数值分析第一次作业

韦俊林 (201928016029023)

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1 计算题

1.1 求矩阵 $A = \begin{pmatrix} 3 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 1 & -1 \\ & & -1 & 3 \end{pmatrix}$ 的 Cholesky 分解。

解： Cholesky 分解是将 A 分解为 $L \times L^T$ 的形式。由题知 A 为 4×4 的矩阵。因此，

$$L = \begin{pmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \\ L_{31} & L_{32} & L_{33} & \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \quad L^T = \begin{pmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ & L_{22} & L_{32} & L_{42} \\ & & L_{33} & L_{43} \\ & & & L_{44} \end{pmatrix}$$

由此得，

1. $a_{11} = L_{11}^2 = 3 \quad \Rightarrow \quad L_{11} = \sqrt{3}$
2. $a_{21} = L_{21}L_{11} = -1 \quad \Rightarrow \quad L_{21} = \frac{a_{21}}{L_{11}} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
3. $a_{31} = L_{31}L_{11} = 0 \quad \Rightarrow \quad L_{31} = \frac{a_{31}}{L_{11}} = \frac{0}{\sqrt{3}} = 0$
4. $a_{41} = L_{41}L_{11} = 4 \quad \Rightarrow \quad L_{41} = \frac{a_{41}}{L_{11}} = \frac{0}{\sqrt{3}} = 0$
5. $a_{22} = L_{21}^2 + L_{22}^2 = 2 \quad \Rightarrow \quad L_{22} = \sqrt{a_{22} - L_{21}^2} = \sqrt{2 - (-\frac{\sqrt{3}}{3})^2} = \frac{\sqrt{15}}{3}$
6. $a_{32} = L_{31}L_{21} + L_{32}L_{22} = -1 \quad \Rightarrow \quad L_{32} = \frac{a_{32} - L_{31}L_{21}}{L_{22}} = \frac{-1-0\times(-\frac{\sqrt{3}}{3})}{\frac{\sqrt{15}}{3}} = -\frac{\sqrt{15}}{5}$
7. $a_{42} = L_{41}L_{21} + L_{42}L_{22} = 0 \quad \Rightarrow \quad L_{42} = \frac{a_{42} - L_{41}L_{21}}{L_{22}} = \frac{0-0\times(-\frac{\sqrt{3}}{3})}{\frac{\sqrt{15}}{3}} = 0$
8. $a_{33} = L_{31}^2 + L_{32}^2 + L_{33}^2 = 1 \quad \Rightarrow \quad L_{33} = \sqrt{a_{33} - L_{31}^2 - L_{32}^2} = \sqrt{1 - 0^2 - (-\frac{\sqrt{15}}{5})^2} = \frac{\sqrt{10}}{5}$
9. $a_{43} = L_{41}L_{31} + L_{42}L_{32} + L_{43}L_{33} = -1 \quad \Rightarrow \quad L_{43} = \frac{a_{43} - L_{41}L_{31} - L_{42}L_{32}}{L_{33}} = \frac{-1-0\times0-0\times(-\frac{\sqrt{15}}{5})}{\frac{\sqrt{10}}{5}} = -\frac{\sqrt{10}}{2}$
10. $a_{44} = L_{41}^2 + L_{42}^2 + L_{43}^2 + L_{44}^2 = 3 \Rightarrow L_{44} = \sqrt{a_{44} - L_{41}^2 - L_{42}^2 - L_{43}^2} = \sqrt{3 - 0^2 - 0^2 - (-\frac{\sqrt{10}}{2})^2} = \frac{\sqrt{2}}{2}$

综上得，

$$A = L \times L^T = \begin{pmatrix} \sqrt{3} & & & \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{15}}{3} & & \\ 0 & -\frac{\sqrt{15}}{5} & \frac{\sqrt{10}}{5} & \\ 0 & 0 & -\frac{\sqrt{10}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{3} & 0 & 0 \\ & \frac{\sqrt{15}}{3} & -\frac{\sqrt{15}}{5} & 0 \\ & & \frac{\sqrt{10}}{5} & -\frac{\sqrt{10}}{2} \\ & & & \frac{\sqrt{2}}{2} \end{pmatrix}$$

1.2 研究 Jacobi 迭代和 Gauss-Seidel 迭代应用于以解以 $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ -1 & 2 & -2 \end{pmatrix}$ 为系数矩阵的线性方程组时的收敛性。

解：由题得，A 为 3×3 的矩阵，

Jacobi： Jacobi 迭代是将 A 分解为 $D - L - U$ 的形式，其中 D, L, U 分别为：

$$D = \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & & \\ & 2 & \\ & & -2 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & & \\ -a_{21} & 0 & \\ -a_{31} & -a_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 1 & -2 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & -a_{12} & -a_{13} \\ & 0 & -a_{23} \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 0 \end{pmatrix}$$

因此，$D^{-1} = \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix}$ 令 $J = D^{-1}(L+U)$ 得：

$$J = \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & -\frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 0 & & \\ -1 & 0 & \\ 1 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{pmatrix}$$

令

$$det(\lambda E - J) = \begin{vmatrix} \lambda & -\frac{1}{2} & 0 \\ \frac{1}{2} & \lambda & -1 \\ \frac{1}{2} & -1 & \lambda \end{vmatrix} = (\lambda - \frac{1}{2})^2(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \frac{1}{2}, \quad \lambda_3 = -1$$

其中 λ_i $(i = 1, 2, 3)$, 为矩阵 J 的特征值，所以矩阵 J 的谱半径有 $\rho(J) = \max\limits_{i=1,2,3} |\lambda_i| = 1$ 。
由定理，用 Jacobi 迭代法解以该矩阵为系数的线性方程组不收敛。

Gauss-Seidel： 以矩阵 A 为系数的线性方程组的 Gauss-Seidel 迭代法的迭代矩阵为，

$$G = (D-L)^{-1}U = \left(\begin{pmatrix} 2 & & \\ & 2 & \\ & & -2 \end{pmatrix} - \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 1 & -2 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 1 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

令

$$det(\lambda E - G) = \begin{vmatrix} \lambda & -\frac{1}{2} & 0 \\ 0 & \lambda + \frac{1}{4} & -1 \\ 0 & \frac{1}{2} & \lambda - 1 \end{vmatrix} = \lambda(4\lambda^2 - 3\lambda + 1) = 0$$

解得：$\lambda_1 = 0, \lambda_2 = \frac{6+\sqrt{7}i}{8}, \lambda_3 = \frac{6-\sqrt{7}i}{8}$，所以 $\rho(G) = \frac{\sqrt{43}}{8} < 1$。
由定理，用 Gauss-Seidel 迭代法解以矩阵 A 为系数的线性方程组收敛。

2 计算机编程

2.1 请使用 MATLAB 编写程序，用 SOR 迭代解方程组。

$$Ax = b\,, \quad A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

迭代初值选为 $x^{(0)} = (0,0,0)^T$，ω 分别取 $1, 1.03$ 和 1.1。用 Gauss 消去求方程组真解 x^*，以此判断 SOR 迭代误差，当 $\|x^{(k)} - x^*\|_\infty \leqslant 5 \times 10^{-6}$ 时停止迭代，列表给出每步迭代数据 $x^{(k)}$，并写出用每一个 ω 时达到题给精度所需的迭代次数。

实验报告：

1. matlab 实现高斯消去法，图 1。

```
function x=MyGauss(A,b)
    n=length(A);
    m=numel(A)/n;
    x=zeros(n,1);
    for i = 1:n
        if (A(i,i)==0)
            disp('wrong input!');
            return;
        end
        for j = i+1:m
            r=A(j,i)/A(i,i);
            A(j,i:n)=A(j,i:n)-r*A(i,i:n);
            b(j)=b(j)-r*b(i);
        end
    end
    x(n)=b(n)/A(n,n);
    for i=n-1:-1:1
        x(i)=(b(i)-sum(A(i,i+1:n)*x(i+1:n)))/A(i,i);
    end
    disp('Gauss elimination:');
    disp(x.');
```

图 1: gauss Elimination

2. matlab 实现基于公式，

$$x_i^{(k+1)} = x_i^{(k)} + \omega(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)})/a_{ii}, \quad i = 1, 2, ..., n$$

的 SOR 迭代法，图 2。

```
function x_k1=MySOR(A,b,w,x_gauss,error_limit)
    n=length(A);
    m=numel(A)/n;
    x_k=zeros(n,1);
    x_k1=zeros(n,1);
    disp('Successive Over Relaxation:');
    disp(['w= ',num2str(w)]);
%     for iter=1:1:inf
    for iter=1:1:150
        for i=1:1:n
            sum_1=0;
            sum_2=0;
            for j=1:1:n
                if((j>=1) && (j<=(i-1)))
                    sum_1=sum_1+A(i,j)*x_k1(j);
                end
                if((j>=i) && (j<=n))
                    sum_2=sum_2+A(i,j)*x_k(j);
                end
            end
            x_k1(i)=x_k(i)+w*(b(i)-sum_1-sum_2)/A(i,i);
        end
        error_value=x_k1-x_gauss;
        error_norm=norm(error_value,inf);
        disp(['iterations=',num2str(iter)]);
        disp(x_k1.');
        disp('-------------------');
        if (error_norm <= error_limit)
            disp(['iterations= ',num2str(iter),'    error_norm= ',num2str(error_norm)]);
            disp('iteration done.');
            return
        end
        x_k=x_k1;
    end
```

图 2: Successive Over Relaxation

3. 松弛因子 $\omega = 1$，实验结果，图 3。

4. 松弛因子 $\omega = 1.03$，实验结果，图 4。

5. 松弛因子 $\omega = 1.1$，实验结果，图 5。

6. 由上述知，要达到题给精度，ω 分别取 $1, 1.03, 1.1$ 时所需的迭代次数分别为 $6, 5, 6$。

```
Gauss elimination:
   0.500000000000000   1.000000000000000  -0.500000000000000

Successive Over Relaxation:
w= 1
iterations=1
   0.250000000000000   1.062500000000000  -0.484375000000000


--------------------
iterations=2
   0.515625000000000   1.007812500000000  -0.498046875000000


--------------------
iterations=3
   0.501953125000000   1.000976562500000  -0.499755859375000


--------------------
iterations=4
   0.500244140625000   1.000122070312500  -0.499969482421875


--------------------
iterations=5
   0.500030517578125   1.000015258789063  -0.499996185302734


--------------------
iterations=6
   0.500003814697266   1.000001907348633  -0.499999523162842


iterations= 6   error_norm= 3.8147e-06
iteration done.
```

图 3: $\omega = 1$

```
Gauss elimination:
   0.500000000000000   1.000000000000000  -0.500000000000000

Successive Over Relaxation:
w= 1.03
iterations=1
   0.257500000000000   1.096306250000000  -0.490201140625000


--------------------
iterations=2
   0.532073859375000   1.007893037578125  -0.498261508604883


--------------------
iterations=3
   0.501070241395117   1.000486457566142  -0.499926891918572


--------------------
iterations=4
   0.500093155581428   1.000028219166201  -0.499994926807146


--------------------
iterations=5
   0.500004471767854   1.000001611252396  -0.499999737298294


--------------------
iterations= 5   error_norm= 4.4718e-06
iteration done.
>>
```

图 4: $\omega = 1.03$

```
Gauss elimination:
    0. 500000000000000    1. 000000000000000   -0. 500000000000000

Successive Over Relaxation:
w= 1. 1
iterations=1
    0. 275000000000000    1. 175625000000000   -0. 501703125000000


--------------------
iterations=2
    0. 570796875000000    1. 001438281250000   -0. 499434160156250


--------------------
iterations=3
    0. 493315839843750    0. 998173633789063   -0. 500558834692383


--------------------
iterations=4
    0. 500166165307617    1. 000074652540283   -0. 499923587082184


--------------------
iterations=5
    0. 500003912917816    1. 000014624350771   -0. 500003619595320


--------------------
iterations=6
    0. 500003630404680    0. 999998540537497   -0. 500000039392656


--------------------
iterations= 6    error_norm= 3. 6304e-06
iteration done.
```

图 5: $\omega = 1.1$