

TSDT14 Signal Theory

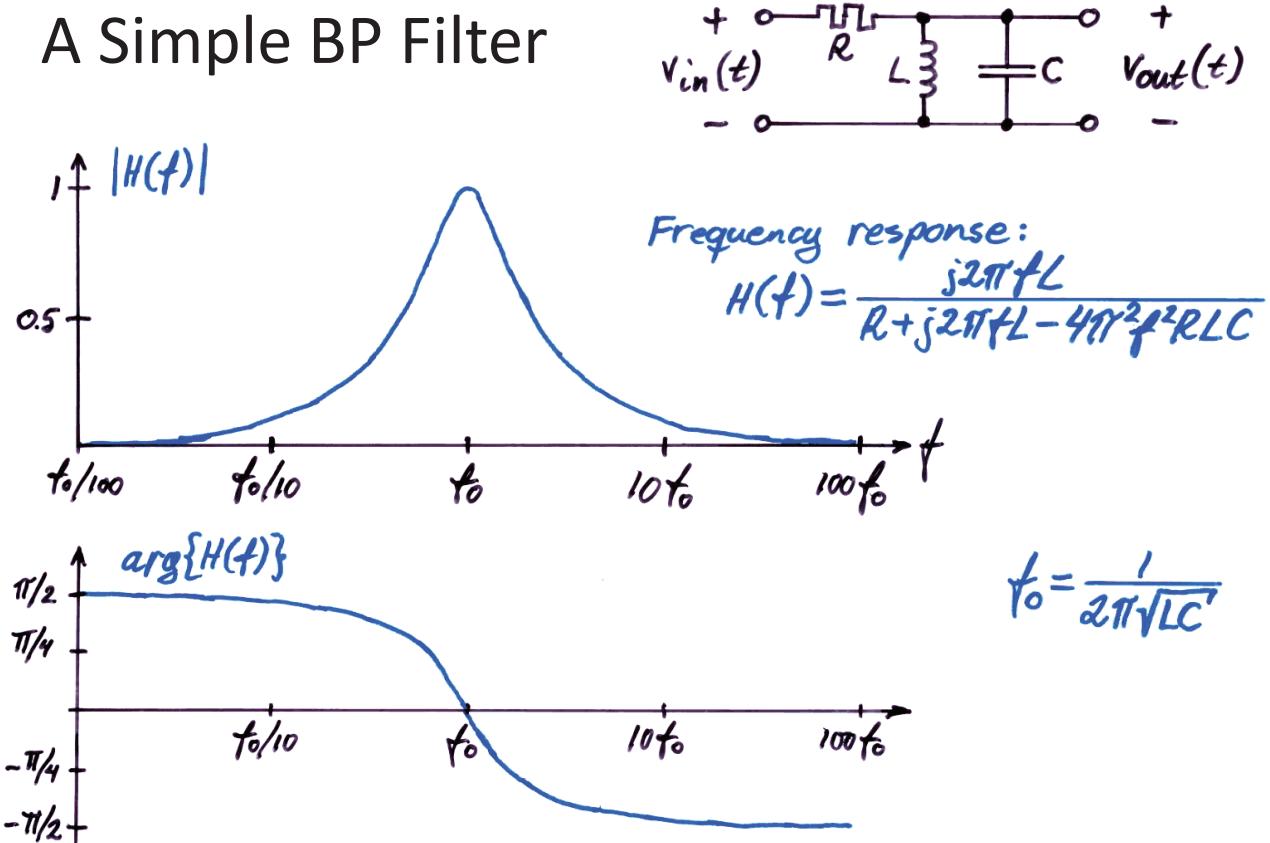
Lecture 7

Analog Modulation

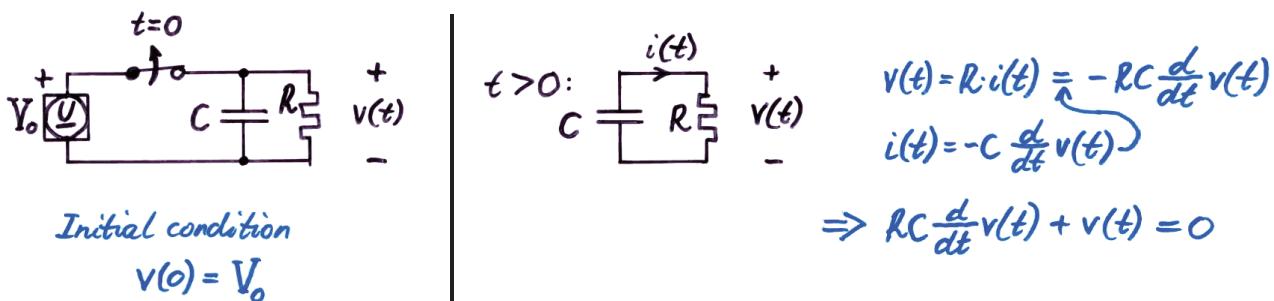
Mikael Olofsson
Department of EE (ISY)
Div. of Communication Systems



A Simple BP Filter

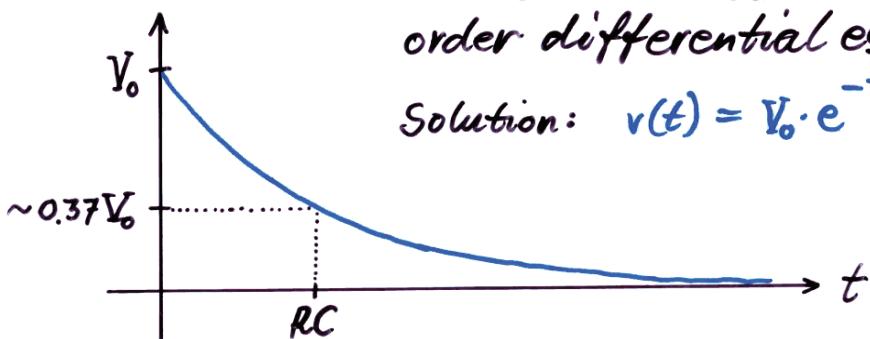


Discharging a Capacitor



This is the standard example of a first order differential equation.

Solution: $v(t) = V_0 \cdot e^{-t/RC}$ for $t \geq 0$



Amplitude Modulation

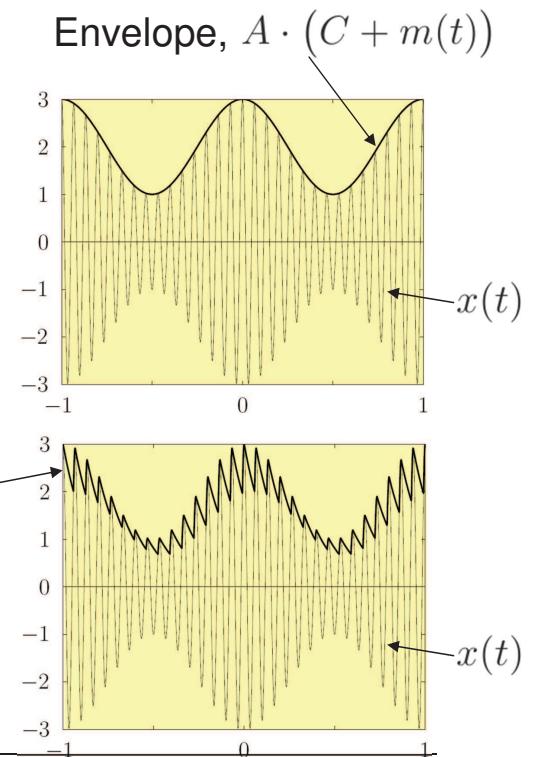
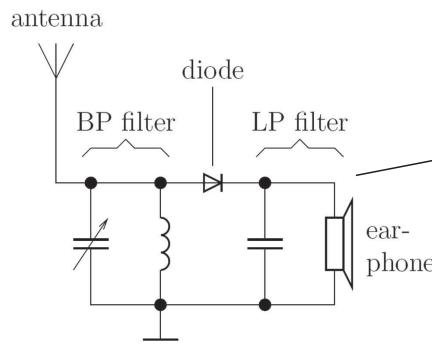
- The first technique used for radio broadcasts.
- A linear modulation technique.
- Simple to analyze.
- Simple demodulation.
- Noise sensitive.

Amplitude Modulation – Deterministic Case

Standard AM:

$$x(t) = A \cdot (C + m(t)) \cos(2\pi f_c t)$$

Crystal receiver, an envelope detector, first demodulator of standard AM:



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Spectrum of Standard AM

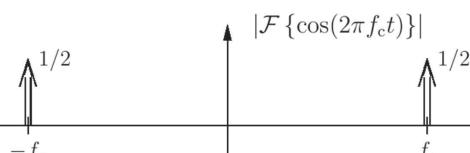
$$x(t) = A \cdot (C + m(t)) \cos(2\pi f_c t)$$

$$\begin{aligned} X(f) &= \mathcal{F}\{AC \cos(2\pi f_c t)\} + \mathcal{F}\{A m(t) \cos(2\pi f_c t)\} \\ &= \frac{AC}{2}(\delta(f - f_c) + \delta(f + f_c)) + \frac{A}{2}(M(f - f_c) + M(f + f_c)), \end{aligned}$$

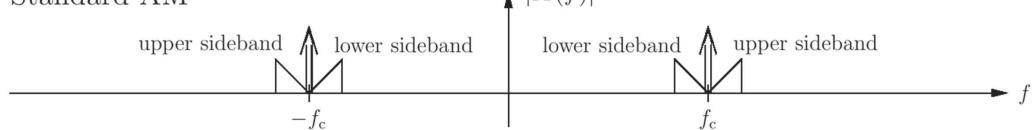
Message



Carrier

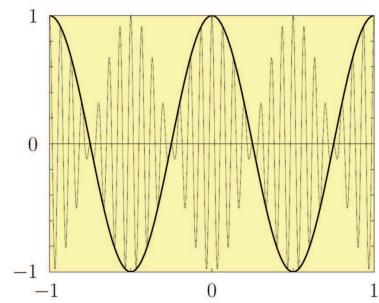


Standard AM



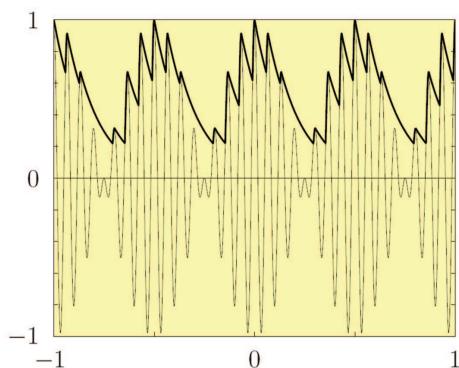
AM-SC – Suppressed Carrier

AM-SC: $x(t) = A m(t) \cos(2\pi f_c t)$

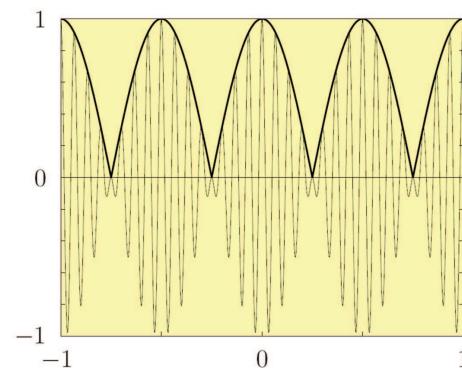


Demodulating AM-SC with an envelope detector:

Crystal receiver output



Envelope



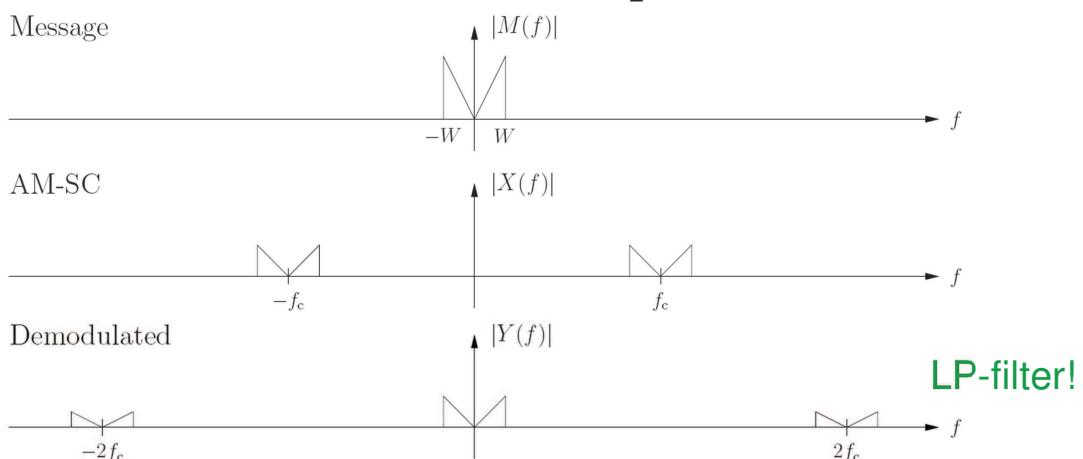
Spectrum of AM-SC – and Demodulation

$$x(t) = A m(t) \cos(2\pi f_c t) \quad \Rightarrow \quad X(f) = \frac{A}{2} (M(f - f_c) + M(f + f_c))$$

Coherent demodulation:

$$y(t) = 2 x(t) \cos(2\pi f_c t) = 2A m(t) \cos^2(2\pi f_c t) = A m(t) (1 + \cos(4\pi f_c t)),$$

$$Y(f) = A (X(f - f_c) + X(f + f_c)) = A \cdot M(f) + \frac{A}{2} (M(f - 2f_c) + M(f + 2f_c)).$$



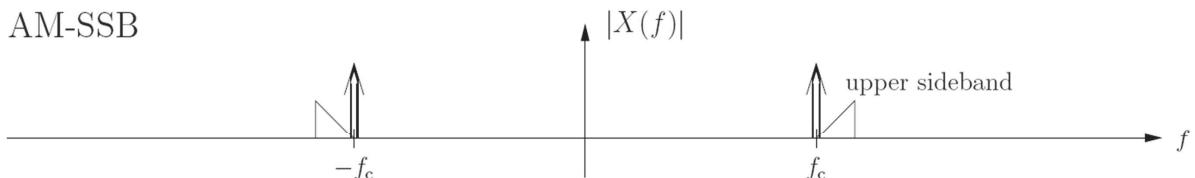
AM-SSB – Single Side-Band

AM (SC) uses twice as much bandwidth as needed.

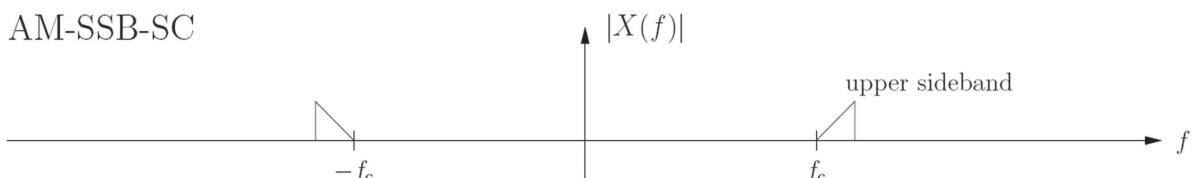
Each sideband contains all the information.

Filter out one of the sidebands.

AM-SSB

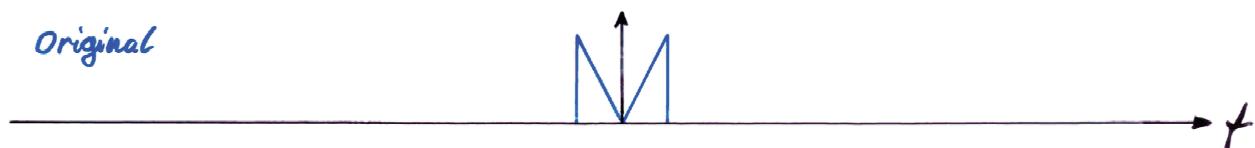


AM-SSB-SC

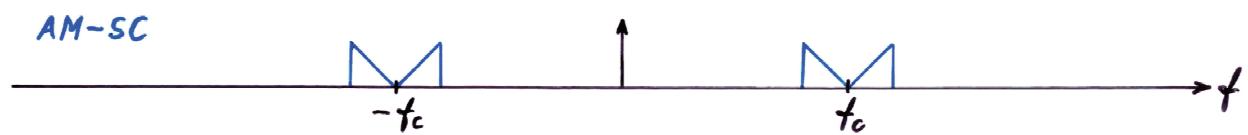


Demodulate SSB

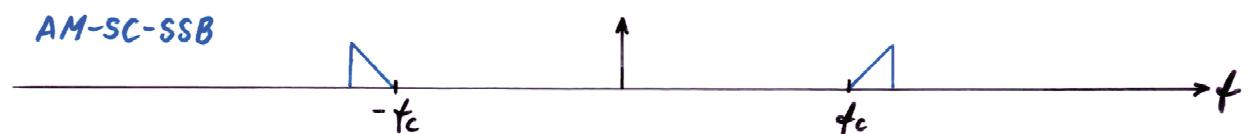
Original



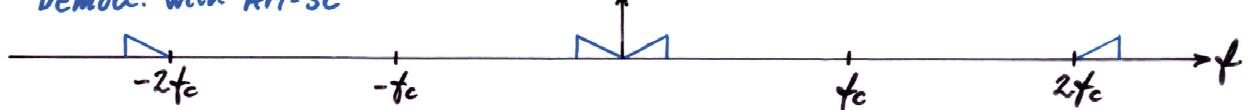
AM-SC



AM-SC-SSB



Demod. with AM-SC



LP-filter again!

AM of Stochastic Processes 1(3)

As for deterministic signals: $X(t) = A \cdot (C + M(t)) \cos(2\pi f_c t)$

Question: Is $X(t)$ stationary in any sense if $M(t)$ is stationary?

ACF:

$$\begin{aligned} r_X(t, t + \tau) &= \\ &= E \{ A(C + M(t)) \cos(2\pi f_c t) A(C + M(t + \tau)) \cos(2\pi f_c(t + \tau)) \}, \\ &= A^2 E \{ C^2 + C(M(t) + M(t + \tau)) + M(t)M(t + \tau) \} \cos(2\pi f_c t) \cos(2\pi f_c(t + \tau)) \\ &= \frac{A^2}{2} (C^2 + 2Cm_M + r_M(\tau)) (\cos(2\pi f_c(2t + \tau)) + \cos(2\pi f_c\tau)), \end{aligned}$$

Dependent on t . Non-stationary.

AM of Stochastic Processes 2(3)

Adjust the situation:

$$X(t) = A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi)$$

Question: Is $X(t)$ stationary now?

Unif. on $[0, 2\pi]$.
Indep of $X(t)$.

$$\begin{aligned} \text{Mean: } m_X(t) &= E\{X(t)\} = E\{A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi)\} \\ &= E\{A \cdot (C + M(t))\} \cdot E\{\cos(2\pi f_c t + \Psi)\}, \end{aligned}$$

$$\text{Note: } E\{\cos(2\pi f_c t + \Psi)\} = \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi f_c t + \psi) d\psi = 0,$$

Result: $m_X(t) = 0$

OK! Independent of t .
What about the ACF?

AM of Stochastic Processes 3(3)

Still this situation:

$$X(t) = A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi)$$

ACF:

$$\begin{aligned} r_X(t, t + \tau) &= E \{ A \cdot (C + M(t)) \cos(2\pi f_c t + \Psi) A \cdot (C + M(t + \tau)) \cos(2\pi f_c(t + \tau) + \Psi) \}, \\ &= A^2 E \{ C^2 + C(M(t) + M(t + \tau)) + M(t)M(t + \tau) \} \cdot \\ &\quad \cdot E \{ \cos(2\pi f_c t + \Psi) \cos(2\pi f_c(t + \tau) + \Psi) \}, \\ &= \frac{A^2}{2} (C^2 + 2Cm_M + r_M(\tau)) \left(E \{ \cos(2\pi f_c(2t + \tau) + 2\Psi) \} + \cos(2\pi f_c \tau) \right), \end{aligned}$$

Note: $E \{ \cos(2\pi f_c(2t + \tau) + 2\Psi) \} = \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi f_c(2t + \tau) + 2\psi) d\psi = 0$.

Result: $r_X(t, t + \tau) = \frac{A^2}{2} (C^2 + 2Cm_M + r_M(\tau)) \cos(2\pi f_c \tau)$,

OK! Independent of t . Stationary in the wide sense.

PSD of AM

Often we have: $m_M = 0$.

ACF: $r_X(\tau) = \frac{A^2}{2} (C^2 + r_M(\tau)) \cos(2\pi f_c \tau)$

PSD: $R_X(f) = \frac{A^2 C^2}{4} (\delta(f + f_c) + \delta(f - f_c)) + \frac{A^2}{4} (R_M(f + f_c) + R_M(f - f_c))$

Power: $P_X = r_X(0) = \frac{A^2}{2} (C^2 + r_M(0)) \cos(0) = \frac{A^2}{2} (C^2 + r_M(0)) = \frac{A^2}{2} (C^2 + P_M)$

For AM-SC:

$$r_X(\tau) = \frac{A^2}{2} r_M(\tau) \cos(2\pi f_c \tau),$$

$$R_X(f) = \frac{A^2}{4} (R_M(f + f_c) + R_M(f - f_c)),$$

$$P_X = \frac{A^2}{2} P_M.$$

Coherent Demodulation of AM-SC

Coherent demodulation:

$$Y(t) = 2X(t) \cos(2\pi f_c t + \Psi)$$

↑
Same Ψ as for modulation.

Demodulated signal:

$$Y(t) = 2A M(t) \cos^2(2\pi f_c t + \Psi) = A M(t) (1 + \cos(4\pi f_c t + 2\Psi)),$$

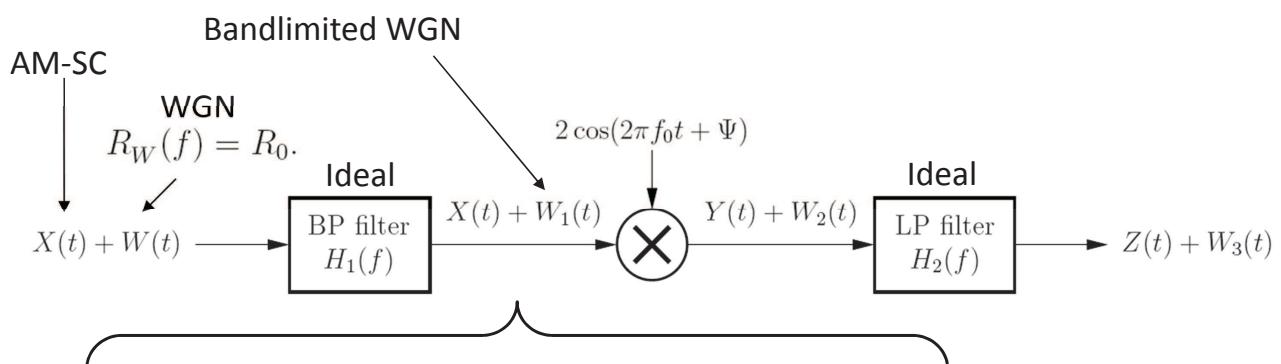
Result:

$$\begin{aligned} r_Y(\tau) &= A^2 r_M(\tau) + \frac{A^2}{2} r_M(\tau) \cos(4\pi f_c \tau), \\ R_Y(f) &= A^2 R_M(f) + \frac{A^2}{4} (R_M(f + 2f_c) + R_M(f - 2f_c)), \\ P_Y &= \frac{3A^2}{2} P_M. \end{aligned}$$

After ideal LP-filter:

$$r_Z(\tau) = A^2 r_M(\tau), \quad R_Z(f) = A^2 R_M(f), \quad P_Z = A^2 P_M.$$

Noise Analysis of AM-SC 1(4)



$$H_1(f) = \begin{cases} 1, & |f| - f_c | < B, \\ 0, & \text{elsewhere.} \end{cases}$$

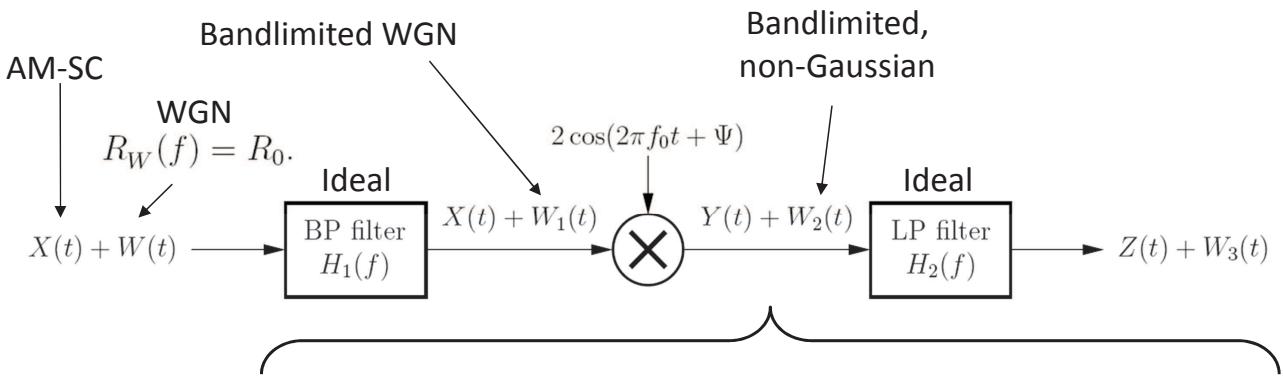
$$R_{W_1}(f) = |H_1(f)|^2 R_W(f) = \begin{cases} R_0, & |f| - f_c | < B, \\ 0, & \text{elsewhere,} \end{cases}$$

$$P_{W_1} = \int_{-\infty}^{\infty} R_{W_1}(f) df = 2 \int_{f_c-B}^{f_c+B} R_0 df = 4BR_0.$$

Input SNR

$$\frac{P_X}{P_{W_1}} = \frac{A^2 P_M / 2}{4BR_0} = \frac{A^2 P_M}{8BR_0}$$

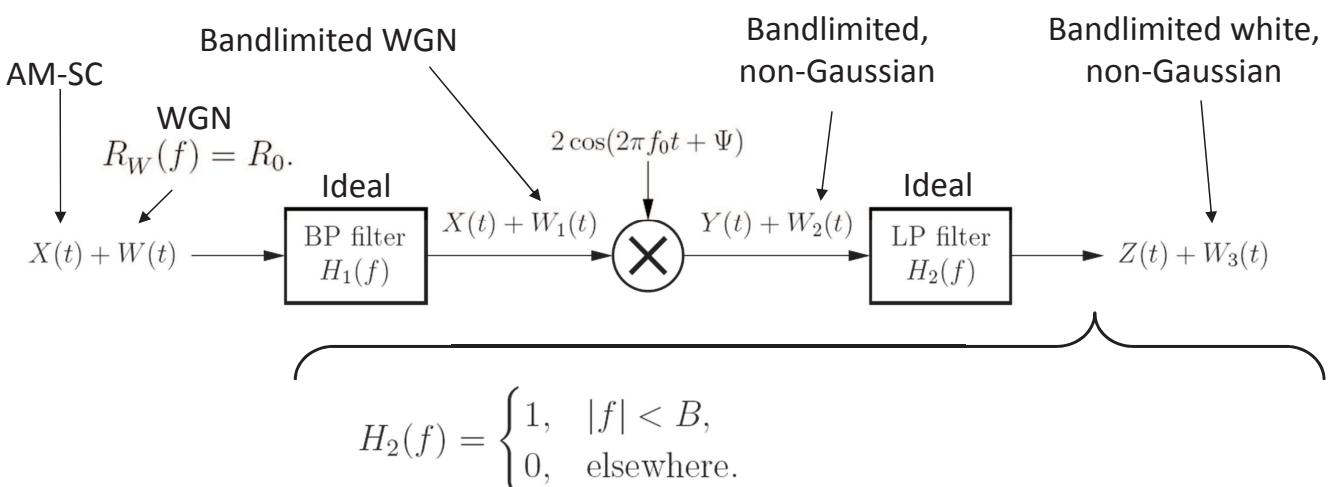
Noise Analysis of AM-SC 2(4)



$$W_2(t) = 2W_1(t) \cos(2\pi f_c t + \Psi)$$

$$R_{W_2}(f) = R_{W_1}(f + f_c) + R_{W_1}(f - f_c) = \begin{cases} 2R_0, & |f| < B, \\ R_0, & ||f| - 2f_c| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

Noise Analysis of AM-SC 3(4)



$$H_2(f) = \begin{cases} 1, & |f| < B, \\ 0, & \text{elsewhere.} \end{cases}$$

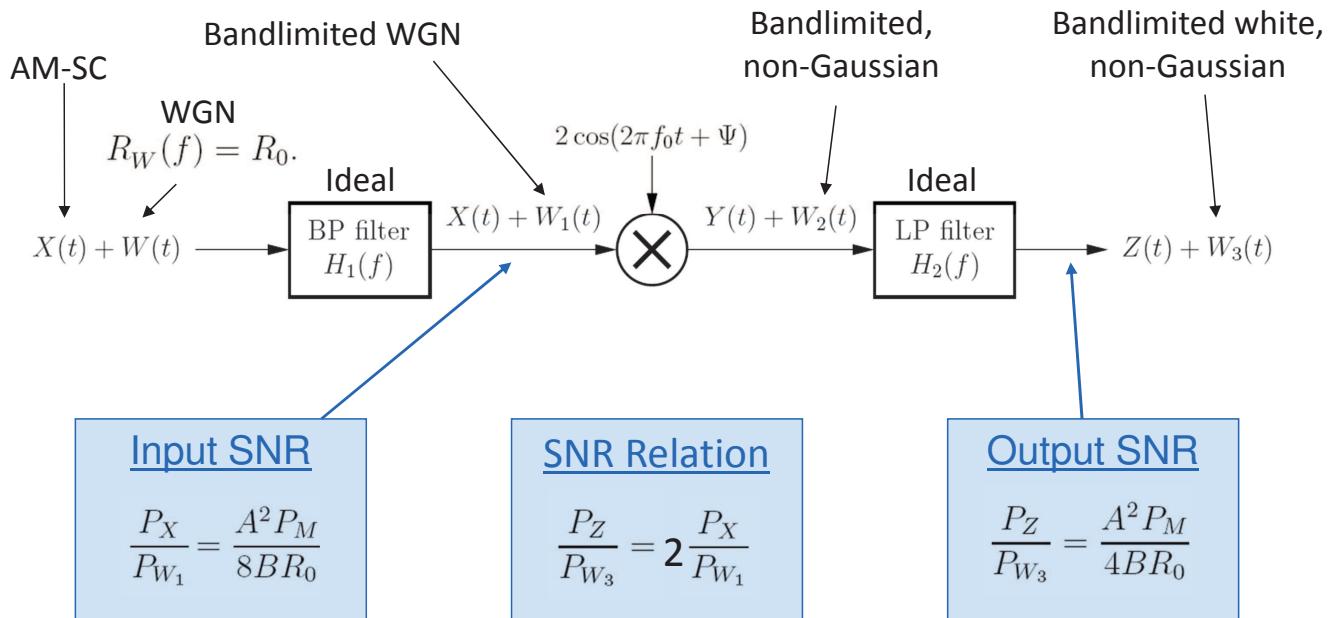
Output SNR

$$\frac{P_Z}{P_{W_3}} = \frac{A^2 P_M}{4BR_0}$$

$$R_{W_3}(f) = |H_2(f)|^2 R_{W_2}(f) = \begin{cases} 2R_0, & |f| < B, \\ 0, & \text{elsewhere,} \end{cases}$$

$$P_{W_3} = \int_{-\infty}^{\infty} R_{W_3}(f) df = \int_{-B}^{B} 2R_0 df = 4BR_0.$$

Noise Analysis of AM-SC 4(4)



Twice the bandwidth – Twice the SNR.

Angle Modulation

- The major modulation techniques used in radio broadcasts today are examples of angle modulation.
 - FM – Frequency Modulation
 - PM – Phase Modulation
- Nonlinear modulation techniques.
- Complicated to analyze.
- Still fairly simple demodulation.
- Less sensitive to noise than AM.

Angle Modulation – Modulation Indices

Angle modulation: $x(t) = A \cdot \cos \left(2\pi f_c t + \underbrace{\phi\{m(t)\}}_{\text{Momentary phase}} \right)$

Phase deviation: $\phi_d(t) = \phi\{m(t)\}$, for mean 0.

Phase modulation index: $\mu_p = \phi_{d,\max} = \max |\phi_d(t)|$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt} (2\pi f_c t + \phi\{m(t)\}) = f_c + \frac{1}{2\pi} \cdot \frac{d}{dt} \phi\{m(t)\}$

Frequency deviation: $f_d(t) = f_{\text{mom}}(t) - f_c = \frac{1}{2\pi} \cdot \frac{d}{dt} \phi\{m(t)\}$

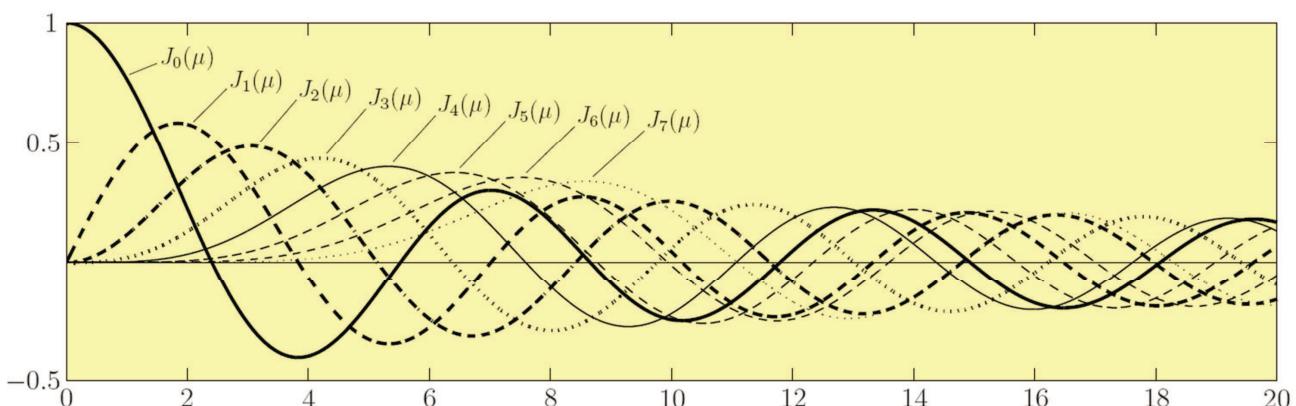
Frequency modulation idx: $\mu_f = \frac{f_{d,\max}}{B}$ $f_{d,\max} = \max |f_d(t)|$

Spectrum of Angle Modulation 1(2)

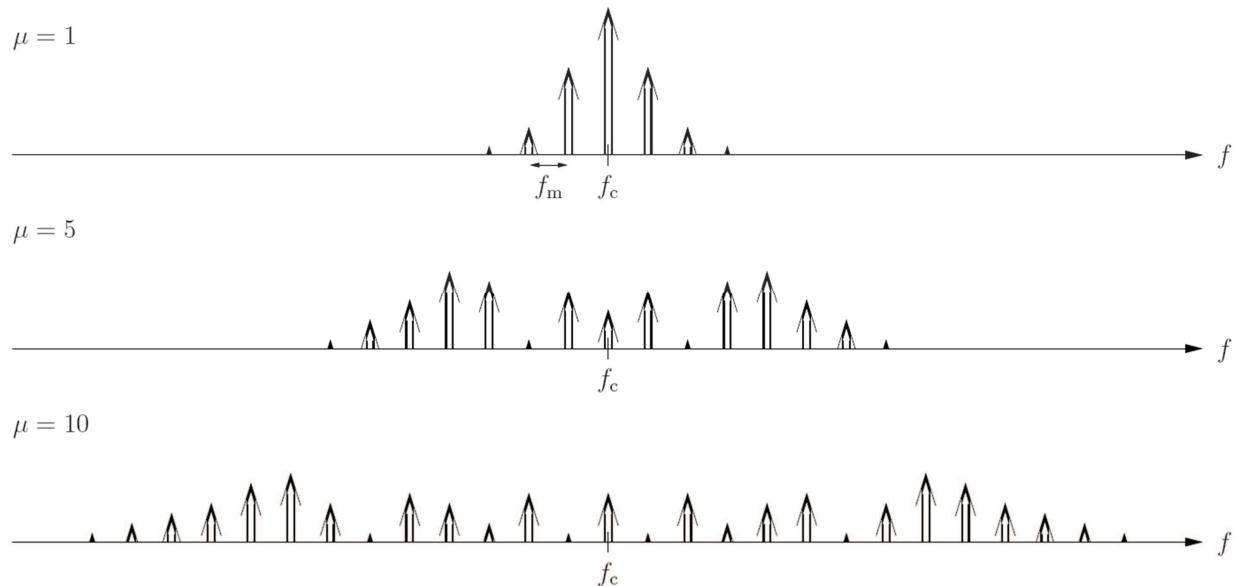
Example: $x(t) = A \cdot \cos (2\pi f_c t + \mu \sin(2\pi f_m t)) = \sum_{n=-\infty}^{\infty} A \cdot J_n(\mu) \cos (2\pi(f_c + n f_m)t)$

Spectrum: $X(f) = \sum_{n=-\infty}^{\infty} \frac{A \cdot J_n(\mu)}{2} (\delta(f + f_c + n f_m) + \delta(f - f_c - n f_m)).$

Bessel functions of the first kind of order n : $J_n(\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{\mu}{2}\right)^{n+2k}$



Spectrum of Angle Modulation 2(2)



Carsons rule: Bandwidth $\approx 2(\mu+1)f_m = 2\left(1 + \frac{1}{\mu}\right) f_{d,\max}$.

PM – Phase Modulation

Momentary phase: $\phi\{m(t)\} = a \cdot m(t),$

Signal: $x(t) = A \cdot \cos(2\pi f_c t + a \cdot m(t)).$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt}(2\pi f_c t + a \cdot m(t)) = f_c + \frac{a}{2\pi} \cdot \frac{d}{dt}m(t),$

Frequency deviation: $f_d(t) = \frac{a}{2\pi} \cdot \frac{d}{dt}m(t).$

Peak frequency dev.: $f_{d,\max} = \frac{a}{2\pi} \cdot \max \left| \frac{d}{dt}m(t) \right|$

Frequency mod. idx: $\mu_f = \frac{a}{2\pi B} \cdot \max \left| \frac{d}{dt}m(t) \right|$

FM – Frequency Modulation

Momentary phase:

$$\phi\{m(t)\} = a \int m(t) dt, \quad \phi\{m(t)\} = a \int_{t_0}^t m(\tau) d\tau,$$

Signal:

$$x(t) = \cos \left(2\pi f_c t + a \int m(t) dt \right)$$

Momentary frequency: $f_{\text{mom}}(t) = \frac{1}{2\pi} \cdot \frac{d}{dt} \left(2\pi f_c t + a \int m(t) dt \right) = f_c + \frac{a}{2\pi} \cdot m(t).$

Frequency deviation:

$$f_d(t) = \frac{a}{2\pi} \cdot m(t).$$

Peak frequency dev.:

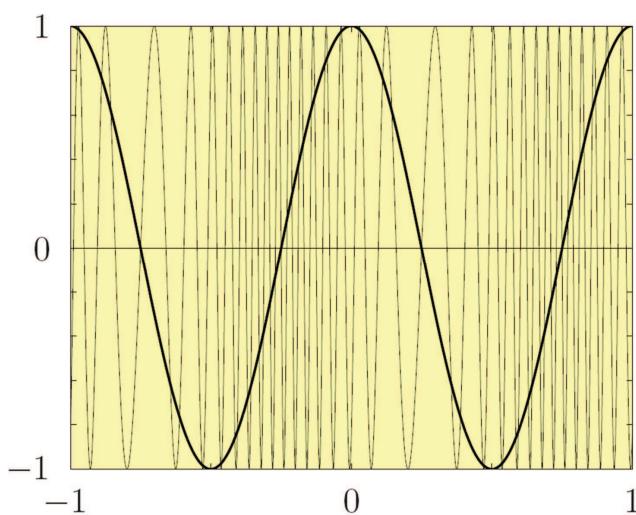
$$f_{d,\max} = \frac{a}{2\pi} \cdot \max |m(t)|$$

Frequency mod. idx:

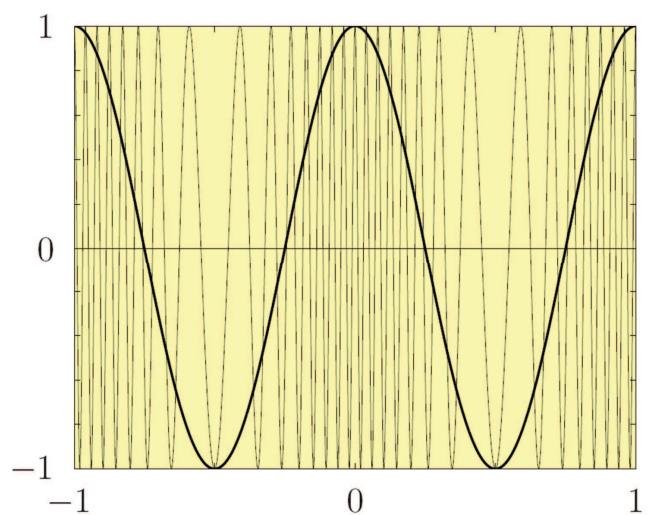
$$\mu_f = \frac{a}{2\pi B} \cdot \max |m(t)|$$

Angle Modulation in the Time Domain

Phase Modulation



Frequency Modulation



Demodulating Phase Modulation

Sent signal: $x(t) = A \cdot \cos(2\pi f_c t + \phi\{m(t)\})$

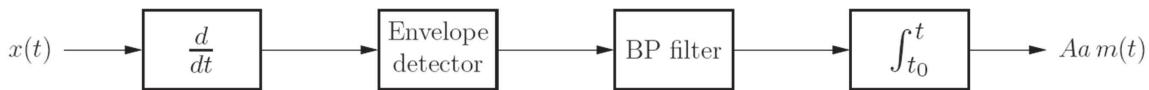
Derivative: $\frac{d}{dt}x(t) = -A \left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\} \right) \sin(2\pi f_c t + \phi\{m(t)\})$

Envelope detector gives us: $A \left(2\pi f_c + \frac{d}{dt}\phi\{m(t)\} \right)$

BP or HP filter gives us: $A \frac{d}{dt}\phi\{m(t)\}$

For PM: $\phi\{m(t)\} = am(t).$

Integrate: $\int_{t_0}^t A \frac{d}{d\tau}\phi\{m(\tau)\} d\tau = Aa(m(t) - m(t_0))$ Filter!



Demodulating Frequency Modulation

As for PM: Derivative, envelope detection and HP filter gives us:

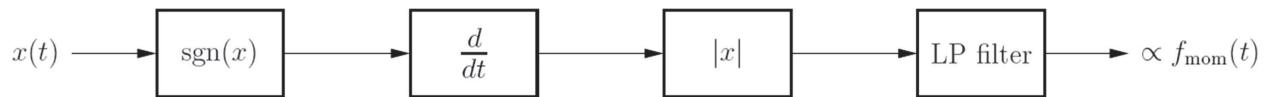
$$A \frac{d}{dt}\phi\{m(t)\}$$

For FM: $\phi\{m(t)\} = a \int m(t) dt.$

Then: $A \frac{d}{dt}\phi\{m(t)\} = A \frac{d}{dt} a \int m(t) dt = Aa m(t)$ Done!



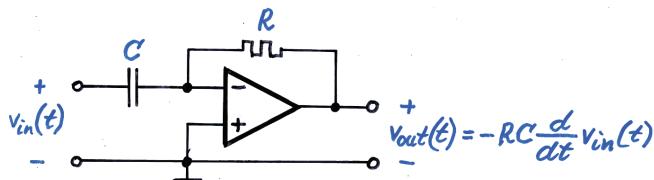
Alternative Demodulation



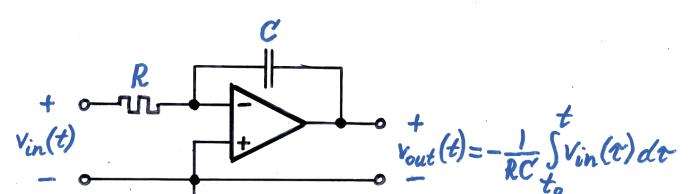
$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Building Blocks

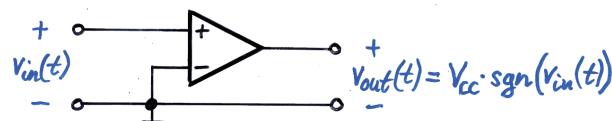
Derivative



Integration



The Sign function



Mikael Olofsson
ISY/CommSys

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