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UNIVERSITÄT
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ZUKUNFT
SEIT 1386

Advanced Macroeconomics

Tutorial 8: Empirical Business Cycles & Stochastic Processes

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Where are we?

1. Introduction (Problem Set 1)
2. Growth (Problem Sets 2 - 5)
   - Solow Model: Romer (2018) Chapter 1
   - The Ramsey Model: Romer Chapter 2.A
   - The AK Model: Barro and Sala-i-Martin (2003) Chapter 4.1-4.2
3. **Real Business Cycle Theory** (Problem Sets 8 - 11)
   - Basic concepts (Problem Sets 6 and 7)
   - **Introduction to Real Business Cycle Theory**: Romer Chapter 5 (Problem Set 8)
   - Solving the model (e.g., Linearization and Method of undetermined coefficients, Problem Set 9 - 11)
   - Empirics (Problem Set 10, Exercise 2)
4. New Keynesian Economics (Problem Set 12)
   - Workhorse Model: Galí Chapter 1 and Chapter 3
   - Monetary Policy, Empirics: Christiano et al. (1999)
   - Monetary Policy, Theory: Galí (2002)

Some Notation for Time Series

A time series is a collection of observations indexed by the date of each observation. Usually, we have collected data beginning at some particular date (say, $t = 1$) and ending at another (say, $t = T$):

$$(y_1, y_2, \ldots, y_T)$$

- The first lag of a time series Y_t is Y_{t-1}; its j th lag is Y_{t-j}.
- The first difference of a series, ΔY_t, is its change between periods $t-1$ and t, that is, $\Delta Y_t = Y_t - Y_{t-1}$.
- The first difference of the logarithm of Y_t is $\Delta \ln(Y_t) = \log(Y_t) - \log(Y_{t-1})$.

Covariance Stationarity

Covariance Stationarity (Weak Stationarity)

A time series Y_t is weakly stationary if:

1. $\mathbb{E}[Y_t] = \mu_t = \mu < \infty$ for all t constant mean over time
2. $\text{Cov}\left(Y_t, Y_{t-j}\right) = \gamma_{j,t} = \gamma_j < \infty$ for all t and all j. → variance is constant

That is, a covariance stationary time series has no trend.

Example: Consider the AR(1) $Y_t = \rho Y_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim \mathcal{N}(0,1)$ and $\rho < 1$.
persistence

When allowing for a deterministic trend, a process is called trend stationary

$$Y_t = \mu + \beta t + \varepsilon_t$$

since taking the deterministic trend off Y_t, we are left with a stationary model.

Example: Three AR(1) processes

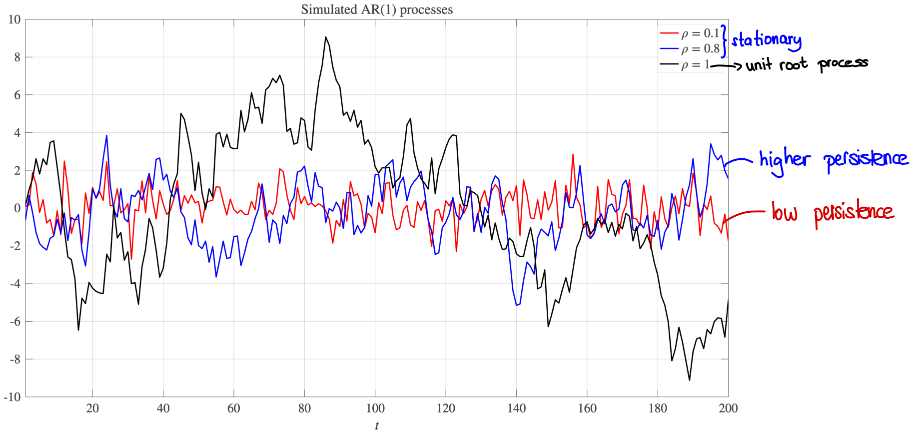

Figure 1: Three AR(1) processes $Y_t = \rho Y_{t-1} + \varepsilon_t$ obtained from the same shock series $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Question 1

Download data for GDP, consumption, and investment for Germany. Choose quarterly data for at least 10 years (for example, from 1996 to 2025). A reliable source for aggregate time series is provided by FRED or Eurostat. Make sure to choose *real* (at constant prices) and *seasonally-adjusted* time series. For the following exercises, you might use the software package of your choice (e.g., Matlab, Stata, R).

Transformation of time series: Why look at $\log(GDP_t)$ and not GDP_t?

Question 1.1

Take the logarithm of the series before detrending the data. Why is the logarithm a useful transformation?

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Take the logarithm of the series before detrending the data. Why is the logarithm a useful transformation?

Suppose the growth rate of GDP is g: $Y_{t+1} = (1+g)Y_t$

A simple derivation yields a useful approximation: The log difference is (approximately) equal to the growth rate

$$\begin{aligned}\log(Y_{t+1}) &= \log((1+g)Y_t) \\ &= \log(1+g) + \log(Y_t) \\ &\approx g + \log(Y_t) \Rightarrow \log(Y_{t+1}) - \log(Y_t) \approx g\end{aligned}$$

holds for small g

Motivation: How can we see the business cycle in the data?

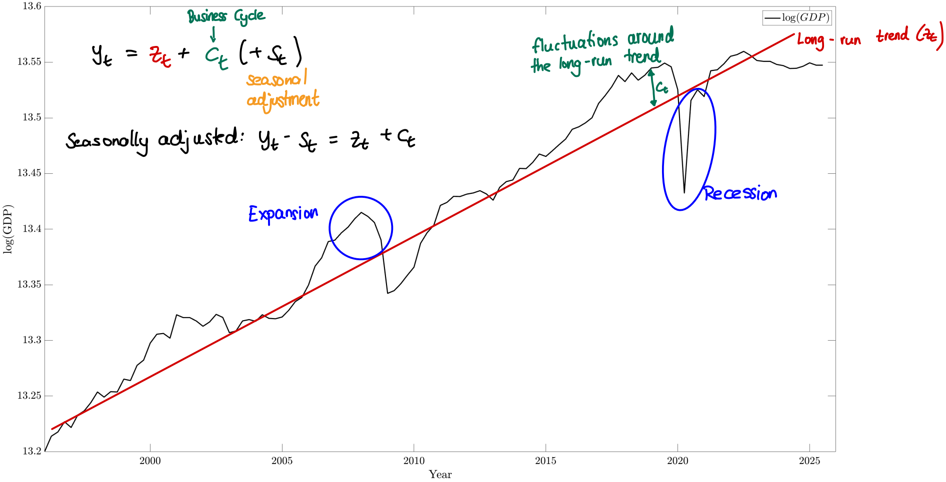

Figure 2: Real German GDP (seasonally-adjusted) as a measure of real activity.

Estimating a linear trend

Question 1.2

Using time series data on German GDP, estimate a linear trend model using ordinary least squares (OLS). Once you have estimated the linear trend, plot the observed GDP values and the fitted trend line on the same graph. Briefly discuss the limitations of using a linear trend model to capture GDP fluctuations over time

Question 1.2

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Decompose log(GDP) into a trend (z_t) and cyclical (c_t) component:

$$\log(GDP_t) = y_t = z_t + c_t$$

We could simply estimate a linear (deterministic) trend

$$\log\left(GDP_t\right) = \beta_0 + \beta_1 t + \varepsilon_t \qquad t = 1, 2, \dots, T$$

where β_1 is the trend growth rate and ε_t are the deviations from the trend.

OLS residual

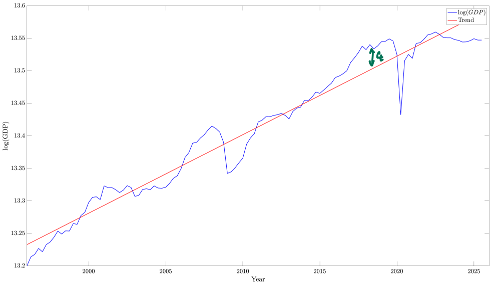


Figure 3: Estimated linear trend for German GDP.

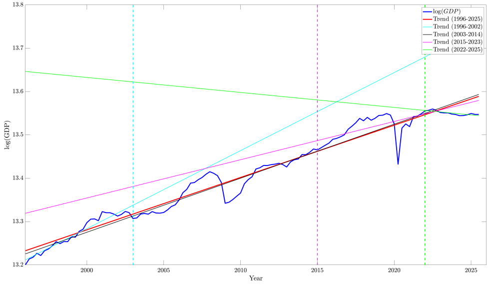


Figure 4: Different estimated linear trends for German GDP.

- Structural changes in the economy can affect the trend growth rate (e.g., population growth may change, pandemic)
- To capture structural factors, we need to allow for time variation in the trend growth rate
- Define the trend growth rate

$$\frac{Z_t - Z_{t-1}}{Z_{t-1}} = \frac{Z_t}{Z_{t-1}} - 1 \simeq \log \frac{Z_t}{Z_{t-1}} = z_t - z_{t-1}$$

Denote the change of the trend growth rate by v_t

$$(z_t - z_{t-1}) - (z_{t-1} - z_{t-2}) = v_t$$

Smoothing parameter: penalizes variations in growth rate of the trend (λ=1600 for quarterly data)
The larger λ, the higher the penalty

→ Paper on Moodle

Hodrick–Prescott filter

The Hodrick–Prescott filter filter formally captures the tradeoff between a smooth and a time varying trend growth rate:

$$\min \sum_t c_t^2 + \lambda \sum_t v_t^2 = \min \left\{ \underbrace{\sum_{t=1}^{T} (y_t - \tau_t)^2}_{\text{penalizes cyclical component}} + \lambda \underbrace{\sum_{t=1}^{T} (\Delta z_t - \Delta z_{t-1})^2}_{\text{Penalizes variations in the growth rate of the trend component}} \right\}$$

where λ captures how much one cares about the smoothness of the trend and $\Delta z_t = z_t - z_{t-1}$.

Allowing for a change in trend growth lowers $\sum_t c_t^2$ but increases $\sum_t v_t^2$

Application: Hodrick–Prescott filter

Question 1.3 and 1.4

Use the Hodrick-Prescott (1997) filter to detrend the data with a smoothing coefficient of $\lambda = 1600$ (built-in function in Matlab/Stata/EViews, otherwise available here). Plot the detrended series, the trend component, and the raw data for all three variables.

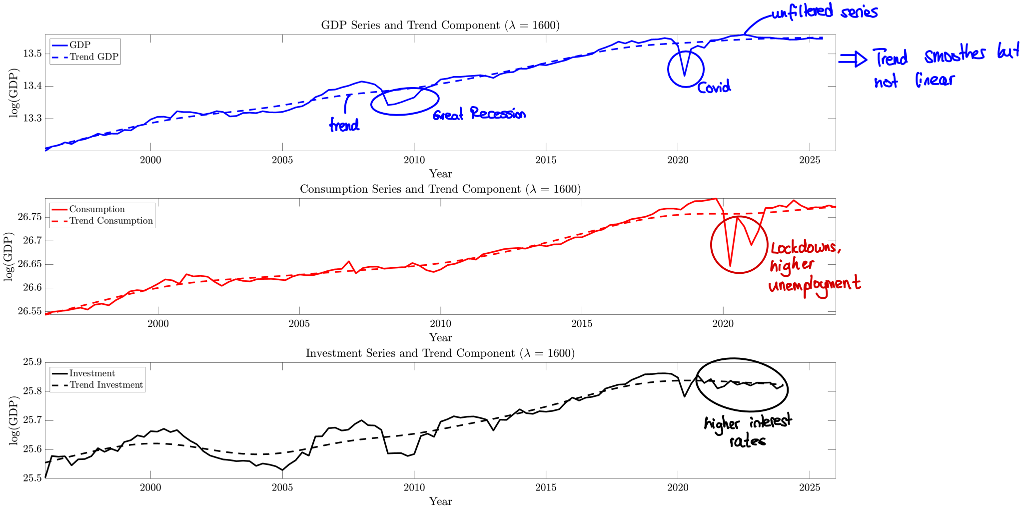
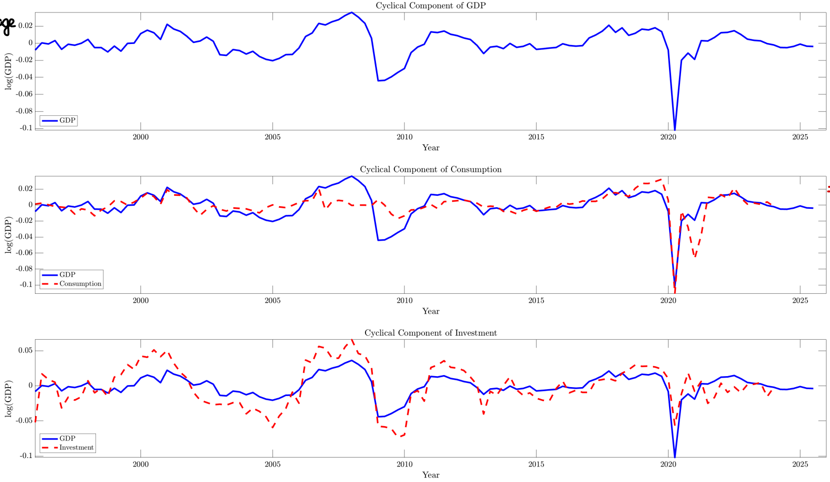


Figure 5: Time series and estimated trend.

Measured in percentage deviations from the trend because we use log (·)



⇒ Consumption is less volatile than GDP

⇒ investment is more volatile than GDP

Figure 6: Estimated cyclical component.

Question 1.5

Briefly discuss the role of λ for the estimated trend. What are the potential disadvantages of the Hodrick-Prescott filter?

The HP filter formally captures the tradeoff between a smooth and a time-varying trend growth rate:

$$\min \sum_t c_t^2 + \lambda \sum_t v_t^2 = \min \left\{ \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=1}^{T} (\Delta z_t - \Delta z_{t-1})^2 \right\}$$

where λ captures how much one cares about the smoothness of the trend and $\Delta z_t = z_t - z_{t-1}$.

- For $\lambda \to 0$ we get

$$\sum_t \left(\log\left(Y_t\right) - z_t\right)^2$$

The time trend z_t that minimizes this loss function is simply $\log\left(Y_t\right) = z_t \Rightarrow$ There is no cycle! $\rightarrow$ only trend

- As $\lambda \to \infty$, the Hodrick-Prescott filter is a linear time trend.
- Ravn and Uhlig (2002) suggest to adjust the HP filter to the frequency of the observations: $\lambda_{\text{quarterly}} = 1600$, $\lambda_{\text{annual}} = 1600/4^4 = 6.25$ and $\lambda_{\text{monthly}} = 1600 \cdot 3^4$.

$\Rightarrow$ Carefully think about choice of λ!

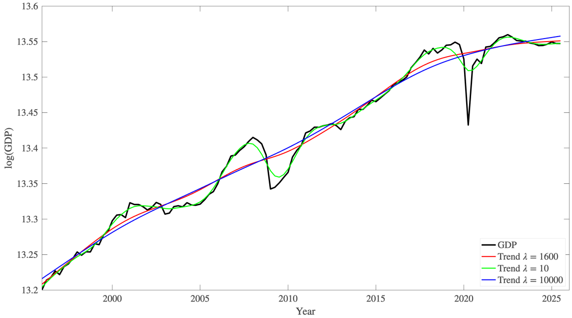


Figure 7: The effect of different λ on the estimated trend

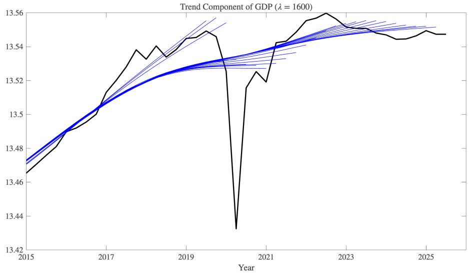


Figure 8: Instability of the HP trend at the margin - GDP trend for Germany

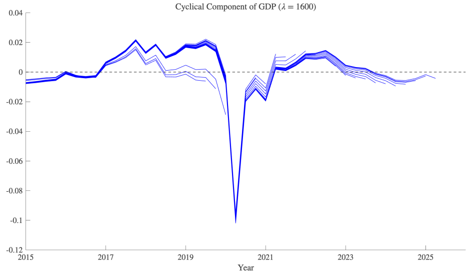


Figure 9: Instability of the HP cycle at the margin - GDP cycle for Germany

Discussion of the Hodrick–Prescott filter

- HP filter offers a reasonable approach to detrending GDP and is widely used but there exist other filters.
- HP filter uses unknowable future values that could not be recognized in real-time and the filter has an end-of-sample bias.
- Hamilton (2018, *Why You Should Never Use the Hodrick-Prescott Filter*) argues that the HP filter introduces spurious dynamic relations that have no basis in the underlying DGP
  - Detrending a random walk can generate spurious dynamics
  - Suppose GDP_t is $I(1)$, and the c_t would be white noise; the cycle would be random and exhibit no smooth cycles
- Hamilton (2018) proposes an alternative to isolate the cyclical component of a non-stationary time series, avoiding its drawbacks, but it can alter the variances of the different frequencies captured in an estimated cyclical component (Schüler, 2024).

How does consumption vary with the business cycle?

Question 1.6

Calculate the standard deviations of consumption and investment relative to that of GDP.

- We get $\sigma_i = 0.029 > \sigma_y = 0.017 > \sigma_c = 0.016$ which yields the following ratios:

$$\frac{\sigma_c}{\sigma_y} = 0.916 \qquad \frac{\sigma_i}{\sigma_y} = 1.706$$

- Consumption and investment are procyclical.
- Consumption is less volatile than output, and investment is more volatile than output.

non-stationary for $g \neq 0$
$\rightarrow$ upward-trend if $g > 0$

If $|\rho_A| < 1$: stationary technology shock process

Question 2.1

Consider the stochastic process for technology in the baseline RBC model from the lecture given by

$$\ln A_t = \bar{A} + gt + \tilde{A}_t \quad \text{with} \quad \tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t}, \quad \Leftarrow \text{our business cycle} \quad AR(1)$$

where $\bar{A}$ is a constant, g denotes the deterministic growth rate and $\varepsilon_{A,t} \overset{i.i.d.}{\sim} N(0, \sigma_A^2)$ is a technology shock. A_0 is given.
↑ starting value

Express $\ln A_1$, $\ln A_2$ and $\ln A_3$ in terms of $\ln A_0$, $\varepsilon_{A,1}$, $\varepsilon_{A,2}$, $\varepsilon_{A,3}$, $\bar{A}$ and g.

$\boxed{t=0:}$ $\ln(A_0) = \bar{A} + g \cdot t + \tilde{A}_0 = \bar{A} + \tilde{A}_0 \iff \tilde{A}_0 = \ln(A_0) - \bar{A}$ starting value of known quantities

use recursive structure

shock

$\boxed{t=1:}$ $\ln(A_1) = \bar{A} + g + \tilde{A}_1 = \bar{A} + g + \rho_A \tilde{A}_0 + \varepsilon_{A,1} = \bar{A} + g + \rho_A [\ln(A_0) - \bar{A}] + \varepsilon_{A,1}$

$\boxed{t=2:}$ $\ln(A_2) = \bar{A} + 2g + \tilde{A}_2 = \bar{A} + 2g + \rho_A \tilde{A}_1 + \varepsilon_{A,2} = \bar{A} + 2g + \rho_A (\rho_A \tilde{A}_0 + \varepsilon_{A,1}) + \varepsilon_{A,2} = \bar{A} + 2g + \rho_A^2 \tilde{A}_0 + \underbrace{\rho_A \varepsilon_{A,1} + \varepsilon_{A,2}}_{\text{history of shocks}}$

↑ starting value

$\boxed{t=3:}$

$$\ln(A_3) = \bar{A} + 3g + \tilde{A}_3 = \bar{A} + 3g + \rho_A^3 \tilde{A}_0 + \underbrace{\rho_A^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3}}_{\text{history of shocks}}$$

$\boxed{\text{for } t:}$

$$\ln(A_t) = \bar{A} + \underbrace{t \cdot g}_{\text{trend}} + \underbrace{\rho_A^t \left(\ln(A_0) - \bar{A}\right)}_{\text{If } \rho_A < 1, \text{ starting value vanishes for } t \to \infty} + \underbrace{\sum_{j=0}^{t-1} \rho_A^j \varepsilon_{A,t-j}}_{\text{shocks from } t=0 \text{ up to } t \text{ where the most recent shock has the highest weight}}$$

$\mathbb{E}_t[x_t] = \mathbb{E}[x_t \mid \mathcal{F}_t]$ where $\mathcal{F}_t = \sigma(X(r,\omega), r \leq t)$ smallest σ-field generated by past of st. process

Question 2.2

What are the time-zero expectations of $\ln A_1$, $\ln A_2$, $\ln A_3$ and $\ln A_\infty$?

Apply expectations operator to results from 2.1.

$\mathbb{E}_0[\ln(A_0)] = \bar{A} + \tilde{A}_0$ $\quad$ $\mathbb{E}_0[\cdot]$ has no effect

Linearity of expectations

$$\mathbb{E}_0[\ln(A_1)] = \mathbb{E}_0\Big[\bar{A} + g + \rho_A[\ln(A_0) - \bar{A}] + \underbrace{\varepsilon_{A,1}}_{=\text{stochastic}}^{\text{Shock}}\Big] = \mathbb{E}_0\Big[\underbrace{\bar{A} + g + \rho_A[\ln(A_0) - \bar{A}]}_{\text{Constants}}\Big] + \underbrace{\mathbb{E}_0\big[\overset{\text{Shock}}{\varepsilon_{A,1}}\big]}_{=0} = \bar{A} + g + \rho_A[\ln(A_0) - \bar{A}]$$

$$\mathbb{E}_0[\ln(A_2)] = \mathbb{E}_0\Big[\bar{A} + 2g + \rho_A^2 \underset{\substack{\uparrow \\ \text{Starting} \\ \text{value}}}{\tilde{A}_0} + \underbrace{\rho_A \varepsilon_{A,1} + \varepsilon_{A,2}}_{\text{history of shocks}}\Big] = \bar{A} + 2g + \rho_A^2 \tilde{A}_0$$

$$\mathbb{E}_0[\ln(A_3)] = \bar{A} + 3g + \rho_A^3 \tilde{A}_0$$

$$\mathbb{E}_0\left[\ln(A_\infty)\right] = \bar{A} + \lim_{k\to\infty}\left\{ k\cdot g + \rho_A^k\left(\ln(A_0) - \bar{A}\right)\right\}$$

Importance of ρ_A (governs persistence)

- $\rho_A > 1$ & $k \to \infty$ $\lim\{\cdot\}$ explodes
- $\rho_A < 1$ & $k \to \infty$ starting value fades out

Question 2.3

Suppose that technology is only driven by the stochastic term $\tilde{A}_t$, i.e. set $\bar{A} = g = 0$, and assume $A_0 = 1$ and $\rho_A = 0.5$. Compute $\ln A_t$ for $t = \{1, ..., 5\}$ given that

a) no shock hits the economy.

b) a one-time unity shock hits the economy in period 1, i.e. $\varepsilon_{A,1} = 1$.

Assumptions: $\bar{A} = g = 0$, $A_0 = 1$, $\rho_A = 0.5$ (stationary process).

a) $\varepsilon_{A,t} = 0 \ \forall t \iff \ln A_t = 0 \ \forall t$

b) $\varepsilon_{A,1} = 1, \ \varepsilon_{A,t} = 0 \ \forall t \neq 1$:

$$\ln A_1 = \varepsilon_{A,1} = 1$$
$$\ln A_2 = \rho_A \varepsilon_{A,1} + \varepsilon_{A,2} = 0.5 \cdot 1 = 0.5$$
$$\ln A_3 = \rho_A^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3} = 0.5^2 \cdot 1 = 0.25$$
$$\ln A_4 = \rho_A^3 \varepsilon_{A,1} + \rho_A^2 \varepsilon_{A,2} + \rho_A \varepsilon_{A,3} + \varepsilon_{A,4} = 0.5^3 \cdot 1 = 0.125$$
$$\ln A_5 = \rho_A^4 \varepsilon_{A,1} + \rho_A^3 \varepsilon_{A,2} + \rho_A^2 \varepsilon_{A,3} + \rho_A \varepsilon_{A,4} + \varepsilon_{A,5} = 0.5^4 \cdot 1 = 0.0625$$

Role of persistence ρ_A: shock fades out over time if $0 \leq \rho_A \leq 1$.

Stochastic Processes: One-time unity shock when $rho_A = 0.5$

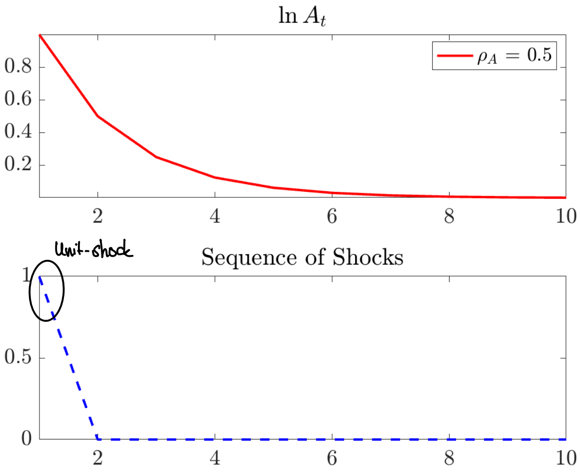


Figure 10: Stochastic process for technology for $\rho_A = 0.5$.

Question 2.4

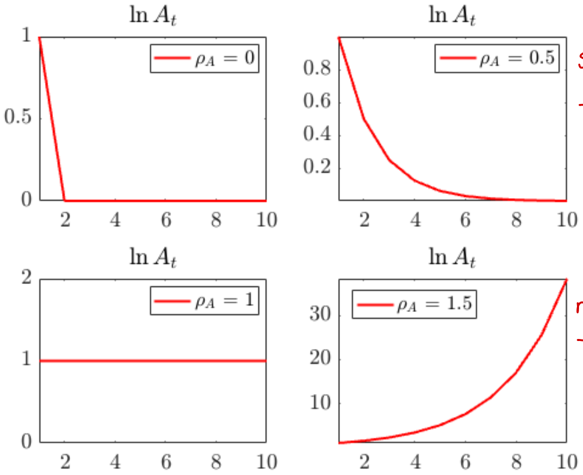
Repeat 3b) for the cases $\rho_A = 0, \;\; \rho_A = 1, \;\; \rho_A = 1.5.$

$\ln A_t$	$\rho_A = 0$	$\rho_A = 1$	$\rho_A = 1.5$
$\ln A_1$	1	1	1
$\ln A_2$	0	1	1.5
$\ln A_3$	0	1	2.25
$\ln A_4$	0	1	3.375
$\ln A_5$	0	1	5.0625

Question 2.4.: Role of persistence in the AR-technology process

one-time shock
without persistence
(Random-Walk)

stationary
-> shock fades out



$\ln(A_t) = \ln(A_{t-1}) + \varepsilon_{A,t}$
unit-root process

non-stationary
-> technology explodes

Figure 11: Stochastic process for technology for $\rho_A = 0.0$, $\rho_A = 0.5$, $\rho_A = 1$, and $\rho_A = 1.5$.

Summary and Outlook

- Economic fluctuations do not underly simple repeating patterns.
- The trend observed in the data doesn't seem to be as rigid as a deterministic trend.
- We can use the HP filter to extract the trend and cycle, but you need to be careful using the HP filter.
- Investment is more volatile than output. Output is more volatile than consumption.
- In the RBC Model (Problem sets 10 and 11), we want to match the cyclicality of the data using a simple model.

Further Reading:

- Romer (2019): Chapter 5.1-5.3.
- Mitman, Kurt (2024): Real Business Cycles. In: Macroeconomics. Chapter 13.1. to 13.3.2. for a discussion of the HP filter and a more detailed discussion of business cycles.
- Hansen (2023): Econometrics. Chapters 14.1-14.21 and 14.41 are for a technical introduction of AR processes and detrending.