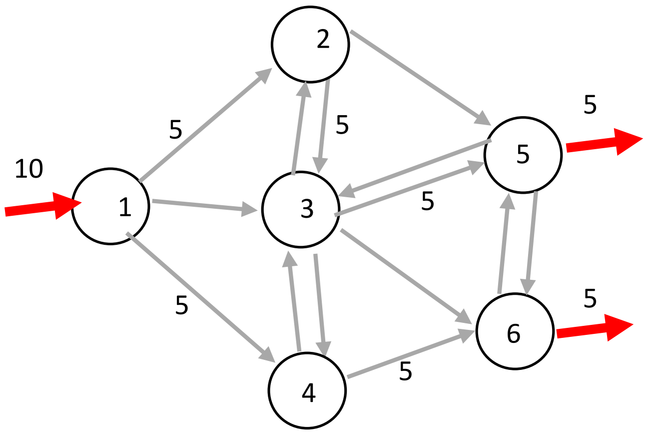
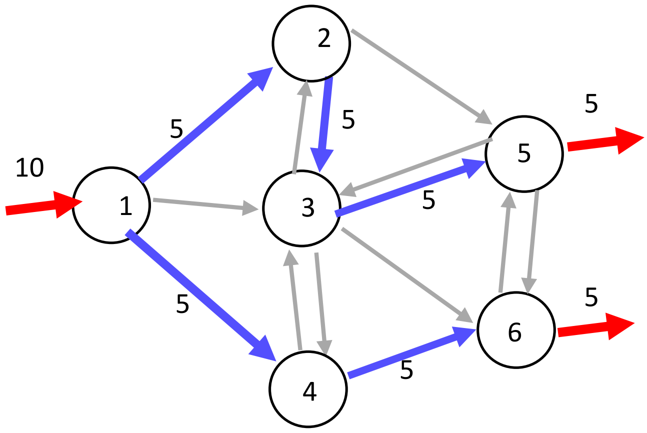
P1.) Consider the following min-cost flow problem with no upper bounds of the link flows and costs cij given by: c_{ij} =1, if i > j and c_{ij} =2, if i < j . Also consider the feasible solution depicted on the figure below.

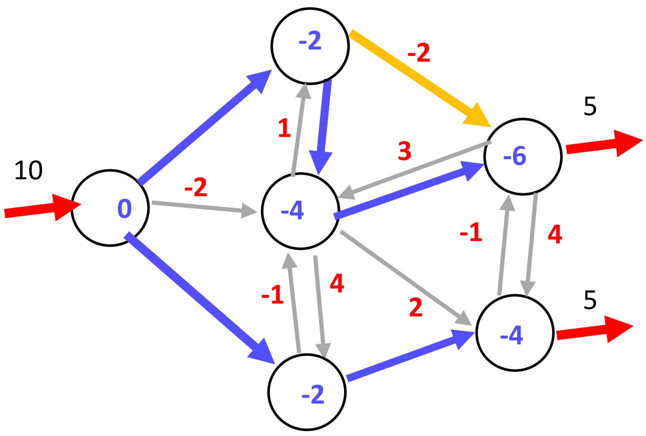
a) Check that it corresponds to a basic feasible solution.
b) Starting from that solution use the simplex algorithm to find a solution of the min-cost flow problem
c) If there exist alternative optimal solutions find one using also the simplex algorithm.



SOLUTION.

A) It is a basic feasible solution since it is obviously feasible. Notice also that there are 5 links with positive cost and that these links are candidates to form a basic index set I_B with no need to complete this set using links with zero flow. When building the associated non-directed graph, the corresponding subgraph is a spanning tree.
B) Obj.f value = 50. I_B = { (1,2), (1,4), (2,3), (3,5), (4,6) }, I_N = the remaining links





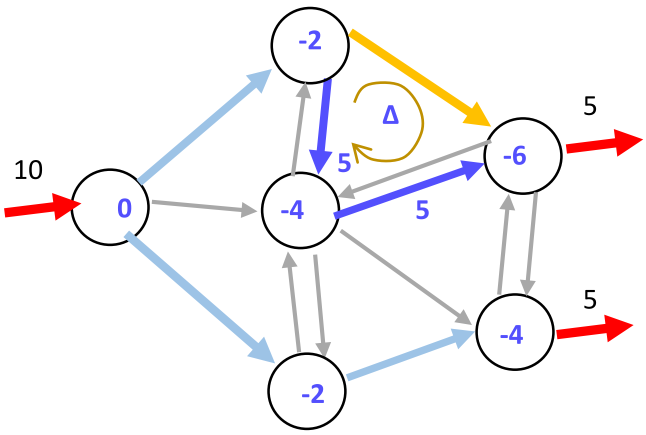
Dual variables λ_i (blue) and reduced costs r_{ij} (red) for the current basis.

Remember that $r_{ij} = c_{ij} - (\lambda_i - \lambda_j)$

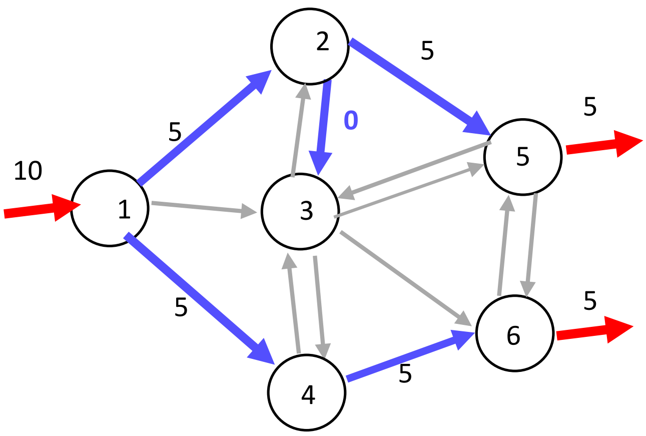
Link (2,5) is chosen to leave I_N.

Link (2,5) is chosen to leave I_N. On the cycle created by links (2,5), (3,5), (2,3) the maximum possible recirculating flow Δ is evaluated. This recirculating flow Δ will be the maximum value that flow $x_{2,5}$ will reach and will be subtracted from the current flows on links (2,3) and (3,5) oriented counterclockwise to the recirculating flow. $\Delta = \text{Min}\{5, 5\} = 5$.

New obj.function value= old value + $r_{2,5} \times (\Delta) = 50 + (-2) \times (5) = 40$

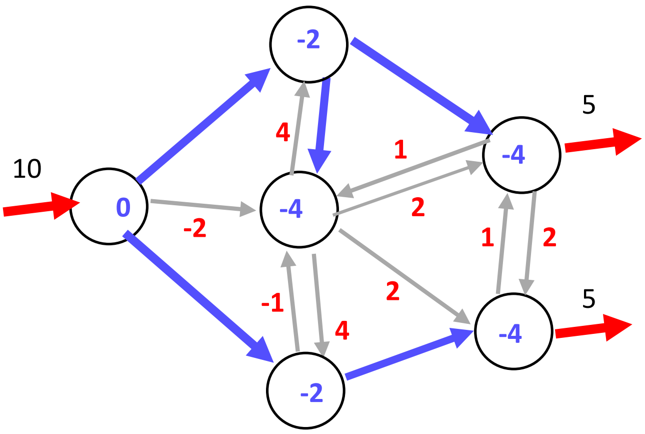


New iteration: starts with New basic feasible solution with $I_B = \{ (1,2), (1,4), (2,5), (3,5), (4,6) \}$



Dual variables λ_i and reduced costs r_{ij} for the current basis.

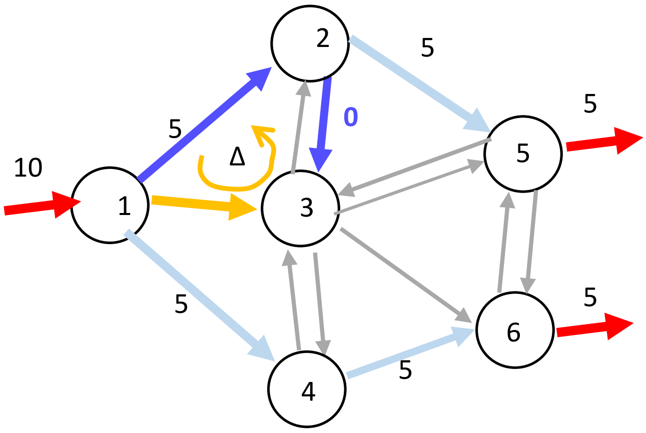
Remember that $r_{ij} = c_{ij} - (\lambda_i - \lambda_j)$

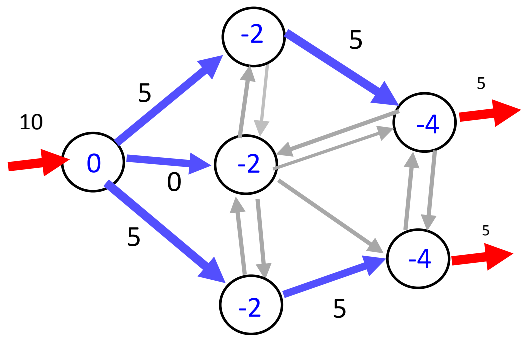


There exist non-basic links with negative reduced cost: links (1,3) and (4,3) have $r_{ij} < 0$;

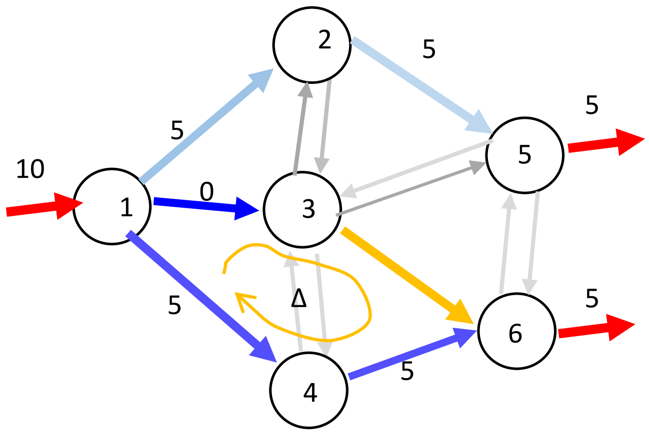
Link (1,3) with the smallest reduced cost is chosen

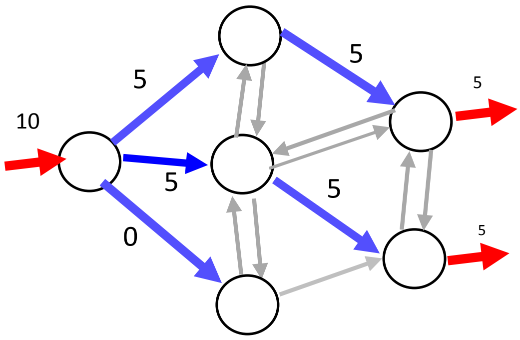
Notice that if (1,3) is chosen no actual change in link flows occur: $\Delta = \text{Min}\{5, 0\} = 0$. (Hence, there is no change in the objective function value). All flows in the cycle remain unaltered. Simply, there is another set of indices corresponding to the same flows:





Notice that links (3,6) and (3,5) are non-basic links with zero reduced cost and all other non-basic links have positive reduced cost → OPTIMAL SOLUTION FOUND with optimal objective function value = 40 (although it is not the only solution). To obtain another optimal basic feasible solution, link (3,6) is chosen to enter the basis.





Δ = min{ 5, 5 } = 5

The vectors of flows is listed accordingly to the emerging links of the nodes, in increasing order.

$x = (x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{32}, x_{34}, x_{35}, x_{36}, x_{43}, x_{46}, x_{53}, x_{56}, x_{65})$

Two optimal basic feasible solutions have been found: x^1 and x^2

$x^1 = (5, 0, 5, 0, 5, 0, 0, 0, 0, 0, 5, 0, 0, 0)$, $\quad x^2 = (5, 5, 0, 0, 5, 0, 0, 0, 5, 0, 0, 0, 0, 0)$

Notice that, for instance a "mixture" "50%, 50%" of x^1 and x^2 results also in a feasible flow:

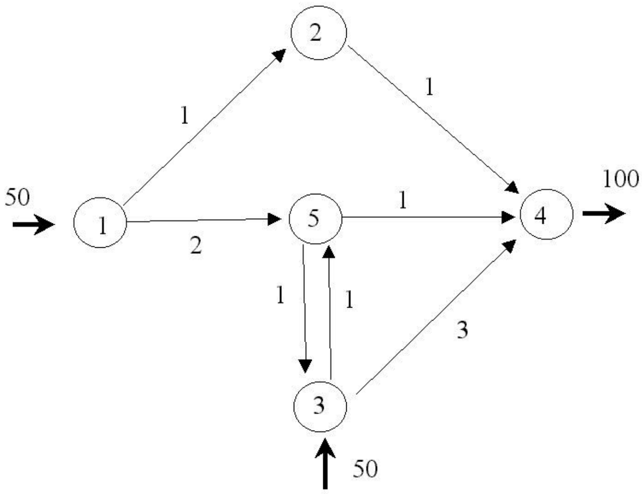
Thus, $x' = 0.5\, x^1 + 0.5x^2 = (5, 2.5, 5, 0, 5, 0, 0, 0, 2.5, 0, 2.5, 0, 0, 0)$. Remember that, in general, if $x^1, x^2, \ldots, x^n$ feasible vector flows had been found, then the "mixture" of these solutions with non-negative weights $\alpha_1 \geq 0$, $\alpha_2 \geq 0, \ldots, \alpha_n \geq 0$ summing up to 1 ($\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$),

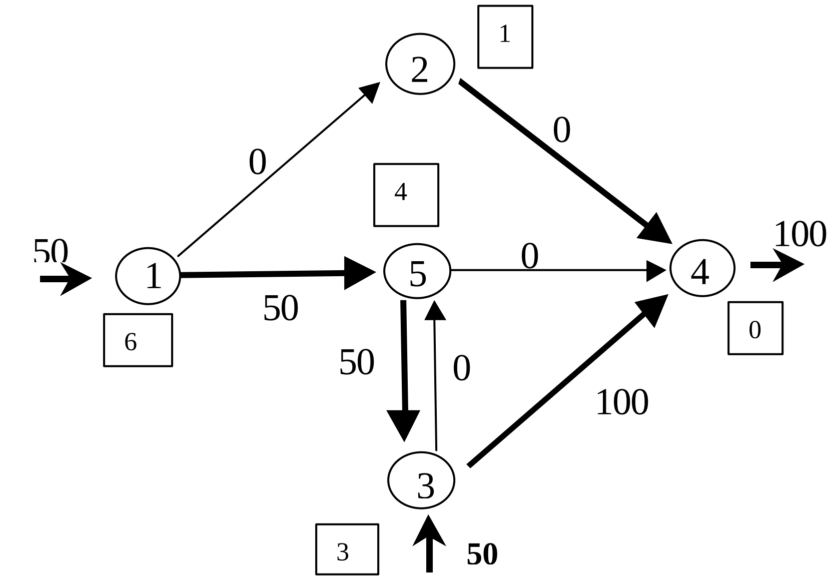
$$x = \alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_n x^n$$

would be also feasible for the problem and that, if $x^1, x^2, \ldots, x^n$ are all them optimal, so will be x.

P2) The transportation network of a distribution company is described by the picture below. Freight is shipped from nodes 1 and 3 whereas main warehouses are located at nodes 2 and 4. For a given period the number of tons shipped from each shipping node is 50 and the transportation costs per ton appear on each network link.

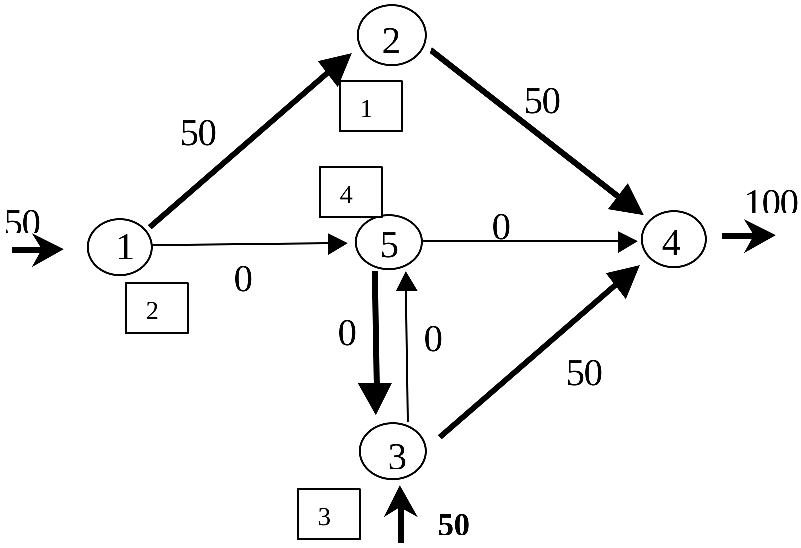
1. Starting from the basic feasible solution made up by $I_B = \{(1,5),(5,3),(3,4),(2,4)\}$ use the simplex algorithm to determine the minumum total transportation cost. Which connections would result useless?.
2. A new transportation mode would have per-unit transportation costs of $3/2$, independently of the link length and other physical characteristics. Which are the nodes i, j on which this new connection should operate?. This service must be rented. Which would be the maximum rental cost per period of this new transportation mode?
3. For a certain period, nodes 1 and 3 have passed to receive shipments of up to 75 tonnes each, while node 4 continues admitting only 100. Knowing that the cost per ton not sent (canceled) in node 1 is 2 units and node 3 is $3/2$. It is required: a) to write the formulation of the resulting linear programming problem with the help of a node-link incident matrix and represent the graph associated with this matrix of incidences. b) determine the volumes transmitted in tons on the transport network and that can not be sent, so that the total cost (transmission + cancellation) of the network operation is minimized.



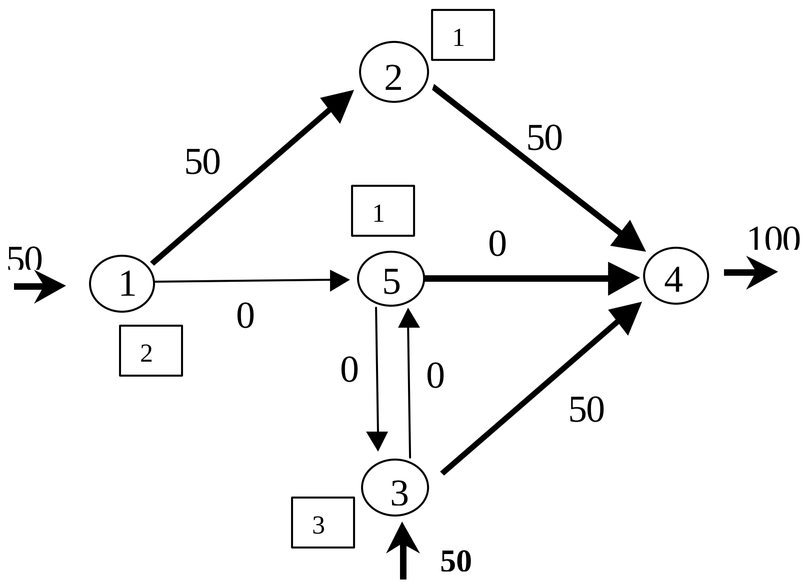


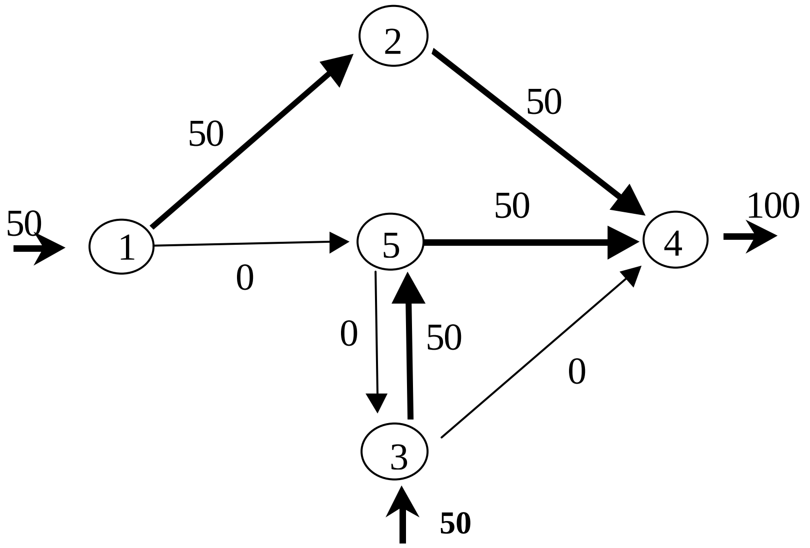
r_{12}=1-(6-1) = -4 <0
Min {50, 50,100}=50
(1,2) exits from I_N,
(1,5) exits from I_B

r_{54}=1-(5-0) = -4 <0
Min {50,0}=0
degenerate pivoting
(5,4) exits from I_N ,
(5,3) exits from I_B



r_{25}=2-(2-1) = 1 > 0
r_{35}=1-(3-1) = -1 < 0
Min {50}=50
(3,5) exits from I_N ,
(3,4) exits from I_B



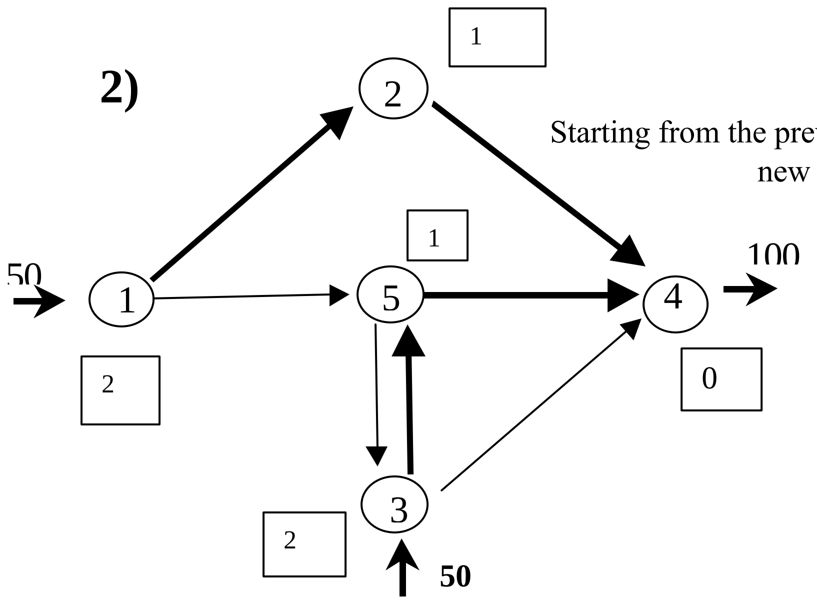


rij >= 0

optimal solution found

transportion cost = 200

2)

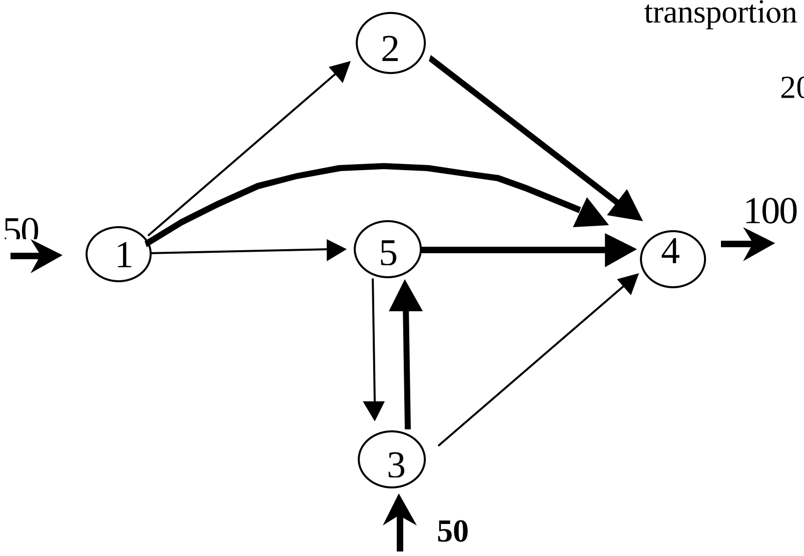


Starting from the previous optimal solution found. The new link must verify: 3/2 - (λi-λj) < 0:

λi > 3/2 + λj

The only possibility is (1,4)

r_{14}=3/2-(2-0) = -1/2 < 0

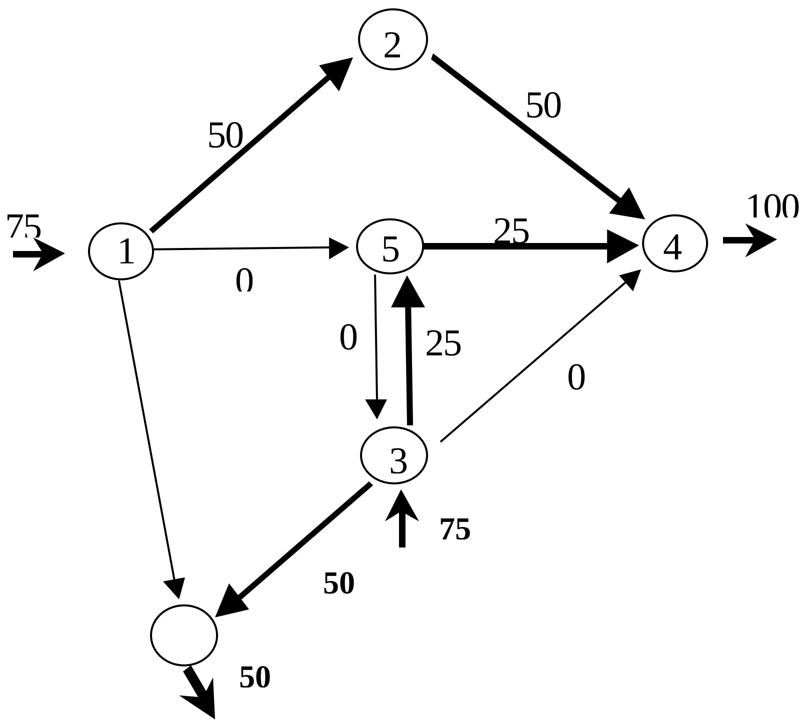


Rij > 0; optimal solution

transportion cost: 3/2 · 50 + 50 · 1+ 50 · 1= 175

Previous transportation cost 200;

200 -175= 25 → it is possible to rent 25 units for the new line

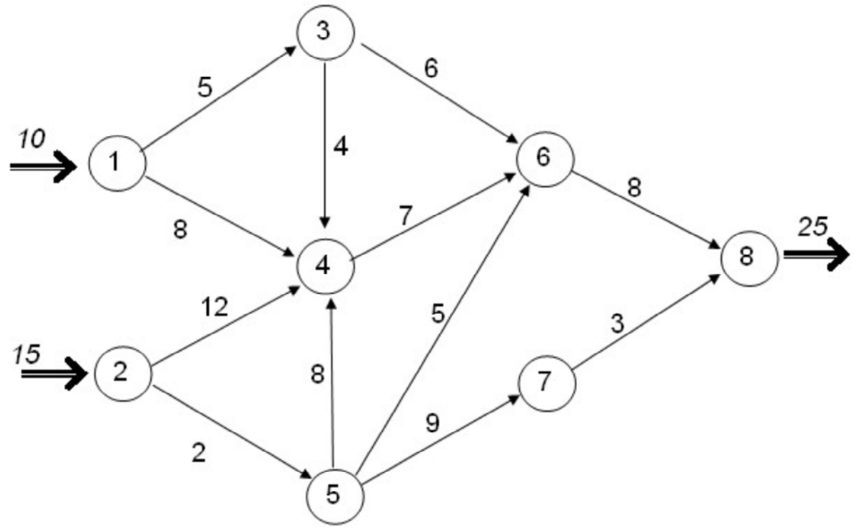


3)

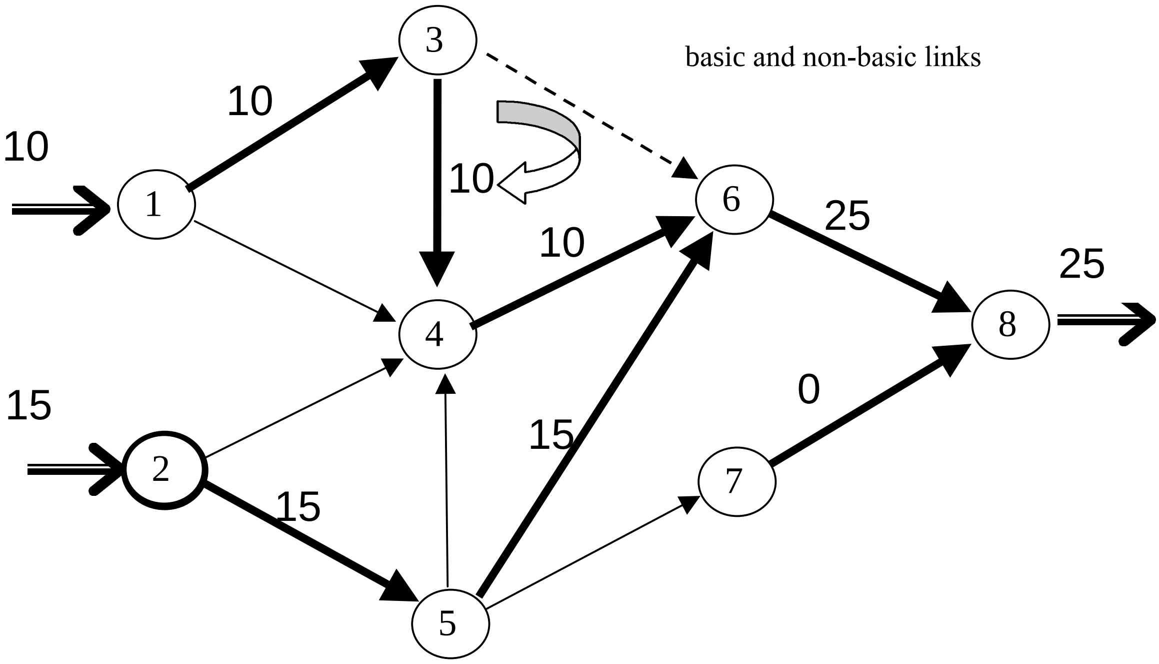
r_{ij} >=0;
optimal
slution found

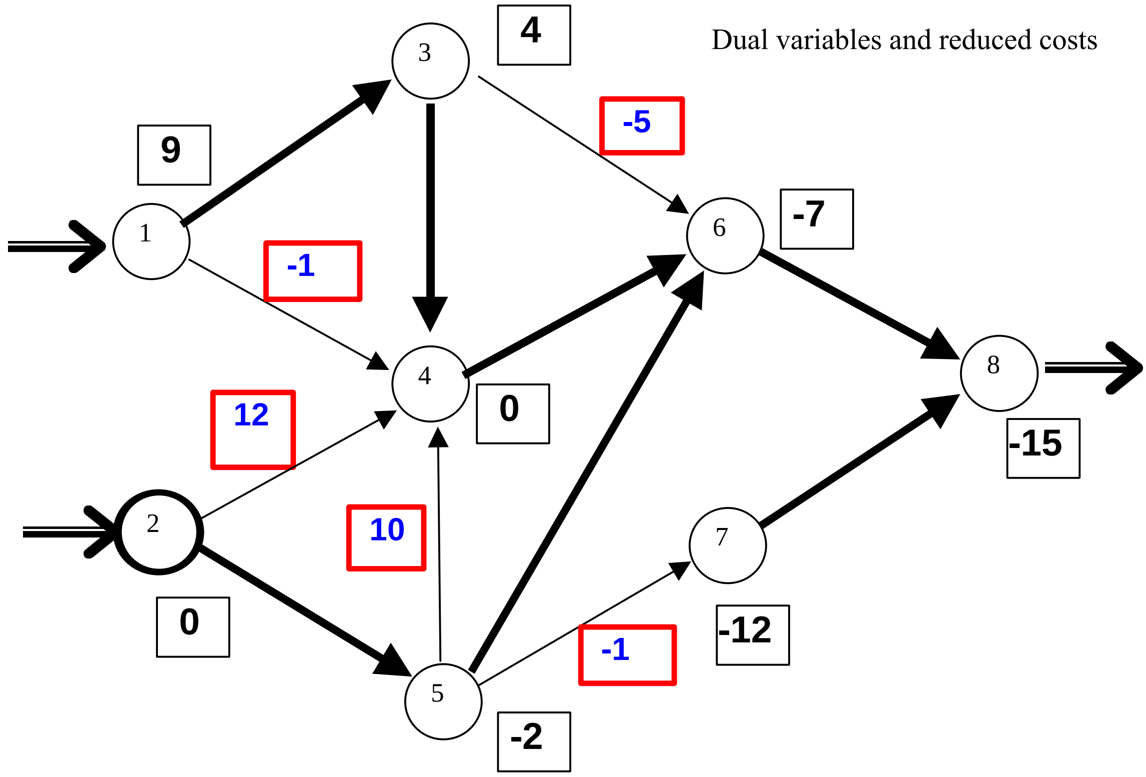
P3) The transport network of a company's products is illustrated by the graph below where unit transportation costs are given on the same figure:

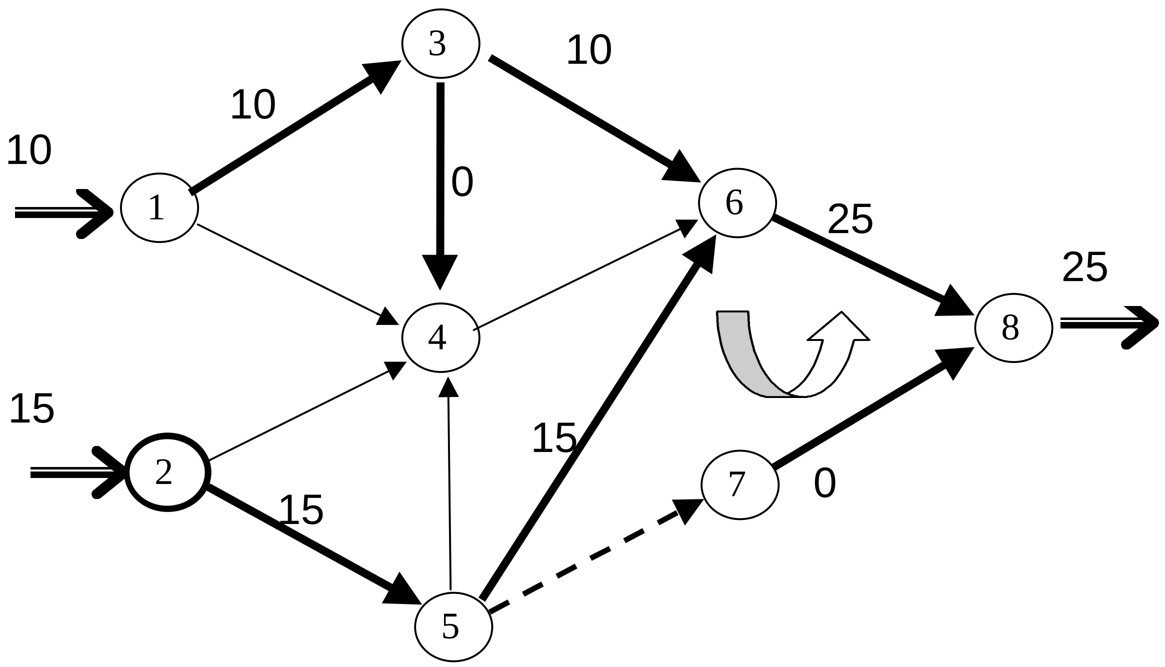
1. Starting from the feasible basic set given by the links (1; 3), (3; 4), (4, 6), (6, 8), (2, 5), (5, 6), (7, 8), determine the optimal distribution of flows so that the total transportation cost is minimized.
2. A new link joining directly nodes 1 to 7 is considered to be put in operation. Which should be the maximum unit transportation cost on that link so that it could be profitable using it.
3. Finally a transportaton cost of 15 is fixed for the link (1,7). How will flows distribute once it is in operation?



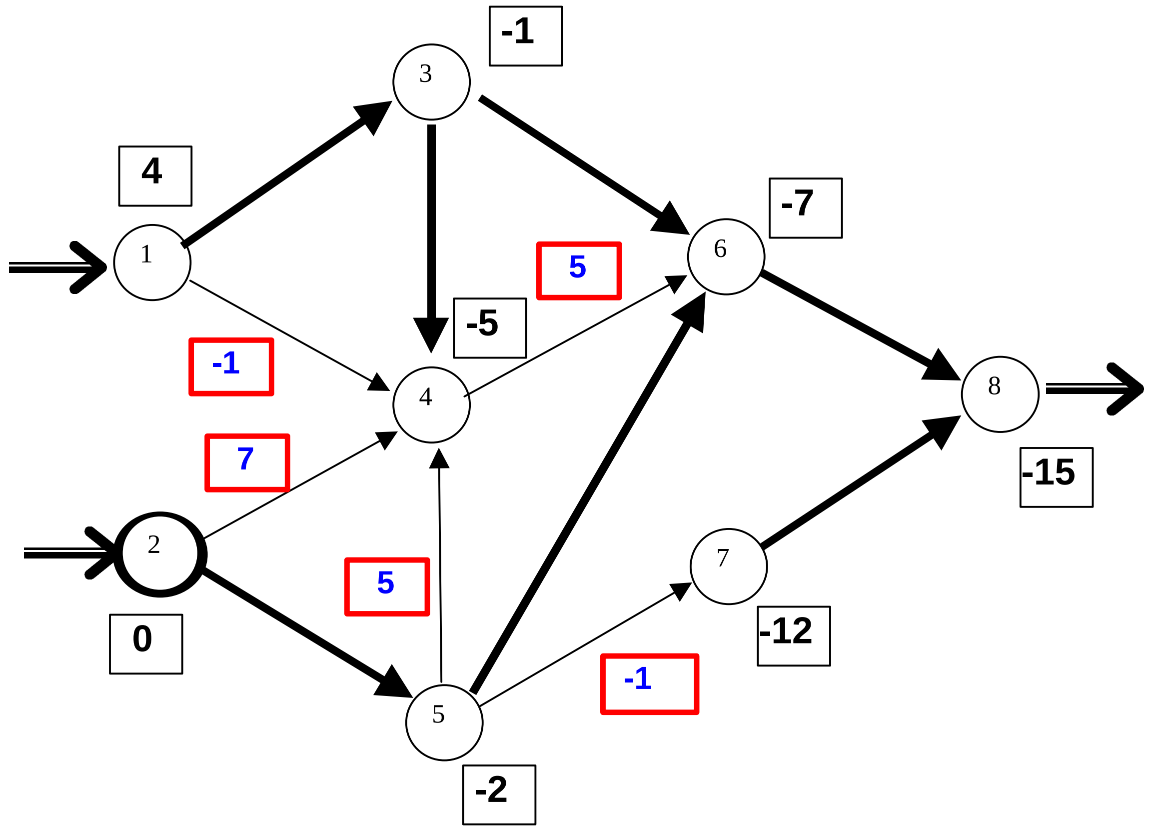
Problem 4. Solution



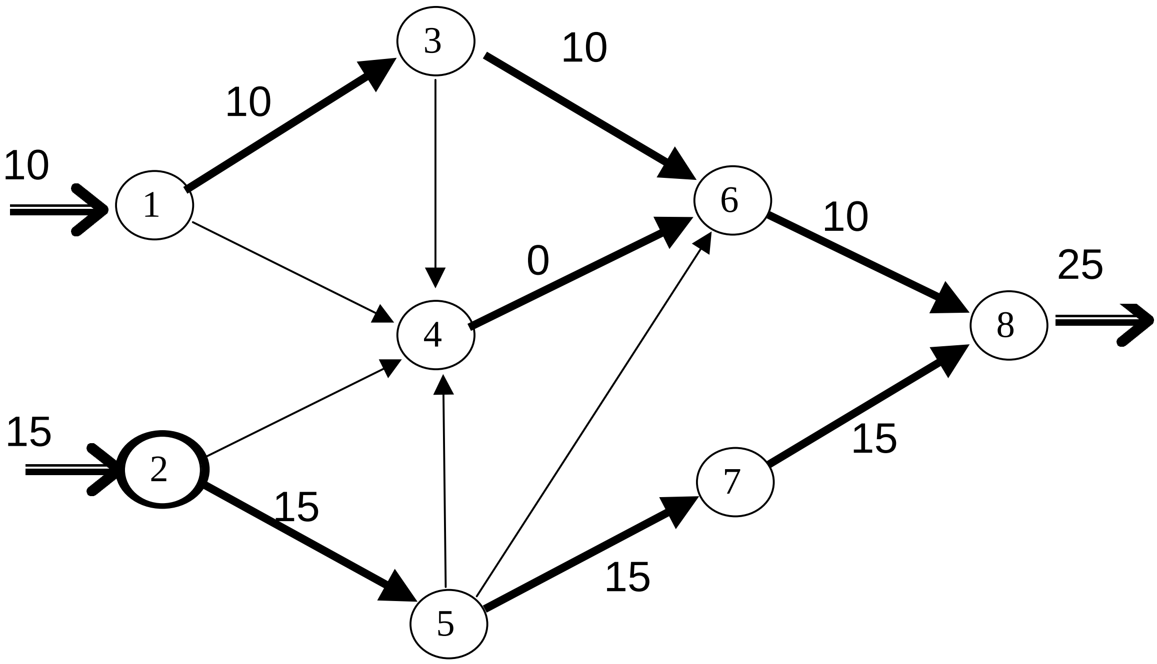


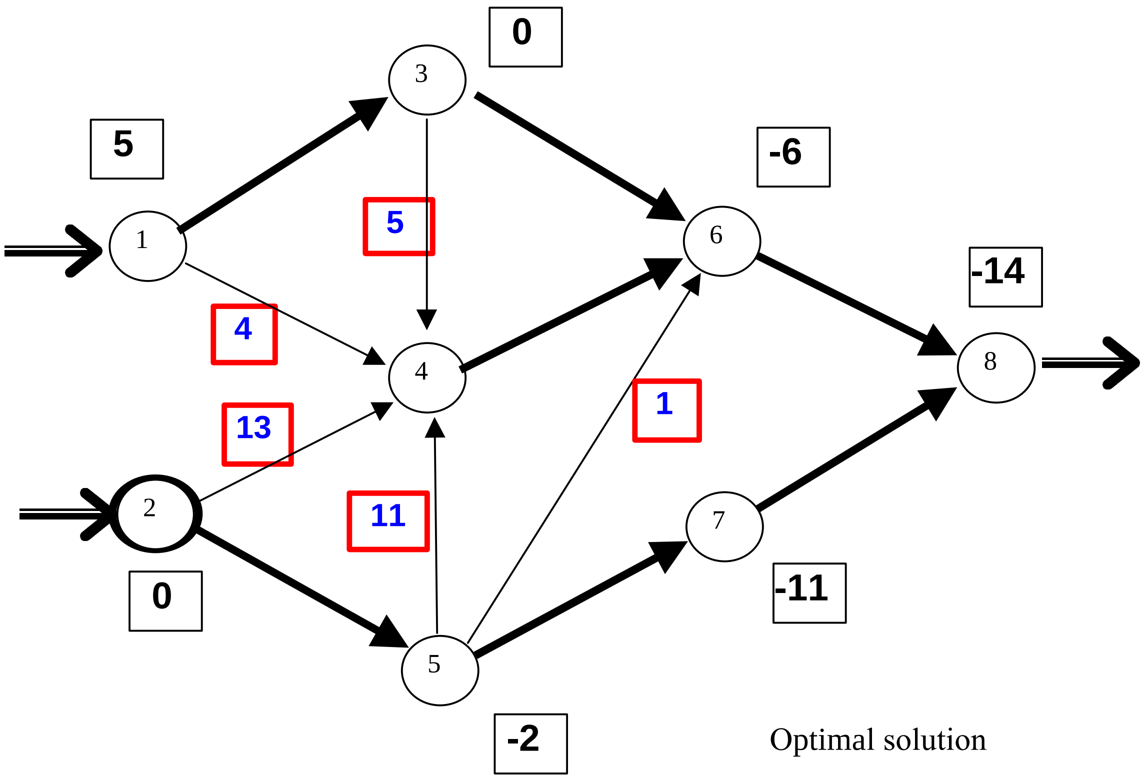


10
10
10
0
25
25
15
15
15
0
1
2
3
4
5
6
7
8



-1
4
-7
5
-5
-1
7
-15
5
0
-12
-1
-2
1
2
3
4
5
6
7
8





Optimal solution

2) Using dual variables for the optimal solution, the maximum unit transportation cost for link (1, 7) must be 16.
3) If the cost is 15, then the new arc will enter the basis. The optimal basic solution obtained will be degenerate:

$$x_{14} = x_{54} = x_{68} = x_{36} = x_{15} = x_{56} = x_{46} = x_{24} = x_{14} = 0,\ x_{17} = 10,\ x_{25} = 15,\ x_{57} = 15,\ x_{78} = 25.$$

P4) Using the road network schematically shown in the figure below, a company needs to transport 100 tones of their products from node 1 to node 4. Unit transportation costs are shown on the network links.

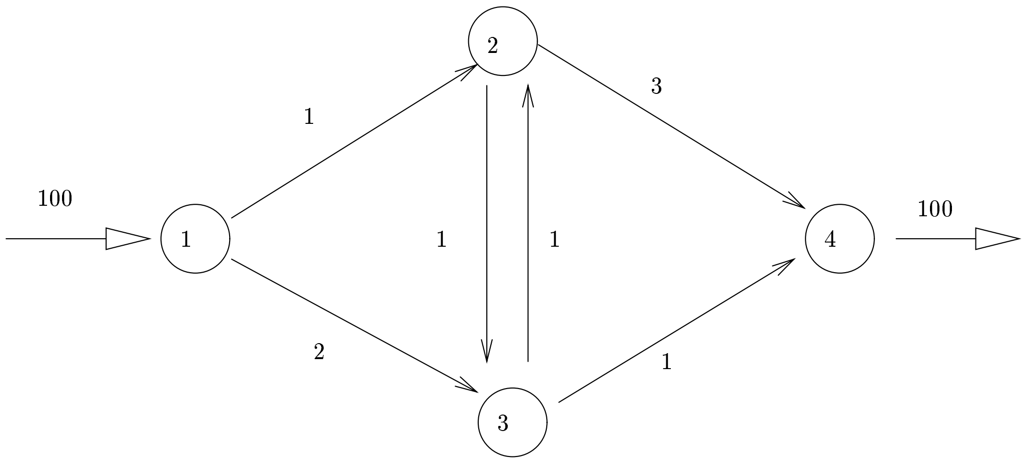
1. Formulate a min-cost flow problem so that the overall transportation costs are minimized using the following order of network links:

1	2	3	4	5	6
$(1,2)$	$(1,3)$	$(2,4)$	$(3,4)$	$(2,3)$	$(3,2)$

2. Starting from the feasible flows determined by the basic set $I_B = \{(1,2),(1,3),(2,4)\}$ carry out a complete iteration of the simplex algorithm. Determine clearly which are the dual variables at the beginning and at the end of the iteration, as well as the basic index set I_B, the reduced costs and the objective function values.

3. For the optimal basis $I_B = \{(1,3),(3,4),(1,2)\}$, calculate the range of values for the unit cost of link (1,3), $c_{1,3}$, so that the basis $I_B = \{(1,3),(3,4),(1,2)\}$ remains unchanged. And for the unit cost $c_{2,3}$ of link (2,3).?

4. A new road is planned to open joining nodes 1 to 4 with a unit transportation cost of $c_{1,4} = 2$. How would this affect to the distribution of optimal flows? justify the answer using arguments based on the simplex algorithm.

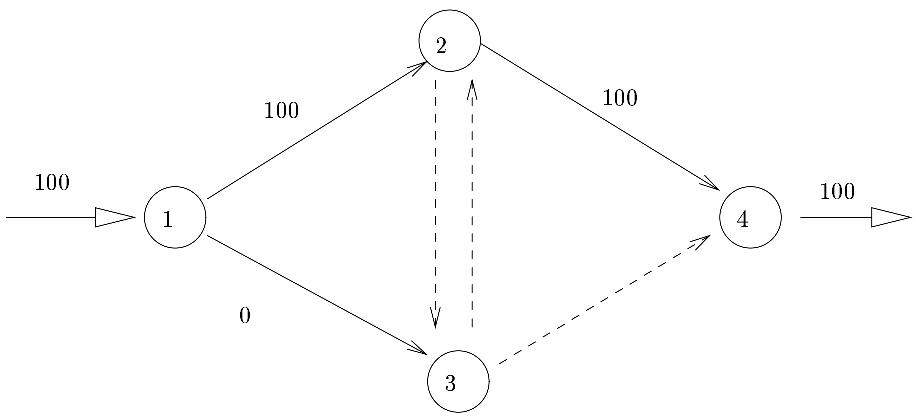


P4) Solution

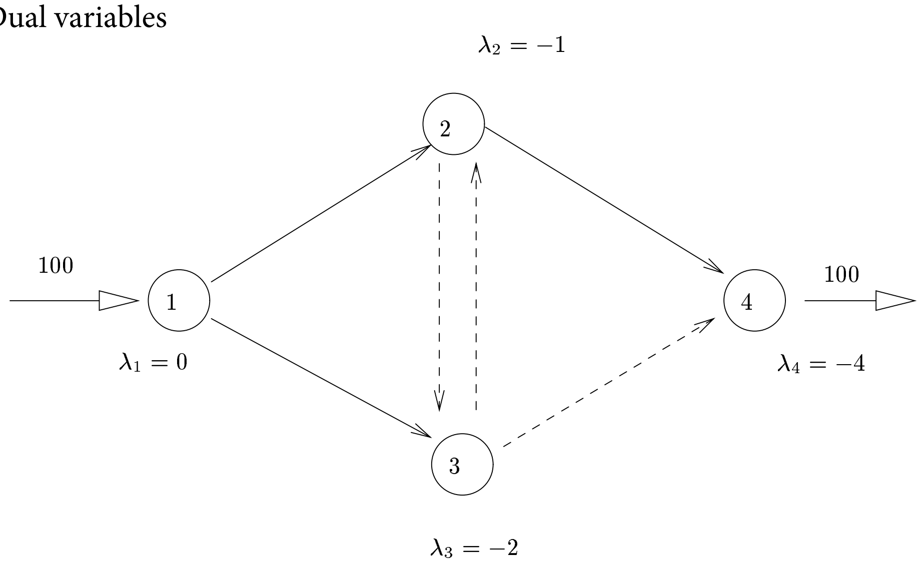
$$\begin{array}{ll} Min & c^\top x \\ & Dx = p \ , \ c = \begin{pmatrix} c_1 \\ \vdots \\ c_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ p = \begin{pmatrix} p_1 \\ \vdots \\ p_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ -100 \end{pmatrix} \\ & x \geq 0 \end{array}$$

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

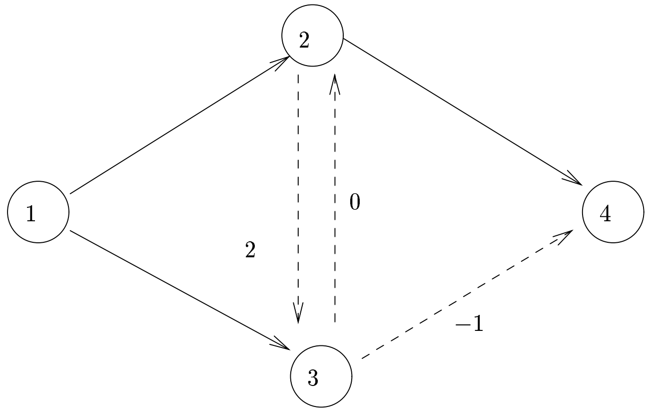
Initial basic feasible solution



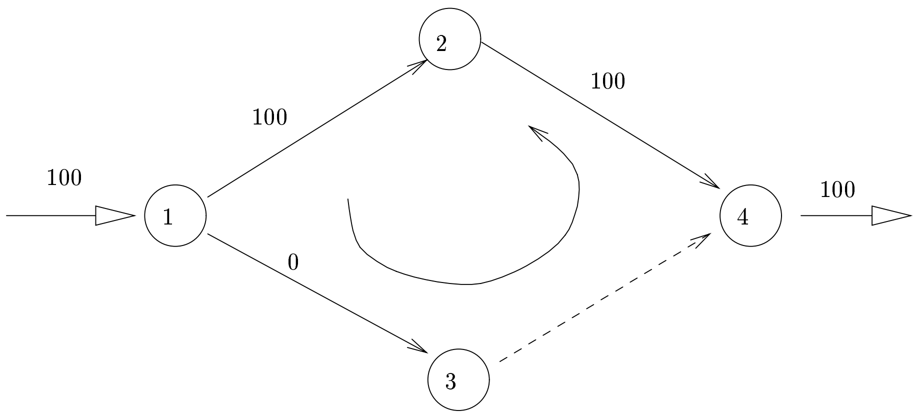
Dual variables



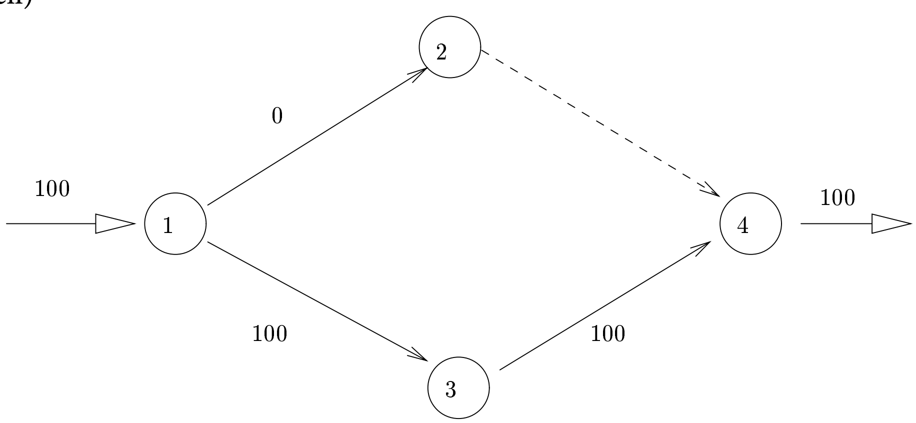
reduced costs $r_{ij} = c_{ij}$ - (lambda$_i$ - lambda$_j$)



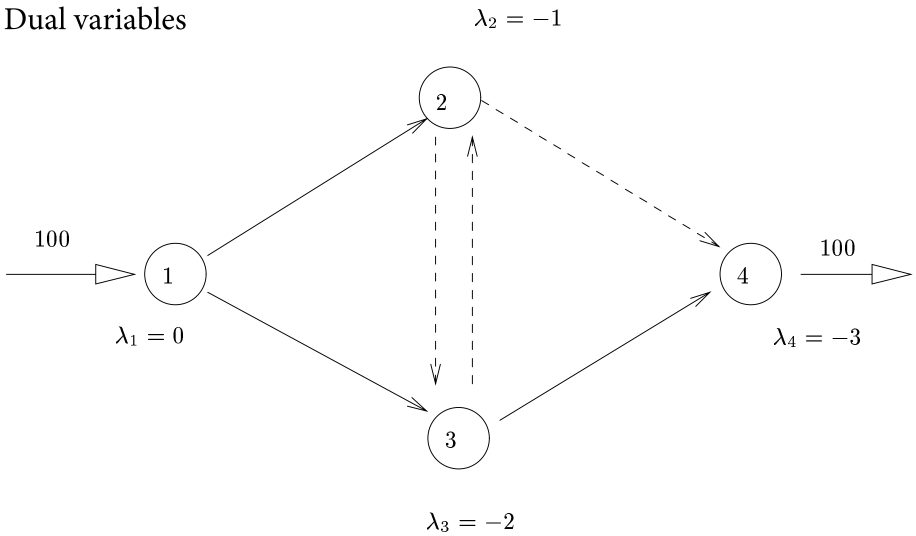
Link (3,4) enters in I_B; Min{100,100}=100



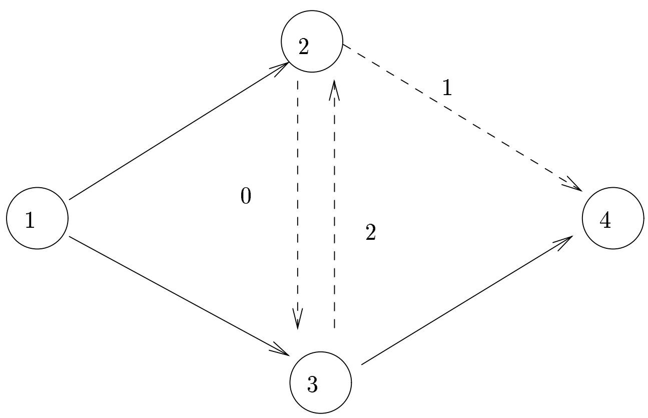
New basic feasible solution (Link (2,4) exited from I_B, although (1,2) could have exited as well)



Dual variables

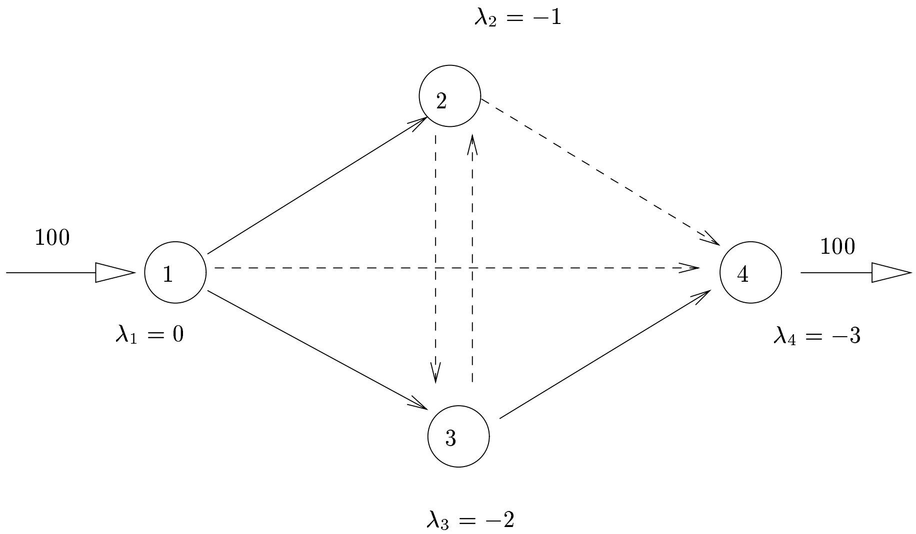


Optimal solution; there exist alternative optima: clearly IB ={1,5,4} is also optimal

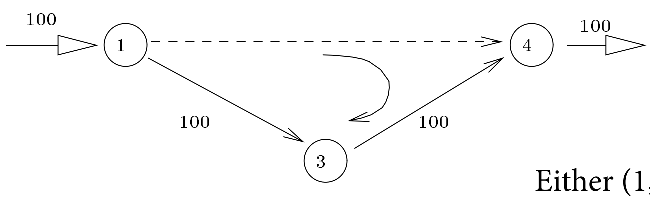


3) Check that it is $0 <= c_{13} < 2$. For link (2,3)
(non-basic) it is $c_{13} >= 1$

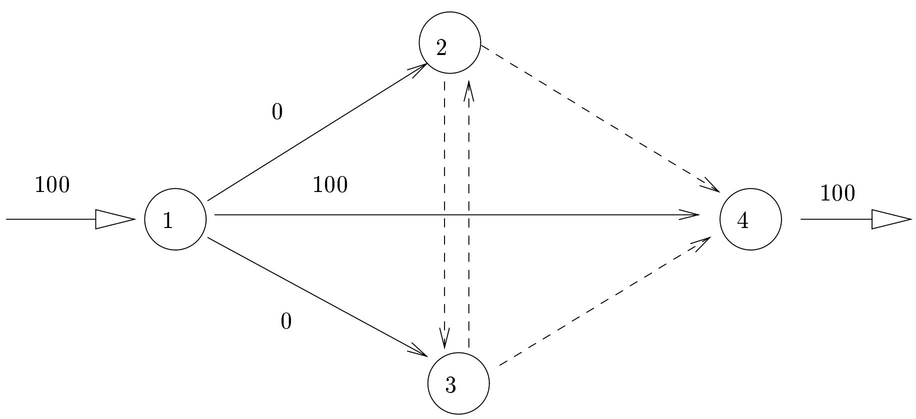
Solution obtained in 2). New link (1,4) with reduced cost r34 = 2- (0+3) = -1. It will enter the basis



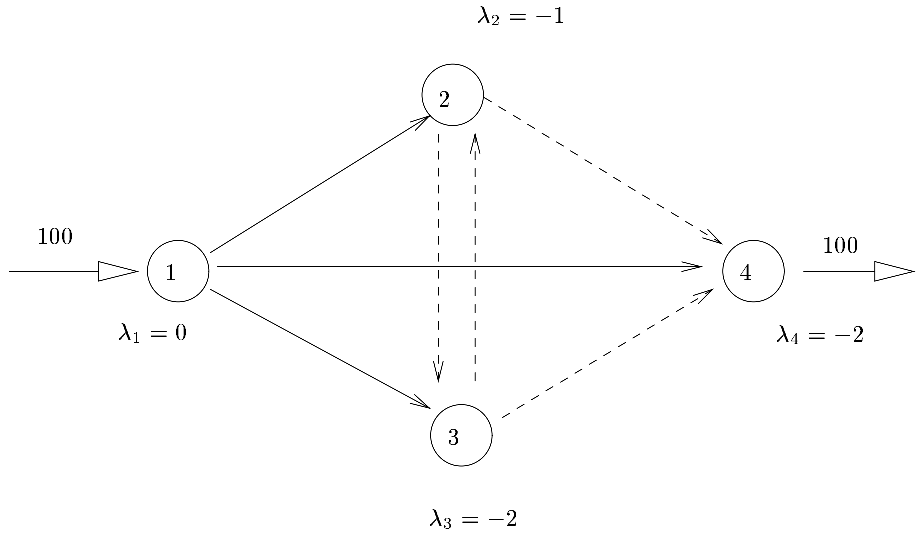
$\hat{x}_{1,4} \;=\; Min\{100, 100\} \;=\; 100$



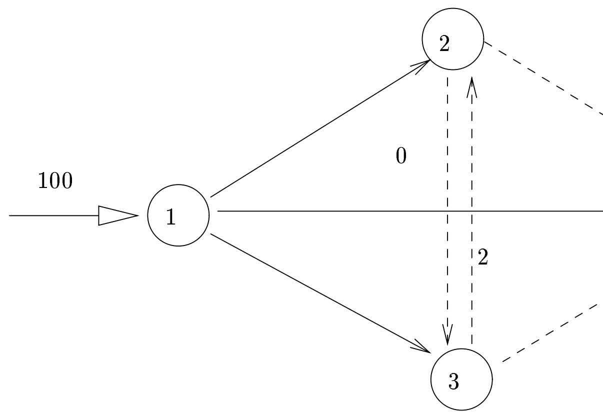
Either (1,3) or (3,4) may exit. New basic feasible solution if (3,4) exists the basis.



Dual variables



Reduced costs and new optimal solution

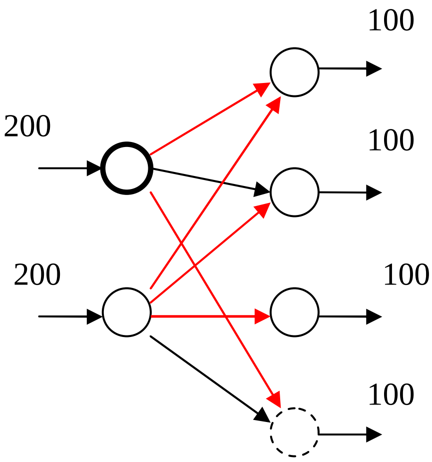


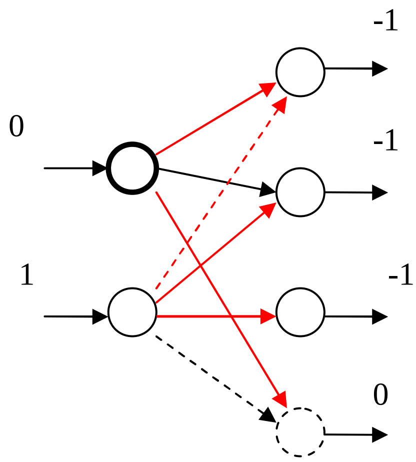
P5) A company has two production centers that deliver their products to three cities. The maximum amount that each of the production centres may generate is 200 units. The demand of each city is only 10 units. The unit transportation costs have the following value:

	1	2	3
1	1	1	X
2	2	2	2

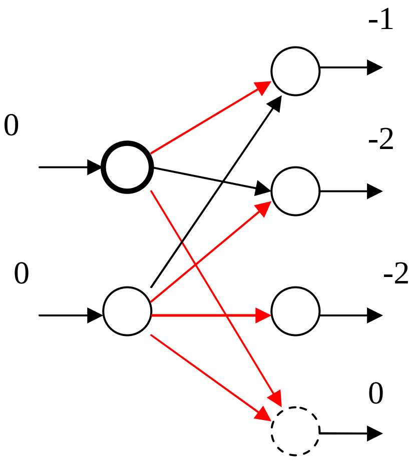
Staring from the basic set $I_B = \{(1,1),(1,0),(2,1),(2,2),(2,3)\}$, find the distribution of flows that minimizes the total transportation costs of the company. (Hint: use an additional node 0 to absorb the excess of production)

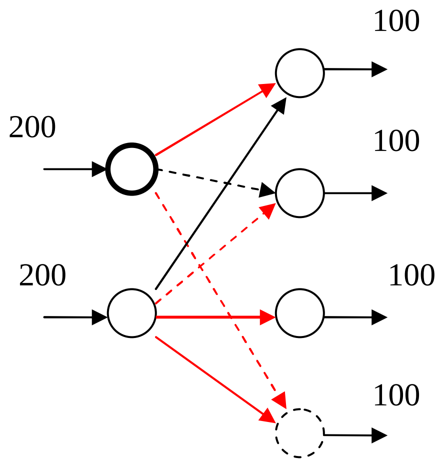
Solution:
P5



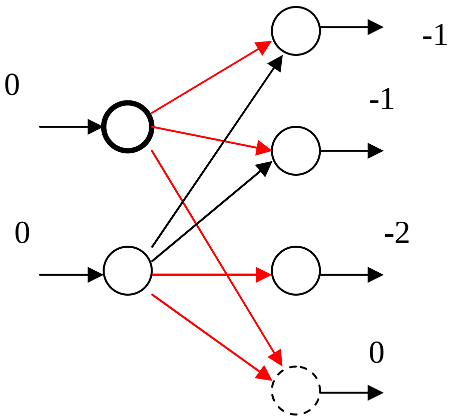


r_{12} =1- (0+1) = 0; r_{20} = 0 – (1-0) = -1 <0. Link (2,0) enters the basis. A cycle is formed (2,0), (1,0), (2,1), (1,1). Link (2,1). exits the network x_{20} = Min {100, 0}=0

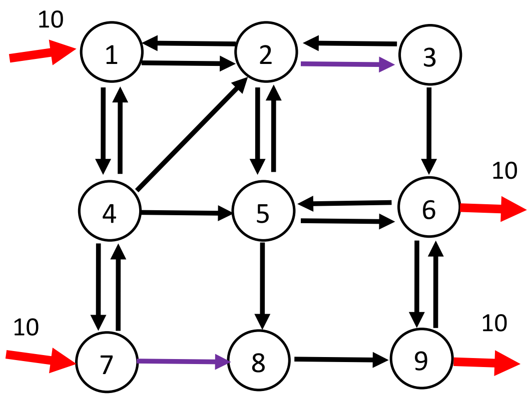




r_{21} = 2 – (0+1) = 1 >0 ; r_{12}= 1 – (0+2) = -1 <0. Link (1,2) enters the basis. A cycle is formed (1,2), (1,0), (2,2), (2,0). x_{12} = Min{100, 100} = 100. Either link (2,2) or link (1,0) may exit the basis. Link (2,2) is arbitrarily chose



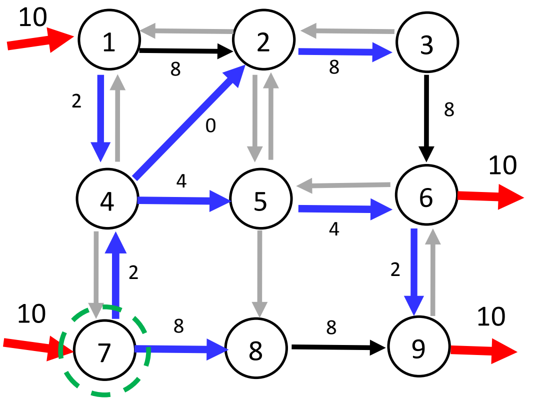
r_{21} = 2 – (0+1) = 1 >0
r_{22} = 2 – (0+1) = 1 >0 Optimal solution.



P6) Consider the network in the figure with a capacity limit at each link of u_{ij} = 8 and all costs of transportation c_{ij} =1, excluding links (2,3) and (7,8) for which these costs are 3. Find the distribution of flows on the network that minimizes the total transportation cost, using the SIMPLEX algorithm. Start with the feasible solution determined by the sets :

I_B = {(1,4), (6,9), (4,5), (7,4), (5,6), (7,8), (4,2), (2,3)}

I_{N+} = {(1,2), (3,6), (8,9)}; I_{N-} = the remaining links.



I_B = {(1,4), (6,9), (4,5), (7,4), (5,6), (7,8), (4,2), (2,3)}

I_{N+} = {(1,2), (3,6), (8,9)}; I_{N-} = the remaining links.

As the basic sets are specified, the only possible distribution of link flows on the graph is shown. With it, the corresponding objective function value can be calculated.

f.obj = 86

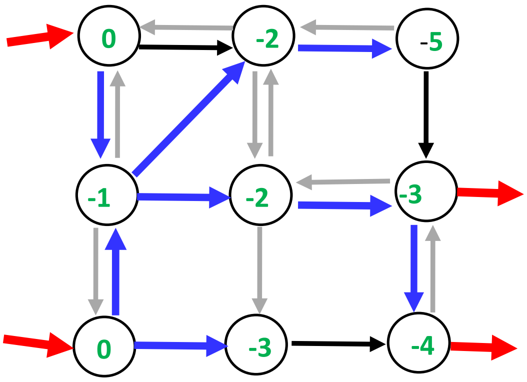
Also, the corresponding dual variables (in green in the 2nd figure) and reduced costs r_{ij} can calculated:

r_{12} = 1-(0+2) = -1, r_{21} = 1-(-2) = -3, r_{32} = 1-(-5+2) = 4

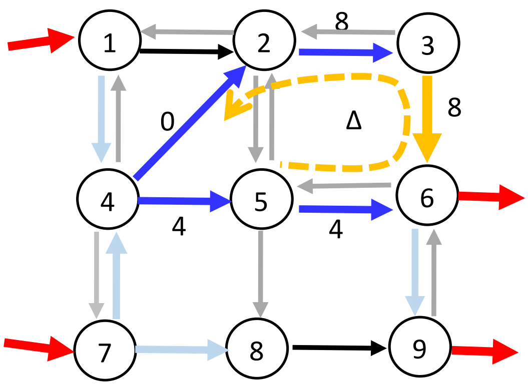
r_{36} = 1-(-5+3) = 3, r_{52} = 1-(-2+2) = 1, r_{25} = 1,

r_{41} = 1-(-1-0) = 2, r_{47} = 1-(-1+0) = 2, r_{58} = 1-(-2+3) = 0

r_{96} = 1-(-4+3) = 2



Notice that, since (3,6) ∊ I_{N+} and its reduced cost is positive, then link (3,6) is a candidate to leave I_{N+} for an interchange (either with I_{N-} or I_B). As link (3,6) is at capacity, the only possibility is to decrease its flow:



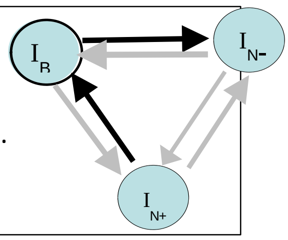
The recirculating flow Δ will be taken into account following the anticlockwise sense marked in the figure (in opposition to the link (3,6)). The maximum value of Δ will be determined by variations of flow in the links comprised by the cycle formed when (3,6) is considered.

Δ = Min { 8, 8, 0, 4, 4 } = 0

This implies no actual change in flows and only in the index sets. There are two possibilities:

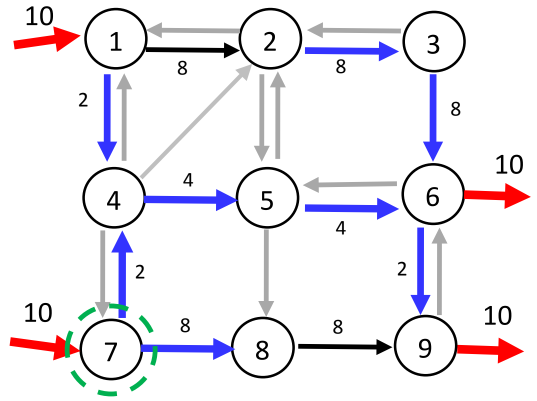
(a) (4,2) exits from I_B and enters in I_{N-} , then (3,6) enters I_B

(b) (2,3) exits from I_B and enters in I_{N+} , then (3,6) enters I_B



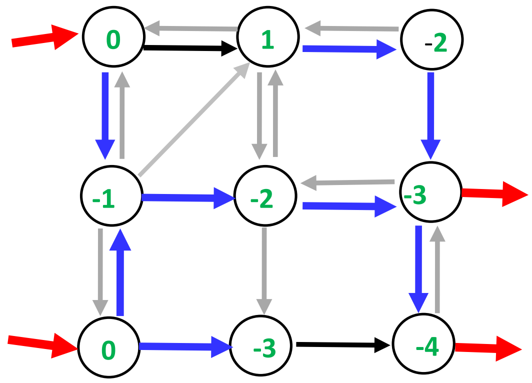
Anyone of these two options is valid and will lead to a solution of the problem.

Option (a) is chosen



I_B = {(1,4), (6,9), (4,5), (7,4), (5,6), (7,8), (2,3), (3,6)}

I_{N+} = {(1,2), (8,9)}; I_{N-} = the remaining links (this includes (4,2))



No actual in link flows occurs. The corresponding objective function value will be the same:

f.obj = 86

Also, the corresponding dual variables (in green in the 2nd figure) and reduced costs r_{ij} can calculated:

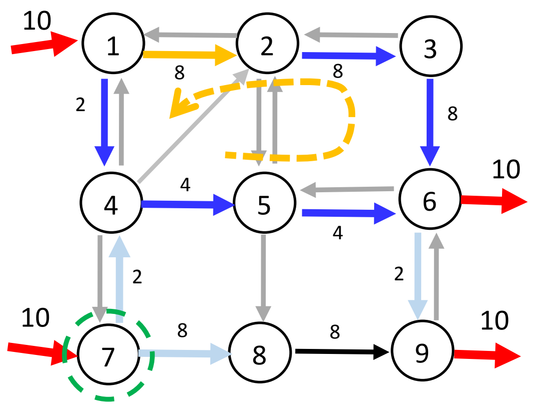
r_{12} = 1-(0-1) = 2, r_{21} = 1-(1-0) = 0, r_{32} = 1-(-5+2) = 4

r_{52} = 1-(-2-1) = 4, r_{25} = 1 – (1+2) = -2,

r_{42} = 1-(-1+1) = 1, r_{41} = 1-(-1+0) = 2, r_{58} = 1-(-2+3) = 0

r_{96} = 1-(-4+3) = 2, r_{47} = 1-(-1+0) = 2, r_{89} = 1-(-3+4) = 0

Link (1,2) is chosen.



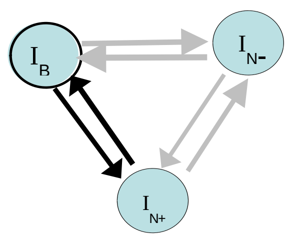
The recirculating flow Δ will be taken into account following the counterclockwise sense marked in the figure, which is in opposition to the link (1,2) forming the cycle. The maximum value of Δ will be determined by variations of flow in the links comprised by the cycle formed when (1,2) is considered.

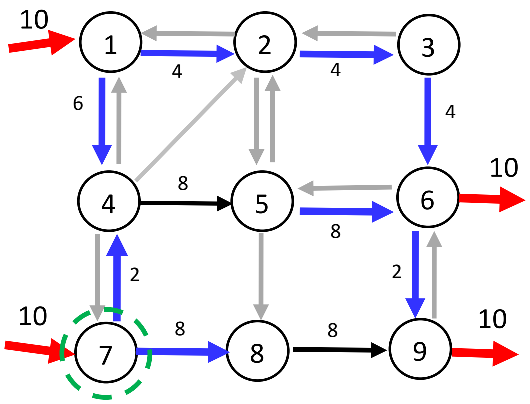
Δ = Min { 8, 6, 4, 4, 8, 8 } = 4

This implies *that flows change*. The new objective function value will be given by:

New obj.F = Old obj.F. + r_{12} (-Δ) = 86 -2 × 4 = 78

Notice that flow on link (1,2) changes from 8 (upper bound) to 4 (neither upper nor lower); this implies that, mandatorily, link (1,2) will enter the set I_B and that links (2,3), (3,6) and (1,4) will remain in I_B. This implies that either (4,5) exits or (5,6) exits from I_B and will enter I_{N+}. Link (4,5) is chosen arbitrarily. Notice that the strong leaving arc rule is not followed (see, "Network Flows", Ahuja, Magnanti, Orlin, p. 423)





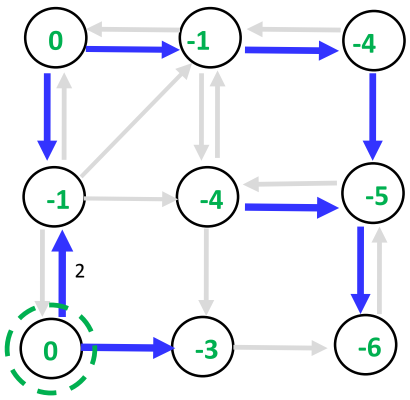
I_B = {(1,4), (6,9), (1,2), (7,4), (5,6), (7,8), (2,3), (3,6)}

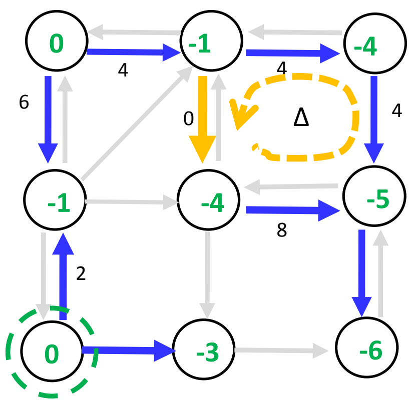
I_{N+} = {(4,5), (8,9)}; I_{N-} = the remaining links

By simple inspection it can be seen that the only pair of nodes where there is a "potential drop" and possibility of increasing flow is for link (2,5) and that its reduced cost is r_{25} = -3

Δ = Min {4, 4, 0} = 0

No change in flows but in basic indices. The objective function value remains as before: 78



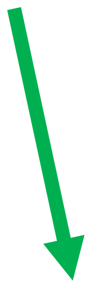


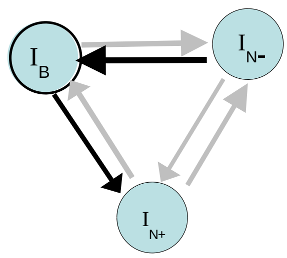
Optimal basic solution:

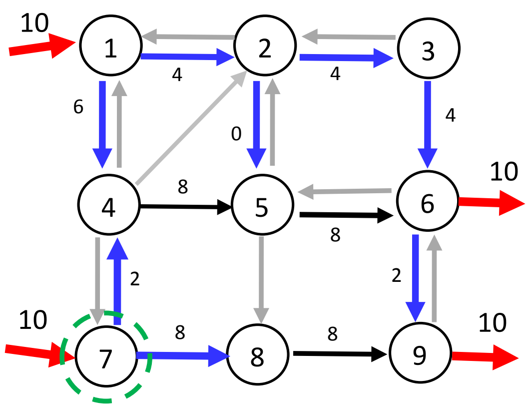
I_B = {(1,4), (6,9), (1,2), (7,4), (2,5), (7,8), (2,3), (3,6)}

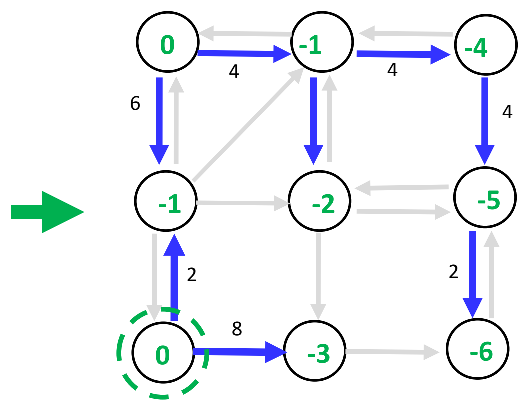
I_{N+} = {(4,5), (8,9), (5,6)}; I_{N-} = the remaining links

By inspection, it can be seen that **this is an optimal solution of the problem**, since neither negative reduced costs exist for links in I_{N-}, nor positive reduced costs exist for links in I_{N+}

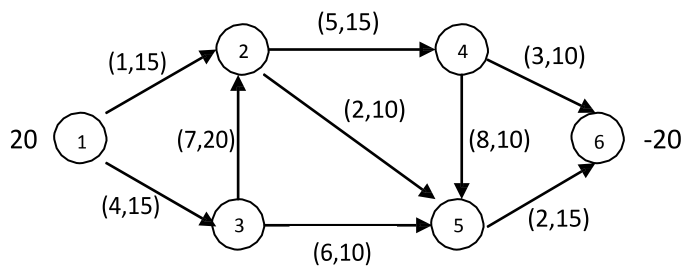








P7) Consider the following problem of network flows, in which there is a single production node (node 1), and a single demand node (node 6). Suppose we want to send 20 flow units from node 1 to node 6. The values that appear on the arcs indicate the unit cost of utilization and its maximum capacity (cij, uij), respectively.



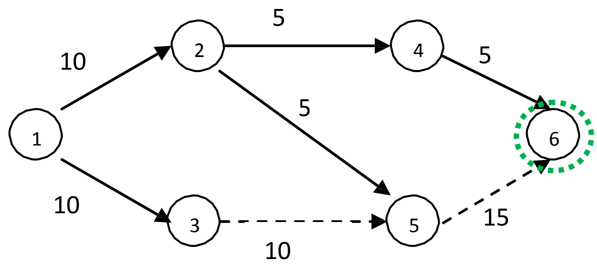
a. ¿Which is the node-link incidence matrix of the graph?
b. ¿ Which is the basic solution associated with the basic index set I_B = {(1,2), (1,3), (2,4), (2,5), (4,6)}, in which the set of non-basic variables at their lower bound is I_{N-} = {(3,2), (4,5)} and the set of non-basic variables at their upper bound is I_{N+} = {(3,5), (5,6)}?
c. Is optimal the solution associated to the previous indices sets? If not, use the SIMPLEX method to find the optimal flows.

SOLUTION:

a)

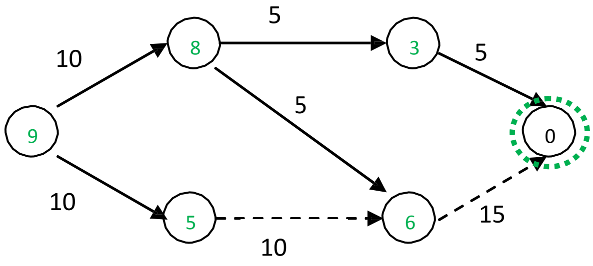
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

b)



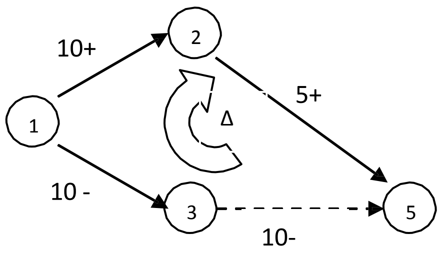
c) Objective function value= 190

Dual variables taking 6 as root node.



Reduced costs will be $r_{ij} = c_{ij}\ (u_i - u_j)$:
R1 < 0 ? $r_{32} = 7 - (5 -8) = 10$, $r_{45} = 8 - (3 -6) = 11$

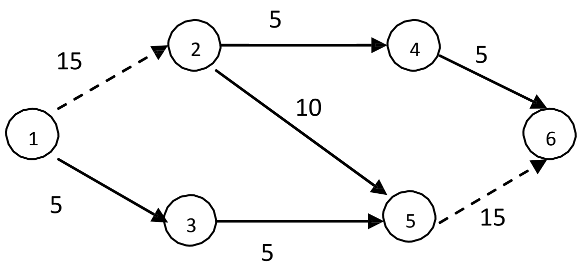
R2 > 0 ? $r_{35} = 6 - (5 -6) = 7$ ✱ Link (3,5) will exit I_{N+}
$r_{56} = 2 - (6 -0) = -4$



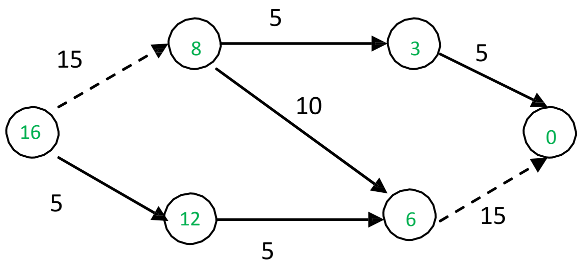
Δ = Min {10, 10, 5, 5 } = 5

Objective function value= 190 + 7 x (-5) = 155
Links (1,2) and (2,5) are at capacity and may both exit the basis. Link (1,2) is chosen.

I_B = {(1,3), (2,4), (2,5), (3,5), (4,6)}. I_{N-} = {(3,2),(4,5)}, I_{N+} ={(1,2), (5,6)}:
Flows on links redistribute as follows:



Dual variables:



Reduced costs:
R1 < 0 ? $r_{32} = 7 - (12 -8) = 3$
$r_{45} = 8 - (3 -6) = 11$

R2 > 0 ? $r_{12} = 1 - (16 -8) = -7$
$r_{56} = 2 - (6 - 0) = -4$ Optimal solution found; Objective functions 1 + 15 + 4 *5 + 5* 5 + 2* 10 + 6* 5 + 3* 5 + 2 *15 = 155 (as we knew)