Home work solutions: HW1 HW2 HW3 HW4.

Lecture 1 - Intro

Put Option

Option to sell a share at strike price K, at maturity T. (Also before maturity if american option).

$$Payoff = (K - S)^+,\ P\&L = (K - S)^+ - p$$

Call option

Option to buy a share at strike price K, at maturity T. (Also before if american).

$$Payoff = (S - K)^+\ P\&L = (S - K)^+ - c$$

Forward Contract

Agree to buy or sell an asset at time T for price K. The price of the forward is that which makes the contract value equal zero.

- *Long* - buy forward. Payoff: $S - K$
- *Short* - Sell forward. Payoff: $K - S$

Lecture 2 - Bonds

Time Value of Money and Rents

- Rate of return: $\frac{B-A}{A}$
- Compound interest: $(1 + r)^t$ for annual compounding, e^{rt} for continuous.
- Repo: Rate implied by financial institution selling and buying back securities.
- Basis point: 0.01%
- Discount Factor: e^{-rt}

Annuity

Series of payments made at equal intervals. A - payment amount, N - number of payments.

- Future value: F

- Present Value: P
- rate/period: i
- $P = \frac{A}{i}\left(1-(1+i)^{-N}\right)$
- $F = \frac{A}{i}\left((1+i)^{N}-1\right)$
- $A = \frac{P*i}{1-(1+i)^{-N}}$

Zero Coupon Bonds

Sold at price lower than face value, and pays no coupons. Return derived from price difference. **Zero rate**(spot rate) is the interest rate implied by a zero-coupon bond.

$$P = e^{-rT}F,\ r = \frac{1}{T}ln(F/P)$$

Coupon Bonds

Bootstrap methods - Discount coupons and finally face value to figure out equivalent zero rates.

Bond Yield - Constant interest rate implied by bond so that present value of future payments is equal to the current bond price.

$$P = c_1 e^{-yt_1} + ... + c_n e^{-yt_n} \quad \text{Find y}$$

Duration - Weighted average of payment times.

$$D = \sum_{i=1}^{n} t_i \left(\frac{c_i e^{-yt_i}}{P}\right),\ P = \sum_{i=1}^{n} c_i e^{-yt_i}$$

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i} = -DP,\ \Delta P \approx -DP\Delta y$$

Convexity - Weighted average of squared payment times

$$C = \sum_{i=1}^{n} t_i^2 \left(\frac{c_i e^{-yt_i}}{P}\right) \quad \text{Measures the curvature of the price-yield curve}$$

$$\Delta P = \frac{\partial P}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\Delta y^2 = -DP\Delta y + \frac{1}{2}CP\Delta y^2$$

Lecture 3 - Forward Contract

Forward Price: Specific delivery price K so that forward contract is of zero cost. The value is determined so that no arbitrage is possible.

- $t = 0, T$. Inception and maturity
- S_0, S_T. Underlying asset prices
- F_0: Forward price
- r: Risk free interest rate per year.
- $F_0 = S_0 e^{rT}$

Investment with discrete income

- $c_0, ...c_n$
- Effective cost: $S_0 - D_0$
- $D_0 = c_1 e^{-r_1 t_1} + ... + c_n e^{-r_n t_n}$
- $F_0 = (S_0 - D_0)e^{rT}$

Investment with Continuous Yield

- Underlying asset rate: q
- 1 unit = e^{qT} at time T
- Time zero cost: $S_0 e^{-qT}$ (in USD if q for e.g. GBP)
- $F_0 = S_0 e^{-qT} e^{rT} = S_0 e^{(r-q)T}$

Time t Forward Price

- No income asset: $F_t = e^{r(T-t)} S_t$
- Discrete income: $F_t = e^{r(T-t)}(S_t - D_t)$
- Continuous yield: $F_t = e^{(r-q)(T-t)} S_t$
- At time t, value of forward is 0 if $K = F_t$

- V_t: Value at time t of long forward with delivery price K.
- $V_t = e^{-r(T-t)}(F_t - K)$

Forward Rate Agreement (FRA)

- FRA: Agreement to borrow/lend an amount of L at a certain rate r_K
- For no arbitrage: $r_K = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$
- Value of RFA (after revealed market rate r_M): $V(t_2) = L\left(e^{r_M(t_2-t_1)} - e^{r_K(t_2-t_1)}\right)$
- At time t_1, the value is the present value of $V(t_2)$: $V(t_1) = e^{-r(t_2-t_1)}V(t_2)$
- Lender pays borrower $V(t_1)$ if $r_M > r_K$ and borrower pays lender $-V(t_1)$ otherwise.

Lecture 4 - Options

European and American options

- American can be exercised any time before or at maturity
- European - only at maturity
- American price $\geq$ European price
- Intrinsic Value: call: $(S-K)^+$, put: $(K-S)^+$
- Time Value = Option Value - Intrinsic Value
- Call price is decreasing function of K
- Put price is increasing function of K

European put-call Parity

$$c + Ke^{-rT} = p + S_0$$

Both portfolios have same payoff at time T, and should therefore have same value at $T = 0$.

General case (depending on income type from asset):

$$c + e^{-rT}K = p + \begin{cases} S_0 \\ S_0 - D_0 \\ S_0 e^{-qT} \end{cases}$$

Option Price bounds

- European Call option price bound: $(S_0e^{-qT} - Ke^{-rT}) \leq c \leq S_0e^{-qT}$
- Upper bound from payoff: $(S_T - K)^+ \leq S_T \implies c \leq S_0e^{-qT}$
- Lower bound from parity: $c = p + S_0e^{-qT} - Ke^{-rT} \geq S_0e^{-qT} - Ke^{-rT}$
- European Put option bounds:
- Upper bound: $(K - S_T)^+ \leq K \implies p \leq Ke^{-rT}$
- Lower bound: From parity: $p = c + Ke^{-rT} - S_0e^{-qT} \geq Ke^{-rT} - S_0e^{-qT}$

Option Trading Strategies

- Covered Spread: Sell call, buy asset. Cost: $S - c > 0$, payoff: $S_t - (S_T - K)^+$
- Bull Spread: Buy call with strike K_1, sell call with strike $K_2 > K_1$. cost: $c_1 - c_2 > 0$
- Bear Spread: Buy put with strike K_2, sell put with strike K_1, cost: $p_2 - p_1 > 0$.
- Box Spread: Bull (call) spread + bear (put) spread
- Butterfly Spreads: Buy calls with strikes K_1 and K_3 and sell two calls with strike $K_2 \in (K_1, K_3)$, cost: $c_1 + c_3 - 2c_2$.
- Calendar Spread: Sell call (maturity T), buy call (maturity $T_1 > T$), cost: $c_1 - c$.
- Straddle: Buy call & Put with strike K, cost: $c + p > 0$
- Strangles: Buy put with strike K_1 and call with strike $K_2 > K_1$, cost: $c + p > 0$

Binomial Model

One-step binomial model

Stock price over period $[0, \delta]$: $S = (S_0, S_\delta)$.

$$S_\delta = \begin{cases} uS_0, & \text{with prob p} \\ dS_0, & \text{with prob 1-p} \end{cases}$$

$$p \sim Ber$$

For risk free asset, we have: $t = 0 : 1, t = \delta : e^{r\delta}$. Thus to avoid arbitrage: $d < e^{r\delta} < u$.

If $d < u \leq e^{r\delta}$, short stock and deposit at rate r. If $u > d \geq e^{r\delta}$, borrow at rate r and buy stock.

Replicating Portfolio

Find a replicating portfolio for the contract (how many shares, and how much to borrow). There is no arbitrage when the value of the portfolio equals the value of the contract.

- Derivative Payoff: $f(S_\delta) : f_u = f(uS_0), f_d = f(dS_0)$
- For calls: $f_u = (uS_0 - K)^+, f_d = (dS_0 - K)^+$
- Replicating portfolio: Buy Δ shares, and borrow Ψ dollars.
- Value at time δ: $\Delta S_\delta - \Psi e^{r\delta}$
- $\Delta * uS_0 - \Psi e^{r\delta} = f_u,\ \Delta * dS_0 - \Psi e^{r\delta} = f_d$
- $\Delta = \frac{f_u - f_d}{S_0(u-d)},\ \Psi = \frac{df_u - uf_d}{e^{r\delta}(u-d)}$
- Value of derivative at time 0 should be: $f_0 = \Delta S_0 - \Psi = \frac{f_u - f_d}{u-d} - \frac{df_u - uf_d}{e^{r\delta}(u-d)}$

Risk Neutral Pricing

$$p^* = \frac{e^{r\delta} - d}{u - d}$$

p^* is the risk neutral probability. Risk neutral pricing: Derivative price = risk neutral expectation of the payoff discounted at the risk free rate.

$$f_0 = e^{-r\delta}\mathbb{E}^*\left[f(S_\delta)\right] = e^{-r\delta}\left(p^* f_u + (1-p^*)f_d\right)$$

Delta Hedging

How to hedge a short position in a derivative contract. Trade the underlying asset and risk free investment. Thus the replicating portfolio contains $\Delta = \frac{f_u - f_d}{S_0(u-d)}$ shares. Sell the derivative and buy Δ shares to cancel the risk.