TSDT14 Signal Theory

Lecture 11

Multi-Dimensional Processes – Primarily 2-D

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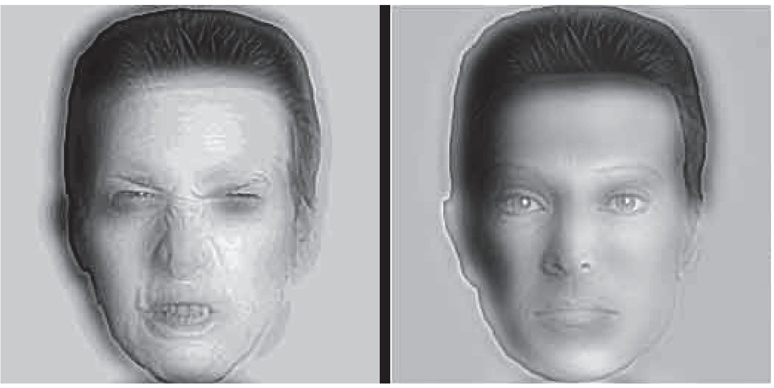
Example: 2-D Filtering

Mr. Angry



Mrs. Calm

Source: http://www.grand-illusions.com/opticalillusions/angry_and_calm/

Example: 2-D Filtering

Mr. Angry ?

? Mrs. Calm

Source: http://www.grand-illusions.com/opticalillusions/angry_and_calm/

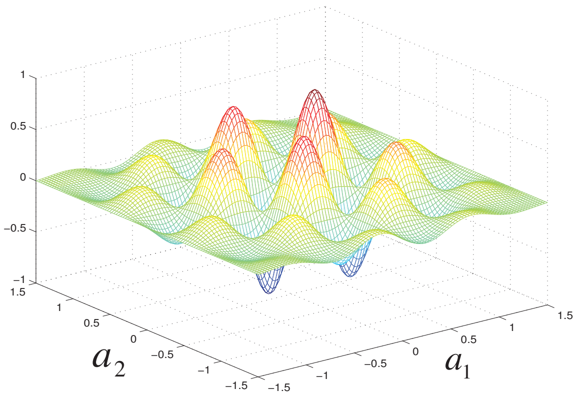
Multi-Dimensional Signals & Systems

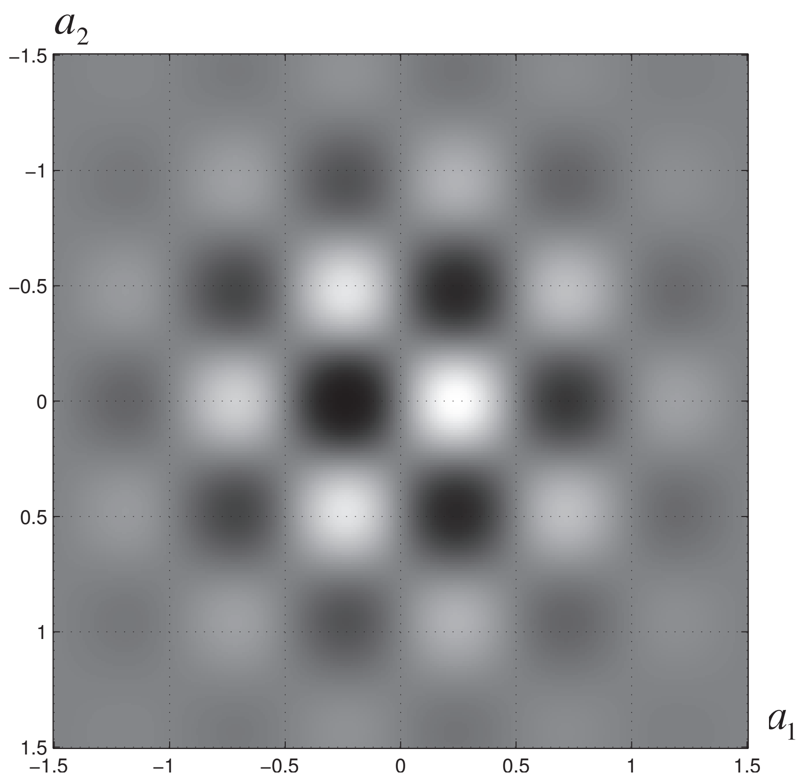
Two-Dimensional Signals

A function of two variables

$x(a_1, a_2)$ (space-continuous)

$x[n_1, n_2]$ (space-discrete)





Separable signals:

$$x(a_1, a_2) = x_1(a_1) \cdot x_2(a_2)$$

$$x[n_1, n_2] = x_1[n_1] \cdot x_2[n_2]$$

Two-Dimensional Stochastic Processes

Mean: $m_X(a_1, a_2) = \mathrm{E}\{X(a_1, a_2)\}$

Auto-correlation: $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2) = \mathrm{E}\{X(a_1, a_2) X(a_1 + b_1, a_2 + b_2)\}$

Wide-sense stationarity:

Both $m_X(a_1, a_2)$ and $r_X(a_1, a_2; a_1 + b_1, a_2 + b_2)$ independent of (a_1, a_2).

Simplified notation: m_X and $r_X(b_1, b_2)$

Classification of Systems

Linearity: Input $x_1(a_1,a_2)$ gives output $y_1(a_1,a_2)$.
Input $x_2(a_1,a_2)$ gives output $y_2(a_1,a_2)$.

Then input $a\cdot x_1(a_1,a_2)+b\cdot x_2(a_1,a_2)$
gives output $a\cdot y_1(a_1,a_2)+b\cdot y_2(a_1,a_2)$.

Space-invariance: Input $x(a_1,a_2)$ gives output $y(a_1,a_2)$.
Then input $x(a_1-b_1,a_2-b_2)$
gives output $y(a_1-b_1,a_2-b_2)$.

LSI: Both linear and space-invariant.

Similarily for space-discrete systems

Two-Dimensional Convolution

For LSI systems, the output is given by a two-dimensional convolution of the input and the impulse response of the filter.

Definition:

$$(x\circledast h)(a_1,a_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(b_1,b_2)h(a_1-b_1,a_2-b_2)\,db_1db_2$$

Separable signals:

$$(x\circledast h)(a_1,a_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1(b_1)x_2(b_2)\,h_1(a_1-b_1)\,h_2(a_2-b_2)\,db_1db_2$$

$$=\int_{-\infty}^{\infty}x_1(b_1)h_1(a_1-b_1)\,db_1\int_{-\infty}^{\infty}x_2(b_2)h_2(a_2-b_2)\,db_2$$

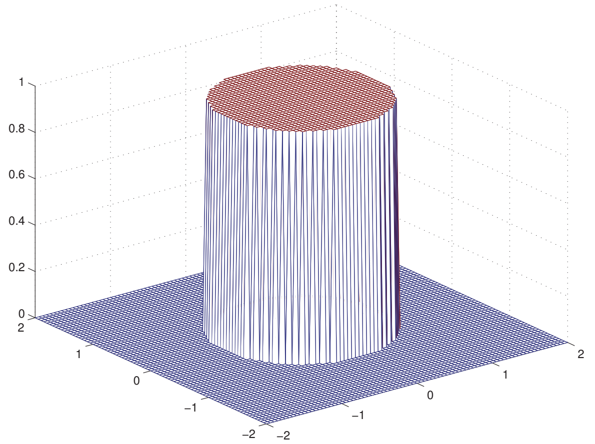
$$=(x_1*h_1)(a_1)\cdot(x_2*h_2)(a_2)$$

Space-discrete:

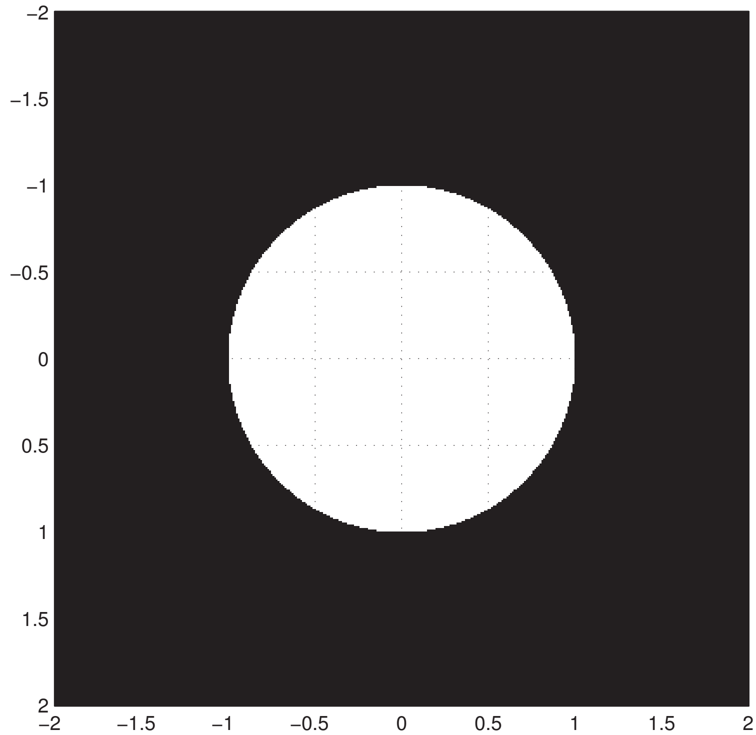
$$(x\circledast h)[n_1,n_2]=\sum_{k_1}\sum_{k_2}x[k_1,k_2]\cdot h[n_1-k_1,n_2-k_2]$$

Non-Separable Convolution 1(2)

$$x(a_1,a_2)=h(a_1,a_2)=\begin{cases}1, & a_1^2+a_2^2<1\\ 0, & \text{elsewhere}\end{cases}$$



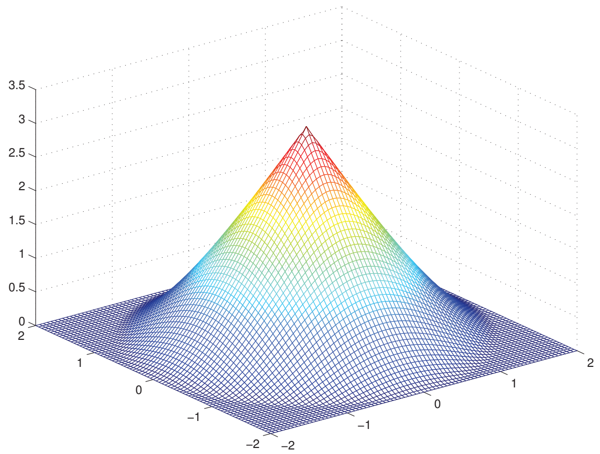
Non-separable signals

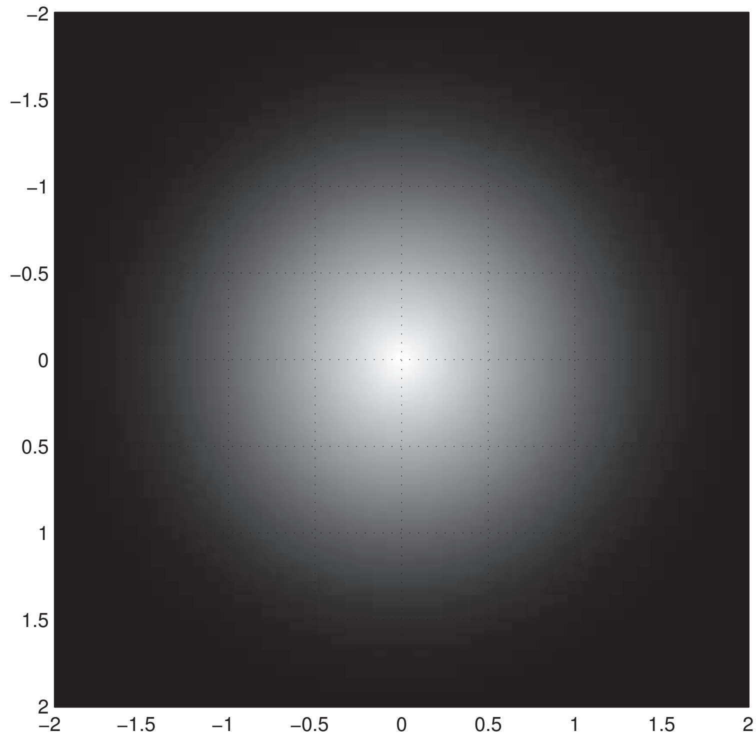


Non-Separable Convolution 2(2)

$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$

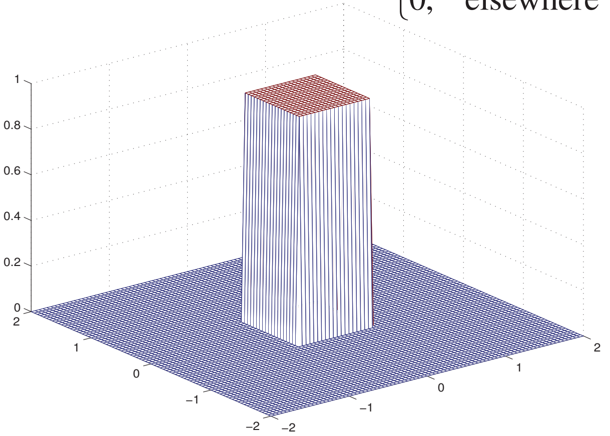
$$=\begin{cases}2\arccos(a/2)-a\sqrt{1-(a/2)^2}, & a^2=a_1^2+a_2^2<4\\ 0, & \text{elsewhere}\end{cases}$$



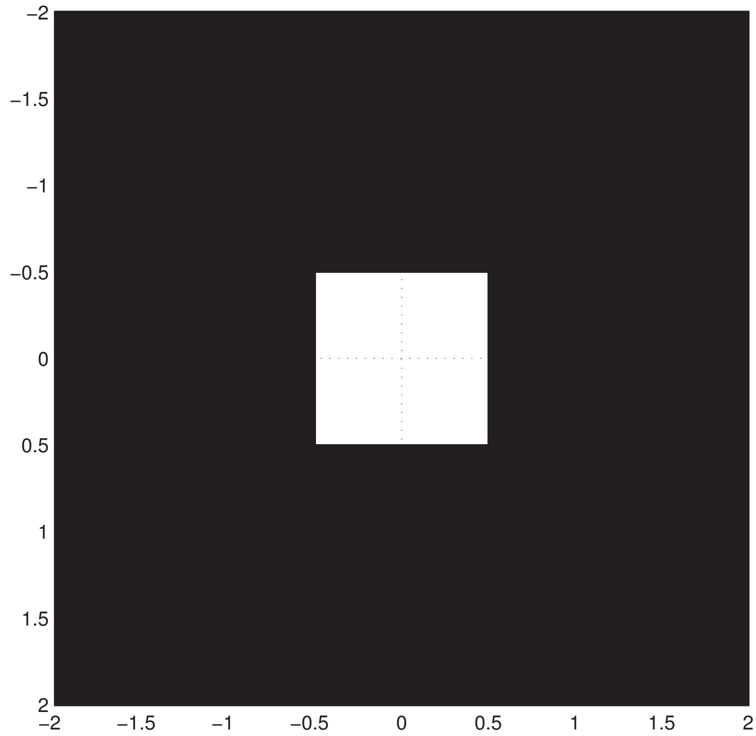


Separable Convolution 1(3)

$$x(a_1,a_2)=h(a_1,a_2)=$$
$$=\begin{cases}1, & |a_1|<1/2 \quad \text{and} \quad |a_2|<1/2\\ 0, & \text{elsewhere}\end{cases}$$
$$=z(a_1)\cdot z(a_2)$$
with $z(a)=\text{rect}(a)=\begin{cases}1, & |a|<1/2\\ 0, & \text{elsewhere}\end{cases}$

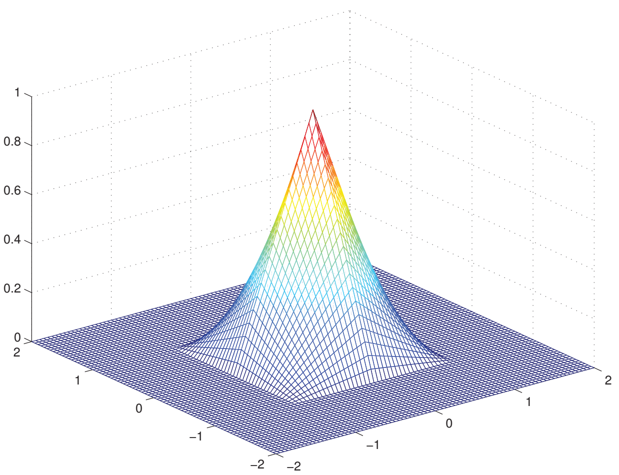


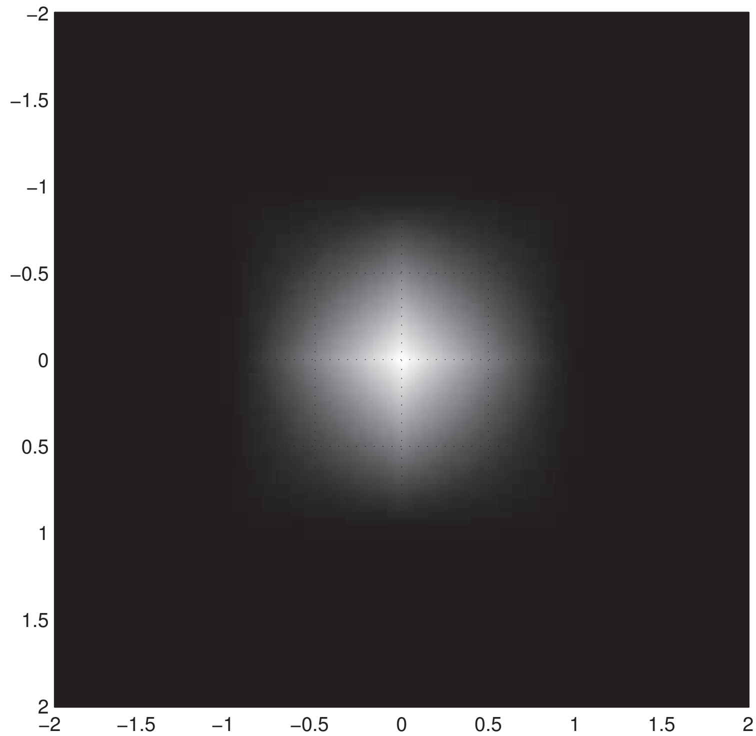
Separable signals



Separable Convolution 2(3)

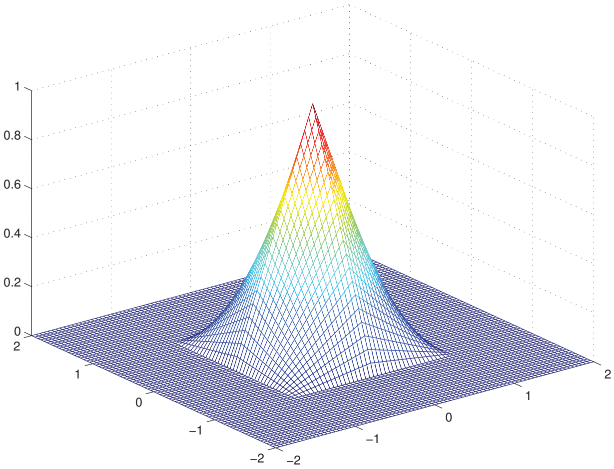
$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$
$$=\max\{0,1-|a_1|\}\cdot\max\{0,1-|a_2|\}$$
$$=\text{triangle}(a_1)\cdot\text{triangle}(a_2)$$

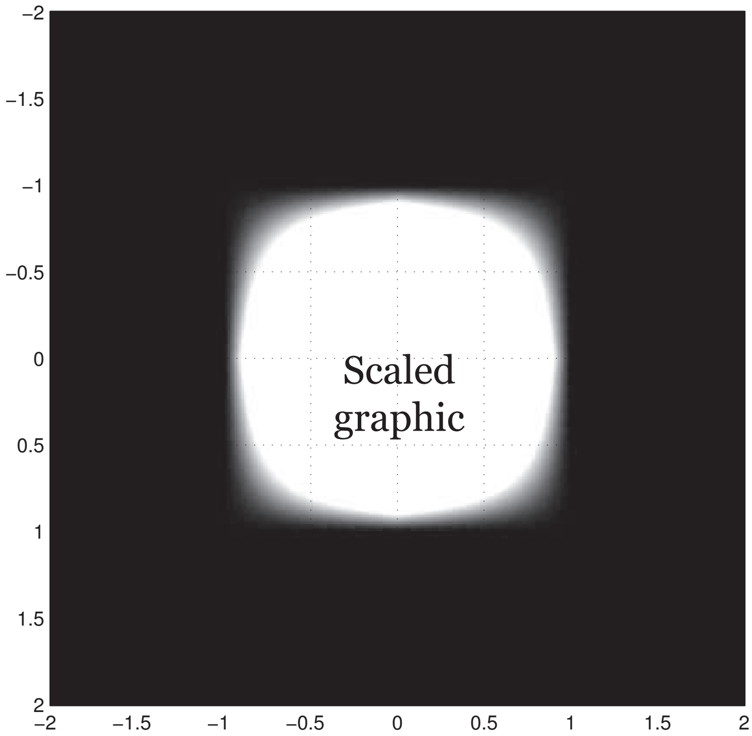




Separable Convolution 3(3)

$$y(a_1,a_2)=(x\circledast h)(a_1,a_2)$$
$$=\max\{0,1-|a_1|\}\cdot\max\{0,1-|a_2|\}$$
$$=\text{triangle}(a_1)\cdot\text{triangle}(a_2)$$





2-D Space-Continuous Fourier Transform

Definition: $$X(f_1,f_2)=\mathcal{F}\{x(a_1,a_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(a_1,a_2)e^{-j2\pi(f_1a_1+f_2a_2)}da_1da_2$$

Inverse: $$x(a_1,a_2)=\mathcal{F}^{-1}\{X(f_1,f_2)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X(f_1,f_2)e^{j2\pi(f_1a_1+f_2a_2)}df_1df_2$$

Properties: $$\mathcal{F}\{x(a_1,a_2)\}=\mathcal{F}_1\{\mathcal{F}_2\{x(a_1,a_2)\}\}=\mathcal{F}_2\{\mathcal{F}_1\{x(a_1,a_2)\}\}$$

$$\mathcal{F}\{x_1(a_1)\cdot x_2(a_2)\}=\mathcal{F}_1\{x_1(a_1)\}\cdot\mathcal{F}_2\{x_2(a_2)\}$$

$$\mathcal{F}\{(x\circledast h)(a_1,a_2)\}=X(f_1,f_2)\cdot H(f_1,f_2)$$

$$\mathcal{F}\{x(a_1,a_2)\cdot h(a_1,a_2)\}=(X\circledast H)(f_1,f_2)$$

2-D Space-Discrete Fourier Transform

Definition: $X[\theta_1,\theta_2] = \mathcal{F}\{x[n_1,n_2]\} = \sum_{n_1}\sum_{n_2} x[n_1,n_2]e^{-j2\pi(\theta_1 n_1+\theta_2 n_2)}$

Inverse: $x[a_1,a_2] = \mathcal{F}^{-1}\{X[\theta_1,\theta_2]\} = \int_0^1\int_0^1 X[\theta_1,\theta_2]e^{j2\pi(\theta_1 n_1+\theta_2 n_2)}\,d\theta_1 d\theta_2$

Properties: $X[\theta_1,\theta_2] = X[\theta_1+m_1,\theta_2+m_2]$ for integers m_1 and m_2

$\mathcal{F}\{x[n_1,n_2]\} = \mathcal{F}_1\{\mathcal{F}_2\{x[n_1,n_2]\}\} = \mathcal{F}_2\{\mathcal{F}_1\{x[n_1,n_2]\}\}$

$\mathcal{F}\{x_1[n_1]\cdot x_2[n_2]\} = \mathcal{F}_1\{x_1[n_1]\}\cdot\mathcal{F}_2\{x_2[n_2]\}$

$\mathcal{F}\{(x \circledast h)[n_1,n_2]\} = X[\theta_1,\theta_2]\cdot H[\theta_1,\theta_2]$

$\mathcal{F}\{x[n_1,n_2]\cdot h[n_1,n_2]\} = (X \circledast H)[\theta_1,\theta_2]$ (periodic conv)

Power and Power Spectral Density

PSD: $R_X(f_1,f_2) = \mathcal{F}\{r_X(b_1,b_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_X(b_1,b_2)e^{-j2\pi(f_1 b_1+f_2 b_2)}\,db_1 db_2$

Power: $P_X = E\{X^2(a_1,a_2)\} = r_X(0,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(f_1,f_2)\,df_1 df_2$

Output of LSI-system if input is WSS:

Output: $Y(a_1,a_2) = (X \circledast h)(a_1,a_2)$

Mean: $m_Y = m_X \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(a_1,a_2)\,da_1 da_2 = m_X \cdot H(0,0)$

ACF: $r_Y(b_1,b_2) = (h \circledast \tilde{h} \circledast r_X)(b_1,b_2)$ with $\tilde{h}(a_1,a_2) = h(-a_1,-a_2)$

PSD: $R_Y(f_1,f_2) = |H(f_1,f_2)|^2 R_X(f_1,f_2)$

Two-Dimensional Sampling

Deterministic (for rectangular grid):

Sampling: $y[n_1, n_2] = x(n_1 A_1, n_2 A_2)$

Sampling periods: A_1 and A_2

Spectrum: $Y[\theta_1, \theta_2] = \frac{1}{A_1 A_2} \sum_{k_1} \sum_{k_2} X\left(\frac{\theta_1 - k_1}{A_1}, \frac{\theta_2 - k_2}{A_2} \right)$

Probabilistic (for rectangular grid):

Sampling: $Y[n_1, n_2] = X(n_1 A_1, n_2 A_2)$

PSD: $R_Y[\theta_1, \theta_2] = \frac{1}{A_1 A_2} \sum_{k_1} \sum_{k_2} R_X\left(\frac{\theta_1 - k_1}{A_1}, \frac{\theta_2 - k_2}{A_2} \right)$

Two-Dimensional PAM

Deterministic (for rectangular grid):

PAM: $z(a_1, a_2) = \sum_{n_1} \sum_{n_2} y[n_1, n_2] p(a_1 - n_1 A_1, a_2 - n_2 A_2)$

Spectrum: $Z(f_1, f_2) = P(f_1, f_2) \cdot Y[f_1 A_1, f_2 A_2]$

Probabilistic (for rectangular grid):

PAM: $Z(a_1, a_2) = \sum_{n_1} \sum_{n_2} Y[n_1, n_2] p(a_1 - n_1 A_1 - \Psi_1, a_2 - n_2 A_2 - \Psi_2)$

Ψ_1 uniform on $[0, A_1)$ and Ψ_2 uniform on $[0, A_2)$ both independent of $Y[n_1, n_2]$ and of each other.

PSD: $R_Z(f_1, f_2) = \frac{1}{A_1 A_2} |P(f_1, f_2)|^2 R_Y[f_1 A_1, f_2 A_2]$

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Rounding Up the Course

Stochastic processes: Stationarity, ergodicity, mean, ACF, PSD...

LTI filtering: Mean, ACF, PSD.

Cross-correlation and cross-spectrum. Joint stationarity.

Poisson processes.

Prediction.

Non-linearities: Squaring and such, saturation, quantization.

Modulation: AM, FM, PM, noise.

Estimation (only on laborations).

Linear mappings: Sampling, PAM, reconstruction.

Two-dimensional: Signals, systems,...

Written Examination

When: Friday 2015-10-30, 14.00-18.00. Sign up!

Allowed aids:

Olofsson: Tables and Formulas for Signal Theory

~~Henriksson/Lindman: Formelsamling i Signalteori~~

Pocket calculator with empty memory

A German 10 mark note of the fourth series (1991-2001)

What:

A three-part introductory task (simple, 2/3 must be OK).

Five problems – 5 points each, pass is 10 points.

Written Examination – cont'd

A German 10 mark note of the fourth series (1991-2001)

Good Practices at Exams

Rules according to the exam cover:

- Only one task on the same piece of paper.
- Use only one side of the paper.
- Number the pages.
  (see common sense →)
- Do not use a red pen(cil).
  (that's my color)

Let me add:

- Hand in readable solutions.
- Do not hand in scriblings!

Common sense:

1. Solve the exam problems.
2. Sort the papers according to task numbering.
3. Number the pages last!
4. Now hand in your exam.

Do not do it in any other order!

Finally:

- Always provide solid arguments for steps taken in your solutions.