ELEC4402 Formula Sheet

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Abstract

Key formulae and helpers for ELEC4402 - Communication Systems. Hopefully this helps to carry!!!

Contents

1 Handy Formulae

1.1 Rectangle Functions

$$\text{rect}(t) \rightarrow \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{rect}\left(\frac{t}{T}\right) \rightarrow \begin{cases} 1, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\text{rect}\left(\frac{t-T}{T}\right) \rightarrow \begin{cases} 1, & T - \frac{T}{2} \leq t \leq T + \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

1.2 Sinusoid Integration

$$\int \sin(at)\, dt = -\frac{\cos(at)}{a} + C$$

$$\int \cos(at)\, dt = \frac{\sin(at)}{a} + C$$

$$\int e^{at}\, dt = \frac{e^{at}}{a} + C$$

2 Signal Characteristics

2.1 (Average) Power of a Signal

$$P_x = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\, dt$$

For sinusoidal signals, the power can be simplified to:

$$P_x = \frac{A_c^2}{2}$$

2.2 Energy of a Signal

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

2.3   Representation of Narrowband (NB) Signals

$$g(t) = a(t) \cos(\underbrace{2\pi f_c t}_{A} + \underbrace{\phi(t)}_{B})$$

Using the identity $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\tilde{g}(t) = \underbrace{a(t)\cos(\phi(t))}_{\text{in-phase}} - \underbrace{a(t)\sin(\phi(t))}_{\text{quadrature}}$$

$$g_i(t) = a(t)\cos(\omega_c t + \phi(t))$$
$$g_q(t) = a(t)\sin(\omega_c t + \phi(t))$$

2.4   Complex Envelope

Given $g_i(t)$ and $g_q(t)$ and $a(t)$

$$g(t) = g(t) + j \cdot g(t)$$
$$a(t) = e^{j\phi(t)}$$

3   Amplitude Modulation (AM) Formulae

3.1   Conventional AM Signal

$$s(t) = A_c \left[1 + \mu \cos(2\pi f_m t)\right] \cos(2\pi f_c t)$$

3.2   AM Modulation Index

$$\mu = k_a \cdot A_m$$

Where:

- $0 \leq \mu \leq 1$

- $\mu = 1 \Rightarrow$ overmodulated

3.3   AM Power Efficiency

$$\eta = \frac{\text{Baseband Power}}{\text{Total Power}} = \frac{\mu^2}{2 + \mu^2}$$

3.4   AM Signal Modulation Index

$$\mu = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}}$$

Where:

- Maximum Amplitude: $A_{\max} = A_c(1 + \mu)$

- Minimum Amplitude: $A_{\max} = A_c(1 - \mu)$

4 Frequency Modulation (FM) Formulae

4.1 FM Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Where:

- Narrowband: $\beta < 1$

- Wideband: $\beta > 1$

4.2 Maximum Frequency Deviation

$$\Delta f = k_f \cdot A_m$$
$$= \beta \cdot f_m$$

Where:

- k_f: Frequency Sensitivity; typically the VCO Specification

- A_m: Message Amplitude; the message signal is usually injected into the VCO to drive the signal

- β: The FM Modulation Index

4.3 FM Complex Envelope

$$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$
$$= Re\left[A_c \cdot e^{2\pi f_c t j + j\beta \sin(2\pi f_m t)}\right]$$
$$= Re\left[A_c \cdot e^{2\pi f_c t j} \cdot e^{j\beta \cdot \sin(2\pi f_m t)}\right]$$
$$= Re\left[\tilde{s}(t) \cdot e^{j\beta \cdot \sin(2\pi f_c t)}\right]$$

Where:

- $\tilde{s}(t) = A_c \cdot e^{j\beta \cdot \sin(2\pi f_m t))}$: The complex envelope of the FM signal

4.4 Single Tone FM Signal

4.4.1 Time Domain Representation

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[2\pi(f_c + nf_m)t\right]$$

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{2\pi \cdot n f_m t \cdot j}$$

4.4.2 Frequency Domain Representation

$$s(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} \underbrace{J_n(\beta)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]}_{\text{Bessel Function of Delta Functions}}$$

4.5 Carson's Rule

4.5.1 Narrow Band Signals ($\beta < 1$)

$$B = 2f_m$$

4.5.2 Wideband Signals ($\beta > 1$)

$$B = 2\Delta f + 2f_m$$

4.5.3 For Arbitrary Signals

$$W = 2DW + 2W$$

$$W = 2DW(1 + \frac{1}{D})$$

$$= 2 \cdot \underbrace{\max_t |k_f m(t)|}_{k_f \text{ x message bw}} + 2f_m$$

Where:

- $D = \frac{\Delta f}{W}$: Frequency Deviation of the arbitrary signal

4.6 Arbitrary Signal Helpers for FM

$$f(t) = \frac{1}{2\pi} \cdot \frac{d\theta(t)}{d(t)} = f_c + k_f m(t)$$

Generic form of $s(t)$:
$$s(t) = A_c \cos(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\beta m(t)}_{\text{baseband}})$$

Base-Band Frequency of a Signal:

$$f(t) = f_c + k_f m(t)$$

Power Spectral Density (PSD)

$$G(f) = \lim_{T \to \infty} \frac{|X_T(f)|^2}{T}$$

4.7   PSD through an LTI System

$$G_y(f) = |H(f)|^2 \cdot G_x(f)$$

Where:

- $H(f)$: Fourier Transform of the LTI System's Transfer. By squaring it, you then get the Power Spectral Density

4.8   PSD of a Random Process

$$G_X(f) = \lim_{T \to \infty} \frac{\mathbb{E}\left\{|X_T(f, \varepsilon_i)|^2\right\}}{T}$$

NOTE: very similar for deterministic signals PSD
NOTE: The PSD is an ensemble average of the powers of the of captured samples

4.9   Power of a PSD

$$P = \int_{-\infty}^{\infty} G_X(f)\, df$$

4.10   PSD Of Narrowband Random Processes

The PSD of random processes (noise) are the same in both, the in-phase and quadrature components (i.e. $G_{x_s} == G_{x_c}$).

$$x(t) = x_c \cos(2\pi f_c t) - x_s \sin(2\pi f_c t)$$

Where:

- $x(t)$: Is a random process

- $x_c \cos(2\pi f_c t)$: In-Phase component of $x(t)$.

- $x_s \sin(2\pi f_c t)$: Quadratur component of $x(t)$.

In this case, the upconverted PSD of the signals are also equal within the bandwidth of the upconverted channel.

$$G_{x_s} = G_{x_c} = \begin{cases} G_x(f + f_c) + G_x(f - f_c) & \text{if } |f| < B \\ 0 & \text{otherwise} \end{cases}$$

5 Sampling and Quantization

5.1 Pulse Train Fourier Transform

The fourier transform of a periodid pulse train in the tiem domain leads to a series of impulses in the frequency domain:

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \xrightarrow{\mathcal{FT}} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Where:

- $f_0 = 1/T_0$: is the sampling frequency of the pulse train

$\therefore$ the Fourier Transform of a periodic pulse train results in a series of delta functions, representing the discrete frequency components present in the signal. The spectrm of $x(f)$ - the signal being sampled - is replicated around mulitples of f_0.

5.2 Quantizer Step Size

$$\Delta = \frac{2V}{N} = \frac{2V}{2^m}$$

Where:

- V: The amplitude of the signal

- N: Number of Quantization Steps

- m: Bit depth of quantization $(m = \log_2 N)$

6 Pulse Amplitude Modulation (PAM)

- T: Symbol period (duration of each symbol).

- $D = \frac{1}{T}$: Symbol rate or pulse rate.

- Note: A single symbol can represent multiple bits, depending on the modulation scheme.

7   PAM Signal Representation

The PAM signal for the m-th symbol, $s_m(t)$, is given by:

$$s_m(t) = A_m \cdot V(t)$$

where:

- A_m: Amplitude corresponding to the m-th symbol.

- $V(t)$: Fixed pulse shape.

- $m = 1, 2, \ldots, M$, where M is the number of symbols, typically based on the quantization levels.

7.1   Bit Duration and Bit Rate

- T_b: Bit duration, or time taken to transmit one bit.

- R_b: Bit rate, or number of bits transmitted per second $1/T_b$.

7.2   Relationship between Bit Rate, Symbol Rate, and Bits per Symbol

The relationship between bit duration, bit rate, symbol rate, and the number of bits per symbol is given by:

$$T_b = \frac{1}{R_b} = \frac{1}{kD} = \frac{T}{k} = \frac{T}{\log_2 M}$$

where:

- k: Number of bits per symbol.

- M: Number of distinct symbols in the signal set, with $M = 2^k$, so that each symbol represents $k = \log_2 M$ bits.

8   Spectra of Linearly Modulated Digital Signals

8.1   Signal Model

The signal $s(t)$ for a linearly modulated digital signal is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} A_n V(t + \Delta - nT)$$

where:

- A_n: Sequence of scalar symbols (data).

- $V(t)$: Pulse shape (basis function).

- T: Symbol period.

- Δ: Random channel delay, uniformly distributed over $[0, T]$.

8.2 Power Spectral Density (PSD)

The Power Spectral Density $G(f)$ of $s(t)$ is:

$$G(f) = \frac{|V(f)|^2}{T} \sum_{\ell=-\infty}^{\infty} R(\ell) e^{-j2\pi fT\ell}$$

where:

- $|V(f)|^2$: Magnitude squared of the Fourier Transform of $V(t)$.

- $R(\ell)$: Autocorrelation function of the sequence $\{A_n\}$ at lag ℓ.

- $e^{-j2\pi fT\ell}$: Frequency shift term.

8.3 Frequency Domain Summation of Delta Functions

The periodic nature of the spectrum due to sampling can be represented as:

$$\sum_{\ell=-\infty}^{\infty} e^{-j2\pi fT\ell} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

where:

- Each delta function at $f = \frac{k}{T}$ represents an impulse in the spectrum, showing periodic repetition at multiples of the symbol rate $\frac{1}{T}$.

9 Symbol Distance and Basis Functions

Basis functions play a critical role in transforming vector spaces into a form that facilitates simpler calculations, especially for distance metrics. By mapping an n-dimensional vector space into an n-dimensional Euclidean space, each axis becomes orthogonal and has unit length. This transformation provides a clear framework for calculating distances and simplifies operations on vectors.

9.1 Mathematical Representation

Let a vector be represented in terms of basis functions as follows:

$$\mathbf{v} = A\phi_1 + B\phi_2 + \ldots + Z\phi_m$$

where:

- ϕ_i represents the basis functions.

- $A, B, \ldots, Z$ are scalar multiples corresponding to each basis function ϕ_i.

10 Distance Calculation

In this representation, the Euclidean distance (or the magnitude of the vector $\mathbf{v}$) can be computed as:

$$d = \sqrt{(A\phi_1)^2 + (B\phi_2)^2 + \ldots + (Z\phi_m)^2}$$

This formula leverages the orthogonality of the basis functions, simplifying the computation of distances in the Euclidean space.

10.1 Benefits of Euclidean Transformation

Mapping a vector to its Euclidean representation using orthogonal basis functions offers several advantages:

- **Simplified Distance Computation**: In a Euclidean space, the distance between points is computed using the Pythagorean theorem, which is easier and more intuitive.

- **Ease of Representation**: Each vector component aligns with an orthogonal axis, allowing for straightforward interpretation and manipulation of vectors in terms of individual components.

- **Application in Signal Processing**: Basis functions are often used in signal processing to break down complex signals into simpler components, facilitating analysis and processing.

In summary, basis functions enable the transformation of complex spaces into manageable Euclidean forms, making calculations like distances more intuitive and easier to compute.

11 Matched Filter Overview

A matched filter is designed to maximize the signal-to-noise ratio (SNR) at the output for a known signal in the presence of additive white Gaussian noise (AWGN). Given a signal $s(t)$, the matched filter $h(t)$ is defined as:

$$h(t) = s(T - t)$$

where T is the duration of the signal. This filter aligns with the signal $s(t)$ and maximizes its correlation with the received signal $r(t) = s(t) + n(t)$, thereby improving detection performance in noisy conditions.

11.1 Deriving the Matched Filter

The process of deriving the matched filter involves the following steps:

1. **Maximizing SNR**: To maximize the output SNR, we choose a filter $h(t)$ that aligns with the signal $s(t)$.

2. **Impulse Response**: The impulse response $h(t)$ that achieves this SNR maximization is $h(t) = s(T - t)$, which is the time-reversed version of the signal.

3. **Convolution with Received Signal**: Convolve the received signal $r(t) = s(t) + n(t)$ with $h(t)$:

$$y(t) = r(t) * h(t) = \int_{-\infty}^{\infty} r(\tau)h(t - \tau) \, d\tau$$

4. **Decision Based on Output Peak**: Sample the output $y(t)$ at $t = T$. A high output value at $t = T$ indicates the presence of $s(t)$ in the received signal.

11.2   Differential Matched Filter for Binary Detection

A differential matched filter is commonly used for binary detection (i.e., $M = 2$), where there are only two possible signals, $s_1(t)$ and $s_2(t)$. Instead of constructing separate matched filters for each signal, a single differential matched filter can be used, defined as:

$$h(t) = s_1(T - t) - s_2(T - t)$$

The differential matched filter is derived based on the following principles:

1. **Difference Between Signals**: By taking the difference between $s_1(t)$ and $s_2(t)$, the filter is tuned to maximize the difference in response for the two symbols.

2. **Convolution and Decision Rule**: The received signal $r(t)$ (which is either $s_1(t) + n(t)$ or $s_2(t) + n(t)$) is convolved with $h(t)$:

$$y(t) = r(t) * h(t)$$

This convolution produces a positive output if $r(t) \approx s_1(t)$ and a negative output if $r(t) \approx s_2(t)$. The detection decision can be made by simply checking the sign of $y(T)$:

$$\text{Detected symbol} = \begin{cases} s_1, & \text{if } y(T) > 0 \\ s_2, & \text{if } y(T) < 0 \end{cases}$$

12   Matched Filter for Multiple Symbols (M-ary Detection)

For M-ary detection (where $M > 2$), the differential matched filter approach does not generalize effectively. Instead, a separate matched filter is constructed for each symbol $s_k(t)$ as follows:

$$h_k(t) = s_k(T - t)$$

To detect which symbol was transmitted:

1. **Convolution for Each Symbol**: Convolve the received signal $r(t)$ with each matched filter $h_k(t)$, resulting in outputs $y_k(T)$ for each filter.

2. **Decision Rule**: The transmitted symbol is determined by selecting the filter with the highest output:
$$\hat{k} = \arg \max_k y_k(T)$$

12.0.1 Why Differential Matched Filters Do Not Generalize for $M > 2$

For binary detection $(M = 2)$, a differential matched filter $h(t) = s_1(T-t) - s_2(T-t)$ works well because there are only two hypotheses, and the output sign can be used to distinguish between the two symbols. However, this approach does not generalize for $M > 2$ because:

- A single differential filter cannot uniquely represent multiple symbols.

- Each symbol requires a distinct matched filter for reliable identification in M-ary systems.

12.1 Optimality of Matched Filters

The matched filter is optimal in Gaussian noise as it maximizes the SNR at the output, achieving maximum likelihood detection. This makes matched filtering an effective tool for symbol detection in noisy environments.

13 Bit Error Rate

$$\text{BER} = Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

- d: The average distance between symbols.

- N_o: Noise Figure

13.1 Calulating BER or Matched Filters

To calcualte the bit error rate of matched filters, you will need to derive d^2 from scratch. This is where the following comes into play:

$$d^2 = \int_{-\infty}^{\infty} \underbrace{|s_1(t) - s_2(t)|^2}_{|\text{matched filter}|^2} \, dt$$

In this case, the piecewise function of the matched filter $h(t)$ can be substituted into the function.

   To calculate the BER, make sure to also have a target bit rate R_b usually stated in the question/problem. Remember that $T_b = 1/R_b$

14 Entropy

$$H(X) = -\sum p(x) \log_2 p(x)$$

15 Joint Entropy

$$H(X,Y) = -\sum \sum p(x,y) \log_2 p(x,y)$$