

ELEC4402 Formula Sheet

Henry Chen

November 4, 2024

Abstract

Key formulae and helpers for ELEC4402 - Communication Systems. Hopefully this helps to carry!!!

Contents

1	Handy Formulae	3
1.1	Rectangle Functions	3
1.2	Sinusoid Integration	3
2	Signal Characteristics	3
2.1	(Average) Power of a Signal	3
2.2	Energy of a Signal	3
2.3	Representation of Narrowband (NB) Signals	3
2.4	Complex Envelope	4
3	Amplitude Modulation (AM) Formulae	4
3.1	Conventional AM Signal	4
3.2	AM Modulation Index	4
3.3	AM Power Efficiency	4
3.4	AM Signal Modulation Index	4
4	Frequency Modulation (FM) Formulae	4
4.1	FM Modulation Index	4
4.2	Maximum Frequency Deviation	5
4.3	FM Complex Envelope	5
4.4	Single Tone FM Signal	5
4.4.1	Time Domain Representation	5
4.4.2	Frequency Domain Representation	5
4.5	Carson's Rule	5
4.5.1	Narrow Band Signals ($\beta < 1$)	5
4.5.2	Wideband Signals ($\beta > 1$)	5
4.5.3	For Arbitrary Signals	6
4.6	Arbitrary Signal Helpers for FM	6
5	Power Spectral Density (PSD)	6
5.1	PSD through an LTI System	6
5.2	PSD of a Random Process	6
5.3	Power of a PSD	6
5.4	PSD Of Narrowband Random Processes	7

6 Sampling and Quantization	7
6.1 Nyquist Rate	7
6.2 Pulse Train Fourier Transform	7
6.3 Quantizer Step Size	7
7 Pulse Amplitude Modulation (PAM)	8
8 PAM Signal Representation	8
8.1 Bit Duration and Bit Rate	8
8.2 Relationship between Bit Rate, Symbol Rate, and Bits per Symbol	8
9 Spectra of Linearly Modulated Digital Signals	8
9.1 Signal Model	8
9.2 Power Spectral Density (PSD)	9
9.3 Frequency Domain Summation of Delta Functions	9
10 Symbol Distance and Basis Functions	9
10.1 Mathematical Representation	9
11 Distance Calculation	9
11.1 Benefits of Euclidean Transformation	10
12 Matched Filter Overview	10
12.1 Deriving the Matched Filter	10
12.2 Differential Matched Filter for Binary Detection	10
12.3 Matched Filter for Multiple Symbols (M-ary Detection)	11
12.3.1 Why Differential Matched Filters Do Not Generalize for $M > 2$	11
12.4 Optimality of Matched Filters	11
13 Bit Error Rate	11
13.1 Calculating BER or Matched Filters	12
14 Information Theory	12
14.1 Entropy	12
14.2 Joint Entropy	12
14.3 Weakly Symmetric Channel Capacity	12
14.4 Code Rate R_c	12
14.4.1 Example Calculation	13
14.4.2 Trade-Off Between Rate and Reliability	13
15 Intersymbol Interference (ISI) Channels	13
15.1 Causes and Effects of ISI	13
15.2 Nyquist Criterion for ISI-Free Channels	14
15.2.1 Explanation of the Nyquist Criterion	14
15.2.2 Practical Interpretation	14
15.3 Raised Cosine Filtering	14
15.4 Calculating Symbol Rates	14
15.5 Calculating Bit Rate	15

1 Handy Formulae

1.1 Rectangle Functions

$$\begin{aligned}\text{rect}(t) &\rightarrow \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \\ \text{rect}\left(\frac{t}{T}\right) &\rightarrow \begin{cases} 1, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \\ \text{rect}\left(\frac{t-T}{T}\right) &\rightarrow \begin{cases} 1, & T - \frac{T}{2} \leq t \leq T + \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

1.2 Sinusoid Integration

$$\begin{aligned}\int \sin(at) dt &= -\frac{\cos(at)}{a} + C \\ \int \cos(at) dt &= \frac{\sin(at)}{a} + C \\ \int e^{at} dt &= \frac{e^{at}}{a} + C\end{aligned}$$

2 Signal Characteristics

2.1 (Average) Power of a Signal

$$P_x = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) dt$$

For sinusoidal signals, the power can be simplified to:

$$P_x = \frac{A_c^2}{2}$$

2.2 Energy of a Signal

$$E = \int_{-\infty}^{\infty} x^2(t) dt$$

2.3 Representation of Narrowband (NB) Signals

$$g(t) = a(t) \cos(\underbrace{2\pi f_c t}_A + \underbrace{\phi(t)}_B)$$

Using the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\begin{aligned}\tilde{g}(t) &= \underbrace{a(t) \cos(\phi(t))}_{\text{in-phase}} - \underbrace{a(t) \sin(\phi(t))}_{\text{quadrature}} \\ g_i(t) &= a(t) \cos(\omega_c t + \phi(t)) \\ g_q(t) &= a(t) \sin(\omega_c t + \phi(t))\end{aligned}$$

2.4 Complex Envelope

Given $g_i(t)$ and $g_q(t)$ and $a(t)$

$$\begin{aligned} g(t) &= g_i(t) + j \cdot g_q(t) \\ a(t) &= e^{j\phi(t)} \end{aligned}$$

3 Amplitude Modulation (AM) Formulae

3.1 Conventional AM Signal

$$s(t) = A_c [1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

3.2 AM Modulation Index

$$\mu = k_a \cdot A_m$$

Where:

- $0 \leq \mu \leq 1$
- $\mu = 1 \Rightarrow$ overmodulated

3.3 AM Power Efficiency

$$\eta = \frac{\text{Baseband Power}}{\text{Total Power}} = \frac{\mu^2}{2 + \mu^2}$$

3.4 AM Signal Modulation Index

$$\mu = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}}$$

Where:

- Maximum Amplitude: $A_{\max} = A_c(1 + \mu)$
- Minimum Amplitude: $A_{\min} = A_c(1 - \mu)$

4 Frequency Modulation (FM) Formulae

4.1 FM Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Where:

- Narrowband: $\beta < 1$
- Wideband: $\beta > 1$

4.2 Maximum Frequency Deviation

$$\begin{aligned}\Delta f &= k_f \cdot A_m \\ &= \beta \cdot f_m\end{aligned}$$

Where:

- k_f : Frequency Sensitivity; typically the VCO Specification
- A_m : Message Amplitude; the message signal is usually injected into the VCO to drive the signal
- β : The FM Modulation Index

4.3 FM Complex Envelope

$$\begin{aligned}s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= \text{Re} \left[A_c \cdot e^{2\pi f_c t j + j\beta \sin(2\pi f_m t)} \right] \\ &= \text{Re} \left[A_c \cdot e^{2\pi f_c t j} \cdot e^{j\beta \sin(2\pi f_m t)} \right] \\ &= \text{Re} \left[\tilde{s}(t) \cdot e^{j\beta \sin(2\pi f_m t)} \right]\end{aligned}$$

Where:

- $\tilde{s}(t) = A_c \cdot e^{j\beta \sin(2\pi f_m t)}$: The complex envelope of the FM signal

4.4 Single Tone FM Signal

4.4.1 Time Domain Representation

$$\begin{aligned}s(t) &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t] \\ \tilde{s}(t) &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{2\pi n f_m t j}\end{aligned}$$

4.4.2 Frequency Domain Representation

$$s(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} \underbrace{J_n(\beta) [\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)]}_{\text{Bessel Function of Delta Functions}}$$

4.5 Carson's Rule

4.5.1 Narrow Band Signals ($\beta < 1$)

$$B = 2f_m$$

4.5.2 Wideband Signals ($\beta > 1$)

$$B = 2\Delta f + 2f_m$$

4.5.3 For Arbitrary Signals

$$\begin{aligned}
 W &= 2DW + 2W \\
 W &= 2DW\left(1 + \frac{1}{D}\right) \\
 &= 2 \cdot \underbrace{\max_t |k_f m(t)|}_{k_f \times \text{message bw}} + 2f_m
 \end{aligned}$$

Where:

- $D = \frac{\Delta f}{W}$: Frequency Deviation of the arbitrary signal

4.6 Arbitrary Signal Helpers for FM

$$f(t) = \frac{1}{2\pi} \cdot \frac{d\theta(t)}{dt} = f_c + k_f m(t)$$

Generic form of $s(t)$:

$$s(t) = A_c \cos(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\beta m(t)}_{\text{baseband}})$$

Base-Band Frequency of a Signal:

$$f(t) = f_c + k_f m(t)$$

5 Power Spectral Density (PSD)

$$G(f) = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{T}$$

5.1 PSD through an LTI System

$$G_y(f) = |H(f)|^2 \cdot G_x(f)$$

Where:

- $H(f)$: Fourier Transform of the LTI System's Transfer. By squaring it, you then get the Power Spectral Density

5.2 PSD of a Random Process

$$G_X(f) = \lim_{T \rightarrow \infty} \frac{\mathbb{E}\{|X_T(f, \varepsilon_i)|^2\}}{T}$$

NOTE: very similar for deterministic signals PSD

NOTE: The PSD is an ensemble average of the powers of the of captured samples

5.3 Power of a PSD

$$P = \int_{-\infty}^{\infty} G_X(f) df$$

5.4 PSD Of Narrowband Random Processes

The PSD of random processes (noise) are the same in both, the in-phase and quadrature components (i.e. $G_{x_s} = G_{x_c}$).

$$x(t) = x_c \cos(2\pi f_c t) - x_s \sin(2\pi f_c t)$$

Where:

- $x(t)$: Is a random process
- $x_c \cos(2\pi f_c t)$: In-Phase component of $x(t)$.
- $x_s \sin(2\pi f_c t)$: Quadratur component of $x(t)$.

In this case, the upconverted PSD of the signals are also equal within the bandwidth of the upconverted channel.

$$G_{x_s} = G_{x_c} = \begin{cases} G_x(f + f_c) + G_x(f - f_c) & \text{if } |f| < B \\ 0 & \text{otherwise} \end{cases}$$

6 Sampling and Quantization

6.1 Nyquist Rate

The Nyquist Rate determines the minimum sampling rate required to accurately capture the information of a continuous-time signal without introducing aliasing.

$$f_s \geq 2 \times f_{sig}$$

Additionally, for signals that do have higher frequency noise components, a lowpass filter is applied to the signal to band limit it — preventing distortions to the sampled signal. The cutoff of the lowpass filter is the same as the signal's bandwidth f_s .

6.2 Pulse Train Fourier Transform

The Fourier transform of a periodic pulse train in the time domain leads to a series of impulses in the frequency domain:

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \xrightarrow{FT} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Where:

- $f_0 = 1/T_0$: is the sampling frequency of the pulse train

\therefore the Fourier Transform of a periodic pulse train results in a series of delta functions, representing the discrete frequency components present in the signal. The spectrum of $x(f)$ - the signal being sampled - is replicated around multiples of f_0 .

6.3 Quantizer Step Size

$$\Delta = \frac{2V}{N} = \frac{2V}{2^m}$$

Where:

- V : The amplitude of the signal
- N : Number of Quantization Steps
- m : Bit depth of quantization ($m = \log_2 N$)

7 Pulse Amplitude Modulation (PAM)

- T : Symbol period (duration of each symbol).
- $D = \frac{1}{T}$: Symbol rate or pulse rate.
- Note: A single symbol can represent multiple bits, depending on the modulation scheme.

8 PAM Signal Representation

The PAM signal for the m -th symbol, $s_m(t)$, is given by:

$$s_m(t) = A_m \cdot V(t)$$

where:

- A_m : Amplitude corresponding to the m -th symbol.
- $V(t)$: Fixed pulse shape.
- $m = 1, 2, \dots, M$, where M is the number of symbols, typically based on the quantization levels.

8.1 Bit Duration and Bit Rate

- T_b : Bit duration, or time taken to transmit one bit.
- R_b : Bit rate, or number of bits transmitted per second $1/T_b$.

8.2 Relationship between Bit Rate, Symbol Rate, and Bits per Symbol

The relationship between bit duration, bit rate, symbol rate, and the number of bits per symbol is given by:

$$T_b = \frac{1}{R_b} = \frac{1}{kD} = \frac{T}{k} = \frac{T}{\log_2 M}$$

where:

- k : Number of bits per symbol.
- M : Number of distinct symbols in the signal set, with $M = 2^k$, so that each symbol represents $k = \log_2 M$ bits.

9 Spectra of Linearly Modulated Digital Signals

9.1 Signal Model

The signal $s(t)$ for a linearly modulated digital signal is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} A_n V(t + \Delta - nT)$$

where:

- A_n : Sequence of scalar symbols (data).
- $V(t)$: Pulse shape (basis function).
- T : Symbol period.
- Δ : Random channel delay, uniformly distributed over $[0, T]$.

9.2 Power Spectral Density (PSD)

The Power Spectral Density $G(f)$ of $s(t)$ is:

$$G(f) = \frac{|V(f)|^2}{T} \sum_{\ell=-\infty}^{\infty} R(\ell) e^{-j2\pi f T \ell}$$

where:

- $|V(f)|^2$: Magnitude squared of the Fourier Transform of $V(t)$.
- $R(\ell)$: Autocorrelation function of the sequence $\{A_n\}$ at lag ℓ .
- $e^{-j2\pi f T \ell}$: Frequency shift term.

9.3 Frequency Domain Summation of Delta Functions

The periodic nature of the spectrum due to sampling can be represented as:

$$\sum_{\ell=-\infty}^{\infty} e^{-j2\pi f T \ell} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

where:

- Each delta function at $f = \frac{k}{T}$ represents an impulse in the spectrum, showing periodic repetition at multiples of the symbol rate $\frac{1}{T}$.

10 Symbol Distance and Basis Functions

Basis functions play a critical role in transforming vector spaces into a form that facilitates simpler calculations, especially for distance metrics. By mapping an n -dimensional vector space into an n -dimensional Euclidean space, each axis becomes orthogonal and has unit length. This transformation provides a clear framework for calculating distances and simplifies operations on vectors.

10.1 Mathematical Representation

Let a vector be represented in terms of basis functions as follows:

$$\mathbf{v} = A\phi_1 + B\phi_2 + \dots + Z\phi_m$$

where:

- ϕ_i represents the basis functions.
- A, B, \dots, Z are scalar multiples corresponding to each basis function ϕ_i .

11 Distance Calculation

In this representation, the Euclidean distance (or the magnitude of the vector \mathbf{v}) can be computed as:

$$d = \sqrt{(A\phi_1)^2 + (B\phi_2)^2 + \dots + (Z\phi_m)^2}$$

This formula leverages the orthogonality of the basis functions, simplifying the computation of distances in the Euclidean space.

11.1 Benefits of Euclidean Transformation

Mapping a vector to its Euclidean representation using orthogonal basis functions offers several advantages:

- **Simplified Distance Computation:** In a Euclidean space, the distance between points is computed using the Pythagorean theorem, which is easier and more intuitive.
- **Ease of Representation:** Each vector component aligns with an orthogonal axis, allowing for straightforward interpretation and manipulation of vectors in terms of individual components.
- **Application in Signal Processing:** Basis functions are often used in signal processing to break down complex signals into simpler components, facilitating analysis and processing.

In summary, basis functions enable the transformation of complex spaces into manageable Euclidean forms, making calculations like distances more intuitive and easier to compute.

12 Matched Filter Overview

A matched filter is designed to maximize the signal-to-noise ratio (SNR) at the output for a known signal in the presence of additive white Gaussian noise (AWGN). Given a signal $s(t)$, the matched filter $h(t)$ is defined as:

$$h(t) = s(T - t)$$

where T is the duration of the signal. This filter aligns with the signal $s(t)$ and maximizes its correlation with the received signal $r(t) = s(t) + n(t)$, thereby improving detection performance in noisy conditions.

12.1 Deriving the Matched Filter

The process of deriving the matched filter involves the following steps:

1. **Maximizing SNR:** To maximize the output SNR, we choose a filter $h(t)$ that aligns with the signal $s(t)$.
2. **Impulse Response:** The impulse response $h(t)$ that achieves this SNR maximization is $h(t) = s(T - t)$, which is the time-reversed version of the signal.
3. **Convolution with Received Signal:** Convolve the received signal $r(t) = s(t) + n(t)$ with $h(t)$:

$$y(t) = r(t) * h(t) = \int_{-\infty}^{\infty} r(\tau) h(t - \tau) d\tau$$

4. **Decision Based on Output Peak:** Sample the output $y(t)$ at $t = T$. A high output value at $t = T$ indicates the presence of $s(t)$ in the received signal.

12.2 Differential Matched Filter for Binary Detection

A differential matched filter is commonly used for binary detection (i.e., $M = 2$), where there are only two possible signals, $s_1(t)$ and $s_2(t)$. Instead of constructing separate matched filters for each signal, a single differential matched filter can be used, defined as:

$$h(t) = s_1(T - t) - s_2(T - t)$$

The differential matched filter is derived based on the following principles:

1. **Difference Between Signals:** By taking the difference between $s_1(t)$ and $s_2(t)$, the filter is tuned to maximize the difference in response for the two symbols.

2. **Convolution and Decision Rule:** The received signal $r(t)$ (which is either $s_1(t)+n(t)$ or $s_2(t)+n(t)$) is convolved with $h(t)$:

$$y(t) = r(t) * h(t)$$

This convolution produces a positive output if $r(t) \approx s_1(t)$ and a negative output if $r(t) \approx s_2(t)$. The detection decision can be made by simply checking the sign of $y(T)$:

$$\text{Detected symbol} = \begin{cases} s_1, & \text{if } y(T) > 0 \\ s_2, & \text{if } y(T) < 0 \end{cases}$$

12.3 Matched Filter for Multiple Symbols (M-ary Detection)

For M-ary detection (where $M > 2$), the differential matched filter approach does not generalize effectively. Instead, a separate matched filter is constructed for each symbol $s_k(t)$ as follows:

$$h_k(t) = s_k(T - t)$$

To detect which symbol was transmitted:

1. **Convolution for Each Symbol:** Convolve the received signal $r(t)$ with each matched filter $h_k(t)$, resulting in outputs $y_k(T)$ for each filter.
2. **Decision Rule:** The transmitted symbol is determined by selecting the filter with the highest output:

$$\hat{k} = \arg \max_k y_k(T)$$

12.3.1 Why Differential Matched Filters Do Not Generalize for $M > 2$

For binary detection ($M = 2$), a differential matched filter $h(t) = s_1(T - t) - s_2(T - t)$ works well because there are only two hypotheses, and the output sign can be used to distinguish between the two symbols. However, this approach does not generalize for $M > 2$ because:

- A single differential filter cannot uniquely represent multiple symbols.
- Each symbol requires a distinct matched filter for reliable identification in M-ary systems.

12.4 Optimality of Matched Filters

The matched filter is optimal in Gaussian noise as it maximizes the SNR at the output, achieving maximum likelihood detection. This makes matched filtering an effective tool for symbol detection in noisy environments.

13 Bit Error Rate

$$\text{BER} = Q \left(\sqrt{\frac{d^2}{2N_o}} \right)$$

- d : The average distance between symbols.
- N_o : Noise Figure

13.1 Calculating BER or Matched Filters

To calculate the bit error rate of matched filters, you will need to derive d^2 from scratch. This is where the following comes into play:

$$d^2 = \int_{-\infty}^{\infty} \underbrace{|s_1(t) - s_2(t)|^2}_{|\text{matched filter}|^2} dt$$

In this case, the piecewise function of the matched filter $h(t)$ can be substituted into the function.

To calculate the BER, make sure to also have a target bit rate R_b usually stated in the question/problem. Remember that $T_b = 1/R_b$

14 Information Theory

14.1 Entropy

$$H(p) = - \sum_{i=1}^N p_i \log_2(p_i)$$

14.2 Joint Entropy

$$H(X, Y) = - \sum \sum p(x, y) \log_2 p(x, y)$$

14.3 Weakly Symmetric Channel Capacity

The channel capacity C for a weakly symmetric channel can be defined as:

$$C = \log_2(N) - H(p)$$

where:

- N : The number of symbols that the channel can output.
- p : Probability vector representing the probability distribution of symbols, which is used to calculate the symbol entropy $H(p)$.

The term $H(p)$ denotes the entropy of the probability distribution p , and it can be expanded as follows:

$$H(p) = - \sum_{i=1}^N p_i \log_2(p_i)$$

where p_i represents the probability of each symbol i in the output set. The capacity C represents the maximum achievable rate of information transfer through the channel, measured in bits per symbol.

14.4 Code Rate R_c

The *code rate* R_c is a measure of the efficiency of an error-correcting code. It represents the ratio of useful information to the total transmitted data, including the redundant bits added for error correction. Mathematically, the code rate R_c is defined as:

$$R_c = \frac{k}{n} = \frac{\log_2 M}{n}$$

where:

- k : Number of information bits (useful data) per codeword.

- n : Total number of bits per codeword, which includes both information bits and redundant bits for error correction.
- m : Number of unique symbols in the frame.

The code rate R_c ranges between 0 and 1:

- When $R_c = 1$, all bits transmitted are information bits, with no redundancy, meaning no error correction is applied.
- When R_c is closer to 0, a larger proportion of bits are dedicated to error correction, increasing reliability but reducing the effective data rate.

14.4.1 Example Calculation

Consider a code with:

- $k = 4$: 4 information bits per codeword
- $n = 7$: 7 total bits per codeword (4 information bits + 3 redundant bits)

Then the code rate R_c is calculated as:

$$R_c = \frac{4}{7} \approx 0.57$$

This means that 57% of the transmitted bits are information, while the remaining 43% are used for error correction.

14.4.2 Trade-Off Between Rate and Reliability

The code rate R_c is chosen based on channel conditions. In noisy channels, a lower code rate (more redundancy) is preferred to enhance error resilience. In clear channels, a higher code rate is typically chosen to maximize data throughput.

15 Intersymbol Interference (ISI) Channels

Intersymbol Interference (ISI) occurs in communication channels when signal reflections, multipath propagation, or channel bandwidth limitations cause multiple instances of the same signal to arrive at different times. These delayed versions of the transmitted signal interfere with each other, causing distortion in the received signal. This interference results in a "smearing" or "smoothing" of the signal, where symbols are no longer distinguishable, leading to errors in symbol detection.

15.1 Causes and Effects of ISI

ISI can arise due to several physical and design-related factors, including:

- **Multipath Propagation:** In wireless communication, signals may take multiple paths to reach the receiver due to reflections off buildings, the ground, or other obstacles. Each path has a different delay, causing overlapping symbols at the receiver.
- **Limited Channel Bandwidth:** When the channel bandwidth is insufficient to pass the full frequency range of the transmitted signal, it leads to signal distortion. Sharp transitions in the signal are "rounded," resulting in spreading that overlaps with adjacent symbols.
- **Reflection in Wired Media:** In wired channels, such as coaxial cables or twisted-pair lines, reflections can occur due to impedance mismatches at connectors, which send portions of the signal back along the path, causing interference with subsequent symbols.

To mitigate ISI, the transmitted signal must be shaped or designed such that it does not interfere with adjacent symbols at the sampling instants. This can be achieved by designing the system so that the received pulse shape $p_r(t)$ has zero crossings at all sampling instants other than its own, i.e., $p_r[n] = 0$ for $n \neq 0$.

15.2 Nyquist Criterion for ISI-Free Channels

A channel is considered ISI-free if it satisfies the **Nyquist Criterion**. This criterion ensures that there is no interference between adjacent symbols at the sampling instants. Specifically, for a system with symbol period T , the Nyquist criterion states that the pulse shape $P(f)$ in the frequency domain must satisfy:

$$\sum_{n=-\infty}^{\infty} \underbrace{P_r \left(f + \frac{n}{T} \right)}_{\text{folded function}} = \text{constant}, \quad |f| \leq \frac{1}{2T}$$

where $P_r(f)$ is the frequency response of the received pulse $p_r(t)$.

15.2.1 Explanation of the Nyquist Criterion

The Nyquist criterion requires that the combined, “folded” frequency response of the channel and the pulse shape does not vary within the frequency band $|f| \leq \frac{1}{2T}$. This folding (summation over shifted frequency bands) is necessary because the bandwidth of a symbol is repeated periodically in the frequency domain due to sampling. By ensuring that this folded spectrum is constant, the Nyquist criterion guarantees that there is no ISI at the sampling instants.

15.2.2 Practical Interpretation

If the Nyquist criterion is satisfied, The symbols at the receiver will not interfere with each other, as each pulse $p_r(t)$ has zero crossings at the sampling instants of other symbols.

To achieve an ISI-free channel, practical pulse shapes, such as the **raised cosine pulse**, are often used because they satisfy the Nyquist criterion in practical systems.

15.3 Raised Cosine Filtering

One commonly used pulse that satisfies the Nyquist criterion is the **raised cosine pulse**. The raised cosine filter has a roll-off factor α that controls the excess bandwidth, making it a flexible choice for designing ISI-free systems. The frequency response of the raised cosine filter is given by:

$$P(f) = \begin{cases} T, & |f| \leq \frac{1-\alpha}{2T} \\ \frac{T}{2} \left[1 + \cos \left(\frac{\pi T}{\alpha} \left(|f| - \frac{1-\alpha}{2T} \right) \right) \right], & \frac{1-\alpha}{2T} < |f| \leq \frac{1+\alpha}{2T} \\ 0, & \text{otherwise} \end{cases}$$

where α is the roll-off factor, with $0 \leq \alpha \leq 1$. When $\alpha = 0$, the raised cosine filter becomes a sinc pulse, which perfectly satisfies the Nyquist criterion. As α increases, the filter allows more excess bandwidth, making it more robust to timing errors and practical imperfections.

15.4 Calculating Symbol Rates

$$D \leq \frac{2W}{1 + \alpha}$$

Where:

- D : Symbol Rate
- W : Channel Bandwidth
- α : RC Rolloff

15.5 Calculating Bit Rate

$$\begin{aligned}R_b &= \log_2 M \cdot D \\ &= k \cdot D\end{aligned}$$

Where:

- R_b : Bit rate
- D : Symbol Rate
- M : Number of Unique Symbols
- k : Number of Bits in a Symbol