TSDT14 Signal Theory

Lecture 5
Prediction and Non-Linear Systems

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Prediction

WSS TD process $X[n]$.

Task: Predict next sample.

Method: Filter the signal.

$$X[n] \longrightarrow \boxed{h[n]} \longrightarrow Y[n]$$

Prediction of $X[n+1]$

Prediction error:

$$Z[n] = Y[n] - X[n+1].$$

Interpretation:

$$X[n] \longrightarrow \boxed{h_1[n]} \longrightarrow Z[n]$$

Prediction error filter:

$$h_1[n] = h[n] - \delta[n+1]$$
$$H_1[\theta] = H[\theta] - e^{j2\pi\theta}$$

Mean: $m_Z = (H[0] - 1)m_X$

Unbiased: $m_Z = 0$

$$\Rightarrow H[0] = 1 \text{ or } m_X = 0$$

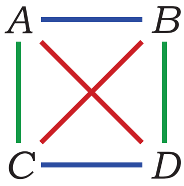
Quality measure (distorsion):

$$\varepsilon^2 = r_Z[0] = \left(h_1 * \tilde{h}_1 * r_X\right)[0]$$
$$= \int_0^1 |H[\theta]|^2 \, R_X[\theta] \, d\theta$$

$$\tilde{h}_1[n] = h_1[-n]$$

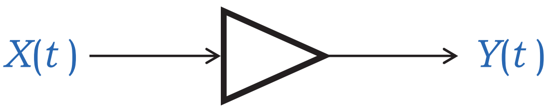
3- & 4-Dim. Jointly Gaussian Variables

$$\mathrm{E}\{ABCD\} = \mathrm{E}\{AB\} \cdot \mathrm{E}\{CD\} + \mathrm{E}\{AC\} \cdot \mathrm{E}\{BD\} + \mathrm{E}\{AD\} \cdot \mathrm{E}\{BC\} \\ - 2 \cdot \mathrm{E}\{A\} \cdot \mathrm{E}\{B\} \cdot \mathrm{E}\{C\} \cdot \mathrm{E}\{D\}$$



$$\mathrm{E}\{ABC\} = \mathrm{E}\{AB\} \cdot \mathrm{E}\{C\} + \mathrm{E}\{AC\} \cdot \mathrm{E}\{B\} + \mathrm{E}\{A\} \cdot \mathrm{E}\{BC\} \\ - 2 \cdot \mathrm{E}\{A\} \cdot \mathrm{E}\{B\} \cdot \mathrm{E}\{C\}$$

Example: Amplifier 1(3)



Input: $X(t)$ stationary Gaussian process with $\mathrm{m}_X = 0$.

Output: $Y(t) = X(t) + \varepsilon X^2(t)$. Quadratic distorsion.

Mean: $\mathrm{m}_Y = \mathrm{E}\{X(t) + \varepsilon X^2(t)\} \underset{\uparrow \; \mathrm{m}_X = 0}{=} \varepsilon \cdot \mathrm{E}\{X^2(t)\} = \varepsilon \cdot r_X(0)$

ACF: $\mathrm{r}_Y(\tau) = \mathrm{E}\{Y(t)\,Y(t+\tau)\} = \mathrm{E}\{(X(t)+\varepsilon X^2(t))(X(t+\tau)+\varepsilon X^2(t+\tau))\}$

$$= \underbrace{\mathrm{E}\{X(t)X(t+\tau)\}}_{r_X(\tau)} + \varepsilon \cdot \underbrace{\mathrm{E}\{X^2(t)X(t+\tau)\}}_{=\,?} + \varepsilon \cdot \underbrace{\mathrm{E}\{X(t)X^2(t+\tau)\}}_{=\,?} + \varepsilon^2 \cdot \underbrace{\mathrm{E}\{X^2(t)X^2(t+\tau)\}}_{=\,?}$$

Example: Amplifier 2(3)

$$E\{X^2(t)X(t+\tau)\} =$$

$$= E\{X^2(t)\}\cdot E\{X(t+\tau)\} + 2E\{X(t)\}\cdot E\{X(t)X(t+\tau)\} - 2E^2\{X(t)\}\cdot E\{X(t+\tau)\}$$

$$= r_X(0)\cdot m_X + 2\cdot m_X\cdot r_X(\tau) - 2\cdot m_X^3 = m_X\cdot\left(r_X(0) + r_X(\tau) - 2m_X^2\right) = 0$$

$$E\{X(t)X^2(t+\tau)\} = \ldots = m_X\cdot\left(r_X(0) + r_X(\tau) - 2m_X^2\right) = 0 \leftarrow$$ Since $m_X=0$

$$E\{X^2(t)X^2(t+\tau)\} =$$

$$= E\{X^2(t)\}\cdot E\{X^2(t+\tau)\} + 2E^2\{X(t)X(t+\tau)\} - 2E^2\{X(t)\}\cdot E^2\{X(t+\tau)\}$$

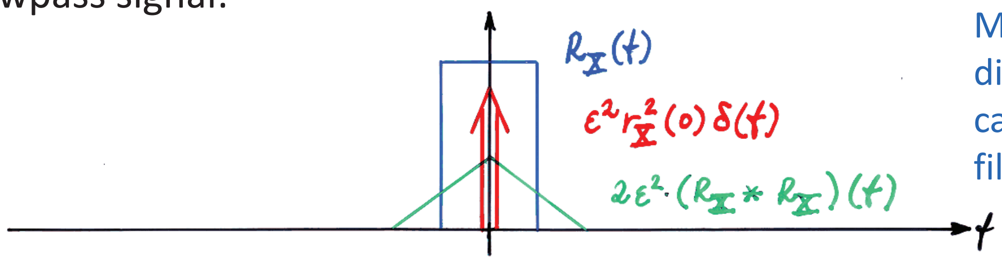
$$= r_X^2(0) + 2\cdot r_X^2(\tau) - 2\cdot m_X^4 = r_X^2(0) + 2\cdot r_X^2(\tau) \leftarrow$$ Since $m_X=0$

Totally: $r_Y(\tau) = r_X(\tau) + \varepsilon^2\cdot\left(r_X^2(0) + 2\cdot r_X^2(\tau)\right)$

$$R_Y(f) = R_X(f) + \varepsilon^2\cdot\left(r_X^2(0)\cdot\delta(f) + 2\cdot(R_X * R_X)(f)\right)$$

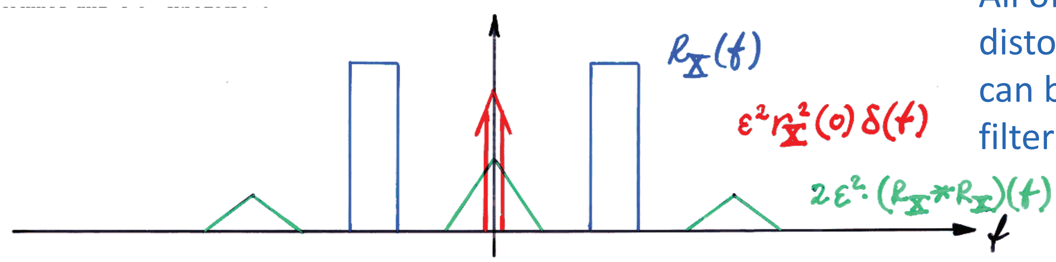
Example: Amplifier 3(3)

Lowpass signal:



Most of the distorsion cannot be filtered out

Bandpass signal:



All of the distorsion can be filtered out

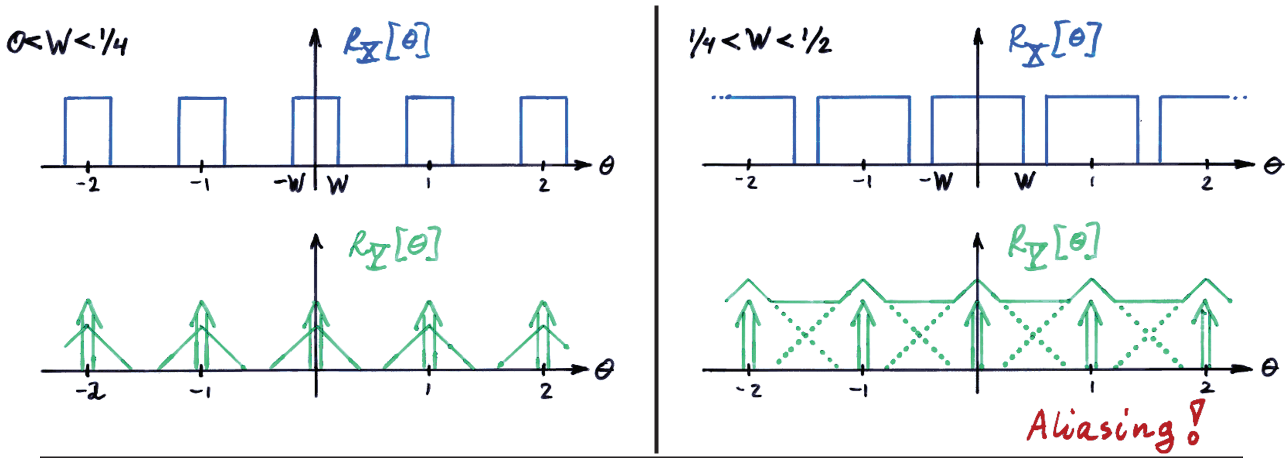
Squaring a Time-Discrete Gaussian Process

Input: $X[n]$ stationary Gaussian with $m_X = 0$. (Bandlimited)

Output: $Y[n] = X^2[n]$ $\qquad r_Y[k] = r_X^2[0] + 2\,r_X^2[k]$

$$R_Y[\theta] = r_X^2[0] \cdot \sum_m \delta(\theta - m) + 2(R_X \otimes R_X)[\theta]$$

Periodic convolution: $(R_X \otimes R_X)[\theta] = \int_0^1 R_X[\phi]\, R_X[\theta - \phi]\, d\phi$



Price's Theorem

Useful to determine the ACF after a nonlinear mapping of Gaussian processes.

Prerequisites: (A,B) is jointly Gaussian with mean $(0,0)$, $f(a)$ and $g(b)$ are functions, usually nonlinear, and $\rho = \mathrm{E}\{AB\}$.

Then we have: $\frac{\partial^n}{\partial \rho^n} E\{f(A) \cdot g(B)\} = E\left\{\frac{\partial^n}{\partial A^n} f(A) \cdot \frac{\partial^n}{\partial B^n} g(B)\right\}$

Example Price's Theorem 1(2)

Assumptions: $X(t)$ strictly stationary Gaussian Process with $m_X = 0$.
$Y(t) = X^2(t)$

Objective: Express $r_Y(\tau)$ in $r_X(\tau)$.

Method: Price's theorem with $A = X(t)$, $B = X(t+\tau)$,
$f(a) = g(a) = a^2$ and $\rho = r_X(\tau)$.

We have $r_Y(\tau) = E\{Y(t)Y(t+\tau)\} = E\{X^2(t)\cdot X^2(t+\tau)\} = E\{f(X(t))\cdot g(X(t+\tau))\}$

Price's theorem for $n=1$ gives us

$$\frac{\partial}{\partial r_X(\tau)} r_Y(\tau) = E\left\{\frac{\partial}{\partial A}f(A)\cdot\frac{\partial}{\partial B}g(B)\right\} = E\left\{\frac{\partial A^2}{\partial A}\cdot\frac{\partial B^2}{\partial B}\right\} = E\{2A\cdot 2B\}$$
$$= 4\cdot E\{X(t)\cdot X(t+\tau)\} = 4\, r_X(\tau)$$

This gives us

$$\partial r_Y(\tau) = 4 r_X(\tau)\cdot\partial r_X(\tau) \Rightarrow \int \partial r_Y(\tau) = \int 4 r_X(\tau)\,\partial r_X(\tau) \Rightarrow$$
$$\Rightarrow r_Y(\tau) + C_1 = 2 r_X^2(\tau) + C_2 \Rightarrow r_Y(\tau) = 2 r_X^2(\tau) + C$$

Example Price's Theorem 2(2)

We had: $r_Y(\tau) = 2 r_X^2(\tau) + C$

Objective: Determine the constant C.

How? Choose a τ such that $r_X(\tau) = 0$ holds.

Then: $X(t)$ & $X(t+\tau)$ are uncorrelated.

But: $X(t)$ & $X(t+\tau)$ jointly Gaussian

$\Rightarrow X(t)$ & $X(t+\tau)$ indep.

So: $C \underset{\uparrow\, r_X(\tau)=0}{=} r_Y(\tau) = E\{X^2(t)X^2(t+\tau)\} \underset{\uparrow\,\text{Indep.}}{=} \underbrace{E\{X^2(t)\}}_{r_X(0)}\cdot \underbrace{E\{X^2(t+\tau)\}}_{r_X(0)} = r_X^2(0)$

Totally: $r_Y(\tau) = 2 r_X^2(\tau) + r_X^2(0)$

More Non-Linearities

From
Tables & Formulas,
Page 15.

Situation

The input to a momentary non-linearity is a Gaussian process $X(t)$ with mean $m_X = 0$ and auto-correlation function $r_X(\tau)$.

Identities

$Y(t)$	$r_Y(\tau)$
$X^2(t)$	$2r_X^2(\tau) + r_X^2(0)$
$X^3(t)$	$6r_X^3(\tau) + 9r_X^2(0)r_X(\tau)$
$X^4(t)$	$24r_X^4(\tau) + 72r_X^2(0)r_X^2(\tau) + 9r_X^4(0)$
$X^5(t)$	$120r_X^5(\tau) + 600r_X^2(0)r_X^3(\tau) + 225r_X^4(0)r_X(\tau)$
$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$	$\frac{r_X(\tau)}{4} + \frac{1}{2\pi}\left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau)\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)\right]$ $= \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \ldots$
$\operatorname{sgn}(X)$	$\frac{2}{\pi}\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)$

Half-Wave Rectifier

From the table:

$\begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases}$	$\frac{r_X(\tau)}{4} + \frac{1}{2\pi}\left[\sqrt{r_X^2(0) - r_X^2(\tau)} + r_X(\tau)\arcsin\left(\frac{r_X(\tau)}{r_X(0)}\right)\right]$ $= \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \ldots$

Complete Maclaurin expansion:

$$r_Y(\tau) = \frac{r_X(0)}{2\pi} + \frac{r_X(\tau)}{4} + \frac{r_X^2(\tau)}{4\pi r_X(0)} + \sum_{n=2}^{\infty} \frac{(2n-3)!! \cdot r_X(0)}{2\pi \cdot (2n-1) \cdot (2n)!!} \left(\frac{r_X(\tau)}{r_X(0)}\right)^{2n}$$

$n!!$ is semi-factorial (product of every second positive integer):

$$6!! = 2 \cdot 4 \cdot 6 = 48 \qquad \text{and} \qquad 7!! = 1 \cdot 3 \cdot 5 \cdot 7 = 105.$$

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