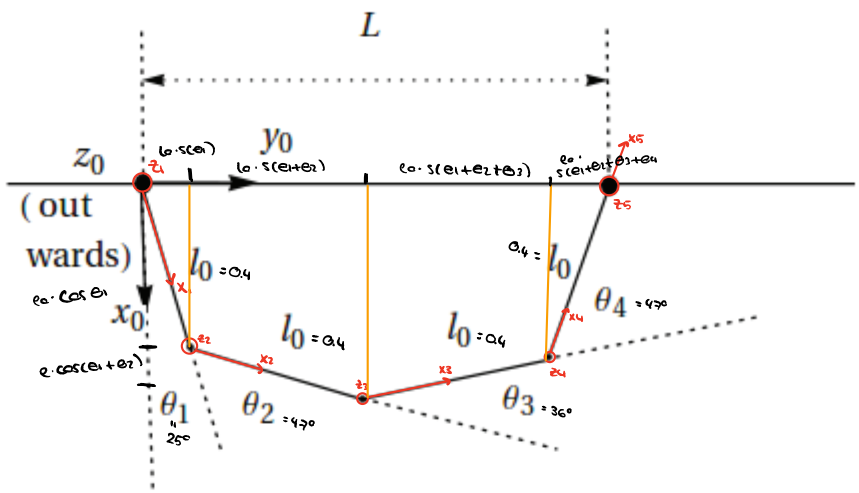
Problem 1

A system with 4 degrees of freedom $\{\theta_1,\ \theta_2,\ \theta_3,\ \theta_4\}$ is hanging from a horizontal bar, as shown in the figure.



Tx	Rx	Tz	Rz		
0	0	0	θ_1	$\{0\} \to \{1\}$	$Rz(\theta_1)$
l_0	0	0	θ_2	$\{1\} \to \{2\}$	$Tx(l_0)\, Rz(\theta_2)$
l_0	0	0	θ_3	$\{2\} \to \{3\}$	$Tx(l_0)\, Rz(\theta_3)$
l_0	0	0	θ_4	$\{3\} \to \{4\}$	$Tx(l_0)\, Rz(\theta_4)$
l_0	0	0	0	$\{4\} \to \{5\}$	$Tx(l_0)$

(Data: $\theta_1 = 25^{\text{o}},\ \theta_2 = 47^{\text{o}},\ \theta_3 = 36^{\text{o}},\ \theta_4 = 47^{\text{o}},\ l_0 = 0.4\,m$)

The subjection points, separated a distance L, are to be kept fixed.

1) Under this restriction, if the system performs a generic infinitesimal movement

$$\{\theta_1 \to \Delta\theta_1 + \theta_1,\ \theta_2 \to \Delta\theta_2 + \theta_2,\ \theta_3 \to \Delta\theta_3 + \theta_3,\ \theta_4 \to \Delta\theta_4 + \theta_4\}$$

the values $\{\Delta\theta_1,\ \Delta\theta_2,\ \Delta\theta_3,\ \Delta\theta_4\}$ are subject to a restriction of the form

$$0 = J \ . \begin{pmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{pmatrix}$$

$J = \frac{\partial x_i}{\theta_j}$

$x_1 = L = l_0 s(\theta_1) + l_0 s(\theta_1+\theta_2) + l_0 s(\theta_1+\theta_2+\theta_3) + l_0 s(\theta_1+\theta_2+\theta_3+\theta_4)$
$x_2 = l_0 s(\theta_1+\theta_2) + l_0 s(\theta_1+\theta_2+\theta_3) + l_0 s(\theta_1+\theta_2+\theta_3+\theta_4)$
$x_3 = l_0 s(\theta_1+\theta_2+\theta_3) + l_0 s(\theta_1+\theta_2+\theta_3+\theta_4)$
$x_4 = l_0 s(\theta_1+\theta_2+\theta_3+\theta_4)$

$$\left.\begin{aligned} J_1 &= l_0 c(\theta_1) + l_0 c(\theta_1+\theta_2) + l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4) \\ J_2 &= l_0 c(\theta_1+\theta_2) + l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4) \\ J_3 &= l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4) \\ J_4 &= l_0 c(\theta_1+\theta_2+\theta_3+\theta_4) \end{aligned}\right\} \to \text{substituint:}$$

$J_1 = 0$
$J_2 = -0.362523$
$J_3 = -0.48613$
$J_4 = -0.362523$

where the jacobian J is a matrix 1×4 . Find the numerical value of the 4 jacobian matrix elements.

$\hookrightarrow J = \frac{\partial x_i}{\partial j}$

Finalment, ens queda: $0 = -0.362523\,\Delta\theta_2 - 0.48613\,\Delta\theta_3 - 0.362523\,\Delta\theta_4$

2) If we impose an additional second condition consisting in raising the heigth of the central point P by 0.10 m (i.e. $\Delta x_2 = -0.10\,m$, since x_0 points downwards), in addition to keeping the separation distance L fixed (i.e., $\Delta y_1 = 0$), find the new jacobian matrix 2×4 element values relating $(\Delta y_1,\ \Delta x_2)$ and $\{\Delta\theta_1,\ \Delta\theta_2,\ \Delta\theta_3,\ \Delta\theta_4\}$:

for x_1:
$J_1 = l_0 c(\theta_1) + l_0 c(\theta_1+\theta_2) + l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4)$
$J_2 = l_0 c(\theta_1+\theta_2) + l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4)$
$J_3 = l_0 c(\theta_1+\theta_2+\theta_3) + l_0 c(\theta_1+\theta_2+\theta_3+\theta_4)$
$J_4 = l_0 c(\theta_1+\theta_2+\theta_3+\theta_4)$

$$\begin{array}{r} l_0 s(\theta_1) + l_0 s(\theta_1+\theta_2) + l_0 s(\theta_1+\theta_2+\theta_3) + l_0 s(\theta_1+\theta_2+\theta_3+\theta_4) \leftarrow x_1 \\ l_0 \cdot c(\theta_1) + l_0 c(\theta_1+\theta_2) \leftarrow x_2 \end{array} \begin{pmatrix} \Delta y_1 \\ \Delta x_2 \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} 0 & -0.362523 & -0.48613 & -0.362523 \end{array} \\ \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \\ \begin{array}{cccc} -0.54947 & -0.380423 & 0 & 0 \end{array} \end{array} \begin{pmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta\theta_4 \end{pmatrix}$$

for x_2

$\hookrightarrow x_1 = l_0 c(\theta_1)$
$x_2 = l_0 \cdot c(\theta_1) + l_0 c(\theta_1+\theta_2)$

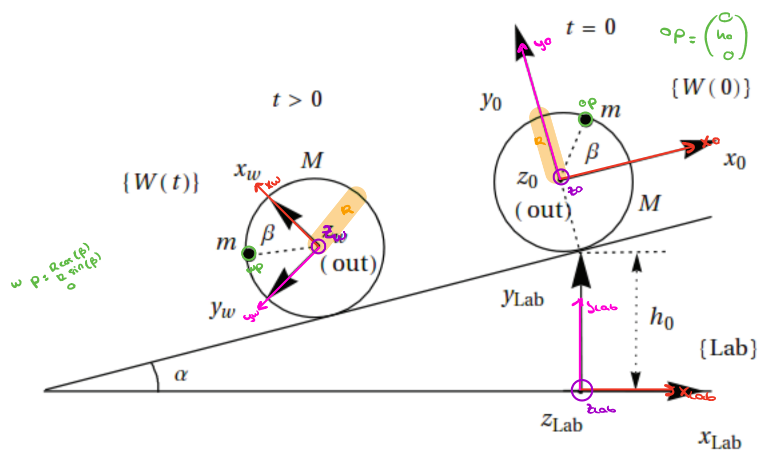
$\to J_1 = -l_0 s(\theta_1)$
$\to J_2 = -l_0 s(\theta_1) - l_0 s(\theta_1+\theta_2)$
$J_3 = 0$
$J_4 = 0$

$\to$ substituim
$\hookrightarrow -0.54947$
$\hookrightarrow -0.380423$
$\hookrightarrow 0$
$\hookrightarrow 0$

Problem 2

Consider a ring of radius R and mass M. We attach a system of reference $\{W(t)\} \equiv \{x_w, y_w, z_w\}$ with origin at its center. The ring contains an additional point mass m at the position ${}^{W}P = \{R\,cos(\beta),\, R\,sin(\beta),\, 0\}$.

At $t = 0$ we release the ring, at rest, from the position $\{W(0)\}$ shown in the figure, with the x_w axis paral.lel to the ground and the point making contact with the ground being ${}^{Lab}P = \{0,\, h_0,\, 0\}$.



Ⓟ

$${}^{Lab}_{W(0)}T = T_y(h_0)\, R_z(\alpha)\, T_y(R)$$

$${}^{W(0)}_{W(t)}T = T_x(-R\cdot\theta)\, R_z(\theta)$$

$${}^{Lab}_{W(t)}T = T_y(h_0)\, R_z(\alpha)\, T_y(R)\, T_x(-R\cdot\theta)\, R_z(\theta)$$

$$= \begin{pmatrix} c(\alpha+\theta) & -s(\alpha+\theta) & 0 & -R\,s(\alpha) - R\theta\, c(\alpha) \\ s(\alpha+\theta) & c(\alpha+\theta) & 0 & h_0 + R\,c(\alpha) - R\theta\, s(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Data: Assume that $(g,\, R,\, M,\, m,\, \alpha,\, \beta)$ are all known values.

The movement takes place without squidding, just rolling, i.e, with the point in contact with the ground at any time having always $v = 0$, and the distance travelled by the center of the ring being $d(t) = R\,\theta(t)$, where $\theta(t)$ measures the amount of rotation of the ring between $t = 0$ and the instant t (i.e. $\theta(t = 0) = 0$)

✓ 1) Using $\theta = \theta(t)$ as the degree of freedom of the system, write the transformation matrices ${}^{Lab}_{W(0)}T$, ${}^{W(0)}_{W(t)}T$ and ${}^{Lab}_{W(t)}T$ (it' s ok if you let the results as a products of matrices, provided you give the contents of each matrix. Don't perform the matrix multiplications).

2) Find the origin of the system $W(t)$ in terms of $\theta = \theta(t)$, as seen from the{Lab} reference frame, ${}^{Lab}P_{W(t)}$.

Which is the transformation matrix ${}^{Lab}_{S}T$ for a system S with origin and at ${}^{Lab}P_{W(t)}$ and the x axis pointing to m? Give the complete contents ${}^{Lab}_{S}T$, with the rotation part written in

Origen de W(t) : $\quad {}^{Lab}_{W(0)}T(\theta) \cdot \begin{pmatrix} -R\theta \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -R\,s(\alpha) - R\theta\, c(\alpha) \\ h_0 + R\,c(\alpha) - R\theta\, s(\alpha) \\ 0 \\ 1 \end{pmatrix}$

$${}^{Lab}_{S}T(t) = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare & \\ \blacksquare & {}^{Lab}_{S}R & \blacksquare & {}^{Las}_{S}P \\ \blacksquare & \blacksquare & \blacksquare & \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c(\alpha+\theta+\beta) & -s(\alpha+\theta+\beta) & 0 & -R\,s(\alpha) - R\theta\,c(\alpha) \\ s(\alpha+\theta+\beta) & c(\alpha+\theta+\beta) & 0 & h_0 + Rc(\alpha) - R\theta\,s(\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

m position in Lab system $= {}^{Lab}_{S}T(t) \cdot \begin{pmatrix} R \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -R\,s(\alpha) - R\theta\,c(\alpha) + Rc(\alpha+\theta+\beta) \\ h_0 + Rc(\alpha) - R\theta\,s(\alpha) + Rs(\alpha+\theta+\beta) \\ 0 \\ 1 \end{pmatrix}$

terms of $(\alpha + \beta + \theta)$. Making use of ${}^{Lab}_{S}T$, find the Lab coordinates of the position of the point mass m.

3) Compute, as a function of θ, the gravitational potential energy of the system $V = M\,g\,y_{CM} + m\,g\,y(m)$ where y_{CM} and $y(m)$ are the vertical components of the positions of the ring center and of the point mass m in the Lab reference frame. Show that $V = \text{ct} + g\,m\,R\,\sin(\alpha+\beta+\theta) - g\,\theta\,m\,R\,\sin(\alpha) - g\,\theta\,M\,R\,\sin(\alpha)$ and find the corresponding generalized force $F_\theta = \frac{\partial W}{\partial \theta} = -\frac{\partial V}{\partial \theta}$.

4) Show that the kinetic energy of the system is

$$K(\theta, \dot{\theta}) = (M + m + m\,\sin(\theta+\beta))\,R^2\,\dot{\theta}^2$$

(you may proceed to the next point using of this result, and do this calculation at the end).

5) Compute both the generalized momentum P_θ and its time derivative $\dot{P}_\theta$ at $t = 0$ as a function of $(g,\,R,\,M,\,m,\,\alpha,\,\beta,\,\theta,\,\dot{\theta})$. Find the value of β that makes the system not to start rolling downwards, by making the system stay at its initial position (Hint: find which condition β must satisfy so that $\dot{P}_\theta = 0$ at $t = 0$). Can you provide an explanation for that particular β value in terms of the center of mass of the full system?

3 center ring position, in Lab system: ${}^{Lab}_{S}T(t) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -R\,s(\alpha) - R\theta\,c(\alpha) \\ h_0 + Rc(\alpha) - R\theta\,s(\alpha) \\ 0 \\ 1 \end{pmatrix}$

force??

Gravitational potential energy is then: $V_{pot} = m \cdot g \cdot y_m + Mg\,y_M$
$= mg\left[h_0 + Rc(\alpha) - R\theta\,s(\alpha) + Rs(\alpha+\theta+\beta)\right] +$
$+ Mg\left[h_0 + Rc(\alpha) - R\theta\,s(\alpha)\right]$

4 Show that $K(\theta,\dot\theta) = (M + m + m\,s(\theta+\beta))\,R^2\,\dot\theta^2$

$\underline{\underline{M}}$

$K_{cin}(M) = \frac{1}{2} M V_{CM}^2 + \sum_i \frac{1}{2} m_i V_i'^2$
$= \frac{1}{2} M (R^2 \theta'(t)^2) + \sum_i \frac{1}{2} m_i (R^2 \theta'(t)^2)$
$= \frac{1}{2} M R^2 \theta'(t)^2 + \frac{1}{2} M R^2 \theta'(t)^2$
$= M \underbrace{R^2 \theta'(t)^2}_{\equiv}$

$\underline{\underline{m}}$ (in the θ system) $\xrightarrow{\alpha=0}$

Position $\begin{pmatrix} R\theta + Rc(\theta+\beta) \\ Rs(\theta+\beta) \\ 0 \\ 1 \end{pmatrix}$

Velocity $\begin{pmatrix} -R\dot\theta - R\dot\theta\,s(\theta+\beta) \\ R\dot\theta\,c(\theta+\beta) \\ 0 \\ 1 \end{pmatrix}$

$K_{cin}(m) = \frac{1}{2} m v^2 =$
$= \frac{1}{2} m (R\dot\theta)^2 + m (R\dot\theta)^2 s(\beta+\theta) + \frac{1}{2} m (R\dot\theta)^2 =$
$= m (R\dot\theta)^2 + m (R\dot\theta)^2 s(\beta+\theta) =$
$= (R\dot\theta)^2 (m + m\,s(\beta+\theta))$

$K_{TOTAL} = (R\dot\theta)^2 (M + m + m\,s(\beta+\theta))$
$= (M + m + m\,s(\theta+\beta))\,R^2\,\dot\theta^2$

5 Generalized momentum of P_θ

$P_\theta = \frac{\partial K}{\partial \dot\theta} = 2\,(M + m + m\,s(\theta+\beta))\,R^2\,\dot\theta$

$\dot P_\theta$:

we have $\frac{\partial}{\partial t}\left(\frac{\partial K_{cin}}{\partial \dot\theta}\right) - \frac{\partial K_{cin}}{\partial \theta} = F_\theta \rightarrow \frac{\partial}{\partial t} P_\theta = \frac{\partial K_{cin}}{\partial \theta} + F_\theta \rightarrow \dot P_\theta = \frac{\partial K_{cin}}{\partial \theta} + F_\theta$

we have $\frac{\partial K_{cin}}{\partial \theta} = m\cos(\beta+\theta)\,R^2\dot\theta^2$, $F_\theta = \frac{\partial V_{pot}}{\partial \theta} = -gmR\,c(\alpha+\beta+\theta) + gmR\,s(\alpha) + gMR\,s(\alpha)$

At $t=0 \rightarrow \theta = 0,\ \dot\theta = 0$

$\dot P_\theta = \frac{\partial K_{cin}}{\partial \theta} + F_\theta = -gmR\,c(\alpha+\beta) + gmR\,s(\alpha) + gMR\,s(\alpha)$