ARTIN PRESENTATIONS AND CLOSED 4-MANIFOLDS

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1. Introduction

The classification of manifolds has been a fundamental yet complicated problem in topology. This problem is especially difficult for smooth 4-manifolds because of the limitation of both geometric tools and algebraic tools in dimension 4. Geometrically, 4-manifolds neither are known to admit geometry like lower dimensional manifolds, nor can their topological structures be classified by surgery theory, which requires manifolds to be of dimension 5 and higher.

Another algebraic difficulty is due to the undecidability of the isomorphism problem, which along with the word problem and the conjugacy problem, is one of the three fundamental decision problems first posed by Max Dehn in 1911. The isomorphism problem decides if two group presentations represent the same group. This is an essential problem in the abstract classification of high-dimensional (4 and above) manifolds. The intuition behind the existence of such undecidable problems is that the number of languages (math problems) is uncountably infinite as it's the power set of all Turing Machines, which have a countably infinite number. Unfortunately the isomorphism problem is one of the problems that does not have an effective Turing Machine with which to match. The study of the classification of 4-manifolds is thus restricted by these problems. However, certain subsets of these problems are computable, which gives motivation to the study of special 4-manifolds if connections with these subsets exist.

In 1912 Dehn solved the word problem for closed orientable surface groups. In 1926 Emil Artin solved the word problem for braid groups. More recently, the word and isomorphism problems for closed orientable 3-manifold groups were solved building on spectacular work of Thurston, Perelman, and others. Every finitely presentable group appears as the fundamental group of a smooth, closed 4-manifold. Using this, Markov [Mar58] showed that there is no algorithm to decide whether two arbitrary smooth closed orientable 4-manifolds are diffeomorphic. This problem persists in dimensions greater than four as well. With this in mind, it is common to study simply-connected 4-manifolds as their topology is already very rich.

In this paper we focus on the study of smooth closed simply-connected 4-manifolds. In particular, we study such manifolds arising from Artin presentations. Artin presentations are intimately related to the pure braid group, and they characterize the fundamental groups of closed orientable 3-manifolds. Each Artin presentation r gives rise to a 4-manifold $W^4(r)$, whose boundary is a closed orientable 3-manifold

$M^3(r)$ with fundamental group presented by r. In case the 4-manifold has boundary S^3, we may close up $W^4(r)$ by attaching a 4-handle to obtain $W^4(r) \cup_{S^3} D^4$. This closed 4-manifold has quadratic form represented by the exponent sum matrix of r.

Artin presentations arising from 2-strand pure braids are particularly tractable. Further, we show that they are naturally related to von Dyck (triangle) groups. These two facts play crucial roles in our determination of all closed 4-manifolds arising from Artin presentations on two generators.

Section 2 introduces the open book decomposition, which is an important construction of closed orientable 3-manifolds that leads to their correspondence with Artin presentations. Section 3 explains how Artin presentations give rise to 4-manifolds, specifically how the 3-manifold boundary of a 4-manifold can be studied with an Artin presentation group as their fundamental group. Then in section 4 we find all possible closed, smooth, simply-connected 4-manifolds from Artin presentations on two generators.

2. Open Book Decompositions

Open book decompositions are special decompositions of manifolds. They provide a nice way to study a manifold by breaking it down to a lower dimensional manifold, although with a compromise of a complicated diffeomorphism in the construction compared to decompositions that maintain the manifold dimension, such as a triangulation.

Definition 2.1. An *open book* decomposition of a smooth closed n-manifold M^n is constructed as follows. Let V be a smooth compact $(n-1)$-manifold with $\partial V \neq \emptyset$. Let $h : V \to V$ be a diffeomorphism with $h|\partial V = \mathrm{id}_{\partial V}$ (pointwise). The mapping torus V_h of h is the space $V \times [0, 1]$ with $(x, 1)$ identified to $(h(x), 0)$. The boundary of V_h is $\partial V \times S^1$, since h is the identity on ∂V. Now there are two equivalent ways to construct M^n: glue $\partial V \times D^2$ to V_h by the identity on their boundaries (which are both $\partial V \times S^1$), or in V_h crush each $p \times S^1$, $p \in \partial V$, to a point. The two resulting smooth closed manifolds M^n are diffeomorphic. Each of M^n is called an open book. The map h is called the monodromy.

For some variations on the definition of an open book, see [Etn06].

In high dimensions, many but not all manifolds admit open book decompositions. The interested readers should see the appendix to Ranicki's book by Winkelnkemper [Ran98]. In this paper, we focus on 3-dimensional open books, where we have the following result.

Theorem 2.2 (Alexander, 1923)**.** *Every closed orientable 3-manifold M^3 has an open book decomposition.*

Proof. The manifold M^3 has a triangulation by Moise [Moi77, p. 252]. This triangulation gives a Heegaard decomposition $M^3 = H_g \cup_\phi H'_g$ of M^3 as in Rolfsen [Rol90, p. 240]. Recall that the solid handlebody H_g is a regular neighborhood of the 1-skeleton in the triangulation of M^3, H'_g is the closure in M^3 of the complement $M^3 - H_g$, and $\phi : \Sigma_g \to \Sigma'_g$ is a homeomorphism of their boundaries.

By Lickorish [Lic62, Lic64], ϕ is isotopic to a finite product of Dehn twists about $3g - 1$ specific curves on Σ_g. To construct M^3, Lickorish [Lic62] showed that each

Dehn twist may be replaced by ± 1 Dehn surgery along a parallel curve in the interior of H_g. Lickorish [Lic65] further showed that the union of the Dehn surgery curves form the closure of a pure braid in the 3-sphere S^3 (see also Rolfsen [Rol90, p. 279]). This yields an open book decomposition of M^3 with a planar page. $\square$

3. 4-MANIFOLDS FROM ARTIN PRESENTATIONS

3.1. **Group presentations.** Group presentations were first studied systematically by von Dyck in the 1880s. A *presentation* of a group has the structure: $\langle x_1, x_2, \ldots | r_1, r_2, \ldots \rangle$ where $x_1, x_2, \ldots$ are the generators, and $r_1, r_2, \ldots$ are the relators that are words in the generators, where $r_i = 1$ for all i. Throughout this paper, let F_n denote the free group on n generators $x_1, x_2, \ldots, x_n$. A *finite presentation* is a presentation with a finite number of generators and relators. Every group has a presentation, in fact, infinitely many presentations. And a presentation with n generators presents the quotient group $\pi(r) = F_n/N$, where N is the smallest normal subgroup of F_n containing the words r_i for all i (see Stillwell [Stil93, Ch. 0.5]). A finite presentation is *balanced* provided the number of generators equals to the number of relators.

Given a finite presentation $r = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$, the *exponent sum matrix* of r is defined to be the $n \times m$ integer matrix $A(r)$ where $A(r)_{ij}$ equals the exponent sum of x_i in r_j. This matrix yields a linear map $\mathbb{Z}^m \to \mathbb{Z}^n$. The cokernel of this map is isomorphic to the abelianization of $\pi(r)$.

3.2. **Artin presentations.** Artin presentations, as a restricted form of group presentations, were first systematically studied by Winkelnkemper [Win02]. They characterize the fundamental groups of closed orientable 3-manifolds. Winkelnkemper discovered that the exponent sum matrix of each Artin presentation is symmetric. As any matrix that represents the quadratic form of a closed simply-connected 4-manifold is symmetric, Winkelnkemper was led to the discovery of a 4-manifold for each Artin presentation. We will explain these constructions below.

Definition 3.1. An *Artin presentation* r is a finite, balanced presentation:
$$\langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle$$
such that the following equation is satisfied in F_n:
$$x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n).$$
We use $\mathcal{R}_n$ to denote the set of Artin presentations with n generators (and thus n relators).

Example 3.2. Artin presentations arise naturally as we will explain below. First, we give some examples of Artin presentations (by convention, $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ in the examples below; in particular, $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_1 = \{0\}$):

- $r \in \mathcal{R}_0 \Leftrightarrow r = \langle | \rangle$, where $\pi(r) = 1$ is the trivial group. This is the only $r \in \mathcal{R}_0$.
- $r \in \mathcal{R}_1 \Leftrightarrow r = \langle x_1 | x_1^n \rangle, n \in \mathbb{Z}$, where $\pi(r) \cong \mathbb{Z}_n$. These are all the $r \in \mathcal{R}_1$.
- $r \in \mathcal{R}_2 \Leftrightarrow r = \langle x_1, x_2 | x_1^a (x_1 x_2)^c, x_2^b (x_1 x_2)^c \rangle$, where $a, b, c \in \mathbb{Z}$. These are all the $r \in \mathcal{R}_2$. Note that each can also be written in the form
$$r = \langle x_1, x_2 | x_1^{a-c}(x_1 x_2)^c, x_2^{b-c}(x_1 x_2)^c \rangle, \text{ where } a, b, c \in \mathbb{Z}.$$
  For this latter form, $A(r) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$.
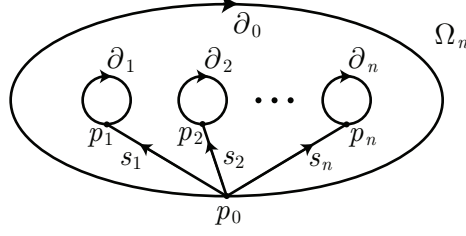
FIGURE 3.1. The compact 2-disk with n holes Ω_n.

- $r \in \mathcal{R}_2 : r = \langle x_1, x_2 | x_1^3(x_1x_2)^{-2}, x_2^5(x_1x_2)^{-2}\rangle$, where $\pi(r) \cong I(120)$, the binary icosahedral group.
- $r \in \mathcal{R}_n : r = \langle x_1, x_2, \ldots, x_n | x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}\rangle$ where $\pi(r) \cong \mathbb{Z}_{a_1} * \mathbb{Z}_{a_2} * \cdots * \mathbb{Z}_{a_n}$. These are the trivial examples in $\mathcal{R}_n$.
- $r \in \mathcal{R}_n : r = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle, r_1 = r_2 = \cdots = r_n = (x_1 x_2 \cdots x_n)^k$ for any fixed k. Note that when $k = 0$, the resulting presentation with all the n relators $= 1$ is always an Artin presentation of the free group F_n.

3.3. **Artin presentations from topology.** Artin presentations arise naturally in the following manner (see also [Cal04]). Let Ω_n denote the compact 2-disk with n holes, as in figure. Let $\partial_0, \partial_1, \ldots, \partial_n$ denote the oriented boundary components of Ω_n. Let s_i denote the oriented segment from the base point $p_0 \in \partial_0$ to the base point $p_i \in \partial_i$. Note that $\pi_1(\Omega_n, p_0) \cong F_n$, with each generator x_i defined as the loop $s_i \partial_i s_i^{-1}$.

Fix a homeomorphism $h : \Omega_n \to \Omega_n$ with $h|_{\partial\Omega_n} = \mathrm{id}_{\partial\Omega_n}$ (pointwise). Define each r_i as the loop $s_i h(s_i^{-1})$. Notice that $r_i^{-1} = h(s_i)s_i^{-1}$. We claim that

$$r = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle$$

is an Artin presentation.

To prove this claim, first notice that the induced map $H_\# : \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0)$ satisfies $h_\#(x_i) = r_i^{-1} x_i r_i$ [Cal04, p.12]. Thus $x_1 x_2 \cdots x_n = h_\#(x_1 x_2 \cdots x_n)$ (since h is the identity on boundary) and this equals to $r_1^{-1} x_1 r_1 r_2^{-1} x_2 r_2 \cdots r_n^{-1} x_n r_n$, as desired.

Thus a homeomorphism $h : \Omega_n \to \Omega_n$ with $h|_{\partial\Omega_n} = \mathrm{id}_{\partial\Omega_n}$ (pointwise) determines an Artin presentation $r \in \mathcal{R}_n$.

More interestingly, every $r \in \mathcal{R}_n$ also determines such a homeomorphism h (unique up to isotopy rel$\partial\Omega_n$). The idea of a proof is sketched below.

Given $r \in \mathcal{R}_n$, where $r = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle$ with $x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)$, we can define a group homomorphism $\psi : F_n \to F_n$ by $\psi(x_i) = r_i^{-1} x_i r_i$ for each i. By [Art65] [Bir74], ψ is a braid group automorphism, thus $\psi = h_\# : \pi_1(\Omega_n) \to \pi_1(\Omega_n)$ for some homeomorphism $h|_{\partial\Omega_n} = \mathrm{id}_{\partial\Omega_n}$, where h is unique up to isotopy rel $\partial\Omega_n$. This construction also gives a group isomorphism $\mathcal{R}_n \cong P_n \times \mathbb{Z}^n$, where P_n is the n-strand pure braid group and $\mathbb{Z}^n$ represents the twist on each strand. We can and will take $h = h(r)$ to be a diffeomorphism as explained in [Cal04].

A 1-1 correspondence between $r \in \mathcal{R}_n$ and a homeomorphism $h : \Omega_n \to \Omega_n :$ $h|_{\partial\Omega_n} = \mathrm{id}_{\partial\Omega_n}$ unique up to isotopy is thus determined.

3.4. **Artin presentations and 3-manifolds.** An Artin presentation determines a closed orientable 3-manifold $M^3(r)$ by applying the open book construction to the diffeomorphism $h(r)$. By **Theorem 2.2**, every closed oriented 3-manifold arises in this manner.

Example 3.3. Some 3-manifolds corresponding to the Artin presentations from *Example 3.2* are described as follows:

- $r = \langle | \rangle \in \mathcal{R}_0 \Rightarrow M^3(r) = S^3$.
- $r = \langle x_1 | x_1^n \rangle \in \mathcal{R}_1$ for $n \in \mathbb{Z} \Rightarrow M^3(r)$ is the Lens space $L(n, 1)$.
- $r = \langle x_1, x_2 | x_1^3 (x_1 x_2)^{-2}, x_2^5 (x_1 x_2)^{-2} \rangle \in \mathcal{R}_2 \Rightarrow M^3(r)$ is the Poincaré homology 3-sphere.
- $r = \langle x_1, x_2, \ldots, x_n | x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n} \rangle \in \mathcal{R}_n \Rightarrow M^3(r)$ is a connnected sum of Lens spaces.

Proposition 3.4. $\pi_1(M^3(r))$ *is isomorphic to the group $\pi(r)$ presented by r.*

Proof. First consider the fundamental group of the mapping torus, which was constructed by an HNN construction using van Kampen's Theorem [Win02]:

$$\pi_1(\Omega_n(h),)p_0) \cong$$

$$\langle x_1, x_2, \ldots, x_n, \beta | \beta x_1 \beta^{-1} = h_\#(x_1), \beta x_2 \beta^{-1} = h_\#(x_2), \ldots, \beta x_n \beta^{-1} = h_\#(x_n) \rangle.$$

Now use van Kampen's theorem again for the glued in $\partial \Omega_n \times D$, Winkelnkemper showed that the resulting $M^3(r)$ has fundamental group isomorphic to exactly $\pi(r)$. $\qquad \square$

As will be discussed in section 4.2, the 3-manifold $M^3(r)$ is an integral homology 3-sphere if and only if $\det(A(r)) = \pm 1$ [Win02].

3.5. **Artin presentations and 4-manifolds.**

Proposition 3.5. *Every Artin presentations gives rise to a 4-manifolds.*

Recall that r determines $M^3(r)$ by giving a diffeomorphism that decides Dehn surgery curves from $\hat{\beta}$, the closure of a pure braid, which is also a link in S^3 with unknots. Now construct a 4-manifold $W^4(r)$ by attaching 2-handles on a 4-ball along the neighbourhood of these unknots, i.e. attaching n copies of $D^2 \times D^2$ along $S^1 \times D^2$ on D^4 according to $\hat{\beta}$. Note that $\partial D^4 = S^3$ and $\partial(\bigcup D^2 \times D^2) = \bigcup S^1 \times D^2$, thus $\partial W^4(r) = M^3(r)$. An equivalent way to construct $W^4(r)$ is by a sort of relative open book on D^3 with a diffeomorphism H extended from the homeomorphism h [Win02, p.250]. The above construction leads to the following theorem which was originally proved by Rohlin [Roh51].

Proposition 3.6. *Every closed, orientable 3-manifold is the boundary of a 4-manifold.*

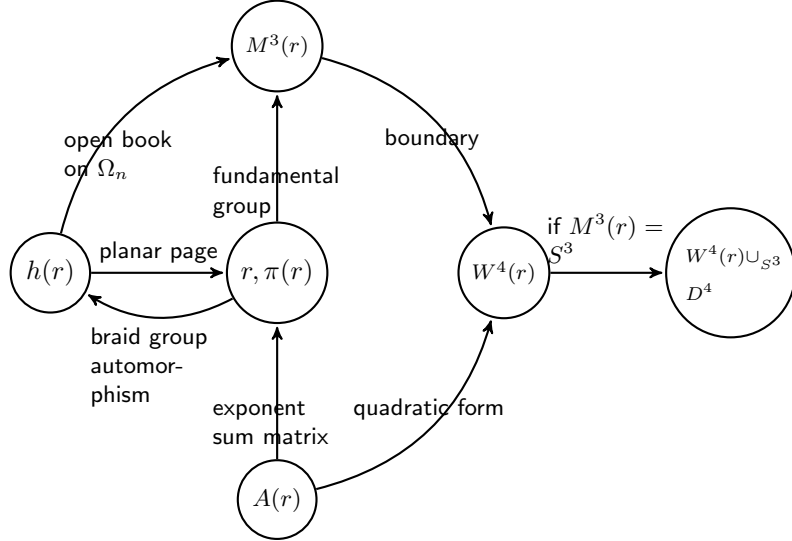Thus an Artin presentation r also gives rise to a 4-manifold, which can be studied using the properties of r.

Suppose a 4-manifold has the property $\partial W^4(r) = M^3(r) = S^3$, then by Cerf's Theorem[Cer68], there is an essentially unique diffeomorphism ψ to glue on a D^4 along its boundary S^3, where ψ is isotopic to either the identity or the reflection on S^3. The resulting 4-manifold $W^4(r) \cup_{\partial = S^3} D^4$ is thus a closed smooth simply-connected 4-manifold, with nicer properties that resemble those of $W^4(r)$. Whether

all closed, smooth, simply-connected 4-manifold can be constructed this way is an open question as neither a counter example nor a proof has been discovered so far.

While the question remains open, a tip of the iceberg can be revealed if restricted to the closed up $W^4(r)$ obtained by Artin presentations $\mathcal{R}_n$. The cases for $\mathcal{R}_0$ and $\mathcal{R}_1$ are trivial:

- In $\mathcal{R}_0$, where $r = \langle | \rangle$ is the only presentation, S^4 is obtained by closing up $W^4(r) = D^4$.
- In $\mathcal{R}_1$, where $r = \langle x_1 | x_1^k \rangle \in \mathcal{R}_1$ for $k \in \mathbb{Z}$. Only $k = \pm 1$ give simply-connected boundary. When $k = +1$ we get $\mathbb{CP}^2$ (upon closing with a 4-handle), and when $k = -1$ we get $\overline{\mathbb{CP}^2}$ (upon closing with a 4-handle).

When $n \geq 2$ the problem is much more complicated. We studied and found all such closed 4-manifolds in $\mathcal{R}_2$ using von Dyck triangle groups, which was inspired by the resemblance between the abelianization groups by $\mathcal{R}_2$ and the triangle groups. Details will be discussed in the next section.

3.6. Summary.

To sum up, an Artin presentation r gives rise to the following:

- $\pi(r)$: The group presented by r.
- $h(r)$: A diffeomorphism such that $h|_{\partial \Omega_n} = \mathrm{id}_{\partial \Omega_n}$. Unique up to isotopy.
- $M^3(r)$: An open book constructed by Ω_n and h, with $\pi(r) \cong \pi_1(M^3(r))$.
- $W^4(r)$: A 4-manifold bounded by $M^3(r)$.
- $A(r)$: An $n \times n$ symmetric integer matrix that is the exponent sum matrix of r. This gives $H_1(M^3(r))$.
- $W^4(r) \cup_{S^3} D^4$: A closed 4-manifold whenever $M^3(r) = S^3$. The quadratic form of this closed 4-manifold is represented by $A(r)$.

4. Closed 4-Manifolds from $\mathcal{R}_2$

4.1. Triangle groups.

Definition 4.1. Let ΔABC be a triangle with interior angles $A = \pi/l, B = \pi/m, C = \pi/n$ where l, m, n are integers ≥ 2. Let a, b, c be reflections along the three sides opposite of A, B, C, respectively. Then the group of motions on ΔABC generated by a, b, c in a 2-dimensional geometric space (Euclidean/spherical/hyperbolic)

is the triangle group $\Delta(l, m, n)$. ΔABC is the fundamental region, also called a Mobius triangle, giving a tessellation in the corresponding space by motions from $\Delta(l, m, n)$.

Note that the group has the following relators based on its definition. Since a, b, c are reflections we have $a^2 = b^2 = c^2 = 1$. Moreover, since the reflections are on a triangle, the product of 2 reflections is always a rotation twice the angle between the reflection edges in the other direction, i.e. ab and ba are rotations of twice the angle C about C in opposite directions (this also means $\Delta(l, m, n)$ is not abelian). Similarly, ab, bc, ca are rotations of $2\pi/m, 2\pi/n, 2\pi/l$ about C, A, B, respectively. This gives the property that $(ab)^m = (bc)^n = (ca)^l = 1$ since l, m, n are integers ≥ 2.

Thus $\Delta(l, m, n)$ has the presentation $\langle a, b, c | a^2, b^2, c^2, (ab)^m, (bc)^n, (ca)^l \rangle$.

The tuple (l, m, n) has different relationships corresponding to the angle sum properties in each of the geometric spaces. Namely, $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$ on a Euclidean plane, $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$ on a 2-sphere, $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ on a hyperbolic plane. Note that a triangle group is finite if and only if it's the spherical case.

Now consider its subgroup with only the orientation-preserving elements. Such a subgroup is also known as a "von Dyck group". Since each a, b, c changes the orientation, all orientation-preserving elements have even lengths, thus one way to construct the generators is to consider all words of length 2 in a regular triangle group. We can eliminate a^2, b^2, c^2 since all the generators have order 2. We can further eliminate ba, cb, ca as these are the same as the inverses of ab, bc, ac. Finally we can eliminate ac since $(ab)(bc) = ac$. Thus it leaves us with 2 generators $x = ab, y = bc$ for the corresponding von Dyck group, with same relators by rotations from the triangle group. The cardinality of a von Dyck group is the same as the corresponding triangle group in the infinite cases (Euclidean and hyperbolic), and half the size in the finite case (spherical).

A von Dyck group thus have presentation $\langle x, y | x^l, y^n, (xy)^m \rangle$, where l, m, n are integers ≥ 2.

4.2. S^3 from $\mathcal{R}_2$.

Lemma 4.2. *Each Artin presentation $r = \langle x_1, x_2 | r_1, r_2 \rangle \in \mathcal{R}_2$ corresponds to an (a, b, c) tuple of the form: $r_1 = x_1^{a-c}(x_1 x_2)^c, r_2 = x_2^{b-c}(x_1 x_2)^c, a, b, c \in \mathbb{Z}$ [Cal04].*

In order to close up $W^4(r)$ with a D^4 to get a closed smooth simply-connected 4-manifold, $M^3(r) = \partial W^4(r)$ needs to be $S^3 = \partial D^4$, one condition needs to be checked is the consistency of the fundamental groups, specifically $\pi(r) = \pi_1(M^3(r)) = \pi_1(S^3) = \{0\}$.

We first check when $M^3(r)$ is the integral homology 3-sphere. By the construction of $M^3(r)$, $H_0(M^3(r)) = H_3(M^3(r)) = \mathbb{Z}$, which are consistent with S^3. Note that $H_1(M^3(r))$ is isomorphic to the abelianized $\pi_1(M^3(r))$, and once $H_1(M^3(r))$ is trivial we get $H_2(M^3(r))$ is also trivial by Poincaré duality. Thus we aim to find r such that its abelianization $Ab(r) = \pi(r)/[\pi(r), \pi(r)]$ is the trivial group as $Ab(\pi_1(S^3)) = \{0\}$.

We can use the exponent sum matrix $A(r)$ to generate the abelianization of $\pi(r)$, with the fact that $Ab(r) \cong \mathbb{Z}^2/ImA(r)$. Note that $\mathbb{Z}^2/ImA(r) = \{0\}$ if and only if $ImA(r) = \mathbb{Z}^2$ so that everything gets killed, furthermore $ImA(r) = \mathbb{Z}^2$ if and only if $\det(A) = \pm 1$ since A is an integer matrix that preserves length. Thus it remains to find all (a, b, c) tuples such that $\det(A) = \pm 1$, which are the cases when $M^3(r)$

is the integral homology 3-sphere. We only need to check within theses cases for $M^3(r)$ is exactly S^3.

Moreover, we check the quotient group

$$H(r) = \langle x_1, x_2 | x_1^{a-c}(x_1 x_2)^c, x_2^{b-c}(x_1 x_2)^c, x_1^{a-c} \rangle$$

to be trivial using its structural resemblance with the von Dyck group. This allows us to eliminate most cases as any von Dyck group is non-trivial.

Specifically, note that

$$H(r) \cong \langle x_1, x_2 | x_1^{a-c}, x_2^{b-c}, (x_1 x_2)^c \rangle$$

$$\cong \langle x_1, x_2 | x_1^{|a-c|}, x_2^{|b-c|}, (x_1 x_2)^c \rangle$$

Recall that a von Dyck group has presentation

$$\langle x, y | x^l, y^n, (xy)^m \rangle, \text{ where } l, m, n \text{ are integers } \geq 2.$$

Thus when $|a-c|, |b-c|, c \geq 2$, the quotient $H(r) \cong$ von Dyck group is non trivial, which means $\pi(r)$ is not trivial.

Below we find all the (a, b, c) tuples that gives an Artin presentation $r \in \mathcal{R}_2$ such that $M^3(r)$ is the integral homology 3-sphere, which are exactly the cases $\det(A) = \det \begin{bmatrix} a & c \\ c & b \end{bmatrix} = ab - c^2 = \pm 1$. In addition, we check if the quotient group $H(r)$ in each of these cases is isomorphic to the von Dyck group, and eliminate those (a, b, c) combinations. The rest of the cases leads to the trivial group cases for $\pi(r)$, furthermore the S^3 cases for $M^3(r)$.

Note that we can switch the values of a, b without changing the group by the symmetry of a, b in $Ab(r)$. We also assume $c > 0$ since for each $r(a, b, c)$ there is $r(-a, -b, -c) \cong r(a, b, c)$. There are basically 2 cases $ab = c^2 - 1$ and $ab = c^2 + 1$. We first check the trivial cases when ab is small or c is small.

The simple cases for $\det(A) = ab - c^2 = \pm 1/ab = c^2 \pm 1$ (within each check if $|a-c| \geq 2, |b-c| \geq 2, c \geq 2$ hold):

- $c = 0$:
  then $ab = c^2 \pm 1 = \pm 1$. This gives
  $a, b, c = 1, 1, 0$ (not a von Dyck group since $c < 2$), or
  $a, b, c = -1, -1, 0$ (not a von Dyck group since $c < 2$), or
  $a, b, c = 1, -1, 0$ (not a von Dyck group since $c < 2$).
- $ab = 0$:
  then $c^2 = ab \pm 1 = \pm 1$. Since assumed $c > 0$ we have $c = 1$. Since the symmetry between a, b all the cases are:
  $a, b, c = 0, n, 1$ (not a von Dyck group since $c < 2$).
- $c = 1$:
  then $ab = c^2 \pm 1 = 1 \pm 1$. If $ab = 0$ then we have the previous case; if $ab = 2$ then we have:
  $a, b, c = 1, 2, 1$ (not a von Dyck group since $c < 2$), or
  $a, b, c = -1, -2, 1$ (not a von Dyck group since $c < 2$).

Now we consider the rest of the general cases. First note the following lemma.

Lemma 4.3. *Suppose $ab = c^2 + 1$ and $c \geq 2$(since $c = 0$ and $c = 1$ have been checked), in order for H not to be a von Dyck group, the difference between a and*

c needs to be small. When $a = c \pm 1$ or *c*, there are only two integer solutions: $a, b, c = 1, 5, 2$ or $a, b, c = 2, 5, 3$.

Proof. Case 1: $a = c$. Then $ab = bc = c^2 + 1 \Rightarrow b = c + \frac{1}{c}$, since we assumed $c \geq 2$, no integer *b* exists.

Case 2: $a = c \pm 1$. Since *a* divides $c^2 + 1 = (c-1)(c+1) + 2$, in either case $a = c \pm 1$ also divides 2, which gives $a = 1$ or $a = 2$. When $a = c + 1$ either case gives $c < 2$, thus the corresponding values for $c \geq 2$ are within $c = a + 1$, specifically $c = 2$ or $c = 3$. Thus we have two combinations $a, b, c = 1, 5, 2$ and $a, b, c = 2, 5, 3$. □

The two cases in general:

- $ab = c^2 + 1, a, b \neq 0, c \geq 2$:
  Case 1. $a(\text{or } b) = c \pm 1$ or *c*: by **Lemma 4.3**
  $a, b, c = 1, 5, 2$ (not a von Dyck group since $|a - c| < 2$), or
  $a, b, c = 2, 5, 3$ (not a von Dyck group since $|a - c| < 2$)
  Case 2. $a(\text{and } b) \neq c \pm 1, c$:
  then we have $|a - c| \geq 2$ and $|b - c| \geq 2$, by assumption $c \geq 2$. Thus $H(r)$ is always a von Dyck group.
- $ab = c^2 - 1, a, b \neq 0, c \geq 2$:
  Case 1. $a = c$:
  then $b = c - \frac{1}{c}$ has no integer solution.
  Case 2. $ab = (c-1)(c+1)$:
  $a, b, c = c - 1, c + 1, c$ (not a von Dyck group since $|a - c| = |b - c| = 1$).
  Case 3. $a, b \neq c$ or $c \pm 1$:
  then $|a - c| \geq 2, |b - c| \geq 2$, by assumption $c \geq 2$. Thus $H(r)$ is always a von Dyck group.

Now we sum up the cases for $M^3(r)$ is the homology 3-sphere($\det(A) = \pm 1$) and $H(r)$ is not a von Dyck group. Then check within these cases for $\pi(r) = \{0\}$, that is, in each $\pi(r)$ check if $x_1 = x_2 = 0$.

a, b, c	$r(a, b, c)$	$\pi(r)$
$1, 1, 0$	$\langle x_1, x_2 \| x_1, x_2 \rangle$	*trivial group*
$1, -1, 0$	$\langle x_1, x_2 \| x_1, x_2^{-1} \rangle$	*trivial group*
$-1, -1, 0$	$\langle x_1, x_2 \| x_1^{-1}, x_2^{-1} \rangle$	*trivial group*
$0, n \in \mathbb{Z}, 1$	$\langle x_1, x_2 \| x_2, x_2^{n-1} x_1 x_2 \rangle$	*trivial group*
$1, 2, 1$	$\langle x_1, x_2 \| x_1 x_2, x_2 x_1 x_2 \rangle$	*trivial group*
$-1, -2, 1$	$\langle x_1, x_2 \| x_1^{-1} x_2, x_2^{-3} x_1 x_2 \rangle$	*trivial group*
$1, 5, 2$	$\langle x_1, x_2 \| x_2 x_1 x_2, x_2^3 x_1 \rangle$	*trivial group*
$2, 5, 3$	$\langle x_1, x_2 \| x_2 x_1 x_2 x_1 x_2, x_2^2 x_1 \rangle$	*trivial group*
$c - 1, c + 1, c (c \geq 1)$	$\langle x_1, x_2 \| x_1^{-1} (x_1 x_2)^c, x_2 (x_1 x_2)^c \rangle$	*trivial group*

4.3. Closed 4-manifolds from $\mathcal{R}_2$.

Recall that $r \in R_2$ with $A(r) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ determines the 4-manifold (with boundary) $W^4(r)$ shown in Figure 4.1. We wish to identify each of these manifolds for the list at the end of the previous subsection. We proceed to do so avoiding the difficult Poincaré conjecture. Our approach uses the Kirby calculus.

First, we dispense with some simple cases. These follow from [Gom99, p.144].
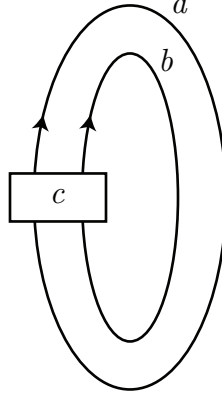
FIGURE 4.1. Framed link diagram for $W^4(r)$ with $r \in \mathcal{R}_2$, where $A(r) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$.
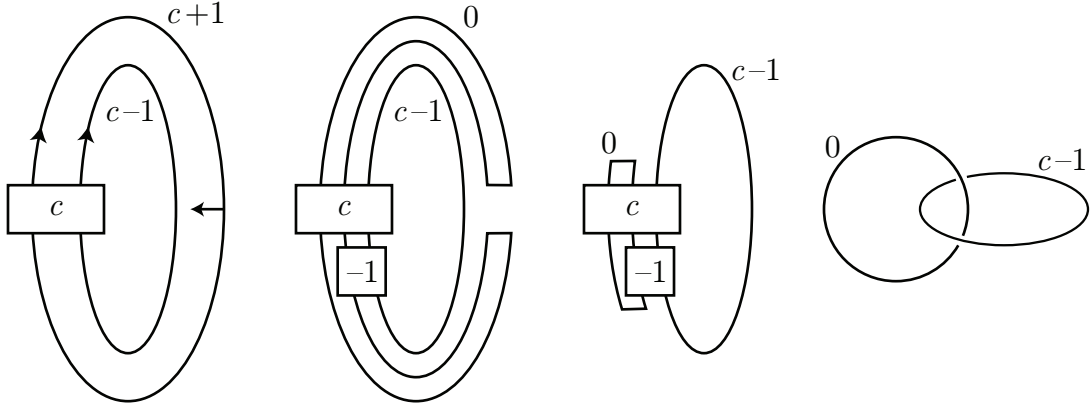

FIGURE 4.2. Identifying a 4-manifold (far left) by a handle subtraction followed by two isotopies.

- The case $1, 1, 0$ corresponds to the diagram with two geometrically unlinked $+1$ framed unknots. This is well known to give $\mathbb{CP}^2 \# \mathbb{CP}^2$.

The following three cases can be obtained using the similar idea.

- The case $1, -1, 0$ gives $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.
- The case $-1, -1, 0$ gives $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$.
- The case $0, n, 1$ gives $S^2 \times S^2$ if n is even, and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ if n is odd.

Next, we study the case $c - 1, c + 1, c$. The corresponding 4-manifold is shown on the far left in Figure 4.2.

- To identify the 4-manifold given by the $c - 1, c + 1, c$ case, we proceed as shown in Figure 4.2. Slide the $c + 1$ framed component over the $c - 1$ framed component using a trivial band and handle subtraction (using the indicated orientations). Then, perform two isotopies as shown. The

resulting manifolds are (upon closing with a 4-handle) $S^2 \times S^2$ if c is odd, and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ if c is even.

The rest of the manifolds can be found mainly using the similar handle subtraction and isotopy.

- For the case $1, 2, 1$, a handle subtraction and some isotopy gives $\mathbb{CP}^2 \# \mathbb{CP}^2$.
- The case $1, 5, 2$ can be shown using the same operations as above, which also gives $\mathbb{CP}^2 \# \mathbb{CP}^2$.
- For the case $-1, -2, 1$, we can change the signs of the framings and reverse each crossing to get the case $1, 2, -1$ (the resulting manifold has its orientation reversed). By isotopy, $1, 2, -1$ yields the same manifold as $1, 2, 1$, which we already identified. Thus, $-1, -2, 1$ yields $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$.
- Finally, for the case $2, 5, 3$, a similar handle subtraction yields $1, 2, -1$. This is the same manifold as the $1, 2, 1$ case, namely $\mathbb{CP}^2 \# \mathbb{CP}^2$.

In summary, the only closed 4-manifolds in Artin presentation theory arising from $r \in R_2$ are the expected ones, namely $S^2 \times S^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, and $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$.

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