

Aula 9

2024-10-31

Regras de Derivação

A seguir, são apresentadas algumas regras gerais, obtidas sempre a partir da definição usando o limite. São elas:

1^a

$$f(x) = k; k \in \mathbb{R}$$

$$f'(x) = 0 \quad \text{ou} \quad \frac{df(x)}{dx} = 0$$

- $f'(x) \rightarrow$ linha dex
- $\frac{df(x)}{dx} \rightarrow$ derivada da f em relação a x

2^a

$$f(x) = x, \text{ então}$$

$$f'(x) = 1 \quad \text{ou} \quad \frac{df(x)}{dx} = 1$$

3^a

$$f(x) = x^n \quad ; \quad n \in \mathbb{Z},$$

$$\text{então } f'(x) = n \cdot x^{n-1} \quad \text{ou}$$

$$\frac{df(x)}{dx} = n \cdot x^{n-1}$$

Exemplo 1

$$g(x) = x^2$$

$$\begin{aligned} g'(x) &= 2x^{2-1} \\ &= 2x \end{aligned}$$

Exemplo 2

$$h(x) = x^{-7}, \text{ então}$$

$$h'(x) = -7x^{-7-1} = -7x^{-8}$$

4^a

$$h(x) = f(x) \pm g(x), \text{ então}$$

$$h'(x) = f'(x) \pm g'(x)$$

5^a

$$f(x) = \sin x, \text{ então}$$

$$f'(x) = \cos x$$

6^a

$$f(x) = \cos x, \text{ então} \\ f'(x) = -\sin x$$

7^a

$$f(x) = \operatorname{tg} x, \text{ então } f'(x) = \sec^2 x$$

Lembrando que $\sec x = \frac{1}{\cos x}$

8^a

$$f(x) = \operatorname{ctg} x, \text{ então } f'(x) = \csc^2 x \\ \text{onde } \csc x = \frac{1}{\sin x}$$

9^a

$$f(x) = \sec x, \text{ então } f'(x) = \sec x \cdot \operatorname{tg} x$$

10^a

$$f(x) = \csc x, \text{ então} \\ f'(x) = -\csc x \cdot \operatorname{ctg} x \quad ; \quad \operatorname{ctg} x = \frac{1}{\operatorname{tg} x}$$

11^a

$$f(x) = e^x, \text{ então } f'(x) = e^x$$

e = número de Euler

12^a

$$f(x) = a^x, \text{ então} \\ f'(x) = a^x \cdot \ln a$$

13^a

$$f(x) = \ln x, \text{ então} \\ f'(x) = \frac{1}{x}$$

14^a

$$f(x) = \log_a^x, \text{ então} \\ f'(x) = \frac{1}{x} \cdot \ln a$$

Exemplo 1

$$\begin{aligned}f(x) &= e^x \cos x = u \cdot v \\u &= e^x \Rightarrow u' = e^x \\v &= \cos x \Rightarrow v' = -\sin x \\(u \cdot v)' &= u' \cdot v + u \cdot v' \\&= e^x \cos x + e^x \cdot (-\sin x) \\&= e^x \cdot (\cos x - \sin x)\end{aligned}$$

Exemplo 2

$$\begin{aligned}g(x) &= x^4 \cdot \ln x = u \cdot v \Rightarrow \\u &= x^4 \Rightarrow u' = 4x^{4-1} = 4x^3 \\v &= \ln x \Rightarrow v' = \frac{1}{x} \\(u \cdot v)' &= u'v + u \cdot v' \\&= 4x^3 \cdot \ln x + x^4 \cdot \frac{1}{x} \\&= 4x^3 \cdot \ln x + x^3 \\&= x^3 \cdot (4\ln x + 1)\end{aligned}$$

Exemplo 3

$$h(x) = \sin(2x) = \underbrace{2 \sin x}_u \cdot \underbrace{\cos x}_v = u \cdot v$$

$$\left. \begin{aligned}f(x) &= c \cdot f(x) \quad ; \quad c \in \mathbb{R} \\f'(x) &= c f'(x)\end{aligned} \right\} \text{regra}$$

$$\begin{aligned}u &= 2 \sin x \Rightarrow u' = 2 \cdot \cos x \\v &= \cos x \Rightarrow v' = -\sin x \\(u \cdot v)' &= u'v + u \cdot v' \quad (\cos x)^2 \\&= 2 \cos x \cdot \cos x + 2 \sin x \cdot (-\sin x) \\&= 2 \cos^2 x - 2 \sin^2 x \\&= 2(\cos^2 x - \sin^2 x) \\&= 2 \cos(2x)\end{aligned}$$

$$(*) \quad \sin(a + b) = \sin a \cdot \cos b + \sin b \cdot \cos a$$

$$\sin(x + x) = \sin x \cdot \cos x + \sin x \cdot \cos x$$

$$\sin(2x) = 2 \sin x \cdot \cos x$$

$$(*) \quad \cos(a + b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$

$$\cos(x + x) = \cos x \cdot \cos x - \sin x \cdot \sin x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Exemplo 4

$$m(x) \underbrace{\operatorname{ctg} x}_u \cdot \underbrace{\sin x}_v = \frac{\cos x}{\sin x} \cdot \cancel{\sin x} = \cos x$$

$$u = \operatorname{ctg} x \Rightarrow u' = -\operatorname{csc}^2 x$$

$$v = \sin x \Rightarrow v' = \cos x$$

$$m(x) = u \cdot v$$

$$'(x) = (u \cdot v)' = u'v + u \cdot v'$$

$$= -\operatorname{csc}^2 x \cdot \sin x + \operatorname{ctg} x \cdot \cos x$$

$$= -\left(\frac{1}{\sin x}\right)^2 \cdot \sin x + \frac{\cos x}{\sin x} \cdot \cos x$$

$$= -\frac{1}{\sin^2 x} \cdot \cancel{\sin x} + \frac{\cos^2 x}{\sin x}$$

$$= \frac{-1}{\sin x} + \frac{\cos^2 x}{\sin x}$$

$$= \frac{\cos^2 x - 1}{\sin x}$$

$$= \frac{-\cancel{\sin^2 x}}{\cancel{\sin x}} = -\sin x$$

Regra do Quociente

Sejam as funções $f(x)$ e $g(x)$ tais que se definem uma função quociente $q(x) = \frac{f(x)}{g(x)}$; $g(x) \neq 0$.

Para derivar funções desse tipo usamos a seguinte regra:

$$q'(x) = \left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

Usando como variáveis auxiliares $u = f(x)$ e $v = g(x)$, temos:

$$\left(\frac{u}{v} \right)' = \frac{u'v - u \cdot v'}{v^2} \quad ; \quad v \neq 0$$

Exemplo 1

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$$

Usando a Regra do Quociente

$$\begin{aligned}
 u &= 1 \Rightarrow u' = 0 \\
 u &= 1 \cdot x^0 \Rightarrow u' = 1 \cdot 0 \cdot x^{-1} = 0 \\
 v &= x^2 \Rightarrow v' = 2x \\
 \left(\frac{u}{v}\right)' &= \frac{u'v - u \cdot v'}{v^2} \\
 &= \frac{0 \cdot x^2 - 1 \cdot 2x}{(x^2)^2} \\
 &= \frac{-2x}{x^4} \\
 &= \frac{-2}{x^3}
 \end{aligned}$$

Exemplo 2

$$\begin{aligned}
 f(x) &= \operatorname{tg} x = \frac{\sin x}{\cos x} (\operatorname{tg} x) = \sec^2 x \\
 u &= \sin x \Rightarrow u' = \cos x \\
 v &= \cos x \Rightarrow v' = -\sin x \\
 \operatorname{tg} x &= \frac{\sin x}{\cos x} = \frac{u}{v} \Rightarrow \left(\frac{u}{v}\right)' = \frac{u'v - u \cdot v'}{v^2} \\
 &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 \\
 &= (\sec x)^2 \\
 &= \sec^2 x
 \end{aligned}$$

Resolução da Lista

Exercício 1

$$\begin{aligned}
 y &= -x^2 + 3 \\
 y' &= -2x + 0 = -2x \\
 y'' &= (-2x)' = -2 \cdot (x)' = -2 \cdot 1 = -2 \\
 \underbrace{y''}_{y \text{ 2 linhas}} &\equiv \underbrace{\frac{d^2 f}{dx^2}}_{\substack{\text{derivada segunda} \\ \text{de } f \text{ em relação} \\ \text{a } x}} = \frac{d}{dx} \left(\frac{df}{dx} \right)
 \end{aligned}$$