Chapter 3

Exercise Set 3.1

1\. (a) $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = (2x5) - (1x3) = 7.$ (b) $\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = (3x2) - (-2x1) = 8.$

(c) $\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = (4x3) - (1x-2) = 14.$ (d) $\begin{vmatrix} 5 & -2 \\ -3 & -4 \end{vmatrix} = (5x-4) - (-2x-3) = -26.$

2\. (a) $\begin{vmatrix} 1 & -5 \\ 0 & 3 \end{vmatrix} = (1x3) - (-5x0) = 3.$ (b) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1x4) - (2x3) = -2.$

(c) $\begin{vmatrix} -3 & 1 \\ 2 & -5 \end{vmatrix} = (-3x-5) - (1x2) = 13.$ (d) $\begin{vmatrix} 3 & -2 \\ -1 & 0 \end{vmatrix} = (3x0) - (-2x-1) = -2.$

3\. (a) $M_{11} = \begin{vmatrix} 0 & 6 \\ 1 & -4 \end{vmatrix} = (0x-4) - (6x1) = -6.$ $C_{11} = (-1)^{1+1} M_{11} = (-1)^2(-6) = -6.$

(b) $M_{21} = \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix} = (2x-4) - (-3x1) = -5.$ $C_{21} = (-1)^{2+1} M_{21} = (-1)^3(-5) = 5.$

(c) $M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 1 \end{vmatrix} = (1x1) - (2x7) = -13.$ $C_{23} = (-1)^{2+3} M_{23} = (-1)^5(-13) = 13.$

(d) $M_{33} = \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = (1x0) - (2x5) = -10.$ $C_{33} = (-1)^{3+3} M_{33} = (-1)^6(-10) = -10.$

4\. (a) $M_{13} = \begin{vmatrix} -2 & 3 \\ 0 & -6 \end{vmatrix} = (-2x-6) - (3x0) = 12.$ $C_{13} = (-1)^{1+3} M_{13} = (-1)^4(12) = 12.$

(b) $M_{22} = \begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} = (5x2) - (1x0) = 10.$ $C_{22} = (-1)^{2+2} M_{22} = (-1)^4(10) = 10.$

(c) $M_{31} = \begin{vmatrix} 0 & 1 \\ 3 & 7 \end{vmatrix} = (0x7) - (1x3) = -3.$ $C_{31} = (-1)^{3+1} M_{31} = (-1)^4(-3) = -3.$

(d) $M_{33} = \begin{vmatrix} 5 & 0 \\ -2 & 3 \end{vmatrix} = (5x3) - (0x-2) = 15.$ $C_{33} = (-1)^{3+3} M_{33} = (-1)^6(15) = 15.$

5. (a) $M_{12} = \begin{vmatrix} 8 & 2 & 1 \\ 4 & -5 & 0 \\ 1 & 8 & 2 \end{vmatrix}$

$= (8\text{x}-5\text{x}2) + (2\text{x}0\text{x}1) + (1\text{x}4\text{x}8) - (1\text{x}-5\text{x}1) - (8\text{x}0\text{x}8) - (2\text{x}4\text{x}2)$

$= -80 + 0 + 32 - (-5) - 0 - 16 = -59.$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3(-59) = 59.$

(b) $M_{24} = \begin{vmatrix} 2 & 0 & 1 \\ 4 & -3 & -5 \\ 1 & 4 & 8 \end{vmatrix}$

$= (2\text{x}-3\text{x}8) + (0\text{x}-5\text{x}1) + (1\text{x}4\text{x}4) - (1\text{x}-3\text{x}1) - (2\text{x}-5\text{x}4) - (0\text{x}4\text{x}8)$

$= -48 + 0 + 16 - (-3) - (-40) - 0 = 11.$

$C_{24} = (-1)^{2+4} M_{24} = (-1)^6(11) = 11.$

(c) $M_{33} = \begin{vmatrix} 2 & 0 & -5 \\ 8 & -1 & 1 \\ 1 & 4 & 2 \end{vmatrix}$

$= (2\text{x}-1\text{x}2) + (0\text{x}1\text{x}1) + (-5\text{x}8\text{x}4) - (-5\text{x}-1\text{x}1) - (2\text{x}1\text{x}4) - (0\text{x}8\text{x}2)$

$= -4 + 0 + (-160) - 5 - 8 - 0 = -177.$

$C_{33} = (-1)^{3+3} M_{33} = (-1)^6(-177) = -177.$

(d) $M_{43} = \begin{vmatrix} 2 & 0 & -5 \\ 8 & -1 & 1 \\ 4 & -3 & 0 \end{vmatrix}$

$= (2\text{x}-1\text{x}0) + (0\text{x}1\text{x}4) + (-5\text{x}8\text{x}-3) - (-5\text{x}-1\text{x}4) - (2\text{x}1\text{x}-3) - (0\text{x}8\text{x}0)$

$= 0 + 0 + 120 - 20 - (-6) - 0 = 106.$

$C_{43} = (-1)^{4+3} M_{43} = (-1)^7(106) = -106.$

6. (a) $\begin{vmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ -2 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 5 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} + 4\begin{vmatrix} 4 & -1 \\ -2 & 2 \end{vmatrix}$

= [(−1x1) − (5x2)] − 2[(4x1) − (5x−2)] + 4[(4x2) − (−1x−2)]

= −11 − 2(14) + 4(6) = −15.

"diagonals" method:

(1x−1x1) + (2x5x−2) + (4x4x2) − (4x−1x−2) − (1x5x2) − (2x4x1)

= −1 + (−20) + 32 − 8 − 10 − 8 = −15.

(b) $$\begin{vmatrix} 3 & 1 & 4 \\ -7 & -2 & 1 \\ 9 & 1 & -1 \end{vmatrix} = 3\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} -7 & 1 \\ 9 & -1 \end{vmatrix} + 4\begin{vmatrix} -7 & -2 \\ 9 & 1 \end{vmatrix}$$

= 3[(−2x−1) − (1x1)] − [(−7x−1) − (1x9)] + 4[(−7x1) − (−2x9)]

= 3 − (−2) + 4(11) = 49.

"diagonals" method:

(3x−2x−1) + (1x1x9) + (4x−7x1) − (4x−2x9) − (3x1x1) − (1x−7x−1)

= 6 + 9 + (−28) − (−72) − 3 − 7 = 49.

(c) $$\begin{vmatrix} 4 & 1 & -2 \\ 5 & 3 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 5 & -1 \\ 2 & 1 \end{vmatrix} + (-2)\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}$$

= 4[(3x1) − (−1x4)] − [(5x1) − (−1x2)] + (−2)[(5x4) − (3x2)]

= 4(7)− 7 + (−2)(14) = −7.

"diagonals" method:

(4x3x1) + (1x−1x2) + (−2x5x4) − (−2x3x2) − (4x−1x4) − (1x5x1)

= 12 + (−2) + (−40) − (−12) − (−16) − 5 = −7.

7. (a) $$\begin{vmatrix} 2 & 0 & 7 \\ 8 & -1 & -2 \\ 5 & 6 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & -2 \\ 6 & 1 \end{vmatrix} - 0\begin{vmatrix} 8 & -2 \\ 5 & 1 \end{vmatrix} + 7\begin{vmatrix} 8 & -1 \\ 5 & 6 \end{vmatrix}$$

= 2[(−1x1) − (−2x6)] − 0+ 7[(8x6) − (−1x5)] = 2(11)− 0 + 7(53) = 393.

"diagonals" method:

(2x−1x1) + (0x−2x5) + (7x8x6) − (7x−1x5) − (2x−2x6) − (0x8x1)

= −2 + 0 + (336) − (−35) − (−24) − 0 = 393.

(b) $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$

$= 2[(0x1) - (2x1)] - (-1)[(4x1) - (2x1)] + 3[(4x1) - (0x1)]$

$= 2(-2) - (-1)(2) + 3(4) = 10.$

"diagonals" method:

$(2x0x1) + (-1x2x1) + (3x4x1) - (3x0x1) - (2x2x1) - (-1x4x1)$

$= 0 + (-2) + 12 - 0 - (4) - (-4) = 10.$

(c) $\begin{vmatrix} 0 & 0 & 5 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

$= 0 - 0 + 5[(1x2) - (1x2)] = 0$

"diagonals" method:

$(0x1x2) + (0x1x2) + (5x1x2) - (5x1x2) - (0x1x2) - (0x1x2) = 0.$

8. (a) $\begin{vmatrix} 0 & 3 & 2 \\ 1 & 5 & 7 \\ -2 & -6 & -1 \end{vmatrix}$

using row 2:

$= -\begin{vmatrix} 3 & 2 \\ -6 & -1 \end{vmatrix} + 5\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} - 7\begin{vmatrix} 0 & 3 \\ -2 & -6 \end{vmatrix} = -9 + 5x4 - 7x6 = -31.$

using column 1:

$= 0\begin{vmatrix} 5 & 7 \\ -6 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ -6 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix} = 0 - 9 + (-2)(11) = -31.$

(b) $\begin{vmatrix} 4 & 2 & 1 \\ -6 & 3 & -2 \\ 7 & 1 & -1 \end{vmatrix}$

using row 3:

$= 7\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ -6 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 2 \\ -6 & 3 \end{vmatrix} = -49 - (-2) + (-24) = -71.$

using column 2:

$$= -2\begin{vmatrix} -6 & -2 \\ 7 & -1 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 7 & -1 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ -6 & -2 \end{vmatrix} = -40 + (-33) - (-2) = -71.$$

(c) $\begin{vmatrix} 5 & -1 & 2 \\ 3 & 0 & 6 \\ -4 & 3 & 1 \end{vmatrix}$

using row 1:

$$= 5\begin{vmatrix} 0 & 6 \\ 3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 6 \\ -4 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ -4 & 3 \end{vmatrix} = -90 - (-27) + 18 = -45.$$

using row 3:

$$= (-4)\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & 0 \end{vmatrix} = 24 - (72) + (3) = -45.$$

(d) $\begin{vmatrix} 6 & 3 & 0 \\ -2 & -1 & 5 \\ 4 & 6 & -2 \end{vmatrix}$

using column 2:

$$= -3\begin{vmatrix} -2 & 5 \\ 4 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 6 & 0 \\ 4 & -2 \end{vmatrix} - 6\begin{vmatrix} 6 & 0 \\ -2 & 5 \end{vmatrix} = 48 + 12 - 180 = -120.$$

using column 3:

$$= 0\begin{vmatrix} -2 & -1 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 6 & 3 \\ 4 & 6 \end{vmatrix} + (-2)\begin{vmatrix} 6 & 3 \\ -2 & -1 \end{vmatrix} = 0 - 120 + 0 = -120.$$

9. (a) $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 1 & 4 & 3 \end{vmatrix}$

using row 2:

$$= -2\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -26 + 0 - 5 = -31.$$

using column 1:

$$= \begin{vmatrix} 0 & 5 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix} = -20 - 26 + 15 = -31.$$

(b) $\begin{vmatrix} -2 & -1 & 1 \\ 9 & 3 & 2 \\ 4 & 0 & 0 \end{vmatrix}$

using row 1:

$$= (-2)\begin{vmatrix} 3 & 2 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 9 & 2 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 9 & 3 \\ 4 & 0 \end{vmatrix} = 0 - 8 + (-12) = -20.$$

using column 3:

$$= \begin{vmatrix} 9 & 3 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} -2 & -1 \\ 4 & 0 \end{vmatrix} + 0\begin{vmatrix} -2 & -1 \\ 9 & 3 \end{vmatrix} = -12 - 8 + 0 = -20.$$

(c) $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \\ 4 & 0 & 2 \end{vmatrix}$

using column 1:

$$= \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} = -4 - 0 + 16 = 12.$$

using column 2:

$$= -0\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 0 + 12 - 0 = 12.$$

(d) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & -4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$

using row 3:

$$= 0\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} = 0 - 0 + (-8) = -8.$$

using column 1:

$$= \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix} = -16 - (-8) + 0 = -8.$$

10. (a) Using column 3, $\begin{vmatrix} 1 & -2 & 3 \\ 1 & 4 & 0 \\ 2 & -1 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = -27.$

(b) Using row 2, $\begin{vmatrix} 3 & -1 & 2 \\ 0 & 4 & 0 \\ -5 & 1 & 9 \end{vmatrix} = 4 \begin{vmatrix} 3 & 2 \\ -5 & 9 \end{vmatrix} = 148.$

(c) Using row 3, $\begin{vmatrix} 9 & 2 & 1 \\ -3 & 2 & 6 \\ 0 & 0 & -3 \end{vmatrix} = (-3) \begin{vmatrix} 9 & 2 \\ -3 & 2 \end{vmatrix} = -72.$

11. (a) Using column 4, $\begin{vmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 5 & 0 \\ 7 & -3 & 8 & 4 \\ -3 & 0 & 4 & 0 \end{vmatrix} = -4 \begin{vmatrix} 1 & -2 & 3 \\ 4 & 0 & 5 \\ -3 & 0 & 4 \end{vmatrix}$

(using column 2 of the 3x3 matrix) $= -4(-(-2)) \begin{vmatrix} 4 & 5 \\ -3 & 4 \end{vmatrix} = -8 \times 31 = -248.$

(b) Using row 3, $\begin{vmatrix} 1 & 4 & 5 & 9 \\ 2 & 3 & -7 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 8 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 4 & 5 \\ 2 & 3 & -7 \\ 0 & 1 & 0 \end{vmatrix}$

(using row 3 of the 3x3 matrix) $= -(-3)(-1) \begin{vmatrix} 1 & 5 \\ 2 & -7 \end{vmatrix} = -3 \times (-17) = 51.$

(c) Using column 2, $\begin{vmatrix} 9 & 3 & 7 & -8 \\ 1 & 0 & 4 & 2 \\ 1 & 0 & 0 & -1 \\ -2 & 0 & -1 & 3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ -2 & -1 & 3 \end{vmatrix}$

(using row 2 of the 3x3 matrix) $= -3 \left(- \begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 4 \\ -2 & -1 \end{vmatrix} \right)$

$= -3(-14 - (-7)) = 21.$

12. $\begin{vmatrix} x+1 & x \\ 3 & x-2 \end{vmatrix} = (x+1)(x-2) - 3x = x^2 - x - 2 - 3x = 3$, so $x^2 - 4x - 5 = 0$.

$(x-5)(x+1) = 0$, so there are two solutions, $x = 5$ and $x = -1$.

13. $\begin{vmatrix} 2x & -3 \\ x-1 & x+2 \end{vmatrix} = (2x)(x+2) - (-3)(x-1) = 2x^2 + 4x - (-3x) - 3 = 1$, so $2x^2 + 7x - 4 = 0$.

$(2x-1)(x+4) = 0$, so there are two solutions, $x = 1/2$ and $x = -4$.

14. $\begin{vmatrix} x-1 & -2 \\ x-2 & x-1 \end{vmatrix} = (x-1)(x-1) - (-2)(x-2) = x^2 - 2x + 1 - (-2x) - 4 = 0$,

so $x^2 - 3 = 0$, and there are two solutions, $\sqrt{3}$ and $-\sqrt{3}$.

15. $\begin{vmatrix} x & 0 & 2 \\ 2x & x-1 & 4 \\ -x & x-1 & x+1 \end{vmatrix} = x(x-1)(x+1) + 0 + 2(2x)(x-1) - 2(x-1)(-x) - x(4)(x-1) - 0$

$= x^3 - x + 4x^2 - 4x + 2x^2 - 2x - 4x^2 + 4x = x^3 + 2x^2 - 3x = x(x+3)(x-1) = 0$, so there are three solutions, $x = 0$, $x = -3$, and $x = 1$.

16. The cofactor expansion of each determinant using the third column gives $-3\begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix}$.

17. The cofactor expansion using the third row is independent of the numbers a and b.

18. (a) 4213 → 4123 → 1423 → 1243 → 1234; even

(b) 3142 → 1342 → 1324 → 1234; odd

(c) 3214 → 3124 → 1324 → 1234; odd

(d) 2413 → 2143 → 1243 → 1234; odd

(e) 4321 → 4312 → 3412 → 3142 → 1342 → 1324 → 1234; even

19. (a) 35241 → 32541 → 32451 → 32415 → 32145 → 31245 → 13245 → 12345; odd

(b) 43152 → 41352 → 14352 → 13452 → 13425 → 13245 → 12345; even

(c) 54312 → 45312 → 43512 → 43152 → 43125 → 34125 → 31425 → 31245 → 13245 → 12345; odd

(d) 25143 → 21543 → 12543 → 12453 → 12435 → 12345; odd

(e) 32514 → 23514 → 23154 → 21354 → 12354 → 12345; odd

20.

1234	even	2134	odd
1243	odd	2143	even
1324	odd	2314	even
1342	even	2341	odd
1423	even	2413	odd
1432	odd	2431	even
3124	even	4123	odd
3142	odd	4132	even
3214	odd	4213	even
3241	even	4231	odd
3412	even	4312	odd
3421	odd	4321	even

21. Cofactor expansion gives $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$.

The permutations of 1,2,3 are 123(even), 132(odd), 213(odd), 231(even), 312(even), and 321(odd), so the second definition gives $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$. The two expressions are the same.

22. (a) (1)(-1)(1)-(1)(5(2)-(2)(4)(1)+(2)(5)(-2)+(4)(4)(2)-(4)(-1)(-2) = -15.

(b) (3)(-2)(-1)-(3)(1)(1)-(1)(-7)(-1)+(1)(1)(9)+(4)(-7)(1)-(4)(-2)(9) = 49.

(c) (4)(3)(1)-(4)(-1)(4)-(1)(5)(1)+(1)(-1)(2)+(-2)(5)(4)-(-2)(3)(2) = -7.

<u>Exercise Set 3.2</u>

1. (a) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & -1 & 1 \end{vmatrix} \underset{C2+(-2)C1}{=} \begin{vmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5.$

(b) $\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 6 \\ 2 & 2 & 7 \end{vmatrix} \underset{C2+(-1)C1}{=} \begin{vmatrix} 0 & 1 & 5 \\ 1 & 0 & 6 \\ 2 & 0 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix} = 5.$

(c) $\begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 1 & 4 & -4 \end{vmatrix} \underset{C2+C3}{=} \begin{vmatrix} 2 & 0 & -1 \\ 3 & 0 & 1 \\ 1 & 0 & -4 \end{vmatrix} = 0.$

(d) $\begin{vmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \underset{R3+(-2)R2}{=} \begin{vmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \\ -7 & -3 & 0 \end{vmatrix} = - \begin{vmatrix} 3 & -1 \\ -7 & -3 \end{vmatrix} = 16.$

2. (a) $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & 1 & 2 \end{vmatrix} \underset{C3+(2)C2}{=} \begin{vmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 4 \end{vmatrix} = -4 \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 20.$

(b) $\begin{vmatrix} 5 & 1 & 3 \\ 1 & 2 & 4 \\ -1 & 1 & -4 \end{vmatrix} \underset{R2+R3}{=} \begin{vmatrix} 5 & 1 & 3 \\ 0 & 3 & 0 \\ -1 & 1 & -4 \end{vmatrix} = 3 \begin{vmatrix} 5 & 3 \\ -1 & -4 \end{vmatrix} = -51.$

(c) $\begin{vmatrix} 1 & -2 & 3 \\ -3 & 6 & -9 \\ 4 & 5 & 7 \end{vmatrix} \underset{R2+(3)R1}{=} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 7 \end{vmatrix} = 0.$

(d) $\begin{vmatrix} -1 & 3 & 2 \\ 2 & 5 & -4 \\ 4 & 1 & -8 \end{vmatrix} \underset{C3+(2)C1}{=} \begin{vmatrix} -1 & 3 & 0 \\ 2 & 5 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 0.$

3. (a) The given matrix can be obtained from A by multiplying the third row by 2, so its determinant is $2\,|A| = -4$.

(b) The given matrix can be obtained from A by interchanging rows 1 and 2, so its determinant is $-|A| = 2$.

(c) The given matrix can be obtained from A by adding twice row 1 to row 2, so its determinant is $|A| = -2$.

4. (a) The given matrix can be obtained from A by interchanging columns 2 and 3, so its determinant is $-|A| = -5$.

(b) The given matrix can be obtained from A by adding -2 times column 3 to column 2, so its determinant is $|A| = 5$.

(c) The given matrix is the transpose of A, and $|A^t| = |A| = 5$.

5. The second answer is correct. 2R1 - R3 does not preserve the value of the determinant. It multiplies the determinant by 2 before subtracting R3.

6. (a) Row 3 is all zeros.

(b) Columns 2 and 3 are equal.

(c) Row 3 is -3 times row 1.

(d) Row 3 is all zeros.

7. (a) Column 3 is all zeros. (b) Row 3 is all zeros.
   (c) Row 3 is 3 times row 1. (d) Column 3 is –3/2 times column 1.

8. (a) $|2A| = (2)^2\,|A| = 12.$ (b) $|3A^t| = (3)^2\,|A^t| = 9|A| = 27.$
   (c) $|A^2| = |A||A| = 9.$ (d) $|(A^t)^2| = |A^t||A^t| = |A||A| = 9.$
   (e) $|(A^2)^t| = |A^2| = 9.$ (f) $|4A^{-1}| = (4)^2\,|A^{-1}| = 16/|A| = 16/3.$

9. (a) $|AB| = |A||B| = -6.$ (b) $|AA^t| = |A||A^t| = |A||A| = 9.$
   (c) $|A^t B| = |A^t||B| = |A||B| = -6.$ (d) $|3A^2 B| = (3)^3\,|A||A||B| = 486.$
   (e) $|2AB^{-1}| = (2)^3\,|A||B^{-1}| = 8|A|/|B| = -12.$ (f) $|(A^2 B^{-1})^t| = |A^2 B^{-1}| = |A||A|/|B| = 9/2.$

10. (a) The given matrix can be obtained from A by interchanging rows 1 and 2 and then interchanging rows 2 and 3. Thus its determinant is $(-1)(-1)|A| = 3$.
    (b) The given matrix can be obtained from A by interchanging rows 1 and 2 and then taking the transpose. Thus its determinant is $(-1)|A| = -3$.
    (c) The given matrix can be obtained from A by interchanging rows 1 and 2 and then interchanging columns 2 and 3 in the resulting matrix. Thus the determinant of the given matrix is $(-1)(-1)|A| = 3$.
    (d) The given matrix can be obtained from A by interchanging rows 1 and 2 of A and multiplying row 3 by 2. Thus the determinant is $(-1)(2)|A| = -6$.

11. $$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = (a+b)\begin{vmatrix} q & r \\ v & w \end{vmatrix} - (c+d)\begin{vmatrix} p & r \\ u & w \end{vmatrix} + (e+f)\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

$$= a\begin{vmatrix} q & r \\ v & w \end{vmatrix} - c\begin{vmatrix} p & r \\ u & w \end{vmatrix} + e\begin{vmatrix} p & q \\ u & v \end{vmatrix} + b\begin{vmatrix} q & r \\ v & w \end{vmatrix} - d\begin{vmatrix} p & r \\ u & w \end{vmatrix} + f\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

$$= \begin{vmatrix} a & c & e \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} b & d & f \\ p & q & r \\ u & v & w \end{vmatrix}.$$

12. (a) 3x−1x4 = −12 (b) 2x3x5x−2 = −60 (c) 3x−2x5x1 = −30

13. (a) $\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 1 & 0 \end{vmatrix} \underset{R3+(-1)R2}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{vmatrix} = -4.$

(b) $\begin{vmatrix} 1 & -2 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1}}{=} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & 4 \\ 0 & 5 & -3 \end{vmatrix} \underset{R2\leftrightarrow R3}{=} -\begin{vmatrix} 1 & -2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & 4 \end{vmatrix} = -20.$

(c) $\begin{vmatrix} 2 & 3 & 8 \\ -2 & -3 & 4 \\ 4 & 6 & -2 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1}}{=} \begin{vmatrix} 2 & 3 & 8 \\ 0 & 0 & 12 \\ 0 & 0 & -18 \end{vmatrix} = 0.$

(d) $\begin{vmatrix} 2 & -1 & 4 \\ -2 & 1 & -3 \\ 0 & 3 & -2 \end{vmatrix} \underset{R2+R1}{=} \begin{vmatrix} 2 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 3 & -2 \end{vmatrix} \underset{R2\leftrightarrow R3}{=} -\begin{vmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} = -6.$

14. (a) $\begin{vmatrix} 2 & 1 & 3 & 1 \\ -2 & 3 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -4 & -2 & 0 & -1 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-1)R1 \\ R4+(2)R1}}{=} \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 6 & 1 \end{vmatrix} \underset{R4+(6)R3}{=} \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 13 \end{vmatrix} = -104.$

(b) $\begin{vmatrix} 1 & 2 & -1 & 0 \\ 1 & 4 & 2 & 1 \\ -1 & 2 & 6 & 6 \\ 2 & 2 & -4 & -2 \end{vmatrix} \underset{\substack{R2+(-1)R1 \\ R3+R1 \\ R4+(-2)R1}}{=} \begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 5 & 6 \\ 0 & -2 & -2 & -2 \end{vmatrix} \underset{\substack{R3+(-2)R2 \\ R4+R2}}{=} \begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{vmatrix}$

$\underset{R4+R3}{=} \begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{vmatrix} = -6.$

(c) $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 \\ 3 & 1 & 5 & -1 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-2)R1 \\ R4+(-3)R1}}{=} \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 4 & 5 & -7 \end{vmatrix} \underset{R2\leftrightarrow R4}{=} - \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 4 & 5 & -7 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 0.$

15. Expand by row (or column) 1 at each stage.

$$\begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & \dots & 0 \\ 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & a_{nn} \end{vmatrix}$$

$$= \dots = a_{11}\, a_{22}\, a_{33} \dots a_{nn}$$

16. Suppose row i of B is row i of A plus k times row j of A, where k is any real number.

The cofactors of the ith rows of A and B are the same since only the elements in the ith rows are different.

$|B| = b_{i1} C_{i1} + b_{i2} C_{i2} + \dots + b_{in} C_{in} = (a_{i1} + ka_{j1})C_{i1} + (a_{i2} + ka_{j2})C_{i2} + \dots + (a_{in} + ka_{jn})C_{in}$

$= a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in} + k(a_{j1} C_{i1} + a_{j2} C_{i2} + \dots + a_{jn} C_{in}) = |A| + k|D|$, where D is

the matrix obtained from A by replacing the ith row of A by the jth row of A. Thus the ith and jth rows of D are equal, so $|D| = 0$ and $|B| = |A|$.

The proof for columns is similar.

17. Suppose row j is k times row i. Let B be the matrix obtained from A by interchanging rows i and j. Call the cofactors of the ith row of A $C_{i1}, C_{i2}, \dots, C_{in}$ and the cofactors of the ith row of B $D_{i1}, D_{i2}, \dots, D_{in}$. Each cofactor of A is k times the corresponding cofactor of B, because row j in A is k times row j in B.

$|A| = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in} = a_{i1} kD_{i1} + a_{i2} kD_{i2} + \dots + a_{in} kD_{in}$

and $|B| = a_{j1} D_{i1} + a_{j2} D_{i2} + \dots + a_{jn} D_{in} = ka_{i1} D_{i1} + ka_{i2} D_{i2} + \dots + ka_{in} D_{in}$,
so $|A| = |B|$.

But $|B| = -|A|$ because B was obtained from A by interchanging two rows, so $|A| = -|A|$, and therefore $|A| = 0$.

The proof for proportional columns is similar.

18. Suppose the sum of the elements in each column is zero.

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{R1+R2}{=} \begin{vmatrix} a_{11}+a_{21} & \cdots & a_{1n}+a_{2n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{R1+R3}{=} \begin{vmatrix} a_{11}+a_{21}+a_{31} & \cdots & a_{1n}+a_{2n}+a_{3n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$\underset{R1+R4}{=} \cdots \underset{R1+Rn}{=} \begin{vmatrix} a_{11}+a_{21}+a_{31}+\ldots+a_{n1} & \cdots & a_{1n}+a_{2n}+a_{3n}+\ldots+a_{nn} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0,$$

because the first row is all zeros.

19. $|A^n| = |A||A|\ldots|A| = |A|^n$. If $A^n = O$, then $|A^n| = 0$, so $|A|^n = 0$, but that means $|A| = 0$.

20. $|AB| = |A||B| = |B||A| = |BA|$.

21. $|ABC| = |AB||C| = |A||B||C|$.

22. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 4 & -1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

$|A| = |B| = 0$ and $|A+B| = -6$, so $|A+B| \neq |A| + |B|$ for this example.

23. If B is obtained from A using an elementary row operation, then $|B| = |A|$ or $|B| = -|A|$ or $|B| = c|A|$. Since $c \neq 0$ by definition, $|B| \neq 0$ if and only if $|A| \neq 0$.

24. (a) Let E be the reduced echelon form of a 2x2 matrix A and suppose E has no zero rows. Since E is a reduced echelon matrix, the first position in row 2 that can be nonzero is the (2,2) position, and it must be 1 in order for E to have no zero rows. This means the (1,2) position is zero and the (1,1) position is 1. So $E = I_2$.

(b) Let E be the reduced echelon form of a 3x3 matrix A and suppose E has no zero rows. Since E is a reduced echelon matrix, the first position in row 3 that can be nonzero is the (3,3) position, and it must be 1 in order for E to have no zero rows. This means the (1,3) and (2,3) positions are zero and the (2,2) position is 1, since it is now both the first and last position in row 2 that can be nonzero. Therefore the (1,2) position is zero and the (1,1) position is 1. So $E = I_3$.

(c) Let E be the reduced echelon form of an nxn matrix A and suppose E has no zero rows. Since E is a reduced echelon matrix, the first position in row n that can be nonzero is the (n,n) position, and it must be 1 in order for E to have no zero rows. This means the (1,n), (2,n),. . .,(n−1,n) positions are zero and the (n−1,n−1) position is 1, since it is now both the first and last position in row n−1 that can be nonzero. Therefore the (1,n−1), (2,n−1), . . . , (n−2,n−1) positions are zero and the (n−2,n−2) position is 1, ... Continuing in this way, we see that all diagonal elements of E must be 1 to avoid having a zero row, and therefore all other elements of E are zero. Thus $E = I_n$.

(d) E is obtained from A by performing a sequence of elementary row operations. $|A| = (-1)^k c|E|$, where k is the number of row interchanges required and c is 1 divided by the product of all constants multiplied by rows. Thus if $|A| \neq 0$ then $|E| \neq 0$, but if $|E| \neq 0$, then E has no zero rows, so from part (c), $E = I_n$. On the other hand, if $E = I_n$, then $|E| \neq 0$, so $|A| \neq 0$.

25. Let $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$. Then $|A|=1$, $|B|=1$. A and B are nonsingular. $A+B=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

A+B is singular. S is not closed under addition. Further, $0A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. 0A is singular. S is not closed under scalar multiplication.

<u>Exercise Set 3.3</u>

1. (a) The determinant is 5. The matrix is invertible.

   (b) The determinant is zero. The matrix is singular. The inverse does not exist.

   (c) The determinant is zero. The matrix is singular. The inverse does not exist.

   (d) The determinant is 1. The matrix is invertible.

2. (a) The determinant is −6. The matrix is invertible.

   (b) The determinant is zero. The matrix is singular. The inverse does not exist.

(c) The determinant is 7. The matrix is invertible.

(d) The determinant is zero. The matrix is singular. The inverse does not exist.

3. (a) The determinant is 18. The matrix is invertible.

(b) The determinant is zero. The matrix is singular. The inverse does not exist.

(c) The determinant is –105. The matrix is invertible.

(d) The determinant is zero. The matrix is singular. The inverse does not exist.

4. (a) The determinant is zero. The matrix is singular. The inverse does not exist.

(b) The determinant is 42. The matrix is invertible.

(c) The determinant is –27. The matrix is invertible.

(d) The determinant is zero. The matrix is singular. The inverse does not exist.

5. (a) The determinant is –10.

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{-1}{10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}.$$

(b) The determinant is 1.

$$\begin{bmatrix} -2 & -1 \\ 7 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ -7 & -2 \end{bmatrix}.$$

(c) The determinant is zero. The inverse does not exist.

(d) The determinant is 2.

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}.$$

6. (a) The determinant is –3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 3 \end{bmatrix}^{-1} = \frac{-1}{3}\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} -7 & 9 & 1 \\ 8 & -9 & -2 \\ -4 & 3 & 1 \end{bmatrix}.$$

(b) The determinant is –3.

$$\begin{bmatrix} 0 & 3 & 3 \\ 1 & 2 & 3 \\ 1 & 4 & 6 \end{bmatrix}^{-1} = \frac{-1}{3}\begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} 0 & -6 & 3 \\ -3 & -3 & 3 \\ 2 & 3 & -3 \end{bmatrix}.$$

(c) The determinant is –4.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \\ 1 & -2 & 0 \end{bmatrix}^{-1} = \frac{-1}{4}\begin{bmatrix} -6 & 2 & -2 \\ -3 & 1 & 1 \\ -8 & 4 & 0 \end{bmatrix}.$$

(d) The determinant is zero.
The inverse does not exist.

7. (a) The determinant is –1.

$$\begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 5 & 2 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = -\begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}.$$

(b) The determinant is 2.

$$\begin{bmatrix} -3 & -2 & -5 \\ 3 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -3 & 14 \\ 0 & 2 & -6 \\ -1 & 1 & -6 \end{bmatrix}.$$

(c) The determinant is zero. The inverse does not exist.

(d) The determinant is –3.

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & -1 & 4 \\ 7 & 4 & 5 \end{bmatrix}^{-1} = \frac{-1}{3}\begin{bmatrix} -21 & -6 & 9 \\ 8 & 3 & -4 \\ 23 & 6 & -10 \end{bmatrix}.$$

8. (a) $x_1 = \dfrac{\begin{vmatrix} 8 & 2 \\ 19 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}} = \dfrac{2}{1} = 2.$ $\quad x_2 = \dfrac{\begin{vmatrix} 1 & 8 \\ 2 & 19 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}} = \dfrac{3}{1} = 3.$

(b) $x_1 = \dfrac{\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \dfrac{5}{-5} = -1.$ $\quad x_2 = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \dfrac{-10}{-5} = 2.$

(c) $x_1 = \dfrac{\begin{vmatrix} 11 & 3 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}} = \dfrac{14}{7} = 2.$ $\quad x_2 = \dfrac{\begin{vmatrix} 1 & 11 \\ -2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}} = \dfrac{21}{7} = 3.$

9. (a) $x_1 = \dfrac{\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{-4}{2} = -2.$ $\quad x_2 = \dfrac{\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{10}{2} = 5.$

(b) $x_1 = \dfrac{\begin{vmatrix} 11 & 2 \\ 14 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \dfrac{5}{5} = 1.$ $\quad x_2 = \dfrac{\begin{vmatrix} 3 & 11 \\ 2 & 14 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \dfrac{20}{5} = 4.$

(c) $x_1 = \dfrac{\begin{vmatrix} -1 & 1 \\ 10 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}} = \dfrac{-12}{6} = -2.$ $\quad x_2 = \dfrac{\begin{vmatrix} 2 & -1 \\ -2 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}} = \dfrac{18}{6} = 3.$

10. (a) $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 8,\ |A_1| = \begin{vmatrix} 3 & 3 & 4 \\ 5 & 6 & 9 \\ 7 & 1 & -2 \end{vmatrix} = 8,\ |A_2| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 5 & 9 \\ 3 & 7 & -2 \end{vmatrix} = 16,$

$$|A_3| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 6 & 5 \\ 3 & 1 & 7 \end{vmatrix} = -8, \text{ so } x_1 = \frac{8}{8} = 1,\ x_2 = \frac{16}{8} = 2,\ x_3 = \frac{-8}{8} = -1.$$

(b) $$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{vmatrix} = 2,\ |A_1| = \begin{vmatrix} 9 & 2 & 1 \\ 4 & 3 & -1 \\ 7 & 4 & -1 \end{vmatrix} = -2,\ |A_2| = \begin{vmatrix} 1 & 9 & 1 \\ 1 & 4 & -1 \\ 1 & 7 & -1 \end{vmatrix} = 6,$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 9 \\ 1 & 3 & 4 \\ 1 & 4 & 7 \end{vmatrix} = 8, \text{ so } x_1 = \frac{-2}{2} = -1,\ x_2 = \frac{6}{2} = 3,\ x_3 = \frac{8}{2} = 4.$$

(c) $$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -2 & 4 \\ 1 & 4 & -2 \end{vmatrix} = 28,\ |A_1| = \begin{vmatrix} 2 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & 4 & -2 \end{vmatrix} = 14,\ |A_2| = \begin{vmatrix} 2 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 7,$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 2 \\ 3 & -2 & 2 \\ 1 & 4 & 1 \end{vmatrix} = 7, \text{ so } x_1 = \frac{14}{28} = \frac{1}{2},\ x_2 = \frac{7}{28} = \frac{1}{4},\ x_3 = \frac{7}{28} = \frac{1}{4}.$$

11. (a) $$|A| = \begin{vmatrix} 1 & 4 & 2 \\ 1 & 4 & -1 \\ 2 & 6 & 1 \end{vmatrix} = -6,\ |A_1| = \begin{vmatrix} 5 & 4 & 2 \\ 2 & 4 & -1 \\ 7 & 6 & 1 \end{vmatrix} = -18,\ |A_2| = \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & -1 \\ 2 & 7 & 1 \end{vmatrix} = 0,$$

$$|A_3| = \begin{vmatrix} 1 & 4 & 5 \\ 1 & 4 & 2 \\ 2 & 6 & 7 \end{vmatrix} = -6, \text{ so } x_1 = \frac{-18}{-6} = 3,\ x_2 = \frac{0}{-6} = 0,\ x_3 = \frac{-6}{-6} = 1.$$

(b) $$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix} = -35,\ |A_1| = \begin{vmatrix} 7 & -1 & 3 \\ 10 & 4 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 70,\ |A_2| = \begin{vmatrix} 2 & 7 & 3 \\ 1 & 10 & 2 \\ 3 & 0 & 1 \end{vmatrix} = -35,$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 7 \\ 1 & 4 & 10 \\ 3 & 2 & 0 \end{vmatrix} = -140, \text{ so } x_1 = \frac{70}{-35} = -2,\ x_2 = \frac{-35}{-35} = 1,\ x_3 = \frac{-140}{-35} = 4.$$

(c) $$|A| = \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 6 \\ 6 & 1 & 4 \end{vmatrix} = -128,\ |A_1| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & 6 \\ 3 & 1 & 4 \end{vmatrix} = -16,\ |A_2| = \begin{vmatrix} 8 & 1 & 1 \\ 2 & 3 & 6 \\ 6 & 3 & 4 \end{vmatrix} = -32,$$

$$|A_3| = \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 6 & 1 & 3 \end{vmatrix} = -64, \text{ so } x_1 = \frac{-16}{-128} = \frac{1}{8},\ x_2 = \frac{-32}{-128} = \frac{1}{4},\ x_3 = \frac{-64}{-128} = \frac{1}{2}.$$

12. (a) $|A| = 0$ so this system of equations cannot be solved using Cramer's rule.

(b) $$|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 6 & 0 \end{vmatrix} = -18,\ |A_1| = \begin{vmatrix} 7 & 1 & -1 \\ 3 & 2 & 1 \\ -4 & 6 & 0 \end{vmatrix} = -72,\ |A_2| = \begin{vmatrix} 3 & 7 & -1 \\ 1 & 3 & 1 \\ 2 & -4 & 0 \end{vmatrix} = 36,$$

$$|A_3| = \begin{vmatrix} 3 & 1 & 7 \\ 1 & 2 & 3 \\ 2 & 6 & -4 \end{vmatrix} = -54, \text{ so } x_1 = \frac{-72}{-18} = 4,\ x_2 = \frac{36}{-18} = -2,\ x_3 = \frac{-54}{-18} = 3.$$

(c) $$|A| = \begin{vmatrix} 3 & 6 & -1 \\ 1 & -2 & 3 \\ 4 & -2 & 5 \end{vmatrix} = 24,\ |A_1| = \begin{vmatrix} 3 & 6 & -1 \\ 2 & -2 & 3 \\ 5 & -2 & 5 \end{vmatrix} = 12,\ |A_2| = \begin{vmatrix} 3 & 3 & -1 \\ 1 & 2 & 3 \\ 4 & 5 & 5 \end{vmatrix} = 9,$$

$$|A_3| = \begin{vmatrix} 3 & 6 & 3 \\ 1 & -2 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 18, \text{ so } x_1 = \frac{12}{24} = \frac{1}{2},\ x_2 = \frac{9}{24} = \frac{3}{8},\ x_3 = \frac{18}{24} = \frac{3}{4}.$$

13. (a) The determinant of the coefficient matrix is zero, so there is not a unique solution.

(b) The determinant of the coefficient matrix is -10, so there is a unique solution.

(c) The determinant of the coefficient matrix is zero, so there is not a unique solution.

14. (a) The determinant of the coefficient matrix is 42, so there is a unique solution.

(b) The determinant of the coefficient matrix is zero, so there is not a unique solution.

(c) The determinant of the coefficient matrix is zero, so there is not a unique solution.

15. The system of equations will have nontrivial solutions if the determinant of the coefficient matrix is zero, i.e., if

$$\begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = 0.$$

$$\begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 30 = 2 - 3\lambda + \lambda^2 - 30 = \lambda^2 - 3\lambda - 28 = 0, \text{ so}$$

$(\lambda-7)(\lambda+4) = 0$ and $\lambda = 7$ or $\lambda = -4$. Substituting $\lambda = 7$ in the given equations, one finds that the general solution is $x_1 = x_2 = r$. For $\lambda = -4$, the general solution is $x_1 = -6r/5$, $x_2 = r$.

16. The system of equations will have nontrivial solutions if the determinant of the coefficient matrix is zero, i.e., if

$$\begin{vmatrix} \lambda+4 & \lambda-2 \\ 4 & \lambda-3 \end{vmatrix} = 0.$$

$$\begin{vmatrix} \lambda+4 & \lambda-2 \\ 4 & \lambda-3 \end{vmatrix} = (\lambda+4)(\lambda-3) - (\lambda-2)4 = \lambda^2 + \lambda - 12 - 4\lambda + 8 = \lambda^2 - 3\lambda - 4 = 0,$$

so $(\lambda-4)(\lambda+1) = 0$ and $\lambda = 4$ or $\lambda = -1$. For $\lambda = 4$, the general solution is $x_1 = -r/4$, $x_2 = r$. For $\lambda = -1$, the general solution is $x_1 = x_2 = r$.

17. The system of equations will have nontrivial solutions if the determinant of the coefficient matrix is zero, i.e., if

$$\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0.$$

$$\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda)(2-\lambda) + 16 + 16 - 4(5-\lambda) - 4(5-\lambda) - 16(2-\lambda)$$

$= 10 - 21\lambda + 12\lambda^2 - \lambda^3 = (1-\lambda)(1-\lambda)(10-\lambda) = 0$, so $\lambda = 1$ or $\lambda = 10$. For $\lambda = 1$, the general solution is $x_1 = -s - r/2$, $x_2 = s$, $x_3 = r$. For $\lambda = 10$, the general solution is $x_1 = x_2 = 2r$, $x_3 = r$.

18. $AX = \lambda X = \lambda I_n X$, so $AX - \lambda I_n X = 0$. Thus $(A - \lambda I_n)X = 0$, and this system of equations has a nontrivial solution if and only if $|A - \lambda I_n| = 0$.

19. If $A = A^t$, then the matrix obtained by deleting the ith row and jth column of A is the transpose of the matrix obtained by deleting the jth row and ith column. The determinant of the first of these matrices is C_{ij}, the (i,j)th element of $(\mathrm{adj}(A))^t$, and the determinant of

the second is C_{ji}, the (i,j)th element of adj(A). Since these determinants are equal, $\mathrm{adj}(A) = (\mathrm{adj}(A))^t$, and so adj(A) is symmetric.

20. If A is the zero matrix then adj(A) is the zero matrix also. If A is not the zero matrix and if A is not invertible then $|A| = 0$ and $A\,\mathrm{adj}(A) = |A|\, I_n = O$. If adj(A) is invertible then $A = A\, I_n = A\,[\mathrm{adj}(A)\,(\mathrm{adj}(A))^{-1}] = [A\,\mathrm{adj}(A)]\,(\mathrm{adj}(A))^{-1} = O\,(\mathrm{adj}(A))^{-1} = O$. Since A is not the zero matrix, this means that adj(A) has no inverse.

21. If A is invertible then $\frac{1}{|A|}\,\mathrm{adj}(A) = A^{-1}$, so that $A\,\frac{1}{|A|}\,\mathrm{adj}(A) = AA^{-1} = I_n$. Thus $\frac{1}{|A|}\,A = [\mathrm{adj}(A)]^{-1}$.

22. $A = (A^{-1})^{-1} = \frac{1}{|A^{-1}|}\,\mathrm{adj}(A^{-1})$, so $\mathrm{adj}(A^{-1}) = |A^{-1}|\,A = \frac{1}{|A|}\,A = [\mathrm{adj}(A)]^{-1}$ from Exercise 21

23. If AB is invertible then $|AB| \neq 0$. $|A||B| = |AB|$, so $|A| \neq 0$ and $|B| \neq 0$ and both A and B are invertible. The converse is also true. If A and B are invertible then $|AB| = |A||B| \neq 0$, so AB is invertible.

24. Let $B = A^{-1}$. The (i,j)th element of the product $AB = I_n$ is $a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + \ldots + a_{in}\,b_{nj}$.
For $i = n$, $a_{n1}\,b_{1j} + a_{n2}\,b_{2j} + \ldots + a_{nn}\,b_{nj} = a_{nn}\,b_{nj}$, because $a_{n1} = a_{n2} = \ldots = a_{nn-1} = 0$. This means that $a_{nn}\,b_{nj} = 0$ for $j < n$. But $a_{nn} \neq 0$ so $b_{nj} = 0$ for $j < n$.
For $i = n-1$, $a_{n-11}\,b_{1j} + a_{n-12}\,b_{2j} + \ldots + a_{n-1n-1}\,b_{n-1j} + a_{n-1n}\,b_{nj} = a_{n-1n-1}\,b_{n-1j}$, for $j < n$, because $a_{n-11} = a_{n-12} = \ldots = a_{n-2n-1} = b_{nj} = 0$. This means that $a_{n-1n-1}\,b_{n-1j} = 0$ for $j < n-1$. But $a_{n-1n-1} \neq 0$ so $b_{n-1j} = 0$ for $j < n-1$.
In the same manner each row of B is shown to have zero in all positions for which the column number is less than the row number. Thus B is upper triangular.

25. If $|A| = \pm 1$, then $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) = \pm\mathrm{adj}(A)$, and since all elements of A are integers, all elements of adj(A) are integers (because adding and multiplying integers gives integer results).

26. If $AX = 0$ has only the trivial solution, then $|A| \neq 0$ so that $|A^k| = |A|^k \neq 0$. Thus $A^k X = 0$ has only the trivial solution.

27. $AX = B_2$ has a unique solution if and only if $|A| \neq 0$ if and only if $AX = B_1$ has a unique solution.

28. (a) True: $|A^2| = |AA| = |A||A| = (|A|)^2$.

(b) True: Determinant of diagonal matrix is the product of diagonal elements. If this product is zero at least one of the elements must be zero.

(c) True: $A^{-1} = \mathrm{adj}(A)/|A|$. Thus if $|A| = 1$, $A^{-1} = \mathrm{adj}(A)$.

(d) False: e.g., Let $A-B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, so that $|A-B|=0$. Find two matrices A and B such that $A-B= \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$. Chances are they have distinct determinants. Say $A= \begin{bmatrix} 4 & 9 \\ 2 & 6 \end{bmatrix}$, $B= \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$. $|A|= 6$ and $|B|=3$.

(e) True: If A is nonsingular then it is invertible. Thus it is row equivalent to I (section 2.4).

Exercise Set 3.4

1. $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda-6)(\lambda-1)$, so the eigenvalues are $\lambda = 6$ and $\lambda = 1$. For $\lambda = 6$, the eigenvectors are the solutions of $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

2. $\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$, so the eigenvalues are $\lambda = 2$ and $\lambda = 3$. For $\lambda = 2$, the eigenvectors are the solutions of $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For $\lambda = 3$, the eigenvectors are the solutions of $\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3. $\begin{vmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{vmatrix} = (5-\lambda)(-2-\lambda) + 12 = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$, so the eigenvalues are $\lambda = 2$ and $\lambda = 1$. For $\lambda = 2$, the eigenvectors are the solutions of $\begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

4. $\begin{vmatrix} 5-\lambda & 2 \\ -8 & -3-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda) + 16 = \lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1)$, so the only eigenvalue is $\lambda = 1$. The eigenvectors are the solutions of $\begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

5. $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$, so the eigenvalues are $\lambda = 3$ and $\lambda = -1$. For $\lambda = 3$, the eigenvectors are the solutions of $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = -1$, the eigenvectors are the solutions of $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

6. $\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)(\lambda-3)$, so the only eigenvalue is $\lambda = 3$. The eigenvectors are the solutions of $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are

vectors of the form $r\begin{bmatrix}1\\1\end{bmatrix}$.

7. $\begin{vmatrix}3-\lambda & -1\\ 2 & -\lambda\end{vmatrix} = (3-\lambda)(-\lambda)+2 = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$, so the eigenvalues are $\lambda = 2$ and $\lambda = 1$. For $\lambda = 2$, the eigenvectors are the solutions of $\begin{bmatrix}1 & -1\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix}1\\1\end{bmatrix}$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix}2 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix}1\\2\end{bmatrix}$.

8. $\begin{vmatrix}2-\lambda & -4\\ -1 & 2-\lambda\end{vmatrix} = (2-\lambda)(2-\lambda) - 4 = \lambda^2 - 4\lambda = \lambda(\lambda-4)$, so the eigenvalues are $\lambda = 0$ and $\lambda = 4$. For $\lambda = 0$, the eigenvectors are the solutions of $\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix}2\\1\end{bmatrix}$. For $\lambda = 4$, the eigenvectors are the solutions of $\begin{bmatrix}-2 & -4\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix}-2\\1\end{bmatrix}$.

9. $\begin{vmatrix}3-\lambda & 2 & -2\\ -3 & -1-\lambda & 3\\ 1 & 2 & -\lambda\end{vmatrix} = (3-\lambda)(-1-\lambda)(-\lambda) + 18 + 2(-1-\lambda) - 6(3-\lambda) - 6\lambda$

$= -\lambda^3 + 2\lambda^2 + \lambda - 2 = (1-\lambda^2)(\lambda-2)$, so the eigenvalues are $\lambda = 1$, $\lambda = -1$, and $\lambda = 2$.

For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix}2 & 2 & -2\\ -3 & -2 & 3\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form r $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = -1$, the eigenvectors are the solutions of $\begin{bmatrix} 4 & 2 & -2 \\ -3 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form s $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, the eigenvectors are the solutions of $\begin{bmatrix} 1 & 2 & -2 \\ -3 & -3 & 3 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form t $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

10. $\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & 2 \\ -2 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda)$, so the eigenvalues are $\lambda = 1$ and $\lambda = 3$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form r $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 3$, the eigenvectors are the solutions of $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form s $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

11. $\begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & 1-\lambda & 2 \\ -2 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda)$, so the eigenvalues are $\lambda = 1$ and $\lambda = 3$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are

vectors of the form $r\begin{bmatrix}1\\0\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\end{bmatrix}$. For $\lambda=3$, the eigenvectors are the solutions of

$\begin{bmatrix}-2&0&0\\-2&-2&2\\-2&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$, so the eigenvectors are vectors of the form $t\begin{bmatrix}0\\1\\1\end{bmatrix}$.

12. $\begin{vmatrix}1-\lambda&0&0\\-2&5-\lambda&-2\\-2&4&-1-\lambda\end{vmatrix}=(1-\lambda)(5-\lambda)(-1-\lambda)\ +8(1-\lambda)=(1-\lambda)^2(3-\lambda)$, so the eigenvalues are

$\lambda=1$ and $\lambda=3$. For $\lambda=1$, the eigenvectors are the solutions of $\begin{bmatrix}0&0&0\\-2&4&-2\\-2&4&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$,

so the eigenvectors are vectors of the form $r\begin{bmatrix}-1\\0\\1\end{bmatrix}+s\begin{bmatrix}1\\1\\1\end{bmatrix}$. For $\lambda=3$, the eigenvectors

are the solutions of $\begin{bmatrix}-2&0&0\\-2&2&-2\\-2&4&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$, so the eigenvectors are vectors of the form

$t\begin{bmatrix}0\\1\\1\end{bmatrix}$.

13. $\begin{vmatrix}15-\lambda&7&-7\\-1&1-\lambda&1\\13&7&-5-\lambda\end{vmatrix}=(1-\lambda)(16-10\lambda+\lambda^2)=(1-\lambda)(2-\lambda)(8-\lambda)$, so the eigenvalues are

$\lambda=1$, $\lambda=2$, and $\lambda=8$. For $\lambda=1$, the eigenvectors are the solutions of

$\begin{bmatrix}14&7&-7\\-1&0&1\\13&7&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix}1\\-1\\1\end{bmatrix}$. For $\lambda=2$, the eigenvectors are the solutions of $\begin{bmatrix} 13 & 7 & -7 \\ -1 & -1 & 1 \\ 13 & 7 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form s $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 8$, the eigenvectors are the solutions of $\begin{bmatrix} 7 & 7 & -7 \\ -1 & -7 & 1 \\ 13 & 7 & -13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form t $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

14. $\begin{vmatrix} 5-\lambda & -2 & 2 \\ 4 & -3-\lambda & 4 \\ 4 & -6 & 7-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(1-\lambda)$, so the eigenvalues are $\lambda = 5$, $\lambda = 3$, and $\lambda = 1$.

For $\lambda = 5$, the eigenvectors are the solutions of $\begin{bmatrix} 0 & -2 & 2 \\ 4 & -8 & 4 \\ 4 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form r $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 3$, the eigenvectors are the solutions of $\begin{bmatrix} 2 & -2 & 2 \\ 4 & -6 & 4 \\ 4 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form s $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 4 & -2 & 2 \\ 4 & -4 & 4 \\ 4 & -6 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form t $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

15. $\begin{vmatrix} 4-\lambda & 2 & -2 & 2 \\ 1 & 3-\lambda & 1 & -1 \\ 0 & 0 & 2-\lambda & 0 \\ 1 & 1 & -3 & 5-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & 2 & 2 \\ 1 & 3-\lambda & -1 \\ 1 & 1 & 5-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(2-\lambda)(6-\lambda)$, so the eigenvalues are $\lambda = 2$, $\lambda = 4$, and $\lambda = 6$. For $\lambda = 2$, the eigenvectors are the solutions of

$\begin{bmatrix} 2 & 2 & -2 & 2 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form

$r\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 4$, the eigenvectors are the solutions of

$\begin{bmatrix} 0 & 2 & -2 & 2 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 1 & 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $t\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

For $\lambda = 6$, the eigenvectors are the solutions of $\begin{bmatrix} -2 & 2 & -2 & 2 \\ 1 & -3 & 1 & -1 \\ 0 & 0 & -4 & 0 \\ 1 & 1 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $p\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

16. $\begin{vmatrix} 3-\lambda & 5 & -5 & 5 \\ 3 & 1-\lambda & 3 & -3 \\ -2 & 2 & -\lambda & 2 \\ 0 & 4 & -6 & 8-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 5 & -5 & 8-\lambda \\ 3 & 1-\lambda & 3 & 0 \\ -2 & 2 & -\lambda & 0 \\ 0 & 4 & -6 & 8-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 1 & 0 \\ 3 & 1-\lambda & 3 & 0 \\ -2 & 2 & -\lambda & 0 \\ 0 & 4 & -6 & 8-\lambda \end{vmatrix}$

$= (8-\lambda)\begin{vmatrix} 3-\lambda & 1 & 1 \\ 3 & 1-\lambda & 3 \\ -2 & 2 & -\lambda \end{vmatrix} = (8-\lambda)(4-\lambda)(2-\lambda)(2+\lambda)$, so the eigenvalues are $\lambda = 8$, $\lambda = 4$,

$\lambda = 2$, and $\lambda = -2$. For $\lambda = 2$, the eigenvectors are the solutions of

$\begin{bmatrix} 1 & 5 & -5 & 5 \\ 3 & -1 & 3 & -3 \\ -2 & 2 & -2 & 2 \\ 0 & 4 & -6 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. For

$\lambda = 4$, the eigenvectors are the solutions of $\begin{bmatrix} -1 & 5 & -5 & 5 \\ 3 & -3 & 3 & -3 \\ -2 & 2 & -4 & 2 \\ 0 & 4 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the

eigenvectors are vectors of the form $s\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. For $\lambda = 8$, the eigenvectors are the

solutions of $\begin{bmatrix} -5 & 5 & -5 & 5 \\ 3 & -7 & 3 & -3 \\ -2 & 2 & -8 & 2 \\ 0 & 4 & -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form

$t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda = -2$, the eigenvectors are the solutions of $\begin{bmatrix} 5 & 5 & -5 & 5 \\ 3 & 3 & 3 & -3 \\ -2 & 2 & 2 & 2 \\ 0 & 4 & -6 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$,

so the eigenvectors are vectors of the form $p\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

17. $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$, so the only eigenvalue is $\lambda = 1$. The eigenvectors are the

solutions of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are all the vectors in $\mathbf{R}^2$. The

transformation represented by the identity matrix is the identity transformation that maps each vector in $\mathbf{R}^2$ into itself.

18. $\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$, so the only eigenvalue is $\lambda = 3$. The eigenvectors are the

solutions of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are all the vectors in $\mathbf{R}^2$. The

transformation represented by the given matrix is an expansion transformation that maps each vector **v** in $\mathbf{R}^2$ into the vector 3**v**. Thus each image in $\mathbf{R}^2$ has the same direction as the original vector.

19. $\begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2$, so the only eigenvalue is $\lambda = -2$. The eigenvectors are the

solutions of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are all the vectors in $\mathbf{R}^2$. The

transformation represented by the given matrix maps each vector **v** in $\mathbf{R}^2$ into the vector −2**v**. Thus each image in $\mathbf{R}^2$ has the direction opposite the original vector.

20. $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \neq 0$ for any real value of λ, so there are no real eigenvalues.

The given matrix is a rotation matrix that rotates each vector in $\mathbf{R}^2$ through a 90° angle. Thus no vector has the same or opposite direction as its image.

21. $\begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 2 = \lambda^2 + 1 \neq 0$ for any real value of λ, so there are no

real eigenvalues. Thus no eigenvectors. The image of every vector lies on a line different from the line on which the original vector lies.

22. $$|I_n - \lambda I_n| = \left| \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 0 & \dots & 0 \\ 0 & 1-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} = (1-\lambda)^n |I_n| = (1-\lambda)^n.$$

Eigenvalues of I_n are given by $|I_n - I I_n| = 0$. Thus eigenvalue $\lambda=1$, of multiplicity n. $I_n\mathbf{x} = 1\mathbf{x}$ is true for all x in $\mathbf{R}^n$. Thus the eigenspace is $\mathbf{R}^n$. Note that if we multiply any vector in $\mathbf{R}^n$ by I_n it remains unchanged in direction and magnitude, confirming this result. Every vector is an eigenvector. Geometrically, the matrix maps every vector into itself.

23. $$|A - \lambda I_n| = \left| \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 1 & \dots & 1 \\ 1 & 1-\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1-\lambda \end{vmatrix} =$$

$$\begin{vmatrix} n-\lambda & n-\lambda & \dots & n-\lambda \\ 1 & 1-\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1-\lambda \end{vmatrix} = (n-\lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1-\lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1-\lambda \end{vmatrix} = (n-\lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda \end{vmatrix} =$$

(on adding each of the following rows to the first) (on subtracting the first row from each of the following rows)

$$(n-\lambda) \begin{vmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda \end{vmatrix} = (n-\lambda)\lambda^{n-1} = 0.$$ Thus $\lambda = n$ or 0.

$\lambda = n$: Eigenvectors given by $A\mathbf{x} = n\mathbf{x}$. Let $\mathbf{x} = (a, b, \dots, z)$. Then $a+b+\dots+z = na$; $a+b+\dots+z = nb$; ... ; $a+b+\dots+z = nz$. Thus $na=nb=,\dots,=nz$. $a=b=\dots=z$. Eigenvectors are of the form $k(1,1,\dots,1)$.
$\lambda = n$: Eigenvectors given by $A\mathbf{x}=0\mathbf{x}$. $A\mathbf{x} = 0$. Let $\mathbf{x} = (a, b, \dots, z)$. Thus $a+b+\dots+z = 0$. $z=-(a+b+\dots)$. Eigenvectors are of the form $(a, b, \dots, -(a+b+\dots))$.

24. If A is a diagonal matrix with diagonal elements a_{ii}, then $A - \lambda I_n$ is also a diagonal matrix with diagonal elements $a_{ii} - \lambda$. Thus $|A - \lambda I_n|$ is the product of the terms $a_{ii} - \lambda$, and the solutions of the equation $|A - \lambda I_n| = 0$ are the values $\lambda = a_{ii}$, the diagonal elements of A.

25. If A is an upper triangular matrix with diagonal elements a_{ii}, then $A - \lambda I_n$ is also an upper triangular matrix with diagonal elements $a_{ii} - \lambda$. $|A - \lambda I_n|$ is the product of the terms $a_{ii} - \lambda$, and the solutions of the equation $|A - \lambda I_n| = 0$ are the values $\lambda = a_{ii}$, the diagonal elements of A.

26. $(A - \lambda I_n)^t = A^t - (\lambda I_n)^t = A^t - \lambda I_n$, so $|A - \lambda I_n| = |(A - \lambda I_n)^t| = |A^t - \lambda I_n|$, that is, A and A^t have the same characteristic polynomial and therefore the same eigenvalues.

27. $|A - 0I_n| = |A|$, so $|A - 0I_n| = 0$ if and only if $|A| = 0$.

28. If λ is an eigenvalue of A and $\mathbf{x}$ is an eigenvector corresponding to λ, then $A\mathbf{x} = \lambda\mathbf{x}$,

$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$ (i.e., λ^2 is an eigenvalue of A^2 with corresponding eigenvector $\mathbf{x}$), and $A^m \mathbf{x} = A^{m-1}(A\mathbf{x}) = A^{m-1}(\lambda\mathbf{x}) = \lambda A^{m-1}\mathbf{x} = \lambda A^{m-2}(A\mathbf{x})$

$= \lambda A^{m-2}(\lambda\mathbf{x}) = \lambda^2 A^{m-2}\mathbf{x} = \lambda^2 A^{m-3}(A\mathbf{x}) = \ldots = \lambda^{m-1} A(A\mathbf{x}) = \lambda^{m-1} A(\lambda\mathbf{x}) = \lambda^m \mathbf{x}$, so that λ^m is an eigenvalue of A^m with corresponding eigenvector $\mathbf{x}$.

29. Have that $A\mathbf{x} = \lambda\mathbf{x}$. Thus $A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x})$. $(A^{-1}A)\mathbf{x} = \lambda A^{-1}\mathbf{x}$. $(I)\mathbf{x} = \lambda A^{-1}\mathbf{x}$.

$\mathbf{x} = \lambda A^{-1}\mathbf{x}$. $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$.

30. Have that $A\mathbf{x} = \lambda\mathbf{x}$. $A\mathbf{x} - c\mathbf{x} = \lambda\mathbf{x} - c\mathbf{x}$. $(A - cI)\mathbf{x} = (\lambda - c)\mathbf{x}$.
Thus $\lambda - c$ is an eigenvalue of $A - cI$ with corresponding eigenvector $\mathbf{x}$.

31. The characteristic polynomial for the nxn zero matrix O_n is λ^n. Thus the only eigenvalue of O_n is $\lambda = 0$. However if λ is an eigenvalue of A then λ^k is an eigenvalue of $A^k = O_n$ (see Exercise 26). Thus $\lambda^k = 0$ so $\lambda = 0$.

32. The characteristic polynomial of A is $|A - \lambda I_n| = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$.
Substituting $\lambda = 0$, this equation becomes $|A| = c_0$.

33. $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$, so the eigenvalues are $\lambda = 2$ and $\lambda = -1$. The corresponding eigenvectors (written as row vectors) are [2 1] and [1 −1].

$\begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - 3 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$, so the eigenvalues are $\lambda = 3$ and $\lambda = -1$. The corresponding eigenvectors are [3 1] and [1 −1].

$\begin{vmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$, so the eigenvalues are $\lambda = 4$ and $\lambda = -1$. The corresponding eigenvectors are [4 1] and [1 −1].

$\begin{vmatrix} 4-\lambda & 5 \\ 1 & -\lambda \end{vmatrix} = (4-\lambda)(-\lambda) - 5 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$, so the eigenvalues are $\lambda = 5$ and $\lambda = -1$. The corresponding eigenvectors are [5 1] and [1 −1].

Conjecture: The matrix $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda = a+1$ and $\lambda = -1$ and corresponding eigenvectors [a+1 1] and [1 −1].

Proof: $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a+1 \\ 1 \end{bmatrix} = (a+1)\begin{bmatrix} a+1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} a & a+1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

34. (a) $\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = (-\lambda)(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2.$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) $\begin{vmatrix} 8-\lambda & -10 \\ 5 & -7-\lambda \end{vmatrix} = (8-\lambda)(-7-\lambda) + 50 = \lambda^2 - \lambda - 6.$

$$\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}^2 - \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} -14 & 10 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) $\begin{vmatrix} 6-\lambda & -8 \\ 4 & -6-\lambda \end{vmatrix} = (6-\lambda)(-6-\lambda) + 32 = \lambda^2 - 4.$

$$\begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}^2 - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(d) $\begin{vmatrix} -1-\lambda & 5 \\ -10 & 14-\lambda \end{vmatrix} = (-1-\lambda)(14-\lambda) + 50 = \lambda^2 - 13\lambda + 36.$

$$\begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}^2 - 13\begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -49 & 65 \\ -130 & 146 \end{bmatrix} + \begin{bmatrix} 49 & -65 \\ 130 & -146 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

35. (a) False: Let A be a 3x3 matrix. The characteristic equation of A is $|A - \lambda I_n| = 0$. This will be a polynomial of degree 3 in λ. Thus 3, 2, or 1 distinct roots. 3, 2, or 1 distinct eigenvalues. In general nxn matrix has n, n-1, ..., 2, or 1 distinct eigenvalues.

(b) True: Suppose $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$. Then $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$, $\lambda_1\mathbf{x} - \lambda_2\mathbf{x} = \mathbf{0}$, $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$. Since eigenvector $\mathbf{x}$ is nonzero, $(\lambda_1 - \lambda_2) = 0$, $\lambda_1 = \lambda_2$.

(c) False: Set of all eigenvectors for a given eigenvalue λ lie in a subspace. Thus the sum of any two of these is an eigenvector in that subspace. However sum of two eigenvectors from different eigenspaces is not an eigenvector - Let $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$. Add, $A\mathbf{x}_1+A\mathbf{x}_2 = \lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2$, $A(\mathbf{x}_1+\mathbf{x}_2) = \lambda_1\mathbf{x}_1+\lambda_2\mathbf{x}_2 \neq \lambda(\mathbf{x}_1+\mathbf{x}_2)$ for any value of λ since $\lambda_1 \neq \lambda_2$. Thus in general, the sum of two eigenvectors is not an eigenevector.

(d) True: e.g. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$. Then $|A-\lambda I| = (2-\lambda)(-2-\lambda)=0$, $\lambda=2$ or -2. $|B-\lambda I| = (-2-\lambda)(2-\lambda)=0$, $\lambda=-2$ or 2. A and B have the same eigenvalues.

Exercise Set 3.5

1. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. If there is no change in total population $2r + r = 245 + 52 = 297$, so $r = 297/3$. Thus the long-term prediction is that population in metropolitan areas will be $2r=198$ million and population in nonmetropolitan areas will be $r=99$ million.

2. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} .7 \\ 1.3 \\ 1 \end{bmatrix}$. If there is no change in total population $.7r + 1.3r + r = 82 + 163 + 52 = 297$, so $r = 297/3$. Thus the long-term prediction is that population in the cities will be $.7r = 60.3$ million, population in the suburbs will be $1.3r = 128.7$ million, and population in nonmetropolitan areas will be $r=99$ million.

3. $P^2 = \begin{bmatrix} .375 & .25 & .125 \\ .5 & .5 & .5 \\ .125 & .25 & .375 \end{bmatrix}$, and since all terms are positive, P is regular. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. The powers of P approach the stochastic matrix $Q = \begin{bmatrix} s & s & s \\ 2s & 2s & 2s \\ s & s & s \end{bmatrix}$, so $s = .25$ and $Q = \begin{bmatrix} .25 & .25 & .25 \\ .5 & .5 & .5 \\ .25 & .25 & .25 \end{bmatrix}$.

The columns of Q indicate that when guinea pigs are bred with hybrids, only the long-term distribution of types AA, Aa, and aa will be 1:2:1. That is, the long-term probabilities of Types AA, Aa, and aa are .25, .5, and .25.

4. $213/326=.65$, $117/511=.23$ (to 2 dec places). $P = \begin{array}{cc} \text{wet} & \text{dry} \\ \end{array}$ $\begin{array}{l} \\ \left[\begin{array}{cc} .65 & .23 \\ .35 & .77 \end{array}\right] \begin{array}{l} \text{wet} \\ \text{dry} \end{array} \end{array}$

(a) $P^2 = \begin{bmatrix} .5 & .33 \\ .5 & .67 \end{bmatrix}$. If Thursday is dry, the probability that Saturday will also be dry is .67, the (2,2) term in P^2.

(b) The eigenvectors of P corresponding to $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 23 \\ 35 \end{bmatrix}$.

Thus the powers of P approach the stochastic matrix $Q = \begin{bmatrix} 23s & 23s \\ 35s & 35s \end{bmatrix}$, so

$(23+35)s = 1$ and $s = \frac{1}{58}$. $\frac{23}{58} = .4$, $\frac{35}{58} = .6$, and $Q = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}$, so the long-term probability for a wet day in December is .4 and for a dry day is .6.

5. P is regular since all terms of P^2 are positive. The eigenvectors of P corresponding to $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$; thus the distribution of rats in rooms 1, 2, and 3 is 3:3:2. The powers of P approach the stochastic matrix $Q = \begin{bmatrix} 3s & 3s & 3s \\ 3s & 3s & 3s \\ 2s & 2s & 2s \end{bmatrix}$, so $3s + 3s + 2s = 1$ and $s = 1/8$. The long-term probability that a given rat will be in room 2 is therefore 3/8.

6.

$$\begin{array}{cc} & \begin{array}{cccc} \text{room } 1 & 2 & 3 & 4 \end{array} \\ P = & \begin{bmatrix} 0 & 1/3 & 0 & 1/4 \\ 1/2 & 0 & 1/3 & 1/4 \\ 0 & 1/3 & 0 & 1/2 \\ 1/2 & 1/3 & 2/3 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

P is regular since every term in P^2 is positive.

The eigenvectors of P corresponding to $\lambda = 1$ are vectors of the form

$r\begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$; thus the distribution of rats in rooms 1, 2, 3, and 4 is 2:3:3:4. The powers of P approach the stochastic matrix $Q = \begin{bmatrix} 2s & 2s & 2s & 2s \\ 3s & 3s & 3s & 3s \\ 3s & 3s & 3s & 3s \\ 4s & 4s & 4s & 4s \end{bmatrix}$, so $2s + 3s + 3s + 4s = 1$ and $s = 1/12$. The long-term probability that a given rat will be in room 4 is therefore $4/12 = 1/3$.

7. $P = \begin{bmatrix} .75 & .20 \\ .25 & .80 \end{bmatrix}$. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 4 \\ 5 \end{bmatrix}$. The powers of P approach $Q = \begin{bmatrix} 4s & 4s \\ 5s & 5s \end{bmatrix}$, so $4s + 5s = 1$ and $s = 1/9$. If current trends continue, the eventual distribution will be $4/9 = 44.4\%$ using company A and $5/9 = 55.6\%$ using company B.

8. $P = \begin{bmatrix} .8 & .2 & .05 \\ .05 & .75 & .05 \\ .15 & .05 & .9 \end{bmatrix}$. The eigenvectors of $\lambda = 1$ are vectors of the form $r\begin{bmatrix} 9 \\ 5 \\ 16 \end{bmatrix}$. The powers of P approach $Q = \begin{bmatrix} 9s & 9s & 9s \\ 5s & 5s & 5s \\ 16s & 16s & 16s \end{bmatrix}$, so $9s + 5s + 16s = 1$ and $s = \frac{1}{30}$. If current buying patterns continue, the eventual distribution will be $9/30 = 30\%$ using product I, $5/30 = 16\frac{2}{3}\%$ using product II, and $16/30 = 53\frac{1}{3}\%$ using product III.

9. The sum of the terms in each column of a stochastic matrix A is 1, so the sum of the terms in each column of $A - I$ is zero. It has previously been proved (Exercise 18, Section 3.2) that if the sum of the terms in each column of a matrix is zero, the determinant of the matrix is zero. Thus $|A - 1I| = |A - I| = 0$, and 1 is an eigenvalue of A.

<u>Chapter 3 Review Exercises</u>

1. (a) $3\text{x}1 - 2\text{x}5 = -7$. (b) $-3\text{x}6 - 0\text{x}1 = -18$. (c) $9\text{x}4 - 7\text{x}1 = 29$.

2. (a) $M_{12} = \begin{vmatrix} -3 & 1 \\ 7 & 2 \end{vmatrix} = -3x2 - 1x7 = -13.$ $\quad C_{12} = (-1)^{1+2} M_{12} = 13.$

(b) $M_{31} = \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} = 1x1 - 0x4 = 1.$ $\quad C_{31} = (-1)^{3+1} M_{31} = 1.$

(c) $M_{22} = \begin{vmatrix} 2 & 0 \\ 7 & 2 \end{vmatrix} = 2x2 - 0x4 = 4.$ $\quad C_{22} = (-1)^{2+2} M_{22} = 4.$

3. (a) using row 1:

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & 5 \\ 4 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 4 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -1 + 40 + 24 = 63.$$

using column 1:

$$= 1\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 2 & 5 \end{vmatrix} = -1 + 0 + 64 = 63.$$

(b) using row 3:

$$\begin{vmatrix} 0 & 5 & 3 \\ 2 & -3 & 1 \\ 2 & 7 & 3 \end{vmatrix} = 2\begin{vmatrix} 5 & 3 \\ -3 & 1 \end{vmatrix} - 7\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 5 \\ 2 & -3 \end{vmatrix} = 28 + 42 - 30 = 40.$$

using column 2:

$$= -5\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + (-3)\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix} - 7\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = -20 + 18 + 42 = 40.$$

4. $\begin{vmatrix} x & x \\ 2 & x-3 \end{vmatrix} = x(x-3) - 2x = x^2 - 5x = -6,\ x^2 - 5x + 6 = 0.$ Thus $(x-3)(x-2) = 0$, so $x = 3$ or $x = 2$.

5. (a) $$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 2 & 4 & 1 \end{vmatrix} \underset{R3+(-2)R1}{=} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -15.$$

(b) $$\begin{vmatrix} 5 & 3 & 4 \\ 4 & 6 & 1 \\ 2 & -3 & 7 \end{vmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{=} \begin{vmatrix} 5 & 3 & 4 \\ -6 & 0 & -7 \\ 7 & 0 & 11 \end{vmatrix} = -3\begin{vmatrix} -6 & -7 \\ 7 & 11 \end{vmatrix} = 51.$$

(c) $\begin{vmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ -1 & 5 & 6 \end{vmatrix} \underset{\substack{R2+(-2)R1 \\ R3+R1}}{=} \begin{vmatrix} 1 & 4 & -2 \\ 0 & -5 & 5 \\ 0 & 9 & 4 \end{vmatrix} = \begin{vmatrix} -5 & 5 \\ 9 & 4 \end{vmatrix} = -65.$

6. (a) This matrix can be obtained from A by multiplying row 2 by 3, so its determinant is $3|A| = 6$.

   (b) This matrix can be obtained from A by adding −2 times row 1 to row 2, so its determinant is $|A| = 2$.

   (c) This matrix can be obtained from A by multiplying row 1 by 2, row 2 by −1, and row 3 by 3, so its determinant is $2 \times -1 \times 3 \times |A| = -12$.

7. (a) $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 4 & 3 \\ 2 & 0 & 5 \end{vmatrix} \underset{R2+(-2)R1}{=} \begin{vmatrix} 1 & 2 & 4 \\ -3 & 0 & -5 \\ 2 & 0 & 5 \end{vmatrix} = -2\begin{vmatrix} -3 & -5 \\ 2 & 5 \end{vmatrix} = 10.$

   (b) $\begin{vmatrix} -1 & 3 & 2 \\ 0 & 5 & 2 \\ 1 & 7 & 6 \end{vmatrix} \underset{R3+R1}{=} \begin{vmatrix} -1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & 10 & 8 \end{vmatrix} = -1\begin{vmatrix} 5 & 2 \\ 10 & 8 \end{vmatrix} = -20.$

   (c) $\begin{vmatrix} 2 & -3 & 5 \\ 4 & 0 & 6 \\ 1 & 2 & 7 \end{vmatrix} \underset{R3+(2/3)R1}{=} \begin{vmatrix} 2 & -3 & 5 \\ 4 & 0 & 6 \\ 7/3 & 0 & 31/3 \end{vmatrix} = -(-3)\begin{vmatrix} 4 & 6 \\ 7/3 & 31/3 \end{vmatrix} = 82.$

8. (a) $|3A| = 3^3\,|A| = 27 \times -2 = -54.$ (b) $|2AA^t| = 2^3\,|A||A^t| = 8|A||A| = 32.$

   (c) $|A^3| = |A|^3 = (-2)^3 = -8.$

   (d) $|(A^t A)^2| = (|A^t A|)^2 = (|A^t||A|)^2 = (|A||A|)^2 = |A|^4 = 16.$

   (e) $|(A^t)^3| = (|A^t|)^3 = (|A|)^3 = (-2)^3 = -8.$

   (f) $|2A^t(A^{-1})^2| = (2)^3\,|A^t||(A^{-1})^2| = 8|A^t||A^{-1}|^2 = 8|A|(1/|A|)^2 = 8/|A| = -4.$

9. $|B| \neq 0$ so B^{-1} exists, and A and B^{-1} can be multiplied. Let $C = AB^{-1}$. Then $CB = AB^{-1}B = A$.

10. $|C^{-1}AC| = |C^{-1}||AC| = |C^{-1}||A||C| = |C^{-1}||C||A| = |C^{-1}C||A| = |A|$

11. If A is upper triangular, then any element in A is zero if its row number is greater than its column number. If $i > j$, then for every k with $1 \le k \le n$, $a_{ik}a_{kj} = 0$ because either $i \ge k$ so that $a_{ik} = 0$, or $k \ge i > j$ so that $a_{kj} = 0$. Thus if $i > j$, the (i,j)th term $a_{i1}a_{1j} + a_{i2}a_{2j} + \ldots + a_{in}a_{nj}$ of A^2 is zero because each summand is zero. So A^2 is upper triangular. The proof is similar for lower triangular matrices.

12. If $A^2 = A$, then $|A||A| = |A|$ so $|A| = 1$ or zero. If A is also invertible, then $|A| \ne 0$ so $|A| = 1$.

13. (a) $\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 1$, so $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

(b) $\begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix} = 17$, so $\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

(c) $\begin{vmatrix} 1 & 4 & -1 \\ 0 & 2 & 0 \\ 1 & 6 & -1 \end{vmatrix} = 0$, so the inverse does not exist.

(d) $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 9 \\ 4 & 2 & 11 \end{vmatrix} = 20$, so $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 9 \\ 4 & 2 & 11 \end{bmatrix}^{-1}$

$$= \frac{1}{20}\begin{bmatrix} \begin{vmatrix} 2 & 9 \\ 2 & 11 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} \\ -\begin{vmatrix} 0 & 9 \\ 4 & 11 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 11 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 9 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -5 & 3 \\ 36 & 10 & -18 \\ -8 & 0 & 4 \end{bmatrix}.$$

14. (a) $x_1 = \dfrac{\begin{vmatrix} -1 & 1 \\ 18 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix}} = \dfrac{-13}{-13} = 1,\quad x_2 = \dfrac{\begin{vmatrix} 2 & -1 \\ 3 & 18 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix}} = \dfrac{39}{-13} = -3.$

(b) $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 5 & 1 \end{vmatrix} = 8$, $|A_1| = \begin{vmatrix} 1 & 1 & 1 \\ 5 & -1 & 3 \\ 3 & 5 & 1 \end{vmatrix} = 16$, $|A_2| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 3 \\ 4 & 3 & 1 \end{vmatrix} = -8$,

$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 4 & 5 & 3 \end{vmatrix} = 0$, so $x_1 = \frac{16}{8} = 2$, $x_2 = \frac{-8}{8} = -1$, and $x_3 = \frac{0}{8} = 0$.

15. If A is not invertible, $|A| = 0$. The (i,j)th term of A[adj(A)] is $a_{i1}C_{j1} + a_{i2}C_{j2} + \ldots + a_{in}C_{jn}$. If $i = j$, this term is $|A| = 0$ and if $i \neq j$ it is the determinant of the matrix obtained from A by replacing row j with row i; that is, it is the determinant of a matrix having two equal rows, and so it is zero. Thus A[adj(A)] is the zero matrix.

16. $|A|$ is the product of the diagonal elements, and since $|A| \neq 0$ all diagonal elements must be nonzero.

17. If $|A| = \pm 1$ then $A^{-1} = \pm\, \mathrm{adj}(A)$, so $X = A^{-1}AX = A^{-1}B = \pm\, \mathrm{adj}(A)B$. If all the elements of A and of B are integers, then all the elements of adj(A) are integers and all the elements of the product adj(A)B are integers, so X has all integer components.

18. $\begin{vmatrix} 5-\lambda & -7 & 7 \\ 4 & -3-\lambda & 4 \\ 4 & -1 & 2-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)(-2-\lambda)$, so the eigenvalues are $\lambda = 5$, $\lambda = 1$, and $\lambda = -2$. For $\lambda = 5$, the eigenvectors are the solutions of $\begin{bmatrix} 0 & -7 & 7 \\ 4 & -8 & 4 \\ 4 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $r\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. For $\lambda = 1$, the eigenvectors are the solutions of $\begin{bmatrix} 4 & -7 & 7 \\ 4 & -4 & 4 \\ 4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = -2$, the eigenvectors are the solutions of $\begin{bmatrix} 7 & -7 & 7 \\ 4 & -1 & 4 \\ 4 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$, so the eigenvectors are vectors of the form $t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

19. Let λ be an eigenvalue of A with eigenvector $\mathbf{x}$. A is invertible so $\lambda \neq 0$. $A\mathbf{x} = \lambda\mathbf{x}$, so $\mathbf{x} = A^{-1} A\mathbf{x} = A^{-1} \lambda\mathbf{x} = \lambda A^{-1} \mathbf{x}$. Thus $\frac{1}{\lambda}\ \mathbf{x} = A^{-1}\mathbf{x}$, so the eigenvalues for A^{-1} are the inverses of the eigenvalues for A and the corresponding eigenvectors are the same.

20. $A\mathbf{x} = \lambda\mathbf{x}$, so $A\mathbf{x} - kI\mathbf{x} = \lambda\mathbf{x} - kI\mathbf{x} = \lambda\mathbf{x} - k\mathbf{x}$, so $(A - kI)\mathbf{x} = (\lambda - k)\mathbf{x}$. Thus $\lambda - k$ is an eigenvalue for $A - kI$ with corresponding eigenvector $\mathbf{x}$.