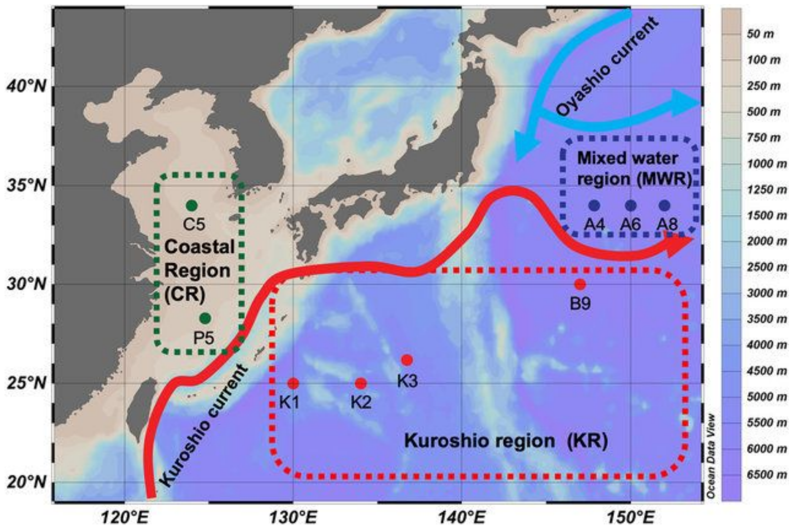
Climate as a complex system

Project 1: “Toy stochastic models”

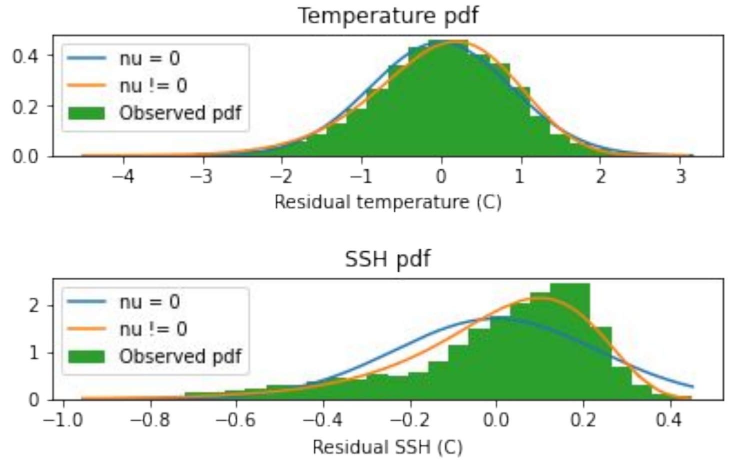
Talk Summary

- Analysis of SST and SSH data (Julia)
- SDE Model and Stationary Fokker Planck Equation (Calvin)
- Parameter Fitting (Jamie)

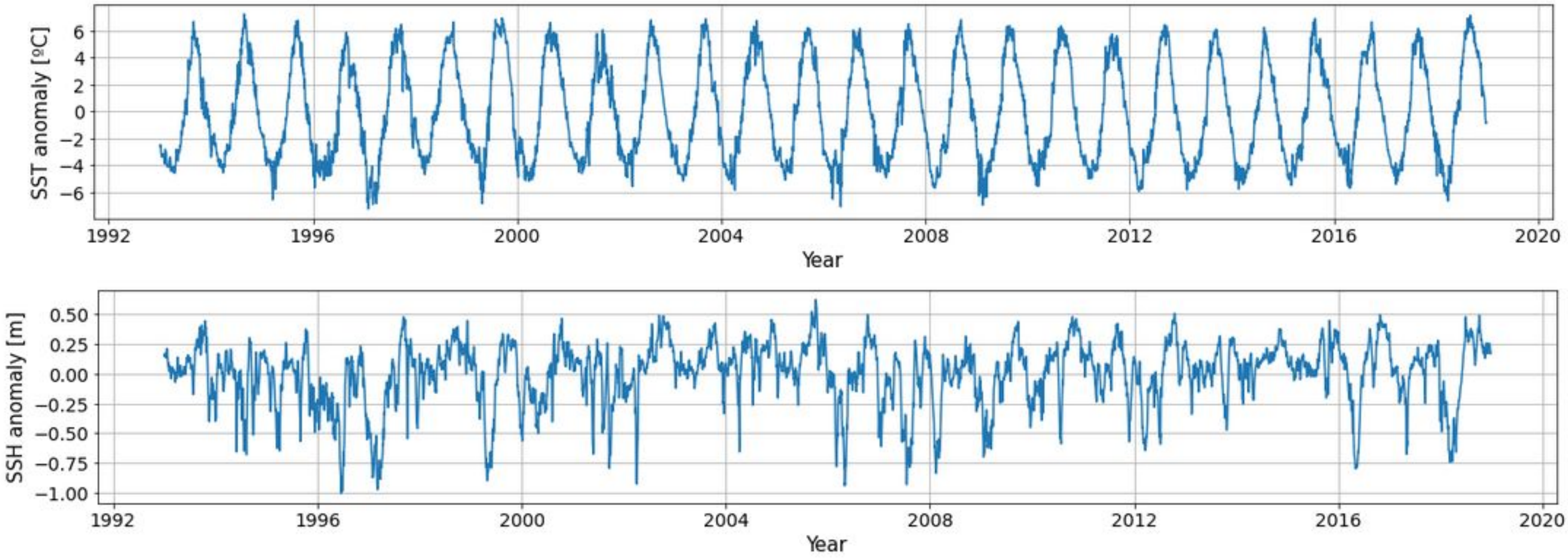


Wu et al., (2019)

$$dX_t = \lambda X_t dt + (\mu + \upsilon X_t) dW_t$$

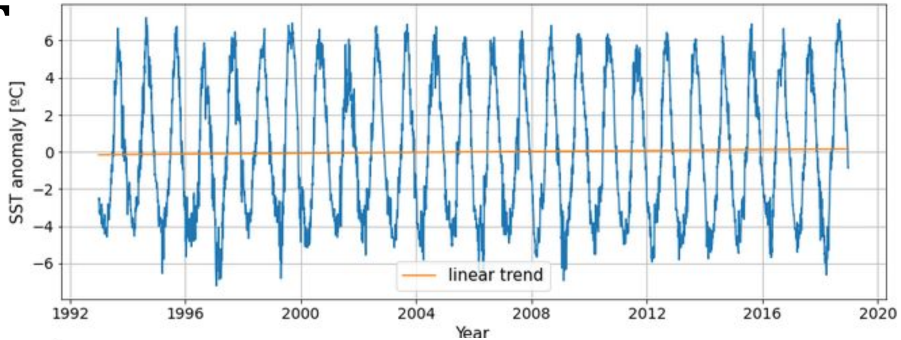


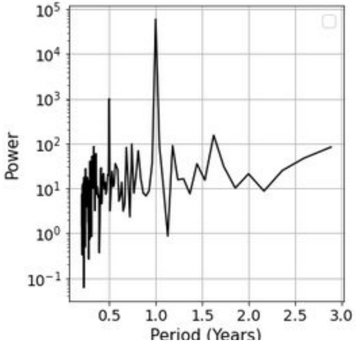
SST and SSH anomalies



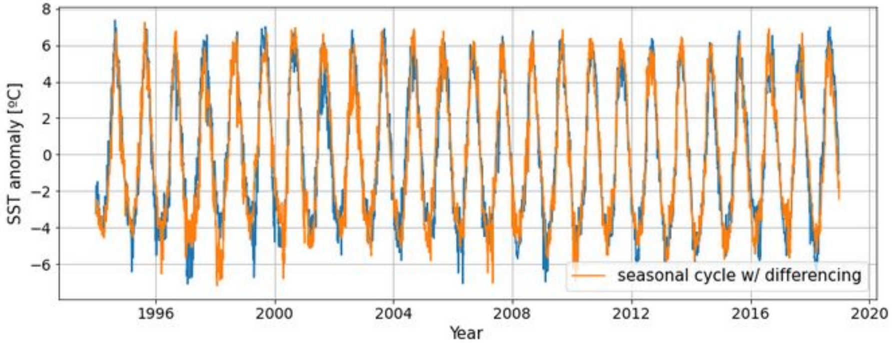
Sea surface temperature and sea surface height at 153ºE, 33ªN, Kuroshio Current

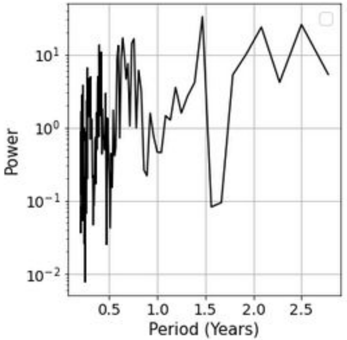
SST



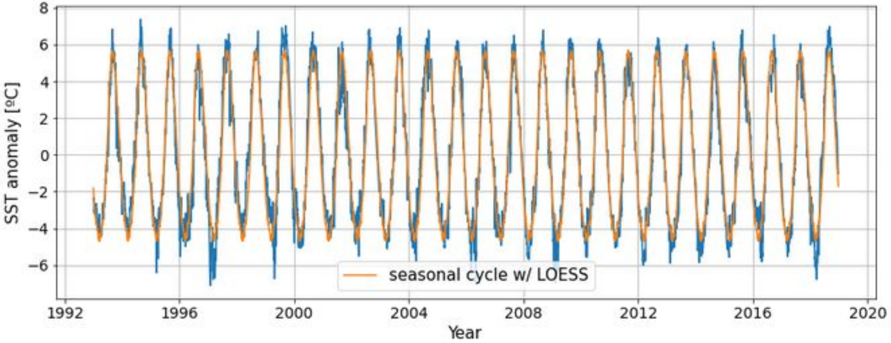


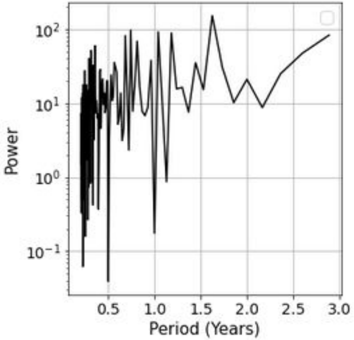
Linear trend





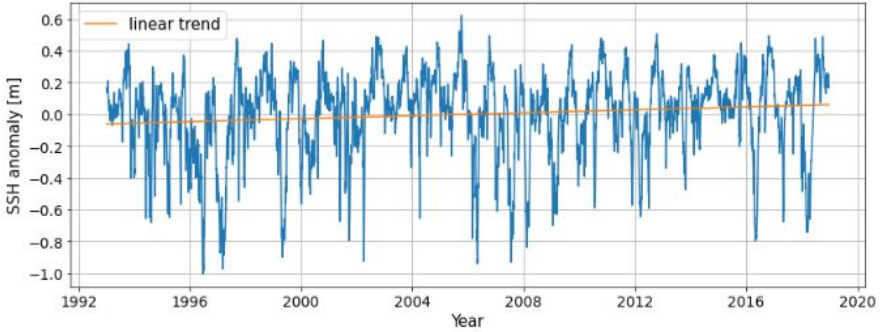
Seasonal cycle removed applying differencing

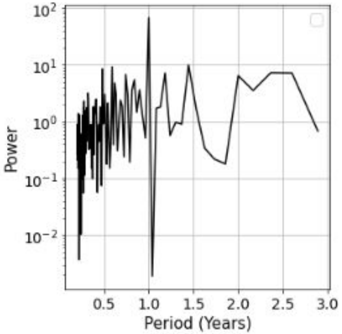




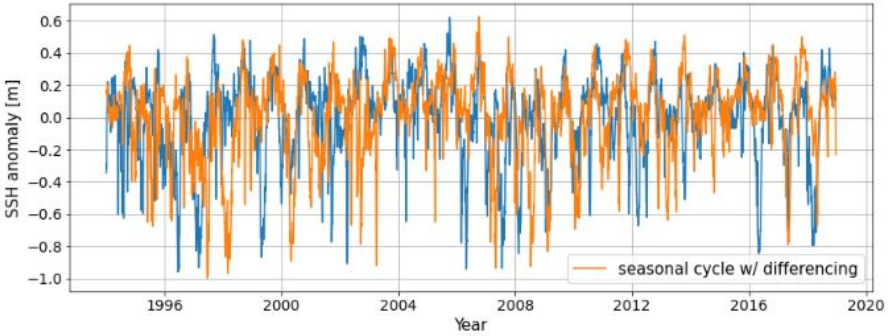
Seasonal cycle removed with the “seasonal_decompose” module, which applies Locally Weighted Scatterplot Smoothing (**LOESS**)

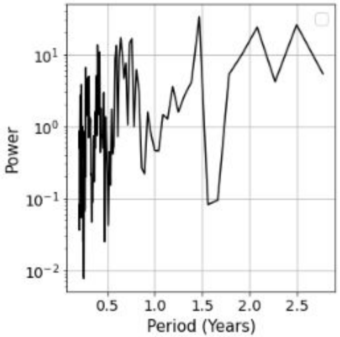
SSH





Linear trend





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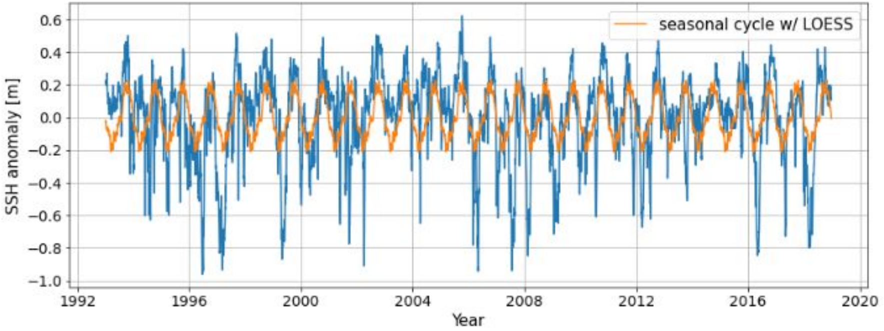
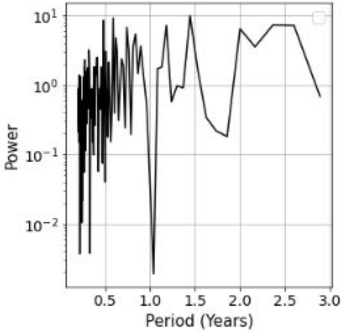
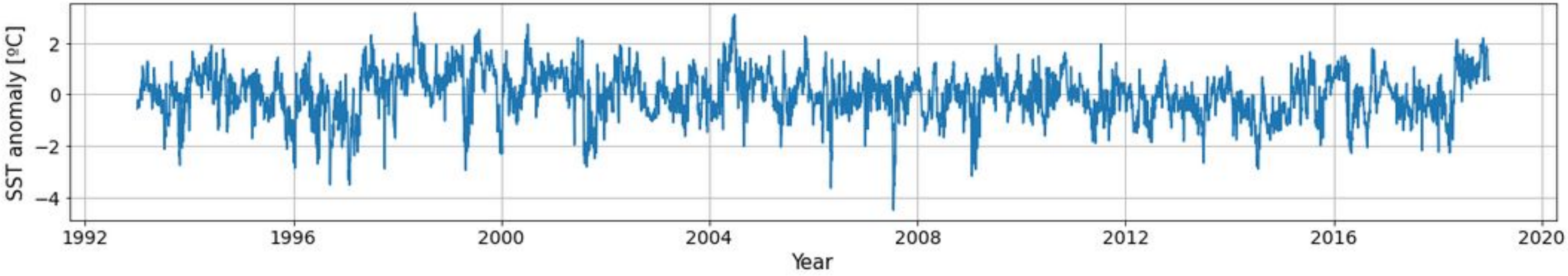
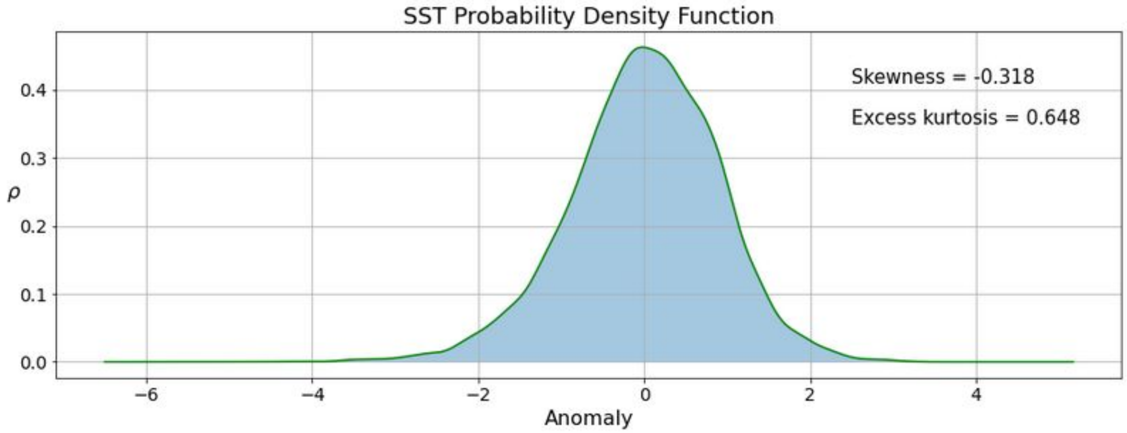




Seasonal cycle removed with the “seasonal_decompose” module, which applies Locally Weighted Scatterplot Smoothing (**LOESS**)

Sea surface temperature anomalies





Kernel Density Estimation using the module “Gaussian_kde” from “scipy.stats”

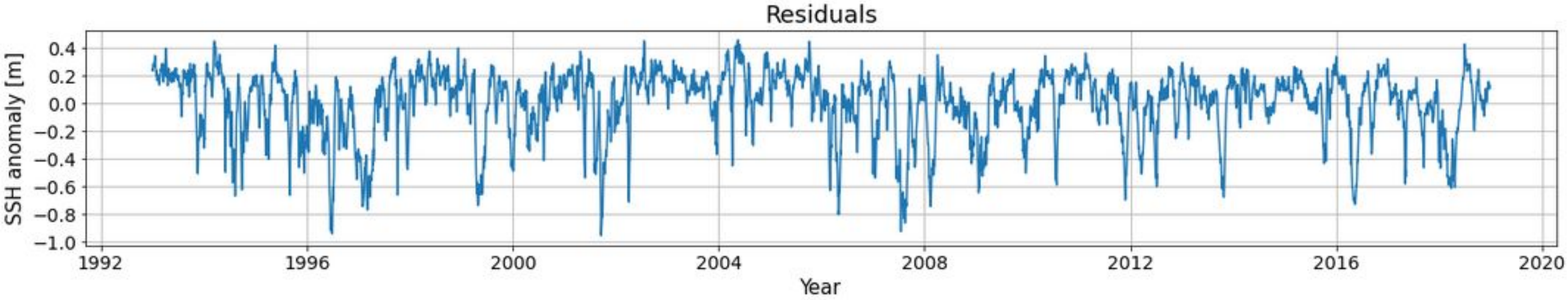
$$S = \frac{\overline{h'^3}}{\sigma^3}$$

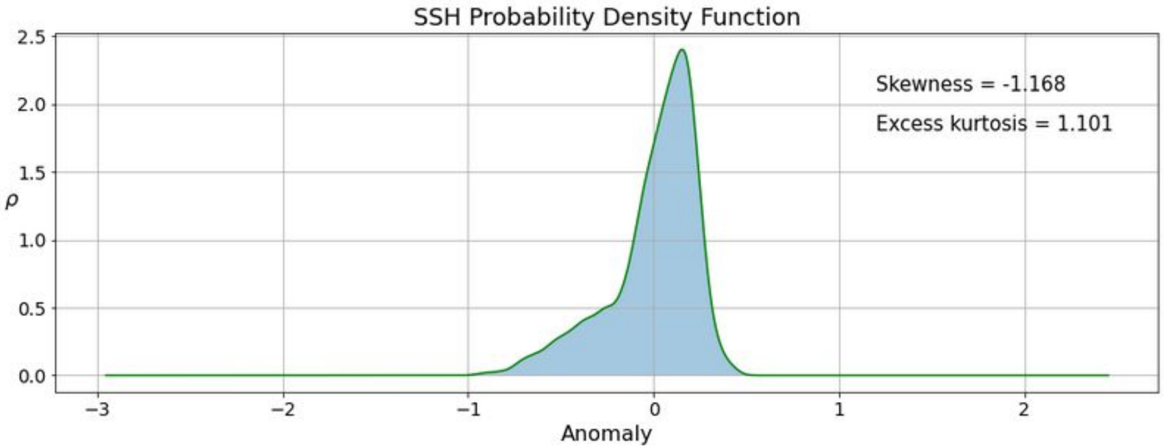
$$K = \frac{\overline{h'^4}}{\sigma^4} - 3$$

Where h’ are the anomalies, S is skewness, K is the “excess kurtosis”

K > 1.5 S^2 like in Sura et al., 2008

Sea Surface Height anomalies



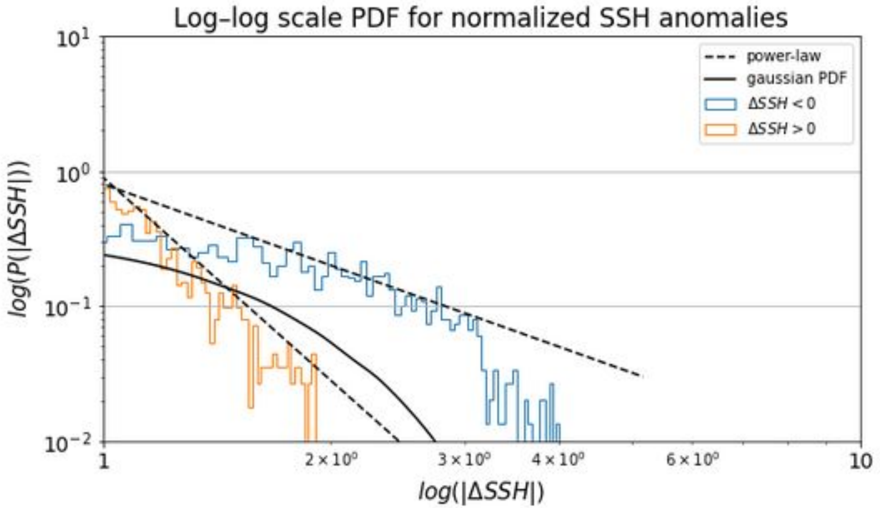


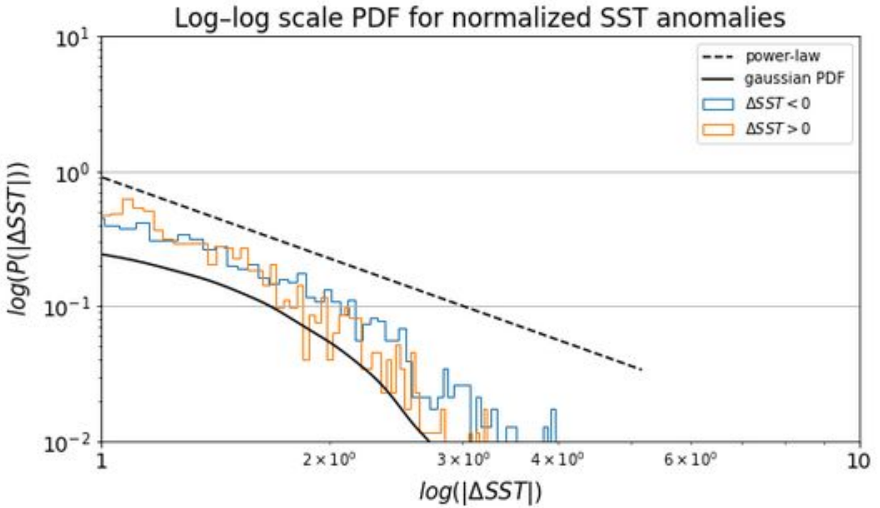
- Skewness represents the asymmetry of the distribution.
- High kurtosis indicates numerous extreme events

Example: 1997 in the Kuroshio Current

K > 1.5 S^2 - 1.5, like in Sura et al. (2010)

Non-gaussian power-law tails

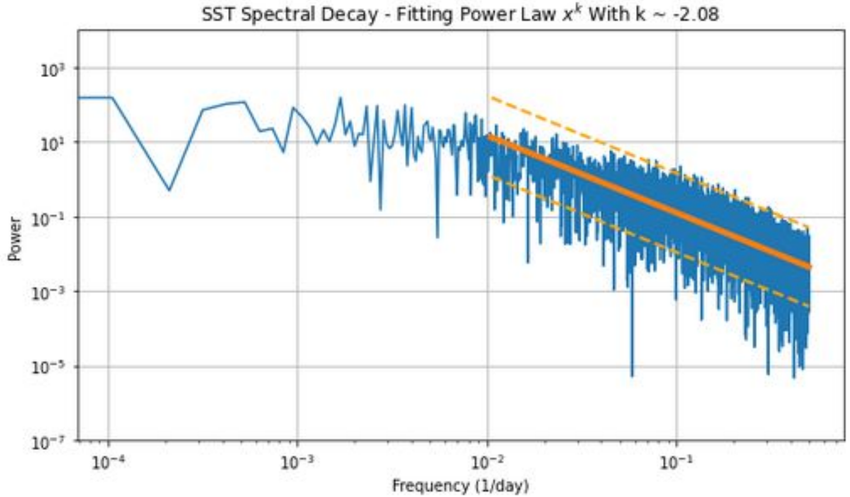


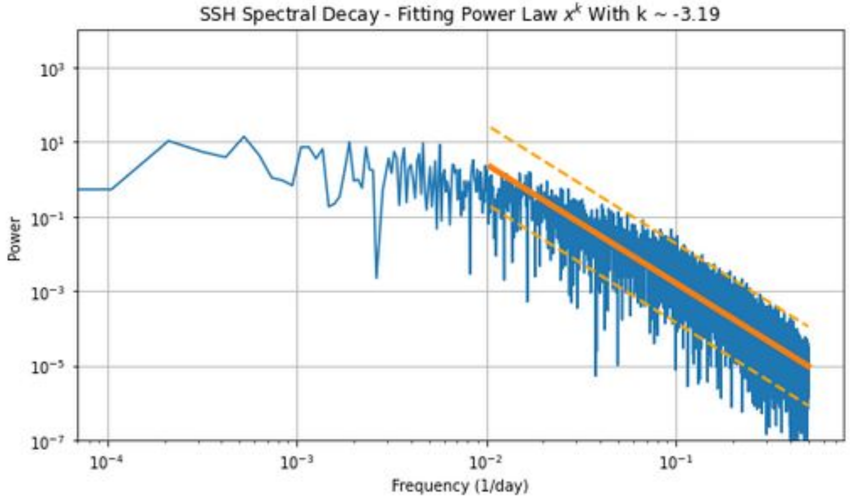


Log-log plot of the PDF tails, we show the PDF of the two normalized tails separately, in absolute value. We can observe deviations from Gaussian behaviour in the SSH data, not too clear for SSTs.

- A highly non-Gaussian power-law tail in this plot will appear as a straight line. $(P(h') \propto h'^{-\alpha})$
- Different decay rate in SSH positive and negative anomalies.
- Scale invariance, indicates self-similarity, this self-similarity is observed in many non-Gaussian natural phenomena

Decay in Power spectral density





$$\tilde{S}_{xx}(\omega) = \frac{(\Delta t)^2}{T} \left| \sum_{n=1}^{N} x_n e^{-i\omega n \Delta t} \right|^2 .$$

$$x_n = x(n\Delta t)$$

$$T = N\Delta t.$$

Multiplicative noise in SST and SSH

$$\frac{\partial T'_o}{\partial t} = -\lambda T'_o - \phi F' T'_o + F' + R' + \phi \overline{F' T'},$$

Sura & Sardeshmukh (2008)

In this article they conclude that the observed non-Gaussianity of SST anomalies is due to **(weak)** multiplicative noise rather than to nonlinearities in the deterministic part of the SST equation.

$$\frac{dx}{dt} = Ax + (Ex + G)\eta - \frac{1}{2}EG + B\zeta,$$

From a general form for the governing equations, applying a **fast-slow time-scale separation.**

Studying the Fokker-Planck equation for this SDE, they find:

- The property that the (excess) kurtosis K is always greater than 1.5 times the square of the skewness S, K >= 1.5 S^2
- A PDF with power-law tails, $P(x) \propto x^{-\alpha}$.

Sura & Guille (2010)

SDE Model With Multiplicative Noise

$$dX_t = \lambda X_t dt + (\mu + \upsilon X_t) dW_t$$

- Compare with Ornstein Uhlenbeck where there is no x dependence in the noise
- Similar to well known geometric Brownian motion

Writing the Stationary Fokker Planck Equation

Drift

$$\lambda x$$

Diffusion

$$\frac{(\mu + \nu x)^2}{2}$$

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t) p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t) p(x,t)\right]$$

Thus the stationary Fokker-Planck equation is

$$\lambda \partial_x (px) - \frac{1}{2}\partial_{xx}(p(\nu x + \mu)^2) = 0.$$

Solving the Stationary Fokker Planck Equation (i)

$$\lambda\partial_x(px) - \frac{1}{2}\partial_{xx}(p(\nu x + \mu)^2) = 0.$$

Integrating with respect to x and using the boundary conditions on p (it's a density) we arrive at:

$$\lambda px - \frac{1}{2}\partial_x(p(\nu x + \mu)^2) = 0.$$

Solving the Stationary Fokker Planck Equation (ii)

$$\lambda p x - \frac{1}{2}\partial_x(p(\nu x+\mu)^2) = 0.$$

This is a separable ODE in x which has solution

$$\boxed{p(x) = C_0(\nu x+\mu)^F \exp\left\{\frac{-2\lambda x}{\nu(\nu x+\mu)}\right\}}$$

where $F = \frac{2\lambda}{\nu^2} - 2$ and C_0 is a constant of integration.

Making Sense of the Solution

$$\boxed{p(x) = C_0(\nu x + \mu)^F \exp\left\{\frac{-2\lambda x}{\nu(\nu x + \mu)}\right\}} \qquad F = \frac{2\lambda}{\nu^2} - 2$$

- We want a finite mean $$< x >:= \int_{\mathbf{R}} x(\nu x + \mu)^F \exp\left\{\frac{-2\lambda x}{\nu(\nu x + \mu)}\right\} \mathrm{d}x$$

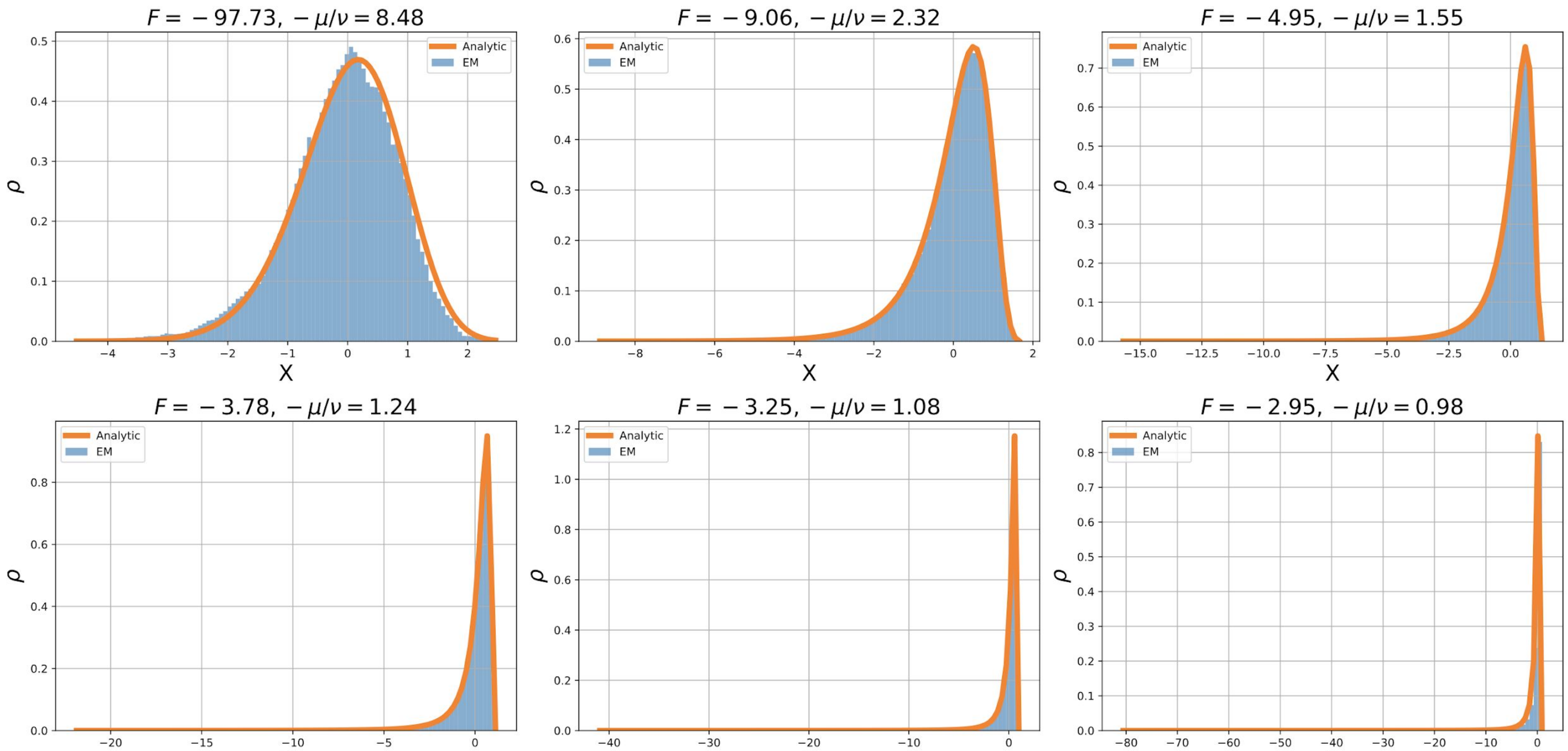
- So would expect integrand to be 0 at the boundaries
- Exponential terms tends to a constant
- Thus we require F + 1 < 0:

$$\lambda < \frac{\nu^2}{2}$$

- Singularity at $$x = -\frac{\mu}{\nu}$$

Characterising Behaviour (i): PDFs

$$p(x) = C_0(\nu x + \mu)^F \exp\left\{\frac{-2\lambda x}{\nu(\nu x + \mu)}\right\}$$



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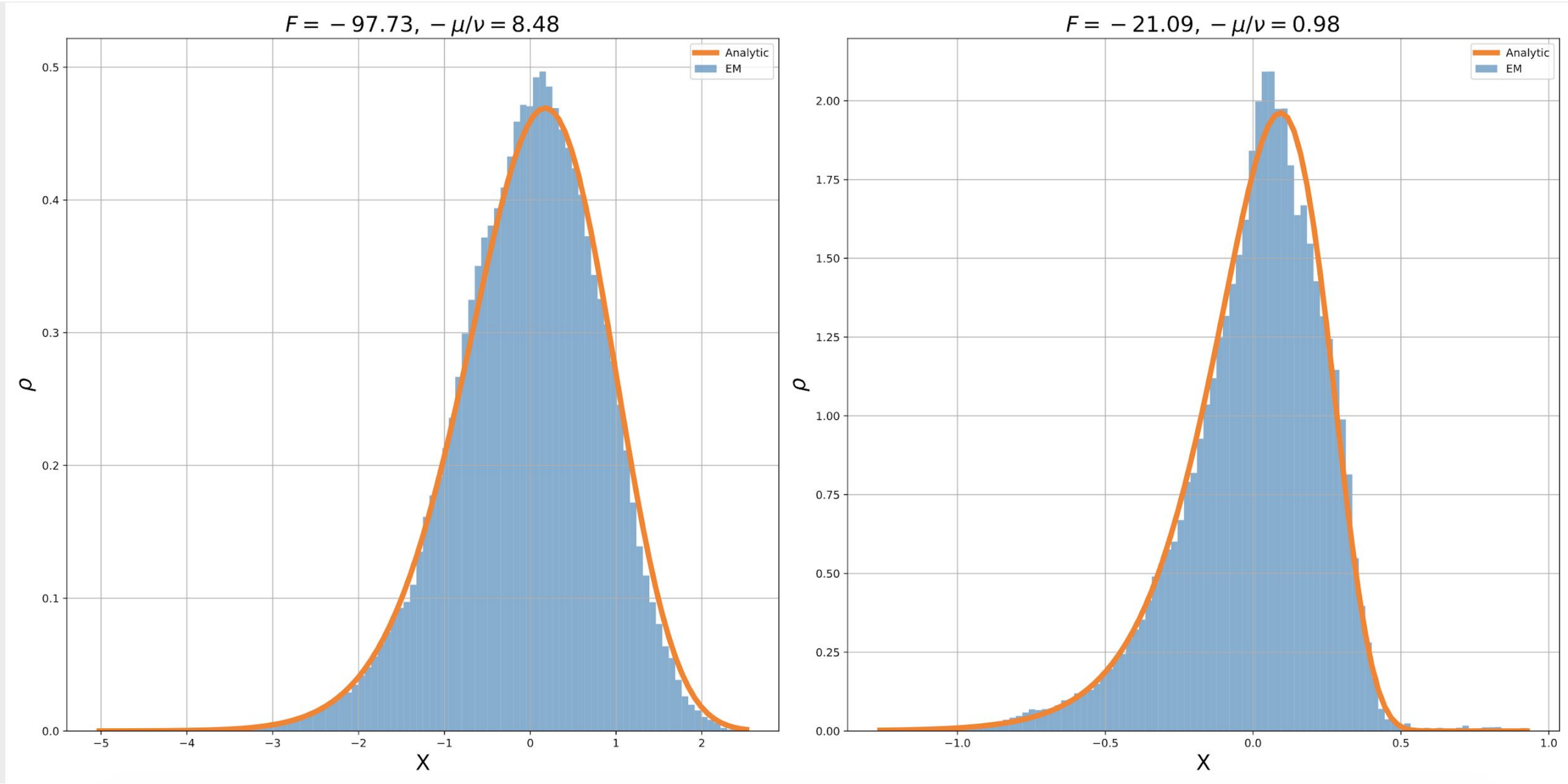
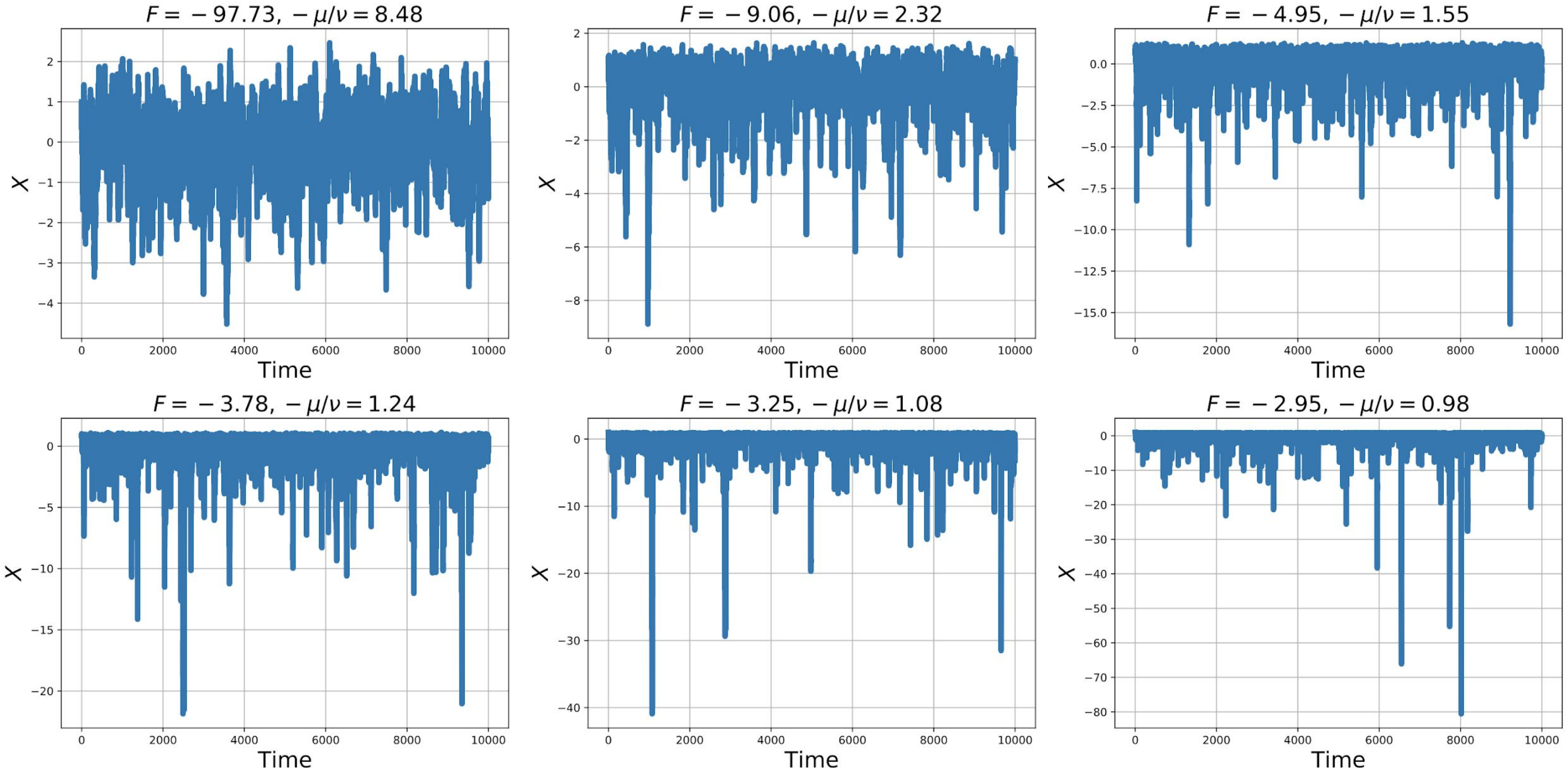


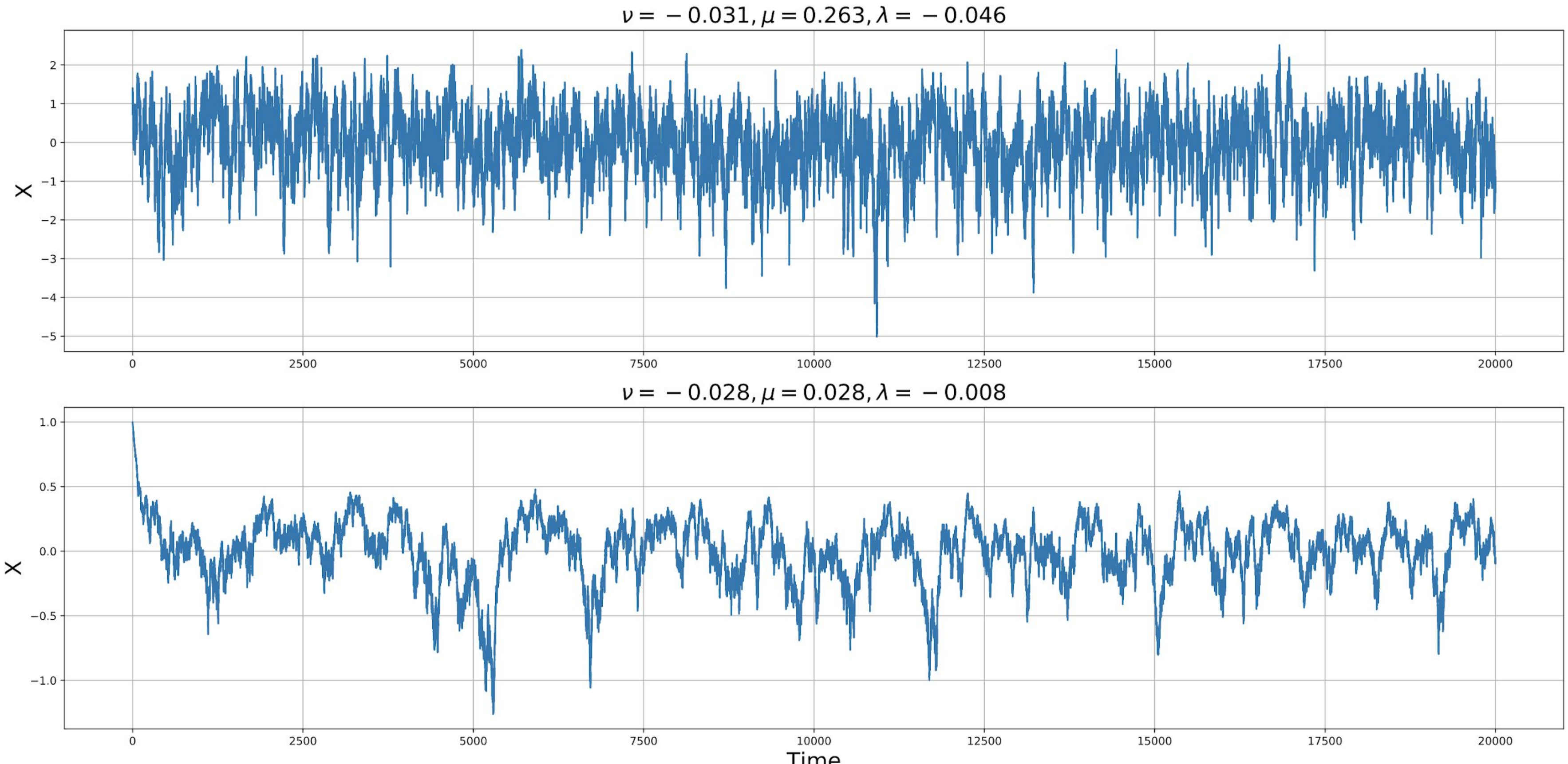
$$dX_t = \lambda X_t dt + (\mu + \upsilon X_t) dW_t$$

Characterising Behaviour (ii): Timeseries

$$F = \frac{2\lambda}{\nu^2} - 2$$



Characterising Behaviour (ii): Timeseries



Parameter Estimation - Maximum Likelihood Estimation

Have state observations, x_k , at N times.

Assume model $dX_t = \lambda X_t dt + (\mu + \nu X) dW_t$

Probability density for observed trajectory $f_0(X_0|\theta) \prod_{k=0}^{N-1} f(X_{k+1}|X_k;\theta)$

Maximise w.r.t. Parameters θ={λ,ν,μ}.

Minimise negative of log-likelihood function (Jeisman, 2005) $-\sum_{k=0}^{N-1} \log(f(X_{k+1}|X_k;\theta))$

Estimate transition probability densities using Euler-Maruyama method

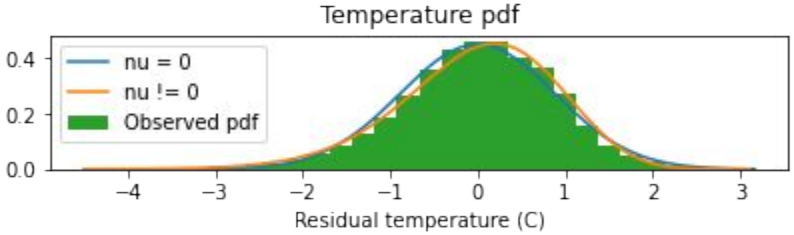
$dt = 1$ day , $X_{k+1} \approx X_k(1+\lambda) + (\mu + \nu X_k)\xi$, $\xi \sim N(0,1)$

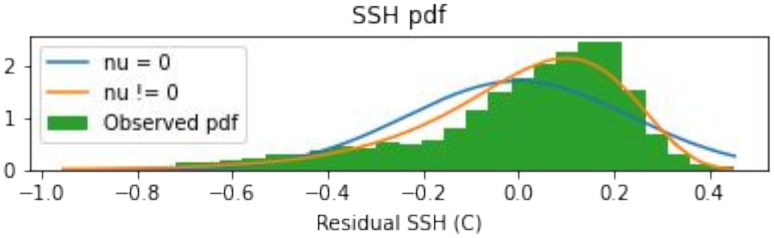
$$\Rightarrow X_{k+1}|X_k \sim N(X_k(1+\lambda), (\mu + \nu X_k))$$

$$\Rightarrow f(X_{k+1} = x|X_k = y; \theta)) \approx \frac{1}{\sqrt{2\pi}(\mu + \nu y)} \exp\left(-\frac{1}{2}\left(\frac{x - y(1+\lambda)}{\mu + \nu y}\right)^2\right)$$

Have analytic form of log-likelihood function for given data

Results





Ornstein-Uhlenbeck (ν=0):

SST: $\mu = 0.265^\circ\,\mathrm{C}\,\mathrm{day}^{-1/2},\ \lambda = -0.044\,\mathrm{day}^{-1}$

SSH: $\mu = 0.0290\,\mathrm{m}\,\mathrm{day}^{-1/2},\ \lambda = -0.0076\,\mathrm{day}^{-1}$

Multiplicative Noise (ν≠0):

SST: $\mu = 0.262^\circ\,\mathrm{C}\,\mathrm{day}^{-1/2},\ \lambda = -0.042\,\mathrm{day}^{-1},\ \nu = -0.032\,\mathrm{day}^{-1/2}$

SSH: $\mu = 0.0276\,\mathrm{m}\,\mathrm{day}^{-1/2},\ \lambda = -0.0083\,\mathrm{day}^{-1},\ \nu = -0.0348\,\mathrm{day}^{-1/2}$

Conclusions / Summary

SSH is significantly skewed compared to SST

Potentially explained by baroclinic instability - affects SSH more than SST

SST is modeled well by red noise whereas SSH requires multiplicative noise

Validated using comparison between spectral powers, parameter estimation for an SDE and comparison between analytic Fokker-Planck and observed PDF

References

Cleveland, R. B., Cleveland, W. S., McRae, J. E., and Terpenning, I. (1990) STL: A Seasonal-Trend Decomposition Procedure Based on LOESS. Journal of Official Statistics, 6, 3-73.

Sura, P., Guille, S. (2010) Stochastic Dynamics of Sea Surface Height Variability, Journal of Physical Oceanography, 40(7):1582-1596

Sura, P. and P. D. Sardeshmukh, 2008: A global view of non-Gaussian SST variability. J. Phys. Oceanogr., 38,639–647.

Vaughan, S., (2005) A simple test for periodic signals in red noise, Astronomy & Astrophysics 431, 391-403

Wu, P-F. et al., (2019) The diversity and biogeography of microeukaryotes in the euphotic zone of the northwestern Pacific Ocean, Science of The Total Environment 698:134289

Joseph Jeisman, (2005) Estimation of the Parameters of Stochastic Differential Equations