ELEC4402 Formula Sheet

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Abstract

Key formulae and helpers for ELEC4402 - Communication Systems. Hopefully this helps to carry!!!

Contents

1 Handy Formulae

1.1 Rectangle Functions

$$\mathrm{rect}(t) \rightarrow \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{rect}\left(\frac{t}{T}\right) \rightarrow \begin{cases} 1, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{rect}\left(\frac{t-T}{T}\right) \rightarrow \begin{cases} 1, & T-\frac{T}{2} \leq t \leq T+\frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

1.2 Sinusoid Integration

$$\int \sin(at)\,dt = -\frac{\cos(at)}{a} + C$$

$$\int \cos(at)\,dt = \frac{\sin(at)}{a} + C$$

$$\int e^{at}\,dt = \frac{e^{at}}{a} + C$$

2 Signal Characteristics

2.1 (Average) Power of a Signal

$$P_x = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,dt$$

For sinusoidal signals, the power can be simplified to:

$$P_x = \frac{A_c^2}{2}$$

2.2 Energy of a Signal

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

2.3 Representation of Narrowband (NB) Signals

$$g(t) = a(t)\cos(\underbrace{2\pi f_c t}_{\text{A}} + \underbrace{\phi(t)}_{\text{B}})$$

Using the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\tilde{g}(t) = \underbrace{a(t)\cos(\phi(t))}_{\text{in-phase}} - \underbrace{a(t)\sin(\phi(t))}_{\text{quadrature}}$$

$$g_i(t) = a(t)\cos(\omega_c t + \phi(t))$$
$$g_q(t) = a(t)\sin(\omega_c t + \phi(t))$$

2.4 Complex Envelope

Given $g_i(t)$ and $g_q(t)$ and $a(t)$

$$g(t) = g(t) + j \cdot g(t)$$
$$a(t) = e^{j\phi(t)}$$

3 Amplitude Modulation (AM) Formulae

3.1 Conventional AM Signal

$$s(t) = A_c\left[1 + \mu\cos(2\pi f_m t)\right]\cos(2\pi f_c t)$$

3.2 AM Modulation Index

$$\mu = k_a \cdot A_m$$

Where:

- $0 \leq \mu \leq 1$
- $\mu = 1 \Rightarrow$ overmodulated

3.3 AM Power Efficiency

$$\eta = \frac{\text{Baseband Power}}{\text{Total Power}} = \frac{\mu^2}{2 + \mu^2}$$

3.4 AM Signal Modulation Index

$$\mu = \frac{A_{\text{max}} - A_{\text{min}}}{A_{\text{max}} + A_{\text{min}}}$$

Where:

- Maximum Amplitude: $A_{\text{max}} = A_c(1 + \mu)$
- Minimum Amplitude: $A_{\text{max}} = A_c(1 - \mu)$

4 Frequency Modulation (FM) Formulae

4.1 FM Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Where:

- Narrowband: $\beta < 1$
- Wideband: $\beta > 1$

4.2 Maximum Frequency Deviation

$$\begin{aligned} \Delta f &= k_f \cdot A_m \\ &= \beta \cdot f_m \end{aligned}$$

Where:

- k_f: Frequency Sensitivity; typically the VCO Specification
- A_m: Message Amplitude; the message signal is usually injected into the VCO to drive the signal
- β: The FM Modulation Index

4.3 FM Complex Envelope

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= Re\left[A_c \cdot e^{2\pi f_c tj + j\beta \sin(2\pi f_m t)}\right] \\ &= Re\left[A_c \cdot e^{2\pi f_c tj} \cdot e^{j\beta \cdot \sin(2\pi f_m t)}\right] \\ &= Re\left[\tilde{s}(t) \cdot e^{j\beta \cdot \sin(2\pi f_c t)}\right] \end{aligned}$$

Where:

- $\tilde{s}(t) = A_c \cdot e^{j\beta \cdot \sin(2\pi f_m t))}$: The complex envelope of the FM signal

4.4 Single Tone FM Signal

4.4.1 Time Domain Representation

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[2\pi(f_c + nf_m)t\right]$$
$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{2\pi \cdot nf_m t \cdot j}$$

4.4.2 Frequency Domain Representation

$$s(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} \underbrace{J_n(\beta)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]}_{\text{Bessel Function of Delta Functions}}$$

4.5 Carson's Rule

4.5.1 Narrow Band Signals ($\beta < 1$)

$$B = 2f_m$$

4.5.2 Wideband Signals ($\beta > 1$)

$$B = 2\Delta f + 2f_m$$

4.5.3 For Arbitrary Signals

$$\begin{aligned} W &= 2DW + 2W \\ W &= 2DW(1 + \frac{1}{D}) \\ &= 2 \cdot \underbrace{\max_t |k_f m(t)|}_{k_f \text{ x message bw}} + 2f_m \end{aligned}$$

Where:

- $D = \frac{\Delta f}{W}$: Frequency Deviation of the arbitrary signal

4.6 Arbitrary Signal Helpers for FM

$$f(t) = \frac{1}{2\pi} \cdot \frac{d\theta(t)}{d(t)} = f_c + k_f m(t)$$

Generic form of $s(t)$:

$$s(t) = A_c \cos(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\beta m(t)}_{\text{baseband}})$$

Base-Band Frequency of a Signal:

$$f(t) = f_c + k_f m(t)$$

Power Spectral Density (PSD)

$$G(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

4.7 PSD through an LTI System

$$G_y(f) = |H(f)|^2 \cdot G_x(f)$$

Where:

- $H(f)$: Fourier Transform of the LTI System's Transfer. By squaring it, you then get the Power Spectral Density

4.8 PSD of a Random Process

$$G_X(f) = \lim_{T\to\infty} \frac{\mathbb{E}\left\{|X_T(f, \varepsilon_i)|^2\right\}}{T}$$

NOTE: very similar for deterministic signals PSD
NOTE: The PSD is an ensemble average of the powers of the of captured samples

4.9 Power of a PSD

$$P = \int_{-\infty}^{\infty} G_X(f)\, df$$

4.10 PSD Of Narrowband Random Processes

The PSD of random processes (noise) are the same in both, the in-phase and quadrature components (i.e. $G_{x_s} == G_{x_c}$).

$$x(t) = x_c \cos(2\pi f_c t) - x_s \sin(2\pi f_c t)$$

Where:

- $x(t)$: Is a random process
- $x_c \cos(2\pi f_c t)$: In-Phase component of $x(t)$.

- $x_s \sin(2\pi f_c t)$: Quadratur component of $x(t)$.

In this case, the upconverted PSD of the signals are also equal within the bandwidth of the upconverted channel.

$$G_{x_s} = G_{x_c} = \begin{cases} G_x(f+f_c) + G_x(f-f_c) & \text{if } |f| < B \\ 0 & \text{otherwise} \end{cases}$$

5 Sampling and Quantization

5.1 Pulse Train Fourier Transform

The fourier transform of a periodid pulse train in the tiem domain leads to a series of impulses in the frequency domain:

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \xrightarrow{\mathcal{FT}} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Where:

- $f_0 = 1/T_0$: is the sampling frequency of the pulse train

$\therefore$ the Fourier Transform of a periodic pulse train results in a series of delta functions, representing the discrete frequency components present in the signal. The spectrm of $x(f)$ - the signal being sampled - is replicated around mulitples of f_0.

5.2 Quantizer Step Size

$$\Delta = \frac{2V}{N} = \frac{2V}{2^m}$$

Where:

- V: The amplitude of the signal
- N: Number of Quantization Steps
- m: Bit depth of quantization ($m = \log_2 N$)

6 Pulse Amplitude Modulation (PAM)

- T: Symbol period (duration of each symbol).
- $D = \frac{1}{T}$: Symbol rate or pulse rate.
- Note: A single symbol can represent multiple bits, depending on the modulation scheme.

7 PAM Signal Representation

The PAM signal for the m-th symbol, $s_m(t)$, is given by:

$$s_m(t) = A_m \cdot V(t)$$

where:

- A_m: Amplitude corresponding to the m-th symbol.
- $V(t)$: Fixed pulse shape.
- $m = 1, 2, \ldots, M$, where M is the number of symbols, typically based on the quantization levels.

7.1 Bit Duration and Bit Rate

- T_b: Bit duration, or time taken to transmit one bit.
- R_b: Bit rate, or number of bits transmitted per second $1/T_b$.

7.2 Relationship between Bit Rate, Symbol Rate, and Bits per Symbol

The relationship between bit duration, bit rate, symbol rate, and the number of bits per symbol is given by:

$$T_b = \frac{1}{R_b} = \frac{1}{kD} = \frac{T}{k} = \frac{T}{\log_2 M}$$

where:

- k: Number of bits per symbol.
- M: Number of distinct symbols in the signal set, with $M = 2^k$, so that each symbol represents $k = \log_2 M$ bits.

8 Spectra of Linearly Modulated Digital Signals

8.1 Signal Model

The signal $s(t)$ for a linearly modulated digital signal is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} A_n V(t + \Delta - nT)$$

where:

- A_n: Sequence of scalar symbols (data).

- $V(t)$: Pulse shape (basis function).
- T: Symbol period.
- Δ: Random channel delay, uniformly distributed over $[0, T]$.

8.2 Power Spectral Density (PSD)

The Power Spectral Density $G(f)$ of $s(t)$ is:

$$G(f) = \frac{|V(f)|^2}{T} \sum_{\ell=-\infty}^{\infty} R(\ell) e^{-j2\pi fT\ell}$$

where:

- $|V(f)|^2$: Magnitude squared of the Fourier Transform of $V(t)$.
- $R(\ell)$: Autocorrelation function of the sequence $\{A_n\}$ at lag ℓ.
- $e^{-j2\pi fT\ell}$: Frequency shift term.

8.3 Frequency Domain Summation of Delta Functions

The periodic nature of the spectrum due to sampling can be represented as:

$$\sum_{\ell=-\infty}^{\infty} e^{-j2\pi fT\ell} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

where:

- Each delta function at $f = \frac{k}{T}$ represents an impulse in the spectrum, showing periodic repetition at multiples of the symbol rate $\frac{1}{T}$.

9 Bit Error Rate

$$\text{BER} = Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

- d: The average distance between symbols.
- N_o: Noise Figure

9.1 Calulating BER or Matched Filters

To calcualte the bit error rate of matched filters, you will need to derive d^2 from scratch. This is where the following comes into play:

$$d^2 = \int_{-\infty}^{\infty} |s_1(t) - s_2(t)|^2 \, dt$$

10 Entropy

$$H(X) = -\sum p(x) \log_2 p(x)$$

11 Joint Entropy

$$H(X, Y) = -\sum \sum p(x, y) \log_2 p(x, y)$$