Linear Programming

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1 Linear-inequality Feasibility

1.1 Problem Description

Given a set of m linear inequalities on n variables $x_1, x_2, ..., x_n$, the **linear-inequality feasi-bility problem** asks if there is a setting of the variables that simultaneously satisfies each of the inequalities.

Show that if we have an algorithm for linear programming, we can use it to solve the linear-inequality feasibility problem. The number of variables and constraints that you use in the linear-programming problem should be polynomial of n and m.

1.2 Solution

不失一般性，设 linear-inequality feasibility problem 的一般形式：

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leqslant b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leqslant b_2$$

......

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leqslant b_m$$

加入一个变量 x_0，设目标函数为 $max - x_0$，可以构造出一个线性规划的形式化原问题：

$$max - x_0$$

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n - x_0 \leqslant b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n - x_0 \leqslant b_2$$

......

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n - x_0 \leqslant b_m$$

2 Reformulation Problems with Absolute Values

2.1 Problem Description

Consider th problem

$$\text{minimize } 2|x_1| + x_2$$
$$\text{subject to } x_1 + x_2 \geqslant 4$$

Reformulate this problem as a LP.

2.2 Solution

由题得：

$$\text{minimize } 2x_3 + x_2$$
$$\text{subject to } x_1 + x_2 \geqslant 4$$
$$x_3 \geqslant x_1$$
$$x_3 \geqslant -x_1$$

3 Gas Station Placement

3.1 Problem Description

Let's consider a long, quiet country road with towns scattered very sparsely along it. Sinopec, largest oil refiner in China, wants to place gas stations along the road. Each gas station is assigned to a nearby town, and the distance between any two gas stations being as small as possible. Suppose there are n towns with distances from one endpoint of the road being $d_1, d_2, ..., d_n$. n gas stations are to be placed along the road, one station for one town. Besides, each station is at most r far away from its correspond town. $d_1, ..., d_n$ and r have been given and satisfied $d_1 < d_2 < ... < d_n$, $0 < r < d_1$ and $d_i + r < d_{i+1} - r$ for all i. The objective is to find the optimal placement such that the maximal distance between two successive gas stations is minimized.

Please formulate this problem as a LP, construct an instance, and try to solve both primal and dual problem using GLPK or Gurobi or other similar tools.

3.2 Solution

由题意得，设 x_i 为第 i 个加油站的位置，y 为各个 x 之差的最大值，因此有：

$$min \quad y$$

$$s.t. \begin{cases} x_i - x_{i-1} \leqslant y, & i = 2, 3, .., n \\ d_i - r \leqslant x_i \leqslant d_i + r, & i = 1, 2, ..., n \\ d_1 < d_2 < ... < d_n \\ 0 < r < d_1 \\ d_i + r < d + i + 1 - r, & i = 1, 2, ..., n-1 \end{cases}$$

第一个约束，找出加油站之间的最大距离差 y。
第二个约束，保证加油站 x_i 在小镇 d_i 的 r 范围内。
剩余三个约束是问题描述中给出。
举例：
假设由 4 个城镇，$[d_1, d_2, d_3] = [10, 23, 30]$，$r = 2$。如上所述，则有：

$$min\{y\}$$

$$s.t. \begin{cases} x_2 - x_1 \leqslant y \\ x_3 - x_2 \leqslant y \\ 8 \leqslant x_1 \leqslant 12 \\ 21 \leqslant x_2 \leqslant 25 \\ 28 \leqslant x_3 \leqslant 32 \end{cases}$$

用 GLPK 解得：

$$min \quad y = 8, x_1 = 12, x_2 = 20, x_3 = 28, .$$

4 Volunteer Recruitment

4.1 Problem Description

Suppose you will recruit a group of volunteers for a coming event. It is estimated that this event will take N days to complete, and the $i(th)$ day needs at least A_i volunteers. The number of kinds of volunteers is M. The volunteers of $i(th)$ kind can volunteer from the S_i day to the F_i day and the recruit fee is C_i. In order to do his job well, you hope to recruit enough volunteers with least money. Please formulate this problem as an ILP.

4.2 Solution

设招募第 i 类志愿的数量为 x_i，其中 $x_i \in Z, x_i \geqslant 0$。则目标函数为：

$$min \quad C_1x_1 + C_2x_2 + ... + C_Mx_M$$

建立 $M \times N$ 矩阵表示每一类志愿者与活动天数之间的关系，对应的第 j 行第 i 列用函数 $f_j(i)$ 表示，其中 j 表示志愿者类型，i 表示活动的第 i 天。从而 $f_j(i)$ 表示第 j 类志愿者在第 i 天能否提供志愿服务（0 表示不能，1 表示可以）。则，

$$f_j(i) = \begin{cases} 1, & i \in Z \cap [S_j, F_j] \\ 0, & i \in Z \cap ([1, N] - [S_j, F_j]) \end{cases}$$

从而，每一天人数的约束可表示为：

$$\sum_{j=1}^{M} x_j f_j(i) \geqslant A_i, \quad i = 1, 2, ..., N$$

综合上述条件，该问题可形式化为：

$$min \sum_{j=1}^{M} C_j x_j$$

s.t.

$$\sum_{j=1}^{M} x_j f_j(i) \geqslant A_i$$

其中，

$$f_j(i) = \begin{cases} 1, & i \in Z \cap [S_j, F_j] \\ 0, & i \in Z \cap ([1, N] - [S_j, F_j]) \end{cases}$$

$x_j \geqslant 0$ 并且 $x_j \in Z$，$i = 1, 2, ..., N$，$j = 1, 2, ..., M$。

5 Stable Matching Problem

5.1 Problem Description

n men $(m_1, m_2, ..., m_n)$ and n women $(w_1, w_2; ...; w_n)$, where each person has ranked all members of the opposite gender, have to make pairs. You need to give a stable matching of the men and women such that there is no unstable pair. (A matching is *unstable* if: there is an element A of the first matched set which prefers some given element B of the second matched set over the element to which a is already matched, and B also prefers A over the element to which B is already matched.) Please choose one of the two following known conditions, formulate the problem as an ILP.

1. You have known that for every two possible pairs (man m_i and woman w_j , man m_k and woman w_l), whether they are stable or not. If they are stable, then $S_{i,j,k,l} = 1$; if not, $S_{i,j,k,l} = 0$. $(i, j, k, l \in \{1, 2, , ..., n\})$
2. You have known that for every man m_i, whether m_i likes woman w_j more than w_k. If he does, then $p_{i,j,k} = 1$; if not, $p_{i,j,k} = 0$. Similarly, if woman w_i likes man m_j more than m_k, then $q_{i,j,k} = 1$, else $q_{i,j,k} = 0$. $(i, j, k \in \{1, 2, ..., n\})$

5.2 Solution

(1) 记 $x_{ij} = 1$ 表示男士 i 与女士 j 配对，反之 $x_{ij} = 0$ 表示男士 i 与女士 j 不配对，于是目标函数可表示为：

$$max \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

每位男士仅能与一位女士匹配：

$$s.t. \quad \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, .., n$$

同理，

$$s.t. \quad \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, .., n$$

同时，对任意的 $i \neq k, j \neq l$，只有男士 i 与女士 j、男士 k 与女士 l 这两个不匹配时稳定的，即 $S_{i,j,k,l}=1$，他们之间的两组匹配才可以进行，否则不能两组匹配不能同时成立：

$$s.t. \quad x_{ij}+x_{kl} \leqslant S_{i,j,k,l}+1, \quad i,j,k,l=1,2,...,n; i \neq k, j \neq l$$

综上所述，该问题可形式化为：

$$max \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}$$

s.t.

$$\begin{cases} x_{ij}=0,1, & i,j=1,2,...,n \\ \sum_{j=1}^{n} x_{ij}=1, & i=1,2,...,n \\ \sum_{i=1}^{n} x_{ij}=1, & j=1,2,...,n \\ x_{ij}+x_{kl} \leqslant S_{i,j,k,l}+1, & i,j,k,l=1,2,...,n; i \neq k, j \neq l \end{cases}$$

(2) 根据第 2 问条件，该问题形式化如下：

$$max \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}$$

s.t.

$$\begin{cases} x_{ij}=0,1, & i,j=1,2,...,n \\ \sum_{j=1}^{n} x_{ij}=1, & i=1,2,...,n \\ \sum_{i=1}^{n} x_{ij}=1, & j=1,2,...,n \\ x_{ij}+x_{kl} \leqslant 3-p_{i,l,j}-q_{l,j,k}, & i,j,k,l=1,2,...,n; i \neq k, j \neq l \end{cases}$$