Mesh Simplification

COMP 557

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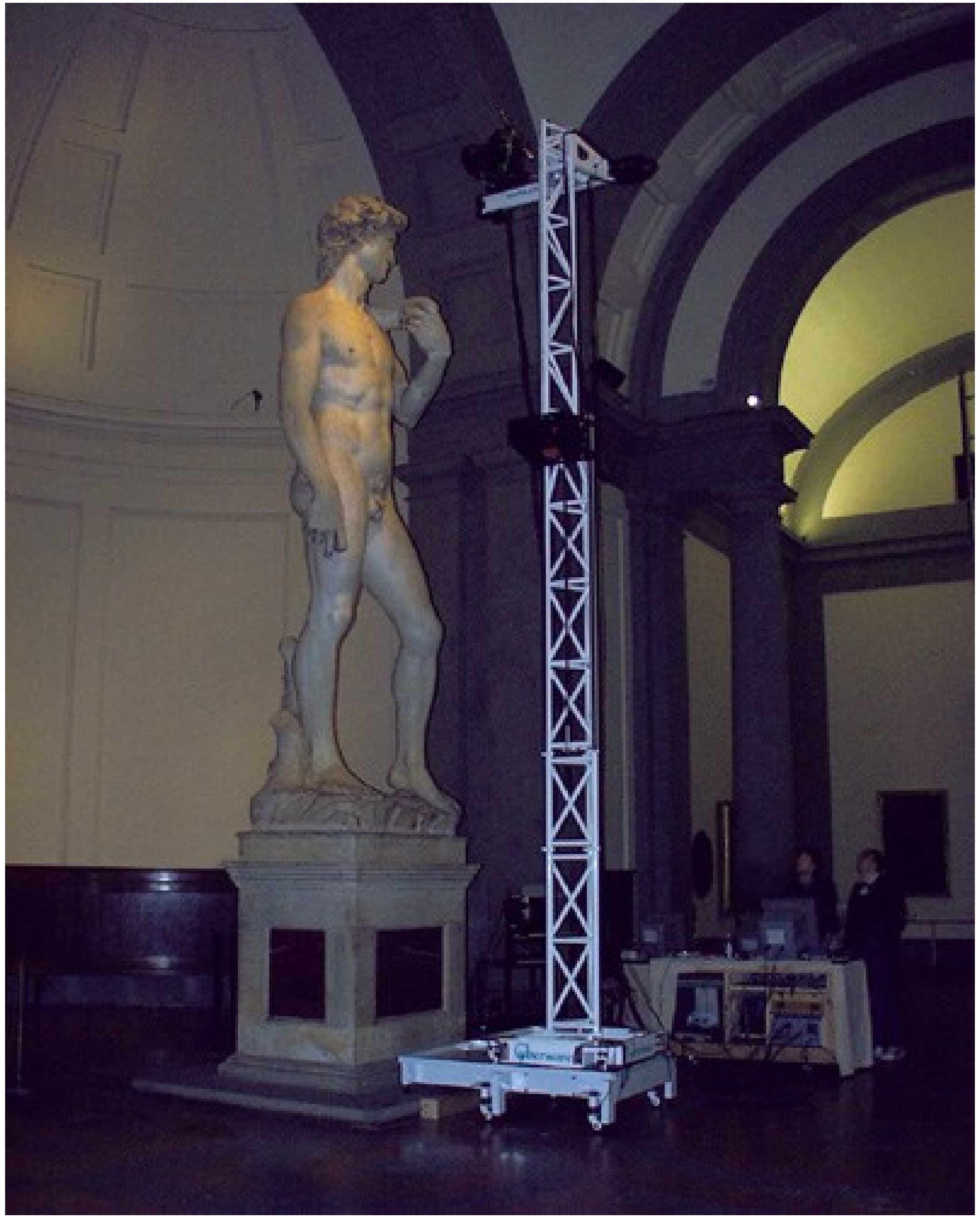
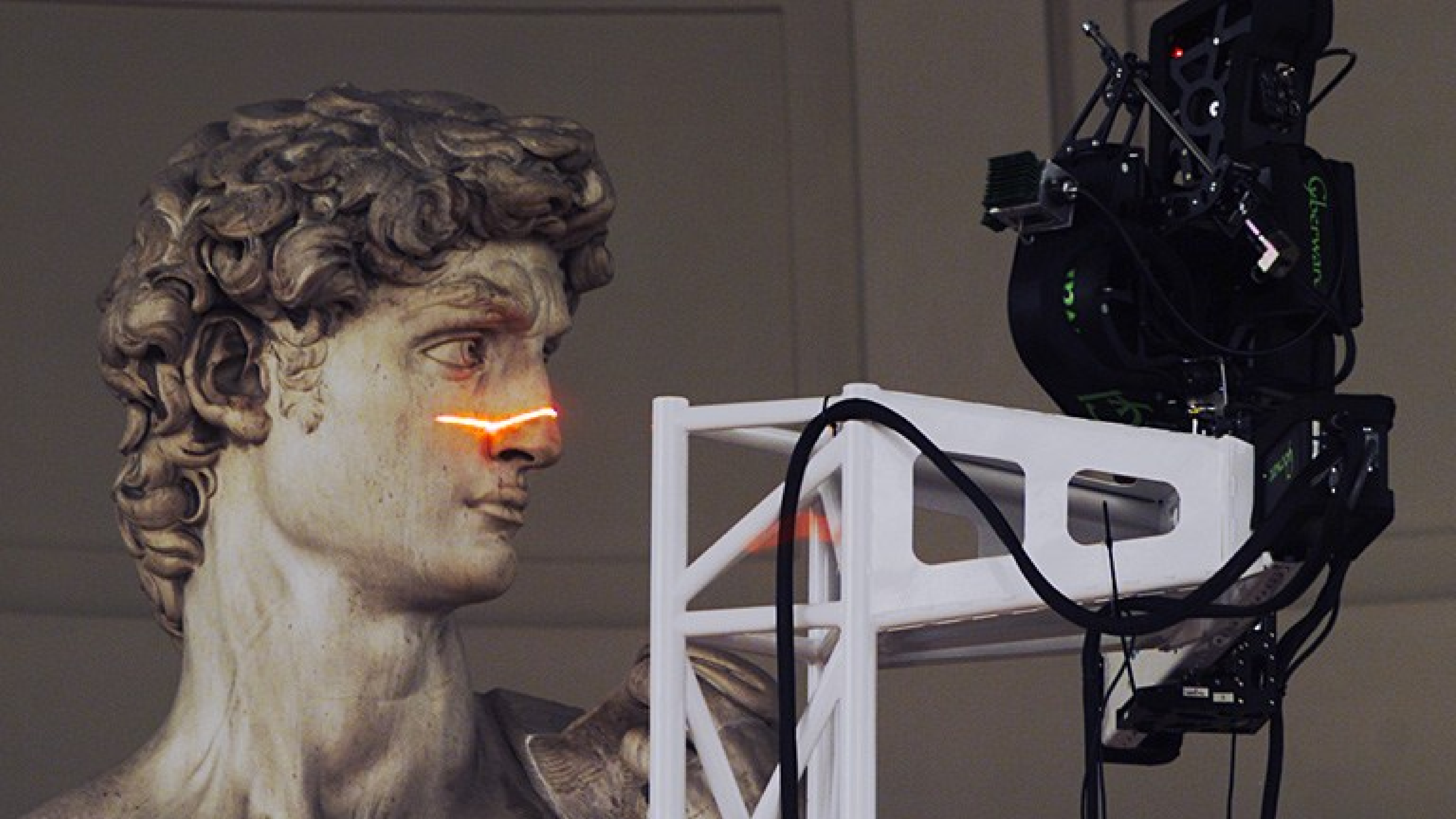
Big Meshes, Big Problems

- Example: the Digital Michelangelo Project
- Team of 30 from Stanford University and the University of Washington spent a year digitizing the sculptures and architecture
- Largest dataset is of the David,
  - 4 weeks of acquisition scanning 16 hours a day
  - 400 individually aimed scans
  - 2 billion polygons
  - 7,000 color images.
  - 60 gigabytes without compression

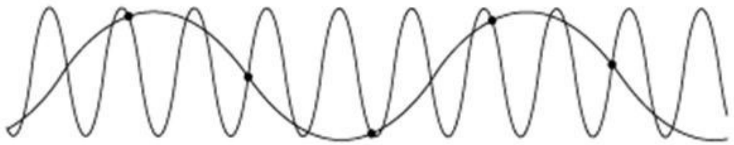




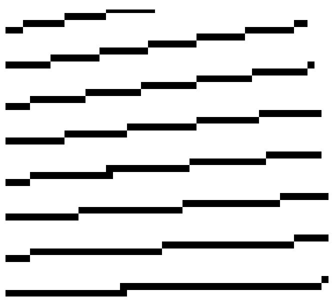
Level of Detail (LOD)

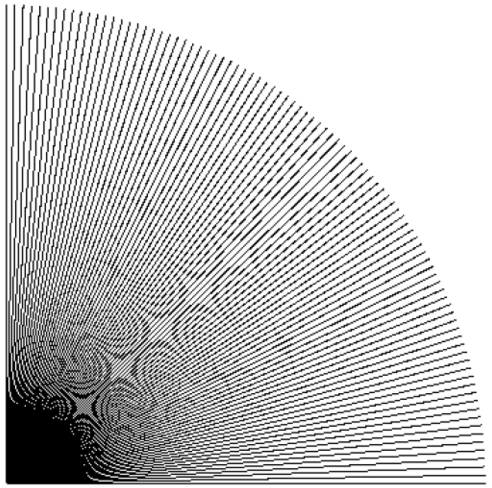
- Loosely defined as the size of the polygons
  - For example, length of shortest edge
- Resolution at which a model is displayed can be too coarse or too fine
  - ***Aliasing*** occurs when high-frequency details sampled at too low a frequency
  - Examples:
    - Jaggies on rasterized lines
    - Moiré patterns (i.e., we see low frequency patterns)
    - Details may also appear as noise rather than a mean value

Adequately sampled signal



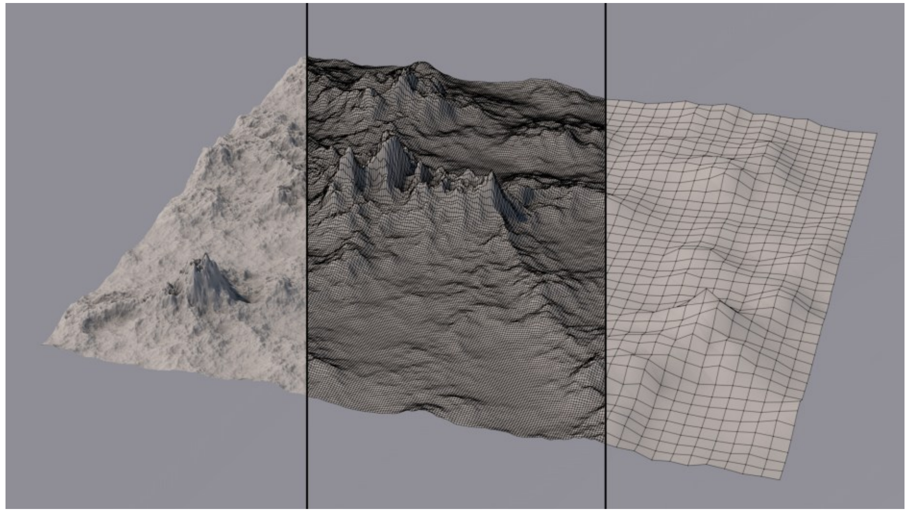
Aliased signal due to under sampling





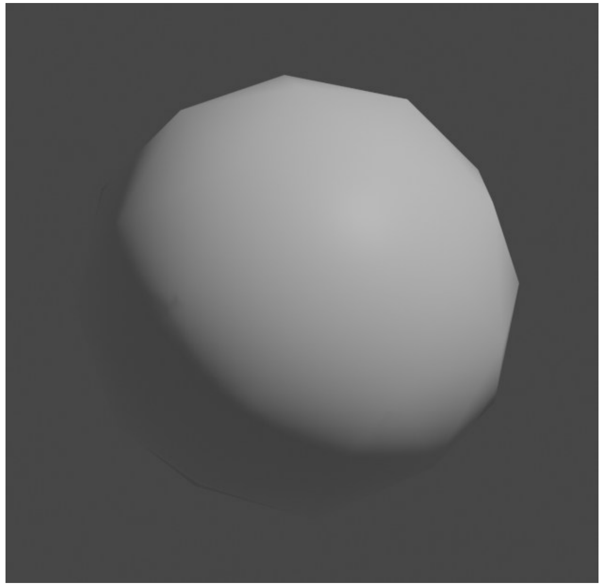
LOD examples

- Example: David model
  - David has approximately 2 billion polygons
  - High-definition screen has about 2 million pixels
  - What should you see at each pixel?
- Another important example: terrain
  - Can see far mountains?
  - Can see close hills?
  - May need multiple approximations
    at the same time for the same view



Level of Detail

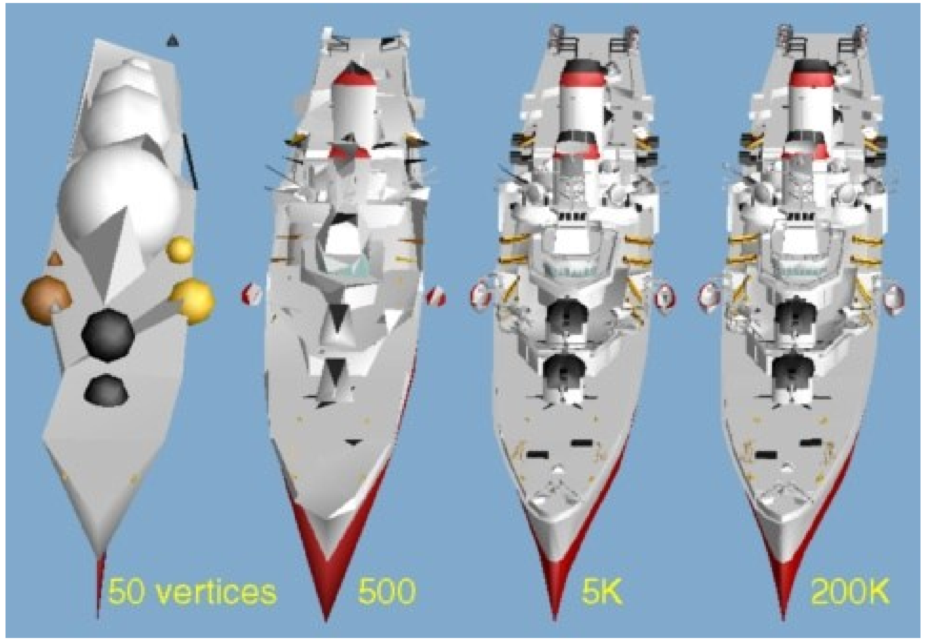
- Solution: compute several different coarse approximations
- How to choose which model to use?
  - # pixels use in screen space?
  - Are continuous levels of details needed to reduce popping when switching between levels or models
  - Is there viewpoint dependence?
    - e.g., terrain, but also consider silhouette edges (low poly sphere)
  - Might need multiple levels simultaneously
    - e.g., terrain, coarse geometry for far, fine geometry for near
- Typically, decision decisions will dictate the approach



Mesh Simplification

- Main idea: reduce the number of polygons
  - Faster rendering
  - Less storage
  - Simpler manipulation
- Find "good" approximation
  - Visual approximation
  - Geometric approximation
  - Data approximation
- Other desirable qualities
  - Applicability (works on all meshes?)
  - Efficiency, Scalability,
  - Preservation of attributes
    (texture coordinates, normals, etc.)

[Garland and Heckbert 1997]



[Hoppe 1996]

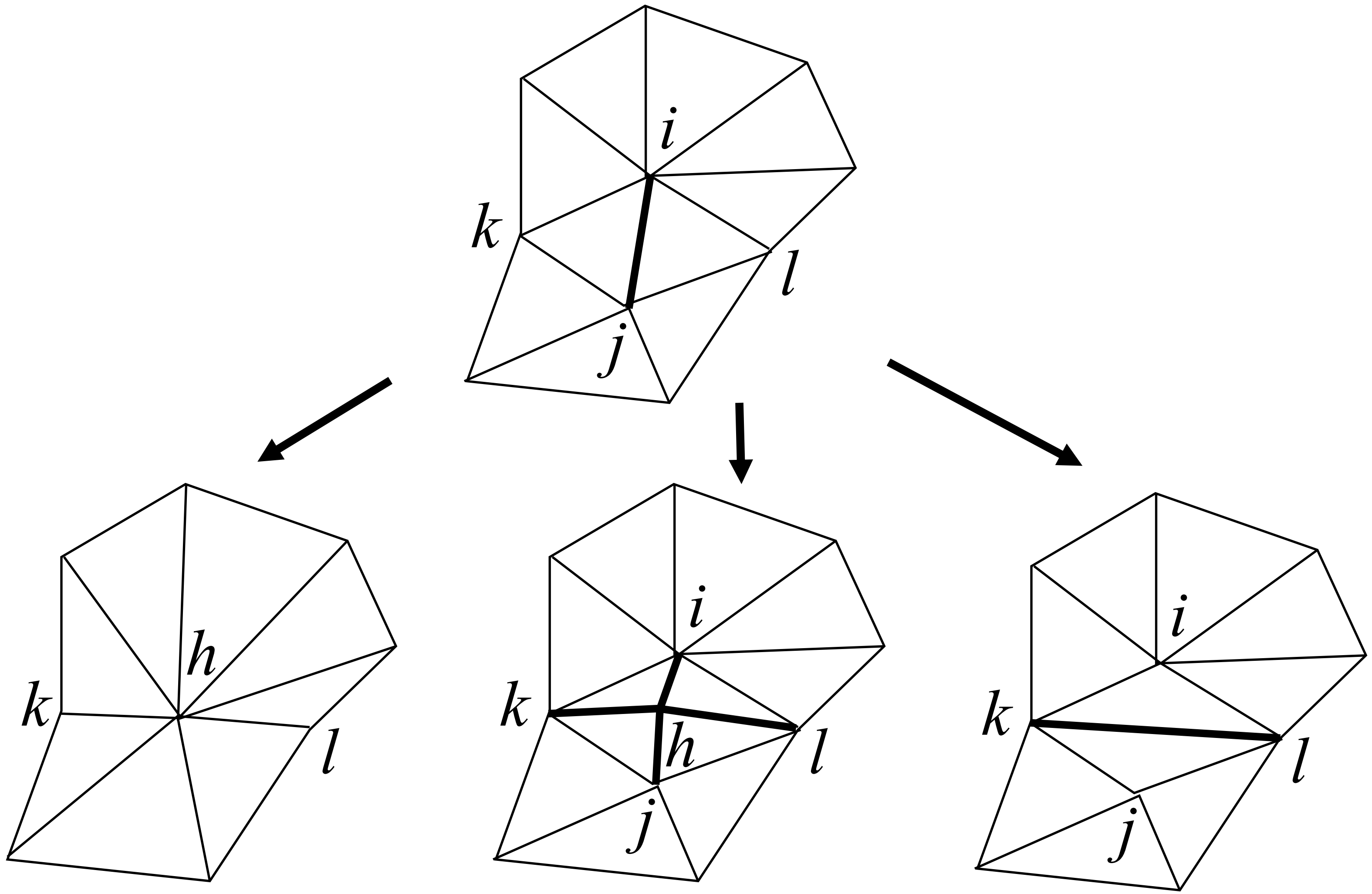
Data Sources Influence Simplification

- Measurement can generate lots of data, and may need simplification
  - Models from laser range finder
  - Iso-surface generation from 3D MRI or CT
  - Terrain from Satellite, Radar, Sonar
- Measurements can also include noise, leading to other problems!
  - Noise can be high frequency details, but can also manifest as topological issues (e.g., high genus) in reconstructed surfaces (more soon…)

Geometry Modification Approaches

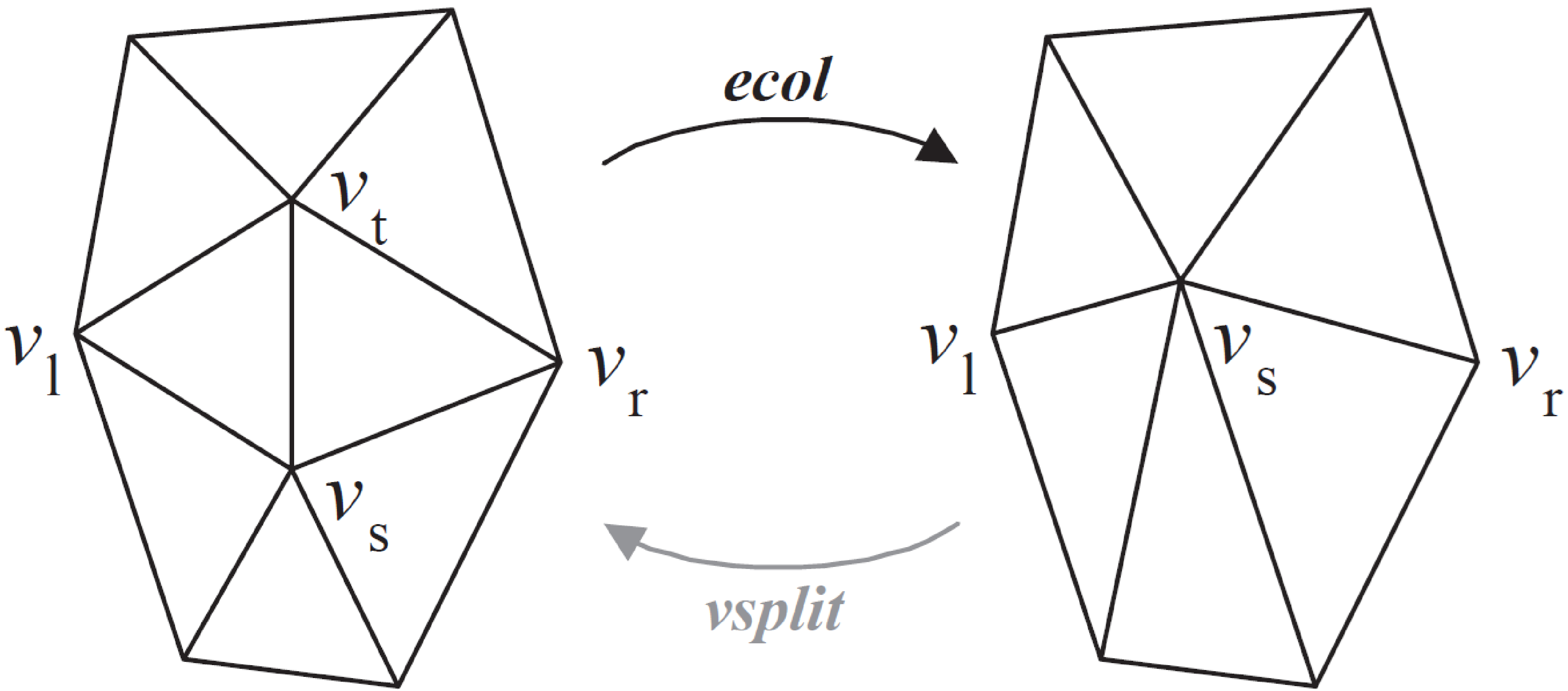
- Geometry refinement
  - If mesh came from adaptive subdivision, there is a natural coarser version
- Geometry resampling
  - Remeshing, or mesh re-tiling
  - Find a set of geometric proxies that fit the data
- Geometry decimation
  - Vertex decimation
    - Remove vertices in planar regions and fill hole with triangles
  - Edge contraction
  - Vertex merging

Local Modification (invertible)



edge collapse | edge split | edge swap

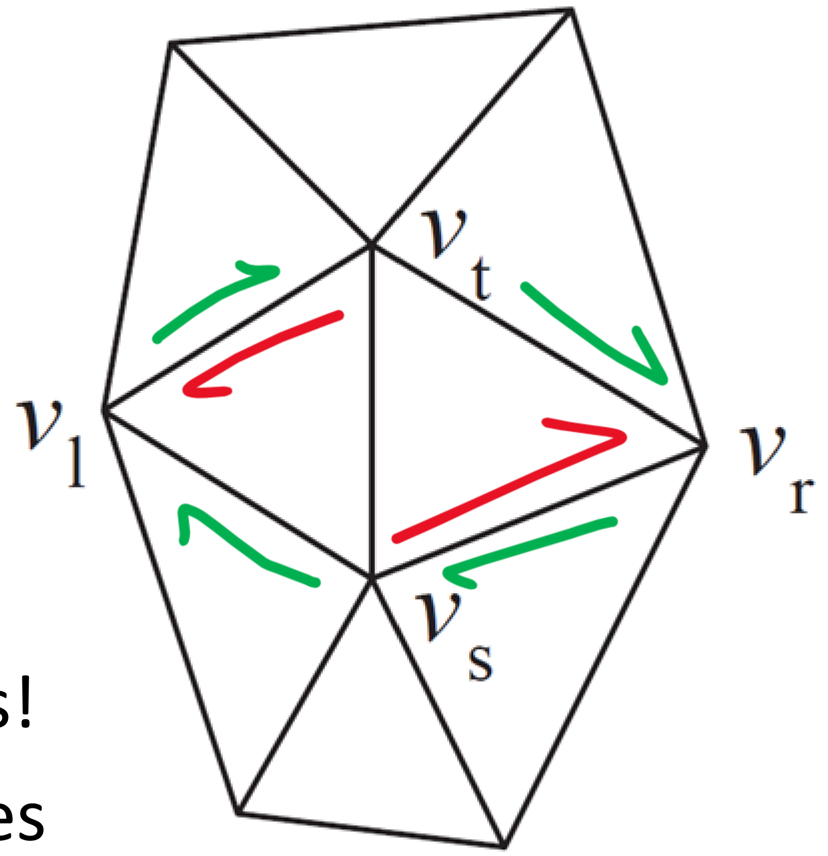
Local Modification (invertible)



How do you do an edge collapse with a half edge data structure?

Edge Collapse

- Get the 4 half edges that need to become twins
- Set them to be each other's twins
- If vertices hold an example half edge, then make sure that v_l and v_r do not point to any removed half edges!
- Create a new vertex and set all the incoming half edges around the collapsed vertices to point to the new vertex
- The two faces and 6 half edges around the collapse are no longer accessible!
- Mesh has 1 fewer vertex, 2 fewer faces, 3 fewer edges
- How do these steps change when there are boundaries?

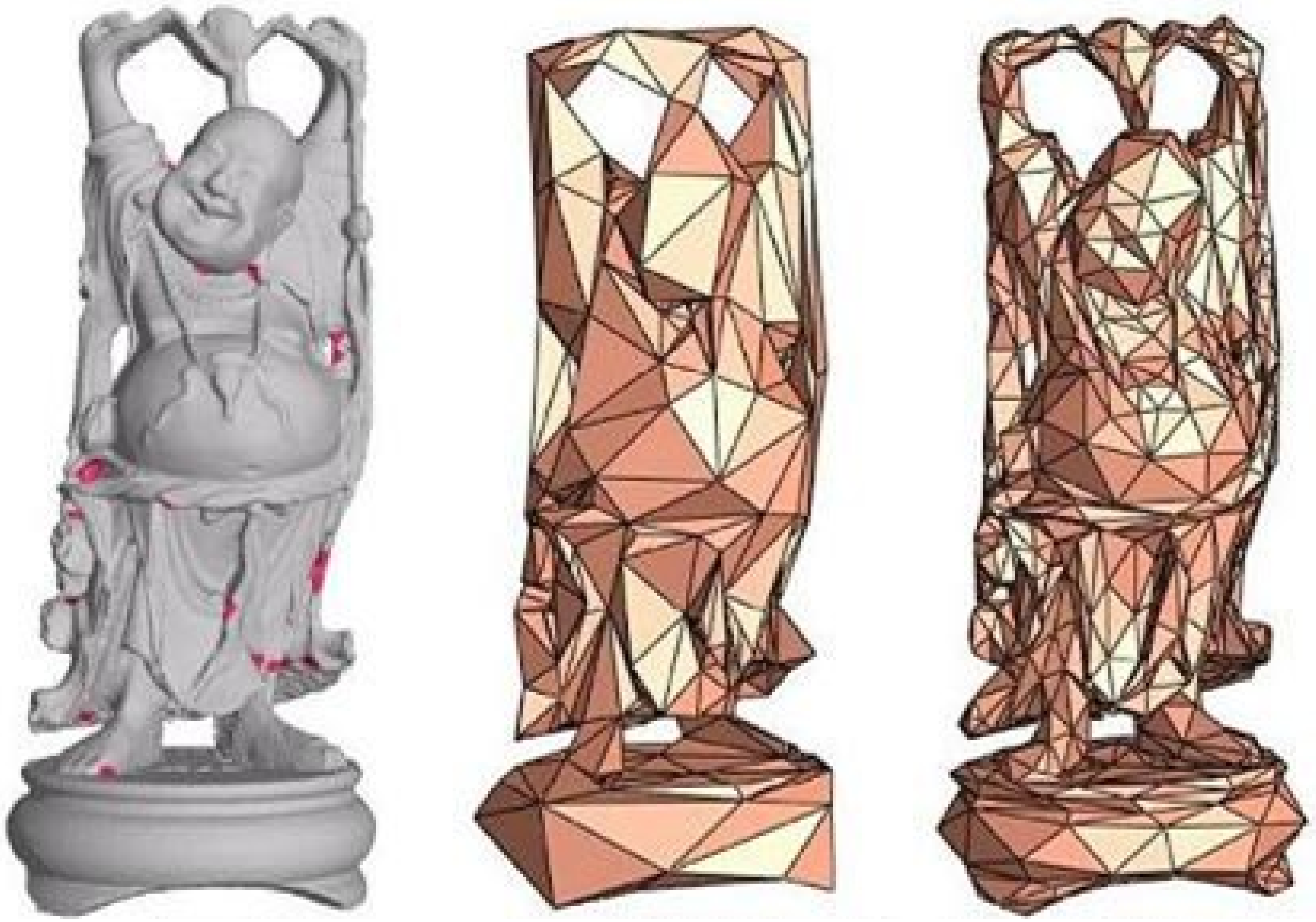


Characteristics

- Speed vs quality?
- Type of Mesh
  - Terrain / Manifold / Polygon soup
- Modifies topology?
- Continuous LOD?
- View-Dependent refinement?
- Simplification of meshes with texture?
- Simplify topology?
  - David head with genus 340?

Fixing Topology

- Preserving topology during simplification is not always a good idea
- Simplifying topology not necessarily a simple task
  - Tricky to find and “repair” all the small handles and tunnels
  - Combining erosion and dilation can be a solution

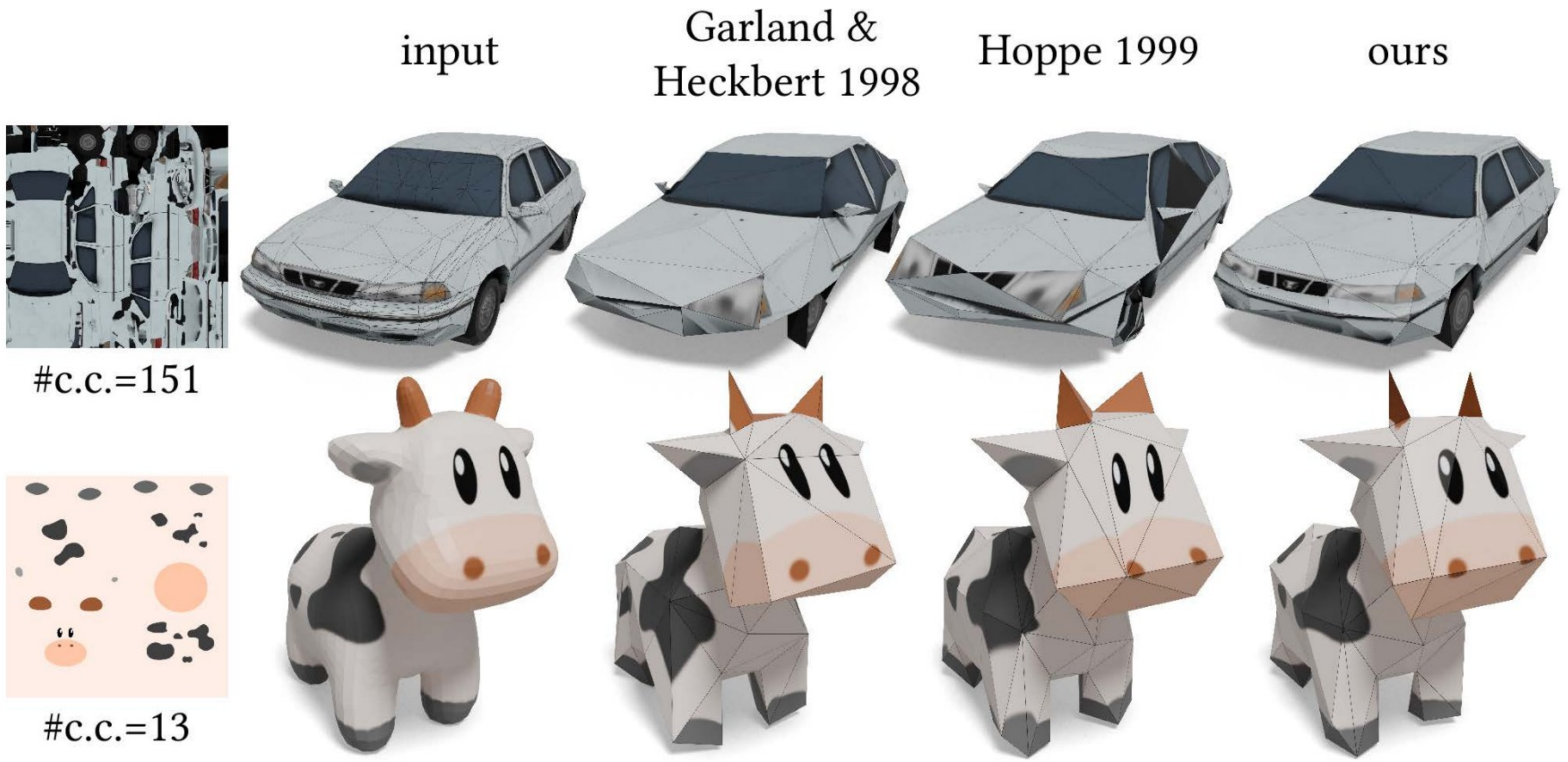


Genus 104

Genus 104
2K triangles

Genus 6
2K triangles
topologically simplified

Simplifying textured meshes in the wild
(SIGGRAPH Asia 2025)

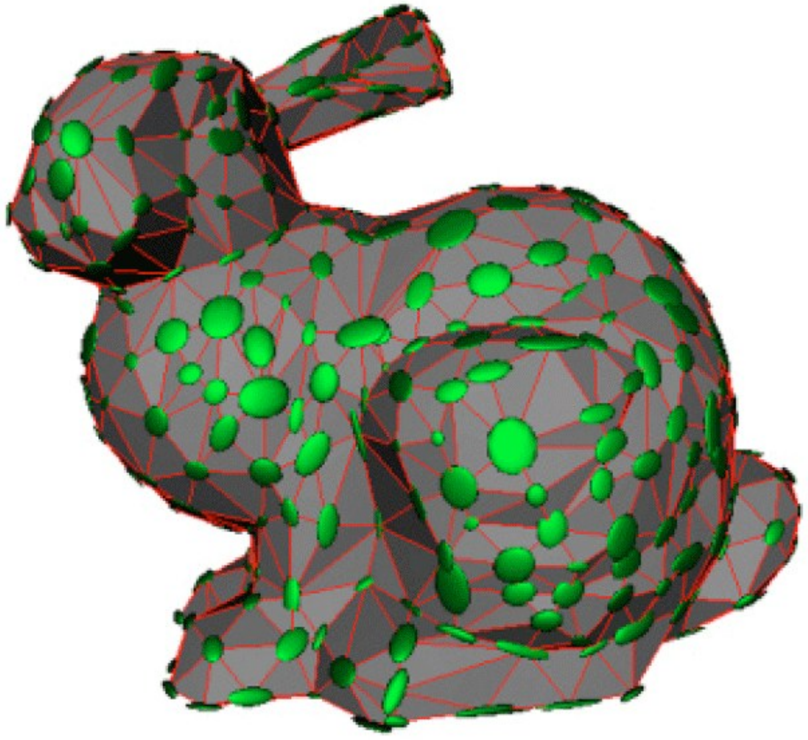


Quadric Mesh Simplificaiton

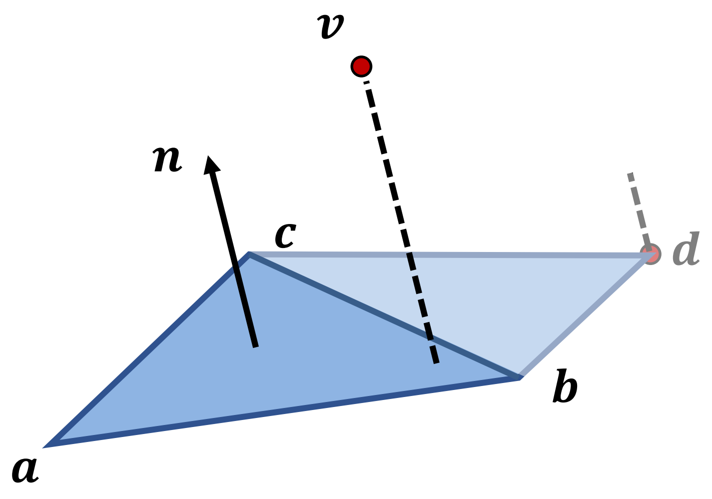
Quadric error metric for mesh simplification

- Let us look more closely at a specific method
- Only use edge collapses
- Choose a good location for collapsed edges
- Always collapse edge with minimal error

SIGGRAPH 1997, Garland, Heckbert
Surface simplification using quadric error metrics



Signed distance to triangle's plane



- Implicit plane equation
$$Ax + By + Cz + D = 0$$
    - Plane normal $\boldsymbol{n} = (A, B, C)^T$
    - With unit length $\boldsymbol{n}$, equation gives the signed distance of point $\boldsymbol{v} = (x, y, z, 1)^T$ from the plane
- Letting $\boldsymbol{p} = (A, B, C, D)^T$, can compute signed distance d using homogenous coordinates
$$d = \boldsymbol{p}^T \boldsymbol{v}$$

$$\boldsymbol{n} = \frac{(\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})}{\|(\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})\|}$$

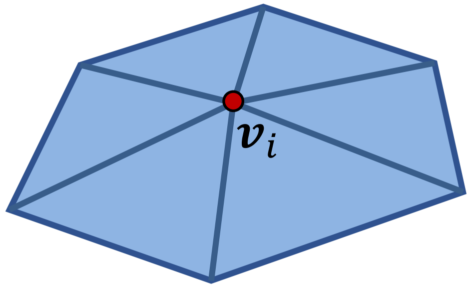
$$D = -\boldsymbol{n}_x \boldsymbol{a}_x - \boldsymbol{n}_y \boldsymbol{a}_y - \boldsymbol{n}_z \boldsymbol{a}_z$$

Squared distance to plane

$$\|\boldsymbol{p}^T\boldsymbol{v}\|^2 = (\boldsymbol{p}^T\boldsymbol{v})\cdot(\boldsymbol{p}^T\boldsymbol{v})$$
$$= \boldsymbol{v}^T\boldsymbol{p}\boldsymbol{p}^T\boldsymbol{v}$$
$$= \boldsymbol{v}^T K_{\boldsymbol{p}}\boldsymbol{v} \quad \text{where } K_{\boldsymbol{p}} = \boldsymbol{p}\boldsymbol{p}^T$$

$$K_{\boldsymbol{p}} = \begin{pmatrix} A^2 & \mathrm{AB} & \mathrm{AC} & AD \\ \mathrm{BA} & B^2 & \mathrm{BC} & BD \\ \mathrm{CA} & \mathrm{CB} & C^2 & CD \\ DA & DB & DC & D^2 \end{pmatrix}$$

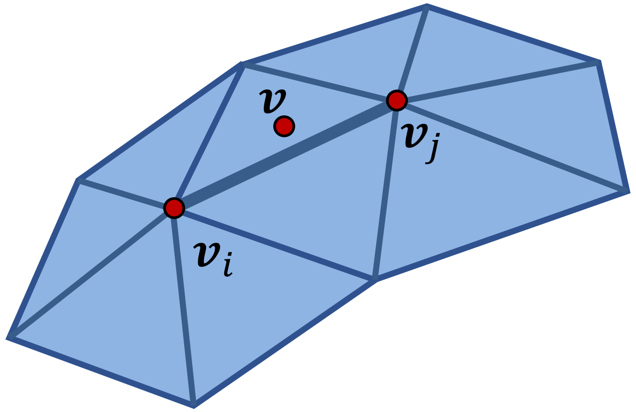
- Now, consider the sum of squared distances to the planes defined by the triangles adjacent to a vertex



$$\sum_{\boldsymbol{p}\in\mathrm{adj}(\boldsymbol{v}_i)} \boldsymbol{v}^T K_{\boldsymbol{p}}\boldsymbol{v} = \boldsymbol{v}^T Q_i \boldsymbol{v}$$

Where $Q_i = \sum_{\boldsymbol{p}\in\mathrm{adj}(\boldsymbol{v}_i)} K_{\boldsymbol{p}}$

Quadric Error Metric



- Each face defines a plane
- Each vertex lies in the planes of all its adjacent faces
- Consider moving a vertex to a new position $\boldsymbol{v}$
  - How well does vertex $\boldsymbol{v}$ lie in a set of planes?
  - Sum of squared distances to adjacent planes, $\min_{\boldsymbol{v}} \boldsymbol{v}^T Q_i \boldsymbol{v}$
- To collapse an edge, we ask this question for both vertices on either side of the edge, specifically, minimizing for best choice of position v for both ends simultaneously

$$e = \min_{v} v^T (Q_i + Q_j) v$$

Solving $e = \min_{\boldsymbol{v}} v^T(Q_i + Q_j)v$

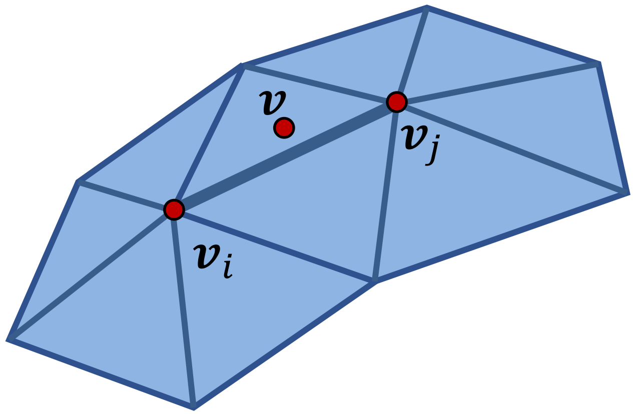
- Take care in minimizing the quadratic function because $\boldsymbol{v}$ is in homogeneous coordinates
- Can rewrite as (nonhomogeneous)

$$\min_{\boldsymbol{v}} \boldsymbol{v}^T A\boldsymbol{v} + 2\boldsymbol{b}^T\boldsymbol{v} + c$$

- Here, A, $\boldsymbol{b}$, and c come from

$$(Q_i + Q_j) = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} & b_x \\ A_{yx} & A_{yy} & A_{yz} & b_y \\ A_{zx} & A_{zy} & A_{zz} & b_z \\ b_x & b_y & b_z & c \end{pmatrix}$$

Take derivative and solve for the zero, can use https://www.matrixcalculus.org/, or sympy, or by hand, or…



Matrix A is symmetric, minimum occurs where gradient is zero

$$2A\boldsymbol{v} + 2\boldsymbol{b} = 0$$

That is, solve for v in

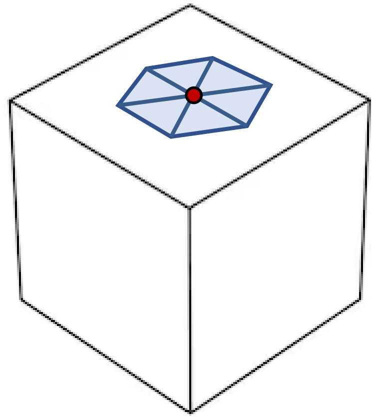
$$A\boldsymbol{v} = -\boldsymbol{b}$$

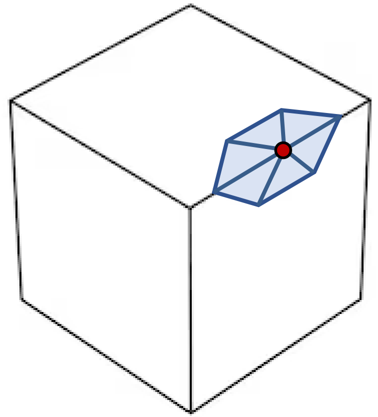
What can go wrong???

When is A not invertible?

Rank of a matrix is the dimension of the vector space generated (spanned) by its columns. Equal to the number of non-zero eigenvalues. The matrix is invertible only if it has full rank (i.e., non-zero determinant)

- Geometrically, what explains rank deficiency of A?
  - Consider the rank of $K_{\boldsymbol{p}} = \boldsymbol{p}\boldsymbol{p}^T$
    - All vectors orthogonal to $\boldsymbol{p}$ are in the null space, i.e., the matrix has rank 1
  - All faces adjacent to vertex i lie in the same plane?
    - Then Q_i will have rank 1
  - All faces adjacent to vertex i lie in 2 planes?
    - Then Q_i will have rank 2
  - Same is true for $Q_i + Q_j$ when considering face planes adjacent to both vertices





What to do when A not invertible?

- Solve a different problem?
- A common trick is to regularize the problem

$$\min_{\boldsymbol{v}} \boldsymbol{v}^T A \boldsymbol{v} + 2\boldsymbol{b}^T \boldsymbol{v} + c + \gamma\, \boldsymbol{v}^T I \boldsymbol{v}$$

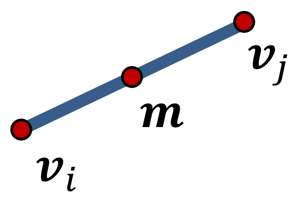
  - With small γ, this will favour a solution with small $\boldsymbol{v}$ when there is no unique local min
  - Leads to the problem $(A + \gamma I)\boldsymbol{v} = -\boldsymbol{b}$ which will have a solution

- Must select some reasonable (small) γ that has minimal influence when eigenvalues of A are large (what are the eigenvalues of A?)

- Geometrically, minimum norm $\boldsymbol{v}$ is ***not what we want***!
  - What if the local mesh we are simplifying is far from the origin?

Better Regularization

$$Q_{reg} = \begin{pmatrix} 1 & 0 & 0 & -\boldsymbol{m}_x \\ 0 & 1 & 0 & -\boldsymbol{m}_y \\ 0 & 0 & 1 & -\boldsymbol{m}_z \\ -\boldsymbol{m}_x & -\boldsymbol{m}_y & -\boldsymbol{m}_z & \boldsymbol{m}^T\boldsymbol{m} \end{pmatrix}$$



- If all planes of triangles adjacent to the two vertices of an edge are coplanar, then any point in the plane has zero error
- A good location for this case is the edge midpoint, $\boldsymbol{m}$
- With $\boldsymbol{m} = \frac{1}{2}(\boldsymbol{v}_i + \boldsymbol{v}_j)$, distance squared between $\boldsymbol{v}$ and $\boldsymbol{m}$ is

$$(\boldsymbol{v} - \boldsymbol{m})^{\boldsymbol{T}}(\boldsymbol{v} - \boldsymbol{m}) = \boldsymbol{v}^{\boldsymbol{T}} I \boldsymbol{v} - 2\boldsymbol{m}^T\boldsymbol{v} + \boldsymbol{m}^T\boldsymbol{m}$$

- Use this to regularize the problem with small factor γ
- Instead, solve $e = \min_{\boldsymbol{v}} \boldsymbol{v}^T\left(Q_i + Q_j + \gamma Q_{reg}\right)\boldsymbol{v}$

Alternative to regularization

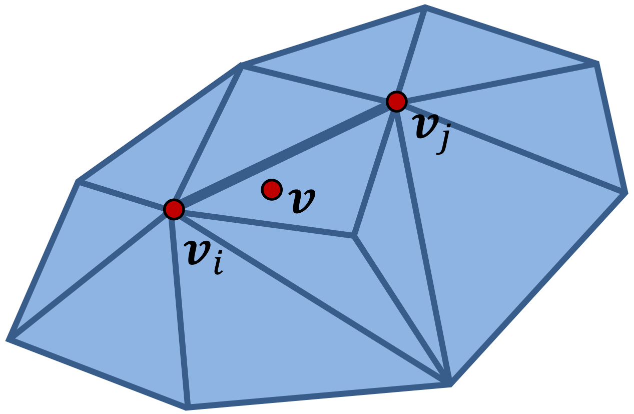
- If $|\det(A)|$ is less than some threshold, e.g., 1e-6 then simply choose the midpoint as the optimal $\boldsymbol{v}^*$ position
  - Compute the error $e = {\boldsymbol{v}^*}^T A \boldsymbol{v}^* + 2\boldsymbol{b}^T \boldsymbol{v}^* + c$ and continue as normal
- This is easy, but might not be as "nice" as regularization
  - For $\det(A)$ just at the threshold, the optimal solution from solving almost rank deficient A might be bad
    - Could this push a vertex far from the model? Perhaps!
  - Instead, our better regularization will ***smoothly*** move the solution toward the midpoint as the eigenvalues of A go to zero (for example, as the adjacent faces become co-planar)

Quadratic Error Metric Implementation Issues

- We have now discussed the following issues
  - Regularization of the minimization problem
    - Use distance squared to point halfway along an edge
  - Computation of the error for each edge
    - Solving the minimum of the quadratic equation
    - That is, take the derivative and solve for the zero
- How to simplify? Must collapse many edges (always choose the best)
  - Use a data structure to keep track of which edge should be collapsed next
    - For instance, sorted list, heap, priority queue
    - Each collapse requires adjacent edges to be revisited!
    - Remove, recompute error, and re-insert

Edge Collapse Problems

- Does it preserve the genus?
- Does it preserve the manifold?
- Does it create self-intersections in geometry?
- Use heuristics to avoid creating non-manifold topology
  - Number of common adjacent vertices to collapsing edge should be 2
  - If {i} and {j} are both ***boundary*** vertices, only collapse if {i,j} is a boundary edge



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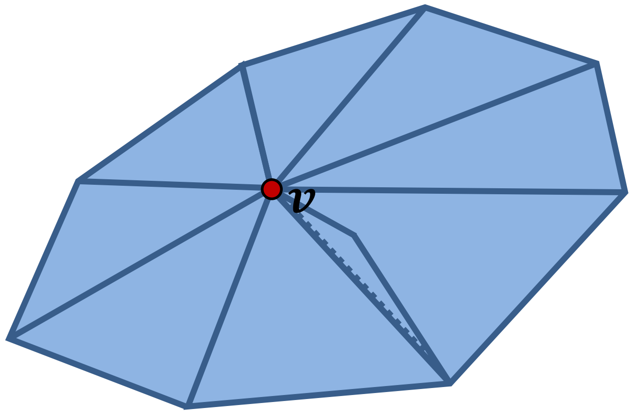

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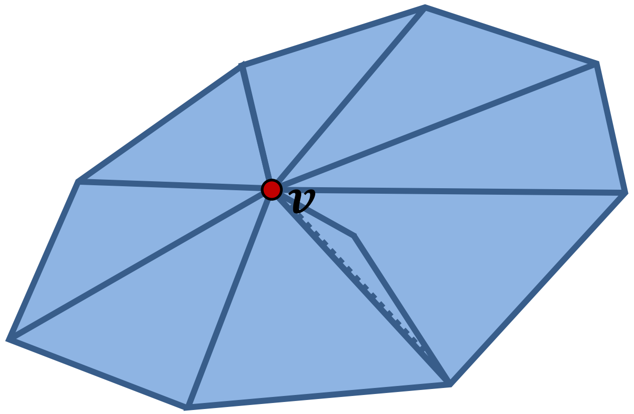
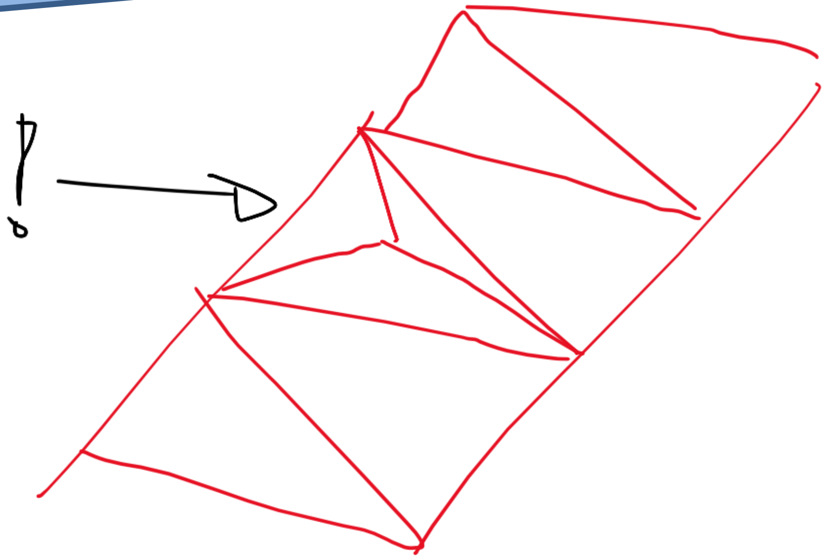




Might also worry if these problems happen together!

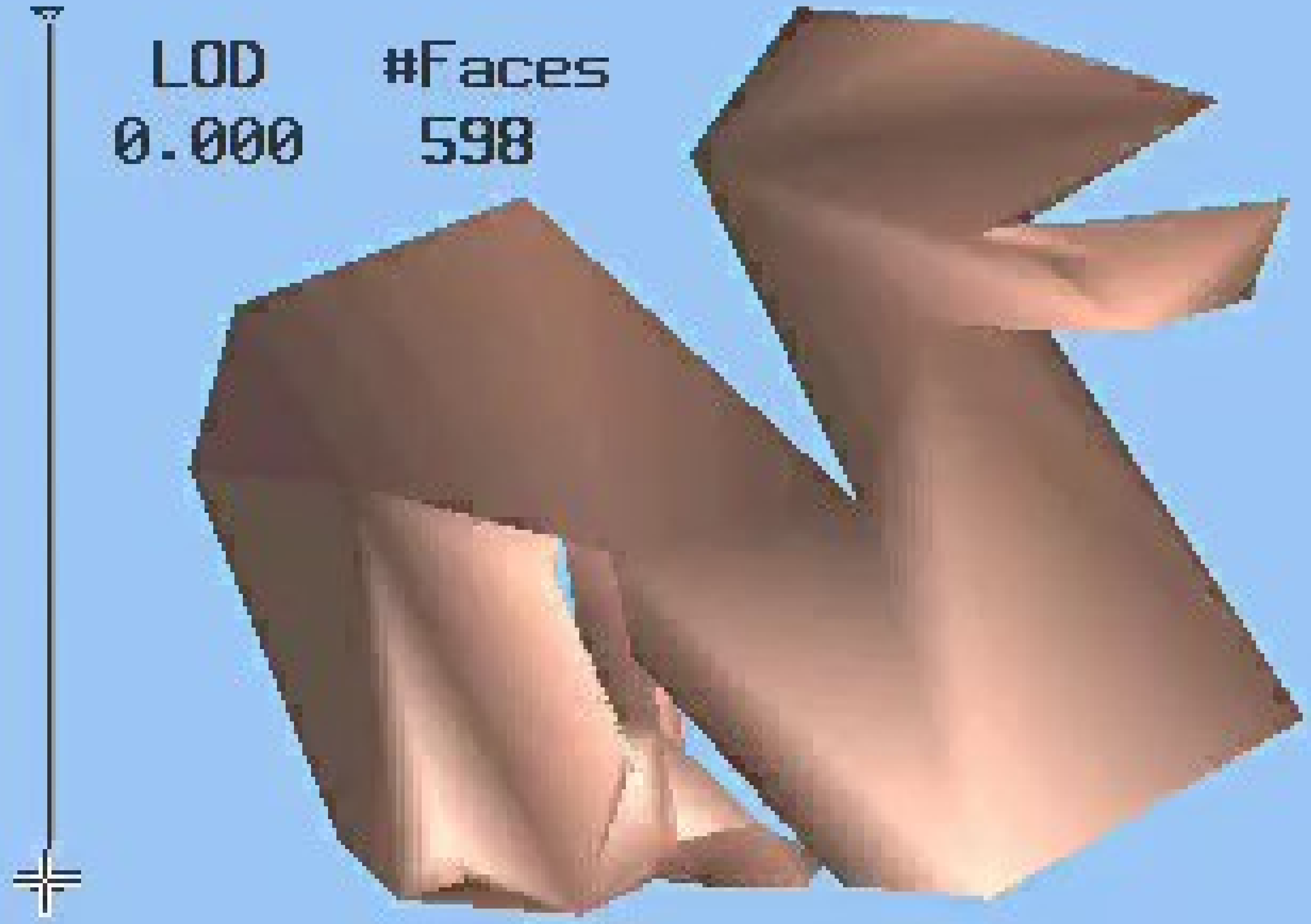
Quadratic Error Metric - Preparation

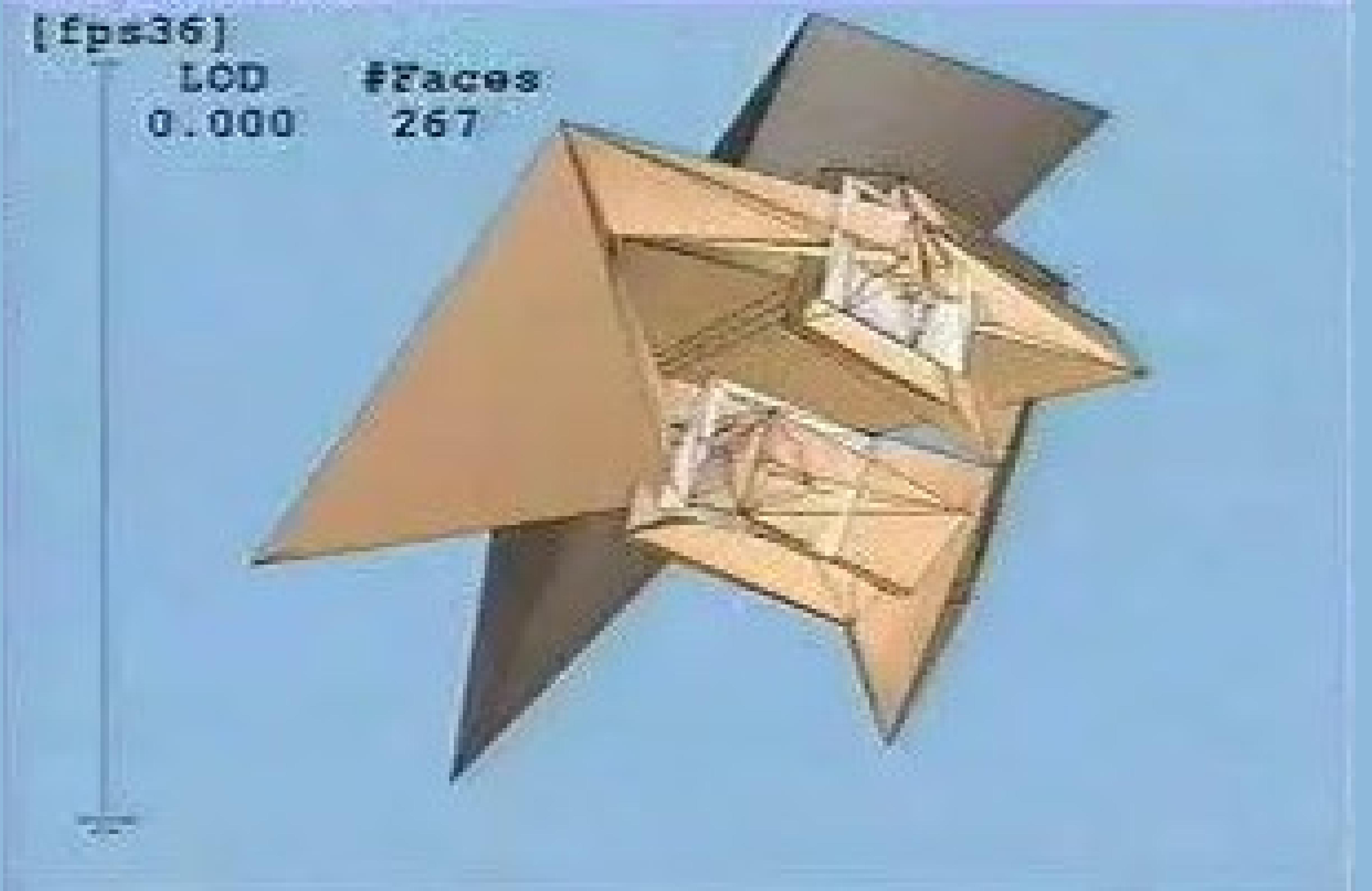
- For each face, compute planes $\boldsymbol{p}$ and quadratic function coefficient $\boldsymbol{K}$
- Compute quadratic function coefficient for each vertex, $\boldsymbol{Q}$
- For each edge (i, j), using $\boldsymbol{Q}_i + \boldsymbol{Q}_j$, solve optimal vertex position and error, possibly including a regularization term in the solve
- Insert all edge errors into a sorted list

Quadratic Error Metric - Simplification

- Pop min cost edge until finding an edge for which collapse does not cause problems (avoid bad topology, perhaps check geometry)
- Collapse the edge to optimal vertex position, set quadratic function coefficient of this vertex as $\boldsymbol{Q}_i + \boldsymbol{Q}_j$
- Ensure all pre-collapse edges adjacent to the collapsed vertices are removed from the sorted list
- Re-compute edge collapse error for adjacent edges to new vertex
- Re-insert these edges into the sorted list
- Repeat until desired level of detail is reached.


LOD #Faces
0.000 48


LOD
#Faces
0.000
598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LOD #Faces
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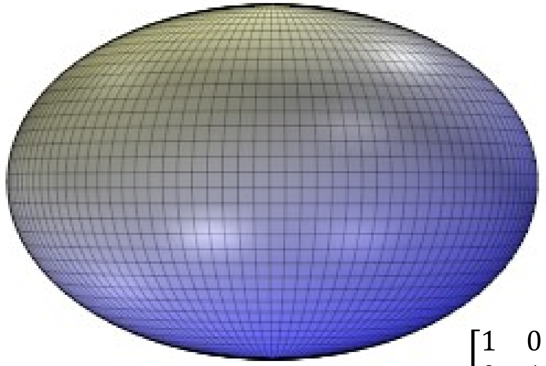
More on Quadrics

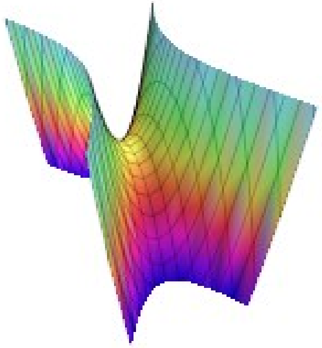
What are Quadrics?

http://en.wikipedia.org/wiki/Quadric

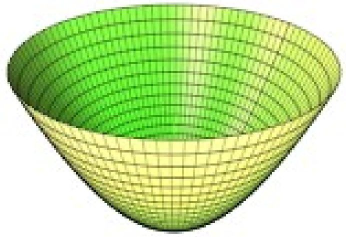


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

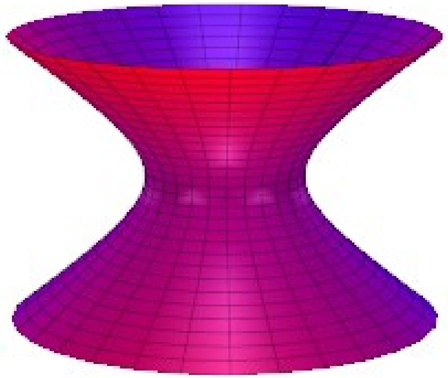
Ellipsoid



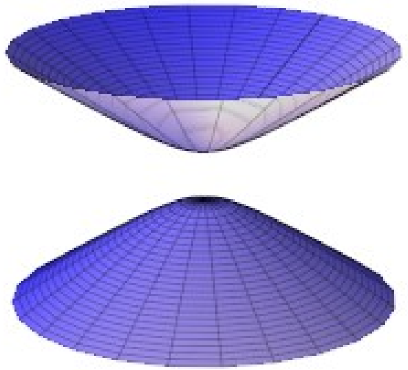
Hyperbolic paraboloid



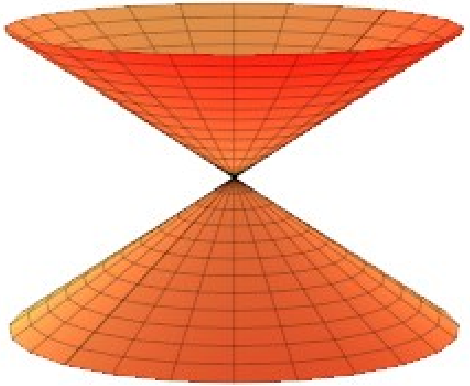
Elliptic paraboloid



Hyperboloid of one sheet



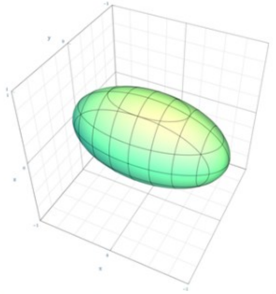
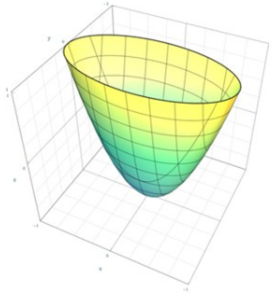
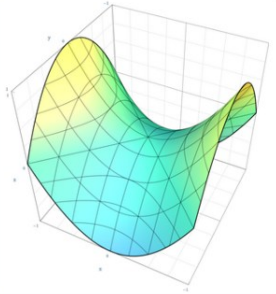
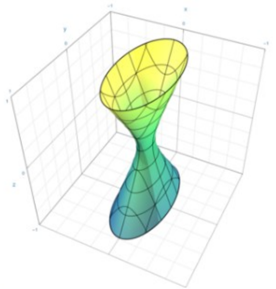
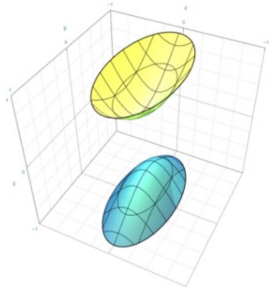
Hyperboloid of two sheets

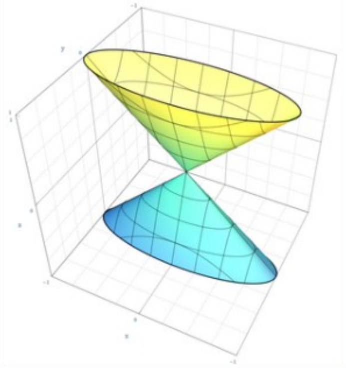
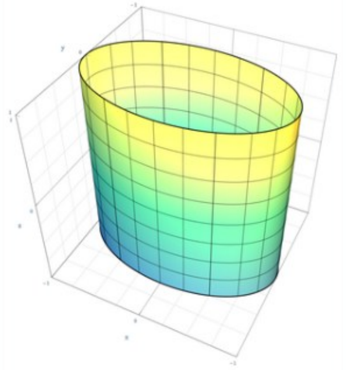
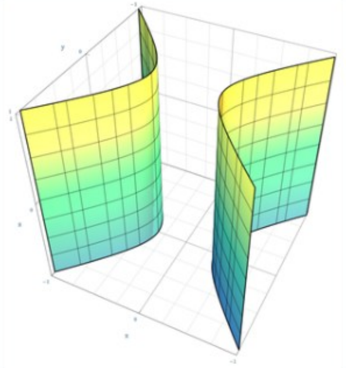
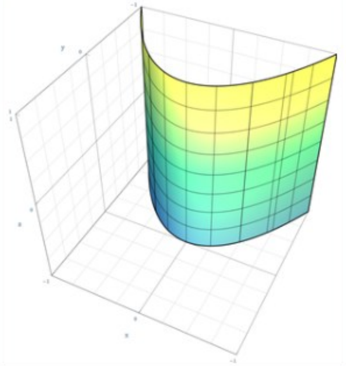


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Cone

More examples

Non-degenerate real quadric surfaces		
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$	
Hyperbolic paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$	
Hyperboloid of one sheet or Hyperbolic hyperboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of two sheets or Elliptic hyperboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	

Degenerate real quadric surfaces		
Elliptic cone or Conical quadric	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	
Elliptic cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	
Hyperbolic cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
Parabolic cylinder	$x^2 + 2ay = 0$	

Quadrics are quadratic implicit surfaces

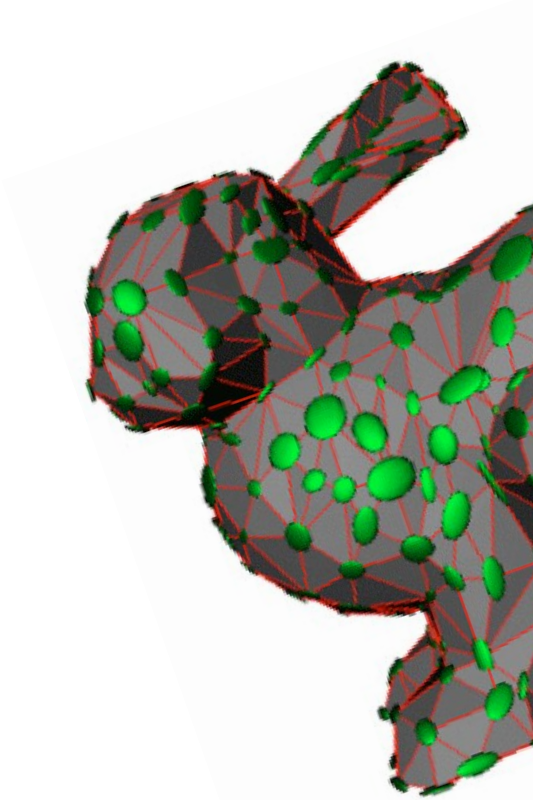
- In homogenous coordinates where Q is a symmetric 4-by-4 matrix, we can write the function
$$v^T Q v = 0$$
- In non-homogenous coordinates, we can write as
$$\{\, \boldsymbol{v} \mid \boldsymbol{v}^T A \boldsymbol{v} + 2\boldsymbol{b}^T \boldsymbol{v} + c = 0 \,\}$$
- Here, A, $\boldsymbol{b}$, and c come from
$$Q = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} & b_x \\ A_{yx} & A_{yy} & A_{yz} & b_y \\ A_{zx} & A_{zy} & A_{zz} & b_z \\ b_x & b_y & b_z & c \end{pmatrix}$$

How to interpret the quadric error metric?

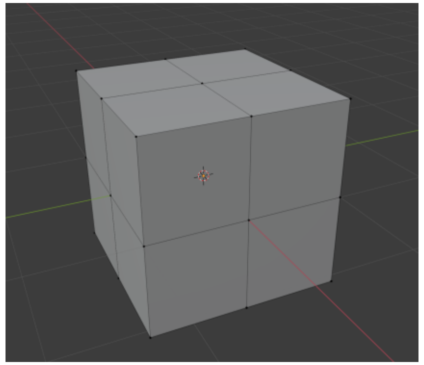
- At a given vertex and at a given level of simplification, have Q
- For a fixed amount of error e, consider the implicit surface

$$v^T Q v - e = 0$$

  - If $e < \min_v v^T Q v$, then there is no surface
  - If $e = \min_v v^T Q v$, then the surface is a single point
  - If $e > \min_v v^T Q v$, then the surface is a quadric



Questions



- What quadric shape do you get at the corner of a cube?
    - What is Q_i in this case?
    - What value of errors will give you that shape?
- What quadric shape do you get for Q_i at vertex along the edge of a subdivided cube? What is Q_i in this case?
- What quadric shape do you get for a vertex in a flat region (middle of cube face)? What is Q_i in this case?
- Why do we use regularization?
- What does regularization do to these shapes?

More information

- Mesh Notes PDF in MyCourses resources
- Surface simplification using quadric error metrics
  - Garland and Heckbert, 1997
    http://dl.acm.org/citation.cfm?id=258849
- Progressive meshes
  - Hoppe, 1996
    http://research.microsoft.com/en-us/um/people/hoppe/proj/pm/