

## ARTIN PRESENTATIONS, TRIANGLE GROUPS, AND 4-MANIFOLDS

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ABSTRACT. González-Acuña showed that Artin presentations characterize closed, orientable 3-manifold groups. Winkelnkemper later discovered that each Artin presentation determines a smooth, compact, simply-connected 4-manifold. In this note, we utilize triangle groups to find all Artin presentations on two generators that present the trivial group. We then determine all smooth, closed, simply-connected 4-manifolds with second betti number at most two that appear in Artin presentation theory.

## 1. INTRODUCTION

An *Artin presentation* is a group presentation  $\langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n \rangle$  such that the following holds in the free group  $F_n = \langle x_1, x_2, \dots, x_n \rangle$

$$x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1) (r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)$$

González-Acuña [Gon75, Thm. 4] showed that every closed, orientable 3-manifold admits an open book decomposition with planar page. As a corollary, he obtained the following algebraic characterization of 3-manifold groups.

**Theorem** (González-Acuña [Gon75, Thm. 6]). *A group  $G$  is the fundamental group of a closed, orientable 3-manifold if and only if  $G$  admits an Artin presentation for some  $n$ .*

Winkelnkemper [Win02, p. 250] discovered that each Artin presentation  $r$  determines not only a closed, orientable 3-manifold  $M^3(r)$  but also a smooth, compact, simply-connected 4-manifold  $W^4(r)$  such that  $\partial W^4(r) = M^3(r)$ . All intersection forms are represented by some  $W^4(r)$  [Win02, pp. 248–250]. If  $M^3(r)$  is the 3-sphere, then we consider the smooth, closed, simply-connected 4-manifold  $X^4(r) = W^4(r) \cup_{\partial} D^4$  obtained from  $W^4(r)$  by closing up with a 4-handle.

While all closed, orientable 3-manifolds appear in Artin presentation theory, it is unknown which 4-manifolds appear as a  $W^4(r)$  or an  $X^4(r)$ . The only contractible manifold  $W^4(r)$  is  $D^4$  (when  $r = \langle \mid \rangle$  is the empty Artin presentation). So, no Mazur manifold appears as a  $W^4(r)$ . Nevertheless, there are no known smooth, closed, simply-connected 4-manifolds that do not appear as an  $X^4(r)$ ; many interesting closed 4-manifolds are known to appear this way including all elliptic surfaces  $E(n)$  where  $E(2)$  is diffeomorphic to the Kummer surface  $K3$  [CW04].

In this note, we determine all closed 4-manifolds  $X^4(r)$  where  $r$  is an Artin presentation on two generators. (For  $n = 0$  and  $n = 1$  the problem is straightforward: only  $S^4$ ,  $\mathbb{C}P^2$ ,

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and  $\overline{\mathbb{C}P^2}$  appear.) Theorem 4.2 gives the complete list of these manifolds:  $\mathbb{C}P^2 \# \mathbb{C}P^2$ ,  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ ,  $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ , and  $S^2 \times S^2$ . Exotic simply-connected, closed 4-manifolds are currently not known to exist with second betti number  $\leq 2$ . Theorem 4.2 shows that such manifolds, whether or not they exist in general, do not appear in Artin presentation theory. Exotic 4-manifolds do appear in Artin presentation theory with second betti number  $\geq 10$  [Cal08]. We conjecture that closed, exotic 4-manifolds appear in Artin presentation theory with second betti number three, and that this relates to the Torelli subgroup in Artin presentation theory (see [Win02, p. 250] and [Cal08]).

Our proof of Theorem 4.2 uses the classification of Artin presentations on two generators and properties of classical triangle groups to find all Artin presentations on two generators that present the trivial group. We then introduce a move on Artin presentations on two generators which preserves the 4-manifolds. Using this move and the Kirby calculus, we then identify the 4-manifolds. It seems to be an interesting problem to find other such moves in Artin presentation theory. Armas-Sanabria has some interesting results showing certain three generator Artin presentations present nontrivial groups [Arm12].

Throughout,  $X \approx Y$  means that  $X$  is orientation preserving diffeomorphic to  $Y$ . If  $X$  is an oriented manifold, then  $\overline{X}$  denotes the same manifold with the opposite orientation.

## 2. ARTIN PRESENTATIONS

In this section, we review some fundamental properties of Artin presentations and fix notation. We begin by recalling how each Artin presentation arises naturally from a homeomorphism of a compact 2-disk with holes.

Let  $\Omega_n$  denote the compact 2-disk with  $n$  holes as in Figure 2.1. The boundary compo-

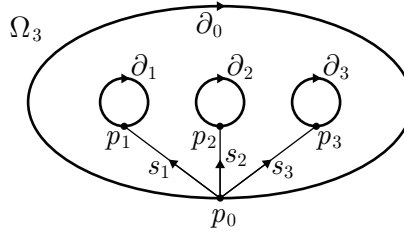
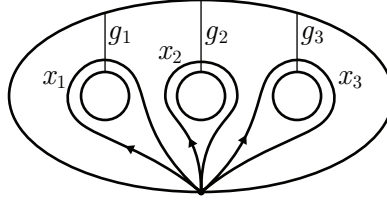


FIGURE 2.1. Compact 2-disk with  $n$  holes denoted  $\Omega_n$  (the case  $n = 3$  is depicted).

nents  $\partial_0, \partial_1, \dots, \partial_n$  of  $\Omega_n$  are oriented clockwise and are based at  $p_0, p_1, \dots, p_n$  respectively. For each  $1 \leq i \leq n$ , let  $s_i$  be an oriented segment from  $p_0$  to  $p_i$  as in Figure 2.1. Given a path  $\alpha$ , let  $\overline{\alpha}$  denote the reverse path and let  $[\alpha]$  denote the path homotopy class of  $\alpha$ . Concatenation of paths and the induced operation on classes will be denoted by juxtaposition. For each  $1 \leq i \leq n$ , let  $x_i = [s_i \partial_i \overline{s_i}]$  as in Figure 2.2. So,  $\pi_1(\Omega_n, p_0) \cong F_n = \langle x_1, x_2, \dots, x_n \rangle$  is free of rank  $n$ .

Let  $h : \Omega_n \rightarrow \Omega_n$  be a homeomorphism that equals the identity (point-wise) on the boundary of  $\Omega_n$ . Then,  $h_\# : \pi_1(\Omega_n, p_0) \rightarrow \pi_1(\Omega_n, p_0)$  is an automorphism. For each  $1 \leq i \leq n$ , define  $r_i = [s_i(h \circ \overline{s_i})] \in F_n$ . Define the presentation  $r = r(h) = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n \rangle$ .

FIGURE 2.2. Generators  $x_i$  of  $\pi_1(\Omega_3, p_0)$ .

**Claim 2.1.** *For each  $1 \leq i \leq n$ , we have  $h_{\#}(x_i) = r_i^{-1}x_i r_i$ .*

*Proof.* Note that  $h \circ \overline{s_i} = \overline{h \circ s_i}$ ,  $r_i^{-1} = [(h \circ s_i)\overline{s_i}]$ , and  $h \circ \partial_i = \partial_i$ . Therefore

$$\begin{aligned} h_{\#}(x_i) &= [h \circ (s_i \partial_i \overline{s_i})] = [h \circ s_i] [h \circ \partial_i] [h \circ \overline{s_i}] \\ &= [h \circ s_i] [\overline{s_i}] [s_i] [\partial_i] [\overline{s_i}] [s_i] [h \circ \overline{s_i}] \\ &= r_i^{-1} x_i r_i \end{aligned}$$

□

**Claim 2.2.** *The presentation  $r = r(h)$  determined by  $h$  is an Artin presentation.*

*Proof.* The following holds in  $F_n$

$$\begin{aligned} x_1 x_2 \cdots x_n &= [\partial_0] = [h \circ \partial_0] \\ &= h_{\#}(x_1 x_2 \cdots x_n) = h_{\#}(x_1) h_{\#}(x_2) \cdots h_{\#}(x_n) \\ &= (r_1^{-1} x_1 r_1) (r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n) \end{aligned}$$

where the last equality used Claim 2.1. □

**Remarks 2.3.**

- (1) Let  $h$  and  $h'$  be homeomorphisms of  $\Omega_n$  that are the identity on  $\partial\Omega_n$ . If  $h$  is homotopic to  $h'$  relative to  $p_0$ , then  $h_{\#} = h'_{\#}$  and so  $r(h) = r(h')$ .
- (2) The converse of Claim 2.2 holds in the following sense. If  $r = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_n \rangle$  is an Artin presentation, then there exists a self-homeomorphism  $h$  of  $\Omega_n$  equal to the identity on  $\partial\Omega_n$  such that  $r = r(h)$  (this follows from the Dehn-Nielsen theorem for planar surfaces with boundary). Furthermore,  $h$  is unique up to isotopy relative to  $\partial\Omega_n$ , and  $h$  may be assumed to be a diffeomorphism. See Artin [Art25], Birman [Bir75, pp. 30–34], and Zieschang, Vogt, and Coldewey [ZVC80, Ch. 5].

Let  $\mathcal{R}_n$  denote the set of Artin presentations on  $n$  generators.

**Examples 2.4.**

- (1) The empty presentation  $\langle \rangle$  of the trivial group is the unique Artin presentation in  $\mathcal{R}_0$ .
- (2) Each  $r \in \mathcal{R}_1$  has the form  $r = \langle x_1 \mid x_1^a \rangle$  for some integer  $a$ .
- (3) Let  $a_1, a_2, \dots, a_n$  be any integers. Then,  $r = \langle x_1, x_2, \dots, x_n \mid x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n} \rangle$  is an Artin presentation in  $\mathcal{R}_n$  that presents the free product of cyclic groups  $\mathbb{Z}/|a_i|\mathbb{Z}$ . In particular, if each  $a_i = 0$ , then  $r$  presents the free group of rank  $n$ .
- (4) Each  $r = \langle x_1, x_2 \mid r_1, r_2 \rangle \in \mathcal{R}_2$  has the form  $r_1 = x_1^{a-c}(x_1 x_2)^c$  and  $r_2 = x_2^{b-c}(x_1 x_2)^c$  for some integers  $a, b$ , and  $c$  (see [Cal07]). We denote such an Artin presentation by  $r(a, b, c)$ . For instance,  $r(-1, -3, 2)$  presents the binary icosahedral group  $I(120)$ .

Each Artin presentation  $r \in \mathcal{R}_n$  determines the following.

$\pi(r)$	the group presented by $r$
$A(r)$	the exponent sum matrix of $r$
$h(r)$	a self-diffeomorphism of $\Omega_n$
$M^3(r)$	a closed, oriented 3-manifold
$W^4(r)$	a smooth, compact, simply-connected, oriented 4-manifold

Here  $[A(r)]_{ij}$  equals the exponent sum of  $x_i$  in  $r_j$ . For example,  $A(\langle x_1 \mid x_1^a \rangle) = [a]$  and  $A(r(a, b, c)) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ . So,  $A(r)$  is an  $n \times n$  integer matrix, and the abelianization of  $\pi(r)$  is isomorphic to  $\mathbb{Z}^n / \text{Im} A$  where  $\text{Im} A$  denotes the image of  $A : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ . Therefore, the group  $\pi(r)$  is perfect if and only if  $A(r)$  is unimodular (that is,  $\det A = \pm 1$ ). By Remarks 2.3,  $r$  determines a self-diffeomorphism  $h = h(r)$  of  $\Omega_n$  equal to the identity on  $\partial\Omega_n$  and unique up to isotopy relative to  $\partial\Omega_n$ . The 3-manifold  $M^3(r)$  is defined by Winkelnkemper's open book construction with planar page  $\Omega_n$  (see González-Acuña [Gon75] and Winkelnkemper [Win02]). Namely, consider the mapping torus  $\Omega(h)$  of  $h$  which is obtained from  $\Omega_n \times [0, 1]$  by identifying  $(x, 1)$  with  $(h(x), 0)$  for each  $x \in \Omega_n$ . The boundary of  $\Omega(h)$  equals  $(\partial\Omega_n) \times S^1$ , and  $M^3(r)$  is obtained from  $\Omega(h)$  by gluing on  $(\partial\Omega_n) \times D^2$  using the identity function on  $(\partial\Omega_n) \times S^1$ . The fundamental group of  $M^3(r)$  is isomorphic to  $\pi(r)$  (see [Gon75, p. 10] or [Win02, p. 247]). In particular,  $M^3(r)$  is an integer homology 3-sphere if and only if  $A(r)$  is unimodular. Using the symplectic property of closed surface homeomorphisms, Winkelnkemper observed that  $A(r)$  is always symmetric for an Artin presentation  $r$  (see [Win02, p. 250], or see [Cal07] for an algebraic proof of this fact). This led Winkelnkemper to discover that  $r$  determines a 4-manifold using a sort of relative open book construction as follows. Embed  $\Omega_n$  in  $S^2$ , and let  $C$  be the closure in  $S^2$  of the complement of  $\Omega_n$  (so,  $C$  is the disjoint union of  $n+1$  smooth 2-disks). Extend  $h$  to  $S^2$  then to  $D^3$ , and let  $H$  be the resulting self-diffeomorphism of  $D^3$ . The mapping torus  $W(H)$  of  $H$  contains  $C \times S^1$  in its boundary. Then,  $W^4(r)$  is obtained from  $W(H)$  by gluing on  $C \times D^2$  in the canonical way. In particular,  $\partial W^4(r) = M^3(r)$  and the quadratic form of  $W^4(r)$  is given by  $A(r)$ . If  $M^3(r)$  is the 3-sphere, then we define  $X^4(r) = W^4(r) \cup_{\partial} D^4$  a smooth, closed, simply-connected, oriented 4-manifold (that is, we close up with a 4-handle). By Cerf's theorem [Cer68], a 4-handle may be added in an essentially unique way, and so  $X^4(r)$  is well-defined.

An alternative definition of  $W^4(r)$  is the following (see [CW04, §2]). Let  $r \in \mathcal{R}_n$  be an Artin presentation. Then,  $r$  determines a (pure) braid group automorphism of  $F_n$  defined by  $x_i \mapsto r_i^{-1} x_i r_i$  for  $1 \leq i \leq n$ . This automorphism determines an  $n$ -strand pure braid [Bir75, pp. 30–34]. Let  $L(r)$  be the integer framed pure link in  $S^3 = \partial D^4$  obtained as the closure of this pure braid; the framing of the  $i$ th component is  $[A(r)]_{ii}$ . Define  $W^4(r)$  to be  $D^4$  union  $n$  2-handles attached along  $L(r)$ . So,  $L(r)$  is a Kirby diagram for  $W^4(r)$  (see [GS99, p. 115] for an introduction to Kirby diagrams). Figure 2.3 gives Kirby diagrams for  $W^4(r)$  where  $r \in \mathcal{R}_1$  and  $r \in \mathcal{R}_2$ . For example,  $X^4(\langle x_1 \mid x_1 \rangle) \approx \mathbb{C}P^2$  and  $X^4(\langle x_1 \mid x_1^{-1} \rangle) \approx \overline{\mathbb{C}P^2}$ .

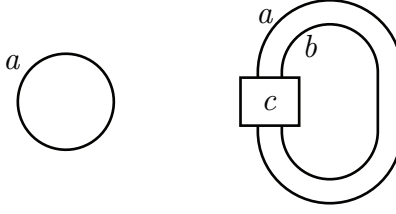


FIGURE 2.3. Kirby diagrams for  $W^4(r)$  where  $r = \langle x_1 \mid x_1^a \rangle \in \mathcal{R}_1$  (left) and  $r = r(a, b, c) \in \mathcal{R}_2$  (right).

**Remark 2.5.** Three basic diffeomorphisms between the 4-manifolds  $W^4(r(a, b, c))$  are as follows where  $a$ ,  $b$ , and  $c$  are any integers.

$$(2.1) \quad W^4(r(a, b, c)) \approx W^4(r(b, a, c))$$

$$(2.2) \quad W^4(r(a, b, 1)) \approx W^4(r(a, b, -1))$$

$$(2.3) \quad W^4(r(-a, -b, -c)) \approx \overline{W^4(r(a, b, c))}$$

The first two diffeomorphisms are given by simple isotopies of the Kirby diagrams: interchange the two link components and flip one component to switch the sign of the single crossing respectively. The third diffeomorphism is a special case of the fact that given a Kirby diagram for a 2-handlebody  $Y$ , one obtains a Kirby diagram for  $\overline{Y}$  by switching all crossings (that is, take a mirror of the link) and multiplying each framing coefficient by  $-1$ . If  $M^3(r)$  is  $S^3$ , then all three diffeomorphisms also hold with  $X^4$  in place of  $W^4$ .

### 3. TRIANGLE GROUPS AND ARTIN PRESENTATIONS

We recall basic facts about triangle groups (for details, see Magnus [Mag74, Ch. II] and Ratcliffe [Rat06, §7.2]). Let  $l$ ,  $m$ , and  $n$  be integers greater than or equal to 2. Let  $\Delta = \Delta(l, m, n)$  be a triangle with angles  $\pi/l$ ,  $\pi/m$ , and  $\pi/n$ . Define

$$\delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n}$$

The triangle  $\Delta$  is: *spherical* and lies in  $X = S^2$  if  $\delta > 1$ , *euclidean* and lies in  $X = \mathbb{R}^2$  if  $\delta = 1$ , and *hyperbolic* and lies in  $X = \mathbb{H}^2$  if  $\delta < 1$ . The *triangle reflection group*  $T^*(l, m, n)$  is the group generated by the reflections of  $X$  in the lines containing the sides of  $\Delta$ . The *triangle group*  $T(l, m, n)$  (sometimes called a *von Dyck group*) is the index 2 subgroup of  $T^*(l, m, n)$  consisting of orientation preserving isometries of  $X$ . Geometrically,  $T(l, m, n)$  is generated by the rotations of  $X$  about the vertices of  $\Delta$  by  $2\pi/l$ ,  $2\pi/m$ , and  $2\pi/n$  respectively. The triangle group  $T(l, m, n)$  is presented by  $\langle x, y \mid x^l, y^m, (xy)^n \rangle$ . Notice that  $T(l, m, n)$  is independent up to isomorphism of the order in which the integers  $l$ ,  $m$ , and  $n$  are listed. The triangle group  $T(l, m, n)$  is: *spherical* and finite (but nontrivial) if  $\delta > 1$ , *euclidean* and infinite if  $\delta = 1$ , and *hyperbolic* and infinite if  $\delta < 1$ . For example,  $T(2, 3, 5)$  is the *icosahedral group* isomorphic to the order 60 alternating group  $A_5$  on five letters. The infinite groups  $T(3, 3, 3)$  and  $T(3, 3, 4)$  correspond respectively to triangular tilings of the euclidean and hyperbolic planes.

**Lemma 3.1.** *Let  $r = r(a, b, c) \in \mathcal{R}_2$ . If  $|a - c|$ ,  $|b - c|$ , and  $|c|$  are all greater than or equal to 2, then  $\pi(r)$  is nontrivial. If in addition  $1/|a - c| + 1/|b - c| + 1/|c| \leq 1$ , then  $\pi(r)$  is infinite.*

*Proof.* We construct a surjective group homomorphism  $\pi(r) \twoheadrightarrow T(|a-c|, |b-c|, |c|)$ . Add the relation  $(x_1x_2)^c$  to  $r$  to obtain

$$\begin{aligned} \pi(r) &\twoheadrightarrow \langle x_1, x_2 \mid x_1^{a-c}(x_1x_2)^c, x_2^{b-c}(x_1x_2)^c, (x_1x_2)^c \rangle \\ &\cong \langle x_1, x_2 \mid x_1^{a-c}, x_2^{b-c}, (x_1x_2)^c \rangle \\ &\cong \langle x_1, x_2 \mid x_1^{|a-c|}, x_2^{|b-c|}, (x_1x_2)^{|c|} \rangle \\ &= T(|a-c|, |b-c|, |c|) \end{aligned}$$

Now, apply properties of triangle groups recalled above.  $\square$

**Examples 3.2.** Consider the groups  $\pi(r(-1, -3, 2))$  and  $\pi(r(10, 1, 3))$ . Both groups are perfect since their exponent sum matrices are unimodular. Lemma 3.1 implies that  $\pi(r(-1, -3, 2))$  is nontrivial and  $\pi(r(10, 1, 3))$  is infinite. The proof of Lemma 3.1 shows that  $\pi(r(-1, -3, 2))$  surjects onto  $T(3, 5, 2) \cong A_5$ .

**Theorem 3.3.** *Let  $r = r(a, b, c) \in \mathcal{R}_2$ . If  $\pi(r)$  is trivial, then the 3-tuple  $(a, b, c)$  lies in the following list where  $-(a, b, c) = (-a, -b, -c)$ .*

(3.1)  $(\pm 1, \pm 1, 0)$  (four 3-tuples)

(3.2)  $\pm(2, 1, \pm 1)$  and  $\pm(1, 2, \pm 1)$  (eight 3-tuples)

(3.3)  $\pm(1, 5, 2), \pm(5, 1, 2), \pm(2, 5, 3), \pm(5, 2, 3)$  (eight 3-tuples)

(3.4)  $(a, 0, \pm 1)$  and  $(0, b, \pm 1)$  where  $a, b \in \mathbb{Z}$

(3.5)  $(c \pm 1, c \mp 1, c)$  where  $c \in \mathbb{Z}$

*Proof.* As  $\pi(r)$  is trivial,  $A(r)$  must be unimodular and  $ab - c^2 = \pm 1$ . Now, the basic idea is that either  $|c| \leq 1$  is small and  $ab = c^2 \pm 1$  determines  $a$  and  $b$ , or  $|c| > 1$  is larger and Lemma 3.1 forces  $a$  or  $b$  to be close to  $c$ . We have  $ab = c^2 \pm 1$  and, by Lemma 3.1,  $|a - c| \leq 1$ ,  $|b - c| \leq 1$ , or  $|c| \leq 1$ . Notice that  $(a, b, c)$  appears in the given list if and only if  $-(a, b, c)$  appears. Indeed, as  $\pi(r(a, b, c)) \cong \pi(r(-a, -b, -c))$ , our list must have this property. So, it suffices to assume  $c \geq 0$  for the rest of the proof. If  $c = 0$ , then  $ab = \pm 1$ , which gives the tuples (3.1). If  $c = 1$ , then  $ab = 0$  or  $ab = 2$ . The former gives the tuples (3.4), and the latter gives the tuples (3.2).

Assume now that  $c > 1$ . Then  $a$  or  $b$  equals  $c - 1$ ,  $c$ , or  $c + 1$ . If  $a = c$ , then  $cb = c^2 \pm 1$  implies that  $c \mid \pm 1$ , a contradiction. Similarly,  $b \neq c$ . Thus,  $a = c \pm 1$  or  $b = c \pm 1$ .

**Case 1:**  $ab = c^2 + 1$ . Suppose  $a = c \pm 1$ . Then,  $a \mid c^2 - 1$  and  $a \mid c^2 + 1$ . So,  $a \mid 2$  and, as  $a = c \pm 1$  and  $c > 1$ , we have  $a = 1$  or  $a = 2$ . This gives the tuples  $(1, 5, 2)$  and  $(2, 5, 3)$ . Similarly,  $b = c \pm 1$  gives the tuples  $(5, 1, 2)$  and  $(5, 2, 3)$ .

**Case 2:**  $ab = c^2 - 1$ . Then,  $a = c \pm 1$  if and only if  $b = c \mp 1$ . This gives the tuples (3.5).  $\square$

**Remark 3.4.** It is not difficult to verify the converse of Theorem 3.3 directly using Tietze transformations. This converse also follows from the Kirby calculus arguments in the next section. Hence, Theorem 3.3 lists exactly the Artin presentations on two generators that present the trivial group.

## 4. 4-MANIFOLDS

In this section, we show that  $M^3(r)$  is  $S^3$  for each  $r$  listed in Theorem 3.3, and we identify the corresponding closed 4-manifolds  $X^4(r) = W^4(r) \cup_{\partial} D^4$ . First, we present a useful operation.

**Lemma 4.1.** *Let  $r = r(a, b, c) \in \mathcal{R}_2$ . There are diffeomorphisms*

$$(\dagger) \quad W^4(r(a, b, c)) \approx W^4(r(a + b - 2c, b, b - c))$$

$$(\ddagger) \quad W^4(r(a, b, c)) \approx W^4(r(a, a + b - 2c, a - c))$$

*In particular, the corresponding 3-manifolds  $M^3(r)$  are diffeomorphic, and the corresponding groups  $\pi(r)$  are isomorphic.*

*Proof.* For the first diffeomorphism, proceed as shown in Figure 4.1. In the second diagram

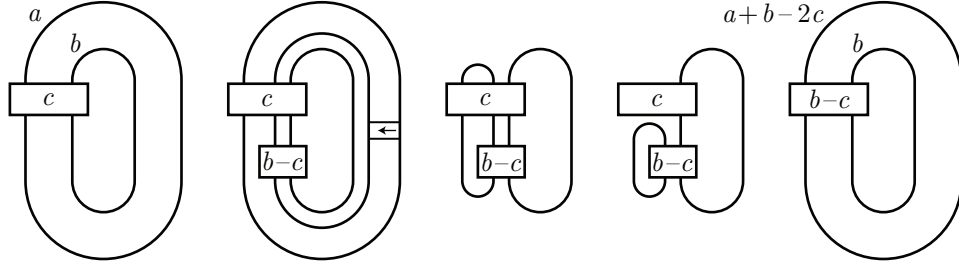


FIGURE 4.1. From the left: Kirby diagram for  $W^4(r(a, b, c))$ , a 2-handle slide, and two isotopies.

in Figure 4.1, the middle circle is parallel to the  $b$ -framed circle and has linking number  $b$  with it. If the  $a$ - and  $b$ -framed circles are oriented clockwise, then the indicated 2-handle slide is a handle subtraction; the framing of the  $a$ -framed circle changes to  $a + b - 2c$  (see [GS99, p. 141]). The result of Figure 4.1 is a Kirby diagram for  $W^4(r(a + b - 2c, b, b - c))$ . For  $(\ddagger)$ , instead slide the  $b$ -framed circle over the  $a$ -framed circle in a similar manner. The remaining claims in the lemma follow from  $(\dagger)$  and  $(\ddagger)$  by taking boundaries.  $\square$

**Theorem 4.2.** *For each Artin presentation  $r$  listed in Theorem 3.3,  $M^3(r)$  is  $S^3$ . Furthermore, the corresponding closed 4-manifolds  $X^4(r)$  are as follows.*

$$(4.1) \quad X^4(r(1, 1, 0)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2 \text{ and } X^4(r(1, -1, 0)) \approx \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$$

$$(4.2) \quad X^4(r(2, 1, 1)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2$$

$$(4.3) \quad X^4(r(5, 1, 2)) \approx X^4(r(5, 2, 3)) \approx \mathbb{C}P^2 \# \mathbb{C}P^2$$

$$(4.4) \quad X^4(r(a, 0, 1)) \approx \begin{cases} S^2 \times S^2 & \text{if } a \text{ is even} \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{if } a \text{ is odd} \end{cases}$$

$$(4.5) \quad X^4(r(c + 1, c - 1, c)) \approx \begin{cases} S^2 \times S^2 & \text{if } c \text{ is odd} \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{if } c \text{ is even} \end{cases}$$

*The 4-manifolds for the remaining 3-tuples in Theorem 3.3 are determined immediately from those just listed and Remark 2.5.*



*Proof.* First, (4.1) is clear since the Kirby diagrams are two-component unlinks with framings  $\pm 1$ . By  $(\dagger)$ ,  $W^4(r(2, 1, 1)) \approx W^4(r(1, 1, 0))$ , and (4.2) now follows from (4.1). Next,  $W^4(r(2, 1, -1)) \approx W^4(r(2, 1, 1))$  by Remark 2.5,  $W^4(r(2, 1, -1)) \approx W^4(r(5, 1, 2))$  by  $(\dagger)$ , and  $W^4(r(5, 1, 2)) \approx W^4(r(5, 2, 3))$  by  $(\ddagger)$ . So, (4.3) now follows from (4.2). The Kirby diagram for  $W^4(r(a, 0, 1))$  is a Hopf link with framings  $a$  and  $0$ ; by [GS99, pp. 127, 130, & 144], the corresponding 4-manifold may be closed up with a 4-handle yielding  $S^2 \times S^2$  if  $a$  is even and  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  if  $a$  is odd. This proves (4.4). Lastly,  $(\ddagger)$  gives  $W^4(r(c+1, c-1, c)) \approx W^4(r(c+1, 0, 1))$ , and (4.5) now follows from (4.4).  $\square$

**Corollary 4.3.** *The closed 4-manifolds appearing as  $X^4(r)$  for an Artin presentation  $r$  on  $n$ -generators for  $n = 0, 1, 2$  are exactly:  $S^4$  for  $n = 0$ ,  $\mathbb{C}P^2$  and  $\overline{\mathbb{C}P^2}$  for  $n = 1$ , and  $\mathbb{C}P^2 \# \mathbb{C}P^2$ ,  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ ,  $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ , and  $S^2 \times S^2$  for  $n = 2$ .*

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