TSDT14 Signal Theory

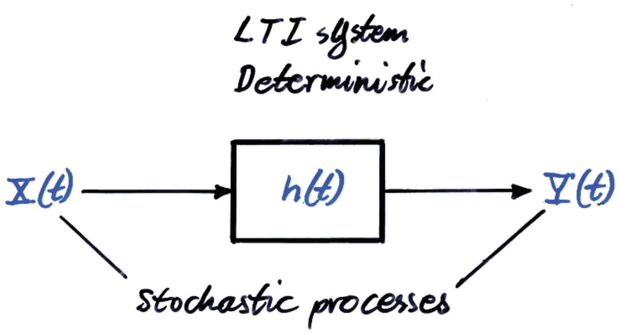
Lecture 3

LTI Filtering, White Noise, Colored Noise

Mikael Olofsson
Department of EE (ISY)
Div. of Communication Systems

LINKÖPINGS UNIVERSITET

Filtering Stochastic Processes



$$Y(t) = (X * h)(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Demand: stability : $\int_{-\infty}^{\infty} |h(t)|\, dt$ convergent.

Holds regardless of stationarity.

Expectation of the Output

Notation: $H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t}\, dt$

Expectation: $m_Y(t) = E\{Y(t)\} = E\left\{\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right\}$

Expectation is linear

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} h(\tau) \cdot E\{X(t-\tau)\}\, d\tau = \int_{-\infty}^{\infty} h(\tau) \cdot m_X(t-\tau)\, d\tau$$

X(t) WSS — Identify

$$\overset{\downarrow}{=} m_X \cdot \int_{-\infty}^{\infty} h(\tau)\, d\tau \overset{\downarrow}{=} m_X \cdot H(0)$$

Thus: $m_Y(t)$ is independent of t.

ACF of the Output

ACF: $r_Y(t, t+\tau) = E\{Y(t)Y(t+\tau)\} = E\left\{\int_{-\infty}^{\infty} h(\tau_1) X(t-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, X(t+\tau-\tau_2)\, d\tau_2\right\}$

E{...} Linear

$$\overset{\downarrow}{=} \iint_{-\infty}^{\infty} h(\tau_1) h(\tau_2)\, E\{X(t-\tau_1) X(t+\tau-\tau_2)\}\, d\tau_1\, d\tau_2$$

X(t) WSS

$$\overset{\downarrow}{=} \iint_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, r_X(\tau+\tau_1-\tau_2)\, d\tau_1\, d\tau_2 = \Big/ \begin{array}{l} \tilde{h}(t) = h(-t) \\ \tau_3 = -\tau_1 \end{array} \Big/$$

$$= \iint_{-\infty}^{\infty} \tilde{h}(\tau_3)\, h(\tau_2)\, r_X(\tau-\tau_3-\tau_2)\, d\tau_3\, d\tau_2 = (h * \tilde{h} * r_X)(\tau)$$

Thus: $r_Y(t, t+\tau)$ independent of t, and we write

$$r_Y(\tau) = (h * \tilde{h} * r_X)(\tau)$$

PSD: $R_Y(f) = H(f) \cdot H^*(f) \cdot R_X(f) = |H(f)|^2\, R_X(f)$

Example Filtering

Let $X(t)$ be a wide sense stationary process with $r_X(\tau) = e^{-|\tau|}$

$$H(f) = \begin{cases} 1, & |f| \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the output power $E\{Y^2(t)\}$.

$$E\{Y^2(t)\} = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 \cdot R_X(f)\,df$$

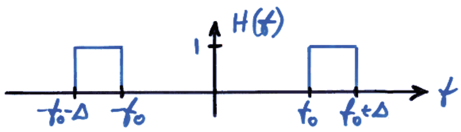
$$= \int_{-1}^{1} R_X(f)\,df \underset{\text{Table}}{=} \int_{-1}^{1} \frac{2}{1+(2\pi f)^2}\,df = \left/ \begin{array}{l} \omega = 2\pi f \\ d\omega = 2\pi\,df \end{array} \right/$$

$$= \frac{1}{\pi} \int_{-2\pi}^{2\pi} \frac{1}{1+\omega^2}\,d\omega \underset{\text{Std integral}}{=} \frac{1}{\pi}\Big[\arctan(\omega)\Big]_{-2\pi}^{2\pi} = \frac{2}{\pi}\arctan(2\pi)$$

Argument for $R_X(f) \ge 0$

$$H(f) = \begin{cases} 1, & f_0 \le |f| \le f_0 + \Delta \\ 0, & \text{elsewhere} \end{cases}$$

$\Delta \ge 0$



$$0 \le E\{Y^2(t)\} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot R_X(f)\,df = 2 \cdot \int_{f_0}^{f_0+\Delta} R_X(f)\,df$$

$$\underset{\Delta \text{ small}}{\approx} 2 \cdot \Delta \cdot R_X(f_0) \quad \Rightarrow \quad R_X(f_0) \approx \frac{E\{Y^2(t)\}}{2 \cdot \Delta} \ge 0$$

More precisely:

$$R_X(f_0) = \lim_{\Delta \to 0} \frac{E\{Y^2(t)\}}{2\Delta} \ge 0$$

Normalized Filters

Normalized filter: $\int_{-\infty}^{\infty} |h(t)|^2 dt = 1$

Input: $X(t)$, WGN, $R_X(f) = R_0$, $m_X = 0$.

Output: $R_Y(f) = |H(f)|^2 R_X(f)$

$m_Y = H(0) \cdot m_X = 0$

$\sigma_Y^2 = E\{Y^2(t)\} = r_Y(0) = \int_{-\infty}^{\infty} R_Y(f)\, df$

$= R_0 \cdot \int_{-\infty}^{\infty} |H(f)|^2 df \underset{\text{Parseval}}{=} R_0 \cdot \int_{-\infty}^{\infty} |h(t)|^2 dt \underset{\text{Normalized filter}}{=} R_0$

And LTI + Gaussian input $\Rightarrow$ $Y(t)$ Gaussian.

Cross-Correlation

Two Processes: $X(t)$ & $Y(t)$

Cross-Correlation: $r_{X,Y}(t_1, t_2) = E\{X(t_1) Y(t_2)\}$

Properties: $r_{X,X}(t_1, t_2) = r_X(t_1, t_2)$

$r_{X,Y}(t_1, t_2) = r_{Y,X}(t_2, t_1)$

$r_{X,Y}^2(t_1, t_2) \leq r_X(t_1, t_1) \cdot r_Y(t_2, t_2)$

Uncorrelated and Independent Processes

Definition: Consider two processes $X(t)$ and $Y(t)$ and sample them in the time instances $\bar{t}_1 = (t_{11}, \dots, t_{1N})$ and $t_2 = (t_{21}, \dots, t_{2N})$, respectively. The processes are said to be <u>independent</u> if $X(\bar{t}_1)$ and $Y(\bar{t}_2)$ are independent, i.e. if

$$F_{X(\bar{t}_1), Y(\bar{t}_2)}(\bar{x}, \bar{y}) = F_{X(\bar{t}_1)}(\bar{x}) \cdot F_{Y(\bar{t}_2)}(\bar{y})$$

holds for every N, every $\bar{t}_1$ and every $\bar{t}_2$.

Definition: Two processes $X(t)$ and $Y(t)$ are said to be uncorrelated if

$$r_{X,Y}(t_1, t_2) = m_X(t_1) \cdot m_Y(t_2)$$

holds for all t_1 and t_2.

Relation: Independent $\Rightarrow$ Uncorrelated.

Jointly Gaussian Processes

Definition: Consider two processes $X(t)$ and $Y(t)$ and sample them in the time instances $\bar{t}_1 = (t_{11}, \dots, t_{1N})$ and $\bar{t}_2 = (t_{21}, \dots, t_{2N})$, respectively. The processes are said to be <u>jointly Gaussian</u> if $[X(\bar{t}_1), Y(\bar{t}_2)]$ are jointly Gaussian for every N, every $\bar{t}_1$ and every $\bar{t}_2$.

Theorem: If $X(t)$ and $Y(t)$ are uncorrelated and jointly Gaussian, they are also independent.

Joint Stationarity

Definition: The processes $X(t)$ and $Y(t)$ are said to be <u>jointly stationary in the wide sense</u> if

$X(t)$ is stationary in the wide sense,
$Y(t)$ is stationary in the wide sense,
$r_{X,Y}(t_1, t_2)$ depends only on $t_1 - t_2$.

Notation:

$$r_{X,Y}(t_1 - t_2) = r_{X,Y}(t_1, t_2)$$

$$r_{X,Y}(\tau) = r_{X,Y}(t+\tau, t)$$

Cross spectrum:

$$R_{X,Y}(f) = \mathcal{F}\{r_{X,Y}(\tau)\}$$

LTI Filtering

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = (X * h)(t) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau)\, d\tau$$

$$r_{Y,X}(t_1, t_2) = E\{Y(t_1) X(t_2)\} = E\left\{\int_{-\infty}^{\infty} h(\tau) X(t_1 - \tau)\, d\tau \cdot X(t_2)\right\}$$

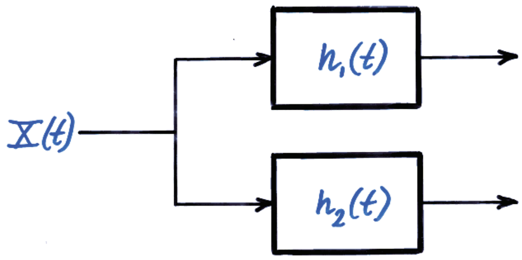
$$= \int_{-\infty}^{\infty} h(\tau)\, E\{X(t_1 - \tau) X(t_2)\}\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, r_X(t_1 - \tau, t_2)\, d\tau$$

$$\overset{\text{w.s.stat.}}{=} \int_{-\infty}^{\infty} h(\tau)\, r_X(t_1 - t_2 - \tau)\, d\tau = (h * r_X)(t_1 - t_2)$$

Jointly stationary in the wide sense!

$$r_{Y,X}(\tau) = (h * r_X)(\tau) \qquad R_{Y,X}(f) = H(f) \cdot R_X(f)$$

Filtering with Orthogonal Filters 1(2)



$$Y_1(t) = (X * h_1)(t) = \int_{-\infty}^{\infty} h_1(\tau_1) X(t-\tau_1)\, d\tau_1$$

$$Y_2(t) = (X * h_2)(t) = \int_{-\infty}^{\infty} h_2(\tau_2) X(t-\tau_2)\, d\tau_2$$

Orthogonal: $\int_{-\infty}^{\infty} h_1(t) h_2(t)\, dt = 0$

Input: $X(t)$, WGN, $R_X(f) = R_0$, $m_X = 0$.

Means: $m_{Y_k} = H_k(0)\, m_X = 0$

Question: Are they correlated? (same time).

Filtering with Orthogonal Filters 2(2)

$$r_{Y_1 Y_2}(t,t) = E\{Y_1(t) Y_2(t)\} = E\left\{ \int_{-\infty}^{\infty} h_1(\tau_1) X(t-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h_2(\tau_2) X(t-\tau_2)\, d\tau_2 \right\}$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot E\{X(t-\tau_1) X(t-\tau_2)\}\, d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot r_X(\tau_1 - \tau_2)\, d\tau_2\, d\tau_1$$

White

$$\overset{\downarrow}{=} \int_{-\infty}^{\infty} h_1(\tau_1) \cdot \int_{-\infty}^{\infty} h_2(\tau_2) \cdot R_0 \delta(\tau_1 - \tau_2)\, d\tau_2\, d\tau_1 = R_0 \cdot \int_{-\infty}^{\infty} h_1(\tau_1) \cdot (h_2 * \delta)(\tau_1)\, d\tau_1$$

orthogonal

$$= R_0 \cdot \int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_1)\, d\tau_1 \overset{\downarrow}{=} 0 = m_{Y_1} \cdot m_{Y_2}$$

Uncorrelated + jointly Gaussian ⇒ Independent.

DFT – Signal Analysis

Time-discrete signal with limited duration:

$$x[n] = 0 \quad \text{for} \quad n \notin \{0, 1, \dots, N-1\}$$

Fourier transform: $X[\theta] = \sum_n x[n] e^{-j2\pi\theta n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\theta n}$

cont. w. period 1.

DFT of length *L*: $X_L[k] = X[k/L] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{L}n} \quad \text{for } k \in \{0, 1, \dots, L-1\}$

IDFT (inverse): $x_L[n] = \frac{1}{L} \cdot \sum_{k=0}^{L-1} X_L[k] e^{j2\pi\frac{k}{L}n} \Rightarrow x_L[n+L] = x_L[n]$

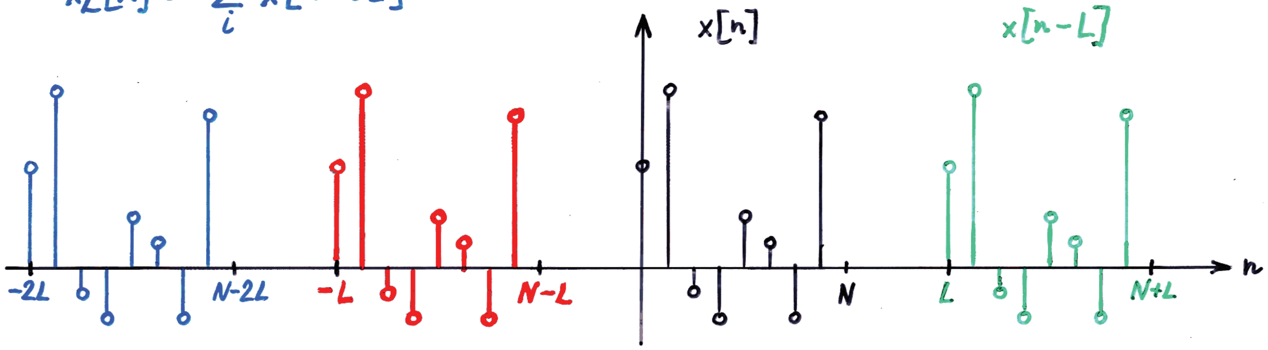
since $e^{j2\pi Ln/L} = e^{j2\pi \cdot n} = 1$

Relation to *x*[*n*]: $x_L[n] = \frac{1}{L} \sum_{k=0}^{L-1} \sum_{m=0}^{N-1} x[m] e^{-j2\pi\frac{k}{L}m} \cdot e^{j2\pi\frac{k}{L}n}$

$$= \sum_{m=0}^{N-1} x[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} e^{-j2\pi(m-n)\frac{k}{L}}}_{= \begin{cases} L, & m-n = 0 \bmod L \\ 0, & \text{elsewhere} \end{cases}} = \sum_{i=-\infty}^{\infty} x[n - i \cdot L]$$

DFT – Avoiding Aliasing

$$x_L[n] = \sum_i x[n - iL]$$



If $L < N$, then we get overlap and aliasing in the time domain.

Therefore: Demand $L \geq N$.

Note: $X_L[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{k}{L}n} = \sum_{n=0}^{L-1} x_L[n] e^{-j2\pi\frac{k}{L}n}$

DFT – Periodic Convolution

We are used to: $y[n] = (x * h)[n] \iff Y[\theta] = X[\theta] \cdot H[\theta]$

But we have: $y[n] = x[n] \cdot h[n] \iff Y[\theta] = \int_0^1 X[\phi]\, H[\theta-\phi]\, d\phi$

With DFT: $Y_L[k] = X_L[k] \cdot H_L[k] \iff$

$$y_L[n] = IDFT\{X_L[k]\, H_L[k]\} = \frac{1}{L}\sum_{k=0}^{L-1} X_L[k]\, H_L[k]\, e^{j2\pi \frac{k}{L} \cdot n}$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} X_L[k] \cdot \sum_{m=0}^{L-1} h_L[m] \cdot e^{-j2\pi \frac{k}{L} m} \cdot e^{j2\pi \frac{k}{L} n}$$

$$= \sum_{m=0}^{L-1} h_L[m] \cdot \underbrace{\frac{1}{L}\sum_{k=0}^{L-1} X_L[k]\, e^{j2\pi \frac{k}{L}(n-m)}}_{x_L[n-m]}$$

$$= \sum_{m=0}^{L-1} h_L[m]\, x_L[n-m]$$

And also: $y_L[n] = x_L[n] \cdot h_L[n] \iff Y_L[k] = \frac{1}{L}\sum_{m=0}^{L-1} X_L[m]\, H_L[k-m]$

Mikael Olofsson

ISY/CommSys

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