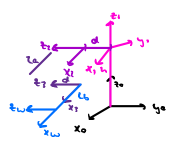
Fomar Final exam

Problem 1

For a 3 degree of freedom (θ_1, θ_2, θ_3) manipulator, whose Denavit-Hartenberg table is the following:

	Rx	Tx	Tz	Rz
0 → 1	0	0	h	θ1
1 → 2	90 º	0	d	θ2
2 → 3	0	La	-d	θ3
3 → W	0	Lb	0	0

${}^0_1T$ = Tz(h) . Rz(θ_1)
${}^1_2T$ = Rx(90°) Tz(d) Rz(θ_2)
${}^2_3T$ = Tx(La) Tz(-d) Rz(θ_3)
${}^3_wT$ = tx(Lb)



✓ 1) Draw, for the values $\theta_1 = 30^o$, $\theta_2 = 20^o$, $\theta_3 = -30^o$, the various reference frames {1}, {2}, {3}, {W} attached to each mobile part. Take the {0} reference frame with z_0 pointing vertically upwards, and (x_0, y_0) in the horizontal plane.

2) Making use of the composition of elementary transformations, compute in the reference frame {1} the position of the origin of the {W} system, for the (θ_1, θ_2, θ_3) values of point 1. Check the result you get against the figure displaying the various reference frames.

${}^1_2T$ = Rx(90°) Tz(d) Rz(θ_2)
${}^2_3T$ = Tx(La) Tz(-d) Rz(θ_3)
${}^3_wT$ = tx(Lb)

${}^1_wT = {}^1_2T\,{}^2_3T\,{}^3_wT$ = Rx(90°) Tz(d) Rz(θ_2) Tx(La) Tz(-d) Rz(θ_3) tx(Lb)

3) We define $X \equiv$ vertical difference of the heigth of the {W} and {0} origins, in the reference frame {0}. If the actual values of (θ_1, θ_2, θ_3) change to ($\theta_1 + \Delta\theta_1$, $\theta_2 + \Delta\theta_2$, $\theta_3 + \Delta\theta_3$), the corresponding change in X can be written as

$$\Delta X = (J_1,\, J_2,\, J_3) \begin{pmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{pmatrix}$$

for small but otherwise arbitrary ($\Delta\theta_1$, $\Delta\theta_2$, $\Delta\theta_3$) , with (J_1, J_2, J_3) being the jacobian values which are independent of ($\Delta\theta_1$, $\Delta\theta_2$, $\Delta\theta_3$) .

Describe which operations involving the transformations appearing in the Denavit-Hartenberg table we should perform in order to obtain the (J_1, J_2, J_3) values. Using the values of point 1, compute the numerical value of J_3 .

$${}^0_wT = \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

wo ${}^0_wT$ = Tz(h) . Rz(θ_1) Rx(90°) Tz(d) Rz(θ_2) Tx(La) Tz(-d) Rz(θ_3) tx(Lb)

$$J_i = \frac{\partial x}{\partial \theta_i}$$

```
J1 = (tz[h].Drz[θ1].rx[90 º].tz[d].rz[θ2].tx[La].tz[-d].rz[θ3].tx[Lb])_{3,4}

J2 = (tz[h].rz[θ1].rx[90 º].tz[d].Drz[θ2].tx[La].tz[-d].rz[θ3].tx[Lb])_{3,4}

J3 = (tz[h].rz[θ1].rx[90 º].tz[d].rz[θ2].tx[La].tz[-d].Drz[θ3].tx[Lb])_{3,4}
```

where

$$\text{Drz}[\theta] = \frac{\partial\, \text{rz}[\theta]}{\partial \theta} = \begin{pmatrix} -\text{Sin}[\theta] & -\text{Cos}[\theta] & 0 & 0 \\ \text{Cos}[\theta] & -\text{Sin}[\theta] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
J3 = (tz[h].rz[θ1].rx[90 º].tz[d].rz[θ2].tx[La].tz[-d].Drz[θ3].tx[Lb])_{3,4} = Lb Cos[θ2 + θ3] = 29.54 (cm / rad)
```

or numerically

$$J_3 = \left(\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&40\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 0.866&-0.5&0&0\\0.5&0.866&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&5\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 0.9397&-0.3420&0&0\\0.3420&0.9397&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 1&0&0&35\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&-5\\0&0&0&1 \end{pmatrix} . \begin{pmatrix} 0.5&-0.8660&0&0\\0.8660&0.5&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} . \begin{pmatrix} 1&0&0&30\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right)_{3,4}$$

$$= \begin{pmatrix} \blacksquare&\blacksquare&\blacksquare&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&\blacksquare\\ \blacksquare&\blacksquare&\blacksquare&29.54\\ \blacksquare&\blacksquare&\blacksquare&\blacksquare \end{pmatrix}_{3,4} = 29.54\ (\text{cm / rad})$$

Problem 2

bala cos

A point-like mass $m_A = 0.2\,Kg$ is shot at high speed against a body B with total mass $M_B = 15\,Kg$, with a mass distribution given by its inertia matrix

1.45 2.13 0.93

matriu d'inèrcia
$$I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \quad [Kg \cdot m^2]$$

with $(I_{11},\, I_{22},\, I_{33}) = (1.45,\, 2.13,\, 0.95)$ (in units of $Kg\,m^2$) when described in a coordinate system {B} attached to the body, with origin located at the centre of mass.

impact

The impact leaves a mark at a point P with coordinates ${}^{B}P = (0.254,\, 0.157,\, -0.669)\,m$ (when measured in the {B} reference frame). The data extracted from the film footage which has recorded the collision reveals that at impact time :

a) the angular velocity of the body changes from $\vec{w} = (0.15,\, 0.31,\, 0.39)$ rad/s just before the impact to $\vec{w} = (5.35,\, 8.13,\, 10.37)$ rad/s just after the impact (values with respect to the lab reference frame {0}), and

b) the orientation of the body reference frame {B} with respect to system {0} at impact time is given by the rotation matrix

$${}^{O}_{B}R = \begin{pmatrix} 0.95713 & -0.14691 & 0.249632 \\ 0.22218 & 0.925289 & -0.307364 \\ -0.18583 & 0.349652 & 0.918265 \end{pmatrix}$$

1) Compute the angular momentum of the point mass m_A with respect to the body CM system just before impact, in the system {0}, in terms of the components in the {0} system of the velocity ${}^{0}\vec{v} = (v_x,\, v_y,\, v_z)$ with which it was shot.

2) Compute the numeric results for the difference of the angular momentum of the body {B} immediately after and before the impact, ${}^{B}\vec{L}(after) - {}^{B}\vec{L}(before)$, expressed in the {B} system.

3) Assuming that the angular momentum of the point mass m_A after the impact is negligible, find a set of 3 equations relating the components of the velocity ${}^{0}\vec{v} = (v_x,\, v_y,\, v_z)$ used in point 1 to the results found in point 2. Discuss whether or not the velocity of m_A ${}^{0}\vec{v} = (v_x,\, v_y,\, v_z)$ with which it was shot can be uniquely determined from the data extracted from the film footage.

(1) $\vec{L}(m_A) = m_A(\vec{r}_p - \vec{r}_{cm}) \times \vec{v}_A$

$${}^{0}(\vec{r}_p - \vec{r}_{cm}) = {}^{0}_{B}R \cdot {}^{B}P_{impact} = \begin{pmatrix} 0.95713 & -0.14691 & 0.249632 \\ 0.22218 & 0.925289 & -0.307364 \\ -0.18583 & 0.349652 & 0.918265 \end{pmatrix} \begin{pmatrix} 0.254 \\ 0.157 \\ -0.669 \end{pmatrix} = \begin{pmatrix} 0.05204 \\ 0.40933 \\ -0.606625 \end{pmatrix}$$

$$\vec{v}_A = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\vec{L}(m_A) = m_A(\vec{r}_p - \vec{r}_{cm}) \times \vec{v}_A$$

$$= 0.2 \cdot \begin{pmatrix} 0.05204 \\ 0.40933 \\ -0.606625 \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0.12132\, vy + 0.0814661\, vz \\ -0.12132\, vx - 0.0106085\, vz \\ -0.0814661\, vx + 0.0106085\, vy \end{pmatrix}$$

(2) ${}^{B}\vec{L}(\text{after}) - {}^{B}\vec{L}(\text{before})$

$$\vec{L}(M_B) = I_{inertia} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\rightarrow \quad {}^{B}L(M_B) = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \cdot \begin{pmatrix} {}^{B}\omega_x \\ {}^{B}\omega_y \\ {}^{B}\omega_z \end{pmatrix}$$

$$\begin{pmatrix} {}^{B}\omega_x \\ {}^{B}\omega_y \\ {}^{B}\omega_z \end{pmatrix} = {}^{B}_{0}R \cdot \begin{pmatrix} {}^{0}\omega_x \\ {}^{0}\omega_y \\ {}^{0}\omega_z \end{pmatrix} = ({}^{0}_{B}R)^T \begin{pmatrix} {}^{0}\omega_x \\ {}^{0}\omega_y \\ {}^{0}\omega_z \end{pmatrix}$$

Before impact 1B & system:

$${}^{B}\vec{L}(M_B) = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} ({}^{0}_{B}R)^T \begin{pmatrix} 0.15 \\ 0.31 \\ 0.39 \end{pmatrix} = \begin{pmatrix} 0.202959 \\ 0.854486 \\ 0.235271 \end{pmatrix} \frac{kg\, m^2}{s}$$

After impact 1B & system:

$${}^{B}\vec{L}(M_B) = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} ({}^{0}_{B}R)^T \begin{pmatrix} 5.35 \\ 8.13 \\ 10.39 \end{pmatrix} = \begin{pmatrix} 7.2502 \\ 22.0713 \\ 7.9401 \end{pmatrix} \frac{kg\, m^2}{s}$$

$${}^{B}\vec{L}(\text{after}) - {}^{B}\vec{L}(\text{before}) = \begin{pmatrix} 7.2502 \\ 22.0713 \\ 7.9401 \end{pmatrix} - \begin{pmatrix} 0.202959 \\ 0.854486 \\ 0.235271 \end{pmatrix} = \begin{pmatrix} 7.0472 \\ 21.2168 \\ 7.6548 \end{pmatrix} \frac{kg\, m^2}{s}$$

(3) ${}^{0}\vec{L}_{before}(M_A) + {}^{0}\vec{L}_{before}(M_B) = {}^{0}\vec{L}_{after}(M_A) + {}^{0}\vec{L}_{after}(M_B) \simeq \vec{L}_{after}(M_B)$

$${}^{0}\vec{L}_{before}(M_A) = {}^{0}_{B}R \left({}^{0}\vec{L}_{after}(M_B) - {}^{0}\vec{L}_{before}(M_B) \right) = \boxed{\begin{pmatrix} 5.53904 \\ 18.8446 \\ 13.1386 \end{pmatrix} = \begin{pmatrix} 0.12132\, vy + 0.0814661\, vz \\ -0.12132\, vx - 0.0106085\, vz \\ -0.0814661\, vx + 0.0106085\, vy \end{pmatrix}}$$