

HW LOG

CME241

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HW – Jan 11

- Write out the MP/MRP definitions and MRP Value Function definition (in LaTeX) in your own style/notation (so you really internalize these concepts)
- Think about the data structures/class design (in Python 3) to represent MP/MRP and implement them with clear type declarations
- Remember your data structure/code design must resemble the Mathematical/notational formalism as much as possible
- Specifically the data structure/code design of MRP should be incremental (and not independent) to that of MP
- Separately implement the $r(s, s')$ and the $R(s) = \sum_{s'} p(s, s') * r(s, s')$ definitions of MRP
- Write code to convert/cast the $r(s, s')$ definition of MRP to the $R(s)$ definition of MRP (put some thought into code design here)
- Write code to generate the stationary distribution for an MP

MP/MRP definition

MP: A markov process is a chain that is memory less, i.e. it only cares about about the current state and not the past. The mathematical definition is

$$\mathbb{P}(S_{t+h} = s_{t+h} | S_0 = s_0, S_1 = s_1, \dots, S_{t-1} = s_{t-1}, S_t = s_t) = \mathbb{P}(S_{t+h} = s_{t+h} | S_t = s_t)$$

or equivalently

$$\mathbb{E}[S_{t+h} | S_0 = s_0, S_1 = s_1, \dots, S_{t-1} = s_{t-1}, S_t = s_t] = \mathbb{E}[S_{t+h} | S_t = s_t].$$

The Markov process is defined as $\{s, P_s\}$ where $s \in \{s_0, \dots, s_k\}$ is the state spaces and P_s is the probability distribution in each state.

MRP: A Markov reward process is a Markov process that has a reward $R(s)$ associated with each state and some discounting factor $\gamma \in [0, 1]$.

Value function: The value function is the accumulated expected reward associated with the current known state s . It defines as

$$v(s) = \mathbb{E}[\sum_{i=0}^T R(s_{t+i})\gamma^i | S_t = s],$$

where T is the time of termination for the process.

Data structures

- *State* – TypeVar('State')
- *States* – List[State]
- $\mathcal{R}(s)$ – List[float] (*)
- $r(ss')$ – List[List[float]] (*)
- P_{MP} – Dict[State, Tuple[State, float]]
- P_{MRP_A} – Dict[P_{MP} , float]
- P_{MRP_B} – Dict[State, Dict[State, Tuple[float, float]]]
- γ – float.

Thus we see that

$$\begin{aligned}
\mathcal{R}(s) &= \mathbb{E}[R_t | S_{t-1} = s] \\
&= \sum_{s'} R_t(\{\text{reward after state } s'\}) \mathbb{P}(S_t = s' | S_{t-1} = s) \\
&= \sum_{s'} \mathbb{E}[R_t | S_{t-1} = s \cap S_t = s'] \mathbb{P}(S_t = s' | S_{t-1} = s) \\
&= \sum_{s'} r(s, s') p(s, s')
\end{aligned}$$

HW – Jan 16

- Write the Bellman equation for MRP Value Function and code to calculate MRP Value Function (based on Matrix inversion method you learnt in this lecture)
- Write out the MDP definition, Policy definition and MDP Value Function definition (in LaTeX) in your own style/notation (so you really internalize these concepts)
- Think about the data structure/class design (in Python 3) to represent MDP, Policy, Value Function, and implement them with clear type definitions
- The data structure/code design of MDP should be incremental (and not independent) to that of MRP
- Separately implement the $r(s, s', a)$ and $R(s, a) = \sum_{s'} p(s, s', a) * r(s, s', a)$ definitions of MDP
- Write code to convert/cast the $r(s, s', a)$ definition of MDP to the $R(s, a)$ definition of MDP (put some thought into code design here)
- Write code to create a MRP given a MDP and a Policy
- Write out all 8 MDP Bellman Equations and also the transformation from Optimal Action-Value function to Optimal Policy (in LaTeX)

Data structures

- *Action* – TypeVar('Action')
- *Policy* – Dict[State, Tuple[Action, (float or int)]]
- MDP_A – Dict[State, Dict[Action, Dict[Tuple[State, (float or int)], (float or int)]]]
- MDP_B – Dict[State, Dict[Action, Tuple[State, Tuple[float, float]]]]

Bellman Equations

- (1) Basic Bellman for MRP (A)

$$\begin{aligned} v(s) &= \mathbb{E}[\sum_{i=0}^T R_{t+i+1} \gamma^i | S_t = s] \\ &= \mathbb{E}[R_{t+1} | S_t = s] + \gamma \mathbb{E}[v(S_{t+1}) | S_t = s] \\ &= \mathcal{R}_s + \gamma \sum_{s'} v(s') \mathbb{P}(S_{t+1} = s' | S_t = s). \end{aligned}$$

- (2) Basic Bellman for MRP (A) in matrix form is then

$$v = \mathcal{R} + \gamma \mathcal{P}v.$$

- (3) For the action-value function with policy π we have

$$q_\pi(s, a) = \mathbb{E}[\sum_{i=0}^T R_{t+i+1} \gamma^i | S_t = s \cap A_t = a]$$

which have the same solution in

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}[R_{t+1} | S_t = s \cap A_t = a] + \gamma \mathbb{E}[q_\pi(S_{t+1}, A_{t+1}) | S_t = s \cap A_t = a] \\ &= \mathcal{R}_s^a + \gamma \sum_{s', a'} q_\pi(s', a') \mathbb{P}(S_{t+1} = s' \cap A_{t+1} = a' | S_t = s \cap A_t = a) \end{aligned}$$

where \mathcal{R}_s^a is \mathcal{R}_s for action a .

(4) Likewise for a MDP with a policy π we can create a value MRP with value function

$$v_\pi(s) = \sum_{a'} \pi(a'|s) q_\pi(s, a')$$

where $\pi(a'|s)$ is the probability of taking action a' in state s .

(5) Now, we can combine (3) and (4) to express $v_\pi(s)$ as

$$v_\pi(s) = \sum_{a'} \pi(a'|s) \left(\mathcal{R}_s^a + \gamma \sum_{s'} \mathbb{P}(S_t = s' | S_t = s \cap A_t = a') v_\pi(s') \right)$$

(6) Combining (4) and (5) we can express $q_\pi(s, a)$ as

$$q_\pi(s, a) = \mathcal{R}_s^a + \gamma \sum_{s'} \mathbb{P}(S_{t+1} = s' | S_t = s \cap A_t = a) \sum_{a'} \pi(a'|s') q_\pi(s', a').$$

HW – Jan 18

- Write code for Policy Evaluation (tabular) algorithm
- Write code for Policy Iteration (tabular) algorithm
- Write code for Value Iteration (tabular) algorithm
- Those familiar with function approximation (deep networks, or simply linear in features) can try writing code for the above algorithms with function approximation (a.k.a. Approximate DP)

The first three are pretty much done (at least for MDP_A)

HW – Jan 23

- Work out (in LaTeX) the equations for Absolute/Relative Risk Premia for CARA/CRRA respectively
- Write the solutions to Portfolio Applications covered in class with precise notation (in LaTeX)

CARA

For CARA we have

$$U(x) = -\frac{1}{a}e^{-ax}, \quad a \neq 0.$$

Thus we have

$$\frac{dU(x)}{dx} = e^{-ax} \text{ and } \frac{d^2U(x)}{dx^2} = -ae^{-ax}.$$

For the Arrow-Pratt risk aversion coefficient A we have

$$\begin{aligned} A &= -\frac{U''(x)}{U'(x)} \\ &= a. \end{aligned}$$

CRRA

For CRRA we have

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1.$$

Thus we have

$$\frac{dU(x)}{dx} = x^{-\gamma} \text{ and } \frac{d^2U(x)}{dx^2} = -\gamma x^{-\gamma-1}$$

For the relative Arrow-Pratt risk aversion coefficient A we have

$$\begin{aligned} A &= -\frac{xU''(x)}{U'(x)} \\ &= \gamma. \end{aligned}$$

Portfolio Application Solution

CARA: We have two assets $r_a \sim N(\mu, \sigma^2)$ and $r_f \sim N(r, 0)$. We invest a fraction ρ_a in r_a and ρ_f in r_f . The objective is then to

$$\begin{aligned} \max \quad & \mathbb{E}[U(N(\rho_a\mu + \rho_fr, \rho_a^2\sigma^2))] \\ \text{s.t.} \quad & \rho_a + \rho_f = 1. \end{aligned}$$

Now, substituting $\rho_f = 1 - \rho_a$ and using the PDF of the normal distribution we can set this up as

$$\max_{\rho_a} \left\{ -\frac{1}{a} \int_{\mathbb{R}} \exp(-ax) \frac{1}{\sqrt{2\pi\rho_a\sigma^2}} \exp\left(-\frac{(x - (\rho_a\mu + (1-\rho_a)r))^2}{2\rho_a^2\sigma^2}\right) dx \right\}. \quad (1)$$

Differentiating (1) wrt ρ_a and setting to zero gives

$$\rho_a^* = \frac{\mu - r}{a\sigma^2}.$$

CRRA: The setup is very similar but now we assume that $\log(r_a) \sim N(\mu, \sigma^2)$ instead. This gives the solution

$$\rho_a^* = \frac{\mu - r}{\gamma\sigma^2}$$

as optimal allocation.

HW – Jan 25

- Model Merton's Portfolio problem as an MDP (write the model in LaTeX)
- Implement this MDP model in code
- Try recovering the closed-form solution with a DP algorithm that you implemented previously

Discretization of the model:

For a stock S we have that

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Brownian motion, i.e. a simple random walk with infinitely small steps size. Solving this sde gives for a future time T given a current time t gives

$$S_T = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T - t) + \sigma W_{(T-t)}\right)$$

or more simply

$$S_T = S_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{T - t}Z\right) \quad (2)$$

where $Z \sim N(0, 1)$. Now, if we have the wealth W_t at time t , we consume c_t thus and out of the remaining wealth $W_t - c_t$ we invest π_t fractions in a portfolio of risky assets with homoscedastic variance¹ σ^2 and return μ . Consequently we invest $1 - \pi_t$ fractions in a risk free asset with constant return r and no variance. With the logic of (2) we can set up the distribution of the future wealth as a time-discrete function of the current wealth and the actions c_t and π_t as

$$W_{t+\tau} = (W_t - c_t) \exp\left(\left(\pi_t \mu + (1 - \pi_t)r - \frac{\pi_t^2 \sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right). \quad (3)$$

This will be the discrete setup for the development of the wealth. Furthermore the goal is to find the optimal π_t and c_t at each time, i.e. the π_t and c_t that maximizes the expected utility of the consumption. Thus our goal is to

$$\max_{\pi_t, c_t} \mathbb{E} \left[\sum_{\tau=0}^T e^{-\rho\tau} \frac{c_\tau^{1-\gamma}}{1-\gamma} \right].$$

Setup:

- States – The state is a tuple $\langle t, W_t \rangle$ of time t and wealth W_t at time t . The times is here discrete with equidistant time partition $\tau = t_i - t_{i-1}$ up until time T of maturity. The wealth is also discrete but it follows the distribution in (3) – more on that later
- Action(s) – The action is also a tuple $\langle c_t, \pi_t \rangle$ of consumption c_t and fraction of risky assets π_t . To make this discrete I will define c_t as $c_t \in \{0, 0.01W_t, 0.02W_t, \dots, 1.99W_t, 2W_t\}$ with a fixed upper boundary at $2W_t$. The fraction π_t is supposed to be unconstrained, but in order of modelling this with a finite state space, I will probably have to set some upper and lower boundary, e.g. $\pi_t \in \{-2, -1.99, \dots, 1.99, 2\}$.
- Transitions

¹Perhaps these parameters should not be constant to make the setup more realistic.

HW – Jan 30

- Model a real-world Portfolio Allocation+Consumption problem as an MDP (including real-world frictions and constraints)
- Exam Practice Problem: Optimal Asset Allocation in Discrete Time

Step 1:

The continuous state space is a tuple $\langle t, W(t) \rangle$ where t denotes the time and $W(t)$ the value of the portfolio. In the discrete case we would have that

$$\begin{aligned} \frac{W_t}{W_{t-1}} - 1 &\sim N\left(\mu \frac{x_{t-1}}{W_{t-1}} + r \frac{W_{t-1} - x_{t-1}}{W_{t-1}}, \left(\frac{x_{t-1}}{W_{t-1}}\right)^2 \sigma^2\right) \\ \Rightarrow W_t &\sim N\left((W_{t-1} + \mu x_{t-1} + r(W_{t-1} - x_{t-1})), x_{t-1}^2 \sigma^2\right) \\ \Rightarrow W_t &\sim N\left((\mu - r)x_{t-1} + (1 + r)W_{t-1}, x_{t-1}^2 \sigma^2\right) \end{aligned} \quad (4)$$

In the continuous case this corresponds to

$$S(t) = S(0) \exp(\mu t + \sqrt{t} \sigma Z)$$

where Z is a standard normal distributed variable. The action is $x_t \in \mathbb{R}$ and the discount factor is γ .

Step 2:

For the Bellman optimally equation we have that

$$V_*(\langle t, W_t \rangle) = \max_{x_t} \left\{ \mathcal{R}_{\langle t, W_t \rangle}^{x_t} + \gamma \int_{\langle t+1, W'_{t+1} \rangle} \mathcal{P}_{\langle t, W_t \rangle \langle t+1, W'_{t+1} \rangle} V_*(\langle t+1, W'_{t+1} \rangle) dW_{t+1} \right\}.$$

However, the rewards at each time $t \in \{0, 1, \dots, T-1\}$ are all zero. Hence $\mathcal{R}_{\langle t, W_t \rangle}^{x_t} = 0 \forall t \in \{0, 1, \dots, T-1\}$. We also need to take the utility into account. Thus we have that

$$V_*(\langle t, W_t \rangle) = \gamma \max_{x_t} \{ \mathbb{E}[V_*(\langle t+1, W_{t+1} \rangle)] \}$$

where the expected value only depends on the action x_t in each $t \in \{0, 1, \dots, T-1\}$.

Step 3:

If we know that

$$V_*(\langle t, W_t \rangle) = -b_t e^{-c_t W_t},$$

we can now use this and (4) to solve this expected value as

$$\begin{aligned} &-b_{t+1} \int_{\mathbb{R}} e^{-c_{t+1} W_{t+1}} \frac{1}{\sqrt{2\pi x_t^2 \sigma^2}} \exp\left(-\frac{(W_{t+1} - ((\mu - r)x_t + (1 + r)W_t))^2}{2x_t^2 \sigma^2}\right) dW_{t+1} \\ &= -b_{t+1} \int_{\mathbb{R}} \exp\left(-\frac{(W_{t+1} - ((\mu - r)x_t + (1 + r)W_t))^2 + 2W_{t+1}c_{t+1}x_t^2\sigma^2}{2x_t^2 \sigma^2}\right) / \sqrt{2\pi x_t^2 \sigma^2} dW_{t+1} \\ &= \{\text{Completing the square: } (y - c)^2 + 2yq = (y - (c - q))^2 + 2cq - q^2\} \\ &= -b_{t+1} \exp\left(\frac{-2((\mu - r)x_t + (1 + r)W_t)(c_{t+1}x_t^2\sigma^2) + (c_{t+1}x_t^2\sigma^2)^2}{2x_t^2 \sigma^2}\right) \\ &= -b_{t+1} \exp\left(-((\mu - r)x_t + (1 + r)W_t)c_{t+1} + \frac{1}{2}c_{t+1}^2 x_t^2 \sigma^2\right). \end{aligned} \quad (5)$$

Step 4:

Differentiating the final step of (5) wrt x_t now gives

$$\frac{\partial V_*(\langle t, W_t \rangle)}{\partial x_t} = (c_{t+1}^2 x_t \sigma^2 - (\mu - r)c_{t+1}) V_*(\langle t, W_t \rangle)$$

and setting to zero gives that

$$x_t^* = \frac{\mu - r}{c_{t+1} \sigma^2}.$$

Step 5:

Now, using the definitions

$$V_*(\langle t, W_t \rangle) = -b_t e^{-c_t W_t}$$

and

$$\begin{aligned} V_*(\langle t, W_t \rangle) &= -\gamma b_{t+1} e^{-c_{t+1} W_{t+1}} \\ &= -\gamma b_{t+1} \exp \left(-((\mu - r)x_t^* + (1 + r)W_t)c_{t+1} + \frac{1}{2}c_{t+1}^2 x_t^{*2} \sigma^2 \right) \\ &= -\gamma b_{t+1} \exp \left(-c_{t+1}(1 + r)W_t - \frac{(\mu - r)^2}{2\sigma^2} \right) \end{aligned}$$

we see that

$$c_{t+1}(1 + r) = c_t$$

and

$$b_t = \gamma b_{t+1} \exp \left(\frac{-(\mu - r)^2}{2\sigma^2} \right).$$

Step 6:

Since

$$U(W_T) = -\frac{e^{-aW_T}}{a}$$

we know that

$$c_T = a$$

and

$$b_T = \frac{1}{a}.$$

Thus we can find a general expression for c_t as

$$c_t = a(1 + r)^{T-t}$$

whereas for b_t we have that

$$b_t = \frac{1}{a} \gamma^{T-t} \exp \left(\frac{-(\mu - r)^2 (T - t)}{2\sigma^2} \right).$$

To conclude, this yields the optimal policy

$$x_t^* = \frac{\mu - r}{a\sigma^2} (1 + r)^{t+1-T}.$$

HW – Feb 1

- Work out (in LaTeX) the solution to the Linear Impact model we covered in class
- Model a real-world Optimal Trade Order Execution problem as an MDP (with complete order book included in the State)

HW – Feb 6

- Write code for the interface for tabular RL algorithms. The core of this interface should be a mapping from a (state, action) pair to a sampling of the (next state, reward) pair. It is important that this interface doesn't present the state-transition probability model or the reward model.
- Implement a tabular Monte-Carlo algorithm for Value Function prediction
- Implement a tabular TD algorithm for Value Function prediction
- Test the above implementation of Monte-Carlo and TD VF prediction algorithms versus DP Policy Evaluation algorithm on an example MDP