GROTHENDIECK MAPS ASSOCIATED TO BRAIDS AND THEIR KHOVANOV-ROZANSKY HOMOLOGIES

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Chapter 2

Basics on objects

In this section, we introduce the objects we study throughout this paper. I copied the parts that are relevant to the topic of this paper from STZ.

2.1 Braids and the associated front projection

We write Br_n for the Artin Braid group on n strands, i.e.

$$Br_n =< s_1^{\pm}, s_2^{\pm}, \cdots, s_{n-1}^{\pm} > / \{s_i s_j = s_j s_i \ for \ |i-j| \neq 1, \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\}$$

Geometrically we take an element of Br_n to be an isotopy class of the following sort of object : an n-tuple of disjoint smooth sections of the projection $[0,1]_x \times \mathbb{R}^2_{y,z} \rightarrow [0,1]_x$ such that the i^{th} section has z-coordinate some $1, \cdots, n$ in a neighborhood of $x = 0$ and of $x = 1$.

An isotopy allows one to ensure that the images of the sections under projection to the xz-plane are immersed and coincide only transversely and in pairs. We may take the sections to have constant y and z coordinates-$y = 0$ and $z \in \{1, 2, \cdots, n\}$-except for x in a neighborhood of these coincidences; this gives the standard generators of the braid group by half twists. We write s_i for the positive(counterclockwise) half twist between the strands with z coordinate i and $i+1$, and s_i^{-1} for its inverse.

We say a braid $\beta \in Br_n$ is positive if it can be expressed as a product of the s_i, and write Br_n^+ for the set of positive braids. Taking a geometric description as above of a positive braid, view the projection to the xz plane as a front diagram(there are no vertical tangents since the projection to the xz plane is an immersion). Then the associated Legendrian is isotopic to the original braid, and since the braid relation is a Legendrian Reidemeister iiimove, this construction gives a well-defined Hamiltonian isotopy class of Legendrian.(Define coorientation also!!!)

For each generator s_i, we once and for all fix an n-tuple of disjoint sections of $[0,1]_x \times \mathbb{R}^2_{y,z} \to [0,1]_x$ so that the other section except i^{th} and $i+1^{th}$ strands are constant sections and

$$i^{th} \text{ section} := \begin{cases} (x,0,i) & \text{if } x \leq \frac{1}{3} \\ (x,\ \frac{1}{2}cos(\pi f(x) - \frac{\pi}{2}),\ i + \frac{1}{2} + \frac{1}{2}sin(\pi f(x) - \frac{\pi}{2})) & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ (x,0,i+1) & \text{if } \frac{2}{3} < x \end{cases}$$

$$i+1^{th} \text{ section} := \begin{cases} (x,0,i+1) & \text{if } x \leq \frac{1}{3} \\ (x,\ \frac{1}{2}cos(\pi f(x) + \frac{\pi}{2}),\ i + \frac{1}{2} + \frac{1}{2}sin(\pi f(x) + \frac{\pi}{2})) & \text{if } \frac{1}{3} < x \leq \frac{2}{3} \\ (x,0,i) & \text{if } \frac{2}{3} < x \end{cases}$$

where f is a C^∞ real-valued function on $[0,1]$ such that

- For $x \leq \frac{1}{3}$, $f(x) = 0$
- For $x \geq \frac{2}{3}$, $f(x) = 1$
- f is strictly increasing on $[\frac{1}{3}, \frac{2}{3}]$

Then for each positive braid word ω, we get n-sections by concatenating s_i's along x-coordinate in the same order as the braid word expression and then suitably scaling by the factor of length$(\omega)^{-1}$ so that x-coordinates of the sections to fit into the range $[0,1]_x$. Projecting these sections onto xz-plane strip $[0,1]_x \times \mathbb{R}_z$, we get a braid projection. We will call this to be the front projection of the braid word.

One obtains a knot from a braid by joining the ends in some way. There are several of these; one we will not consider is the plat closure where, at each end, the first and second; third and fourth; etc., strands are joined together by cusps. In fact, one can see using the Reidemeister iimove that all Legendrian knots arise as plat closures of positive braids.

We will instead consider the braid closure. Topologically, the braid closure amounts to joining the highest strand on the right to the highest strand on the left, and so on. We will consider two different variants on this. The first, which we call the cylindrical closure, simply identifies the right and left sides of the front diagram, giving a front diagram in the cylinder $S^1_x \times \mathbb{R}_z$ and hence a Legendrian knot in $T^{\infty,-}(S^1_x \times \mathbb{R}_z)$.

2.2 Sheaves microsupported along braids

We begin by studying the local picture: sheaves microsupported along the braid closure to a rectangle containing all crossings of the braid β i.e., a picture as in Figure 6.1.1.

In this context we will be interested in sheaves with acyclic stalks in the connected component of $z \to -\infty$; we denote this full subcategory by $\mathbf{Sh}^\bullet_\beta(\mathbb{R}^\mathbf{2}, \Bbbk)_\mathbf{0}$, and in the Maslov potential which is identically zero on the braid.

Proposition 1. Let ω be a Legendrian whose front diagram is a positive braid word. Fix the Maslov potential which is everywhere zero. Let $\mathcal{F} \in \mathbf{Sh}^\bullet_\omega(\mathbb{R}^\mathbf{2}, \Bbbk)_\mathbf{0}$ be such that $\mu mon(\mathcal{F})$ is concentrated in degree zero, and assume the same for the stalk of $\mathcal{F}$ at a point of $\mathbb{R}^2$. Then $\mathcal{F}$ is quasi-isomorphic to its zeroeth cohomology sheaf.

Combining this with Proposition 3.22 in STZ, we see that objects of $\mathbf{Sh}^{\bullet}_{\omega}(\mathbb{R}^{\mathbf{2}}, \Bbbk)_{\mathbf{0}}$ can be described by legible diagrams(in the sense of Section 3.4 in STZ), and moreover that every region is assigned a $\Bbbk$-module, rather than a complex of them. That is,

Proposition 2. Let ω be a braid word; fix the zero Maslov potential on its front diagram. Let Q_ω be the quiver with one vertex for each region in the front diagram, and one arrow $S \to N$ for each arc a separating a region N above from a region S below. Then $\mathbf{Sh}^{\bullet}_{\beta}(\mathbb{R}^{\mathbf{2}}, \Bbbk)_{\mathbf{0}}$ is equivalent to the full subcategory of representations of Q in which

- The vertex corresponding to the connected component of $z \to -\infty$ is sent to zero.
- All maps are injective
- If N, E, S, W are the north, east, south, and west regions at a crossing, then the sequence $0 \to F(S) \to F(E) \oplus F(W) \to F(N) \to 0$ is exact.

For a positive braid word ω, we define $\mathcal{C}(\beta) := \mathbf{Sh}^{\bullet}_{\beta}(\mathbb{R}^{\mathbf{2}}, \mathbf{k})_{\mathbf{0}}$. We write $\mathcal{C}_r(\beta)$ for the corresponding subcategories of objects of microlocal rank r with respect to the zero Maslov potential.
We write '$\equiv_n$' for the identity braid word with n strands, and we omit the subscript when no confusion will arise. By cutting the front diagram into overlapping vertical strips, each of which contains a single crossing, and such that the overlap contain trivial braids, we find from the sheaf axiom that

$$\mathcal{C}(s_{i_1} \cdots s_{i_w}) = \mathcal{C}(s_{i_1}) \times_{\mathcal{C}(\equiv)} \mathcal{C}(s_{i_2}) \times_{\mathcal{C}(\equiv)} \cdots \times_{\mathcal{C}(\equiv)} \mathcal{C}(s_{i_w}) \tag{2.1}$$

and likewise, for moduli spaces,

$$\mathcal{M}_r(s_{i_1} \cdots s_{i_w}) = \mathcal{M}_r(s_{i_1}) \times_{\mathcal{M}_r(\equiv)} \mathcal{M}_r(s_{i_2}) \times_{\mathcal{M}_r(\equiv)} \cdots \times_{\mathcal{M}_r(\equiv)} \mathcal{M}_r(s_{i_w}) \tag{2.2}$$

Remark 3. The above moduli spaces are Artin stacks, and the fiber products should be understood in the sense of such stacks. For a fixed r, it is possible to work equivariantly with schemes instead by framing appropriately.

To calculate the $\mathcal{C}_r$ and $\mathcal{M}_r$ in general, it now suffices to determine these for the trivial braid and one-crossing braids(and to understand the maps between these). As special cases of Proposition 6.2 in STZ, we have:

Corollary 4. $\mathcal{C}_r(\equiv_n)$ is the subcategory of representations of the A_n quiver $\bullet \to \bullet \to \cdots \to \bullet$ which take the k^{th} vertex to a rk dimensional vector space, and all arrows to injections. Writing $P_{r,n} \subseteq GL_{rn}$ for the group of $r \times r$ block upper triangular matrices, $\mathcal{M}_r(\equiv_n) = pt/P_{r,n}$.

Corollary 5. Let s_i be the interchange of the i^{st} and $i+1^{st}$ strands in Br_n. An object of $\mathcal{C}_r(s_i)$ is determined by two flags $L_\bullet \Bbbk^{\oplus rn}$ and $R_\bullet \Bbbk^{\oplus rn}$ such that $dim_{\Bbbk}(L_k/L_{k-1}) = r = dim_{\Bbbk}(R_k/R_{k-1})$ and $L_k = R_k$ except possibly for $k = i$. Moreover, for such an object, the following are equivalent:

- $L_{i-1} = L_i \cap R_i = R_{i-1}$
- $L_{i+1} = L_i + R_i = R_{i+1}$
- $F \in \mathcal{C}_r(s_i)$

Two such pair $(L_\bullet, R_\bullet)$ and $(L'_\bullet, R'_\bullet)$ are isomorphic if and only if there is a linear automorphism of k^{rn} carrying one to the other, and moreover all isomorphisms arise in this manner.

Remark 6. If two braid words are connected by a sequence of Reidemiester moves, their corresponding moduli spaces $\mathcal{M}_n(\omega)$ are canonically isomorphic. Therefore, for a braid β, $\mathcal{M}_n(\beta)$ is well-defined upto canonical isomorphism. Our moduli spaces $\mathcal{M}_n(\beta)$ appear explicitly in the work of Broué and Michel[10], where they are called $\mathcal{B}(\beta)$. They are sometimes called *open* $Bott-Samelson$ *varieties*; the spaces $\mathcal{M}_r(\beta)$

are corresponding closed Bott-Samelson varieties. It was shown by Deligne[18] that the association $\beta \mapsto \mathcal{B}(\beta)$ gives a categorical representation of the positive braids. Here we have seen that, in type A, these spaces arise as moduli of objects in the Fukaya category which end on the given braid. We note that in our presentation, all the data of a categorical positive braid representation(i.e. the higher homotopies etc.) are automatically present because isotopies of positive braids can be chosen to be Legendrian isotopies, for which all desired categorical data is furnished by(appropriate family versions of) Theorem 4.10. That is, for us, $\beta \mapsto \mathcal{M}_r(\beta)$ was a categorical braid invariant *for a priori geometric reasons*, which subsequently we checked combinatorially to be equivalent to a classical construction.

We can similarly calculate the moduli space for the cylindrical closure of a braid:

Definition 7. For ω a positive braid word, we can write ω° for the Legendrian knot in $T^\infty(S^1_x \times \mathbb{R}_z)$ whose front diagram in the annulus is obtained by gluing the $x = 0$ and $x = 1$ boundaries of the front plane.

Note this gluing creates no vertical(i.e. parallel to the z-axis) tangents. The gluing data for taking a sheaf on strip to a sheaf on the cylinder is just a choice of isomorphism between the restriction to the right boundary and the restriction to the left. Thus

$$\mathcal{C}_r(\omega^\circ) = \mathcal{C}_r(\omega) \times_{\mathcal{C}_r(\equiv)\times\mathcal{C}_r(\equiv)} \mathcal{C}_r(\equiv) \tag{2.3}$$

$$\mathcal{M}_r(\omega^\circ) = \mathcal{M}_r(\omega) \times_{\mathcal{M}_r(\equiv)\times\mathcal{M}_r(\equiv)} \mathcal{M}_r(\equiv) \tag{2.4}$$

2.3 Moduli spaces associated to braids

braids(1 regular, 1 irregular) on a cylinder, constructible sheaves on a cylinder singular supported along the braid, moduli space of constructible sheaves singular supported along the braid, combinatorial model, open Bott-Samelson type varieties.

2.4 cluster charts

Chapter 3

Multivalent braids and wild character varieties of Sibuya type

3.1 Basic terminologies

Definition 8. Let ω be a braid word representing a braid $\beta \in Br_n^+$ i.e. $[\omega] = \beta$. Then we define Q_ω to be the quiver

- whose vertices are labeled by the regions of the front projection(note that this front projection is defined on $\mathbb{R}^2$ not on a cylinder)
- whose arrows are labeled by pairs of vertices whose corresponding regions are adjacent(bordered by the front projection of the braid) subject to the condition that the arrows always go against the co-orientation(hairs).

There are two distinguished vertices of Q_ω. We denote the vertex corresponding to the region $z \to \infty$(resp. $z \to -\infty$) as U(resp. D).

Maybe good to have an example after each definition.

Definition 9. Locally for each crossing c, there is a region all the hairs are pointing outward, we call this n_c(read north of c). Starting from n_c, as we move counterclockwise about the crossing, we call the corresponding regions n_c, e_c, s_c, w_c respectively(read north of, east of, south of, west of c).

Theorem 10. Suppose we have a fixed braid word ω and v is a vertex corresponding to a region given by the front projection of ω. Any path from d to a vertex v have same length.

Proof. we prove the statement by the induction on the length of the braid words. The statement is trivial for trivial braid word because there is only one path from d to any vertex. Suppose the statement is true for all quivers associated to braid words whose length is less than n. Suppose ω is a braid word of length n. Suppose there are two distinct paths p_1 and p_2 from d to v. Let $\omega = s_{i_1} \cdots s_{i_n}$. There are two cases to consider :

(case1) v is the region east of the new crossing generated by s_{i_n}. Then the paths to v must have passed the region right below the region of v because that's the only possible way to get to v under the constraint that the arrow always go against the co-orientation. If we remove the last edge and vertex from the paths p_1, p_2, we have paths p_1', p_2' ending at the same vertex(i.e. south of the crossing generated by s_{i_n}) and are entirely contained in the subquiver $Q_{\omega'}$ where $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Therefore, by the induction hypothesis the lengths of p_1' and p_2' are the same which immediately implies $length(p_1) = length(p_1') + 1 = length(p_2') + 1 = length(p_2)$.

(case2) Suppose v is not the region that is the east of the crossing generated by s_{i_n}. Without loss of generality, we can assume that two paths do not pass through the region east of the crossing, because we can always replace the part of the path $s_c \to e_c \to n_c$ with $s_c \to w_c \to n_c$ having the same length. Then once we know that two paths p_1 and p_2 do not pass through the region east of the crossing, we know that they are entirely contained in the subquiver $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Then the $length(p_1) = length(p_2)$ by the induction hypothesis. □

Definition 11. The valency of the vertex of a quiver is the number of incoming arrows. The height of the vertex of a quiver is the length of a path starting from d ending at that vertex which is well-defined by the previous theorem. Note that for every crossing c, e_c and w_c have the same height.

Definition 12. We say two vertices are adjacent if there is a crossing c such that two vertices are e_c and w_c of this crossing. Let k be a positive integer, then we define a natural ordering on the set of all height k vertices generated by the following relations : For each crossing $w_c \leq e_c$.

Theorem 13. The above ordering is well-defined and is a total ordering on the set of all height k vertices.

Proof. To prove the claim, we have to prove the following facts :

(i) For any two distinct points of height k, there is a chain of crossings connecting two points.

(ii) there is no chain of crossings starting at a point and ending at the same point.

We prove the claim by induction on the length of braid words. For the trivial braid, (i) holds because there is only one vertex for each height and (ii) holds because there is no arrow starting from that unique and ending at the point.
Now suppose the claim holds for all $length < n$ braids and suppose $Rn1$ does not hold for the braid word $\omega = s_{i_1} \cdots s_{i_n}$. Then one of the two vertices should be the vertex corresponding to e_c of the crossing generated by s_{i_n}. Let's call this vertex v and the other as v'. Then we know that by the induction hypothesis there is a chain of crossing connecting v' and w_c. Since w_c is connected by the crossing c to $e_c = v$ and v,v' are connected by a chain of crossings which is a contradiction.
Now suppose the claim holds for all $length < n$ braids and suppose $Rn2$. Suppose there is a point where there is a chain of crossing starting and ending at the same point. By the induction hypothesis, the chain of crossing generated by s_{i_n} along the

way. Without loss of generality, using cyclic shift we can assume the chain of crossings starts from e_c which is a contradiction because e_c is not the w'_c of any crossing c'. $\square$

Definition 14. There are two distinguished type A sub-quivers of Q_ω. We denote R_ω(resp. L_ω) to be the full sub-quiver containing all the vertices corresponding to the rightmost(resp. leftmost) regions. Alternatively, the path following the largest(resp. smallest) arrows at each step. We denote the vertex of R_ω(resp. L_ω) of height k by R^k_ω(resp. L^k_ω).

Definition 15. Let $\mathcal{M}^{fr}_\omega$ be the framed moduli space classifying pairs (F, g) where F is the representation of Q_ω and $g \in GL_n(\mathbb{C})$ subject to the following conditions:

- For each vertex the vectorspace associated to it is a subspace of $\mathbb{C}^n$ of dimension equal to its height.
- All maps are inclusion maps.
- $gF(R^k_\omega) = F(L^k_\omega)$ for all $k = 0, 1, \cdots, n$.
- For each crossing c, then the sequence $0 \to F(s_c) \to F(e_c) \oplus F(w_c) \to F(n_c) \to 0$ is a short exact sequence where maps are induced by the inclusion maps.

Remark 16. There is a natural left action of $x \in GL_n(\mathbb{C})$ on $(F, g)\mathcal{M}^{fr}_\omega$, that is, $x \cdot (F, g) = (xF, xGx^{-1})$ where xF is left translation on quiver representation and xgx^{-1} is conjugation.

Theorem 17. $\mathcal{M}(\beta^\circ) \cong GL_n(\mathbb{C}) \backslash \mathcal{M}^{fr}_\omega$. By $\mathcal{M}(\beta^\circ)$ I mean the moduli space defined in STZ.

Proof. maybe in STZ or STWZ?? $\square$

Now consider the following map

Definition 18. Suppose we have the braid word ω of a braid β. Let $\{v_1, v_2, \cdots, v_m\}$ be the complete list of all height 1 vertices of Q_ω such that $v_i < v_j$ if and only if $i < j$. We define ι_ω to be the forgetful map

$$\iota_\omega \;:\; \mathcal{M}^{fr}_\omega \to (\mathbb{P}^{n-1})^m \times GL_n(\mathbb{C})$$
$$(F, g) \mapsto ([F(v_1)], \cdots, [F(v_m)], g)$$

Definition 19. Let ω be a braid word. We denote the infinite cyclic copies of the original braid word ω by ω^∞. More precisely, suppose ω is a braid word given by a collection of sections $\{\sigma_i : [0,1]_x \to [0,1]_x \times \mathbb{R}^2_{y,z}\}_{i=1,\dots,n}$, then ω^∞ is given by the collection $\{\overline{\sigma_i} : \mathbb{R}_x \to \mathbb{R}^3_{x,y,z} \mid \overline{\sigma}(x,y,z) = \sigma(x - [x], y, z)\}_{i=1,\dots,n}$. Again, I will abuse the notation ω^∞ to denote its front projection onto $\mathbb{R}^2_{x,z}$.

Definition 20. Let ω be a braid word, then we have the quiver Q_ω associated to it. We define the quiver Q^∞_ω to be the quotient of $\prod_{i\in\mathbb{Z}} Q_\omega$ by the relations $\{R_{\omega,i} = L_{\omega,i+1}\}_{i\in\mathbb{Z}}$ where $R_{\omega,i}$(resp. $L_{\omega,i+1}$) is the subquiver R_ω(resp. L_ω) in the i^{th}(resp. $i+1^{th}$) copy of Q_ω in $\prod_{i\in\mathbb{Z}} Q_\omega$. Therefore, we have the quotient map of quivers $q : \prod_{i\in\mathbb{Z}} Q_\omega \to Q^\infty_\omega$. Let the signature function $\sigma' : Vert(\prod_{i\in\mathbb{Z}} Q_\omega) \to \mathbb{Z}$ be $\sigma'(v) = i$ if v is in the i^{th} copy of Q_ω in $\prod_{i\in\mathbb{Z}} Q_\omega$. Define $\sigma : Q^\infty_\omega \to \mathbb{Z}$ to be $\sigma(v) = min_{w\in q^{-1}(v)}\sigma'(w)$ if $|q^{-1}(v)| < \infty$ and 0 otherwise. Note that for $v \in Vert(Q^\infty_\omega)$, $q^{-1}(v)$ is infinite if and only if $R^{height(v)}_\omega = L^{height(v)}_\omega$ i.e. there is a unique vertex of $height(v)$ in Q_ω. We can think of Q_ω as the full subquiver of Q^∞_ω spanned by the signature 0 vertices.

Definition 21. Suppose we have a quiver representation F_ω of Q_ω, then we define the induces quiver representation F^∞_ω of the quiver Q^∞_ω to be $F^\infty_\omega(v) := g^{\sigma(v)} \cdot F_\omega(v - \sigma(v))$.

Definition 22. Υ^k_0 is the set of all height k, signature zero vertices in Q^∞_ω. For each vertex v of Q^∞_ω, we define $I_k(v)$ to be the set of all the height k vertices that have paths to v.

Definition 23. $\mathcal{M}^{fr,\infty}_\omega, \mathcal{M}^{fr,\infty,+}_\omega$

Theorem 24. $\mathcal{M}_\omega^{fr} \cong \mathcal{M}_\omega^{fr,\infty}$

3.2 Multivalent braid words

Definition 25. A braid word ω is multivalent if and only if the valencies are all greater than 1 for vertices in Q_ω^∞.

Theorem 26. If the braid word ω is multivalent, then ι_ω is an embedding.

Proof. It is enough to prove that once we specify vectorspaces to height 1 vertices of Q_ω and $g \in GL_n(\mathbb{C})$, the quiver representation of Q_ω extending the above datum is unique. Since $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$, it is enough to prove once we specify vectorspaces to signature 0 height 1 vertices of Q_ω^∞ and $g \in GL_n(\mathbb{C})$, the quiver representation F_ω^∞ of Q_ω^∞ extending the above datum is unique. Since $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, for $v \in Vert(Q_\omega^\infty)$, $F_\omega^\infty(v) = g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))$. If $height(v) = 1$, then $v - \sigma(v)$ is height 1 signature 0 vertex. Therefore, $F_\omega^\infty(v)$ is uniquely determined.
Now we prove the statement by induction on the heights of vertices. Assi,e $\forall v \in Vert(Q_\omega^\infty)$ with $height(v) < h$, $F_\omega^\infty(v)$ are determined. Suppose $v \in Vert(Q_\omega^\infty)$ such that $height(v) = h > 1$, then there are at least two vertices of height $h-1$ that have arrows to v, say v' and v''. Without loss of generality, $v' \leq v''$. Then by $\omega^\infty, Q_\omega^\infty$-analogue of Theorem 13, there is a chain of crossings $c_1, \ldots, c_k$ and $v' = v_0, \ldots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i. Therefore, if we choose any c_i to be c, then $v = N_c$ i.e. v is the north of the crossing c. By the induction hypothesis, $F_\omega^\infty(W_c)$, $F_\omega^\infty(E_c)$.and $F_\omega^\infty(S_c)$ have already been specified because heights of W_c,E_c,and S_c are $h-1$, $h-1$,and $h-2$ respectively. By the crossing condition,

$$0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$$

$F_\omega^\infty(N_c)$ is uniquely determined. $\square$

Theorem 27. If the braid word ω is multivalent, then the image of ι_ω is

$$\begin{aligned} X_\omega = \{((z_v)_{v\in\Upsilon_0^1}, g) \in \prod_{v\in\Upsilon_0^1} \mathbb{P}_v^{n-1} \times GL_n(\mathbb{C}) \mid \\ \forall u \in Vert(Q_\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u), \\ \forall c \in Cross(\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c)\} \end{aligned}$$

where $\mathbb{P}_v^{n-1}$ is a copy of $\mathbb{P}^{n-1}$ labeled by $v \in \Upsilon_0^1$.

Proof. Instead of proving $Im(\iota_\omega) = X_\omega$, I will prove that for ι', the map obtained by pre-composing the canonical isomorphism $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$ to ι_ω, $Im(\iota') = X_\omega$.
First, let's prove $Im(\iota') \subset X_\omega$. Recall that

$$\begin{aligned} \mathcal{M}_\omega^{fr,\infty} = \{(F_\omega^\infty, g) \in Rep(Q_\omega^\infty) \times GL_n(\mathbb{C}) \mid \textit{All the maps of } F_\omega^\infty \textit{ are inclusion maps}, \\ \forall v \in Vert(Q_\omega^\infty),\ dim_{\mathbb{C}}(F_\omega^\infty(v)) = height(v), \\ \forall v \in Vert(Q_\omega^\infty),\ F_\omega^\infty(v+n) = g^n \cdot F_\omega^\infty(v), \\ \forall c \in Cross(\omega^\infty),\ 0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0 \textit{ are short exact sequ} \end{aligned}$$

We need to show that for $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, $\iota'_\omega((F_\omega^\infty, g)) = ((F_\omega^\infty(v))_{v\in\Upsilon_0^1}, g) \in X_\omega$ i.e.

- $\forall u \in Vert(Q_\omega^\infty)$, $dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v-\sigma(v))) = height(u)$
- $\forall c \in Cross(\omega^\infty)$, $dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v-\sigma(v))) = height(N_c)$

It is enough to prove that

- $\forall u \in Vert(Q_\omega^\infty)$, $\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v-\sigma(v)) = F_\omega^\infty(u)$
- $\forall c \in Cross(\omega^\infty)$, $\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v-\sigma(v)) = F_\omega^\infty(N_c)$

because of the condition $\forall u \in Vert(Q_\omega^\infty),\ dim_{\mathbb{C}}(F_\omega^\infty(u)) = height(u)$ defining $\mathcal{M}_\omega^{fr,\infty}$.
Moreover, $g^{\sigma(v)} \cdot F_\omega^\infty(v-\sigma(v)) = F_\omega^\infty(v)$ by the definition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, we

need to prove that,

(i) $\sum_{v\in I_1(u)} F^{\infty}_{\omega}(v) = F^{\infty}_{\omega}(u)$

(ii) $\sum_{v\in I_1(E_c)\cup I_1(W_c)} F^{\infty}_{\omega}(v) = F^{\infty}_{\omega}(N_c)$

Proof of (i) : For each $v \in I_1(u)$, there is a path from v to u, say $v = v_0 \to \cdots \to v_k = u$. Since $F^{\infty}_{\omega}(v_i) \subset F^{\infty}_{\omega}(v_{i+1})$, $F^{\infty}_{\omega}(v) \subset F^{\infty}_{\omega}(u)$. Therefore, $\sum_{v\in I_1(u)} F^{\infty}_{\omega}(v) \subset F^{\infty}_{\omega}(u)$. Conversely, we prove $F^{\infty}_{\omega}(u) \subset \sum_{v\in I_1(u)} F^{\infty}_{\omega}(v)$ by induction on the height of u. If $height(u) = 1$, the statement is trivial. Now suppose the statement holds for all $u \in Vert(Q^{\infty}_{\omega})$ of $height(u) = h$. Since ω is multivalent, u has at least two vertices of height $h-1$ that have arrows to u, say v' and v''. Without loss of generality $v' \leq v''$. Then by $\omega^{\infty}, Q^{\infty}_{\omega}$-analogue of Theorem 13, there is a chain of crossings $c_1, \dots, c_k$ and $v' = v_0, \dots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i. Therefore, if we choose any c_i to be c, then $u = N_c$ i.e. v is the north of the crossing c. Then by the crossing condition of $M^{fr,\infty}_{\omega}$

$$0 \to F^{\infty}_{\omega}(S_c) \to F^{\infty}_{\omega}(E_c) \oplus F^{\infty}_{\omega}(W_c) \to F^{\infty}_{\omega}(N_c) \to 0$$

we have $F^{\infty}_{\omega}(u) = F^{\infty}_{\omega}(N_c) = F^{\infty}_{\omega}(E_c) + F^{\infty}_{\omega}(W_c)$ by the induction hypothesis,

$$\begin{aligned} F^{\infty}_{\omega}(E_c) = \sum_{v\in I_1(E_c)} F^{\infty}_{\omega}(v),\ F^{\infty}_{\omega}(W_c) = \sum_{v\in I_1(W_c)} F^{\infty}_{\omega}(v) \\ \Rightarrow F^{\infty}_{\omega}(u) = \sum_{v\in I_1(E_c)\cup I_1(W_c)} F^{\infty}_{\omega}(v) \end{aligned}$$

because E_c and W_c have arrows to $u = N_c \;\Rightarrow\; I_1(E_C) \cup I_1(W_c) \subset I_1(u)$. Therefore,

$$F^{\infty}_{\omega}(u) = \sum_{v\in I_1(E_c)\cup I_1(W_c)} F^{\infty}_{\omega}(v) \subset \sum_{v\in I_1(u)} F^{\infty}_{\omega}(v)$$

Proof of (ii) : By (i), we have

$$\begin{aligned} & \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \\ = & \sum_{v \in I_1(E_c)} F_\omega^\infty(v) \ + \sum_{v \in I_1(W_c)} F_\omega^\infty(v) \\ = & \ F_\omega^\infty(E_c) \ + \ F_\omega^\infty(W_c) \end{aligned}$$

This is equal to $F_\omega^\infty(N_c)$ by the crossing condition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, the proof of (ii) is complete.

Now let's prove $X_\omega \subset Im(\iota'_\omega)$. Let $((z_v)_{v \in \Upsilon_0^1}, g)$ be an arbitrary point of X_ω. I will define a point (F_ω^∞, g) such that $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$. We define a quiver representation F_ω^∞ to be

$$F_\omega^\infty(u) := \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}$$

and all the arrows of F_ω^∞ are inclusion maps. The inclusion maps are well-defined because if there is an arrow from u to u', then $I_1(u) \subset I_1(u')$.

Note that if $u \in \Upsilon_0^1$ i.e. $height(u) = 1$ and $\sigma(u) = 0$, then

$$F_\omega^\infty(u) := \sum_{v \in \{u\}} g^{\sigma(v)} \cdot z_{v-\sigma(v)} = g^{\sigma(u)} \cdot z_{u-\sigma(u)}$$

If (F_ω^∞, g) is indeed a point of $\mathcal{M}_\omega^{fr,\infty}$, then $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$. Thus, it is enough to prove that $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$ i.e.

(i) All maps of F_ω^∞ are inclusion maps

(ii) $\forall v \in Vert(Q_\omega^\infty)$, $dim_{\mathbb{C}}(F_\omega^\infty(v)) = height(v)$

(iii) $\forall v \in Vert(Q_\omega^\infty)$ and $n \in \mathbb{Z}$, $F_\omega^\infty(v+n) = F_\omega^\infty(v)$

(iv) $\forall c \in Cross(\omega^\infty)$, $0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$ are short exact sequences

We get (i) immediately from the definition of F_ω^∞.

To prove (ii), note that $dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u)$ because $((z_v)_{v \in \Upsilon_0^1}, g) \in X_\omega$. Therefore, $dim_{\mathbb{C}}(F_\omega^\infty(u)) = dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u)$.

To prove (iii), note that $\forall u \in Vert(Q_\omega^\infty)$ and $n \in \mathbb{Z}$, $I_1(u+n) = I_1(u)+n$. Therefore,

$$\begin{aligned} F_\omega^\infty(u+n) &= \sum_{v \in I_1(u+n)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v \in I_1(u)+n} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v-n \in I_1(u)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v' \in I_1(u)} (g^{\sigma(v'+n)} \cdot z_{(v'+n)-\sigma(v'+n)}) \\ &= g^n \cdot (\sum_{v' \in I_1(u)} (g^{\sigma(v')} \cdot z_{v'-\sigma(v')})) \\ &= g^n \cdot F_\omega^\infty(u) \end{aligned}$$

Finally, let's prove (iv). Let $c \in Cross(\omega^\infty)$, then

$$\begin{aligned} & height(S_c) + 2 = height(E_c) + 1 = height(W_c) + 1 = height(N_c) \\ \Rightarrow\ & dim_{\mathbb{C}}(F_\omega^\infty(S_c)) + 2 = dim_{\mathbb{C}}(F_\omega^\infty(E_c)) + 1 = dim_{\mathbb{C}}(F_\omega^\infty(W_c)) + 1 = dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \end{aligned}$$

Since there are arrows from S_c to E_c, W_c and from E_c, W_c to N_c, $I_1(S_c) \subset I_1(E_c), I_1(W_c) \subset I_1(N_c)$. Therefore, $F_\omega^\infty(S_c) \subset F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c)$ and $F_\omega^\infty(E_c) \cup F_\omega^\infty(W_c) \subset F_\omega^\infty(N_c)$.

By the condition that

$$
\begin{aligned}
& dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)}\cdot z_{v-\sigma(v)}) = height(N_c) \\
\Rightarrow\ & dim_{\mathbb{C}}(F_\omega^\infty(E_c)+F_\omega^\infty(W_c)) = height(N_c) = dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \\
\Rightarrow\ & F_\omega^\infty(E_c)+F_\omega^\infty(W_c) = F_\omega^\infty(N_c)
\end{aligned}
$$

we have surjection part of the short exact sequence i.e. $F_\omega^\infty(E_c)\oplus F_\omega^\infty(W_c)\to F_\omega^\infty(N_c)\to 0$ is exact. Thus we have a short exact sequence

$$
0\to F_\omega^\infty(E_c)\cap F_\omega^\infty(W_c)\to F_\omega^\infty(E_c)\oplus F_\omega^\infty(W_c)\to F_\omega^\infty(N_c)\to 0
$$

I claim that $F_\omega^\infty(E_c)\cap F_\omega^\infty(W_c) = F_\omega^\infty(S_c)$. Since $F_\omega^\infty(E_c)$, $F_\omega^\infty(W_c)$,and $F_\omega^\infty(S_c)$ are codimension 1, 1,and 2 inside of $F_\omega^\infty(N_c)$ respectively, it is enough to show that $F_\omega^\infty(E_c)$ and $F_\omega^\infty(W_c)$ intersect transversely in $F_\omega^\infty(N_c)$ i.e. $F_\omega^\infty(E_c)\neq F_\omega^\infty(W_c)$. This is true because otherwise

$$
\begin{aligned}
height(N_c) &= dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \\
&= dim_{\mathbb{C}}(F_\omega^\infty(E_c)+F_\omega^\infty(W_c)) \\
&= dim_{\mathbb{C}}(F_\omega^\infty(E_c)) \\
&= height(E_c)
\end{aligned}
$$

which is a contradiction.

Therefore, (F_ω^∞, g) satisfy all the condition (i)-(iv) i.e. $(F_\omega^\infty, g)\in\mathcal{M}_\omega^{fr,\infty}$. □

Theorem 28. Combinatorial characterization of multivalent braid words. More precisely, the braid word is multivalent if and only if in between s_i's there is at least on s_{i-1} upto cyclic shifts for all i's.

Lemma 29. there is a one to one correspondence between the set of region of height

k except the smallest one and s_k's in the braid word.

Proof. by induction on the length □

Lemma 30. the valency of a region correspoding to a certain s_k is equal to the number of s_{k-1} in between that certain s_k and the next s_k

Proof. by induction on the length □

(examples : powers of Coxeter braid)

Theorem 31. Theorems terminologies involving braid words and terminologies involving the associated quivers.

3.3 Bivalent braid words

Definition 32. A braid word ω is bivalent if it is multivalent and the valencies of all the vertices of height $1, \ldots, n-1$ in Q_ω^∞ are equal to 2 where n is the number of strands of ω.

Lemma 33. Let ω be a braid word and $v \in Vert(Q_\omega(Vert(Q_\omega^\infty)$ resp.), then there is a path from v to the unique height n vertex U. The path passes through a vertex of height $n-1$.

Proof. If we prove the statement for Q^ω, then the proof for the Q_ω^∞ case follows immediately. Let's prove the claim by induction on the length of the braid word ω. If $length(\omega) = 0$ i.e. trivial braid word, there is a path from D to U and along the way it passes through all the points. Therefore, restricting this path to start from the point that we are interested in gives the desired path. Now let's assume that the statement holds for all braid words of length less than k. Let $\omega = s_{i_1} \cdots s_{i_k}$, then define $\omega' = s_{i_1} \cdots s_{i_{k-1}}$. We get that $Q_{\omega'}$ is the subquiver of Q_ω where $Vert(Q_\omega) - Vert(Q_{\omega'}) = \{v\}$ is a singleton where v is the east of the crossing added by s_{i_k}, say c. Note that $v = E_c$

has an arrow to N_c and N_c has a path to U by the induction hypothesis because $N_c \in Vert(Q_{\omega'})$. Therefore, we can extend the path from N_c to U to start from E_c i.e. $E_c \rightarrow (N_c \rightarrow \cdots \rightarrow U)$. $\square$

Lemma 34. Let ω be a bivalent braid word and $u \in Vert(Q_\omega^\infty)$ of $height(u) < n$, then $|I_1(u)| = height(u)$. In particular, if $c \in Cross(\omega^\infty)$, then $I_1(S_c) = height(S_c)$, $I_1(E_c) = height(E_c)$, and $I_1(W_c) = height(W_c)$.

Proof. We prove the statement by induction on the height of u. If $height(u) = 1$, then $I_1(u) = \{u\} \Rightarrow |I_1(u)| = 1 = height(u)$ holds. Now suppose the statement holds for vertices of heights less than h and $height(u) = h$ where $h < n$. Since ω is bivalent, we have exactly two vertices of height $h-1$ and a crossing c, where those two vertices are the east and west of the crossing c. By the induction hypothesis, $|I_1(E_c)| = |I_1(W_c)| = h-1$, $|I_1(S_c)| = h-2$ and $I_1(S_c) \subset I_1(E_c), I_1(W_c)$. Let

$$I_1(E_c) - I_1(S_c) = \{v_1\}$$
$$I_1(W_c) - I_1(S_c) = \{v_2\}$$

Since ω is bivalent,

$$\begin{aligned} I_1(N_c) &= I_1(E_c) \cup I_1(W_c) \\ &= \{v_1, v_2\} \cup I_1(S_c) \end{aligned}$$

If $v_1 \neq v_2 \Leftrightarrow I_1(E_c) \neq I_1(W_c)$, then $|I_1(N_c)| = 2 + (h-2) = h$. Therefore, it is enough to prove that $\forall c \in Cross(\omega^\infty)$, $I_1(E_c) \neq I_1(W_c)$. This follows from Lemma 35 below. $\square$

Lemma 35. Let ω be a bivalent braid word and u, v be distinct vertices of Q_ω^∞ of the same height, then $I_1(u) \neq I_1(v)$. Note that the height of u, v cannot be n because there is only one vertex of height n, say U.

Proof. We prove the claim by the induction on the height of u and v. If $height(u) = height(v) = 1$, then the claim holds because $I_1(u) = \{u\}$ and $I_1(v) = \{v\}$.

Now suppose the claim holds for vertices of heights less than h. Then there are exactly two height $h-1$ vertices for each, say $\{u_1, u_2\}$, $\{v_1, v_2\}$, that have arrows to u and v. Note that there are crossings c, c' such that $\{u_1, u_2\} = \{E_c, W_c\}$, $\{v_1, v_2\} = \{E_{c'}, W_{c'}\}$. $\{u_1, u_2\} \neq \{v_1, v_2\}$, otherwise $c = c' \Rightarrow u = N_c = N_{c'} = v$ which is a contradiction. Therefore, by the induction hypothesis, $I_1(u) = I_1(u_1) \cup I_1(u_2) \neq I_1(v_1) \cup I_1(v_2) = I_1(v)$. □

Lemma 36. Let ω be a bivalent braid word and $c \in Cross(\omega^\infty)$, then $I_1(E_c) \cap I_1(W_c) = I_1(S_c)$ and $|I_1(E_c) \cup I_1(W_c)| = height(N_c)$.

Proof. $I_1(S_c) \subset I_1(E_c) \cap I_1(W_c)$ because there are arrows from S_c to E_c, W_c. Since $E_c \neq W_c$, by Lemma 35, $I_1(E_c) \neq I_1(W_c)$. Therefore,

$$\begin{aligned} &|I_1(E_c) \cap I_1(W_c)| \leq |I_1(E_c)| - 1 = |I_1(S_c)| \\ \Rightarrow\ & I_1(E_c) \cap I_1(W_c) = I_1(S_c) \end{aligned}$$

As a consequence,

$$\begin{aligned} |I_1(E_c) \cup I_1(W_c)| &= |I_1(E_c)| + |I_1(W_c)| - |I_1(E_c) \cap I_1(W_c)| \\ &= |I_1(E_c)| + |I_1(W_c)| - |I_1(S_c)| \\ &= (h-1) + (h-1) - (h-2) = h \\ &= |I_1(N_c)| \\ &= height(N_c) \end{aligned}$$

□

Definition 37. Suppose $L_1, L_2, \cdots, L_k \subset \mathbb{C}^n$ are lines(1 dimensional subspaces). We say $\{L_1, L_2, \cdots, L_k\}$ are linearly independent if and only if for nonzero $v_i \in L_i$,

$\{v_1, v_2, \cdots, v_k\}$ are linearly independent. Note that the definition does not depend on the choice of v_i's.

Theorem 38. Let ω be a bivalent braid word, then ι_ω is an open embedding whose image is

$$\begin{aligned} X'_\omega = \{((z_v)_{v\in\Upsilon_0^1}, g) \in \prod_{v\in\Upsilon_0^1} \mathbb{P}^{n-1} \times GL_n(\mathbb{C}) \mid \\ \forall c \in Cross(\omega) \subset Cross(\omega^\infty) \text{ with } height(N_c) = n, \\ dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n\} \end{aligned}$$

Proof. Since bivalent braids are multivalent, by Theorem 27, the image of ι_ω is given by

$$\begin{aligned} X_\omega \;=\; \{((z_v)_{v\in\Upsilon_0^1}, g) \in \prod_{v\in\Upsilon_0^1} \mathbb{P}_v^{n-1} \times GL_n(\mathbb{C}) \mid \\ \forall u \in Vert(Q_\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u), \\ \forall c \in Cross(\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c)\} \end{aligned}$$

I claim that $X_\omega = X'_\omega$. Since the condition

$$\forall c \in Cross(\omega) \subset Cross(\omega^\infty) \text{ with } height(N_c) = n,\ dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n$$

defining X'_ω is subsumed in the condition

$$\forall c \in Cross(\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c)$$

defining X_ω, $X_\omega \subset X'_\omega$.

Now let's prove $X'_\omega \subset X_\omega$ i.e. suppose $((z_v)_{v\in\Upsilon_0^1}, g)$ satisfy

(i) $\forall c \in Cross(\omega) \subset Cross(\omega^\infty)$ with $height(N_c) = n$, $dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n$

then it also satisfy

(ii) $\forall u \in Vert(Q_\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u)$

(iii) $\forall c \in Cross(\omega^\infty),\ dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c)$

Let $u \in Vert(Q_\omega^\infty)$ and $c \in Cross(\omega^\infty)$. First let's assume $height(u) < n$. Since ω is bivalent, by Lemma 34, $|I_1(u)| = height(u)$ and by Lemma 34 and Lemma 36, $|I_1(E_c) \cup I_1(W_c)| = |I_1(N_c)| = height(N_c)$. Therefore, the condition (i) is equivalent to saying that

$\forall c \in Cross(\omega)$ with $height(N_c) = n$, $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent

Likewise, the conditions (ii),(iii) are equivalent to saying that

$$\forall u \in Vert(Q_\omega^\infty),\ \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)} \text{ are linearly independent}$$
$$\forall c \in Cross(\omega^\infty),\ \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)} \text{ are linearly independent}$$

Let's prove (i) implies (ii) and (iii) using these paraphrased statements. Assume (i) holds for $((z_v)_{v \in \Upsilon_0^1}, g)$. Suppose for some $u \in Vert(Q_\omega^\infty)$, $\{g_{v \in I_1(u)}^{\sigma(v) \cdot z_{v-\sigma(v)}}\}$ are linearly dependent. Let $u' = u - \sigma(u)$, then $I_1(u') = I_1(u) - \sigma(u)$. Therefore,

$$\begin{aligned}
\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u')} &= \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v+\sigma(u) \in I_1(u)} \\
&= \{g^{\sigma(v')-\sigma(u)} \cdot z_{v'-\sigma(v')}\}_{v' \in I_1(u)} \\
&= g^{-\sigma(u)} \cdot \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}
\end{aligned}$$

are also linearly dependent. Therefore, without loss of generality, we can assume $\sigma(u) = 0$ i.e. $u \in Vert(Q_\omega) \subset Vert(Q_\omega^\infty)$. By Lemma 33, there is a path from u to a height $n-1$ vertex p. Thus, $I_1(u) \subset I_1(p)$. Since ω is bivalent, for some $c \in Cross(\omega)$, $p = E_c$ or W_c. By the condition (i), $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent, thus its subset $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$ are linearly independent as well. Therefore, $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (ii).

Now let's show that $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (iii). Suppose $c \in Cross(\omega^\infty)$. If $height(N_c) = n$, then the condition (i) is equal to the condition (iii), there is nothing to prove. Suppose $height(N_c) < n$, then $I_1(E_c) \cup I_1(W_c) = I_1(N_c)$. Thus, the condition (iii) translates to $\{g^{\sigma(v) \cdot z_{v-\sigma(v)}}\}_{v \in I_1(N_c)}$ are linearly independent, which follows from (ii) that we already proved.

Now let's show that (iii) holds when $height(u) = n$ i.e. u is the unique height n point U. Since ω is bivalent, there is a crossing c such that $u = N_c$. Therefore,

$$dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c) = n$$

Since $I_1(E_c) \cup I_1(W_c) \subset I_1(N_c) = I_1(u)$,

$$dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n = height(u)$$

□

Theorem 39. Combinatorial characterization of bivalent braid words. More precisely, the braid word is multivalent if and only if in between s_i's there is at least on s_{i-1} upto cyclic shifts for all i's.

(examples : powers of Coxeter braid)

Definition 40. Wild character varieties of Sibuya type are wild character varieties associated to bivalent braids.

Remark 41. microlocal monodromy as generalized cross-ratio. Symplectic leaves.

3.4 Examples : wild character varieties of Sibuya type

By nequation, I mean a formula that expresses the non-equality of two expressions.

Whenever I denote capital X with subscript i.e. X_j, I mean an element of some projective space.

Lower case x's with subscripts are used to denote the homogeneous coordinates of X_j's.

For example $X_j = [x_{1,j} : \cdots : x_{n,j}]$.

I will also denote

$$(X_1, \cdots, X_{n-1}) = ([x_{1,1} : \cdots : x_{n,1}], [x_{1,2} : \cdots : x_{n,2}], \cdots, [x_{1,n-1} : \cdots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

We will use

$$\left(\begin{array}{c|c|c|c} & & & \\ X_1 & X_2 & \cdots & X_n \\ & & & \end{array} \right) = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

to denote the matrix whose entries are $x_{i,j}$ which are sections of the line bundle $O_{\mathbb{P}^k}(1)$.

Thereby

$$det\left(\begin{array}{c|c|c|c} & & & \\ X_1 & X_2 & \cdots & X_n \\ & & & \end{array}\right) = det\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

is a section of the line bundle $\underbrace{O_{\mathbb{P}^k}(1) \boxtimes \cdots \boxtimes O_{\mathbb{P}^k}(1)}_{n-copies}$ on $\underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{n-copies}$.

Example 42. (**Empty Set**) In this section we compute 2 kinds of moduli spaces that are empty.

In the first case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than the number of strands with unipotent monodromy $u = I$

By the theorem ??, the framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \cdots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k-copies} \mid det \left(\begin{array}{c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k} \end{array} \right) \neq 0,$$

$$det \left(\begin{array}{c|c|c|c|c|c} X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k+1} \end{array} \right) \neq 0, \cdots ,$$

$$det \left(\begin{array}{c|c|c|c|c|c|c} X_k & X_1 & \cdots & X_k & X_1 & \cdots & X_{n-k-1} \end{array} \right) \neq 0\}$$

Take the first nequation and subtract 1^{st}column from the $(k+1)^{th}$column we get

$$det \left(\begin{array}{c|c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & 0 & X_2 & \cdots & X_{n-k} \end{array} \right) \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the moduli space.

In the second case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than $(the\ number\ of\ strands) - 1$ with unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

By the theorem ??, the framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \cdots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k-copies} \mid det \left(\begin{array}{c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k} \end{array} \right) \neq 0,$$

$$det \left(\begin{array}{c|c|c|c|c|c} X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k+1} \end{array} \right) \neq 0, \cdots ,$$

$$det \left(\begin{array}{c|c|c|c|c|c|c} X_k & uX_1 & \cdots & uX_k & u^2X_1 & \cdots & u^2X_{n-k-1} \end{array} \right) \neq 0$$

Take the first nequation and subtract 1^{st}(resp. 2^{nd}) column from the $(k+1)^{th}$(resp. $(k+2)^{th}$) column we get

$$det \left(\begin{array}{c|c|c|c|c|c|c|c|c} & & & & 0 & 0 & & & \\ & & & & \vdots & \vdots & & & \\ X_1 & X_2 & \cdots & X_k & 0 & 0 & uX_3 & \cdots & uX_{n-k} \\ & & & & x_{n,1} & x_{n,2} & & & \\ & & & & 0 & 0 & & & \end{array} \right) \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the moduli space.

Example 43. (**Points**) In this section, we compute the moduli space of braids represented by the braid word

$$(s_1 s_2 \cdots s_{n-1})^{n-1}$$

and the unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

i.e. a unipotent matrix given by the partition $(\underbrace{1, 1, \cdots, 1}_{n-1}, 2)$.

Therefore by theorem??, the framed moduli space associated to the above data is given as follows

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid det \left(\begin{array}{c|c|c|c|c} X_1 & X_2 & \cdots & X_{n-1} & uX_1 \end{array} \right) \neq 0,$$

$$det \left(\begin{array}{c|c|c|c|c} X_2 & \cdots & X_n & uX_1 & uX_2 \end{array} \right) \neq 0, \cdots, det \left(\begin{array}{c|c|c|c|c} X_{n-1} & uX_1 & \cdots & uX_{n-2} & uX_{n-1} \end{array} \right) \neq 0\}$$

Using the elementary column operation of subtracting the first column from the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_1 & X_2 & \cdots & X_{n-1} & 0 \\ & & & & x_{n,1} \\ & & & & 0 \end{array} \right) \neq 0,$$

$$det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_2 & \cdots & X_{n-1} & uX_1 & 0 \\ & & & & x_{n,2} \\ & & & & 0 \end{array} \right) \neq 0, \cdots, det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_{n-1} & uX_1 & \cdots & uX_{n-2} & 0 \\ & & & & x_{n,n-1} \\ & & & & 0 \end{array} \right) \neq 0, \}$$

Applying the cofactor expansion formula with respect to the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{n-copies} \mid x_{n,1} \cdot det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0,$$

$$x_{n,2} \cdot det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \cdots, x_{n,n-1} \cdot det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \}$$

$$= \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{n-copies} \mid x_{n,i} \neq 0 \ (for\ i = 1, 2, \cdots n-1)$$

$$, det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0\}$$

To get the moduli space out of the framed moduli space above, we have to quotient it out by the centralizer subgroup of u , that is,

$$C_{GL_n(\mathbb{C})}(u) = \{ \left(\begin{array}{ccc|cc} c_{1,1} & \cdots & c_{1,n-2} & 0 & c_{1,n} \\ \vdots & \ddots & \vdots & 0 & c_{1,n} \\ c_{n-2,1} & \cdots & c_{n-2,n-2} & 0 & c_{n-2,n} \\ \hline 0 & \cdots & 0 & 0 & c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-2} & c_{n,n-1} & c_{n,n} \end{array}\right) \in GL_n(\mathbb{C}) |$$

$$det \begin{pmatrix} c_{1,1} & \cdots & c_{1,n-2} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n-2} \end{pmatrix}, c_{n,n-1}, c_{n-1,n} \neq 0 \}$$

It acts diagonally on $\underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$ where the action on each coordinate is given by left multiplication.

To simplify the notation, I will denote

$$(X_1, \cdots, X_{n-1}) = ([x_{1,1} : \cdots : x_{n,1}], [x_{1,2} : \cdots : x_{n,2}], \cdots, [x_{1,n-1} : \cdots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

I claim that for any $(X_1, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$, there exists $A \in C_{GL_n(\mathbb{C})}(u)$

such that

$$A \cdot X = \left[\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline & & & \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

Let

$$A_1 = \left(\begin{array}{c|cc} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-2} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-2} \end{pmatrix}^{-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & 0 & 1 \\ \begin{matrix} a_1 & \cdots & a_{n-2} \end{matrix} & a_{n-1} & a_n \end{array}\right)$$

with $a_{n-1} \neq 0$ such that

$$\sum_{i=1}^{n} a_i \cdot (x_{i,1}, \cdots, x_{i,n}) = (1, \cdots, 1)$$

We can always find such because we know that

$$det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0$$

Then we get

$$A_1 \cdot X = \left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ x_{n,1} & \cdots & x_{n,n-2} & x_{n,n-1} \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

Since we know that $x_{n,i} \neq 0$ for $i = 1, \cdots, n-1$, without loss of generality we put $x_{n,i} = 1$ for $i = 1, \cdots, n-1$. Then we get

$$A_1 \cdot X = \left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

such that $b_1 + \cdots + b_{n-1} + 1 \neq 0$. Now let

$$A_2 = \left(\begin{array}{ccc|cc} a_{1,1} & \cdots & a_{1,n-2} & 0 & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,1} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\ \hline & & & & \\ 0 & \cdots & 0 & & \\ & & & \multicolumn{2}{c}{I} \\ 0 & \cdots & 0 & & \end{array}\right)$$

such that

$$a_{i,1} \cdot (1, 0, \cdots, 0, b_1) + a_{i,2} \cdot (0, 1, 0, \cdots, 0, b_2) + \cdots + a_{i,n-2} \cdot (0, \cdots, 0, 1, b_{n-2}) + a_{i,n} \cdot (1, \cdots, 1) = (0, \cdot, 0, \overset{\mathrm{i}^{th}}{\overset{\downarrow}{1}}, 0, \cdot$$

Again, we can find such because

$$det \left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ 1 & \cdots & 1 & 1 \end{array}\right] \neq 0$$

Then $A_2 \cdot A_1$ is the desired A.

Example 44. (**Regular Unipotent Fibers**) Finite type integral schemes. Also, it has a stratification into rational varieties at most 1 stratum in each dimension. $(s_1)^3$, $(s_1 s_2)^2$

Example1

In this section, we compute the moduli space associated to the braid word : $(s_1 s_2)^2$ with regular unipotent monodromy. By the theorem ??, the framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x:y:z],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0,\ det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$C := C_{GL_3(\mathbb{C})}(u) = \{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in GL_3(\mathbb{C}) \mid \alpha \neq 0 \}$$

We can cover $\mathcal{M}^{fr}$ with open subsets U_1, U_2 where

$$U_1 = \{([x:y:z],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0,\ det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, z \neq 0\}$$

$$U_2 = \{([x:y:z],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0,\ det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, c \neq 0\}$$

Therefore, we have a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

Quotienting out by the centralizer subgroup C, we get

$$\begin{tikzcd} \overline{U}_1 := C\backslash U_1 \arrow[r, hook] & \mathcal{M} = C\backslash \mathcal{M}^{fr} \\ \overline{U_1 \cap U_2} := C\backslash U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & \overline{U}_2 := C\backslash U_2 \arrow[u, hook] \end{tikzcd}$$

Also note that, for the action of C, the centralizer subgroup of any element of $\mathcal{M}^{fr}$ is the set of scalar multiplication matrices.

First let's simplify, $\overline{U}_1$. Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\begin{aligned} \alpha &= 1 \\ \beta &= -\frac{y}{z} \\ \gamma &= \frac{y^2}{z^2} - \frac{x}{z} \end{aligned}$$

This expression makes sense because $z \neq 0$ in U_1. If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\overline{U}_1 \cong \{([0:0:1],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid det\begin{pmatrix} 0 & a & 0 \\ 0 & b & 1 \\ 1 & c & 0 \end{pmatrix} \neq 0,\ det\begin{pmatrix} 0 & a & b \\ 0 & b & c \\ 1 & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\}$$

$$\cong \{[a:b:c] \in \mathbb{P}^2 \mid a \neq 0,\ det\begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0\}$$

$$\cong \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}$$

Under this identification,

$$\overline{U_1 \cap U_2} \cong \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

Now let's simplify, $\overline{U}_2$. Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{b}{c}$$
$$\gamma = \frac{b^2}{c^2} - \frac{a}{c}$$

This expression makes sense because $c \neq 0$ in U_1. If we take an element of U_2 with

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned}
\overline{U}_2 &\cong \{([x:y:z],[0:0:1]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0,\ det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\
&\cong \{[x:y:z] \in \mathbb{P}^2 \mid x \neq 0,\ det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \neq 0\} \\
&\cong \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}
\end{aligned}$$

Under these identifications, the pushout square above becomes

$$\begin{tikzcd}
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \arrow[r, hook] & \mathcal{M} \\
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \arrow[u, hook] \arrow[r, hook, "f"] & \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \arrow[u, hook]
\end{tikzcd}$$

where

$$f : \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \to \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}$$
$$(b,c) \mapsto (-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c})$$

Now define a variety V to be

$$V := \{(Y,Z,W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0\} - \{(0,0,0)\}$$

I claim that V is isomorphic to $\mathcal{M}$. More precisely, I claim that

$$\begin{tikzcd}
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \arrow[rr, hook, "i"] & & V \\
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \arrow[u, hook, "g"] \arrow[r, hook, "f"] & \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \arrow[ur, hook, "\iota"] &
\end{tikzcd}$$

is a pushout square where

$$i : \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \to V$$
$$(b,c) \mapsto (-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}, b)$$

$$\iota : \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \to V$$
$$(y,z) \mapsto (y, z, \frac{yz}{z - y^2})$$

It is easy to check the square commutes. f, g are inclusion maps by construction. ι is also an inclusion map because we can recover (y,z) from $\iota(y,z)$ by projecting onto the $1^{st}\&2^{nd}$ coordinates.

For i, we can recover b from $i(b,c)$ by projecting onto the 3^{rd} coordinate. We can recover c from $i(b,c,)$ by multiplying $2^{nd}\&3^{rd}$ coordinates and dividing with -(the 1^{st} coordinate).

Let's check that the images of i and ι form an open cover of V. The images of i and ι are

$$i(\{(b,c) \in \mathbb{A}^2 \mid c-b^2 \neq 0\}) = \{(Y,Z,W) \in \mathbb{A}^3 | Y^2W-ZW+YZ = 0, YW+Z+W^2 \neq 0\}$$

$$\begin{aligned}\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\}) &= \{(Y,Z,W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, Y^2 - Z \neq 0\} \\ &= \{(Y,Z,W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z-Y^2}, Y^2 - Z \neq 0\}\end{aligned}$$

Clearly, they are open subsets of V.

Let's check that if $(Y,Z,W) \in V$ and $Y^2 = Z$, then $YW + Z + W^2 \neq 0$. If $Y^2 = Z$, then the equation $Y^2W - ZW + YZ = 0$ becomes $YZ = 0$. Therefore, we get $Y = Z = 0$. Since $(0,0,0)$ is not contained in V, W can only take non-zero values. Therefore, $YW + Z + W^2 = W^2 \neq 0$. We conclude that the images of i and ι cover V.

Now let's check that

$$i^{-1}(\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\})) = \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

The image of ι is

$$\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\}) = \{(Y,Z,W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z-Y^2}, Y^2 - Z \neq 0\}$$

we have

$$\begin{aligned}
& i(b,c) \in \iota(\{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) \\
\Longleftrightarrow & (-\frac{bc}{b^2-c})^2 \neq \frac{c^2}{b^2-c} \\
\Longleftrightarrow & \; b^2c^2 \neq c^2(b^2-c) \\
\Longleftrightarrow & \; c^3 \neq 0 \\
\Longleftrightarrow & \; c \neq 0
\end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

as desired.

Therefore, V is isomorphic to $\mathcal{M}$.

Example2

In this section, we compute the moduli space associated to the braid word : s_1^3 with regular unipotent monodromy. By the theorem ??, the framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x:y],[z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid det \begin{pmatrix} x & z \\ y & w \end{pmatrix} \neq 0,\ det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0,\ det \begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is

$$C := C_{GL_2(\mathbb{C})}(u) = \{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in GL_2(\mathbb{C}) \mid \alpha \neq 0\}$$

From this point on, we will use the following notation

$$\infty := [1 : 0]$$

$$X := [x : y]$$

$$Z := [z : w]$$

$$A := [a : b]$$

We can cover $\mathcal{M}^{fr}$ with open subsets U_1, U_2 where

$$U_1 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid X \neq \infty\}$$

$$U_2 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid Z \neq \infty\}$$

Therefore, we have a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

Quotienting out by the centralizer subgroup C, we get

$$\begin{tikzcd} \overline{U}_1 := C \backslash U_1 \arrow[r, hook] & \mathcal{M} = C \backslash \mathcal{M}^{fr} \\ \overline{U_1 \cap U_2} := C \backslash U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & \overline{U}_2 := C \backslash U_2 \arrow[u, hook] \end{tikzcd}$$

Also note that, for the action of C, the centralizer subgroup of any element of $\mathcal{M}^{fr}$

is the set of scalar multiplication matrices.

First let's simplify, $\overline{U}_1$. Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{x}{y}$$

This expression makes sense because $y \neq 0$ in U_1. If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned}
\overline{U}_1 &\cong \{([0:1],[z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid det\begin{pmatrix} 0 & z \\ 1 & w \end{pmatrix} \neq 0,\ det\begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0,\ det\begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \neq 0\} \\
&\cong \{([z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z \neq 0,\ bz - aw \neq 0,\ a - b \neq 0\} \\
&\cong \{(w,[a:b]) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid b - aw \neq 0,\ a - b \neq 0\}
\end{aligned}$$

Change variables $a' := a + b, b' := a - b$ we get

$$\begin{aligned}
\overline{U_1} &\cong \{(w,[a':b']) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid \frac{a'-b'}{2} - \frac{(a'+b')w}{2} \neq 0,\ b' \neq 0\} \\
&\cong \{(w,a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a'-1) - (a'+1)w \neq 0\} \\
&\cong \{(w,a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a''+1)(1-w) - 2 \neq 0\} \\
&\cong \{(w',a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1\}
\end{aligned}$$

Under this identification,

$$\overline{U_1 \cap U_2} \cong \{(w',a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1, 2w' \neq 1\}$$

Now let's simplify, $\overline{U}_2$. Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{z}{w}$$

This expression makes sense because $w \neq 0$ in U_2. If we take an element of U_2 with

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned}
\overline{U_2} &\cong \{([x:y],[0:1],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid det \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \neq 0,\ det \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \neq 0,\ det \begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0\} \\
&\cong \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x \neq 0,\ a \neq 0,\ ay - b(x+y) \neq 0\} \\
&\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y - b(1+y) \neq 0\} \\
&\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y(1-b) - b \neq 0\}
\end{aligned}$$

Change variables $b' := 1-b, y' := y+1$ we get

$$\overline{U_2} \cong \{(y',b') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid yb' \neq 1\}$$

Under these identifications, the pushout square above becomes

$$\begin{tikzcd}
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} \arrow[r, hook] & \mathcal{M} \\
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} \arrow[u, hook] \arrow[r, hook, "f"] & \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \arrow[u, hook]
\end{tikzcd}$$

where

$$f : \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} \to \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\}$$
$$(x,y) \mapsto (2x, \frac{4x+y-4}{2xy-2})$$

Now define a variety V to be

$$V := \{(A,B,C) \in \mathbb{A}^3 \mid (AC-2)B = 2A+C-4\} - \{(1,1,2)\}$$

I claim that V is isomorphic to $\mathcal{M}$. More precisely, I claim that

$$\begin{tikzcd}
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} \arrow[r, hook, "i"] & V \\
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} \arrow[u, hook, "g"] \arrow[r, hook, "f"] & \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \arrow[u, hook, "\iota"']
\end{tikzcd}$$

is a pushout square where

$$i : \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} \to V$$
$$(x,y) \mapsto (2x, \frac{4x+y-4}{2xy-2}, y)$$
$$\iota : \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \to V$$
$$(a,b) \mapsto (a, b, \frac{2a+2b-4}{ab-1})$$

It is easy to see that the square commutes and f, g, i, ι are inclusion maps.

Let's check that the images of i and ι form an open cover of V.

The image of i and ι are

$$i(\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\}) = \{(A,B,C) \in V | AC \neq 2\}$$

$$\iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A,B,C) \in V | AB \neq 1\}$$

Clearly, they are open subsets of V.

Let's check that if $(A,B,C) \in V$ and $AC = 2$, then $AB \neq 1$. If $AC = 2$, then the left hand side of the equation $(AC-2)B = 2A + C - 4$ becomes zero. Therefore, we get $AC = 2$ and $2A + C = 4$ which implies $2A + \frac{2}{A} = 4 \Leftrightarrow A^2 - 2A + 1 = 0$. Solving the quadratic equation, we get $A = 1, C = 2$. Since $(1,1,2)$ is not contained in V, B can take any value except 1. Therefore, $AB \neq 1$. We conclude that the images of i and ι cover V.

Now let's check that

$$i^{-1}(\iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\}$$

The image of ι is

$$\iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A,B,C) \in V | AB \neq 1\}$$

we have

$$\begin{aligned}
i(x,y) \in \iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\}) \\
\Longleftrightarrow 2x \cdot (\frac{4x+y-4}{2xy-2}) \neq 1 \\
\Longleftrightarrow 2x \cdot (4x+y-4) \neq 2xy-2 \\
\Longleftrightarrow 8x^2 - 8x + 2 \neq 0 \\
\Longleftrightarrow 2(2x-1)^2 \neq 0 \\
\Longleftrightarrow 2x1 \neq 1
\end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x,y) \in \overline{U}_1 \mid 2x \neq 1\}$$

as desired.

Therefore, V is isomorphic to $\mathcal{M}$.

Example3

In this section, we prove that the moduli space of a bivalent braid with regular unipotent monodromy is finite type integral scheme over $\mathbb{C}$ not necessarily separated. Suppose we have a bivalent braid word with n-strands and a unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

By theorem??, the framed moduli space is given as

$$\mathcal{M}^{fr} = \{X = (X^1, X^2, \cdots, X^k) \in (\mathbb{P}^{n-1})^k \mid f_1(X) \neq 0, \cdots f_m(X) \neq 0\}$$

where f_i 's are determinants with column vectors of the form $u^s \cdot X_j$. Note that the entries of the last row(i.e. the n^{th} row) is one of $x_{n,i}$ $(i = 1, \cdots, k)$. Thus $x_{n,i}$'s $(i = 1, \cdots, k)$ cannot be identically zero otherwise all of the f_r's will vanish. Therefore, we have an open cover of $\mathcal{M}^{fr}$, i.e. $\{U_i\}_{i=1,\cdots,k}$ where $U_i := \{X \in \mathcal{M}^{fr} \mid x_{n,i} \neq 0\}$.
To get the moduli space, we quotient the framed moduli space with the centralizer subgroup of u in $GL_n(\mathbb{C})$ i.e.

$$C := C_{GL_n(\mathbb{C})} = \{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in GL_n(\mathbb{C}) \mid \alpha_1 \in \mathbb{C}^*, \ \alpha_i \in \mathbb{C} \; for \; i = 2, \cdots, n\}$$

Suppose we have an element

$$X = (X^1, \cdots, X^k) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \in U_i$$

Then there exists a

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \cdot \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

That is, recursively,

$$\alpha_1 = 1$$

$$\alpha_s = -\frac{1}{x_{n,i}}(x_{n-1,i}\alpha_{s-1} + x_{n-2,i}\alpha_{s-2} + \cdots x_{n-s+1,i}\alpha_1) = -\frac{1}{x_{n,i}}(\sum_{t=1}^{s-1} x_{n-t,i} \cdot \alpha_{s-t})$$

This expression makes sense because $x_{n,i} \neq 0$ in U_i. If we take an element of U_i with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\overline{U}_i := C\backslash U_i \cong \{ \begin{array}{c} \begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix} \\ \uparrow \\ \text{i}^{th} column \end{array} \in (\mathbb{P}^{n-1})^k \mid f_1(\begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix}) \neq 0, \cdots ,$$

$$f_m(\begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix}) \neq 0\}$$

which are finite type scheme over $\mathbb{C}$. In summary, we have found an finite open cover of $\mathcal{M}$ i.e. $\{\overline{U}_i\}_{i=1,\cdots,k}$ such that each open is a finite type scheme over $\mathbb{C}$. Thus, $\mathcal{M}$ is also a finite type scheme over $\mathcal{C}$. Now we have a smooth surjective map $\pi : \mathcal{M}^{fr} \to \mathcal{M}$. $\mathcal{M}$ is irreducible because $\mathcal{M}^{fr}$ is irreducible and π is surjective. $\mathcal{M}$ is reduced because $\mathcal{M}^{fr}$ is reduced and π is smooth. Therefore, we conclude that the moduli space associated attached to bivalent braid with regular unipotent monodromy is finite type integral scheme over $\mathbb{C}$. But it may not be separated.

Example4

In this section, I will provide an example of the moduli space associated to bivalent braid with regular unipotent monodromy that is non-separated. Consider a 2-strand braid given by the braid word s_1^2 and a regular unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

By the theorem ??, the framed moduli space is given as follows

$$\begin{aligned}\mathcal{M}^{fr} &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], u \cdot [a:b] \neq [x:y]\} \\ &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], [a+b:b] \neq [x:y]\} \\ &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid ay \neq bx, (a+b)y \neq bx\}\end{aligned}$$

Then we have an open cover $\{U_i\}_{i=1,2}$ where

$$U_1 := \{([x:y],[a:b]) \in \mathcal{M}^{fr} | b \neq 0\}$$
$$U_2 := \{([x:y],[a:b]) \in \mathcal{M}^{fr} | y \neq 0\}$$

We get a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

we take quotients of these opens with respect to the centralizer subgroup of u i.e.

$$C := C_{GL_2(\mathbb{C})} = \{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in GL_2(\mathbb{C}) \mid \alpha \neq 0\}$$

we get

$$\begin{aligned}
\overline{U}_1 := C\backslash U_1 &\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x\} \\
&\cong \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \\
\overline{U}_2 := C\backslash U_2 &\cong \{([0:1],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid a \neq 0, (a+b) \neq 0\} \\
&\cong \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \\
\overline{U}_1 \cap \overline{U}_2 := C\backslash U_1 &\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
&\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
&\cong \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\}
\end{aligned}$$

and a pushout square

$$\begin{tikzcd}
\{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \arrow[r, hook] & \mathcal{M} \\
\{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \arrow[u, hook] \arrow[r, hook, "f"] & \cong \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \arrow[u, hook]
\end{tikzcd}$$

where

$$f : \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \to \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}$$

$$([1:y],[0:1]) \mapsto ([0:1],[1:-y])$$

Now consider the map

$$\begin{aligned}
g : \mathbb{A}^1 - \{0,1\} &\longrightarrow \overline{U}_1 \cap \overline{U}_2 \subseteq \mathcal{M} \\
y &\mapsto ([1:y],[0:1])
\end{aligned}$$

This map extends to $\mathbb{A}^1$ in two different ways i.e. we have two distinct h_1, h_2 that fit into the following commutative square

$$\begin{tikzcd} \mathbb{A}^1 - \{0,1\} \arrow[r, hook] \arrow[dr, "g"] & \mathbb{A}^1 - \{1\} \arrow[d, "h_i"] \\ & \mathcal{M} \end{tikzcd}$$

which are

$$h_1 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_1 \subseteq \mathcal{M}$$
$$y \mapsto ([1:y],[0:1])$$

$$h_2 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_2 \subseteq \mathcal{M}$$
$$y \mapsto ([0:1],[1:-y])$$

h_1 and h_2 are distinct because $h_1(0) \in U_1 - U2$. Therefore, by the valuative criterion for separatedness, $\mathcal{M}$ is non-separated.

Chapter 4

Classification of Cluster Coordinates

4.1 natural alternating diagram

Suppose we have a Riemann sphere C with two punctures at 0 and ∞. Topologically, C is homeomorphic to the boundaryless cylinder(figure below)

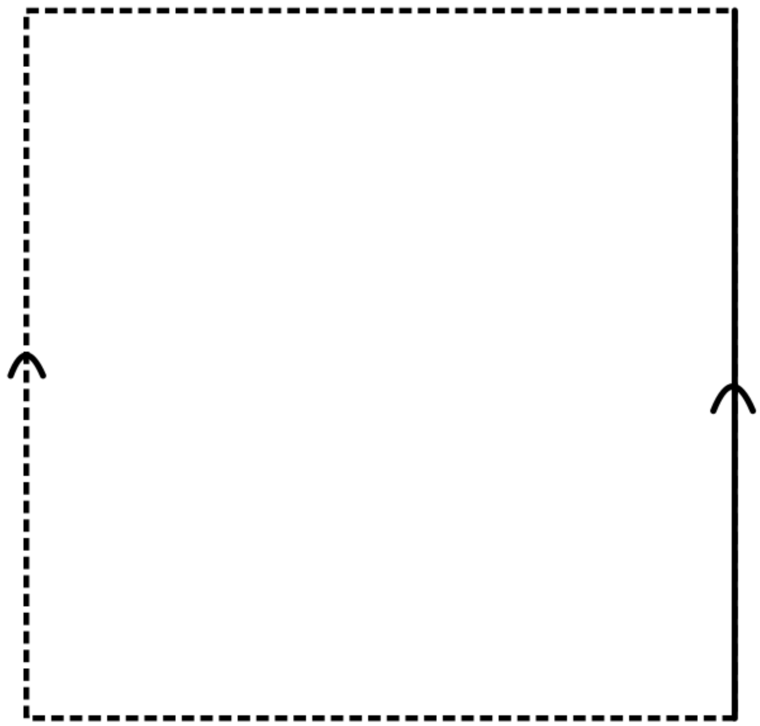


Figure 4.1: Your caption here

Suppose we have a braid word ω, then we can draw the associated braid diagram$(i_1, \cdots, i_{n-1} : [0,1] \to [0,1] \times (0,1))$ on $[0,1] \times (0,1)$ and its cylindrical closure on $S^1 \times (0,1)$ as shown below.

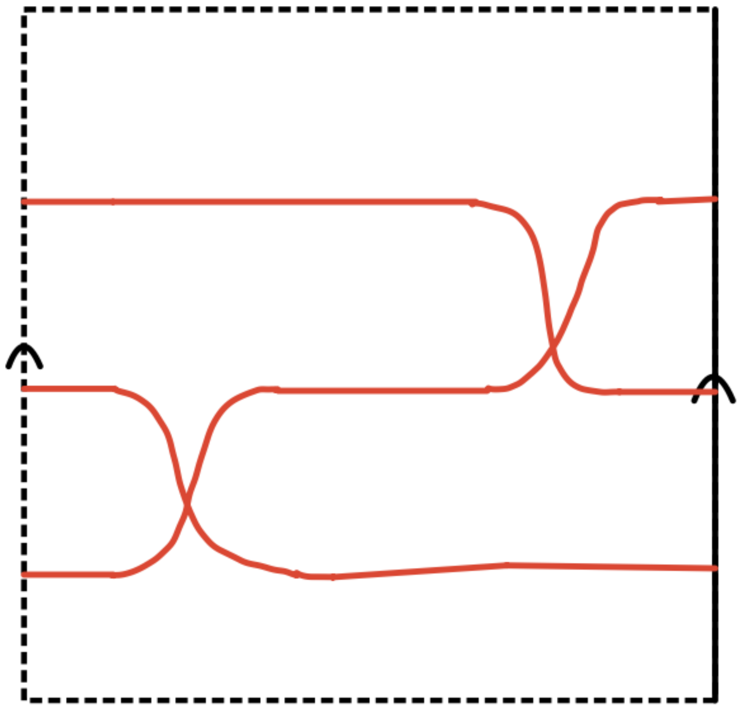


Figure 4.2: Your caption here

We will specify co-orientations of $i_1, \cdots, i_{n-1}$ so that we can think of the cylindrical closure of the braid word as the the front projection Legendrian knot living inside the co-circle bundle of the cylinder.

we define the co-orientation at $i_k(t_0)$ to be $\xi = adx + bdy$ so that ξ vanishes at $\frac{di_k}{dt}(t_0)$,$||(a,b)|| = 1$, and $b > 0$. This can be visually represented as hairs pointing upward($b > 0$).

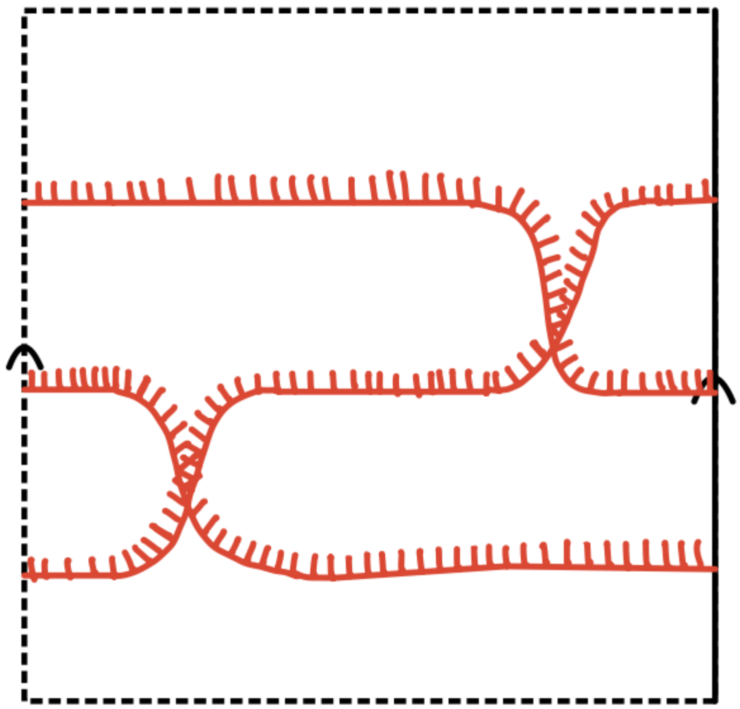


Figure 4.3: Your caption here

There are two different ways of embedding this cylindrical closure into C. We can embed this cylindrical closure onto the hemisphere containing $0(\infty$ resp.) in such a way that the embedding extends

(roman*) to $S^1 \times \{0\})$ as an isomophism onto the equator of C

(roman*) to $S^1 \times \{1\}$ as a constant map to $0(\infty$ resp.)

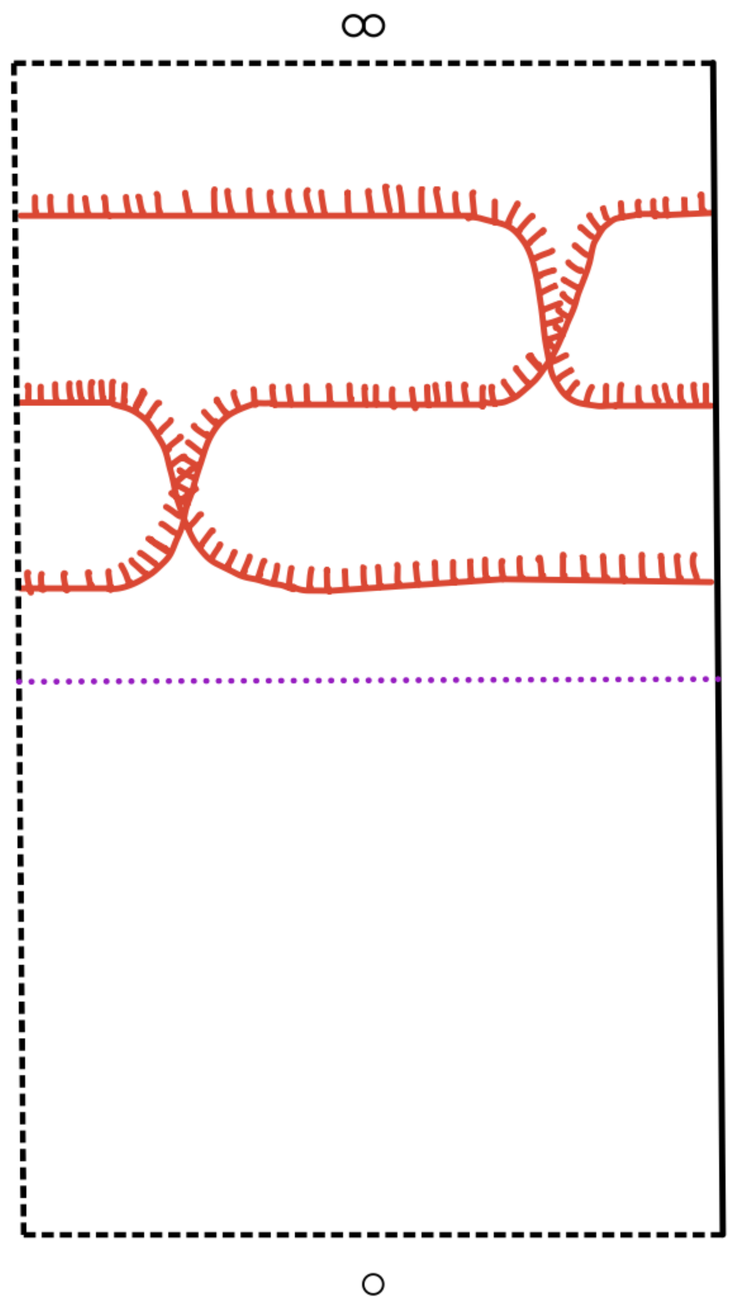


Figure 4.4: embedding of the cylindrical closure onto the hemisphere containing ∞

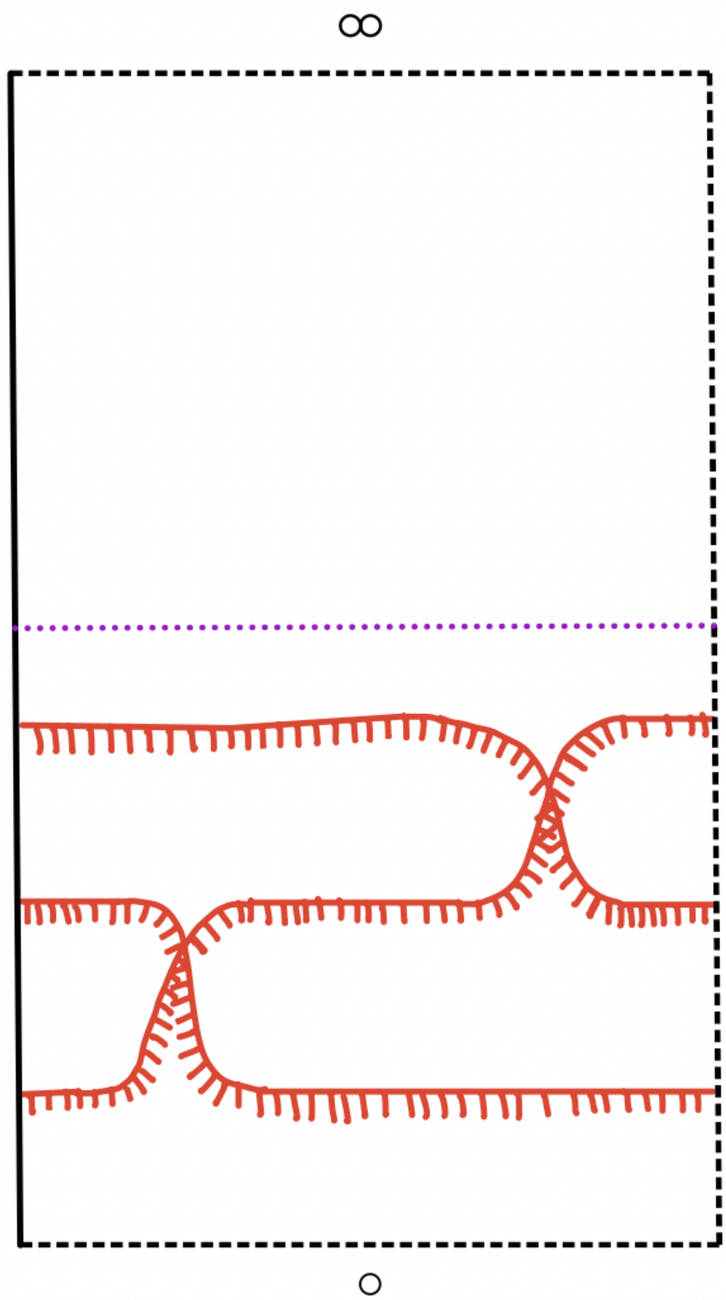


Figure 4.5: embedding of the cylindrical closure onto the hemisphere containing 0

Suppose we have a braid word ω on n strands. Consider the following associated object :

- C : A Riemann sphere with two punctures at 0 and ∞
- $\iota_0 : (S^1)^n -> C$ the link given by the embedding of the cylindrical closure of

the trivial braid word onto the hemisphere containing 0

- ξ_0 : co-orientation of ι_0
- $\iota_\infty : (S^1)^m- > C$ (where $m \leq n$) the link given by the embedding of the cylindrical closure of the braid word ω onto the hemisphere containing ∞
- ξ_∞ : co-orientation of ι_∞

we will denote the union of

- two embeddings ι_0 and ι_∞ as ι
- ξ_0 and ξ_∞ as ξ

Now fix a braid word ω and the object (C, ι, ξ) associated with it. I will define a natural alternating braid diagram (C, ι', ξ') whose associated Legendrian is Legendrian isotopic to the Legendrian associated with (C, ι, ξ) by drawing associated diagram on C. The proof of it will be the main contents of this section. The isotopy will be only applied to ι_0, so the ι_∞ will remain fixed i.e. $\iota_\infty = \iota'_\infty$.

First, let's draw ι'_∞ as in red on C as follows :

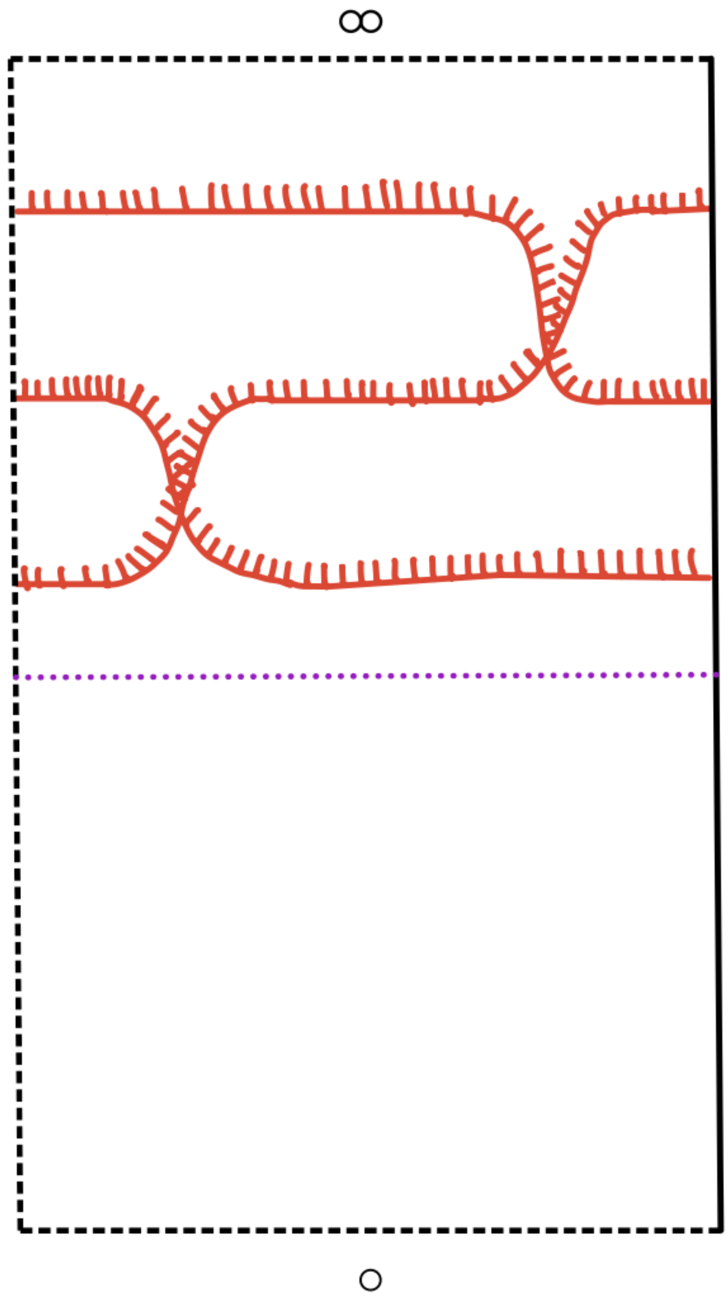


Figure 4.6: Your caption here

Now on the above diagram, let's draw (ι_0') the part that is isotopic to ι_0 in blue. We need some definitions. Suppose $\omega = s_{1_1}, ..., s_{i_k}$, then the cylindirical closure can be parsed into concatenation of k mutually disjoint regions where i^{th} region containing a part of the braid diagram corresponding to the generator s_{i_j}(figure below). We call the region corresponding to s_{i_j} as the j^{th} generator region(also its image inside under the embedding into C).

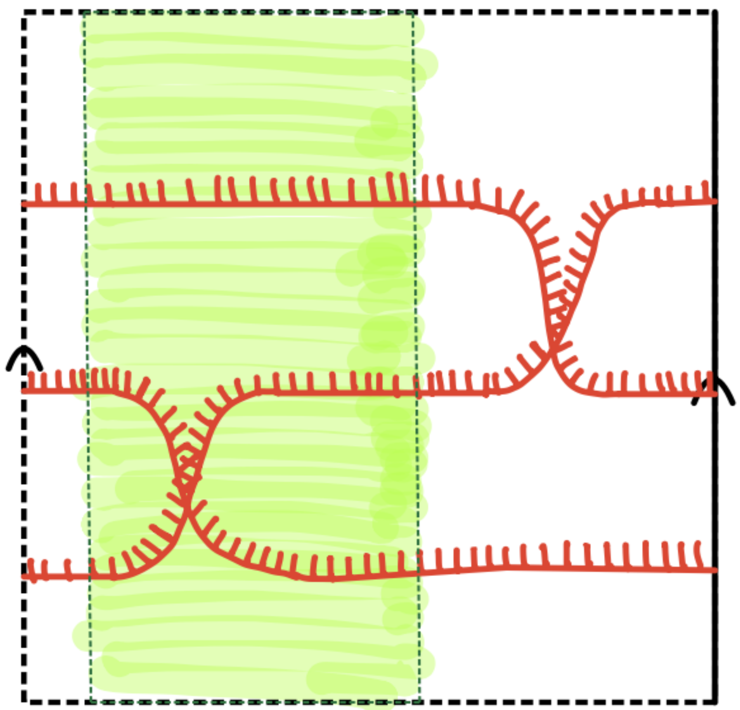


Figure 4.7: 1st generator region

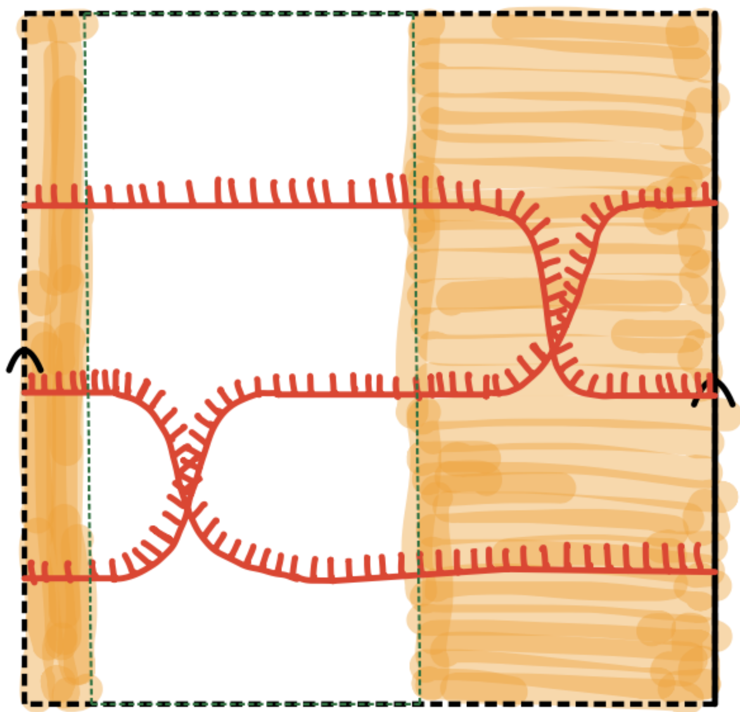


Figure 4.8: 2nd generator region

Suppose we set-theoretically subtract the union of all generator regions from the cylinder, we get k connected components. That is, for each $j = 1, ..., k$, we have one component in between j^{th} and $j+1^{th}$(modulo k) regions. We call the neighborhood of this component inside the cylinder as j^{th} inter-generator region(also its image inside the cylinder under the embedding into C).

- inter-generator regions do not contain any crossing
- inter-generator regions are mutually disjoint
- j^{th} intergenerator region intersects with j^{th} and $j+1^{th}$(modulo k) generator region

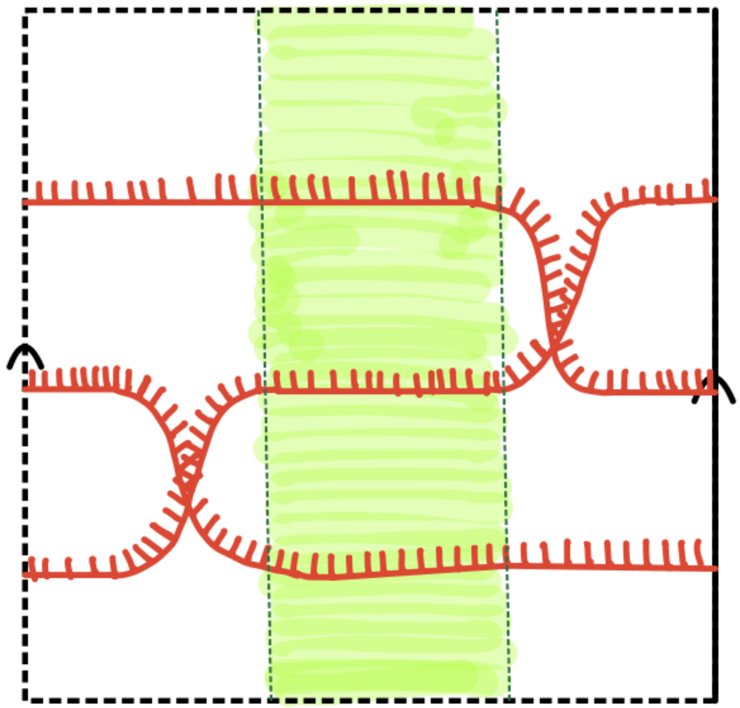


Figure 4.9: 1st inter-generator region

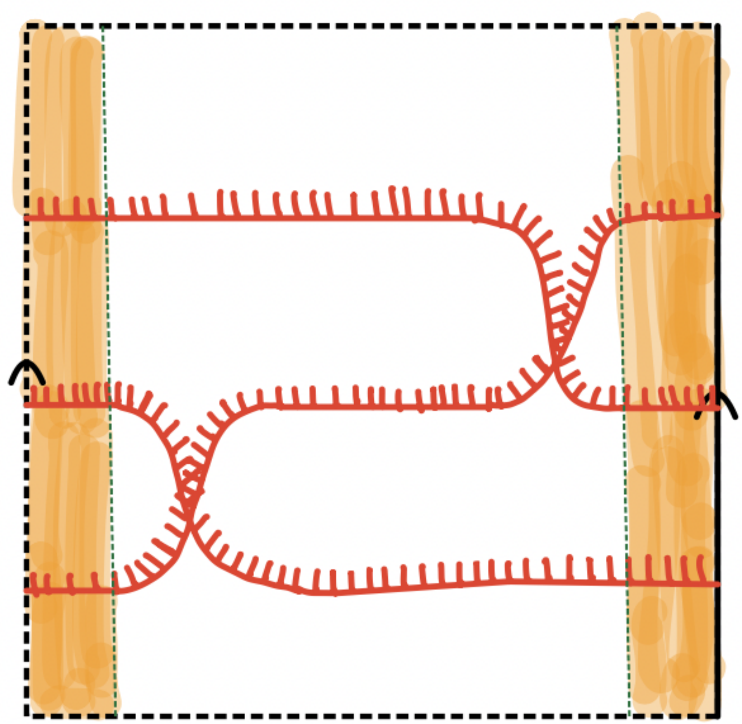


Figure 4.10: 2nd inter-generator region

I will draw ι_0' for each generator region so that they glue up to the whole ι_0'.

First, we restrict the diagram to j^{th} generator region, we have the following diagram:

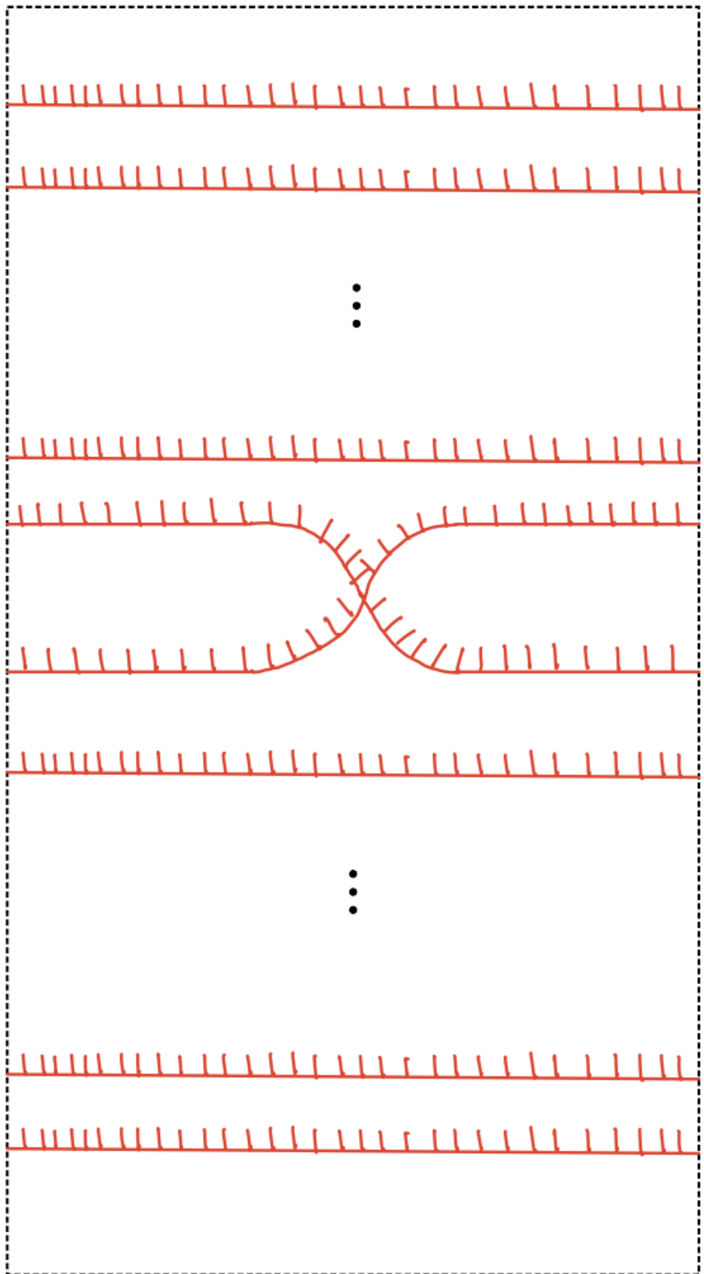


Figure 4.11: Your caption here

notation : we number the strands from top to bottom from $1, \cdots, n$ with reference to the left end points.

I will draw ι_0' as blue strand on it as follows :

- lth blue strand starts from the midpoint of the starting points of lth and l+1th red strands and ends at the midpoint of the end points of lth and l+1th red strands
- if $l \neq i_j$ and $i \neq i_j + 1$, then along the way the l^{th} blue strand crosses up and down twice
- if $l = i_j$, l^{th} blue strand crosses l^{th} red strand up in the part before the crossing and then crosses $l + 1^{th}$ red strand down in the part after the crossing.
- if $l = i_j + 1$, l^{th} blue strand crosses $l + 1^{th}$ red strand up and down in the part before the crossing and then crosses l^{th} red strand up and down in the part after the crossing.

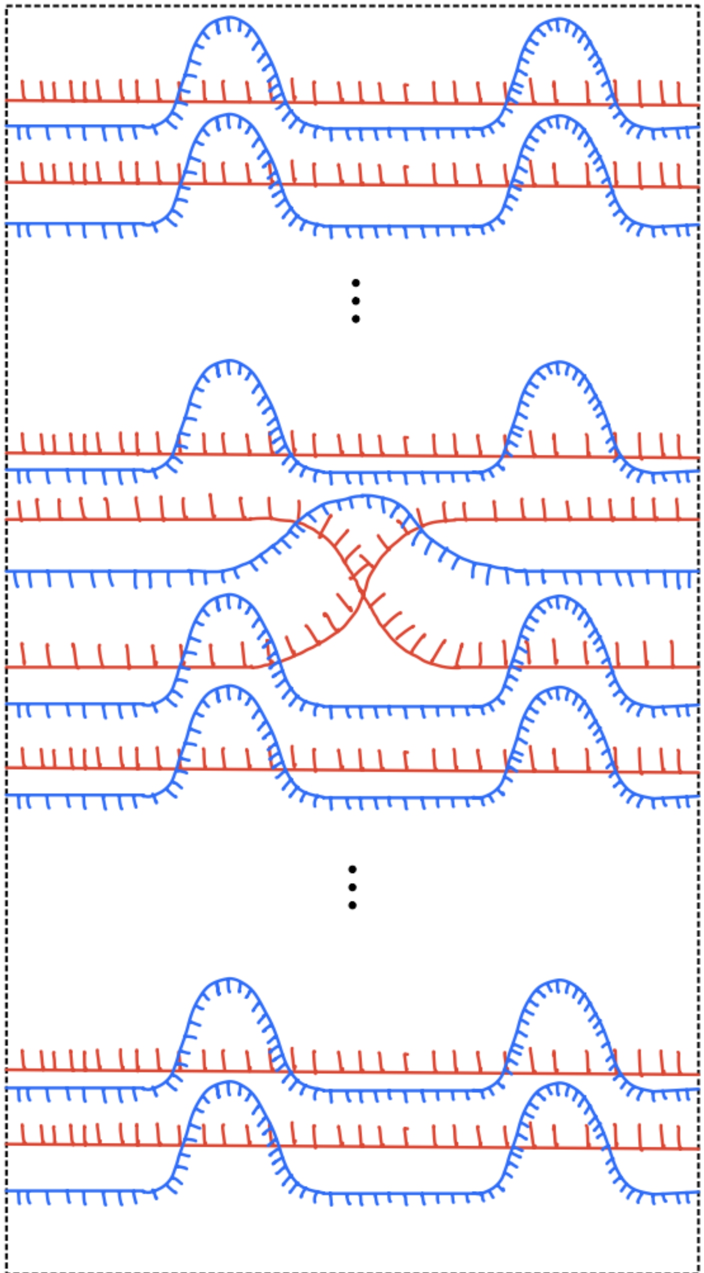


Figure 4.12: Your caption here

For the full alternating strand diagram, we take the closure of blue strands from the generator regions so that the end points from the bordering regions coincide.

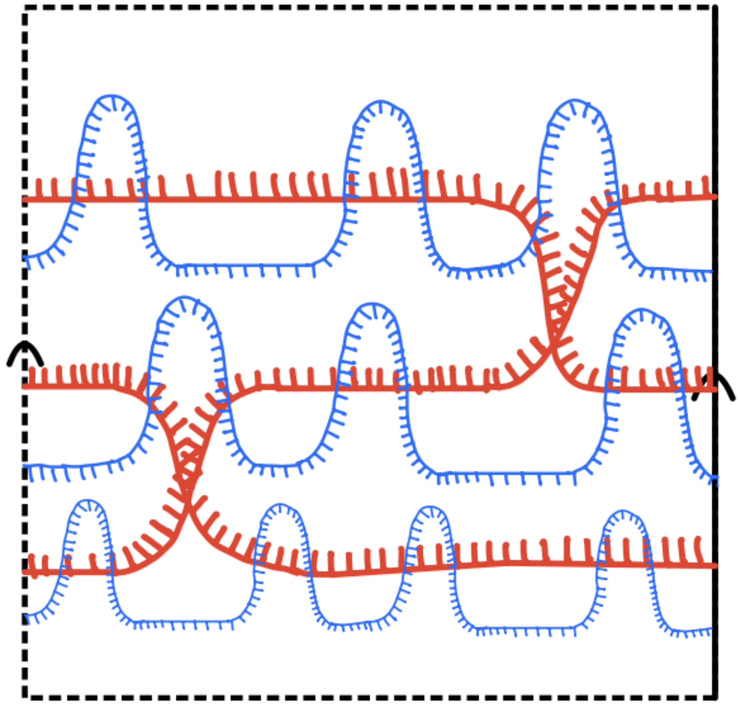


Figure 4.13: Your caption here

Theorem 45. The above defined strand diagram is alternating

(proof) we will denote

- the region with all the hairs pointing outward as $\circ$
- the region with all the hairs pointing inward as $\triangle$
- else with $\times$

for the generator region we have the following figure :

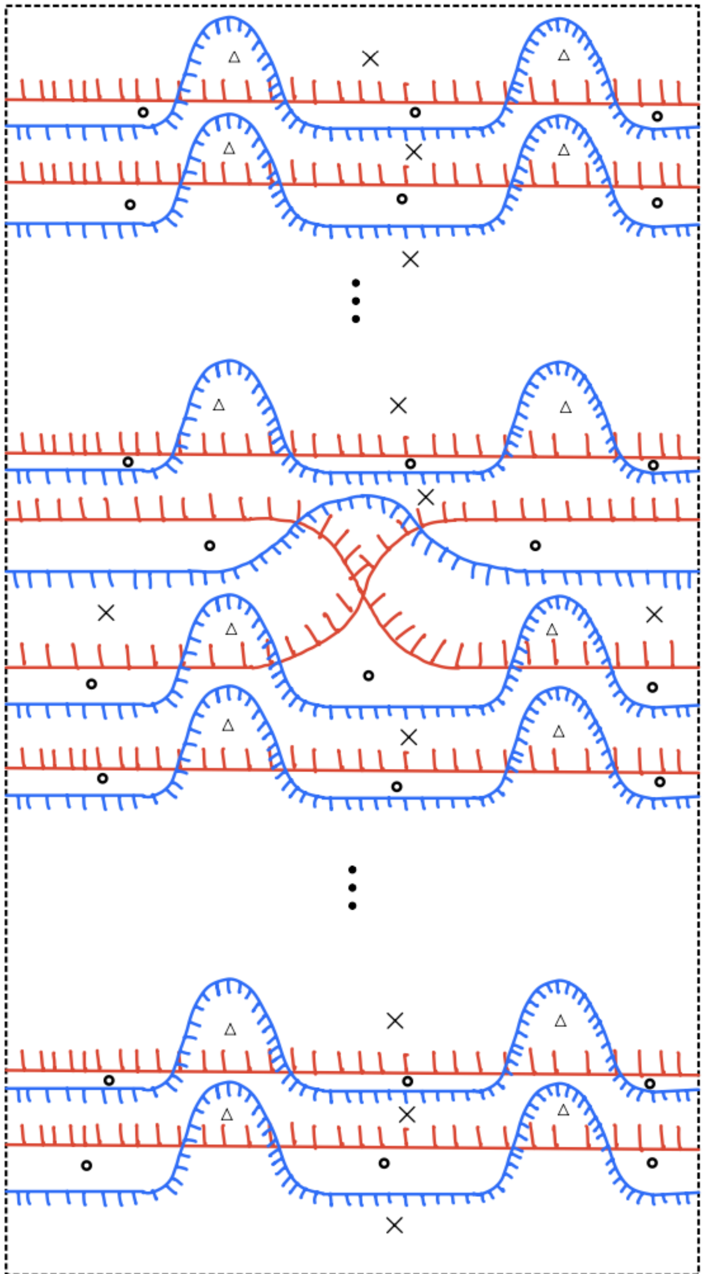


Figure 4.14: Your caption here

for the inter-generator region we have the following figure:

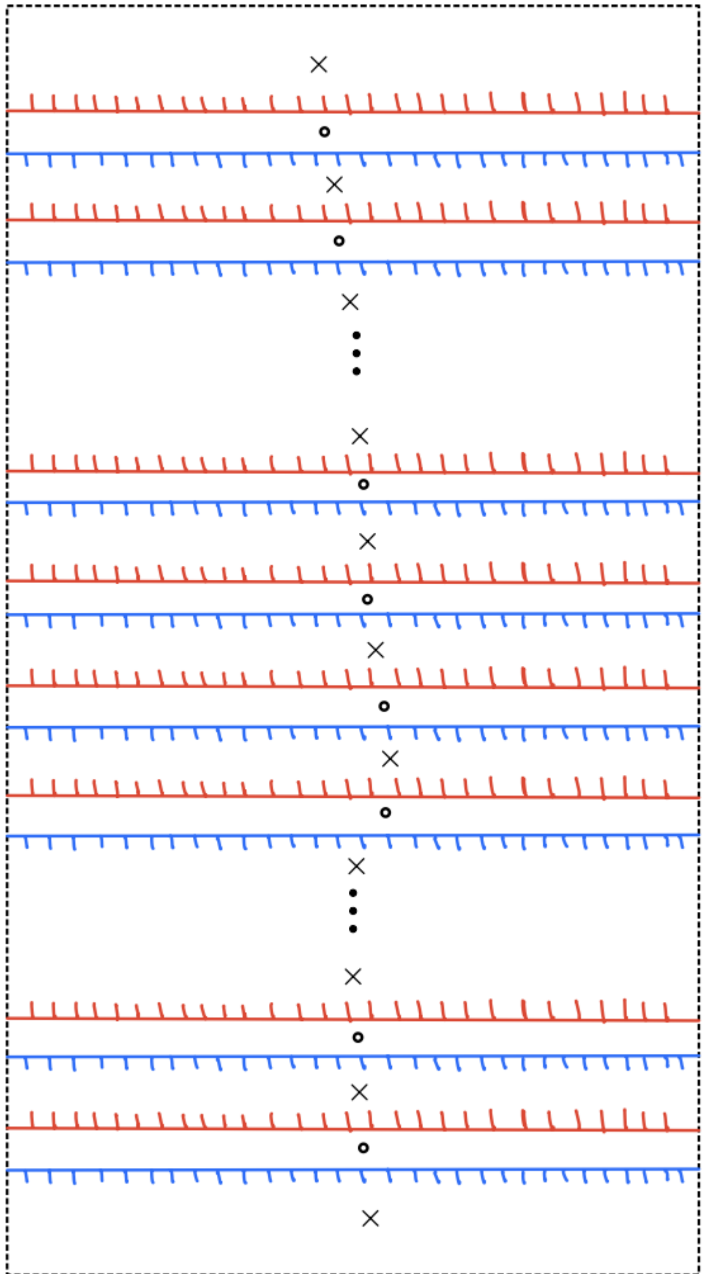


Figure 4.15: Your caption here

for each crossing, it satisfy the alternating condition. This diagram is indeed alternating.

4.2 local systems on natural alternating diagrams

Suppose we have a braid word ω then we have the associated natural alternating diagram (C, ι', ξ') defined in the previous section.

We can associate a quiver Q to the alternating diagram in such a way that :

- we have one vertex for regions where all hairs are pointing outward/inward
- for each crossing, we have an arrow from the vertex corresponding to the region where all hairs pointing outward to inward

Once we have an alternating strand diagram, we have the associated spectral curve SC. Furthermore, we can embedd the underlying undirected graph of the quiver Q in SC in such a way that SC deformation retracts to Q(with abuse of notation I will denote this underlying undirected graph as Q)(STZ) Suppose we have a local system on the spectral curve associated with (C, ι', ξ'), then the restricting to Q, we get a local system on Q. Note that the pullback, induced by the restriction map, between the space of local systems $H^1(SC, \mathbb{C}^*) \to H^1(Q, \mathbb{C}^*)$ is an isomorphism.

$H^1(Q, \mathbb{C}^*)$ is isomorphic to $(\mathbb{C}^*)^{|Arr(Q)|}//(\mathbb{C}^*)^{|Vert(Q)|}$ here the group action is defined as the following : let $g_v in (C^*)^{|Vert(Q)|}$ what I mean by g_v, only supported at index v with value g_v. For other indices the entries are 1. then $g_v \cdot (x_a)_{a \in Arr(Q)}$ is

- for entries with index a such that the source of a is v i.e. $s(a) = v$, we have $g_v.x_a$
- for entries with index a such that the target of a is v i.e. $t(a) = v$, we have $g_v^{-1} \cdot x_a$

Now we define the associated constructible sheaf on some regular cell complex refinement of the natural alternating strand diagram associated with local systems on Q.

First, I will describe the special kind of regular cell complex associated with the alternating strand diagram.

Suppose we fix a generator region for the alternating strand diagram. Then we label j^{th} crossing(numbering starts from left to right) the i^{th} blue strand(numbering starts from top to bottom) crosses red strands as $c_{i,j}$. We will call the crossing between i^{th} and $i+1^{th}$ red strand as c.

Suppose we have an alternating diagram. Then for each crossing $c_{i,j}$ we add - two 0 dimensional strata at the interiors of the 1 dimensional strata at the northwest(northeast resp.) and southwest(southeast resp.) of the crossing when j is even(odd resp.) - one 1 dimensional stratum at the interior of the west region of the crossing connecting the above two 0 dimensional strata. We will draw this line in a squiggly line Also for the crossing c, we add - two 0 dimensional strata at the interiors of the 1 dimensional strata at the northwest and southwest of the crossing.

Below is the picture of a generator region of a natural alternating diagram(figure1), labeling of the crossings(figure2), and its regular cell complex refinement(figure3)

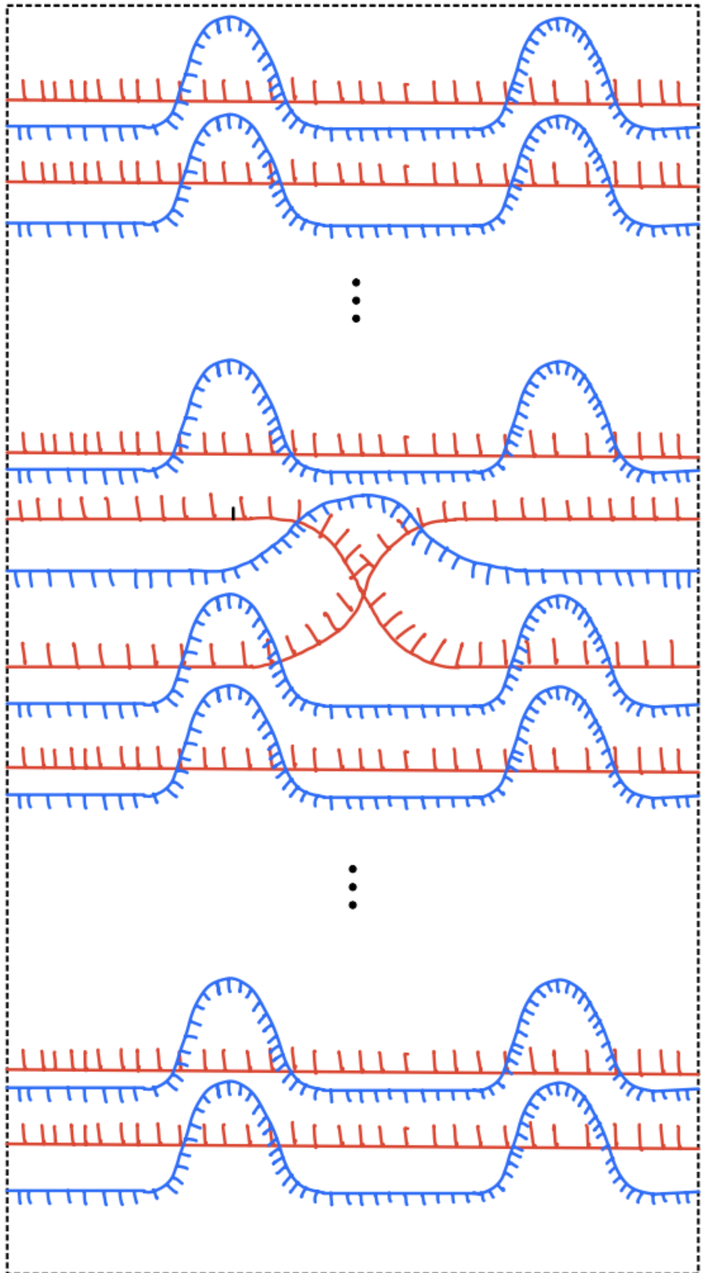


Figure 4.16: Your caption here

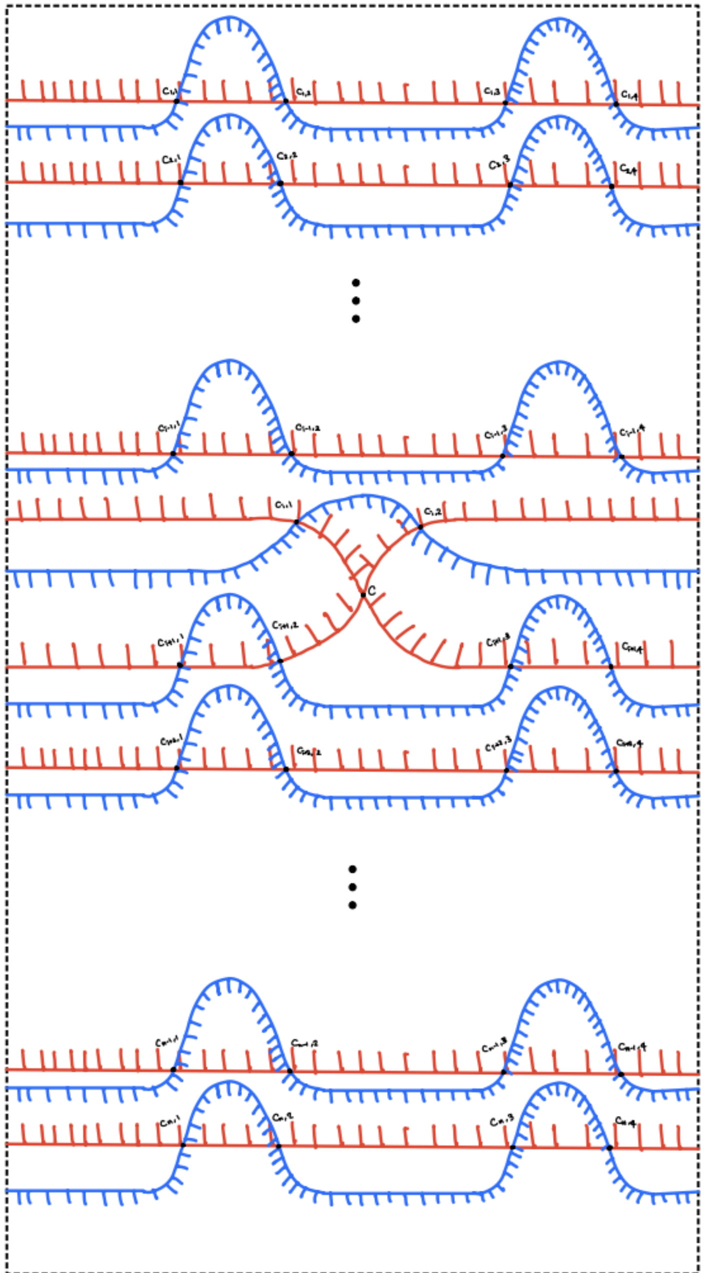


Figure 4.17: Your caption here

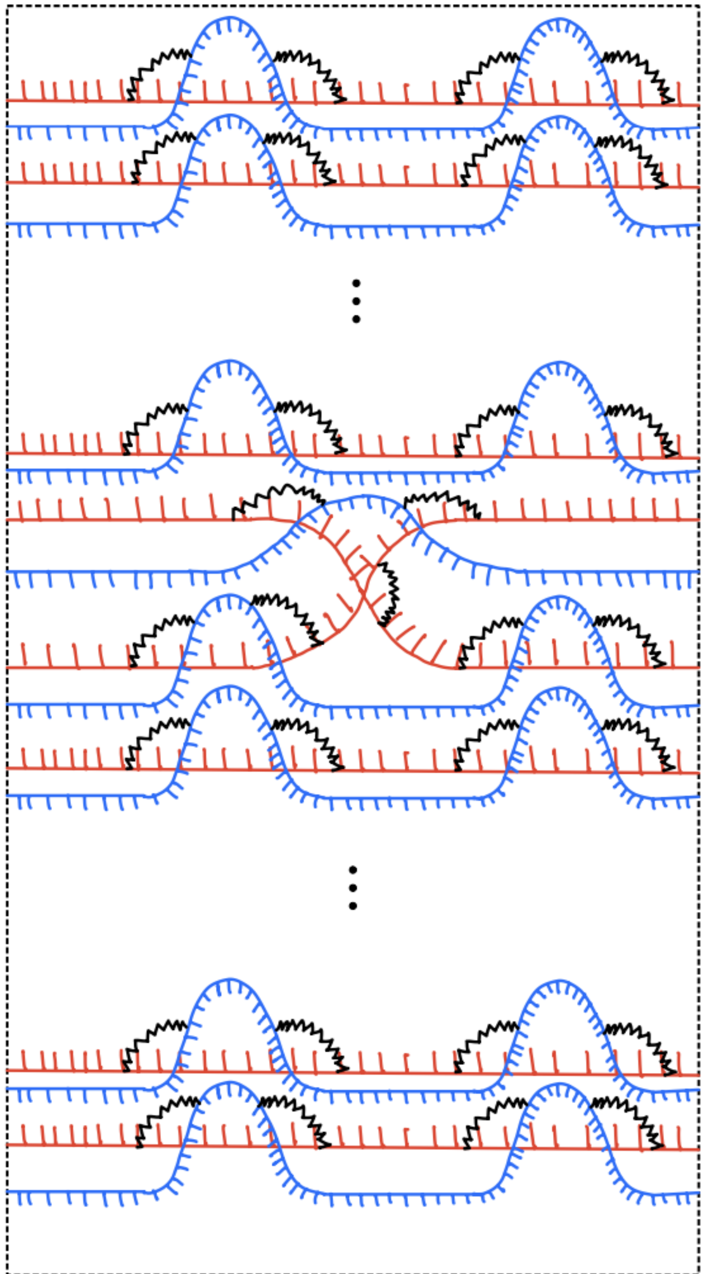


Figure 4.18: Your caption here

Now I will describe a way to specify a constructible sheaf on the above regular cell complex refinement associated with the local system on Q. suppose we have a local system on Q which can be represented as an element of $(\mathbb{C}^*)^{|Arr(Q)|}$:

(i) stalk of the region where all the hairs are pointing outward is $\mathbb{C}[-1]$

(ii) stalk of the region where all the hairs are pointing inward is $\mathbb{C}$

(iii) stalk of the region that has crossing at its boundary and surrounded by a squiggly line is $\mathbb{C} \to \mathbb{C}$ where the map is multiplication by g_a where a is the arrow goes from the south of the crossing to the north of the crossing

(iv) the only nonzero genrization maps are region of type (iii) to (i) or (i) to (ii)

In the first case, the map is

$$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

In the latter case, the map is

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

For example, suppose we have a regular cell complex refinement of a generator region and the associated quiver Q as follows:

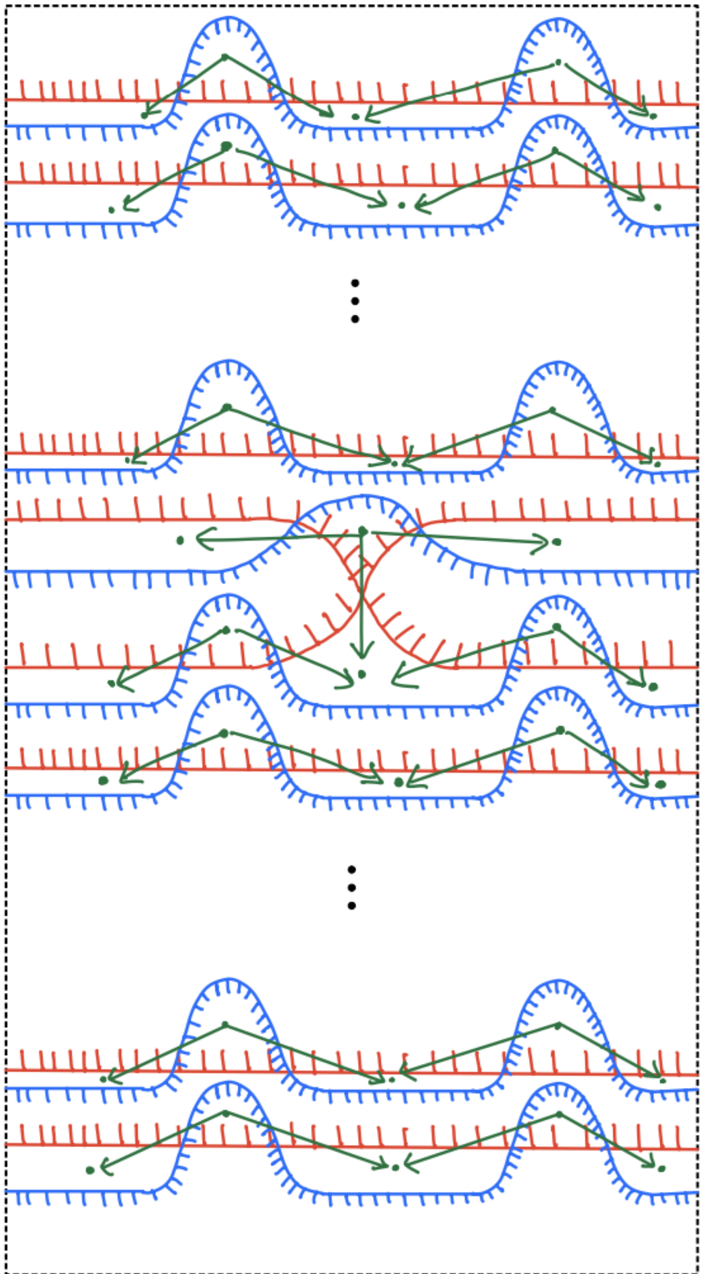


Figure 4.19: Your caption here

and a local system represented by $(g_{a_i})_{a_i \in Arr(Q)}$ then we get the associated constructible sheaf:

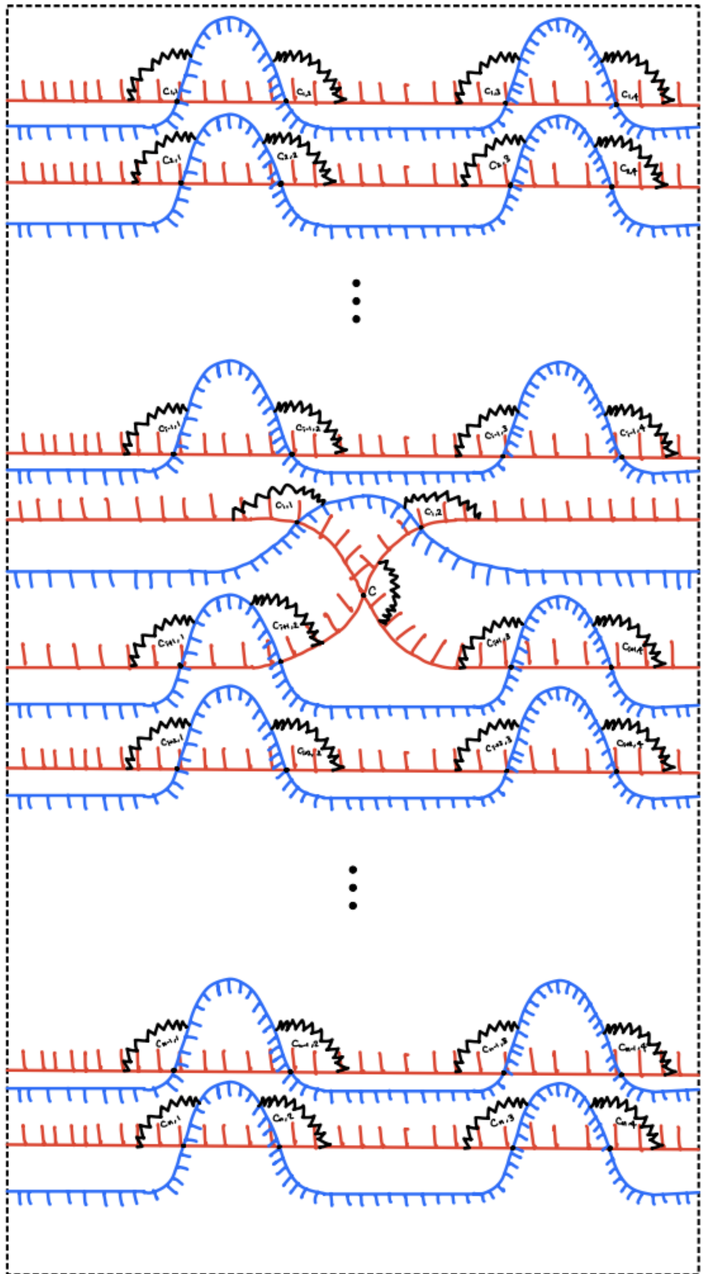


Figure 4.20: Your caption here

Note that the group action maps a constructible sheaf to the isomorphic constructible sheaf. Therefore, we have a well defined map $H^1(SC, \mathbb{C}^*) \to \mathcal{M}_{\overline{(C,\iota',\xi')}}(C)$ where $\overline{(C, \iota', \xi')}$ is the regular cell complex refinement of the natural alternating diagram (C, ι', ξ').

4.3 lemma1

Suppose we have a Riemann sphere C and two diagrams (C, ι, ξ) and (C, ι', ξ') where they differ only in a small disk $D \subset C$. Let D' be a slightly smaller co-centric disk inside D. Suppose there is a diffeomorphism between D and $\{(x, y) \mid x^2 + y^2 < 4\}$ such that under this diffeomorphism

- D' is sent to $\{(x, y) \mid x^2 + y^2 < 1\}$
- the blue strand is sent to a strand that is isotopic to $\{(x, y) \mid y = \frac{1}{2}\}(\{(x, y) \mid y = \frac{5}{3}x^2 - \frac{3}{4}\}$ resp.)
- the red strand is sent to a strand that is isotopic to $\{(x, y) \mid y = -\frac{1}{2}\}(\{(x, y) \mid y = \frac{1}{2}\}$ resp.)

Below figures show the diagrams on D.

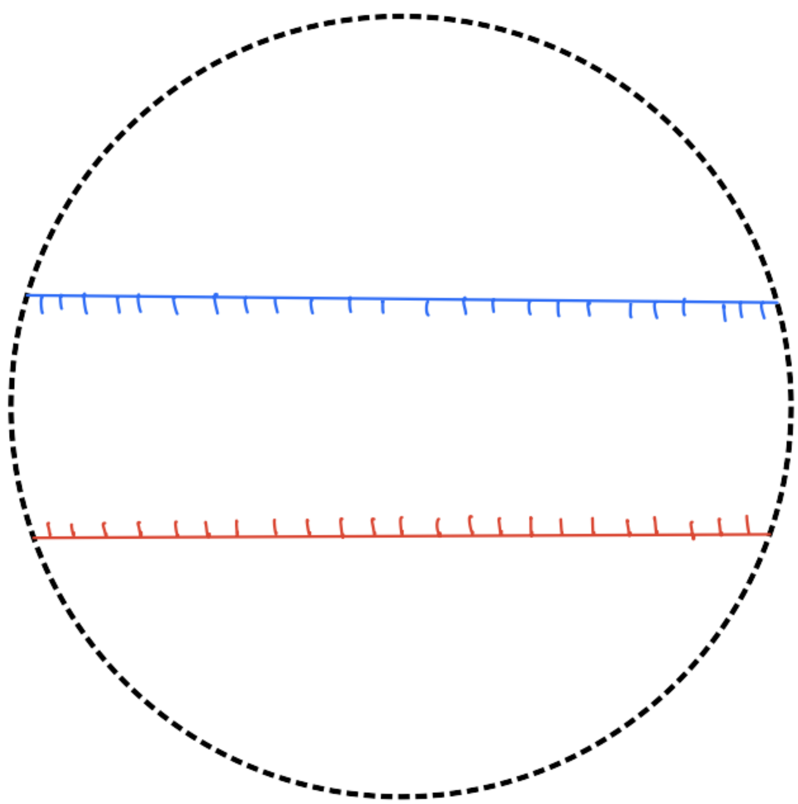


Figure 4.21: Your caption here

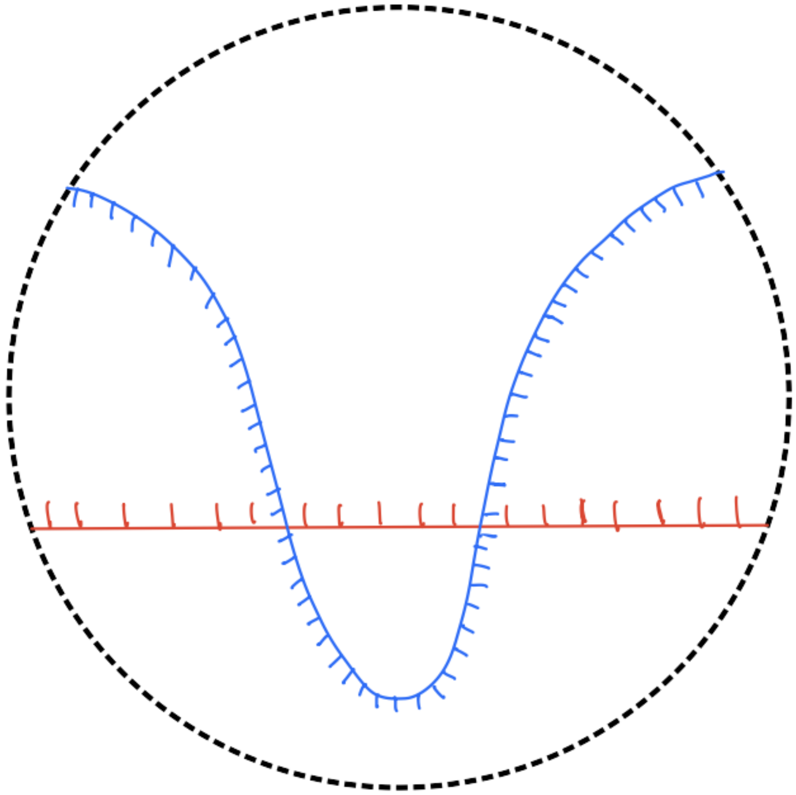


Figure 4.22: Your caption here

isotopy$_1$ defined in Definition1 defines an isotopy from (C, ι, ξ) to (C, ι', ξ'). It defines a map $\mathcal{M}_{(C,\iota,\xi)} \to \mathcal{M}_{(C,\iota',\xi')}$. Under this map, a sheaf on (C, ι, ξ) described in the figure3 below is mapped to the sheaf on (C, ι', ξ') described in the figure4 below(figures below show only on D where they differ because the stalks and generization maps away from the region remains the same).

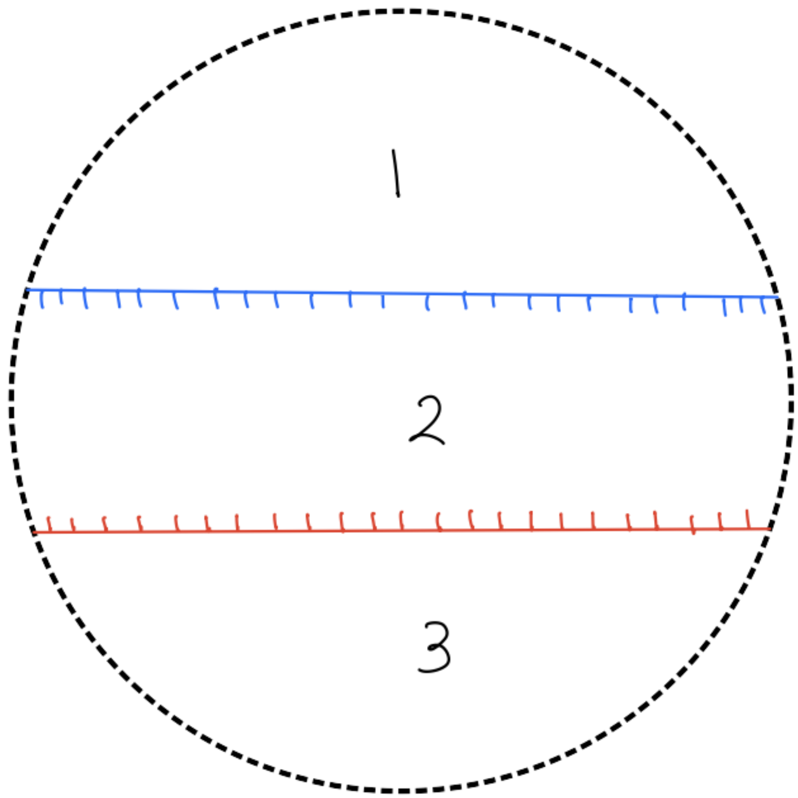


Figure 4.23: Your caption here

Stalks :

- 1 : $\mathbb{C}^{m+1}$
- 2 : $\mathbb{C}^{m}$
- 3 : $\mathbb{C}^{m+1}$

Generization maps :

- $2 \to 1$: $\mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_{i+1}$
- $2 \to 3$: $\mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_i$

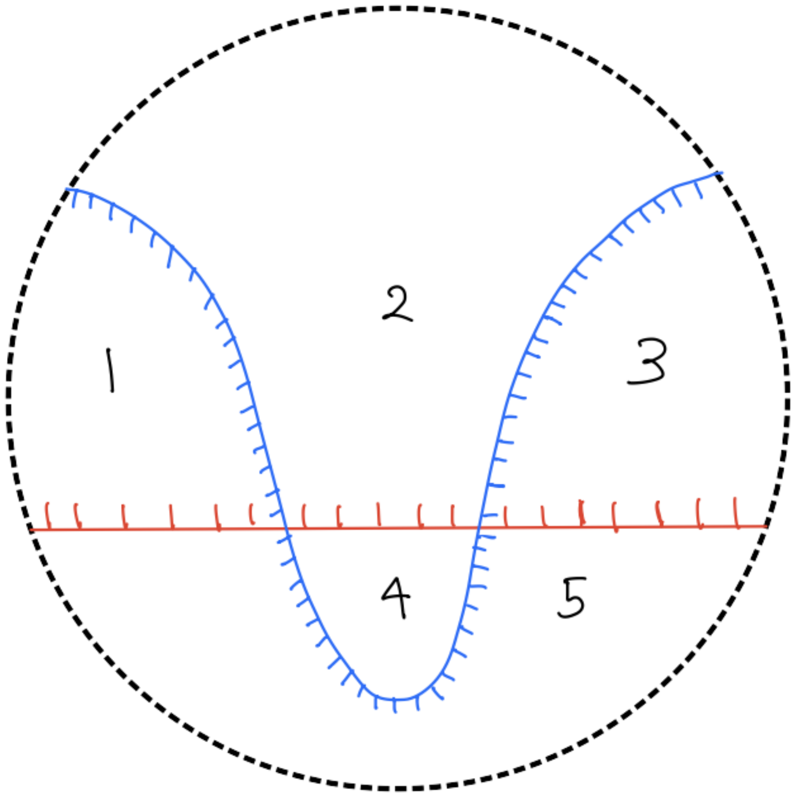


Figure 4.24: Your caption here

Stalks :

- 1 : $\mathbb{C}^m$
- 2 : $\mathbb{C}^{m+1}$
- 3 : $\mathbb{C}^m$
- 4 : $\mathbb{C}^{m+2}$
- 5 : $\mathbb{C}^{m+1}$

Generization maps :

- $1 \to 2 : \mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_{i+1}$
- $3 \to 2 : \mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_{i+1}$

- $1 \to 5 : \mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_i$
- $3 \to 5 : \mathbb{C}^m \to \mathbb{C}^{m+1}$ where $e_i \mapsto e_i$
- $5 \to 4 : \mathbb{C}^{m+1} \to \mathbb{C}^{m+2}$ where $e_i \mapsto e_{i+1}$
- $2 \to 4 : \mathbb{C}^{m+!} \to \mathbb{C}^{m+2}$ where $e_i \mapsto e_i$

(proof)

4.4 lemma2(core)

Lemma 46.

Suppose we have a Riemann Sphere C and two diagrams (C, ι, ξ),(C', ι', ξ'), their regular cell complex refinements $\overline{(C, \iota, \xi)}$,$\overline{(C', \iota', \xi')}$ and sheaves $\mathfrak{F}$,$\mathfrak{F}'$ on them such that they differ only on a small disk $D \subset C$.

Below figures show only $\mathfrak{F}$,$\mathfrak{F}'$ restricted to D.

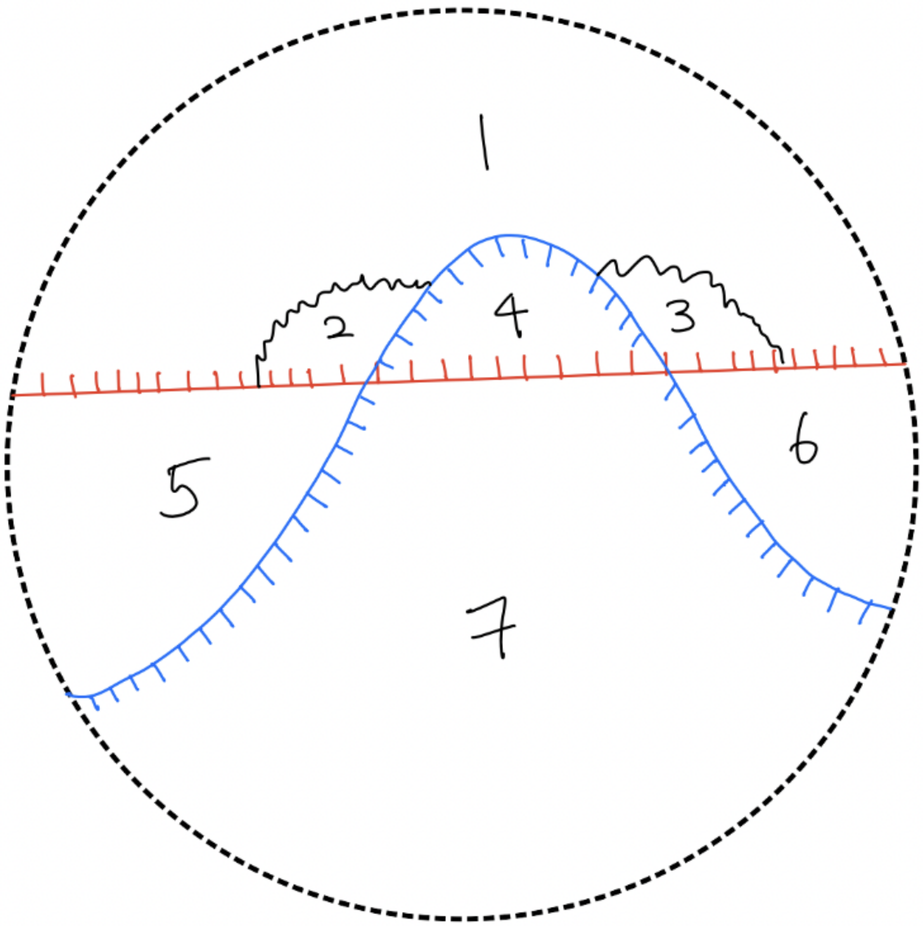


Figure 4.25: Your caption here

Stalks :

- 1 : 0
- 2 : $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- 3 : $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- 4 : $\mathbb{C}[-1]$
- 5 : $\mathbb{C}$
- 6 : $\mathbb{C}$
- 7 : 0

Generization maps :

- 4→ 1 : zero map

- 2→ 1 : zero map

- 3→ 1 : zero map

- 1→ 5 : zero map

- 1→ 6 : zero map

- 7→ 5 : zero map

- 7→ 6 : zero map

- 4→ 2: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$

- 4→ 3: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$

- 2→ 5: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- 3→ 6: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

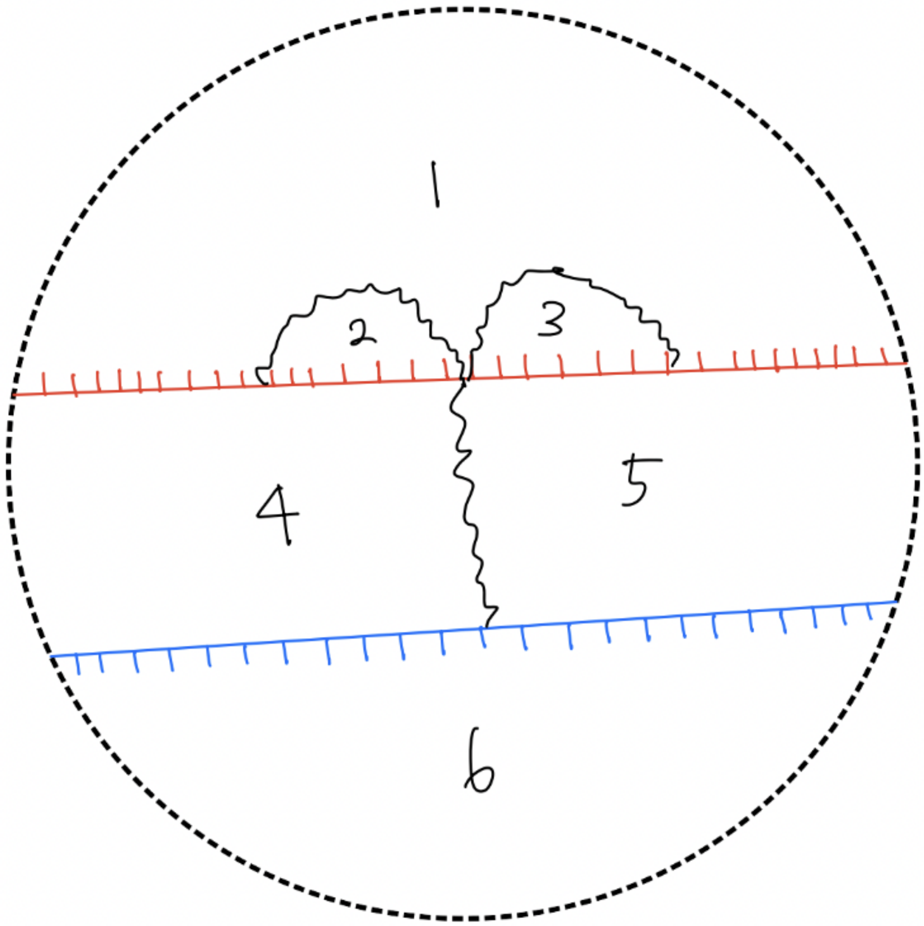


Figure 4.26: Your caption here

Stalks :

$1 : 0$

$2 : \mathbb{C} \xrightarrow{\times a} \mathbb{C}$

$3 : \mathbb{C} \xrightarrow{\times b} \mathbb{C}$

$4 : \mathbb{C}$

$5 : \mathbb{C}$

$6 : 0$

Generization maps :

$2 \to 1$: zero map

3→1 : zero map

6→4 : zero map

6→5 : zero map

2→ 4:
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

3→ 5:
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

4→ 5: $\mathbb{C} \xrightarrow{\times ab^{-1}} \mathbb{C}$

We define an isotopy between $\mathfrak{F}$ and $\mathfrak{F}'$ called $isotopy_2$ as follows :

(i) The trajectory of the red strand in $D \times [0,1]$ is $\{(x,y,t) \in D \times [0,1] \mid y = \frac{1}{2}\}$

(ii) The trajectory of blue strand in $D \times [0,1]$ is $\{(x,y,t) \in D \times [0,1] \mid y = \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}\}$. Note that when $t = t_0$, the above set is a parabola passing through $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(0, \frac{5}{4}t_0 + \frac{3}{4})$

(iii) The trajectory of the left squiggly line is $\{(x,y,t) \in D \times [0,1] \mid y = \sqrt{\frac{1}{16} - (x+\frac{1}{4})^2}, y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}\}$

(iv) The trajectory of the right squiggly line $\{(x,y,t) \in D\times[0,1] \mid y = \sqrt{\frac{1}{16} - (x-\frac{1}{4})^2}, y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}\}$

(v) The trajectory of the middle squiggly line is$\{(x,y,t) \in D \times [0,1] \mid x = 0, y \geq -\frac{1}{2}, y \leq -\frac{5}{4}t + \frac{3}{4}\}$

Now I will define sheaf on $D \times [0,1]$ singular supported on $(i, \cdots, v)$ such that restricted to $D \times \{0\}(D \times \{1\}$ resp.) is $\mathfrak{F}(\mathfrak{F}'$ resp.)

We can describe the sheaf by assigning stalks to the 7 regions separated by $(i, \cdots, v)$ and generization maps between them.

Let's list the seven regions :

region 1.$\{(x,y,t) \in D \times [0,1] \mid y \geq \sqrt{\frac{1}{16} - (x+\frac{1}{4})^2}, y \geq \sqrt{\frac{1}{16} - (x-\frac{1}{4})^2}, y \geq \frac{1}{2}\}$

region2. $\{(x,y,t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \leq \sqrt{\frac{1}{16} - (x+\frac{1}{4})^2}, y \geq \frac{1}{2}\}$

region3. $\{(x,y,t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \leq \sqrt{\frac{1}{16} - (x-\frac{1}{4})^2}, y \geq \frac{1}{2}\}$

region4. $\{(x,y,t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \geq \frac{1}{2}\}$

region5. $\{(x,y,t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \leq \frac{1}{2}, x \leq 0\}$

region6. $\{(x,y,t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \leq \frac{1}{2}, x \geq 0\}$

region7. $\{(x,y,t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4}t + \frac{3}{4}, y \leq \frac{1}{2}\}$

Now let's define

Stalks:

- 1 : 0
- 2 : $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- 3 : $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- 4 : $\mathbb{C}[-1]$
- 5 : $\mathbb{C}$

- 6 : $\mathbb{C}$

- 7 : 0

Generization maps :

- 4→ 1 : zero map

- 2→ 1 : zero map

- 3→ 1 : zero map

- 1→ 5 : zero map

- 1→ 6 : zero map

- 7→ 5 : zero map

- 7→ 6 : zero map

- 4→ 2: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$

- 4→ 3: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$

- 2→ 5: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r, "id"] \arrow[u, "\times a"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- 3→ 6: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r, "id"] \arrow[u, "\times b"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- 5→ 6: $\mathbb{C} \xrightarrow{\times ab^{-1}} \mathbb{C}$

Below we will prove that this is a well-defined isotopy between $\mathfrak{F}$ and $\mathfrak{F}'$.

(proof of well-definedness)

4.5 lemma3(core)

Lemma 47.

Suppose we have a Riemann sphere C and two diagrams (C, ι, ξ),(C, ι', ξ') and their refinements $\overline{(C, \iota, \xi)}$,$\overline{(C, \iota', \xi')}$ and sheaves $\mathfrak{F}$,$\mathfrak{F}'$ on them such that they differ only on a small disk $D \subset C$.

Below figures show only $\mathfrak{F}$, $\mathfrak{F}'$ restricted to D.

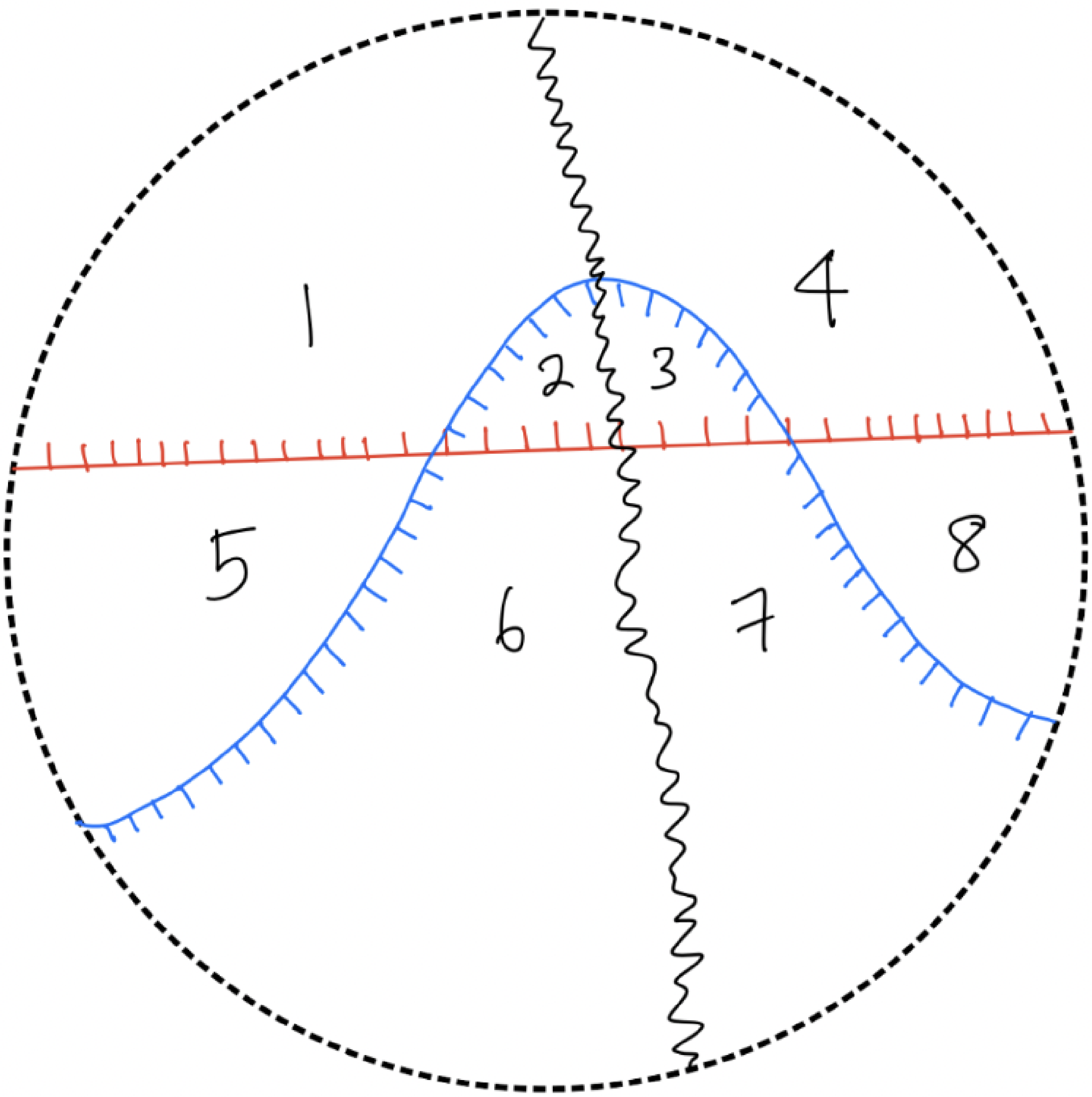


Figure 4.27: Your caption here

Stalks :

- 1 : $\mathbb{C}^{m+1}$
- 2 : $\mathbb{C}^{m}$
- 3 : $\mathbb{C}^{m}$
- 4 : $\mathbb{C}^{m+1}$
- 5 : $\mathbb{C}^{m+2}$
- 6 : $\mathbb{C}^{m+1}$
- 7 : $\mathbb{C}^{m+1}$
- 8 : $\mathbb{C}^{m+2}$

Generization maps :

- 2→1 : ι_l
- 6→5 : ι_l
- 1→4 : ι_l
- 1→5 : ι_f
- 2→6 : ι_f
- 1→4 : f
- 2→3 : g
- 6→7 : h
- 3→4 : h'
- 7→8 : g'
- 3→7 : h''
- 4→8 : f''

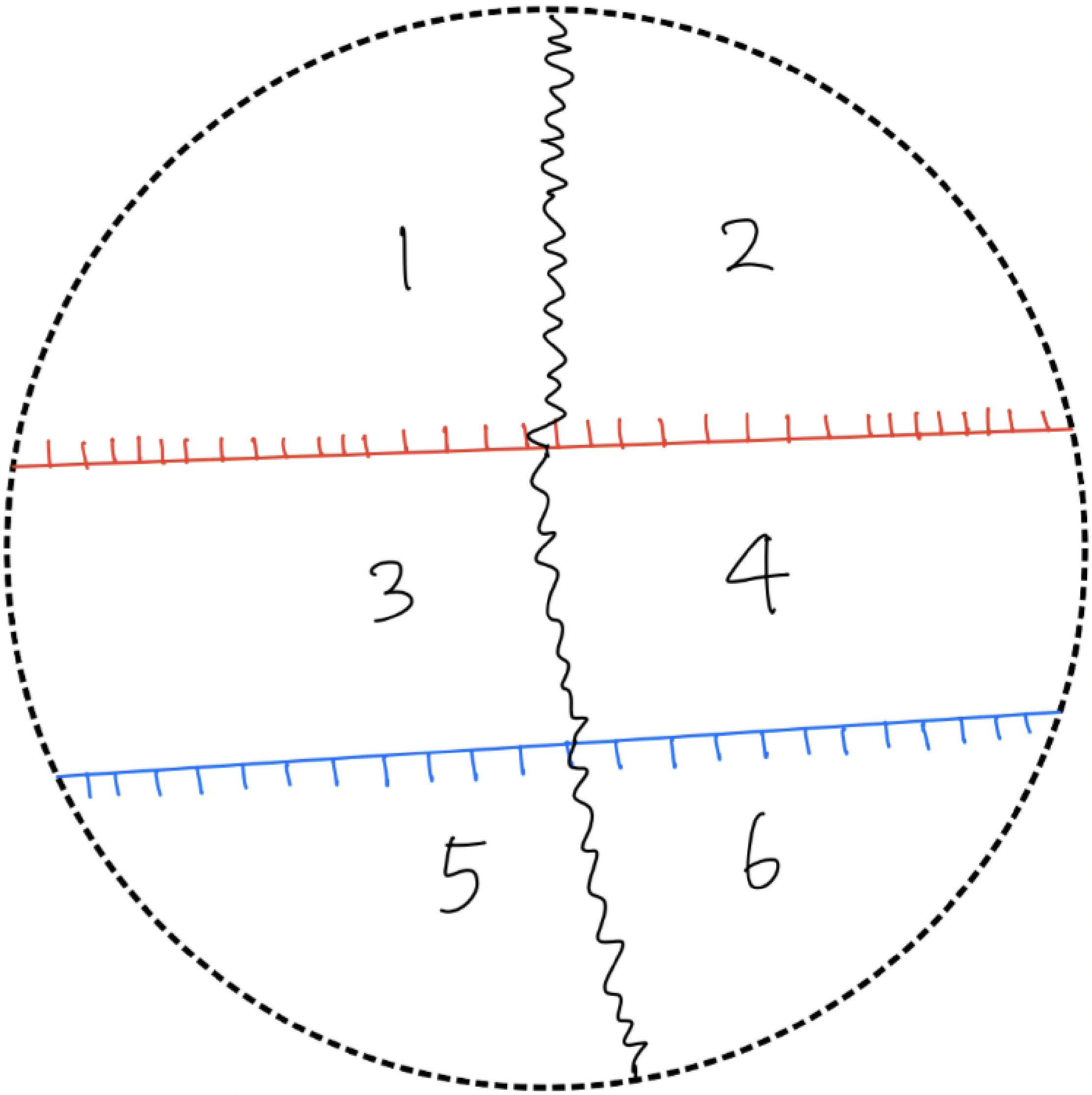


Figure 4.28: Your caption here

Stalks :

- 1 : $\mathbb{C}^{m+1}$

- 2 : $\mathbb{C}^{m+2}$

- 3 : $\mathbb{C}^{m+1}$

- 4 : $\mathbb{C}^{m+1}$

- 5 : $\mathbb{C}^{m+2}$

- 6 : $\mathbb{C}^{m+1}$

Generization maps :

- 1→3 : ι_f
- 2→4 : ι_f
- 5→3 : ι_l
- 6→4 : ι_l
- 1→2 : f
- 5→6 : g
- 2→4 : f'
- 6→4 : g'
- 3→4 : T

where

- $T_{(1,1),(m+2,m+1)} = f' \circ f$
- $T_{(1,2),(m+2,m+2)} = g' \circ g$

We define an isotopy between $\mathfrak{F}$ and $\mathfrak{F}'$ called $isotopy_3$ as follows :

(i) The trajectory of red strand in $D \times [0,1]$ is $\{(x,y,t) \in D \times [0,1] \mid y = \frac{1}{2}\}$

(ii) The trajectory of blue strand in $D \times [0,1]$ is $\{(x,y,t) \in D \times [0,1] \mid y = \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}\}$. When $t = t_0$, the above set is a parabola passing through $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(0, -\frac{5}{4}t_0 + \frac{3}{4})$

(iii) The trajectory of squiggly line in $D \times [0,1]$ is $\{(x,y,t) \in D \times [0,1] \mid x = 0\}$

Now we will describe the sheaf isotopy from $\mathfrak{F}$ to $\mathfrak{F}'$ by assigning stalks to the 8 regions separated by (i, ii, iii) and generization maps between them.

First let's list the 8 regions :

1. $\{(x, y, t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \geq \frac{1}{2}, x \leq 0\}$

2. $\{(x, y, t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \geq \frac{1}{2}, x \geq 0\}$

3. $\{(x, y, t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \geq \frac{1}{2}, x \leq 0\}$

4. $\{(x, y, t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \geq \frac{1}{2}, x \geq 0\}$

5. $\{(x, y, t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \leq \frac{1}{2}, x \leq 0\}$

6. $\{(x, y, t) \in D \times [0,1] \mid y \geq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \leq \frac{1}{2}, x \geq 0\}$
7. $\{(x, y, t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \leq \frac{1}{2}, x \leq 0\}$

8. $\{(x, y, t) \in D \times [0,1] \mid y \leq \frac{5}{3}(t-1)x^2 - \frac{5}{4} + \frac{3}{4}, y \leq \frac{1}{2}, x \geq 0\}$

Now let's define :

Stalks :

- 1 : $\mathbb{C}^{m+1}$
- 2 : $\mathbb{C}^{m+1}$
- 3 : $\mathbb{C}^{m}$
- 4 : $\mathbb{C}^{m}$

- 5 : $\mathbb{C}^{m+2}$

- 6 : $\mathbb{C}^{m+2}$

- 7 : $\mathbb{C}^{m+1}$

- 8 : $\mathbb{C}^{m+1}$

Generization maps :

- 3→1 : ι_l

- 7→5 : ι_l

- 1→5 : ι_f

- 3→7 : ι_f

- 1→2 : f

- 3→4 : h

- 7→8 : g

- 4→2 : h'

- 8→6 : g'

- 4→8 : h''

- 2→6 : f'

- 5→6 : T

Below we will prove that this is a well-defined isotopy between $\mathfrak{F}$ and $\mathfrak{F}'$

(proof of well-definedness)

4.6 lemma4(core)

Lemma 48.

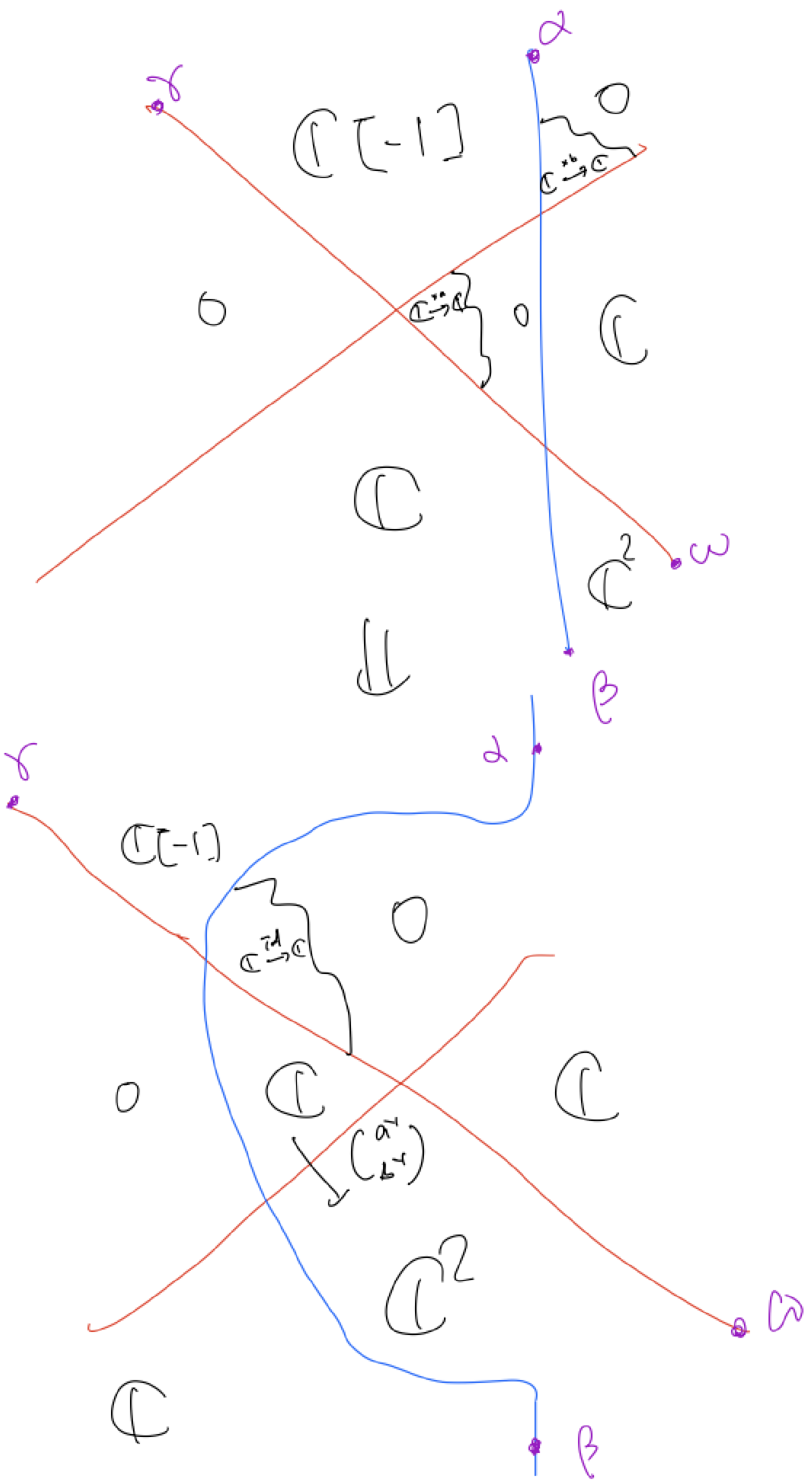


Figure 4.29: Your caption here

4.7 definition5

Definition 49.

Suppose we have the following local diagram:

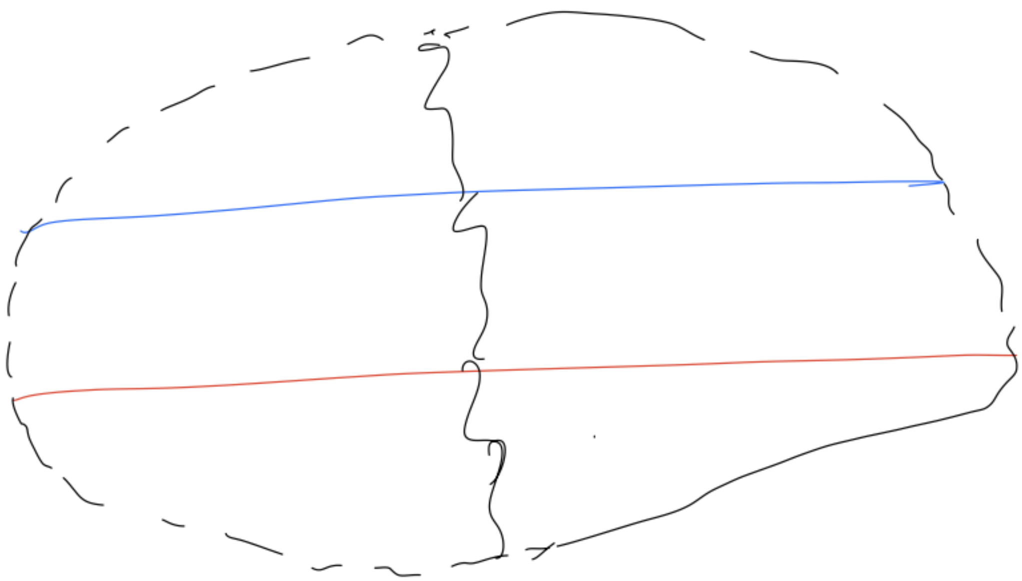


Figure 4.30: Your caption here

Let's apply a collection of moves as follows:

(Step1) Apply MOVE ito the regions surrounded by purple dotted lines:

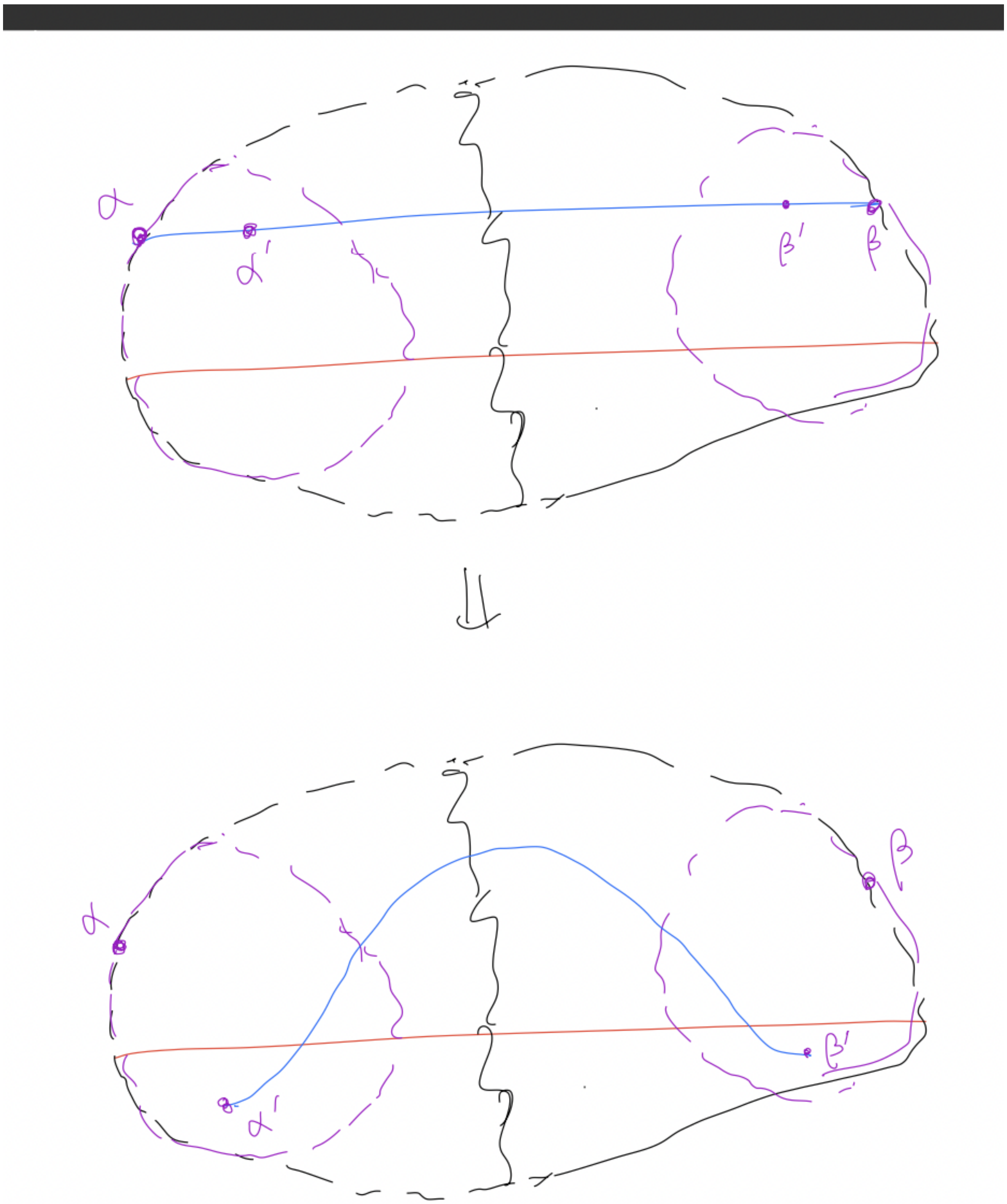


Figure 4.31: Your caption here

(Step2) Apply MOVE iiito the following diagram :

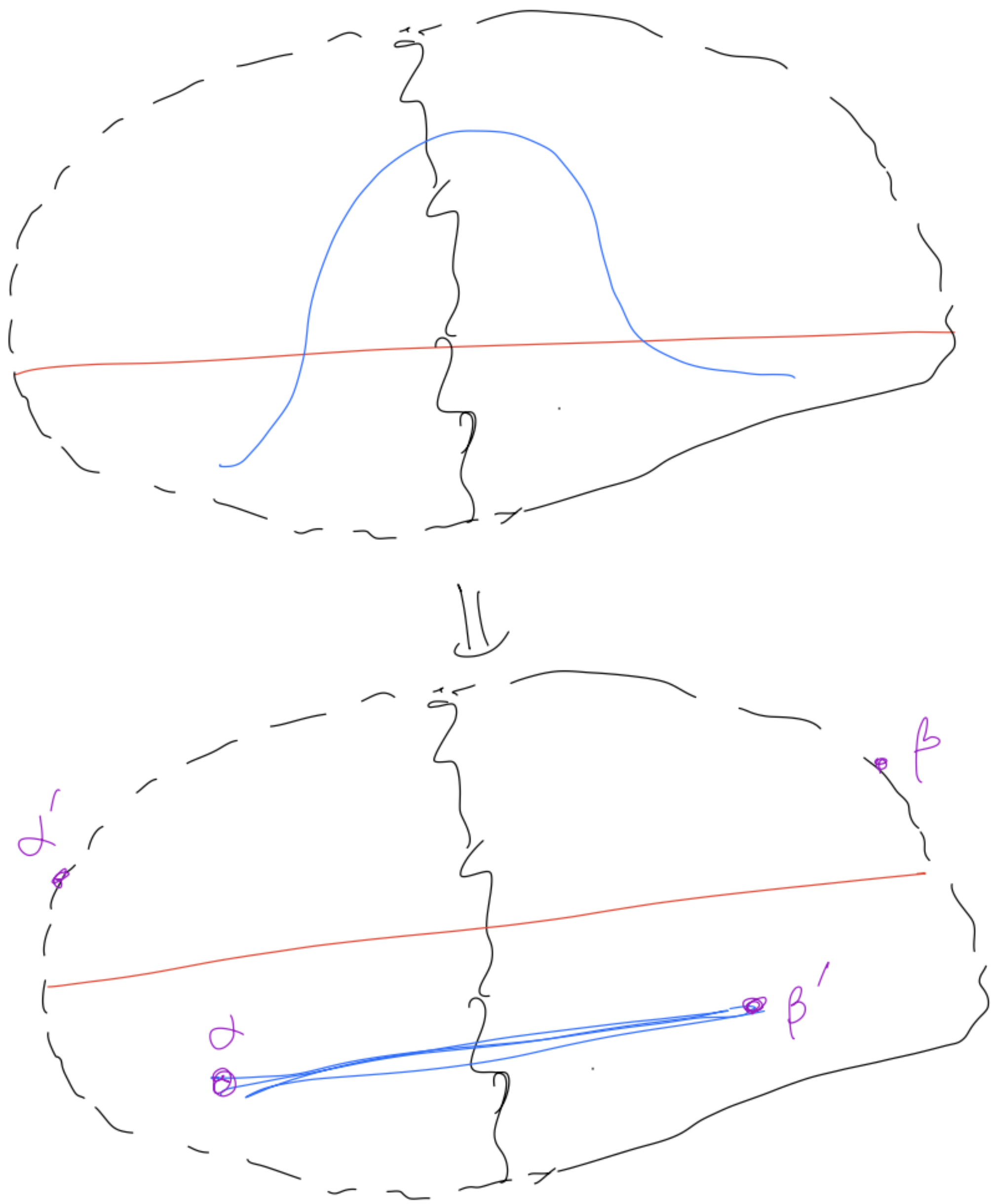


Figure 4.32: Your caption here

We call this sequential application of moves as MOVE v.

4.8 lemma5

Lemma 50.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$ the refinement is as the following figure :

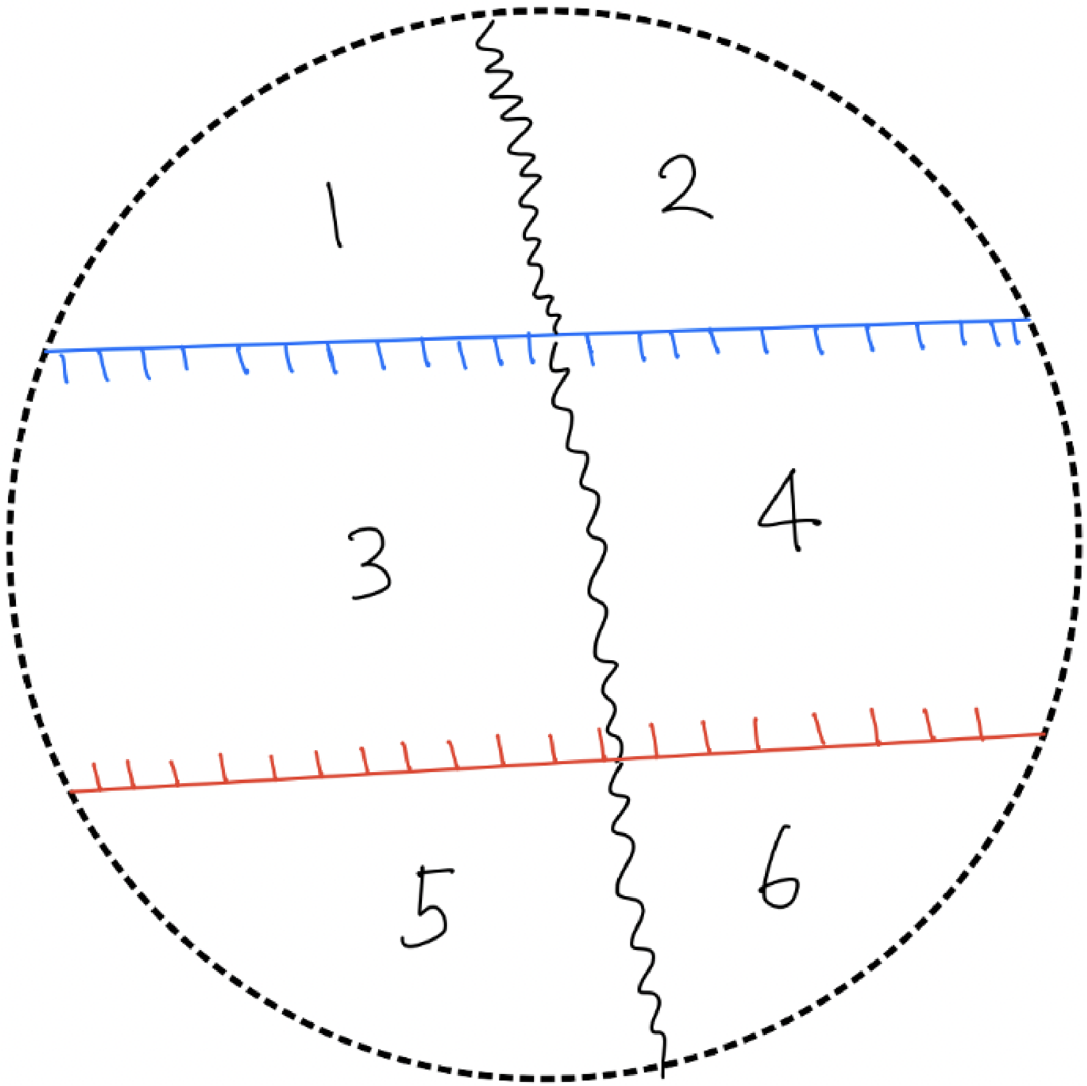


Figure 4.33: Your caption here

Stalks :

- 1 : $\mathbb{C}^{m+1}$

- 2 : $\mathbb{C}^{m+1}$

- 3 : $\mathbb{C}^{m}$

- 4 : $\mathbb{C}^m$

- 5 : $\mathbb{C}^{m+1}$

- 6 : $\mathbb{C}^{m+1}$

Let T be a block diagonal matrix of size $(1, m, 1)$. We denote the submatrix of T with index range $a \leq i \leq b$,$a \leq j \leq b$ to be $T_{a,b}$.

Let m be arbitrary positive integer, then we denote the inclusion map from $\mathbb{C}^m$ to $\mathbb{C}^{m+1}$ that sends e_i to $e_i(e_{i+1}$ resp.) by $\iota_f(\iota_l$ resp.)

Generization maps :

- 1→2 : $T_{1,m+1}$

- 3→4 : $T_{2,m+1}$

- 5→6 : $T_{2,m+1}$

- 3→1 : ι_l

- 4→2 : ι_l

- 3→5 : ι_f

- 4→6 : ι_f

Now we apply a sequence of isotopies to $\mathfrak{F}$ which will be called $isotopy_5$:

(step1) Inside the disk surrounded by purple circle apply $isotopy_1$

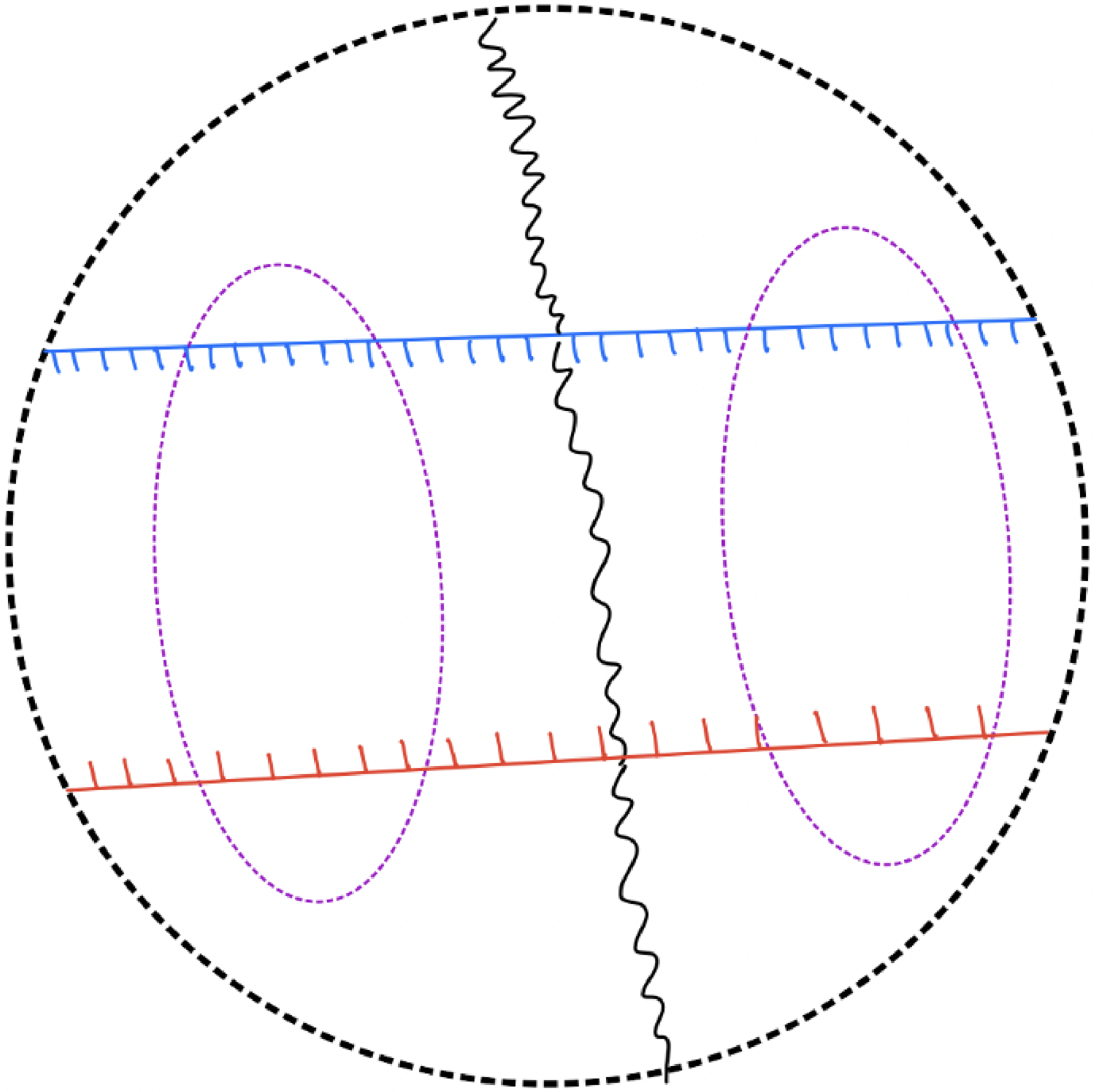


Figure 4.34: Your caption here

we get the following diagram :

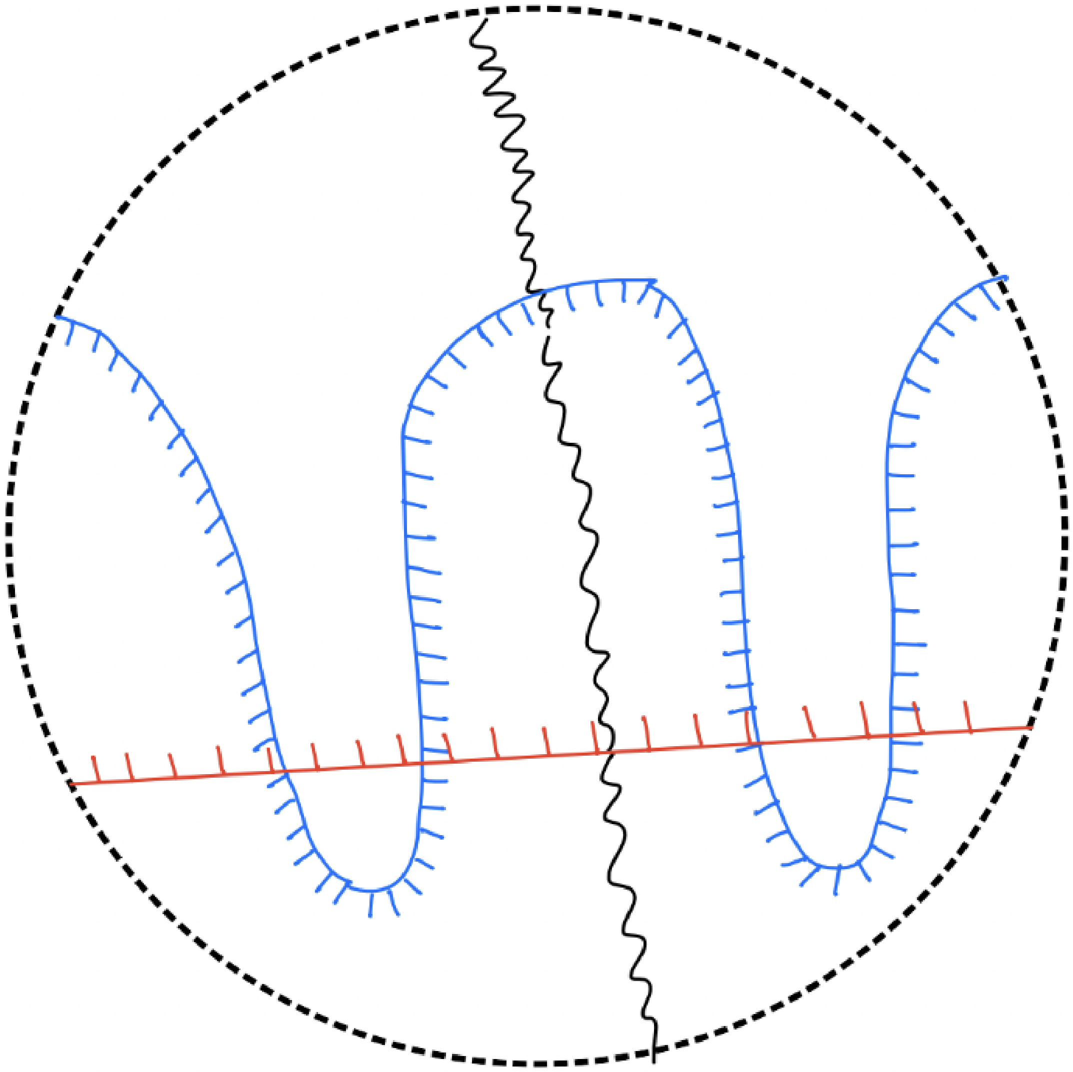


Figure 4.35: Your caption here

(step2) Now apply $isotopy_3$ inside the disk surrounded by the purple circle:

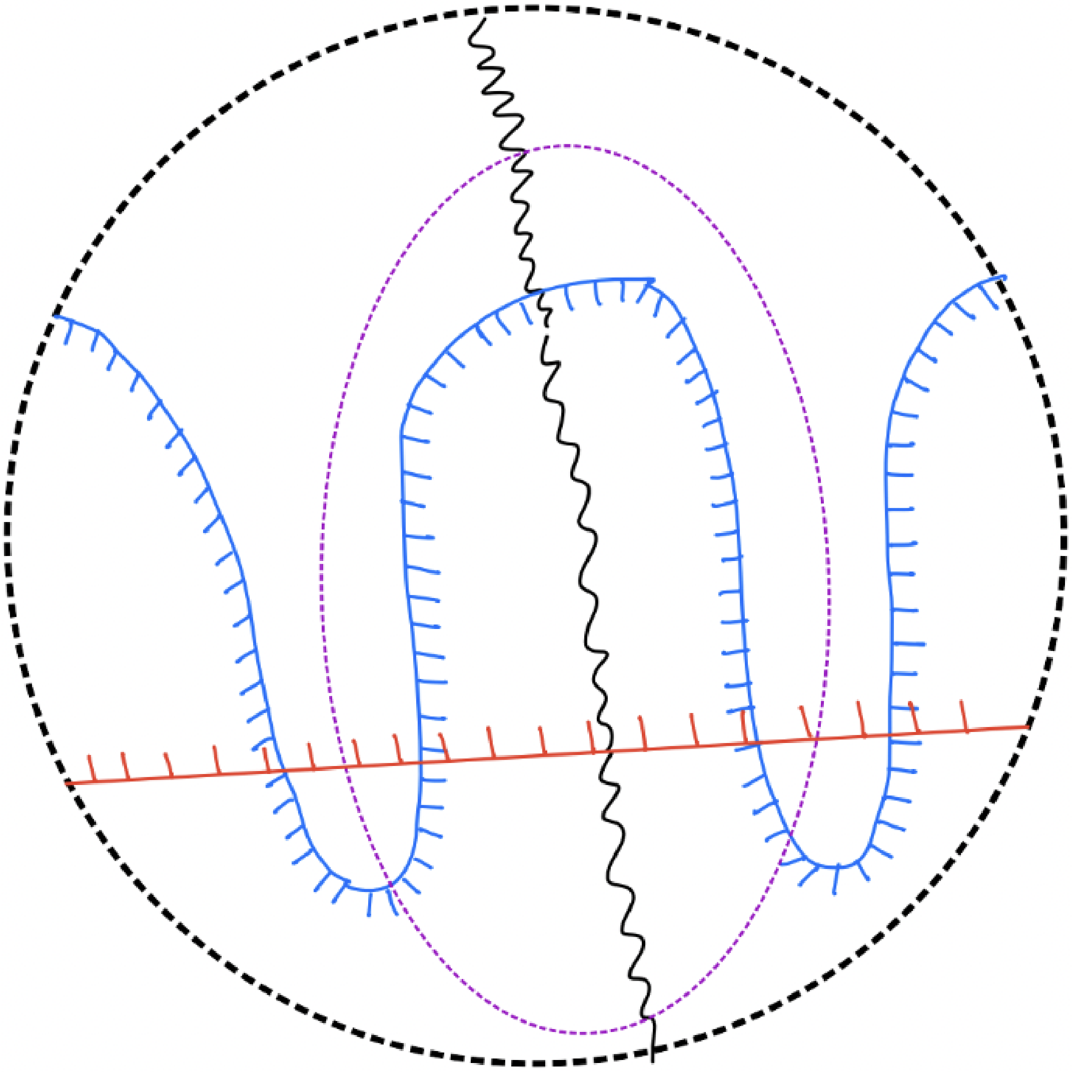


Figure 4.36: Your caption here

We get the following diagram and a sheaf :

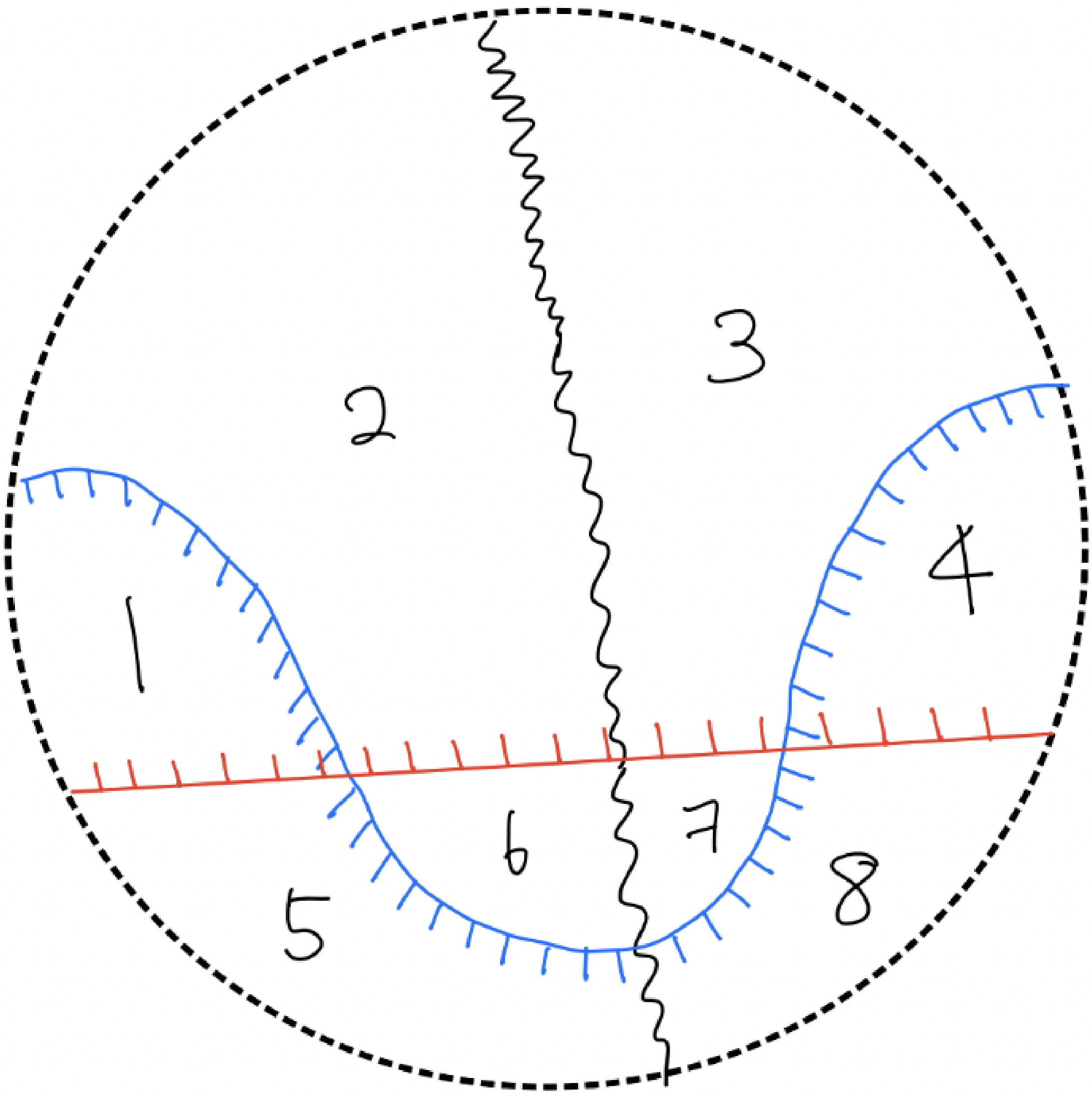


Figure 4.37: Your caption here

Stalks :

- 1 : $\mathbb{C}^m$

- 2 : $\mathbb{C}^{m+1}$

- 3 : $\mathbb{C}^{m+1}$

- 4 : $\mathbb{C}^m$

- 5 : $\mathbb{C}^{m+1}$

- 6 : $\mathbb{C}^{m+2}$

- 7 : $\mathbb{C}^{m+2}$

- 8 : $\mathbb{C}^{m+1}$

Generization maps :

- 1→2 : ι_l

- 4→3 : ι_l

- 1→5 : ι_f

- 4→8 : ι_f

- 2→6 : ι_f

- 3→7 : ι_f

- 5→6 : ι_l

- 8→7 : ι_l

- 2→3 : $T_{1,m+1}$

- 6→7 : $T_{1,m+2}$

- 5→8 : $T_{2,m+2}$

(proof) After (step1), by lemma1, we get the following sheaf :

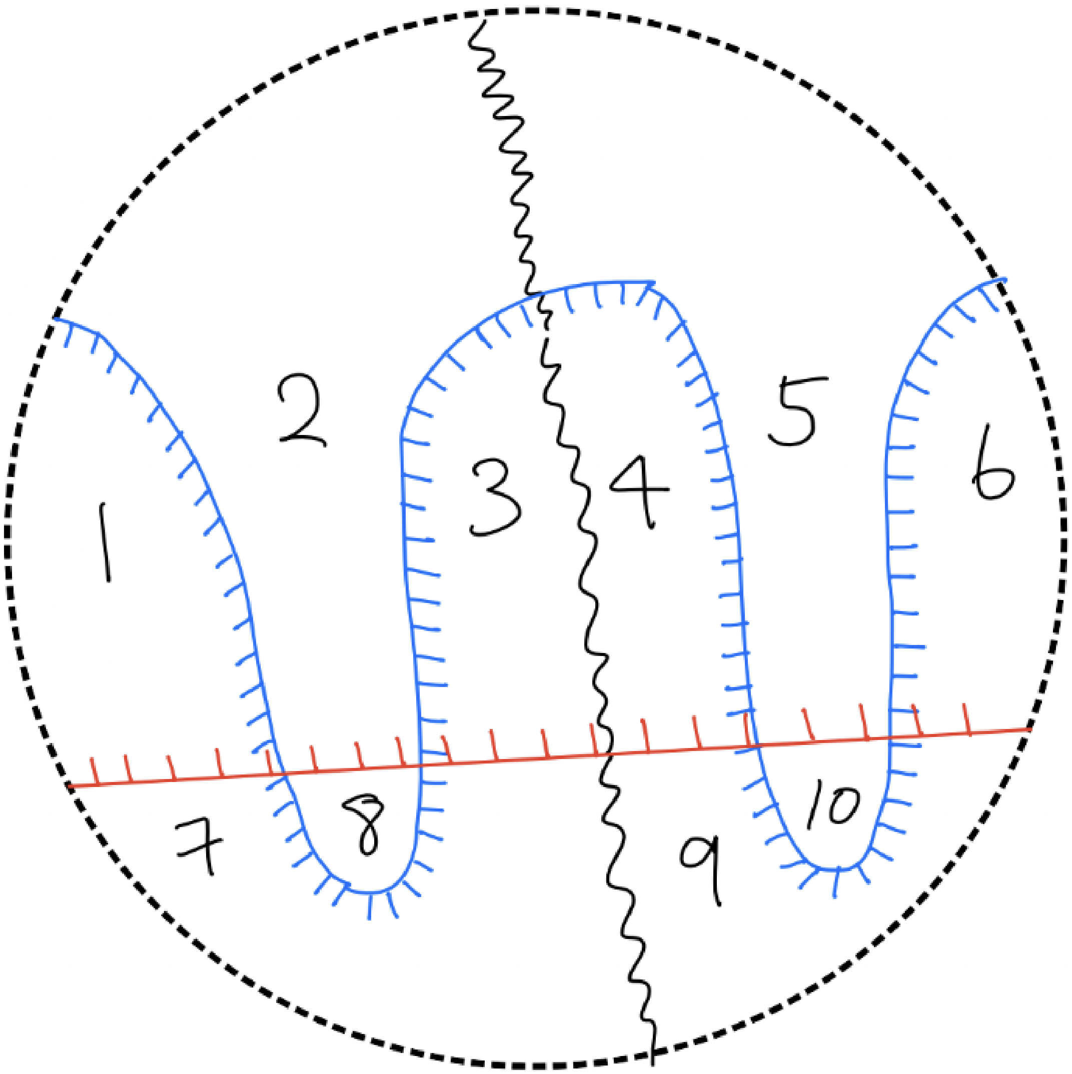


Figure 4.38: Your caption here

Stalks :

- 1 : $\mathbb{C}^m$
- 2 : $\mathbb{C}^{m+1}$
- 3 : $\mathbb{C}^m$
- 4 : $\mathbb{C}^m$
- 5 : $\mathbb{C}^{m+1}$
- 6 : $\mathbb{C}^m$

- 7 : $\mathbb{C}^{m+1}$

- 8 : $\mathbb{C}^{m+2}$

- 9 : $\mathbb{C}^{m+1}$

- 10 : $\mathbb{C}^{m+2}$

Generization maps :

- 1→2 : ι_l
- 3→2 : ι_l
- 4→5 : ι_l
- 6→5 : ι_l
- 1→7 : ι_f
- 3→7 : ι_f
- 4→9 : ι_f
- 6→9 : ι_f
- 2→8 : ι_f
- 7→8 : ι_f
- 5→10 : ι_f
- 9→10 : ι_l
- 2→5 : $T_{1,m+1}$
- 3→4 : $T_{2,m+1}$
- 7→9 : $T_{2,m+2}$

Now to the above sheaf we apply (step2), then by lemma3, we get the final sheaf.

4.9 definition6

Definition 51.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$, the refinement is as following figure where 2 dimensional strata are labeled by tuples

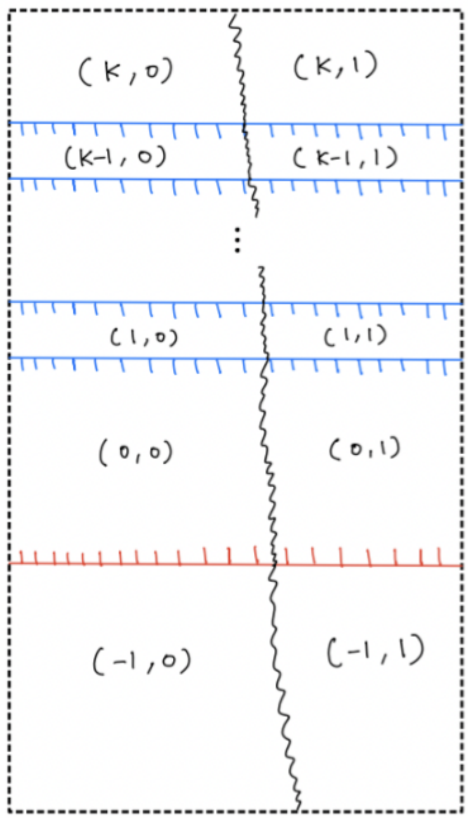


Figure 4.39: Your caption here

Stalks :

- $(i, 0)$: $\mathbb{C}^{m+|i|}$

- $(i, 1)$: $\mathbb{C}^{m+|i|}$

Generization maps

- the maps corresponding to the arrow crossing the red strand is ι_f

- the maps corresponding to the arrow crossing the blue strand is ι_l

- $(i,0) \to (i,1)$ where $i \geq 0$: $\tilde{T}_{k+1-i,m+k}$

- $(i,0) \to (i,1)$ where $i = -1$: $\tilde{T}_{k+1,m+k+1}$

where $\tilde{T}_{1,m+k}$ is an isomorphism preserving the flag $\mathbb{C}^m \subset_l \mathbb{C}^{m+1} \subset_l \cdots \subset_l \mathbb{C}^{m+k}$ and $T_{k+1,m+k+2}$ is an isomorphism preserving the flag $\mathbb{C}^m \subset_f \mathbb{C}^{m+1}$.

Now we will define isotopy starting from the above sheaf $\mathfrak{F}$ inductively on the number of blue strands so that the final sheaf $\mathfrak{F}'$ is

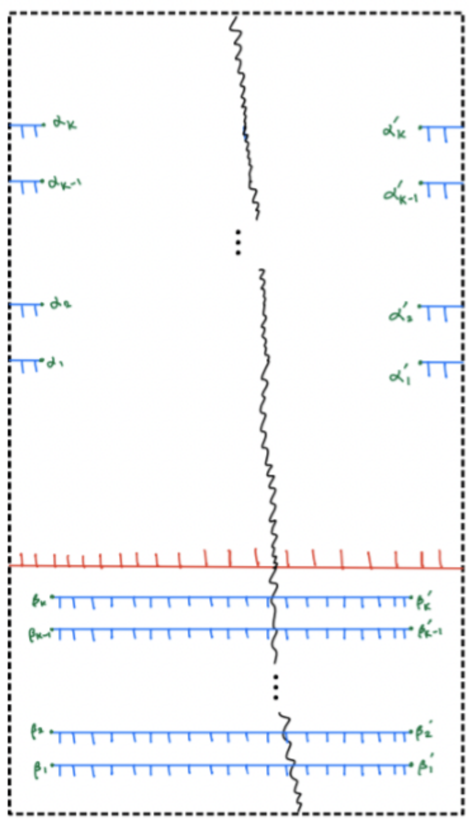


Figure 4.40: Your caption here

In the above diagram, I have intentionally omitted lines connecting - α_i and β_i - α_i' and β_i' for $i = 1, \cdots, k$ so as not to make diagram too messy. These omitted lines are mutually disjoint and crosses red strand only once. Let the crossing of the red strand with $\overline{\alpha_i\beta_i}$ be called c_i and $\overline{\alpha_i'\beta_i'}$ be called c_i'. Let's denote the north, east, west, south of the crossing $c_i(c_i'$ resp.) as $N_i, E_i, W_i, S_i(N_i', E_i', W_i', S_i'$ resp.).

The final sheaf will be described as follows:

Stalks

- $W_i, E_i' : \mathbb{C}^{m+i}$
- $N_i, N_i' : \mathbb{C}^{m+i+1}$
- $S_1, E_1 : \mathbb{C}^m, \mathbb{C}^{m+1}$
- $W_1', S_1' : \mathbb{C}^{m+1}, \mathbb{C}^m$

Generization maps

- maps crossing blue strands are ι_l
- maps crossing red strands are ι_f
- maps crossing squiggly lines are
- $E_1 \to W_1' : T' = \tilde{T}_{k+!,m+k+1}$
- $W_k \to E_k' : T = \tilde{T}_{1,m+k}$
- $N_i \to N_i' : \tilde{T}_{k-i+1,m+k+1}$

when n=1, (skip this part for now)

Suppose $isotopy_6$ is defined up to the number of blue strands less than n. Now we define $isotopy_6$ for the number of blue strands $= n$.

(step1) Apply $isotopy_6$ for the number of blue strands $n-1$ on the disk surrounded by purple dotted line which is well-defined by the induction hypothesis.

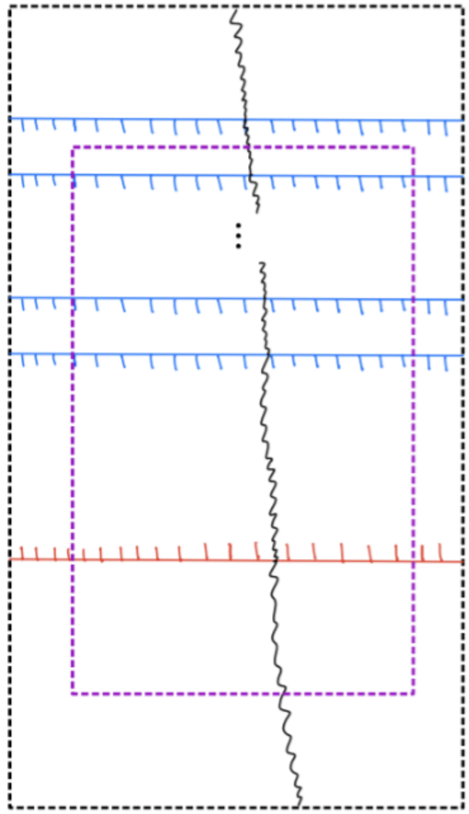


Figure 4.41: Your caption here

we get the following diagram

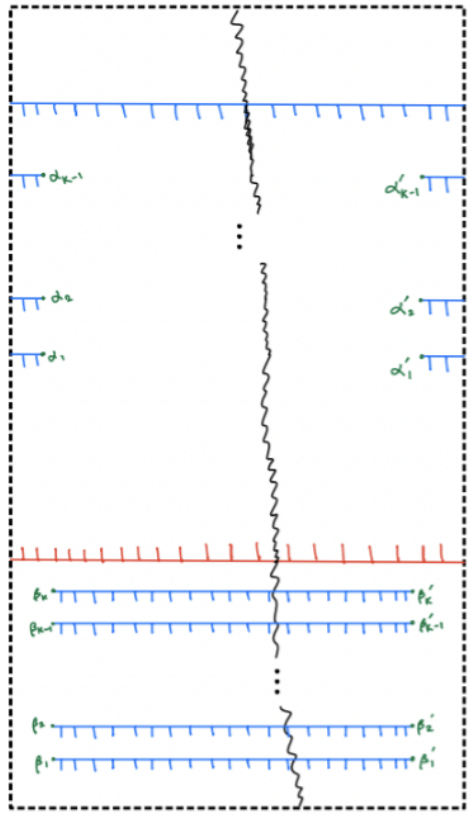


Figure 4.42: Your caption here

(step2) Apply $isotopy_6$ for the number of blue strands = 1 on the disk surrounded by purple dotted line

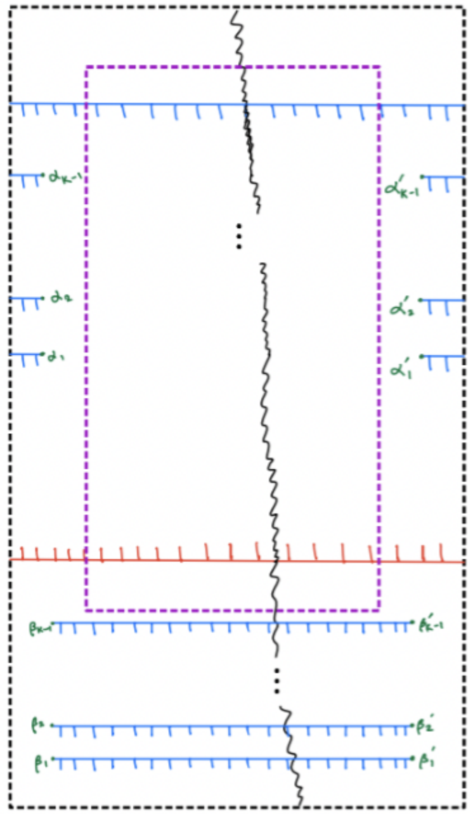


Figure 4.43: Your caption here

we get the final diagram

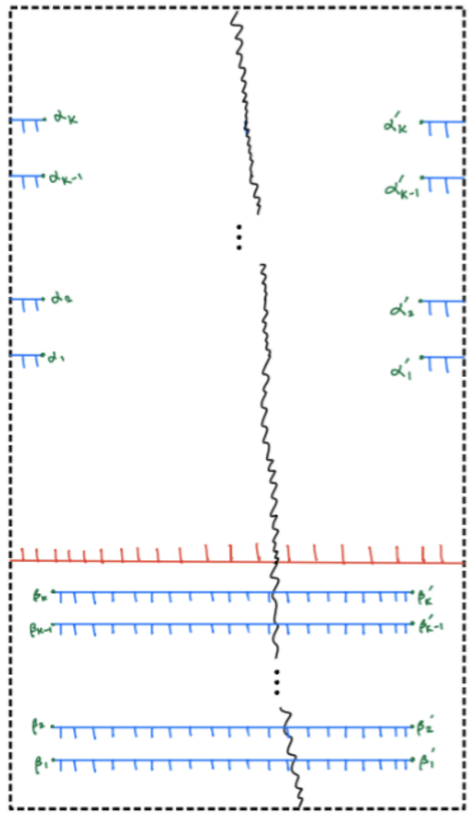


Figure 4.44: Your caption here

with sheaf $\mathfrak{F}$ on it. (proof)

4.10 lemma6

Lemma 52.

Suppose we have a local braid diagram and a sheaf singular supported along the braids represented by the following:

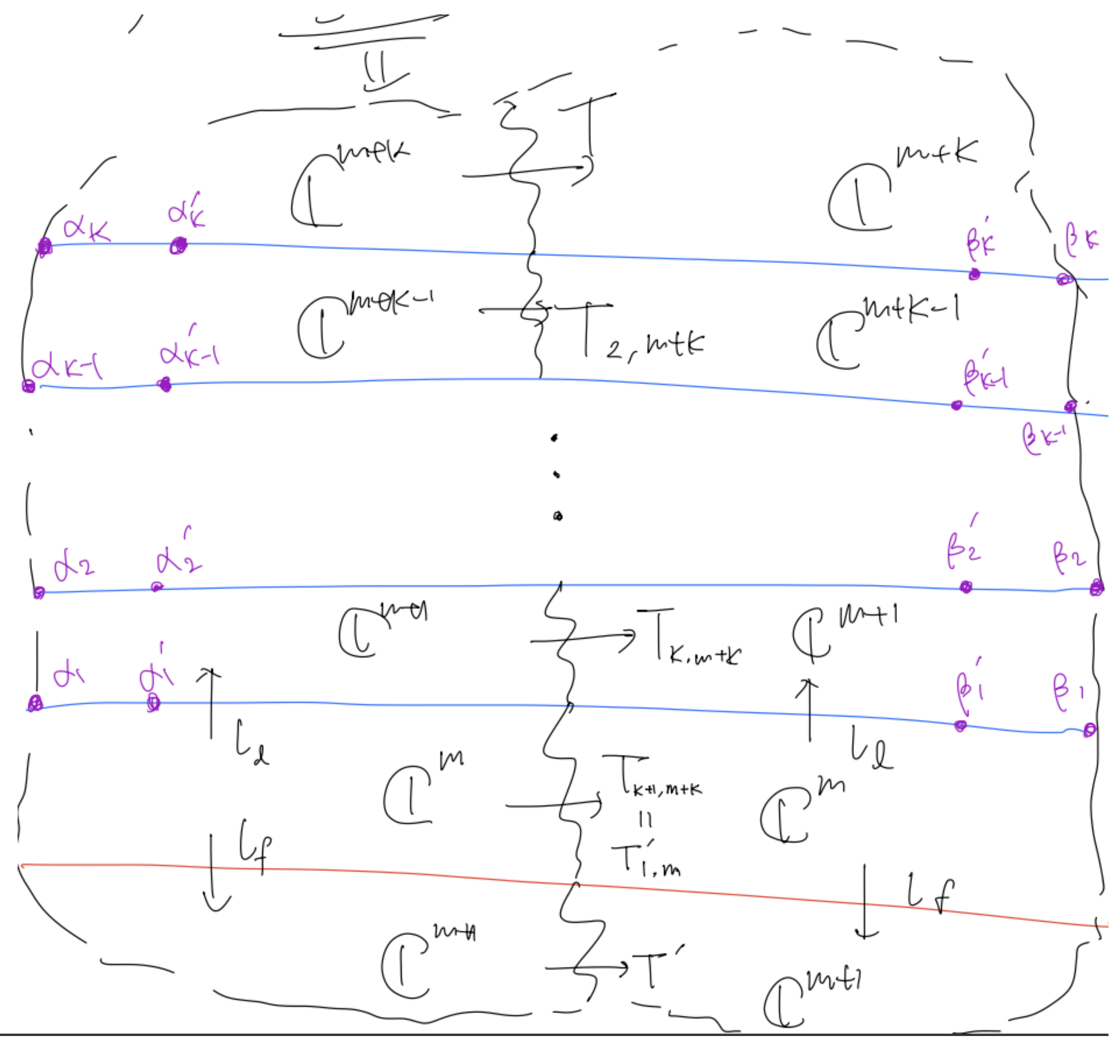


Figure 4.45: Your caption here

where T is an isomorphism preserving the flag $\mathbb{C}^m \subset_l \mathbb{C}^{m+1} \subset_l \cdots \subset_l \mathbb{C}^{m+k}$(here $\subset_l$ denotes inclusion into the last factors), T' is an isomorphism preserving $\mathbb{C}^m \subset_f \mathbb{C}^{m+1}$(here $\subset_f$ means inclusion into the first factors), and $T_{k+1,m+k} = T'_{1,m}$.

If we apply MOVE vito the above diagram and the sheaf on it, we get the following:

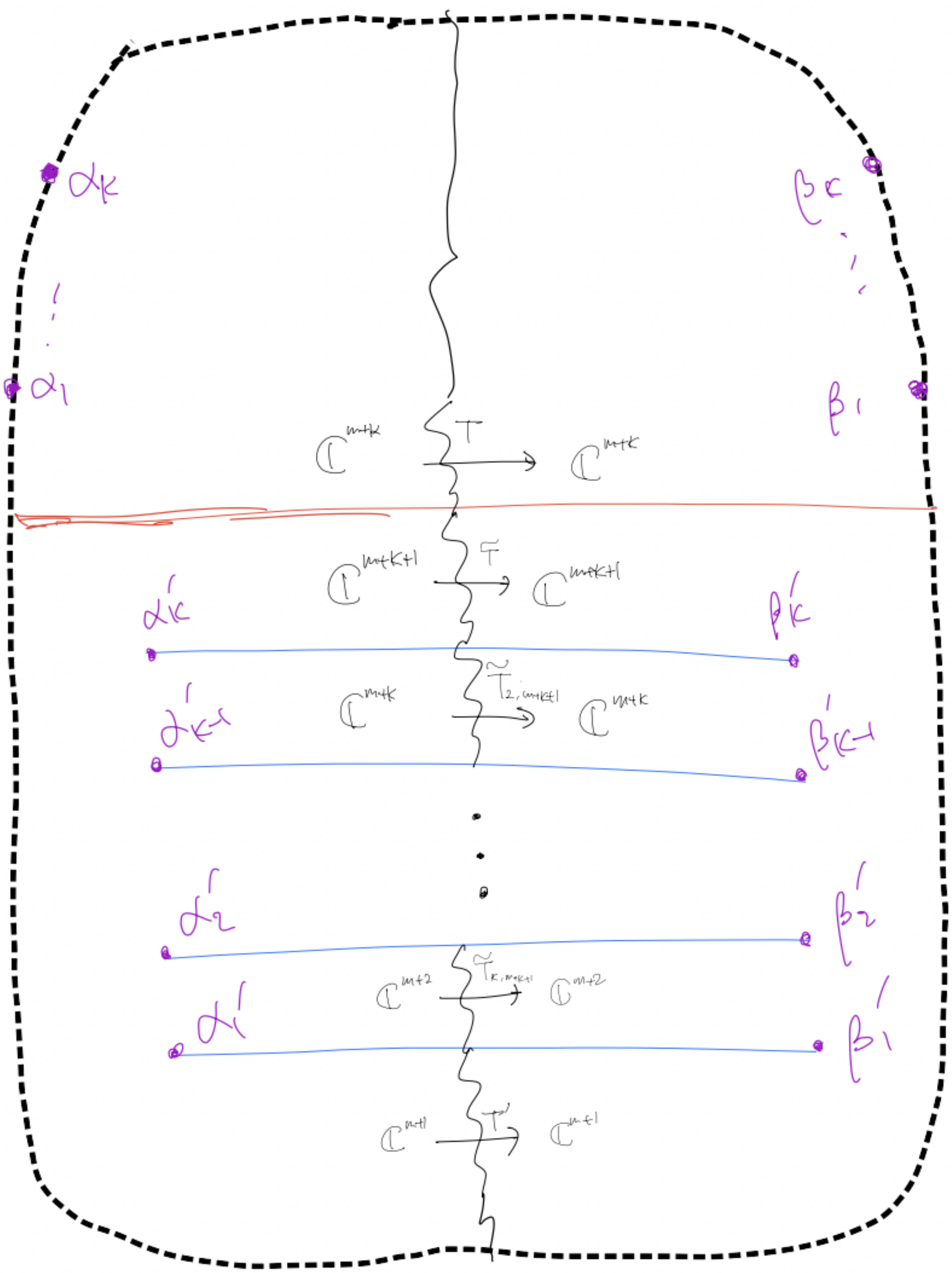


Figure 4.46: Your caption here

where $\tilde{T}_{1,m+k} = T$ and $\tilde{T}_{k+1,m+k+1} = T'$(rest of the entries are 0)

(proof) We prove the claim by induction on k. By the induction hypothesis after $k-1$ application of MOVE v(MOVE vi), we get

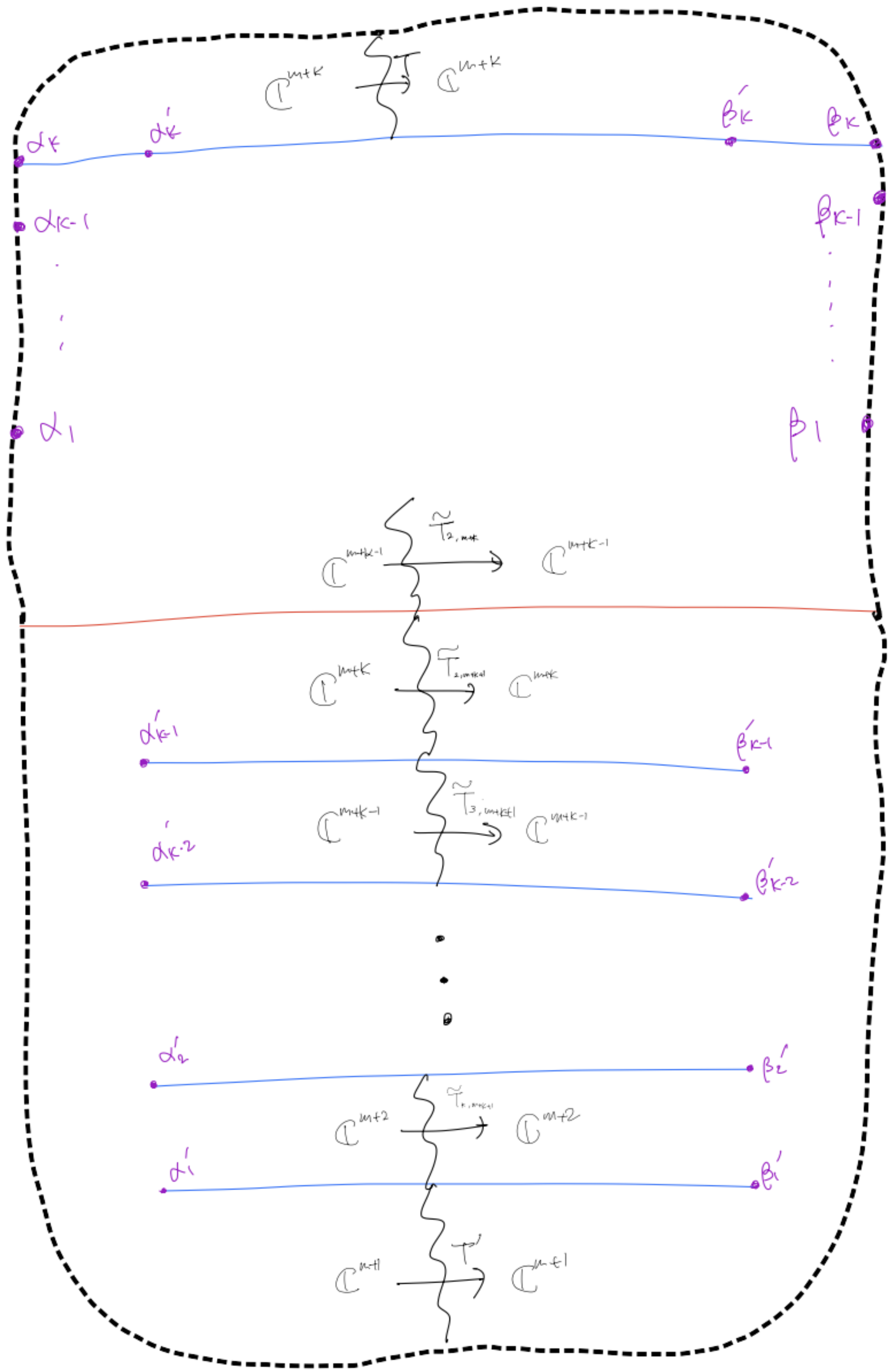


Figure 4.47: Your caption here

Now we apply MOVE vto the uppermost blue strand and the red strand, then by Lemma5, we get

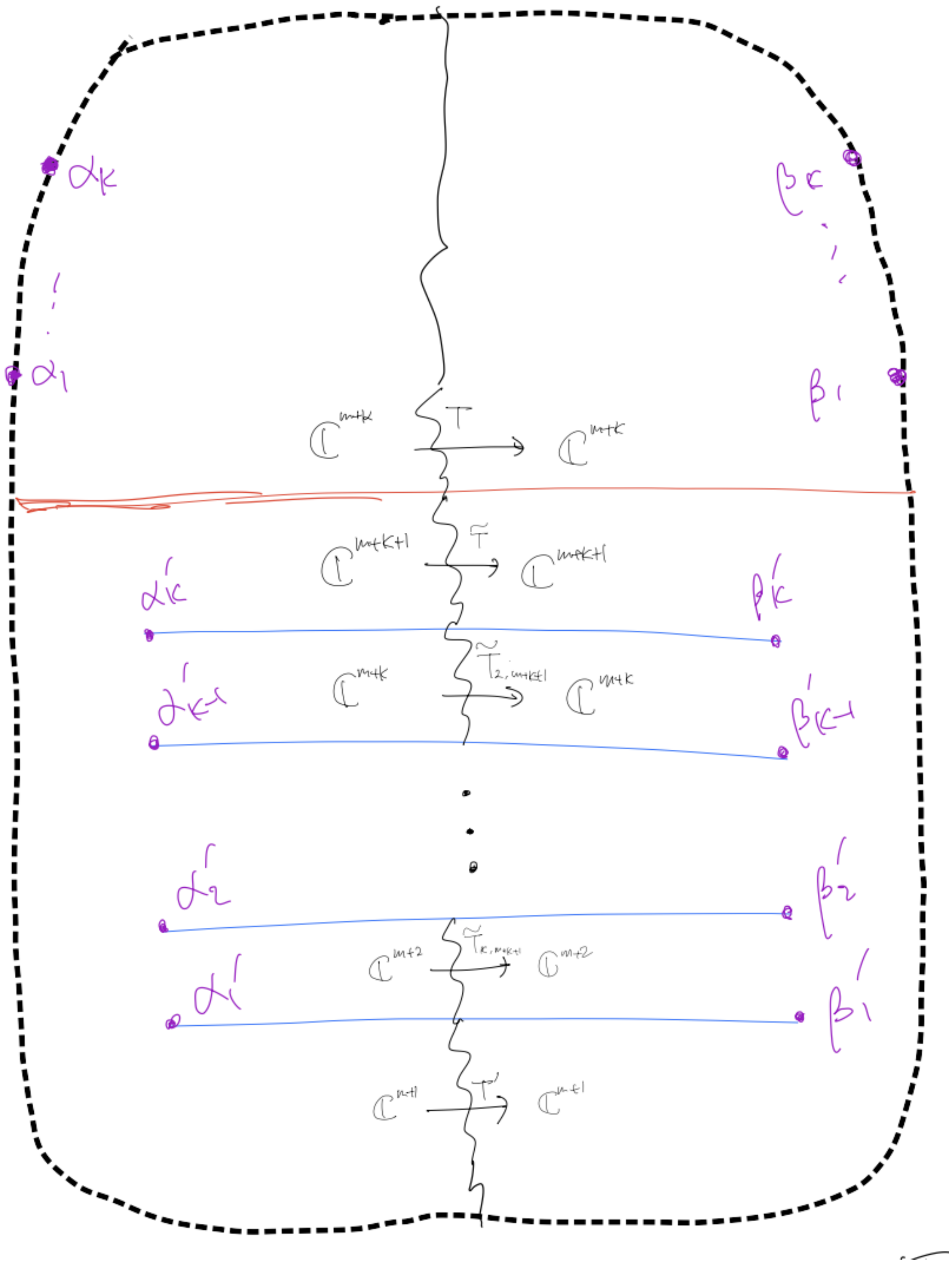


Figure 4.48: Your caption here

4.11 definition7

Definition 53.

Suppose we have the following diagram of alternating m red strands($R_1 - R_m$) and m blue strands($B_1 - B_m$) labeled from the top:

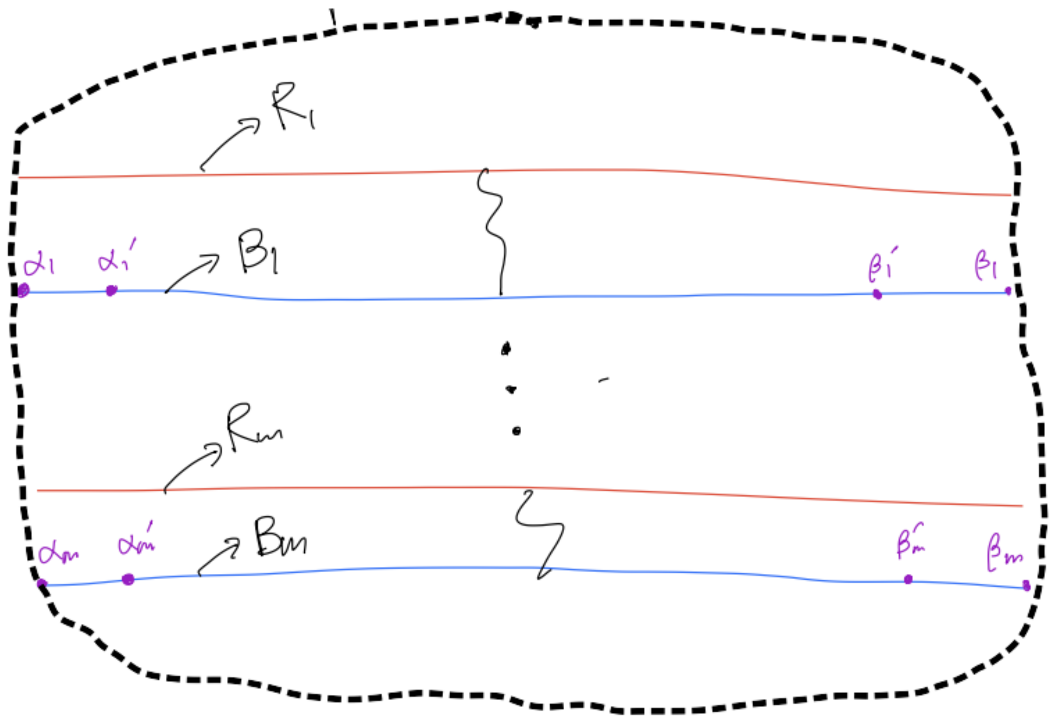


Figure 4.49: Your caption here

We define MOVE vii-(a) inductively so that the final diagram looks as follows:

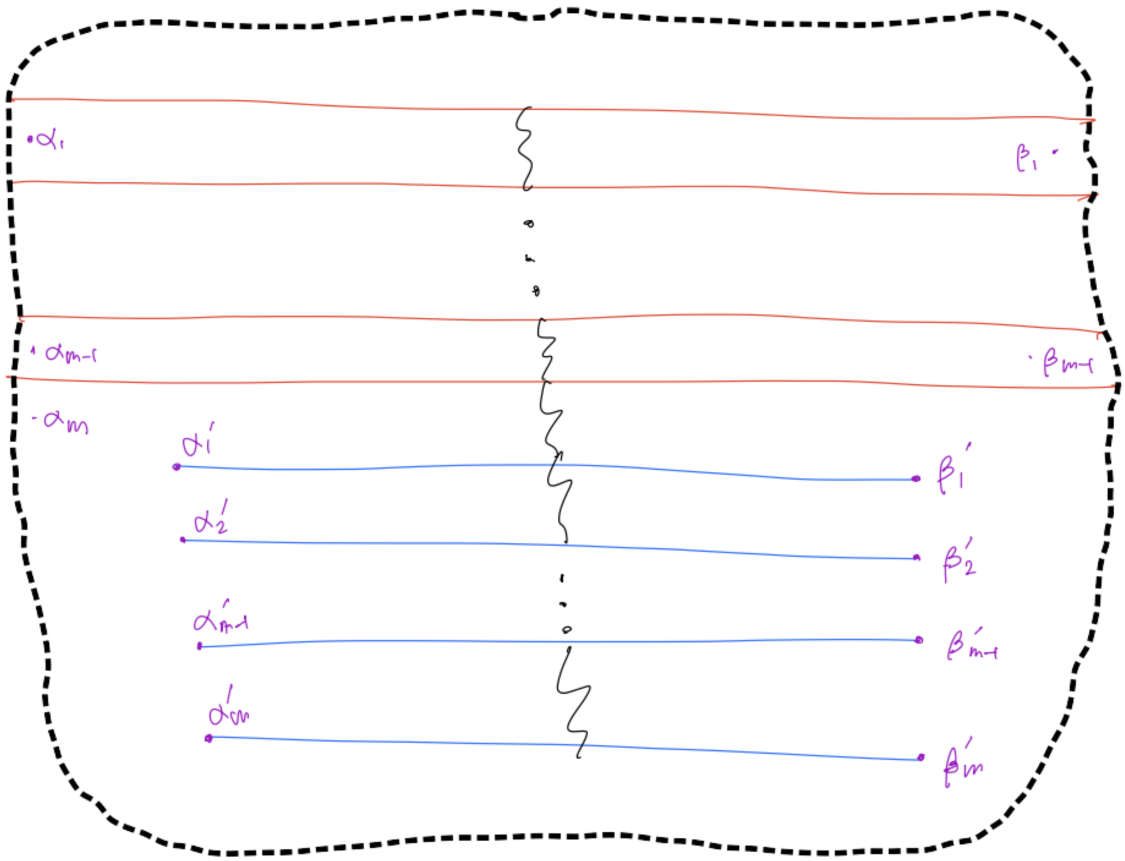


Figure 4.50: Your caption here

We define MOVE vii-(a) inductively on m. If $m = 1$, then MOVEvii-(a) is the null move. If $m > 1$, (Step1) Apply MOVEvii-(a) to the first $m-1$ blue and red strands($B_1 - B_{m-1}$ and $R_1 - R_{m-1}$)(this has been defined by the induction hypothesis), we get :

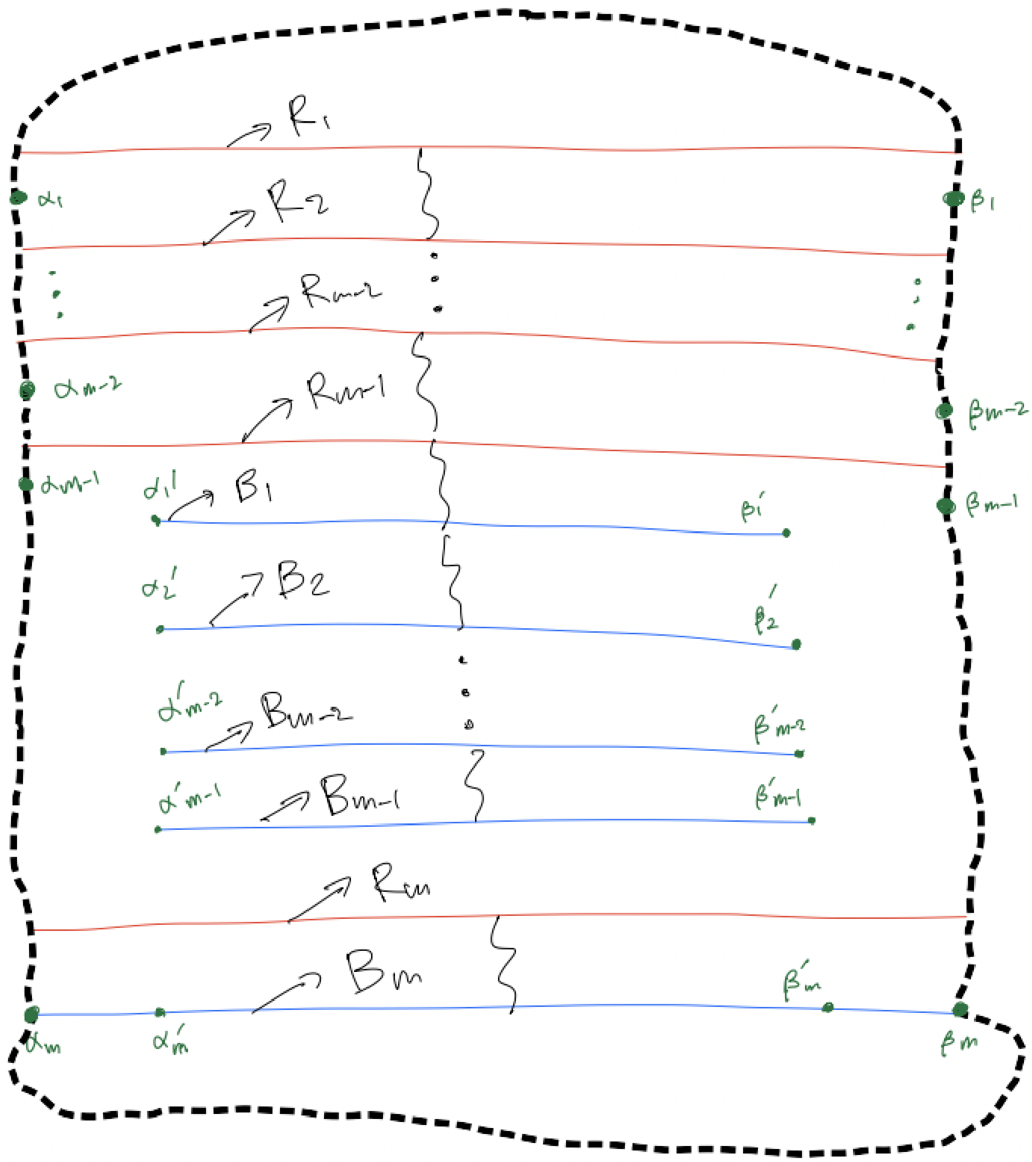


Figure 4.51: Your caption here

(Step2) Apply MOVE vito $B_1 - B_{m-1}$ and R_m, we get :

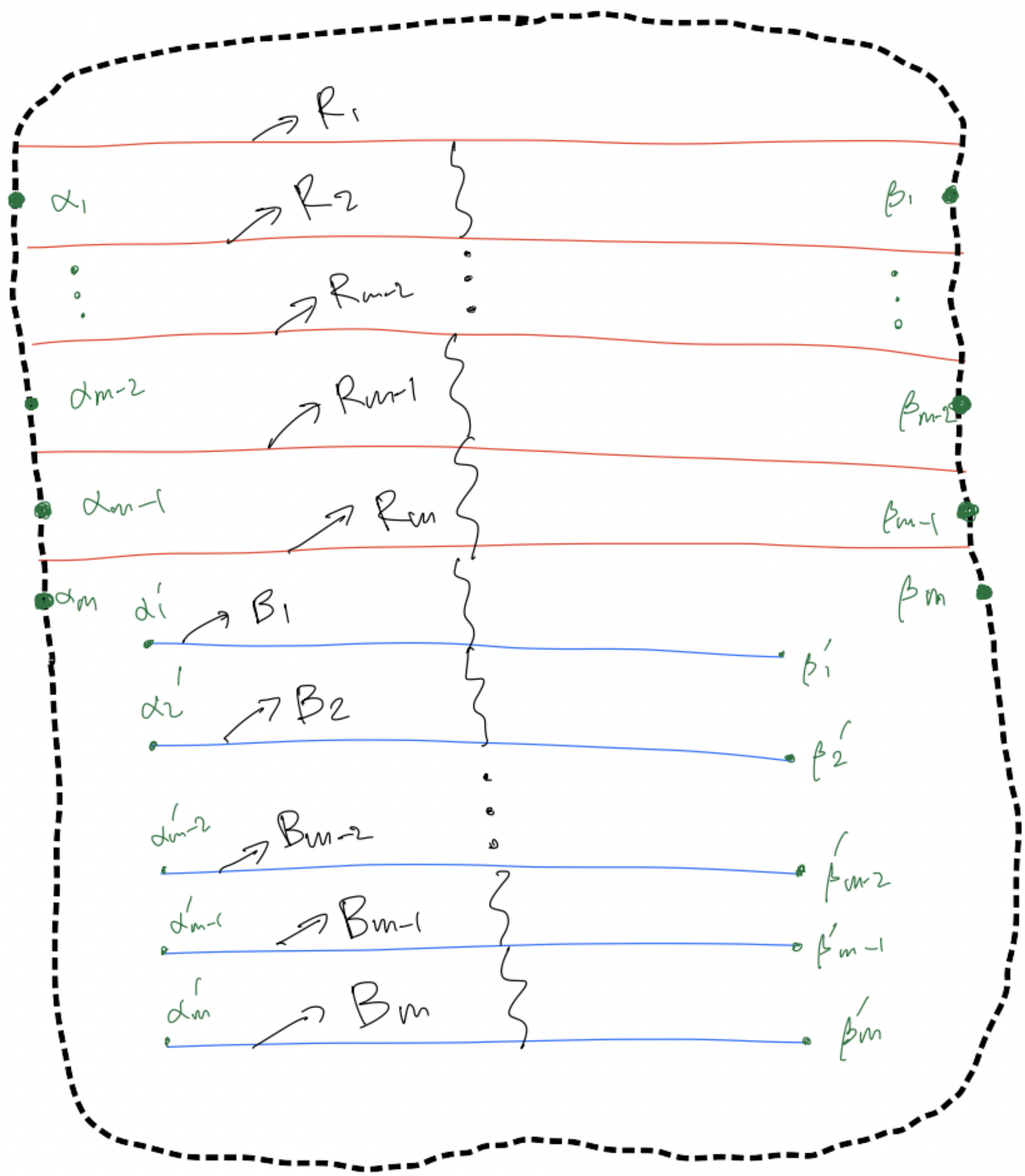


Figure 4.52: Your caption here

By induction, we have defined MOVE vii-(a) for all $m \in \mathbb{M}$.

(b) Suppose we have the following diagram of n red strands($R_1 - R_n$) and m blue strands($B_1 - B_m$) labeled from the top :

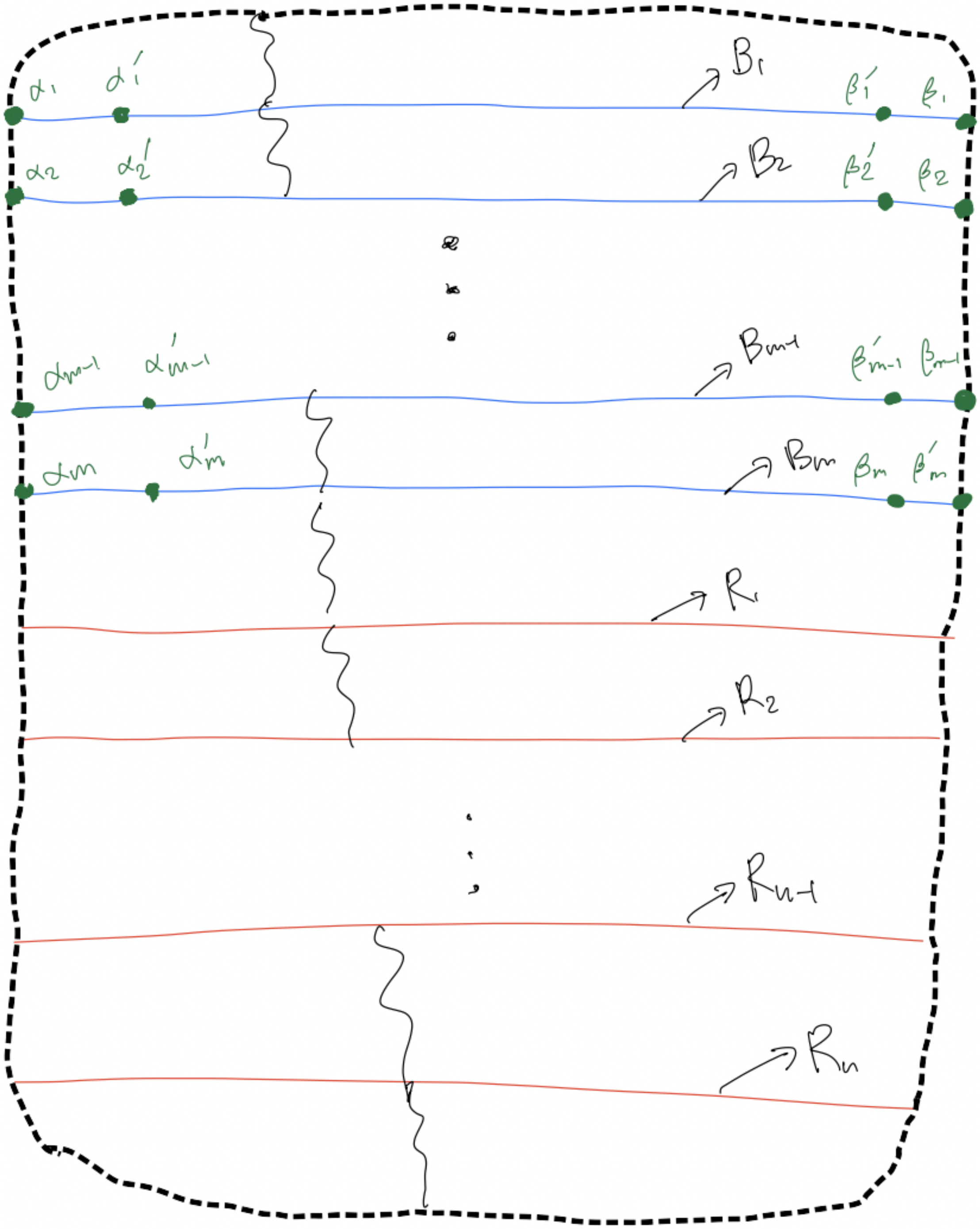


Figure 4.53: Your caption here

We define MOVE vii-(b) inductively so that the final diagram looks as follows:

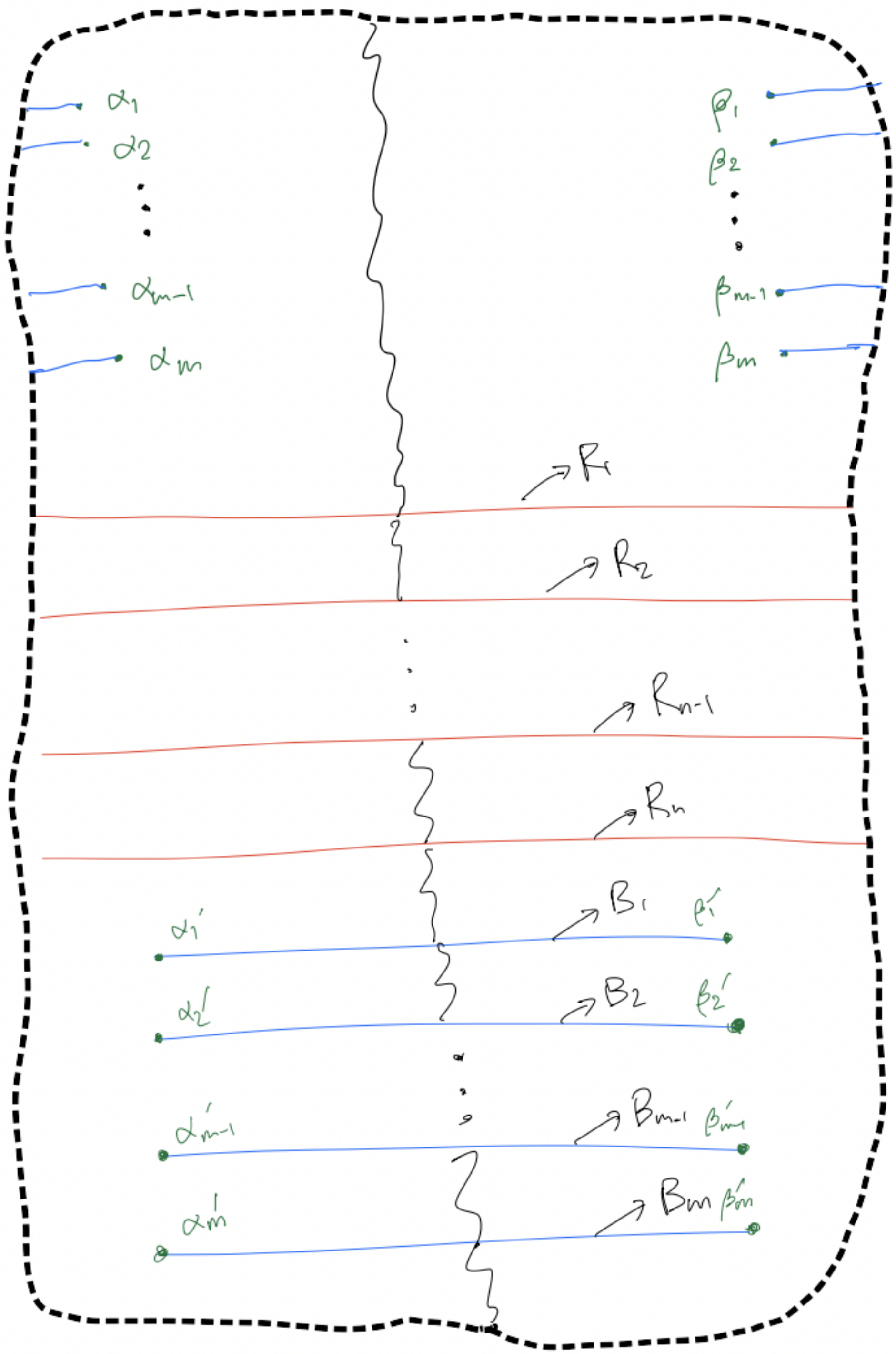


Figure 4.54: Your caption here

We define MOVE vii-(b) inductively on n. If $n = 1$, then MOVE vii-(b) is MOVE vi. If $n > 1$, (Step1) apply MOVE vii-(b) to the first m blue strands($B_1 - B_m$) and $n-1$ red strands($R_1 - R_{n-1}$)(this has been defined by the induction hypothesis), we get :

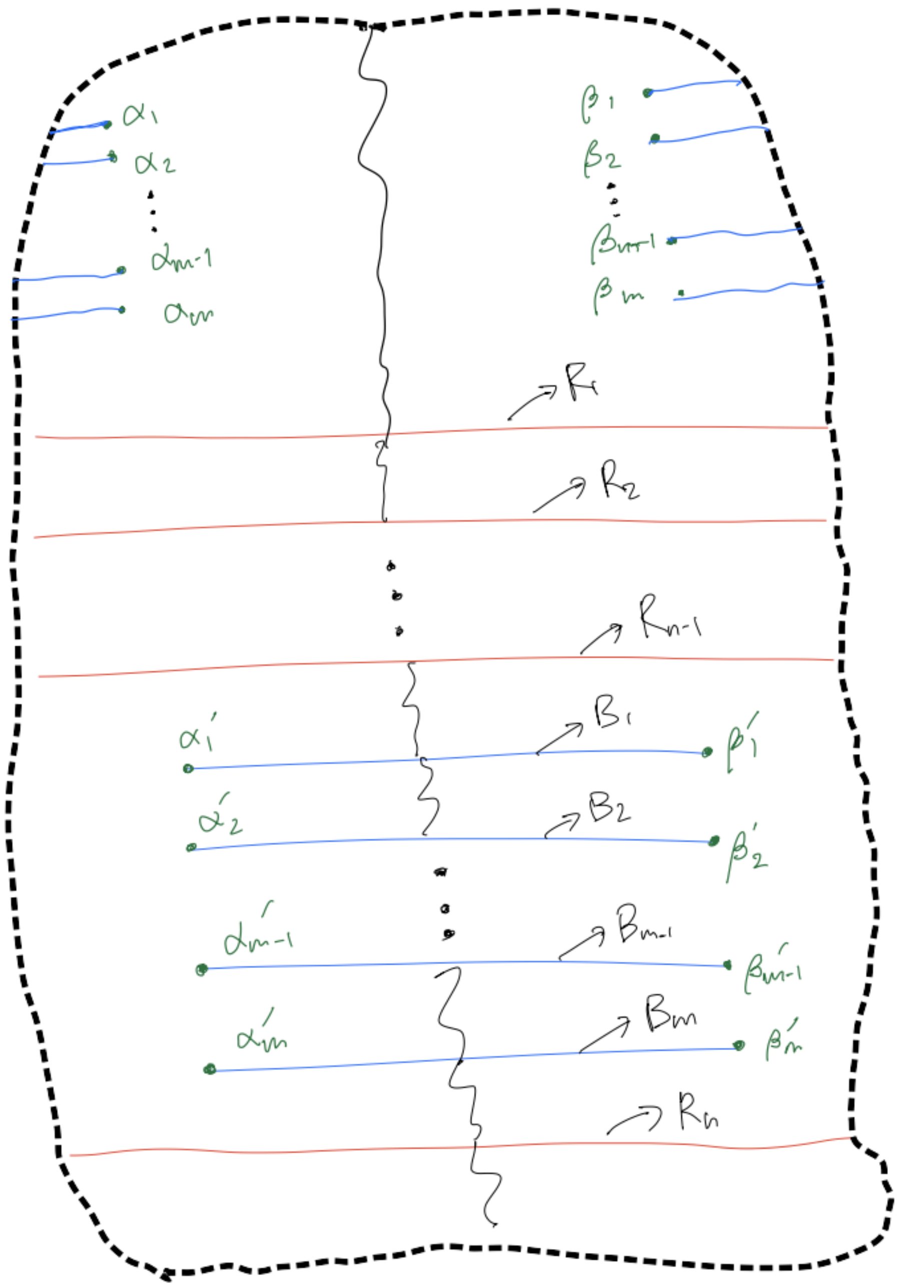


Figure 4.55: Your caption here

(step2) apply MOVE vito be $B_1 - B_m$ and R_n, we get the final diagram :

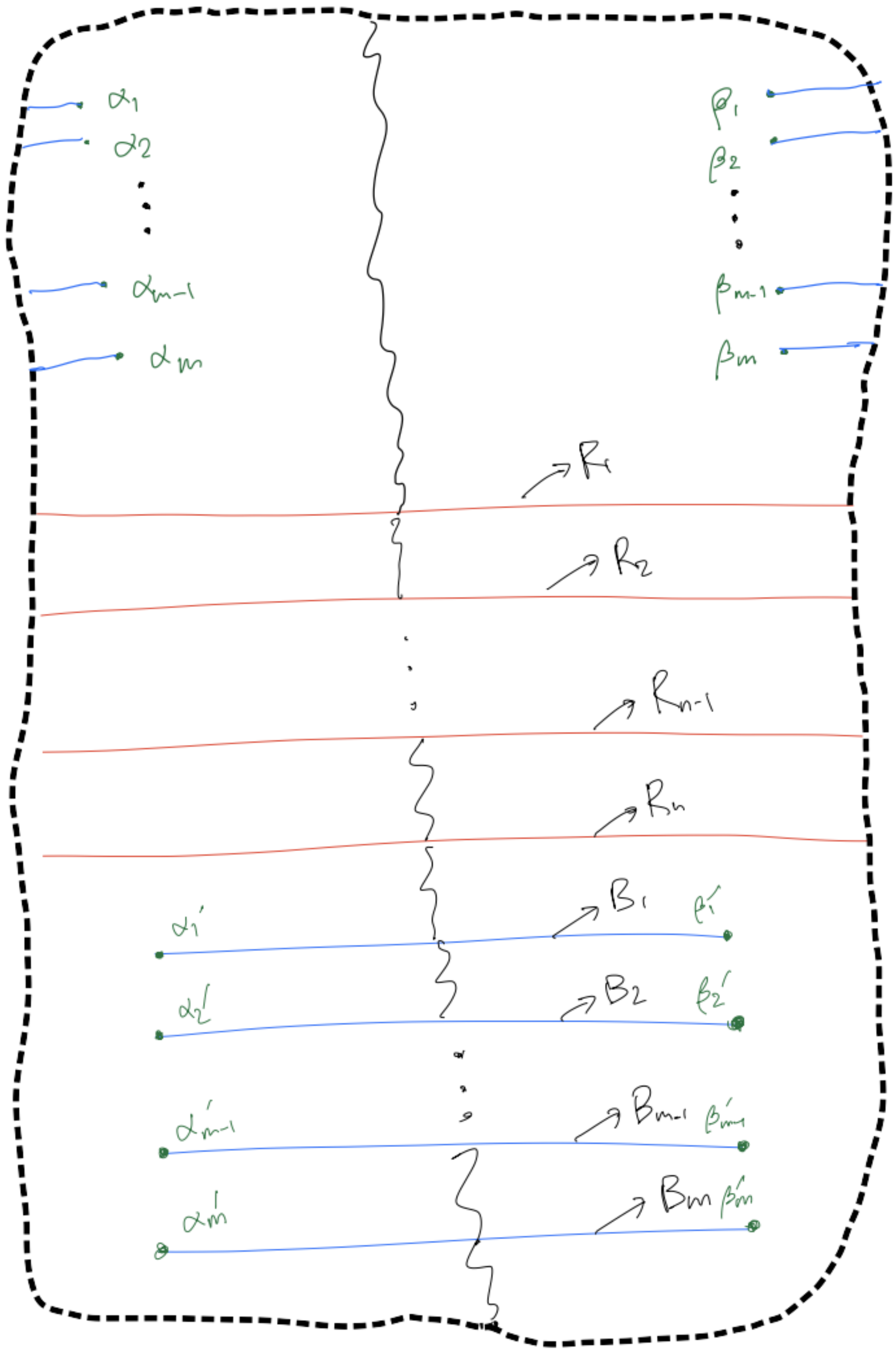


Figure 4.56: Your caption here

By induction, we have defined MOVE vii-(b) for all $m \in \mathbb{N}$.

4.12 lemma7

Lemma 54.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$, the refinement is a the following figure where two dimensional strata are labeld by tuples.

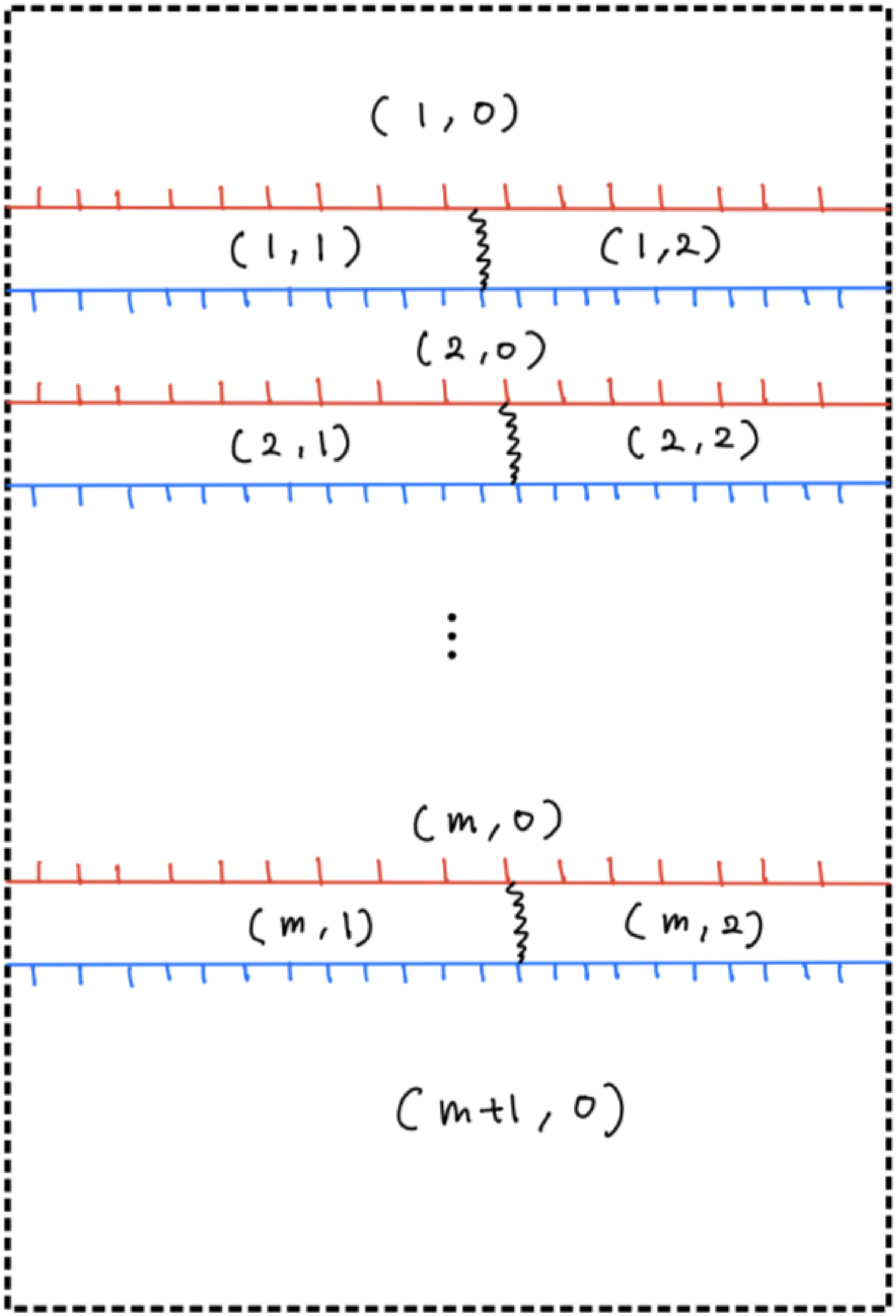


Figure 4.57: Your caption here

Stalks :

$(i, 0)$: 0

$(i, 1), (i, 2))$: $\mathbb{C}$

Generization maps :

$(i, 1) \to (i, 2)$: multiplication by $a_i \in \mathbb{C}$

All the other maps are zero maps.

Now we will define isotopy starting from the above sheaf $\mathfrak{F}$ inductively on the number of blue strands(=the number of red strands) so that the final sheaf $\mathfrak{F}'$ is :

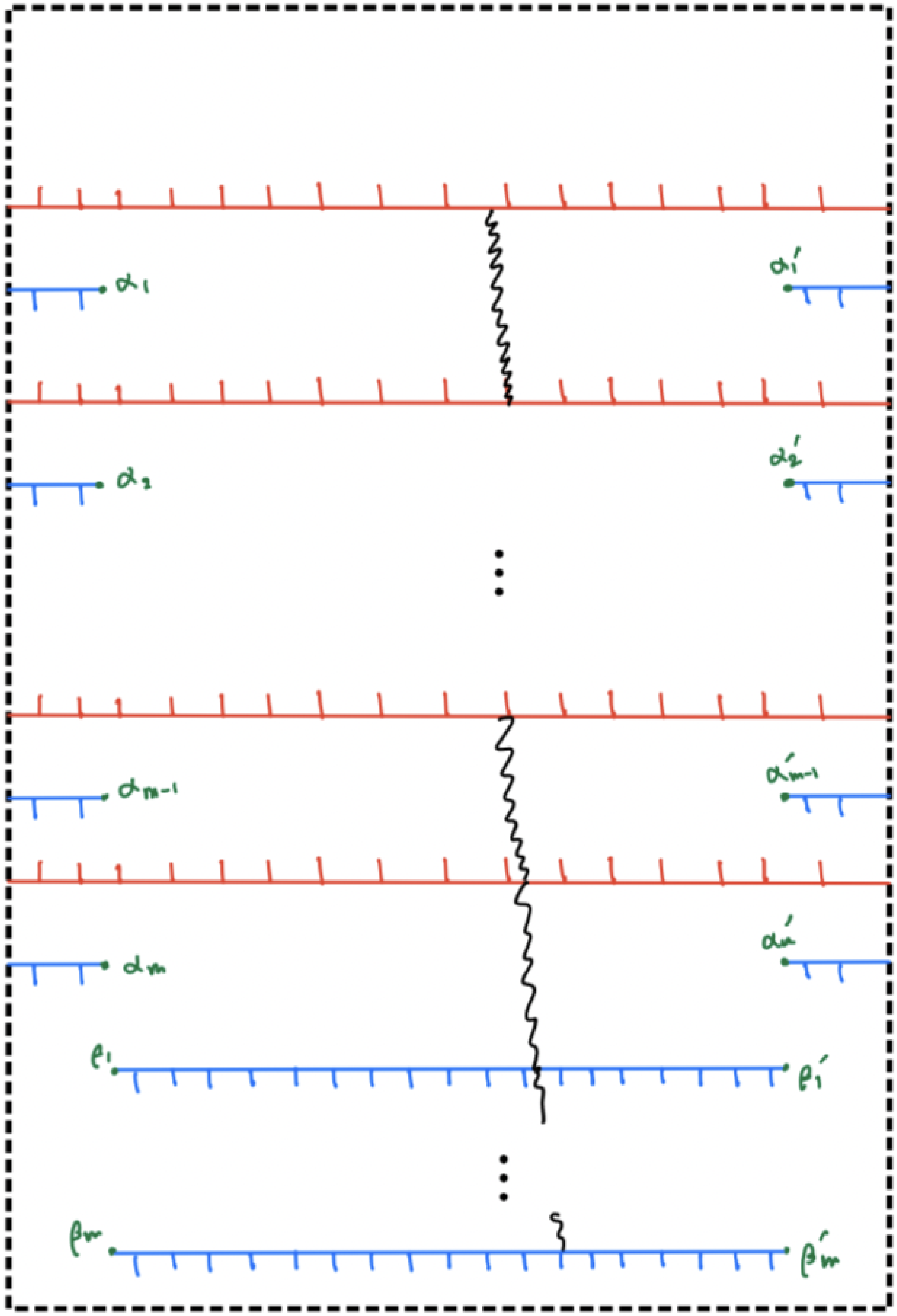


Figure 4.58: Your caption here

In the above diagram, I have intentionally omitted lines connecting

- α_i and β_i for $i = 1, \cdots, m$
- α'_i and β'_i for $i = 1, \cdots, m$

so as not to make diagram too messy. These omitted lines are mutually disjoint and crosses each red strand at most once. Let the crossing of $\overline{\alpha_j \beta_j}$ with the $i^t h$ red strand be called $c_{i,j}$ and $\overline{\alpha'_j \beta'_j}$ with the $i^t h$ red strand be called $c'_{i,j}$.

Let's denote the north, east, west, south of the crossing $c_{i,j}(c_{i,j}$ resp.$)$ as $N_i, E_i, W_i, S_i(N'_i, E'_i, W'_i, S'_i)$.

The final sheaf will be described as follows :

Stalks :

for $i > j$,

- $E_{i,j}$: $\mathbb{C}^{i-j}$
- $S_{i,j}$: $\mathbb{C}^{i-j-1}$
- $W_{i,j}$: $\mathbb{C}^{i-j}$
- $N_{i,j}$: $\mathbb{C}^{i-j+1}$

- $E'_{i,j}$: $\mathbb{C}^{i-j}$
- $S'_{i,j}$: $\mathbb{C}^{i-j-1}$
- $W'_{i,j}$: $\mathbb{C}^{i-j}$
- $N'_{i,j}$: $\mathbb{C}^{i-j+1}$

Generization maps :

- maps crossing the blue strands are ι_l.
- maps crossing the red strands are ι_f.
- maps crossing the squiggly lines :
- for $i \geq 2$, $W_{i,1} \to E'_{i,1}$: $T_{1,i-1}$

- $N_{m,1} \to N'_{m,1} : T_{1,m}$

- $E_{m,j} \to W'_{m,j} : T_{j+1,m}$

where $T = diag(a_1, \cdots, a_m)$

Now let's define an isotopy from $\mathfrak{F}$ to $\mathfrak{F}'$ which we call$isotopy_7$.

If $n = 1$, $isotopy_7$ is a null move.

Suppose we have defined $isotopy_7$ upto the number of blue strands(= the number of red strands) less than m. Now we define $isotopy_7$ for the number of blue strands(= the number of red strands) equals m :

(step1) Apply $isotopy_7$ for the number of blue strands(= the number of red strands) equals $m - 1$ on the disk surrounded by purple dotted line which is well-defined by the induction hypothesis.

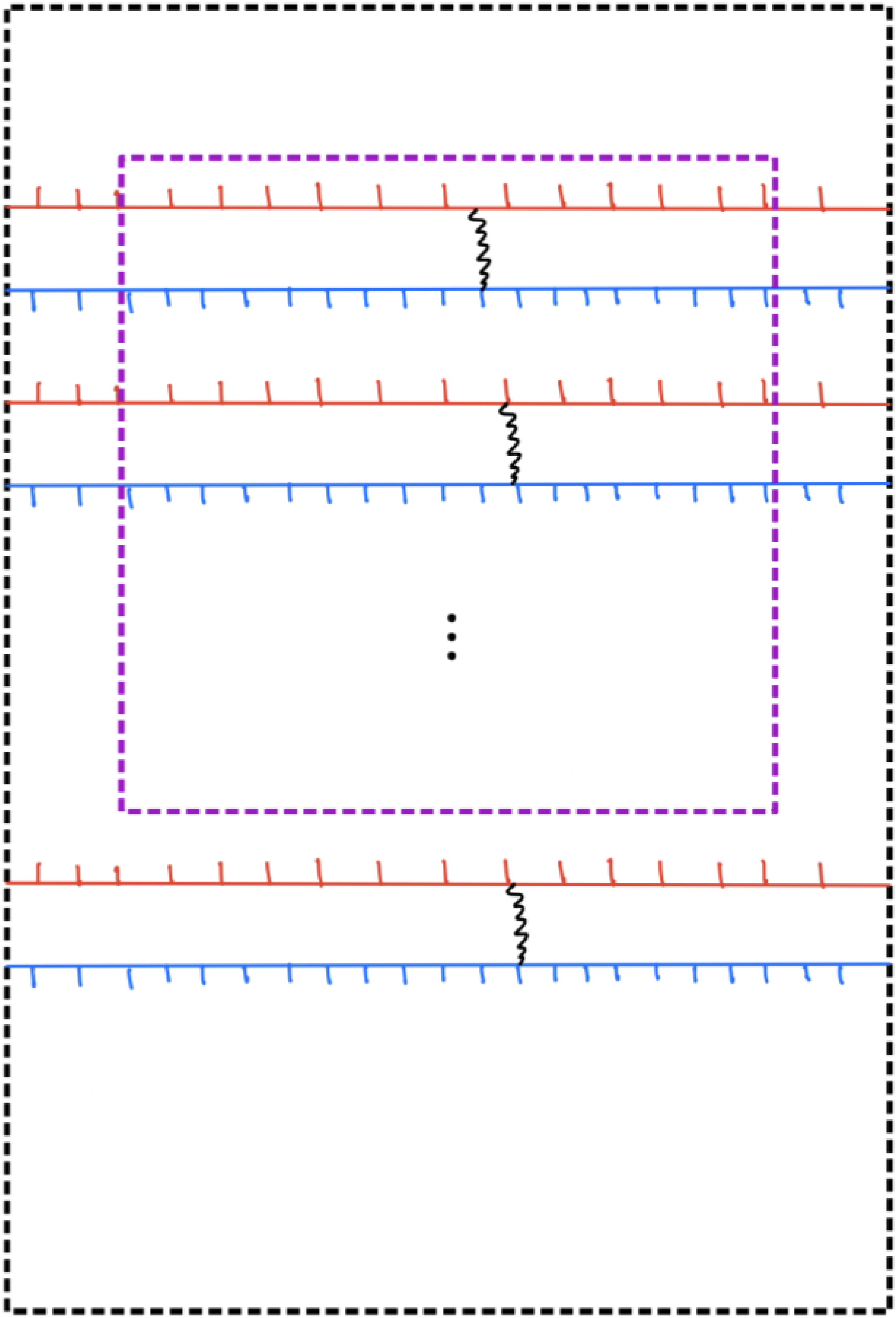


Figure 4.59: Your caption here

We get the following diagram :

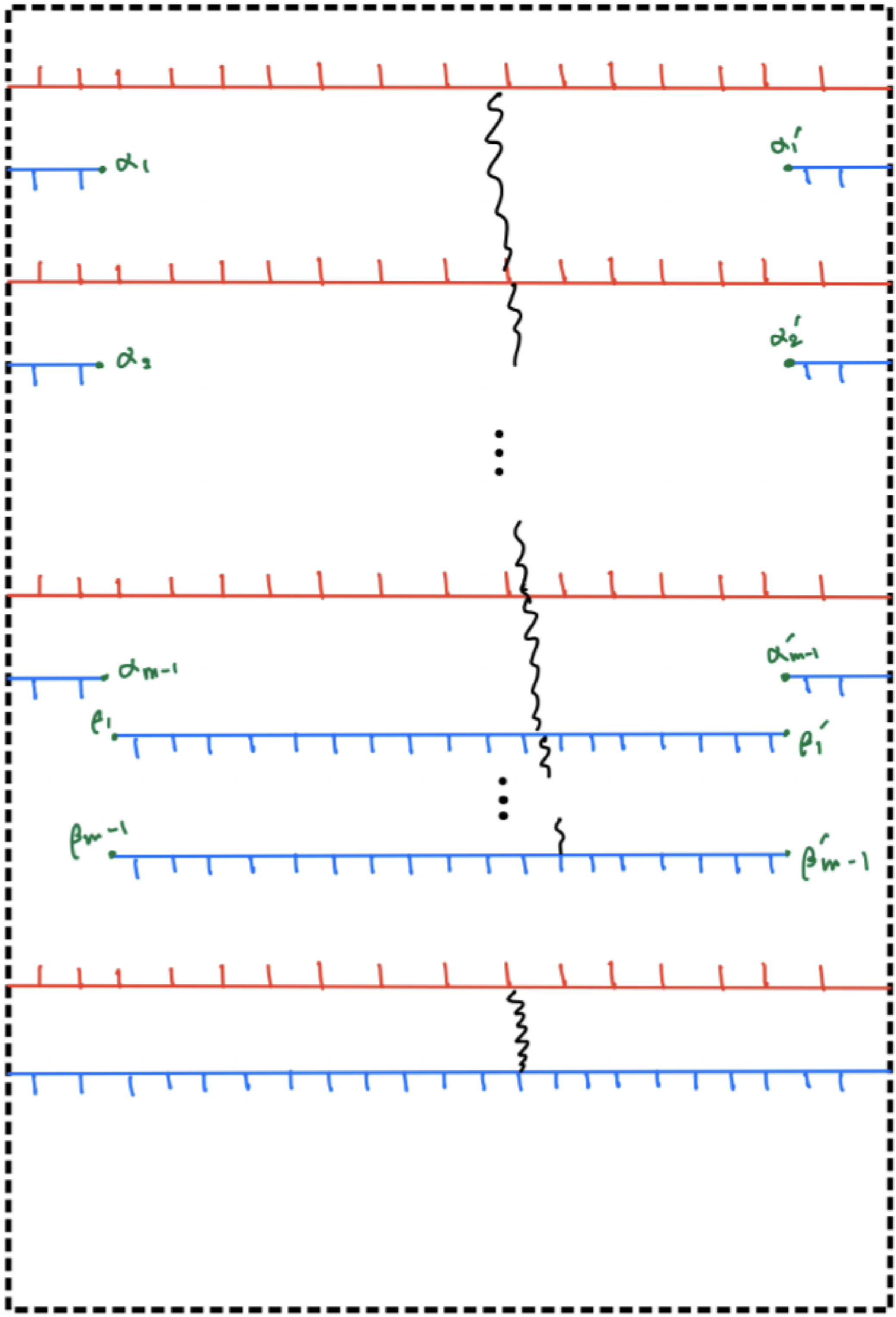


Figure 4.60: Your caption here

(step2) Apply $isotopy_6$ on the disk surrounded by purple dotted line.

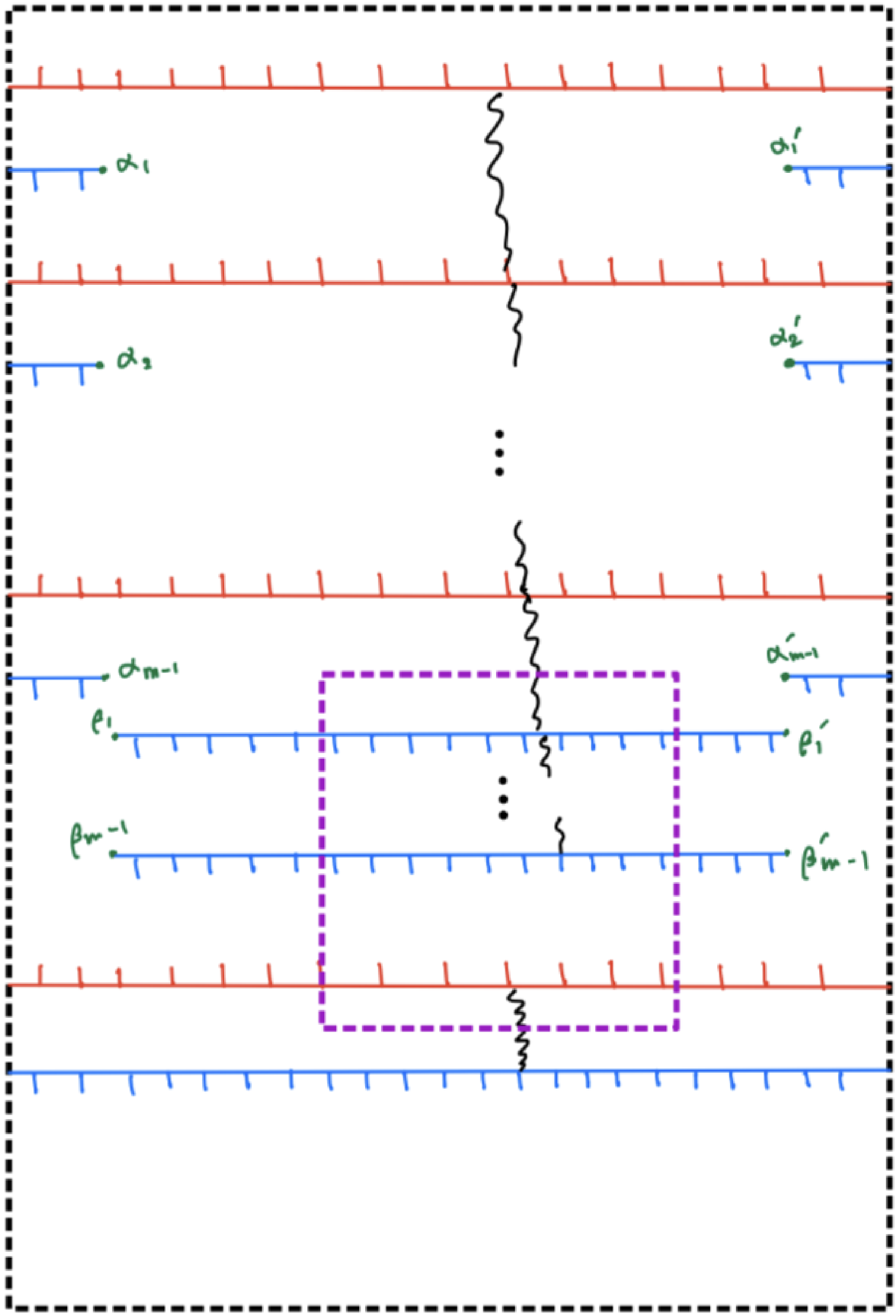


Figure 4.61: Your caption here

we get the final diagram :

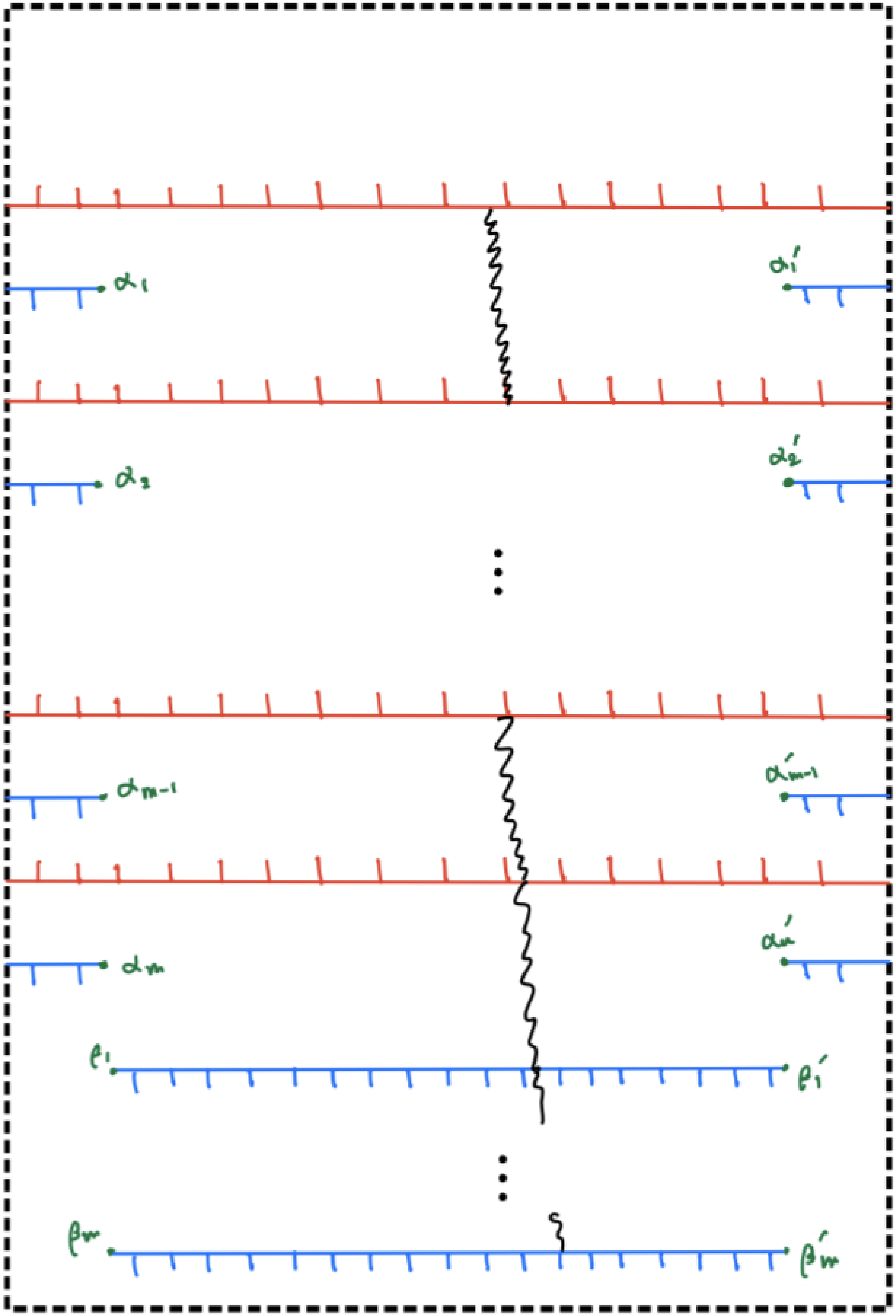


Figure 4.62: Your caption here

4.13 definition8

Definition 55.

Suppose we have the following diagram of 2 red strands(R_1, R) and a blue strand B.

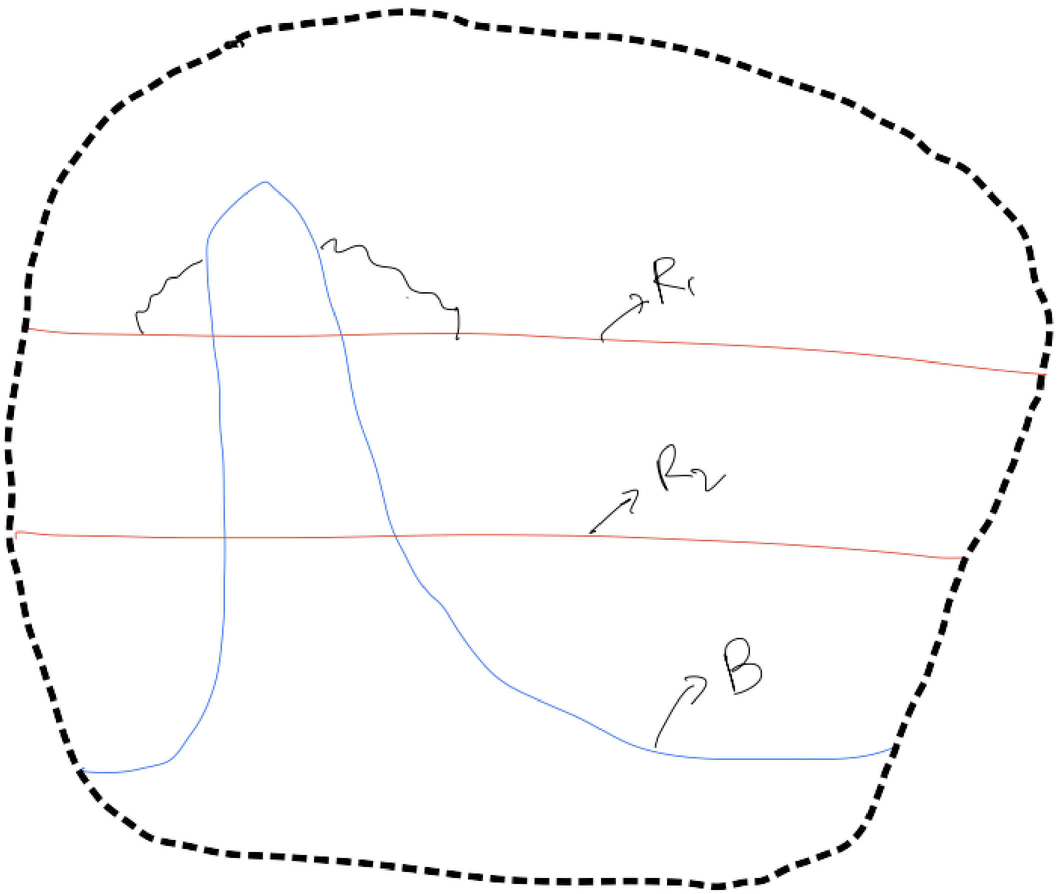


Figure 4.63: Your caption here

We define MOVE viiias follows :

(Step1) First, apply MOVE iito R_1 and B, then we get the following diagram :

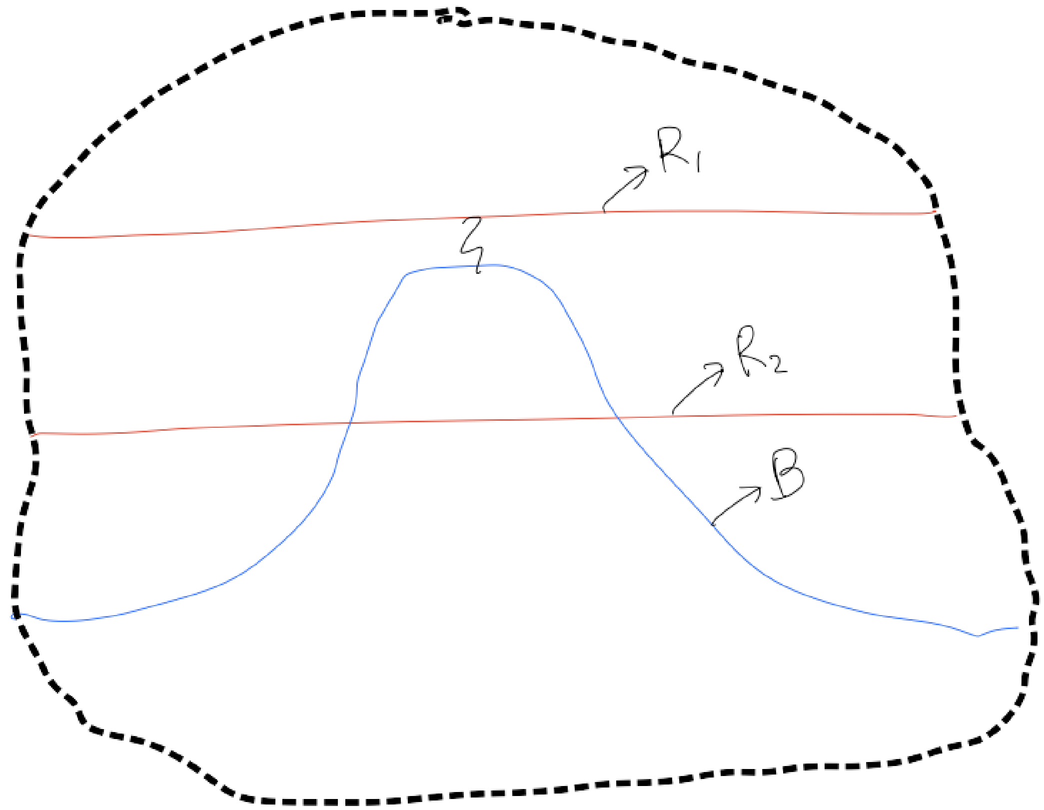


Figure 4.64: Your caption here

(Step2) Apply MOVE v(=MOVE ii??) to R_2 and B, then we get the final diagram

:

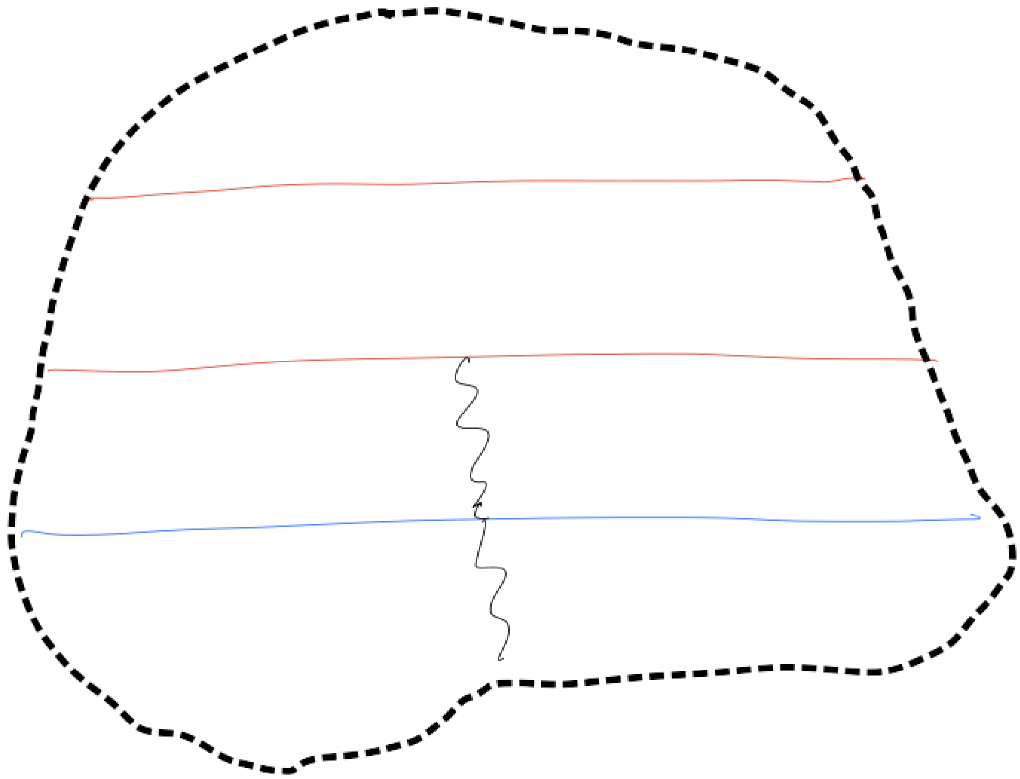


Figure 4.65: Your caption here

4.14 lemma8

Lemma 56.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$, the refinement is as the following figure where two dimensional strata are labeled :

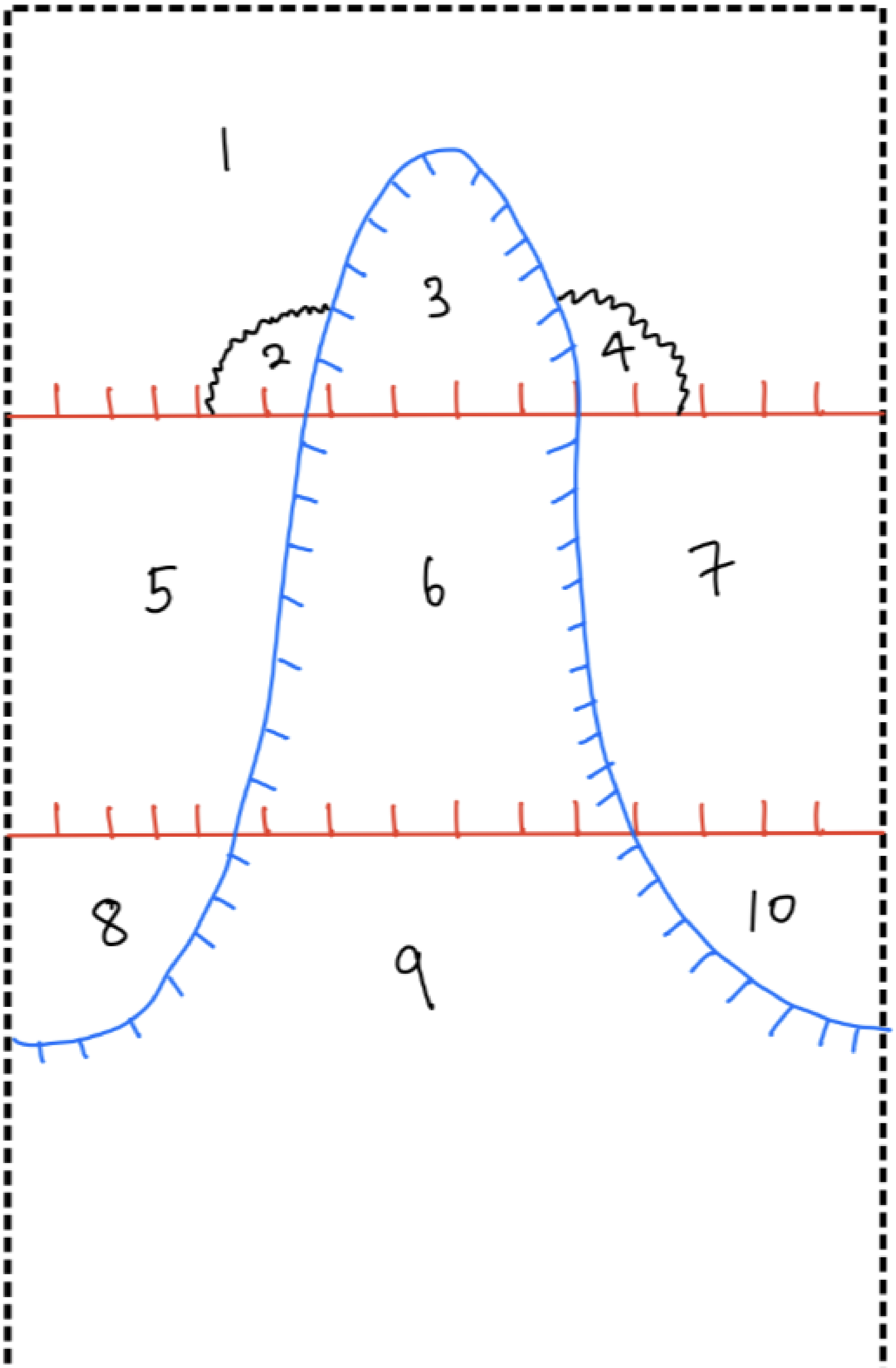


Figure 4.66: Your caption here

Stalks :

- 1 : 0 - 2 : $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$

- 3 : $\mathbb{C}[-1]$

- 4 : $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- 5 : $\mathbb{C}$

- 6 : 0

- 7 : $\mathbb{C}$

- 8 : $\mathbb{C}^2$

- 9 : $\mathbb{C}$

- 10 : $\mathbb{C}^2$

Generization maps :

I won't write down the zero maps.

- 3→2 : $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$

- 3→4 : $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$

- 2→5 : $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- 4→7 : $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- 5→8 : ι_f

- 9→8 ,9→10 : ι_l

- 7→10 : $\mathbb{C} \to \mathbb{C}^2$ where $1 \mapsto (x, y)^T$ where $x \neq 0$

Now we will define isotopy starting from $\mathfrak{F}$ to the final sheaf $\mathfrak{F}'$ which is described as follows:

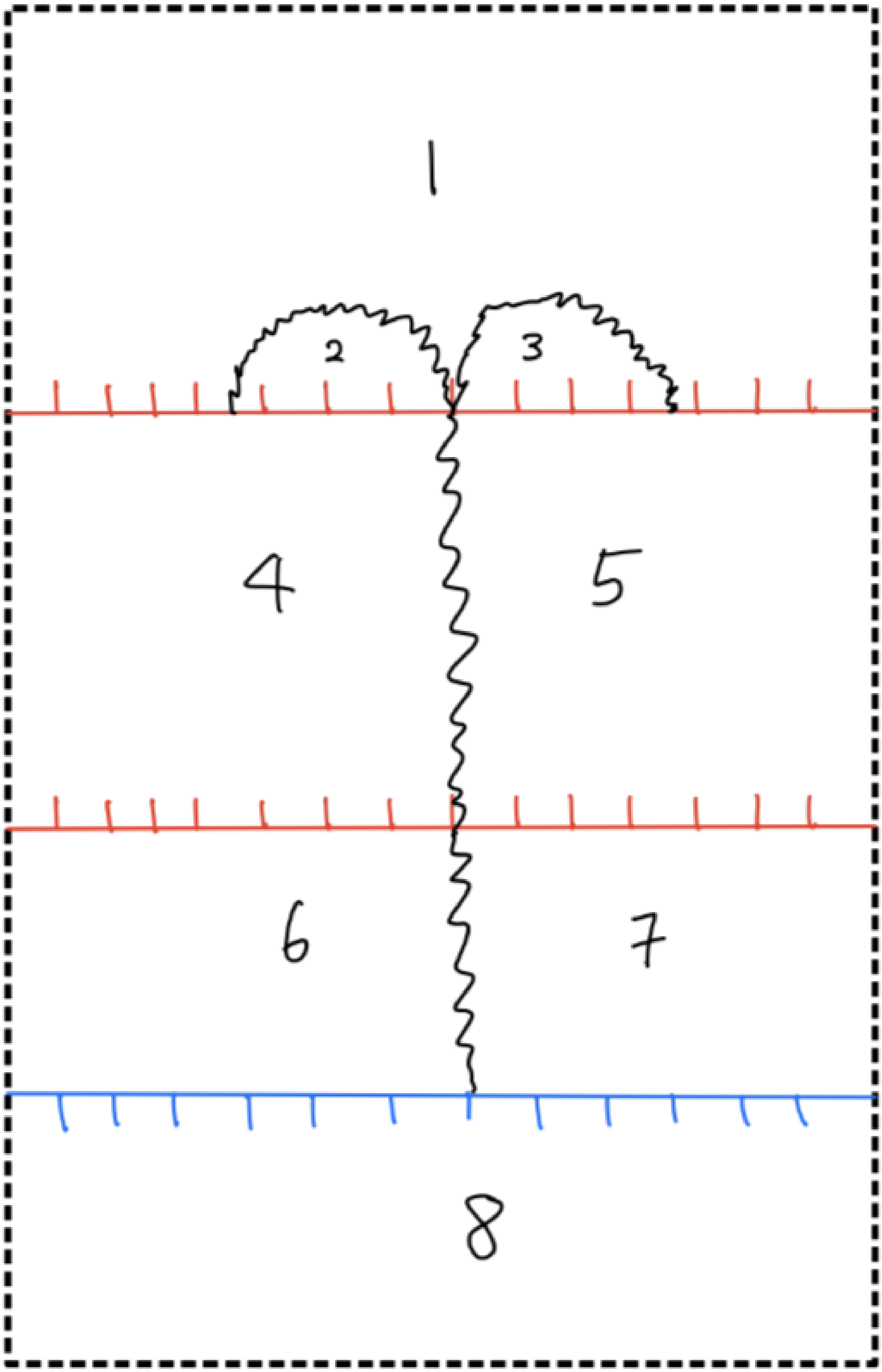


Figure 4.67: Your caption here

Stalks :

- 1: 0

- 2 : $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$

- 3 : $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- 4 : $\mathbb{C}$

- 5 : $\mathbb{C}$

- 6 : $\mathbb{C}$

- 7 : $\mathbb{C}$

- 8 : $\mathbb{C}$

Generization maps :

- 4→5 : muliplication by ab^{-1}

- 4→6 : ι_f

- 5→7 : $\mathbb{C} \to \mathbb{C}^2$ where $1 \mapsto (z, y)^T$

- 6→7 : $\mathbb{C}^2 \to \mathbb{C}^2$ where $e_1 \mapsto (ab^{-1}x, ab^{-1}y)^T$ and $e_2 \mapsto e_2$

-8→6, 8→7 : ι_l

Now we define $isotopy_8$ as follows :

(step1) Apply $isotopy_2$ inside the disk surrounded by the purple dotted line :

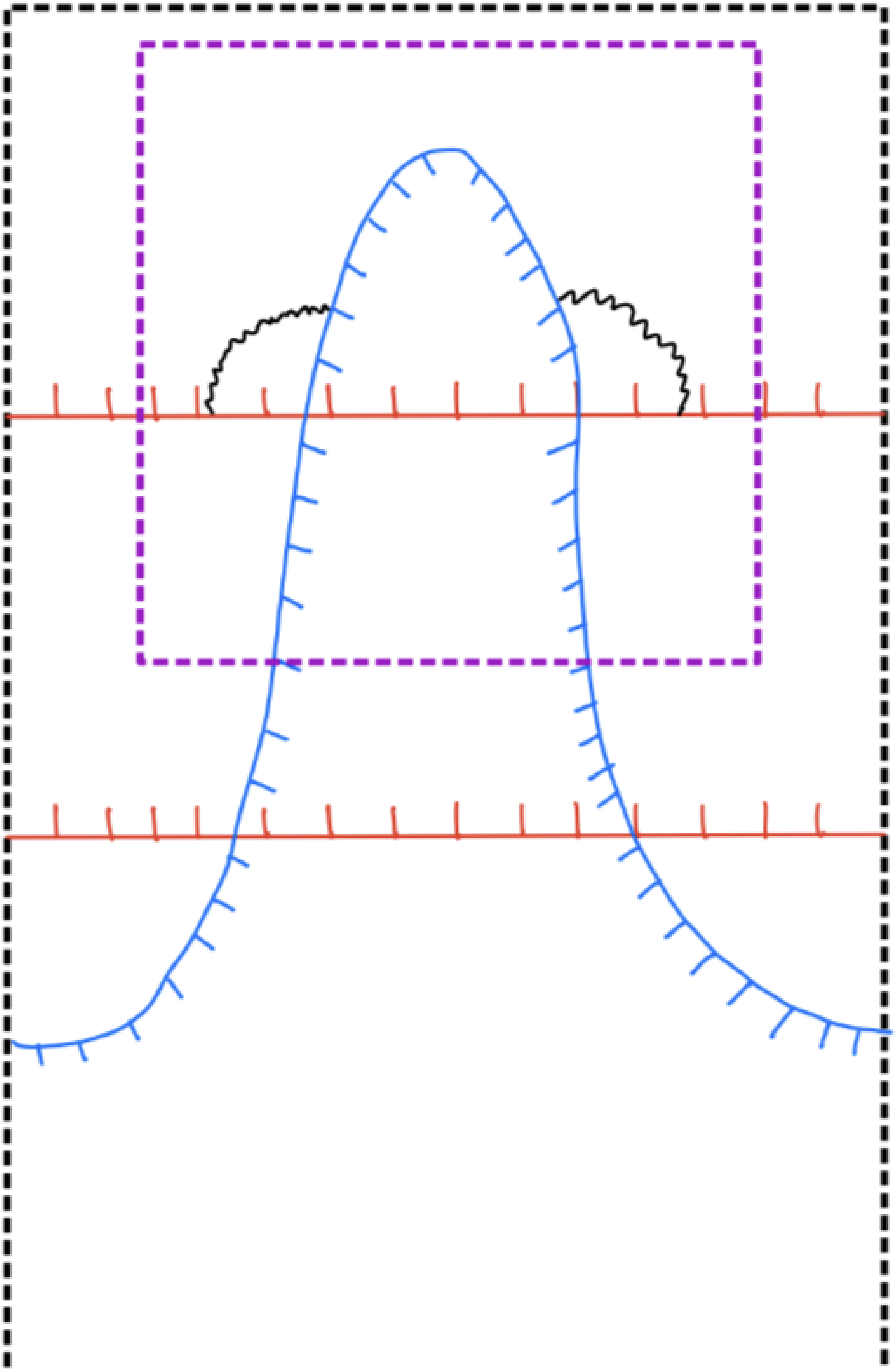


Figure 4.68: Your caption here

We get the following diagram :

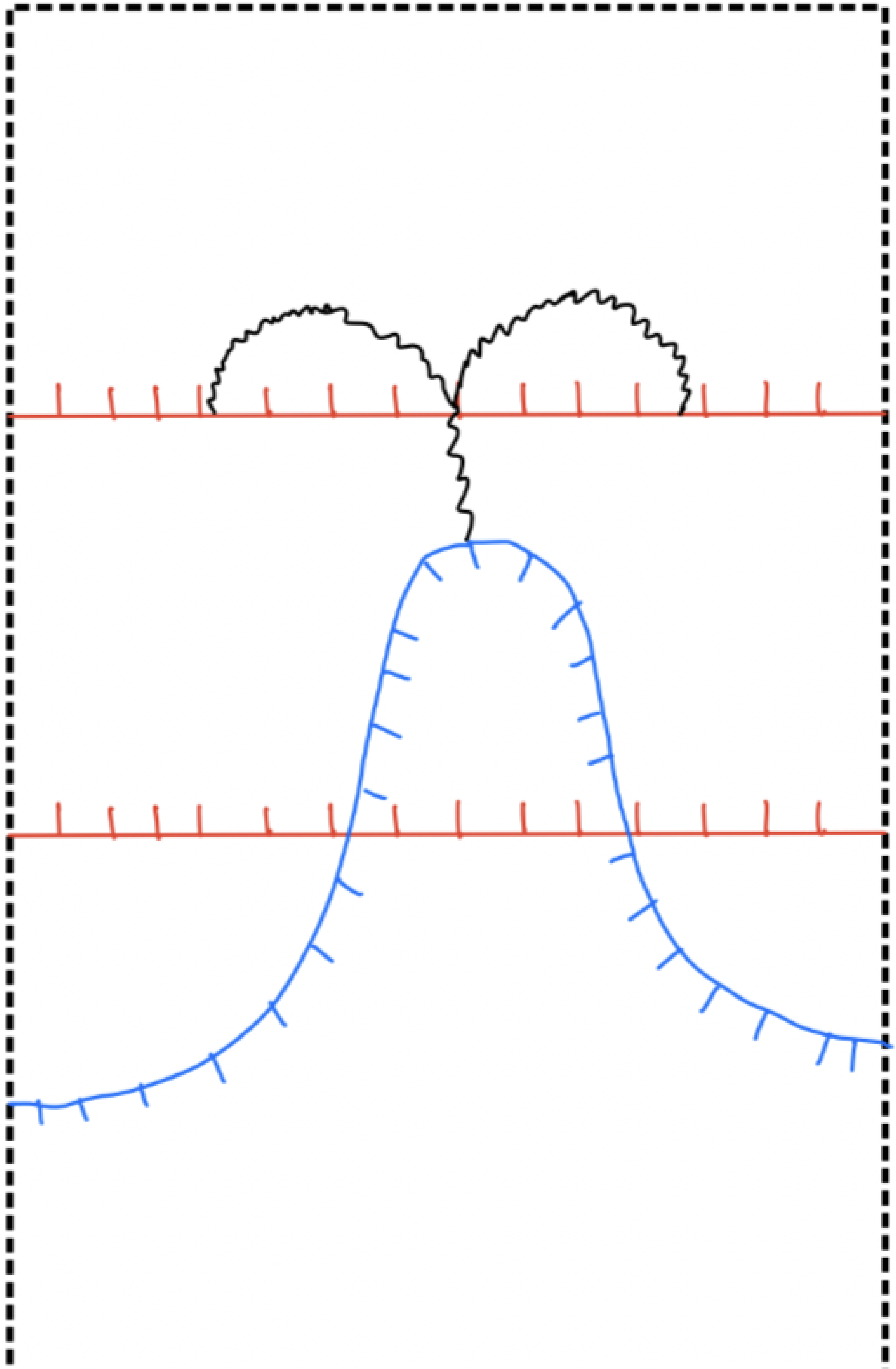


Figure 4.69: Your caption here

(step2) Apply $isotopy_5$ on the disk surrounded by the purple dotted line:

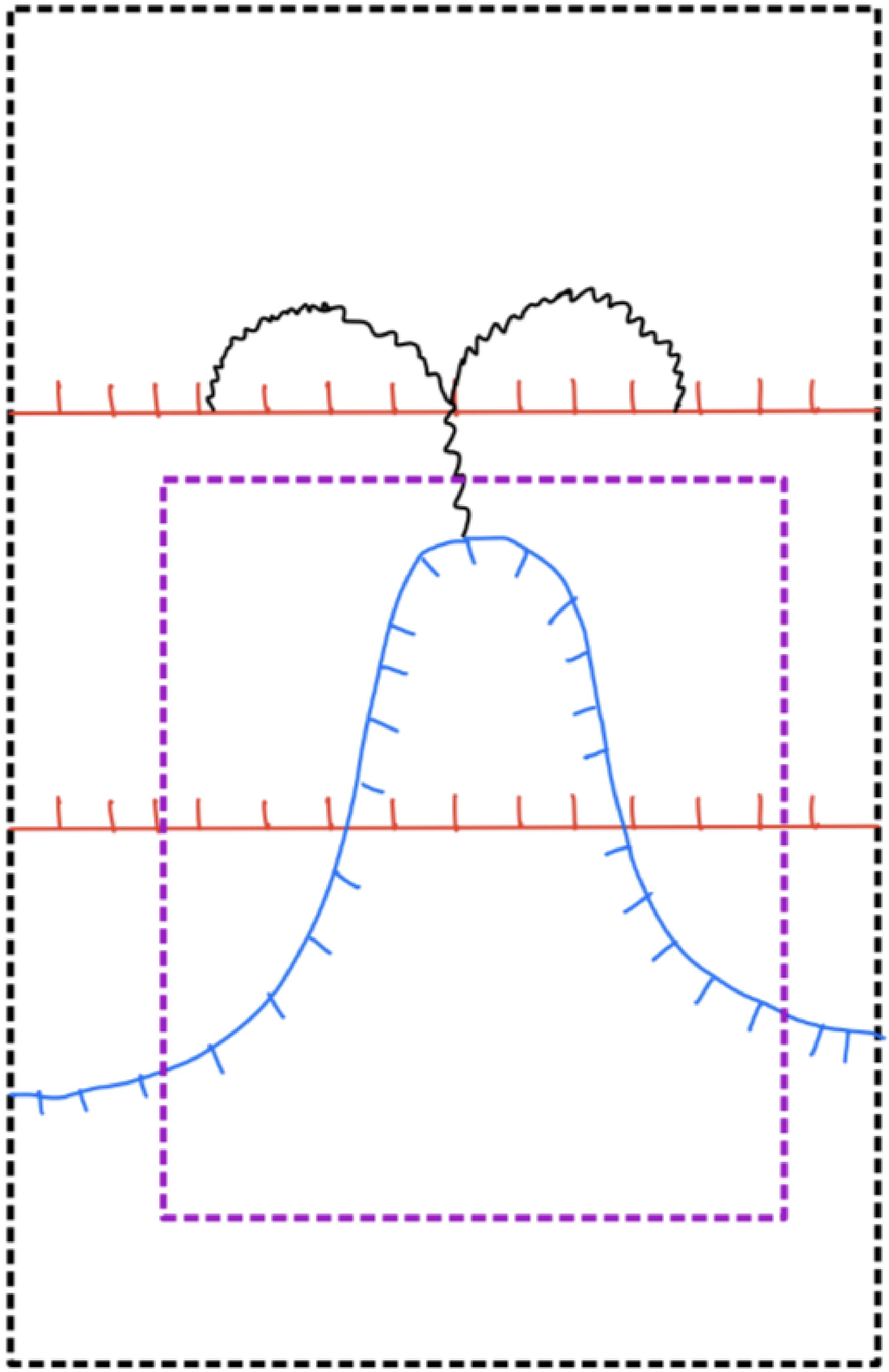


Figure 4.70: Your caption here

We get the final diagram :

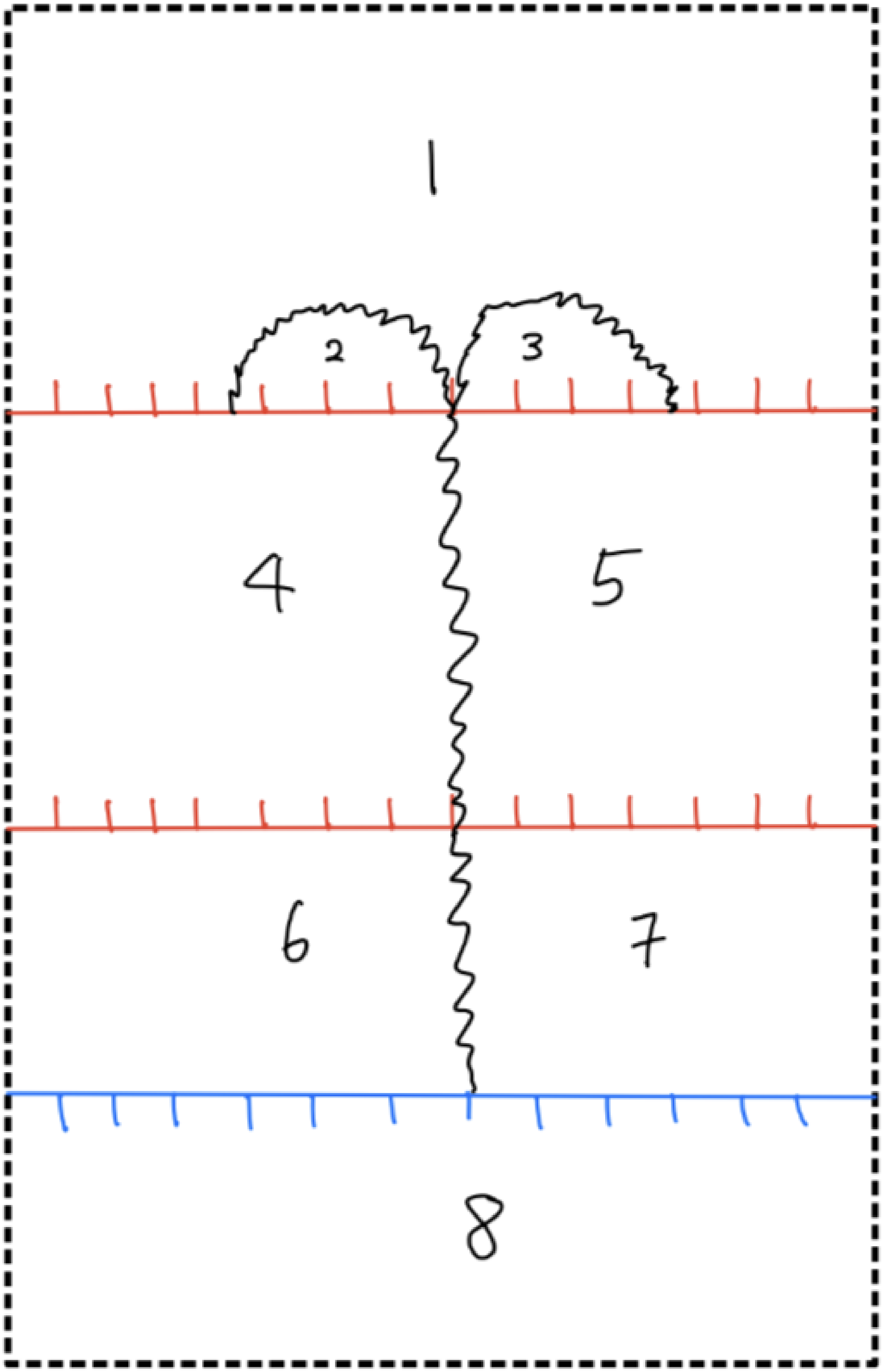


Figure 4.71: Your caption here

with sheaf $\mathfrak{F}'$ on it.

(proof)

4.15 definition9

Definition 57.

Suppose we have the following diagram of 2 red strands(R_1, R_2) and a blude strand B.

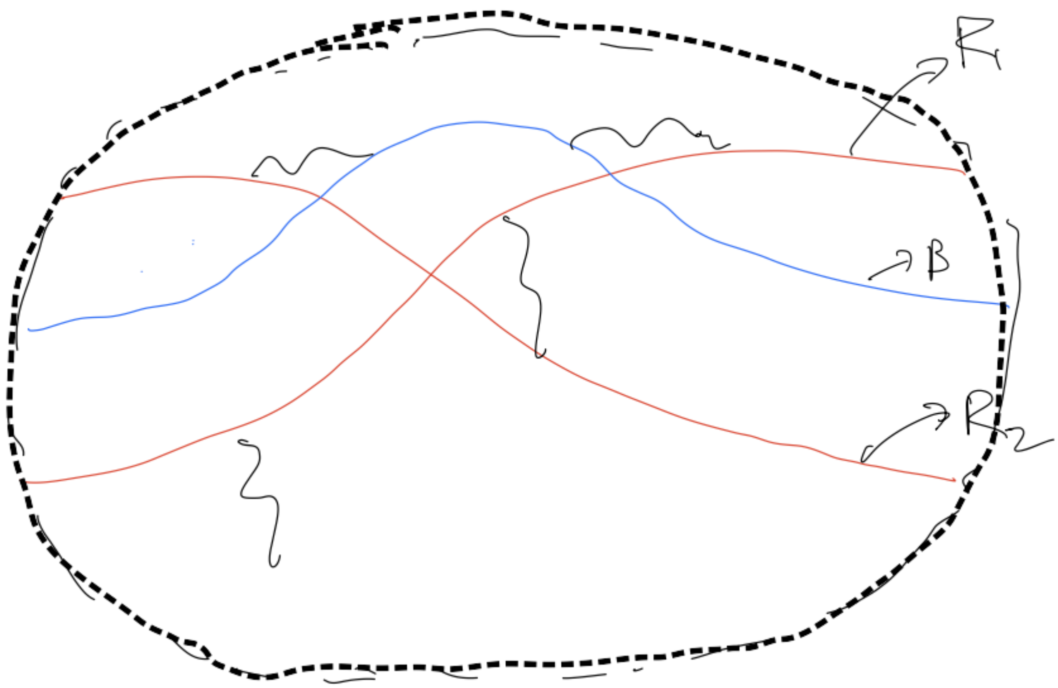


Figure 4.72: Your caption here

We define MOVE ixas follows :

(Step1) Apply MOVE ito pairs (R_1, B) and (R_2, B), we get :

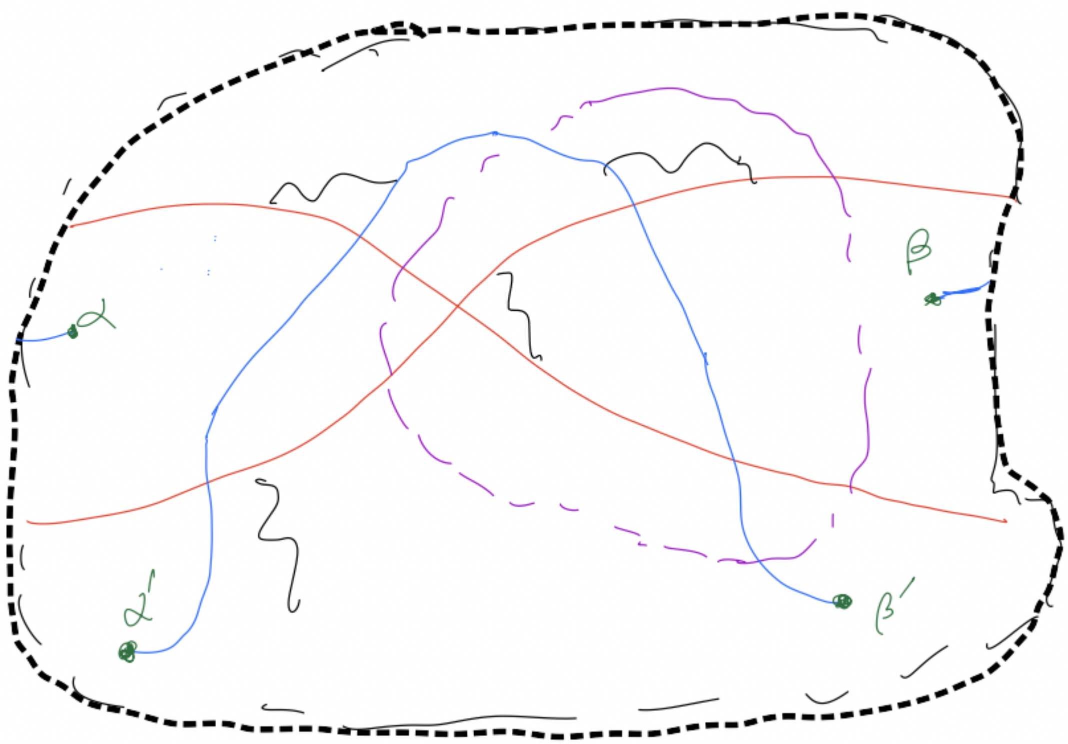


Figure 4.73: Your caption here

(Step2) Apply MOVE ivlocally inside the region surrounded by purple dotted circle, we get :

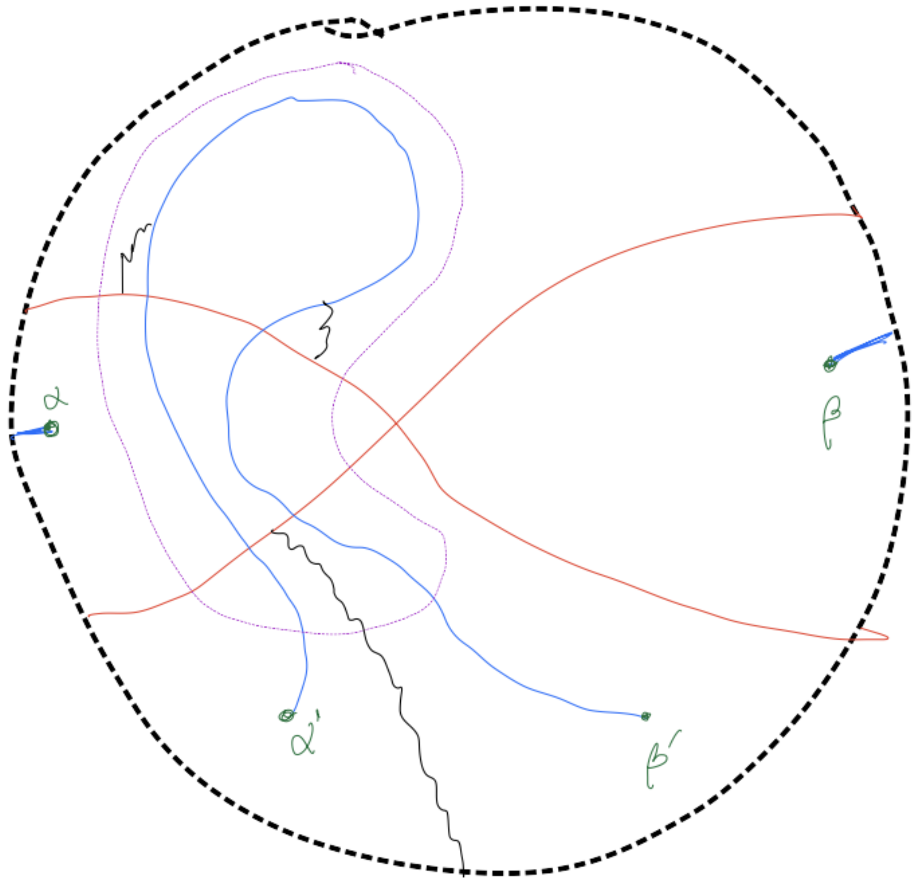


Figure 4.74: Your caption here

(Step3) Apply MOVE iiilocally inside the region surrounded by purple dotted line, we get :

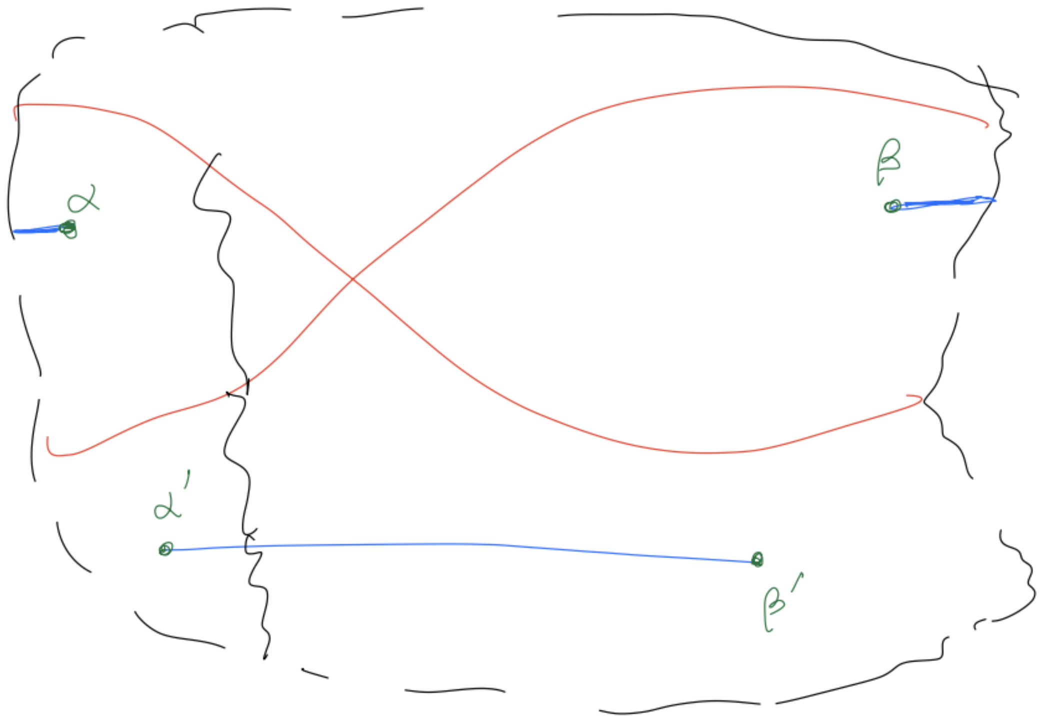


Figure 4.75: Your caption here

4.16 lemma9

Lemma 58.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$ the refinement is as the following figure where two dimensional strata are labeled :

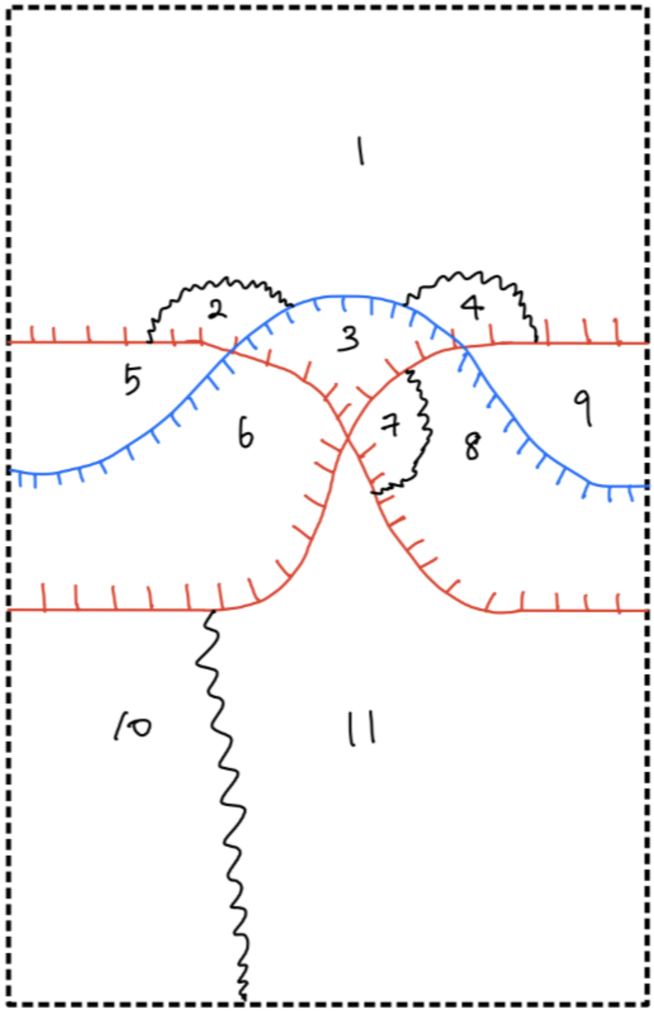


Figure 4.76: Your caption here

Stalks :

- 1 : 0
- 2 : $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- 3 : $\mathbb{C}[-1]$
- 4 : $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- 5 : $\mathbb{C}$
- 6 : 0
- 7 : $\mathbb{C} \xrightarrow{\times c} \mathbb{C}$

- 8 : 0

- 9 : $\mathbb{C}$

- 10 : $\mathbb{C}$

- 11 : $\mathbb{C}$

Generization maps :

I won't write down the zero maps.

- 2→5 : $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

- 3→2 : $$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

- 3→4 : $$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

- 4→9 : $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

- 3→7 : $$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

- 7→11 : $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

- 10→11 : $\mathbb{C} \xrightarrow{\times d} \mathbb{C}$

Now we will define isotopy starting from the above sheaf $\mathfrak{F}$ to the final sheaf $\mathfrak{F}'$. This will be called $isotopy_9$. The following figure is the sheaf $\mathfrak{F}'$ restricted to $D \subset C$.

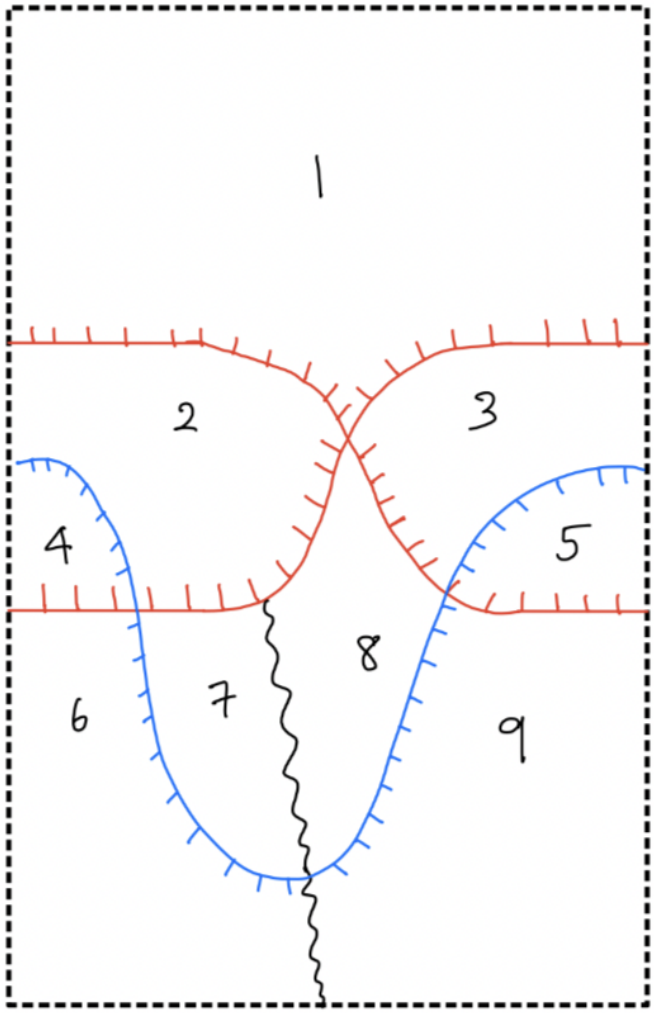


Figure 4.77: Your caption here

Stalks :

- 1 : 0
- 2 : $\mathfrak{C}$
- 3 : $\mathfrak{C}$
- 4 : 0
- 5 : 0
- 6 : $\mathfrak{C}$
- 7 : $\mathfrak{C}^2$

- 8 : $\mathfrak{C}^2$

- 9 : $\mathfrak{C}$

Generization maps :

-7→8 : $\mathbb{C}^2 \to \mathbb{C}^2$ where $e_1 \mapsto (ac^{-1}, ab^{-1})^T$ and $e_2 \mapsto (0, d)^T$

- 6 →9 : $\mathbb{C} \xrightarrow{\times d} \mathbb{C}$

- 2 → : $\mathbb{C} \to \mathbb{C}^2$ where $e_1 \mapsto (aC^{-1}, ab^{-1})^T$

- rest of the maps crossing red strands are ι_f

- rest of the maps crossing blue strands are ι_f

We define $isotopy_9$ as follows :

(step1) Apply $isotopy_1$ inside the disk surrounded by the purple dotted lines:

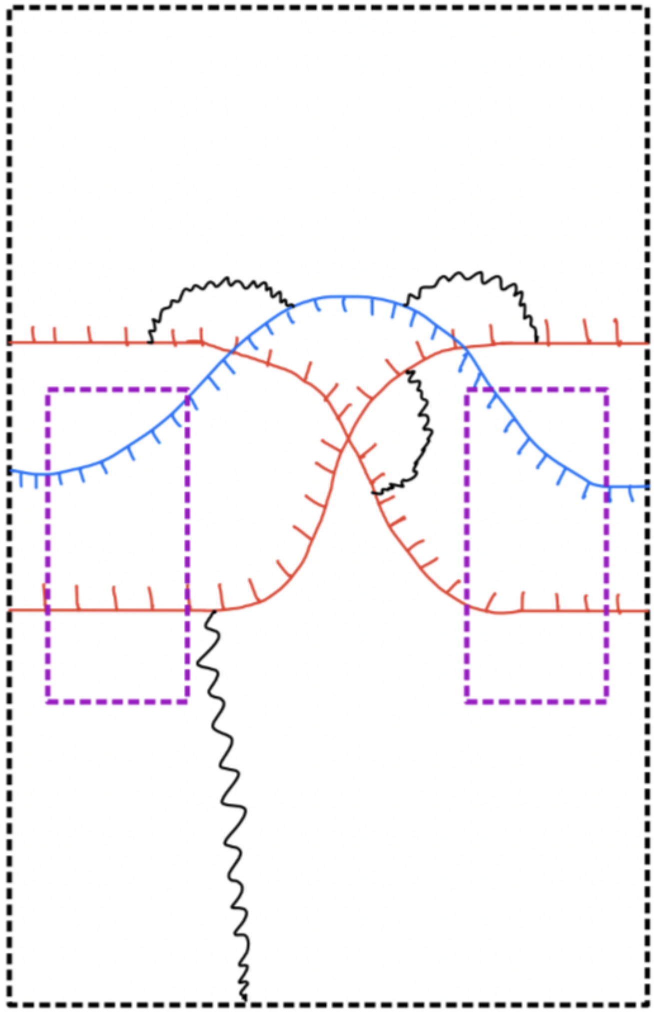


Figure 4.78: Your caption here

We get the following diagram :

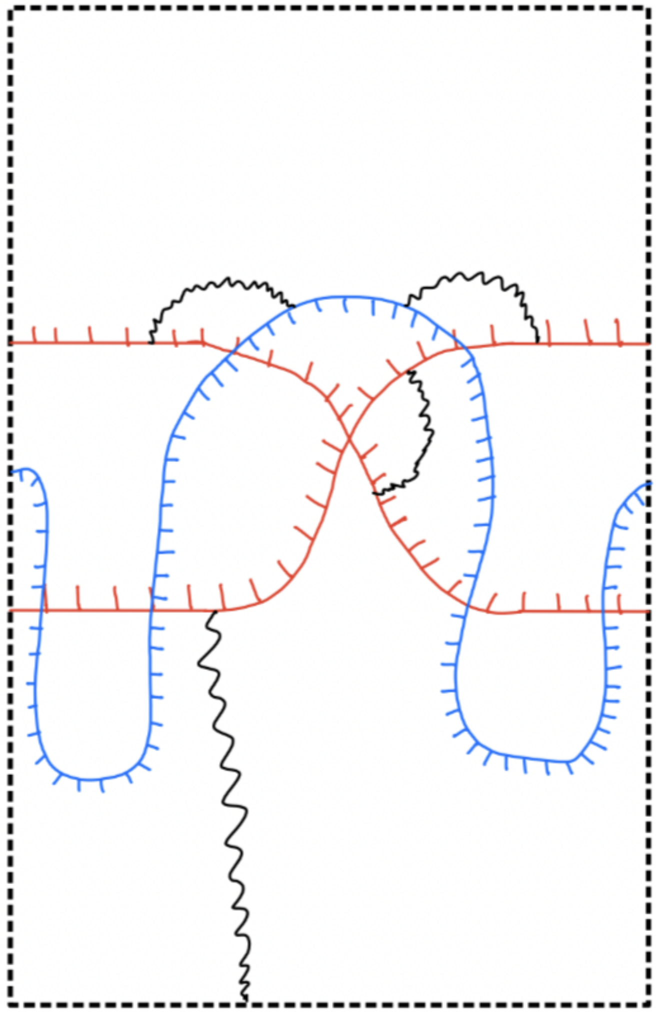


Figure 4.79: Your caption here

(step2) Apply $isotopy_4$ on the disk surrounded by purple dotted line:

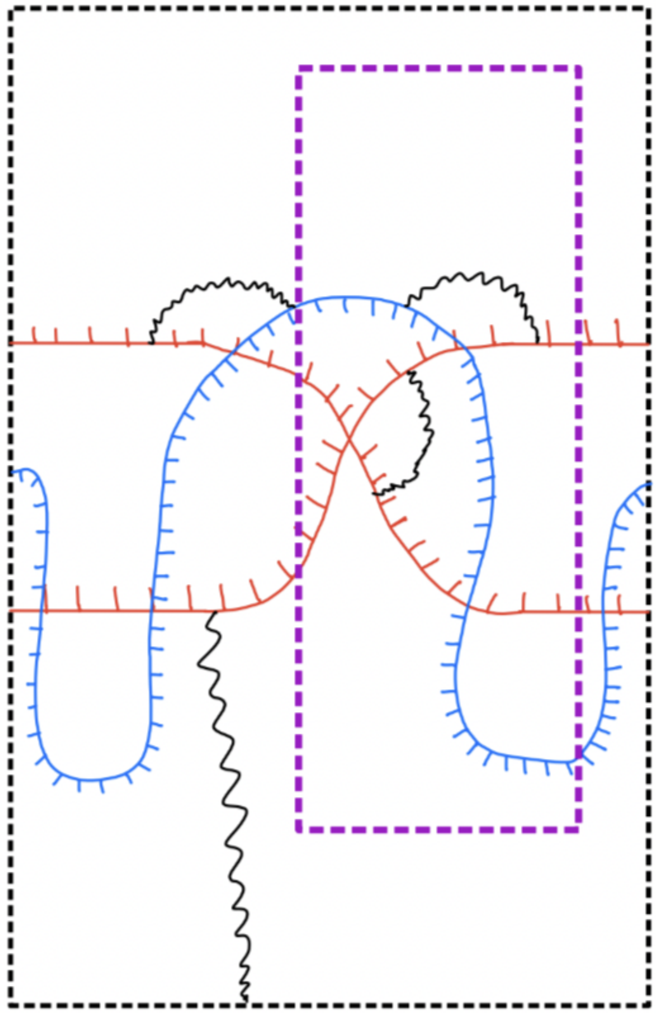


Figure 4.80: Your caption here

We get the following diagram :

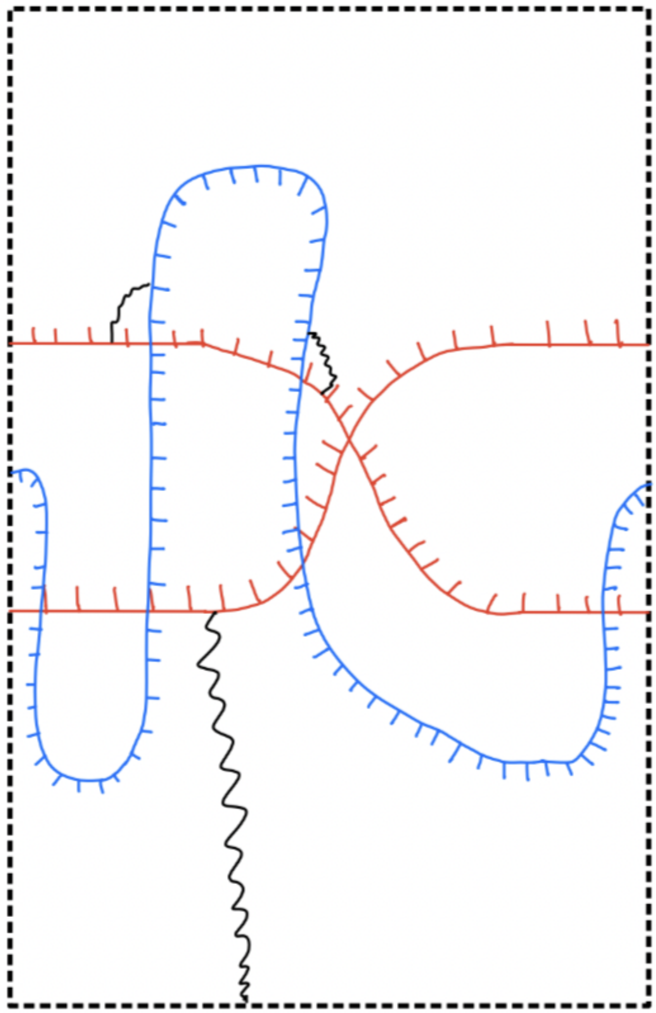


Figure 4.81: Your caption here

(step3) Apply *isotopy*$_8$ on the disk surrounded by purple dotted line:

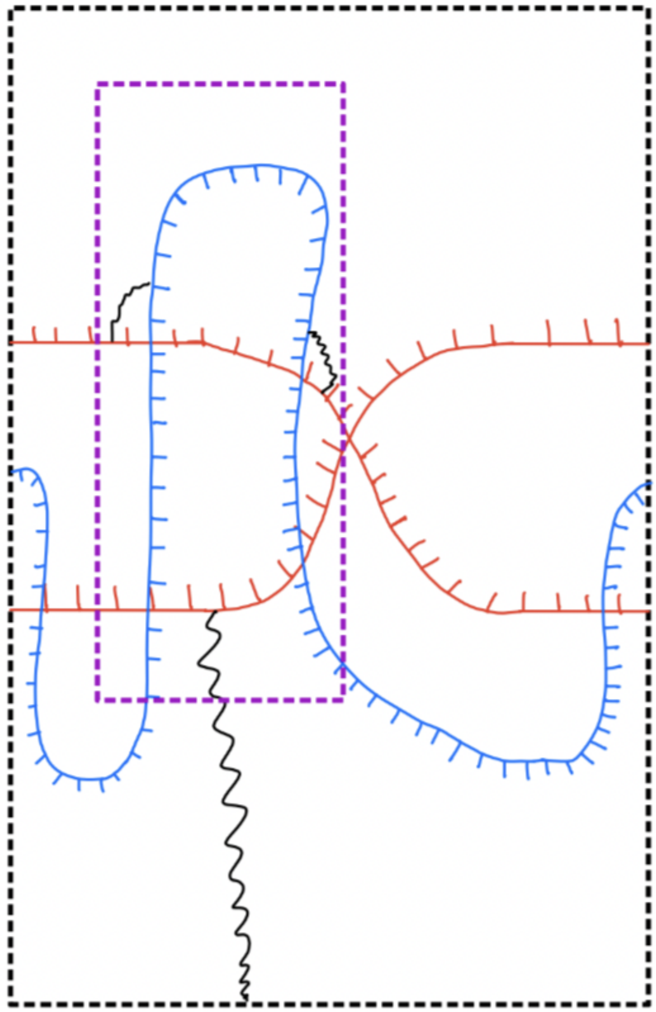


Figure 4.82: Your caption here

we get the following diagram :

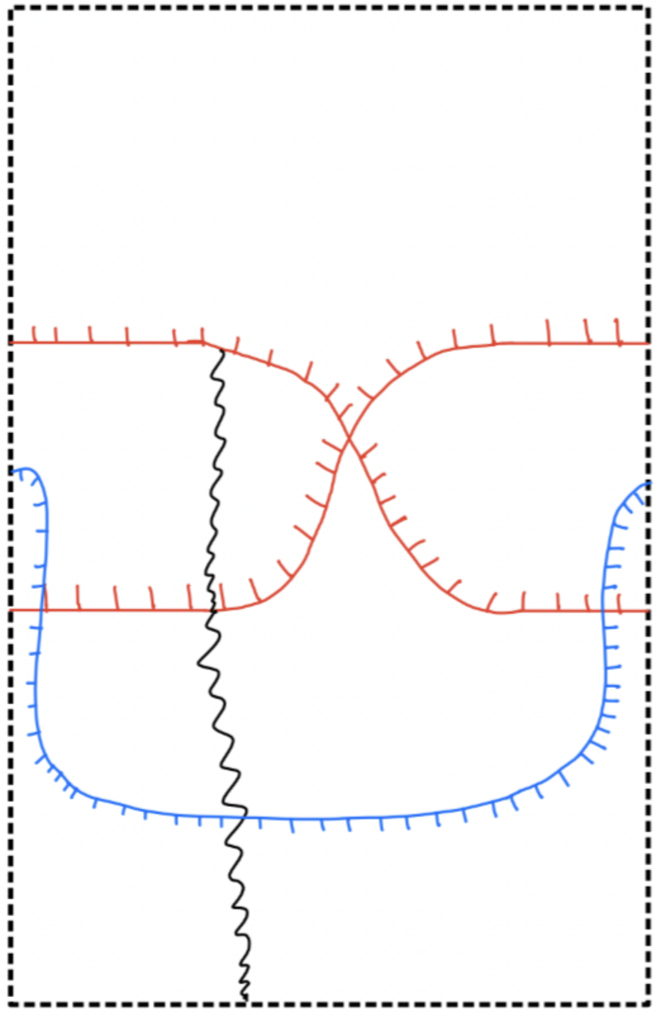


Figure 4.83: Your caption here

(step4) we can change the basis of the stalk at the region marked with purple star so that the generization map corresponding to the squiggly line next to the purple star is the identity map:

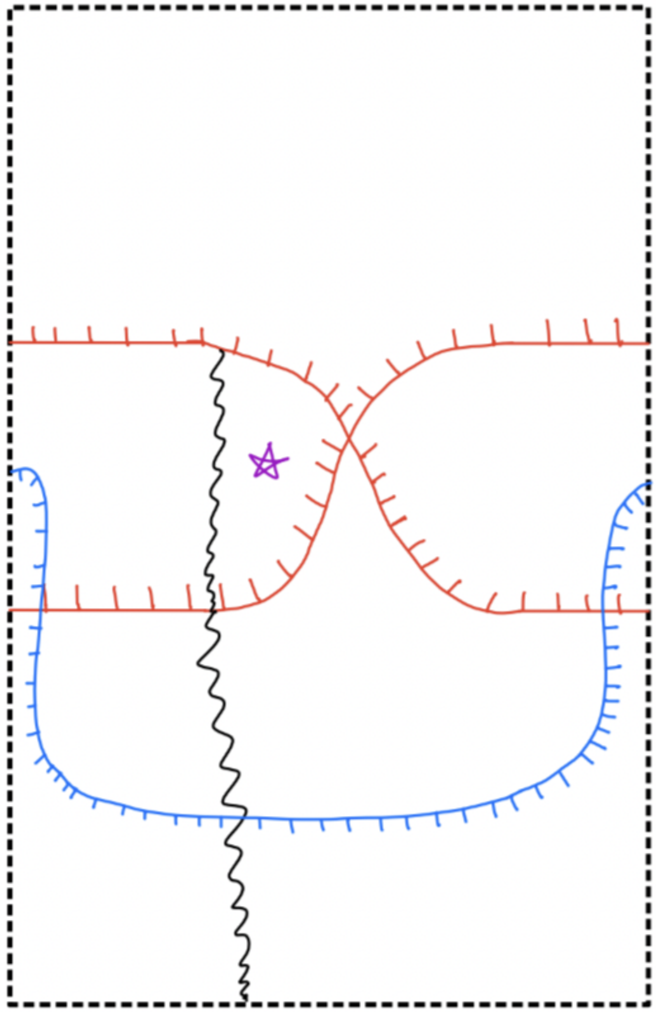


Figure 4.84: Your caption here

Now the sheaf could be thought of as a sheaf singular supported on the below diagram:

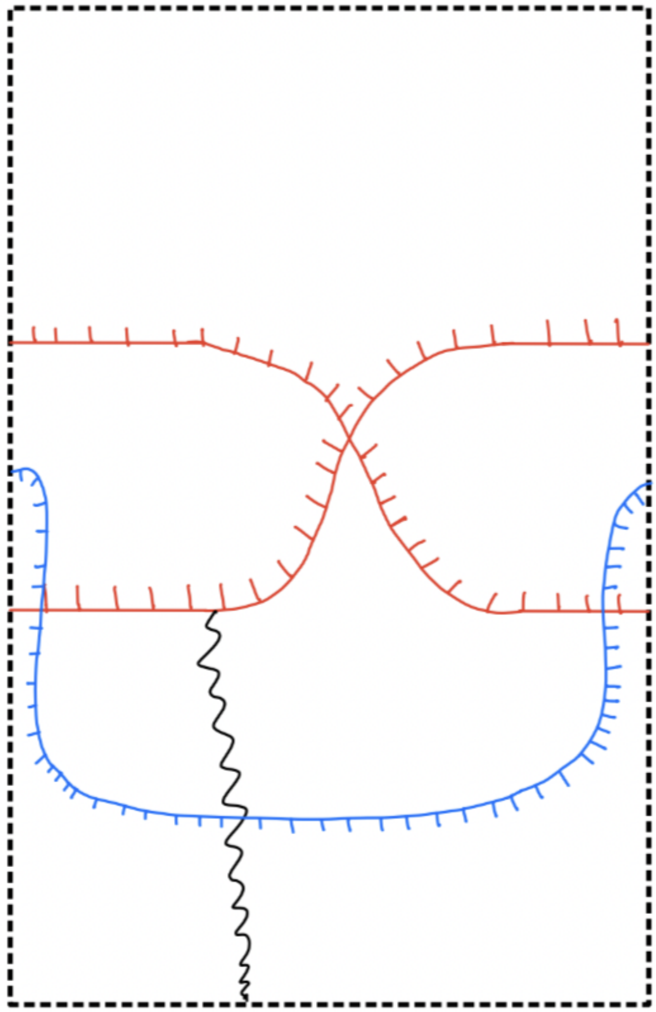


Figure 4.85: Your caption here

and that sheaf is $\mathfrak{F}'$.

(proof) By Lemma1, after (step1) we get the following sheaf

4.17 definition10

Definition 59.

Suppose we have the following diagram :

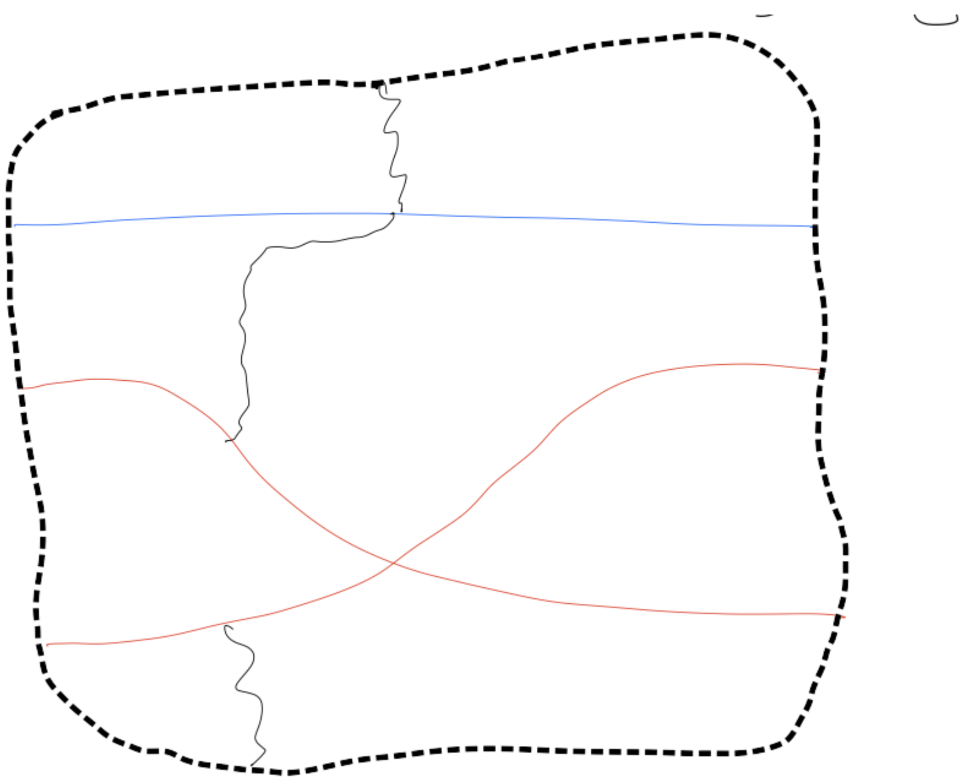


Figure 4.86: Your caption here

We define MOVE xas follows :

(Step1) Apply MOVE ito the region inside of the purple circle :

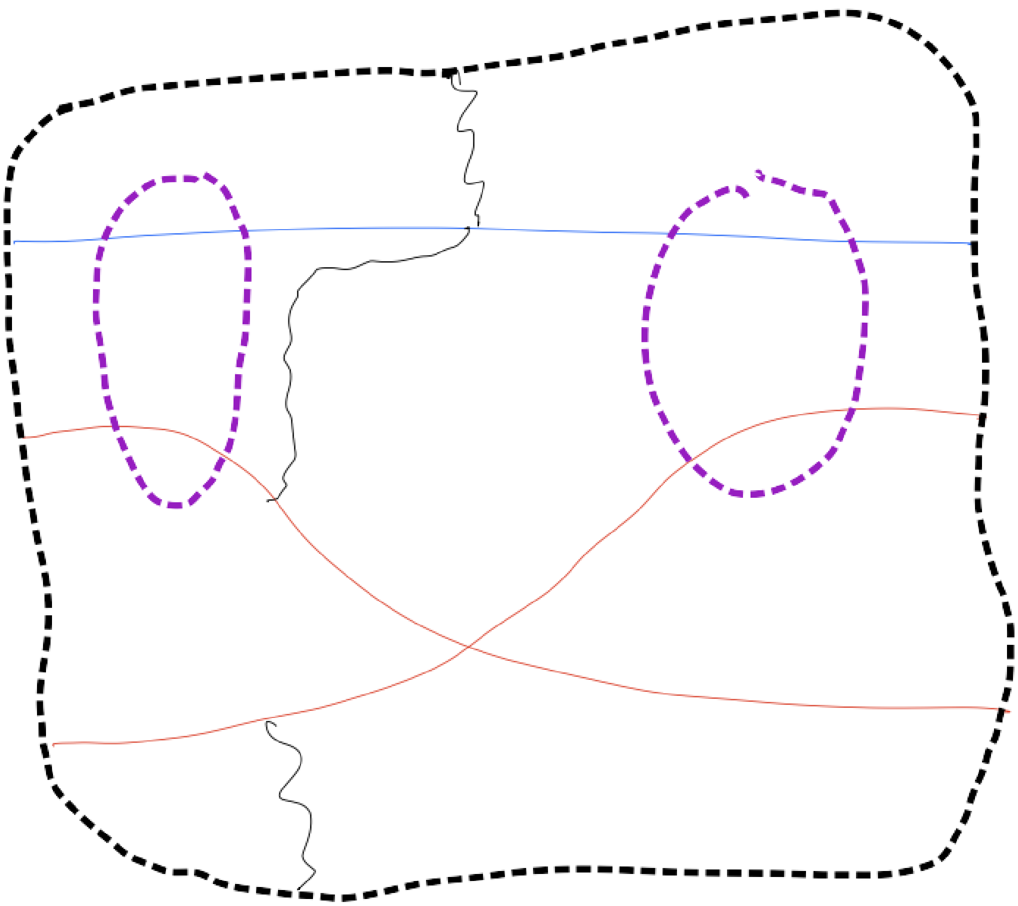


Figure 4.87: Your caption here

we get :

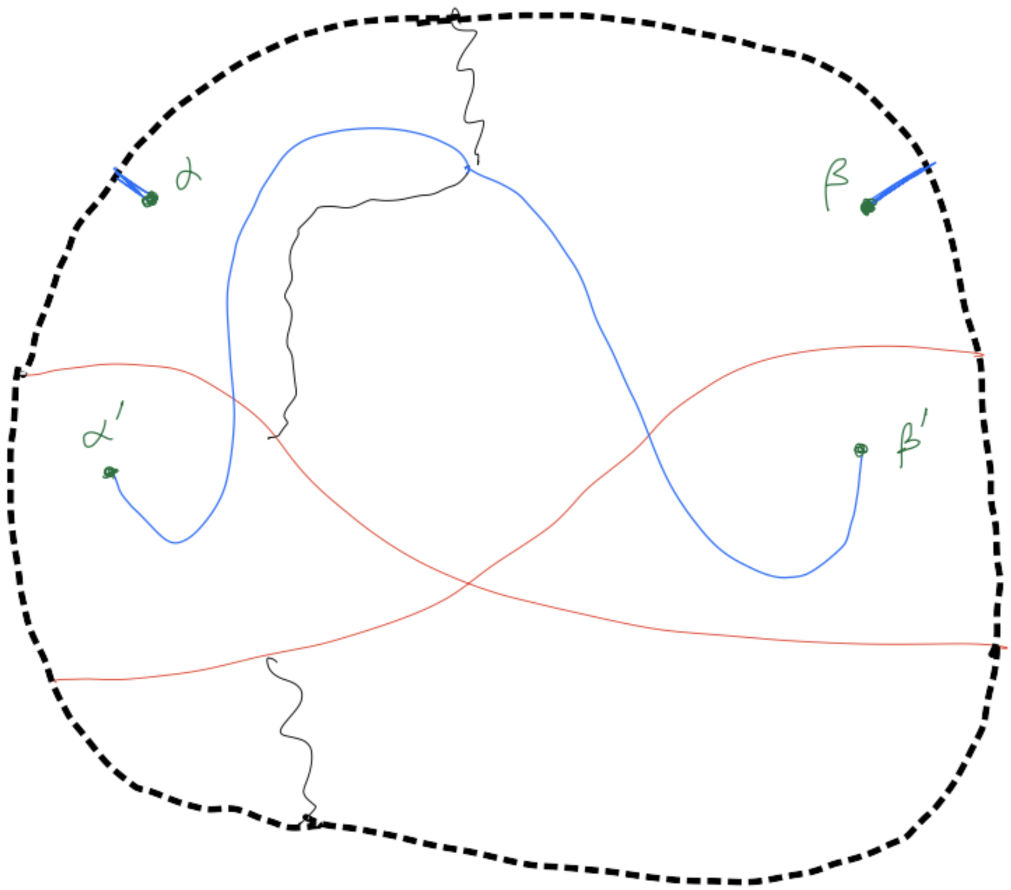


Figure 4.88: Your caption here

(Step2) Apply MOVE ito the region inside the purple circle :

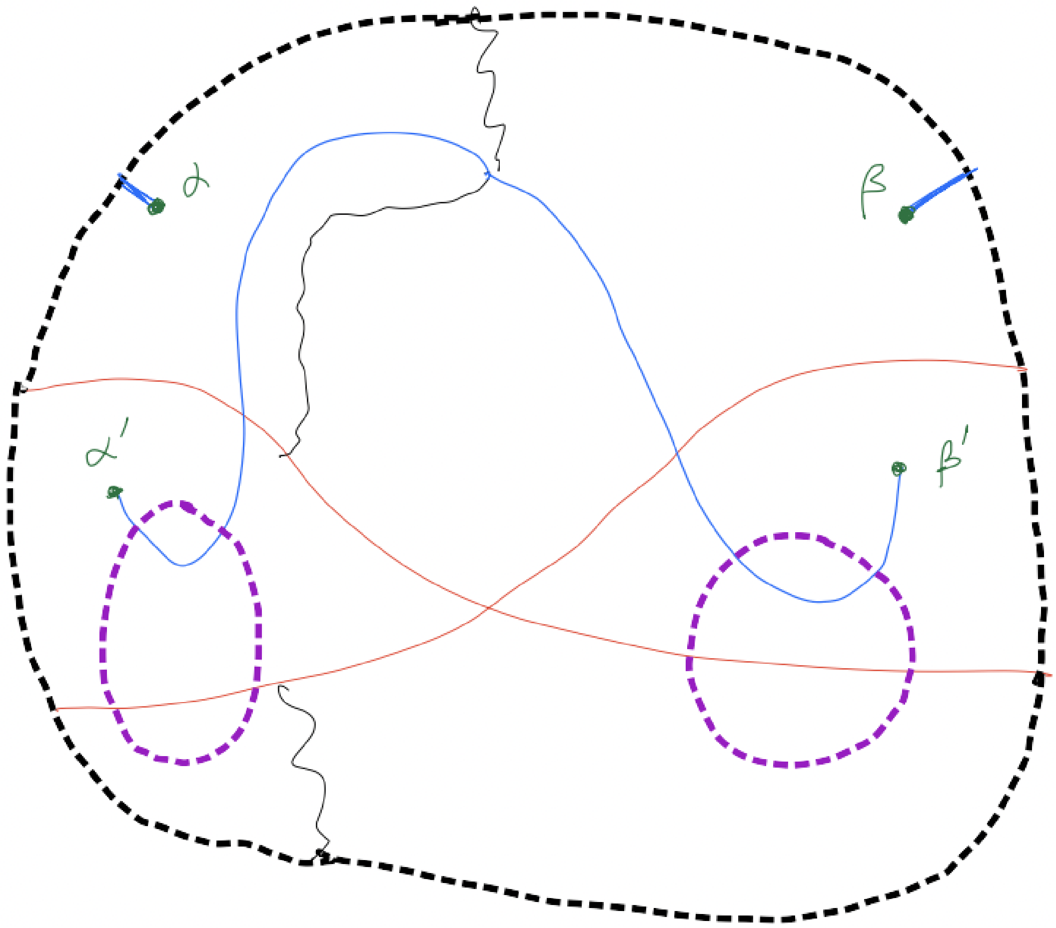


Figure 4.89: Your caption here

we get :

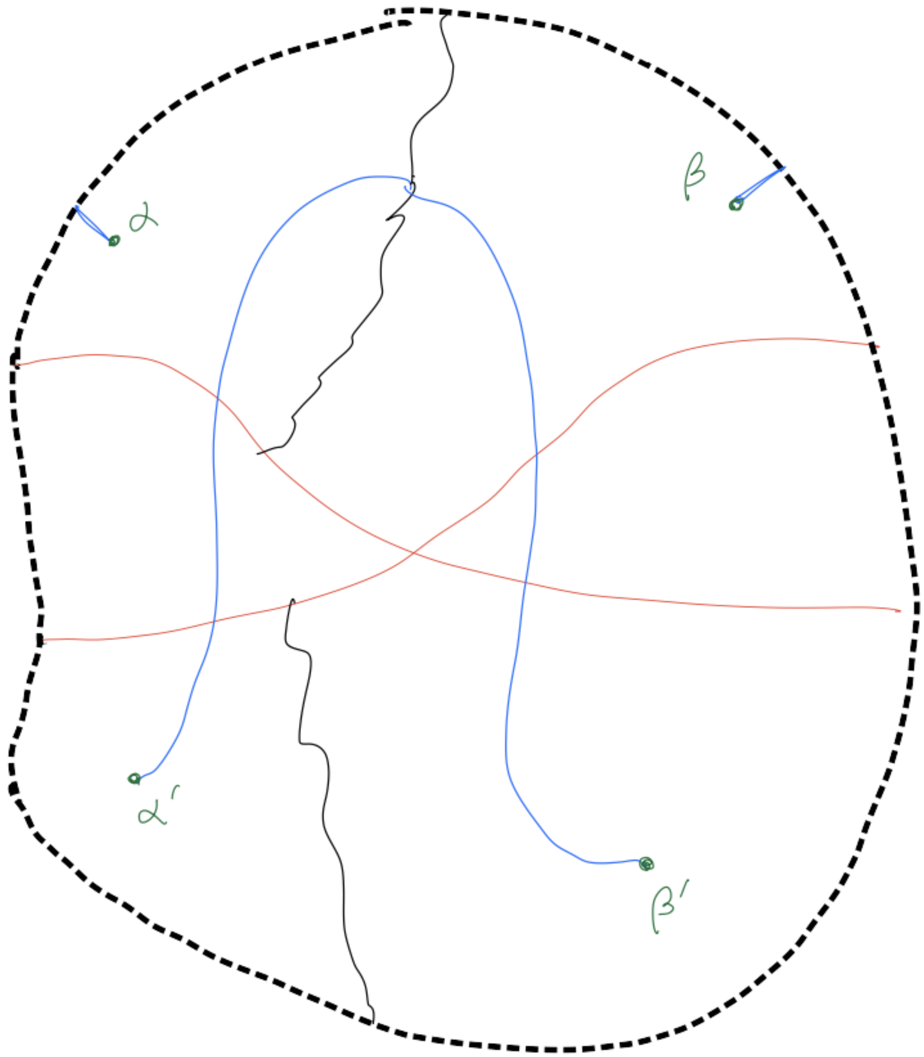


Figure 4.90: Your caption here

(Step3) Apply MOVE ivto the region inside the purple circle :

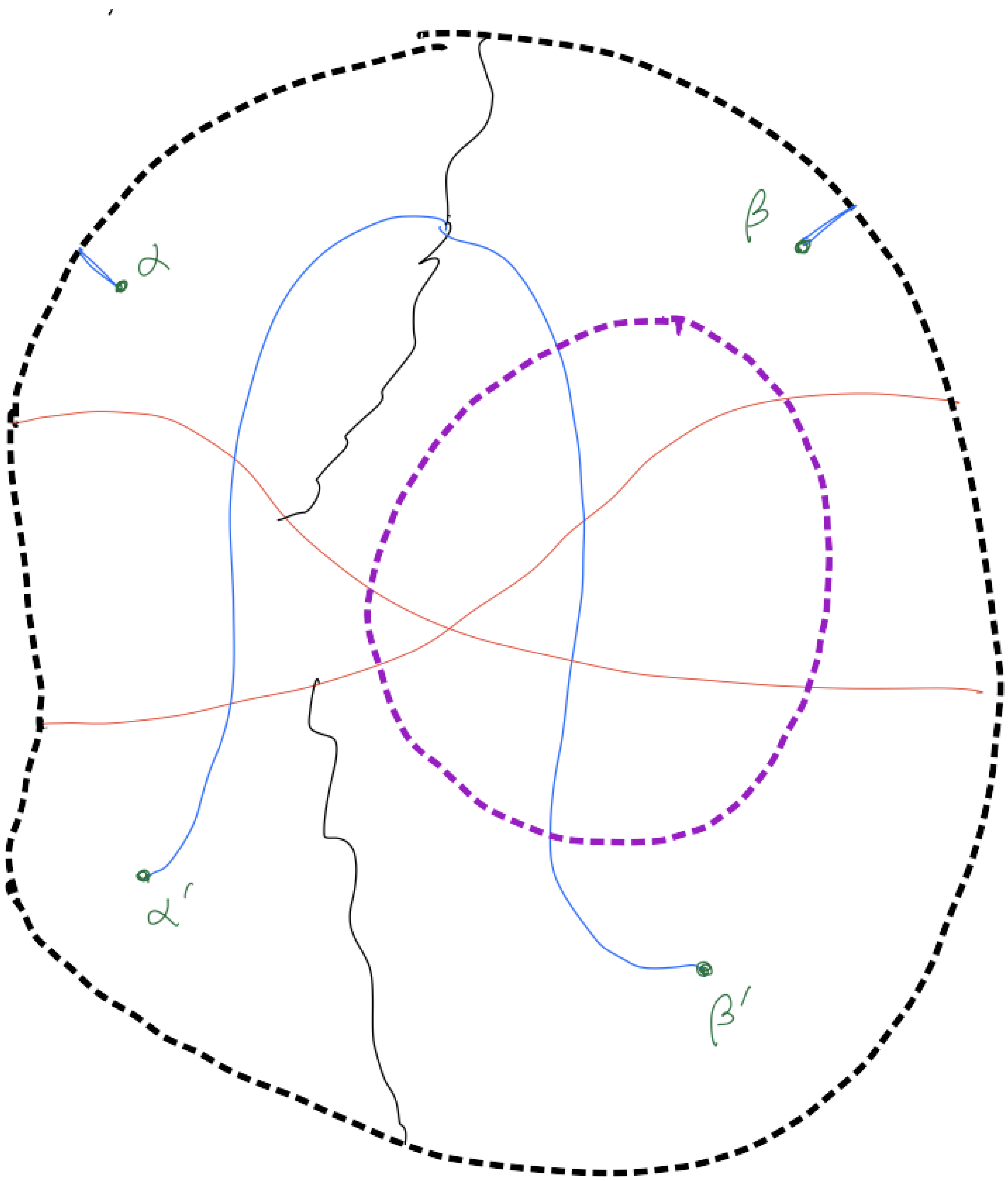


Figure 4.91: Your caption here

we get :

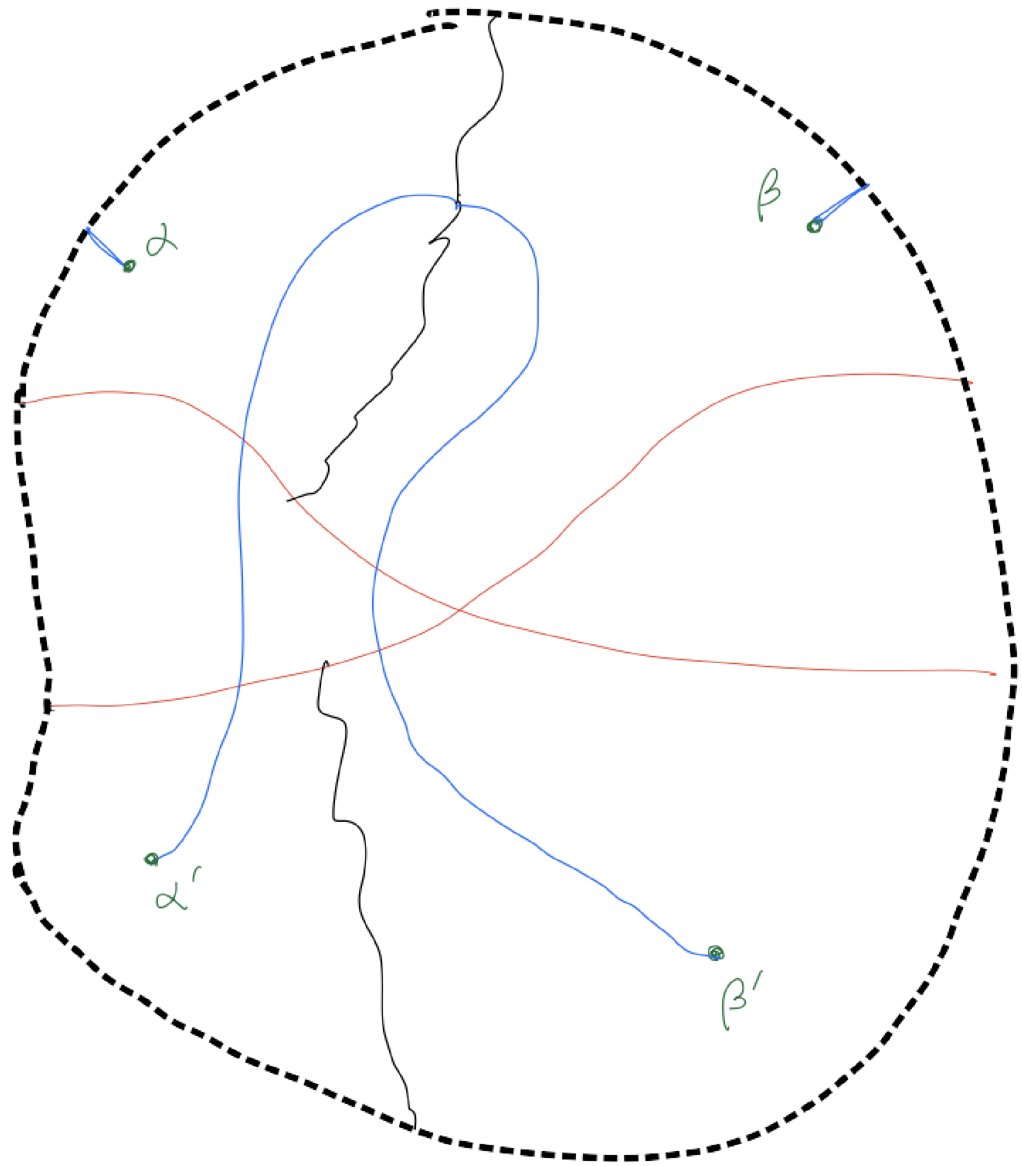


Figure 4.92: Your caption here

(Step4) Apply MOVE iiito the region inside the purple circle :

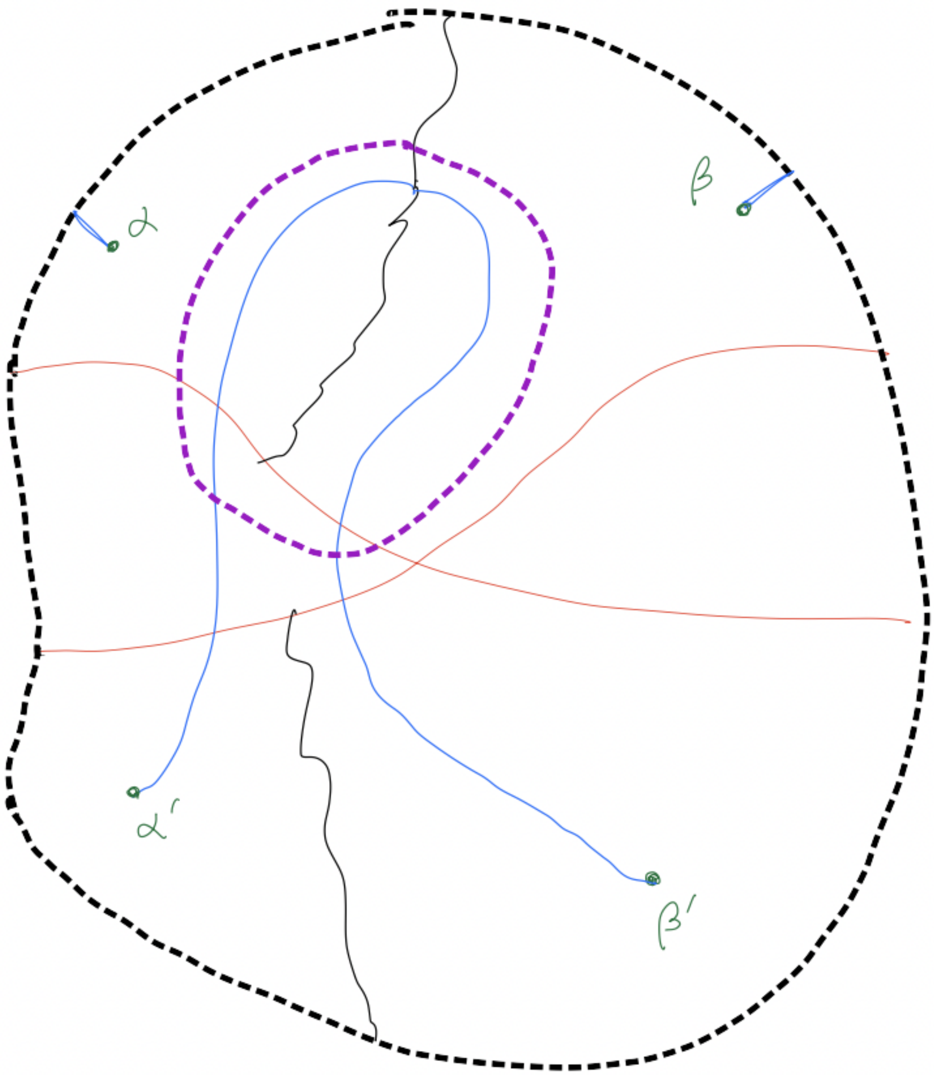


Figure 4.93: Your caption here

we get :

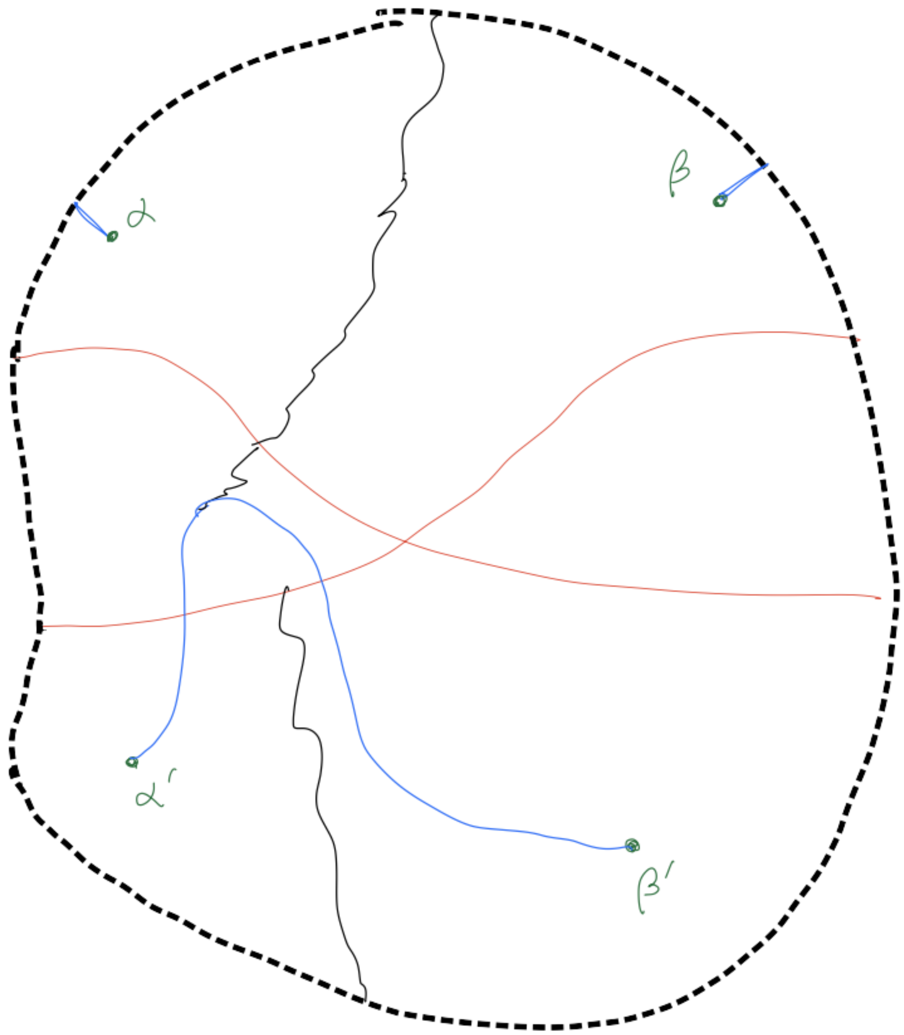


Figure 4.94: Your caption here

(Step5) Apply MOVE vto the region inside the purple circle :

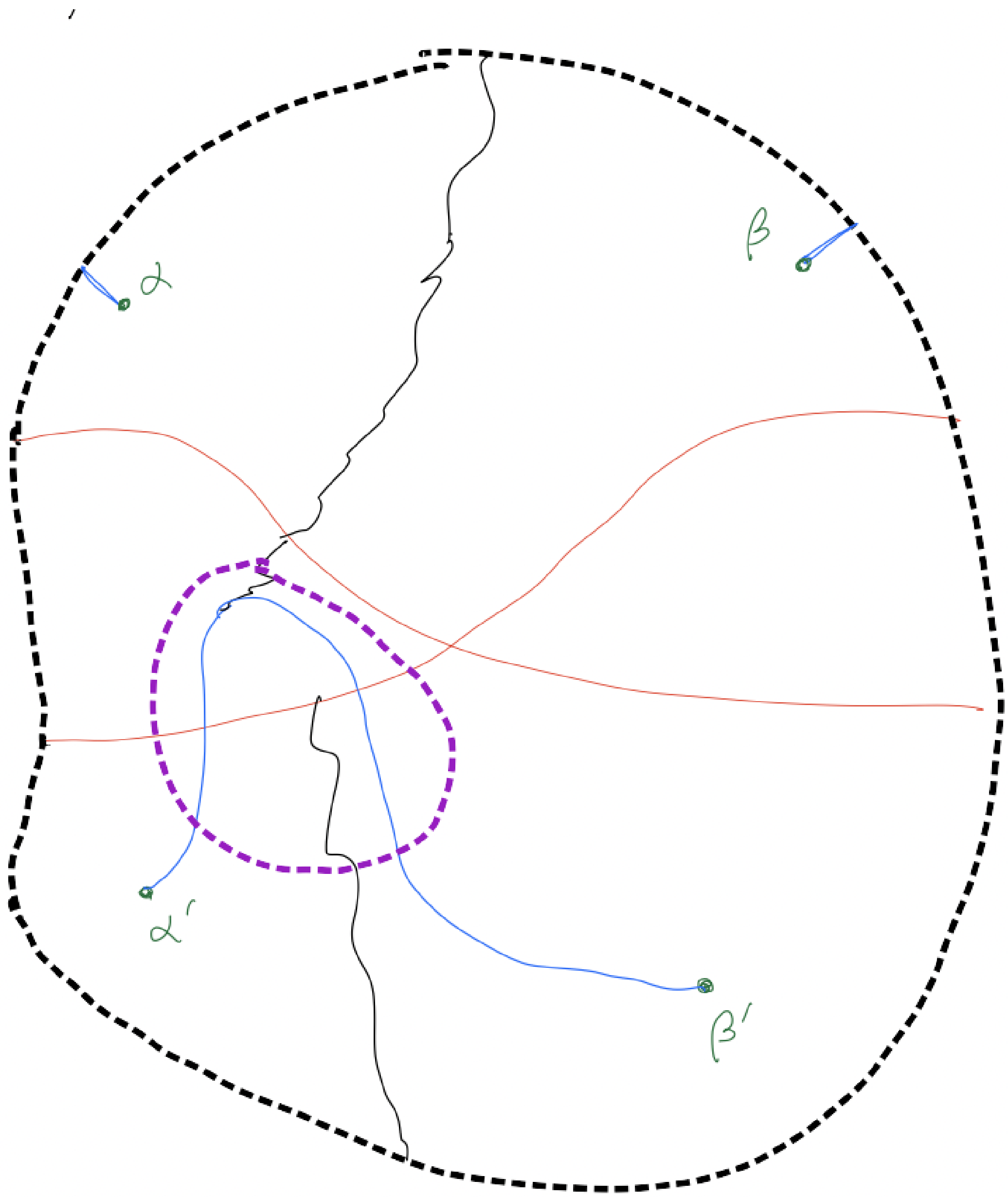


Figure 4.95: Your caption here

we get the final diagram :

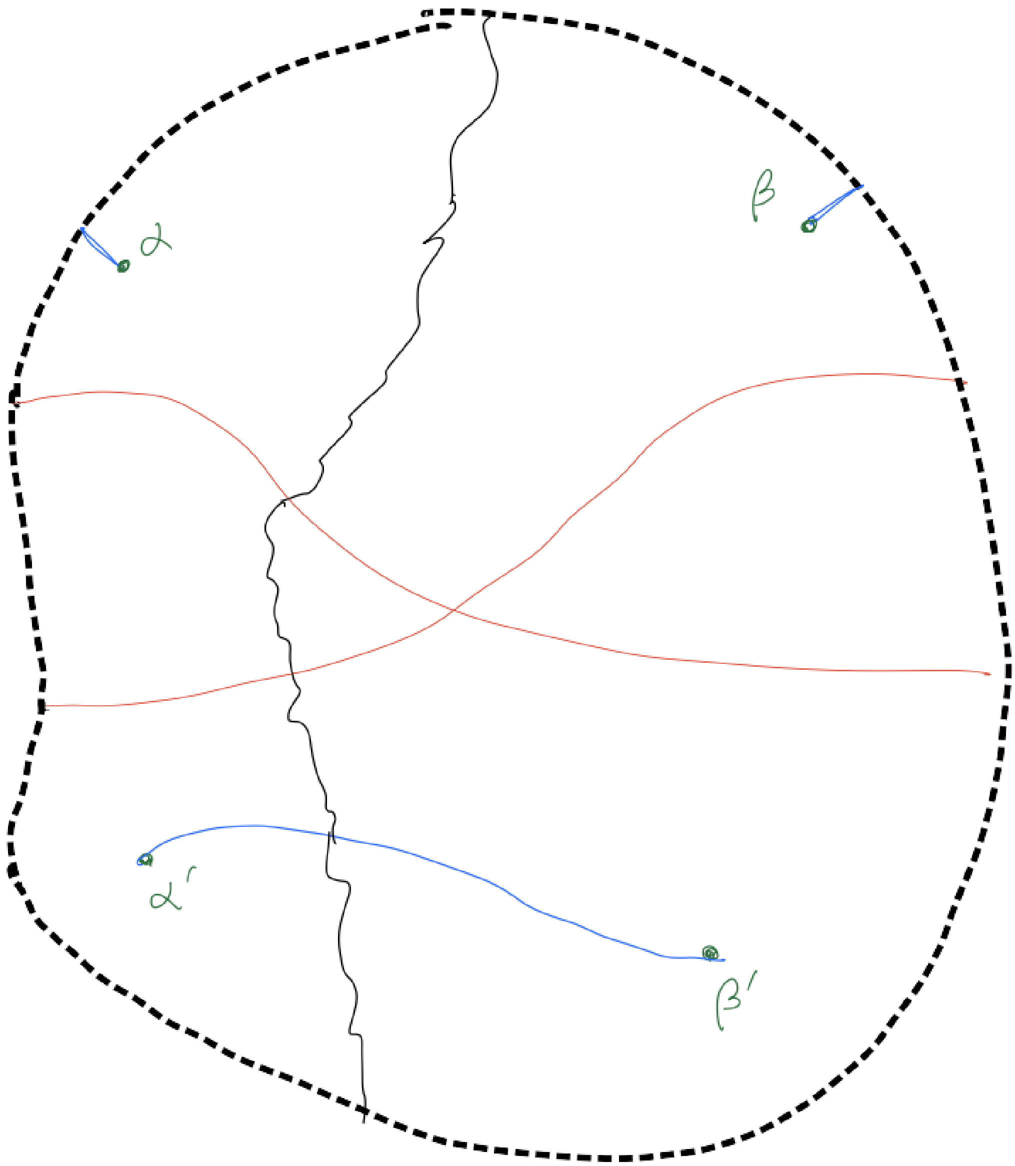


Figure 4.96: Your caption here

4.18 lemma10

Lemma 60.

Suppose we have a Riemann sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$ the refinement is as the following figure where two dimensional strata are labeled:

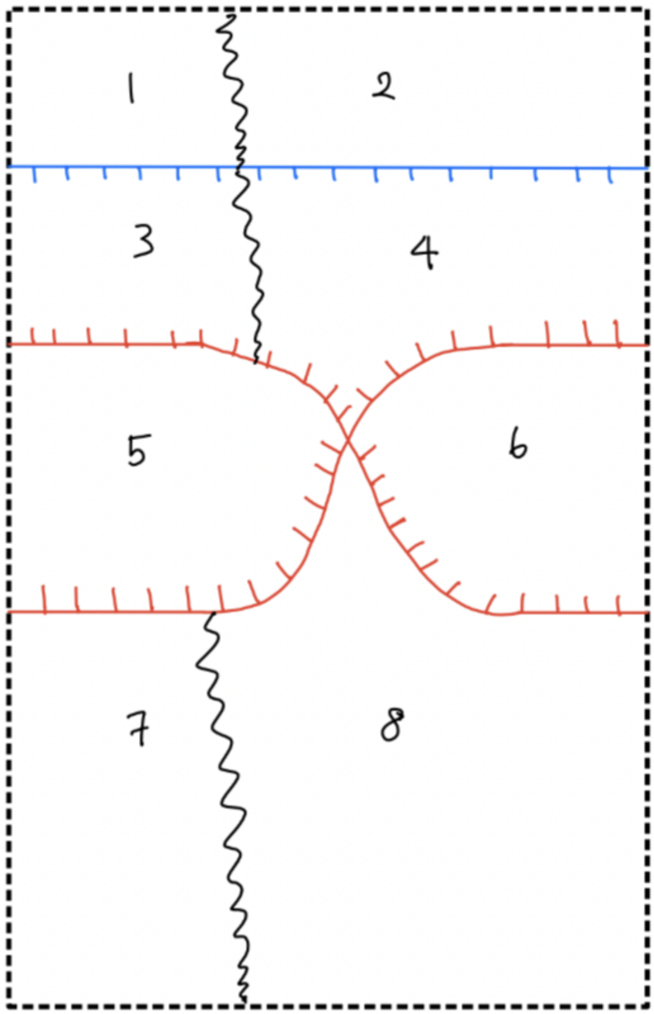


Figure 4.97: Your caption here

Stalks:

- 1 : $\mathbb{C}^{m+1}$
- 2 : $\mathbb{C}^{m+1}$
- 3 : $\mathbb{C}^{m}$
- 4 : $\mathbb{C}^{m}$
- 5 : $\mathbb{C}^{m+1}$

- 6 : $\mathbb{C}^{m+1}$

- 7 : $\mathbb{C}^{m+2}$

- 8 : $\mathbb{C}^{m+2}$

Generization maps:

- 1→2 : $diag(d_0, \cdots, d_m)$

- 3→4 : $diag(d_1, \cdots, d_m)$

- 4→5 : $\left(\begin{array}{ccc} d_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_m^{-1} \\ \hline 0 & \cdots & 0 \end{array}\right)$

- 5→8 : $\left(\begin{array}{ccc|c} d_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d_m & \alpha \\ \hline 0 & \cdots & 0 & \beta \\ 0 & \cdots & 0 & 0 \end{array}\right)$

- 7→8 : $\left(\begin{array}{ccc|cc} d_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_m & \alpha & 0 \\ \hline 0 & \cdots & 0 & \beta & 0 \\ 0 & \cdots & 0 & 0 & d_0' \end{array}\right)$

Now we will define isotopy starting from the above sheaf $\mathfrak{F}$ to the final sheaf $\mathfrak{F}'$.

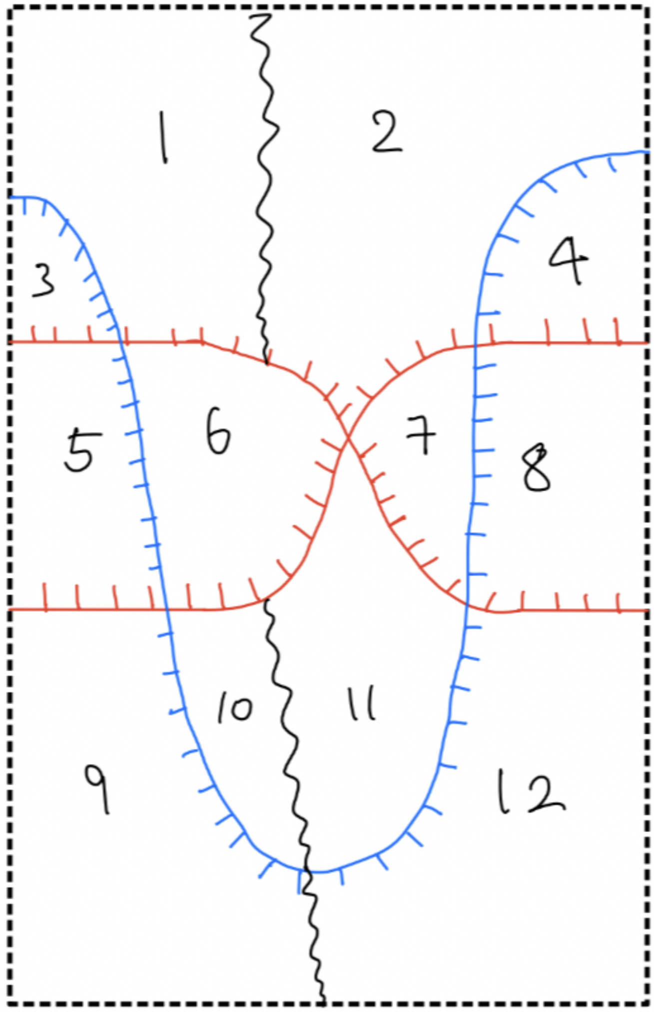


Figure 4.98: Your caption here

Stalks:

- 1 : $\mathbb{C}^{m+1}$
- 2 : $\mathbb{C}^{m+1}$
- 3 : $\mathbb{C}^{m}$
- 4 : $\mathbb{C}^{m}$
- 5 : $\mathbb{C}^{m+1}$
- 6 : $\mathbb{C}^{m+2}$
- 7 : $\mathbb{C}^{m+2}$

- 8 : $\mathbb{C}^{m+1}$

- 9 : $\mathbb{C}^{m+2}$

- 10 : $\mathbb{C}^{m+3}$

- 11 : $\mathbb{C}^{m+3}$

- 12 : $\mathbb{C}^{m+2}$

Generization maps:

- 1→2 : $diag(d_0, \cdots, d_m)$

- 2→6 : $\left(\begin{array}{ccc} d_0^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_m^{-1} \\ \hline 0 & \cdots & 0 \end{array}\right)$

- 6→10 : $\left(\begin{array}{ccc|c} d_0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d_m & \alpha \\ \hline 0 & \cdots & 0 & \beta \\ 0 & \cdots & 0 & 0 \end{array}\right)$

- 10→11 : $\left(\begin{array}{ccc|cc} d_0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_m & \alpha & 0 \\ \hline 0 & \cdots & 0 & \beta & 0 \\ 0 & \cdots & 0 & 0 & d_0' \end{array}\right)$

$$- 10\to 11 : \left(\begin{array}{ccc|cc} d_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_m & \alpha & 0 \\ \hline 0 & \cdots & 0 & \beta & 0 \\ 0 & \cdots & 0 & 0 & d_0' \end{array}\right)$$

We define $isotopy_9$ as follows:

(step1)Apply $isotopy_1$ inside the disks surrounded by purple dotted lines

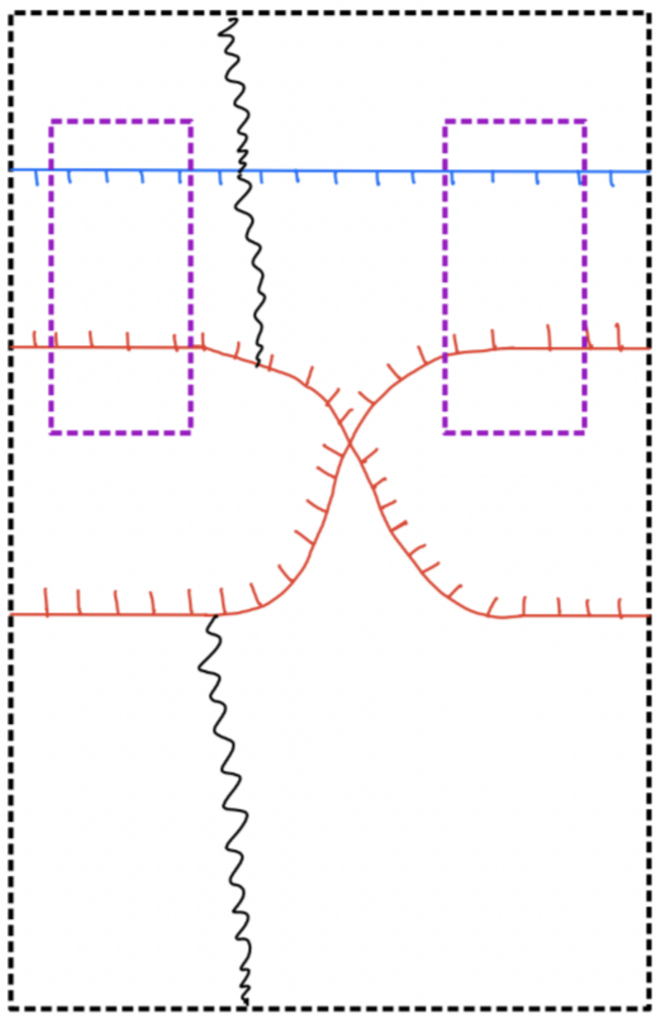


Figure 4.99: Your caption here

We get the following diagram

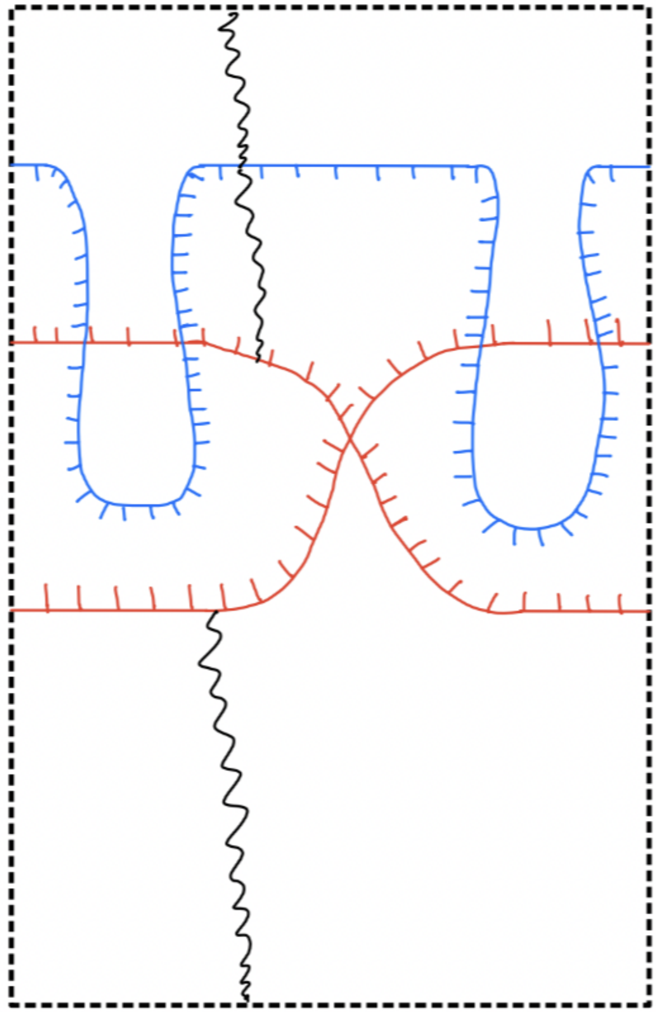


Figure 4.100: Your caption here

(step2) Apply $isotopy_1$ on the disk surrounded by purple dotted lines

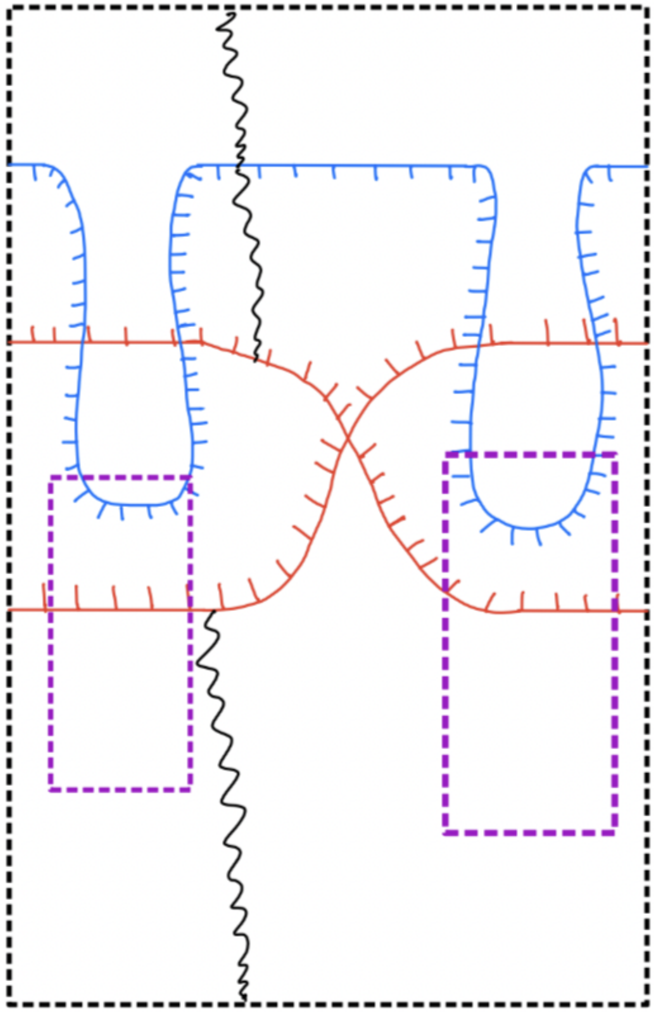


Figure 4.101: Your caption here

We get the following diagram

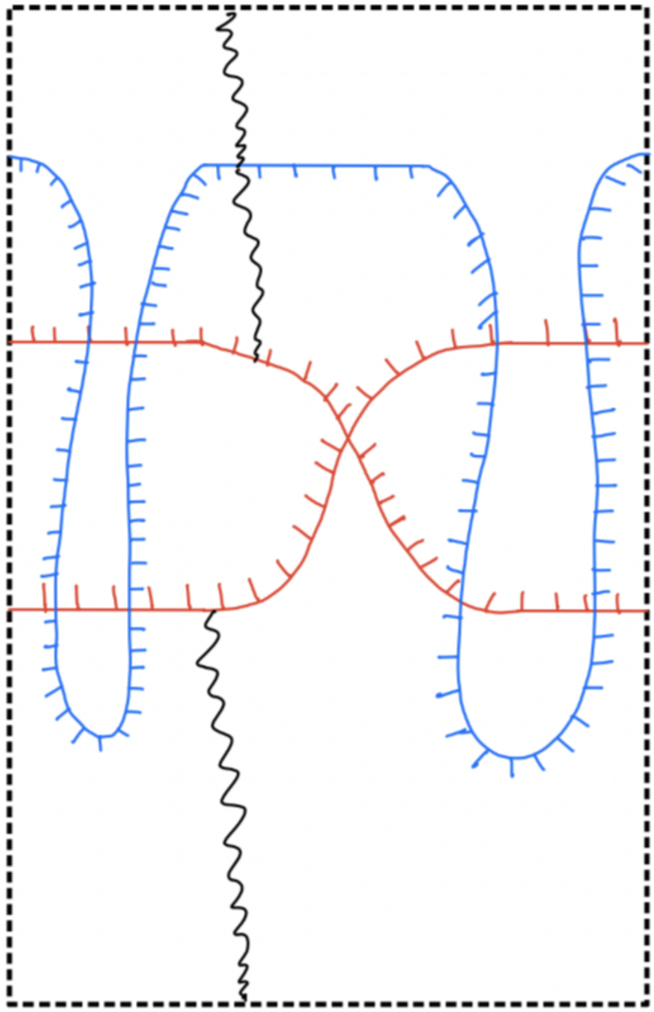


Figure 4.102: Your caption here

(step3) Apply $isotopy_4$ on the disk surrounded by purple dotted line

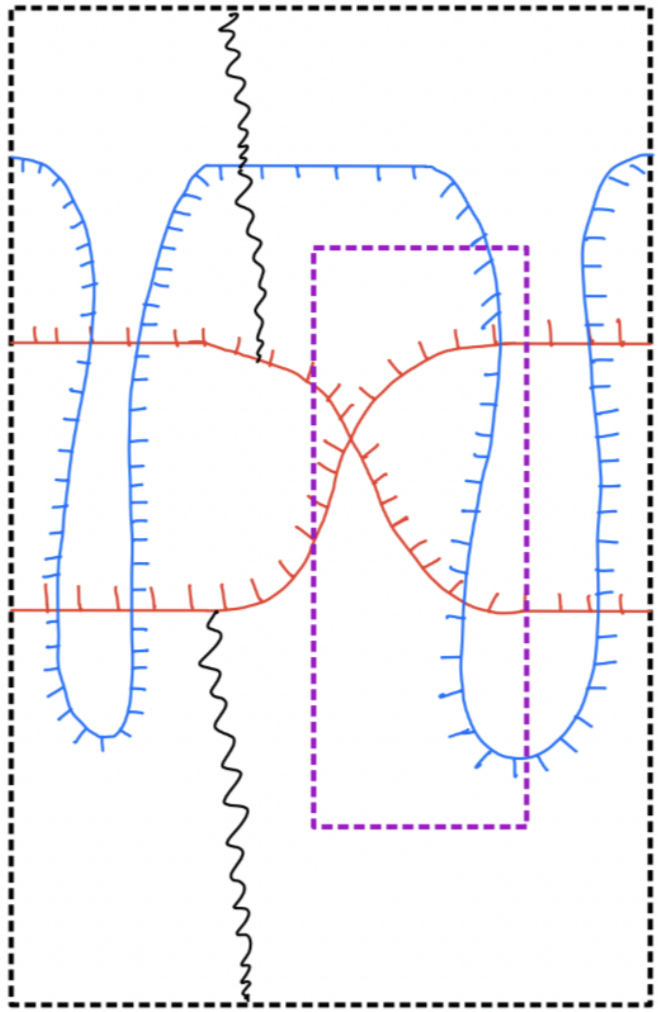


Figure 4.103: Your caption here

We get the following diagram:

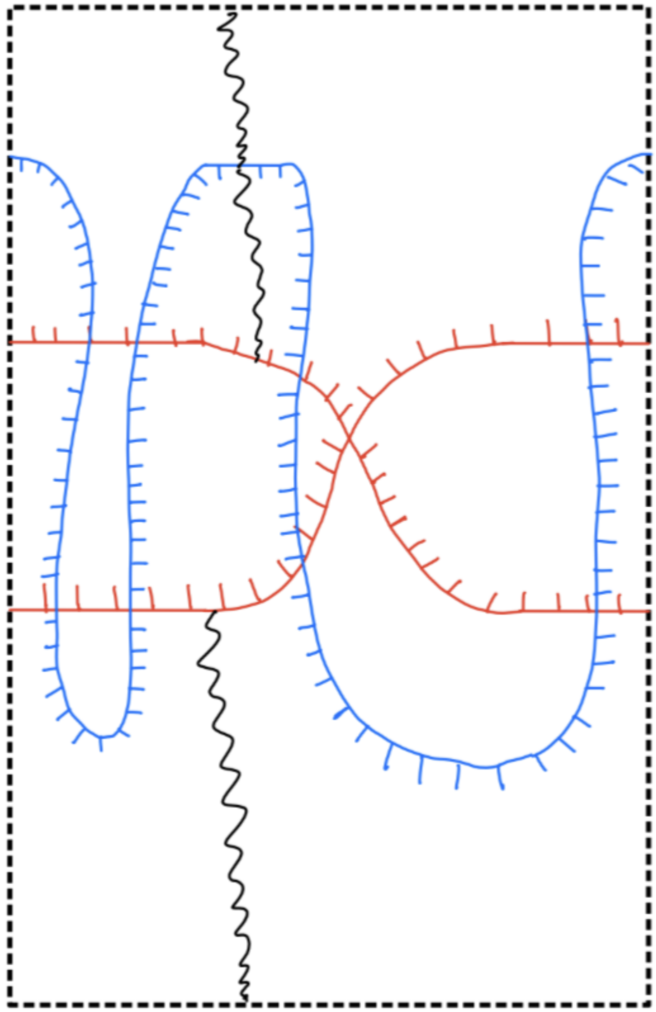


Figure 4.104: Your caption here

(step4) Apply $isotopy_3$ on the disk surrounded by purple dotted line

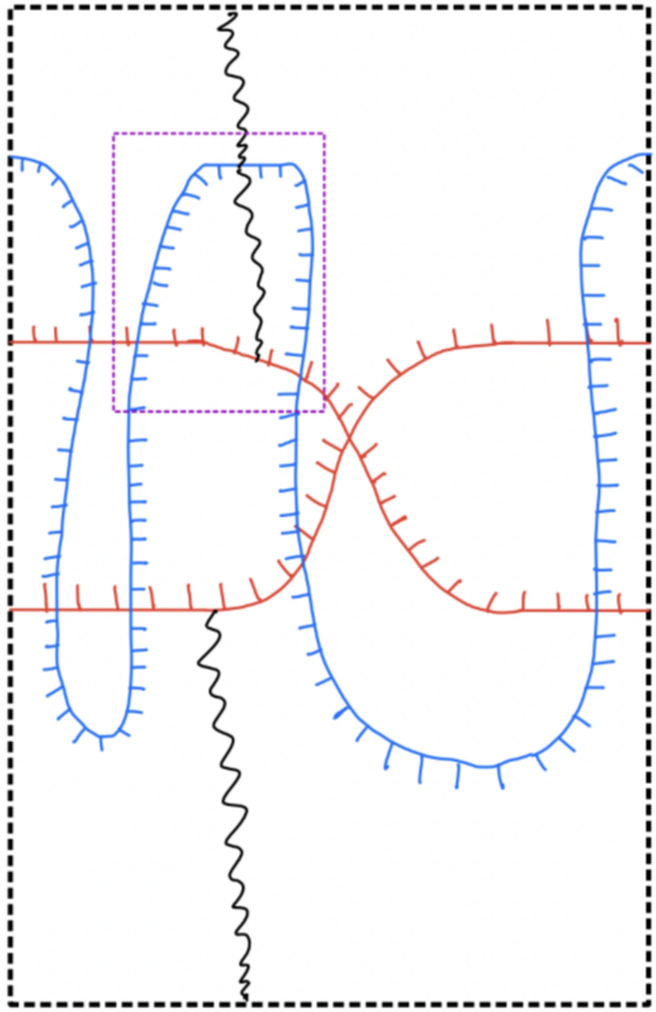


Figure 4.105: Your caption here

We get the following diagram

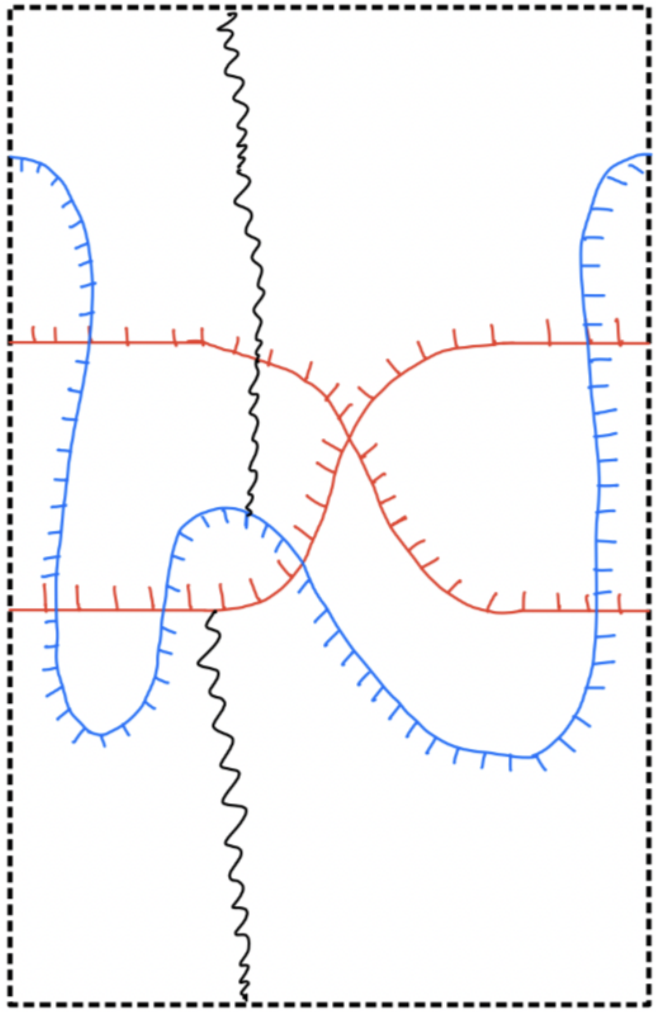


Figure 4.106: Your caption here

(step5) Apply $isotopy_3$ on the disk surrounded by purple dotted line

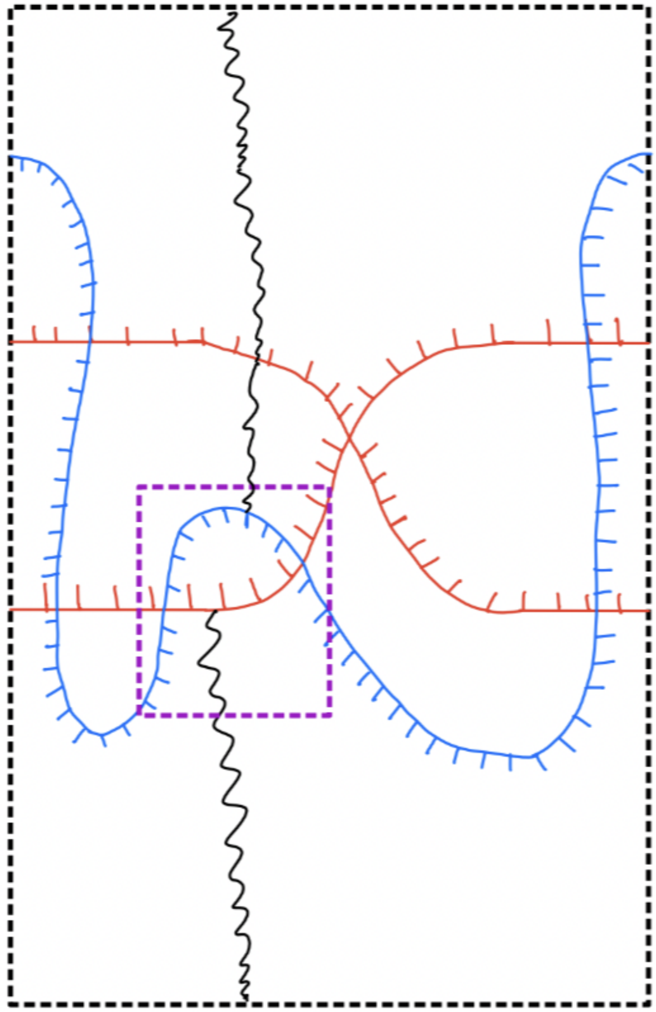


Figure 4.107: Your caption here

We get the following diagram:

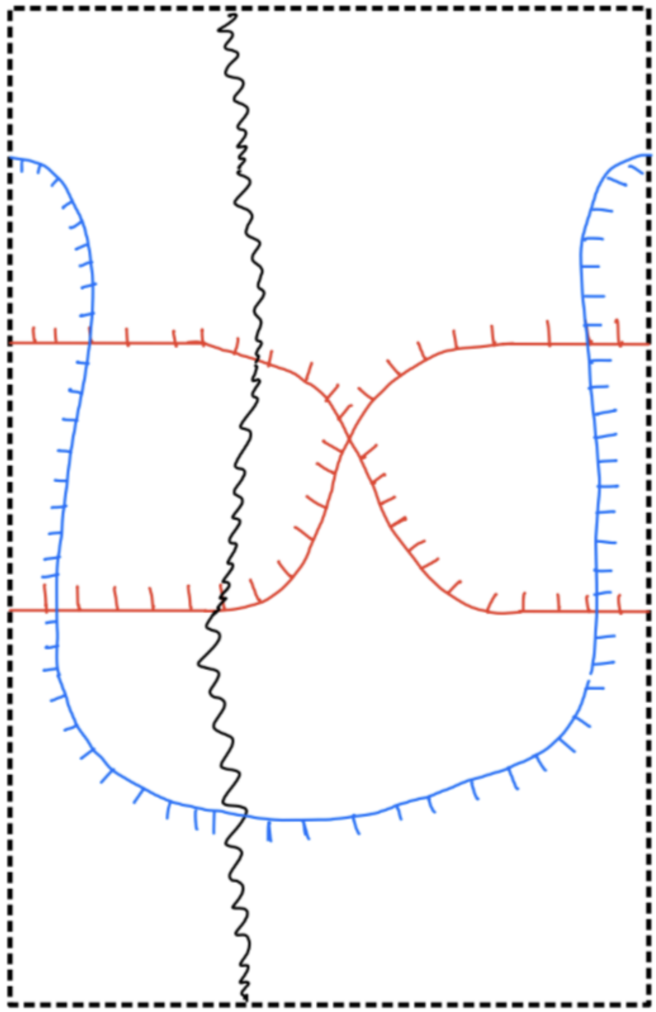


Figure 4.108: Your caption here

(step6) We can change the basis of the stalk at the region marked with purple start so that the generization map corresponding to the squiggly line next to the purple star is the identity map

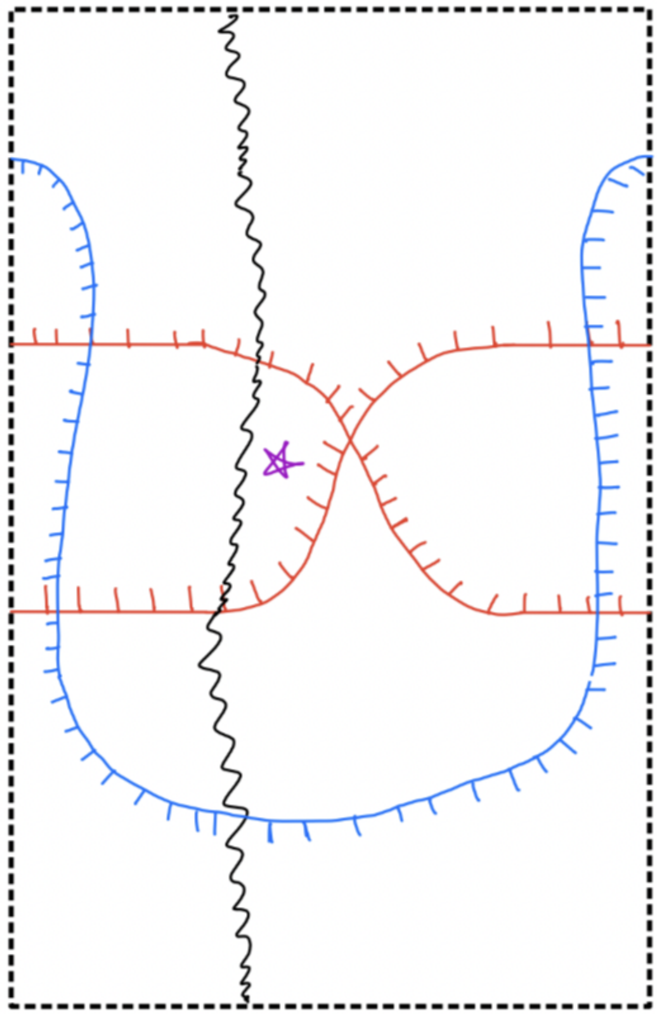


Figure 4.109: Your caption here

Now the sheaf could be thought of as a sheaf singular supported on the below diagram

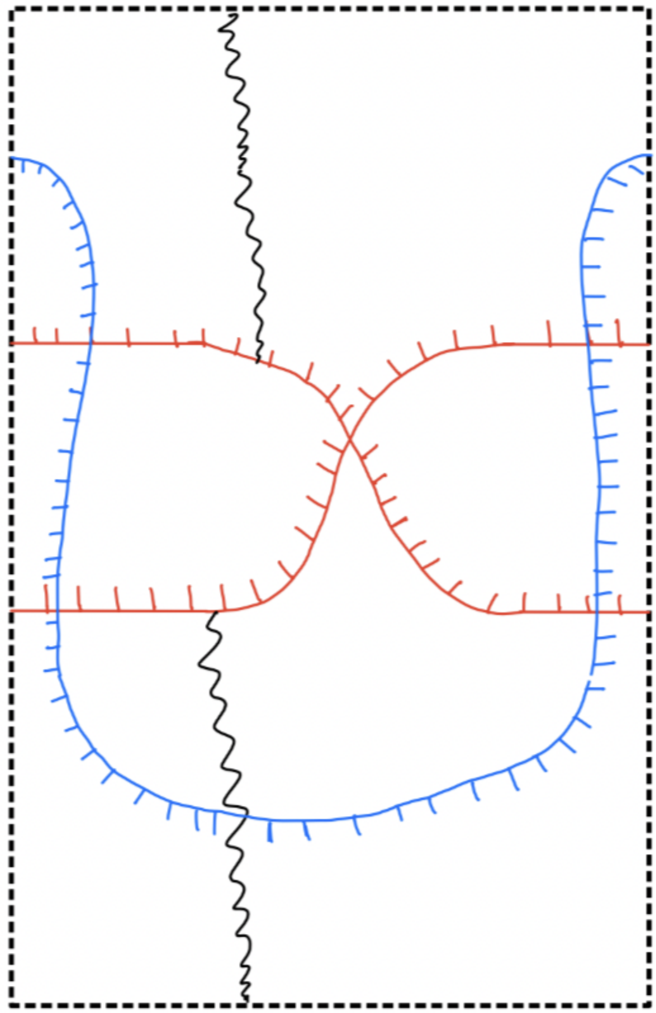


Figure 4.110: Your caption here

and the sheaf is $\mathfrak{F}'$.

(proof) By Lemma1, after (step1) we get the following sheaf

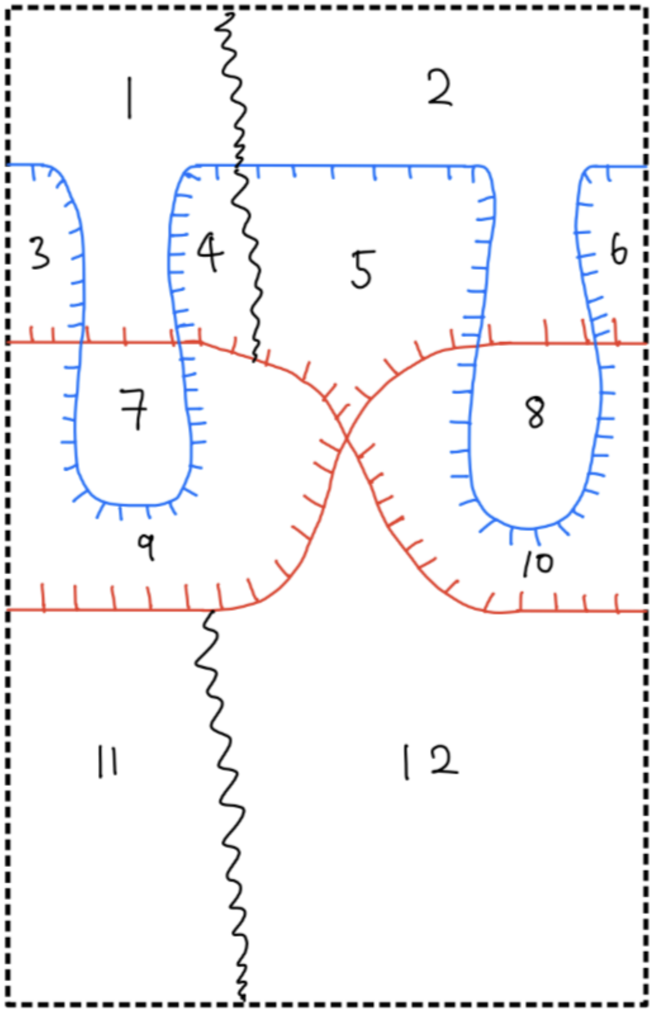


Figure 4.111: Your caption here

Stalks :

4.19 definition11

Definition 61.

Suppose we have a braid diagram as follows :

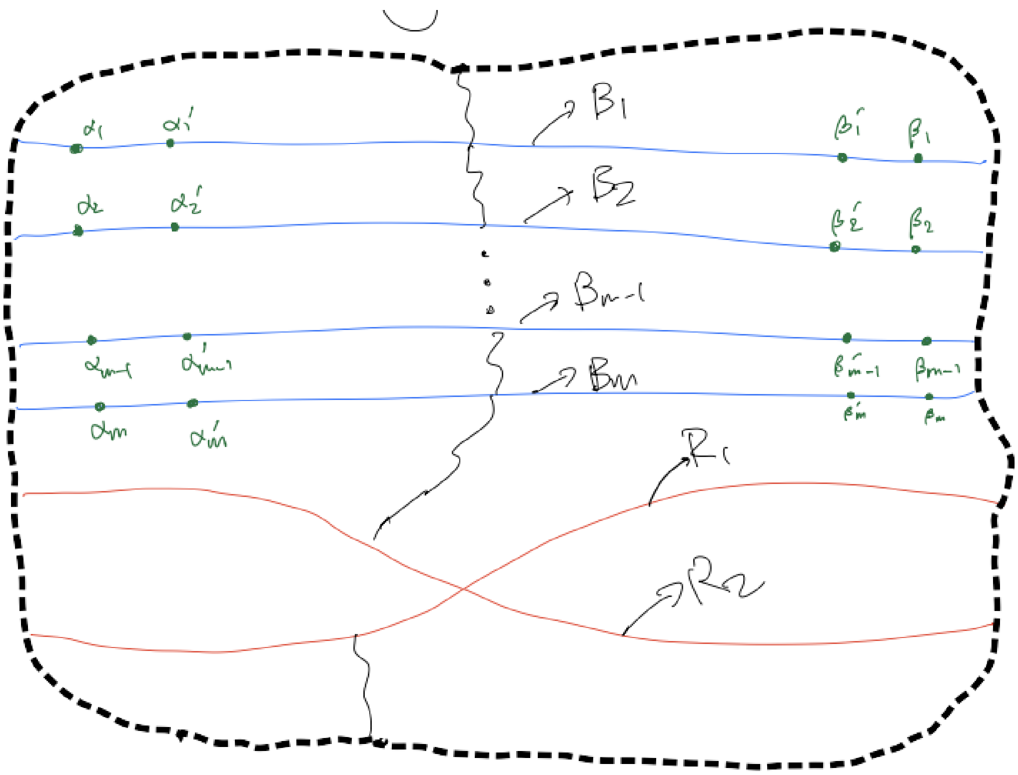


Figure 4.112: Your caption here

we define MOVE xiso that the final diagram looks as follows :

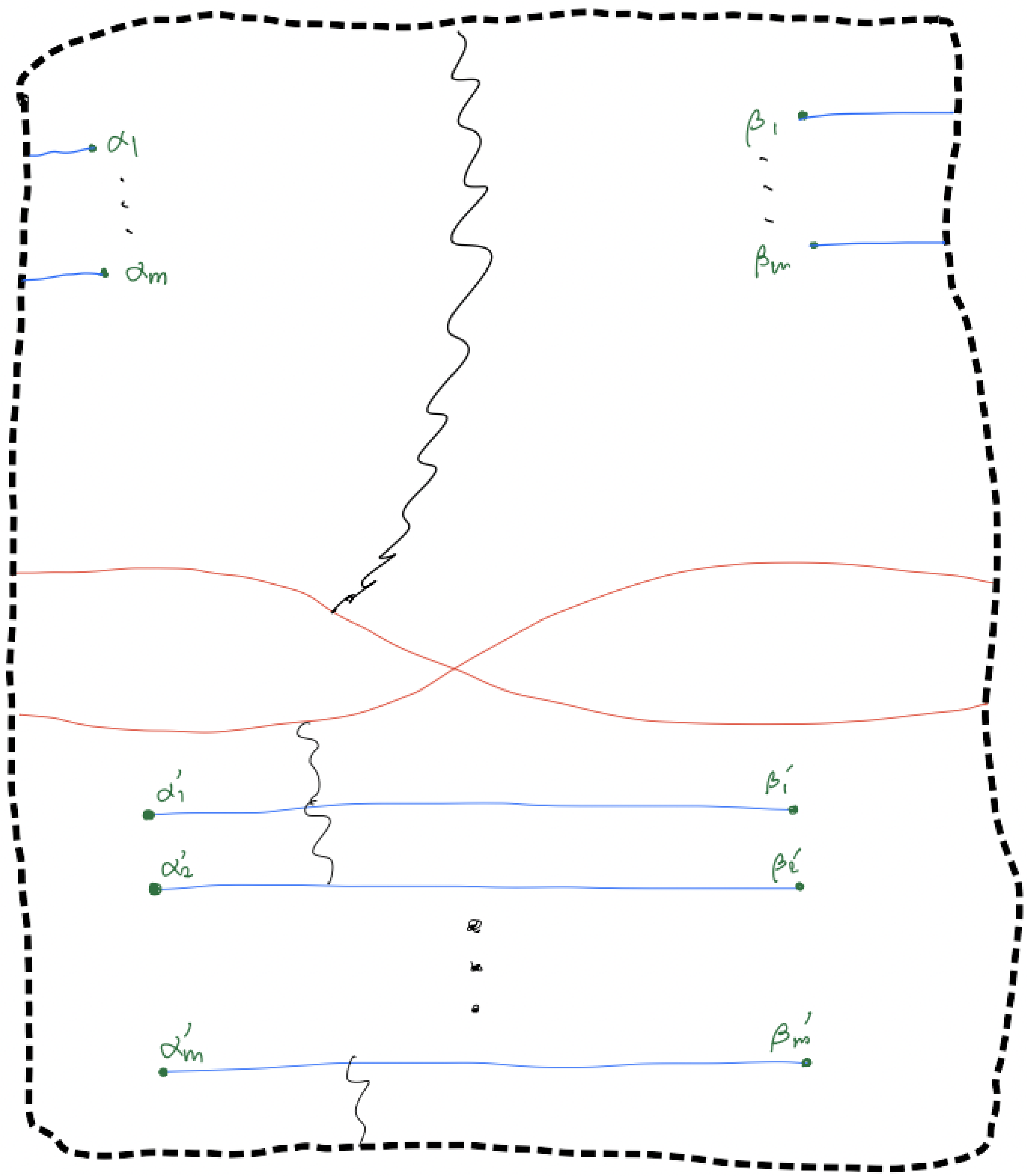


Figure 4.113: Your caption here

We define MOVE xiinductively as follows : If $m = 1$, then MOVE xiis just MOVE x. If $m > 1$, then apply MOVE xito $B_2 - B_m$ and R_1, R_2(this is well-defined by induction hypothesis), we get :

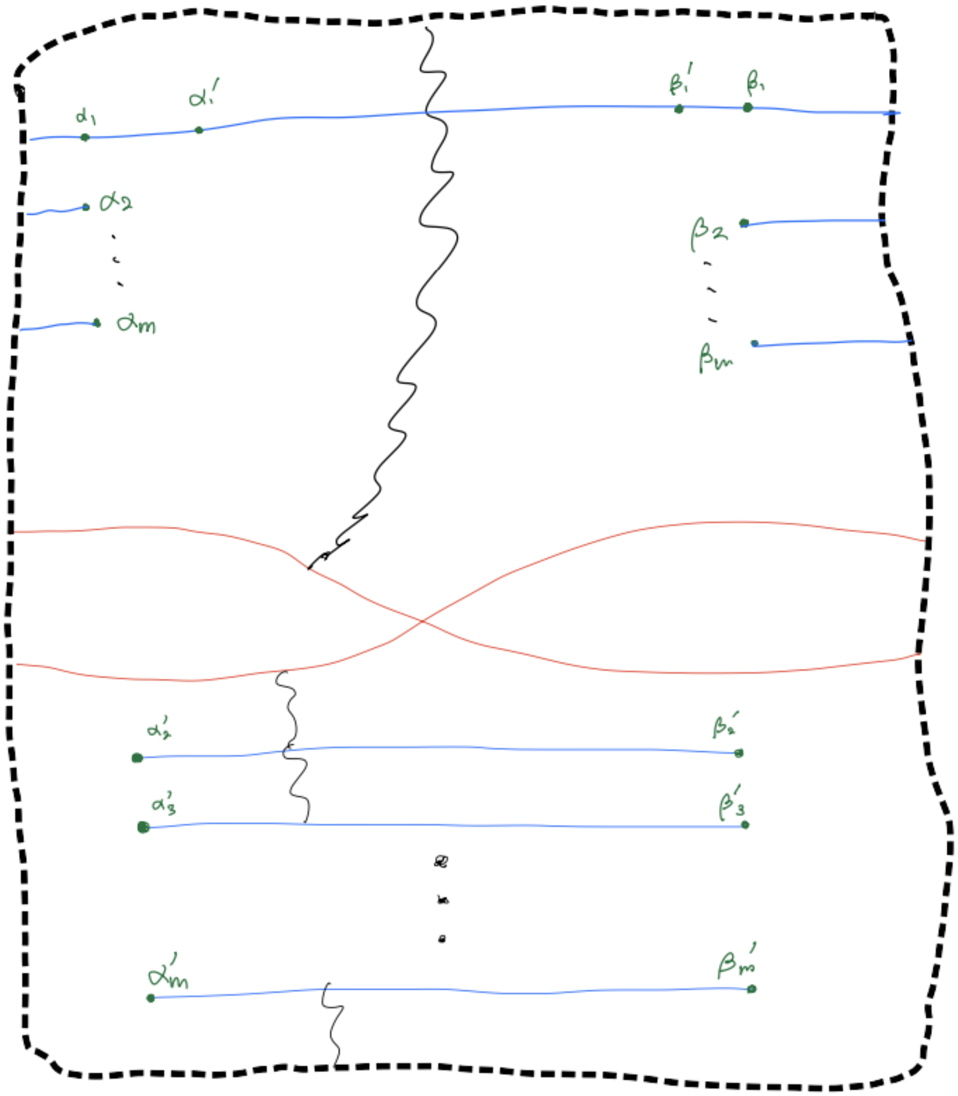


Figure 4.114: Your caption here

Then apply MOVE xto B_1 and R_1, R_2, we get the final diagram :

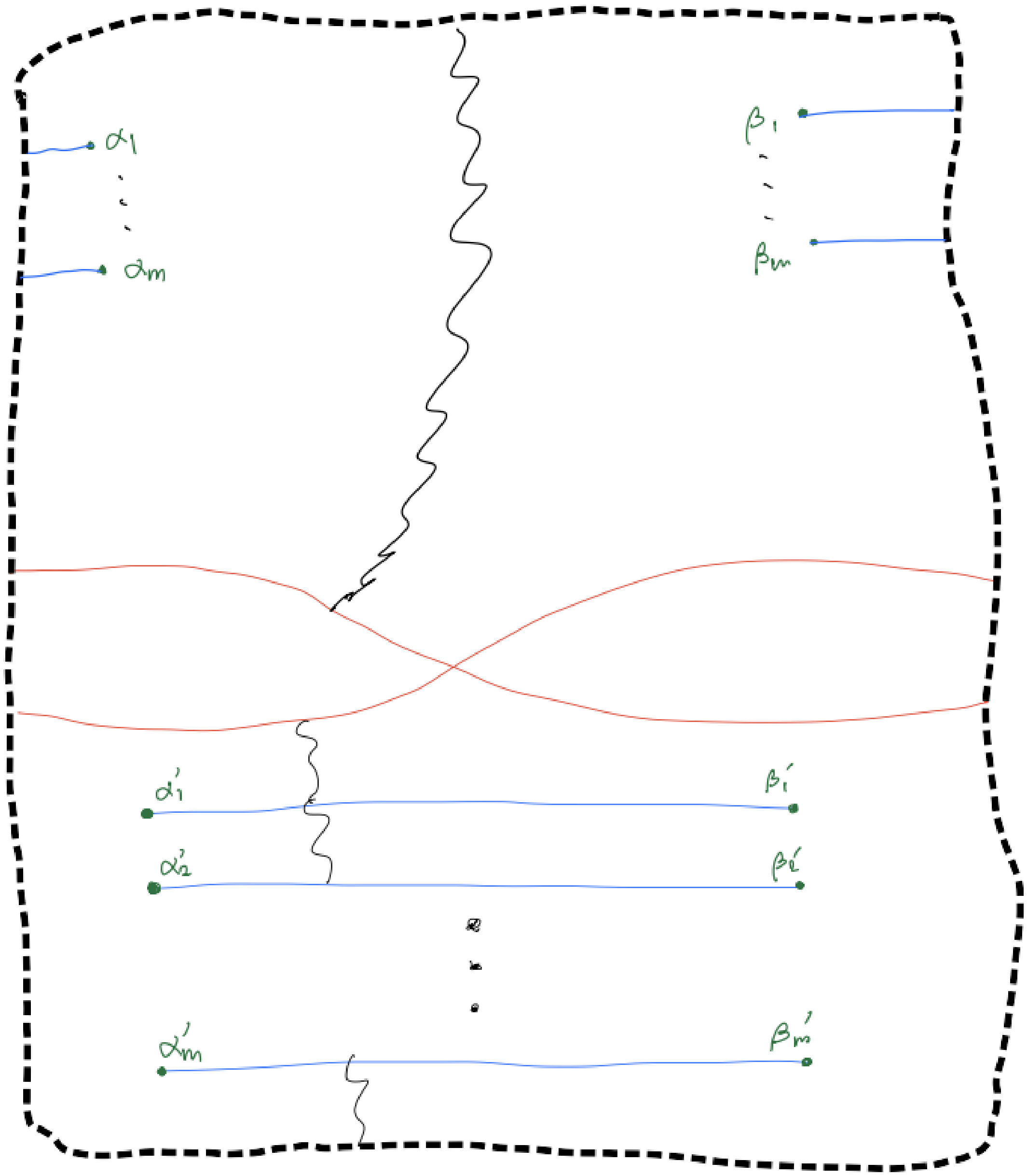


Figure 4.115: Your caption here

4.20 lemma11

Lemma 62.

Suppose we have a Riemann Sphere C and a diagram (C, ι, ξ) and a regular cell complex refinement $\overline{(C, \iota, \xi)}$ and a sheaf $\mathfrak{F}$ singular supported on it such that when restricted to a small disk $D \subset C$ the refinement is as the following figure where two dimensional strata are labeled by tuples and numbers:

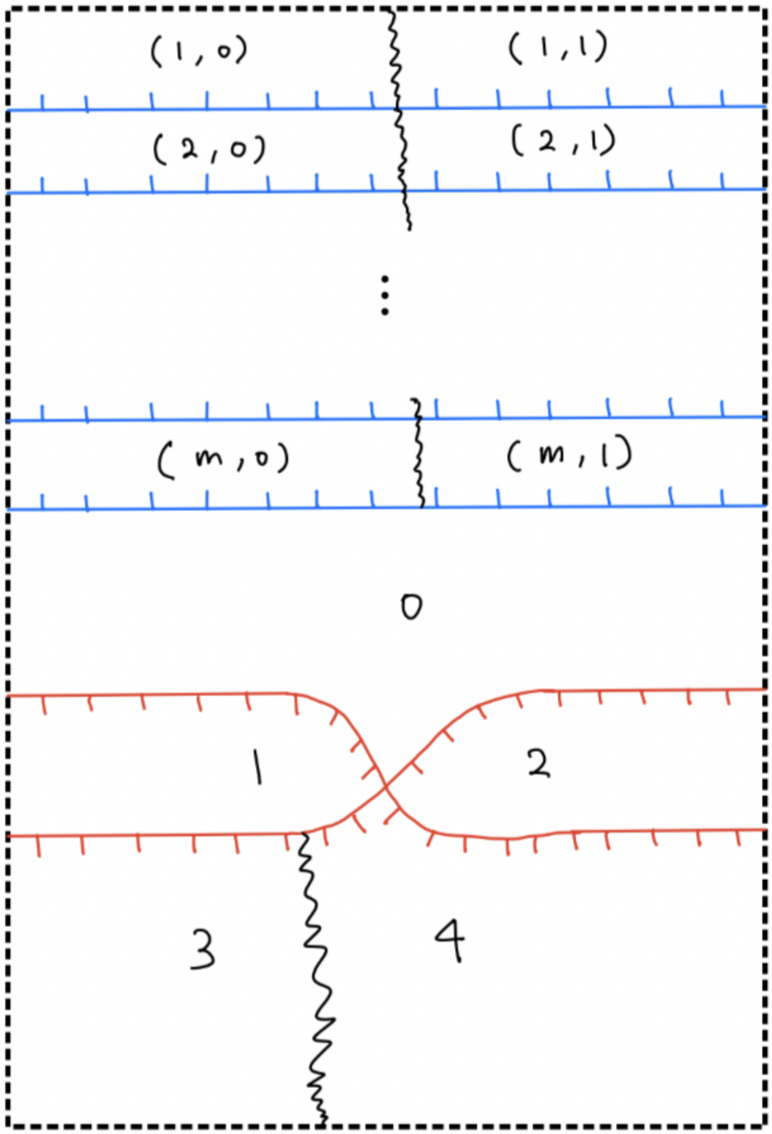


Figure 4.116: Your caption here

Stalks:

- $(i,0),(i,1)$: $\gtrdot + \nVdash - \beth$

- 0 : 0

- 1,2 : $\mathbb{C}$

- 3,4 : $\mathbb{C}^2$

Generization maps :

- $(i,0) \to (i,1)$: $\mathbb{C}^{m-i+1} \to \mathbb{C}^{m-i+1}$, $D_{i,m}$

- 1→4 : $\mathbb{C} \to \mathbb{C}^2$ where $e_1 \mapsto (a,b)^T$

- 3→4 : $\mathbb{C}^2 \to \mathbb{C}^2$ where $e_1 \mapsto (a,b)^T$ and $e_2 \mapsto (0,d)^T$

- all the other maps crossing the red strands are ι_f

- all the other maps crossing the blue strands are ι_l

- rest of the maps are zero maps

Now we will define isotopy starting from the above sheaf $\mathfrak{F}$ inductively on the number of blue strands so that the final sheaf $\mathfrak{F}'$ is

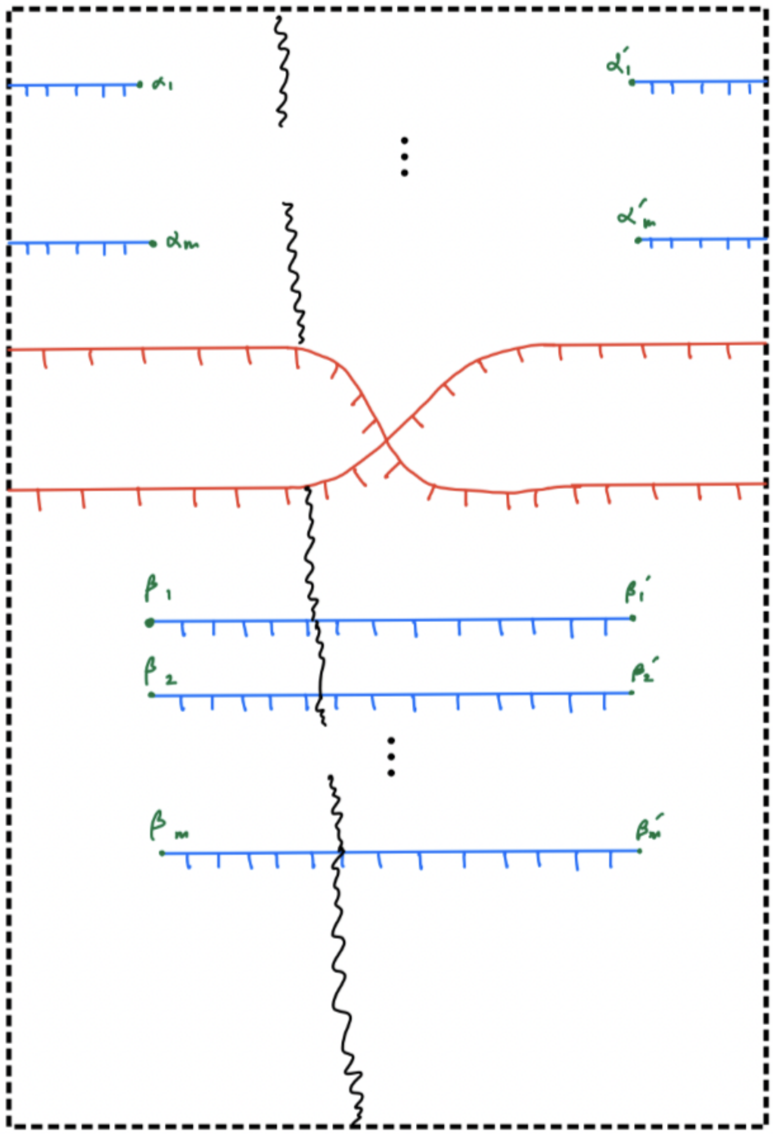


Figure 4.117: Your caption here

In the above diagram I have intentionally omitted lines connecting

- α_i and β_i for $i = 1, \cdots, m$

- α'_i and β'_i for $i = 1, \cdots, m$

so as not to make diagram too messy. These omitted lines are mutually disjoint and crosses each red strand at most once.

Let the crossing of $\overline{\alpha_j\beta_j}$ with the upper(lower resp.) red strand be called $c_{0,j}(c_{1,j}$ resp.) and $\overline{\alpha'_j\beta'_j}$ be called $c_{;0,j}(c'_{1,j}$ resp.)

Let's denote the north, east, west, south of the crossing $c_{i,j}(c'_{i,j}$ resp.) as $N_{i,j}, E_{i,j}, W_{i,j}, S_{i,j}(N'_{i,j}, E'_{i,j},$ resp.)

The final sheaf will be describe as follows:

Stalks :

- $E_{i,j}$: $\mathbb{C}^{m+1+i-j}$
- $S_{i,j}$: $\mathbb{C}^{m+i-j}$
- $W_{i,j}$: $\mathbb{C}^{m+1+i-j}$
- $N_{i,j}$: $\mathbb{C}^{m+2+i-j}$

- $E'_{i,j}$: $\mathbb{C}^{m+1+i-j}$
- $S'_{i,j}$: $\mathbb{C}^{m+i-j}$
- $W'_{i,j}$: $\mathbb{C}^{m+1+i-j}$
- $N'_{i,j}$: $\mathbb{C}^{m+2+i-j}$

Generization maps:

- maps crossing red strands are ι_f and maps crossing blue strands are ι_l unless mentioned otherwise
- So I will describe the maps crossing squiggly lines, non ι_f, ι_l maps only.

- $E_{1,j} \to W'_{1,j}$: $\left(\begin{array}{ccc|cc} d_{j+1} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_m & 0 & 0 \\ \hline 0 & \cdots & 0 & a & 0 \\ 0 & \cdots & 0 & b & d \end{array}\right)$

- $W_{0,1} \to E'_{0,1}$: $D_{1,m}$

- $N_{1,1} \to N'_{1,1}$: $\left(\begin{array}{ccc|cc} d_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & d_m & 0 & 0 \\ \hline 0 & \cdots & 0 & a & 0 \\ 0 & \cdots & 0 & b & d \end{array}\right)$

- $W_{1,1} \to W'_{1,1}$: $\left(\begin{array}{ccc|c} d_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d_m & 0 \\ \hline 0 & \cdots & 0 & a \\ 0 & \cdots & 0 & b \end{array}\right)$

- $E'_{0,1} \to E_{0,1}$: $\left(\begin{array}{ccc} d_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_m^{-1} \\ \hline 0 & \cdots & 0 \end{array}\right)$

If $m = 1$, $isotopy_{11}$ is just $isotopy_{10}$.

Suppose we have defined $isotopy_{11}$ upto the number of blue strands less than m. Let's define $isotopy_{11}$ for the number of blue strands equals m:

(step1) Apply $isotopy_1 1$ for the number of blue strands equals $m-1$ on the disk surrounded by purple dotted line which is well-defined by induction hypothesis

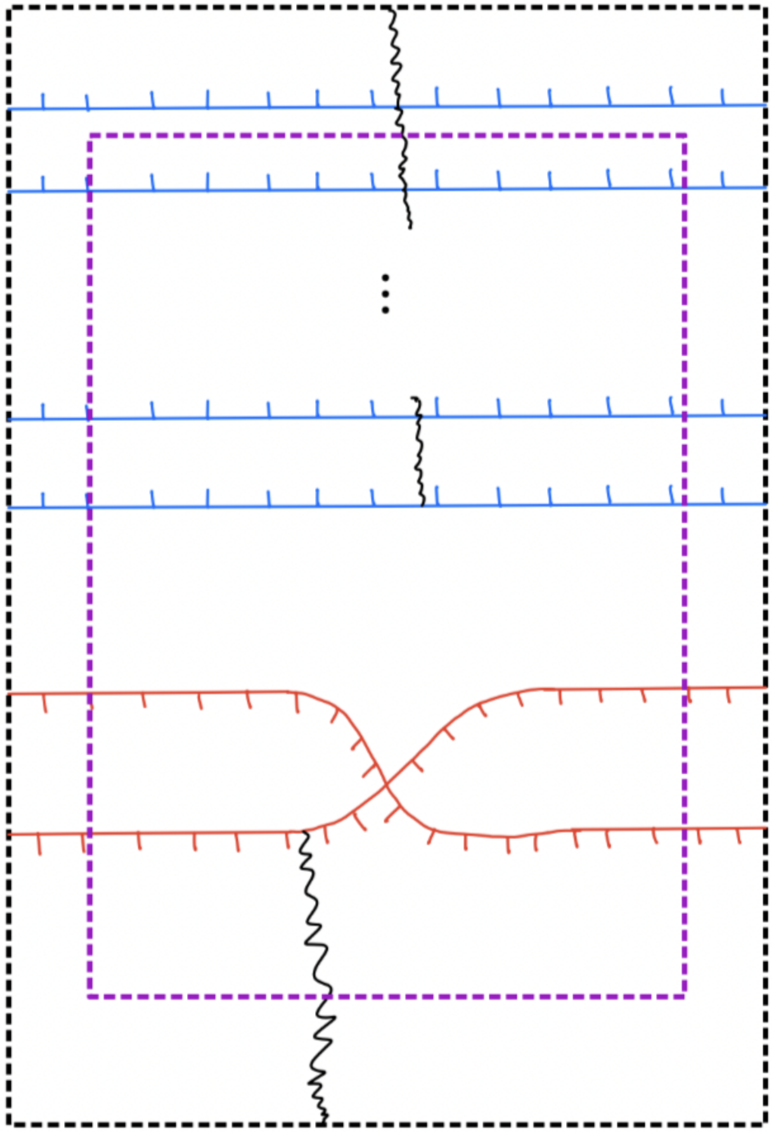


Figure 4.118: Your caption here

We get the following diagram:

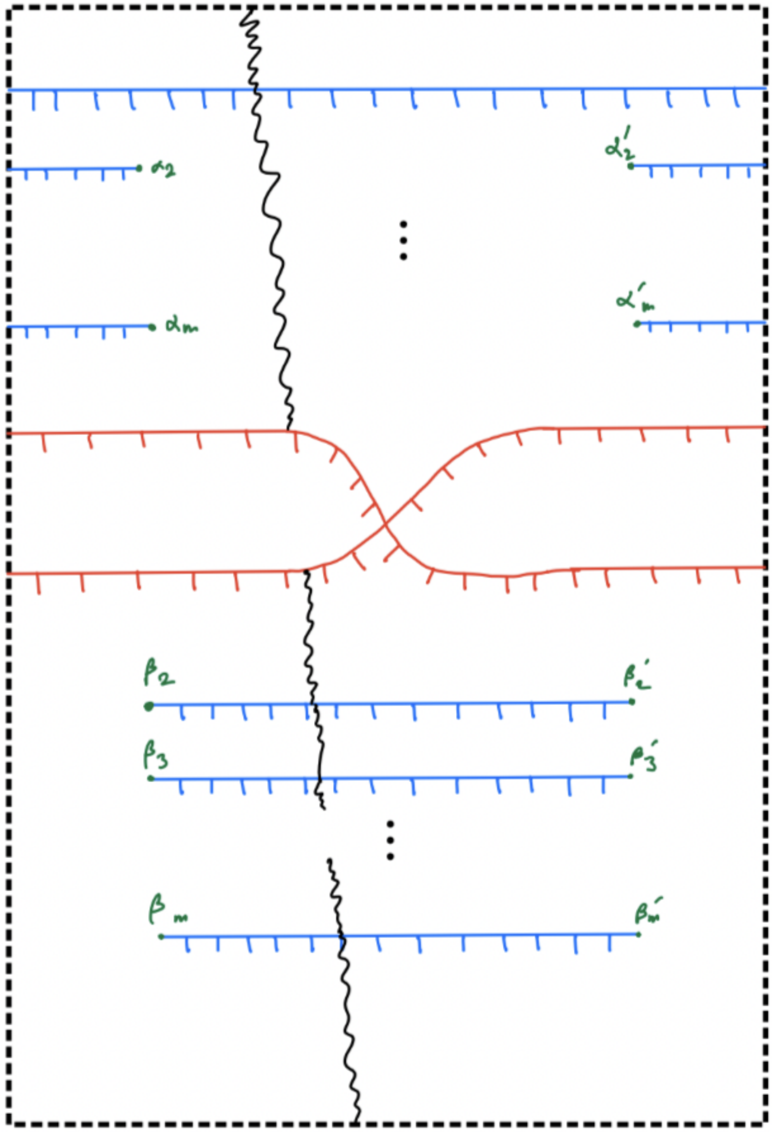


Figure 4.119: Your caption here

(step2) Apply $isotopy_{10}$ on the disk surrounded by purple dotted line:

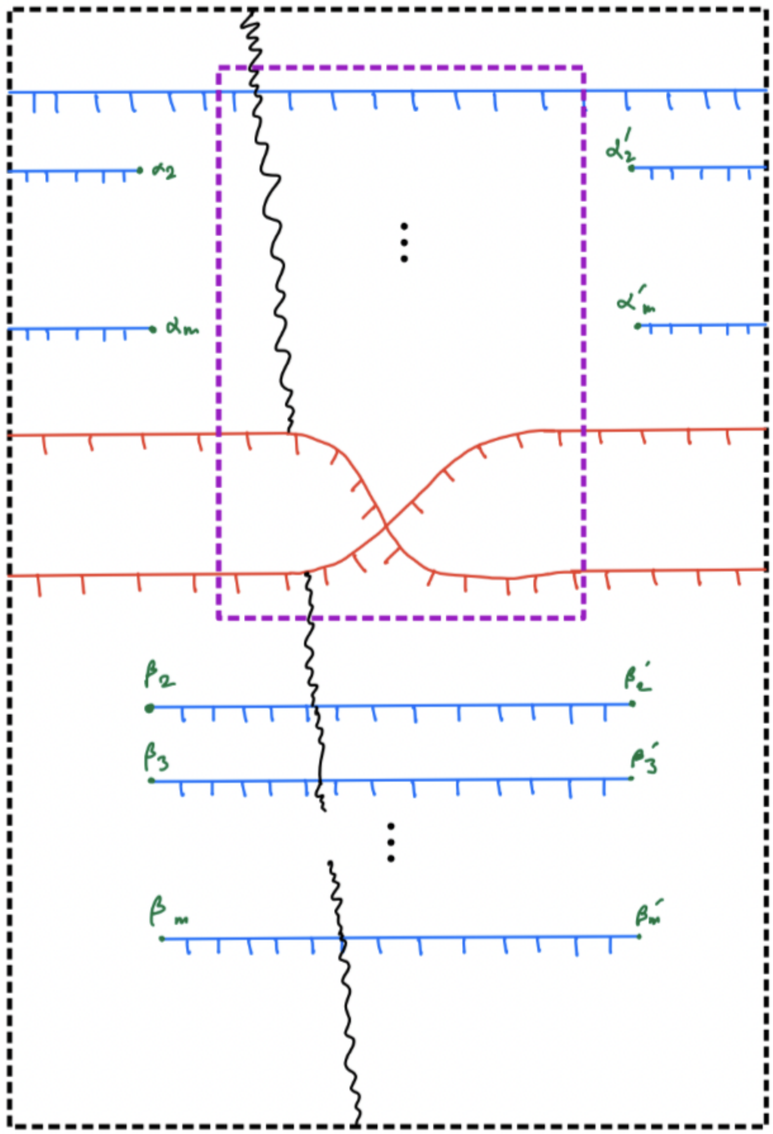


Figure 4.120: Your caption here

We get the final diagram:

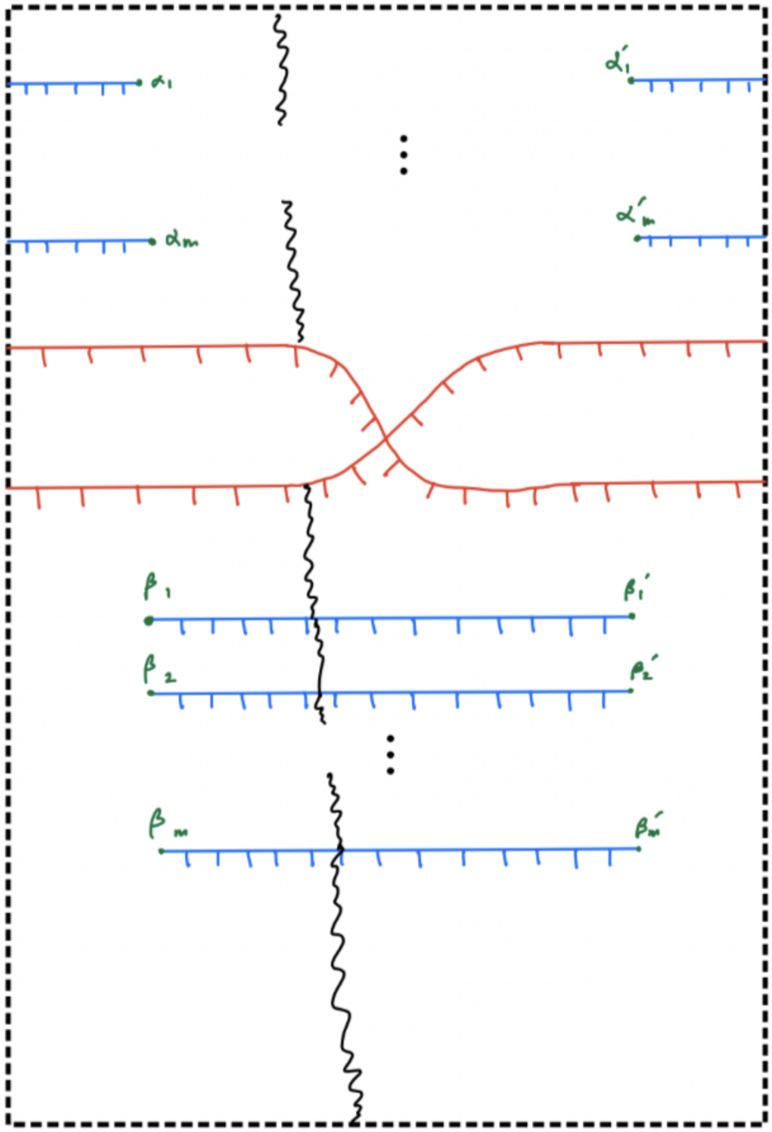


Figure 4.121: Your caption here

with $\mathfrak{F}'$ on it.

(proof)

4.21 definition12(for the main theorem)

Definition 63.

Suppose we have a braid diagram as follows :

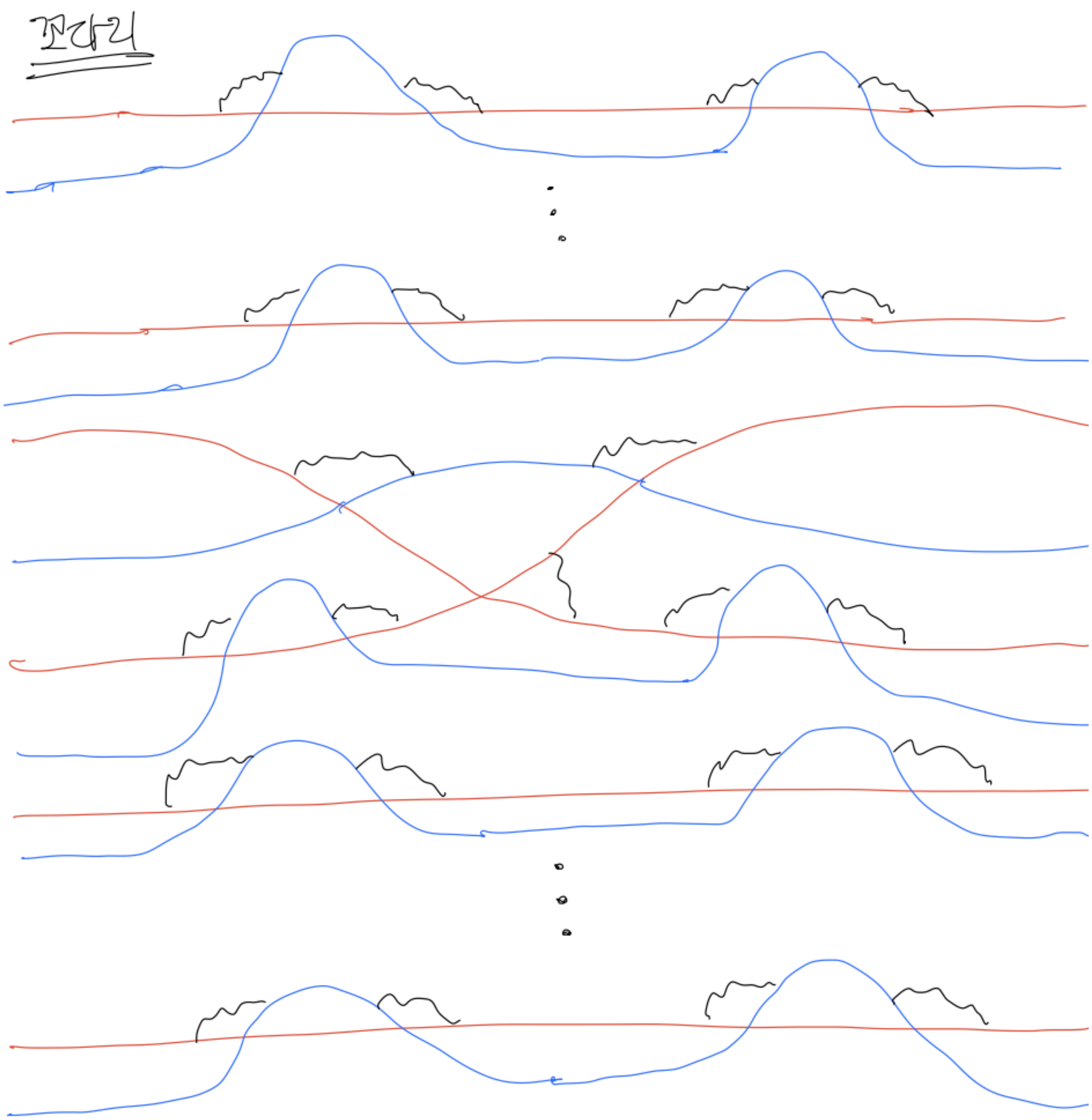


Figure 4.122: Your caption here

We define MOVE xiiso that the final diagram looks as follows :

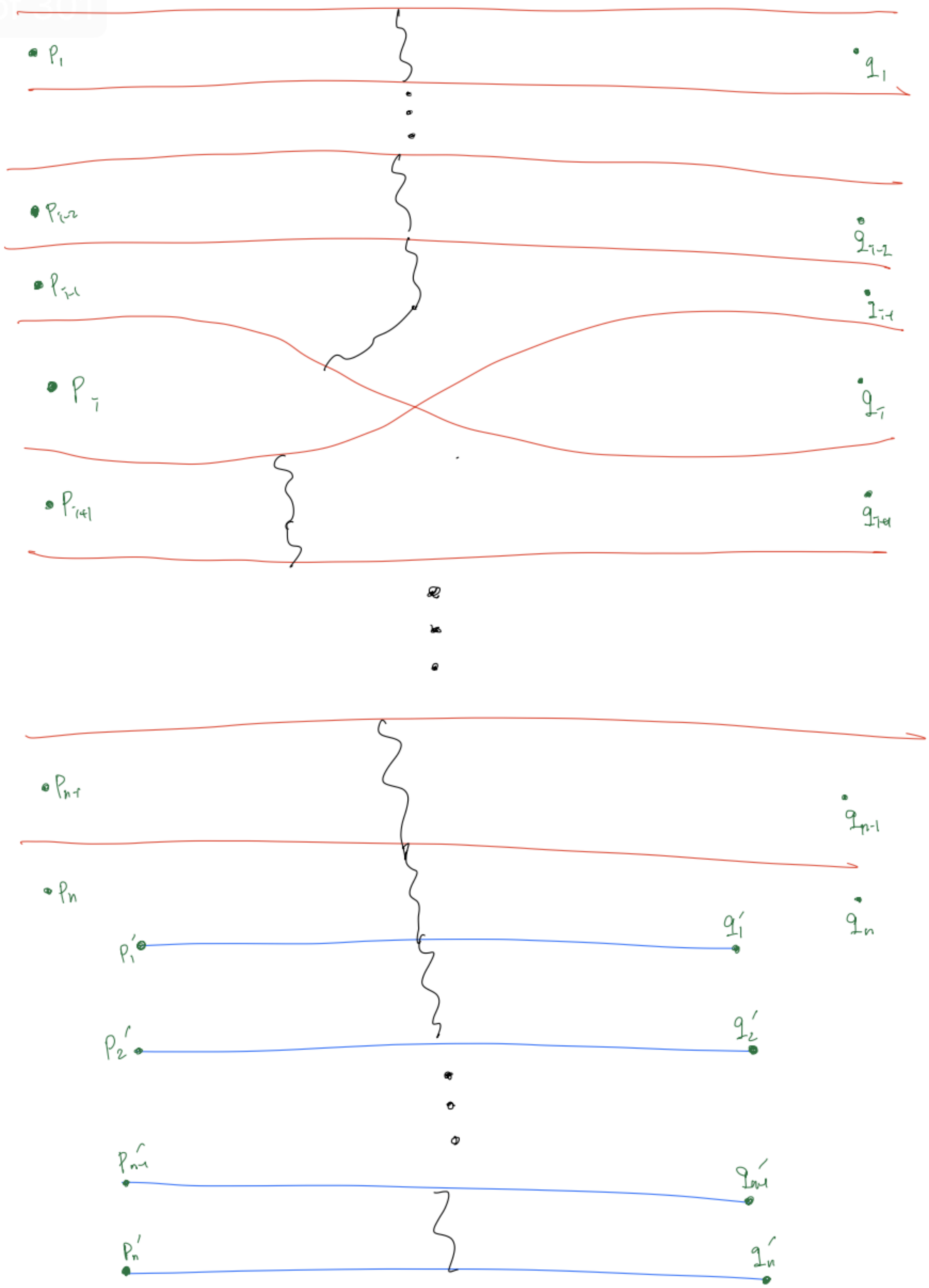


Figure 4.123: Your caption here

We define MOVE xiias follows :

(Step1) Apply MOVE iito the region inside the purple circle :

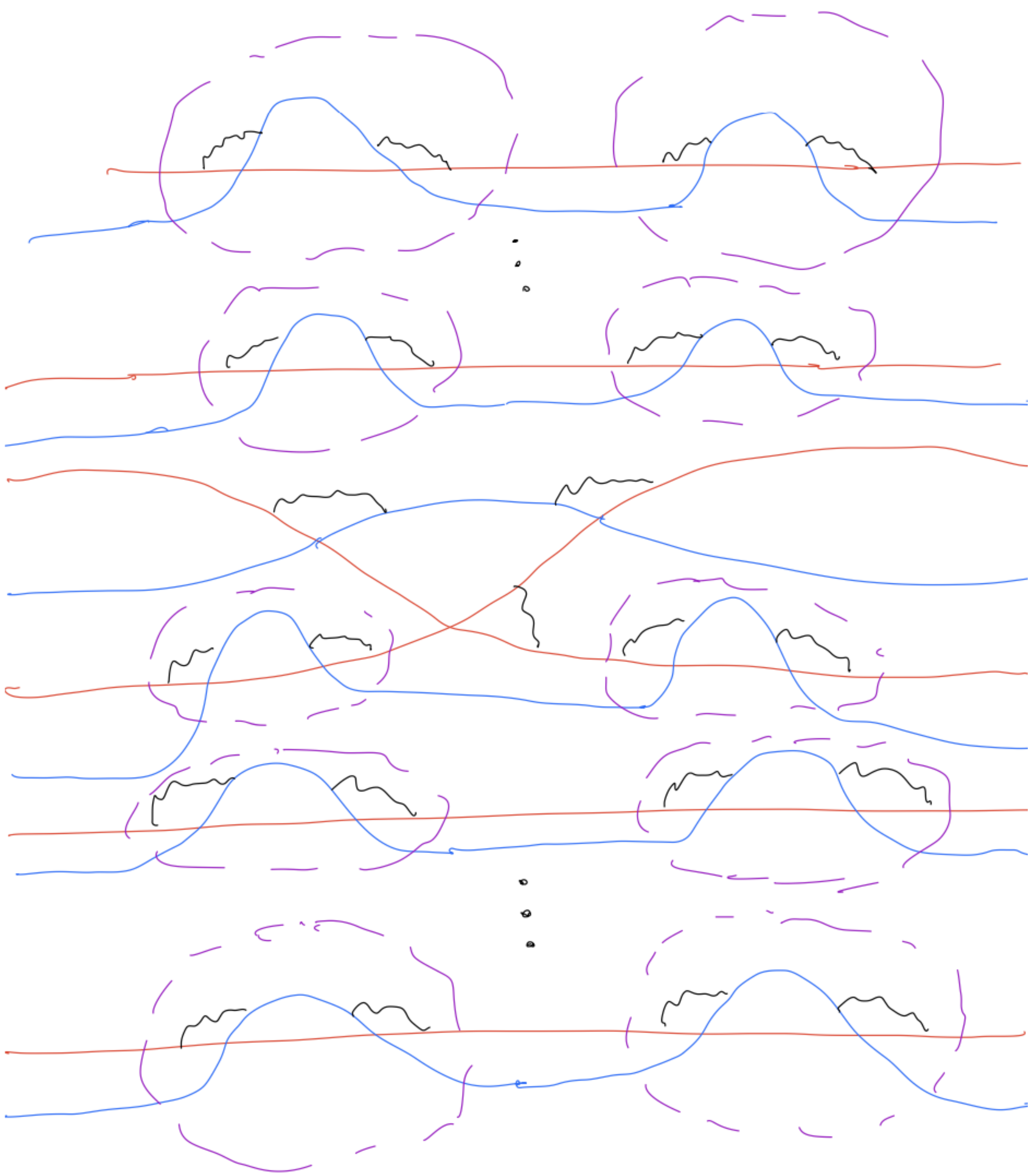


Figure 4.124: Your caption here

we get :

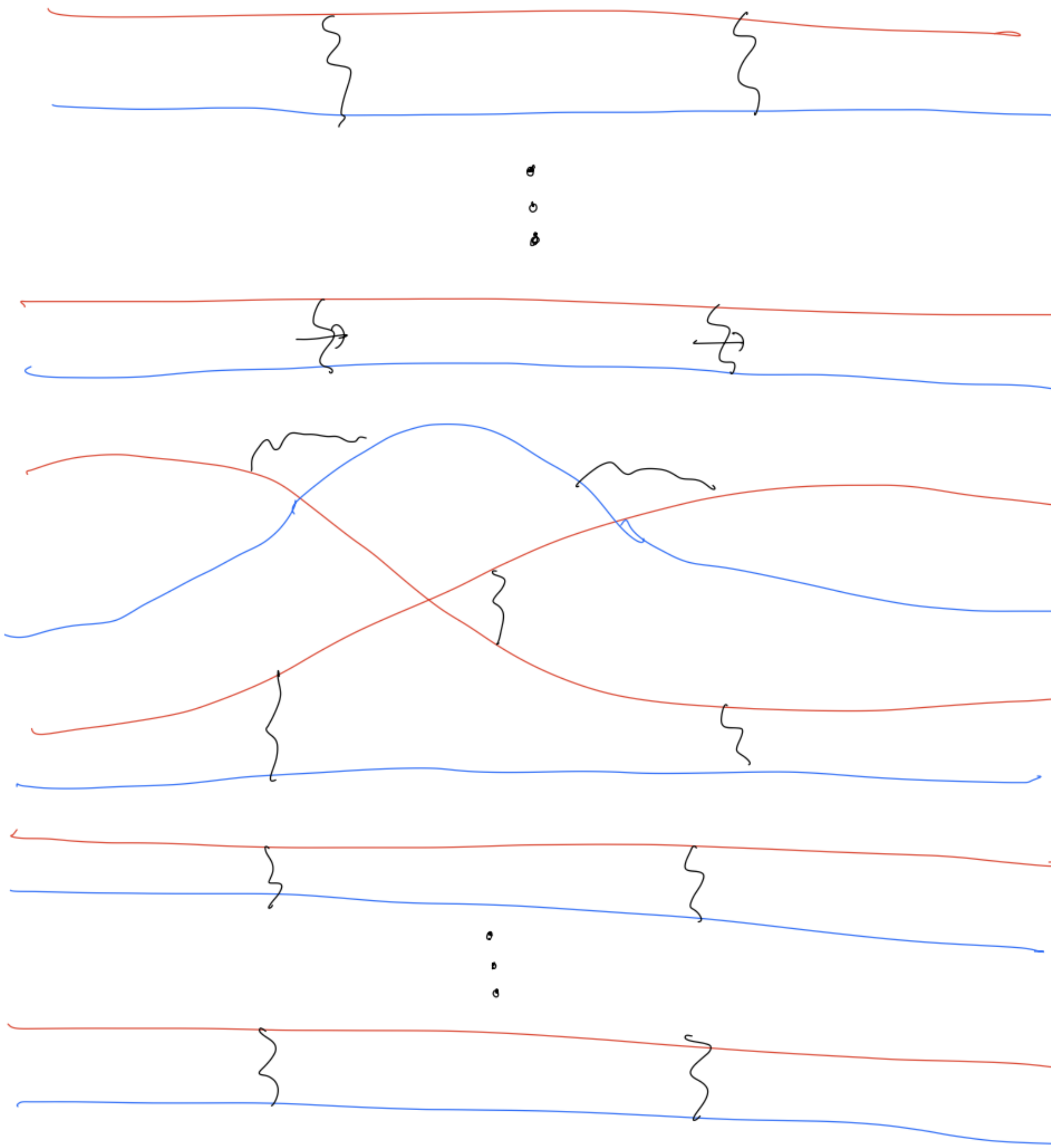


Figure 4.125: Your caption here

(Step2)get rid of the squiggly lines in the region inside the purple circles:

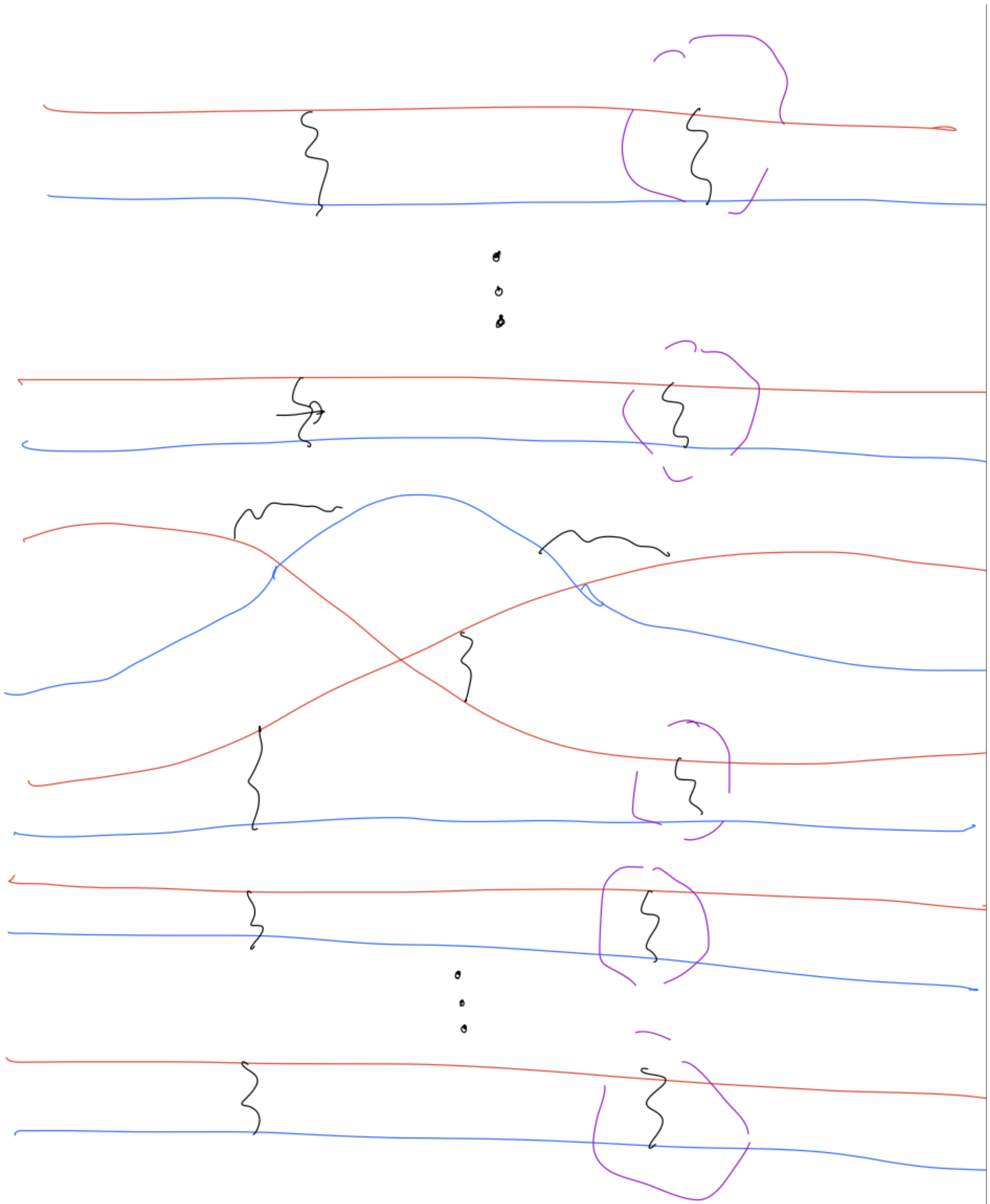


Figure 4.126: Your caption here

we get :

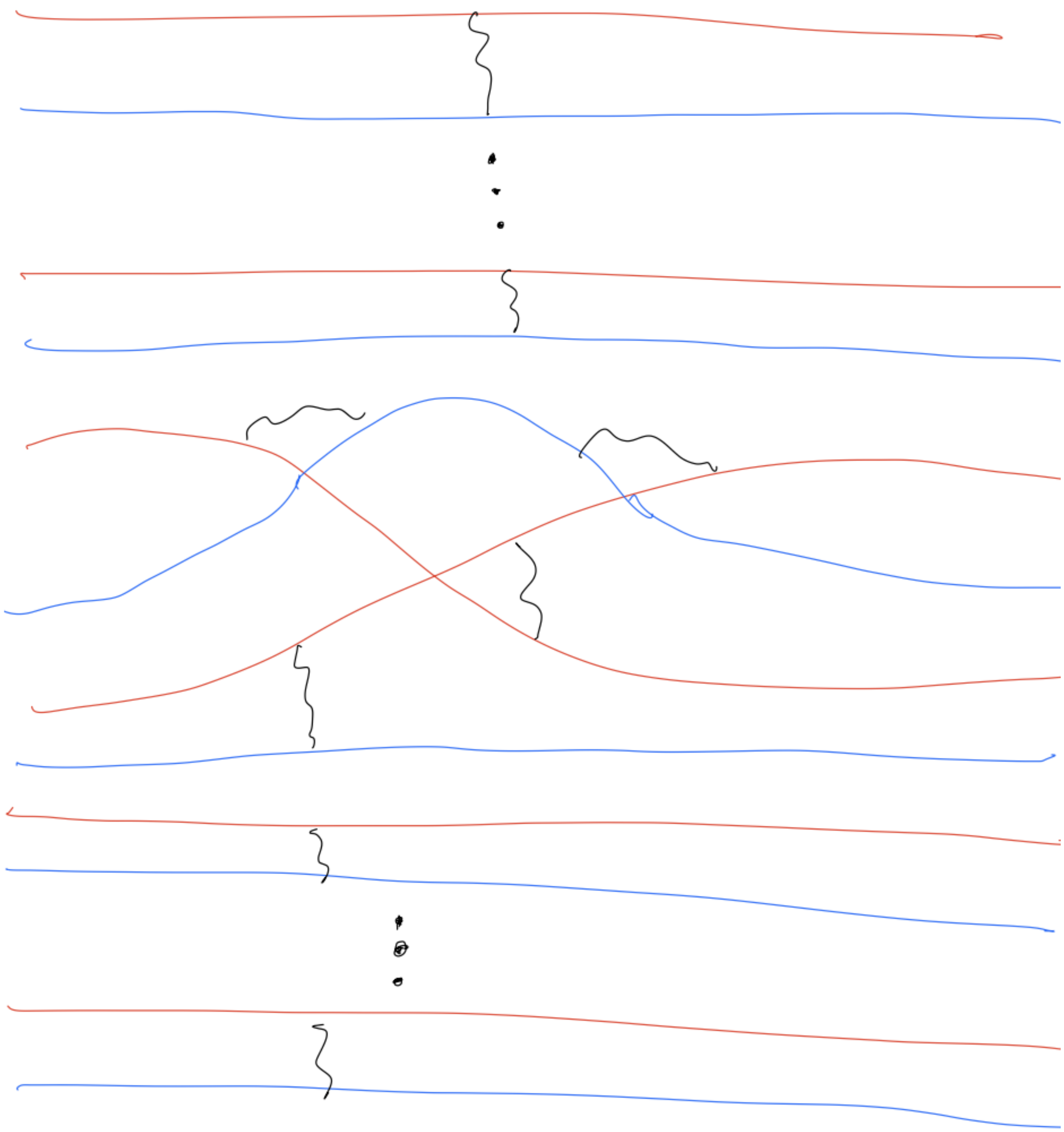


Figure 4.127: Your caption here

(Step3) Apply MOVE vii-(a) to the region inside the purple circles

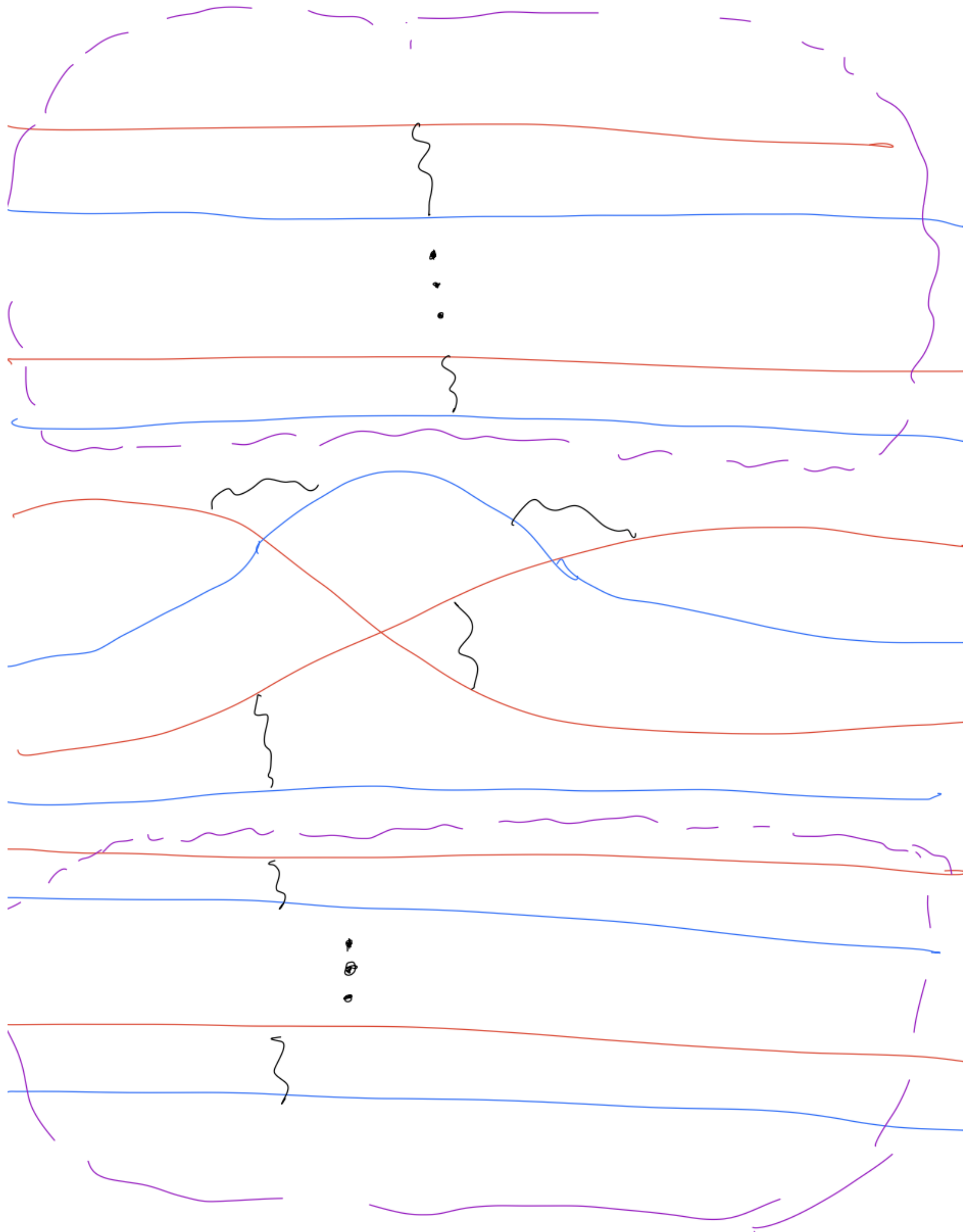


Figure 4.128: Your caption here

we get :

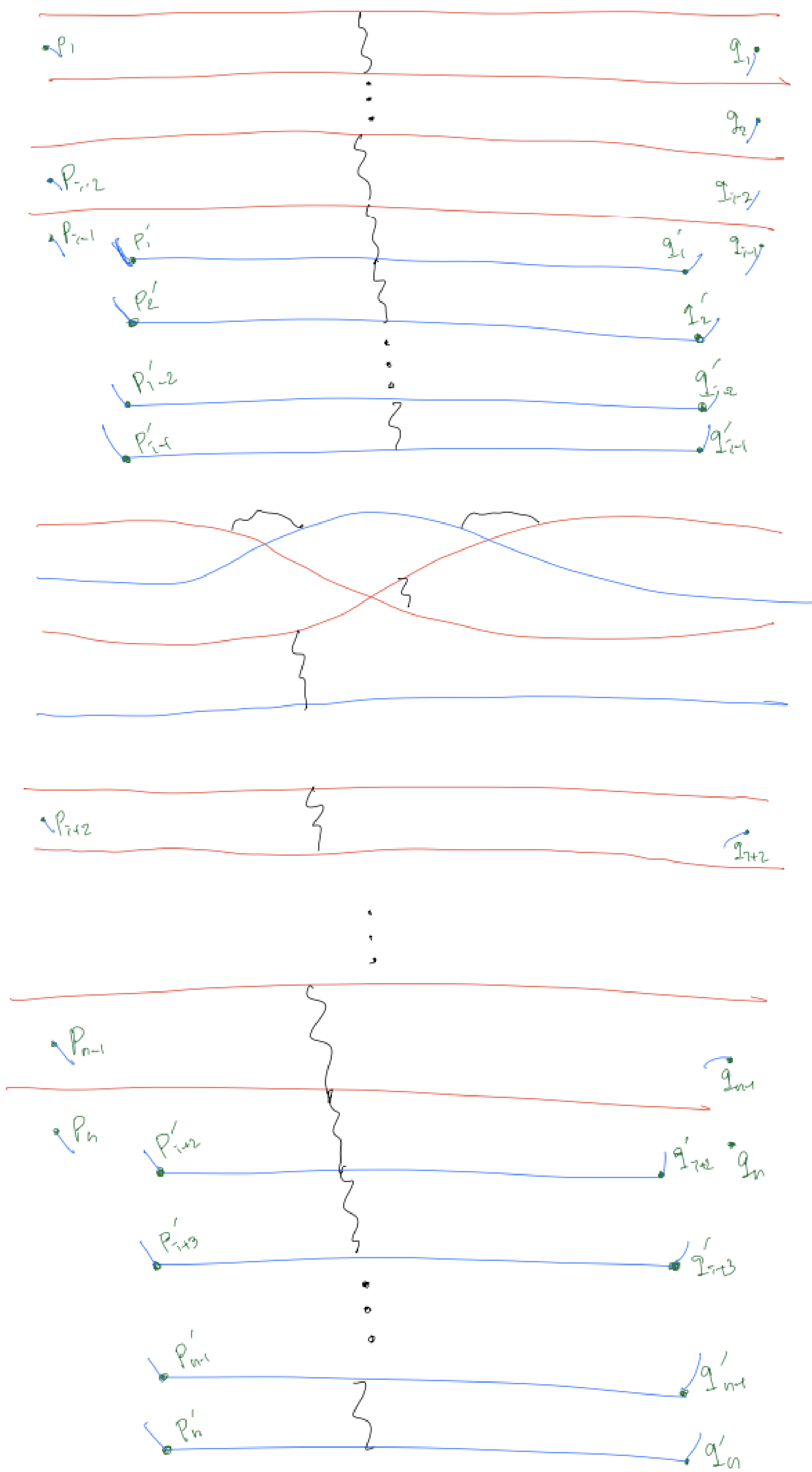
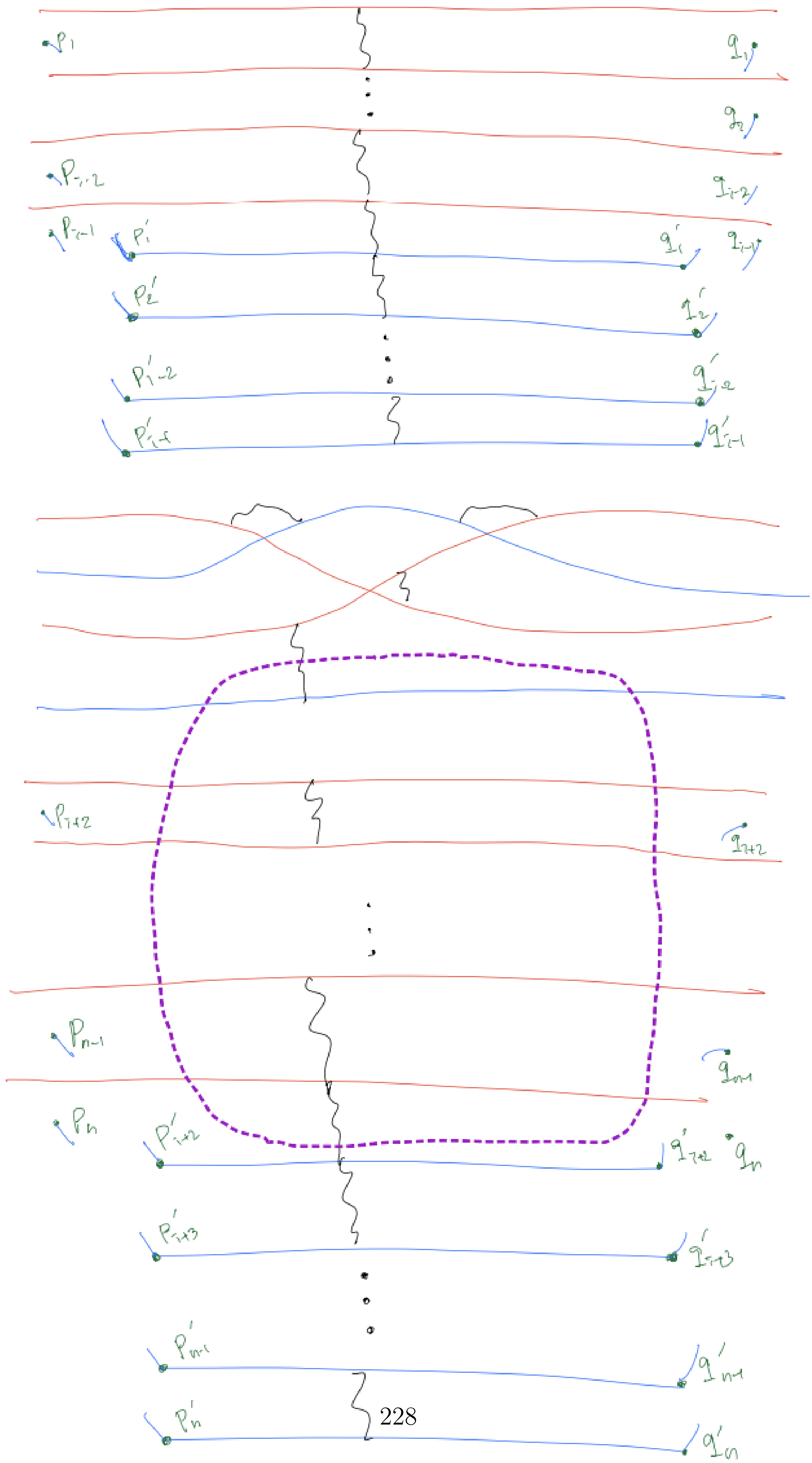


Figure 4.129: Your caption here

(Step3') Apply MOVE vii-(b) to the region inside the purple circle



we get:

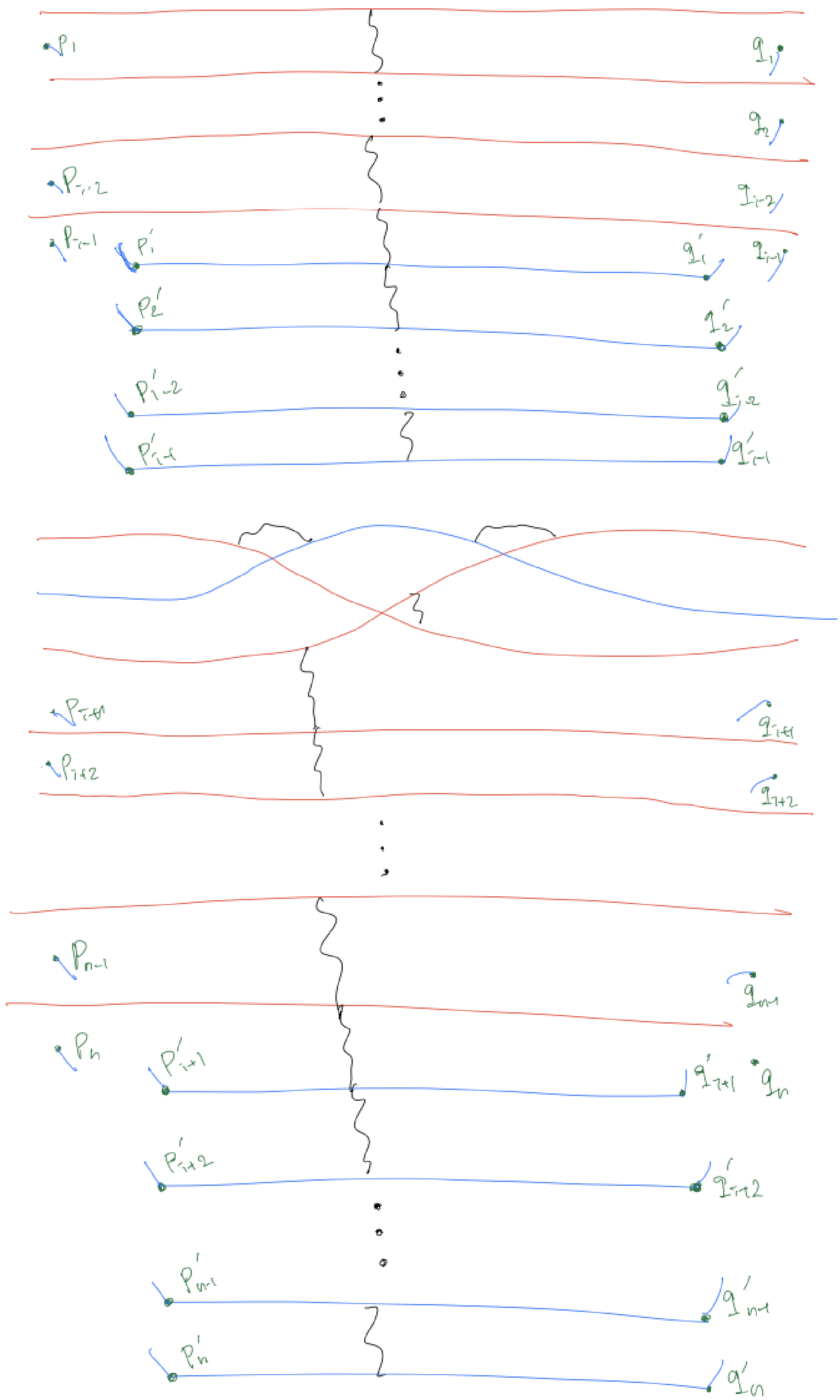


Figure 4.131: Your caption here

(Step4) Appy MOVE ixto the region inside the purple circle

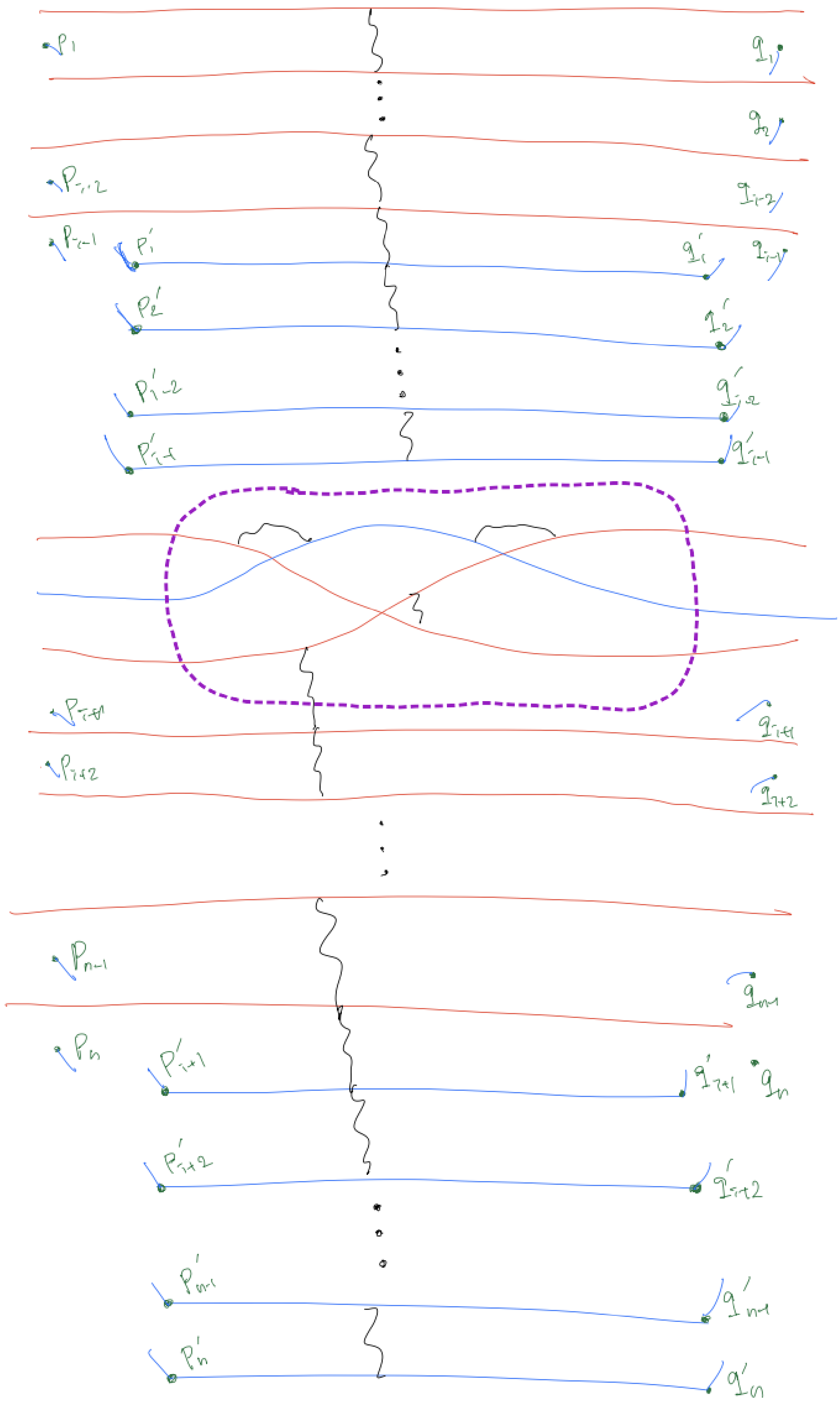


Figure 4.132: Your caption here

we get :

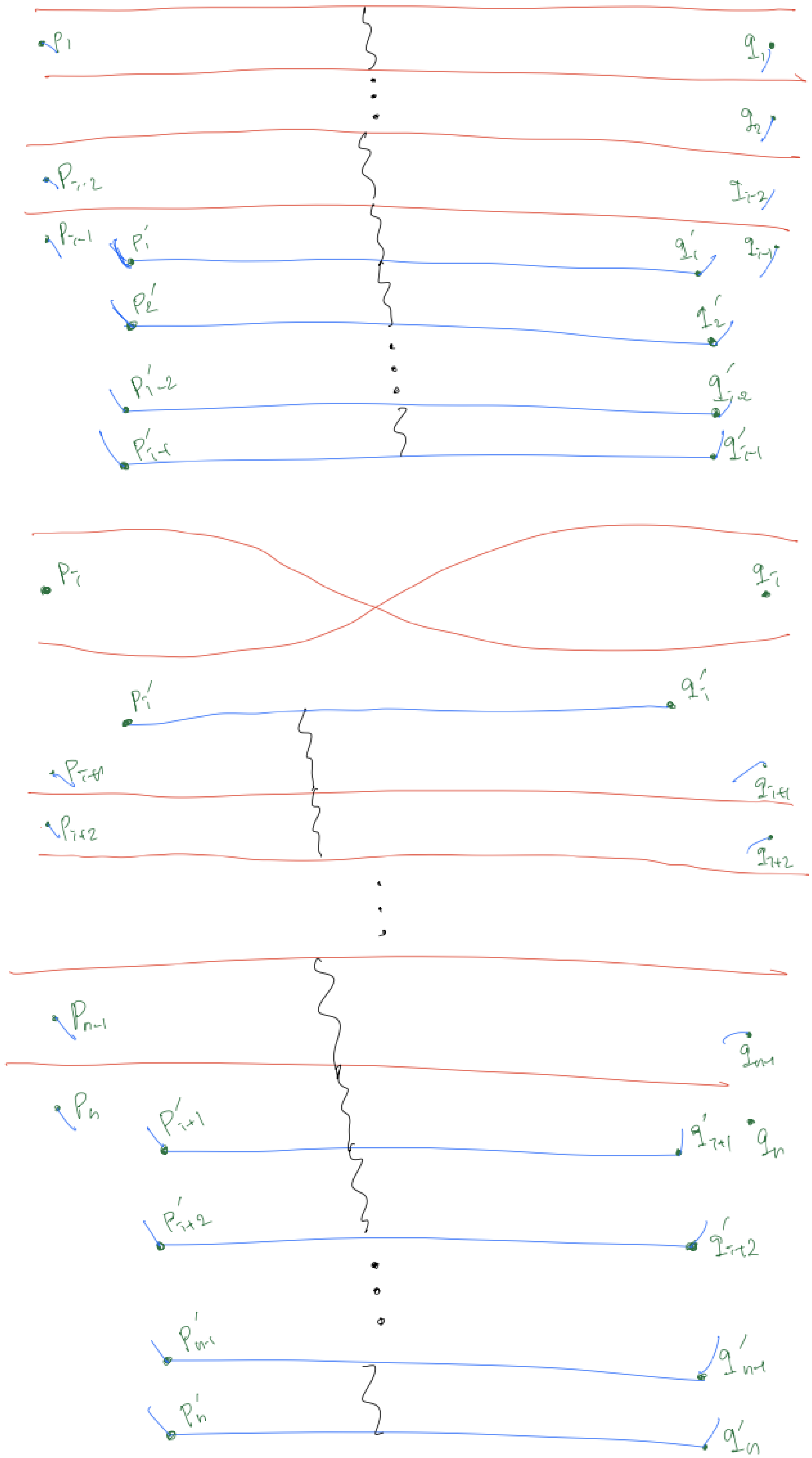


Figure 4.133: Your caption here

(Step5) Apply MOVE xito the region inside the purple circle

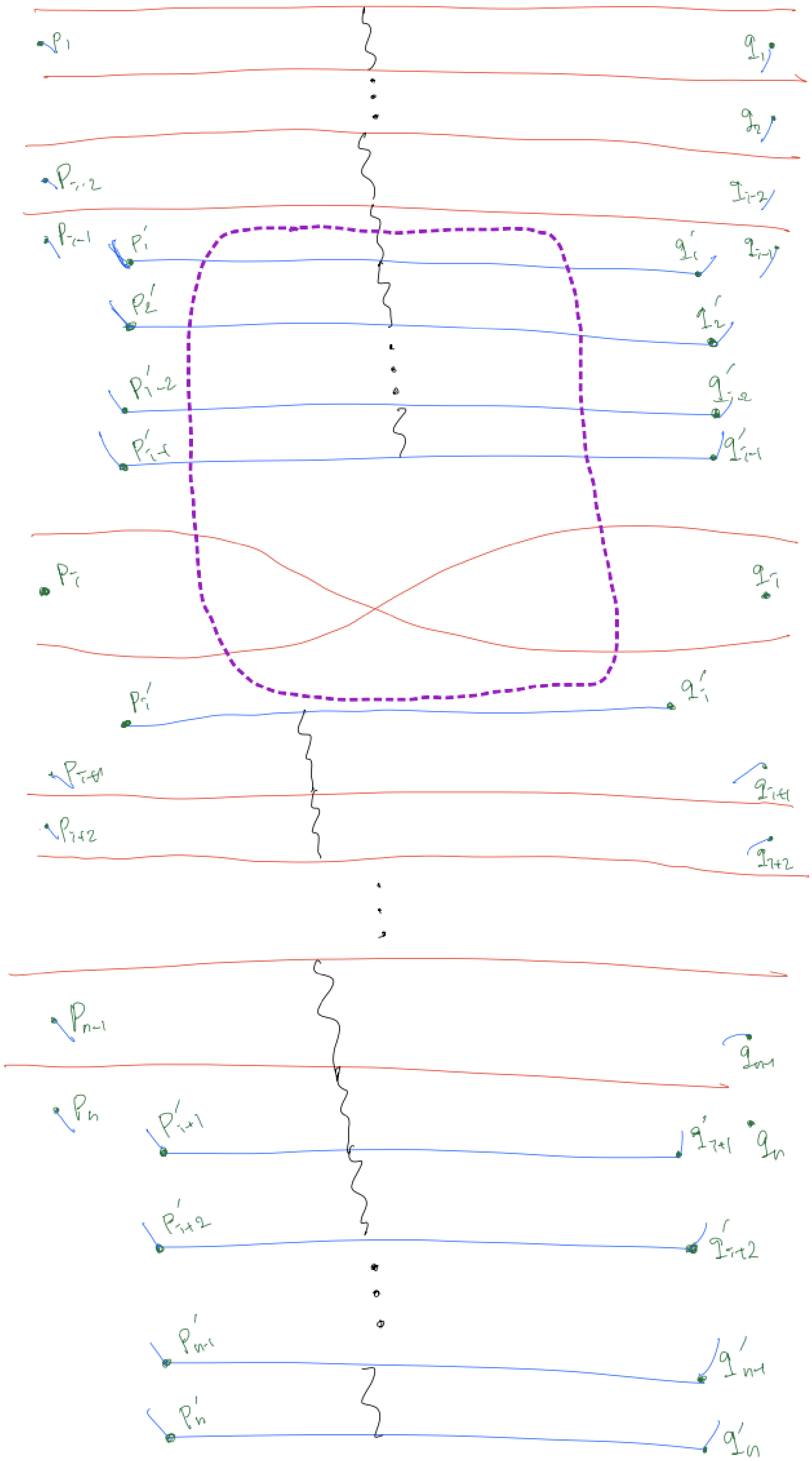


Figure 4.134: Your caption here

we get :

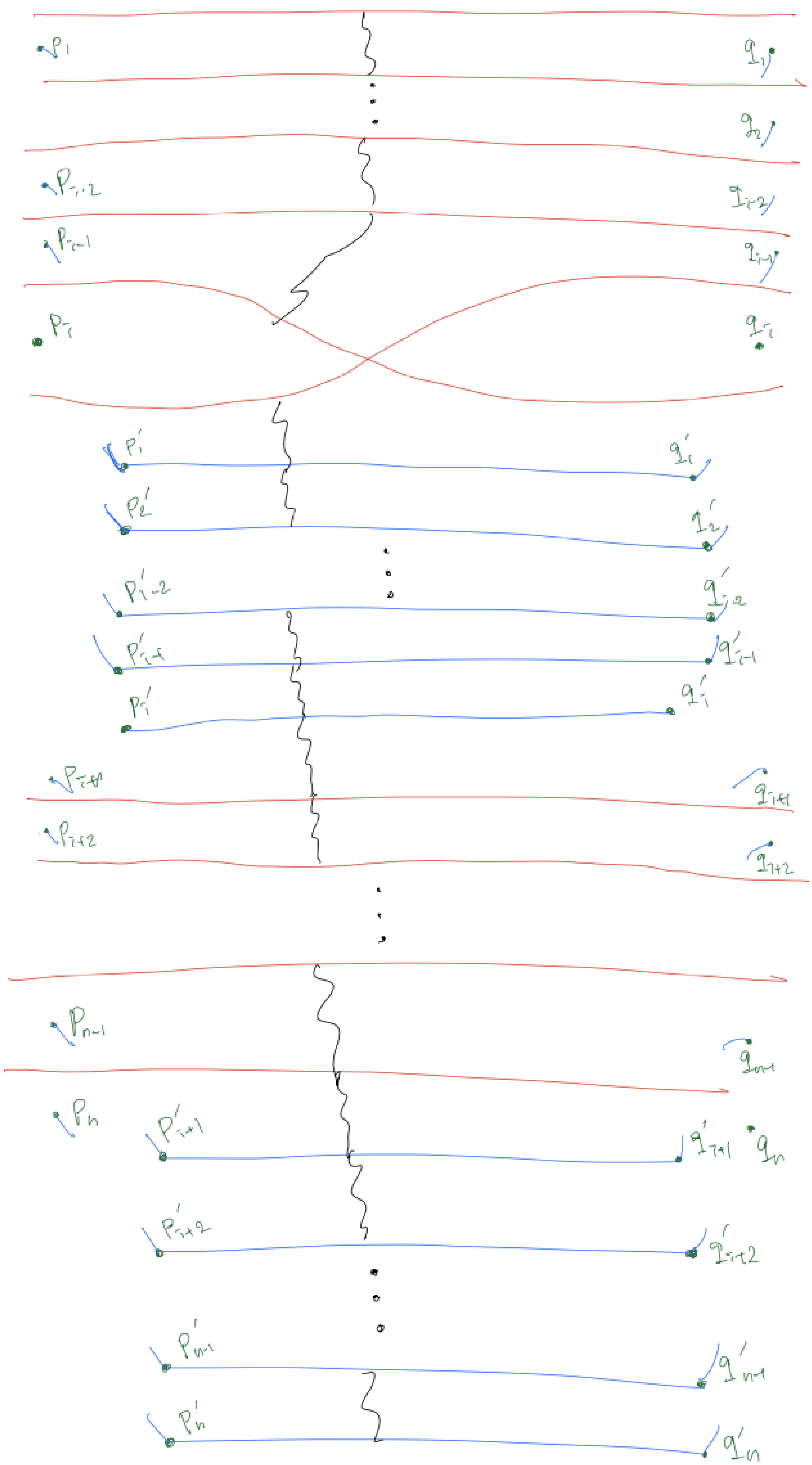


Figure 4.135: Your caption here

(Step6) Apply MOVE vii-(b) to the region inside the purple circle

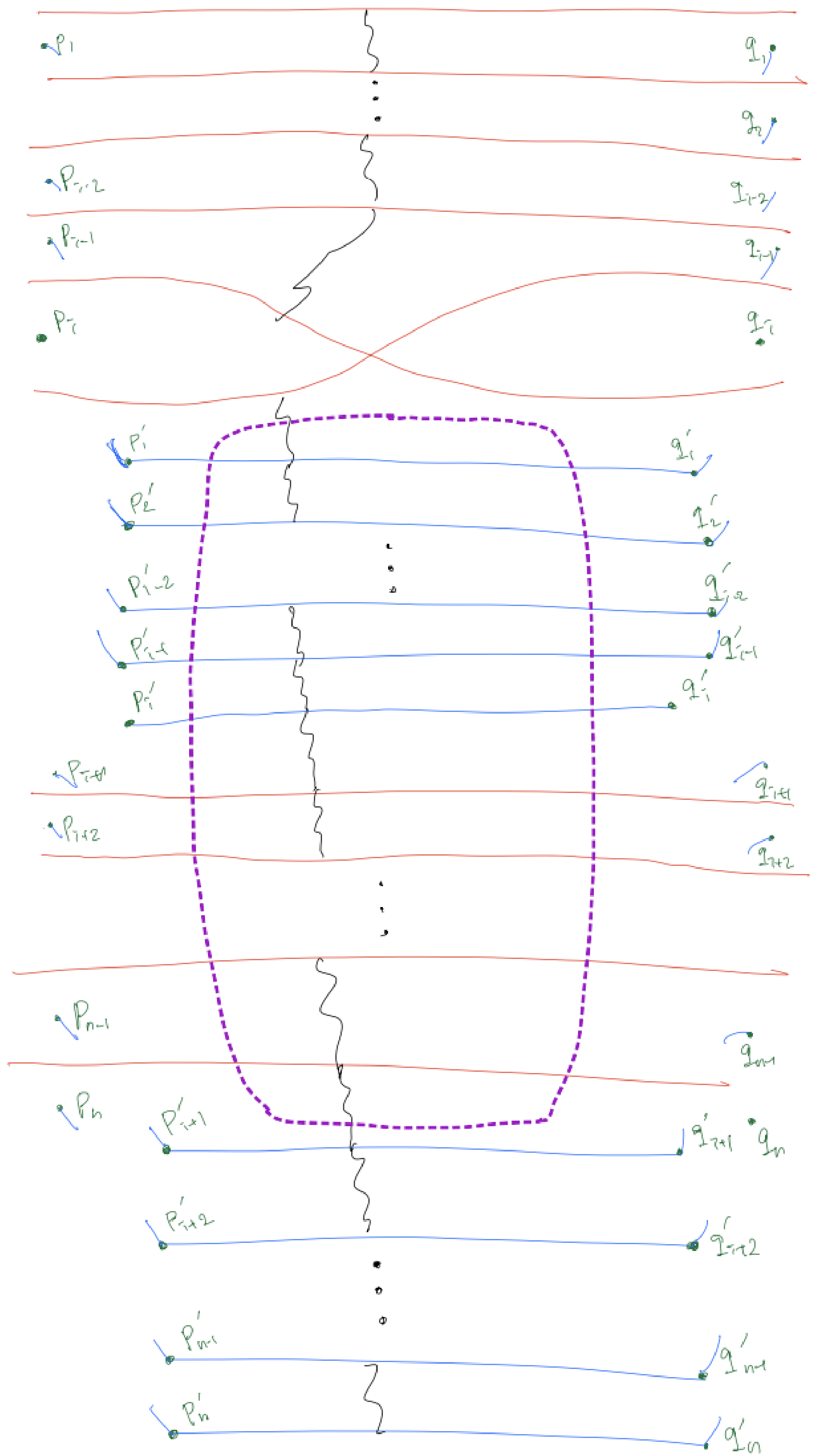


Figure 4.136: Your caption here

we get :

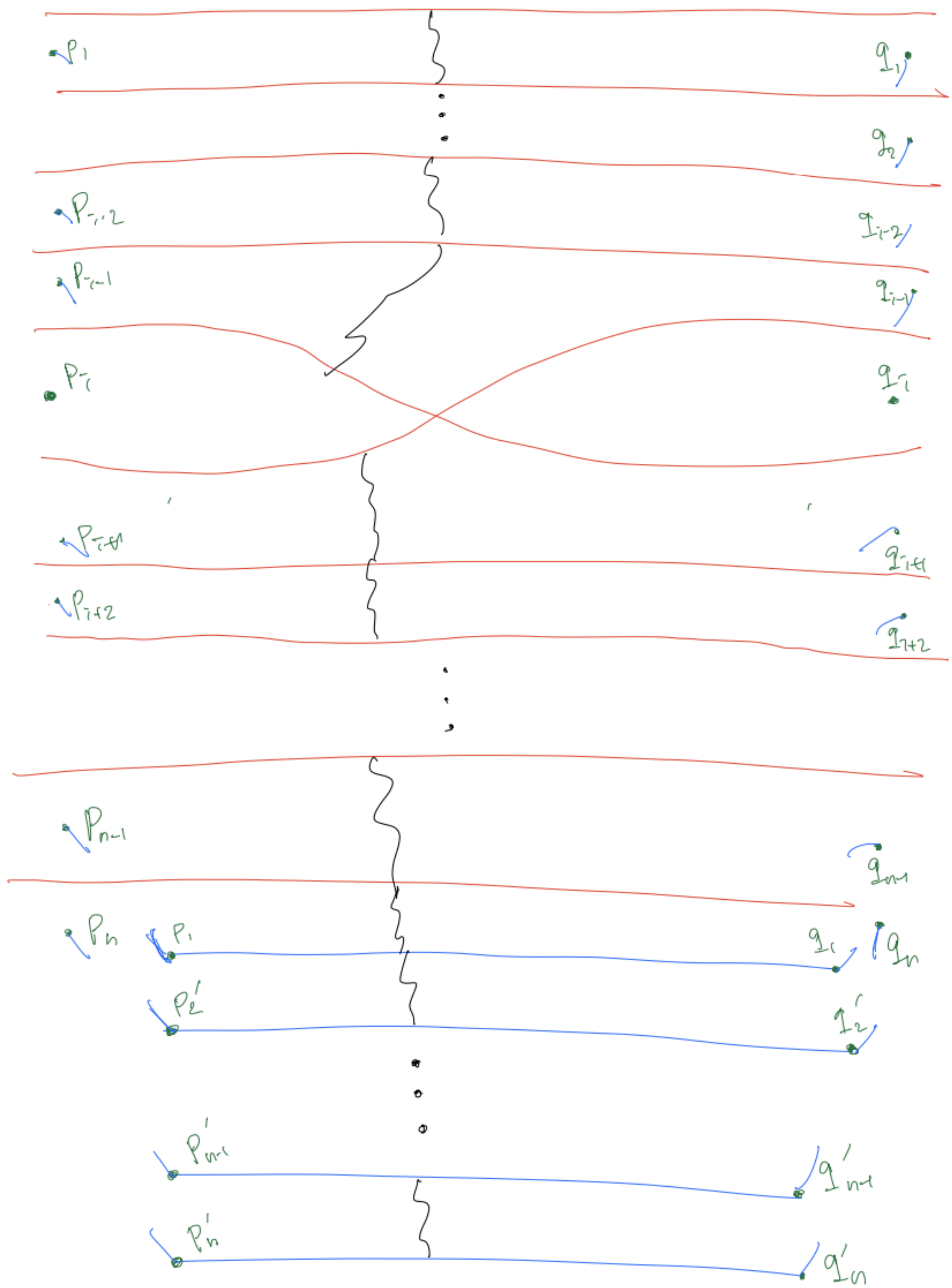


Figure 4.137: Your caption here

4.22 theorem12(the main theorem)

Theorem 64.

Generator Move

Suppose we have a Riemann sphere and a natural alternating diagram and a local system on the associated conjugate surface which could be represented as a sheaf $\mathfrak{F}$ singular supported along the natural alternating strand diagram.
Suppose the sheaf $\mathfrak{F}$ restricted to the generator region $D \subset C$ is as follows :

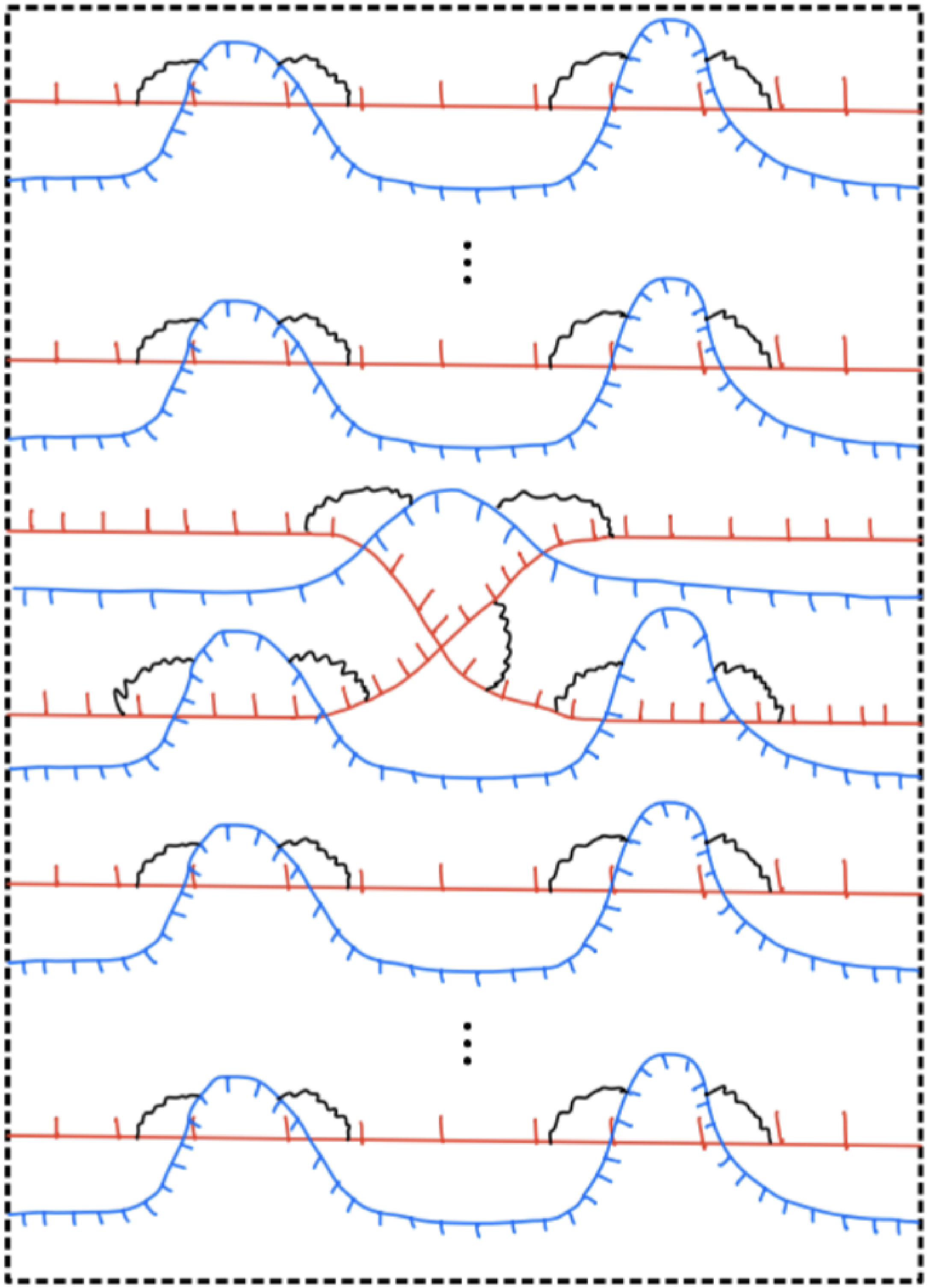


Figure 4.138: Your caption here

Let the l^{th} crossing from the left of the j^{th} blue strand be $c_{j,l}$.

- If $j \neq i$, there are $c_{j,1}, c_{j,2}, c_{j,3}, c_{j,4}$
- If $j = i$, there are $c_{j,1}, c_{j,2}$

Stalks:

- $N_{j,1} : \mathbb{C}$
- $W_{j,1} : 0$
- $E_{j,1} : \mathbb{C} \xrightarrow{\times a_j} \mathbb{C}$
- $S_{j,1} : \mathbb{C}[-1]$
- $N_{j,2} : \mathbb{C}$
- $W_{j,2} : \mathbb{C} \xrightarrow{\times b_j} \mathbb{C}$
- $E_{j,2} : 0$
- $S_{j,2} : \mathbb{C}[-1]$
- $N_{j,3} : \mathbb{C}$
- $W_{j,3} : 0$
- $E_{j,3} : \mathbb{C} \xrightarrow{\times c_j} \mathbb{C}$
- $S_{j,3} : \mathbb{C}[-1]$
- $N_{j,4} : \mathbb{C}$
- $W_{j,4} : \mathbb{C} \xrightarrow{\times d_j} \mathbb{C}$
- $E_{j,4} : 0$
- $S_{j,4} : \mathbb{C}[-1]$

Let the crossing of i^{th} and $i+1^{th}$ red strands be mc, then

- $N : \mathbb{C}$
- $E : \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $W : 0$
- $S : \mathbb{C}[-1]$

Generization maps :

- $S_{j,1} \to E_{j,1}$: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a_j"'] \end{tikzcd}$

- $E_{j,1} \to N_{j,1}$: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a_j"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- $S_{j,2} \to W_{j,2}$: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b_j"'] \end{tikzcd}$

- $W_{j,2} \to N_{j,2}$: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b_j"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- $S_{j,3} \to E_{j,3}$: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c_j"'] \end{tikzcd}$

- $E_{j,3} \to N_{j,3}$: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c_j"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

- $S_{j,4} \to W_{j,4}$: $\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times d_j"'] \end{tikzcd}$

- $W_{j,4} \to N_{j,4}$: $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times d_j"] \arrow[r, "id"] & \mathbb{C} \arrow[u] \end{tikzcd}$

rest of the maps are zero maps.

Now we will define a move called "generator move" to $\mathfrak{F}$ so that the final sheaf is as follows :

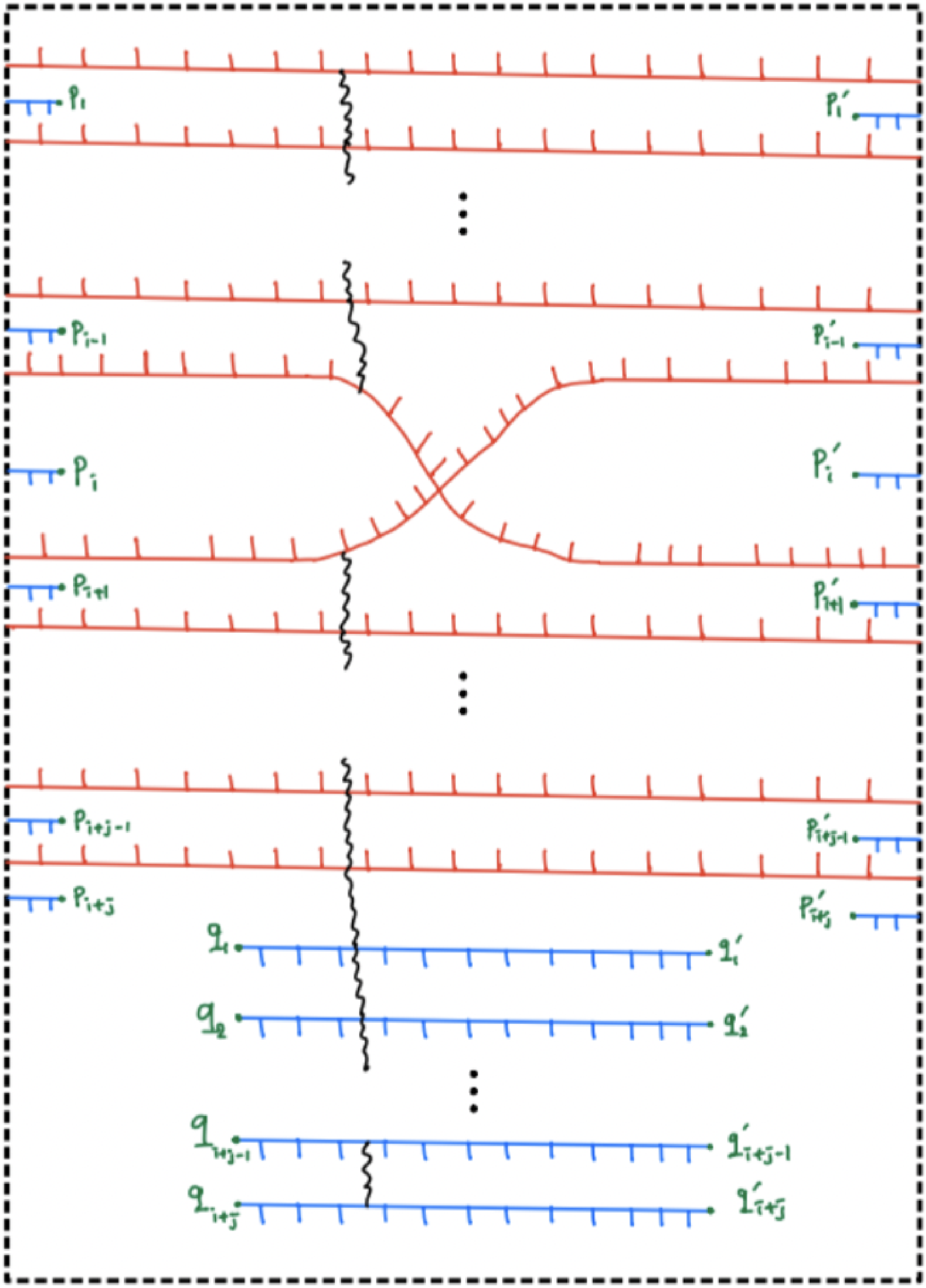


Figure 4.139: Your caption here

Here I intentionally omitted lines connecting:

- p_l and q_l
- p'_l and q'_l

do as not to make diagram too messy.

Let's denote the crossing of $p_{l_1}, q_{l_1}(p'_{l_1}, q'_{l_1}$ resp.$)$ with l^t_2h red strand as $c_{l_1,l_2}(c'_{l_1,l_2}$ resp.).

Let's denote the north, east, west, south of $c_{l_1,l_2}(c'_{l_1,l_2}$ resp.$)$ as $N_{l_1,l_2}, E_{l_1,l_2}, W_{l_1,l_2}, S_{l_1,l_2}(N'_{l_1,l_2}, E'_{l_1,l_2}, W$ resp.)

The final sheaf $\mathfrak{F}'$ can be described as follows:

Stalks:

- $N_{l_1,l_2} : \mathbb{C}^{l_2-l_1+1}$
- $W_{l_1,l_2} : \mathbb{C}^{l_2-l_1}$
- $E_{l_1,l_2} : \mathbb{C}^{l_2-l_1}$
- $S_{l_1,l_2} : \mathbb{C}^{l_2-l_1-1}$
- the stalk of the upper-most and bottom-most regions are 0

Generization maps:

- $W_{1,l_2} \to E'_{1,l_2}$: if $l_2 \geq 2$ and $l_2 \neq i, i+1$,

$$\left(\begin{array}{ccc|cc|ccc}
u_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & u_{i-1} & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & \alpha & 0 & 0 & \cdots & 0 \\
 & & z & & & & & \\
0 & \cdots & 0 & \beta & v_1 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & v_2 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & v_{j-1}
\end{array}\right)_{1,l_2-1}$$

- $W_{1,i} \to E'_{1,i+1}$:

$$\left(\begin{array}{ccc|cc|ccc}
u_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & u_{i-1} & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & \alpha & 0 & 0 & \cdots & 0 \\
 & & z & & & & & \\
0 & \cdots & 0 & \beta & v_1 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & v_2 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & v_{j-1}
\end{array}\right)_{1,i-1}$$

- $N_{l_1,i+j} \to N'_{l_1,i+j}$:

$$\left(\begin{array}{ccc|cc|ccc}
u_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & u_{i-1} & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & \alpha & 0 & 0 & \cdots & 0 \\
 & & z & & & & & \\
0 & \cdots & 0 & \beta & v_1 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & v_2 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & v_{j-1}
\end{array}\right)_{l_1,i+j}$$

- $E'_{1,i+1} \to E_{1,i+1}$: $\left(\begin{array}{ccc} u_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_{i-1}^{-1} \\ \hline 0 & \cdots & 0 \end{array}\right)$

- $E_{1,i+1} \to E'_{1,i+2}$: $\left(\begin{array}{ccc|c} u_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & u_{i-1} & 0 \\ \hline 0 & \cdots & 0 & \alpha \\ 0 & \cdots & 0 & \beta \end{array}\right)$

- All the other maps crossing the blue strands are ι_l.

- All the other maps crossing the red strands are ι_f.

- Rest of the maps are zero maps.

Now let's define the "Generator move" step by step.

(step1) Apply $Move_2$ to the disk surrounded by purple dotted lines:

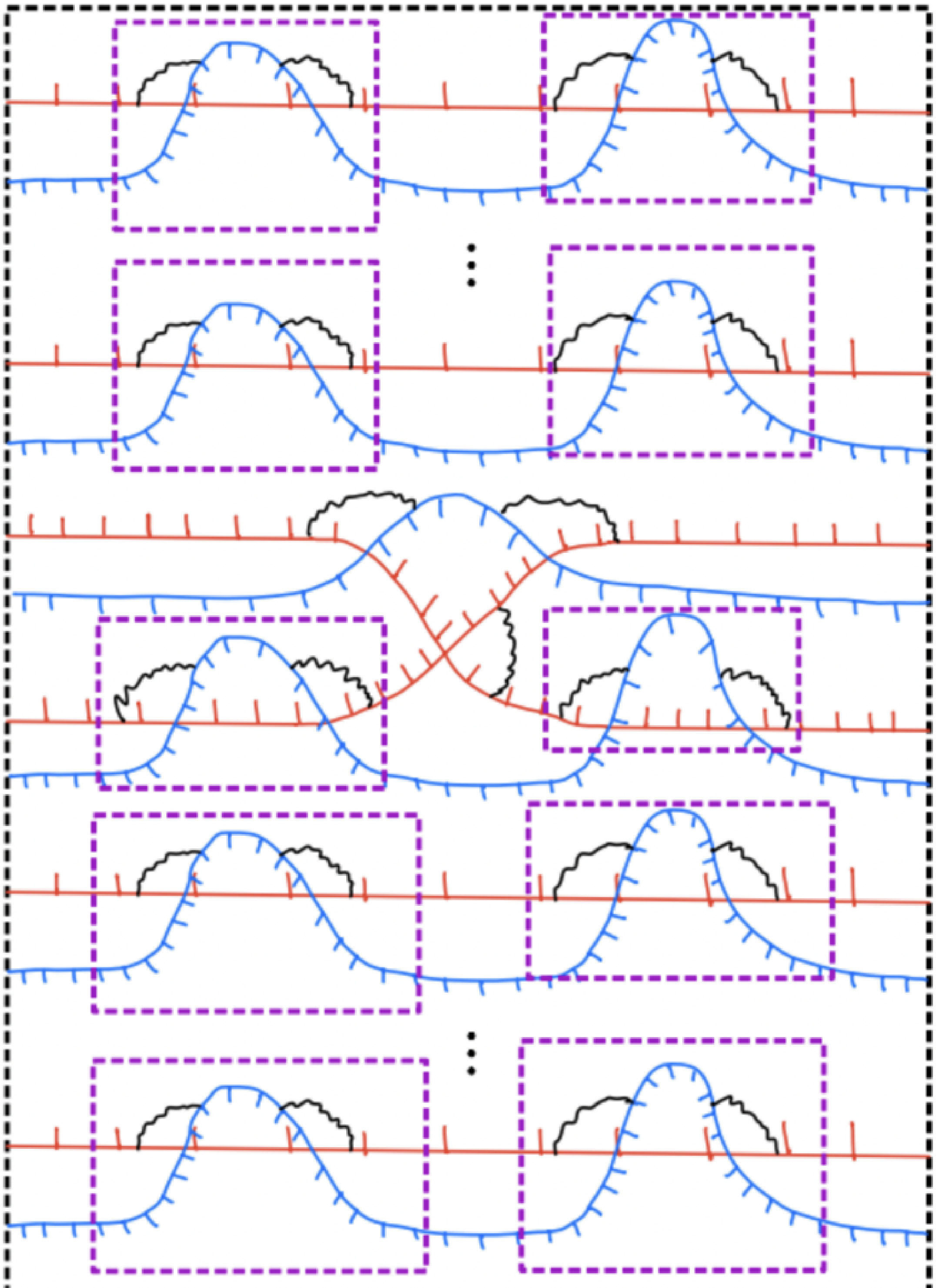


Figure 4.140: Your caption here

We get the following diagram:

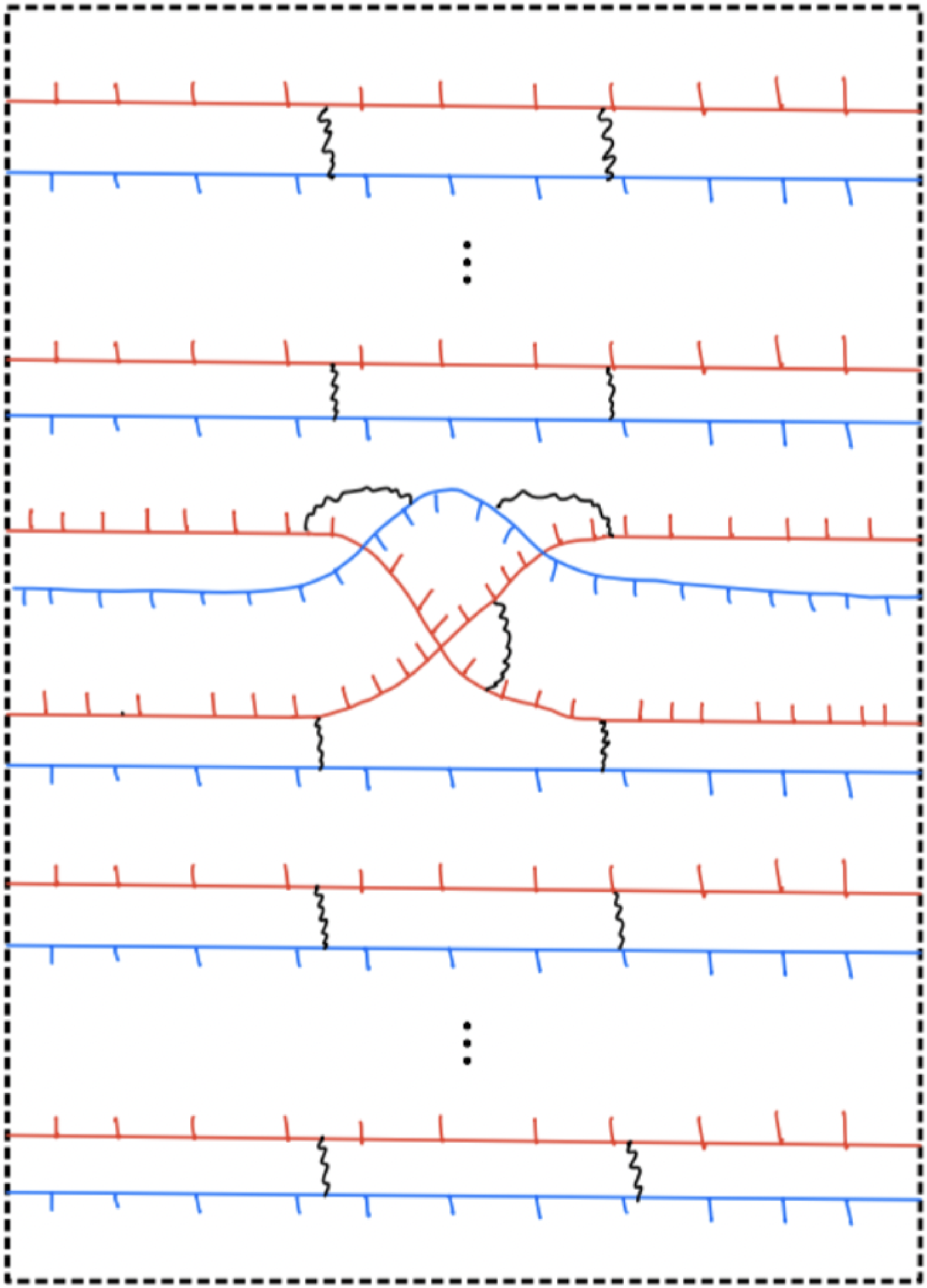


Figure 4.141: Your caption here

(Step2) Now change the basis of the stalks of the regions containing purple start so that the map corresponding to the squiggly lines to the right of the stars are identity maps:

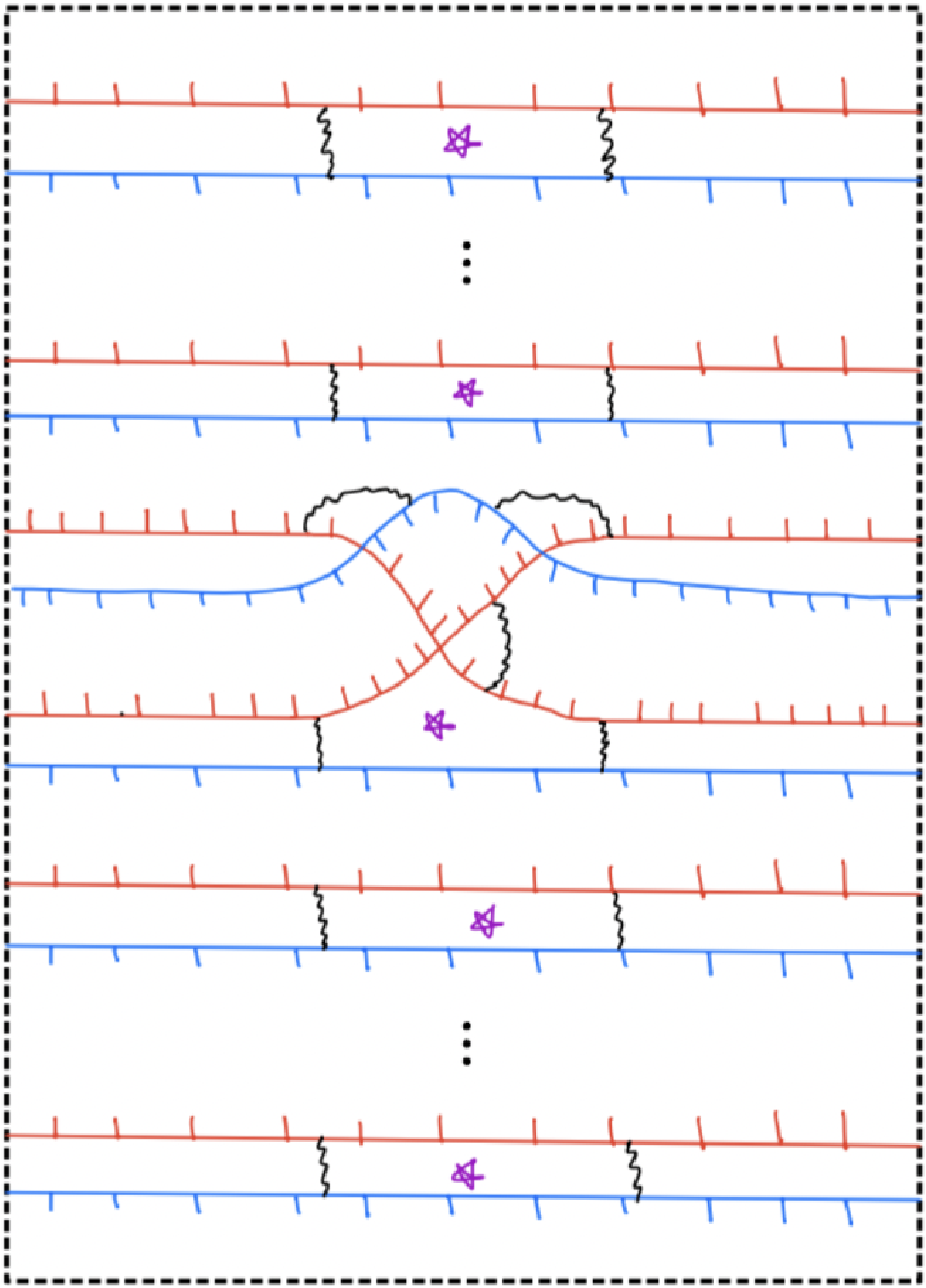


Figure 4.142: Your caption here

We get the following diagram:

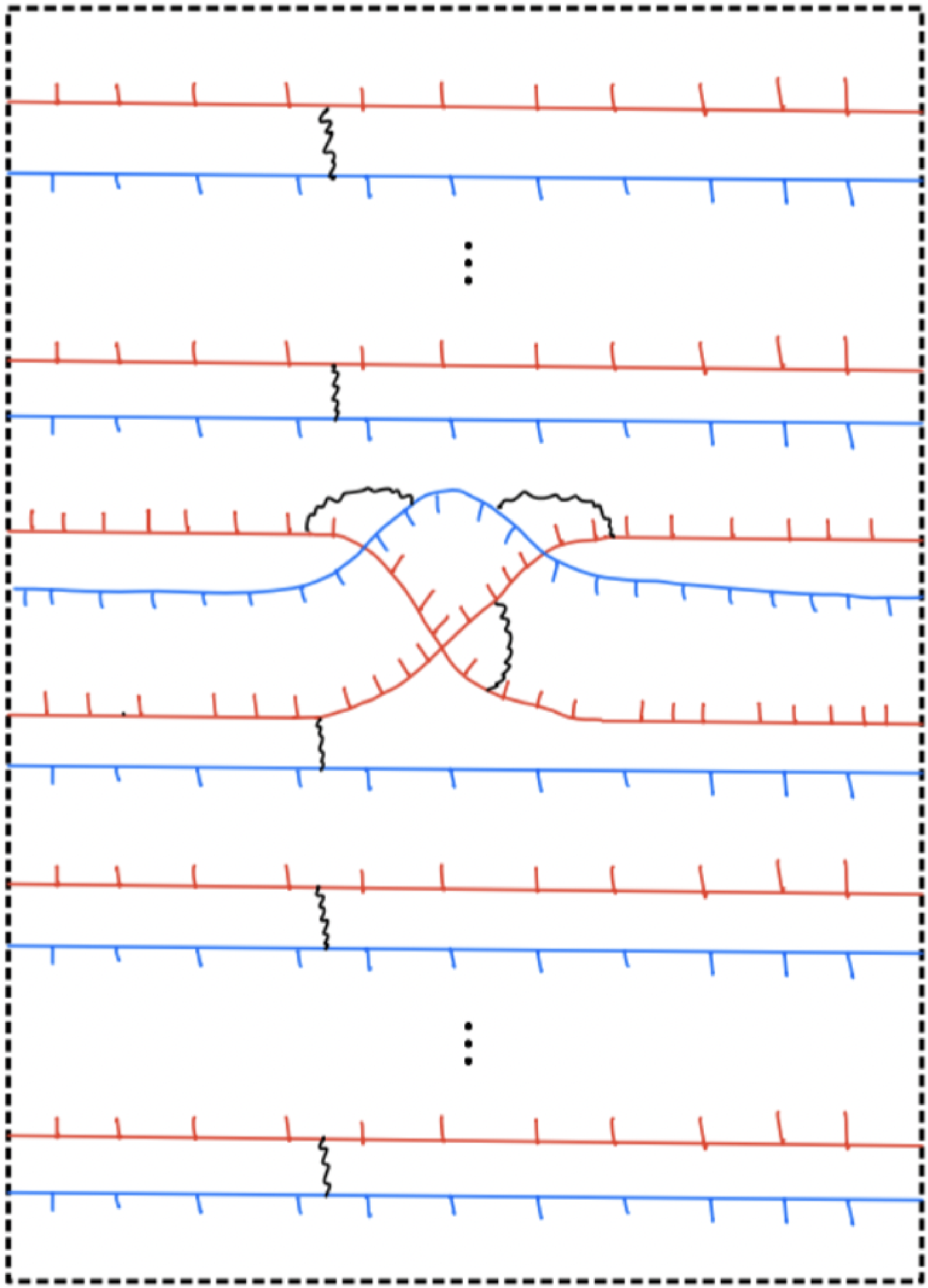


Figure 4.143: Your caption here

(Step3) Now we apply $Move_{7-(a)}$ to the disks surrounded by purple dotted lines:

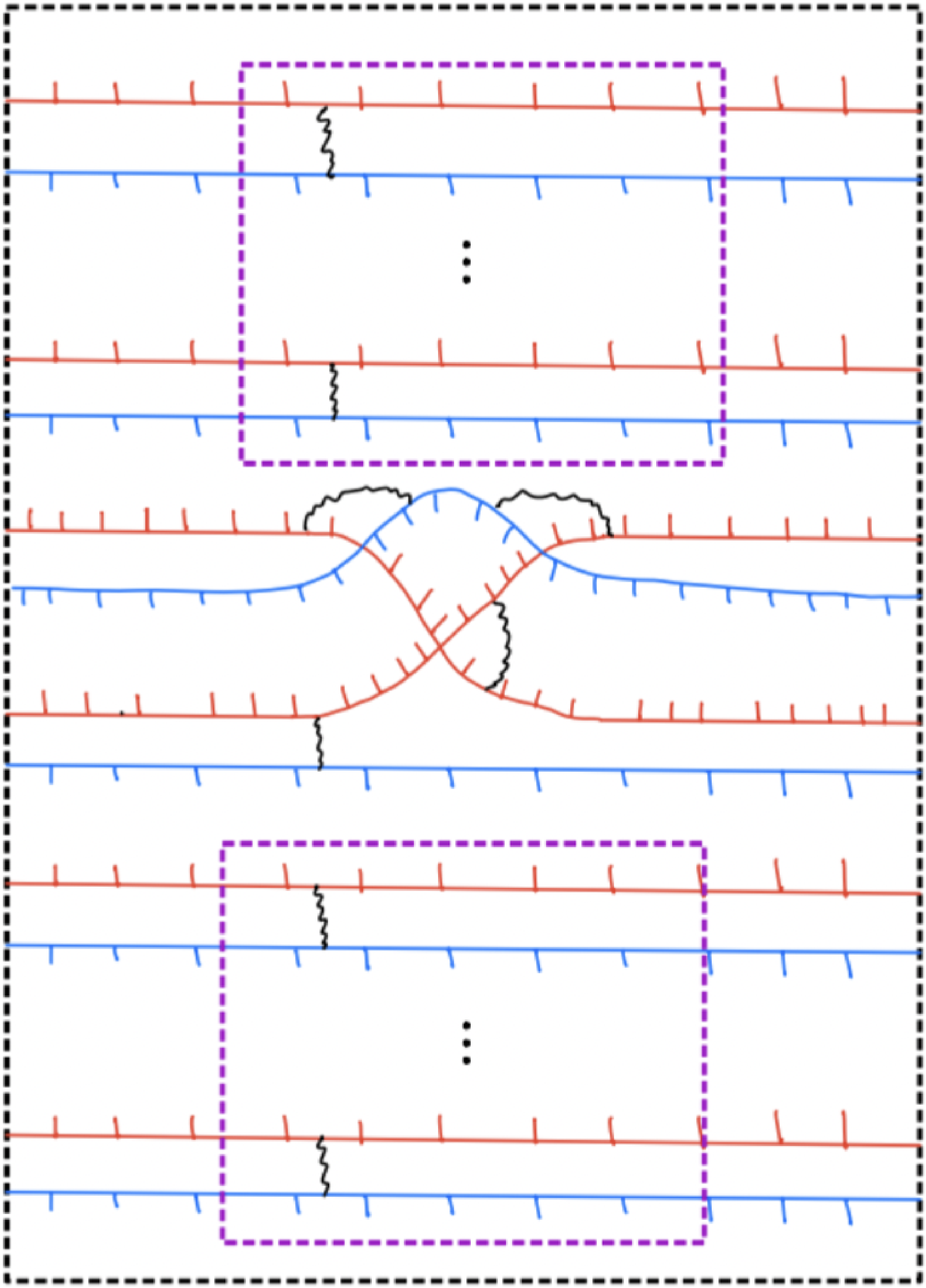


Figure 4.144: Your caption here

We get the following diagram:

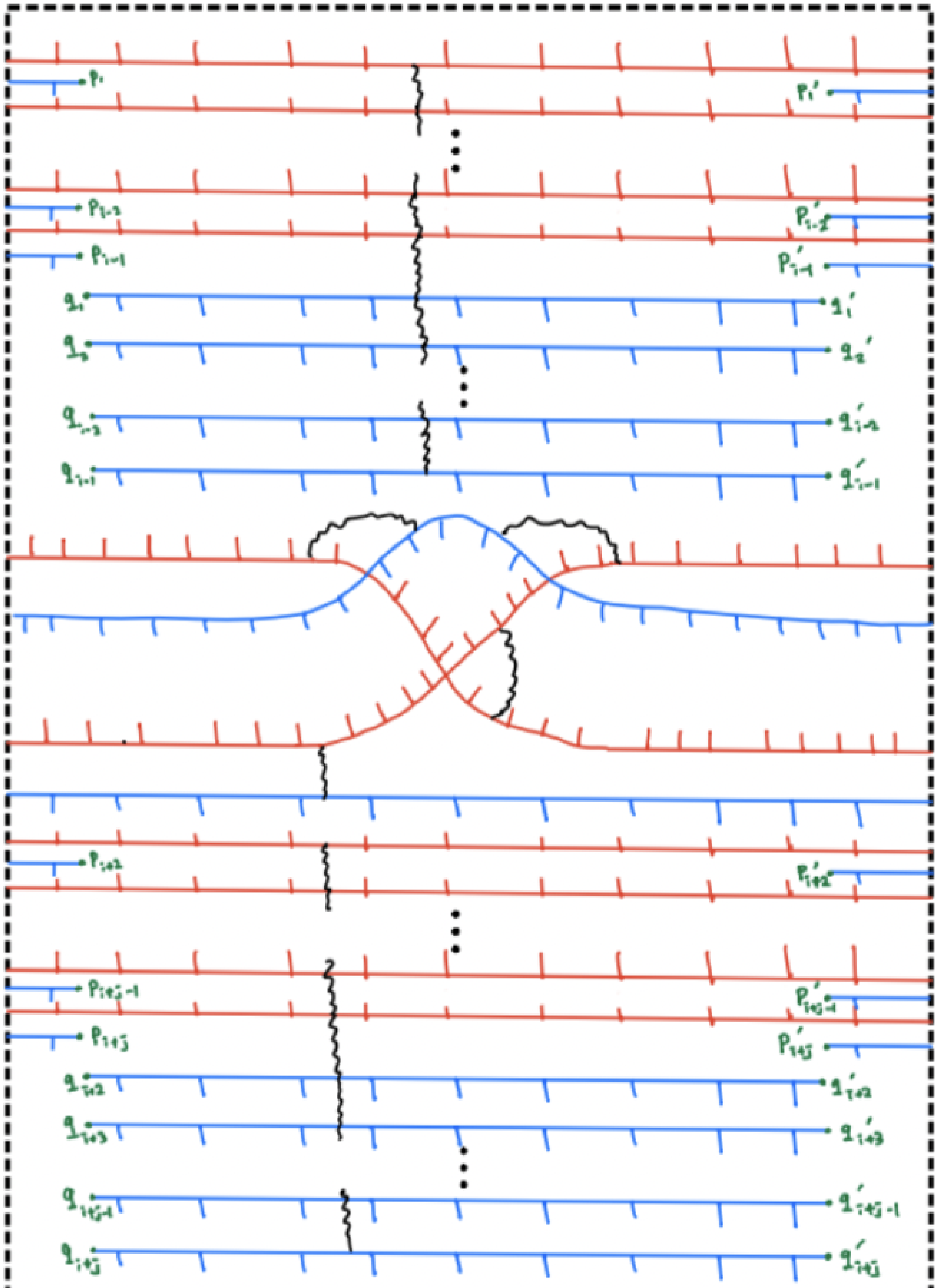


Figure 4.145: Your caption here

(Step4) Now we apply $Move_{7-(b)}$ to the disks surrounded by purple dotted lines:

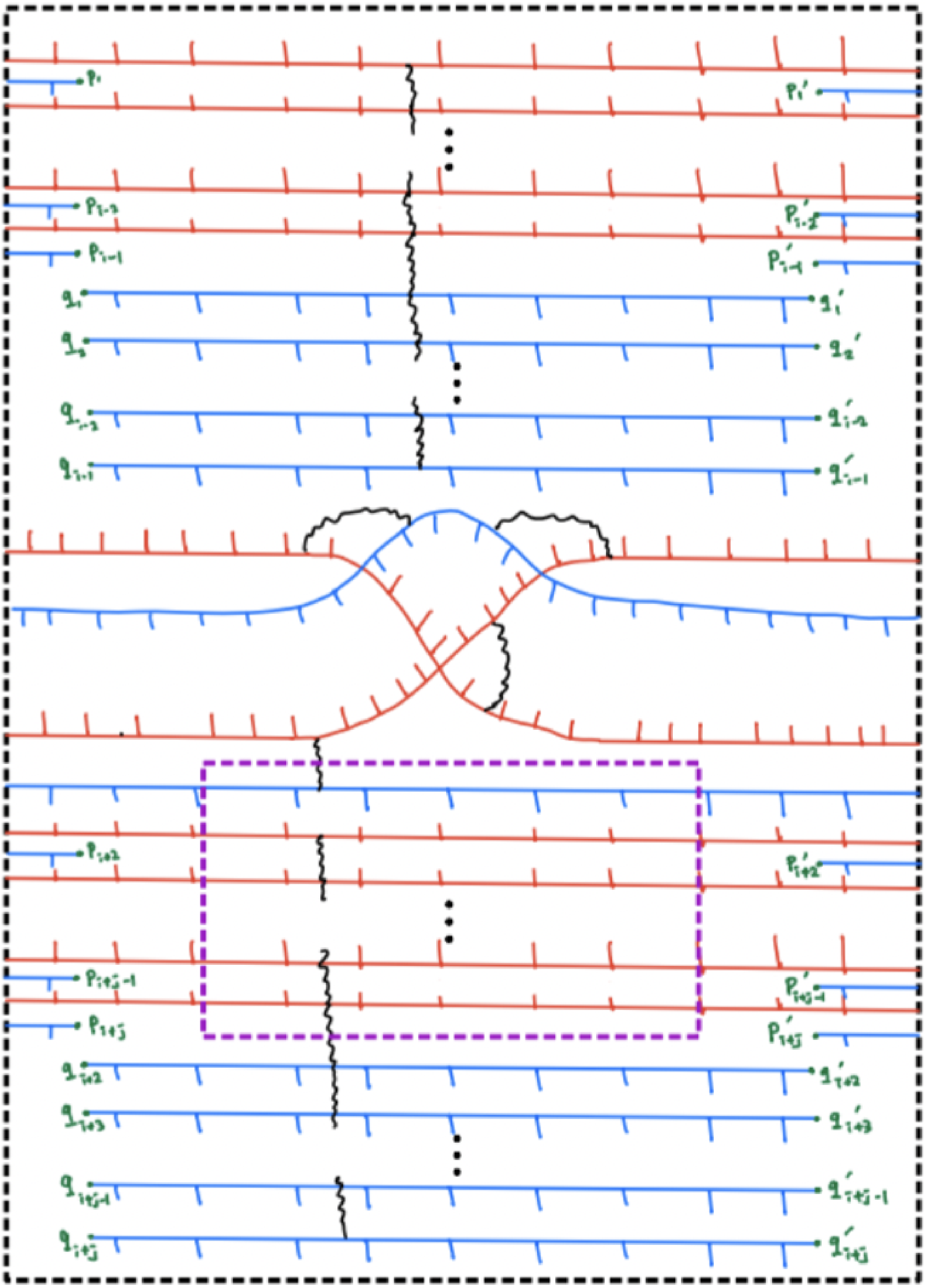


Figure 4.146: Your caption here

We get the following diagram:

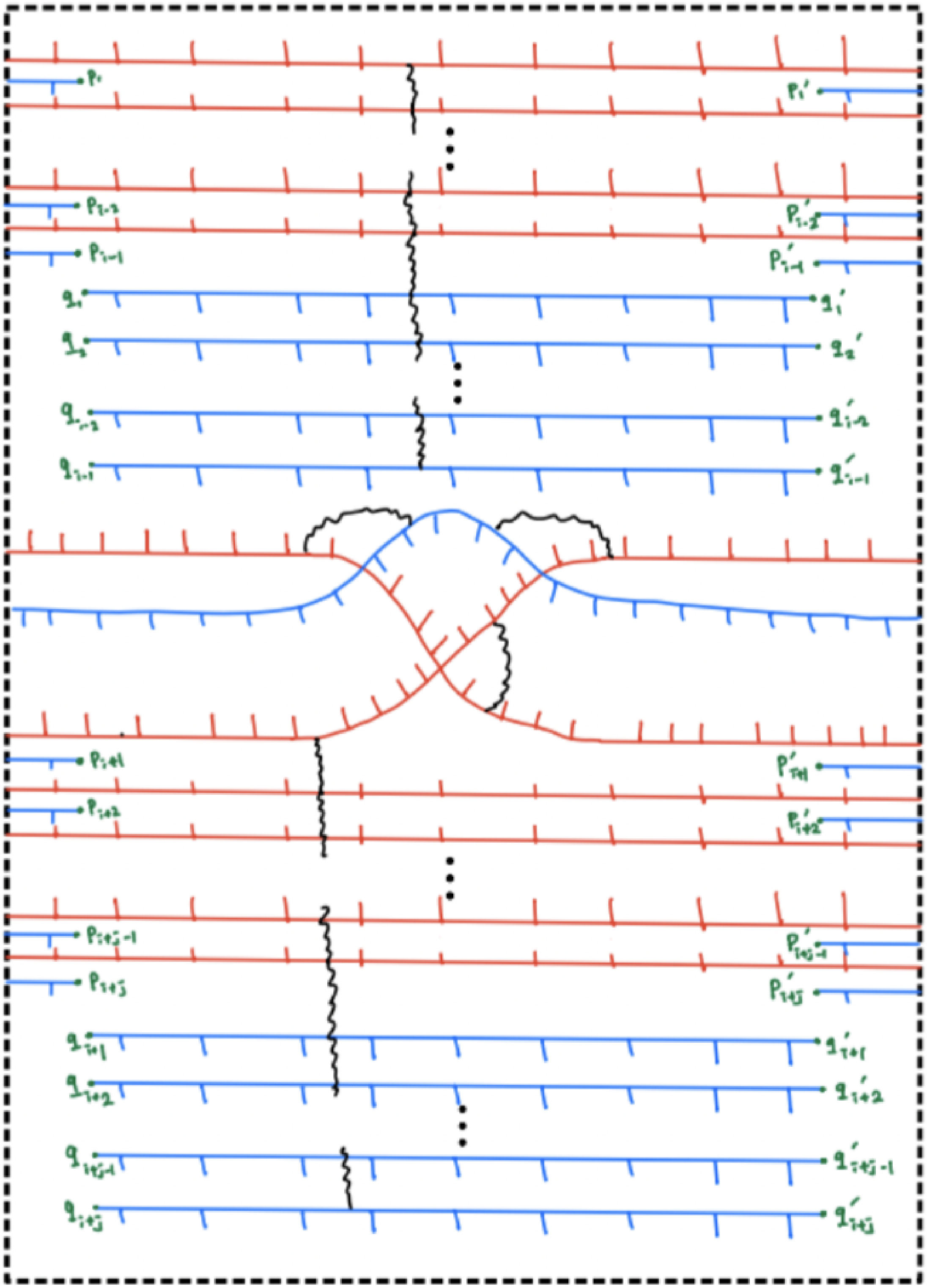


Figure 4.147: Your caption here

(Step5) Now we apply $Move_9$ to the disks surrounded by purple dotted lines:

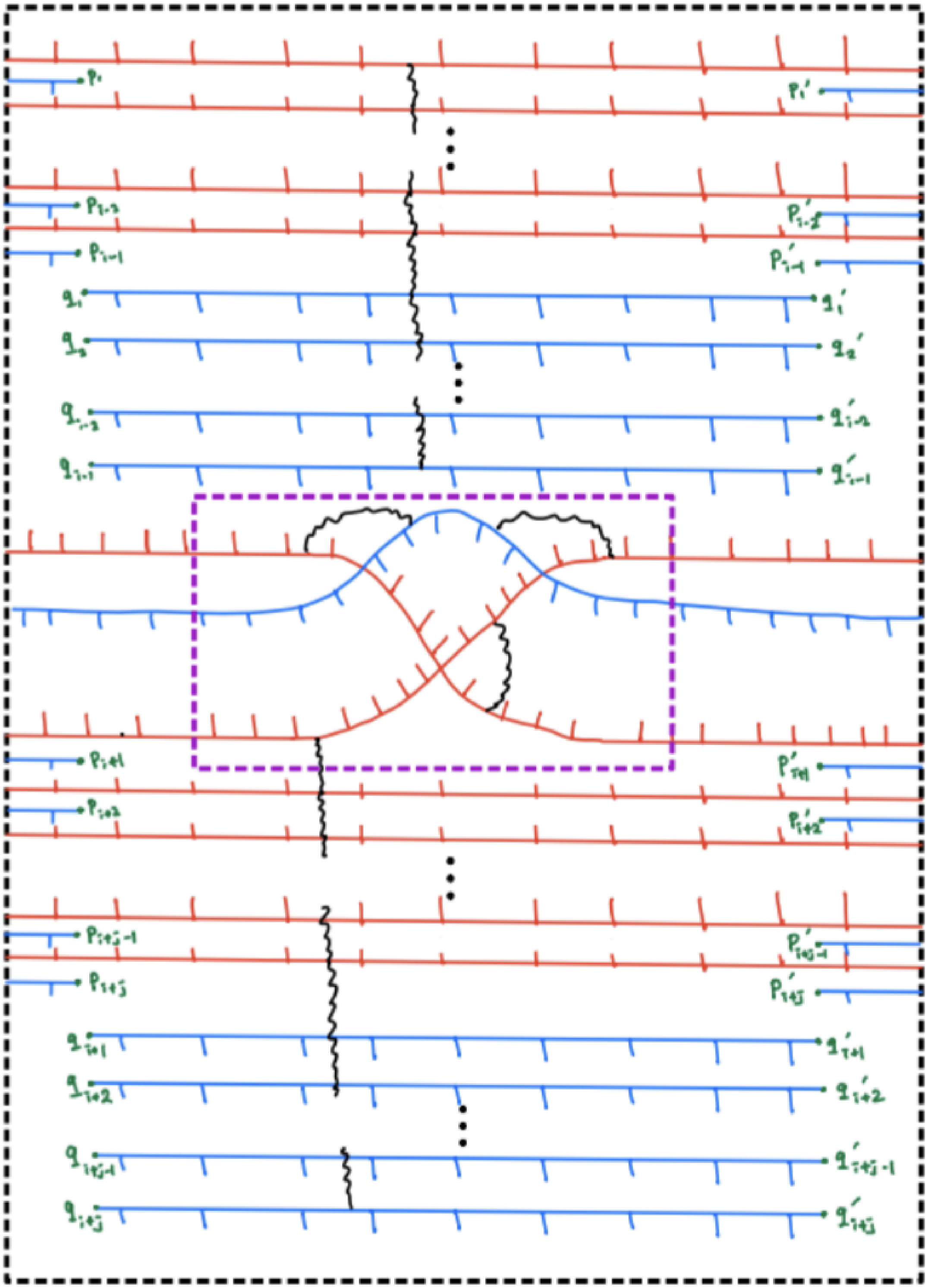


Figure 4.148: Your caption here

We get the following diagram:

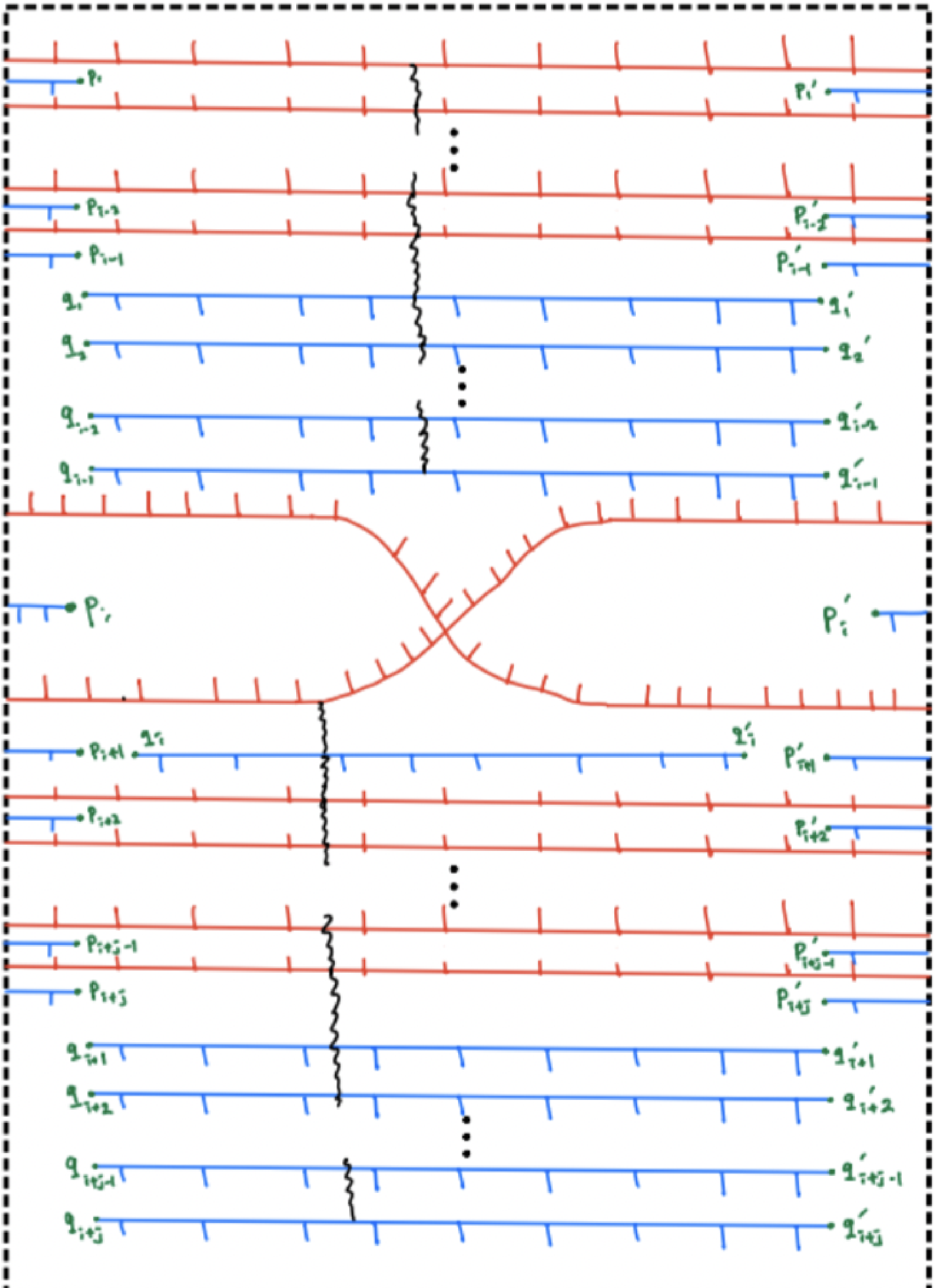


Figure 4.149: Your caption here

(Step6) Now we apply $Move_{11}$ to the disks surrounded by purple dotted lines:

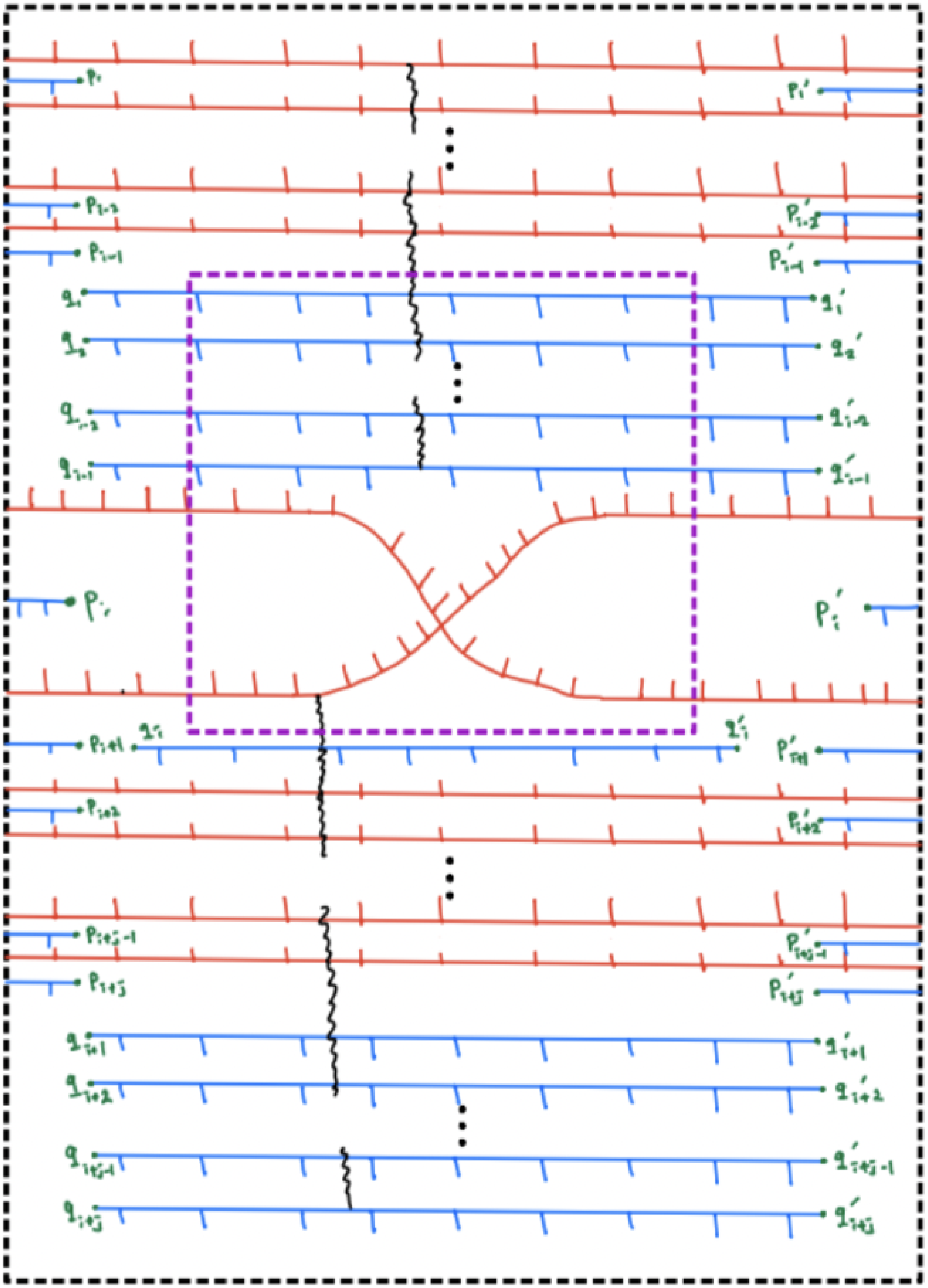


Figure 4.150: Your caption here

We get the following diagram:

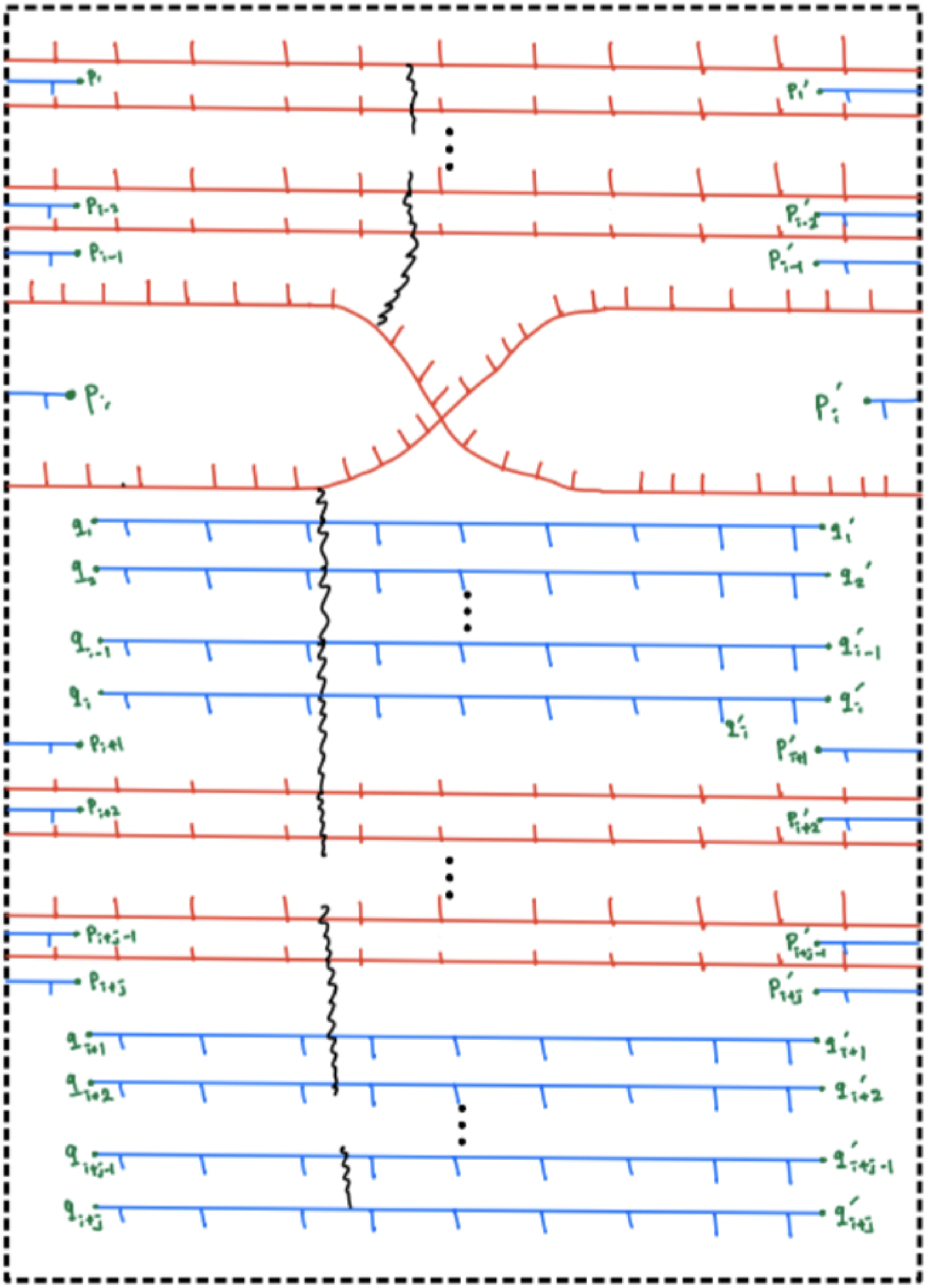


Figure 4.151: Your caption here

(Step7) Now we apply $Move_{7-(b)}$ to the disks surrounded by purple dotted lines:

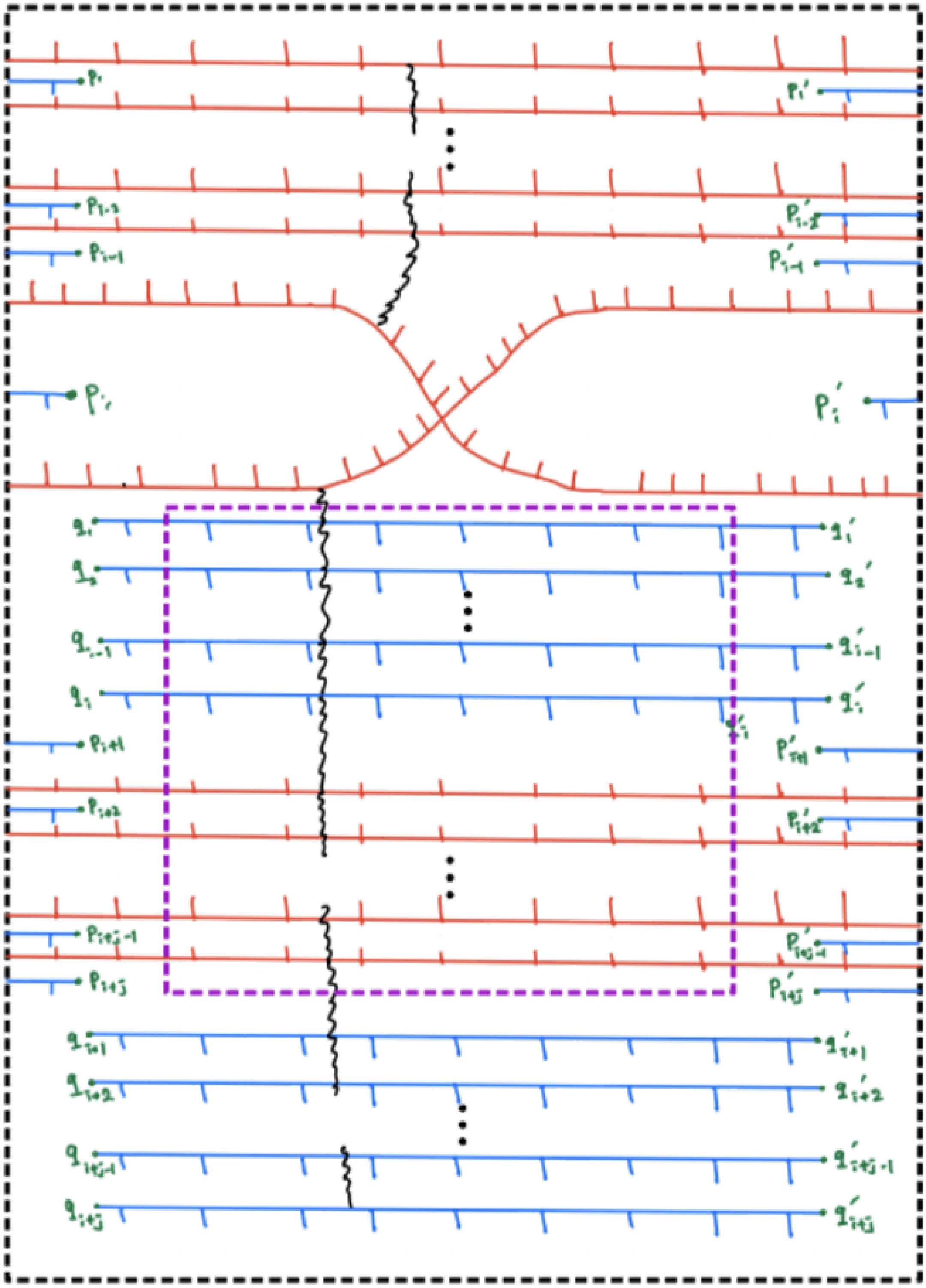


Figure 4.152: Your caption here

We get the final diagram:

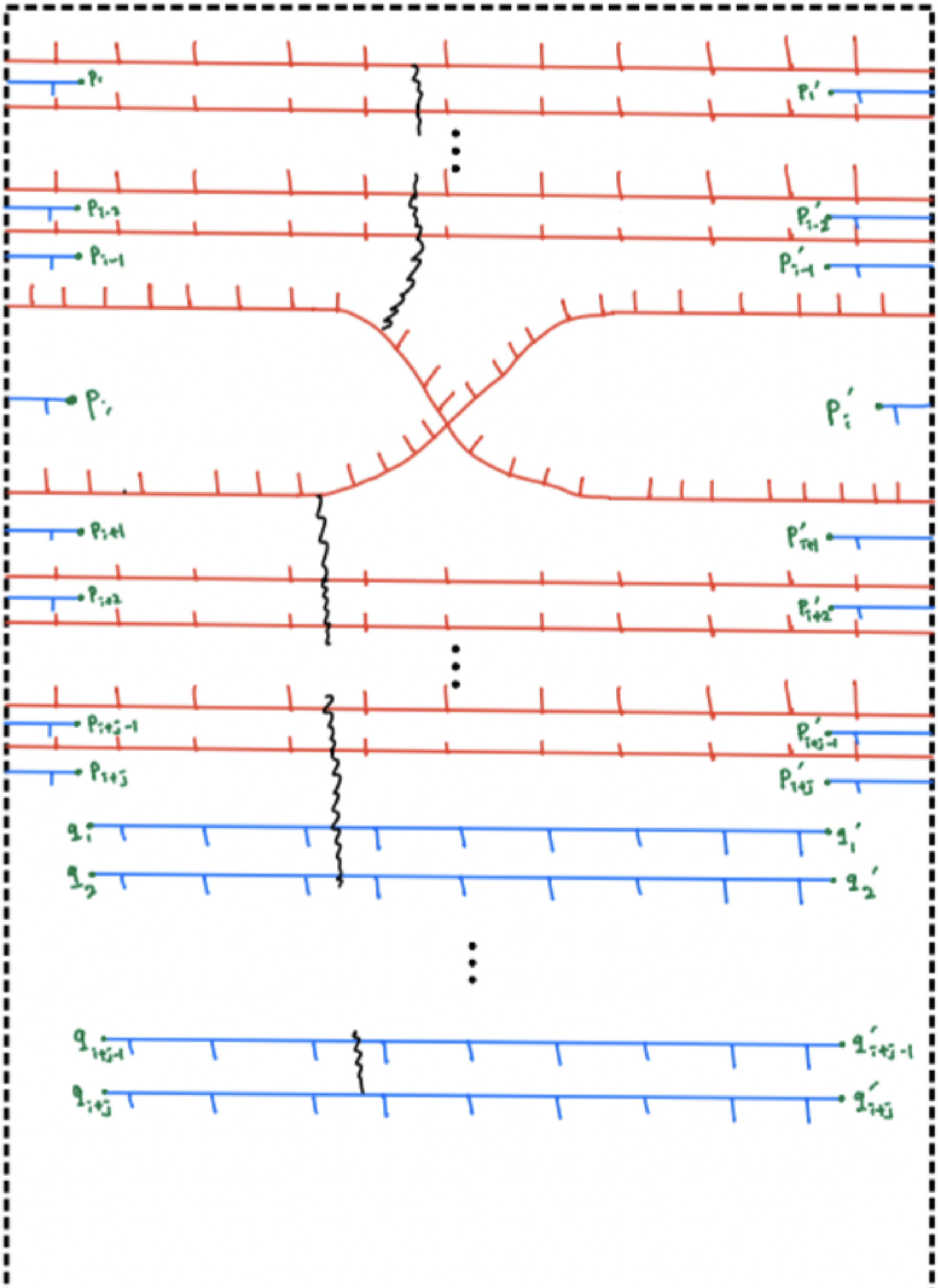


Figure 4.153: Your caption here

4.23 definition13

Definition 65.

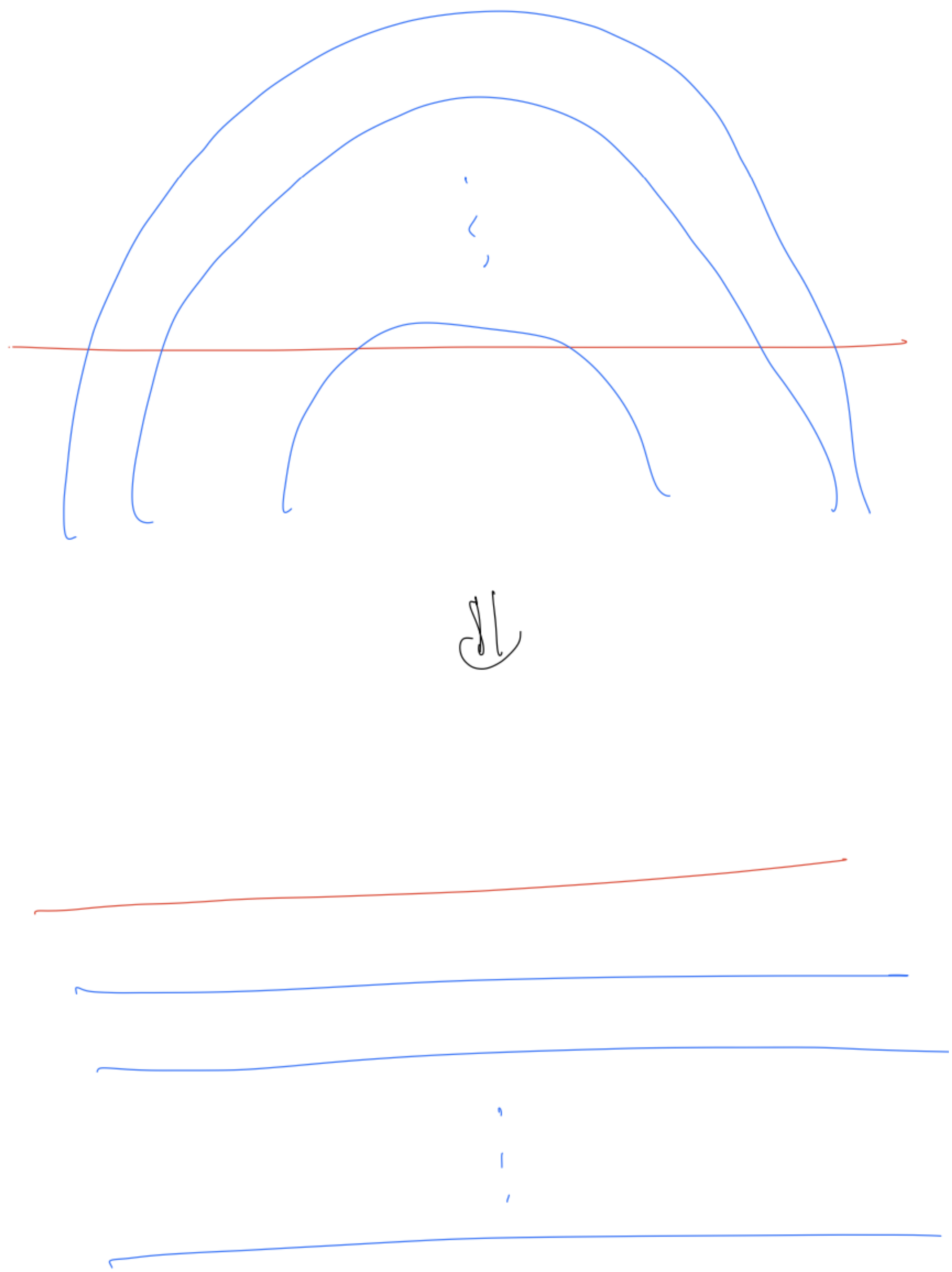


Figure 4.154: Your caption here

4.24 theorem13

Theorem 66.

Suppose we have the following local diagram and a sheaf on it :

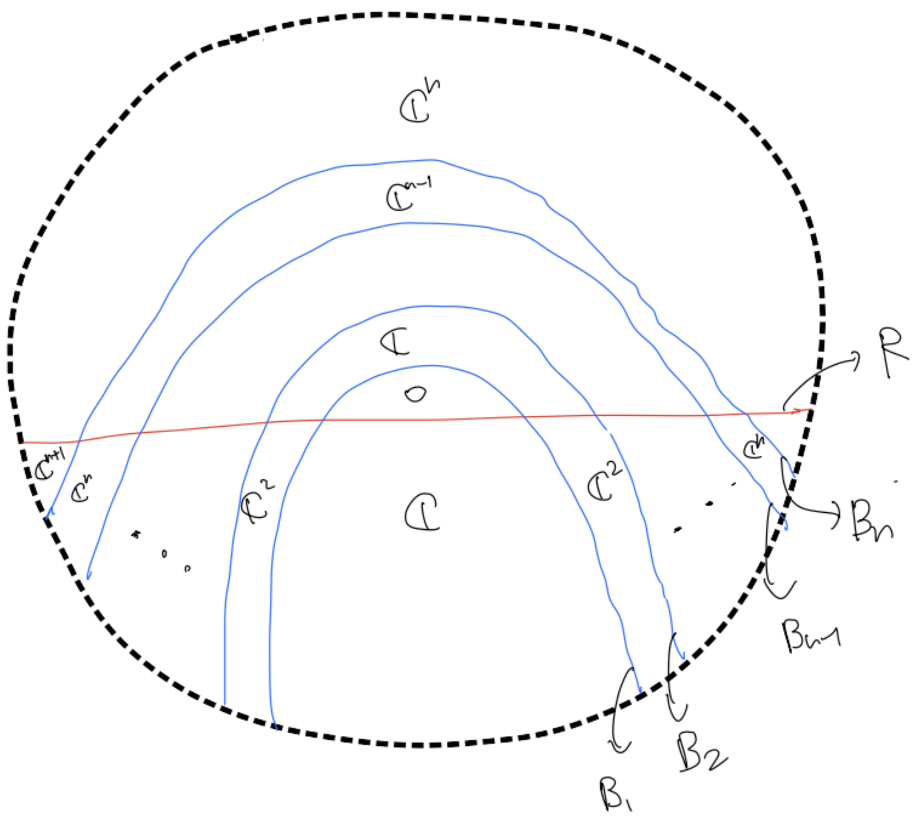


Figure 4.155: Your caption here

If we iteratively appy reverse Reidemeister ii$n$ times(in the i^{th} iteration we apply reverse Redemeister iito B_i and R), we get the following diagram and a sheaf :

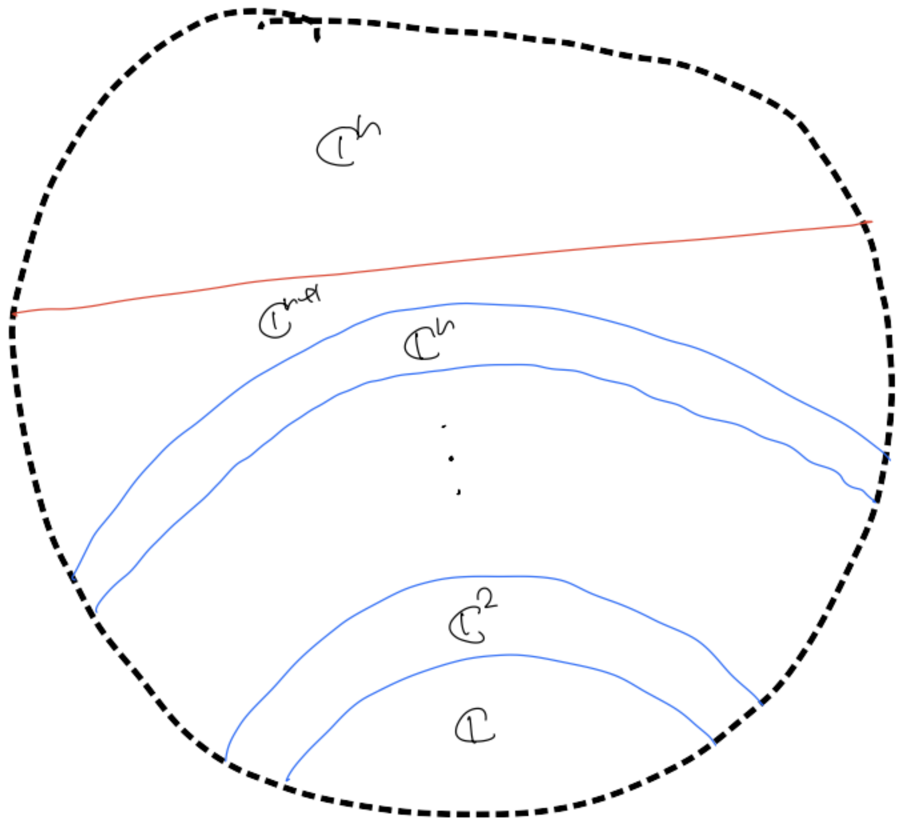


Figure 4.156: Your caption here

(proof) If $n = 1$, then this is just a single reverse Reidemeister iimove. If $n > 1$, then by the induction hypothesis, after applying $n - 1$ reverse Reidemeister iimoves, we get :

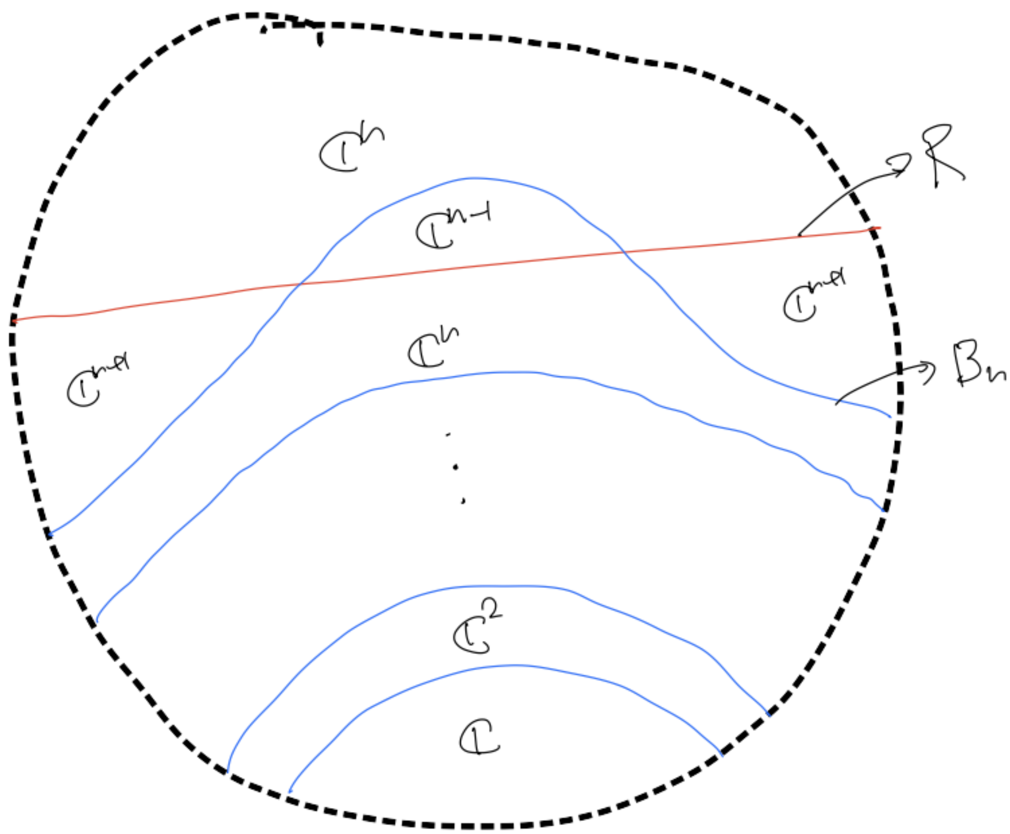


Figure 4.157: Your caption here

To the above diagram and the sheaf, we apply reverse Reidemeister ii(on R and B_n), we get the final diagram:

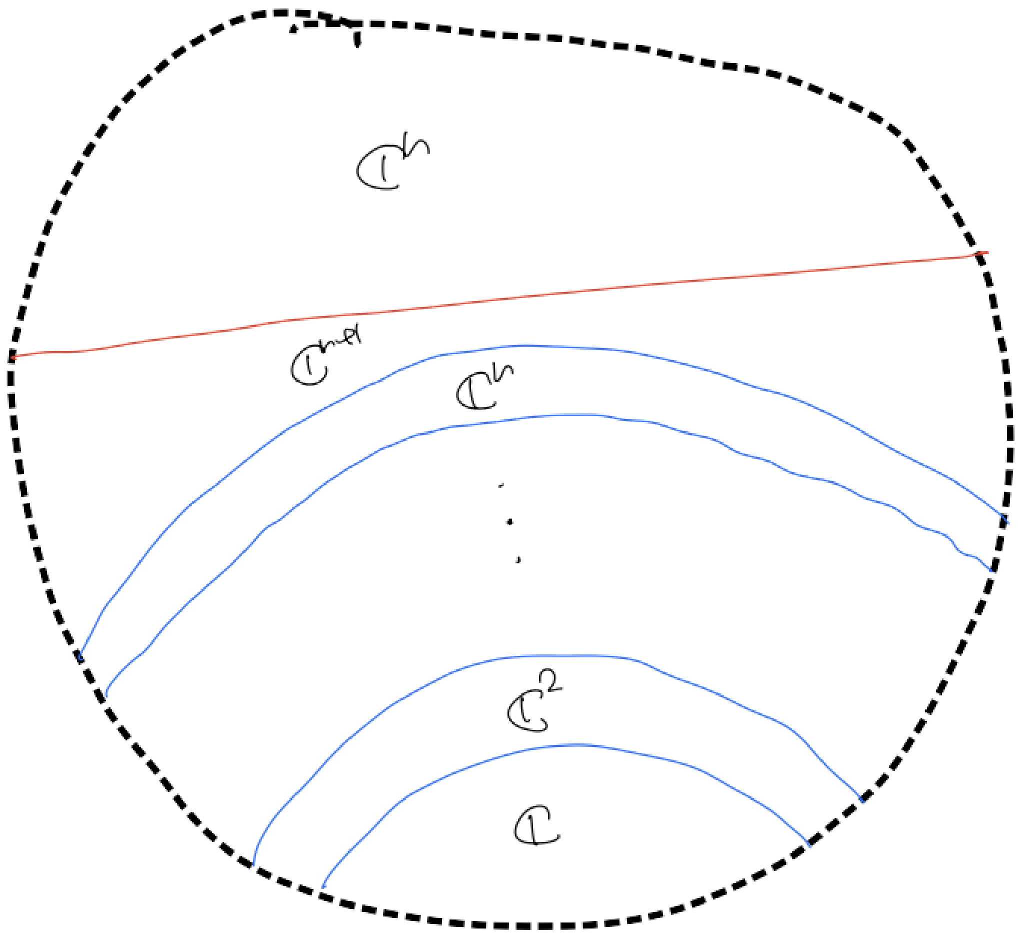


Figure 4.158: Your caption here

4.25 definition14-1(intergenerator diagram)

Definition 67.

For every $n \in \mathbb{N}$, we define a local braid diagram inductively as follows: We will call this braid diagram as inter-generator diagram on n strands. It is a diagram that fits into the smaller disk in the figure below. It has n red strands and n blue strands where n red strands start at $p_1 - p_n$ and end at $p'_1 - p'_n$, n blue strands start at $q_1 - q_n$ and end at $q'_1 - q'_n$.

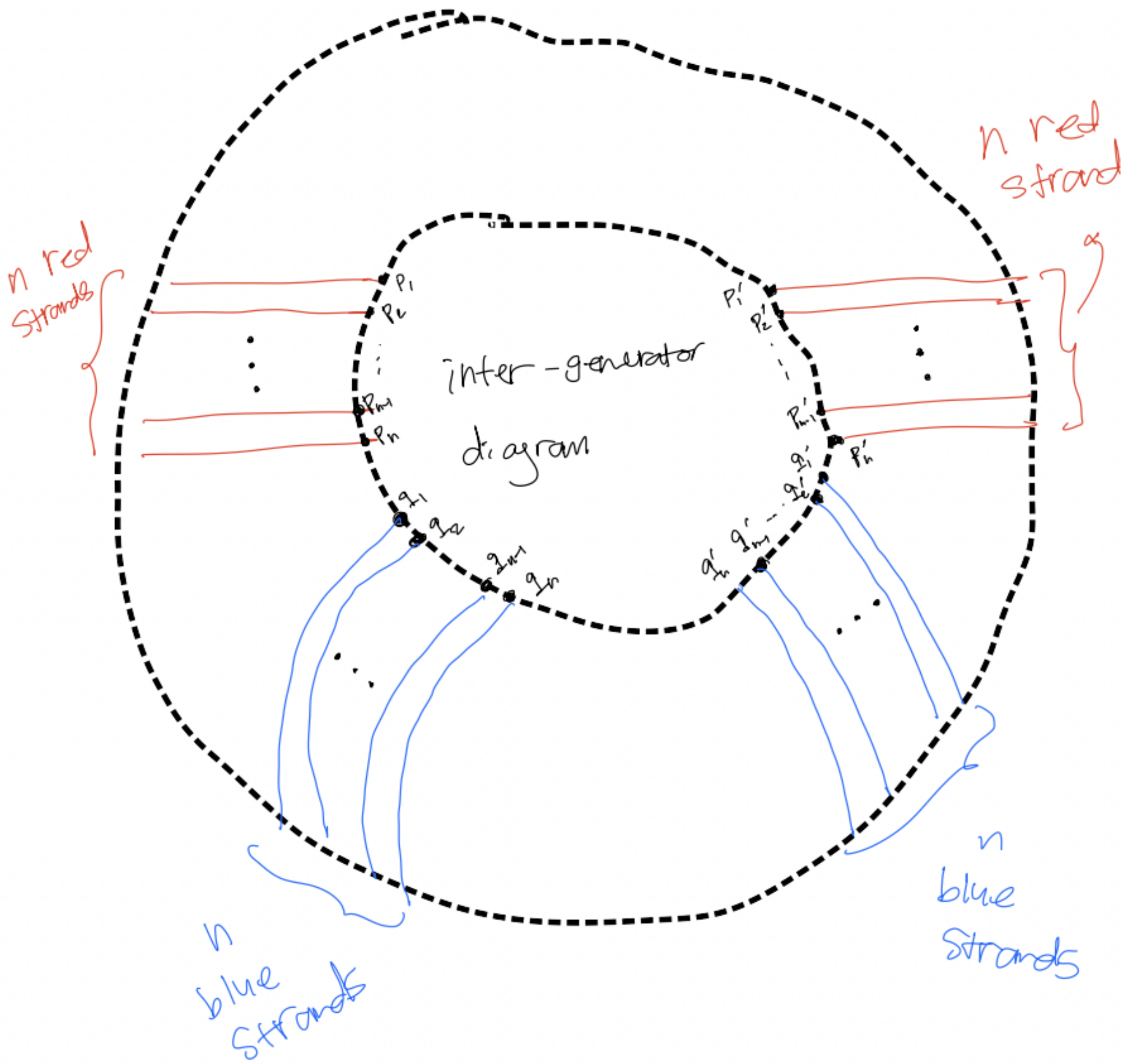


Figure 4.159: Your caption here

If $n = 1$, then the diagram is

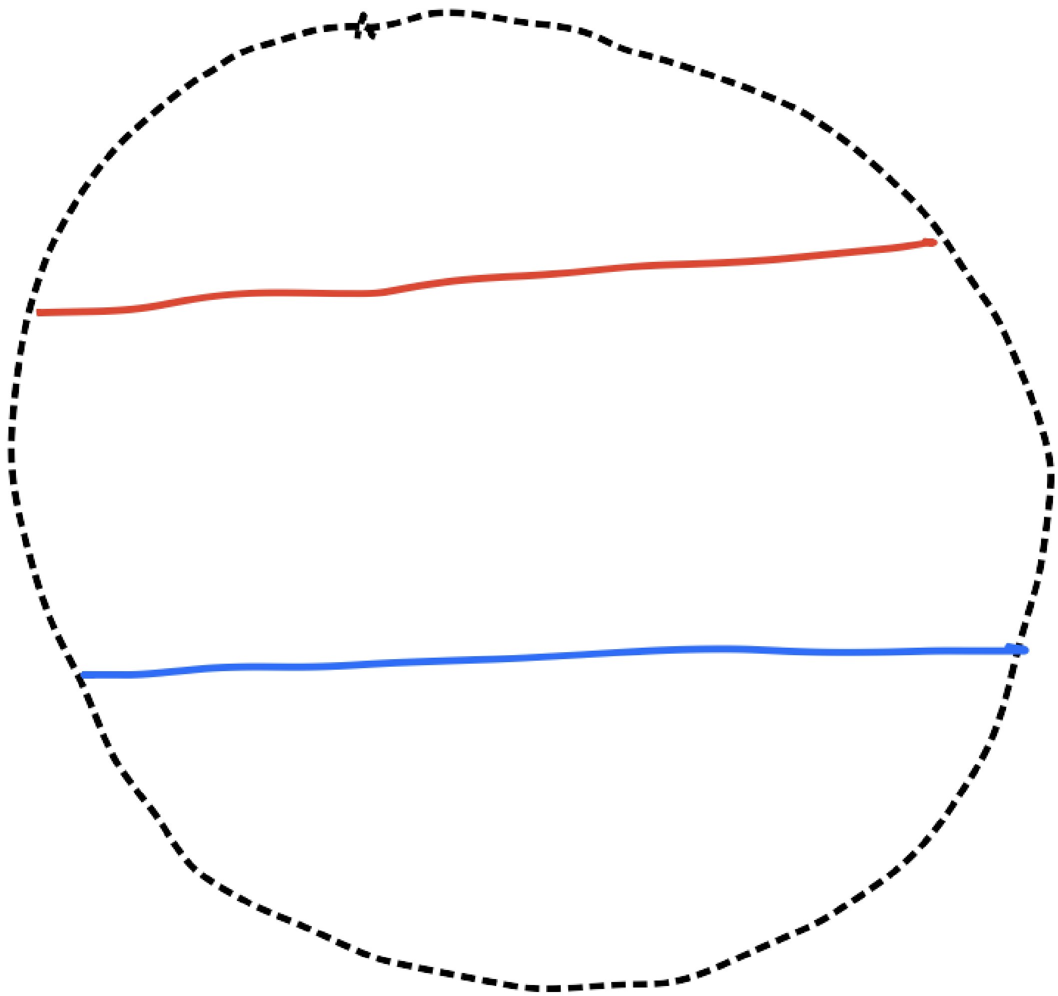


Figure 4.160: Your caption here

Suppose we have defined inter-generator diagram up to $n-1$ strands. We define intergenerator diagram on n strands as follows. To the following diagram :

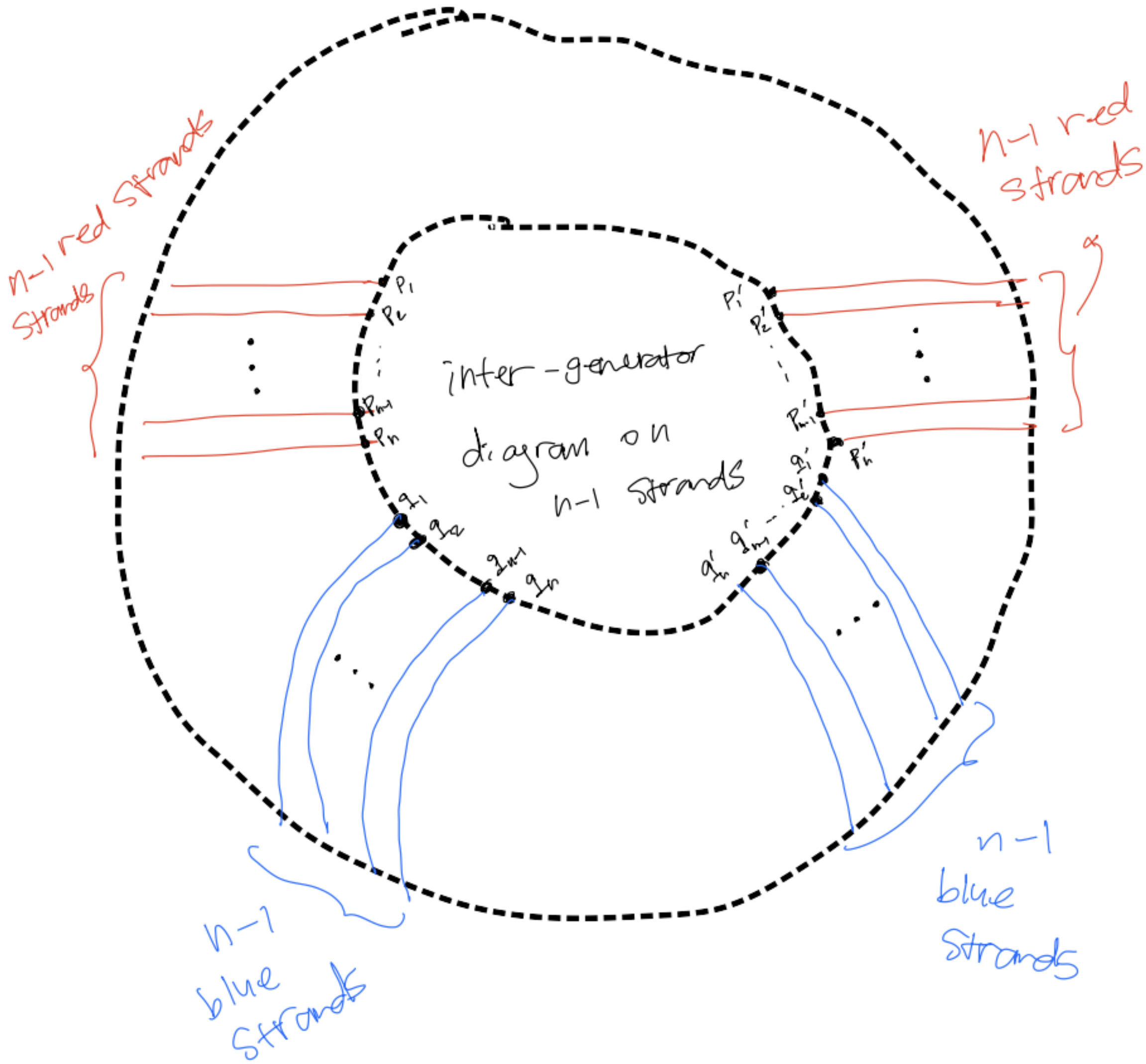


Figure 4.161: Your caption here

we add a red strand and a blue strand as follows(drawn in thick lines)

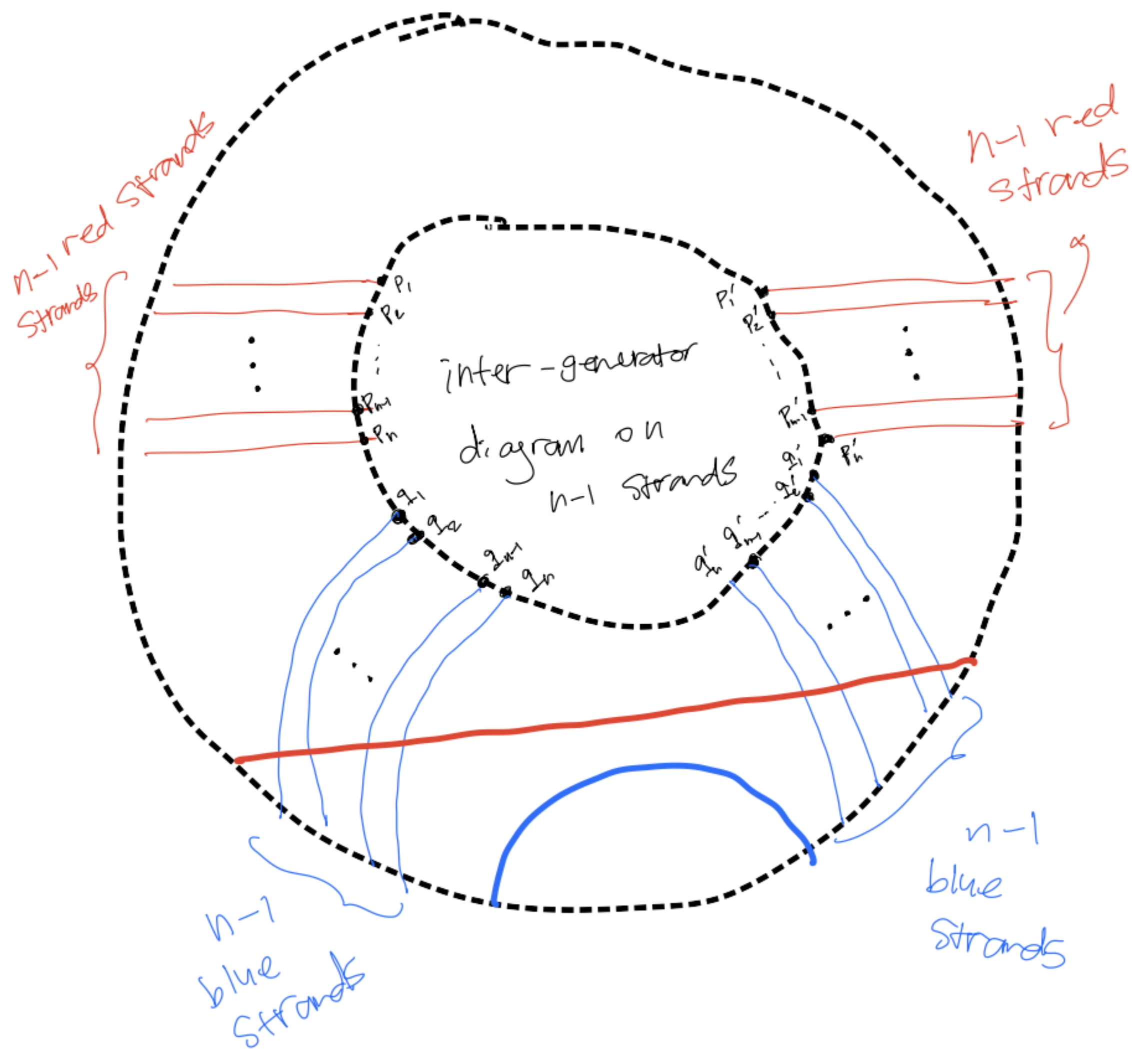


Figure 4.162: Your caption here

NEw red strand is parallel to other red strands and crosses $n-1$ blue strand left and right. New blue strand does not cross any other lines.

4.26 definition14-2(intergenerator move)

Definition 68.

Suppose we have an inter-generator diagram on n strands. Then we define a collection of moves, called inter-generator move, inductively so that the final diagram looks as follows:

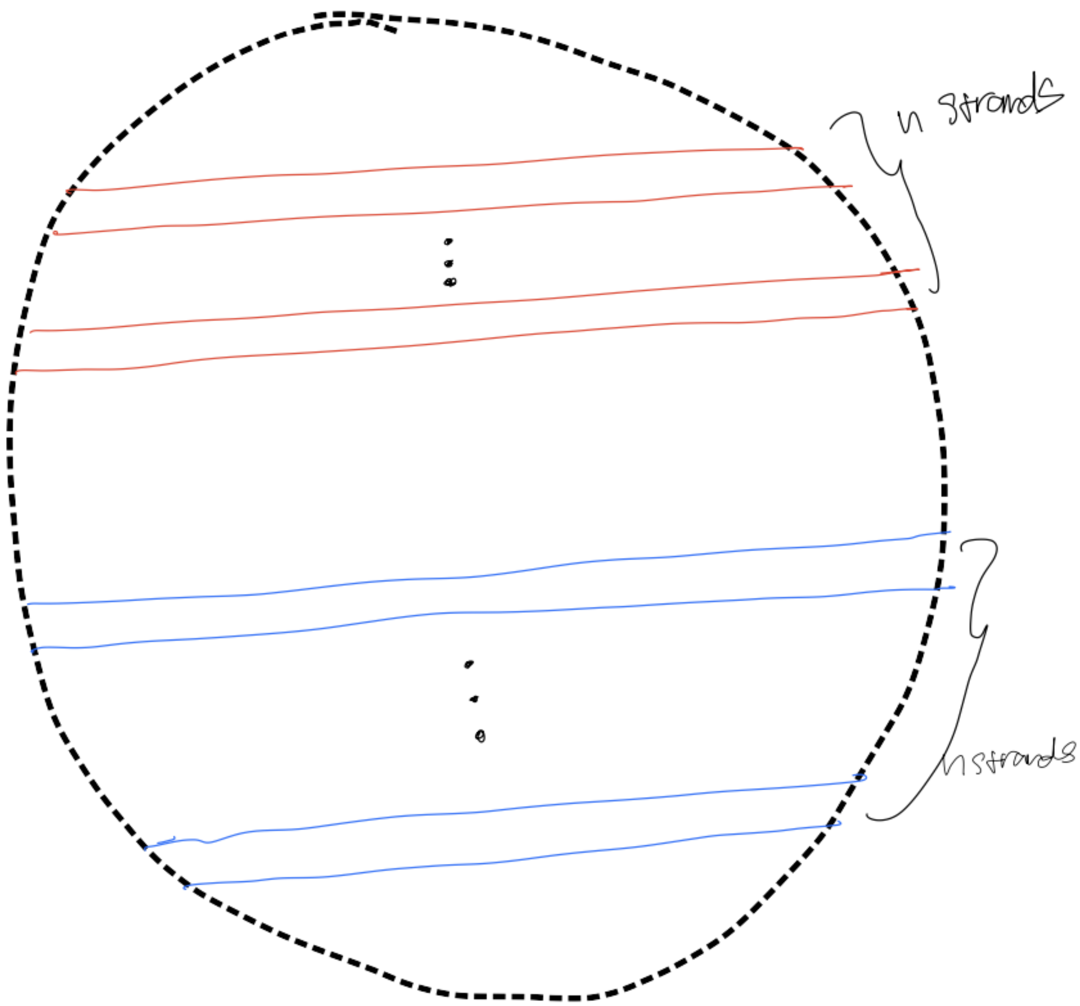


Figure 4.163: Your caption here

If $n = 1$, then intergenerator move is the null move. Suppose $n > 1$ and we have defined intergenerator move up to $n-1$ strands. Suppose we have the following inter-generator diagram on n strands :

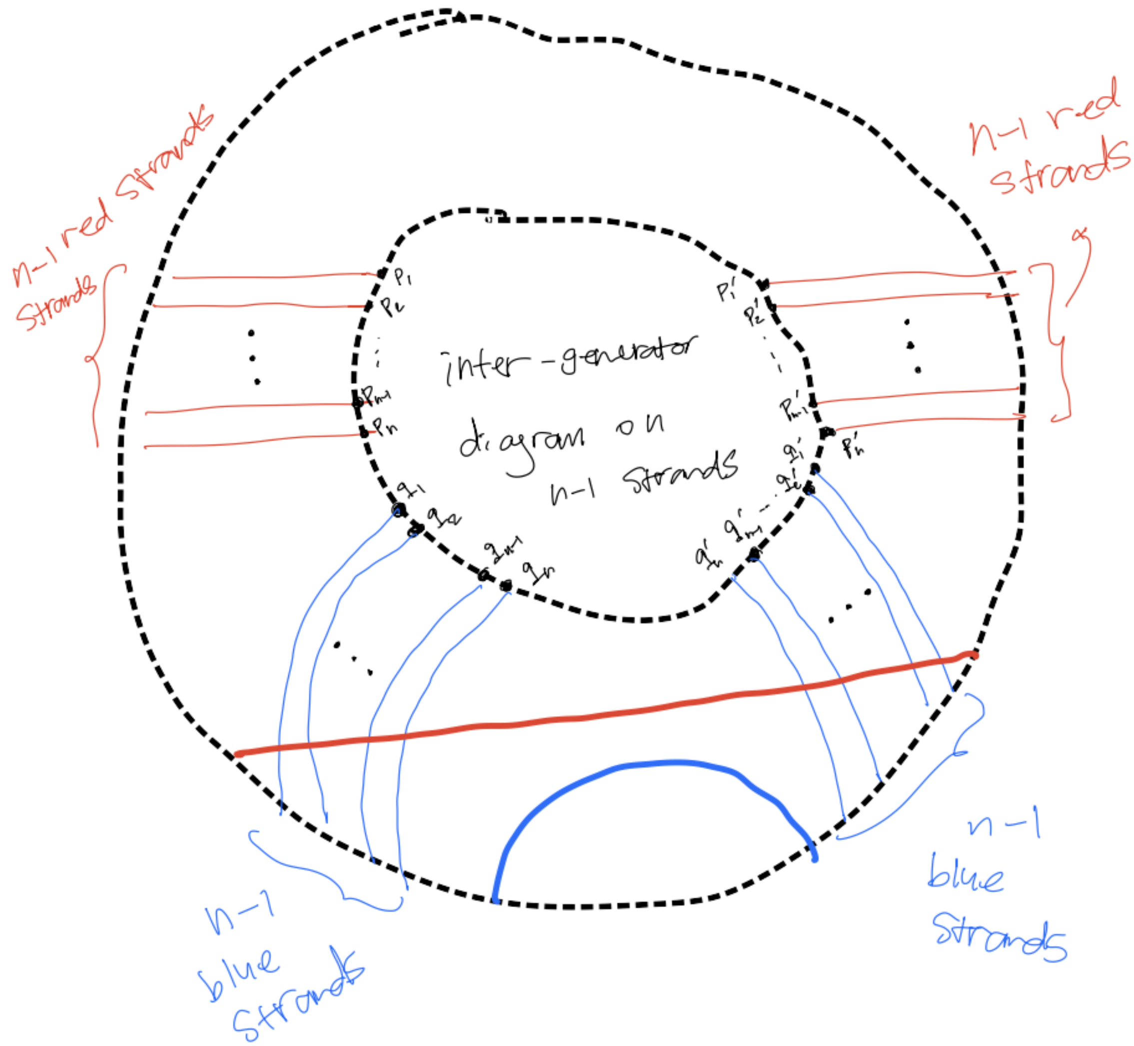


Figure 4.164: Your caption here

(Step1) Apply intergenerator move to the intergenerator diagram on $n-1$ strands inside the smaller disk we get

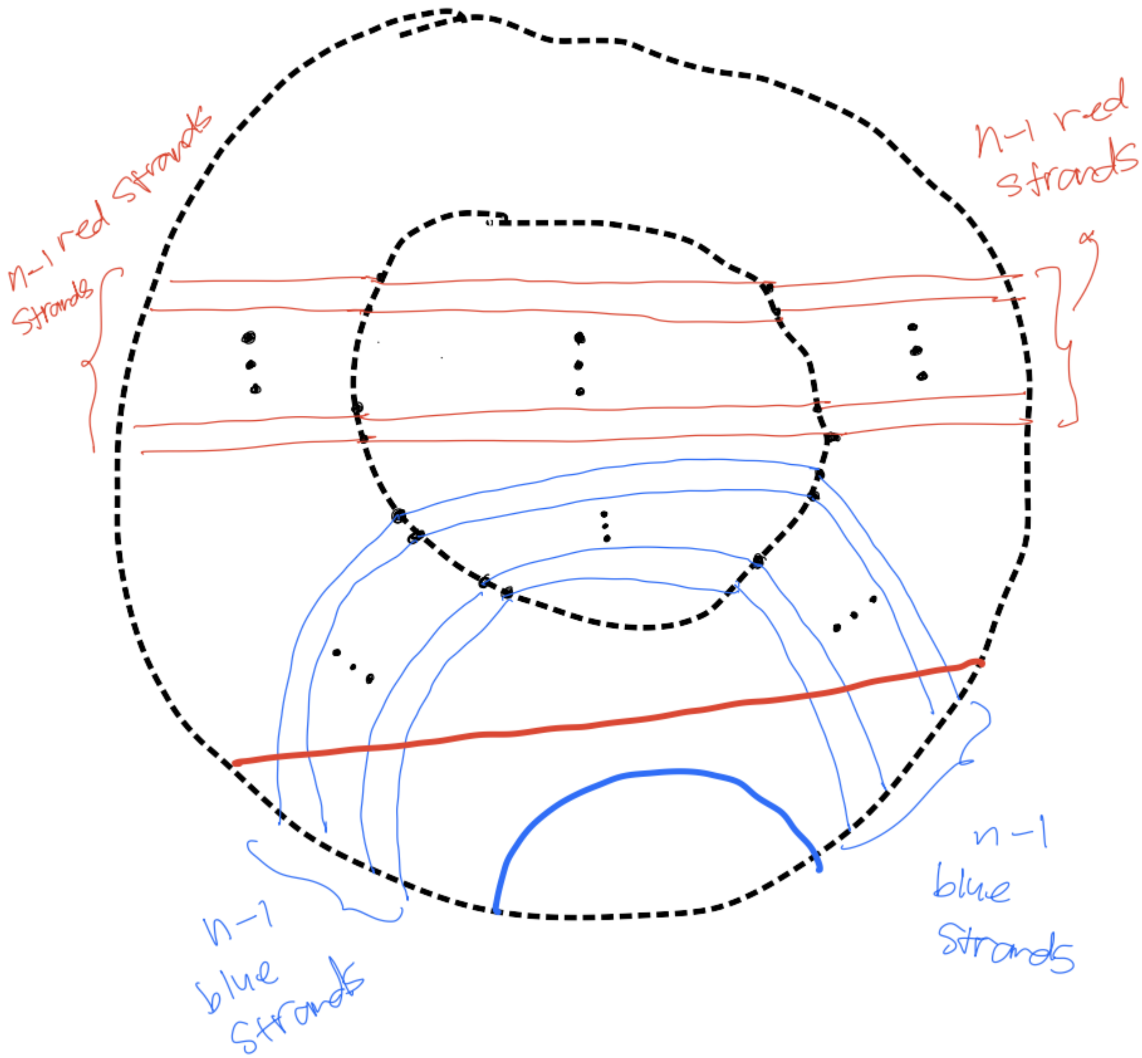


Figure 4.165: Your caption here

(Step2) Apply MOVE vi(successive application of reverse Reidemeister moves) inside the region surrounded by the purple circle

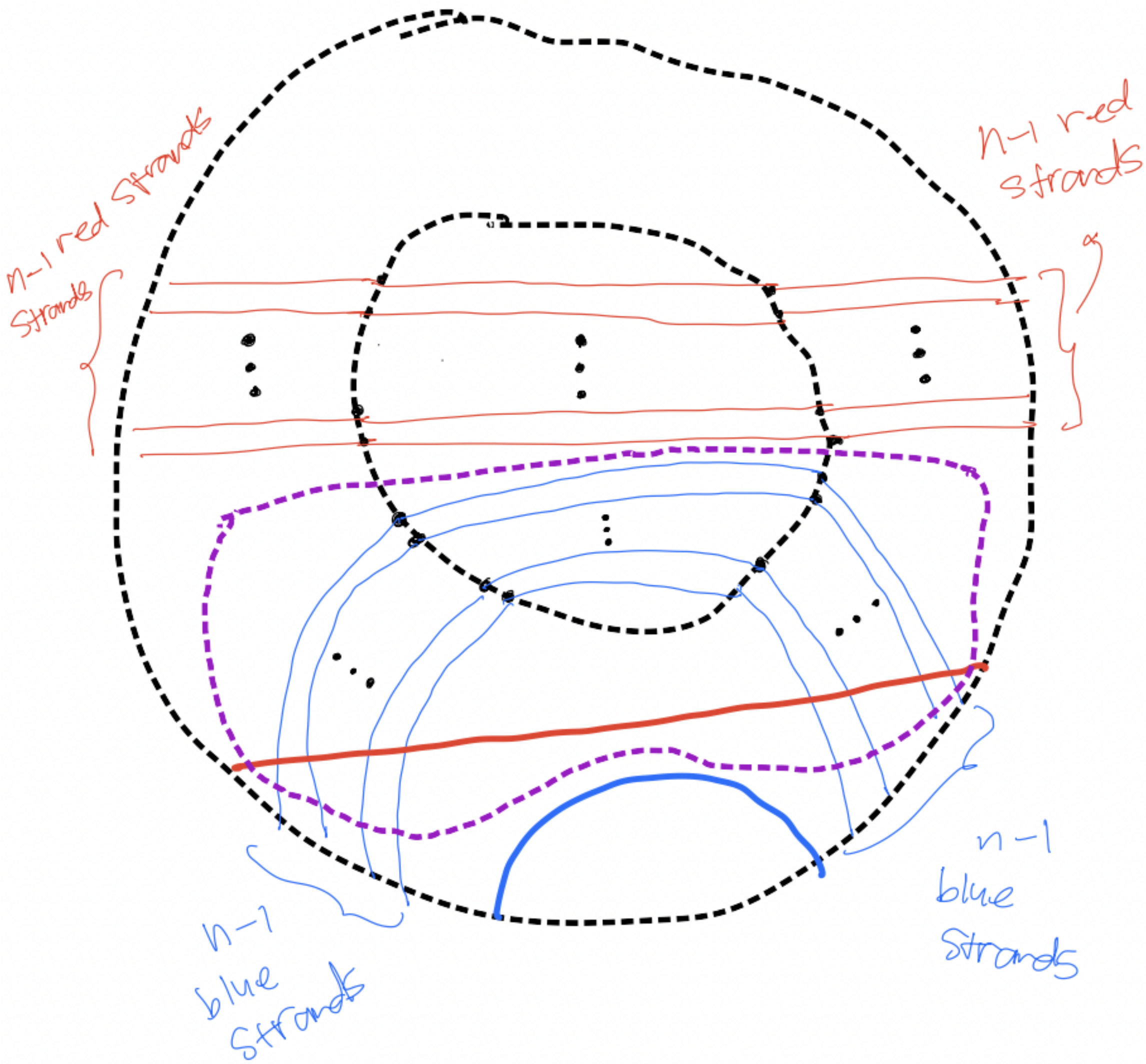


Figure 4.166: Your caption here

we get :

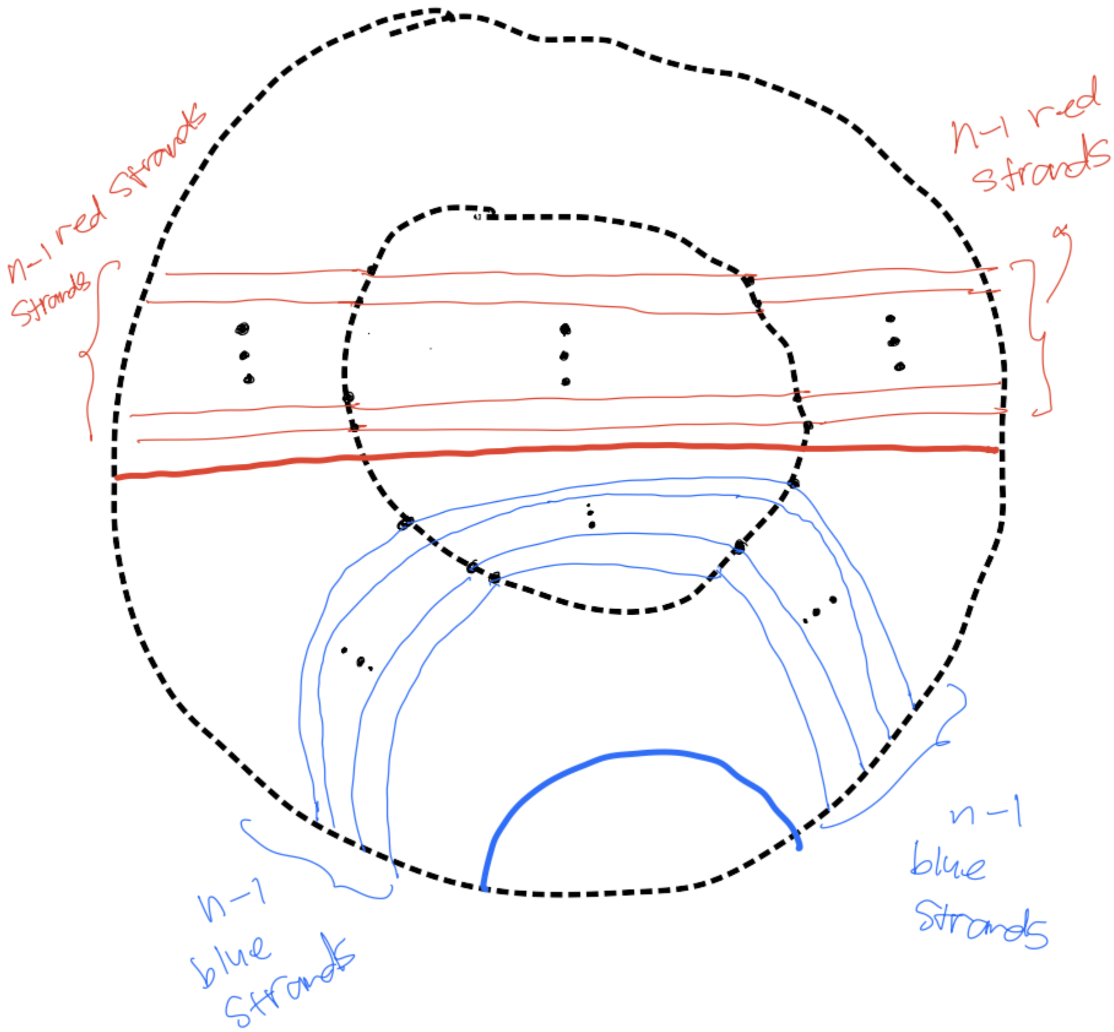


Figure 4.167: Your caption here

4.27 definition14-3(intergenerator sheaf)

Definition 69.

We define a special sheaf, called intergenerator sheaf, a intergeneram on n strands inductively as follows: Suppose $n = 1$, then

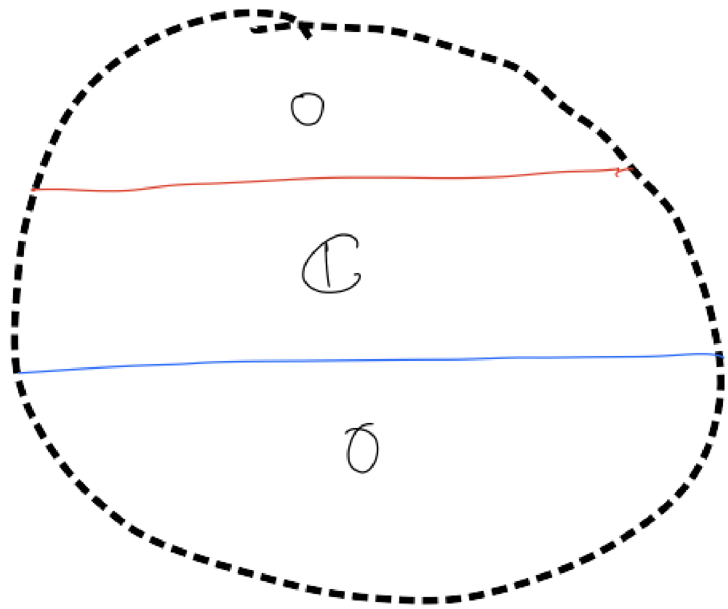


Figure 4.168: Your caption here

Suppose $n >!$, then intergenerator sheaf is the unique sheaf

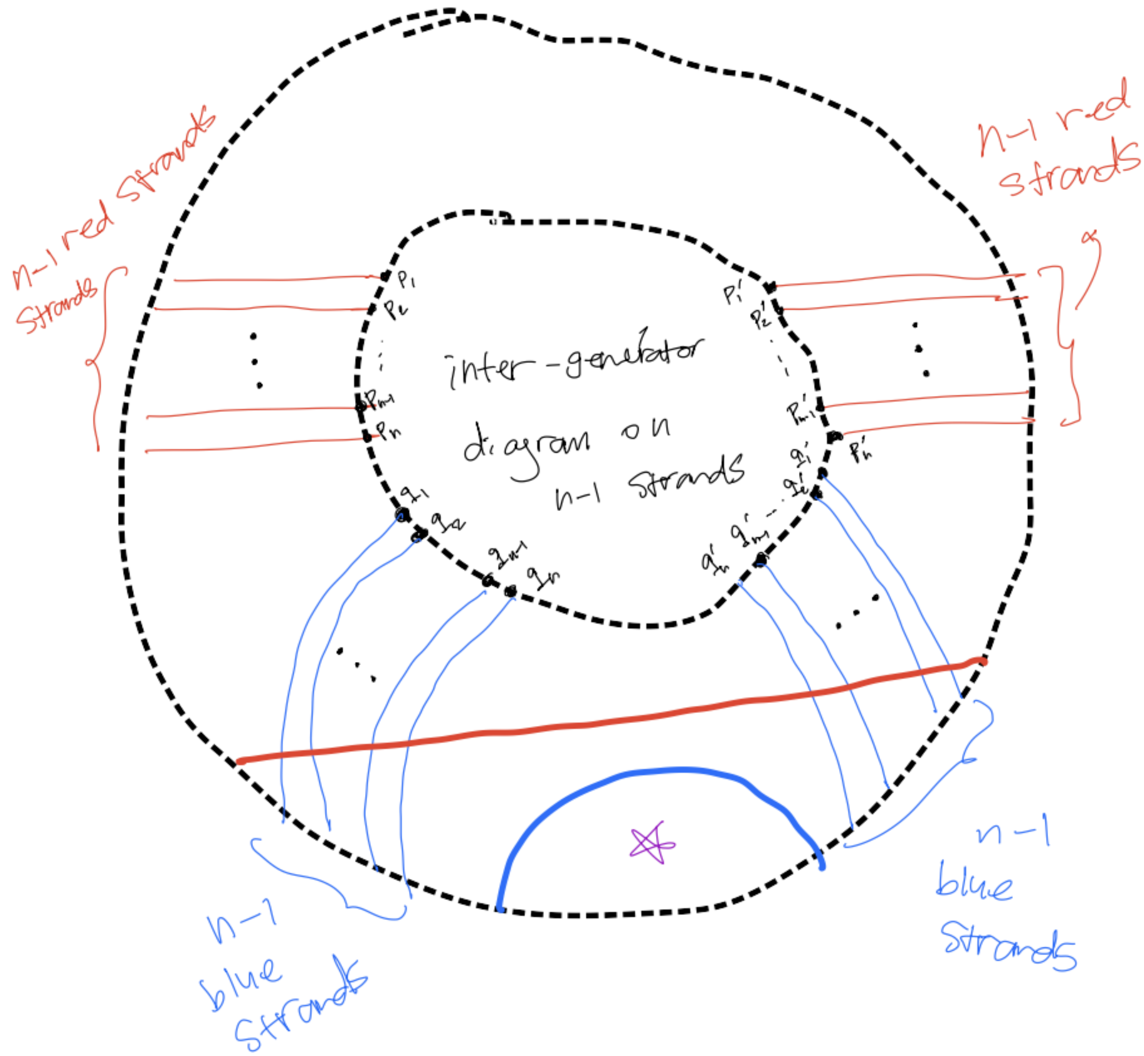


Figure 4.169: Your caption here

extending the intergenerator sheaf on the intergenerator diagram on $n-1$ strands with vanishing stalk at the region containing the purple star.

(proof of well-definedness) Note that in the following diagram:

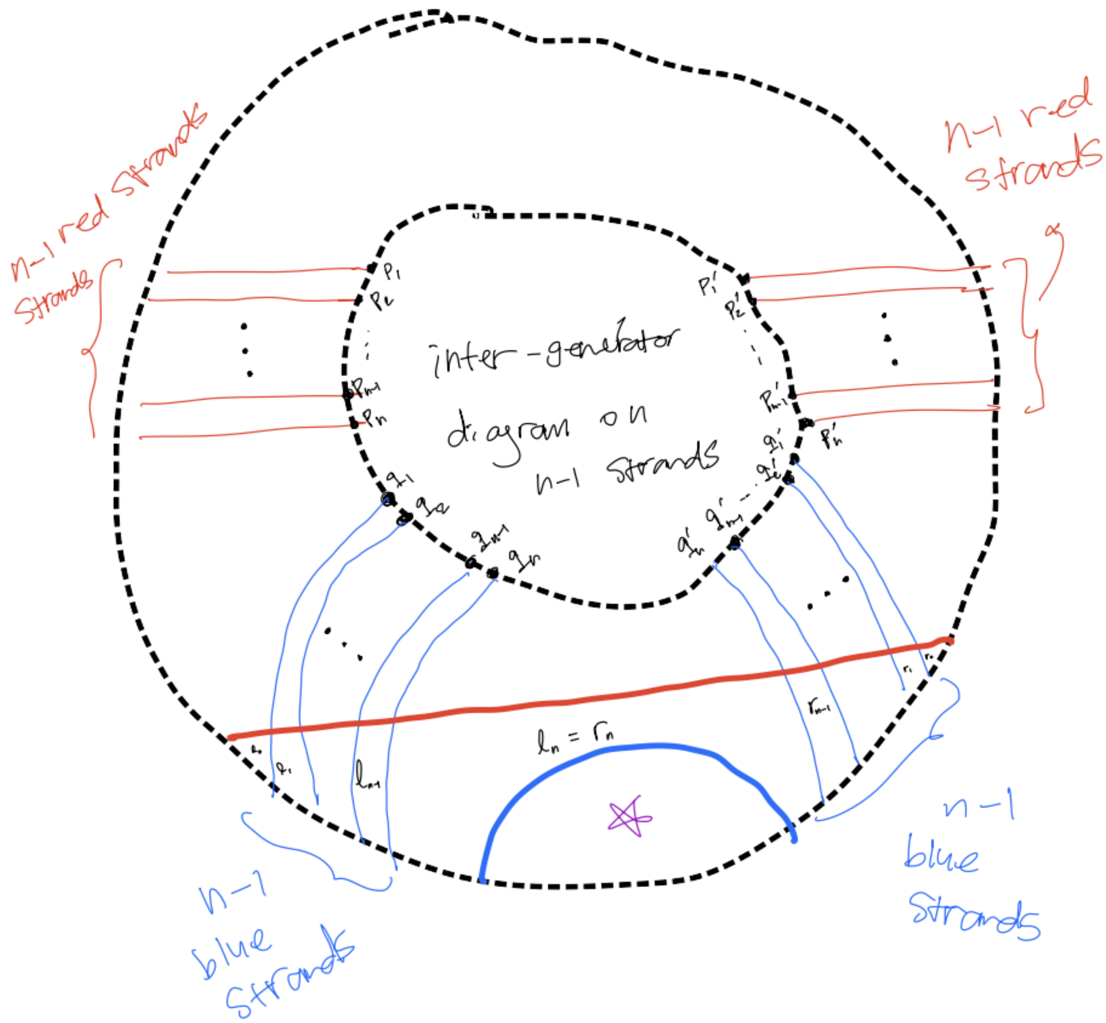


Figure 4.170: Your caption here

The regions except $l_0 = r_0, l_1 - l_n, r_1 - r_n$ are connected to the regions of the intergenerator diagram on $n-1$ strands. Therefore, the stalks and the generization maps between those regions are determined. Furthermore, because the sheaf is of microlocal rank one, the stalk at $l_n = r_n$ is quasi-isomorphic to $\mathbb{C}$ and the stalk at the region bordering $l_n = r_n$ with n^{th} red strand is 0. Therefore, the stalks and generization maps between regions except $l_0 - l_{n-1}$ and $r_0 - r_{n-1}$ are determined. Now using the crossing condition to determine the stalks of $l_0 - l_{n-1}, r_0 - r_{n-1}$ by induction. Suppose the stalks and generization maps between regions except $l_0 - l_k$ and $r_0 - r_k$ are determined. Then by the crossing conditions of the crossings of the $k+1^{th}$ blue strand and the $n+1^{th}$ strand determine the stalks of l_k and r_k and maps into them. Therefore, the proof is complete.

4.28 theorem14(intergenerator theorem)

Theorem 70.

Suppose we have an intergenerator diagram on n strands and an intergenerator sheaf on it. If we apply intergenerator move to the diagram, then we get the following diagram and a sheaf :

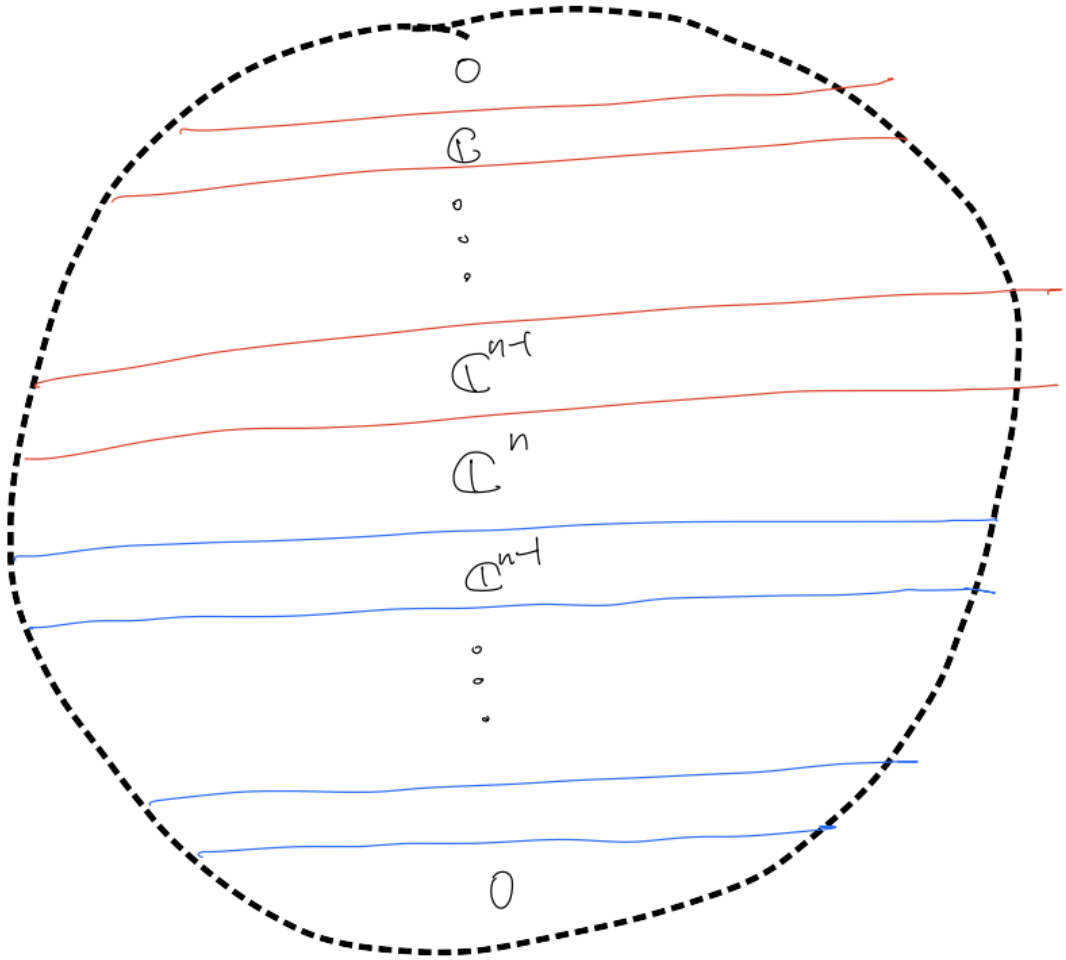


Figure 4.171: Your caption here

(proof) We prove the statement by induction. If $n = 1$, then the intergenerator move is the null move. So the statement holds trivially. If $n > 1$, then by induction hypothesis, after applying intergenerator move to intergenerator diagram on $n-1$ strands inside the middle circle

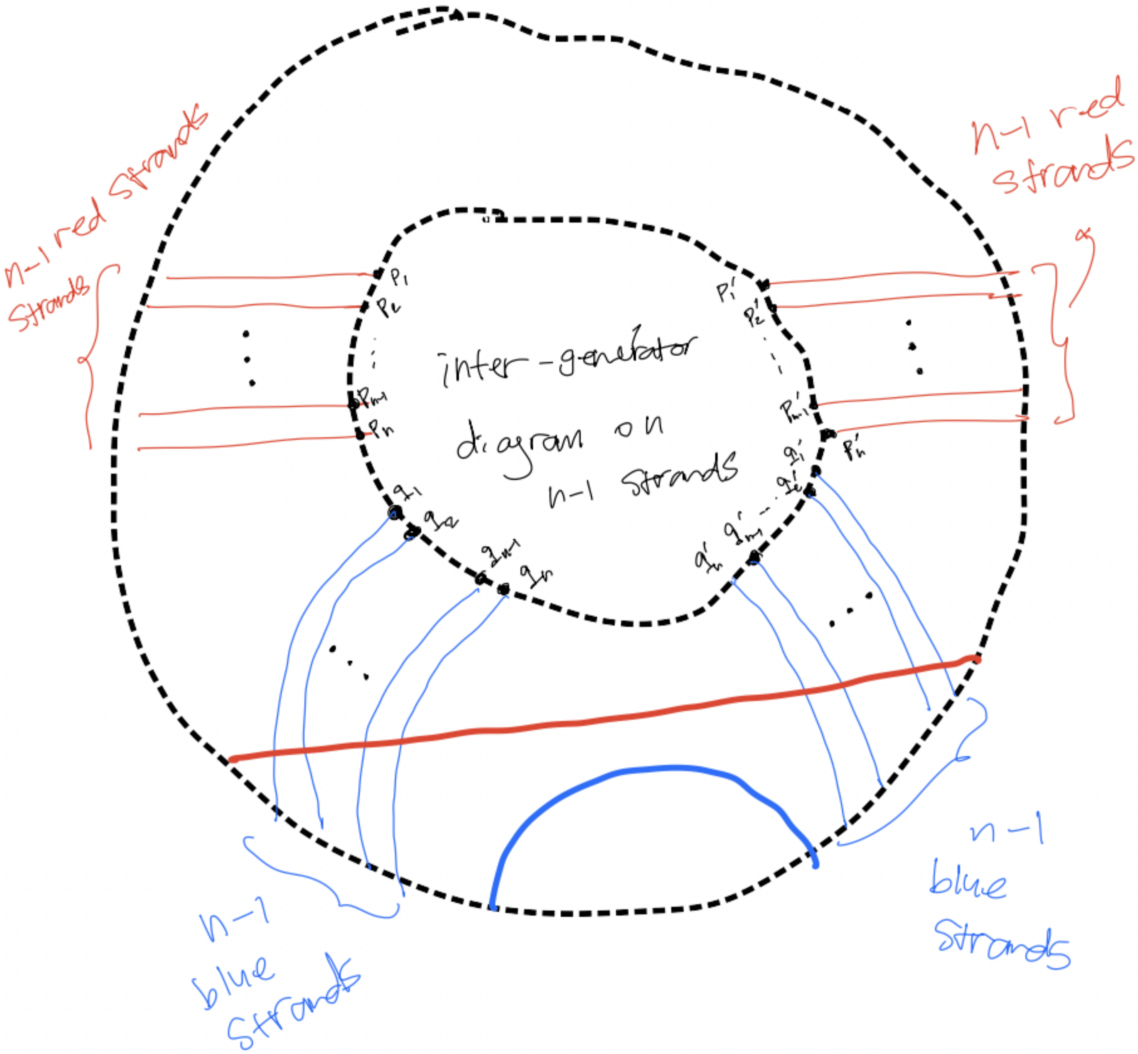


Figure 4.172: Your caption here

we get :

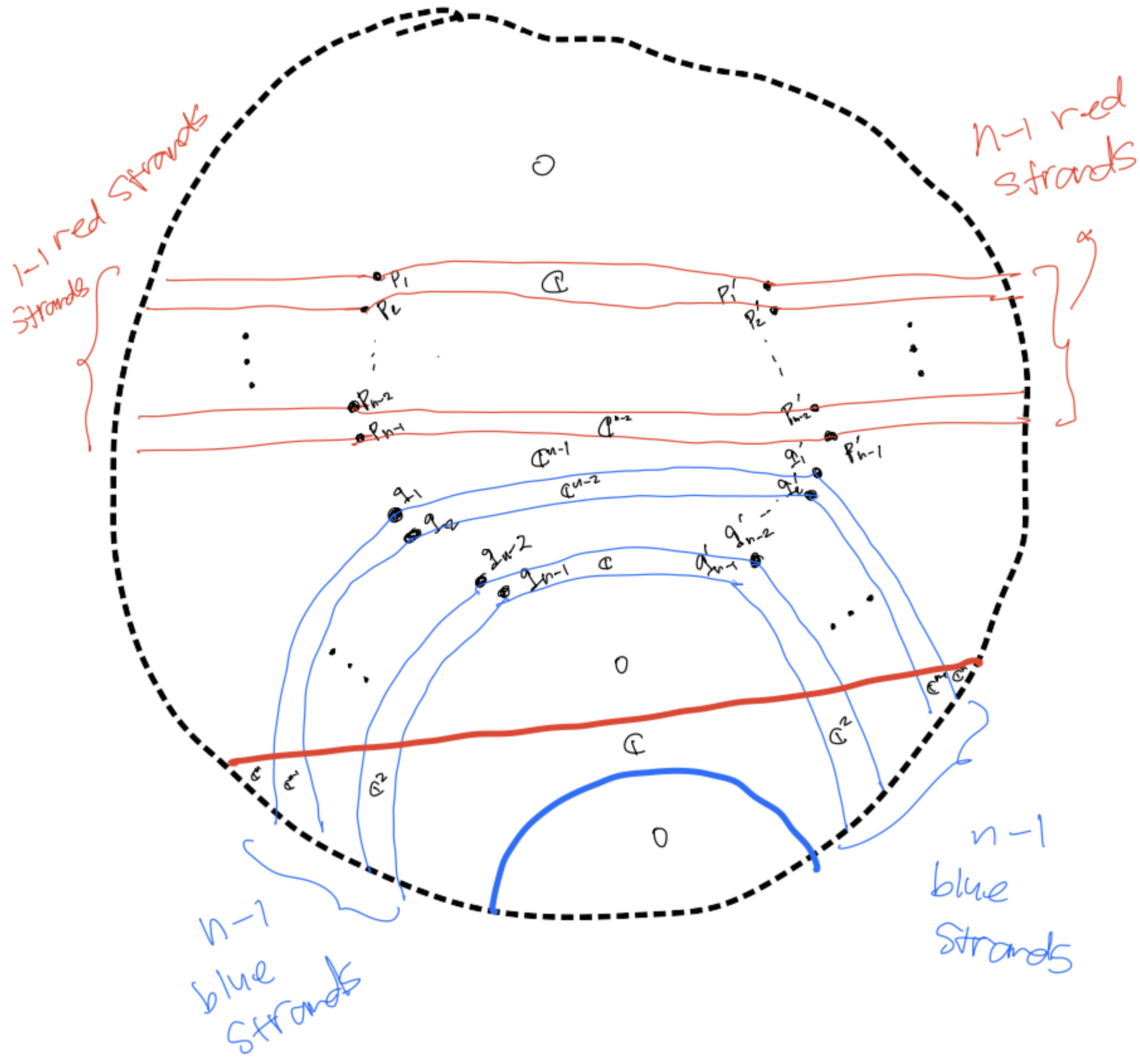


Figure 4.173: Your caption here

After applying MOVE xiii, by Theorem 13, we get

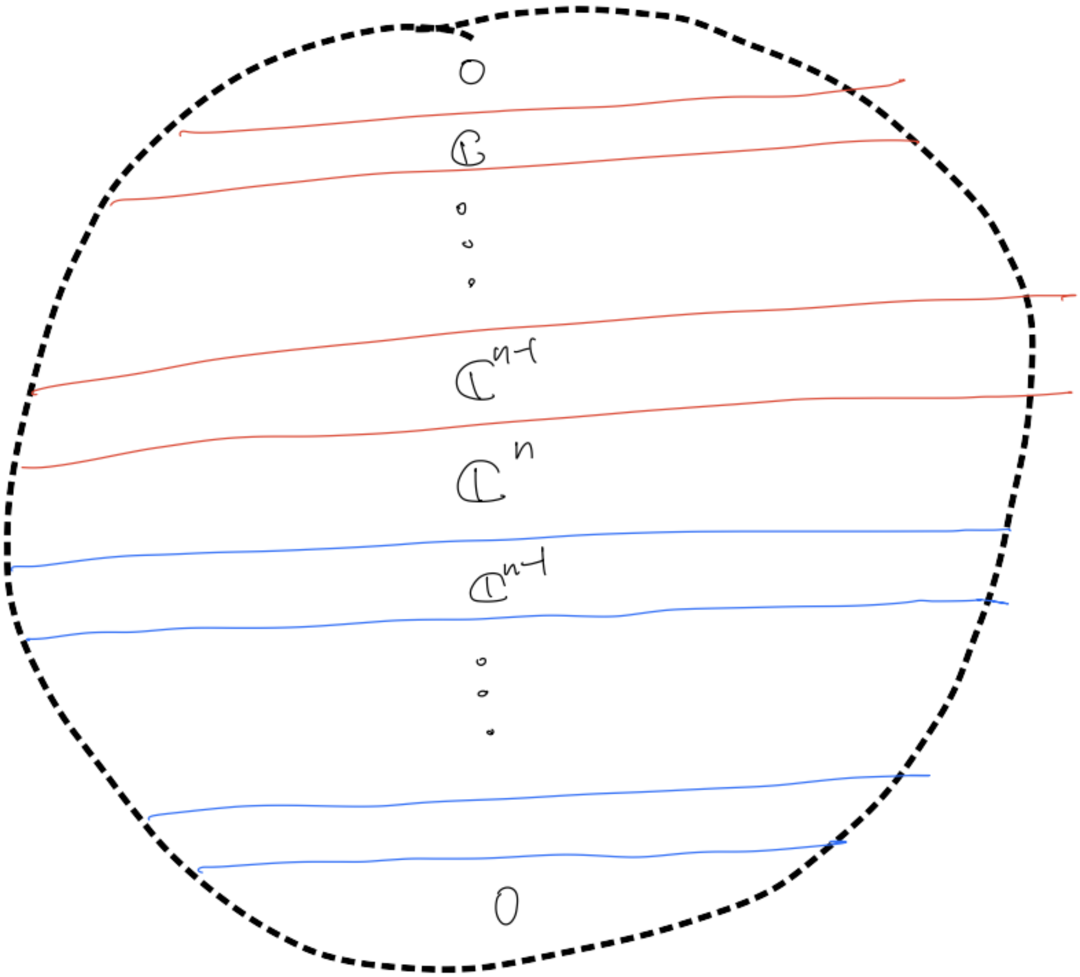


Figure 4.174: Your caption here

4.29 theorem15(the real main theorem)

Theorem 71.

Suppose we have a braid represented by a braid word $\omega = s_{i_1} \cdots s_{i_k}$, then on a two punctured Riemann sphere we have the natural alternating strand diagram associated to the braid word ω. We can cover the diagram with k generator regions and k intergenerator regions.

Suppose we have a sheaf given as follows: on the k^{th} generator region the sheaf looks as follows. Let $i := i_k$

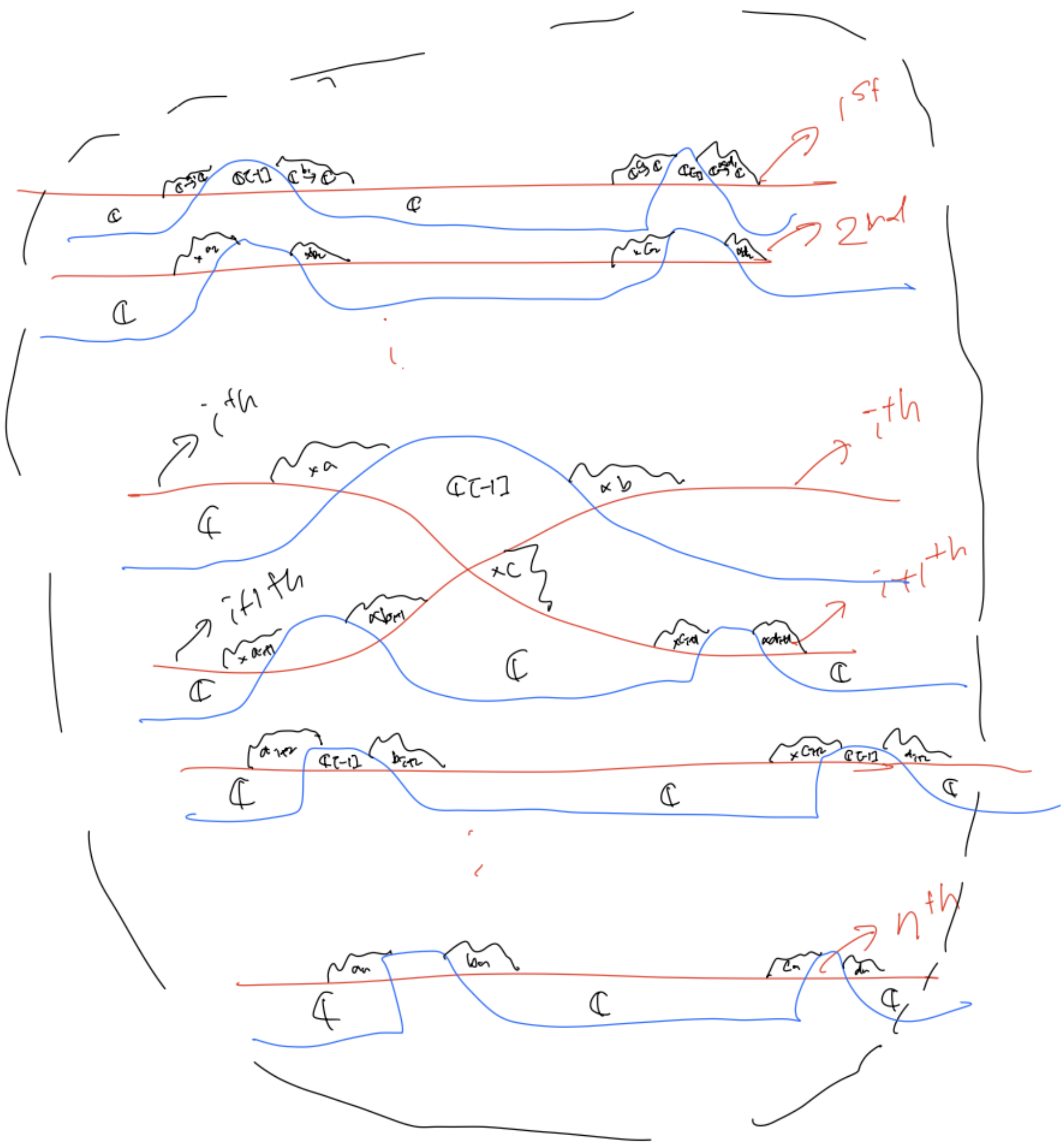


Figure 4.175: Your caption here

On the k^{th} intergenerator region looks as follows

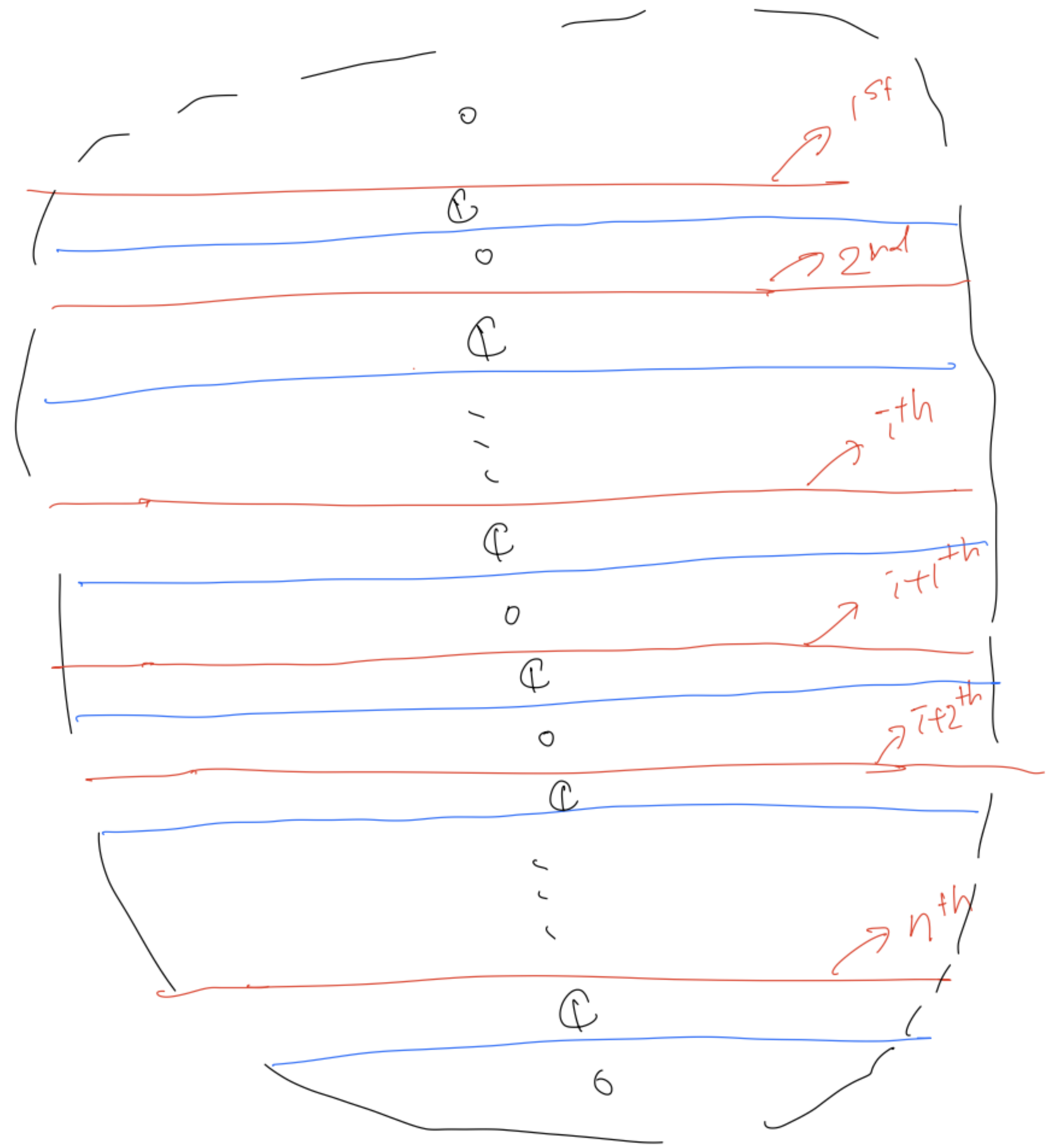


Figure 4.176: Your caption here

Suppose we apply MOVE xiito generator regions and then apply intergenerator moves to intergenerator regions, then we get the following sheaf :

On the k^{th} generator region the sheaf looks as follows: Let $i := i_k$

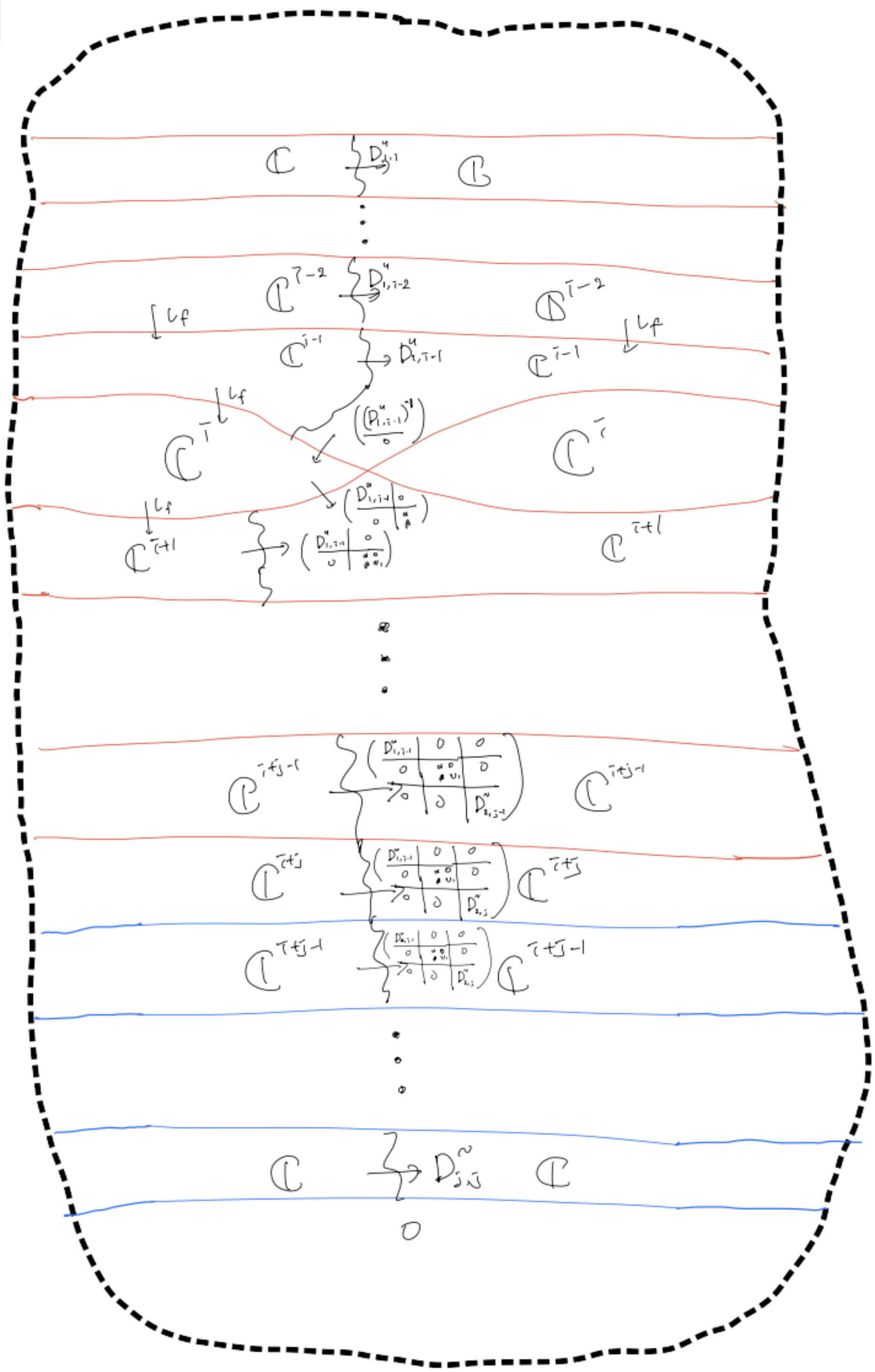


Figure 4.177: Your caption here

where

$$u_k := \frac{a_k \cdot c_k}{b_k \cdot d_k}(1 \leq k < i)$$
$$v_k := \frac{a_k \cdot c_k}{b_k \cdot d_k}(1 < k \leq n)$$
$$j := n - i$$

and $D^u_{l,m} := diag(u_l, \cdots, u_m)$, $D^v_{l,m} := diag(v_l, \cdots, v_m)$

On the k^{th} intergenerator region looks as follows :

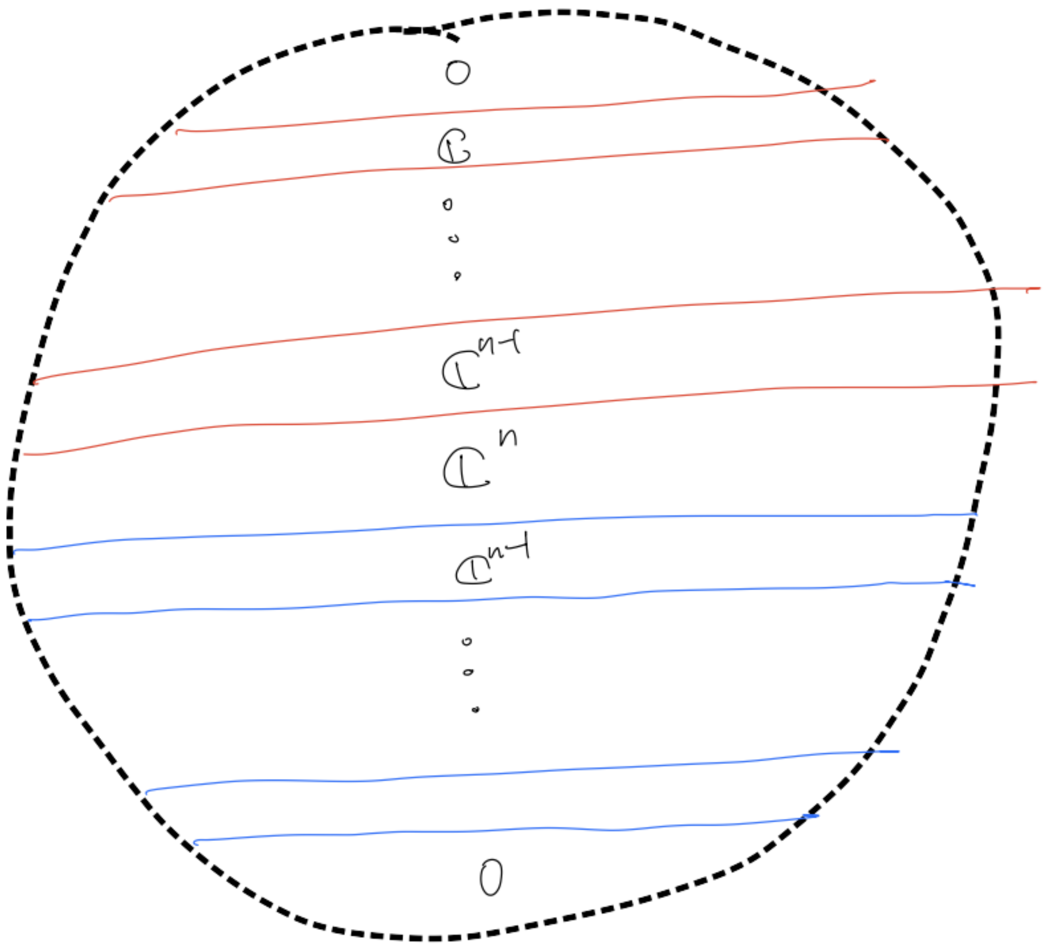


Figure 4.178: Your caption here

(proof) First we apply MOVE xii(generator moves) to the sheaf described above. By Lemma12, we get : On the k^{th} generator region the sheaf looks as follows :

Let $i := i_k$

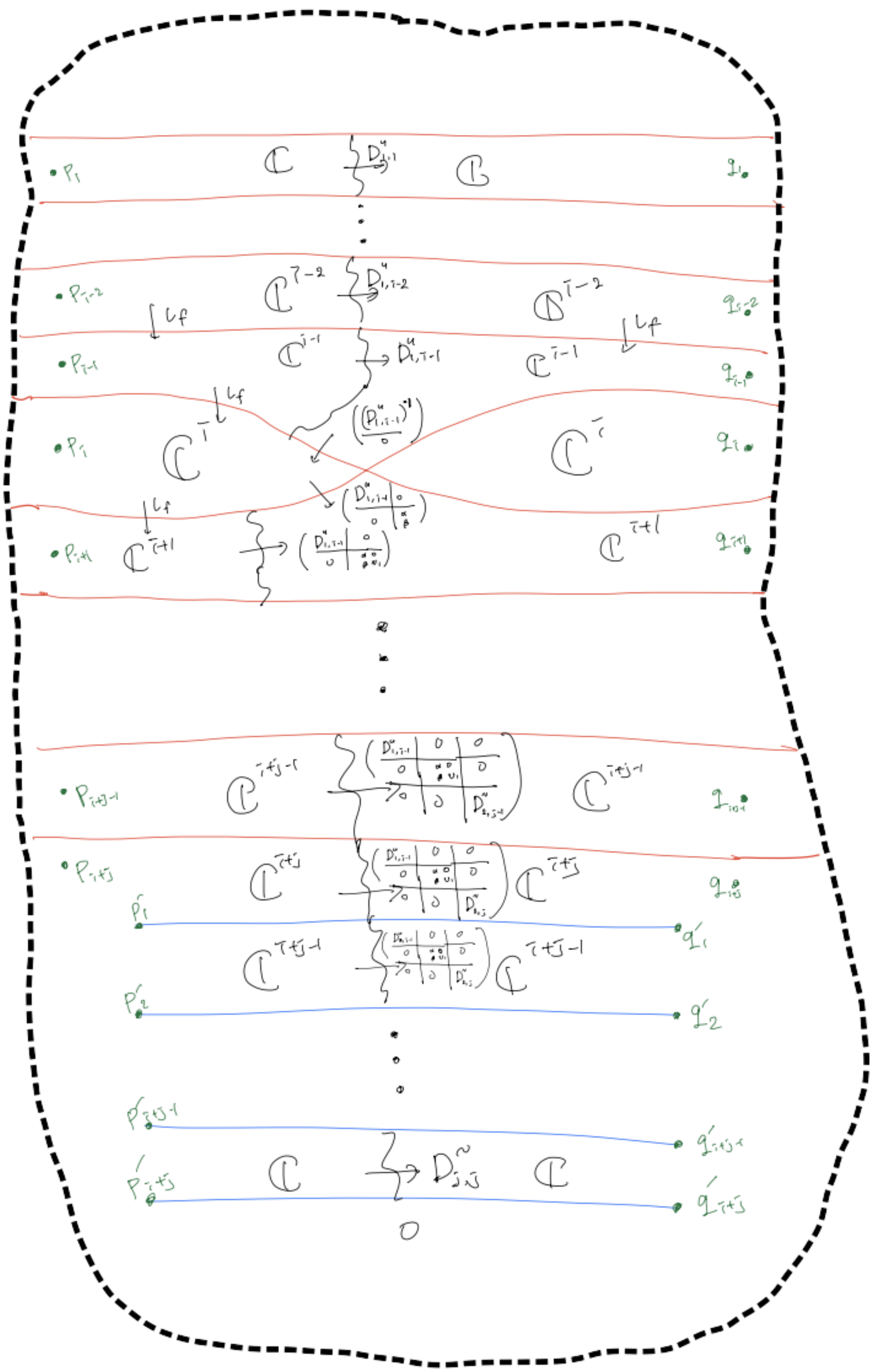


Figure 4.179: Your caption here

where

$$u_k := \frac{a_k \cdot c_k}{b_k \cdot d_k}(1 \le k < i)$$
$$v_k := \frac{a_k \cdot c_k}{b_k \cdot d_k}(1 < k \le n)$$
$$j := n - i$$

and $D^u_{l,m} := diag(u_l, \cdots, u_m)$, $D^v_{l,m} := diag(v_l, \cdots, v_m)$

and the description of the intergenerator region automatically follows.

On the k^{th} intergenerator region, with intergenerator sheaf on a intergenerator diagram on n strands. Then we apply intergenerator move to the intergenerator regions. By the intergenerator theorem, we get :

On the k^{th} generator region, the sheaf looks as follows :

Let $i := i_k$

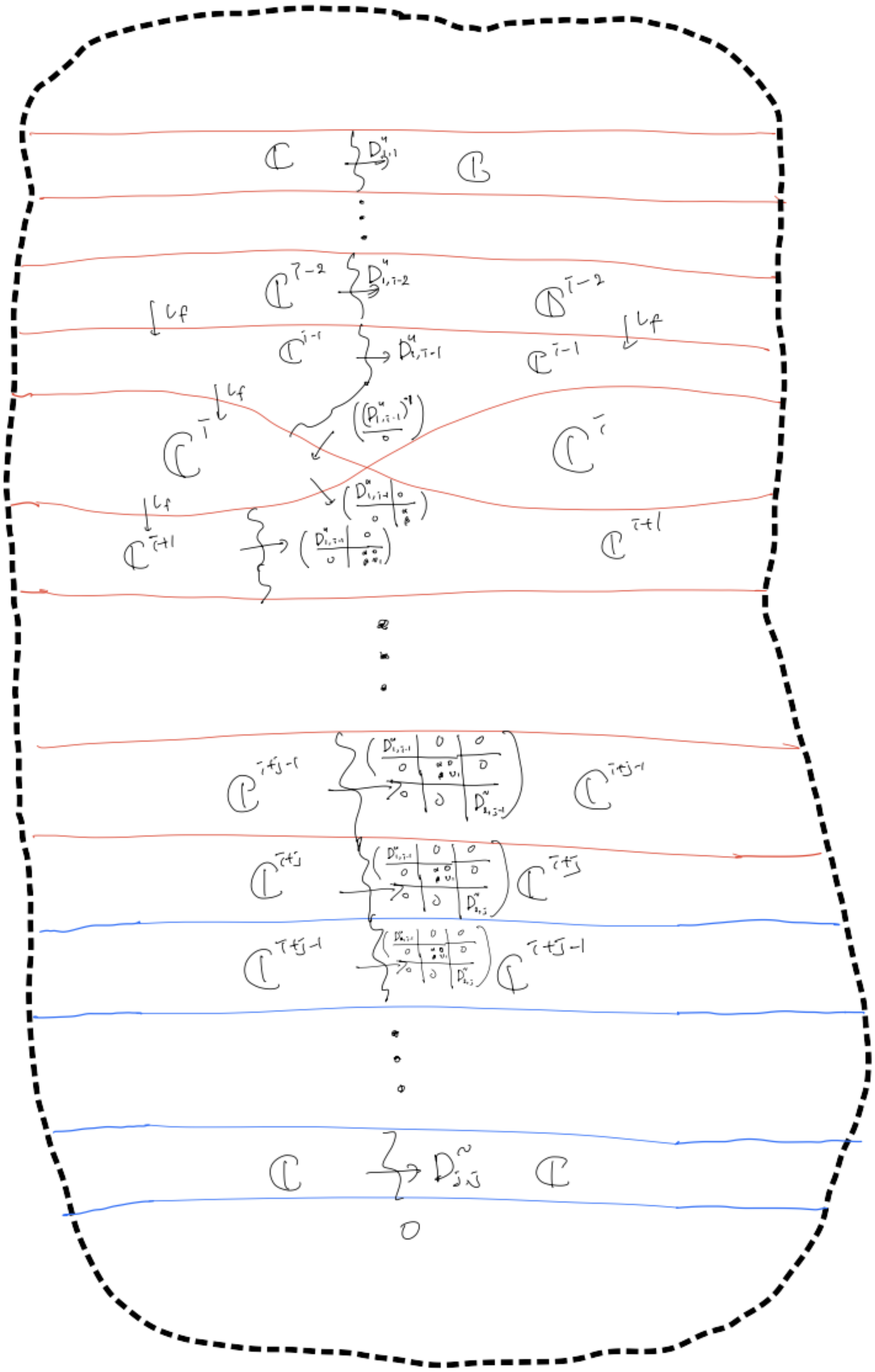


Figure 4.180: Your caption here

On the k^{th} intergenerator region looks as follows :

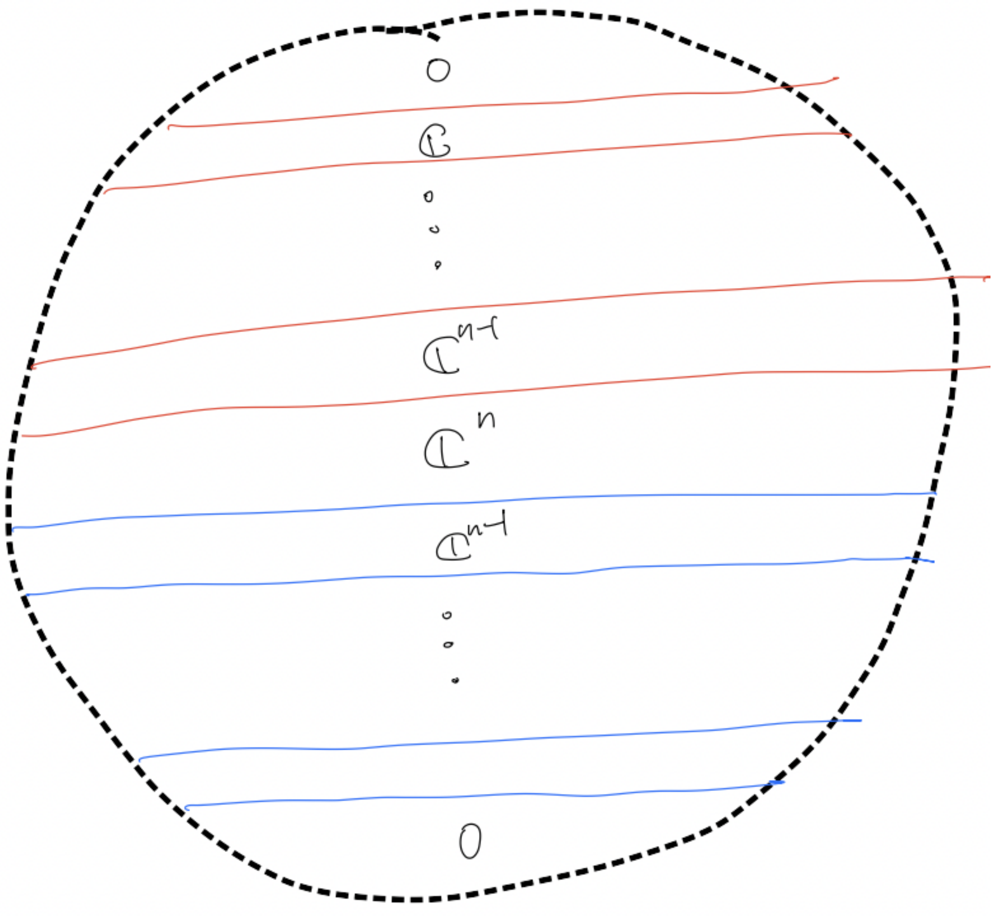


Figure 4.181: Your caption here