

# ELEC4402 Formula Sheet

Henry Chen

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## Abstract

Key formulae and helpers for ELEC4402 - Communication Systems. Hopefully this helps to carry!!!

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# 1 Handy Formulae

## 1.1 Rectangle Functions

$$\begin{aligned}\text{rect}(t) &\rightarrow \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \\ \text{rect}\left(\frac{t}{T}\right) &\rightarrow \begin{cases} 1, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \\ \text{rect}\left(\frac{t-T}{T}\right) &\rightarrow \begin{cases} 1, & T - \frac{T}{2} \leq t \leq T + \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

## 1.2 Sinusoid Integration

$$\begin{aligned}\int \sin(at) dt &= -\frac{\cos(at)}{a} + C \\ \int \cos(at) dt &= \frac{\sin(at)}{a} + C \\ \int e^{at} dt &= \frac{e^{at}}{a} + C\end{aligned}$$

# 2 Signal Characteristics

## 2.1 (Average) Power of a Signal

$$P_x = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) dt$$

For sinusoidal signals, the power can be simplified to:

$$P_x = \frac{A_c^2}{2}$$

## 2.2 Energy of a Signal

$$E = \int_{-\infty}^{\infty} x^2(t) dt$$

## 2.3 Representation of Narrowband (NB) Signals

$$g(t) = a(t) \cos(\underbrace{2\pi f_c t}_A + \underbrace{\phi(t)}_B)$$

Using the identity  $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\tilde{g}(t) = \underbrace{a(t) \cos(\phi(t))}_{\text{in-phase}} - \underbrace{a(t) \sin(\phi(t))}_{\text{quadrature}}$$

$$g_i(t) = a(t) \cos(\omega_c t + \phi(t))$$

$$g_q(t) = a(t) \sin(\omega_c t + \phi(t))$$

## 2.4 Complex Envelope

Given  $g_i(t)$  and  $g_q(t)$  and  $a(t)$

$$g(t) = g_i(t) + j \cdot g_q(t)$$

$$a(t) = e^{j\phi(t)}$$

# 3 Amplitude Modulation (AM) Formulae

## 3.1 Conventional AM Signal

$$s(t) = A_c [1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

## 3.2 AM Modulation Index

$$\mu = k_a \cdot A_m$$

Where:

- $0 \leq \mu \leq 1$
- $\mu = 1 \Rightarrow$  overmodulated

## 3.3 AM Power Efficiency

$$\eta = \frac{\text{Baseband Power}}{\text{Total Power}} = \frac{\mu^2}{2 + \mu^2}$$

## 3.4 AM Signal Modulation Index

$$\mu = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}}$$

Where:

- Maximum Amplitude:  $A_{\max} = A_c(1 + \mu)$
- Minimum Amplitude:  $A_{\min} = A_c(1 - \mu)$

## 4 Frequency Modulation (FM) Formulae

### 4.1 FM Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Where:

- Narrowband:  $\beta < 1$
- Wideband:  $\beta > 1$

### 4.2 Maximum Frequency Deviation

$$\begin{aligned}\Delta f &= k_f \cdot A_m \\ &= \beta \cdot f_m\end{aligned}$$

Where:

- $k_f$ : Frequency Sensitivity; typically the VCO Specification
- $A_m$ : Message Amplitude; the message signal is usually injected into the VCO to drive the signal
- $\beta$ : The FM Modulation Index

### 4.3 FM Complex Envelope

$$\begin{aligned}s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= \text{Re} [A_c \cdot e^{2\pi f_c t j + j\beta \sin(2\pi f_m t)}] \\ &= \text{Re} [A_c \cdot e^{2\pi f_c t j} \cdot e^{j\beta \cdot \sin(2\pi f_m t)}] \\ &= \text{Re} [\tilde{s}(t) \cdot e^{j\beta \cdot \sin(2\pi f_m t)}]\end{aligned}$$

Where:

- $\tilde{s}(t) = A_c \cdot e^{j\beta \cdot \sin(2\pi f_m t)}$ : The complex envelope of the FM signal

## 4.4 Single Tone FM Signal

### 4.4.1 Time Domain Representation

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos [2\pi(f_c + nf_m)t]$$

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{2\pi \cdot nf_m t \cdot j}$$

### 4.4.2 Frequency Domain Representation

$$s(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} \underbrace{J_n(\beta) [\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]}_{\text{Bessel Function of Delta Functions}}$$

## 4.5 Carson's Rule

### 4.5.1 Narrow Band Signals ( $\beta < 1$ )

$$B = 2f_m$$

### 4.5.2 Wideband Signals ( $\beta > 1$ )

$$B = 2\Delta f + 2f_m$$

### 4.5.3 For Arbitrary Signals

$$W = 2DW + 2W$$

$$W = 2DW \left(1 + \frac{1}{D}\right)$$

$$= 2 \cdot \underbrace{\max_t |k_f m(t)|}_{k_f \times \text{message bw}} + 2f_m$$

Where:

- $D = \frac{\Delta f}{W}$ : Frequency Deviation of the arbitrary signal

## 4.6 Arbitrary Signal Helpers for FM

$$f(t) = \frac{1}{2\pi} \cdot \frac{d\theta(t)}{dt} = f_c + k_f m(t)$$

Generic form of  $s(t)$ :

$$s(t) = A_c \cos(\underbrace{2\pi f_c t}_{\text{carrier}} + \underbrace{\beta m(t)}_{\text{baseband}})$$

Base-Band Frequency of a Signal:

$$f(t) = f_c + k_f m(t)$$

## Power Spectral Density (PSD)

$$G(f) = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{T}$$

### 4.7 PSD through an LTI System

$$G_y(f) = |H(f)|^2 \cdot G_x(f)$$

Where:

- $H(f)$ : Fourier Transform of the LTI System's Transfer. By squaring it, you then get the Power Spectral Density

### 4.8 PSD of a Random Process

$$G_X(f) = \lim_{T \rightarrow \infty} \frac{\mathbb{E}\{|X_T(f, \varepsilon_i)|^2\}}{T}$$

**NOTE:** very similar for deterministic signals PSD

**NOTE:** The PSD is an ensemble average of the powers of the of captured samples

### 4.9 Power of a PSD

$$P = \int_{-\infty}^{\infty} G_X(f) df$$

### 4.10 PSD Of Narrowband Random Processes

The PSD of random processes (noise) are the same in both, the in-phase and quadrature components (i.e.  $G_{x_s} = G_{x_c}$ ).

$$x(t) = x_c \cos(2\pi f_c t) - x_s \sin(2\pi f_c t)$$

Where:

- $x(t)$ : Is a random process
- $x_c \cos(2\pi f_c t)$ : In-Phase component of  $x(t)$ .

- $x_s \sin(2\pi f_c t)$ : Quadratur component of  $x(t)$ .

In this case, the upconverted PSD of the signals are also equal within the bandwidth of the upconverted channel.

$$G_{x_s} = G_{x_c} = \begin{cases} G_x(f + f_c) + G_x(f - f_c) & \text{if } |f| < B \\ 0 & \text{otherwise} \end{cases}$$

## 5 Sampling and Quantization

### 5.1 Pulse Train Fourier Transform

The fourier transform of a periodid pulse train in the tiem domain leads to a series of impulses in the frequency domain:

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \xrightarrow{\mathcal{FT}} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Where:

- $f_0 = 1/T_0$ : is the sampling frequency of the pulse train

$\therefore$  the Fourier Transform of a periodic pulse train results in a series of delta functions, representing the discrete frequency components present in the signal. The spectrm of  $x(f)$  - the signal being sampled - is replicated around mulitples of  $f_0$ .

### 5.2 Quantizer Step Size

$$\Delta = \frac{2V}{N} = \frac{2V}{2^m}$$

Where:

- $V$ : The amplitude of the signal
- $N$ : Number of Quantization Steps
- $m$ : Bit depth of quantization ( $m = \log_2 N$ )

## 6 Pulse Amplitude Modulation (PAM)

- $T$ : Symbol period (duration of each symbol).
- $D = \frac{1}{T}$ : Symbol rate or pulse rate.
- Note: A single symbol can represent multiple bits, depending on the modulation scheme.



## 7 PAM Signal Representation

The PAM signal for the  $m$ -th symbol,  $s_m(t)$ , is given by:

$$s_m(t) = A_m \cdot V(t)$$

where:

- $A_m$ : Amplitude corresponding to the  $m$ -th symbol.
- $V(t)$ : Fixed pulse shape.
- $m = 1, 2, \dots, M$ , where  $M$  is the number of symbols, typically based on the quantization levels.

### 7.1 Bit Duration and Bit Rate

- $T_b$ : Bit duration, or time taken to transmit one bit.
- $R_b$ : Bit rate, or number of bits transmitted per second  $1/T_b$ .

### 7.2 Relationship between Bit Rate, Symbol Rate, and Bits per Symbol

The relationship between bit duration, bit rate, symbol rate, and the number of bits per symbol is given by:

$$T_b = \frac{1}{R_b} = \frac{1}{kD} = \frac{T}{k} = \frac{T}{\log_2 M}$$

where:

- $k$ : Number of bits per symbol.
- $M$ : Number of distinct symbols in the signal set, with  $M = 2^k$ , so that each symbol represents  $k = \log_2 M$  bits.

## 8 Spectra of Linearly Modulated Digital Signals

### 8.1 Signal Model

The signal  $s(t)$  for a linearly modulated digital signal is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} A_n V(t + \Delta - nT)$$

where:

- $A_n$ : Sequence of scalar symbols (data).

- $V(t)$ : Pulse shape (basis function).
- $T$ : Symbol period.
- $\Delta$ : Random channel delay, uniformly distributed over  $[0, T]$ .

## 8.2 Power Spectral Density (PSD)

The Power Spectral Density  $G(f)$  of  $s(t)$  is:

$$G(f) = \frac{|V(f)|^2}{T} \sum_{\ell=-\infty}^{\infty} R(\ell) e^{-j2\pi f T \ell}$$

where:

- $|V(f)|^2$ : Magnitude squared of the Fourier Transform of  $V(t)$ .
- $R(\ell)$ : Autocorrelation function of the sequence  $\{A_n\}$  at lag  $\ell$ .
- $e^{-j2\pi f T \ell}$ : Frequency shift term.

## 8.3 Frequency Domain Summation of Delta Functions

The periodic nature of the spectrum due to sampling can be represented as:

$$\sum_{\ell=-\infty}^{\infty} e^{-j2\pi f T \ell} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

where:

- Each delta function at  $f = \frac{k}{T}$  represents an impulse in the spectrum, showing periodic repetition at multiples of the symbol rate  $\frac{1}{T}$ .

# 9 Bit Error Rate

$$\text{BER} = Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

- $d$ : The average distance between symbols.
- $N_o$ : Noise Figure

## 9.1 Calculating BER or Matched Filters

To calculate the bit error rate of matched filters, you will need to derive  $d^2$  from scratch. This is where the following comes into play:

$$d^2 = \int_{-\infty}^{\infty} \underbrace{|s_1(t) - s_2(t)|^2}_{|\text{matched filter}|^2} dt$$

In this case, the piecewise function of the matched filter  $h(t)$  can be substituted into the function.

To calculate the BER, make sure to also have a target bit rate  $R_b$  usually stated in the question/problem. Remember that  $T_b = 1/R_b$

## 10 Entropy

$$H(X) = - \sum p(x) \log_2 p(x)$$

## 11 Joint Entropy

$$H(X, Y) = - \sum \sum p(x, y) \log_2 p(x, y)$$