títols
problemes
conceptes

INVESTIGACIÓ OPERATIVA

teoria

$\max / \min_{(x)} \quad f_0(x_1, x_2, \dots, x_n)$ ⟵ funció objectiu

s.a. $f_1(x_1, x_2, \dots, x_n) \geq b$
...
$f_m(x_1, x_2, \dots, x_n) \leq c$
$x > 0$

⟵ funcions restrictives

→ NOTACIÓ FORMAL
↳ també existeix la matricial

Problema de producció

- s'han de decidir la quantitat $x_i \geq 0 \quad i = 1, \dots, n$ de n productes
- benefici unitari de cada producte c_i, $i = 1, \dots, n$ → Benefici total: $c_1x_1 + \dots + c_nx_n$ (maximitzar)
- tenim m recursos en quantitats b_j, $j = 1, \dots, m$
- cada unitat del producte i consumeix a_{ij} unitats del recurs j → consum del recurs j ⟿ $a_{j1}x_1 + \dots + a_{jn}x_n \leq b_j$ (quantitat del recurs j)

$$\mathrm{Max}_x \sum_{i=1}^{n} c_i x_i$$
$$\text{s.a.} \rightarrow \begin{cases} \forall j \in m \quad \sum_{i=0}^{n} a_{ji} \leq b_j \\ x \geq 0 \end{cases}$$

Problema de pienso para ganado

- b_j = quantitat mínima de nutrient j
- s'ha de comprar n aliments en quantitat x_i $\quad i = 1, \dots, n$
- preu unitari de l'aliment i ⟿ c_i, (minimitzar) $c_1x_1 + \dots + c_nx_n$
- quantitat del nutrient j a l'aliment i és a_{ij}

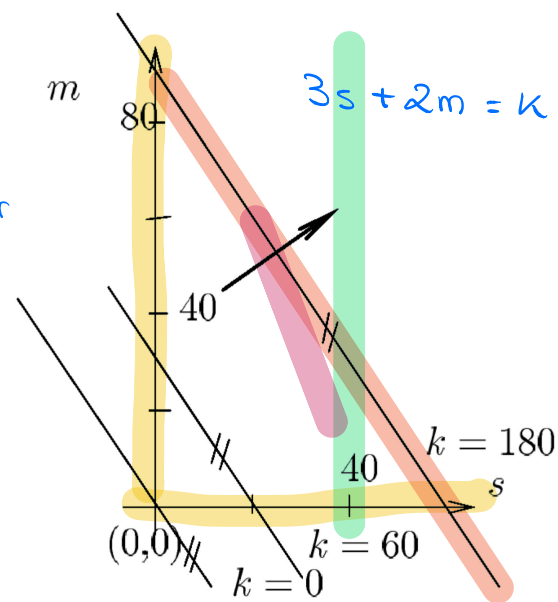
$$\mathrm{Min}_x \sum_{i=0}^{n} c_i x_i \qquad \text{s.a.} \begin{cases} \forall j \in m \quad \sum_{i=0}^{n} x_i a_{ij} \geq b_j \\ x \geq 0 \end{cases}$$

Anàlisi gràfic d'un problema en 2 dimensions

Problema metalls

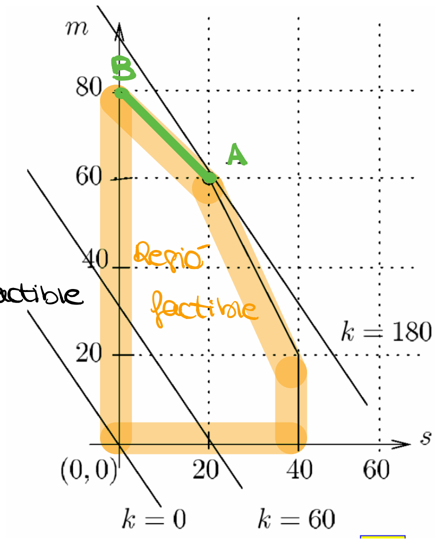
- s = Tm aleació 1, m = Tm aleació 2
- vendes totals en milers de $\boxed{€ = 3s + 2m}$ ⟵ maximitzar

Tm coure ⟿ $2s + m \leq 100$
Tm estany ⟿ $s + m \leq 80$
màxim aleació 1 $\quad s \leq 40$
$s, m \geq 0$



Per a valors de $k > 180$, les corbes de nivell de la funció objectiu no intersecten la regió factible.

- $k = 180$ és el valor màxim que pot tenir la funció objectiu
- aquí, l'obtenim al punt $(20, 60)$

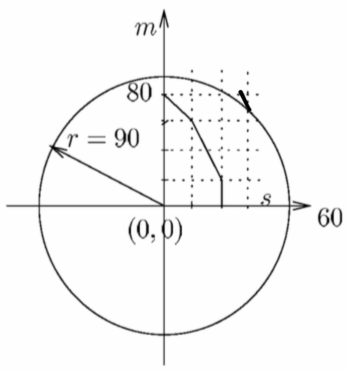


Regió factible (F): en el problema anterior, $F \neq \emptyset \rightsquigarrow$ és un problema factible
↳ quan $F = \emptyset$, el problema és infactible

Òptims alternatius → Els punts del segment AB són la intersecció entre la regió factible de P'' $(s \leq 40)$ i la corba de nivell amb màxim valor de la funció objectiu.

La regió factible és **ACOTADA** si jo hi puc dibuixar un cercle al voltant

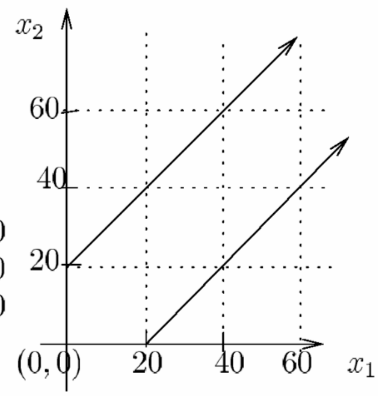
$$\forall (x_1, x_2, \ldots, x_n) \in F \rightarrow (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \leq r$$



ACOTADA

NO ACOTADA

$$\begin{array}{rcr} x_2 - x_1 & \leq & 20 \\ x_2 - x_1 & \leq & -20 \\ x_1, x_2 & \geq & 0 \end{array}$$



Classificació

- Problemes de programació lineal
  - Factible $(F \neq \emptyset)$
    - F acotada ($S \neq \emptyset \rightsquigarrow$ Existeixen òptims únics o alternatius)
    - F no acotada
      - * Existeixen òptims únics o alternatius
      - * Problema no acotat (no existeixen òptims $\rightsquigarrow S = \emptyset$)
  - Infactible $(F = \emptyset)$

Connexivitat de la regió factible

↳ DEF: Donats 2 punts x_1, x_2 : el segment obert $\overline{x_1x_2}$ són tots aquells punts x tals que:

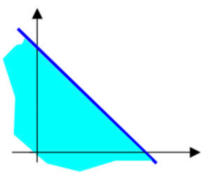
$$x = \alpha x^1 + (1-\alpha)x^2 \qquad (0<\alpha<1)$$

→ Definició de conjunt convexe ⇝ $C \subseteq \mathbb{R}^n$ és convexe si, $\forall x_1, x_2 \in C$, el segment obert $\overline{x_1x_2} \subset C$

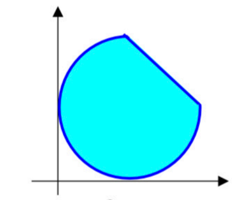
Supongamos $F = \{\, x \in \mathbb{R}^n \mid Ax \geq b \,\}$, (o $Ax \leq b$, o $Ax = b$)

F es un conjunto convexo

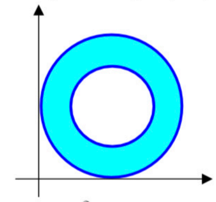
$$\left.\begin{array}{l} Ax^1 \geq b \\ Ax^2 \geq b \end{array}\right\} \quad \begin{array}{l} A(\alpha x^1 + (1-\alpha)x^2) = \\ \qquad = \alpha Ax^1 + (1-\alpha)Ax^2 \geq \alpha b + (1-\alpha)b = b, \quad (\leq, =) \end{array}$$



$x_1 + x_2 \leq 1$

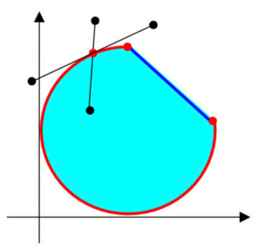


$(x_1-1)^2 + (x_2-1)^2 \leq 1$
$x_1 + x_2 \leq \frac{29}{10}$

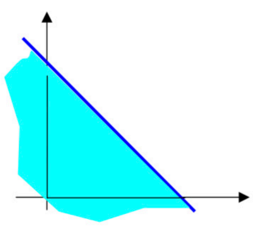


$(x_1-1)^2 + (x_2-1)^2 \leq 1$
$(x_1-1)^2 + (x_2-1)^2 \geq \frac{1}{2}$

→ Vèrtexs d'un conjunt convex

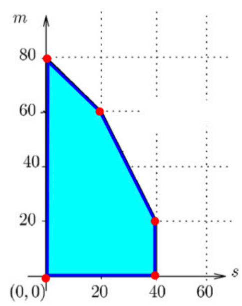


$(x_1-1)^2 + (x_2-1)^2 \leq 1$
$(x_1-1)^2 + (x_2-1)^2 \geq \frac{1}{2}$



$x_1 + x_2 \leq 1$

NO TODOS LOS CONVEXOS TIENEN VÉRTICES



$\hat{x}$ és un vertex de $C \subset \mathbb{R}^n$, convexe, si $\nexists x_1, x_2 \in \mathbb{R}^n$ tals que $\hat{x} \in \overline{x_1x_2}$.

$x_1 \notin C$ OR $x_2 \notin C$

Forma estàndard d'un problema lineal

podem expressar tots els problemes lineals de la següent manera: ⟶

- $\forall x_i$ tal que $i = 1, \dots, n \rightarrow x_i \geq 0$
- $\forall b_j$ tal que $j = 1, \dots, m \rightarrow b_j \geq 0$
- la matriu A és de ple rang

↳ Hi ha m columnes de A tals que, al formar una matriu B amb elles, aquesta és inversible.

$$\left\{\begin{array}{ll} \min_x & c_1x_1 + \dots + c_nx_n \\ \text{s.a.:} & a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ & \dots \\ & a_{m1}x_1 + \dots + a_{mn}x_n = b_m \\ & x_1 \geq 0, \ \dots, \ x_n \geq 0 \end{array}\right\} \rightarrow \begin{array}{ll} \min_x & c^Tx \\ \text{s.a.:} & Ax = b \\ & x \geq 0 \end{array}$$

$\boxed{m \leq n}$

Teorema Fonamental de la P.L.

1. Si $F \neq \emptyset \Rightarrow$ existeix almenys una s.b.f. ← solucions bàsiques factibles
2. Si (P) té solució $\Rightarrow$ existeix una solució de (P) que és s.b.f.

→ ESTRATÈGIA

1. Determinar si $F = \emptyset$
2. En cas contrari, determinar una s.b.f (vèrtice) de F inicial.
3. Visitar s.b.f's fins a trobar un que sigui solució de (P)
4. Determinar si s.b.f trobada és la solució única o n'hi ha més.

Definició de base factible

sistema $Ax = b,\ x \geq 0$

$$\left.\begin{array}{l} x_1 - x_2 + x_3 = 1 \\ -x_1 + \frac{1}{3}x_2 + x_4 = 1 \\ -x_1 + \frac{2}{3}x_2 + x_5 = 4 \\ x_i \geq 0 \quad (i = 1, \dots, 5) \end{array}\right\} \rightarrow A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1/3 & 0 & 1 & 0 \\ -1 & 2/3 & 0 & 0 & 1 \end{pmatrix}$$

$b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$ ----> aleatòriament

B és factible si.... $B^{-1} \cdot b \geq 0$

↳ Anem a comprovar-ho:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \geq 0$$

→ B és una base associada al conjunt de índexos $\{1,4,5\}$ i és factible. $I_B = \{1,4,5\}$

possible solució

$x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 2,\ x_5 = 5$

Les dues variables que no utilitzo (aquí x_2, x_3), les fixo a 0.

Notació:

$$x_{Reord} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \\ \hline x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ \hline 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ \hline 0 \\ 0 \end{pmatrix} \geq 0$$

Forma canònica d'un sistema lineal $Ax = b$

Treballant amb $I_B = \{1,4,5\}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$B\ x_B + N\ x_N = b$$

$$B^{-1}(B\ x_B + N\ x_N) = B^{-1}b$$

$$x_B + B^{-1}N\ x_N = B^{-1}b$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$x_B + Y\ x_N = y_0$$

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{-2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

forma tabular

1	4	5	2	3	
1	0	0	-1	1	1
0	1	0	$-2/3$	1	2
0	0	1	$-1/3$	1	5

→ Per a un conjunt d'índexs associats a una base B, $I_B = \{i_1, i_2, \dots, i_m\}$

$$x_B + Y x_N = y_0 \qquad x_B = \begin{pmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_m} \end{pmatrix}, \quad x_N = \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_{n-m}} \end{pmatrix}$$

$$\begin{array}{ccc|ccc|c} i_1 & \cdots & i_m & j_1 & \cdots & j_{n-m} & 0 \\ \hline 1 & \cdots & 0 & y_{11} & \cdots & y_{1,n-m} & y_{1,0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & y_{m1} & \cdots & y_{m,n-m} & y_{m,0} \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \to \geq 0 \text{ si B és factible}$$

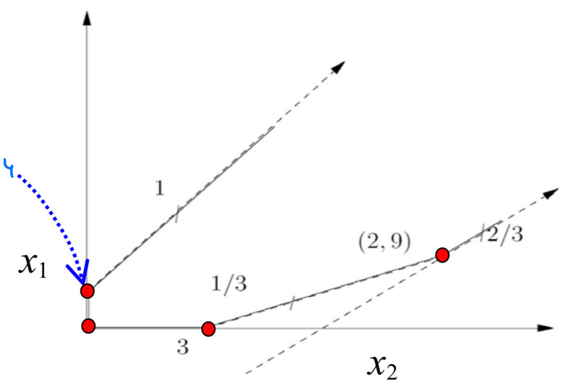
columnes bàsiques — columnes NO bàsiques

Canvi de base amb conservació de factibilitat

$$\begin{aligned} x_1 - x_2 + x_3 \quad\quad &= 1 \\ -x_1 + \tfrac{1}{3}x_2 \quad + x_4 \quad &= 1 \\ -x_1 + \tfrac{2}{3}x_2 \quad\quad + x_5 &= 4 \end{aligned}$$

$x_i \geq 0 \quad (i = 1, \dots, 5)$

$I_B = \{1, 4, 5\}$



La forma canònica expressa la dependència de les variables x_B respecte x_N

↳ $x_B(x_N) = y_0 - Y x_N$

Per a $\boxed{x_N = 0}$

↳ $x_B(0) = y_0$

↳ $x_R = (y_0, 0)$ és un vèrtex del Poliedre

Si B és una base factible: $x_B(0) \geq 0$; ⇝ Incrementant x_N des de 0, trobarem altres punts $x_B(x_N) \geq 0$

$$\begin{array}{ccc|cc|c} 1 & 4 & 5 & 2 & 3 & \\ \hline 1 & 0 & 0 & 4 & -2 & 8 \\ 0 & 1 & 0 & 5 & 3 & 10 \\ 0 & 0 & 1 & 2 & 8 & 2 \end{array}$$

$I_B = \{1, 4, 5\}$

$I_N = \{2, 3\}$

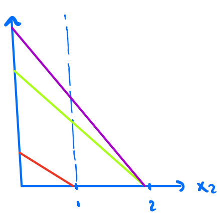
$$\hat{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 10 \\ 2 \end{pmatrix}$$

Fixem $x_3 = 0$ ⇝ $x_2 \uparrow$

$x_1 + 4x_2 = 8$

$x_4 + 5x_2 = 10$

$x_5 + 2x_2 = 2$



$$\hat{x}' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}$$

Si $x_2 = 1$

- $x_1 + 4 \cdot 1 = 8 \to x_1 = 4$
- $x_4 + 5 \cdot 1 = 10 \to x_4 = 5$
- $x_5 + 2 \cdot 1 = 2 \to x_5 = 0$

$$\hat{x}_1 = \text{Min}\left\{ \frac{8}{4}, \frac{10}{5}, \frac{2}{2} \right\} = 1$$

Pivotació

nova $I_B \longrightarrow I_B' = \{1, 2, 4\}$

$I_N' = \{5, 3\}$

$$\begin{array}{ccc|cc|c} 1 & 4 & 5 & 2 & 3 & \\ \hline 1 & 0 & 0 & 4 & -2 & 8 \\ 0 & 1 & 0 & 5 & 3 & 10 \\ \to 0 & 0 & 1 & 2 & 8 & 2 \end{array}$$

$$\begin{array}{ccccc|c} 1 & 4 & 5 & 2 & 3 & \\ \hline 1 & 0 & 0 & 4 & -2 & 8 \\ 0 & 1 & -5/2 & 0 & -17 & 5 \\ 0 & 0 & 1/2 & 1 & 4 & 1 \end{array}$$

$$\begin{array}{ccccc|c} 1 & 4 & 5 & 2 & 3 & \\ \hline 1 & 0 & -2 & 0 & -18 & 4 \\ 0 & 1 & -5/2 & 0 & -17 & 5 \\ 0 & 0 & 1/2 & 1 & 4 & 1 \end{array}$$

Entra una variable NO BÀSICA i surt BÀSICA

$$\begin{array}{ccc|ccc|c} i_1 & \dots\ i_s\ \dots & i_m & j_1 & \dots\ q\ \dots & j_{n-m} & 0 \\ \hline 1 & \dots & 0 & y_{1,1} & \dots\ y_{1,q}\ \dots & y_{1,n-m} & y_{1,0} \\ \vdots & 1 & \vdots & \vdots & y_{s,q} & \vdots & y_{s,0} \\ 0 & \dots & 1 & y_{m,1} & \dots\ y_{m,q}\ \dots & y_{m,n-m} & y_{m,0} \end{array}$$

$$\hat{x}_{j_q} = \text{Min}\left\{ \frac{y_{t,0}}{y_{t,q}} \,\middle|\, y_{t,q} > 0 \right\}$$

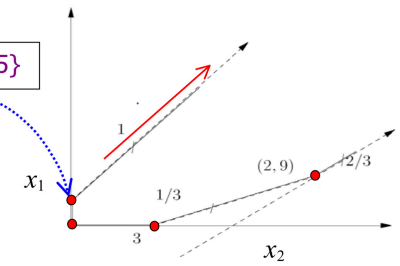
- Extraure i_s de I_B ⟶ $I_B = \{i_1, \dots, i_s, \dots i_m\}$
- Substituir per j_q ⟶ $I_B' = \{i_1, \dots, j_q, \dots i_m\}$

→ Casos singulars

$\boxed{y_{t,q} \leq 0}$

1	2	3	4	5	
1	-1	1	0	0	1
0	-2/3	1	1	0	2
0	-3/3	1	0	1	5

$d^T = (d_1, d_4, d_5, d_2, d_3) = (1, \frac{2}{3}, \frac{1}{3}, 1, 0)$



Incrementant x_2, almenys una v. bàsica creix indefinidament → DETECTEM DIRECCIÓ DE CREIXEMENT IL·LIMITAT DE LA REGIÓ FACTIBLE

$\boxed{\exists y_{t,0} = 0 \land y_{t,q} > 0}$

1	4	5	2	3	
1	0	0	4	-2	8
0	1	0	5	3	0
0	0	1	2	8	2

$I_B = \{1, 4, 5\}$

$I_N =$ s'obté el mateix punt

1	4	5	2	3	
1	-4/5	0	0	-22/5	8
0	1/5	0	1	3/5	0
0	-2/5	1	0	-34/5	2

→ $\hat{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$

Exercici: Calcular solucions bàsiques factibles per al conjunt de restriccions...

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ -x_1 + \frac{1}{3}x_2 + x_4 = 1 \\ -x_1 + \frac{2}{3}x_2 + x_5 = 4 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 0 & 1 \\ -1 & 1/3 & 0 & 1 & 0 & 1 \\ -1 & 2/3 & 0 & 0 & 1 & 4 \end{array}\right)$$

Partim de la base $I_B = \{3, 4, 5\}$

3	4	5	1	2	
1	0	0	1	-1	1
0	1	0	-1	1/3	1
0	0	1	-1	2/3	4

→ provem de treure 4 i incorporar 2.

$I_B = \{1, 4, 5\}$ (1, 2, 3)

$I_2^+ = \{i_4 = 2, i_5 = 3\}$

$\hat{\varepsilon} = \min_{i \in I_2^+} \left\{ \frac{y_{i,0}}{y_{i,2}} \right\} = \min \left\{ \frac{1}{1/3} = 3, \frac{4}{2/3} = 6 \right\} = \frac{1}{1/3} = 3 \rightsquigarrow \begin{matrix} i_p = 2 \\ p = 4 \end{matrix}$

3	4	5	1	2	
1	0	0	1	-1	1
0	1	0	-1	1/3	1
0	0	1	-1	2/3	4

→

3	4	5	1	2	
1	0	0	1	-1	1
0	3	0	-3	1	3
0	0	1	-1	2/3	4

→

3	4	5	1	2	
1	3	0	-2	0	4
0	3	0	-3	1	3
0	-2	1	1	0	2

Nova Base → $I_B = \{3, 2, 5\}$ $b = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

$I_1^+ = \{i_5 = 3\}$ $\hat{\varepsilon} = \min\left(\frac{2}{1}\right) = 2$

Treiem 5 i posem 1 → $I_B = \{3, 2, 1\}$

3	4	5	1	2	
1	-1	2	0	0	8
0	-3	3	0	1	9
0	-2	1	1	0	2

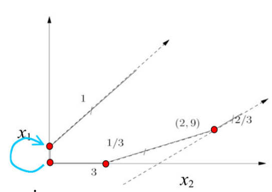
$d^T = (d_3, d_2, d_1, d_4, d_5) = (1, 3, 2, 1, 0)$

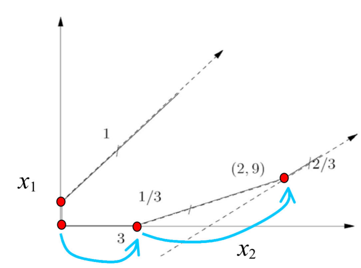
1	2	3	4	5	
1	−1	1	0	0	1
−1	1/3	0	1	0	1
−1	2/3	0	0	1	4

Entra x_1 ⇒ Sale x_3 ⇒

1	2	3	4	5	
1	−1	1	0	0	1
0	−2/3	1	1	0	2
0	−1/3	1	0	1	5

$$d^\top = (d_1, d_4, d_5, d_2, d_3) = (1, 2/3, 1/3, 1, 0)$$





Canvis de base disminuint el valor de la funció objectiu

- És útil trobar les solucions del problema de P.L. $\longrightarrow$ $\boxed{\begin{array}{ll} \min_x & c^T x \\ \text{s.a.} & Ax = b \\ & x \geq 0 \end{array}}$
- Per a una base $I_B = \{i_1, \ldots, i_m\} \rightarrow \hat{x} = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$

i_1	---	i_m	j_1	---	j_{n-m}	0
1	---	0	y_{11}	---	$y_{1,n-m}$	$y_{1,0}$
⋮	⋱	⋮	⋮	⋱	⋮	⋮
0	---	1	y_{m1}	---	$y_{m,n-m}$	$y_{m,0}$

- Valor de la funció objectiu $\boxed{z_0 = c^T \hat{x} = (c_B^T, c_N^T)\begin{pmatrix} y_0 \\ 0 \end{pmatrix} = c_B^T y_0}$

Llavors, tenint que:

$x_B(x_N) = y_0 - Y x_N$, $\quad x_B = \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_m} \end{pmatrix}$, $\quad x_N = \begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_{n-m}} \end{pmatrix}$, $\quad x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$

Anem a calcular $c^T x$ (funció objectiu) $\longrightarrow$

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N \\ &= c_B^T (y_0 - Y x_N) + c_N^T x_N \\ &= c_B^T y_0 - c_B^T Y x_N + c_N^T x_N \\ &= c_B^T y_0 + (-c_B^T Y + c_N^T) x_N \\ &= c_B^T y_0 + (c_N^T - c_B^T Y) x_N \\ &= c_B^T y_0 + (c_N - c_B Y^T)^T x_N \end{aligned}$$

$$\begin{aligned} f.\text{objectiu}(x_N) &= c_B^T y_0 + (\underbrace{c_N - Y^T c_B}_{r})^T x_N \\ &= z_0 + r^T x_N \\ &= z_0 + r_1 x_{j_1} + \ldots + r_{n-m} x_{j_{n-m}} \end{aligned}$$

$$\boxed{r = c_N - Y^T c_B}$$

Per tant, tenim que $\boxed{f.\text{obj}(x_N) = z_0 + r_1 x_{j_1} + \ldots + r_{n-m} x_{j_{n-m}}}$

- si $r \geq 0$, I_B és una base òptima
  - $\longrightarrow r = 0 \rightsquigarrow$ si $\exists r_i = 0 \rightarrow$ existeixen òptims alternatius
  - $\longrightarrow r > 0 \rightsquigarrow$ la solució per la base I_B és la <u>única</u> solució possible
- si $\exists q$ tal que $r_q < 0 \longrightarrow$ hem de disminuir la funció objectiu incrementant el valor de x_{j_q}
  - $\hookrightarrow$ Entrant la variable x_{j_q} i formant una nova base

Exemple:

$$\left.\begin{array}{lllll} \min_x & x_1 - \frac{4}{3}x_2 & & & \\ \text{s.a.} & x_1 - x_2 & + x_3 & = 1 \\ & -x_1 + 1/3\, x_2 & + x_4 & = 1 \\ & -x_1 + 2/3\, x_2 & + x_5 & = 4 \\ & (i = \{1, \ldots, 5\}) & x_i & \geq 0 \end{array}\right\} \rightarrow I_B = \{1, 4, 5\}$$

1	4	5	2	3	
1	0	0	-1	1	1
0	1	0	-2/3	1	2
0	0	1	-1/3	1	5
1	0	0	-4/3	0	0

$\longrightarrow$ funció objectiu

$$r = c_N - Y^T c_B = \begin{pmatrix} -4/3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & -2/3 & -1/3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1 \end{pmatrix}$$

$$-z_0 = -y_0^T c_B = -(1\ 2\ 5)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -1$$

1	4	5	2	3	
1	0	0	-1	1	1
0	1	0	-2/3	1	2
0	0	1	-1/3	1	5
0	0	0	-1/3	-1	-1

$\longrightarrow$ fila 4 ← fila 4 - fila 1

$$\begin{array}{l} 1\ 0\ 0 \mid -\frac{4}{3}\ 0 \mid 0 \\ -[1\ 0\ 0 \mid -1\ 1 \mid 1] \\ \hline 0\ 0\ 0 \mid -1/3\ -1 \mid -1 \end{array}$$

Disposició en forma tabular i fòrmules matricials

$\min_x \quad c^T x$
s.a. $\quad Ax = b$
$\quad x \geq 0$

I	Y	Y_0
0	r_N	$-z_0$

$Y = B^{-1}N$
$Y_0 = B^{-1} b$
$r_N = c_N - N^T B^{-T} c_B$
$z_0 = c_B^T B^{-1} \cdot b$

$$
\begin{array}{ccc|ccc|c}
i_1 & \cdots & i_m & j_1 & \cdots & j_{n-m} & 0 \\
\hline
1 & \cdots & 0 & y_{11} & \cdots & y_{1,n-m} & y_{1,0} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & y_{m1} & \cdots & y_{m,n-m} & y_{m,0}
\end{array}
$$

"Algoritmo"

0) Inicialització → Determinar s.b.f inicial $B_{(0)}$. $k = 0$

1) Calcular costs reduïts per a $B = B_{(k)}$: $r_N = c_N - Y^T c_B = c_N - N^T B^{-T} c_B$

2) Si $r_N \geq 0 \implies B_{(k)}$ és la base òptima → STOP

3) Si $\exists q$ tal que $(r_N)_q < 0 \wedge y_{iq} \leq 0$ $\quad (1 \leq i \leq m) \implies$ Problema no acotat → STOP

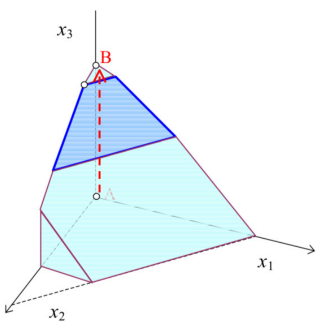
4) Seleccionar variable no bàsica de entrada x_{j_q} $\quad$ (q tal que $r_q = \min\{r_\ell \mid 1 \leq \ell \leq n-m\}$)

5) Encontrar variable básica de salida x_{i_s} y Efectuar cambio de base

Determinar $p = i_s \in I_B$ según: $\begin{cases} \hat{x}_{j_q} = \dfrac{y_{s,0}}{y_{s,q}} = \min\left\{ \dfrac{y_{t,0}}{y_{t,q}} \mid y_{tq} > 0 \right\} \\ I_{B(k+1)} = I_{B(k)} \cup \{j_q\} - \{i_s\} \end{cases}$

6) $k \leftarrow k+1$, volver a 1.

→ EXEMPLE



$$
\begin{array}{lrrrl}
Min & -12x_1 & -12x_2 & -16x_3 & \\
 & x_1 & +x_2 & +x_3 & \leq 1 \\
 & x_1 & +x_2 & +1/3x_3 & \leq 11/9 \\
 & x_1 & +x_2 & +5/3x_3 & \leq 53/36 \\
 & -1/2x_1 & x_2 & & \leq 2/3 \\
 & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$

VÉRTICE A

1	2	3	4	5	6	7	0
1	1	1	1	0	0	0	1
1	1	4/3	0	1	0	0	11/9
1	1	5/3	0	0	1	0	53/36
−1/2	1	0	0	0	0	1	2/3
−12	−12	−16	0	0	0	0	0

$Min\left\{1/1, \frac{11/9}{4/3}, \frac{53/36}{5/3}\right\} = Min\{1, 11/12, 53/60\}$

VÉRTICE B

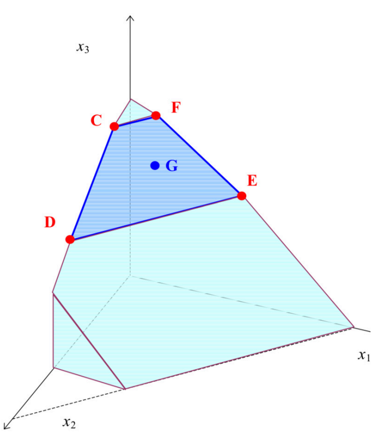
1	2	3	4	5	6	7	0
2/5	2/5	0	1	0	−3/5	0	7/60
1/5	1/5	0	0	1	−4/5	0	2/45
3/5	3/5	1	0	0	3/5	0	53/60
−1/2	1	0	0	0	0	1	2/3
−12/5	−12/5	0	0	0	48/5	0	212/15

$Min\left\{\frac{7/60}{2/5}, \frac{2/45}{1/5}, \frac{53/60}{3/5}, 2/3\right\} = Min\{35/60, 10/45, 265/18($

VÉRTICE C

1	2	3	4	5	6	7	0
0	0	0	1	−2	1	0	1/36
1	1	0	0	5	−4	0	2/9
0	0	1	0	−3	3	0	9/12
−3/2	0	0	0	−5	4	1	4/9
0	0	0	0	12	0	0	220/15

ÓPTIMOS ALTERNATIVOS



Recorrent les diferents bases trobaríem els punts C, D, E, F.
En tots ells, la f. obj té el valor $z^* = 212/15$

$$\boxed{\begin{array}{l} X^G = \alpha_1 X^C + \alpha_2 X^D + \alpha_3 X^E + \alpha_4 X^D \\ \sum_{\ell=1}^{4} \alpha_\ell = 1,\ \alpha_\ell \geq 0,\ \ell = \{1,2,3,4\} \end{array}}$$

Qualsevol punt de G sobre la cara tindrà el mateix valor per la funció objectiu.

Eficàcia del mètode SIMPLEX

- Pitjor cas possible: haver de visitar TOTS els vertexs → no es pot ⇝ Exemple

- Min $z = c^T x$ s.a. $Ax = b$, $x \geq 0$ } → En els problemes reals amb un #variables (n) >> # restriccions (m) (A = ...)

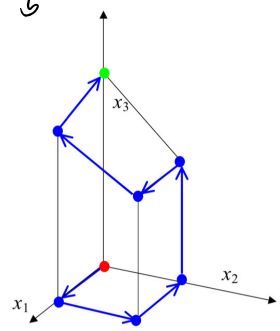
  # mitjà d'iteracions ≈ k·m $(1 \leq k \leq 3)$

Los Problemas de Klee-Minty

$$Max \;\; z = \Sigma_{j=1}^{m} 10^{m-j} x_j$$
$$s.a: \;\; 2\Sigma_{j=1}^{i-1} x_j + x_i \leq 100^{i-j}$$
$$x \geq 0$$

$m = 3$

$$Max \;\; z = 100x_1 + 10x_2 + x_3$$
$$s.a: \;\; x_1 \leq 1$$
$$20x_1 + x_2 \leq 100$$
$$200x_1 + 20x_2 + x_3 \leq 10000$$
$$x \geq 0$$



Exemple:

$$\min_x : -300x_1 - 250x_2$$
$$\text{s.a.:} \quad \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 150 \\ 3x_1 + 2x_2 + x_4 &= 300 \\ 2x_1 + x_5 &= 100 \\ x_i &\geq 0 \end{aligned}\right\} \; i = 1,\dots,5$$

1	2	3	4	5	
1	2	1	0	0	150
3	2	0	1	0	300
2	0	0	0	1	100
-300	-250	0	0	0	0

x_5 és la que té 001

$$\min\left\{\frac{150}{1}, \frac{300}{3}, \frac{100}{2}\right\} = \min\{150, 100, 50\} = 50 \longrightarrow$$ Entra x_1, Surt x_5

1	2	3	4	5	
0	2	1	0	-1/2	100
0	2	0	1	-3/2	150
1	0	0	0	1/2	50
0	-250	0	0	150	0

x_3

$$\min\left\{\frac{100}{2}, \frac{150}{2}\right\} = \min\{50, 75\} = 50 \longrightarrow$$ Entra x_2, Surt x_3

1	2	3	4	5	
0	1	1/2	0	-1/4	50
0	0	-1	1	-1	50
1	0	0	0	1/2	50
0	0	125	0	175/2	27500

→ Taula òptima ⇝ òptim únic del problema
$X_B = \{x_2, x_4, x_1\} = (50, 50, 50)$

Exemple (2)

$$\begin{array}{lll} \min & x_1 - \frac{3}{4}x_2 & \\ \text{s.a.} & x_1 - x_2 + x_3 & = 1 \\ i=\{1,\dots,5\} & -x_1 + \frac{1}{3}x_2 + x_4 & = 1 \\ x_i \geq 0 & -x_1 + \frac{2}{3}x_2 + x_5 & = 4 \end{array}$$

Elements de la taula per la base $I_B\{3,2,5\}$

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/3 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} \rightarrow B^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Càlcul de costos reduïts : $r = c_N - N^T B^{-T} c_B = c_N - Y^T c_B$

$$\begin{pmatrix} r_1 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ -\frac{4}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 & -3 & 1 \\ 3 & 3 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{4}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

3	4	5	1	2	
1	2	3	2	-2	4
0	3	0	-3	1	3
0	-2	1	1	0	2
0	4	0	-3	0	4

Entra x_1
Surt x_5 $\Big\} \rightarrow \text{Min}\left\{\frac{2}{1}\right\} = 2$

Nova Base $I_B = \{3, 2, 1\}$ ⟿ Nova taula :

3	4	5	1	2	
1	-1	2	0	0	8
0	-3	3	0	1	9
0	-2	1	1	0	2
0	-2	3	0	0	10

La columna de x_4 és < 0

PROBLEMA
NO
ACOTAT

Se llega a una s.b.f en la que se detecta una dirección d de crecimiento ilimitado de la región factible.

$$d^\top = (d_3, d_2, d_1, d_4, d_5) = (1, 3, 2, 1, 0)$$

Identificació de bases inicials factibles

→ Quan no pots: crees variables artificials (següent pàgina)

$$\begin{array}{lll} Min & x_1 - \frac{4}{3}x_2 & \\ s.a: & x_1 - x_2 + x_3 & = 1 \\ & -x_1 + \frac{1}{3}x_2 + x_4 & = 1 \\ & -x_1 + \frac{2}{3}x_2 + x_5 & = 4 \\ & x_i \geq 0 & (\, i = 1, \dots 5\,) \end{array}$$

→

3	4	5	1	2	
1	0	0	1	−1	1
0	1	0	−1	1/3	1
0	0	1	−1	2/3	4

$$\begin{array}{ll} Min_x & 5x_1 + x_2 - 7x_3 \\ s.a: & x_1 + x_2 + 3x_3 + s_1 = 3 \\ & 2x_1 + 4x_2 \quad s_2 = 5 \\ & x_1 + 5x_2 - 4x_3 - y_1 = 10 \\ & x_1 + x_2 - y_2 = 6 \\ & x_2 + x_3 = 16 \\ & x_1 + x_2 + x_3 = 5 \\ & x_1,\ x_2,\ x_3 \geq 0, \quad s_1, \quad s_2, \quad y_1, \quad y_2 \quad \geq 0 \end{array}$$

→

x_1	x_2	x_3	s_1	s_2	y_1	y_2	
1	1	3	1	0	0	0	3
2	4	0	0	1	0	0	5
1	5	−4	0	0	−1	0	10
1	1	0	0	0	0	−1	6
0	1	1	0	0	0	0	16
1	1	1	0	0	0	0	5

No es posible identificar una base inicial factible

→ Variables artificials i problema auxiliar

- Construïm un nou problema "problema auxiliar" amb les mateixes variables i coeficients en les restriccions que en el problema original

- Afegim variables artificials (≥ 0)
  - ↳ una per cada fila amb una variable d'excés
  - ↳ una per cada fila sense variable d'excés ni de folga

- La funció objectiu del problema auxiliar és la suma de les variables artificials

$$
\begin{array}{lllllll}
Min_x & 5x_1 + x_2 - 7x_3 & & & & & \\
s.a: & x_1 + x_2 + 3x_3 & +s_1 & & & & = 3 \\
 & 2x_1 + 4x_2 & & s_2 & & & = 5 \\
 & x_1 + 5x_2 - 4x_3 & & & -y_1 & & = 10 \quad \leftarrow a_3 \\
 & x_1 + x_2 & & & & -y_2 & = 6 \quad \leftarrow a_4 \\
 & x_2 + x_3 & & & & & = 16 \quad \leftarrow a_5 \\
 & x_1 + x_2 + x_3 & & & & & = 5 \quad \leftarrow a_6 \\
 & x_1,\ x_2,\ x_3 \geq 0, & s_1, & s_2, & y_1, & y_2 & \geq 0
\end{array}
$$

$$
\begin{array}{lllllllllll}
Min_x & a_3 + a_4 + a_5 + a_6 & & & & & & & & & \\
s.a: & x_1 + x_2 + 3x_3 & +s_1 & & & & & & & & = 3 \\
 & 2x_1 + 4x_2 & & +s_2 & & & & & & & = 5 \\
 & x_1 + 5x_2 - 4x_3 & & & +a_3 & & & & -y_1 & & = 10 \\
 & x_1 + x_2 & & & & +a_4 & & & & -y_2 & = 6 \\
 & x_2 + x_3 & & & & & +a_5 & & & & = 16 \\
 & x_1 + x_2 + x_3 & & & & & & +a_6 & & & = 5 \\
 & x_1,\ x_2,\ x_3 \geq 0, & s_1, & s_2, & y_1, & y_2 & \geq 0 & & & & \\
 & a_3, a_4, a_5, a_6 & \geq 0 & & & & & & & &
\end{array}
$$

x_1	x_2	x_3	s_1	s_2	y_1	y_2	
1	1	3	1	0	0	0	3
2	4	0	0	1	0	0	5
1	5	−4	0	0	−1	0	10
1	1	0	0	0	0	−1	6
0	1	1	0	0	0	0	16
1	1	1	0	0	0	0	5

Para el problema auxiliar se obtiene una base inicial factible de forma inmediata

x_1	x_2	x_3	s_1	s_2	a_3	a_4	a_5	a_6	y_1	y_2	
1	1	3	1	0	0	0	0	0	0	0	3
2	4	0	0	1	0	0	0	0	0	0	5
1	5	−4	0	0	1	0	0	0	−1	0	10
1	1	0	0	0	0	1	0	0	0	−1	6
0	1	1	0	0	0	0	1	0	0	0	16
1	1	1	0	0	0	0	0	1	0	0	5

→ Solució del problema auxiliar

Casos

→ El problema presenta una solució òptima amb les variables artificials $a_i = 0$

↳ s'obté una base inicial factible pel problema original ⟹ REGIÓ FACTIBLE NO BUIDA

→ El problema presenta una solució òptima amb alguna variable artif. $a_i > 0$

↳ no es pot trobar b.i.f pel problema original ⟹ REGIÓ FACTIBLE BUIDA

a	c	s_2	s_3	s_4	a_0	
1	0	1	0	0	0	60
1	1	0	−1	0	1	70
1	2	0	0	1	0	120
0	0	0	0	0	1	0

$\rightarrow$

a	c	s_2	s_3	s_4	a_0	
1	0	1	0	0	0	60
1	1	0	−1	0	1	70
1	2	0	0	1	0	120
−1	−1	0	1	0	0	−70

a	c	s_2	s_3	s_4	a_0	
1	0	1	0	0	0	60
0	1	−1	−1	0	1	10
0	2	−1	0	1	0	60
0	−1	1	1	0	0	−10

$Entra\ c;\quad Min\{\frac{10}{1}, \frac{60}{2}\} = 10$

a	c	s_2	s_3	s_4	a_0	
1	0	1	0	0	0	60
0	1	−1	−1	0	1	10
0	0	1	2	1	−2	40
0	0	0	0	0	1	0

$$
\begin{array}{lll}
Min & -20a - 30c & \\
s.a: & a & \leq 60 \\
 & a + c & \geq 70 \\
 & a + 2c & \leq 120 \\
(P) & a, c \geq 0 &
\end{array}
$$

$$
\begin{array}{lll}
Min & a_0 & \\
s.a: & a + s_2 & = 60 \\
 & a + c \quad - s_3 \quad + a_0 & = 70 \\
 & a + 2c + \quad s_4 & = 120 \\
(P') & a, c, s_2, s_3, s_4, a_0 \geq 0 &
\end{array}
$$

a	c	s_2	s_3	s_4	
1	0	1	0	0	60
0	1	−1	−1	0	10
0	0	1	2	1	40
−20	−30	0	0	0	0

$\rightarrow$

a	c	s_2	s_3	s_4	
1	0	1	0	0	60
0	1	−1	−1	0	10
0	0	1	2	1	40
0	0	−10	−30	0	1500

a	c	s_2	s_3	s_4	
1	0	1	0	0	60
0	1	−1/2	0	1/2	30
0	0	1/2	1	1/2	20
0	0	5	0	15	2100

RESOLVER

$$
\begin{array}{lll}
Min & 18x_1 + 20x_2 + 22x_3 & \\
s.a: & 90x_1 + 65x_2 + 45x_3 & \geq 60 \\
 & 80x_1 + 40x_2 + 10x_3 & \leq 50 \\
 & x_1 + x_2 + x_3 & = 1 \\
 & x_i \geq 0 & (i = 1, \ldots 3)
\end{array}
$$

$$
\begin{array}{lll}
Min & x_6 + x_7 & \\
s.a: & 90x_1 + 65x_2 + 45x_3 - x_4 \quad + x_6 & = 60 \\
 & 80x_1 + 40x_2 + 10x_3 \quad + x_5 & = 50 \\
 & x_1 + x_2 + x_3 \quad + x_7 & = 1 \\
 & x_i \geq 0 & (i = 1, \ldots 7)
\end{array}
$$

$$x_6 + x_7 = 61 - 91x_1 - 66x_2 - 46x_3 + x_4$$

1	2	3	4	5	6	7	
90	65	45	−1	0	1	0	60
80	40	10	0	1	0	0	50
1	1	1	0	0	0	1	1
−91	−66	−46	1	0	0	0	−61

$Min\{1, 5/8, 2/3\} = 5/8$

$Entra\ x_1 \Rightarrow\ Surt\ x_5$

48

	5	2	3	4	
x_6	−9/8	20	135/4	−1	15/4
x_1	1/80	1/2	1/8	0	5/8
x_7	−1/80	1/2	7/8	0	3/8
	91/80	−41/2	−154/8	1	−33/8

$Min\{3/7, 5, 15/135\} = 1/9$

$Entra\ x_3 \Rightarrow\ Surt\ x_6$

	5	2	6	4	
x_3	−1/30	16/27	4/135	−4/135	1/9
x_1	1/60	23/54	−1/270	1/270	11/18
x_7	1/60	−1/54	−7/270	7/270	5/18
	−1/60	1/54	277/270	−7/270	−5/18

$Min\{75/7, 165\} = 1$

$Entra\ x_4 \Rightarrow\ Surt\ x_7$

	5	2	6	7	
x_3	−1/70	4/7	0	8/7	3/7
x_1	1/70	3/7	0	−1/7	4/7
x_4	9/14	−5/7	−1	270/7	75/7
	0	0	1	1	0

$Suma\ d'infactibilitats = 0$

$base\ inicial\ factible\ pel$

$problema\ original$

$I_B = \{3, 1, 4\}$

$$B^{-\top}c_B = \begin{pmatrix} 45 & 90 & 1 \\ 10 & 80 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-\top} \begin{pmatrix} 22 \\ 18 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/70 & 8/7 \\ 0 & 1/70 & -1/7 \\ -1 & 9/14 & 270/7 \end{pmatrix}^{-\top} \begin{pmatrix} 22 \\ 18 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4/70 \\ 158/7 \end{pmatrix}$$

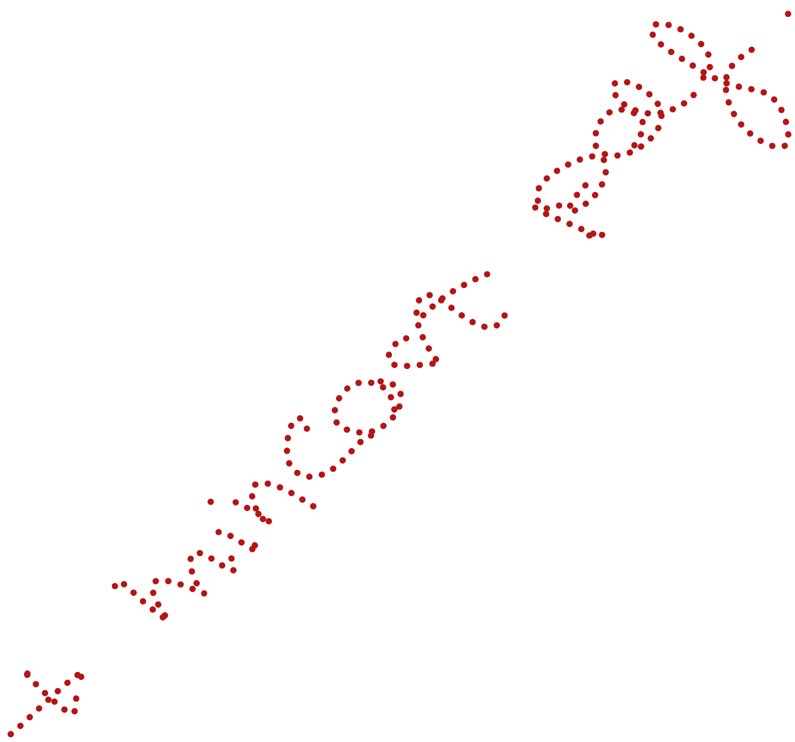
$$\begin{pmatrix} r_5 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 64 & 40 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -4/70 \\ 158/7 \end{pmatrix} = \begin{pmatrix} 4/70 \\ -20/70 \end{pmatrix}$$

3	1	4	5	2	
1	0	0	−1/70	4/7	3/4
0	1	0	1/70	3/7	4/7
0	0	1	9/14	−5/7	75/7
0	0	0	4/70	−20/70	−138/7

$Entra\ x_2\quad Min\{4/3, 3/4\}\ Surt x_3$

2	1	4	5	3	
1	0	0	−1/40	7/4	3/4
0	1	0	1/40	−3/40	1/112
0	0	1	5/8	5/4	45/4
0	0	0	1/20	1/2	−17

Base óptima $I_B = \{2, 1, 4\}$



Programació Lineal Entera

→ Problema de programació lineal ...

pero afegim que totes les variables són enters

per tant, podem considerar que els coeficients c, A, b també són enters

$$\min_x \; c^T x$$
s.a.: $Ax \geq b$
$x \geq 0, \; x \in \mathbb{Z}^n$
c, A, b són enters

A Problemes de recobriment

Exemple:

Un operador de un centro de cálculo necesita reponer 5 ficheros en el disco del ordenador. Los ficheros se hallan en 7 CD's diferentes. Los ficheros contenidos en cada CD vienen dados por la siguiente tabla:

	CD1	CD2	CD3	CD4	CD5	CD6	CD7
Fichero1	X	0	0	X	0	X	X
Fichero2	X	X	X	0	X	X	0
Fichero3	0	X	0	0	0	X	X
Fichero4	0	0	X	0	0	0	X
Fichero5	X	X	0	X	X	0	0

Determinar el número mínimo de CD's que debe utilizar el operador para reponer los cinco ficheros.

Defineixo $x_i \begin{cases} 1 & \text{agafem CD } i \\ 0 & \text{otherwise} \end{cases}$ Func. obj.: $\sum_{i=1}^{7} x_i$

El primer fitxer es reposa si ... $x_1 + x_4 + x_6 + x_7 \geq 1$
" segon " " " " $x_1 + x_2 + x_3 + x_5 + x_6 \geq 1$
" tercer " " " " $x_2 + x_6 + x_7 \geq 1$
" quart " " " " $x_3 + x_7 \geq 1$
" cinquè " " " " $x_1 + x_2 + x_4 + x_5 \geq 1$

Llavors → $\min_x \; \sum_{i=1}^{7} x_i$

$$\text{s.a.:} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$\geq$ ↑ permet redundància

$x_i \in \{0, 1\}$
$i = \{1, \dots, 7\}$

Manera "formalitzada"

- Conjunt d'objectes $I = \{1, 2, \dots, n\}$ que es desitja recomposar ($n = 5$)
- Es disposa de varis subconjunts de I (els COs)

  $P_i \subseteq I, \; i = 1, 2, \dots, m$

  $P_1 = \{1, 2, 5\}, \; P_2 = \{2, 3, 5\}, \; P_3 = \{2, 4\}$, etc.
- Escollir un subconjunt j comporta un cost c_j
- Determinar quins subconjunts → escollir el cost total mínim

$$x_i = \begin{cases} 1 & \text{s'escull el subconjunt} \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, .., m \longrightarrow \boxed{\begin{array}{ll} \text{Min}_x & c^T x \\ \text{s.a.:} & Ax \geq (1) \\ & x_i \in \{0, 1\} \end{array}}$$

$$a_{ij} = \begin{cases} 1 & \text{element } i \in P_j \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m$$

S'exigeix que $\boxed{P_i \cap P_j = \emptyset, \; i \neq j}$

↳ En l'exemple dels fitxers, només hi ha una solució factible $x_5 = 1$, $x_7 = 1$, $x_j = 0$, $j \neq 5, 7$

B) Problemes de caixa fixa

Una fábrica se plantea la producción de dos tipos de productos A y B y tiene garantizada la venta de todas las unidades que produzca. Para la fabricación es necesario un recurso del que dispone de 120 unidades.

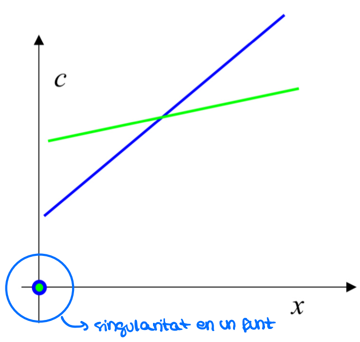
Apareixen quan apareixen costos adicionals que no depenen de la quantitat que produeixis sinó d'altres coses (e.g. nova maquinària per produir)

	A	B
Beneficio Unitario	200	500
# recurso por unidad	3	6
Subvención	2000	1000

$$3x_A + 6x_B \leq 120, \quad x_A \geq 0, \quad x_B \geq 0.$$

$$c_A(x_A) = \begin{cases} 200x_A + 2000 & \text{si } x_A > 0 \\ 0 & \text{si } x_A = 0 \end{cases}$$

$$c_B(x_B) = \begin{cases} 500x_B + 1000 & \text{si } x_B > 0 \\ 0 & \text{si } x_B = 0 \end{cases}$$



PROBLEMA: això no ho entén un solver, ho haig de passar a forma algebraica

Anem al problema:

$$\begin{array}{ll} \text{Max}_x & c_A(x_A) + c_B(x_B) \\ \text{s.a.:} & 3x_A + 6x_B \leq 120 \\ & x_A \geq 0, \; x_B \geq 0 \\ & x_A, x_B \in \mathbb{Z} \\ & d_A, d_B \in \{0, 1\} \end{array}$$

- Definim $d_A, d_B \in \{0, 1\}$

  $x_A > 0 \rightarrow d_A = 1 \qquad x_B > 0 \rightarrow d_B = 1$

  $x_A = 0 \rightarrow d_A = 0 \qquad x_B = 0 \rightarrow d_B = 0$

- Llavors... $c_A(x_A) = 200x_A + 2000 d_A$

  $c_B(x_B) = 500x_B + 1000 d_B$

Haig de reformular ⇝ $C = \{ x_A, x_B \in \mathbb{Z} \mid 3x_A + 6x_B \leq 120 ,\ x_A \geq 0 ,\ x_B \geq 0 \}$

→ Cota superior per a x_A en C ⇝ $\hat{x}_A = 40$ $(3 \cdot 40 = 120)$

→ cota inferior positiva per x_A en C ⇝ $\underline{x_A} = 1$

→ d_A binaria

$d_A \leq x_A \leq 40 d_A$ → $\begin{cases} x_A > 0 \to d_A = 1 \\ x_A = 0 \to d_A = 0 \end{cases}$

$$\boxed{\begin{array}{ll} \text{Max}_x & 200x_A + 2000 d_A + 500 x_B + 1000 d_B \\ \text{s.a.:} & 3x_A + 6x_B \leq 120 ,\ (x_A \geq 0 ,\ x_B \geq 0) \\ & d_A \leq x_A \leq 40 d_A ,\ d_B \leq x_B \leq 20 d_B ,\ 0 \leq d_A, d_B \leq 1 \\ & x_A, x_B, d_A, d_B \in \mathbb{Z} \end{array}}$$

Expressió de condicions lògiques utilitzant variables binàries

$$\begin{array}{llll} \text{Min}_{x \in Z^3} & x_1 + 2x_2 + x_3 & & \\ \text{s.a.:} & x_1 + x_2 + x_3 - 1 \geq 0 & \leftarrow C_1 & \\ & -x_1 + x_2 - x_3 - 3 \geq 0 & \leftarrow C_2 & C_1 \ Y \ C_2 \ Y \ C_3 \ Y \ C_4 \\ & x_1 \geq 0 & \leftarrow C_3 & \\ & x_2 \geq 0 & \leftarrow C_4 & \end{array}$$

$C_1 = x_1 + x_2 + x_3 - 1 \geq 0$, $\overline{C_1} = x_1 + x_2 + x_3 - 1 < 0 \ (\leq -1)$

$\boxed{Y \ O \ \bar{C}}$ ⟶ $C_1 \ O_{ex} \ C_2 \equiv (C_1 \ Y \ \bar{C_2}) \ O \ (C_2 \ Y \ \bar{C_1})$

$C_1 \Rightarrow C_2 \equiv \bar{C_1} \ O \ C_2$

Leyes de Morgan

$\overline{C_1 \ O \ C_2} = \bar{C_1} \ Y \ \bar{C_2}$

$\overline{C_1 \ Y \ C_2} = \bar{C_1} \ O \ \bar{C_2}$

Distributibitat:

$C_1 \ Y \ (C_2 \ O \ C_3) \equiv (C_1 \ Y \ C_2) \ O \ (C_1 \ Y \ C_3)$

$C_1 \ O \ (C_2 \ Y \ C_3) \equiv (C_1 \ O \ C_2) \ Y \ (C_1 \ O \ C_3)$

$\boxed{\text{condició lògica "o"}}$ Suposem el problema

$$\boxed{\begin{array}{ll} \min_{x\in C} & s^T x \\ \text{s.a.:} & g(x) \geq 0 \quad o \quad h(x) \geq 0 \quad o \quad \text{ambas} \end{array}}$$

$$\left.\begin{array}{l} g(x) = a^T x + b \\ h(x) = c^T x + d \end{array}\right\} \quad \begin{array}{l} \underline{g} = \min_{x\in C} g(x) \\ \underline{h} = \min_{x\in C} h(x) \\ \underline{g}, \underline{h} \neq 0 \end{array}$$

$$\boxed{\begin{array}{ll} \min_{x\in C} & s^T x \\ \text{s.a.:} & g(x) \geq \delta \underline{g} \\ & h(x) \geq (1-\delta)\underline{h} \\ & x \in C \\ & \delta \in \{0,1\} \end{array}}$$

- Extensió: almenys k condicions d'entre m.

$$\begin{array}{ll} \min_{x C} & s^T x \\ \text{s.a.:} & \text{al menys k entre: } g_1(x) \geq 0, \dots, g(m) \geq 0 \end{array} \rightsquigarrow \begin{array}{ll} \min_{x,\delta} & s^T x \\ \text{s.a.:} & g_i(x) \geq \delta_i \underline{g}_i, \quad i = 1,2,\dots,m \\ & \sum_{i=1}^{m} \delta_i \leq m-k \\ & x \in C \\ & \delta_i \in \{0,1\}, \quad i = 1,2,\dots,m \end{array}$$

- $h(x) \geq 0 \Rightarrow g(x) \geq 0$

$$\left.\begin{array}{l} h(x) \leq 0 \quad o \quad g(x) \geq 0 \quad o \quad \text{ambas} \\ \downarrow \\ -h(x) \geq 0 \quad o \quad g(x) \geq 0 \quad o \quad \text{ambas} \\ g(x) \geq \delta \underline{g} \\ -h(x) \geq (1-\delta)(-\overline{h}) \rightsquigarrow h(x) \leq -(1-\delta)\overline{h} \\ \delta \in \{0,1\} \end{array}\right\} \text{Restricciones: } \begin{array}{l} g(x) \geq \delta \underline{g} \\ h(x) \leq (1-\delta)\overline{h} \\ \delta \in \{0,1\} \end{array} \quad \begin{array}{l} \underline{g} = \min_{x\in C} g(x) \\ \overline{h} = \max_{x\in C} h(x) \\ \underline{g}, \overline{h} \neq 0 \end{array}$$

Exemple

Deben fabricarse 5 productos en cantidades enteras nonegativas x_i, $i = 1, 2, ..., 5$. Los recursos permiten que $\hat{x}_1 = 100$, $\hat{x}_2 = 50$, $\hat{x}_3 = 20$, $\hat{x}_4 = 50$.

Mediante un acuerdo con el comprador se estipula que:

$$\begin{array}{ccccccc} x_1 > 0, & \text{o} & x_2 > 0 & \Rightarrow & x_3 > 0, & \text{o} & x_4 > 0 \\ C_1 & \text{o} & C_2 & \Rightarrow & C_3 & \text{o} & C_4 \\ & & & ||| & & & \\ (\overline{C_1 \, \text{o} \, C_2}) & & & \text{o} & C_3 & \text{o} & C_4 \\ (\overline{C}_1 \, \text{y} \, \overline{C}_2) & & & \text{o} & C_3 & \text{o} & C_4 \end{array}$$

$\overline{C_1} \equiv x_1 \leq 0$
$\overline{C_2} \equiv x_2 \leq 0$ $\Big\} \rightarrow$ si $C_0 = \overline{C_1}$ y $\overline{C_2}$, llavors almenys C_0, C_3 o C_4 s'haurien de complir

$$C_0 \begin{cases} C_1 & -x_1 \geq 0 \\ C_2 & -x_2 \geq 0 \end{cases}$$
$$C_3 \quad x_3 - 1 \geq 0$$
$$C_4 \quad x_4 - 1 \geq 0$$

$\Rightarrow$

$$\begin{aligned} -x_1 &\geq \delta_0 (-\hat{x}_1) \\ -x_2 &\geq \delta_0 (-\bar{x}_2) \\ x_3 - 1 &\geq \delta_3 (-1) \\ x_4 - 1 &\geq \delta_4 (-1) \\ \delta_0, \delta_3, \delta_4 &\leq 2 \\ \delta_0, \delta_3, \delta_4 &\in \{0, 1\} \end{aligned}$$

$\longrightarrow$

$$\begin{aligned} -x_1 &\leq 100\,\delta_0 \\ -x_2 &\leq 50\,\delta_0 \\ x_3 &\leq 1 - \delta_3 \\ x_4 &\leq 1 - \delta_4 \\ \delta_0 + \delta_3 + \delta_4 &\leq 2 \\ \delta_0, \delta_3, \delta_4 &\in \{0, 1\} \end{aligned}$$

C_0, C_3, C_4
$\uparrow$
$3 - 1 \rightarrow$ almenys 1

Problema de producció de plantes

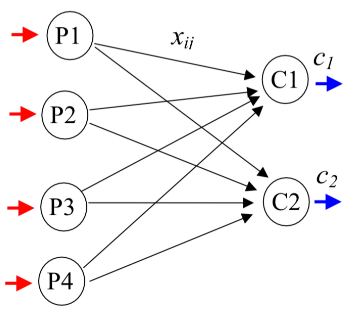
$$Min_{\;x,t} \quad \sum_{i=1}^{n_p} \sum_{j=1}^{n_c} t_{i,j} x_{i,j} + \boxed{\sum_{i=1}^{n_p} d_i \delta_i}$$

$$\sum_{j=1}^{n_c} x_{ij} \leq p_i, \qquad i = 1, 2, ..., n_p$$

$$\sum_{i=1}^{n_p} x_{ij} = c_j, \qquad j = 1, 2, ..., n_c$$

$$x_{i,j} \geq 0$$

$$\boxed{\begin{array}{ll} \delta_i \leq \sum_{j=1}^{n_c} x_{ij} \leq p_i \delta_i & i = 1, ..., n_p \\ \delta_i \in \{0, 1\} & i = 1, ..., n_p \end{array}}$$



(de problemes) ⇝ Dualitat i anàlisi de sensibilitat

Forma estàndard P.P.L. :

$$\left.\begin{array}{llll} \text{Min}_x & c_1x_1 + \dots + c_nx_n & & \\ \text{s.a.:} & a_{11}x_1 + \dots + a_{1n}x_n & = b_1 & \\ & a_{21}x_1 + \dots + a_{2n}x_n & = b_2 & \\ & \dots \qquad\qquad \dots & = \dots & \\ & a_{m1}x_1 + \dots + a_{mn}x_n & = b_m & \\ & x_1 \geq 0, \dots, x_n \geq 0 & & (m \leq n) \end{array}\right\} \longrightarrow \begin{array}{ll} \text{Min}_x & c^Tx \\ \text{s.a.:} & Ax = b \\ & x \geq 0 \end{array}$$

Definició problema dual

solucions

$$\begin{array}{ll} Min_x & c^\top x \\ s.a: & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{lllll} Min_x & c_1 \cdot x_1 + \dots + c_n \cdot x_n & & \\ s.a: & a_{11}\cdot x_1 + \dots + a_{1n} \cdot x_n & = b_1 & \lambda_1 \\ & a_{21}\cdot x_1 + \dots + a_{2n} \cdot x_n & = b_2 & \lambda_2 \\ & \dots & & \\ & a_{m1}\cdot x_1 + \dots + a_{mn} \cdot x_n & = b_m & \lambda_m \\ & x_1 \geq 0, \ \dots x_n \geq 0 & & \end{array}$$

(P) Primal min

Variable dual λ_i, $i = 1, \dots, m$

$$\begin{array}{ll} Max_\lambda & b^\top \lambda \\ s.a: & A^\top \lambda \leq c \end{array} \qquad \begin{array}{lll} Max_\lambda & b_1 \cdot \lambda_1 + \dots + b_m \cdot \lambda_m & \\ s.a: & a_{11} \cdot \lambda_1 + \dots + a_{m1} \cdot \lambda_m & \leq c_1 \\ & a_{12} \cdot \lambda_1 + \dots + a_{m2} \cdot \lambda_m & \leq c_2 \\ & \dots & \\ & a_{1n} \cdot \lambda_1 + \dots + a_{mn} \cdot \lambda_m & \leq c_n \end{array}$$

(D) Dual max

trasposta

Propietat "màgica" ⇝ si el transformo a forma estàndard i faig dual, retorno al principal.

$$D(D(P)) = P$$

Teoremes de dualitat:

→ **Teorema dèbil**

$$\left.\begin{array}{l} x \text{ factible per a (P)}, \ Ax = b, \ x \geq 0 \\ \lambda \text{ factible per a (D)}, \ A^T\lambda \leq c \end{array}\right\} \Rightarrow \lambda^T b = \lambda^T A x \leq c^T x$$

→ **Teorema fort**

a) (P) té solució finita: c^Tx^* ⇒ (D) també: $b^T\lambda^*$ i a més $b^T\lambda^* = c^Tx^*$

b) (P) no és acotat ⇒ (D) és infactible.

⟶ **Conseqüències teorema fort:**

$x_B^* = B^{-1}b$, $I_B = \{i_1, \dots, i_m\}$, $x_N^* = 0$

B: matriu bàsica.

$$\boxed{\begin{array}{l} \text{Max}_c \ b^T\lambda \\ \quad A^T\lambda \leq c \end{array}}$$

Llavors:

$$\left.\begin{array}{l} ① \ \lambda^* = B^{-T}c_B \\ ② \ r = c_N - N^T\lambda^* \end{array}\right\} \begin{array}{l} B^T\lambda^* = c_B \\ N^T\lambda^* + r = c_N \end{array}$$

actua com a folgues pel dual

$$\begin{pmatrix} B^T \\ N^T \end{pmatrix}\lambda^* - 1\begin{pmatrix} 0 \\ r \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

(P) finit: hi ha una solució bàsica òptima $x^* = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$

$r = c_N - N^TB^{-T}c_B \geq 0$

$$\left.\begin{array}{l} B^Tu = c_B \\ r = c_N - N^TB^{-T}c_B \geq 0 \end{array}\right\} \quad \left.\begin{array}{l} B^Tu = c_B \\ N^Tu + r = c_N \end{array}\right\} \quad \begin{pmatrix} B^T \\ N^T \end{pmatrix} u \begin{array}{c} = \\ \leq \end{array} \begin{pmatrix} c_B \\ c_N \end{pmatrix} \Rightarrow A^Tu \leq c$$

u és factible dual: $u^Tb = c_B^TB^{-1}b$ ⇒ u és una solució òptima de (D).

Exemple 1:

$$\left.\begin{array}{llll} \text{Min} & 3x_1 + 4x_2 + 5x_3 & & \\ & x_1 + 2x_2 + 3x_3 & \geq 5 \\ & 2x_1 + 2x_2 + x_3 & \geq 6 \\ & x_i & \geq 0 \end{array}\right\} \xrightarrow{\text{òptima}}$$

1	2	3	4	5	
0	1	5/2	-1	1/2	2
1	0	-2	1	-1	1
0	0	1	1	1	-11

Tabla base òptima: $I_B = \{2, 1\}$

$$B = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

$$\lambda = B^{-T} \cdot c_B = \begin{pmatrix} 1 & -1 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$z_0 = b^T\lambda = (5 \ 6)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 11$$

Exemple 2:

(P) no acotat ⇝ $\text{Min}_x \ -x_1$

s.a.: $x_1 + x_2 - x_3 = 1$

$x_i \geq 0$

⇓

(D) infactible ⇝ $\text{Max}_\lambda \ \lambda$

s.a.: $\lambda \leq -1$

$\lambda \leq 0$

$-\lambda \leq 0$

Si per a una base B els costos reduïts $r \geq 0 \Rightarrow u = B^{-T} c_B$ és una solució factible de (D)

$r = c_N - N^T B^{-T} c_B \geq 0$

$$\left.\begin{array}{l} B^T u = c_B \\ r = c_N - N^T B^{-T} c_B \geq 0 \end{array}\right\} \quad \left.\begin{array}{l} B^T u = c_B \\ N^T u + r = c_N \end{array}\right\} \quad \begin{pmatrix} B^T \\ N^T \end{pmatrix} u \begin{array}{c} = \\ \leq \end{array} \begin{pmatrix} c_B \\ c_N \end{pmatrix} \Rightarrow A^T u \leq c$$

$$\left.\begin{array}{rl} \text{Min} & 3x_1 + 4x_2 + 5x_3 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_i \geq 0 \end{array}\right\}$$

Tabla base óptima : $I_B = \{4, 1\}$

1	2	3	4	5	
0	-1	-5/2	1	-1/2	-2
1	1	1/2	0	-1/2	3
0	1	7/2	0	3/2	-9

$$u = B^{-T} c_B = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{-T} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$$

?????

$$\begin{pmatrix} -1 & 0 \\ 1 & 2 \\ 2 & 2 \\ 3 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 7/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \\ 5 \\ 0 \end{pmatrix} \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array}$$

costos reduïts $\rightsquigarrow$ holgues problema dual

Algoritme SIMPLEX DUAL.

0) Inicialització

Determinar base factible dual inicial B_0 $K = 0$

$(\lambda = B^{-T} c_B , \quad r_N = c_N - N^T \lambda \geq 0)$

1) Si $x_B = B_K^{-1} b \geq 0 \rightarrow B_K$ base òptima STOP.

2) Si $\exists s, 1 \leq s \leq m$ tal que $\{1 \leq t \leq n-m \mid y_{s,t} < 0, x_s < 0\} = \emptyset \Rightarrow$ Problema DUAL no Acotado. STOP. (Problema PRIMAL infactible)

3) Seleccionar variable bàsica de sortida x_{i_s} (la més negativa)

4) Seleccionar variable no bàsica d'entrada x_{j_q}.

$\hookrightarrow$ Trobar $j_q \in I_N$ segons: $\max\limits_{1 \leq t \leq n-m} \left\{ \frac{r_t}{y_{s,t}} \mid y_{s,t} < 0 \right\} = \frac{r_q}{y_{s,q}} = \underline{r}_s$

5) Efectuar canvi de base: $I_{B_{K+1}} = I_{B_K} \cup \{j_q\} - \{i_s\}$

6) $K \leftarrow K+1$ (tornar 1)

1	2	3	4	5	
-1	-2	-3	1	0	-5
-2	-2	-1	0	1	-6
3	4	5	0	0	0

$\max \left\{ \frac{3}{-2}, \frac{4}{-2}, \frac{5}{-1} \right\} = \frac{3}{2}$

Entra x_1, surt x_5

$$\left.\begin{array}{rl} \text{Min} & 3x_1 + 4x_2 + 5x_3 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_i \geq 0 \end{array}\right\}$$

Tabla base óptima : $I_B = \{4, 1\}$

1	2	3	4	5	
-1	-2	-3	1	0	-5
-2	-2	-1	0	1	-6
3	4	5	0	0	0

1	2	3	4	5	
0	-1	-5/2	1	-1/2	-2
1	1	1/2	0	-1/2	3
0	1	7/2	0	3/2	-9

$\max \left\{ \frac{3/2}{-1/2}, \frac{7/2}{-5/2}, \frac{1}{-1} \right\} = \frac{1}{-1}$

Entra x_2, surt x_4

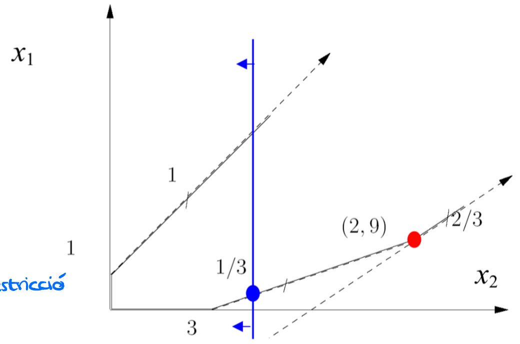
1	2	3	4	5	
0	1	5/2	-1	1/2	2
1	0	-2	1	-1	1
0	0	1	1	1	-11

Base factible (P) + Base factible (D) = Solució bàsica òptima.

ADICIÓN DE RESTRICCIONES Y REOPTIMIZACIÓN

$$Min \quad 2x_1 - x_2$$

$$\begin{array}{llrl} s.a: & x_1 - x_2 & \leq & 1 \\ & x_1 - \frac{1}{3}x_2 & \geq & -1 \\ & x_1 - \frac{2}{3}x_2 & \geq & -4 \\ & & & \\ & x_1, \;\; x_2 \geq 0 & & \end{array}$$



nova restricció

$x_2 \leq 4$

$$x_2 + y = 4, \;\; y \geq 0$$

1	2	3	4	5	
0	0	1	−1	2	8
0	1	0	−3	3	9
1	0	0	−2	1	2
0	0	0	1	1	5

1	2	3	4	5	y	
0	0	1	−1	2	0	8
0	1	0	−3	3	0	9
1	0	0	−2	1	0	2
0	1	0	0	0	1	4
0	0	0	1	1	0	5

$\longrightarrow$

1	2	3	4	5	y	
0	0	1	−1	2	0	8
0	1	0	−3	3	0	9
1	0	0	−2	1	0	2
0	0	0	3	−3	1	−5
0	0	0	1	1	0	5

Programació lineal entera → mètodes.

→ **Introducció:** Recordem...

(Definició P.L.E.) $\quad \text{Min } c^T x$

$$\left.\begin{array}{l} Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n \end{array}\right\} \rightarrow A, b, c \text{ coeficients enters.}$$

→ **Categories d'algoritmes per PLE**

↳ Algoritmes exactes

↳ Asseguren la obtenció de la solució òptima

↳ cost computacional elevat (e^x)

↳ B&B, Plans secants, ...

↳ Algoritmes d'aproximació

↳ Solució subòptima amb estimació de la seva qualitat

↳ cost computacional raonable (polinòmic)

↳ Relaxació Lagrangiana, ...

↳ Heurístiques:

↳ Solució subòptima sense estimació de la seva qualitat

↳ Els més ràpids

↳ Mètodes de busca local, algoritmes genètics, ...

→ **Algoritme exacte - Branch and Bound**

1. Conjunt factible d'un PLE i Relaxació Lineal.
2. Concepte de partició
3. Arbre d'exploració
4. Cotes de la funció objectiu sobre les particions
5. Algoritme bàsic de B&B
6. Exemples.

1. Conjunt factible d'un PLE i Relaxació Lineal.

↳ considerem un PLE de la següent manera: $\quad \text{Min } c^T x$

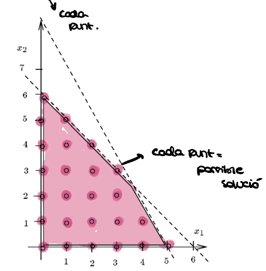
$$\left.\begin{array}{l} Ax = b,\ x \geq 0,\ x \in \mathbb{Z}^n \end{array}\right\} A, b, c \text{ coeficients enters.}$$

↳ El conjunt factible G d'un PLE pot considerar-se com la intersecció d'un poliedre acotat F : $\boxed{F = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}}$ $\boxed{\begin{array}{ll}\text{Min} & c^T x \\ x \in G & \end{array}}$ $\boxed{G = F \cap \mathbb{Z}^n}$

Definición de Relajación Lineal. *Dado un problema de PLE:*

$$\begin{array}{ll} Min & c^\top x \\ x \in G & \end{array} \longrightarrow \begin{array}{ll} Min & c^\top x \\ x \in F & \end{array}$$

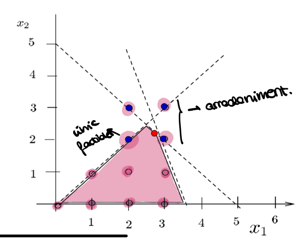
$$\begin{array}{ll} Min & -8x_1 - 5x_2 \\ & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1, x_2 \geq 0 \\ & x \in Z^2 \end{array}$$



La solución de la relajación lineal es el punto $(x_1, x_2) = (11/4, 9/4)$.
Las soluciones obtenidas mediante redondeo son:

$(3,3)$ (infactible)
$(3,2)$ (infactible)
$(2,3)$ (infactible)
$(2,2)$ (factible)

$$\begin{array}{ll} Min & 2x_1 + x_2 \\ & x_1 + x_2 \leq 5 \\ & -x_1 + x_2 \leq 0 \\ & 6x_1 + 2x_2 \leq 21 \\ & x_1, x_2 \geq 0 \\ & x \in Z^2 \end{array}$$



2. Concepte de partició

↳ Donat un conjunt factible G de un PLE, entenem **partició** un conjunt $G_1, G_2, \ldots, G_m$ de subconjunts G en un número finit que verifica:

1) $G_1 \cup G_2 \cup \ldots \cup G_m = G$

2) $G_j \cap G_k = \emptyset,\ \forall j \neq k,\ 1 \leq j, k \leq m.$

↳ suposem un PLE i la seva relaxació lineal:

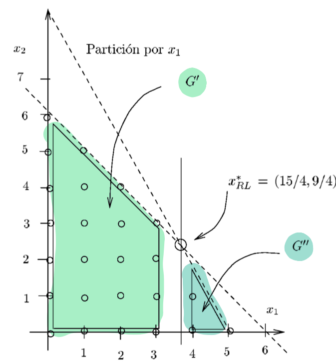
$\boxed{\begin{array}{ll}\text{Min} & c^T x \\ \text{(PLE)} & x \in F \cap \mathbb{Z}^n\end{array}}$ $\boxed{\begin{array}{ll}\text{Min} & c^T x \\ \text{(RL)} & x \in F\end{array}}$

Designem per x^*_{PLE} una solució de PLE i per x^*_{RL} una solució de la seva relaxació

$$\boxed{(x^*_{RL})_e \notin \mathbb{Z} \rightarrow \begin{array}{l} G' = G \cap \{x \in \mathbb{R}^n \mid x_e \leq \lfloor (x^*_{RL})_e \rfloor \} \\ G'' = G \cap \{x \in \mathbb{R}^n \mid x_e \geq \lceil (x^*_{RL})_e \rceil \} \end{array}}$$

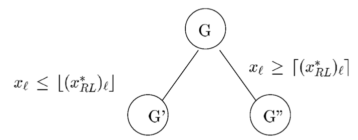
Ejemplo.

$$\begin{array}{ll} Min & -8x_1 - 5x_2 \\ & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1, x_2 \geq 0 \end{array}$$



③ Arbre d'exploració.

Partición según x_ℓ



④ Propietats de les particions de un PLE.

z^*_{PLE} Valor mínimo (óptimo) de la función objetivo para el PLE.
z^*_{RL} Valor mínimo (óptimo) de la función objetivo para la Relajación Lineal.

El conjunto de factible del PLE está contenido dentro del conjunto factible de la relajación RL:

$$G \subseteq F$$

Debido a la anterior inclusión deberá cumplirse:

Propiedad 1.

si estem minimitzant

$$\boxed{c^\top x^*_{RL} = z^*_{RL}} \leq \boxed{z^*_{PLE} = c^\top x^*_{PLE}}$$

El valor mínimo de la función objetivo sobre F (valor óptimo de la relajación lineal) es una cota inferior del valor óptimo del PLE.

Para cada una de las particiones G', G'' también se cumple:

$$G' \subseteq G, \quad G'' \subseteq G$$

Para cada uno de los problemas originados por la partición G', G'':

$$(PLE') \quad \begin{array}{ll} Min & c^\top x \\ & x \in G' \end{array} \qquad \Big| \qquad (PLE'') \quad \begin{array}{ll} Min & c^\top x \\ & x \in G'' \end{array}$$

deberá cumplirse:

Propiedad 2.

$$c^\top x^*_{PLE} = z^*_{PLE} \leq z^*_{PLE'} = c^\top x^*_{PLE'}$$
$$c^\top x^*_{PLE} = z^*_{PLE} \leq z^*_{PLE''} = c^\top x^*_{PLE''}$$

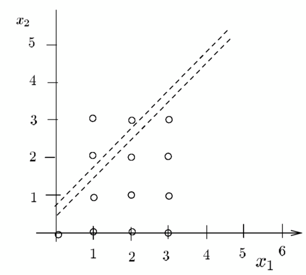
El valor mínimo de la función objetivo sobre G es una cota inferior del valor mínimo de la función objetivo sobre las particiones de G.

Dado que el número de variables es finito, el número de veces qu una partición puede contener igual número de soluciones enteras es finito.

Puesto que el poliedro es acotado podrá contener un número finito de particiones efectuadas mediante el algoritmo por lo cual el algorimo debe terminar en un número finito de iteraciones.

En caso de que el poliedro no sea acotado hay casos en los que el algoritmo propuesto no determina una solución del problema PLE

$$Min \quad 2x_1 - x_2$$
$$\begin{array}{l} 2x_2 - 2x_1 \geq 1 \\ 3x_2 - 3x_1 \leq 2 \\ x_1, x_2 \geq 0, \quad x \in Z^2 \end{array}$$



⑤ Algoritme bàsic de B&B

Algoritmo $B\&B$ para el problema PLE

0) **Inicialización.**
- Situar PLE en la raíz de $\mathcal{T}$. → arbre.
- Incumbente $z_0 \leftarrow \infty$. → fita superior del millor valor de z_0 trobat fins al moment (func. obj).

1) **Mientras** $(\mathcal{T} \neq \emptyset)$:

- Seleccionar un $(PLE)_\ell$ no marcado. → nus
  Sea G su conjunto factible.
- Resolver su relajación $(RL)_\ell$.
  Sea $x^*_{RL\ell}$ una solución de ésta.

→ Es pot donar a b c

a) **Si** $(\ (RL)_\ell$ infactible ó $\lceil z^*_{RL\ell} \rceil \geq z_0\ )$ **entonces** marcar la hoja ℓ.

b) **Si** $(\ x^*_{RL\ell} \in Z^n$ & $(PLE)_\ell$ no marcado $)$ **entonces**
  Si $(\ z^*_{RL\ell} < z_0\ )$ **entonces**
   $x^*_{PLE} \leftarrow x^*_{RL\ell}$ (candidato al óptimo)
   $z_0 \leftarrow z^*_{RLj}$
  Fin si

 - Marcar $(PLE)_\ell$
 Fin si

c) **Si** $(\ (PLE)_\ell$ no marcado $)$ **entonces**
 - Tomar $(x^*_{RL\ell})_j = \omega \notin Z$.
 - Efectuar una partición de $G = G' \cup G''$
  (Añadir la restricción $x_j \leq \lfloor \omega \rfloor$ para formar un nodo sucesor del ℓ y la $x_j \geq \lceil \omega \rceil$ para formar el otro nodo sucesor.)

 Fin si
Fin Mientras

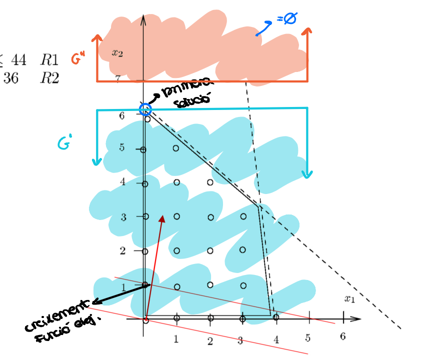
3) Solución: x^*_{PLE}

⑥ Exemples.

Ejemplo.

$$Min \quad -x_1 - 5x_2$$
$$\begin{array}{ll} 6x_1 + 7x_2 \leq 44 & R1 \\ 9x_1 + x_2 \leq 36 & R2 \end{array}$$
$$\begin{array}{l} x_1, x_2 \geq 0 \\ x \in Z^2 \end{array}$$

$x_2 \geq 6$
$x_2 \leq 7$



$(x_3, x_4$ holguras de $R1, R2$ respectivamente)

Primer → resoldre R.L.

nus inicial → $x_{RL} = (\underbrace{0, 6'\ldots}) \notin Z^n$

$z^*_{RL} = 30$ i algo $(\not< \infty)$

Seguint l'algoritme

primer si: $(RL)_\ell$ infactible o $\lceil z^*_{RL} \geq z_0 \rceil$ → marcar ℓ → passem al següent "si"

no · 30 i algo $\not> \infty$

segon si: $x^*_{RL} \in Z^n$ & $(PLE)_\ell$ no marcat

no

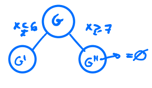
tercer si: $(PLE)_\ell$ no marcat → continuo

- Agafar $(x^*_{RL\ell}) = \omega \notin Z$
- Efectuar partició de $G = G' \cup G''$

⊕ restricció nova $x_j \leq \lfloor \omega \rfloor$; $x_j \geq \lceil \omega \rceil$

$x^*_{2\,RL} = 6'$ algo

$x_2 \leq 6$ $\quad$ $x_2 \geq 7$



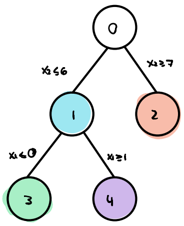
Seguim → buscar PLE no explorat (ara només en dos G', G'' ⇝ agafo G'

Restriccions G':

$6x_1 + 7x_2 \leq 44$
$9x_1 + x_2 \leq 36$
$x_1, x_2 \geq 0, \in Z^2$
$x_2 \leq 6$

$x_{RL}^* = (\underbrace{0'..}_{x_1 \notin Z}, 5)$

$x_1 \leq 0$, $x_1 \geq 1$



Anem ara a G'' → ∅ passo (infactible)

Anem a G''' (3)

$x_3{}^*_{RL} = (0, 6)$

Entrem al repon "si" si $(z_{RL}^* \leq z_0)$ -30 → millor que tenim fins ara

Anem a G^{IV} (4) ⇝ $x_{RL4} = (1, 1'5...)$

$\lceil z_{4RL}^* \rceil = \lceil -27,... \rceil \rightarrow -28$ ↗ $-26 \geq -30$ ↗ $-28 > -30$

Em quedo amb G''' ⇝ (0, 6)

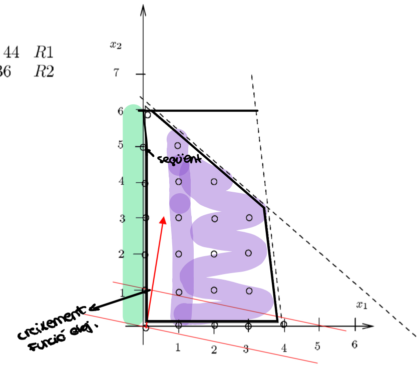
Ejemplo.

$Min \quad -x_1 - 5x_2$

$6x_1 + 7x_2 \leq 44 \quad R1$
$9x_1 + x_2 \leq 36 \quad R2$

$x_1, x_2 \geq 0$
$x \in Z^2$

$x_1 \geq 7$
$x_2 \leq 6$



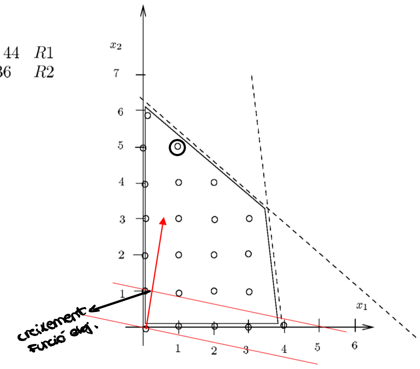
(x_3, x_4 holguras de $R1, R2$ respectivamente)

Ejemplo.

$Min \quad -x_1 - 5x_2$

$6x_1 + 7x_2 \leq 44 \quad R1$
$9x_1 + x_2 \leq 36 \quad R2$

$x_1, x_2 \geq 0$
$x \in Z^2$



(x_3, x_4 holguras de $R1, R2$ respectivamente)

Idea del algoritmo:

Si la solución de la relajación lineal de un PLE no es entera, se añaden un conjunto de restricciones ("cortes") tales que :

a) El nuevo problema ampliado posee igual conjunto de soluciones factibles enteras $F \cap Z^n$

b) La relajación lineal del problema ampliado presenta soluciones enteras.

Derivación de un corte de Gomory

Se dispone de la tabla óptima para la relajación de un PLE:

$$\text{(A)} \quad x_{s_i} + \sum_{q=1}^{n-m} y_{iq} x_{j_q} = y_{i0},$$

$$x_{s_i} + \sum_{q=1}^{n-m} \lfloor y_{iq} \rfloor x_{j_q} \leq y_{i0}$$

i_1	...	i_m	j_1	...	j_{n-m}	0
1	...	0	$y_{1,1}$	...	$y_{1,n-m}$	$y_{1,0}$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	...	1	$y_{m,1}$	...	$y_{m,n-m}$	$y_{m,0}$
0	...	0	r_1	...	r_{n-m}	$-z_0$

Si las componentes básicas son enteras:

$$\text{(B)} \quad x_{s_i} + \sum_{q=1}^{n-m} \lfloor y_{iq} \rfloor x_{j_q} \leq \lfloor y_{i0} \rfloor, \quad i = 0, 1, 2, ..., m$$

$$\text{(A) -(B):} \quad \sum_{q=1}^{n-m} (y_{iq} - \lfloor y_{iq} \rfloor) x_{j_q} \geq y_{i0} - \lfloor y_{i0} \rfloor, \quad i = 0, 1, 2, ..., m$$

Para la base B:

- todas las componentes son enteras
  $\rightarrow$ NINGUNA RESTRICCIÓN (A)-(B) es violada.
- la componente s_i es no entera
  $\rightarrow$ LA RESTRICCIÓN (A)-(B) i-ésima es violada.

$$\sum_{q=1}^{n-m} (y_{iq} - \lfloor y_{iq} \rfloor) x_{j_q} \geq y_{i0} - \lfloor y_{i0} \rfloor, \quad i = 0, 1, 2, ..., m$$

$$\boxed{\sum_{q=1}^{n-m} f_{iq} \, x_{j_q} \geq f_{i0} \quad i = 0, 1, 2, ..., m}$$

$$(\, y_{iq} = \lfloor y_{iq} \rfloor + f_{iq} \,)$$

Algoritmo de Planos Secantes:

1.- Se resuleve la relajación lineal del PLE: x^*_{RL}

2.- Si x^*_{RL} es entera, STOP: $x^*_{RL} \equiv x^*_{PLE}$

3.- Si x^*_{RL} no es entera: añadir a PLE una restricción lineal t.q.:

- **a) Sea satisfecha por todas las soluciones enteras del PLE.**
- **b) Sea violada per x^*_{RL}**

4.- Volver a 1

Programació multiobjectiu.

Fabricación propia + instalación. Importación + instalación.

	d_i	s_i	h_i	m_i	p_i	c_i		b_i	e_i
S_1	3000	20	0,55	1	4	7		16	13
S_2	5000	24	0,4	1	6	7	$\rightarrow$	18	17
S_3	7000	18	0,6	1	7	9		11	9
Recursos	–	–	2400	6000	–	–		–	–

- d_i Demanda de producto instalado en Hm^2
- s_i Precio de venta de producto instalado/Hm^2.
- h_i Tiempo de máquina /Hm^2
- m_i Materia prima/Hm^2
- p_i Coste de producción/Hm^2
- c_i Coste de importación/Hm^2
- $b_i = s_i - p_i$ Beneficio /Hm^2 por fabricación propia.
- $e_i = s_i - c_i$ Beneficio /Hm^2 por importación.

Solución intuitiva: ponderar con pesos $\alpha_1,\ \alpha_2 \geq 0$ las dos funciones objetivo $z_1,\ z_2$ y formar

$$z_P = \alpha_1 z_1 + \alpha_2 z_2$$

$$Max_{\,x,t}\ z_P = \alpha_1(16x_1 + 18x_2 + 11x_3 + 13t_1 + 17t_2 + 9t_3) - \\ -\alpha_2(7t_1 + 7t_2 + 9t_3)$$

$$s.a: \begin{array}{ll} 0{,}55x_1 + 0{,}4x_2 + 0{,}6x_3 & \leq 2400 \\ x_1 + x_2 + x_3 & \leq 6000 \\ x_1 + t_1 & = 3000 \\ x_2 + t_2 & = 5000 \\ x_3 + t_3 & = 7000 \\ x_i \geq 0,\ \ t_i \geq 0, i = 1,2,3 & \end{array}$$

Ensayar diferentes valores de $\alpha_1,\ \alpha_2$ y examinar las soluciones.

$F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$

k funciones objetivo $z_1 = c_1^\top x, ..., z_k = c_k^\top x$ (a minimizar)

Definición: x^* es óptimo Pareto de $z_1, ..., z_k$ sobre F si, $\forall x \in F,\ \nexists \ell,\ 1 \leq \ell \leq k$ t.q: $\ c_\ell^\top x > c_\ell^\top x^*$

A) x^* es mínimo Pareto de $z_1, ..., z_k$ sobre F

$$\exists \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \geq 0, \quad \alpha \neq 0$$

$\Rightarrow x^*$ es solución de (P)

B) Sea $\alpha > 0$.

Si x^* es solución de (P)

$\Downarrow$

x^*es mínimo Pareto

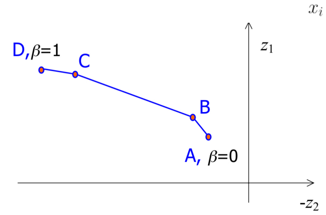
$$\begin{array}{ll} Min_{\,x} & \alpha_1 c_1^\top x + ... + \alpha_k c_k^\top x \\ s.a: & x \in F \qquad\qquad (P) \end{array}$$

$$Max_{\,x,t}\ z_P = \beta(16x_1 + 18x_2 + 11x_3 + 13t_1 + 17t_2 + 9t_3) - \\ -(1-\beta)(7t_1 + 7t_2 + 9t_3)$$

$\alpha_1 + \alpha_2 = 1,\ \ \alpha_1, \alpha_2 \geq 0$

$\alpha_1 = \beta,\ \alpha_2 = 1 - \beta,\ \ \beta \in [0,1]$

$$s.a: \begin{array}{ll} 0{,}55x_1 + 0{,}4x_2 + 0{,}6x_3 & \leq 2400 \\ x_1 + x_2 + x_3 & \leq 6000 \\ x_1 + t_1 & = 3000 \\ x_2 + t_2 & = 5000 \\ x_3 + t_3 & = 7000 \\ x_i \geq 0,\ \ t_i \geq 0, i = 1,2,3 & \end{array}$$

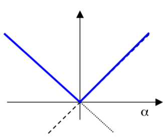


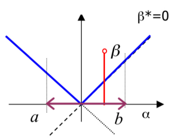
	z_1	z_2
A	193333, 3	78000, 0
B	194181, 8	78909, 1
C	197875, 0	84875, 0
D	198500, 0	86750, 0

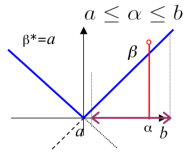
MINIMIZACIÓN DE $|\cdot|$

$|\alpha| = max\{\alpha, -\alpha\}$

$$\begin{array}{ll} Min_{\,\alpha} & |\alpha| \\ s.a: & a \leq \alpha \leq b \end{array} \equiv \begin{array}{ll} Min_{\,\alpha,\beta} & \beta \\ s.a: & \alpha \leq \beta \\ & -\alpha \leq \beta \\ & a \leq \alpha \leq b \end{array}$$



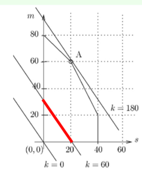




$$\begin{array}{ll} Min_{\,x} & |c^\top x| \\ s.a: & Ax = b \\ & x \geq 0 \end{array} \equiv \begin{array}{ll} Min_{\,x,\beta} & \beta \\ & c^\top x \leq \beta \\ & -c^\top x \leq \beta \\ & Ax = b,\ x \geq 0 \end{array}$$

CONCEPTO DE LA PROGRAMACIÓN POR OBJETIVOS

$$\begin{array}{llll} & Max & 3s + 2m & \\ & s.a & 2s + m & \leq 100 \\ (P) & & s + m & \leq 80 \\ & & s & \leq 40 \\ & & s,\ m & \geq 0 \end{array}$$



Política: Calcular s, m t.q.: beneficio = 60. ← **Valor de referencia**

$3s + 2m + u^+ - u^- = 60, \quad u^+, u^- \geq 0$

$$\begin{array}{lll} Min_{\,s,m,u} & |u^+ - u^-| & \\ s.a: & 3s + 2m + u^+ - u^- & = 60 \\ & 2s + m & \leq 100 \\ & s + m \leq 80, \quad s & \leq 40 \\ & s, m, u^+, u^- & \geq 0 \end{array}$$

$$\begin{array}{lll} Min_{\,s,m,u,\beta} & \beta & \\ s.a: & u^+ - u^- & \leq \beta \\ & -u^+ + u^- & \leq \beta \\ & 3s + 2m + u^+ - u^- & = 60 \\ & 2s + m & \leq 100 \\ & s + m \leq 80, \quad s & \leq 40 \\ & s, m, u^+, u^- & \geq 0 \end{array}$$

$F = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$

$c_1^\top x \ \ ... \ \ c_k^\top x \ \leftarrow$ expresión mediante las v. de decisión

$\theta_1 \ \ ... \ \ \theta_k \ \leftarrow$ valores de referencia

Para el objetivo ℓ:

$A-$ "lo más cerca posible" $c_\ell^\top x \approx \theta_\ell$ $\qquad c_\ell^\top x + u_\ell^+ - u_\ell^- = \theta_\ell,\ \ u_\ell^+,\ u_\ell^- \geq 0$

$B-$ "como mínimo" $c_\ell^\top x \geq \theta_\ell$ y lo mayor posible $\qquad c_\ell^\top x - u_\ell^- = \theta_\ell, \qquad u_\ell^- \geq 0$

$C-$ "como máximo" $c_\ell^\top x \leq \theta_\ell$ y lo menor posible $\qquad c_\ell^\top x + u_\ell^+ = \theta_\ell, \qquad u_\ell^+ \geq 0$

Formar f.obj con k términos según caso A,B,C.

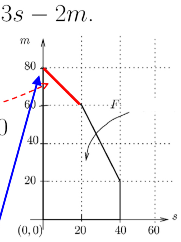
	A	B	C
Término en f.obj.	$\lvert u_\ell^+ - u_\ell^- \rvert$	$-u_\ell^-$	u_ℓ^+

Beneficio: $z_1 = 10s + 10m$, Calidad: $z_2 = 100 - 0,2s - 0,1m$

Recursos sobrantes:

$z_3 = 100 - (2s + m) + 80 - (s + m) = 180 - 3s - 2m.$

$$\left.\begin{array}{ll} Min_{\,s,m} & -(10s + 10m) \\ s.a: & 2s + m \leq 100 \\ & s + m \leq 80 \\ & s \leq 40 \\ & s, m, \geq 0 \end{array}\right\} \rightarrow S_1, z^* = 800$$



$$\left.\begin{array}{ll} Min_{\,s,m} & 0,2s + 0,1m \\ & (s,m) \in S_1 \end{array}\right\} \rightarrow S_2 = \{(0, 80)\}, z_2^* = 92$$

El tercer objetivo ya no puede optimizarse: $S_3 \equiv S_2$

MULTIPLES OBJETIVOS CON PRIORIDAD

k objetivos $z_1 = c_1^\top x, ..., z_k = c_k^\top x$ sobre
$F = \{\, x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0 \,\}$

$$\begin{array}{c} Min_{\,x}\ c_1^\top x \\ x \in F \end{array} \xrightarrow{S_1} \begin{array}{c} Min_{\,x}\ c_2^\top x \\ x \in S_1 \end{array} \xrightarrow{S_2} \dots \xrightarrow{S_{\ell-1}} \begin{array}{c} Min_{\,x}\ c_\ell^\top x \\ x \in S_{\ell-1} \end{array} \xrightarrow{S_\ell} \dots$$

A) Para $\ell \leq k$, $S_\ell = \{x^*\}$

B) S_k no unitario; $\Leftarrow$ Conjunto sol. del Problema.

$$\begin{array}{c} Min_{\,x}\ c_1^\top x \\ Ax = b \\ x \geq 0 \end{array} \xrightarrow{z_1^*} \begin{array}{c} Min_{\,x}\ c_2^\top x \\ c_1^\top x = z_1^* \\ Ax = b \\ x \geq 0 \end{array} \xrightarrow{z_2^*} \begin{array}{c} Min_{\,x}\ c_3^\top x \\ c_1^\top x = z_1^* \\ c_2^\top x = z_2^* \\ Ax = b \\ x \geq 0 \end{array} \xrightarrow{z_3^*} \dots$$

En el problema ℓ:

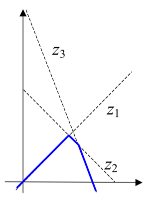
$\exists$ coste reducido $\overset{?}{=} 0$ $\xrightarrow{\text{SI}}$ $\boxed{\text{SIGUIENTE PROBLEMA } \ell + 1 \text{ (Reoptimización)}}$

$\downarrow$ NO

$\boxed{\text{STOP. Solución única } x^* \quad x^* \text{ mínimo jerárquico de } z_1, z_2, ..., z_k}$

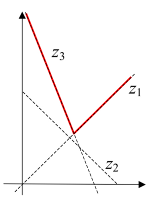
SI NO HAY SOLUCIÓN ÚNICA EN EL PROBLEMA k DEBEN EXAMINARSE SUS SOLUCIONES.

OPTIMIZACIÓN DEL PEOR CASO POSIBLE



$$z_1(x) = x,\ z_2(x) = 1 - x,\ z_3(x) = 2 - 3x$$
$$f(x) = Min\{\ z_1(x),\ z_2(x),\ z_3(x)\ \}$$

$$Max_{\,x} f(x) \ \equiv\ \begin{array}{lll} Max_{\,x,y} & y & \\ s.a: & x & \geq y \\ & 1 - x & \geq y \\ & 2 - 3x & \geq y \end{array}$$



$$g(x) = Max\{\ z_1(x),\ z_2(x),\ z_3(x)\ \}$$

$$Min_{\,x} g(x) \ \equiv\ \begin{array}{lll} Min_{\,x,y} & y & \\ s.a: & x & \leq y \\ & 1 - x & \leq y \\ & 2 - 3x & \leq y \end{array}$$

OPTIMIZACIÓN DEL PEOR CASO POSIBLE. Ejemplo

Cia. con dos productos.

Puede vender toda su producción en los mercados M1 o M2 de dos paises (no en ambos).

Los beneficios por unidad de producto en cada mercado son:

	Prod.1	Prod.2
M1	3	2
M2	5	1

Está en curso la firma de un importante acuerdo económico con los países pero el resultado es incierto. Si la Cia. vende su producción en el mercado con el que se llegue al acuerdo conseguiría una importante posición estratégica. No se puede demorar la producción hasta después de la firma del acuerdo.

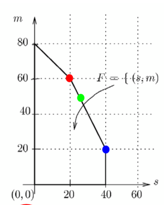
a) Se decide vender en el país con el que se firme el acuerdo. -> No se sabe donde va a venderse la producción.

b) Se decide una opción conservadora: producir en cantidades (s,m) de forma que el peor beneficio posible sea el máximo.

$$F = \left\{ (s,m) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 2s + m \leq 100,\ \ s + m \leq 80 \\ s \leq 40,\ \ s, m \geq 0 \end{array} \right\}$$

$$\boxed{\begin{array}{l} Max \quad z_1 = 3s + 2m \\ \quad (s,m) \in F \end{array} \ \rightarrow\ \begin{array}{c} s_1^* = 20 \\ m_1^* = 60 \\ z_1^* = 180 \end{array}}$$

$$\boxed{\begin{array}{l} Max \quad z_2 = 5s + m \\ \quad (s,m) \in F \end{array} \ \rightarrow\ \begin{array}{c} s_2^* = 40 \\ m_2^* = 20 \\ z_2^* = 220 \end{array}}$$

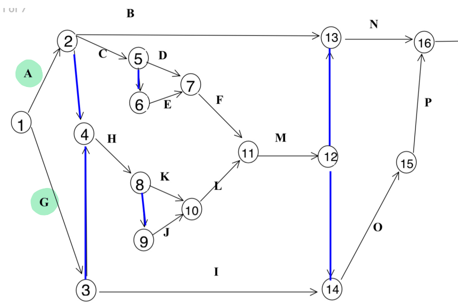


$z_1(40, 20) = 3 \cdot 40 + 2 \cdot 20 = 160$, $\quad z_2(20, 60) = 5 \cdot 20 + \cdot 60 = 160$

$$\begin{array}{l} Max \quad z_P(s,m) = Min\{3s + 2m,\ 5s + m\} \\ \qquad (s,m) \in F \end{array} \ \equiv\ \begin{array}{ll} Max_{\ s,m}\ y & \\ s.a: & 3s + 2m \geq y \\ & 5s + m \geq y \\ & 2s + m \leq 100 \\ & s + m \leq 80 \\ & s \leq 40 \\ & s, m, \geq 0 \end{array}$$

$$\boxed{\begin{array}{c} s_P^* = 25 \\ m_P^* = 50 \\ z_P^* = Min\{3 \cdot 25 + 2 \cdot 50,\ 5 \cdot 25 + 50\} = 175 \end{array}}$$

Critical path method. (CPM)



Grafo Direccional AOA (Activity on Arc)

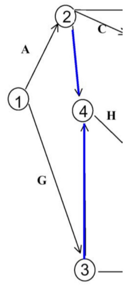
Tarea ficticia. Duración 0

Cada node: un instant.

Tarea	Precedente	Duración
A	—	3
B	A	12
C	A	4
D	C	10
E	C	2
F	D, E	3
G	—	2
H	A, G	4
I	G	3
J	H	16
K	H	2
L	J, K	2
M	F, L	2
N	B, M	2
O	I, M	1
P	O	2
Q	N, P	3

→ Consisteix en determinar durada mínima d'un projecte.

MODELO CPM. FORMULACIÓN EN P.L.



t_j = instante en el que deben finalizar las tareas (i, j).

$\tau_{i,j}$ = duración de la tarea (i, j).

$t_j - t_i - \tau_{i,j}$ tiempo de inactividad para la tarea (i, j).

minimización del tiempo total del proyecto t_n

$$
\begin{aligned}
t_1 + \tau_{12} - t_2 &\leq 0 \\
t_1 + \tau_{13} - t_3 &\leq 0 \\
t_2 + \tau_{24} - t_4 &\leq 0, \quad (\tau_{24} = 0) \\
t_3 + \tau_{34} - t_4 &\leq 0, \quad (\tau_{34} = 0) \\
&\vdots \\
(t_1 = 0)
\end{aligned}
$$

$$\text{Min}_{\ t,\tau} \quad t_n$$

$$t_i + \tau_{i,j} - t_j \leq 0, \quad (i,j) \in A$$

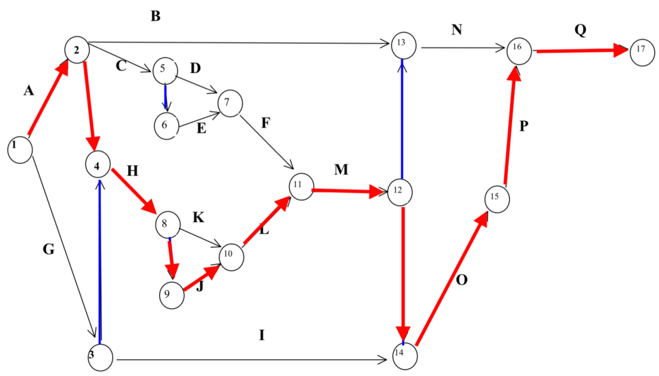
$$\underline{\tau}_{i,j} \leq \tau_{i,j} \leq \hat{\tau}_{i,j}$$

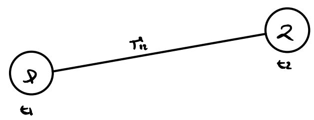
$$(t_1 = 0)$$

Identificación del camíno crítico:
arcos (i, j) con tiempo de inactividad cero

$$t_i + \tau_{i,j} - t_j = 0$$

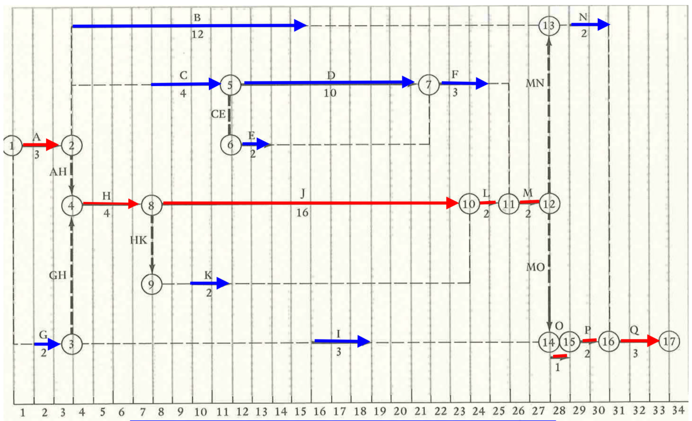
En el resultado





$t_2 \geq t_1 + r_{12}$

$t_1 + r_{12} - t_2 \leq 0$



Solución del CPM: Carta de Gantt (Gantt's Chart)

Coste monetario de la actividad (i, j): $c(\tau_{i,j}) = k_{i,j} - c_{i,j}\,\tau_{i,j}$

$$\text{Min}_{\ t,\tau} \quad -\sum_{(i,j)\in A} c_{i,j}\,\tau_{i,j}$$

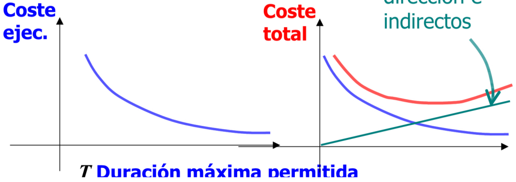
} "no costos d'execució"   cost ↗   diners màxima (constant) ↗   +temps −cost ↗

$$t_i + \tau_{i,j} - t_j \leq 0, \quad (i,j) \in A$$

$$\underline{\tau}_{i,j} \leq \tau_{i,j} \leq \hat{\tau}_{i,j}$$

$$t_n \leq T, \quad (t_1 = 0)$$

variant CPM.



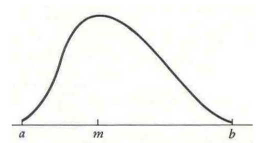
T Duración máxima permitida

MÉTODO PERT. (Program Evaluation & Review Technique)

Las duraciones t_{ij} de las tareas son v. a. independientes entre sí. distribución β.

Se conoce:

- un valor mínimo a.
- un valor máximo b.
- el valor más frecuente m (moda).



$$E[t_{ij}] = \frac{1}{3}\left(2m + \frac{1}{2}(a+b)\right), \quad Var[t_{ij}] = \left[\frac{1}{6}(b-a)\right]^2$$

El método PERT determina el camino crítico usando el modelo CPM tomando como tiempos para cada tarea $E[t_{ij}]$.

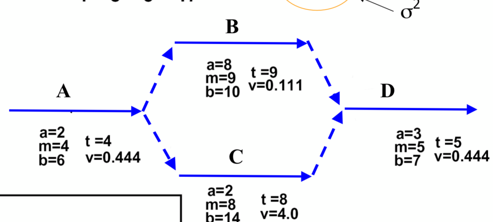
Deben evaluarse entonces las varianzas de los caminos alternativos al crítico.

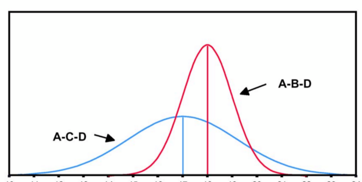
En caso de proyectos con número alto de tareas la distribución de los tiempos de los caminos se toma $\sim N(\mu, \sigma)$

MÉTODO PERT. (Program Evaluation & Review Technique)

A - B - D	4 + 9 + 5 = 18	1.000
A - C - D	4 + 8 + 5 = 17	4.889

σ^2

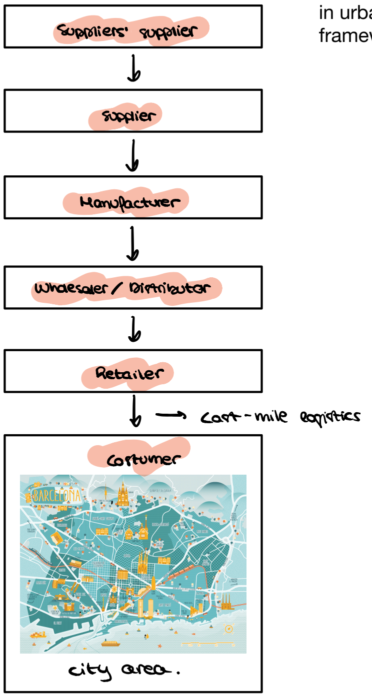




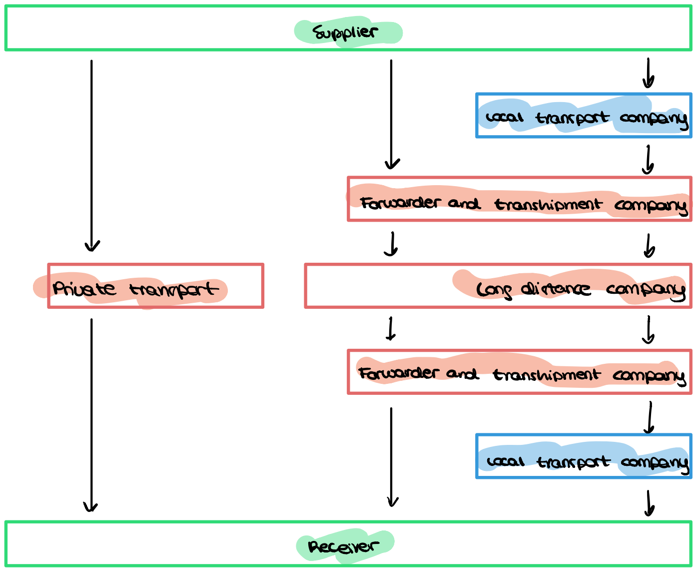
Si se quiere terminar antes de 21 días, ¿cuál es el camino crítico?

Vehicle Routing Models.

Distribution Logistics in Urban Areas



"**City Logistics** is the process of totally optimizing the logistics and transport activities by private companies in urban areas while considering the traffic environment, traffic congestion and energy consumption within the framework of a market economy".

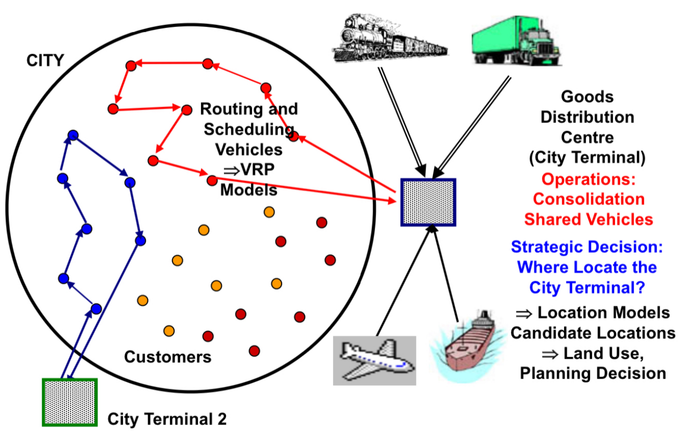


Special features of urban freight transport

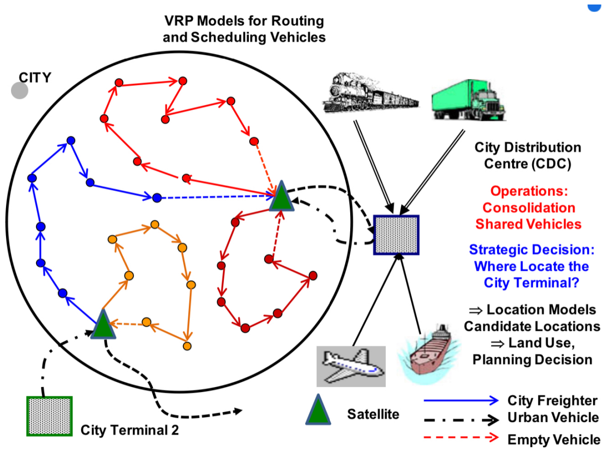
- Spatial restrictions
  - Urban micro-structure
  - Limited vehicle access
  - Small-quantity deliveries
  - High density of delivery points
- Traffic infrastructure
- Environmental concerns and sensitivities
  - Growing role for small specialized urban (green) vehicles
  - Low automation, critical human role
  - High operational and environmental costs

- **⇒Critical to optimize operations to minimize number of vehicles, travel kilometers and idle time**
- **⇒Important role for collaborative logistics strategies**

DECISIONS AND MODELS IN CITY LOGISTICS



TWO-ECHELON, URBAN FREIGHT DISTRIBUTION SYSTEMS AND IMPLIED DECISIONS

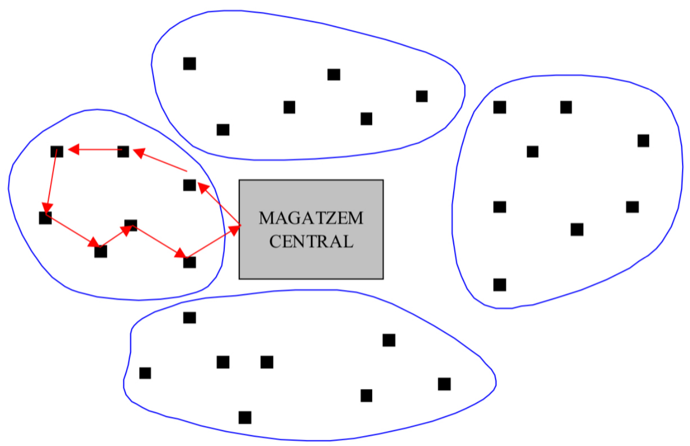


MODELS TO SUPPORT THE DECISION MAKING

- Location Models
  - Where should be located the City Terminals and Urban Satellites
  - How many?
- **Vehicle Routing** and Scheduling Models
- (no els veurem) Transport Planning Models
  - Analyze the mutual impacts between urban planning and logistics planning
  - Conduct accessibility analysis

THE CAPACITATED VEHICLE ROUTING PROBLEM (CVRP)

- The CVRP is to determine K vehicle routes,
- Where a route is a tour beginning and ending at the depot and visiting a subset of customers in a specified sequence.
- Each customer must be assigned to exactly one of the K vehicles, and the total amount of the demand to be serviced by a vehicle must not exceed the vehicle capacity b.
- The routes should be chosen to minimize the total cost.



VEHICLE ROUTING PROBLEMS

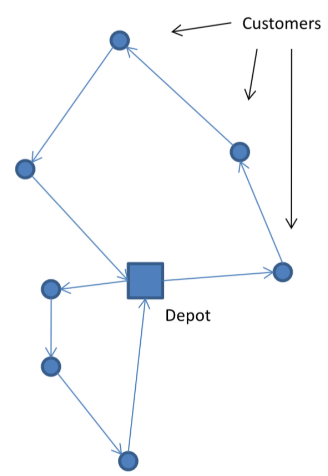
1. Assignment of clients to vehicles
2. Route sequencing

VEHICLE ROUTING PROBLEM OVERVIEW

Problem Overview (General):
- Set of vehicles located in one or more depots.
- Distribution of goods among customers.
- Finite time horizon.
- Usage of a road network.

Objectives (Carriers point of view):
- Minimization of transportation costs.
- Minimization of fleet size.
- Reduction of service failures.
- Route balancing.
- Weighted combination of above objectives.

Objectives (Administration point of view) usually specifics of City Logistics:
- Sustainable urban freight distribution
- Energy consumption
- Emissions (including noise pollution)
- Regulations: loading/unloading, vehicle types, service timings, control access...



The Capacitated Vehicle Routing Problem.

Characteristics:
- Each customer is visited exactly once by one vehicle.
- The route of each vehicle starts and ends at the depot.
- The sum of demands of customers on a route must not exceed vehicle´s capacity.

Grafs direccionals:

Pot ser que $c_{ij} \neq c_{ji}$

$i \underset{c_{ji}}{\overset{c_{ij}}{\rightleftarrows}} j$ (i,j) (j,i

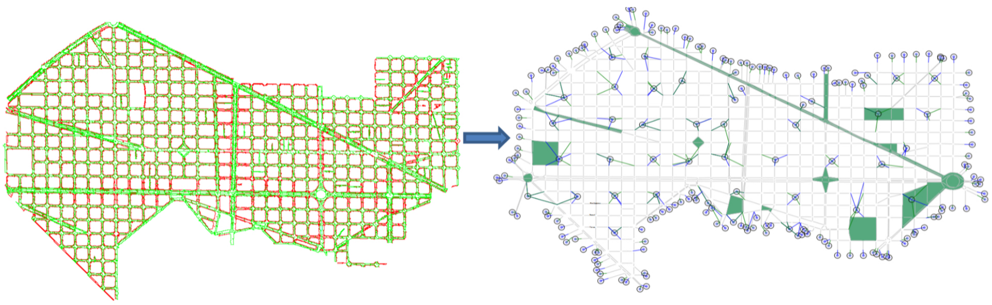
Notation:
- The problem can be defined in a graph G = (V, A), where V = {0,1,...,n} is the set of nodes, and A is the set of arcs.
- cij : cost of travel from node i to node j.
- xij : 1, if arc (i,j) is in the optimal solution, 0, otherwise.
- K : number of vehicles.
- r(S) : minimum number of vehicles.

$x_{ij} = \begin{cases} 1 & \text{ruta men } (i,j) \\ 0 & \text{otherwise} \end{cases}$

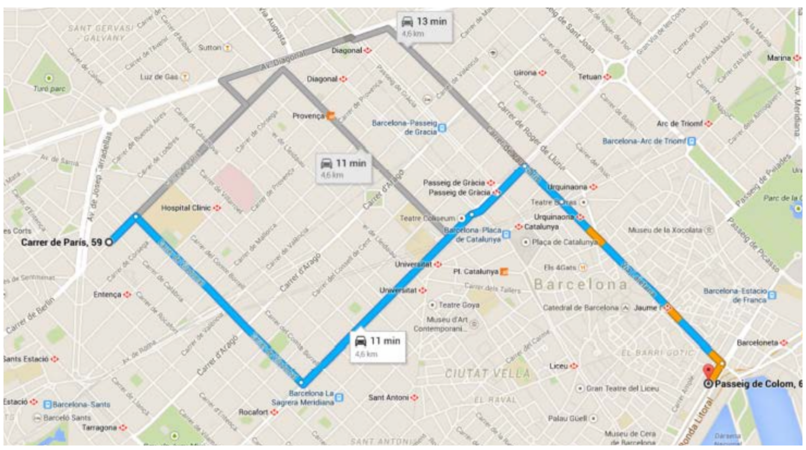
$S \subseteq N$ (nusos) $\rightsquigarrow$ $r(S)$ = # min vehicles per per transport en S

K → # vehicles

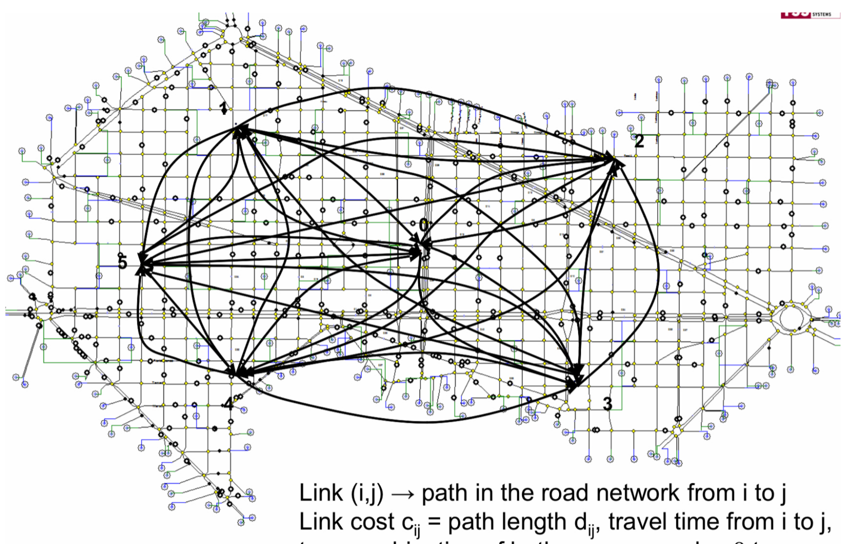
FROM THE DIGITAL MAP TO THE GRAPH REPRESENTATION OF THE ROAD NETWORK



EXAMPLE OF LINK BETWEEN TWO NODES (Customers) DEFINED BY A PATH CONNECTING THEM



EXAMPLE OF GRAPH G=(N,A) FOR VR APPLICATIONS IN AN URBAN SCENARIO

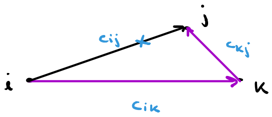


Link (i,j) → path in the road network from i to j
Link cost c_{ij} = path length d_{ij}, travel time from i to j, t_{ij}, a combination of both, e.g. $c_{ij} = \alpha\, d_{ij} + \beta\, t_{ij}$,…..

LINK COSTS ON A URBAN NETWORK

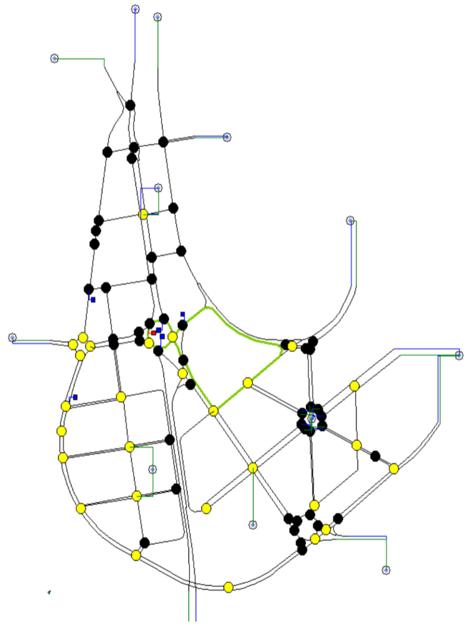
- Depot: red square
- Customers: Blue squares
- Cost c0i: cost of path (green path) from the depot (node 0) to customer i (node i)
- Costs are:
    - Not symmetric
- The graph is not necessarily euclidean
- The triangular property does not hold

Propietat triangular **NO** té perquè complir-se



Assymetric Graph ⇝ $c_{ij} \neq c_{ji}$

$$\boxed{\forall (i,j) \quad c_{ij} \le c_{ik} + c_{kj}}$$



The basic routing model : The traveling Salesman Problem (TSP)

THE TRAVELING SALESMAN PROBLEM

Visitar tots els nusos només un cop

- Problem :
  - Given a complete graph with n vertices Kn, Kn=(V,A), |V|=n, find an hamiltonian circuit of minimum cost.
- Parameters:
  - cij= cost associated with the use of link (i,j)
- Decision Variables:
  - xij that takes value 1 if link (i,j) makes part of the solution and 0 otherwise
- Objective:
  - find an hamiltonian circuit of minimum cost when the cost of a circuit is the sum of the costs of the links defining the circuit.

TSP BASIC FORMULATION

(dirigit)

$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\sum_{j\in A^+(i)} x_{ij} = 1,\ \forall i \in V \quad \left(\text{Alternative formulation} \sum_{1\le j\le n} x_{ij} = 1,\ \forall i\right) \ (1)$$

$$\sum_{j\in A^-(i)} x_{ji} = 1,\ \forall i \in V \quad \left(\text{Alternative formulation} \sum_{1\le i\le n} x_{ij} = 1,\ \forall j\right) \ (2)$$

$$x_{ij} \in \{0,1\},\ \forall (i,j) \in A$$

Where

$$A^+(i) = \{j \mid (i,j) \in A\}$$

$$A^-(i) = \{j \mid (j,i) \in A\}$$

Node degree constraints $\Rightarrow$ Necessary but not sufficient conditions

(1) In-degree =1 (only one positively incident link in each node)

(2) Out-degree =1 (one one negatively incident link in each node)

A MORE COMPACT FORMULATION OF THE SYMMETRIC TSP

Given a symmetric TSP defined in a complete graph K_n=(V,A), with arc costs c_{ij}, $\forall$ (i,j)$\in$A,

Let's denote by : $\delta(v)=A^-(v)\cup A^+(v)$={Set of arcs in K_n with incidence on node v}

The TSP problem consist then in finding the vector x $\in R^A$ such that:

(no dirigit)

$$\text{Min cx}$$

$$\text{such that}: \sum_{u\in A^+(v)} x_{vu} + \sum_{u\in A^-(v)} x_{uv} = 2,\ \forall v \in A$$

$$x_{uv} \in \{0,1\},\ \forall (u,v) \in A \ (0 \le x_{uv} \le 1, \text{strong linear relaxation})$$

Or equivalently:

$$\text{Min cx} \qquad\qquad \text{Min cx}$$

$$x(\delta(v)) = 2 \qquad\qquad Mx = 2$$

$$x_{uv} \in \{0,1\},\ \forall (u,v) \in A \quad (0 \le x_{uv} \le 1)$$

Where M is the node-arc incidence matrix of K_n, and x(δ(v)) represents the summation over the arcs of δ(v).

AMPL MODEL (TSP, necessary conditions: node degree constraints)

```
set NODES;
set ARCS within (NODES cross NODES);
param cost {NODES,NODES} >= 0;
var X {NODES,NODES} binary;
minimize total_cost:
sum {i in NODES, j in NODES:i!=j} cost[i,j] * X[i,j]; # Degree constraints
subject to Degree_Out {i in NODES}:
sum {j in NODES: j != i} X[i,j] = 1; subject to Degree_In {i in NODES}:
sum {j in NODES: j != i} X[j,i] = 1;
```

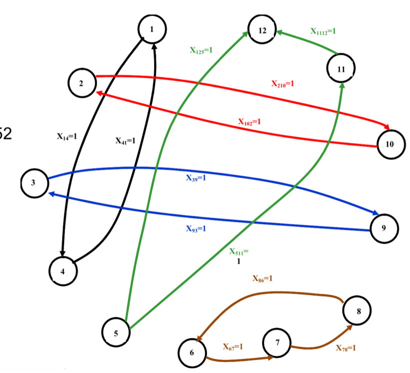
SYMMETRIC TSP: An Example

Symmetric problem c_{ij}= c_{ji} distance matrix between customers

	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	c11	c12
c1		702	454	842	2936	1196	864	772	714	554	2363	2679
c2			324	1093	2136	764	845	764	459	294	2184	2187
c3				1137	2180	798	664	572	284	338	2228	2463
c4					1616	1857	1706	1614	1421	799	1521	2021
c5						2900	2844	2752	2464	1842	95	405
c6							396	424	514	1058	2948	2951
c7								92	386	1002	2892	3032
c8									305	910	2800	2951
c9										622	2512	2646
c10											1890	2125
c11												500

DEGREE CONSTRAINTS AS NECESSARY BUT NOT SUFFICIENT FOR DEFINING A TOUR (I)

Cost = 4752



wasses subtours.

FORMULATION II (Subtour breaking constraints)

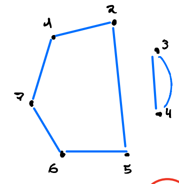
$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\sum_{j\in A^+(i)} x_{ij} = 1,\ \forall i \in V$$

$$\sum_{j\in A^-(i)} x_{ji} = 1,\ \forall i \in V \qquad \text{Where}: \begin{cases} A^+(i) = \{j \mid (i,j)\in A\} \\ A^-(i) = \{j \mid (j,i)\in A\} \end{cases}$$

$$x_{ij} \in \{0,1\},\ \forall (i,j)\in A$$

Noves restriccions que m'impedeixen que subtours m'apareixin



$S = \{1, 2, 5, 6, 7\}$

$\underline{S = V}$

$7 \leq 6$

$\sum x_{ij} \begin{cases} x_{12} + x_{15} + x_{17} + x_{25} + x_{26} + x_{27} + \\ x_{51} + \dots \leq 4 \quad (5-1) \end{cases}$

2^{n-1}

(nombre restriccions)

Subtour breaking constraints

$$\boxed{\sum_{i\in S,\, j\in S} x_{ij} \leq |S|-1,\quad \forall S \neq \emptyset, S \subset V \quad \left(\text{alternatively} \sum_{i\in S,\, j\in V\setminus S} x_{ij} \geq 1\right)}$$

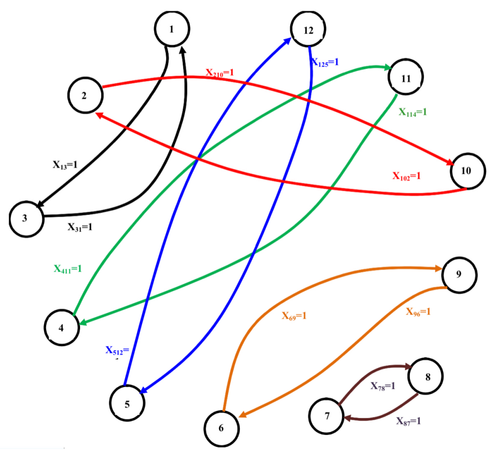
2^{n-1}

AMPL MODEL WITH SUBTOUR BREAKING CONSTRAINTS MANUALLY GENERATED

```
set NODES;
set ARCS within (NODES cross NODES); set NDS1;
set NDS2;
param cost {NODES,NODES} >= 0;
var X {NODES,NODES} binary; minimize total_cost:
sum {i in NODES, j in NODES:i!=j} cost[i,j] * X[i,j]; # Degree constraints
subject to Degree_Out {i in NODES}:
sum {j in NODES: j != i} X[i,j] = 1; subject to Degree_In {i in NODES}:
sum {j in NODES: j != i} X[j,i] = 1; #Subtour breaking constraints subject to Subtour_breaking_1:
sum {i in NDS1, j in NDS1: i!=j} X[i,j]<=2; subject to Subtour_breaking_2:
sum {i in NDS2, j in NDS2: i!=j} X[i,j]<=2;
```

DEGREE CONSTRAINTS AS NECESSARY BUT NOT SUFFICIENT FOR DEFINING A TOUR (II)

Cost = 6560



FORMULATION II-bis (Weak Subtour breaking constraints)

MTZ

$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\sum_{j\in A^+(i)} x_{ij} = 1,\ \forall i \in V$$

$$\sum_{j\in A^-(i)} x_{ji} = 1,\ \forall i \in V \qquad \text{Where}: \begin{cases} A^+(i) = \{j \mid (i,j)\in A\} \\ A^-(i) = \{j \mid (j,i)\in A\} \end{cases}$$

ordre del node dins del tour

$$u_i - u_j \leq n(1-x_{ij}) - 1\ \boxed{i \neq 1}\ i \neq j$$

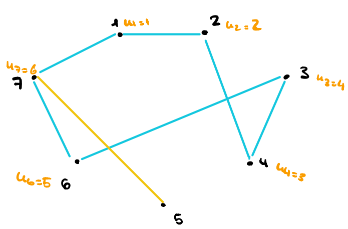
$$x_{ij} \in \{0,1\},\ \forall (i,j)\in A$$

$$u_1 = 1$$

$$1 \leq u_i \leq n, \forall i \in V$$

$u_i - u_j \geq n(1-x_{ij}) - 1 \qquad \forall (i,j)\in A \quad i \neq 1.$ $= |A|$ (nombre restriccions)

$2^n + \frac{n(n+1)}{2} - n + 1$



$S \begin{cases} x_{12} = 1 \\ x_{24} = 1 \\ x_{34} = 1 \\ x_{36} = 1 \\ x_{16} = 1 \end{cases}$

$x_{57} = 1$

$(1,2) \rightarrow u_1 - u_2 \leq 7(1-1) - 1$

$u_2 \geq u_1 + 1$

$(2,4) \rightarrow u_2 - u_4 \leq 7(1-1) - 1$

$u_4 \geq u_2 + 1 \geq u_1 + 2$

$\dots\ u_3 \geq u_1 + 3,\ u_6 \geq u_1 + 4,\ u_1 \geq u_1 + 5 \rightsquigarrow \cancel{5 \leq 0}$ impossible

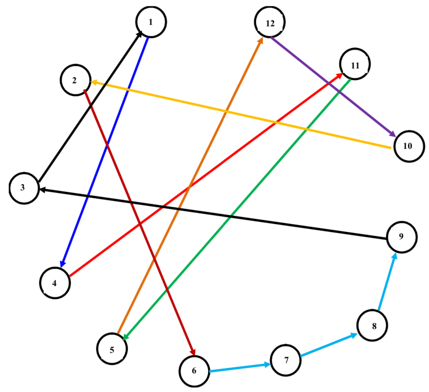
He d'excloure 1 del tour per no trobar contradiccions.

TSP (AMPL MODEL) Weak subtour breaking constraints

```
set NODES;
set ARCS within (NODES cross NODES);
param cost {NODES,NODES} >= 0;
param n >0;
var X {NODES,NODES} binary;
var u {NODES} >=1, <=n;
minimize total_cost:
sum {i in NODES, j in NODES:i!=j} cost[i,j] * X[i,j]; # Degree constraints
subject to Degree_Out {i in NODES}: sum {j in NODES: j != i} X[i,j] = 1;
subject to Degree_In {i in NODES}: sum {j in NODES: j != i} X[j,i] = 1;
#Connectivity constraints (weak subtour breaking constraints) subject to a :
u[1]=1;
subject to Connectivity {i in NODES, j in NODES:i!=j and j!=1}:
u[i]-u[j]<=n*(1- X[i,j])-1;
```

OPTIMAL SOLUTION

Cost = 7577



ALTERNATIVE FORMULATION OF TSP IN TERMS OF MIN COST FLOW

- Associate flow variables y_{ij} to each link (i,j)
- Assume that node 1 is a **source node** that generates n-1 flow units, and that each of the other nodes is a y **sink node absorbing one unit of flow**

$$\text{Min} \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\sum_{1\le j\le n} x_{ij} = 1,\ i=1,\dots,n \qquad (1)$$

$$\sum_{1\le i\le n} x_{ij} = 1,\ j=1,\dots,n \qquad (2)$$

$$Ny = b \qquad (3)$$

$$y_{ij} \le (n-1)x_{ij} \quad \forall (i,j)\in A \qquad (4)$$

$$y_{ij} \ge 0,\ x_{ij}\in\{0,1\},\ \forall (i,j)\in A$$

Fluxes a l'arc (i,j)

INTERPRETATION OF THE ALTERNATIVE FORMULATION

- **Let be: A'={(i,j)|x_{ij}=1} and A''={(i,j)|y_{ij}>0}**
- **Constraints (1) y (2) imply that exactly only on link of A' enters and leaves any node i ⇒ A' is the union of disjoint cycles containing all nodes of V (That is: is a solution composed of subcircuits)**
- **Constraints (3) ensure that A'' is connex (since it is the solution of a minimum cost flow problem in the graph)**
- **Constraints (4) establish a relationship between the two sets of variables (they are redundant when x_{ij}=1)**
- **A''connex and(4) ⇒ A' connex ⇒ A' cannot contain subcircuits ⇒ A' hamiltonian circuit**

Excloc tots els que no siguin necessariament → ja no és TSP.

TSP heurístic solutions.

(sense demostració matemàtica)

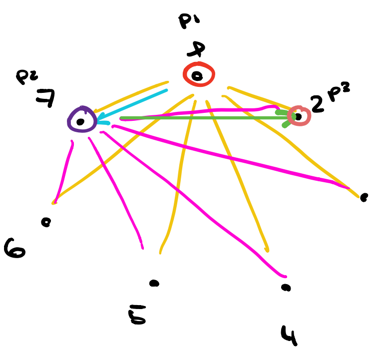
↗ constructives

↘ Millora ⟶ donat un tour fet, millorar-lo

NEAREST NEIGHBOR HEURISTICS

Symmetric problem $c_{ij} = c_{ji}$ distance matrix between customers

	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	c11	c12
c1		702	454	842	2936	1196	864	772	714	554	2363	2679
c2			324	1093	2136	764	845	764	459	294	2184	2187
c3				1137	2180	798	664	572	284	338	2228	2463
c4					1616	1857	1706	1614	1421	799	1521	2021
c5						2900	2844	2752	2464	1842	95	405
c6							396	424	514	1058	2948	2951
c7								92	386	1002	2892	3032
c8									305	910	2800	2951
c9										622	2512	2646
c10											1890	2125
c11												500



marco nus 1

tour està buit

puc anar a tots els altres.

exemple

de tots aquests, quin té menys cost? 7

Agafo el 7 ⟶ tour = {(1,7)}

marco nus 7 → on puc anar? cost mínim 2

(min {c_{7j} | j no marcat})

tour = {(1,7)(7,2)}

marco nus 2

etc.

Nearest Neighbor Heuristic

1. **Select a node i∈N, label it, and make T:=∅ and p:=i**
2. **If all nodes are labeled make T:=T∪{i_p}. STOP, T is a Hamiltonian circuit**
3. **Select a not yet labeled node j such that: c_{pj} = MIN {c_{pk} |k not labeled}**
4. **Make:**

   T:=T∪{c_{pj}}
   Label j
   Let p:=j
   Repeat from 2

SPANNING TREE HEURISTIC

- <u>INPUT</u>: A Complete Graph K_n =(V,A)
- <u>OUTPUT</u>: An Hamiltonian circuit (tour) T in graph K_n.

1. **Find** a "Minimum Spanning Tree" ST of K_n.
2. **Double** all arcs of ST to generate the auxiliary graph (V,ST)
3. Since all nodes of (V,ST) have even degree and (V,ST) is connex, (V,ST) is Eulerian.
   - **Determine** an eulerian circuit C in (V,ST)
   - **Assign** an orientation to C
   - **Select** a node i∈V, **label it** and **make** p:=i, T:=∅
4. **If** all nodes are labeled **make** T:=T∪{(i,p)}. END. T is a hamiltonian circuit .
5. **Otherwise, move** from p, along C, following the orientation in C until an unlabeled node q.
   Make T:=T∪{(p,q)}. **Label** q, **Make** p:=q, **Go to 4**.

1. Trobar min arbre de recobriment.

↳

FINDING A "MINIMUM SPANNING TREE" ST (Prim Algorithm)

1. **Select** an arbitrary node w∈V, **make** ST:= ∅ , W:={w}, V:=V \ W
2. **If** V=∅ END. ST is a "Minimum Spanning Tree"
3. **Select** an arc (u,v) ∈ A with u ∈ W and v ∈V such that:

   c_{uv} = Min { c_e | e ∈ δ (W) }   (δ (W) = {set of arcs of A with a node in W and other in V })
4. **Make**:

   ST := ST ∪ {(u,v)}
   W := W ∪ {v}
   V := V \ {v}

   Go to 2.

2. Doblar arcs sobre l'arbre de recobriment

(subgraf amb multiplicitat mínima de 2)

visitar tots els nusos passant un sol cop per un arc.

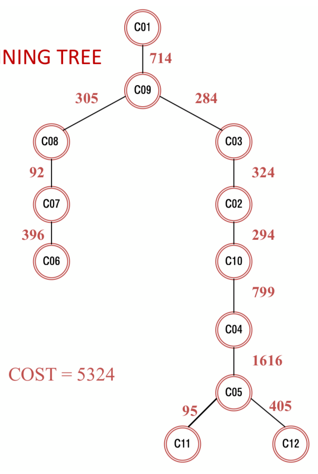
3. Sobre l'arc que s'ha doblat ⟶ passar a circuit eulerià

4. Recorro circuit eulerià

↳ cada cop que visito un nus NO marcat → el marco

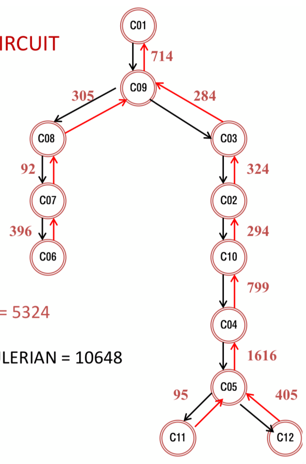
↳ si ja marcat → salto fins a no marcat i agafo aresta des d'últim nus que he marcat al nou que he trobat no marcat.

MINIMUM SPANNING TREE



COST = 5324

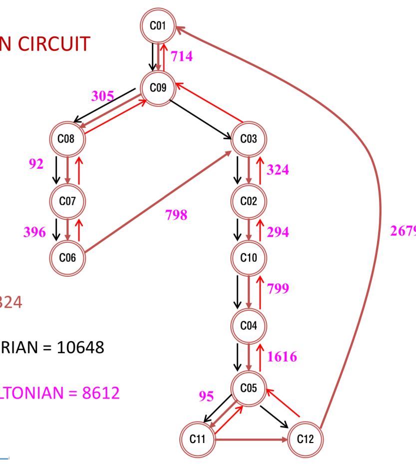
EULERIAN CIRCUIT



COST ST= 5324

COST EULERIAN = 10648

HAMILTONIAN CIRCUIT

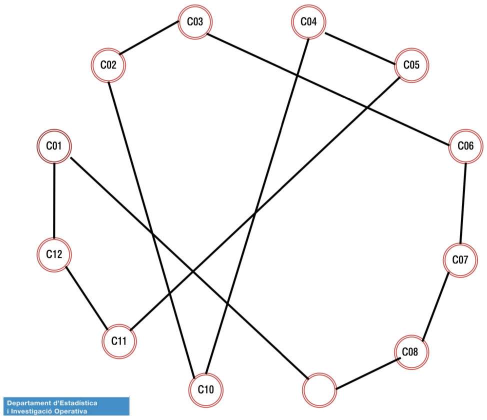


COST ST= 5324

COSTE EULERIAN = 10648

COST HAMILTONIAN = 8612

SPANNING TREE HEURISTIC SOLUTION



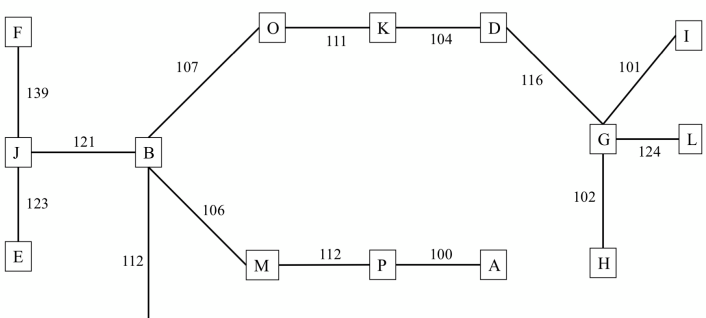
CHRISTOFIDES HEURISTIC

1. **Determine** a Minimum Spanning Tree ST of K_n.
2. Let W be the set of vertices with odd degree in ST. **Find** a Minimum Perfect Matching M in the complete graph K_n over the vertices from M
3. **Combine** the edges of M and ST to form a multigraph H.
4. Form an Eulerian Circuit in H (H is Eulerian because it is connected with only even degree vertices). **Give** H an orientation (arbitrary)
5. Select a node **$i \in$H,** label it and **make p:=i, T:=∅**
6. **If** all nodes are labeled then **make T:=T∪{(i,p)}. END. T is a hamiltonian circuit.**
7. **Otherwise ,** move from p, along H following the orientation, until the first unlabeled node **q. Make T:=T∪{(p,q)}. Label q, make p:=q, Repeat from 4.**

CHRISTOFIDES HEURISTIC (Example)

	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	192	253	115	285	247	143	122	133	229	141	150	275	280	190	100
B		243	281	216	183	212	228	190	121	130	124	106	112	107	144
C			202	181	217	171	205	187	256	210	125	217	120	191	277
D				260	157	116	255	158	200	104	134	214	153	261	129
E					148	210	265	287	123	239	269	180	258	195	177
F						219	299	224	139	267	275	252	219	150	250
G							102	101	245	250	124	188	219	221	185
H								145	142	273	179	234	218	198	253
I									173	245	136	151	193	239	196
J										160	167	258	255	143	291
K											264	230	245	111	183
L												228	217	282	207
M													148	255	112
N														190	113
O															125

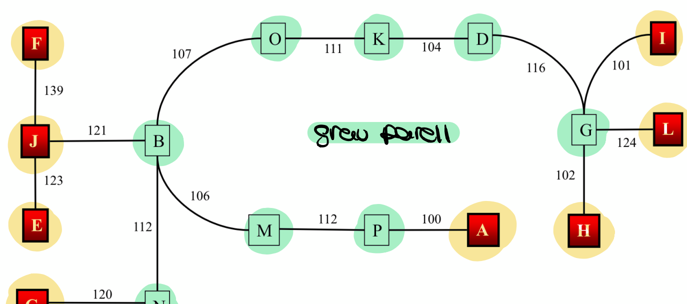
CHRISTOFIDES 1: MINIMUM SPANNING TREE



MST Cost : 1698

CHRISTOFIDES 2: SET OF NODES OF ODD DEGREE

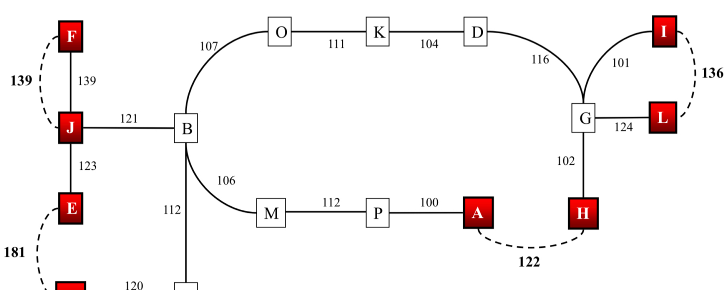
Highlighted in red nodes with odd degree, set W = {A,C,E,F,H,I,J,L} of even cardinality, |W| = 8.



W = {A, C, E, F, H, I, J, L}

CHRISTOFIDES 3: MINIMUM PERFECT MATCHING (MPM)

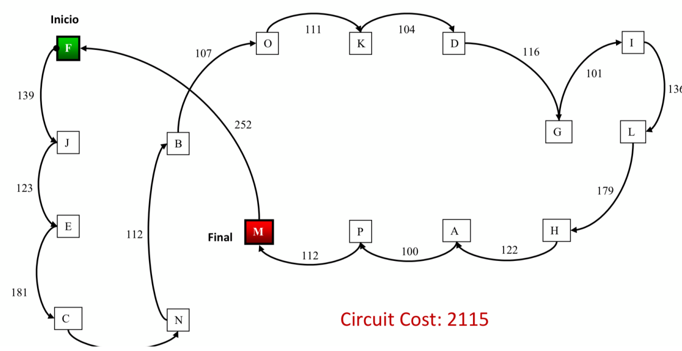
The Minimum Perfect Matching in W is defined by arcs (A,H), (I,L), (F,J) y (E,C) with a cost of 578.
Multigraph H = ST∪M



(A,H)=122 + (I,L)=136 + (F,J)=139 + (E,C)=181 = **578**

CHRISTOFIDES 4: HAMILTONIAN CIRCUIT

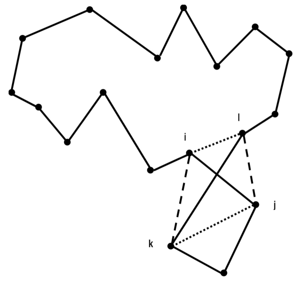
Total Cost:
139+123+181+120+112+107+111+104+116+101+136+179+122+100+112+252 = **2115**.



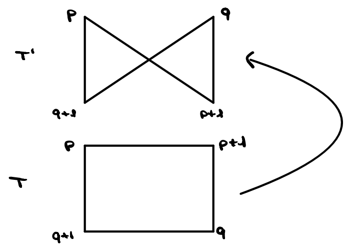
Circuit Cost: 2115

IMPROVEMENT HEURISTICS

- The hamiltonian circuits found by the primal constructive heuristics (discovery heuristics) are, in general, of moderate quality, not enough for certain applications, what raises the question of how can be improved these circuits. A possibility are the improvement heuristics based on exchange procedures on the available hamiltonian circuits.
- Procedure 2-OPT exchange. Is inspired in the following observation for euclidean problems: *If a hamiltonian circuit cuts itself then can be shortened by removing the two arcs that intersect and reconnect the two resulting paths with two non intersecting arcs. This is always possible. The new circuit is shorter.*
- The 2-OPT MOVE procedure consists then on remiving the two arcs and reconnect the two resulting paths in a diferent manner to geta a new circuit. As illustrated in the next slide. In which a better solution is obtained if arcs (i,j) and (k,l) are replaced by (i,k) y (j,l). *Let's remark that there is only one way of reconnecting the paths, since adding arcs (i,l) and (j,k) would result in two subcircuits.*



Diagonals tenen + distància ⇝ pitjor cost.



$c_{ij} \neq k\left((x_i - x_j)^2 + (y_i - y_j)^2\right)^{1/2}$

2-EXCHANGE HEURISTIC

Simètric

- Procedure 2-opt.

  1. **Let** T be the current hamiltonian circuit.
  2. **While** it is still possible for each node i, **do** the following exchange :

     2.1 **Select** a node i

     2.2 **Examine** all 2-OPT MOVE that consider the between i and its successor in the circuit . **If** it is possible to shorten the length of the circuit in this manner, **then select** the best of such 2-OPT MOVE, **otherwise declare** failed the process for node i.
  3. **Repeat** from 1.
- Example:

1. **Let** T = $\{i_1, i_2, \ldots, i_n\}$ **be** the current hamiltonian circuit. **Give** a circulation sense.
2. **Make** Z := $\{ (i_p,i_{p+1}), (i_q,i_{q+1}) \mid p+1 \neq q,\ p \neq q,\ q+1 \neq p,\ 1 \leq p, q \leq n \}$
3. For all pairs of arcs $\{(i_p,i_{p+1}), (i_q,i_{q+1})\} \in Z$ **make**:

$$\textbf{If}\quad c_{pp+1} + c_{qq+1} > c_{i_p i_q} + c_{i_{p+1} i_{q+1}} \quad \textbf{then make:}$$
$$T := \left(T \setminus \{(i_p, i_{p+1}), (i_q, i_{q+1})\}\right) \cup \{(i_p, i_q), (i_{p+1}, i_{q+1})\}$$

4. Go to 2.

IMPROVEMENT HEURISTICS: 2-OPT PROCEDURE

Asimètric

1. Let T be the current Hamiltonian circuit
2. While it is possible for each vertex i, make the next interchange:
2.1 Choose a vertex i
2.2 Evaluate all 2-opt moves setting

$$Z := \{(i_p, i_{p+1}), (i_q, i_{q+1}) \mid p+1 \neq q, p \neq q, q+1 \neq p, 1 \leq p, q \leq n\}$$

2.3 For each pair of edges $\{(i_p, i_{p+1}), (i_q, i_{q+1})\} \in Z$ do:

if $c_{pp+1} + c_{qq+1} > c_{pq} + c_{p+1q+1}$ then:

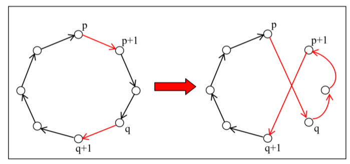
$$T := (T \mid \{(i_p, i_{p+1}), (i_q, i_{q+1})\}) \cup \{(i_p, i_q), (i_{p+1}, i_{q+1})\}$$

2.4 Go to step 2.2

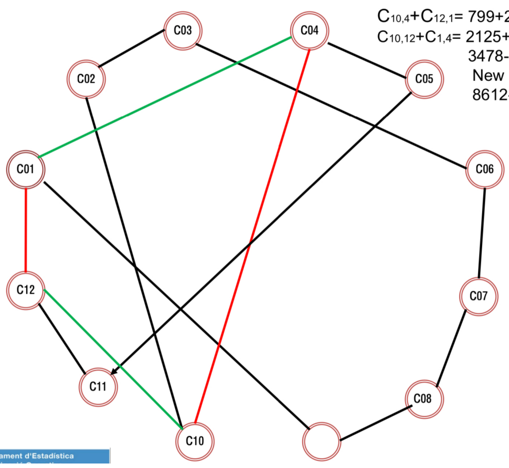
This is the version for symmetrical graphs, so we need to modify this version to work with asymmetrical graphs, where $c_{pq} \neq c_{qp}$, more costs have to be taken into account (in step 2.3) as follows:

$$c_{pp+1} + c_{qq+1} + c_{p+1:q} > c_{pq} + c_{p+1q+1} + c_{q:p+1}$$

Where $c_{p+1:q}$ is the cost to travel from vertex p-1 to vertex q and $c_{q:p+1}$ is the cost to travel from vertex q to vertex q-1



Exchange Heuristics



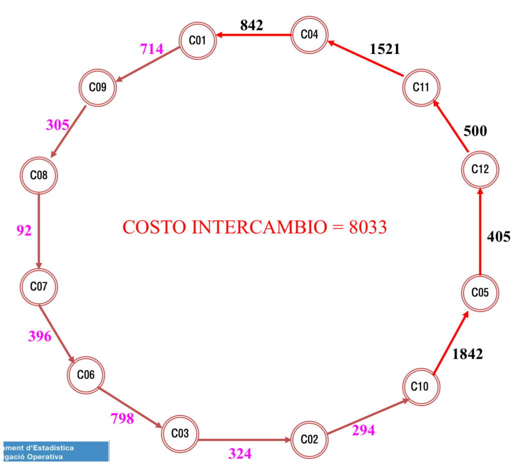
$C_{10,4}$+$C_{12,1}$= 799+2679 = 3478
$C_{10,12}$+$C_{1,4}$= 2125+842 = 2967
3478-2967= 511
New solution
8612-511 = 8101

EVALUATION FOR THE EXCHANGE

INITIAL SOLUTION		
ORIGIN	DESTIN.	COST
01	09	714
09	08	305
08	07	92
07	06	396
06	03	798
03	02	324
02	10	294
10	04	799
04	05	1616
05	11	95
11	12	500
12	01	2679
	TOTAL	8612

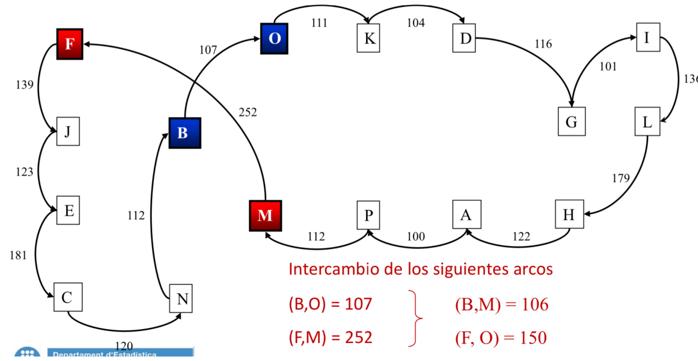
EXCAHNGE IMPROVEMENTS		
ORIGIN	DESTIN.	COST
01	09	714
09	08	305
08	07	92
07	06	396
06	03	798
03	02	324
02	10	294
10	05	1842
05	12	405
12	11	500
11	04	1521
04	01	842
	TOTAL	8033

NEW SOLUTION AFTER EXCHANGE



MEJORA POR PROCEDIMIENTO DE INTERCAMBIO

Para mejorar el costo del circuito por medio de un procedimiento de intercambio, se identifica el arco de mayor coste, en este caso (M,F)=252. Después de estudiar las combinaciones, los nuevos arcos de menor coste son (B,M)=106 y (F,O)=150.

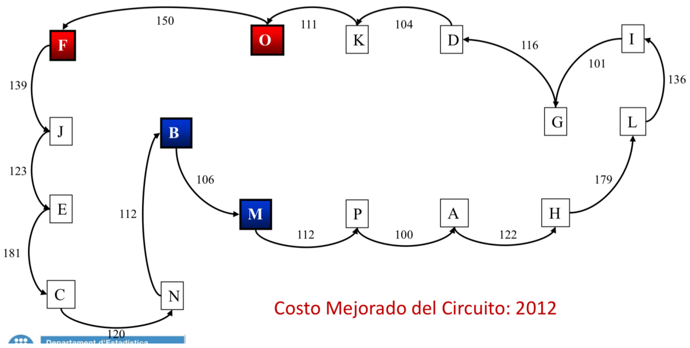


Intercambio de los siguientes arcos

(B,O) = 107 } (B,M) = 106
(F,M) = 252 } (F, O) = 150

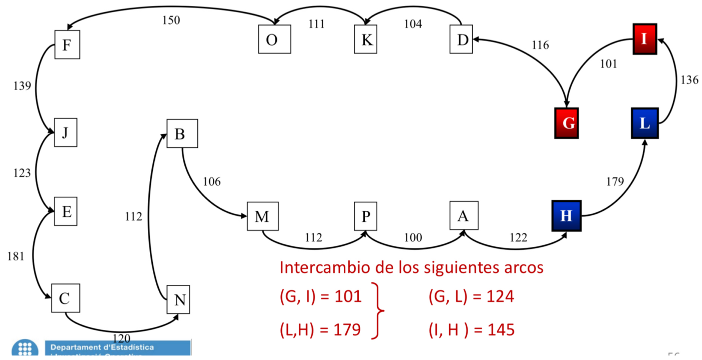
MEJORA POR PROCEDIMIENTO DE INTERCAMBIO

Al cambiar los arcos (B,O)=107 y (M,F)=252 por los nuevos arcos (B,M)=106 y (F,O)=150, el costo del circuito mejora en 103 unidades para quedar con un costo total de 2012.



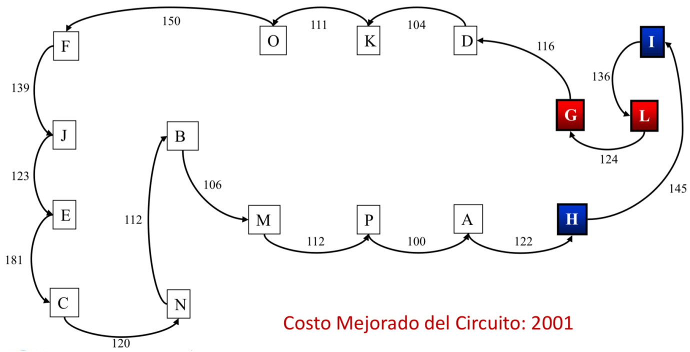
MEJORA POR PROCEDIMIENTO DE INTERCAMBIO

Otro intercambio que se puede implementar es entre los arcos (I,G)=101 y (H,L)=179. Así, tras analizar las combinaciones, los nuevos arcos de menor coste son (G,L)=124 y (H,I)=145.



MEJORA POR PROCEDIMIENTO DE INTERCAMBIO

Al intercambiar los arcos (I,G)=101 y (H,L)=179 por los nuevos arcos (H,I)=145 y (L,G)=124, el costo del circuito mejora en 11 unidades para quedar con un costo total de 2001.



ASYMMETRIC TRAVELING SALESMAN PROBLEM

ASYMMETRIC TRAVELING SALESMAN PROBLEM (PATCHING HEURISTIC)

$$\text{Min} \sum_{(i,j)\in E} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in\delta^+(i)} x_{ji} = 1 \qquad \forall i \in V$$

$$\sum_{j\in\delta^-(i)} x_{ij} = 1 \qquad \forall i \in V$$

$$\sum_{(i,j)\in E(W)} x_{ij} \le |W| - 1 \qquad \forall W \subset V, |W| \ge 3, |W| \le n/2$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in E$$

$$\text{Min} \sum_{(i,j)\in E} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in\delta^+(i)} x_{ji} = 1 \qquad \forall i \in V$$

$$\sum_{j\in\delta^-(i)} x_{ij} = 1 \qquad \forall i \in V$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in E$$

Step 0 (Initialization) Solve the AP (Assignment Problem) Relaxation
If a single circuit is obtained the procedure STOPS. The AP solution is the optimal for the ATSP. Otherwise

Step 1 Let C={C1, C2,….,Cp} be the set of p subcircuits in the AP optimal solution.
Identify the two subcircuits Ch, Ck with the largest number of vertices

Step 2 Merge Ch and Ck in such a way that the cost increase is kept at minimum. Update C and let p:=p-1.
If p=1, STOP, a feasible solution has been obtained.
Otherwise go back to Step 1.

Step 3 Improve the solution applying a 2-Opt exchange

THE HUNGARIAN ALGORITHM FOR THE AP

- Step 1. Find the minimum element in each row of the nxn cost matrix
  Construct a new matrix by subtracting it from each cost in its row
  From this new matrix find the minimum cost in each column
  Construct a new matrix (the reduced costs matrix) by subtracting from each cost the minimum cost in its column
- Step 2. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeroes in the reduced cost matrix. If n lines are required to cover all the zeroes an optimal solution is available among the covered zeroes. If fewer than n lines are required to cover all the zeroes proceed to Step 3.
- Step 3. Find the smallest non zero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in Step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element of the reduced cost matrix that is covered bu two lines. Return to step 2.

HUNGARIAN ALGORITHM: Example (I)

	0	1	2	3	4	5	6	7	Min r
0	∞	5.5	4.2	2.6	2.4	1.3	2.5	4.3	**1.3**
1	4.7	∞	3.7	2.1	5.1	6.0	7.2	9.0	**2.1**
2	4.2	4.5	∞	1.6	3.2	5.5	6.7	8.5	**1.6**
3	2.6	2.9	1.6	∞	3.0	3.9	5.1	6.9	**1.6**
4	3.8	4.1	2.8	1.2	∞	5.1	6.3	8.1	**1.2**
5	3.9	7.4	6.1	4.5	3.3	∞	1.2	3.0	**1.2**
6	3.5	7.0	5.7	4.1	2.9	1.2	∞	2.3	**1.2**
7	5.8	9.3	8.0	6.4	5.7	3.0	2.3	∞	**2.3**

	0	1	2	3	4	5	6	7
0	∞	4.2	2.9	1.3	1.1	0	1.2	3
1	2.6	∞	1.6	0	3.0	3.9	5.1	6.9
2	2.6	2.9	∞	0	1.6	3.9	5.1	6.9
3	1.0	1.3	0	∞	1.4	2.3	3.5	5.3
4	2.6	2.9	1.6	0	∞	3.9	5.1	6.9
5	2.7	6.2	4.9	3.3	2.1	∞	0	1.8
6	2.3	5.8	4.5	2.9	1.7	0	∞	1.1
7	3.5	7.0	5.7	4.1	2.9	0.7	0	∞
Min c	**1.0**	**1.3**	**0**	**0**	**1.1**	**0**	**0**	**1.1**

HUNGARIAN ALGORITHM: Example (II)

	0	1	2	3	4	5	6	7
0	∞	2.9	2.9	1.3	0	0	1.2	1.9
1	1.6	∞	1.6	0	1.9	3.9	5.1	5.8
2	1.6	1.6	∞	0	**0.5**	3.9	5.1	5.8
3	0	0	0	∞	0.3	2.3	3.5	4.2
4	1.6	1.6	1.6	0	∞	3.9	5.1	5.8
5	1.7	4.9	4.9	3.3	1.0	∞	0	0.7
6	1.3	4.5	4.5	2.9	0.6	0	∞	0
7	2.5	5.7	5.7	4.1	1.8	0.7	0	∞

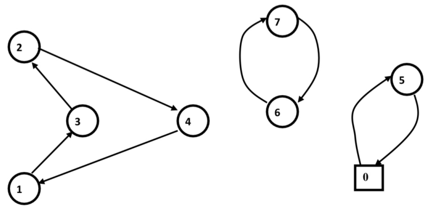
Minimum uncovered element 0.5

	0	1	2	3	4	5	6	7
0	∞	2.9	2.9	1.8	0	0	1.7	1.9
1	1.1	∞	1.1	0	1.4	3.4	5.1	5.3
2	1.1	1.1	∞	0	0	3.4	5.1	5.3
3	0	0	0	∞	0.3	2.3	4.0	4.2
4	1.1	1.1	1.1	0	∞	3.4	5.1	5.3
5	1.2	4.4	4.4	3.3	0.5	∞	0	**0.2**
6	1.3	4.5	4.5	3.4	0.6	0	∞	0
7	2.0	5.2	5.2	4.1	1.3	0.2	0	∞

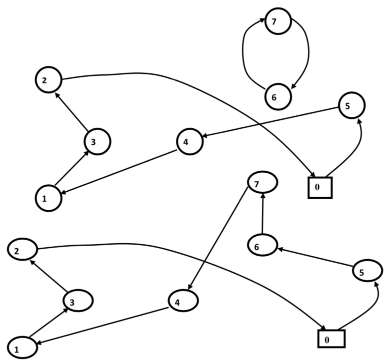
Minum uncovered element 0.2

HUNGARIAN ALGORITHM: Example (III)

	0	1	2	3	4	5	6	7
0	∞	1.8	1.8	1.8	0	0	1.6	1.6
1	0	∞	0	0	1.4	3.4	5.1	5.2
2	0	0	∞	0	**0**	3.2	5.2	5.0
3	0	0	0	∞	1.4	3.4	5.0	5.0
4	0	0	0	0	∞	3.4	5.2	5.2
5	0	3.2	3.2	3.2	0.2	∞	0	**0**
6	0.4	3.5	3.5	3.5	0.7	0.1	∞	0
7	0.8	4.0	4.0	4.0	1.3	0.1	0	∞



HUNGARIAN ALGORITHM: Example (IV)



SMALL EXAMPLE

	A	B	C	D
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	95	115

$$\begin{bmatrix} 90 & 75 & 75 & 80 \\ 35 & 85 & 55 & 65 \\ 125 & 95 & 90 & 105 \\ 45 & 110 & 95 & 115 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ -5 & 45 & 15 & 20 \\ 30 & 0 & -5 & 5 \\ -5 & 60 & 45 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ -5 & 45 & 15 & 20 \\ 30 & 0 & -5 & 5 \\ -5 & 60 & 45 & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix} \sim \begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$