Brief Literature Review

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We apply results from Self and Liang 1987

1 Hypothesis Testing

$$-2\ln\lambda \xrightarrow{d} \inf_{\theta\in\tilde{C}_0} \| \tilde{Z}-\theta \|^2 - \inf_{\theta\in\tilde{C}} \| \tilde{Z}-\theta \|^2 \tag{1}$$

with $\tilde{C} = \{\tilde{\theta} : \tilde{\theta} = \Lambda^{1/2}P \quad \forall \quad \theta \in C_{\Omega}-\theta_0\}$ and $\tilde{C}_0 = \{\tilde{\theta} : \tilde{\theta} = \Lambda^{1/2}P \quad \forall \quad \theta \in C_{\Omega_0}-\theta_0\}$, where $\tilde{Z}$ has a multivariate Gaussian distribution with mean 0 and Identity variance-covariance matrix, $P\Lambda P^T$ represents the spectra decomposition of $I(\theta_0)$

1.1 Case I

$$\begin{aligned} H_0 &: \sigma = 0 \\ H_1 &: \sigma > 0 \end{aligned} \tag{2}$$

$\tilde{C}_0 = \mathbf{R}^{p-1} \times \{0\}, \quad \tilde{C}_1 = \mathbf{R}^{p-1} \times [0,\infty)$. Note that the components $\tilde{Z}$ are uncorrelated since $\tilde{Z}$ follows a standard multivariate normal distribution.

Also, the elements of θ are uncorrelated since $\tilde{C}$ and $\tilde{C}_0$ orthogonal vectors.

$$
\begin{aligned}
\therefore \quad || \tilde{Z} = \theta ||^2 &= \sum_{i=1}^{p} (\tilde{Z}_i - \theta_i)^2 \\
&= (\tilde{Z}_1 - \theta_1)^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \\
\inf_{\theta \in \tilde{C}_0} || \tilde{Z} = \theta ||^2 &= \inf \left[(\tilde{Z}_1 - 0)^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \min \left[\tilde{Z}_1^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \tilde{Z}_1^2 \\
\inf_{\theta \in \tilde{C}} || \tilde{Z} = \theta ||^2 &= \inf \left[(\tilde{Z}_1 - \theta_1)^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \min (\tilde{Z}_1 - \theta_1)^2 \\
&= \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
\therefore \inf_{\theta \in \tilde{C}_0} || \tilde{Z} - \theta ||^2 - \inf_{\theta \in \tilde{C}} || \tilde{Z} - \theta ||^2 &= \tilde{Z}_1^2 - \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
&= \begin{cases} 0, & \tilde{Z}_1 \leq 0 \\ \tilde{Z}_1^2, & \tilde{Z}_1 > 0 \end{cases} \\
\implies 0.5\chi_0^2 + 0.5\chi_1^2 &
\end{aligned}
$$

1.2 Case II

$$
\begin{aligned}
&H_0 : \sigma_{11} > 0, \quad \sigma_{12}, \sigma_{22} = 0 \\
&H_1 : x^T \Sigma x > 0 \quad \forall x \neq 0
\end{aligned}
\tag{3}
$$

$\tilde{C}_0 = \mathbf{R}^{p-2} \times \{0\} \times \{0\}, \quad \tilde{C}_1 = \mathbf{R}^{p-1} \times [0, \infty).$

$$
\begin{aligned}
\therefore \quad || \tilde{Z} = \theta ||^2 &= \sum_{i=1}^{p} (\tilde{Z}_i - \theta_i)^2 \\
&= \min \left[(\tilde{Z}_1 - \theta_1)^2 + (\tilde{Z}_2 - \theta_2)^2 + \sum_{i=3}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \tilde{Z}_1^2 + \tilde{Z}_2^2 \\
\inf_{\theta \in \tilde{C}} || \tilde{Z} = \theta ||^2 &= \inf \left[(\tilde{Z}_1 - \theta_1)^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \min (\tilde{Z}_1 - \theta_1)^2 \\
&= \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
\therefore \inf_{\theta \in \tilde{C}_0} || \tilde{Z} - \theta ||^2 - \inf_{\theta \in \tilde{C}} || \tilde{Z} - \theta ||^2 &= \tilde{Z}_1^2 + \tilde{Z}_2^2 - \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
&= \begin{cases} \tilde{Z}_2^2, & \tilde{Z}_1 \leq 0 \\ \tilde{Z}_1^2 + \tilde{Z}_2^2, & \tilde{Z}_1 > 0 \end{cases} \\
\implies 0.5\chi_1^2 + 0.5\chi_2^2
\end{aligned}
$$

1.3 Case III

$$
\begin{aligned}
&H_0 : \sigma_{11} > 0, \quad \sigma_{12}, \sigma_{22}, \beta = 0 \\
&H_1 : x^T \Sigma x > 0 \quad \forall x \neq 0, \beta \neq 0
\end{aligned} \tag{4}
$$

$\tilde{C}_0 = \mathbf{R}^{p-3} \times \{0\} \times \{0\} \times \{0\}, \quad \tilde{C}_1 = \mathbf{R}^{p-} \times [0, \infty).$

$$\begin{aligned}
\therefore \quad || \tilde{Z} = \theta ||^2 &= \sum_{i=1}^{p} (\tilde{Z}_i - \theta_i)^2 \\
&= \min \left[(\tilde{Z}_1 - \theta_1)^2 + (\tilde{Z}_2 - \theta_2)^2 + (\tilde{Z}_3 - \theta_2)^3 + \sum_{i=4}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2 \\
\inf_{\theta \in \tilde{C}} || \tilde{Z} = \theta ||^2 &= \inf \left[(\tilde{Z}_1 - \theta_1)^2 + \sum_{i=2}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \min (\tilde{Z}_1 - \theta_1)^2 \\
&= \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
\therefore \inf_{\theta \in \tilde{C}_0} || \tilde{Z} - \theta ||^2 - \inf_{\theta \in \tilde{C}} || \tilde{Z} - \theta ||^2 &= \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2 - \begin{cases} \tilde{Z}_1^2, & \tilde{Z}_1 \leq 0 \\ 0, & \tilde{Z}_1 > 0 \end{cases} \\
&= \begin{cases} \tilde{Z}_2^2 + \tilde{Z}_3^2, & \tilde{Z}_1 \leq 0 \\ \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2, & \tilde{Z}_1 > 0 \end{cases} \\
\implies 0.5\chi_2^2 + 0.5\chi_3^2
\end{aligned}$$

1.4 Case IV

$$\begin{aligned}
&H_0 : \sigma_{11} = \sigma_{12} = \sigma_{22} = 0 \\
&H_1 : x^T \Sigma x > 0 \quad \forall x \neq 0
\end{aligned} \tag{5}$$

$\tilde{C}_0 = \mathbf{R}^{p-3} \times \{0\} \times \{0\} \times \{0\}, \quad \tilde{C}_1 = \mathbf{R}^{p-2} \times [0, \infty) \times [0, \infty).$

$$\begin{aligned}
\therefore \quad || \tilde{Z} = \theta ||^2 &= \sum_{i=1}^{p} (\tilde{Z}_i - \theta_i)^2 \\
&= \min \left[(\tilde{Z}_1 - \theta_1)^2 + (\tilde{Z}_2 - \theta_2)^2 + (\tilde{Z}_3 - \theta_2)^3 + \sum_{i=4}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2 \\
\inf_{\theta \in \tilde{C}} || \tilde{Z} = \theta ||^2 &= \inf \left[(\tilde{Z}_1 - \theta_1)^2 + (\tilde{Z}_2 - \theta_2)^2 + \sum_{i=3}^{p} (\tilde{Z}_i - \theta_i)^2 \right] \\
&= \min \left[(\tilde{Z}_1 - \theta_1)^2 + (\tilde{Z}_2 - \theta_2)^2 \right] \\
&= \min((\tilde{Z}_1 - \theta_1)^2) + \min((\tilde{Z}_2 - \theta_2)^2) \\
&= \begin{cases} \tilde{Z}_1^2 + \tilde{Z}_2^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 \le 0 \\ \tilde{Z}_1^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 > 0 \\ \tilde{Z}_2^2, & \tilde{Z}_1 > 0, \tilde{Z}_2 \le 0 \\ 0, & \tilde{Z}_1 > 0, \tilde{Z}_2 > 0 \end{cases} \\
\therefore \inf_{\theta \in \tilde{C}_0} || \tilde{Z} - \theta ||^2 - \inf_{\theta \in \tilde{C}} || \tilde{Z} - \theta ||^2 &= \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2 - \begin{cases} \tilde{Z}_1^2 + \tilde{Z}_2^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 \le 0 \\ \tilde{Z}_1^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 > 0 \\ \tilde{Z}_2^2, & \tilde{Z}_1 > 0, \tilde{Z}_2 \le 0 \\ 0, & \tilde{Z}_1 > 0, \tilde{Z}_2 > 0 \end{cases} \\
&= \begin{cases} \tilde{Z}_3^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 \le 0 \\ \tilde{Z}_2^2 + \tilde{Z}_3^2, & \tilde{Z}_1 \le 0, \tilde{Z}_2 > 0 \\ \tilde{Z}_1^2 + \tilde{Z}_2^2, & \tilde{Z}_1 > 0, \tilde{Z}_2 \le 0 \\ \tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{Z}_3^2, & \tilde{Z}_1 > 0, \tilde{Z}_2 > 0 \end{cases}
\end{aligned}$$

$$\Longrightarrow \pi_1 \chi_1^2 + (\pi_2 + \pi_3) \chi_2^2 + \pi_4 \chi_3^2$$

1.5 Table to Show Comparison to Self & Liang Application

	Parameters of Interest	Nuisance Parameters
Case I	σ	β, ϕ
Case II	$\sigma_{22}, \sigma_{11}, \sigma_{12}$	α, β, ϕ
Case III	$\sigma_{22}, \sigma_{11}, \sigma_{12}, \beta$	α, ϕ
Case IV	$\sigma_{11}, \sigma_{22},\ \sigma_{12}$	α, β, ϕ

Cases	Self & Liang	Parameters of Interest		Nuisance Parameters	
		Boundary	Non-Boundary	Boundary	Non-Boundary
Case I $(1, 0, 0, 2)$ $0.5\chi_0^2 + 0.5\chi_1^2$	Case 5 $(1, 0, 0, p-1)$ $0.5\chi_0^2 + 0.5\chi_1^2$	$\sigma = 0$			β, ϕ
Case II $(1, 2, 0, 3)$ $0.5\chi_1^2 + 0.5\chi_2^2$	Case 6 $(1, 1, 0, p-2)$ $0.5\chi_1^2 + 0.5\chi_2^2$	$\sigma_{22} = 0$	$\sigma_{11} > 0, \sigma_{12} = 0$		α, β, ϕ
Case III $(1, 3, 0, 2)$ $0.5\chi_2^2 + 0.5\chi_3^2$	Case 6 $(1, 1, 0, p-2)$ $0.5\chi_1^2 + 0.5\chi_2^2$	$\sigma_{22} = 0$	$\sigma_{11} > 0, \sigma_{12}, \beta = 0$		α, ϕ
Case IV $(2, 1, 0, 2)$ $\chi_1^2, \chi_2^2, \chi_3^2$	Case 7 $(2, p, 0, 0)$ [1]$\chi_0^2, \chi_1^2, \chi_2^2$	$\sigma_{11} = \sigma_{22} = 0$	$\sigma_{12} = 0$		α, β, ϕ

2 Confidence Interval

Suppose that $\Omega = \Omega_1 \times \cdots \times \Omega_p$ where Ω_i are closed interval in R^p, let $\theta_{01}, \cdots \theta_{0q}$ be left endpoints of $\Omega_1, \cdots, \Omega_q$, and θ_{0i} are interior points of $\Omega_i \quad (q+1 < i < p)$. There are 2^q configurations of the first q coordinates of $\hat{\theta}$, the configuration indicates which coordinates of $\hat{\theta}$ are zero. Each configuration is characterized by q Linear combinations of Z denoted by $\{L_i Z > 0\}$.

[1]This is wrongly configured by Self & Liang and should have been $(2, 0, 0, p-2)$, the misconfiguration is not only in the number of parameters but also in whether 'p' parameters are of interest or not

$\hat{\theta}$ can be expressed as

$$\hat{\theta} = \sum_{i}^{2^q} P_i Z I\{L_i Z > 0\}$$
$$P_i = I - I^{-1}(\theta_0) B_i [B_i' I^{-1}(\theta_0) B_i]^+ B_i'$$
$$F(\cdot) = \sum_{i=1}^{2^q} P[L_i Z_i > 0] P[P_i Z \leq (\cdot) \mid L_i Z_i > 0]$$

B_i is a diagonal matrix with ones as diagonal elements corresponding to coordinates of $\hat{\theta}$ that are zero, and zeros otherwise.

q=1

This means there are 2^1 configurations, So

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix}$$
$$\Omega = [0, \infty) \times R^{p-1}$$
$$\hat{\theta} = \sum_{i=1}^{2} P_i Z I\{L_i Z > 0\}$$
$$I^{-1}(\theta) = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & \sigma_{pp}^2 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

when $i = 1$

$$B_1' I^{-1}(\theta) B_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & \sigma_{pp}^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$\therefore P_1 = I - I^{-1}(\theta_0) B_1 [B_1' I^{-1}(\theta_0) B_1]^{+} B_1' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 1 \end{bmatrix}$$

$$\Longrightarrow P_1 Z I\{L_1 Z\} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} I\{Z_1 > 0\}$$

when $i = 2$

$$I^{-1}(\theta)B_2 = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{pi} & \cdot & \cdots & \sigma_{pp}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ \sigma_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & 0 \end{bmatrix}$$

$$B_2' I^{-1}(\theta)B_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & \sigma_{pp}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$(B_2' I^{-1}(\theta)B_2)^+ = \begin{bmatrix} \frac{1}{\sigma_{11}^2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$I^{-1}(\theta)B_2(B_2' I^{-1}(\theta)B_2)^+ B_2 = \begin{bmatrix} \sigma_{11}^2 & 0 & \cdots & 0 \\ \sigma_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{11}^2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\sigma_{21}}{\sigma_{11}^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sigma_{11}^2} & \cdot & \cdots & 0 \end{bmatrix}$$

$$\therefore P_2 = I - I^{-1}(\theta_0)B_2[B_2' I^{-1}(\theta_0)B_2]^+ B_2'$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\sigma_{21}}{\sigma_{11}^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sigma_{11}^2} & \cdot & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\frac{\sigma_{21}}{\sigma_{11}^2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\sigma_{p1}}{\sigma_{11}^2} & \cdot & \cdots & 1 \end{bmatrix}$$

$$\implies P_2 Z = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\frac{\sigma_{21}}{\sigma_{11}^2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\sigma_{p1}}{\sigma_{11}^2} & \cdot & \cdots & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} I\{Z_1 < 0\}$$

$\therefore \hat{\theta} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} I\{Z_1 > 0\} + \begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} I\{Z_1 < 0\}$ Then according to Self and Liang, the distribution of $\hat{\theta}$ is given as

$$F(\hat{\theta}) = P(L_1 Z > 0) P[P_1 Z \mid L_1 Z > 0] + P[L_2 Z > 0][P_2 Z \mid L_2 Z > 0]$$

$$= P(Z_1 > 0) P \left(\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} \middle| Z_1 > 0 \right) + P(Z_1 < 0) P \left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} \middle| Z_1 < 0 \right)$$

$$LHS : P \left(\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} \middle| Z_1 > 0 \right)$$

follows a truncated normal distribution of the form

$$\frac{(2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\}}{\int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty (2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\} dZ}$$
$$= \frac{(2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\}}{0.5}$$
$$= 2(2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\} = 2\mathcal{N}(0, \Sigma)$$

$$RHS : P\left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} \middle| Z_1 < 0\right)$$

Note that

$$\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} \perp Z_1 < 0$$

$$\therefore P\left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} \middle| Z_1 < 0\right)$$

$$= P\left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix}\right)$$

But this is a Linear combination of a Gaussian multivariate distributed random vector

$$P\left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix}\right) = \mathcal{N}(0, P_2 \Sigma P_2')$$

Where P_2 is a projection matrix defined earlier. Note that the above should no longer be a p-variate but a p-1 variate. Therefore the transformation of the Variance covariance matrix is unneccesary, since the part that remains after deleting the first row and column of the projection matrix is an identity matrix. Thus we simply delete the first row and second column of the variance covariance matrix, and have a p-1 (reduced parameter)

```
Library(GLMMadaptive)

fit <- mixed_model(fixed = count ~ mined, random = ~ 1 + mined |site,
family=negative.binomial, data=Salamanders)

vcov(fit)
              (Intercept)       minedno          D_11          D_12          D_22        phi_1
(Intercept)  0.1226055888 -1.225953e-01 -0.0282212309  0.0277145272 -0.0002444848 -3.969376e-04
minedno     -0.1225953111  1.322643e-01  0.0282097640 -0.0279034493 -0.0059785322  3.928267e-05
D_11        -0.0282212309  2.820976e-02  0.0957609526 -0.0940405816  0.0002676228  4.428928e-04
D_12         0.0277145272 -2.790345e-02 -0.0940405816  0.5459995518 -0.1674378535 -1.626046e-04
D_22        -0.0002444848 -5.978532e-03  0.0002676228 -0.1674378535  0.3764426497  8.467725e-03
phi_1       -0.0003969376  3.928267e-05  0.0004428928 -0.0001626046  0.0084677248  1.381032e-02
```

Figure 1: Minimum working example of GLMMadaptive

Putting it together

From the above, we thus obtain the distribution of $\hat{\theta}$

$$\begin{aligned}
&P[L_1 Z > 0]P[P_1 Z \leq (\cdot) \mid L_1 Z > 0] + P[L_2 Z > 0][P_2 Z \leq (\cdot) \mid L_2 Z > 0] \\
&= \frac{1}{2} 2\mathcal{N}(0, \Sigma') + \frac{1}{2}\mathcal{N}(0, P_2 \Sigma P_2') \\
&= \mathcal{N}(0, \Sigma) + \frac{1}{2}\mathcal{N}(0, P_2 \Sigma P_2') \\
&= \mathcal{N}_p(0, \Sigma) + \frac{1}{2}\mathcal{N}_{p-1}(0, \Sigma)
\end{aligned}$$

q=2

There are 4 configurations in this case

case 1

$$P_1 Z I\{L_1 Z\} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{bmatrix} I\{Z_1 > 0, Z_2 > 0\}$$

$$\begin{aligned} P(L_1 Z) P(P_1 Z I\{L_1 Z\}) &= P(Z_1 > 0, Z_2 > 0) \frac{(2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\}}{\int_0^{\infty^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\} dZ} \\ &= \pi_1 (2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2} Z^T \Sigma^{-1} Z\} \\ &= \pi_1 \mathcal{N}(0, \Sigma) \\ \pi_1 &= \frac{P(Z_1 > 0, Z_2 > 0)}{P(Z_1 > 0, Z_2 > 0, -\infty < Z_3 \cdots Z_p < \infty)} \end{aligned}$$

case 2

$$P(P_2 Z I\{L_2 Z\}) = P\left(\begin{bmatrix} 0 \\ Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 \\ \vdots \\ Z_p - \frac{\sigma_{p1}}{\sigma_{11}^2} Z_1 \end{bmatrix} \middle| Z_1 < 0, Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 > 0 \right)$$

There are two assumptions here

- Just like the case where $q = 1$ we have a degenerate distribution, so this is a $p-1$ variate distribution and the vectors are independent of Z_1
- We can view the $p-1$ vector as a transformation so that we now have

$$\begin{bmatrix} Z_2^* \\ \vdots \\ Z_p^* \end{bmatrix}$$

$$\therefore P(P_2 Z I\{L_2 Z\}) = P\left(\begin{bmatrix} Z_2^* \\ \vdots \\ Z_p^* \end{bmatrix} \middle| Z_2^* > 0 \right)$$

This is another truncated multivariate normal

$$\therefore P\left(\begin{bmatrix} Z_2^* \\ \vdots \\ Z_p^* \end{bmatrix} \middle| Z_2^* > 0\right) = 2\mathcal{N}_{p-1}(0, \Sigma_{-1})$$

[2] [3]

case 3

$$P(P_3 Z I\{L_3 Z\}) = P\left(\begin{bmatrix} Z_1 - \frac{\sigma_{12}}{\sigma_{22}^2} Z_2 \\ 0 \\ Z_3 - \frac{\sigma_{32}}{\sigma_{22}^2} Z_2 \\ \vdots \\ Z_p - \frac{\sigma_{p2}}{\sigma_{22}^2} Z_2 \end{bmatrix} \middle| Z_1 - \frac{\sigma_{12}}{\sigma_{22}^2} Z_2 > 0, Z_2 < 0\right)$$

$$= P\left(\begin{bmatrix} Z_1^+ \\ Z_3^+ \\ \vdots \\ Z_p^+ \end{bmatrix} \middle| Z_1^+ > 0\right) = 2\mathcal{N}_{p-1}(0, \Sigma_{-2})$$

[2] The transformation on this submatrix is an Identity, Σ is unaffected

[3] subscript on Σ indicate what row and column are deleted

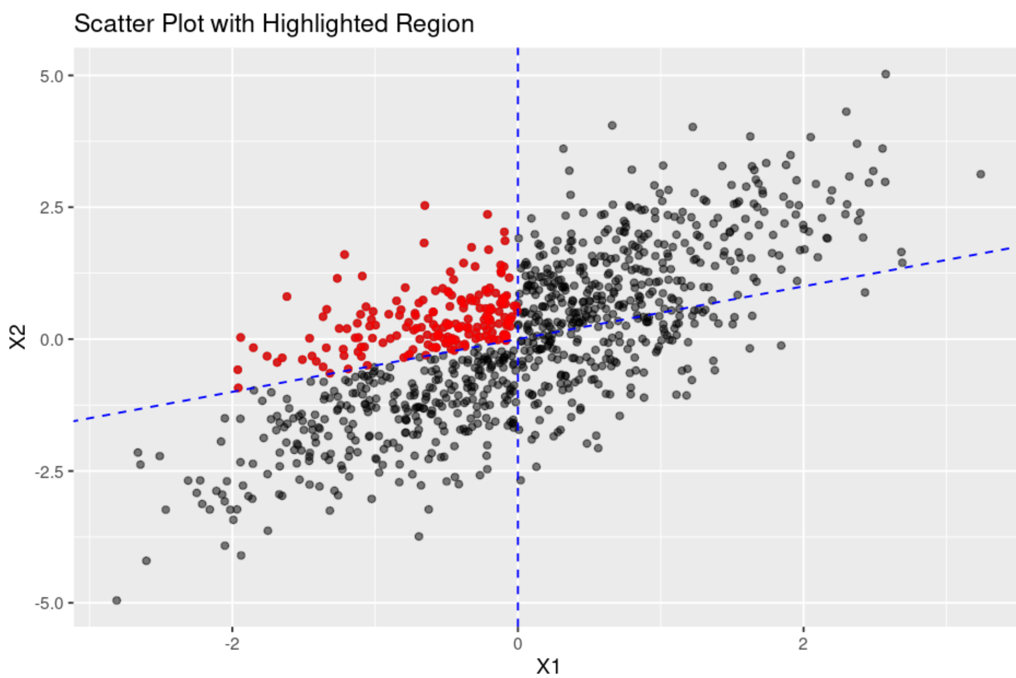

Figure 2: Decorelated bivariate

case 4

This case looks intricate, but let us start with the projection matrix.

$$B_1' I^{-1}(\theta) B_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \cdot & \cdots & \sigma_{pp}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \cdots & 0 \\ \sigma_{21} & \sigma_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

$$(B_1' I^{-1}(\theta) B_1)^+ = \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} & \cdots & 0 \\ -\sigma_{21} & \sigma_{11}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B(B_1' I^{-1}(\theta) B_1)^+ B = \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} & \cdots & 0 \\ -\sigma_{21} & \sigma_{11}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} & \cdots & 0 \\ -\sigma_{21} & \sigma_{11}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$I(\theta_0) B(B_1' I^{-1}(\theta) B_1)^+ B = \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \sigma_{23} & \cdots & \sigma_{2p} \\ \sigma_{31} & \sigma_{32}^2 & \sigma_{33} & \cdots & \sigma_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \sigma_{p3} & \cdots & \sigma_{pp}^2 \end{bmatrix} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} & 0 & \cdots & 0 \\ -\sigma_{21} & \sigma_{11}^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21}} \begin{bmatrix} \sigma_{11}^2\sigma_{22}^2 - \sigma_{12}\sigma_{21} & 0 & 0 & \cdots & 0 \\ 0 & -\sigma_{12}\sigma_{21} + \sigma_{11}^2\sigma_{22}^2 & 0 & \cdots & 0 \\ \sigma_{31}\sigma_{22}^2 - \sigma_{32}\sigma_{21} & -\sigma_{31}\sigma_{12} + \sigma_{32}\sigma_{11}^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1}\sigma_{22}^2 - \sigma_{p2}\sigma_{21} & -\sigma_{p1}\sigma_{12} + \sigma_{p2}\sigma_{11}^2 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \frac{\sigma_{31}\sigma_{22}^2-\sigma_{32}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{-\sigma_{31}\sigma_{12}+\sigma_{32}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}\sigma_{22}^2-\sigma_{p2}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & -\frac{\sigma_{p1}\sigma_{12}+\sigma_{p2}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 0 \end{bmatrix}$$

$$\therefore I - I(\theta_0)B(B_1'I^{-1}(\theta)B_1)^+B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \frac{\sigma_{31}\sigma_{22}^2-\sigma_{32}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{-\sigma_{31}\sigma_{12}+\sigma_{32}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}\sigma_{22}^2-\sigma_{p2}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & -\frac{\sigma_{p1}\sigma_{12}+\sigma_{p2}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-\sigma_{31}\sigma_{22}^2+\sigma_{32}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{\sigma_{31}\sigma_{12}-\sigma_{32}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\sigma_{p1}\sigma_{22}^2+\sigma_{p2}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{\sigma_{p1}\sigma_{12}-\sigma_{p2}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 1 \end{bmatrix}$$

$$P_4Z = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-\sigma_{31}\sigma_{22}^2+\sigma_{32}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{\sigma_{31}\sigma_{12}-\sigma_{32}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\sigma_{p1}\sigma_{22}^2+\sigma_{p2}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & \frac{\sigma_{p1}\sigma_{12}-\sigma_{p2}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_p \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ Z_3 - \frac{\sigma_{31}\sigma_{22}^2+\sigma_{32}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}}Z_1 + \frac{\sigma_{31}\sigma_{12}-\sigma_{32}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}}Z_2 \\ \vdots \\ Z_p - \frac{\sigma_{p1}\sigma_{22}^2+\sigma_{p2}\sigma_{21}}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}}Z_1 + \frac{\sigma_{p1}\sigma_{12}-\sigma_{p2}\sigma_{11}^2}{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}\sigma_{21}}Z_2 \end{bmatrix}$$

$P(P_4ZI\{L_4Z\}) = \mathcal{N}_{p-2}(0,\Sigma)$

Finally, $F(\theta) = \pi_1\mathcal{N}_p(0,\Sigma) + \pi_2\mathcal{N}_{p-1}(0,\Sigma) + \pi_3\mathcal{N}_{p-1}(0,\Sigma) + \pi_4\mathcal{N}_{p-2}(0,\Sigma)$ All Σ's are obtained by deleting the corresponding rows and columns of the component of Z set to zero. Preliminary results from my programming (github) seem to indicate that

$$\pi_2 = \pi_3 = 2P(Z_1 - \frac{\sigma_{12}}{\sigma_{22}^2} Z_2 > 0, Z_2 < 0) = 2P(Z_2 - \frac{\sigma_{12}}{\sigma_{11}^2} Z_1 > 0, Z_1 < 0) = 2\left(\frac{1}{4}\right)$$

$$\pi_1 = \frac{P(Z_1 > 0, Z_2 > 0)}{P(Z_1 > 0, Z_2 > 0, -\infty < Z_3 \cdots Z_p < \infty)}$$

$$\pi_4 = P(Z_1 - \frac{\sigma_{12}}{\sigma_{22}^2} Z_2 < 0, Z_2 - \frac{\sigma_{21}}{\sigma_{11}^2} Z_1 < 0) = \int_{-\infty}^{\frac{\sigma_{12}}{\sigma_{22}^2} Z_2} \int_{-\infty}^{\frac{\sigma_{21}}{\sigma_{11}^2} Z_1}$$

only π_4 looks difficult to compute right now

Attempt at explanation

1. $Z_2^* = Z_2 - \frac{\sigma_{12}}{\sigma_{11}^2} Z_1$ is simply decorrelate Z_2 and Z_1. This is achieved by subtracting a scaled Z_1, the scaling factor is the proportion of the Z_1 that is due to its covariance with Z_2. In the end Z_2^* is uncorrelated with Z_1. To verify

$$\begin{aligned} Cov(Z_2^*, Z_1) &= Cov(Z_2 - \frac{\sigma_{12}}{\sigma_{11}^2} Z_1, Z_1) \\ &= Cov(Z_2, Z_1) - \frac{\sigma_{12}}{\sigma_{11}^2} Cov(Z_1, Z_1) \\ &= \sigma_{12} - \frac{\sigma_{12}}{\sigma_{11}^2} \sigma_{11}^2 \\ &= 0 \end{aligned}$$

2. Now to decorrelate Z_3 from both Z_1 and Z_2, we need to solve

$$\begin{aligned}
Z_3^* &= Z_3 - aZ_1 - bZ_2 \\
&: \quad Cov(Z_3^*, Z_1) = 0 \\
&\& \quad Cov(Z_3^*, Z_2) = 0 \\
\implies Cov(Z_3 - aZ_1 - bZ_2, Z_1) &= 0 \\
Cov(Z_3 - aZ_1 - bZ_2, Z_2) &= 0 \\
\implies Cov(Z_3, Z_1) - aCov(Z_1, Z_1) - bCov(Z_2, Z_1) &= 0 \\
Cov(Z_3, Z_2) - aCov(Z_1, Z_2) - bCov(Z_2, Z_2) &= 0 \\
\implies \sigma_{31} - a\sigma_{11}^2 - b\sigma_{21} &= 0 \\
\sigma_{32} - a\sigma_{12} - b\sigma_{22}^2 &= 0 \\
\implies a\sigma_{11}^2 + b\sigma_{21} &= \sigma_{31} \\
a\sigma_{12} + b\sigma_{22}^2 &= \sigma_{32} \\
\implies a\sigma_{11}^2\sigma_{12} + b\sigma_{21}\sigma_{12} &= \sigma_{31}\sigma_{12} \\
a\sigma_{12}\sigma_{11}^2 + b\sigma_{22}^2\sigma_{11}^2 &= \sigma_{32}\sigma_{11}^2 \\
\implies b = \frac{\sigma_{31}\sigma_{12} - \sigma_{32}\sigma_{11}^2}{\sigma_{21}\sigma_{12} - \sigma_{22}^2\sigma_{11}^2}
\end{aligned}$$

the same result can be obtained for a

$$\begin{aligned}
a\sigma_{11}^2\sigma_{22} + b\sigma_{21}\sigma_{22} &= \sigma_{31}\sigma_{22} \\
a\sigma_{21}\sigma_{12} + b\sigma_{21}\sigma_{22}^2 &= \sigma_{21}\sigma_{32} \\
\implies a = \frac{\sigma_{31}\sigma_{22} - \sigma_{32}\sigma_{21}^2}{\sigma_{11}\sigma_{22} - \sigma_{21}^2\sigma_{12}^2}
\end{aligned}$$

3 Problematic Results

- There seems to be disagreement or lack of understanding of the mixing weights in
  1. Case IV of the Hypothesis Test section
  2. Case 4 of the Confidence Interval section

4 Possible Future Additions

1. Score test

2. Bootstrap confidence interval (parametric or non-parametric)
3. Likelihood-based confidence Interval