TSDT14 Signal Theory

Lecture 4

Estimation

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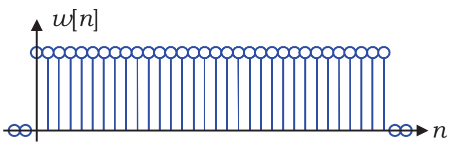
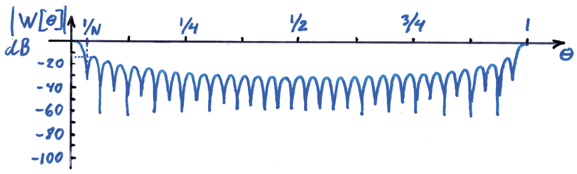
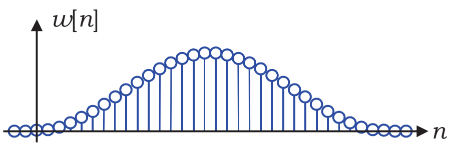
Examples of Windows, $N=32$

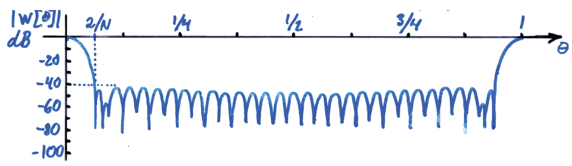
Rectangular window: $w[n] = 1, \quad n \in \{0,1,\ldots,N-1\}$



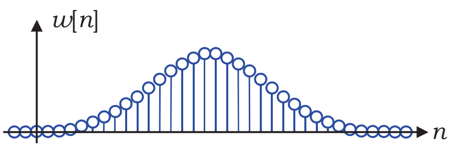


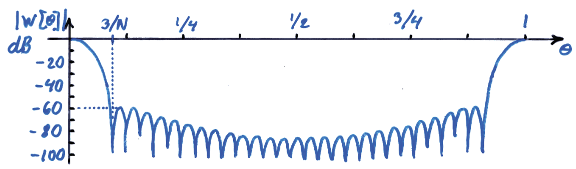
Hamming window: $w[n] = 0.54 - 0.46\cos(\frac{2\pi N}{N-1}), \quad n \in \{0,1,\ldots,N-1\}$





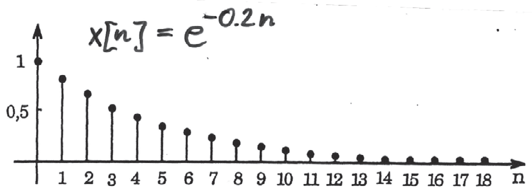
Blackman window: $w[n] = 0.42 - 0.5\cos(\frac{2\pi N}{N-1}) + 0.08\cos(\frac{4\pi N}{N-1}),$
$n \in \{0,1,\ldots,N-1\}$

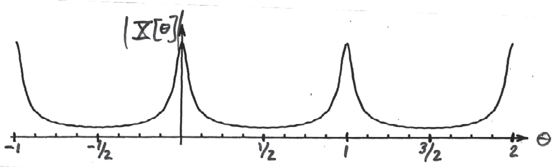




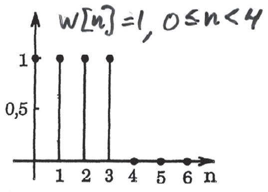
Using Windows

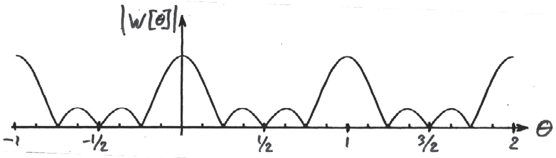
Signal:



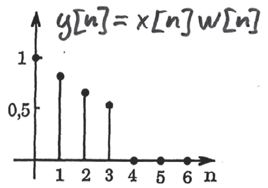


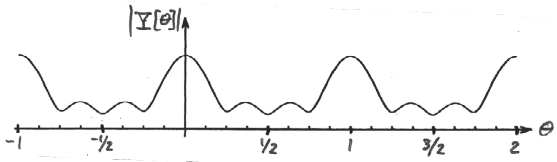
Rectangular window:





Result:





Ergodicity again

A WSS process: $m_X = E\{X(t)\}$

Time average of one realization: $m_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

Time average of the process: $M_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

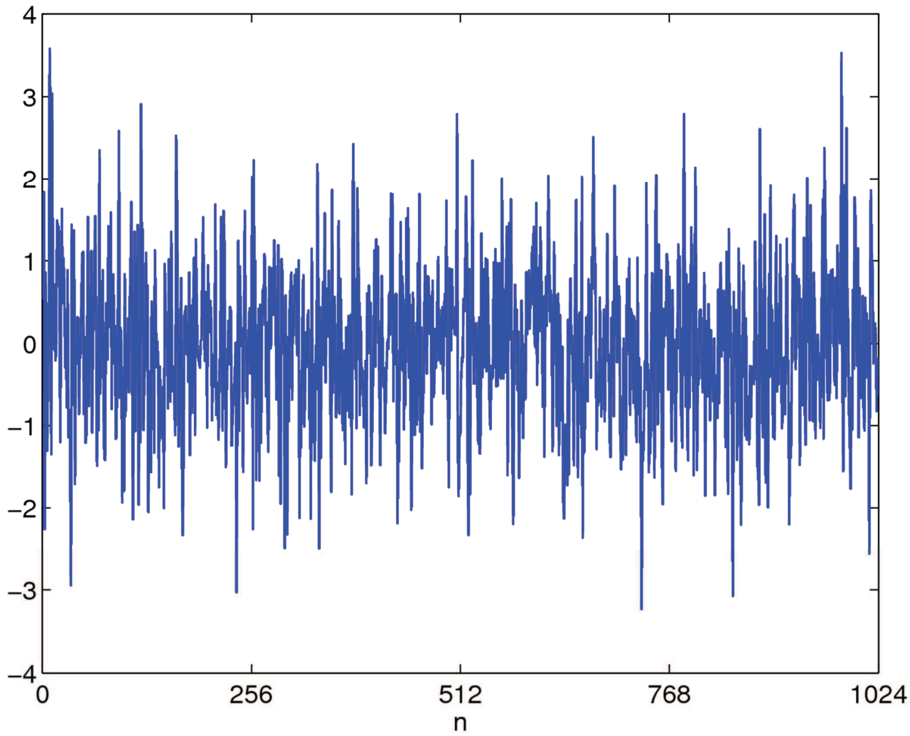
$$E\{M_T\} = E\left\{\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \frac{1}{2T}\int_{-T}^{T} m_X\,dt = m_X$$

Definition: If $\lim_{T\to\infty} E\{(M_T - m_X)^2\} = 0$ then $X(t)$ is said to be ergodic with respect to the mean, and we write

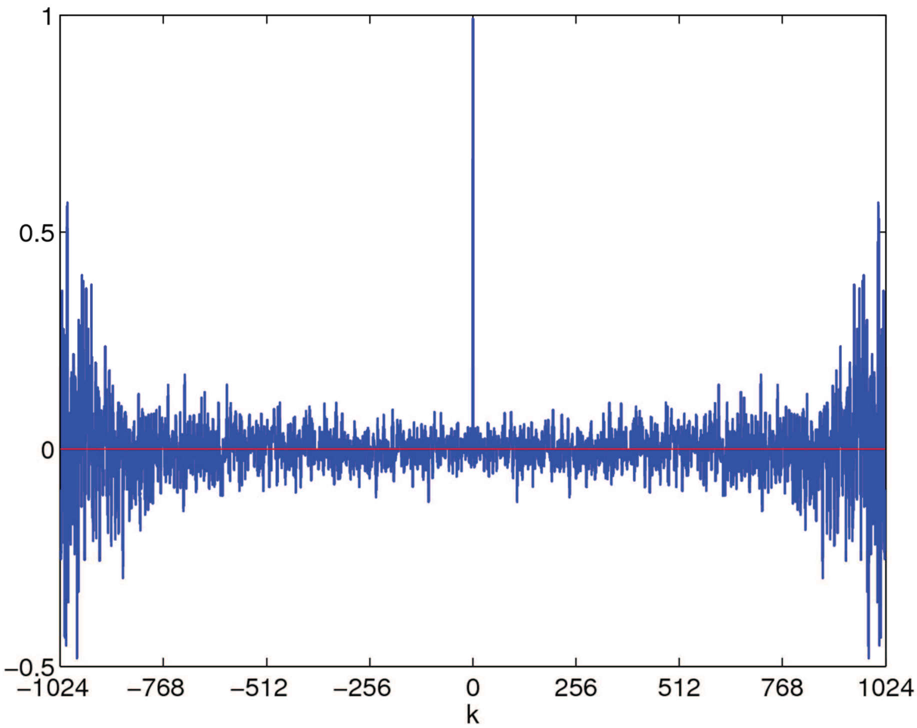
$$m_X = \underset{T\to\infty}{\text{l.i.m.}}\ \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \qquad \text{(limes in mean)}$$

Interpretation: The time average of a realization is very close to the ensemble mean with a probability that is very close to 1 ($\to 1,\ T\to\infty$).

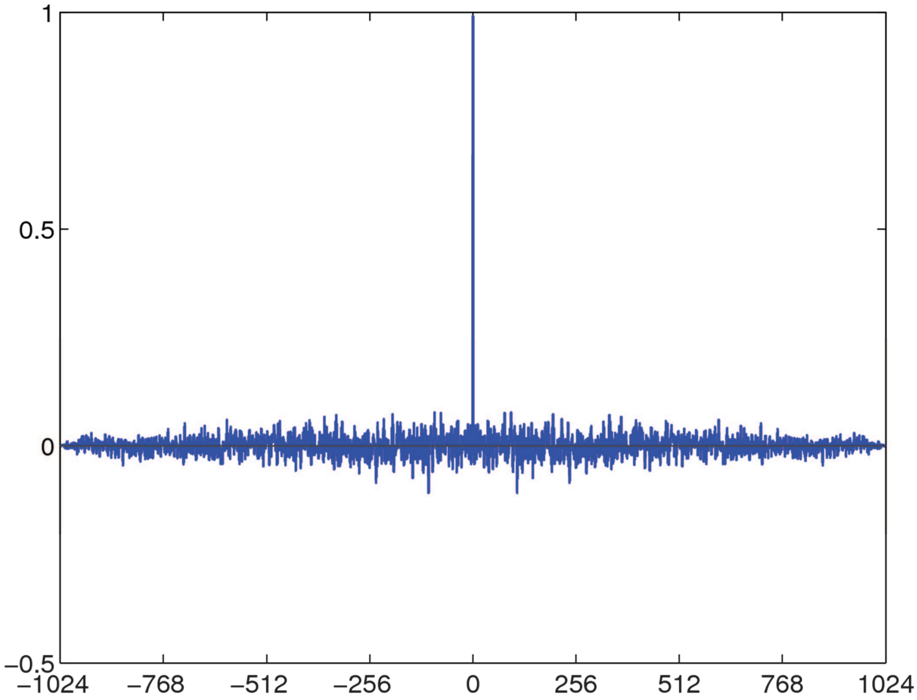
A Signal



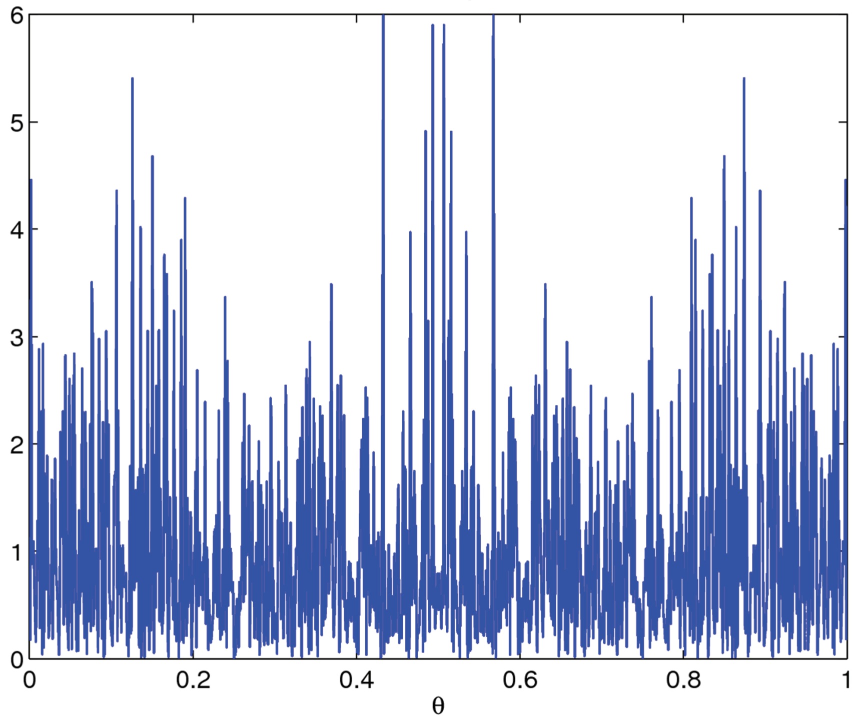
Blackman-Tukey’s Estimate of the ACF



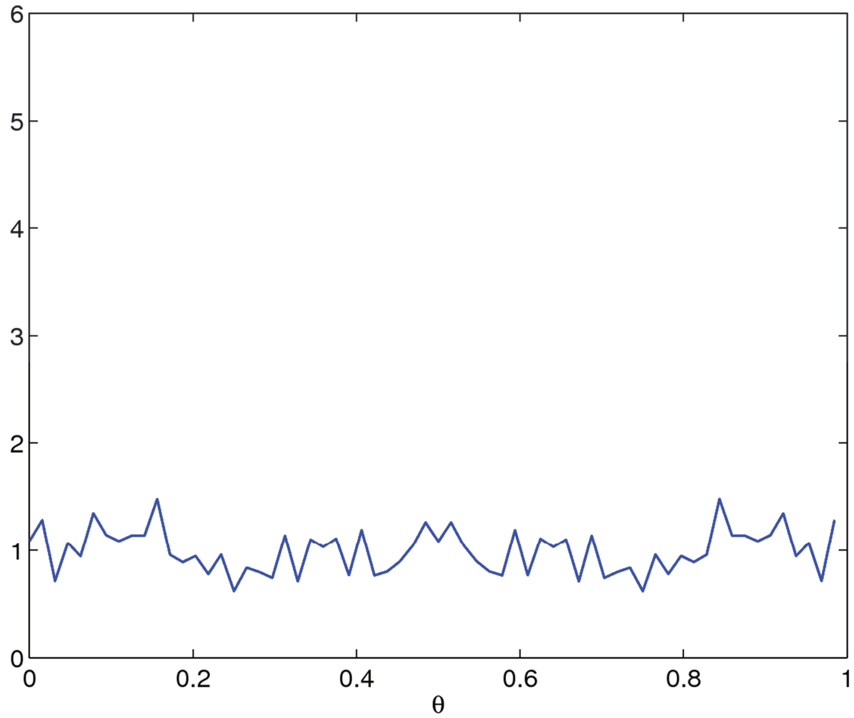
Bartletts Estimate of the ACF



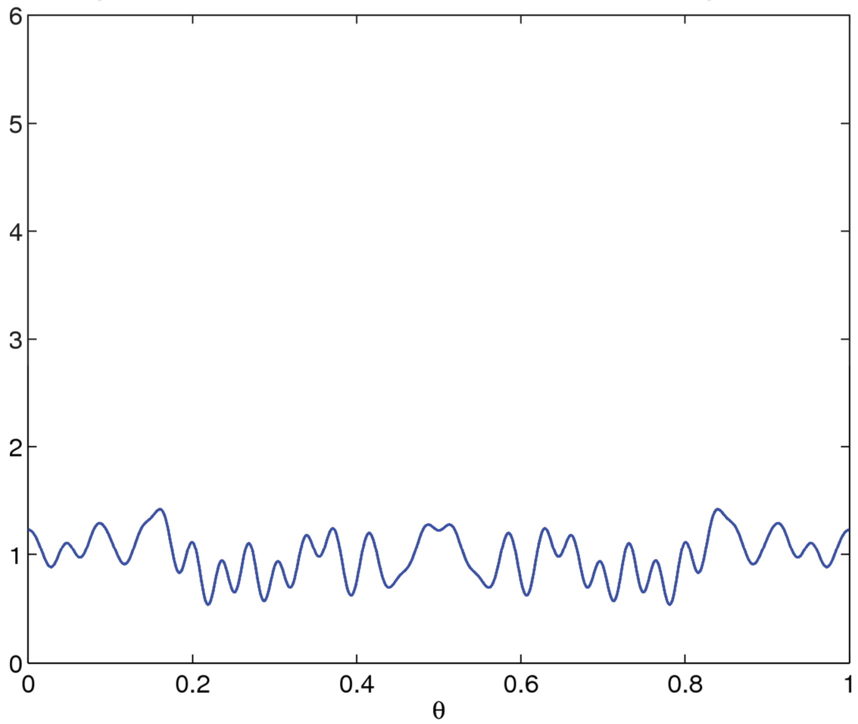
Periodogram



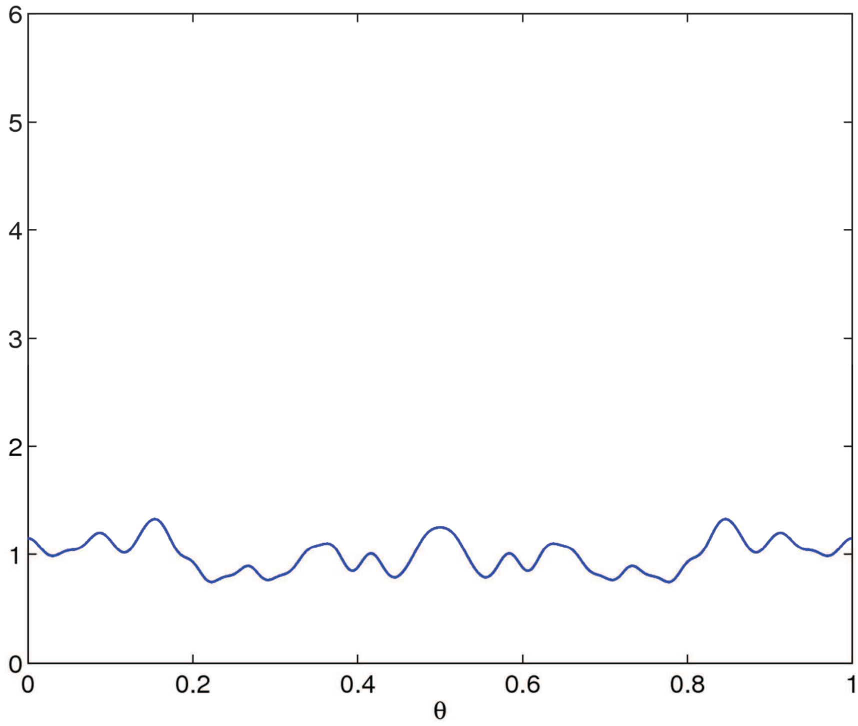
Averaged Periodogram



Estimated PSD using Rectangular Window

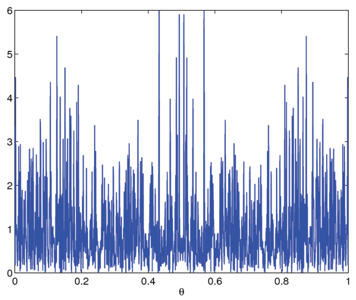


Estimated PSD using Hamming Window

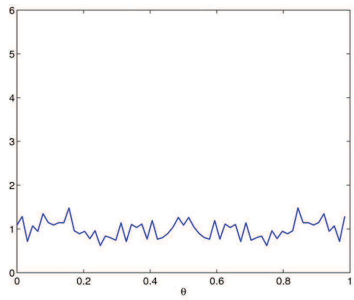


Smoothing – Overview

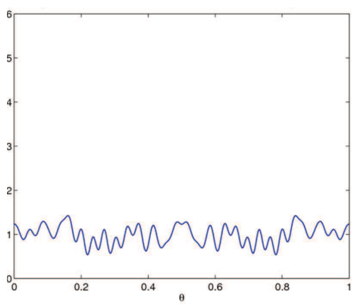
Raw periodogram



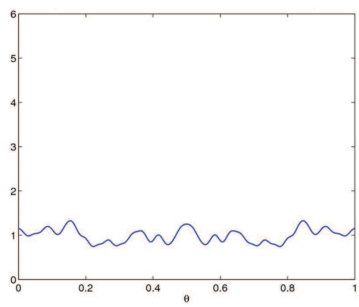
Averaged periodograms



Smoothing - Rectangular



Smoothing - Hamming



DFT – Signal Analysis

Time-discrete signal with limited duration:

$$x[n] = 0 \quad \text{for} \quad n \notin \{0, 1, \ldots, N-1\}$$

Fourier transform: $X[\theta] = \sum_n x[n] e^{-j2\pi\theta n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\theta n}$ cont. w. period 1.

DFT of length L: $X_L[k] = X[k/L] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{L} n}$ for $k \in \{0, 1, \ldots, L-1\}$

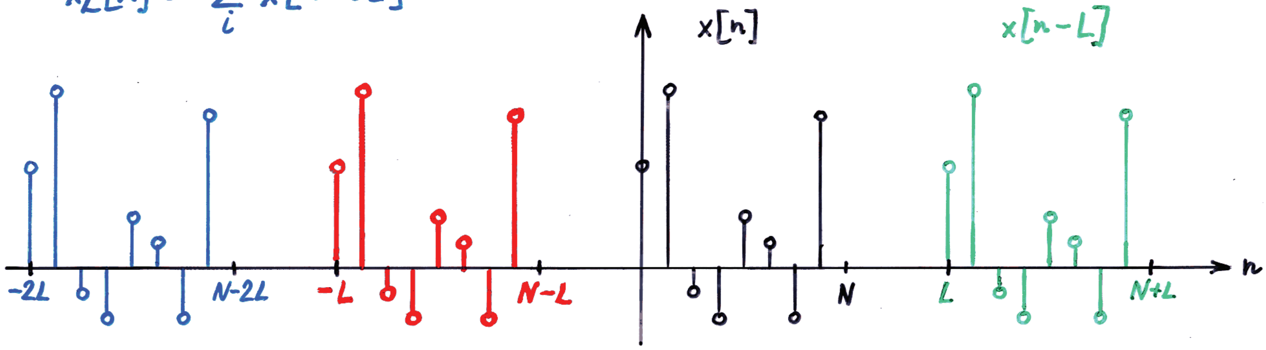
IDFT (inverse): $x_L[n] = \frac{1}{L} \cdot \sum_{k=0}^{L-1} X_L[k] e^{j2\pi \frac{k}{L} n} \Rightarrow x_L[n+L] = x_L[n]$ since $e^{j2\pi Ln/L} = e^{j2\pi \cdot n} = 1$

Relation to $x[n]$: $x_L[n] = \frac{1}{L} \sum_{k=0}^{L-1} \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{k}{L} m} \cdot e^{j2\pi \frac{k}{L} n}$

$$= \sum_{m=0}^{N-1} x[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} e^{-j2\pi(m-n)\frac{k}{L}}}_{= \begin{cases} L, & m-n \equiv 0 \bmod L \\ 0, & \text{elsewhere} \end{cases}} = \sum_{i=-\infty}^{\infty} x[n-i \cdot L]$$

DFT – Avoiding Aliasing

$$x_L[n] = \sum_i x[n-iL]$$



If $L<N$, then we get overlap and aliasing in the time domain.

Therefore: Demand $L \geq N$.

Note: $X_L[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{L} n} = \sum_{n=0}^{L-1} x_L[n] e^{-j2\pi \frac{k}{L} n}$

DFT – Periodic Convolution

We are used to: $y[n] = (x * h)[n] \quad \Leftrightarrow \quad Y[\theta] = X[\theta] \cdot H[\theta]$

But we have: $y[n] = x[n] \cdot h[n] \quad \Leftrightarrow \quad Y[\theta] = \int_0^1 X[\phi] H[\theta - \phi] \, d\phi$

With DFT: $Y_L[k] = X_L[k] \cdot H_L[k] \quad \Leftrightarrow$

$$y_L[n] = IDFT\{X_L[k] H_L[k]\} = \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] H_L[k] e^{j2\pi \frac{k}{L} n}$$

$$= \frac{1}{L} \sum_{k=0}^{L-1} X_L[k] \cdot \sum_{m=0}^{L-1} h_L[m] \cdot e^{-j2\pi \frac{k}{L} m} \cdot e^{j2\pi \frac{k}{L} n}$$

$$= \sum_{m=0}^{L-1} h_L[m] \cdot \underbrace{\frac{1}{L} \sum_{k=0}^{L-1} X_L[k] e^{j2\pi \frac{k}{L}(n-m)}}_{x_L[n-m]}$$

$$= \sum_{m=0}^{L-1} h_L[m] \, x_L[n-m]$$

And also: $y_L[n] = x_L[n] \cdot h_L[n] \quad \Leftrightarrow \quad Y_L[k] = \frac{1}{L} \sum_{m=0}^{L-1} X_L[m] H_L[k-m]$

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