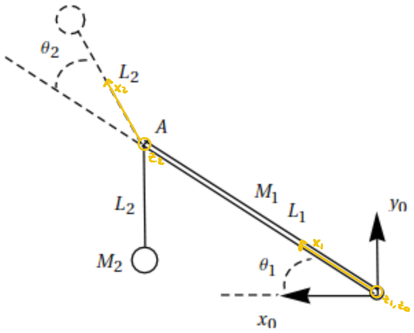
Final exam FoMAR

June 2015

Problem

Consider a catapulting system like the one shown in the scheme:



T_x	R_x	T_z	R_z
0	0	0	θ_1
L_1	0	0	θ_2
~~Z~~	~~Z~~	~~Z~~	~~Z~~

$z_{i+1} \to z_i$ | $x_{i+1} \to x_i$

$\{0 \to 1\}$ $\; {}^0_1T = R_z(\theta_1) = \begin{pmatrix} c(\theta_1) & -s(\theta_1) & 0 & 0 \\ s(\theta_1) & c(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\{1 \to 2\}$ $\; {}^1_2T = T_x(L_1) \cdot R_z(\theta_2) = \begin{pmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c(\theta_2) & -s(\theta_2) & 0 & 0 \\ s(\theta_2) & c(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} c(\theta_2) & -s(\theta_2) & 0 & L_1 \\ s(\theta_2) & c(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\{2 \to 3\}$ $\; \emptyset$

M_2 is the mass of the projectile, hanging from a massless rope of length L_2. M_1 is the mass of a uniform rigid bar of length L_1 extending from the origin to the point where M_2 is suspended.

The system at t = 0 is in its initial conditions at rest, with $\theta_1 = \gamma$, $\theta_2 = -\gamma - \pi/2$ (thus with M_2 hanging vertically, as shown in the figure).

We suddenly and rapidly increase θ_1 by applying a very strong torque at the pivot point. The details of such torque are unimportant; what matters to us is that for t > 0 we are forcing a rapid movement $\theta_1 = \theta_1(t)$ and we are interested in finding how the projectile is accelerated in reaction to our action. Our interest is focussed only on the projectile movement in the acceleration phase, with the rope completely extended in a straight line of length L_2.

✓ 1) Find the Denavit - Hartenberg description ot this system, specifying the reference frames associated to each mobile part and provide the D - H Table. Write explicitely the 4×4 homogeneous transformation matrices ${}^0{}_1T$ and ${}^1{}_2T$.

✓ 2) For arbitrary values of θ_1 and θ_2, compute in the $\{2\}$, $\{1\}$ and $\{0\}$ reference frames the positions of points A and M_2 .

For $\{2\}$ → $A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

→ $M_2 = \begin{pmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\{1\}$ $\; A = {}^1_2T \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1 c(\theta_1) \\ L_1 s(\theta_1) \\ 0 \\ 1 \end{pmatrix}$

$M_2 = {}^1_2T \cdot \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ → molt gran

$\{0\}$ $\; A = {}^0_1T \, {}^1_2T \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$M_2 = {}^0_1T \, {}^1_2T \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3) In the $\{0\}$ system, find the coordinates of the vectors $\vec{r}_{M_2} - \vec{r}_A$ and $\vec{v}_{M_2} - \vec{v}_A$ as a function of $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and find the vector $\vec{w}$ which satisfies the equation

$$\vec{v}_{M_2} - \vec{v}_A = \vec{w} \times (\vec{r}_{M_2} - \vec{r}_A)$$

Is it consistent with the values you find?

$\vec{r}_{M_2} - \vec{r}_A = {}^0_1T \, {}^1_2T \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} - {}^0_1T \, {}^1_2T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_2 \, c(\theta_1 + \theta_2) \\ L_2 \, s(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$

$\vec{v}_{M_2} - \vec{v}_A = \frac{\partial}{\partial t} \left({}^0_1T \, {}^1_2T \begin{pmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{pmatrix} - {}^0_1T \, {}^1_2T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -L_2 \, s(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2) \\ L_2 \, c(\theta_1 + \theta_2) \cdot (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{pmatrix}$

$\vec{w} = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$

4) The kinetic energy of this system can be written as

$$K_{cin} = \frac{1}{2}\left((M_1/3 + M_2)\,L_1^2 + M_2\,L_2^2 + 2\,M_2\,L_1 L_2\,cos(\theta_2)\right)\dot{\theta}_1^2 +$$

$$+(M_2\,L_1 L_2\,cos(\theta_2)\;+\;M_2\,L_2^2\,)\,\dot{\theta}_1\,\dot{\theta}_2 + \frac{1}{2}\,M_2\,L_2^2\,\dot{\theta}_2^2$$

whilst the potential energy is

$$V(\theta_1,\;\theta_2) = V_{Torque}(\theta_1) + \frac{1}{2}\,M_1\,g\,L_1\,sin(\theta_1) +\;M_2\,g\,(\,L_1\,sin(\theta_1) +\;L_2\,sin(\theta_1+\theta_2))$$

(note that the forces F_j can be computed from $F_j = -\partial V/\partial\theta_j$ when the function V is known).

Find the equation of motion for θ_2 (i.e. write the Lagrange equation for θ_2 , and isolate $\ddot{\theta}_2$, assuming $\theta_1(t)$ is known). As a consistency check, verify that in the initial conditions with $\theta_1 = \gamma$, $\theta_2 = -\gamma - \pi/2$, the system at rest ($\dot{\theta}_1 = \dot{\theta}_2 = 0$) if we keep $\theta_1(t) = constant$ then you obtain $\ddot{\theta}_2 = 0$.

5) Derive the above expression for the kinetic energy $K_{cin}(\theta_1,\,\theta_2,\,\dot{\theta}_1,\,\dot{\theta}_2)$ for this system.

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$$K_{cin} = \frac{1}{2}\left(\left(\frac{M_1}{3}+M_2\right)L_1^2 + M_2L_2^2 + 2M_2L_1L_2\cos(\theta_2)\right)\dot{\theta}_1^2 + \left(M_2L_1L_2\cos(\theta_2) + M_2L_2^2\right)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}M_2L_2^2\dot{\theta}_2^2$$

$$\Rightarrow \frac{\partial K_{cin}}{\partial \theta_2} = -\dot{\theta}_1^2\sin(\theta_2)L_1L_2M_2 - \dot{\theta}_1\dot{\theta}_2\sin(\theta_2)L_1L_2M_2$$

$$\Rightarrow \frac{\partial K_{cin}}{\partial \dot{\theta}_2} = \dot{\theta}_2L_2^2M_2 + \dot{\theta}_1(\cos(\theta_2)L_1L_2M_2 + L_2^2M_2)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial K_{cin}}{\partial \dot{\theta}_2}\right) = -\dot{\theta}_1\dot{\theta}_2\sin(\theta_2)L_1L_2M_2 + \ddot{\theta}_2L_2^2M_2 + \ddot{\theta}_1(\cos(\theta_2)L_1L_2M_2 + L_2^2M_2)$$

$$\text{Left hand side} = \frac{\partial}{\partial t}\left(\frac{\partial K_{cin}}{\partial \dot{\theta}_2}\right) - \frac{\partial K_{cin}}{\partial \theta_2} = \ddot{\theta}_1\cos(\theta_2)L_1L_2M_2 + \dot{\theta}_1^2\sin(\theta_2)L_1L_2M_2 + \ddot{\theta}_1L_2^2M_2 + \ddot{\theta}_2L_2^2M_2$$

$$\text{Right Hand Side} = -\frac{\partial V(\theta_1,\theta_2)}{\partial \theta_2} = -M_2\cdot g\cdot L_2\cdot\cos(\theta_1+\theta_2)$$

Equation of motion:

$$\text{Left Hand Side} = \text{Right Hand Side}$$

$$\ddot{\theta}_1\cos(\theta_2)L_1L_2M_2 + \dot{\theta}_1^2\sin(\theta_2)L_1L_2M_2 + \ddot{\theta}_1L_2^2M_2 + \ddot{\theta}_2L_2^2M_2 = -M_2\cdot g\cdot L_2\cdot\cos(\theta_1+\theta_2)$$

$$\ddot{\theta}_2 = -\ddot{\theta}_1\frac{L_1\cos(\theta_2)+L_2}{L_2} - \frac{L_1\sin(\theta_2)\dot{\theta}_1^2 + g\cos(\theta_1+\theta_2)}{L_2}$$

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5)

Show that

$$K_{cin} = \frac{1}{2}\left((M_1/3 + M_2)\,L_1^2 + 2\,M_2\,L_1\,L_2\,Cos[\theta_2] + M_2\,L_2^2\right)\dot{\theta}_1^2 +$$

$$+\left(M_2\,L_1\,L_2\,Cos[\theta_2] + L_2^2\,M_2\right)\dot{\theta}_1\,\dot{\theta}_2 + \frac{1}{2}\,M_2\,L_2^2\,\dot{\theta}_2^2$$

We can start from

$$K_{cin} = \frac{1}{2}\,I_{33}\,(pivot)\;\omega_z^2 + \frac{1}{2}\,M_2\,\vec{v}_{M_2}^{\,2}$$

with

$$I_{33}\,(pivot) = \left(\frac{1}{3}\,M_1\,L_1^2\right),\quad \omega_z = \omega_z[Bar] = \dot{\theta}_1 \quad \text{and}$$

$$\vec{v}_{M_2} = \frac{d}{dt}\begin{pmatrix} Cos[\theta_1]\,L_1 + Cos[\theta_1+\theta_2]\,L_2 \\ Sin[\theta_1]\,L_1 + Sin[\theta_1+\theta_2]\,L_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -Sin[\theta_1]\,L_1\,\dot{\theta}_1 - Sin[\theta_1+\theta_2]\,L_2\,(\dot{\theta}_1+\dot{\theta}_2) \\ Cos[\theta_1]\,L_1\,\dot{\theta}_1 + Cos[\theta_1+\theta_2]\,L_2\,(\dot{\theta}_1+\dot{\theta}_2) \\ 0 \end{pmatrix}$$

$$\vec{v}_{M_2}^{\,2} = L_1^2\,\dot{\theta}_1^2 + L_2^2\,(\dot{\theta}_1+\dot{\theta}_2)^2 + 2\,L_1\,L_2\,Cos[(\theta_1+\theta_2)-\theta_1]\,\dot{\theta}_1\,(\dot{\theta}_1+\dot{\theta}_2)$$

therefore

$$\rightarrow K_{cin} = \frac{1}{2}\left(\frac{1}{3}\,M_1\,L_1^2\right)\dot{\theta}_1^2 + \frac{1}{2}\,M_2\,\left(L_1^2\,\dot{\theta}_1^2 + L_2^2\,(\dot{\theta}_1+\dot{\theta}_2)^2 + 2\,L_1\,L_2\,Cos[(\theta_1+\theta_2)-\theta_1]\,\dot{\theta}_1\,(\dot{\theta}_1+\dot{\theta}_2)\right) =$$

$$\frac{1}{2}\left((M_1/3 + M_2)\,L_1^2 + 2\,M_2\,L_1\,L_2\,Cos[\theta_2] + M_2\,L_2^2\right)\dot{\theta}_1^2 +$$

$$+\left(M_2\,L_1\,L_2\,Cos[\theta_2] + L_2^2\,M_2\right)\dot{\theta}_1\,\dot{\theta}_2 + \frac{1}{2}\,M_2\,L_2^2\,\dot{\theta}_2^2$$

and that's it.