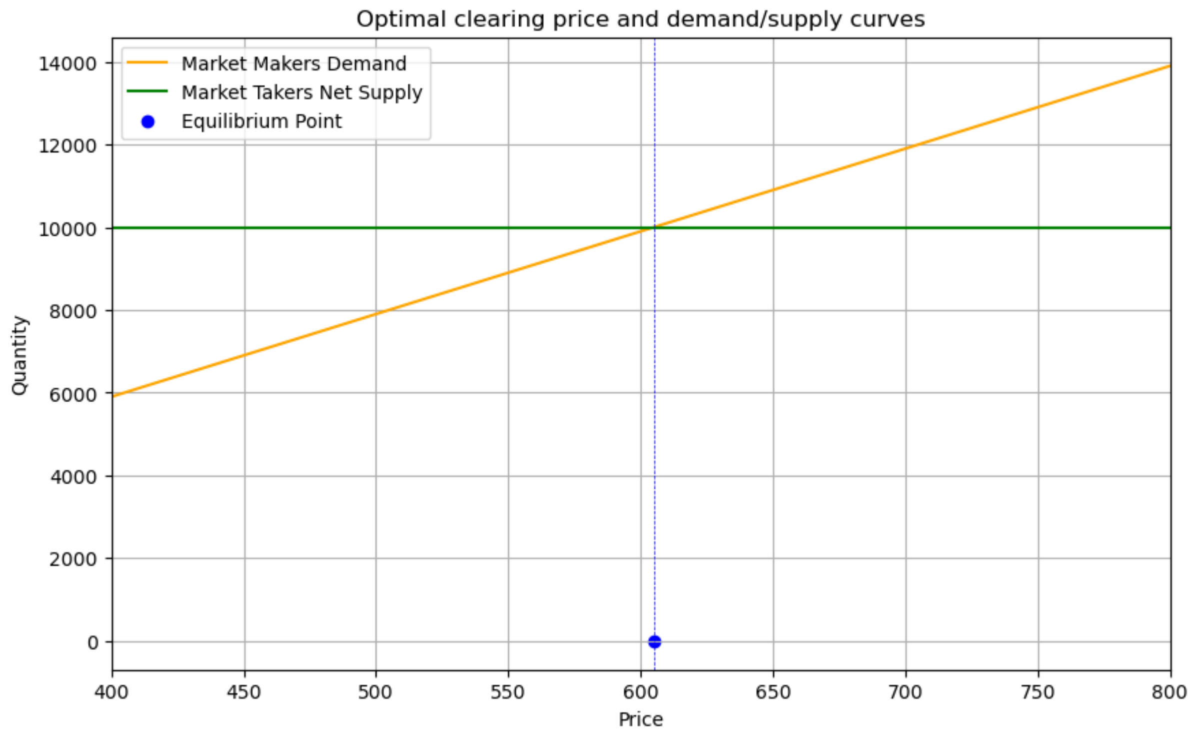


Figure 1: Supply/demand curves and optimal clearing price for $\tilde{P}_1 = 100, \tilde{P}_2 = 110$

2.4 Quality of Price Formation Process Metric

Finally, we need to implement a setup that allows to verify the quality of this clearing price: how close is the clearing price to the efficient price? For which auction duration h are is the clearing price closest to the efficient price? Let $h > 0$. We consider . Let $t \in \mathbb{R}_+$. For $s \in [0, t]$, we set $(\bar{P}^{\mathrm{cl}}_s, \bar{P}_s) = (P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_i}, P_{\tau^{\mathrm{cl}}_i})$ with $i = \sup\{j \geqslant 1 : \tau^{\mathrm{cl}}_j \leqslant s\}$. The quadratic deviation between the clearing price and the efficient price up to time t is:

$$Z^h_t = \int_0^t \left(\bar{P}^{\mathrm{cl}}_s - \bar{P}_s\right)^2 \mathrm{d}s.$$

Definition 2.2. *A stochastic process $\{X_t;\, t \in \mathbb{R}_+\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be regenerative if there exists a non-negative increasing sequence $(t_k)_{k\in\mathbb{N}}$ such that all processes $\{X_{t_k+s};\, s \in \mathbb{R}_+\}$ are distributed as $\{X_{t_0+s};\, s \in \mathbb{R}_+\}$ and are independent of $\{X_s;\, s \in [0, t_k[\}$ for $k \in \mathbb{N}^*$.*

Intuitively, a regenerative process can be decomposed into i.i.d. cycles. In our context, those time events correspond to the auction openings, the cycles being the auction intervals.

Assumption 5. *The process $\{(\bar{P}^{\mathrm{cl}}_s - \bar{P}_s)^2;\, s \in \mathbb{R}_+\}$ is regenerative across auction intervals.*

Under the above assumption, Jusselin et. al (2020) were able to show the following result, leading up to our key quality of price formation metric.

Lemma 2.1. $Z^h_t/t \xrightarrow[t\to+\infty]{\text{a.s.}} \mathbb{E}[(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_1} - P_{\tau^{\mathrm{cl}}_1})^2]$.

We define the function E as $E(h) = \mathbb{E}[(P^{\mathrm{cl}}_{\tau^{\mathrm{cl}}_1} - P_{\tau^{\mathrm{cl}}_1})^2]$ and seek to determine h minimizing the quantity $E(h)$. Our optimization problem is therefore

$$\begin{aligned} \text{minimize} \quad & E(h) \\ \text{s.t.} \quad & h > 0 \end{aligned}$$

with the goal to derive the optimal auction duration h^*. One is able to find a closed form expression for the function E, given by theorem 2.1.

Theorem 2.1. *The quality-of-price-formation metric satisfies*

$$E(h) = E^{\mathrm{mid}}(h) + \frac{\mathbb{E}\big[I^2_{\tau^{\mathrm{cl}}_1}\big]}{K^2}\Big(1 - e^{-\mu h}\frac{\nu}{\nu+\mu}\Big)^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-(\nu+\mu)t} \int_0^{\mu t} \frac{1}{s} \int_0^s \frac{e^u - 1}{u}\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}t,$$

with the mid-price error term

$$E^{\mathrm{mid}}(h) = \Big(1 - e^{-\mu h}\frac{\nu}{\nu+\mu}\Big)^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t}\Big[\big(\sigma_f^2 \frac{t}{6} + \sigma^2\big) e^{-\mu t} \int_0^{\mu t} \frac{e^s - 1}{s}\, \mathrm{d}s + \sigma_f^2 \frac{t}{3}\big(1 - e^{-\mu t}\big)\Big] \mathrm{d}t.$$

Here, the first term $E^{\mathrm{mid}}(h)$ captures the estimation noise of the continuous-time market maker, while the second term featuring $\mathbb{E}[I^2_{\tau^{\mathrm{cl}}_1}]$ quantifies the order-flow imbalance penalty. Crucially, Theorem 2.1 makes explicit how the auction cadence h trades off "averaging" more quotes (reducing estimation error) against the diffusion-driven volatility risk of the efficient price.

Up to this point, all market orders were modeled as exogenous Poisson flows with fixed intensity $\nu > 0$. What if large traders themselves time their orders to minimize expected execution costs? By modeling buyers and sellers as players choosing arrival intensities, they embed the auction design in a two-player game, opening the door to equilibrium analysis of strategic order timing. This is the object of the following section.

3 Strategic Market Takers

In this whole section we fix $0 < \lambda_- \leqslant \lambda_+$. Initially, the intensity of market takers is exogenous: for given intensities, we simply want to find the optimal auction duration and the clearing price. Now, we suppose that market takers control their trading intensities. We consider that market takers aim at minimizing their trading costs by adapting their trading intensities to the market state. Building on the Nash equilibrium framework established in the original paper, we further explore the additional case of a partially strategic setting where only the seller is strategic, for which we formulate and numerically solve the corresponding Hamilton-Jacobi-Bellman (HJB) equation. This extension provides a richer understanding of how strategic behavior and asymmetry impact auction efficiency and optimal timing. We formalize this into a competitive game between buying and selling market takers. We will first tackle the change of measure towards $\mathbb{P}^{\lambda^a,\lambda^b}$, which is crucial to construct a controlled auction market probability. It is important to note that we do not look at the sequential character of auctions in this section. Instead, we look at the optimal control for a specific auction of duration h.

Definition 3.1 (Admissible control)**.** *An admissible control is a $\mathbb{F}$-predictable process with values in $[\lambda_-, \lambda_+]$. The set of admissible controls is denoted $\mathcal{U}$.*

Consider a couple of admissible controls $(\lambda^a, \lambda^b) \in \mathcal{U}^2$. Via Theorem 1 of [5], and as stated in [4], this couple induces a probability measure $\mathbb{P}^{\lambda^a,\lambda^b}$ on $(\Omega, \mathcal{F})$, under which the processes $N_t^a - \int_0^t \lambda_s^a \,\mathrm{d}s$ and $N_t^b - \int_0^t \lambda_s^b \,\mathrm{d}s$ are martingales. Under this new probability measure, the intensities λ^a and λ^b appear as true arrival rates, allowing to study the controlled auction market.

3.1 Non-Homogeneous Poisson Flows

Empirically, liquidity demand is rarely stationary: news releases, macro data, and the auction's own countdown skew the flow of incoming market orders. To capture this, we let buy arrivals accelerate while sell arrivals decelerate as the clearing time approaches, translating a sometimes observed "rush to buy versus reluctance to sell" behavior. In this subsection we thus assume that the market-taker arrivals follow independent non-homogeneous Poisson processes with deterministic intensities

$$\lambda_t^a = e^{c_a t}, \quad \lambda_t^b = e^{-c_b t}, \qquad c_a, c_b > 0.$$

By renewal-theoretic arguments one shows that $E(h)$ decomposes into a mid-price noise term and an order-imbalance penalty:

$$E(h) = E_{\mathrm{mid}}(h) + \frac{G(h)}{K^2}\mathbb{E}\big[I_{\tau+h}^2\big].$$

The variance of the imbalance admits the representation

$$\mathbb{E}\big[I_{\tau+h}^2\big] = \int_0^\infty \left[A(u) + B(u) + (A(u) - B(u))^2\right] f_\tau(u)\mathrm{d}u,$$

where

$$\Lambda(u) = \int_0^u \big(e^{c_a s} + e^{-c_b s}\big)\mathrm{d}s, \quad f_\tau(u) = \big(e^{c_a u} + e^{-c_b u}\big)e^{-\Lambda(u)},$$

and

$$A(u) = \int_u^{u+h} e^{c_a s}\mathrm{d}s = \frac{e^{c_a(u+h)} - e^{c_a u}}{c_a}, \quad B(u) = \int_u^{u+h} e^{-c_b s}\mathrm{d}s = \frac{e^{-c_b u} - e^{-c_b(u+h)}}{c_b}.$$

The two components of the decomposition are given by

$$E_{\mathrm{mid}}(h) = \Big(1 - e^{-\mu h}\frac{\nu}{\nu+\mu}\Big)^{-1} e^{\nu h}\int_h^\infty \nu e^{-\nu t}\Big[\Big(\frac{\sigma_f^2\, t}{6} + \sigma^2\Big)e^{-\mu t}\int_0^{\mu t}\frac{e^s-1}{s}\mathrm{d}s + \frac{\sigma_f^2 t}{3}\big(1-e^{-\mu t}\big)\Big]\mathrm{d}t,$$

$$G(h) = \Big(1 - e^{-\mu h}\frac{\nu}{\nu+\mu}\Big)^{-1} e^{\nu h}\int_h^\infty \nu e^{-\nu t}\int_0^{\mu t}\frac{e^s-1}{s}\mathrm{d}s\mathrm{d}t.$$

3.2 Optimal Strategic Seller

First, we try to understand what happens when we isolate the seller's decision problem and look only at the optimal control of the buyer. We freeze the buyer arrival rate at a deterministic schedule $\lambda_t^a = e^{\kappa t}$ for $\kappa > 0$, and let the seller choose her own intensity $\lambda^b \in \mathcal{U}$. We want to solve

$$\sup_{\lambda^b\in\mathcal{U}} \mathbb{E}_{(\alpha,\beta)}^{\mathbb{P}^{\lambda^a,\lambda^b}}\big[N_h^b(N_h^a - N_h^b)\mid (N_0^a, N_0^b) = (\alpha,\beta)\big].$$

Via the verification theorem, the solution is $v(0,\alpha,\beta)$ for v a classical solution to

$$(\mathbf{C})\begin{cases} \partial_1 v(t,n,m) + \sup_{\lambda\in[\lambda_-,\lambda_+]}(v(t,n+1,m) - v(t,n,m))\lambda_t^a + \lambda(v(t,n,m+1) - v(t,n,m)) = 0\\ v(h,n,m) = m(n-m)\end{cases}$$

on $[0,h]\times\mathbb{N}^2$.

Lemma 3.1 (Itô's Formula for 2D Jump Processes)**.** *Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t,x_1,x_2)$ be a function whose first and second partial derivatives appearing below are defined and continuous. Then*

$$\begin{aligned} f\big(t,X_1(t),X_2(t)\big) = f\big(0,X_1(0),X_2(0)\big) &+ \int_0^t \partial_1 f\big(s,X_1(s),X_2(s)\big)\mathrm{d}s\\ &+ \int_0^t \partial_2 f\big(s,X_1(s),X_2(s)\big)\mathrm{d}X_1^c(s) + \int_0^t \partial_3 f\big(s,X_1(s),X_2(s)\big)\mathrm{d}X_2^c(s)\\ &+ \frac{1}{2}\int_0^t \partial_2^2 f\big(s,X_1(s),X_2(s)\big)\mathrm{d}X_1^c(s)\mathrm{d}X_1^c(s)\\ &+ \int_0^t \partial_{2,3} f\big(s,X_1(s),X_2(s)\big)\mathrm{d}X_1^c(s)\mathrm{d}X_2^c(s)\\ &+ \frac{1}{2}\int_0^t \partial_3^2 f\big(s,X_1(s),X_2(s)\big)\mathrm{d}X_2^c(s)\,\mathrm{d}X_2^c(s)\\ &+ \sum_{0<s\leqslant t}\Big[f\big(s,X_1(s),X_2(s)\big) - f\big(s,X_1(s^-),X_2(s^-)\big)\Big].\end{aligned}$$

The above lemma, taken from Theorem 11.5.4 of [6], yields Itô's formula for 2D jump processes. Applying the lemma to a map $v\in\mathcal{F}([0,h]\times\mathbb{N}^2,\mathbb{R})$ (map from $[0,h]\times\mathbb{N}^2$ into $\mathbb{R}$) and jump processes N^a and N^b gives the following differential form

$$\begin{aligned}\mathrm{d}v(t,N_t^a,N_t^b) =& \partial_1 v(t,N_t^a,N_t^b) + (v(t,N_t^a+1,N_t^b) - v(t,N_t^a,N_t^b)\mathrm{d}N_t^a\\ &+ (v(t,N_t^a,N_t^b+1) - v(t,N_t^a,N_t^b))\mathrm{d}N_t^b\end{aligned}$$

since N^a and N^b jump one at a time (no second order terms). Let $M_t^a = N_t^a - \int_0^t\lambda_s^a\mathrm{d}s$ and $M_t^b = N_t^b - \int_0^t\lambda_s^b\mathrm{d}s$. We know that M^a and M^b are martingales under $\mathbb{P}^{\lambda^a,\lambda^b}$. Thus,

$$\begin{aligned}\mathrm{d}v(t,N_t^a,N_t^b) =& \partial_1 v(t,N_t^a,N_t^b) + (v(t,N_t^a+1,N_t^b) - v(t,N_t^a,N_t^b))e^{\kappa t}\\ &+ (v(t,N_t^a,N_t^b+1) - v(t,N_t^a,N_t^b))\lambda_t^b + \mathrm{d}\tilde{M}_t\end{aligned}$$

where $\tilde{M}$ is a $\mathbb{P}^{\lambda^a,\lambda^b}$-martingale. Consider the map v defined as

$$\forall (t,n,m) \in [0,h] \times \mathbb{N}^2, \quad v(t,n,m) = \sup_{\lambda^b \in \mathcal{U}} \mathbb{E}^{\mathbb{P}^{\lambda^a,\lambda^b}} \left[N_h^b (N_h^a - N_h^b) \mid (N_t^a, N_t^b) = (n,m) \right]$$

The dynamic programming principle states that

$$v(t,n,m) = \sup_{\lambda^b \in \mathcal{U}} \mathbb{E}^{\mathbb{P}^{\lambda^a,\lambda^b}} \left[v(t+\varepsilon, N_{t+\varepsilon}^a, N_{t+\varepsilon}^b) \mid (N_t^a, N_t^b) = (n,m) \right]$$

for some small $\varepsilon > 0$. We approximate the difference $v(t+\varepsilon, N_{t+\varepsilon}^a, N_{t+\varepsilon}^b) - v(t, N_t^a, N_t^b)$ with $\mathrm{d}v(t, N_t^a, N_t^b)$. To satisfy the dynamic programming principle principle, the value function v associated to Player b must verify:

$$\sup_{\lambda \in [\lambda_-, \lambda_+]} \partial_1 v(t,n,m) + (v(t,n+1,m) - v(t,n,m))e^{\kappa t} + (v(t,n,m+1) - v(t,n,m))\lambda = 0$$

for all $t \in [0,h[$ and all $(n,m) \in \mathbb{N}^2$, as well as $v(h,n,m) = m(n-m)$. Thus, v must be a solution of $(\mathbf{C})$. Additionally, by definition, $v(0,\alpha,\beta)$ is the quantity (3.2).

Proposition 3.1. *Define, for all $(t,n,m) \in [0,h[\times \mathbb{N}^2$,*

$$\lambda^\dagger(t,n,m) = \mathrm{proj}_{[\lambda_-,\lambda_+]} \left(\frac{\kappa[e^{\kappa t}(h-t) + n - 2m - 1] + e^{\kappa h} - e^{\kappa t}}{4\kappa(h-t)} \right).$$

and

$$v_\lambda(t,n,m) = m(n-m) + m\kappa^{-1}(e^{\kappa h} - e^{\kappa t}) + \lambda(h-t) \left(\frac{e^{\kappa h} - e^{\kappa t}}{\kappa} + \lambda(h-t) + n - 1 - 2m \right)$$

for $\lambda \in [\lambda_-, \lambda_+]$. Then $v_{\lambda^\dagger}$ is a (viscosity) solution of $(\mathbf{C})$*.*

Proof. Consider the solution v_λ to the equation

$$(\mathbf{C}^\lambda) \begin{cases} \partial_1 v(t,n,m) + (v(t,n+1,m) - v(t,n,m))\lambda_t^a + \lambda(v(t,n,m+1) - v(t,n,m)) = 0 \\ v(h,n,m) = m(n-m) \end{cases}$$

on $[0,h] \times \mathbb{N}^2$. Given the terminal condition, consider the *Ansatz* given by

$$v_\lambda(t,n,m) = A_2^\lambda(t)m^2 + A_1^\lambda(t)m + A_0^\lambda(t)nm + B_1^\lambda(t)n + B_0^\lambda(t).$$

Plugging v_λ into $(\mathbf{C}^\lambda)$ yields $A_2^\lambda = -1$ and $A_0^\lambda = 1$, such that (as $A_1^\lambda(h) = 0$):

$$\forall t \in [0,h], \quad A_1^\lambda(t) = \int_t^h (e^{\kappa s} - 2\lambda)\,\mathrm{d}s = \frac{e^{\kappa h} - e^{\kappa t}}{\kappa} - 2\lambda(h-t).$$

Next, one finds $B_1^\lambda(t) = \lambda(h-t)$ and (via a few computations):

$$\begin{aligned} \forall t \in [0,h], \quad B_0^\lambda(t) &= \int_t^h \left[e^{\kappa s}\lambda(h-s) + \lambda \left(\frac{e^{\kappa h} - e^{\kappa s}}{\kappa} - 2\lambda(h-s) - 1 \right) \right] \mathrm{d}s \\ &= \lambda(h-t) \left(\frac{e^{\kappa h} - e^{\kappa t}}{\kappa} + \lambda(h-t) - 1 \right) \end{aligned}$$

Thus, we consider the solution v_λ for $(\mathbf{C}^\lambda)$ given by

$$v_\lambda(t,n,m) = m(n-m) + m\kappa^{-1}(e^{\kappa h} - e^{\kappa t}) + \lambda(h-t) \left(\frac{e^{\kappa h} - e^{\kappa t}}{\kappa} + \lambda(h-t) + n - 1 - 2m \right).$$

We notice that

$$v_\lambda(t,n+1,m) - v_\lambda(t,n,m) = m + \lambda(h-t)$$
$$v_\lambda(t,n,m+1) - v_\lambda(t,n,m) = n - 2m - 1 + \frac{e^{\kappa h} - e^{\kappa t}}{\kappa} - 2\lambda(h-t)$$

For the control problem (**C**), we must maximize point-wise

$$\begin{aligned} H(\lambda) &= v_\lambda(t,n+1,m) - v_\lambda(t,n,m))e^{\kappa t} + \lambda(v_\lambda(t,n,m+1) - v_\lambda(t,n,m)) \\ &= me^{\kappa t} + \lambda\left(e^{\kappa t}(h-t) + n - 2m - 1 + \frac{e^{\kappa h} - e^{\kappa t}}{\kappa}\right) - 2(h-t)\lambda^2. \end{aligned}$$

Because $-2(h-t) < 0$ for all $t < h$, there exists a unique maximizer of the strictly concave function H (over $\mathbb{R}$), whose projection on $[\lambda_-, \lambda_+]$ writes

$$\lambda^\dagger(t,n,m) = \mathrm{proj}_{[\lambda_-,\lambda_+]}\left(\frac{\kappa(e^{\kappa t}(h-t) + n - 2m - 1) + e^{\kappa h} - e^{\kappa t}}{4\kappa(h-t)}\right).$$

Define $v \in \mathcal{F}([0,h] \times \mathbb{N}^2, \mathbb{R})$ as $v(t,n,m) = v_{\lambda^\dagger(t,n,m)}(t,n,m)$ for all $(t,n,m) \in [0,h[\times\mathbb{N}^2$ and $v(h,n,m) = m(n-m)$. Since $\lambda^\dagger$ maximizes H and $v_{\lambda^\dagger}$ is a solution to $(\mathbf{C}^{\lambda^\dagger})$, we have

$$\begin{aligned} &\partial_1 v(t,n,m) + \sup_{\lambda\in[\lambda_-,\lambda_+]} \big[v(t,n+1,m) - v(t,n,m))e^{\kappa t} + \lambda(v(t,n,m+1) - v(t,n,m)\big] \\ &= \partial_t v(t,n,m) + \sup_{\lambda\in[\lambda_-,\lambda_+]} \Big[\big(v_{\lambda^\dagger}(t,n+1,m) - v_{\lambda^\dagger}(t,n,m)\big)e^{\kappa t} \\ &\qquad\qquad + \lambda\big(v_{\lambda^\dagger}(t,n,m+1) - v_{\lambda^\dagger}(t,n,m)\big)\Big] \\ &= \partial_1 v_{\lambda^\dagger(t,n,m)}(t,n,m) + v_{\lambda^\dagger(t,n,m)}(t,n+1,m) - v_{\lambda^\dagger(t,n,m)}(t,n,m))e^{\kappa t} \\ &\quad + \lambda^\dagger(t,n,m)(v_{\lambda^\dagger(t,n,m)}(t,n,m+1) - v_{\lambda^\dagger(t,n,m)}(t,n,m) \\ &= 0. \end{aligned}$$

This shows that v is a solution of (**C**). However, since v is not $\mathcal{C}^1([0,h[)$ with respect to its first variable, v is not a classical solution, although the only "weak" irregularity allows to show that v is a viscosity solution to (**C**). □

We now move on to the numerical experiments. In Section 4 we will study how the price-formation error $E(h)$ behaves as we vary h and the key model parameters. This will give us a concrete sense of the trade-offs highlighted in Section 2, without getting bogged down in further theoretical detail.

4 Numerical Simulations

All the `Python` code used to compute the following plots and results can be obtained by request by email. This section is not anymore about strategic trader (as was section 3) and we go back to the orginial paper of Jusselin et al. (2021) (the intensity is constant now again in particular).

4.1 Price-Formation Error and Comparison to CLOB

We focus on the case where market takers arrive according to the independent Poisson processes. In particular, both buyers and sellers correspond to the counting processes N^a and N^b, each with intensity $\nu/2$. In this particular case the $\mathbb{E}\left[I^2_{\tau_1^{\mathrm{cl}}}\right]$ term from Theorem 2.1 can be expressed as:

$$\mathbb{E}\left[I^2_{\tau_1^{\mathrm{cl}}}\right] = v^2(\nu h + 1)$$

This setting allows us to derive an explicit formula for calculating the error of an auction from Theorem 2.1. Based on the explicit formula, we can find the optimal duration h^* minimizing the error function $E(h)$. To illustrate the behavior of $E(h)$ under different market conditions, we run simulations with two distinct sets of auction parameters.

Parameter	Simulation 1	Simulation 2	Simulation 3	Simulation 4
ν	1	1	1	0.05
μ	0.05	0.5	0.1	0.1
Common Parameters				
K	50			
σ	0.2			
σ_f	0.3			
v	100			

Table 1: Simulation parameters for four auction scenarios. Only ν and μ vary between simulations.

This set of simulations allows us to highlight different types of market environments by adjusting the arrival intensities of market makers and market takers. As shown in the table, in the first simulation there are significantly more buyers and sellers relative to the number of market makers, compared to the second simulation. In this way, we simulate two very different market conditions. Consequently, we can expect distinct behavior of the error function and different optimal outcomes when choosing between CLOB and an auction.

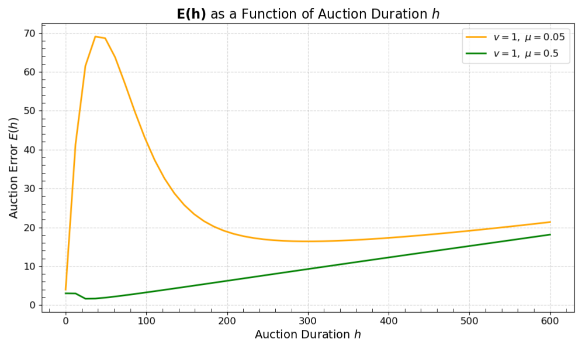


Figure 2: Auction error as a function of duration for Simulations 1 ($\nu = 1, \mu = 0.05$) and 2 ($\nu = 1, \mu = 0.5$).

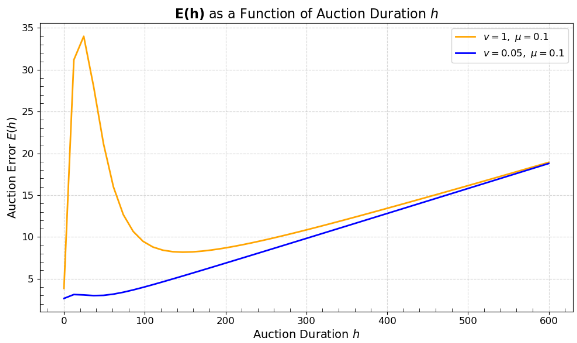


Figure 3: Auction error as a function of duration for Simulations 3 ($\nu = 1, \mu = 0.1$) and 4 ($\nu = 0.05, \mu = 0.1$).

As we can see from Figure 2, for Simulation 2 the optimal duration h^* is positive, which suggests that the auction is the optimal choice. In contrast, in Simulation 1 the optimal duration h^* is zero, indicating that the continuous limit order book (CLOB) is the preferable option. This result could be explained by the trade-off between the averaging effect and volatility risk. In Simulation 2, there are more market makers: a longer auction duration allows a larger number of participants to engage, reducing uncertainty about the efficient price through a significant averaging effect. Therefore, a positive auction duration is optimal. In Simulation 1, a longer auction duration does not attract as many participants but still increases volatility risk. As a result, $h^* = 0$ (i.e., using the CLOB) is preferable in this case. Simulations 3 and 4 highlight a similar effect and trade-off, viewed from the perspective of market takers. Interestingly, in Simulation 4 – where there are fewer market takers (while keeping the intensity of market maker arrivals constant) – the auction performs nearly as well as the CLOB. This suggests that the averaging effect and volatility risk are more balanced in this scenario. In contrast, Simulation 3 shows that volatility risk dominates, making the CLOB ($h^* = 0$) the clearly preferable choice.

Overall, the numerical results provide additional evidence for the hypothesis that *one size does not fit all*: the decision between using an auction or a CLOB depends strongly on the specific characteristics and parameters of the market. For instance, taking $\nu = 1$ and $\mu = 0.5$, one obtains the following plot for the function E.

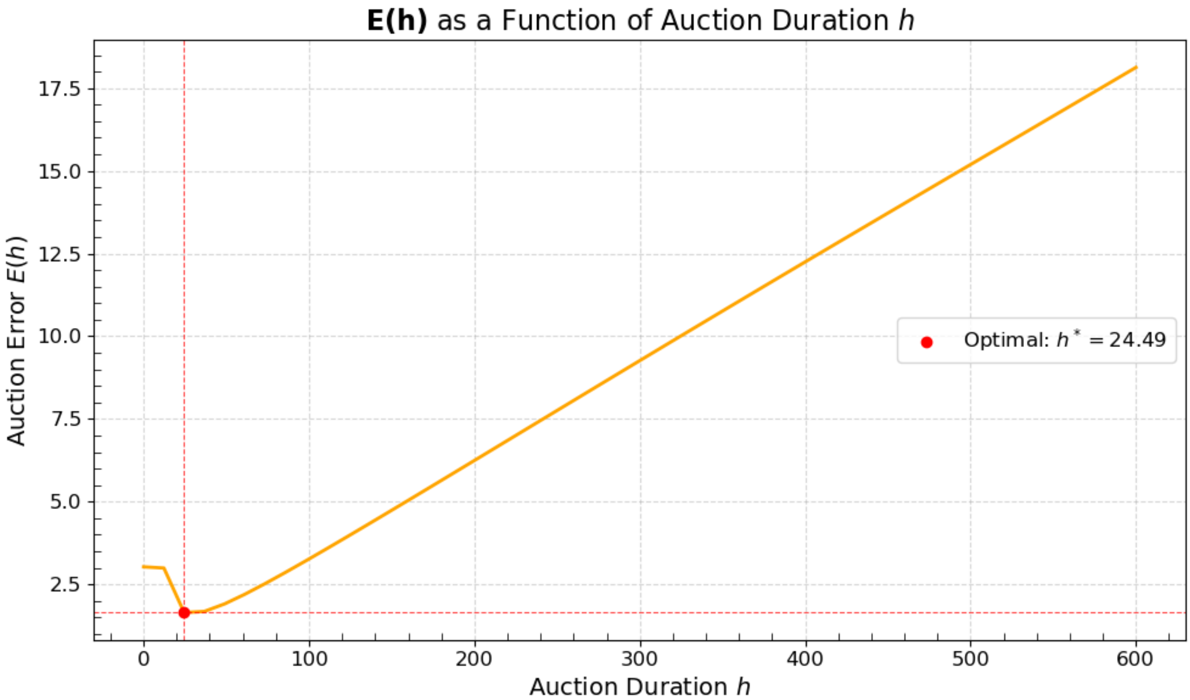


Figure 4: Auction error function for $\nu = 1, \mu = 0.5$

We see that $h^* = 24.49$ and $E(h^*) = 1.65$. Compared to the price-formation error of the CLOB, that is $E(0) = 3.03$, we achieve a 45.59% decrease. To conclude, with typical large-cap equity parameters ($\nu = 1$ and $\mu = 0.5$) our model prescribes batch length 24.5 seconds, shaving 45.6% off price-formation error relative to the CLOB.

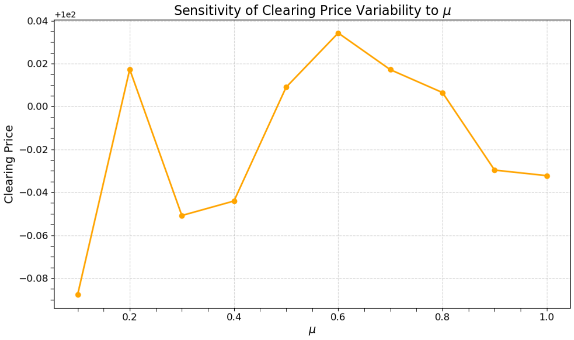
4.2 Clearing Price Sensitivity Analysis

To provide more insights about the clearing price, we implement a simulation of an one round of auction. The simulation follows previous assumptions about N^a and N^b both having intensity $\nu/2$. Moreover, since it is assumed to be the first round of the auction, market makers views on the price are assumed to be efficient prices at the time of their arrival with noise $g_k \sim \mathcal{N}(0, \sigma^2)$, thus we write $\tilde{P}_k = P_{\tau_k^{i,\mathrm{mm}}} + g_k$. The baseline parameters are shown in the Table 2.

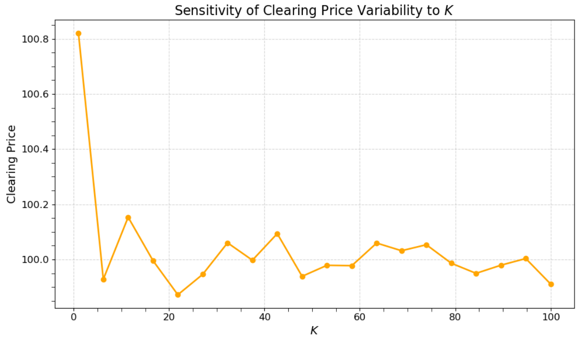
P_0	σ_f	σ	K	μ	ν	v	h
100	0.3	0.2	50	0.5	1	100	50

Table 2: Baseline simulation parameters used for sensitivity analysis.

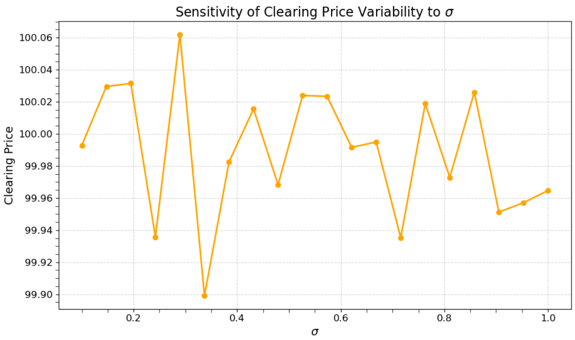
We present 2 sets of results related to the sensitivity with respect to the parameters μ, σ, σ_f, K. First, in Figure 5 we show the average clearing price computed from 1,000 simulations for varying values of the parameters of interest. These plots should be interpreted by observing how the clearing price changes across different parameter values. For example, from Figure 5b, we observe that for smaller values of K, the clearing price is more sensitive to changes in K (i.e., has a steeper slope), whereas for larger K (representing stronger supply from market makers), the clearing price appears much more stable.



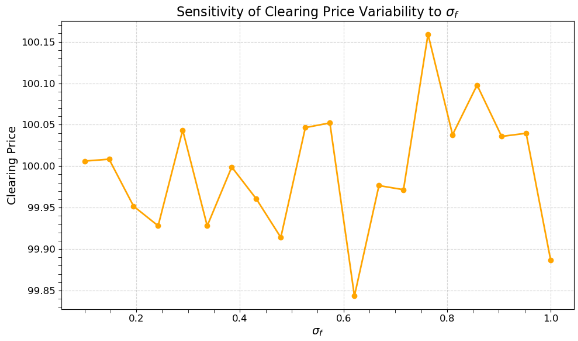
(a) Clearing price sensitivity to μ



(b) Clearing price sensitivity to K



(c) Clearing price sensitivity to σ



(d) Clearing price sensitivity to σ_f

Figure 5: Sensitivity analysis of clearing price (clearing price values)

Figure 6 conveys similar insights in a different form – it displays the derivative of the clearing price with respect to the parameter of interest. This representation captures how the clearing price responds to small changes in a given parameter, similar to the Greeks in option pricing. To compute these values numerically, we use the central difference approximation. For example, for the parameter μ the derivative is estimated using:

$$\frac{\mathrm{d}P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}}}{\mathrm{d}\mu} \approx \frac{P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}}(\mu+\varepsilon) - P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}}(\mu-\varepsilon)]}{2\varepsilon}$$

for some small $\varepsilon > 0$. Note that in the case of sensitivity with respect to K, formula (1) yields that

$$\frac{\mathrm{d}P^{\mathrm{cl}}_{\tau_i^{\mathrm{cl}}}}{\mathrm{d}K} = -\frac{1}{K^2}\frac{I^i_{\Delta_i}}{N^{i,\mathrm{mm}}_{\Delta_i}}$$

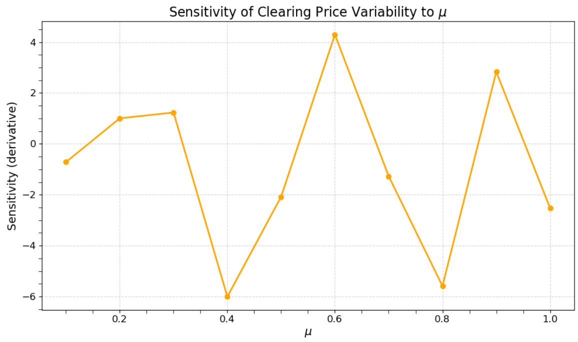
for the i-th auction period. This inverse squared dependence is confirmed by plot 6b.

Figure 6 provides a more direct measure of local sensitivity. For instance, as σ_f (the volatility of the true efficient price) increases, the clearing price becomes naturally more sensitive to perturbations, as reflected by the derivative moving further away from zero.

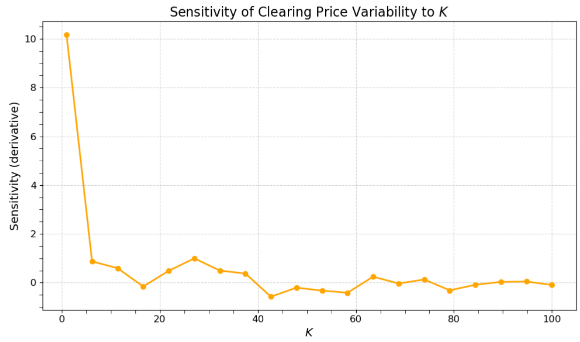
5 Extensions

5.1 Risk-Averse Price-Formation Error

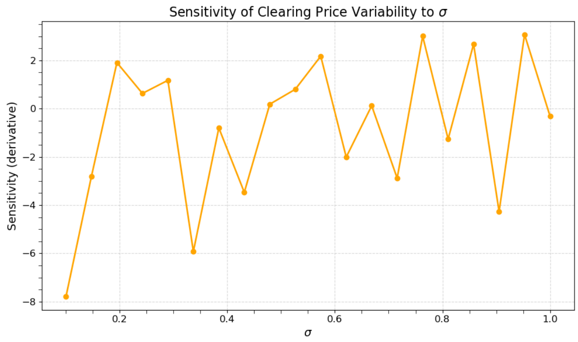
One can generalize the E-function framework by replacing the quadratic loss with a risk-sensitive criterion. Define $E_\rho(h) = \mathbb{E}\left[e^{\rho|P^{\mathrm{cl}}_h - P^*_h|^2}\right]$, where $\rho > 0$ is an aversion parameter. Since the map $x \mapsto \exp(\rho x^2)$ is increasing in x, this metric penalizes large deviations more heavily than the second-moment $E(h)$.



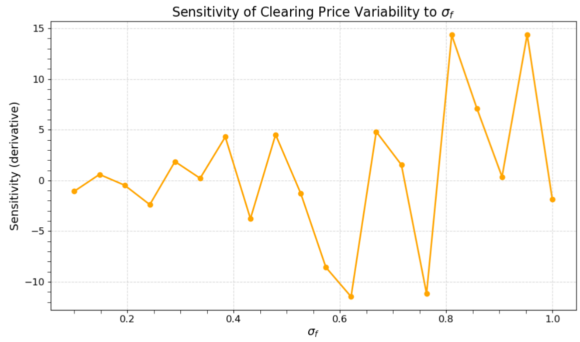
(a) Clearing price sensitivity to μ



(b) Clearing price sensitivity to K



(c) Clearing price sensitivity to σ



(d) Clearing price sensitivity to σ_f

Figure 6: Sensitivity analysis of clearing price (derivative)

In the future, one could work on an analogous decomposition of E_ρ into a risk-adjusted mid-price term and a risk-adjusted imbalance penalty as explicit functions of ρ and the other, usual, parameters.

5.2 Nash Equilibrium

Having characterized the seller's best-response to a fixed buyer flow, we now lift the exogeneity assumption on λ_t^a and allow both buyers and sellers to choose their arrival intensities $(\lambda_t^a, \lambda_t^b)$ strategically. We seek a pair such that each side solves the below HJB problem given the other's strategy.

$$(\mathbf{G}) \begin{cases} \inf_{\lambda^a \in \mathcal{U}} V_h^{a,\alpha,\beta}(\lambda^a, \lambda^b) = \mathbb{E}^{\mathbb{P}^{\lambda^{a,*},\lambda^{b,*}}} \left[N_h^a \left(N_h^a - N_h^b \right) \mid (N_0^a, N_0^b) = (\alpha, \beta) \right] \\ \inf_{\lambda^b \in \mathcal{U}} V_h^{b,\alpha,\beta}(\lambda^a, \lambda^b) = \mathbb{E}^{\mathbb{P}^{\lambda^{a,*},\lambda^{b,*}}} \left[N_h^b \left(N_h^b - N_h^a \right) \mid (N_0^a, N_0^b) = (\alpha, \beta) \right] \end{cases}$$

where we have defined the maps $V_h^{a,\alpha,\beta}(\lambda^a, \lambda^b) = \mathbb{E}^{\mathbb{P}^{\lambda^a,\lambda^b}} \left[N_h^a \left(N_h^a - N_h^b \right) \mid (N_0^a, N_0^b) = (\alpha, \beta) \right]$ and $V_h^{b,\alpha,\beta}(\lambda^a, \lambda^b) = \mathbb{E}^{\mathbb{P}^{\lambda^a,\lambda^b}} \left[N_h^b \left(N_h^b - N_h^a \right) \mid (N_0^a, N_0^b) = (\alpha, \beta) \right]$. We seek to obtain the existence of a Markovian control that is a Nash equilibrium to $(\mathbf{G})$. We refer to the simplified version of Definition 2.10 of [2] of a Markovian control.

Definition 5.1 (Markovian control)**.** *A control process $\lambda^i = \{\lambda_t^i;\, t \in [0, h]\}$ for player $i \in \{a, b\}$ is called Markovian if there exists a measurable function $\phi^i \in \mathcal{F}([0, h] \times \mathbb{N}^2, [\lambda_-, \lambda_+])$ such that for all $t \in [0, h]$, $\mathbb{P}^{\lambda^a,\lambda^b}$-almost surely $\lambda_t^i = \phi^i\big(t, N_t^a, N_t^b\big)$ holds.*

In other words, each player's intensity at time t depends only on the current time and the current state (N_t^a, N_t^b), not on the full past history.

Theorem 5.1. *There exists a Nash equilibrium to the simultaneous optimization problem* (**G**) *given by Markovian controls* $(\lambda^{a,*},\lambda^{b,*})$[1] *satisfying*

$$\inf_{\lambda^a\in\mathcal{U}} V_h^{a,\alpha,\beta}(\lambda^a,\lambda^{b,*}) = \mathbb{E}^{\mathbb{P}^{\lambda^{a,*},\lambda^{b,*}}}\left[N_h^a\left(N_h^a - N_h^b\right) \mid (N_0^a,N_0^b)=(\alpha,\beta)\right],$$
$$\inf_{\lambda^b\in\mathcal{U}} V_h^{b,\alpha,\beta}(\lambda^{a,*},\lambda^b) = \mathbb{E}^{\mathbb{P}^{\lambda^{a,*},\lambda^{b,*}}}\left[N_h^b\left(N_h^b - N_h^a\right) \mid (N_0^a,N_0^b)=(\alpha,\beta)\right].$$

Under the Nash equilibrium $(\lambda^{a,*},\lambda^{b,*})$, we are able to compute $\mathbb{E}\big[I^2_{\tau_1^{\mathrm{op}}+h}\big]$ explicitly, as given by Corollary 5.1. Therefore, circling back to Theorem 2.1, we could compute $E(h)$ explicitly, and thus determine the optimal auction duration under the control of the Nash equilibrium $(\lambda^{a,*},\lambda^{b,*})$.

Corollary 5.1. *Under the Nash equilibrium* $(\lambda^{a,*},\lambda^{b,*})$*, we have*

$$\mathbb{E}\big[I^2_{\tau_1^{\mathrm{op}}+h}\big] = V_h^{a,1,0}\left(\lambda^{a,*},\lambda^{b,*}\right) + V_h^{b,1,0}\left(\lambda^{a,*},\lambda^{b,*}\right).$$

For any $z,\tilde z\in\mathbb{R}^2$ and any $\varepsilon\in[\lambda_-,\lambda_+]$, we set $H^{a,*}(z,\tilde z,\varepsilon) = z_1\lambda^{a,*}(z,\varepsilon)+z_2\lambda^{b,*}(\tilde z,\varepsilon)$ and $H^{b,*}(z,\tilde z,\varepsilon) = z_2\lambda^{b,*}(z,\varepsilon)+z_1\lambda^{a,*}(\tilde z,\varepsilon)$. For $x\in\mathbb{N}^2$ we define $g^a(x)=x_1\big(x_1-x_2\big)$ and $g^b(x)=x_2\big(x_2-x_1\big)$. Finally, let U be a map from $[0,h]\times\mathbb{N}^2$ into $\mathbb{R}$. For any $(s,\alpha,\beta)\in[0,h]\times\mathbb{N}^2$ we set

$$(\mathbf{D})\begin{cases} D_aU(s,\alpha,\beta) = U(s,\alpha+1,\beta)-U(s,\alpha,\beta)\\ D_bU(s,\alpha,\beta) = U(s,\alpha,\beta+1)-U(s,\alpha,\beta)\\ DU(s,\alpha,\beta) = \big(D_aU(s,\alpha,\beta),D_bU(s,\alpha,\beta)\big)^\top\end{cases}.$$

System (**D**) models discrete derivatives, whereas g^i for $i\in\{a,b\}$ models a terminal payoff, and $H^{i,*}$ models the generator for the dynamics of player i. For more context, the two-player game can be characterized as a system of coupled ODEs. The derivation of the system (**S**) below follows from Itô's formula for jump processes, just as in Section 3.2. The maps $\varepsilon^a,\varepsilon^b$ arise because each Hamiltonian is linear in its maximization argument. A linear map is maximized at the extreme λ_+ or λ_-, depending on the sign of the derivative of the linear map. Whenever $DV^i=0$, the optimizer is however no longer unique, and one selects a measurable "threshold" control $\varepsilon^i\in[\lambda_-,\lambda_+]$ to smooth the transition between the two bang-bang regimes. This leads to the following choice of controls:

$$(\mathbf{L})\begin{cases} \lambda^{a,*}(z,\varepsilon^a) = \mathbb{1}_{\{z_1>0\}}\lambda_- + \mathbb{1}_{\{z_1<0\}}\lambda_+ + \varepsilon^a\mathbb{1}_{\{z_1=0\}}\\ \lambda^{b,*}(z,\varepsilon^b) = \mathbb{1}_{\{z_2>0\}}\lambda_- + \mathbb{1}_{\{z_2<0\}}\lambda_+ + \varepsilon^b\mathbb{1}_{\{z_2=0\}}\end{cases}.$$

Moreover, With these definitions and observations in hand, one arrives at the following proposition.

Proposition 5.1. *Assume that there exist two maps* $\varepsilon^a,\varepsilon^b:[0,h]\times\mathbb{N}^2\to[\lambda_-,\lambda_+]$ *such that the following coupled system*

$$(\mathbf{S})\begin{cases} \partial_sV^a + H^{a,*}(DV^a,DV^b,\varepsilon^b)=0, & s\in[0,h[,(\alpha,\beta)\in\mathbb{N}^2\\ V^a(h,\alpha,\beta)=g^a(\alpha,\beta) & \\ \partial_sV^b + H^{b,*}(DV^b,DV^a,\varepsilon^a)=0, & s\in[0,h[,(\alpha,\beta)\in\mathbb{N}^2\\ V^b(h,\alpha,\beta)=g^b(\alpha,\beta) & \end{cases}$$

has a continuously differentiable (in time) solution denoted by (V_a,V_b) *on* $[0,h]\times\mathbb{N}^2$ *and assume moreover that* $DV^i(\cdot,N^a_\cdot,N^b_\cdot)\in\mathcal{H}_h(\mathbb{R}^2)$ *for all* $i\in\{a,b\}$*. Then* $(\lambda^{a,*}(DV^a,\varepsilon^a),\lambda^{b,*}(DV^b,\varepsilon^b))$ *is a Nash equilibrium for* (**G**)*.*

[1] Use [2] to define properly what a Markovian control is.

However, both $\lambda^{a,*}$ and $\lambda^{b,*}$ admit singularities by their definition, and there is a lack of PDE results that allow to solve $(\mathbf{S})$ efficiently. Therefore, Jusselin et al. (2021) decided to introduce a smooth approximation $(\mathbf{S}^n)$ of $(\mathbf{S})$. While the construction is very technical, in essence, one approximates the control function by a smoothed control λ^n (for $n \in \mathbb{N}$) defined as

$$\forall z \in \mathbb{R}, \quad \lambda^n(z) = \lambda_- \mathbb{1}_{]-\infty,1/n]} + \lambda_+ \mathbb{1}_{[1/n,+\infty[} \left(n \frac{\lambda_- - \lambda_+}{2} z + \frac{\lambda_- + \lambda_+}{2} \right) \mathbb{1}_{]-1/n,1/n[}$$

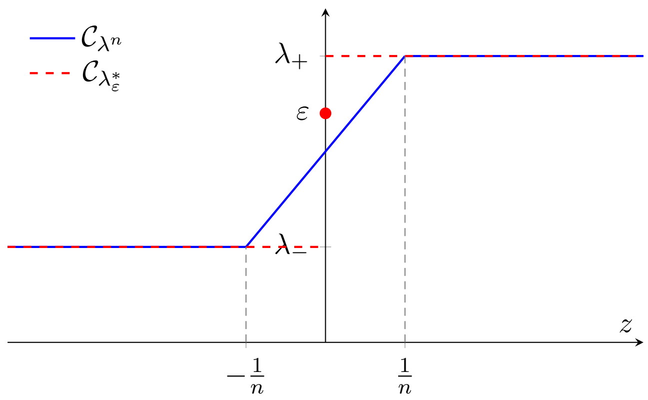


Figure 7: Smoothed control function λ^n and control $\lambda^*_\varepsilon = \lambda_- \mathbb{1}_{\mathbb{R}^*_-} + \varepsilon \mathbb{1}_{\{0\}} + \lambda_+ \mathbb{1}_{\mathbb{R}^*_+}$ for $\varepsilon \in [\lambda_-, \lambda_+]$

It is then possible *to build a sequence of processes converging (up to a subsequence) to a Nash equilibrium for the game* $(\mathbf{G})$ [4]. This result is captured in the following theorem.

Theorem 5.2. *For any $n \in \mathbb{N}$ there exists a unique viscosity solution, denoted $(V^{a,n}, V^{b,n})$, to the following system of integro-PDEs:*

$$(\mathbf{S}^n) \begin{cases} \partial_s V^{a,n} + H^{*,n}\big(D_a V^{a,n}, D_b V^{a,n}, D_b V^{b,n}\big) = 0, & s \in [0,h[, (\alpha,\beta) \in \mathbb{N}^2 \\ V^{a,n}(h,\alpha,\beta) = g^a(\alpha,\beta) & \\ \partial_s V^{b,n} + H^{*,n}\big(D_b V^{b,n}, D_a V^{b,n}, D_a V^{a,n}\big) = 0, & s \in [0,h[, (\alpha,\beta) \in \mathbb{N}^2 \\ V^{b,n}(h,\alpha,\beta) = g^b(\alpha,\beta) & \end{cases}$$

Moreover

(i) The system $(\mathbf{S}^n)$ admits a unique viscosity solution.

(ii) There exists a subsequence $(n_k)_{k \geqslant 0}$ and two measurable maps $V^a, V^b : [0,h] \times \mathbb{N}^2 \to \mathbb{R}$ such that, for every $(s,\alpha,\beta) \in [0,h] \times \mathbb{N}^2$,

$$\lim_{k \to +\infty} V^{i,n_k}(s,\alpha,\beta) = V^i(s,\alpha,\beta), \quad i \in \{a,b\},$$
$$\lim_{n \to +\infty} DV^{i,n}(s,\alpha,\beta) = DV^i(s,\alpha,\beta), \quad i \in \{a,b\}.$$

(iii) The sequences of processes

- $(\lambda^{n_k}\big(D_a V^{a,n_k}(\cdot, N^a, N^b)\big)\mathbb{1}_{\{D_a V^a(\cdot,N^a,N^b)=0\}})_{k \in \mathbb{N}}$
- $(\lambda^{n_k}\big(D_b V^{b,n_k}(\cdot, N^a, N^b)\big)\mathbb{1}_{\{D_b V^b(\cdot,N^a,N^b)=0\}})_{k \in \mathbb{N}}$

converge weakly in $\mathcal{H}^2_r(\mathbb{R}^2)^2$ to progressively measurable, $[\lambda_-, \lambda_+]$-valued processes θ and ϑ, respectively.

[2] $\mathcal{H}^p_r(\mathbb{R}^d) = \left\{ Y : \mathbb{R}^d\text{-valued and } \mathbb{F}\text{-measurable s.t. } \mathbb{E}\left[\left(\int_0^r \|Y_t\|^2 \mathrm{d}t\right)^{p/2} \right] < +\infty \right\}$ for all $d \geqslant 1$, $p > 1$ and $r \geqslant 0$.

Thus the control pair $(\lambda^{a,*}, \lambda^{b,*}) = \Big(\lambda^{a,*}\big(DV^a(s, N_s^a, N_s^b), \theta_s\big), \lambda^{b,*}\big(DV^b(s, N_s^a, N_s^b), \vartheta_s\big)\Big)_{0 \leqslant s \leqslant t}$ *is a Nash equilibrium for game* $(\mathbf{G})$*, and* $V_h^{i,\alpha,\beta}\big(\lambda^{a,*}, \lambda^{b,*}\big) = V^i(0, \alpha, \beta)$ *for* $i \in \{a, b\}$.

The proof of the theorem 5.2 can be found in appendix C.2 of [4]. In summary, Proposition 5.1 and Theorem 5.2 tell us that, even though the "ideal" controls $\lambda^{a,*}$ and $\lambda^{b,*}$ are discontinuous, we can approximate them by a family of smooth problems $(\mathbf{S}^n)$ whose solutions converge to a genuine Markovian Nash equilibrium for the original game $(\mathbf{G})$. In other words, there is a well-defined feedback rule for both buyers and sellers, and it can – in principle – be computed by solving a sequence of regularized integro-PDEs.

6 Conclusion

In this report we have revisited the price-formation framework of Jusselin, Mastrolia & Rosenbaum (2021) through the analytical lens developed in IEOR 222. Starting from first principles, we cast the frequent-batch auction as a regenerative stochastic system, derived the closed-form expression (Theorem 2.1) for the quality-of-price-formation metric $E(h)$ and, by carefully applying Itô's formula for jump processes together with conditional-expectation/PDE arguments covered in IEOR 222, derived the optimal stochastic control for a selling market taker, when buyers arrive increasingly to the market.

Our numerical results highlight what Jusselin et al. already pointed out: *one size does not fit all*. Precisely, taking for example intensities $\nu = 1$ and $\mu = 0.5$, an auction of duration $h^* = 24.49$ s reduces the price-formation error by more than 45% compared to the CLOB, showing that the CLOB is suboptimal in this example case. Furthermore, we investigated the sensitivity of the clearing price with respect to the nominal K and the noise variance σ^2, notably pointing out that first order sensitivity with respect to K decays in $O(1/K^2)$ as $K \to +\infty$.

For future work, it will be interesting to further investigate the extension on the Nash equilibrium we suggested in subsection 5.2, as it remains singular up to this point. More precisely, we extended the model to the case in which all market takers endogenously choose their arrival intensities. By linking the controlled point-process dynamics to a pair of HJB equations, we characterized the seller's optimal "bang-bang" policy and sketched the existence of a Markovian Nash equilibrium via a family of smoothed integro-PDEs. Furthermore, it will be interesting to validate the E function through Monte Carlo simulations, as it has not been done in this report. Finally, adopting a reinforcement lens on the optimal policy market takers can pursue on the market will be a promising avenue for future research.

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