

Moduli Spaces associated to Legendrian Links in the Cocircle Bundle of a Cylinder from Positive Braid Closures

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Abstract

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Chapter 1

Introduction

1.1 Background and context

Contact Geometry

First, we review basics of contact geometry, see [Etn05][Gei08] for details. A contact structure on a $(2n - 1)$ -dimensional manifold X is a maximally nonintegrable distribution of $(2n - 2)$ -planes. A contact structure defined globally as a kernel for a chosen 1-form α is said to be co-oriented and α is said to be its co-orientation. An $(n - 1)$ -dimensional submanifold $\Lambda \subset X$ is said to be Legendrian if its tangent bundle is contained in the contact hyperplanes. Now let's look at several examples of contact manifolds.

- (1) the cotangent bundle T^*M of a manifold M carries a canonical 1-form which is in local coordinates (q_i, p_i) , $\theta := \sum_i p_i dq_i$. This form is invariant under dialation in cotangent directions, therefore, θ descends to a well-defined 1-form (unique up to positive scalar function) on the cosphere bundle $T^\infty M := (T^*M - 0_M)/\mathbb{R}_+$ where 0_M is the zero section and \mathbb{R}_+ -action is the dialation action along cotangent directions. Therefore, $T^\infty M$ is equipped with a natural contact structure.

(2) $\mathbb{R}_{x,y,z}^3$ with the global 1-form $\alpha = dz - ydx$ defines a co-oriented contact structure. We can embed $\mathbb{R}_{x,y,z}^3$ in $T^{\infty,-}\mathbb{R}_{x,z}^2$ as an open contact submanifold as follows:

$$\begin{aligned}\mathbb{R}_{x,y,z}^3 &\rightarrow \mathbb{R}_{x,z}^2 \times S^1 \\ (x, y, z) &\mapsto (x, z; y, -1)\end{aligned}$$

The image is the contact submanifold $T^{\infty,-}\mathbb{R}_{x,z}^2$ of "downward" covectors.

(3) Note that construction (2) is invariant under translation $x \mapsto x + 1$. Let $S_x^1 := \mathbb{R}_x/\mathbb{Z}$, then standard 1-form $dz - ydx$ descends to $S_x^1 \times \mathbb{R}_{y,z}$ and it embeds as the contact submanifold $T^{\infty,-}(S_x^1 \times \mathbb{R}_z)$ in $T^\infty(S_x^1 \times \mathbb{R}_z) \cong S_x^1 \times \mathbb{R}_z \times S_y^1$. The "cocircle bundle" in the title of the thesis refers to $T^\infty(S_x^1 \times \mathbb{R}_z)$.

Front Projection

Now let's restrict our attention to the case when the base manifold M is either $\mathbb{R}_{x,z}^2$ or $S_x^1 \times \mathbb{R}_z$. We call the map $\pi : T^\infty M \rightarrow M$ the "front projection" and $\pi(\Lambda)$ the "front diagram".

- when $M = \mathbb{R}_{x,z}^2$, $\pi : \mathbb{R}_{x,y,z}^3 \subset T^\infty \mathbb{R}_{x,z}^2 \rightarrow \mathbb{R}_{x,z}^2$ is the projection onto x, z coordinates. We call $\mathbb{R}_{x,z}^2$ a front plane.
- similarly, when $M = S_x^1 \times \mathbb{R}_z$, $\pi : S_x^1 \times \mathbb{R}_{y,z}^2 \subset T^\infty(S_x^1 \times \mathbb{R}_z) \rightarrow S_x^1 \times \mathbb{R}_z$ is the projection onto x, z coordinates. We call $S_x^1 \times \mathbb{R}_z$ a front cylinder.

A Legendrian $\Lambda \subset T^\infty M$ at immersed points of $\pi(\Lambda)$ can be recovered from $\pi(\Lambda)$ because Λ vanishes on the contact form $dz - ydx$, we have $y = \frac{dz}{dx}$ as long as it is in general position. Under Hamiltonian isotopy, Λ can be put in general position, i.e.

- $\pi|_\Lambda$ is locally injective.

- there are only finitely many points on M at which $\pi(\Lambda)$ is not an embedded submanifold. These are either
 - cusps where $\pi|_{\Lambda}$ is injective and $\frac{dz}{dx}$ has a well-defined limit of 0.
 - crossing where $\pi(\Lambda)$ is locally a transverse intersection of two smooth curves.

In this paper, we only allow $\pi(\Lambda)$ to have crossings. We won't consider $\pi(\Lambda)$ with cusps. There are no interesting examples in $\mathbb{R}_{x,y,z}^3$ but there are many interesting ones in the cocircle bundle of a cylinder. Suppose we have a smooth parametrized curve Φ in M

- which is an immersion
- which has finitely many self-intersections and they are transverse.

and a choice of preferred side, i.e. a co-orientation, diagrammatically described using hairs normal to the Φ pointing the preferred direction. We say that the region on the preferred side lies beneath the arc and the region on the other side lies above the arc. We can lift it to a Legendrian in $T^\infty M$. Therefore, the Legendrian in $T^\infty M$ amounts to the datum of the co-oriented front diagram.

Constructible sheaves

Let M be a manifold and \mathcal{S} Whitney a stratification of M . A sheaf of \mathbb{C} -vector space \mathcal{F} on M is \mathcal{S} -constructible if the restriction of \mathcal{F} to each stratum of $s \in \mathcal{S}$, $\mathcal{F}|_s$, is a locally constant sheaf. \mathcal{F} is called constructible if there exists a Whitney stratification \mathcal{S} such that \mathcal{F} is \mathcal{S} -constructible. $Sh_{naive}^\bullet(M; \mathbb{C})$ denotes the triangulated dg category whose objects are cochain complexes of sheaves of \mathbb{C} -vector spaces of bounded cohomology whose cohomology sheaves are constructible sheaves. If we localize $Sh_{naive}^\bullet(M; \mathbb{C})$ with respect to acyclic complexes, in the sense of [Dri04], we get

$Sh^\bullet(M; \mathbb{C})$. We denote $Sh_S^\bullet(M; \mathbb{C})$ the full subcategory of $Sh^\bullet(M; \mathbb{C})$ of complexes whose cohomology sheaves are constructible with respect to \mathcal{S} . From now on we will call objects of $Sh^\bullet(M; \mathbb{C})$ simply as sheaves.

Singular Support

To each $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$, we assign a closed conical subset $SS(\mathcal{F}) \subset T^*M$, called the singular support of \mathcal{F} following [GM83][Sch12][STZ17]. For more general treatment, see [KS13]. Fix a Riemannian metric on M and choose an ϵ -ball $B_\epsilon(x)$ around a point $x \in M$. The following constructions are independent of the choice of metrics. Let \mathcal{F} be an \mathcal{S} -constructible sheaf on M . Fix a point $x \in M$ and a smooth function f on $B_\epsilon(x)$. For $\epsilon, \delta > 0$, we define the local Morse group to be

$$Mo_{x,f,\epsilon,\delta}(\mathcal{F}) := H^*(B_\epsilon(x) \cap f^{-1}((-\infty, f(x) + \delta)), B_\epsilon(x) \cap f^{-1}((-\infty, f(x) - \delta)); \mathcal{F})$$

For $\epsilon' < \epsilon$ and $\delta' < \delta$, there is a canonical restriction map

$$Mo_{x,f,\epsilon,\delta}(\mathcal{F}) \rightarrow Mo_{x,f,\epsilon',\delta'}(\mathcal{F})$$

The above restriction is an isomorphism if ϵ and δ are sufficiently small and f is stratified Morse at x . This allows us to define $Mo_{x,f}(\mathcal{F})$ for f suitably generic with respect to \mathcal{S} [KS13, Prop. 7.5.3]. In fact, $Mo_{x,f}(\mathcal{F})$ depends only on the Hessian of f at x . Let $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$, then $(x, \xi) \in T^*M$ is characteristic with respect to \mathcal{F} if for some stratified Morse function f with $df_x = \xi$, the local Morse group $Mo_{x,f}(\mathcal{F})$ is nonzero. Then we define the singular support of \mathcal{F} to be the closure of the set of characteristic covectors for \mathcal{F} in T^*M . Singular support is a conic Lagrangian i.e. stable under \mathbb{R}_+ -dialation along contangent fibers.

Fix a conic Lagrangian $L \subset T^*M$, then we define $Sh_L^\bullet(M; \mathbb{C})$ to be the full subcate-

gory of $Sh_L^\bullet(M; \mathbb{C})$ whose objects are sheaves singular supported in L . If $\Lambda \subset T^\infty M$ is a Legendrian, then we define $Sh_\Lambda^\bullet(M; \mathbb{C}) := Sh_{\mathbb{R}_+ \Lambda \cup 0_M}^\bullet(M; \mathbb{C})$ where $0_M \subset T^*M$ is the zero section.

Combinatorial Model of $Sh_\Lambda^\bullet(M; \mathbb{C})$

The first necessary condition for $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$ to belong to $Sh_L^\bullet(M; \mathbb{C})$ is that \mathcal{F} is constructible with respect to the stratification on M induced by the front projection of Λ . That is the Whitney stratification whose zero dimensional strata are crossings, one dimensional strata are the arcs of $\pi(\Lambda)$ between crossings and two dimensional strata are the connected components of the complement of $\pi(\Lambda)$ in M .

Definition 1. Given a stratification \mathcal{S} , the star of a stratum $s \in \mathcal{S}$ is the union of strata that contain s in their closure. We view \mathcal{S} as a poset category $s \leq t$ if and only if $s \subset \bar{t}$. We call the map from s to t as a generization map. We say \mathcal{S} is a regular cell complex if every stratum is contractible and moreover the star of each stratum is contractible.

Definition 2. We define $Fun_{naive}^\bullet(\mathcal{S}, \mathbb{C})$ to be the dg category of functors from \mathcal{S} to the category whose objects are cochain complexes of \mathbb{C} -vector spaces and whose maps are cochain maps. We define $Fun^\bullet(\mathcal{S}, \mathbb{C})$ the dg quotient [Dri04] of $Fun_{naive}^\bullet(\mathcal{S}, \mathbb{C})$ by the thick subcategory of functors taking values in acyclic complexes.

Proposition 3. [Kas84], [She85], [Nad09, Lemma 2.3.2]. Let \mathcal{S} be a Whitney stratification of M . Consider the functor

$$\Gamma_{\mathcal{S}} : Sh_{\mathcal{S}}^\bullet(M; \mathbb{C}) \rightarrow Fun^\bullet(\mathcal{S}, \mathbb{C})$$

$$\mathcal{F} \mapsto [s \mapsto \Gamma(star(s); \mathcal{F})]$$

If \mathcal{S} is a regular cell complex, then $\Gamma_{\mathcal{S}}$ is a quasi-equivalence.

However, the stratification induced by the front diagram is not necessarily a regular cell complex. Therefore, we choose an \mathcal{S} that refines of the stratification induced by $\pi(\Lambda)$. In figures, we draw the new line segments in squiggly lines.

The restriction of $\Gamma_{\mathcal{S}}$ to $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$ is quasi-fully faithful. We will describe the essential image of $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$ under $\Gamma_{\mathcal{S}}$.

Definition 4. Let \mathcal{S} be a regular cell complex refining the stratification induced by the front diagram. we define $Fun_{\Lambda}^{\bullet}(\mathcal{S}, \mathbb{C})$ to be the full subcategory of $Fun^{\bullet}(\mathcal{S}, \mathbb{C})$ whose objects satisfy the following properties.

- every map from a zero dimensional stratum in \mathcal{S} which is not a crossing is sent to quasi-isomorphism.
- every map from a one dimensional stratum which is not contained in an arc is sent to quasi-isomorphism.
- for each crossing $c \in \mathcal{S}$, Let N, E, S, W be the north, east, south, and west regions adjoining c , and let nw, ne, sw, se be the arcs separating them.

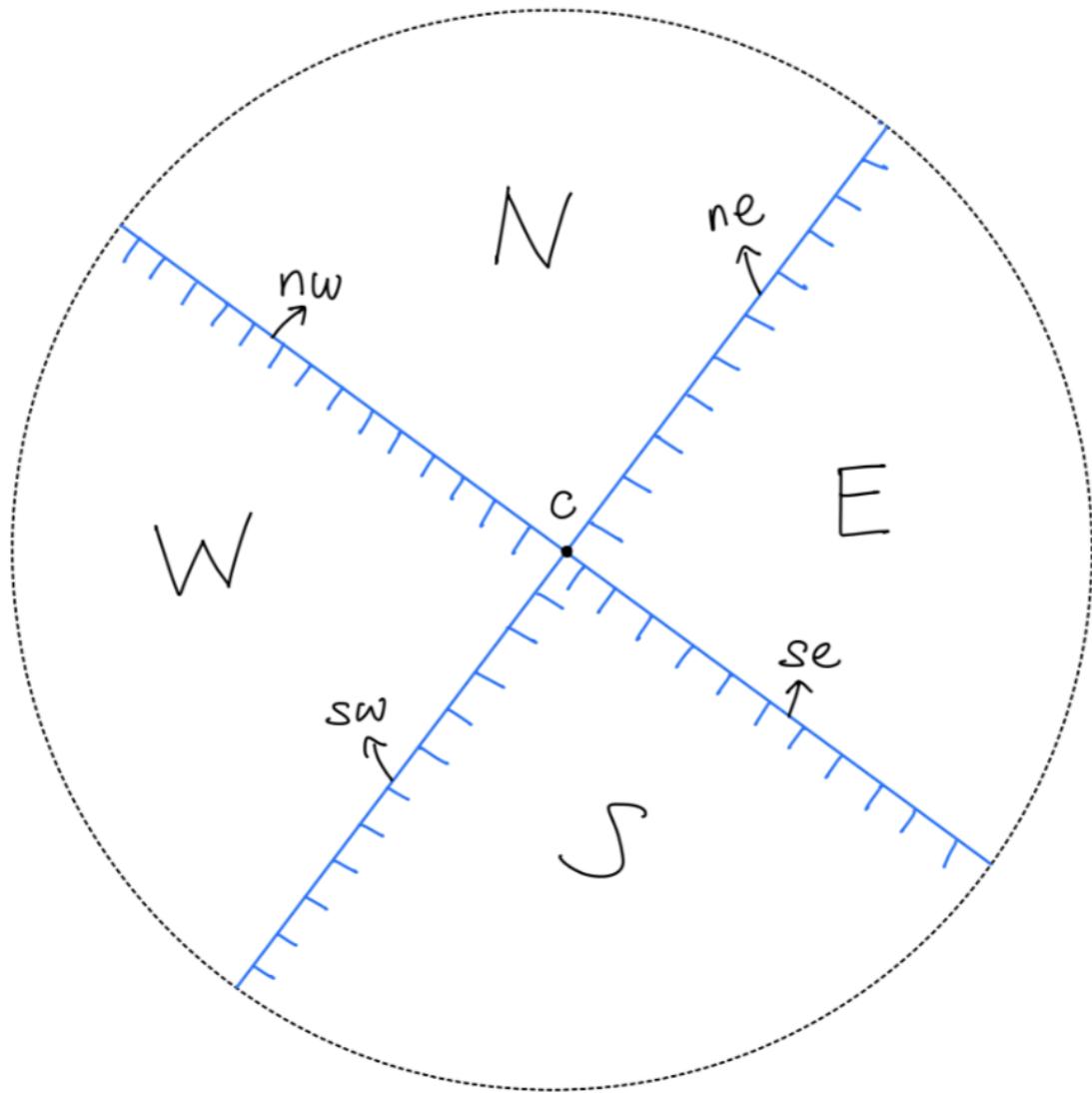
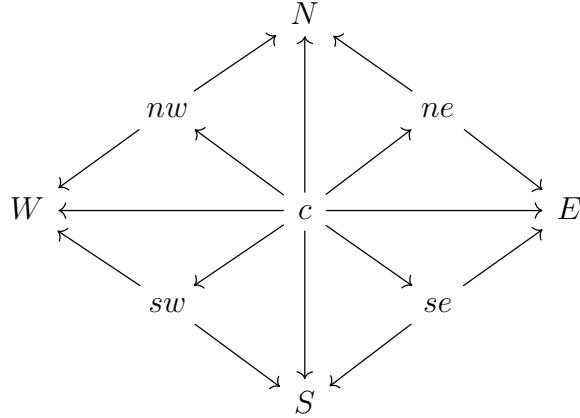


Figure 1.1: Your caption here

Then the following maps are sent to quasi-isomorphisms.

- maps between c , sw , se , and S .
- a map from nw to W .
- a map from ne to E .
- for each crossing $c \in \mathcal{S}$, we restrict the poset structure to the strata containing

c in its closure, we get



Then all triangles in the diagram should commute and the total complex of the following bicomplex

$$F(c) \rightarrow F(nw) \oplus F(ne) \rightarrow F(N)$$

should be acyclic.

A standard argument shows that the essential image of $Sh_{\Lambda}^{\bullet}(M, \mathbb{C})$ under Γ_S is $Fun_{\Lambda}^{\bullet}(\mathcal{S}, \mathbb{C})$ [STZ17].

Legible Object

Let M be a front surface, Φ a front diagram, $\Lambda \subset T^{\infty}M$ the associated Legendrian knot, and \mathcal{S} a refinement of the stratification induced by Φ by adding squiggly lines. In this section, we define a “legible quiver” and a “squiggly legible diagram” to get a better handle on $Fun_{\Lambda}^{\bullet}(M; \mathbb{C})$.

Suppose we impose an auxiliary co-orientation on the squiggly lines as well, then this induces a quiver Q

Definition 5. Suppose we have a regular cell complex \mathcal{S} that refines the stratification

induced by Φ with auxiliary co-orientations on the added squiggly lines segments, then we define the associated quiver Q where

- the vertices of Q are 2 dimensional strata
- the arrows of Q are 1 dimensional strata where the sources and the targets are regions below and above the strata.

For a stratum $s \in \mathcal{S}$, we define Q_s to be the full sub-quiver of Q whose vertices are the regions incident with s .

Definition 6. We say that the quiver associated to \mathcal{S} is *legible* if for every point stratum $p \in \mathcal{S}$, the sub-quiver Q_p is a poset with the smallest element.

Definition 7. A squiggly legible diagram on \mathcal{S} is the representation F^\bullet of the quiver Q associated to \mathcal{S} valued in the cochain complex of \mathbb{C} -vector spaces subject to the condition that

- the maps corresponding to squiggly lines are quasi-isomorphisms.
- for fixed a source and a target vertices s and t , suppose we have two distinct paths in Q (a sequence of arrows) from s to t , say (a_1, a_2, \dots, a_k) and $(a'_1, a'_2, \dots, a'_l)$, then $F^\bullet(a_k) \circ \dots \circ F^\bullet(a_1) = F^\bullet(a'_l) \circ \dots \circ F^\bullet(a'_1)$ i.e. the composition of cochain maps are path independent.
- at each crossing, we have surrounding region N, S, W, E

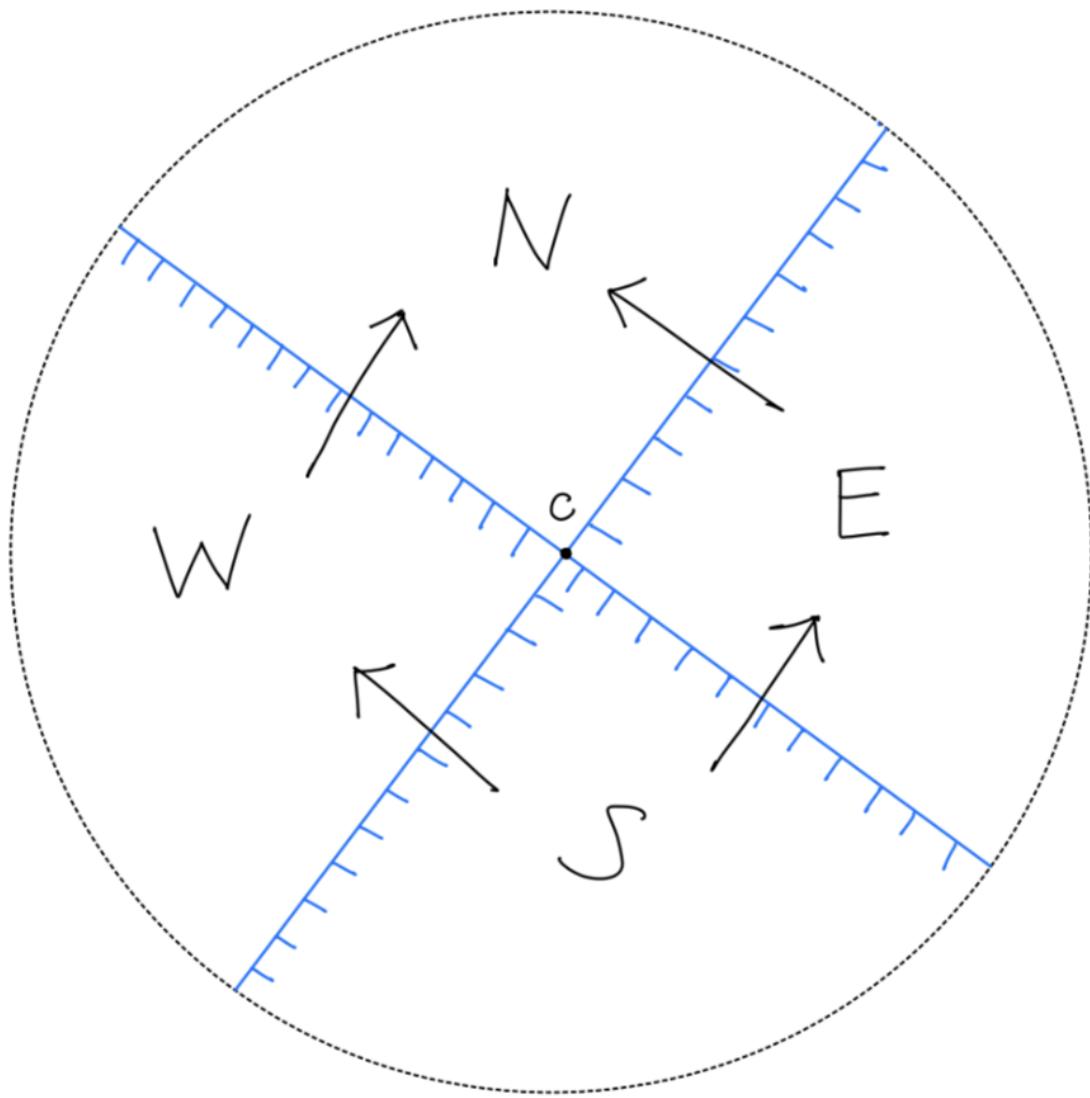


Figure 1.2: Your caption here

then the total complex of

$$F^\bullet(S) \rightarrow F^\bullet(W) \oplus F^\bullet(E) \rightarrow F^\bullet(N)$$

is acyclic.

Suppose we have \mathcal{S} a refinement of the stratification induced by Φ and a squiggly legible diagram F^\bullet , then we can make an object $\bar{F}^\bullet \in \text{Fun}_\Lambda^\bullet(\mathcal{S}, \mathbb{C})$ out of it in the following way.

Definition 8. $\rho : \mathcal{S} \rightarrow \{R \in \mathcal{S} \mid \dim_{\mathbb{R}}(R) = 2\}$ is a function that assigns $s \in \mathcal{S}$ to the smallest element in $\{R \in \text{Vert}(Q) \mid s \subset \bar{R}\}$.

Definition 9. Suppose F^\bullet is a squiggly legible diagram on \mathcal{S} , then we define $\bar{F}^\bullet \in \text{Fun}_\Lambda^\bullet(\mathcal{S}, \mathbb{C})$ where

- for $s \in \mathcal{S}$, $\bar{F}^\bullet(s) := F^\bullet(\rho(s))$
- for $s_1, s_2 \in \mathcal{S}$ such that $s_2 \subset \text{star}(s_1)$, $\bar{F}^\bullet(s_1 \rightarrow s_2) := F^\bullet(a_k) \circ \cdots F^\bullet(a_1)$ where (a_1, a_2, \dots, a_k) is a path from $\rho(s_1)$ to $\rho(s_2)$ in the quiver associated to \mathcal{S} . This is well-defined because the composition of cochain maps are path independent by the definition of squiggly legible diagram and the existence of a path from $\rho(s_1)$ to $\rho(s_2)$ is guaranteed by the fact that Q_{s_2} is a subquiver of Q_{s_1} and $\rho(s_2), \rho(s_1)$ are the smallest elements from each of them.

In this paper, we only consider Φ , Λ , and \mathcal{S} that objects of $\text{Fun}^\bullet(\mathcal{S}, \mathbb{C})$ arises from squiggly legible diagrams. A variant of [STZ17, Prop 3.22.] shows that every objects of $\text{Sh}_\Lambda^\bullet(M; \mathbb{C})$ is quasi-isomorphic to a squiggly legible diagram.

Microllocal Monodromy and Rank 1 Object

In this section, we review the notion of microllocal monodromy of sheaves following [STZ17, Section 5.1]. For more general treatment, see [KS13, Ch. IV]. Suppose on M we have a front diagram Φ and a stratification \mathcal{S} that refines the one induced by Φ by adding squiggly line segments. Suppose we are given a sheaf F^\bullet in terms of squiggly legible diagram, we define the associated sheaf on Λ , called microllocal monodromy denoted $\mu\text{mon}(F^\bullet)$, by assigning cochain complexes of \mathbb{C} -vector spaces for each one dimensional stratum contained in Φ and cochain maps between them for each point stratum contained in Φ as follows:

- if s is a one dimensional stratum contained in Φ , $\mu\text{mon}(F^\bullet)(s) := \text{Cone}(F^\bullet(s))$

- if p is a point stratum in Φ , then p is either a crossing or the endpoint of a squiggly line. We define the cochain map for each case as follows:
 - when p is a crossing, locally near p the diagram looks like

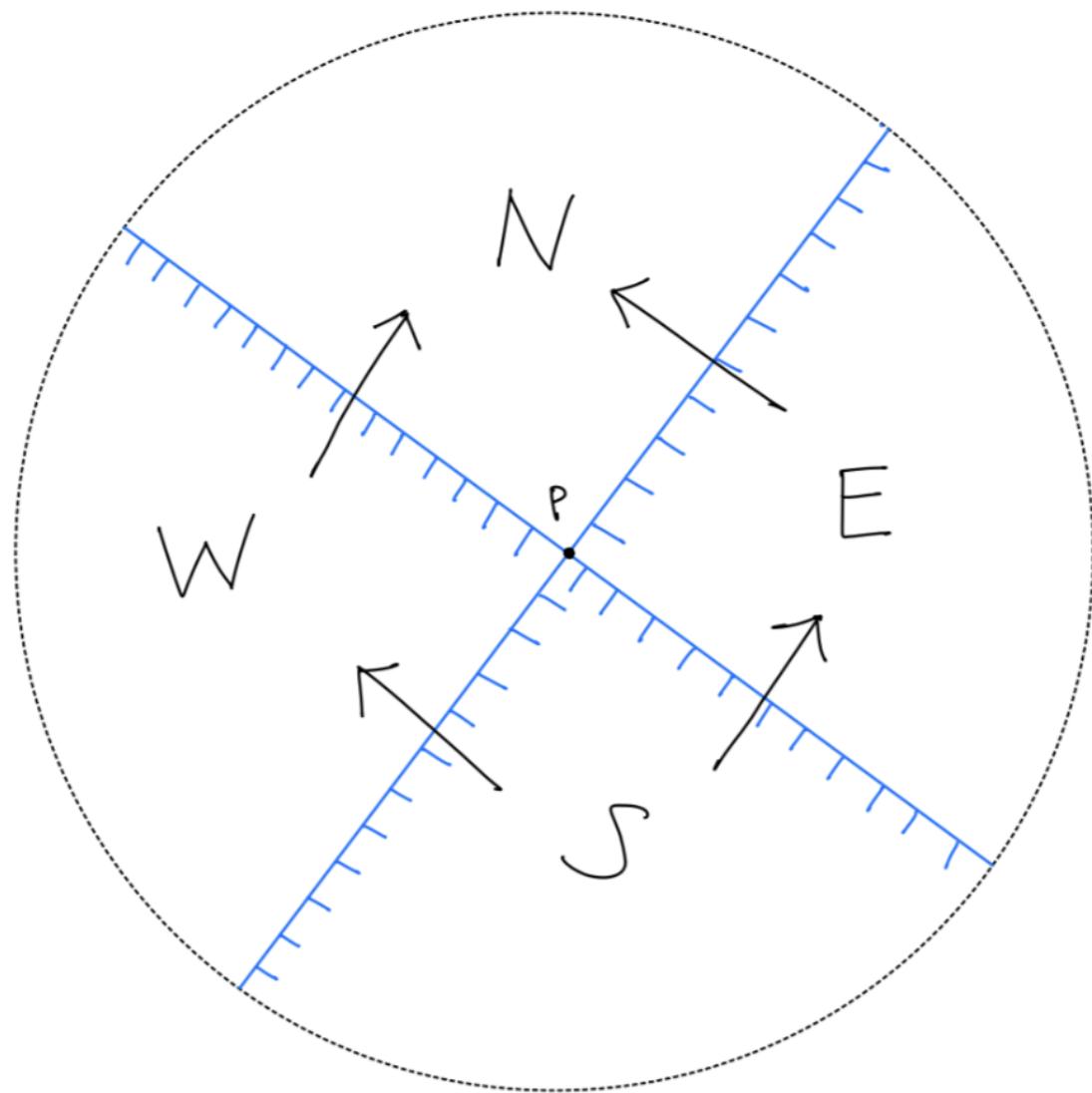


Figure 1.3: Your caption here

then p induces maps

$$\text{Cone}(S \rightarrow W) \rightarrow \text{Cone}(E \rightarrow W)$$

$$\text{Cone}(S \rightarrow E) \rightarrow \text{Cone}(W \rightarrow N)$$

- when p is the endpoint of a squiggly line segment, locally near p the diagram looks like one of the following diagrams

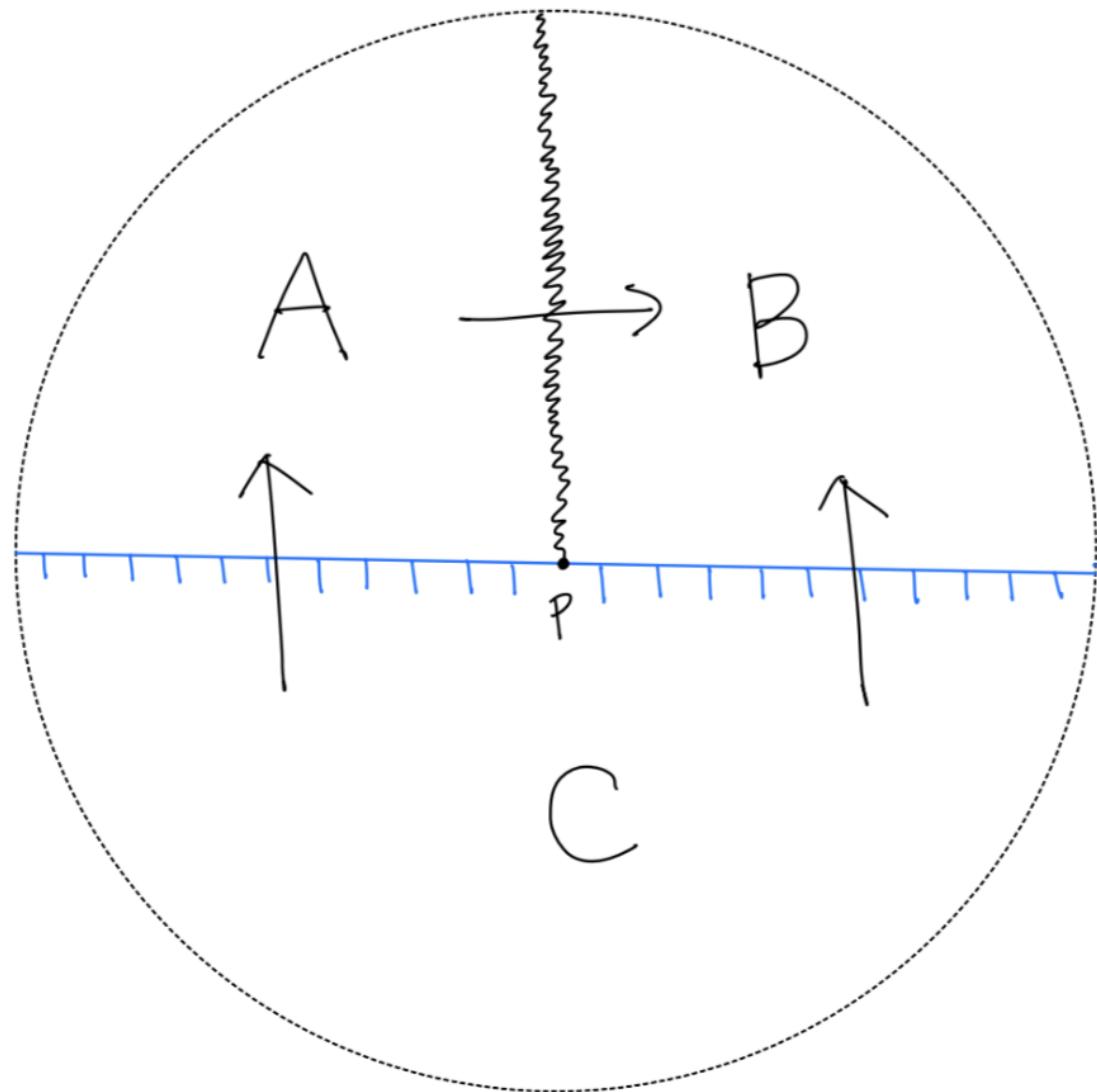


Figure 1.4: Your caption here

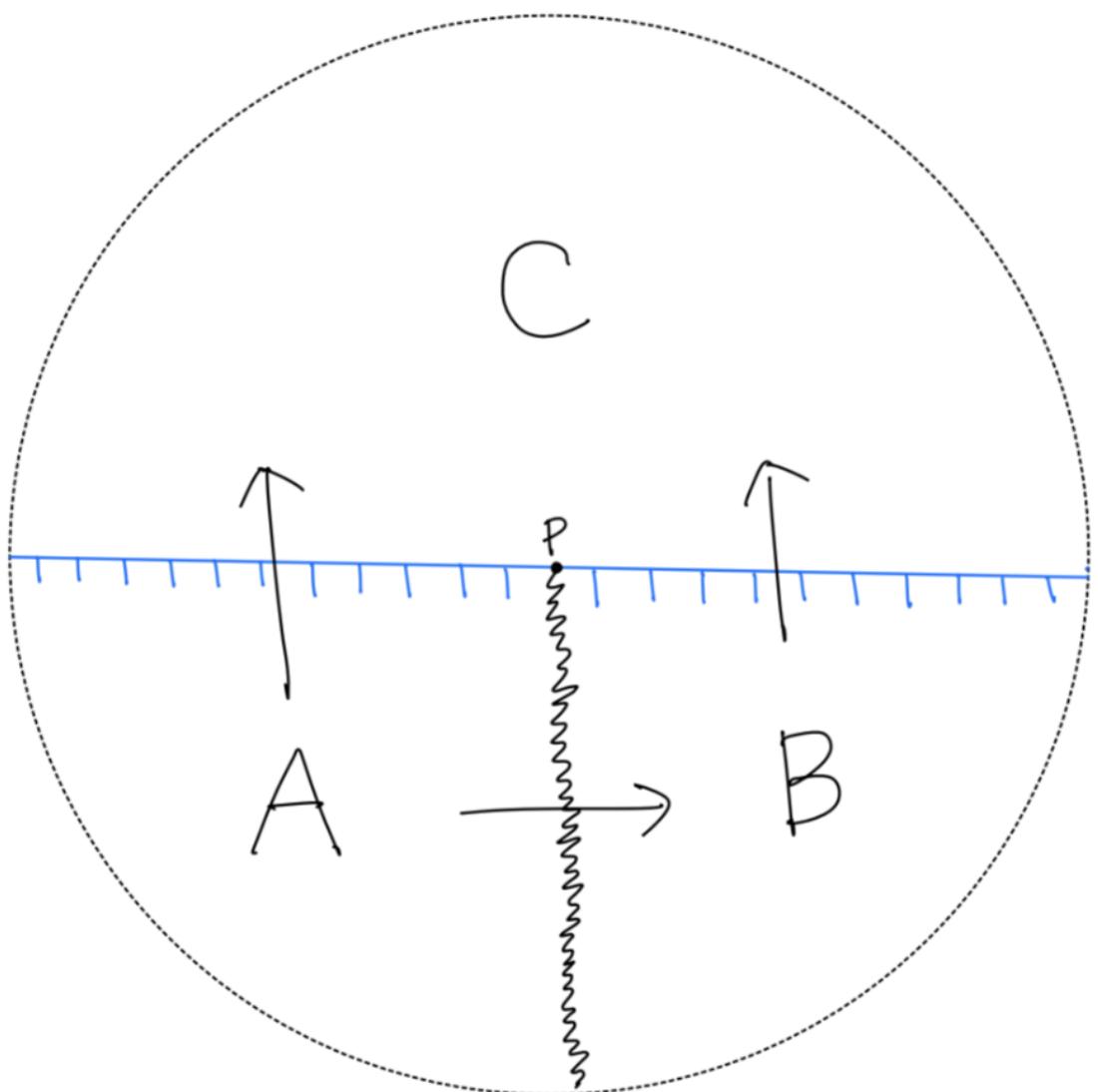


Figure 1.5: Your caption here

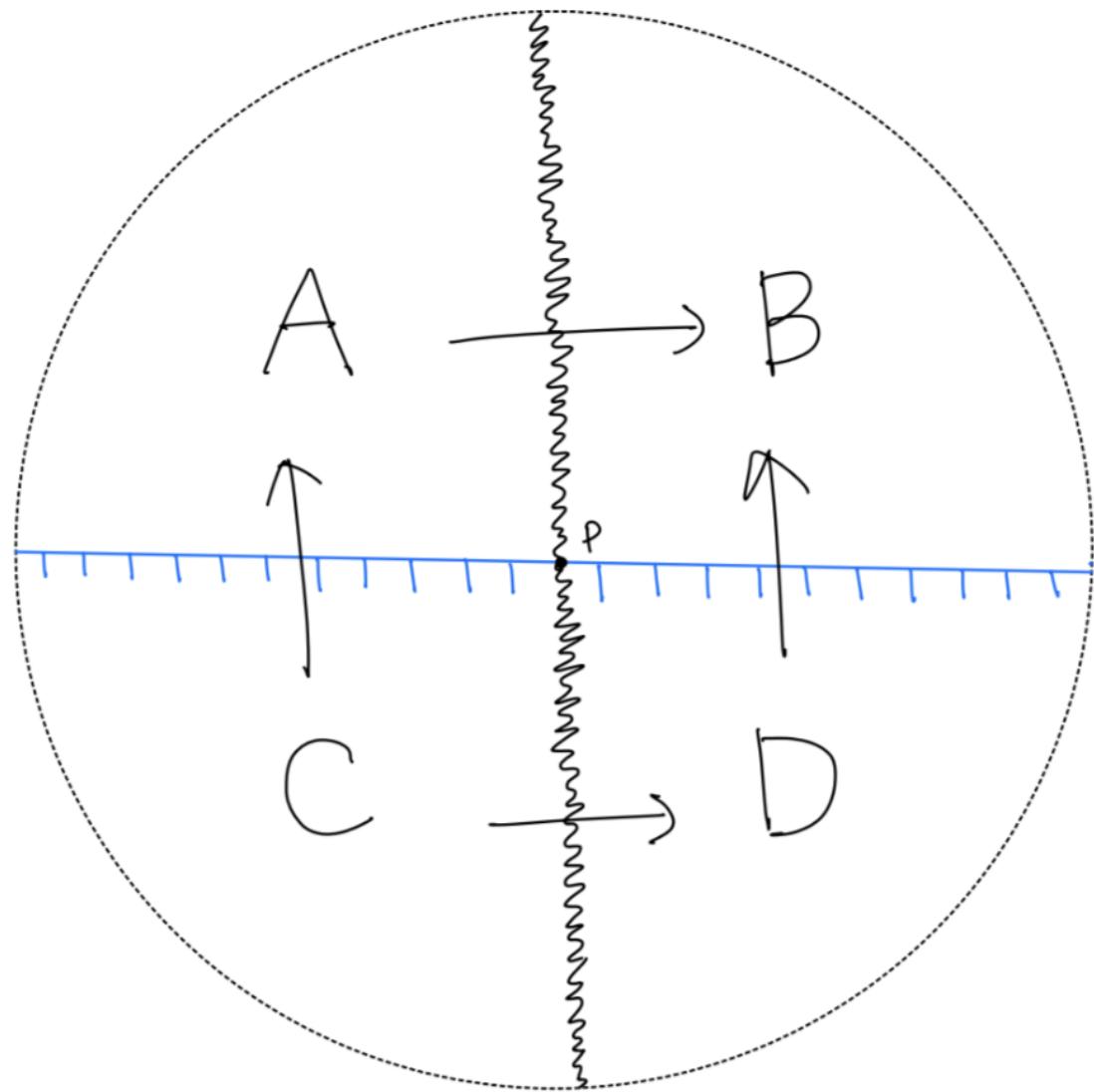


Figure 1.6: Your caption here

, then we have a map

$$\text{Cone}(C \rightarrow A) \rightarrow \text{Cone}(C \rightarrow B)$$

$$\text{Cone}(A \rightarrow C) \rightarrow \text{Cone}(B \rightarrow C)$$

$$\text{Cone}(C \rightarrow A) \rightarrow \text{Cone}(D \rightarrow B)$$

respectively.

The maps between cones are quasi-isomorphisms because of the crossing condition and the fact that the map corresponding to squiggly lines are quasi-isomorphism.

Definition 10. F^\bullet is called *rank n object* if $\mu\text{mon}(F^\bullet)$ is a rank n local system concentrated in degree 0 on Λ .

Definition 11. The full subcategory of $Sh_\Lambda^\bullet(M; \mathbb{C})$ containing rank n objects is denoted $\mathcal{C}_n(M, \Lambda; \mathbb{C})$ and the moduli stack classifying such objects $\mathcal{M}_n(M, \Lambda; \mathbb{C})$. Additionally, let $\{\sigma_1, \dots, \sigma_k\}$ be points in M that are contained in one of the regions separated by Φ . Then we define $\mathcal{C}_n(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$ to be the full subcategory of sheaves with vanishing stalks at $\{\sigma_1, \dots, \sigma_k\}$, $\mathcal{M}(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$ the moduli space classifying such objects.

Moduli Spaces associated to Positive Braids

In this section, we define the moduli spaces associated to positive braids following [STWZ19, Section 3]. For background we refer to the survey [Toë14] and the foundational works [Lur04] [Toë09] [TV04] [TV05] [TV08]. Suppose we have a positive braid word ω of n strands(i.e. a word freely generated by s_1, \dots, s_{n-1}), Then we can think of its cylindrical closure in $S_\theta^1 \times (0, 1)_r = (\mathbb{R}_\theta / \mathbb{Z}) \times (0, 1)_r$. In this paper, we consider two kinds of moduli spaces associated to the braid which are the main objects of study in Chapter2 and Chapter3 respectively.

- (1) First, we embed the cylinder containing cylindrical closure of ω in $M = S_x^1 \times \mathbb{R}_z$ via $(\theta, r) \mapsto (\theta, r - 1)$. We can think of the cylindrical closure of ω in $S_x^1 \times \mathbb{R}_z$ co-oriented downward($z < 0$) to be the front projection of a Legendrian knot Λ_ω living in $T^\infty M$. Let $\sigma_{z \ll 0}$ be a point in the non-compact region where $z \ll 0$. In Chapter2, we consider the moduli space $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z \ll 0}\}; \mathbb{C})$.
- (2) Next, we embed the cylinder containing the cylindrical closure of ω in $M = S_x^1 \times \mathbb{R}_z$ via $(\theta, r) \mapsto (\theta, 1 - r)$ and embed the cylinder containing the cylindrical

closure of the trivial braid ω_\emptyset via $(\theta, r) \mapsto (\theta, r-1)$. We get an embedding of the cylindrical closure of $\omega \coprod \omega_\emptyset$ in $S_x^1 \times \mathbb{R}_z$ where the embedding of the cylindrical closure of ω is co-oriented upward and the embedding of the cylindrical closure of ω_\emptyset is co-oriented downward. We will call the embedding the separated diagram of ω and ω_\emptyset which we consider as the front projection of a Legendrian link $\Lambda_\infty \coprod \Lambda_0 \subset T^\infty M$. Let $\sigma_{z \ll 0}, \sigma_{z \gg 0}$ be points in the non-compact regions where $z \ll 0$ and $z \gg 0$. In Chapter 3, we consider the moduli space $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z \ll 0}, \sigma_{z \gg 0}\}; \mathbb{C})$.

Legendrian Isotopy and Sheaf Cobordism

Invariance of the category $\mathcal{C}_1(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$ under Legendrian isotopy follows from the main theorem of [GKS12].

Theorem 12. [GKS12]. Suppose $\Lambda_0 \subset T^\infty M$ is a Legendrian and $\Lambda_\bullet \subset T^\infty \times [0, 1]$ a Legendrian isotopy where $\Lambda_\bullet|_{T^\infty \times \{0\}}$ is equal to Λ_0 under the natural identification $T^\infty \times \{0\} \cong T^\infty$, then the restriction functor

$$Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C}) \rightarrow Sh_{\Lambda_0}^\bullet(M; \mathbb{C})$$

is a quasi-equivalence. Furthermore, this quasi-equivalence preserves rank n objects.

Definition 13. Suppose we have a Legendrian isotopy $\Lambda_\bullet \subset T^\infty \times [0, 1]$ from $\Lambda_0 = \Lambda_\bullet|_{T^\infty \times \{0\}}$ to $\Lambda_1 = \Lambda_\bullet|_{T^\infty \times \{1\}}$ and a sheaf $\mathcal{F}_\bullet \in Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C})$ that restricts to \mathcal{F}_0 and \mathcal{F}_1 at $M \times \{0\}$ and $M \times \{1\}$, then we say \mathcal{F}_\bullet is a *sheaf cobordism* from \mathcal{F}_0 to \mathcal{F}_1 .

Definition 14. Suppose $\Lambda_0, \Lambda_1 \subset T^\infty M$ are Legendrian links and there is a Legendrian isotopy $\Lambda_\bullet \subset T^{\infty,-} M \times [0, 1]$ where $\Lambda_\bullet|_{T^{\infty,-} \times \{0\}}$ is equal to Λ_0 under the natural identification $T^{\infty,-} \times \{0\} \cong T^{\infty,-}$ and $\Lambda_\bullet|_{T^{\infty,-} \times \{1\}}$ is equal to Λ_1 under the natural

identification $T^{\infty,-} \times \{1\} \cong T^{\infty,-}$, then we have quasi-equivalence

$$Sh_{\Lambda_0}^\bullet(M; \mathbb{C}) \xrightarrow{\sim} Sh_{\Lambda_1}^\bullet(M; \mathbb{C})$$

from the following correspondence

$$Sh_{\Lambda_0}^\bullet(M; \mathbb{C}) \xleftarrow{\sim} Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C}) \xrightarrow{\sim} Sh_{\Lambda_1}^\bullet(M; \mathbb{C})$$

Furthermore, this quasi-equivalence restricts to

$$\mathcal{C}_n(M, \Lambda_0; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}(M, \Lambda_1; \mathbb{C})$$

When we have points $\{\sigma_1, \dots, \sigma_k\} \subset M$ that are disjoint from the front projections of the Legendrian isotopy Λ_\bullet , we can further restrict the quasi-equivalence to

$$\mathcal{C}_n(M, \Lambda_0, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}_n(M, \Lambda_1, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$$

Alternating Legendrian

Definition 15. Let M be a surface and $\Lambda \subset T^\infty M$ a Legendrian link. An *alternating coloring* for Λ is the datum of, for each region in the complement of the front diagram, a label of black, white, or null, subject to the following conditions

- the boundary of a black region is co-oriented inward.
- the boundary of a white region is co-oriented outward.
- the boundary of the null region have both inward and outward co-orientations.
- no black region shares a one dimensional border with white region and no null region shares a one dimensional border with another null region.

An *alternating Legendrian* is a Legendrian equipped with an alternating coloring and their front projection is called an *alternating strand diagram*. The *bipartite graph* of the alternating Legendrian is the graph whose vertices are black and white regions. Edges are connected if their closure intersect and are of distinct color.

Let \hat{M} denote the real blow up of M at the crossings of the front projection of Λ . The blow down map $\hat{M} \rightarrow M$ is a diffeomorphism away from the crossing and the fiber above a crossing is the \mathbb{RP}^1 of lines tangent to the crossing. We define $W \subset M$ ($B \subset M$ resp.) the union of the interiors of the white(black resp.) regions of the complement of the front projection.

Definition 16. \bar{L} denote the closure of the preimage of $W \cup B$ in \hat{M} . It is a smooth surface with boundary and we refer to its interior L as the *conjuate surface* of Λ .

Definition 17. An *alternating sheaf* is an object of $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$ whose support is contained in the closure of the union of the white and black regions.

Theorem 18. [STWZ19, Thm. 4.16][STWZ19, Cor. 4.17]. The full subcategory of $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$ consisting of alternating sheaves is equivalent to the category of locally constant sheaves on L . Under this correspondence, rank 1 local systems of corresponds to rank 1 alternating sheaves.

1.2 Summary of results

Chapter 2

We define a class of braids whose associated moduli spaces $\mathcal{M}_{\omega} := \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_{\omega}, \{\sigma_{z << 0}\}; \mathbb{C})$ have simple descriptions, called Sibuya braids. In stead of providing definition, I will illustrate the concept with an example. Consider a braid word $\omega = (s_1 s_2)^2$ on 3 strands, which is a power of the Coxeter braid. Then we have an embedding of ω into the fundamental domain of $S_x^1 \times \mathbb{R}_z$ given in the below figure

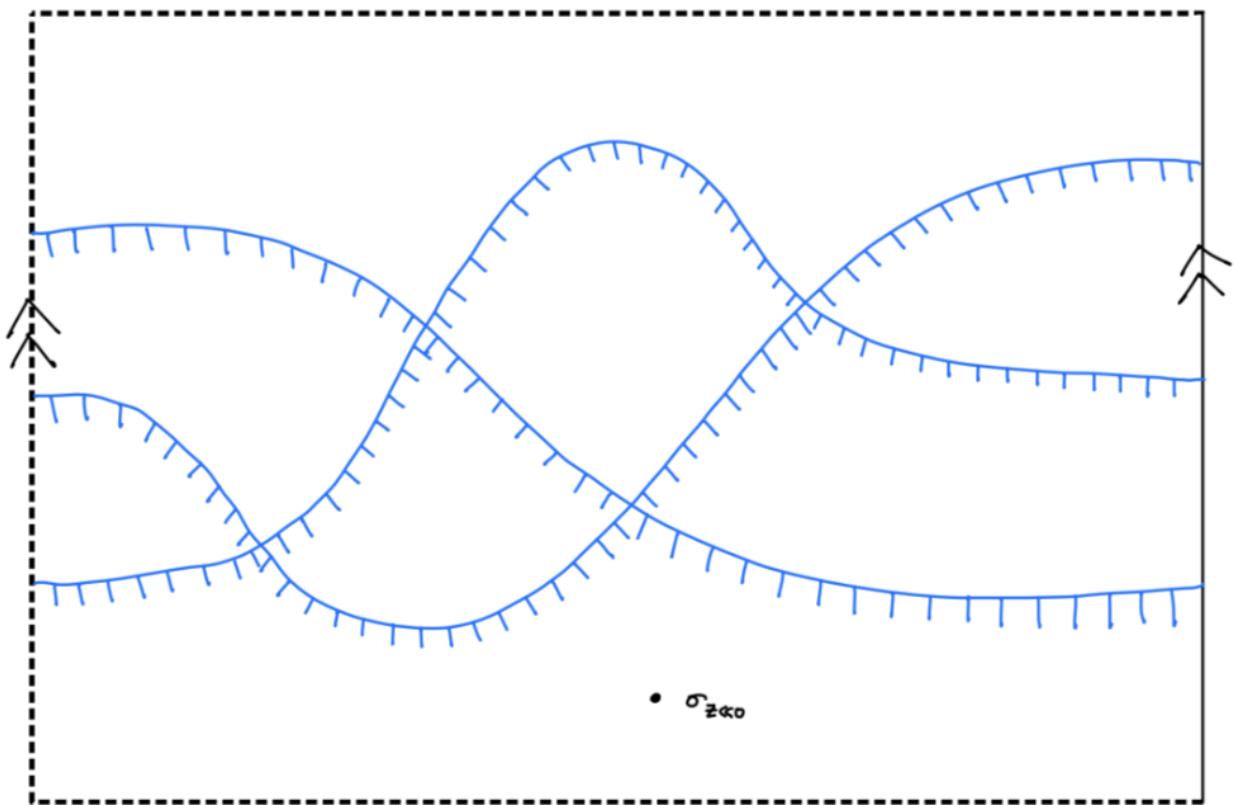


Figure 1.7: Your caption here

Since the stratification induced by ω is not a regular cell complex, for example non-compact regions at the top and the bottom are not contractible, we refine it by adding squiggly lines co-oriented towards left

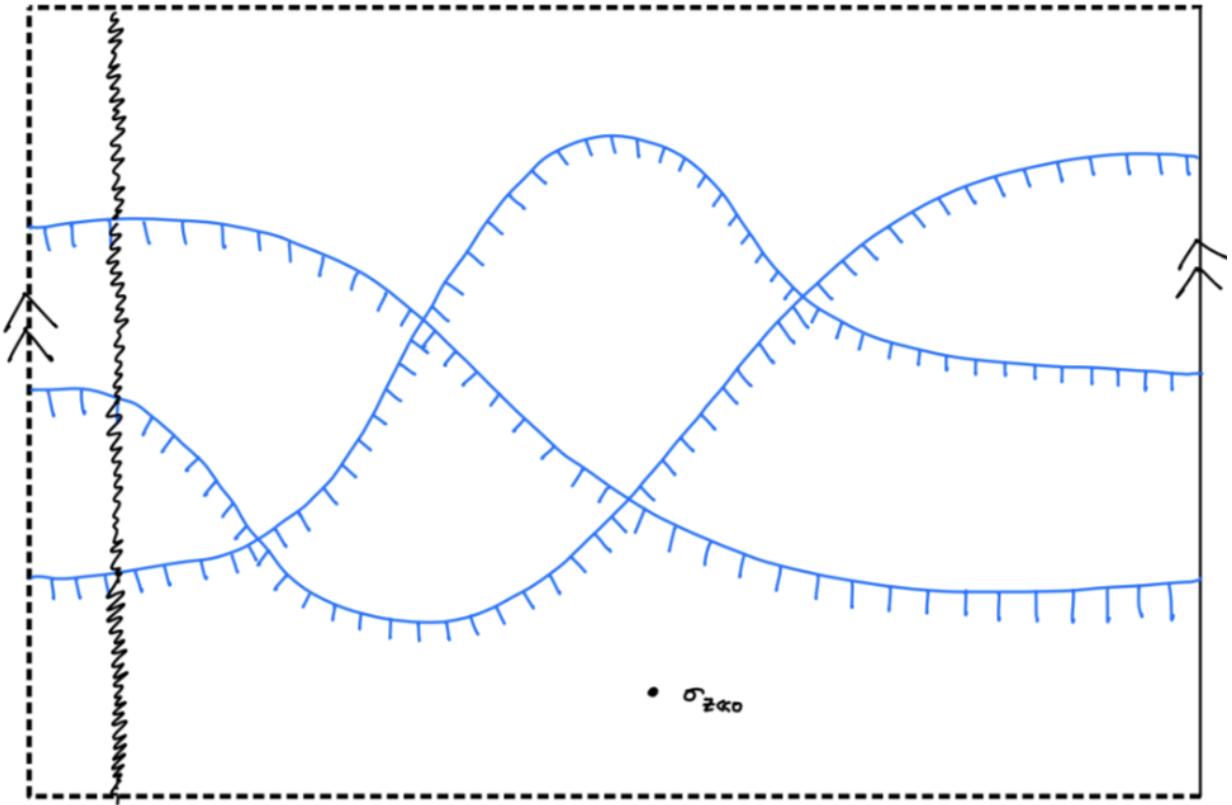


Figure 1.8: Your caption here

Consider an object of $C_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z<<0}\}; \mathbb{C})$ which could be described as a squiggly legible diagram by assigning cochain complexes of \mathbb{C} -vector spaces to regions and cochain maps to arcs subject to the conditions mentioned in (legible object subsection). Let's check how those conditions translates to our situation:

- the cochain complex assigned to the bottom region, which we call height 0 region, is acyclic because of the vanishing condition at $\sigma_{z<<0}$. We are considering cochain complexes of sheaves upto quasi-isomorphism, so we assign 0 stalk to the region.
- the stalk at the regions adjacent to the height 0 region that are not a height 0 region, which we call hieght 1 regions, should be quasi-isomorphic to dimension 1 vector space \mathbb{C} because of the microlocal rank 1 condition on blue arcs separating height 0 region and height 1 regions i.e. the mapping cone of the cochain map

corresponding to blue arcs should be rank 1 vector spaces concentrated in degree 0. Therefore, we assign \mathbb{C} to these height 1 regions and incoming maps from the height 0 region zero maps.

- the stalk at the regions adjacent to the height 1 regions that are not height 0, 1 regions, which we call height 2 regions, should be quasi-isomorphic to dimension 2 vector space \mathbb{C}^2 and incoming maps from the height 1 region to be monomorphisms because of the microlocal rank 1 condition again. Therefore, we assign \mathbb{C}^2 to the height 2 regions and incoming maps from the height 1 regions to be inclusion maps.
- Now, we are left with one non-compact region at the top which we call height 3 region. Again by microlocal rank 1 condition, the stalk should be \mathbb{C}^3 and the incoming maps from height 2 regions are inclusion maps.
- the maps corresponding to squiggly lines are isomorphisms of adjacent vector spaces.
- all diagrams should commute.
- for each crossing the crossing condition translates to

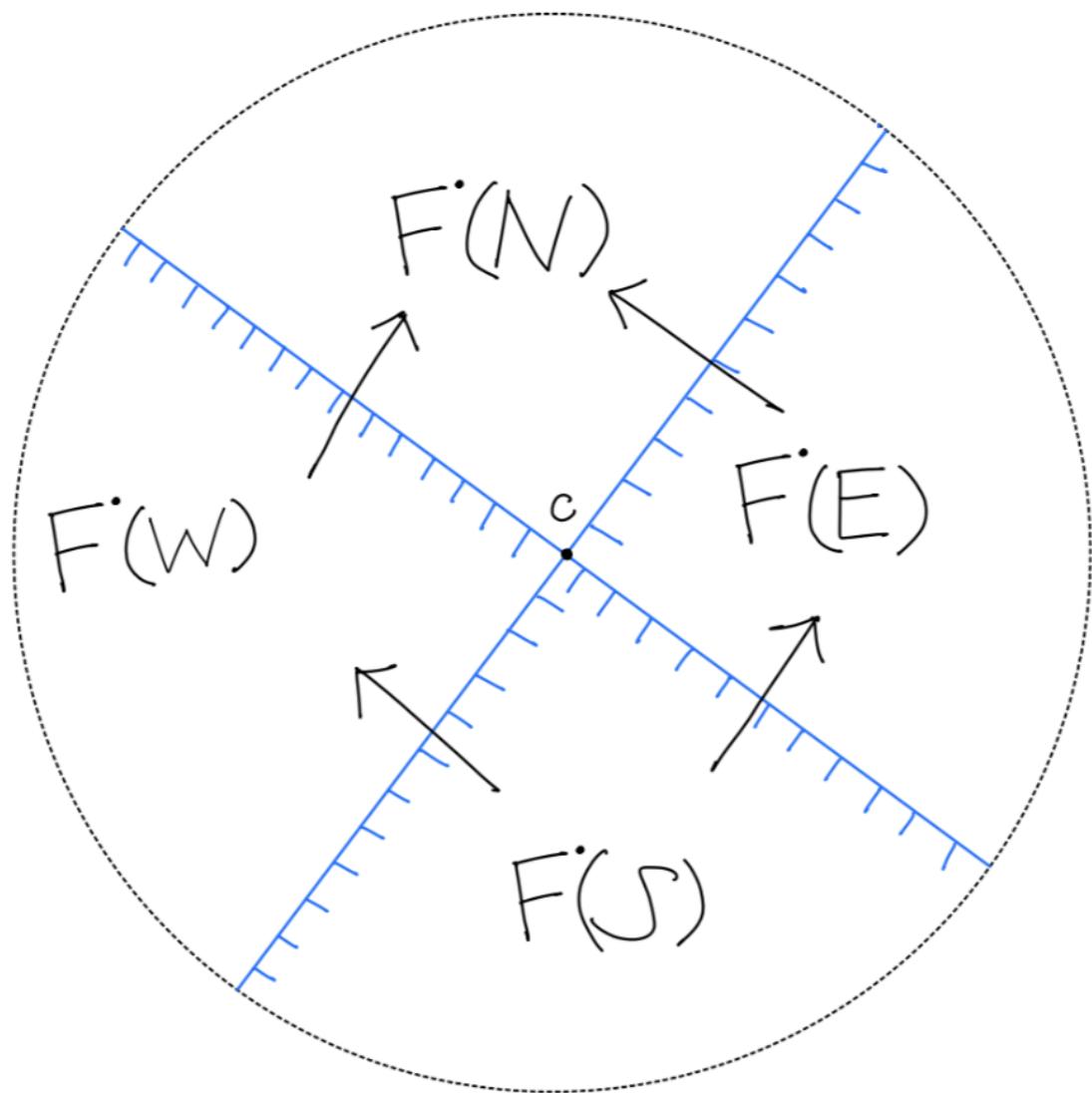


Figure 1.9: Your caption here

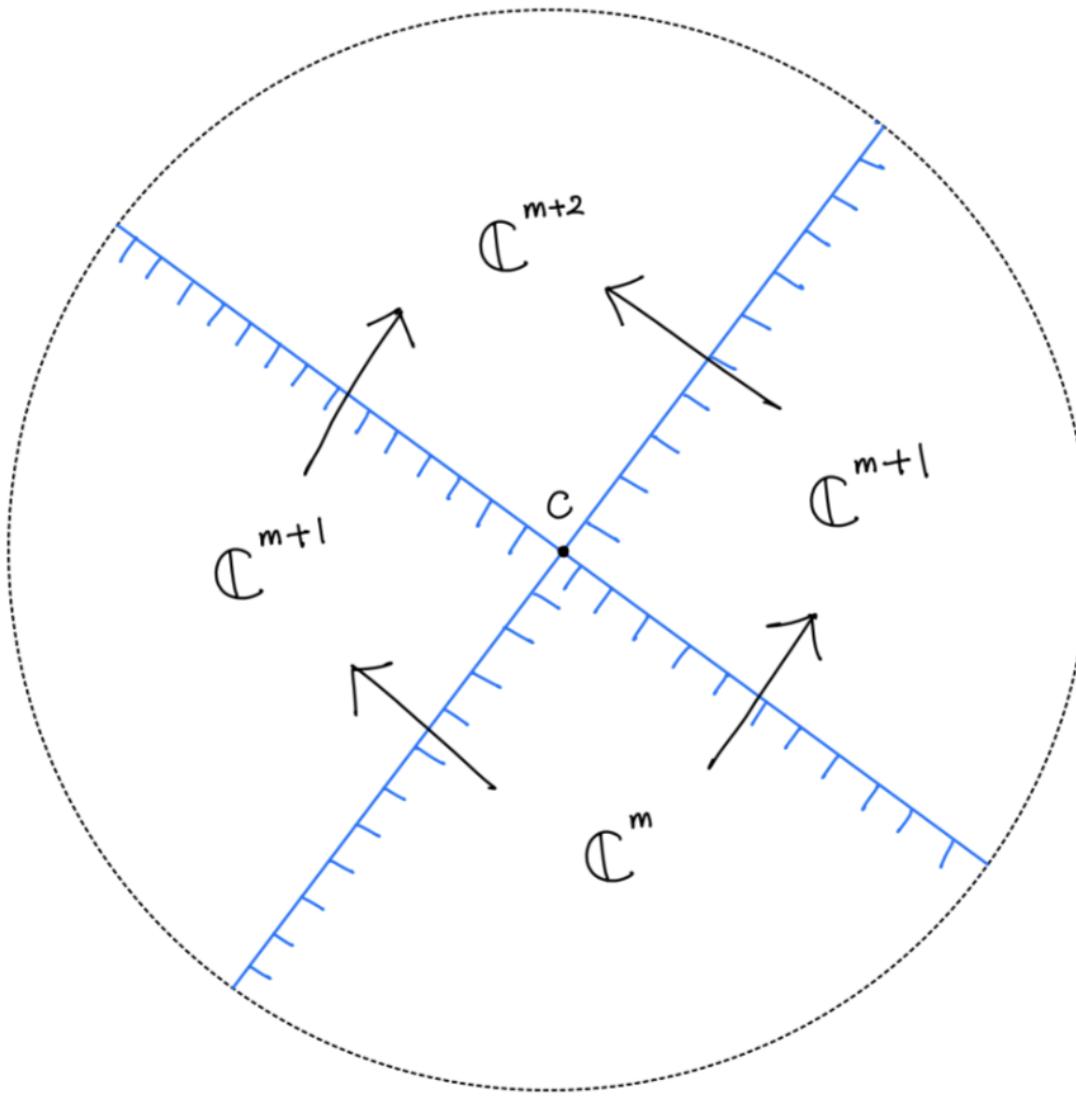


Figure 1.10: Your caption here

the inclusion of $F^\bullet(W)$ and $F^\bullet(E)$ in $F^\bullet(N)$ intersects transversely in $F^\bullet(S)$.

In summary, an object of $C_1(S_x^1 \times \mathbb{R}_z, \Lambda, \{\sigma_{z << 0}\}; \mathbb{C})$ amounts to the data of assigning

- dimension i subspaces of \mathbb{C}^3 to height i regions subject to the condition that $F^\bullet(W) + F^\bullet(E) = F^\bullet(N)$ and $F^\bullet(W) \cap F^\bullet(E) = F^\bullet(S)$.
- an element $g \in GL_3(\mathbb{C})$ that maps the subspaces corresponding to the region to the left of squiggly lines to the subspaces corresponding to the region to the

right of the squiggly lines.

One thing to note in the above example is that once we assign the dimension 1 subspaces to the height 1 regions, all the other subspaces are determined by the crossing conditions. More precisely, we can construct subspaces that are assigned to the regions of heights greater than 1 as follows: Suppose R is a region, then the subspace we assign to R is the subspace of \mathbb{C}^3 spanned by the 1 dimensional subspaces assigned to the height 1 regions that have paths to R . I will describe what this means in Chapter2. This fact(the sheaf is completely determined by its rank 1 stalks) relies on the fact that the diagram induced by ω has the property that every region of height bigger than 1 has more than or equal to 2 different regions of height 1 less adjacent to it. We call a braid word with the above property as Sibuya braid word and a braid with Sibuya braid word representatio as Sibuya braid.

But also note that not all assignments of 1 dimensional subspaces of \mathbb{C}^3 to height 1 regions gives rise to legitimate sheaves. In Chapter2, I will show how to restate the crossing conditions using only rank 1 stalks.

Under this interpretation, the moduli space could be thought of as parametrizing configurations of points on a Projective space subject to certain non-degeneracy conditions and could be thought of as a generalization of Sibuya space that Boalch studied in [Sib75][Boa15].

At the end of the Chapter2, we will explicitly compute examples of the generalized Sibuya space.

Chapter 3

We study a special type of coordinate system of the moduli space $\mathcal{M}_\omega \coprod_{\omega_\emptyset} := \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$ called cluster coordinates.

Let's review how cluster coordinates are constructed. Suppose we have a n alternating Legendrian Λ_{alt} that are Legendrian isotopic to $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset}$ via a Legendrian isotopy

Λ_\bullet whose front projection does not touch $\{\sigma_{z<<0}, \sigma_{z>>0}\}$, then , by GKS thoerem, induces an isomorphism between moduli spaces

$$\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C}) \rightarrow \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

Furthermore, once we have alternating Legendrian, we can construct its conjugate surface L which deformation retracts to the bipartite graph $\Gamma_{\Lambda_{alt}}$ of alternating coloring on Λ_{alt} . Therefore, there is a sequence of isomorphisms

$$H^1(L, \mathbb{C}^*) \cong H^1(\Gamma_{\Lambda_{alt}}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})}$$

where $b_1(\Gamma_{\Lambda_{alt}})$ is the 1st Betti number of $\Gamma_{\Lambda_{alt}}$.

Also, local systems on L corresponds to alternating sheaves in $M_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$ i.e. we have an inclusion

$$H^1(L, \mathbb{C}^*) \hookrightarrow M_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

In conclusion, we have an inclusion

$$(\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})} \hookrightarrow M_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

which is called a cluster coordinate of the moduli space.

In Chapter3,

- (i) we will systematically construct a natural alternating Legendrian(equivalently, alternating strand diagram) that are Legendrian isotopic to the separated Legendrian $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset}$.
- (ii) we will construct a Legendrian isotopy between the alternating Legendrians and the separated Legendrians.

(iii) we will explicitly describe the sheaf cobordism induced by the above isotopy.

Conventions and notation

Chapter 2

Multivalent braids and wild character varieties of Sibuya type

2.1 Basic terminologies

Definition 19. Let ω be a braid word representing a braid $\beta \in Br_n^+$ i.e. $[\omega] = \beta$.

Then we define Q_ω to be the quiver

- whose vertices are labeled by the regions of the front projection(note that this front projection is defined on \mathbb{R}^2 not on a cylinder)
- whose arrows are labeled by pairs of vertices whose corresponding regions are adjacent(bordered by the front projection of the braid) subject to the condition that the arrows always go against the co-orientation(hairs).

There are two distinguished vertices of Q_ω . We denote the vertex corresponding to the region $z \rightarrow \infty$ (resp. $z \rightarrow -\infty$) as U (resp. D).

Maybe good to have an example after each definition.

Definition 20. Locally for each crossing c , there is a region all the hairs are pointing outward, we call this n_c (read north of c). Starting from n_c , as we move counter-

clockwise about the crossing, we call the corresponding regions n_c, e_c, s_c, w_c respectively(read north of, east of, south of, west of c).

Theorem 21. Suppose we have a fixed braid word ω and v is a vertex corresponding to a region given by the front projection of ω . Any path from d to a vertex v have same length.

Proof. we prove the statement by the induction on the length of the braid words. The statement is trivial for trivial braid word because there is only one path from d to any vertex. Suppose the statement is true for all quivers associated to braid words whose length is less than n . Suppose ω is a braid word of length n . Suppose there are two distinct paths p_1 and p_2 from d to v . Let $\omega = s_{i_1} \cdots s_{i_n}$. There are two cases to consider :

(case1) v is the region east of the new crossing generated by s_{i_n} . Then the paths to v must have passed the region right below the region of v because that's the only possible way to get to v under the constraint that the arrow always go against the co-orientation. If we remove the last edge and vertex from the paths p_1, p_2 , we have paths p'_1, p'_2 ending at the same vertex(i.e. south of the crossing generated by s_{i_n}) and are entirely contained in the subquiver $Q_{\omega'}$ where $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Therefore, by the induction hypothesis the lengths of p'_1 and p'_2 are the same which immediately implies $\text{length}(p_1) = \text{length}(p'_1) + 1 = \text{length}(p'_2) + 1 = \text{length}(p_2)$.

(case2) Suppose v is not the region that is the east of the crossing generated by s_{i_n} . Without loss of generality, we can assume that two paths do not pass through the region east of the crossing, because we can always replace the part of the path $s_c \rightarrow e_c \rightarrow n_c$ with $s_c \rightarrow w_c \rightarrow n_c$ having the same length. Then once we know that two paths p_1 and p_2 do not pass through the region east of the crossing, we know that they are entirely contained in the subquiver $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Then the $\text{length}(p_1) = \text{length}(p_2)$ by the induction hypothesis. \square

Definition 22. The valency of the vertex of a quiver is the number of incoming

arrows. The height of the vertex of a quiver is the length of a path starting from d ending at that vertex which is well-defined by the previous theorem. Note that for every crossing c , e_c and w_c have the same height.

Definition 23. We say two vertices are adjacent if there is a crossing c such that two vertices are e_c and w_c of this crossing. Let k be a positive integer, then we define a natural ordering on the set of all height k vertices generated by the following relations

: For each crossing $w_c \leq e_c$.

Theorem 24. The above ordering is well-defined and is a total ordering on the set of all height k vertices.

Proof. To prove the claim, we have to prove the following facts :

- (i) For any two distinct points of height k , there is a chain of crossings connecting two points.
- (ii) there is no chain of crossings starting at a point and ending at the same point.

We prove the claim by induction on the length of braid words. For the trivial braid, (i) holds because there is only one vertex for each height and (ii) holds because there is no arrow starting from that unique and ending at the point.

Now suppose the claim holds for all $\text{length} < n$ braids and suppose $Rn1$ does not hold for the braid word $\omega = s_{i_1} \cdots s_{i_n}$. Then one of the two vertices should be the vertex corresponding to e_c of the crossing generated by s_{i_n} . Let's call this vertex v and the other as v' . Then we know that by the induction hypothesis there is a chain of crossing connecting v' and w_c . Since w_c is connected by the crossing c to $e_c = v$ and v, v' are connected by a chain of crossings which is a contradiction.

Now suppose the claim holds for all $\text{length} < n$ braids and suppose $Rn2$. Suppose there is a point where there is a chain of crossing starting and ending at the same point. By the induction hypothesis, the chain of crossing generated by s_{i_n} along the

way. Without loss of generality, using cyclic shift we can assume the chain of crossings starts from e_c which is a contradiction because e_c is not the w'_c of any crossing c' . \square

Definition 25. There are two distinguished type A sub-quivers of Q_ω . We denote R_ω (resp. L_ω) to be the full sub-quiver containing all the vertices corresponding to the rightmost(resp. leftmost) regions. Alternatively, the path following the largest(resp. smallest) arrows at each step. We denote the vertex of R_ω (resp. L_ω) of height k by R_ω^k (resp. L_ω^k).

Definition 26. Let \mathcal{M}_ω^{fr} be the framed moduli space classifying pairs (F, g) where F is the representation of Q_ω and $g \in GL_n(\mathbb{C})$ subject to the following conditions:

- For each vertex the vectorspace associated to it is a subspace of \mathbb{C}^n of dimension equal to its height.
- All maps are inclusion maps.
- $gF(R_\omega^k) = F(L_\omega^k)$ for all $k = 0, 1, \dots, n$.
- For each crossing c , then the sequence $0 \rightarrow F(s_c) \rightarrow F(e_c) \oplus F(w_c) \rightarrow F(n_c) \rightarrow 0$ is a short exact sequence where maps are induced by the inclusion maps.

Remark 27. There is a natural left action of $x \in GL_n(\mathbb{C})$ on $(F, g)\mathcal{M}_\omega^{fr}$, that is, $x \cdot (F, g) = (xF, xGx^{-1})$ where xF is left translation on quiver representation and xGx^{-1} is conjugation.

Theorem 28. $\mathcal{M}(\beta^\circ) \cong GL_n(\mathbb{C}) \backslash \mathcal{M}_\omega^{fr}$. By $\mathcal{M}(\beta^\circ)$ I mean the moduli space defined in STZ.

Proof. maybe in STZ or STWZ?? \square

Now consider the following map

Definition 29. Suppose we have the braid word ω of a braid β . Let $\{v_1, v_2, \dots, v_m\}$ be the complete list of all height 1 vertices of Q_ω such that $v_i < v_j$ if and only if $i < j$. We define ι_ω to be the forgetful map

$$\begin{aligned}\iota_\omega : \mathcal{M}_\omega^{fr} &\rightarrow (\mathbb{P}^{n-1})^m \times GL_n(\mathbb{C}) \\ (F, g) &\mapsto ([F(v_1)], \dots, [F(v_m)], g)\end{aligned}$$

Definition 30. Let ω be a braid word. We denote the infinite cyclic copies of the original braid word ω by ω^∞ . More precisely, suppose ω is a braid word given by a collection of sections $\{\sigma_i : [0, 1]_x \rightarrow [0, 1]_x \times \mathbb{R}_{y,z}^2\}_{i=1,\dots,n}$, then ω^∞ is given by the collection $\{\bar{\sigma}_i : \mathbb{R}_x \rightarrow \mathbb{R}_{x,y,z}^3 \mid \bar{\sigma}(x, y, z) = \sigma(x - [x], y, z)\}_{i=1,\dots,n}$. Again, I will abuse the notation ω^∞ to denote its front projection onto $\mathbb{R}_{x,z}^2$.

Definition 31. Let ω be a braid word, then we have the quiver Q_ω associated to it. We define the quiver Q_ω^∞ to be the quotient of $\prod_{i \in \mathbb{Z}} Q_\omega$ by the relations $\{R_{\omega,i} = L_{\omega,i+1}\}_{i \in \mathbb{Z}}$ where $R_{\omega,i}$ (resp. $L_{\omega,i+1}$) is the subquiver R_ω (resp. L_ω) in the i^{th} (resp. $i + 1^{th}$) copy of Q_ω in $\prod_{i \in \mathbb{Z}} Q_\omega$. Therefore, we have the quotient map of quivers $q : \prod_{i \in \mathbb{Z}} Q_\omega \rightarrow Q_\omega^\infty$. Let the signature function $\sigma' : Vert(\prod_{i \in \mathbb{Z}} Q_\omega) \rightarrow \mathbb{Z}$ be $\sigma'(v) = i$ if v is in the i^{th} copy of Q_ω in $\prod_{i \in \mathbb{Z}} Q_\omega$. Define $\sigma : Q_\omega^\infty \rightarrow \mathbb{Z}$ to be $\sigma(v) = \min_{w \in q^{-1}(v)} \sigma'(w)$ if $|q^{-1}(v)| < \infty$ and 0 otherwise. Note that for $v \in Vert(Q_\omega^\infty)$, $q^{-1}(v)$ is infinite if and only if $R_\omega^{height(v)} = L_\omega^{height(v)}$ i.e. there is a unique vertex of $height(v)$ in Q_ω . We can think of Q_ω as the full subquiver of Q_ω^∞ spanned by the signature 0 vertices.

Definition 32. Suppose we have a quiver representation F_ω of Q_ω , then we define the induces quiver representation F_ω^∞ of the quiver Q_ω^∞ to be $F_\omega^\infty(v) := g^{\sigma(v)} \cdot F_\omega(v - \sigma(v))$.

Definition 33. Υ_0^k is the set of all height k , signature zero vertices in Q_ω^∞ . For each vertex v of Q_ω^∞ , we define $I_k(v)$ to be the set of all the height k vertices that have paths to v .

Definition 34. $\mathcal{M}_\omega^{fr,\infty}, \mathcal{M}_\omega^{fr,\infty,+}$

Theorem 35. $\mathcal{M}_\omega^{fr} \cong \mathcal{M}_\omega^{fr,\infty}$

2.2 Multivalent braid words

Definition 36. A braid word ω is multivalent if and only if the valencies are all greater than 1 for vertices in Q_ω^∞ .

Theorem 37. If the braid word ω is multivalent, then ι_ω is an embedding.

Proof. It is enough to prove that once we specify vectorspaces to height 1 vertices of Q_ω and $g \in GL_n(\mathbb{C})$, the quiver representation of Q_ω extending the above datum is unique. Since $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$, it is enough to prove once we specify vectorspaces to signature 0 height 1 vertices of Q_ω^∞ and $g \in GL_n(\mathbb{C})$, the quiver representation F_ω^∞ of Q_ω^∞ extending the above datum is unique. Since $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, for $v \in Vert(Q_\omega^\infty)$, $F_\omega^\infty(v) = g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))$. If $height(v) = 1$, then $v - \sigma(v)$ is height 1 signature 0 vertex. Therefore, $F_\omega^\infty(v)$ is uniquely determined.

Now we prove the statement by induction on the heights of vertices. Assume $\forall v \in Vert(Q_\omega^\infty)$ with $height(v) < h$, $F_\omega^\infty(v)$ are determined. Suppose $v \in Vert(Q_\omega^\infty)$ such that $height(v) = h > 1$, then there are at least two vertices of height $h - 1$ that have arrows to v , say v' and v'' . Without loss of generality, $v' \leq v''$. Then by ω^∞ , Q_ω^∞ -analogue of Theorem 13, there is a chain of crossings c_1, \dots, c_k and $v' = v_0, \dots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i . Therefore, if we choose any c_i to be c , then $v = N_c$ i.e. v is the north of the crossing c . By the induction hypothesis, $F_\omega^\infty(W_c)$, $F_\omega^\infty(E_c)$ and $F_\omega^\infty(S_c)$ have already been specified because heights of W_c, E_c , and S_c are $h - 1, h - 1$, and $h - 2$ respectively. By the crossing condition,

$$0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

$F_\omega^\infty(N_c)$ is uniquely determined. \square

Theorem 38. If the braid word ω is multivalent, then the image of ι_ω is

$$X_\omega = \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}_v^{n-1} \times GL_n(\mathbb{C}) \mid$$

$$\forall u \in Vert(Q_\omega^\infty), \dim_{\mathbb{C}} (\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u),$$

$$\forall c \in Cross(\omega^\infty), \dim_{\mathbb{C}} (\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(N_c)\}$$

where \mathbb{P}_v^{n-1} is a copy of \mathbb{P}^{n-1} labeled by $v \in \Upsilon_0^1$.

Proof. Instead of proving $Im(\iota_\omega) = X_\omega$, I will prove that for ι' , the map obtained by pre-composing the canonical isomorphism $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$ to ι_ω , $Im(\iota') = X_\omega$.

First, let's prove $Im(\iota') \subset X_\omega$. Recall that

$$\mathcal{M}_\omega^{fr,\infty} = \{(F_\omega^\infty, g) \in Rep(Q_\omega^\infty) \times GL_n(\mathbb{C}) \mid All \ the \ maps \ of \ F_\omega^\infty \ are \ inclusion \ maps,$$

$$\forall v \in Vert(Q_\omega^\infty), \dim_{\mathbb{C}} (F_\omega^\infty(v)) = height(v),$$

$$\forall v \in Vert(Q_\omega^\infty), F_\omega^\infty(v+n) = g^n \cdot F_\omega^\infty(v),$$

$$\forall c \in Cross(\omega^\infty), 0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0 \text{ are short exact sequences}$$

We need to show that for $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, $\iota'_\omega((F_\omega^\infty, g)) = ((F_\omega^\infty(v))_{v \in \Upsilon_0^1}, g) \in X_\omega$ i.e.

- $\forall u \in Vert(Q_\omega^\infty), \dim_{\mathbb{C}} (\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = height(u)$
- $\forall c \in Cross(\omega^\infty), \dim_{\mathbb{C}} (\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = height(N_c)$

It is enough to prove that

- $\forall u \in Vert(Q_\omega^\infty), \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(u)$
- $\forall c \in Cross(\omega^\infty), \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(N_c)$

because of the condition $\forall u \in Vert(Q_\omega^\infty), \dim_{\mathbb{C}} (F_\omega^\infty(u)) = height(u)$ defining $\mathcal{M}_\omega^{fr,\infty}$.

Moreover, $g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(v)$ by the definition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, we

need to prove that,

- (i) $\sum_{v \in I_1(u)} F_\omega^\infty(v) = F_\omega^\infty(u)$
- (ii) $\sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) = F_\omega^\infty(N_c)$

Proof of (i) : For each $v \in I_1(u)$, there is a path from v to u , say $v = v_0 \rightarrow \dots \rightarrow v_k = u$. Since $F_\omega^\infty(v_i) \subset F_\omega^\infty(v_{i+1})$, $F_\omega^\infty(v) \subset F_\omega^\infty(u)$. Therefore, $\sum_{v \in I_1(u)} F_\omega^\infty(v) \subset F_\omega^\infty(u)$. Conversely, we prove $F_\omega^\infty(u) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$ by induction on the height of u . If $height(u) = 1$, the statement is trivial. Now suppose the statement holds for all $u \in Vert(Q_\omega^\infty)$ of $height(u) = h$. Since ω is multivalent, u has at least two vertices of height $h - 1$ that have arrows to u , say v' and v'' . Without loss of generality $v' \leq v''$. Then by $\omega^\infty, Q_\omega^\infty$ -analogue of Theorem 13, there is a chain of crossings c_1, \dots, c_k and $v' = v_0, \dots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i . Therefore, if we choose any c_i to be c , then $u = N_c$ i.e. v is the north of the crossing c . Then by the crossing condition of $M_\omega^{fr, \infty}$

$$0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

we have $F_\omega^\infty(u) = F_\omega^\infty(N_c) = F_\omega^\infty(E_c) + F_\omega^\infty(W_c)$ by the induction hypothesis,

$$\begin{aligned} F_\omega^\infty(E_c) &= \sum_{v \in I_1(E_c)} F_\omega^\infty(v), \quad F_\omega^\infty(W_c) = \sum_{v \in I_1(W_c)} F_\omega^\infty(v) \\ \Rightarrow F_\omega^\infty(u) &= \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \end{aligned}$$

because E_c and W_c have arrows to $u = N_c \Rightarrow I_1(E_c) \cup I_1(W_c) \subset I_1(u)$. Therefore,

$$F_\omega^\infty(u) = \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$$

Proof of (ii) : By (i), we have

$$\begin{aligned}
 & \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \\
 &= \sum_{v \in I_1(E_c)} F_\omega^\infty(v) + \sum_{v \in I_1(W_c)} F_\omega^\infty(v) \\
 &= F_\omega^\infty(E_c) + F_\omega^\infty(W_c)
 \end{aligned}$$

This is equal to $F_\omega^\infty(N_c)$ by the crossing condition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, the proof of (ii) is complete.

Now let's prove $X_\omega \subset Im(\iota'_\omega)$. Let $((z_v)_{v \in \Upsilon_0^1}, g)$ be an arbitrary point of X_ω . I will define a point (F_ω^∞, g) such that $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$. We define a quiver representation F_ω^∞ to be

$$F_\omega^\infty(u) := \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}$$

and all the arrows of F_ω^∞ are inclusion maps. The inclusion maps are well-defined because if there is an arrow from u to u' , then $I_1(u) \subset I_1(u')$.

Note that if $u \in \Upsilon_0^1$ i.e. $height(u) = 1$ and $\sigma(u) = 0$, then

$$F_\omega^\infty(u) := \sum_{v \in \{u\}} g^{\sigma(v)} \cdot z_{v-\sigma(v)} = g^{\sigma(u)} \cdot z_{u-\sigma(u)}$$

If (F_ω^∞, g) is indeed a point of $\mathcal{M}_\omega^{fr,\infty}$, then $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$. Thus, it is enough to prove that $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$ i.e.

- (i) All maps of F_ω^∞ are inclusion maps
- (ii) $\forall v \in Vert(Q_\omega^\infty), dim_{\mathbb{C}}(F_\omega^\infty(v)) = height(v)$
- (iii) $\forall v \in Vert(Q_\omega^\infty) \text{ and } n \in \mathbb{Z}, F_\omega^\infty(v+n) = F_\omega^\infty(v)$

- (iv) $\forall c \in Cross(\omega^\infty)$, $0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$ are short exact sequences

We get (i) immediately from the definition of F_ω^∞ .

To prove (ii), note that $\dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u)$ because $((z_v)_{v \in \Upsilon_0^1}, g) \in X_\omega$. Therefore, $\dim_{\mathbb{C}}(F_\omega^\infty(u)) = \dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = height(u)$.

To prove (iii), note that $\forall u \in Vert(Q_\omega^\infty)$ and $n \in \mathbb{Z}$, $I_1(u+n) = I_1(u) + n$. Therefore,

$$\begin{aligned} F_\omega^\infty(u+n) &= \sum_{v \in I_1(u+n)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v \in I_1(u)+n} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v-n \in I_1(u)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v' \in I_1(u)} (g^{\sigma(v'+n)} \cdot z_{(v'+n)-\sigma(v'+n)}) \\ &= g^n \cdot \left(\sum_{v' \in I_1(u)} (g^{\sigma(v')} \cdot z_{v'-\sigma(v')}) \right) \\ &= g^n \cdot F_\omega^\infty(u) \end{aligned}$$

Finally, let's prove (iv). Let $c \in Cross(\omega^\infty)$, then

$$\begin{aligned} height(S_c) + 2 &= height(E_c) + 1 = height(W_c) + 1 = height(N_c) \\ \Rightarrow \dim_{\mathbb{C}}(F_\omega^\infty(S_c)) + 2 &= \dim_{\mathbb{C}}(F_\omega^\infty(E_c)) + 1 = \dim_{\mathbb{C}}(F_\omega^\infty(W_c)) + 1 = \dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \end{aligned}$$

Since there are arrows from S_c to E_c , W_c and from E_c , W_c to N_c , $I_1(S_c) \subset I_1(E_c)$, $I_1(W_c) \subset I_1(N_c)$. Therefore, $F_\omega^\infty(S_c) \subset F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c)$ and $F_\omega^\infty(E_c) \cup F_\omega^\infty(W_c) \subset F_\omega^\infty(N_c)$.

By the condition that

$$\begin{aligned} \dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) &= \text{height}(N_c) \\ \Rightarrow \dim_{\mathbb{C}} (F_{\omega}^{\infty}(E_c) + F_{\omega}^{\infty}(W_c)) &= \text{height}(N_c) = \dim_{\mathbb{C}} (F_{\omega}^{\infty}(N_c)) \\ \Rightarrow F_{\omega}^{\infty}(E_c) + F_{\omega}^{\infty}(W_c) &= F_{\omega}^{\infty}(N_c) \end{aligned}$$

we have surjection part of the short exact sequence i.e. $F_{\omega}^{\infty}(E_c) \oplus F_{\omega}^{\infty}(W_c) \rightarrow F_{\omega}^{\infty}(N_c) \rightarrow 0$ is exact. Thus we have a short exact sequence

$$0 \rightarrow F_{\omega}^{\infty}(E_c) \cap F_{\omega}^{\infty}(W_c) \rightarrow F_{\omega}^{\infty}(E_c) \oplus F_{\omega}^{\infty}(W_c) \rightarrow F_{\omega}^{\infty}(N_c) \rightarrow 0$$

I claim that $F_{\omega}^{\infty}(E_c) \cap F_{\omega}^{\infty}(W_c) = F_{\omega}^{\infty}(S_c)$. Since $F_{\omega}^{\infty}(E_c)$, $F_{\omega}^{\infty}(W_c)$, and $F_{\omega}^{\infty}(S_c)$ are codimension 1, 1, and 2 inside of $F_{\omega}^{\infty}(N_c)$ respectively, it is enough to show that $F_{\omega}^{\infty}(E_c)$ and $F_{\omega}^{\infty}(W_c)$ intersect transversely in $F_{\omega}^{\infty}(N_c)$ i.e. $F_{\omega}^{\infty}(E_c) \neq F_{\omega}^{\infty}(W_c)$. This is true because otherwise

$$\begin{aligned} \text{height}(N_c) &= \dim_{\mathbb{C}} (F_{\omega}^{\infty}(N_c)) \\ &= \dim_{\mathbb{C}} (F_{\omega}^{\infty}(E_c) + F_{\omega}^{\infty}(W_c)) \\ &= \dim_{\mathbb{C}} (F_{\omega}^{\infty}(E_c)) \\ &= \text{height}(E_c) \end{aligned}$$

which is a contradiction.

Therefore, (F_{ω}^{∞}, g) satisfy all the condition (i)-(iv) i.e. $(F_{\omega}^{\infty}, g) \in \mathcal{M}_{\omega}^{fr, \infty}$. \square

Theorem 39. Combinatorial characterization of multivalent braid words. More precisely, the braid word is multivalent if and only if in between s_i 's there is at least one s_{i-1} upto cyclic shifts for all i's.

Lemma 40. there is a one to one correspondence between the set of region of height

k except the smallest one and s_k 's in the braid word.

Proof. by induction on the length □

Lemma 41. the valency of a region corresponding to a certain s_k is equal to the number of s_{k-1} in between that certain s_k and the next s_k

Proof. by induction on the length □

(examples : powers of Coxeter braid)

Theorem 42. Theorems terminologies involving braid words and terminologies involving the associated quivers.

2.3 Bivalent braid words

Definition 43. A braid word ω is bivalent if it is multivalent and the valencies of all the vertices of height $1, \dots, n - 1$ in Q_ω^∞ are equal to 2 where n is the number of strands of ω .

Lemma 44. Let ω be a braid word and $v \in Vert(Q_\omega)(Vert(Q_\omega^\infty) \text{ resp.})$, then there is a path from v to the unique height n vertex U . The path passes through a vertex of height $n - 1$.

Proof. If we prove the statement for Q^ω , then the proof for the Q_ω^∞ case follows immediately. Let's prove the claim by induction on the length of the braid word ω . If $length(\omega) = 0$ i.e. trivial braid word, there is a path from D to U and along the way it passes through all the points. Therefore, restricting this path to start from the point that we are interested in gives the desired path. Now let's assume that the statement holds for all braid words of length less than k . Let $\omega = s_{i_1} \cdots s_{i_k}$, then define $\omega' = s_{i_1} \cdots s_{i_{k-1}}$. We get that $Q_{\omega'}$ is the subquiver of Q_ω where $Vert(Q_\omega) - Vert(Q_{\omega'}) = \{v\}$ is a singleton where v is the east of the crossing added by s_{i_k} , say c . Note that $v = E_c$

has an arrow to N_c and N_c has a path to U by the induction hypothesis because $N_c \in Vert(Q_{\omega'})$. Therefore, we can extend the path from N_c to U to start from E_c i.e. $E_c \rightarrow (N_c \rightarrow \dots \rightarrow U)$. \square

Lemma 45. Let ω be a bivalent braid word and $u \in Vert(Q_{\omega}^{\infty})$ of $height(u) < n$, then $|I_1(u)| = height(u)$. In particular, if $c \in Cross(\omega^{\infty})$, then $I_1(S_c) = height(S_c)$, $I_1(E_c) = height(E_c)$, and $I_1(W_c) = height(W_c)$.

Proof. We prove the statement by induction on the height of u . If $height(u) = 1$, then $I_1(u) = \{u\} \Rightarrow |I_1(u)| = 1 = height(u)$ holds. Now suppose the statement holds for vertices of heights less than h and $height(u) = h$ where $h < n$. Since ω is bivalent, we have exactly two vertices of height $h - 1$ and a crossing c , where those two vertices are the east and west of the crossing c . By the induction hypothesis, $|I_1(E_c)| = |I_1(W_c)| = h - 1$, $|I_1(S_c)| = h - 2$ and $I_1(S_c) \subset I_1(E_c), I_1(W_c)$. Let

$$I_1(E_c) - I_1(S_c) = \{v_1\}$$

$$I_1(W_c) - I_1(S_c) = \{v_2\}$$

Since ω is bivalent,

$$\begin{aligned} I_1(N_c) &= I_1(E_c) \cup I_1(W_c) \\ &= \{v_1, v_2\} \cup I_1(S_c) \end{aligned}$$

If $v_1 \neq v_2 \Leftrightarrow I_1(E_c) \neq I_1(W_c)$, then $|I_1(N_c)| = 2 + (h - 2) = h$. Therefore, it is enough to prove that $\forall c \in Cross(\omega^{\infty})$, $I_1(E_c) \neq I_1(W_c)$. This follows from Lemma 35 below. \square

Lemma 46. Let ω be a bivalent braid word and u, v be distinct vertices of Q_{ω}^{∞} of the same height, then $I_1(u) \neq I_1(v)$. Note that the height of u, v cannot be n because there is only one vertex of height n , say U .

Proof. We prove the claim by the induction on the height of u and v . If $\text{height}(u) = \text{height}(v) = 1$, then the claim holds because $I_1(u) = \{u\}$ and $I_1(v) = \{v\}$.

Now suppose the claim holds for vertices of heights less than h . Then there are exactly two height $h - 1$ vertices for each, say $\{u_1, u_2\}, \{v_1, v_2\}$, that have arrows to u and v . Note that there are crossings c, c' such that $\{u_1, u_2\} = \{E_c, W_c\}, \{v_1, v_2\} = \{E_{c'}, W_{c'}\}$. $\{u_1, u_2\} \neq \{v_1, v_2\}$, otherwise $c = c' \Rightarrow u = N_c = N_{c'} = v$ which is a contradiction. Therefore, by the induction hypothesis, $I_1(u) = I_1(u_1) \cup I_1(u_2) \neq I_1(v_1) \cup I_1(v_2) = I_1(v)$. \square

Lemma 47. Let ω be a bivalent braid word and $c \in \text{Cross}(\omega^\infty)$, then $I_1(E_c) \cap I_1(W_c) = I_1(S_c)$ and $|I_1(E_c) \cup I_1(W_c)| = \text{height}(N_c)$.

Proof. $I_1(S_c) \subset I_1(E_c) \cap I_1(W_c)$ because there are arrows from S_c to E_c, W_c . Since $E_c \neq W_c$, by Lemma 35, $I_1(E_c) \neq I_1(W_c)$. Therefore,

$$\begin{aligned} |I_1(E_c) \cap I_1(W_c)| &\leq |I_1(E_c)| - 1 = |I_1(S_c)| \\ \Rightarrow I_1(E_c) \cap I_1(W_c) &= I_1(S_c) \end{aligned}$$

As a consequence,

$$\begin{aligned} |I_1(E_c) \cup I_1(W_c)| &= |I_1(E_c)| + |I_1(W_c)| - |I_1(E_c) \cap I_1(W_c)| \\ &= |I_1(E_c)| + |I_1(W_c)| - |I_1(S_c)| \\ &= (h - 1) + (h - 1) - (h - 2) = h \\ &= |I_1(N_c)| \\ &= \text{height}(N_c) \end{aligned}$$

\square

Definition 48. Suppose $L_1, L_2, \dots, L_k \subset \mathbb{C}^n$ are lines(1 dimensional subspaces). We say $\{L_1, L_2, \dots, L_k\}$ are linearly independent if and only if for nonzero $v_i \in L_i$,

$\{v_1, v_2, \dots, v_k\}$ are linearly independent. Note that the definition does not depend on the choice of v_i 's.

Theorem 49. Let ω be a bivalent braid word, then ι_ω is an open embedding whose image is

$$X'_\omega = \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}^{n-1} \times GL_n(\mathbb{C}) \mid$$

$$\forall c \in Cross(\omega) \subset Cross(\omega^\infty) \text{ with } height(N_c) = n,$$

$$dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n \}$$

Proof. Since bivalent braids are multivalent, by Theorem 27, the image of ι_ω is given by

$$X_\omega = \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}_v^{n-1} \times GL_n(\mathbb{C}) \mid$$

$$\forall u \in Vert(Q_\omega^\infty), dim_{\mathbb{C}} \left(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = height(u),$$

$$\forall c \in Cross(\omega^\infty), dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = height(N_c) \}$$

I claim that $X_\omega = X'_\omega$. Since the condition

$$\forall c \in Cross(\omega) \subset Cross(\omega^\infty) \text{ with } height(N_c) = n, dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n$$

defining X'_ω is subsumed in the condition

$$\forall c \in Cross(\omega^\infty), dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = height(N_c)$$

defining X_ω , $X_\omega \subset X'_\omega$.

Now let's prove $X'_\omega \subset X_\omega$ i.e. suppose $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy

- (i) $\forall c \in \text{Cross}(\omega) \subset \text{Cross}(\omega^\infty)$ with $\text{height}(N_c) = n$, $\dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n$

then it also satisfy

- (ii) $\forall u \in \text{Vert}(Q_\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \text{height}(u)$
- (iii) $\forall c \in \text{Cross}(\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \text{height}(N_c)$

Let $u \in \text{Vert}(Q_\omega^\infty)$ and $c \in \text{Cross}(\omega^\infty)$. First let's assume $\text{height}(u) < n$. Since ω is bivalent, by Lemma 34, $|I_1(u)| = \text{height}(u)$ and by Lemma 34 and Lemma 36, $|I_1(E_c) \cup I_1(W_c)| = |I_1(N_c)| = \text{height}(N_c)$. Therefore, the condition (i) is equivalent to saying that

$\forall c \in \text{Cross}(\omega)$ with $\text{height}(N_c) = n$, $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent

Likewise, the conditions (ii),(iii) are equivalent to saying that

$\forall u \in \text{Vert}(Q_\omega^\infty)$, $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$ are linearly independent

$\forall c \in \text{Cross}(\omega^\infty)$, $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent

Let's prove (i) implies (ii) and (iii) using these paraphrased statements. Assume (i) holds for $((z_v)_{v \in \Upsilon_0^1}, g)$. Suppose for some $u \in \text{Vert}(Q_\omega^\infty)$, $\{g_{v \in I_1(u)}^{\sigma(v) \cdot z_{v-\sigma(v)}}\}$ are linearly dependent. Let $u' = u - \sigma(u)$, then $I_1(u') = I_1(u) - \sigma(u)$. Therefore,

$$\begin{aligned} \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u')} &= \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v+\sigma(u) \in I_1(u)} \\ &= \{g^{\sigma(v')-\sigma(u)} \cdot z_{v'-\sigma(v')}\}_{v' \in I_1(u)} \\ &= g^{-\sigma(u)} \cdot \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)} \end{aligned}$$

are also linearly dependent. Therefore, without loss of generality, we can assume $\sigma(u) = 0$ i.e. $u \in \text{Vert}(Q_\omega) \subset \text{Vert}(Q_\omega^\infty)$. By Lemma 33, there is a path from u to a height $n - 1$ vertex p . Thus, $I_1(u) \subset I_1(p)$. Since ω is bivalent, for some $c \in \text{Cross}(\omega)$, $p = E_c$ or W_c . By the condition (i), $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent, thus its subset $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$ are linearly independent as well. Therefore, $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (ii).

Now let's show that $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (iii). Suppose $c \in \text{Cross}(\omega^\infty)$. If $\text{height}(N_c) = n$, then the condition (i) is equal to the condition (iii), there is nothing to prove. Suppose $\text{height}(N_c) < n$, then $I_1(E_c) \cup I_1(W_c) = I_1(N_c)$. Thus, the condition (iii) translates to $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(N_c)}$ are linearly independent, which follows from (ii) that we already proved.

Now let's show that (iii) holds when $\text{height}(u) = n$ i.e. u is the unique height n point U . Since ω is bivalent, there is a crossing c such that $u = N_c$. Therefore,

$$\dim_{\mathbb{C}} \left(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \text{height}(N_c) = n$$

Since $I_1(E_c) \cup I_1(W_c) \subset I_1(N_c) = I_1(u)$,

$$\dim_{\mathbb{C}} \left(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n = \text{height}(u)$$

□

Theorem 50. Combinatorial characterization of bivalent braid words. More precisely, the braid word is multivalent if and only if in between s_i 's there is at least one s_{i-1} upto cyclic shifts for all i's.

(examples : powers of Coxeter braid)

Definition 51. Wild character varieties of Sibuya type are wild character varieties associated to bivalent braids.

Remark 52. microlocal monodromy as generalized cross-ratio. Symplectic leaves.

2.4 Examples : wild character varieties of Sibuya type

By nequation, I mean a formula that expresses the non-equality of two expressions.

Whenever I denote capital X with subscript i.e. X_j , I mean an element of some projective space.

Lower case x 's with subscripts are used to denote the homogeneous coordinates of X_j 's.

For example $X_j = [x_{1,j} : \dots : x_{n,j}]$.

I will also denote

$$(X_1, \dots, X_{n-1}) = ([x_{1,1} : \dots : x_{n,1}], [x_{1,2} : \dots : x_{n,2}], \dots, [x_{1,n-1} : \dots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

We will use

$$\left(\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_n \end{array} \right) = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

to denote the matrix whose entries are $x_{i,j}$ which are sections of the line bundle $O_{\mathbb{P}^k}(1)$.

Thereby

$$\det \begin{pmatrix} & | & | & | \\ X_1 & | & X_2 & | & \cdots & | & X_n \end{pmatrix} = \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

is a section of the line bundle $\underbrace{\mathcal{O}_{\mathbb{P}^k}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^k}(1)}_{n-\text{copies}}$ on $\underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{n-\text{copies}}$.

Example 53. (Empty Set) In this section we compute 2 kinds of moduli spaces that are empty.

In the first case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than the number of strands with unipotent monodromy $u = I$

By the theorem ??, the framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \dots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{k\text{-copies}} \mid \det \begin{pmatrix} & | & | & | & | & | & | \\ X_1 & X_2 & \dots & X_k & X_1 & \dots & X_{n-k} \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_2 & \dots & X_k & X_1 & \dots & X_{n-k+1} \end{pmatrix} \neq 0, \dots,$$

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_k & X_1 & \dots & X_k & X_1 & \dots & X_{n-k-1} \end{pmatrix} \neq 0\}$$

Take the first nequation and subtract 1^{st} column from the $(k+1)^{th}$ column we get

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_1 & X_2 & \dots & X_k & 0 & X_2 & \dots & X_{n-k} \end{pmatrix} \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the moduli space.

In the second case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than (*the number of strands*) – 1 with unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

By the theorem ??, the framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \dots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{k\text{-copies}} \mid \det \begin{pmatrix} & | & | & | & | & | & | \\ X_1 & | & X_2 & | & \dots & | & X_k & | & uX_1 & | & \dots & | & uX_{n-k} \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_2 & | & \dots & | & X_k & | & uX_1 & | & \dots & | & uX_{n-k+1} \end{pmatrix} \neq 0, \dots,$$

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_k & | & uX_1 & | & \dots & | & uX_k & | & u^2X_1 & | & \dots & | & u^2X_{n-k-1} \end{pmatrix} \neq 0$$

Take the first nequation and subtract 1^{st} (resp. 2^{nd}) column from the $(k+1)^{th}$ (resp. $(k+2)^{th}$) column we get

$$\det \begin{pmatrix} & | & | & | & | & | & | \\ X_1 & | & X_2 & | & \dots & | & X_k & | & 0 & | & 0 & | & \dots & | & 0 \\ & & & & & & & & \vdots & & \vdots & & & & & \\ & & & & & & & & 0 & | & 0 & | & uX_3 & | & \dots & | & uX_{n-k} \\ & & & & & & & & x_{n,1} & | & x_{n,2} & & & & & & \\ & & & & & & & & 0 & | & 0 & & & & & & \end{pmatrix} \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the

moduli space.

Example 54. (Points) In this section, we compute the moduli space of braids represented by the braid word

$$(s_1 s_2 \cdots s_{n-1})^{n-1}$$

and the unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

i.e. a unipotent matrix given by the partition $(\underbrace{1, 1, \cdots, 1}_{n-1}, 2)$.

Therefore by theorem??, the framed moduli space associated to the above data is given as follows

$$\mathcal{M}^{fr} = \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \begin{pmatrix} & | & | & | & | \\ X_1 & | & X_2 & | & \dots & | & X_{n-1} & | & uX_1 \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} & | & | & | & | \\ X_2 & | & \dots & | & X_n & | & uX_1 & | & uX_2 \end{pmatrix} \neq 0, \dots, \det \begin{pmatrix} & | & | & | & | \\ X_{n-1} & | & uX_1 & | & \dots & | & uX_{n-2} & | & uX_{n-1} \end{pmatrix} \neq 0\}$$

Using the elementary column operation of subtracting the first column from the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \begin{pmatrix} & | & | & | & | & | \\ X_1 & | & X_2 & | & \dots & | & X_{n-1} & | & 0 \\ & & & & & & & | & \vdots \\ & & & & & & & | & x_{n,1} \\ & & & & & & & | & 0 \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} & | & | & | & | & | \\ X_2 & | & \dots & | & X_{n-1} & | & uX_1 & | & 0 \\ & & & & & & | & \vdots & | \\ & & & & & & | & x_{n,2} & | \\ & & & & & & | & 0 & | \end{pmatrix} \neq 0, \dots, \det \begin{pmatrix} & | & | & | & | & | \\ X_{n-1} & | & uX_1 & | & \dots & | & uX_{n-2} & | & 0 \\ & & & & & & & | & \vdots \\ & & & & & & & | & x_{n,n-1} \\ & & & & & & & | & 0 \end{pmatrix} \neq 0, \}$$

Applying the cofactor expansion formula with respect to the last column we get

$$\begin{aligned}
 \mathcal{M}^{fr} &= \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{n\text{-copies}} \mid x_{n,1} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \\
 &\quad x_{n,2} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \dots, x_{n,n-1} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \} \\
 &= \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{n\text{-copies}} \mid x_{n,i} \neq 0 \text{ (for } i = 1, 2, \dots, n-1\text{)} \\
 &\quad , \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0\}
 \end{aligned}$$

To get the moduli space out of the framed moduli space above, we have to quotient

it out by the centralizer subgroup of u , that is,

$$C_{GL_n(\mathbb{C})}(u) = \left\{ \begin{array}{c|cc|cc} c_{1,1} & \cdots & c_{1,n-2} & 0 & c_{1,n} \\ \vdots & \ddots & \vdots & 0 & c_{1,n} \\ \hline c_{n-2,1} & \cdots & c_{n-2,n-2} & 0 & c_{n-2,n} \\ \hline 0 & \cdots & 0 & 0 & c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-2} & c_{n,n-1} & c_{n,n} \end{array} \right\} \in GL_n(\mathbb{C}) |$$

$$\det \begin{pmatrix} c_{1,1} & \cdots & c_{1,n-2} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n-2} \end{pmatrix}, c_{n,n-1}, c_{n-1,n} \neq 0 \}$$

It acts diagonally on $\underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$ where the action on each coordinate is given by left multiplication.

To simplify the notation, I will denote

$$(X_1, \dots, X_{n-1}) = ([x_{1,1} : \cdots : x_{n,1}], [x_{1,2} : \cdots : x_{n,2}], \dots, [x_{1,n-1} : \cdots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

I claim that for any $(X_1, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$, there exists $A \in C_{GL_n(\mathbb{C})}(u)$

such that

$$A \cdot X = \left[\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array} \right]$$

Let

$$A_1 = \left(\left. \begin{array}{ccc|cc} x_{1,1} & \cdots & x_{1,n-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-2} & 0 & 0 \end{array} \right) \right| \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ a_{n-1} & a_n \end{array}$$

with $a_{n-1} \neq 0$ such that

$$\sum_{i=1}^n a_i \cdot (x_{i,1}, \dots, x_{i,n}) = (1, \dots, 1)$$

We can always find such because we know that

$$\det \left(\begin{array}{ccc} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{array} \right) \neq 0$$

Then we get

$$A_1 \cdot X = \left[\begin{array}{c|c} I & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ \hline x_{n,1} & \cdots & x_{n,n-2} & x_{n,n-1} \\ 1 & \cdots & 1 & 1 \end{array} \right]$$

Since we know that $x_{n,i} \neq 0$ for $i = 1, \dots, n-1$, without loss of generality we put $x_{n,i} = 1$ for $i = 1, \dots, n-1$. Then we get

$$A_1 \cdot X = \left[\begin{array}{c|c} I & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ \hline 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \end{array} \right]$$

such that $b_1 + \cdots + b_{n-1} + 1 \neq 0$. Now let

$$A_2 = \left(\begin{array}{ccc|cc} a_{1,1} & \cdots & a_{1,n-2} & 0 & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline a_{n-2,1} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\ 0 & \cdots & 0 & & I \\ 0 & \cdots & 0 & & \end{array} \right)$$

such that

$$a_{i,1} \cdot (1, 0, \dots, 0, b_1) + a_{i,2} \cdot (0, 1, 0, \dots, 0, b_2) + \cdots + a_{i,n-2} \cdot (0, \dots, 0, 1, b_{n-2}) + a_{i,n} \cdot (1, \dots, 1) = (0, \overset{i^{th}}{\downarrow}, 0, 1, 0, \dots)$$

Again, we can find such because

$$\det \begin{bmatrix} & & b_1 \\ I & & \vdots \\ & & b_{n-1} \\ \hline & & \\ 1 & \cdots & 1 & 1 \end{bmatrix} \neq 0$$

Then $A_2 \cdot A_1$ is the desired A .

Example 55. (Regular Unipotent Fibers) Finite type integral schemes. Also, it has a stratification into rational varieties at most 1 stratum in each dimension. $(s_1)^3$, $(s_1 s_2)^2$

Example1

In this section, we compute the moduli space associated to the braid word : $(s_1 s_2)^2$ with regular unipotent monodromy. By the theorem ??, the framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x : y : z], [a : b : c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$C := C_{GL_3(\mathbb{C})}(u) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in GL_3(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

We can cover \mathcal{M}^{fr} with open subsets U_1, U_2 where

$$U_1 = \{([x:y:z], [a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, z \neq 0\}$$

$$U_2 = \{([x:y:z], [a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, c \neq 0\}$$

Therefore, we have a pushout square

$$\begin{array}{ccc} U_1 & \longrightarrow & \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ U_1 \cap U_2 & \longrightarrow & U_2 \end{array}$$

Quotienting out by the centralizer subgroup C , we get

$$\begin{array}{ccc} \overline{U}_1 := C \setminus U_1 & \longrightarrow & \mathcal{M} = C \setminus \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ \overline{U}_1 \cap \overline{U}_2 := C \setminus U_1 \cap U_2 & \longrightarrow & \overline{U}_2 := C \setminus U_2 \end{array}$$

Also note that, for the action of C , the centralizer subgroup of any element of \mathcal{M}^{fr} is the set of scalar multiplication matrices.

First let's simplify, \overline{U}_1 . Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{y}{z}$$

$$\gamma = \frac{y^2}{z^2} - \frac{x}{z}$$

This expression makes sense because $z \neq 0$ in U_1 . If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C -orbit, we see that

$$\begin{aligned}\overline{U}_1 &\cong \{([0 : 0 : 1], [a : b : c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} 0 & a & 0 \\ 0 & b & 1 \\ 1 & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} 0 & a & b \\ 0 & b & c \\ 1 & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\ &\cong \{[a : b : c] \in \mathbb{P}^2 \mid a \neq 0, \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0\} \\ &\cong \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}\end{aligned}$$

Under this identification,

$$\overline{U}_1 \cap \overline{U}_2 \cong \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

Now let's simplify, \overline{U}_2 . Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\begin{aligned}\alpha &= 1 \\ \beta &= -\frac{b}{c} \\ \gamma &= \frac{b^2}{c^2} - \frac{a}{c}\end{aligned}$$

This expression makes sense because $c \neq 0$ in U_1 . If we take an element of U_2 with

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C -orbit, we see that

$$\begin{aligned}\overline{U}_2 &\cong \{([x:y:z], [0:0:1]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\ &\cong \{[x:y:z] \in \mathbb{P}^2 \mid x \neq 0, \det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \neq 0\} \\ &\cong \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}\end{aligned}$$

Under these identifications, the pushout square above becomes

$$\begin{array}{ccc}\{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} & \longrightarrow & \mathcal{M} \\ \uparrow & & \uparrow \\ \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} & \xrightarrow{f} & \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}\end{array}$$

where

$$f : \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \rightarrow \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}$$

$$(b, c) \mapsto \left(-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}\right)$$

Now define a variety V to be

$$V := \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0\} - \{(0, 0, 0)\}$$

I claim that V is isomorphic to \mathcal{M} . More precisely, I claim that

$$\begin{array}{ccc} \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} & \xhookrightarrow{i} & V \\ \uparrow g & & \uparrow \iota \\ \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} & \xhookrightarrow{f} & \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \end{array}$$

is a pushout square where

$$i : \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \rightarrow V$$

$$(b, c) \mapsto \left(-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}, b\right)$$

$$\iota : \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \rightarrow V$$

$$(y, z) \mapsto (y, z, \frac{yz}{z - y^2})$$

It is easy to check the square commutes. f, g are inclusion maps by construction.

ι is also an inclusion map because we can recover (y, z) from $\iota(y, z)$ by projecting onto the 1^{st} & 2^{nd} coordinates.

For i , we can recover b from $i(b, c)$ by projecting onto the 3^{rd} coordinate. We can recover c from $i(b, c)$ by multiplying 2^{nd} & 3^{rd} coordinates and dividing with -(the 1^{st} coordinate).

Let's check that the images of i and ι form an open cover of V . The images of i and ι are

$$i(\{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}) = \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, YW + Z + W^2 \neq 0\}$$

$$\begin{aligned} \iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) &= \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, Y^2 - Z \neq 0\} \\ &= \{(Y, Z, W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z - Y^2}, Y^2 - Z \neq 0\} \end{aligned}$$

Clearly, they are open subsets of V .

Let's check that if $(Y, Z, W) \in V$ and $Y^2 = Z$, then $YW + Z + W^2 \neq 0$. If $Y^2 = Z$, then the equation $Y^2W - ZW + YZ = 0$ becomes $YZ = 0$. Therefore, we get $Y = Z = 0$. Since $(0, 0, 0)$ is not contained in V , W can only take non-zero values. Therefore, $YW + Z + W^2 = W^2 \neq 0$. We conclude that the images of i and ι cover V .

Now let's check that

$$i^{-1}(\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

The image of ι is

$$\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) = \{(Y, Z, W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z - Y^2}, Y^2 - Z \neq 0\}$$

we have

$$\begin{aligned}
 i(b, c) \in \iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) \\
 \iff (-\frac{bc}{b^2 - c})^2 \neq \frac{c^2}{b^2 - c} \\
 \iff b^2 c^2 \neq c^2(b^2 - c) \\
 \iff c^3 \neq 0 \\
 \iff c \neq 0
 \end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

as desired.

Therefore, V is isomorphic to \mathcal{M} .

Example2

In this section, we compute the moduli space associated to the braid word : s_1^3 with regular unipotent monodromy. By the theorem ??, the framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x : y], [z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} x & z \\ y & w \end{pmatrix} \neq 0, \det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & x + y \\ b & y \end{pmatrix} \neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is

$$C := C_{GL_2(\mathbb{C})}(u) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in GL_2(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

From this point on, we will use the following notation

$$\infty := [1 : 0]$$

$$X := [x : y]$$

$$Z := [z : w]$$

$$A := [a : b]$$

We can cover \mathcal{M}^{fr} with open subsets U_1, U_2 where

$$U_1 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid X \neq \infty\}$$

$$U_2 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid Z \neq \infty\}$$

Therefore, we have a pushout square

$$\begin{array}{ccc} U_1 & \longrightarrow & \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ U_1 \cap U_2 & \longrightarrow & U_2 \end{array}$$

Quotienting out by the centralizer subgroup C , we get

$$\begin{array}{ccc} \overline{U}_1 := C \setminus U_1 & \longrightarrow & \mathcal{M} = C \setminus \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ \overline{U_1 \cap U_2} := C \setminus U_1 \cap U_2 & \longrightarrow & \overline{U}_2 := C \setminus U_2 \end{array}$$

Also note that, for the action of C , the centralizer subgroup of any element of \mathcal{M}^{fr}

is the set of scalar multiplication matrices.

First let's simplify, \bar{U}_1 . Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{x}{y}$$

This expression makes sense because $y \neq 0$ in U_1 . If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C -orbit, we see that

$$\begin{aligned}\overline{U}_1 &\cong \{([0 : 1], [z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} 0 & z \\ 1 & w \end{pmatrix} \neq 0, \det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \neq 0\} \\ &\cong \{([z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z \neq 0, bz - aw \neq 0, a - b \neq 0\} \\ &\cong \{(w, [a : b]) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid b - aw \neq 0, a - b \neq 0\}\end{aligned}$$

Change variables $a' := a + b, b' := a - b$ we get

$$\begin{aligned}\overline{U}_1 &\cong \{(w, [a' : b']) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid \frac{a' - b'}{2} - \frac{(a' + b')w}{2} \neq 0, b' \neq 0\} \\ &\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a' - 1) - (a' + 1)w \neq 0\} \\ &\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a'' + 1)(1 - w) - 2 \neq 0\} \\ &\cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1\}\end{aligned}$$

Under this identification,

$$\overline{U}_1 \cap \overline{U}_2 \cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1, 2w' \neq 1\}$$

Now let's simplify, \overline{U}_2 . Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{z}{w}$$

This expression makes sense because $w \neq 0$ in U_2 . If we take an element of U_2 with

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C -orbit, we see that

$$\begin{aligned} \overline{U}_2 &\cong \{([x:y], [0:1], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \neq 0, \det \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0\} \\ &\cong \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x \neq 0, a \neq 0, ay - b(x+y) \neq 0\} \\ &\cong \{(y, b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y - b(1+y) \neq 0\} \\ &\cong \{(y, b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y(1-b) - b \neq 0\} \end{aligned}$$

Change variables $b' := 1 - b, y' := y + 1$ we get

$$\overline{U}_2 \cong \{(y', b') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid yb' \neq 1\}$$

Under these identifications, the pushout square above becomes

$$\begin{array}{ccc} \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\} & \xhookrightarrow{\quad} & \mathcal{M} \\ \uparrow & & \uparrow \\ \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} & \xhookrightarrow{f} & \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} \end{array}$$

where

$$\begin{aligned} f : \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} &\rightarrow \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} \\ (x, y) &\mapsto (2x, \frac{4x + y - 4}{2xy - 2}) \end{aligned}$$

Now define a variety V to be

$$V := \{(A, B, C) \in \mathbb{A}^3 \mid (AC - 2)B = 2A + C - 4\} - \{(1, 1, 2)\}$$

I claim that V is isomorphic to \mathcal{M} . More precisely, I claim that

$$\begin{array}{ccc} \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\} & \xhookrightarrow{i} & V \\ \uparrow g & & \uparrow \iota \\ \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} & \xhookrightarrow{f} & \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} \end{array}$$

is a pushout square where

$$\begin{aligned} i : \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\} &\rightarrow V \\ (x, y) &\mapsto (2x, \frac{4x + y - 4}{2xy - 2}, y) \\ \iota : \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} &\rightarrow V \\ (a, b) &\mapsto (a, b, \frac{2a + 2b - 4}{ab - 1}) \end{aligned}$$

It is easy to see that the square commutes and f, g, i, ι are inclusion maps.

Let's check that the images of i and ι form an open cover of V .

The image of i and ι are

$$i(\{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\}) = \{(A, B, C) \in V \mid AC \neq 2\}$$

$$\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V \mid AB \neq 1\}$$

Clearly, they are open subsets of V .

Let's check that if $(A, B, C) \in V$ and $AC = 2$, then $AB \neq 1$. If $AC = 2$, then the left hand side of the equation $(AC - 2)B = 2A + C - 4$ becomes zero. Therefore, we get $AC = 2$ and $2A + C = 4$ which implies $2A + \frac{2}{A} = 4 \Leftrightarrow A^2 - 2A + 1 = 0$. Solving the quadratic equation, we get $A = 1, C = 2$. Since $(1, 1, 2)$ is not contained in V , B can take any value except 1. Therefore, $AB \neq 1$. We conclude that the images of i and ι cover V .

Now let's check that

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\}$$

The image of ι is

$$\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V \mid AB \neq 1\}$$

we have

$$\begin{aligned}
 i(x, y) &\in \iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) \\
 &\iff 2x \cdot \left(\frac{4x + y - 4}{2xy - 2} \right) \neq 1 \\
 &\iff 2x \cdot (4x + y - 4) \neq 2xy - 2 \\
 &\iff 8x^2 - 8x + 2 \neq 0 \\
 &\iff 2(2x - 1)^2 \neq 0 \\
 &\iff 2x \neq 1
 \end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \overline{U}_1 \mid 2x \neq 1\}$$

as desired.

Therefore, V is isomorphic to \mathcal{M} .

Example3

In this section, we prove that the moduli space of a bivalent braid with regular unipotent monodromy is finite type integral scheme over \mathbb{C} not necessarily separated. Suppose we have a bivalent braid word with n -strands and a unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

By theorem??, the framed moduli space is given as

$$\mathcal{M}^{fr} = \{X = (X^1, X^2, \dots, X^k) \in (\mathbb{P}^{n-1})^k \mid f_1(X) \neq 0, \dots, f_m(X) \neq 0\}$$

where f_i 's are determinants with column vectors of the form $u^s \cdot X_j$. Note that the entries of the last row(i.e. the n^{th} row) is one of $x_{n,i}$ ($i = 1, \dots, k$). Thus $x_{n,i}$'s ($i = 1, \dots, k$) cannot be identically zero otherwise all of the f_r 's will vanish. Therefore, we have an open cover of \mathcal{M}^{fr} , i.e. $\{U_i\}_{i=1, \dots, k}$ where $U_i := \{X \in \mathcal{M}^{fr} \mid x_{n,i} \neq 0\}$.

To get the moduli space, we quotient the framed moduli space with the centralizer subgroup of u in $GL_n(\mathbb{C})$ i.e.

$$C := C_{GL_n(\mathbb{C})} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in GL_n(\mathbb{C}) \mid \alpha_1 \in \mathbb{C}^*, \alpha_i \in \mathbb{C} \text{ for } i = 2, \dots, n \right\}$$

Suppose we have an element

$$X = (X^1, \dots, X^k) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \in U_i$$

Then there exists a

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \cdot \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

That is, recursively,

$$\alpha_1 = 1$$

$$\alpha_s = -\frac{1}{x_{n,i}}(x_{n-1,i}\alpha_{s-1} + x_{n-2,i}\alpha_{s-2} + \cdots + x_{n-s+1,i}\alpha_1) = -\frac{1}{x_{n,i}}\left(\sum_{t=1}^{s-1} x_{n-t,i} \cdot \alpha_{s-t}\right)$$

This expression makes sense because $x_{n,i} \neq 0$ in U_i . If we take an element of U_i with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C -orbit, we see that

$$\overline{U}_i := C \setminus U_i \cong \left\{ \begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \in (\mathbb{P}^{n-1})^k \mid f_1 \left(\begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \right) \neq 0, \dots, \right.$$

\uparrow
ith column

$$f_m \left(\begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \right) \neq 0 \}$$

which are finite type scheme over \mathbb{C} . In summary, we have found an finite open cover of \mathcal{M} i.e. $\{\overline{U}_i\}_{i=1,\dots,k}$ such that each open is a finite type scheme over \mathbb{C} . Thus, \mathcal{M} is also a finite type scheme over \mathcal{C} . Now we have a smooth surjective map $\pi : \mathcal{M}^{fr} \rightarrow \mathcal{M}$. \mathcal{M} is irreducible because \mathcal{M}^{fr} is irreducible and π is surjective. \mathcal{M} is reduced because \mathcal{M}^{fr} is reduced and π is smooth. Therefore, we conclude that the moduli space associated attached to bivalent braid with regular unipotent monodromy is finite type integral scheme over \mathbb{C} . But it may not be separated.

Example4

In this section, I will provide an example of the moduli space associated to bivalent braid with regular unipotent monodromy that is non-separated. Consider a 2-strand braid given by the braid word s_1^2 and a regular unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

By the theorem ??, the framed moduli space is given as follows

$$\begin{aligned}
 \mathcal{M}^{fr} &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], u \cdot [a:b] \neq [x:y]\} \\
 &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], [a+b:b] \neq [x:y]\} \\
 &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid ay \neq bx, (a+b)y \neq bx\}
 \end{aligned}$$

Then we have an open cover $\{U_i\}_{i=1,2}$ where

$$\begin{aligned}
 U_1 &:= \{([x:y], [a:b]) \in \mathcal{M}^{fr} \mid b \neq 0\} \\
 U_2 &:= \{([x:y], [a:b]) \in \mathcal{M}^{fr} \mid y \neq 0\}
 \end{aligned}$$

We get a pushout square

$$\begin{array}{ccc}
 U_1 & \longrightarrow & \mathcal{M}^{fr} \\
 \uparrow & & \uparrow \\
 U_1 \cap U_2 & \longrightarrow & U_2
 \end{array}$$

we take quotients of these opens with respect to the centralizer subgroup of u i.e.

$$C := C_{GL_2(\mathbb{C})} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in GL_2(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

we get

$$\begin{aligned}
 \overline{U}_1 &:= C \setminus U_1 \cong \{([x : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x\} \\
 &\cong \{([1 : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \\
 \overline{U}_2 &:= C \setminus U_2 \cong \{([0 : 1], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid a \neq 0, (a + b) \neq 0\} \\
 &\cong \{([0 : 1], [1 : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \\
 \overline{U}_1 \cap \overline{U}_2 &:= C \setminus U_1 \cong \{([x : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
 &\cong \{([x : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
 &\cong \{([1 : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\}
 \end{aligned}$$

and a pushout square

$$\begin{array}{ccc}
 \{([1 : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} & \longrightarrow & \mathcal{M} \\
 \downarrow & & \downarrow \\
 \{([1 : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} & \xrightarrow{f} & \cong \{([0 : 1], [1 : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}
 \end{array}$$

where

$$f : \{([1 : y], [0 : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \rightarrow \{([0 : 1], [1 : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}$$

$$([1 : y], [0 : 1]) \mapsto ([0 : 1], [1 : -y])$$

Now consider the map

$$\begin{aligned}
 g : \mathbb{A}^1 - \{0, 1\} &\longrightarrow \overline{U}_1 \cap \overline{U}_2 \subseteq \mathcal{M} \\
 y &\mapsto ([1 : y], [0 : 1])
 \end{aligned}$$

This map extends to \mathbb{A}^1 in two different ways i.e. we have two distinct h_1, h_2 that fit

into the following commutative square

$$\begin{array}{ccc} \mathbb{A}^1 - \{0, 1\} & \xhookrightarrow{\quad} & \mathbb{A}^1 - \{1\} \\ & \searrow^g & \downarrow h_i \\ & & \mathcal{M} \end{array}$$

which are

$$h_1 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_1 \subseteq \mathcal{M}$$

$$y \mapsto ([1:y], [0:1])$$

$$h_2 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_2 \subseteq \mathcal{M}$$

$$y \mapsto ([0:1], [1:-y])$$

h_1 and h_2 are distinct because $h_1(0) \in U_1 - U_2$. Therefore, by the valuative criterion for separatedness, \mathcal{M} is non-separated.

Chapter 3

Natural Cluster Coordinates on Braid Moduli Spaces

3.1 Local Morse Group Computation

In this section, we compute the local Morse groups of a cochain complex of coordinate-constructible sheaves.

3.1.1 Definitions and Notations

Definition 56. We define $sgn : \mathbb{R} \rightarrow \{-1, 0, 1\}$ such that

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Definition 57. We define $[n]$ to be a set of integers from 1 to n i.e.

$$[n] := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$$

Definition 58. Let X be a set, then we denote the power set of X as $\mathcal{P}(X)$.

Definition 59. We denote the i^{th} standard basis of \mathbb{R}^n to be e_i^n .

Definition 60. Let

$$s^n(sgn_1, \dots, sgn_n) := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \text{for } k = 1, \dots, n, sgn(x_k) = sgn_k\}$$

To simplify the notation, we use + instead of $+1$ and $-$ instead of -1 . For example,

$$s^n(+, -, 0) := s^n(1, -1, 0).$$

Definition 61. 1. We define a coordinate stratification \mathcal{S}^n on \mathbb{R}^n to be

$$\{s^n(sgn_1, \dots, sgn_n) \mid sgn_k \in \{-, 0, +\} \text{ for } k = 1, \dots, n\}$$

2. There is a natural poset structure of \mathcal{S}^n such that $s^n(sgn_1, \dots, sgn_n) \leq s^n(sgn'_1, \dots, sgn'_n)$ if and only if $s^n(sgn'_1, \dots, sgn'_n) \subset \text{star}(s^n(sgn_1, \dots, sgn_n))$. This is equivalent to saying that $|sgn_i| \leq |sgn'_i|$ for all $i \in [n]$.
3. There is also a natural poset structure on n -dimensional strata of \mathcal{S}^n , where $s^n(sgn_1, \dots, sgn_n) \leq s^n(sgn'_1, \dots, sgn'_n)$ if and only if $sgn_i \leq sgn'_i$ for all $i \in [n]$. Note that this is not the poset structure that is inherited from the poset structure mentioned above.
4. We give co-orientations to $(n-1)$ -dimensional stratum as follows: Note that codimension 1 strata can be expressed as $s^n(sgn_1, \dots, sgn_n)$ where only one of arguments is zero, say sgn_i . Then we define the co-orientation of $s^n(sgn_1, \dots, sgn_n)$ to be $-dx_i$ i.e. hairs are pointing the region $s^n(sgn_1, \dots, \underset{i^{th}}{\overset{\uparrow}{-}}, \dots, sgn_n)$
5. Let $x \in \mathbb{R}^n$, then s_x^n is defined as the stratum in \mathcal{S}^n containing x .

Suppose we have \mathcal{F}^\bullet a complex of constructible sheaves on \mathbb{R}^n constructible with respect to the coordinate stratification. In this section, we will compute local Morse groups of \mathcal{F}^\bullet when given $(x, \xi) \in T^*\mathbb{R}^n$ i.e. we will compute the stalk of the microlocalization of $\mathcal{F}^\bullet (= \mu\mathcal{F}^\bullet)$ on $T^*\mathbb{R}^n$. (Fact) The singular support of \mathcal{F}^\bullet , $SS(\mathcal{F}^\bullet)$, is contained in the union of 2^n Lagrangian subspaces that are conormals of strata in \mathcal{S}^n i.e. $\Lambda_{\mathcal{S}^n} = \bigcup_{s \in \mathcal{S}^n} N^*s \subset T^*\mathbb{R}^n$. These Lagrangians intersect with each other so union of them form a singular Lagrangian and it has smooth part in it, say $\Lambda_{\mathcal{S}^n}^{smooth} \subset \Lambda_{\mathcal{S}^n}$. (Question) How do we know $\mu\mathcal{F}^\bullet$ vanishes on $\Lambda_{\mathcal{S}^n}^{singular}$ i.e. singular locus of $\Lambda_{\mathcal{S}^n}$? Now we know that the singular support $SS(\mathcal{F}^\bullet) := supp(\mu\mathcal{F}^\bullet) \subset \Lambda_{\mathcal{S}^n}^{smooth}$ and $\mu\mathcal{F}^\bullet$ is constant on each component of $\Lambda_{\mathcal{S}^n}^{smooth}$, we define the local Morse group of a component of $\Lambda_{\mathcal{S}^n}^{smooth}$ with coefficient \mathcal{F}^\bullet to be the local Morse group of a point in the component with coefficient \mathcal{F}^\bullet . We will see later that the components are contractible so this is well-defined upto unique isomorphism. Note that $\Lambda_{\mathcal{S}^n}^{smooth}$ has 2^{2n} components which are labelled by $\mathcal{P}([n]) \times \mathcal{P}([n])$. More precisely, for each $(I, J) \in \mathcal{P}([n]) \times \mathcal{P}([n])$, we have a component $comp_{(I,J)}^n \subset \Lambda_{\mathcal{S}^n}^{smooth}$ that is the image of

$$\iota_{(I,J)}^n : \mathbb{R}_{>0}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

where $(x_1, \dots, x_n)^T$ is mapped to

$$((\delta_I(1)sgn_J(1)x_1, \dots, \delta_I(n)sgn_J(n)x_n)^T, ((1-\delta_I(1))sgn_J(1)x_1, \dots, (1-\delta_I(n))sgn_J(n)x_n)^T)$$

where

- $\delta_I : [n] \rightarrow \{0, 1\}$

$$\delta_I(k) = \begin{cases} 1 & \text{if } k \in I \\ 0 & \text{if } k \notin I \end{cases}$$

- $sgn_J : [n] \rightarrow \{-1, 1\}$

$$sgn_J(k) = \begin{cases} 1 & \text{if } k \in J \\ -1 & \text{if } k \notin J \end{cases}$$

I will choose the representative of the $comp_{(I,J)}^n$ to be

$$(x_{(I,J)}, \xi_{(I,J)}) := \iota_{(I,J)}^n((1, \dots, 1)^T)$$

Also, we define $G_{(I,J)}(\mathcal{F}^\bullet)$ to be the local Morse group of $comp_{(I,J)}^n$ with coefficient \mathcal{F}^\bullet i.e.

$$H^*(N_x, N_x \cap \xi^{-1}((-\infty, c - \epsilon])); \mathcal{F}^\bullet)$$

where $(x, \xi) \in comp_{(I,J)}^n$

Definition 62. Q^n := the quiver associated to \mathcal{S}^n . We have one vertex for each n -dimensional stratum and one arrow for each $(n - 1)$ -dimensional stratum where the direction goes against the co-orientation.

Definition 63. F^\bullet := the quiver representation associated to \mathcal{F}^\bullet : describe the correspondence between F^\bullet on Q^n , F^\bullet on \mathcal{S}^n , \mathcal{F}^\bullet

Definition 64. Q_s^n := full suquiver of Q^n spanned by vertices whose corresponding strata are contained in the star of $s \in \mathcal{S}^n$.

Definition 65. 1. $s_{(I,J)}^n := s_{x_{(I,J)}}^n$

2. For $k \notin I$, $s_{(I,J)}^n(k) := s_{(x_{(I,J)} - sgn_J(k) \cdot e_k^n)}^n$

Lemma 66. Suppose we have a cochain complex of \mathcal{S}^n -constructible sheaf \mathcal{F}^\bullet on \mathbb{R}^n that could be described by a legible diagram F^\bullet on Q^n , then

$$G_{(I,J)}(\mathcal{F}^\bullet) = \begin{cases} 0 & \text{if } I^c \notin J \\ \text{Tot}(F^\bullet|_{Q_{(I,J)}^n}) & \text{if } k \in J \end{cases}$$

(need more explanation on what the total complex of a quiver representation means)

Proof. We prove the Lemma by induction on n . (skipped the proof of the base case) (i) suppose I is not empty i.e. $x_{(I,J)} \neq 0$. We want to compute $G_{(I,J)}(\mathcal{F}^\bullet) = H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c-\epsilon]); \mathcal{F}^\bullet)$ where $N_{x_{(I,J)}}$ is a regular neighborhood of $x_{(I,J)}$, $c = \xi_{(I,J)}(x_{(I,J)})$ (for this expression to make sense we need to fix an isomorphism $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$), and ϵ a small positive real number. Once and for all fix $i \in I$, consider the following linear map

$$\text{Lin}_i : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

where

$$\text{Lin}_i(e_j) = \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } j \geq i \end{cases}$$

and let $\text{Aff}_i := \text{Lin}_i + e_i^n$ be the affine inclusion map. Define

- $I' = \{k \in I \mid k < i\} \cup \{k - 1 \mid k > i, k \in I\}$
- $J' = \{k \in J \mid k < i\} \cup \{k - 1 \mid k > i, k \in J\}$

and $(x_{(I',J')}, \xi_{(I',J')}) = \iota_{(I',J')}^{n-1}((1, \dots, 1)^T)$, then $x_{(I,J)} \in \text{Im}(\text{Aff}_i)$ and $\alpha_i^{-1}(x_{(I,J)}) = x_{(I',J')}$ and $\xi_{(I,J)} \circ \text{Aff}_i$. We let the regular neighborhood of $x_{(I,J)}$, $N_{x_{(I,J)}}$, to be $\text{star}(s_{(I,J)}^{n-1})$ and the regular neighborhood of $x_{(I',J')}$, $N_{x_{(I',J')}}^*$, to be $\text{star}(s_{(I',J')}^{n-1})$, $c = \xi_{(I,J)}(x_{(I,J)})$, and $c' = \xi_{(I',J')}(x_{(I',J')})$

(claim) $H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c-\epsilon]); \mathcal{F}^\bullet) \cong H^*(N_{x_{(I',J')}}^*, N_{x_{(I',J')}}^* \cap \xi_{(I',J')}^{-1}((-\infty, c'-\epsilon])); \text{Aff}_i^* \mathcal{F}^\bullet)$

(proof) Note that when we have a stratified spaces $(M,)$ and N , a subspace of M , we have an induced stratification \mathcal{S}_N on N . Then we can identify poset \mathcal{S}_N as the subposet of \mathcal{S}_M via $(N \hookrightarrow M)_*$.

Now consider the following cartesian diagram of inclusion maps

$$\begin{array}{ccc} N_{x_{(I,J)}} & \longleftrightarrow & N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c - \epsilon]) \\ \text{Aff}_i \uparrow & & \uparrow \\ N_{x_{(I',J')}} & \longleftrightarrow & N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon]) \end{array}$$

Note that Aff_i induces a poset isomorphism from the poset $\mathcal{S}_{N_{x_{(I',J')}}}$ to the poset $\mathcal{S}_{N_{x_{(I,J)}}}$ where via this isomorphism the subposet $\mathcal{S}_{N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon])}$ gets identified with $\mathcal{S}_{N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c - \epsilon])}$. Furthermore, the representation of the poset $\mathcal{S}_{N_{x_{(I',J')}}}$ that gives rise to the sheaf $\text{Aff}_i^* \mathcal{F}^\bullet$ is exactly the representation of the poset $\mathcal{S}_{N_{x_{(I,J)}}}$ that gives rise to the sheaf \mathcal{F}^\bullet under the poset isomorphism $(\text{Aff}_i)_*$ and the relative cohomology is completely determined by the poset, subpost structures and their representations. Therefore, the claim is proved.

Therefore, $G_{(I,J)}(\mathcal{F}^\bullet) \cong H^*(N_{x_{(I',J')}}, N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon]); \text{Aff}_i^* \mathcal{F}^\bullet)$. By the induction hypothesis if $I'^c \not\subset J'^c$, this is 0 and if $I'^c \subset J'^c$, this is equal to $\text{Tot}(\text{Aff}_i^*(F^\bullet|_{\mathcal{S}_{N_{x_{(I',J')}}}})) = \text{Tot}(F^\bullet|_{\mathcal{S}_{N_{x_{(I,J)}}}}) = \text{Tot}(F^\bullet|_{Q_{s_{(I,J)}}^n})$. Because $I^c \subset J^c$ if and only if $I'^c \subset J'^c$, we proved the case when $I \neq \phi$.

(ii) now consider the case where $I = \phi$.

(case1) $I^c \not\subset J^c$ i.e. $J = \phi$, then we want to prove that $G_{(I,J)}(\mathcal{F}^\bullet) = 0$. Equivalently, we want to prove that the mapping cone of the following restriction map is acyclic.

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c - \epsilon]); \mathcal{F}^\bullet)$$

First, let's compute $R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet)$. We can use a singleton Čech cover $\{N_{x_{(I,J)}} = \text{star}(s_{(I,J)}^n)\}$. Therefore, $R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \cong \text{Tot}(F^\bullet(N_{x_{(I,J)}}))$.

Next, let's compute $R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c - \epsilon])); \mathcal{F}^\bullet$. We can use the following Čech cover

$$\mathcal{U} = \{U_k\}_{k \in [n]}$$

where

$$U_k := \text{star}(s_{x_{(I,J)}}^n(k)) = \text{star}(s^n(0, \dots, 0, \underset{\substack{\uparrow \\ k^{th}}}{(-1)^{\delta_J(k)}}, 0, \dots, 0))$$

We define

$$U_{i_0 \dots i_p} := N_{x_{(I,J)}} \cap U_{i_0} \cap \dots \cap U_{i_p}$$

Note that

- if the index set in the subscript is empty, then $U = N_{x_{I,J}} = \mathbb{R}^n$
- $U_{i_0 \dots i_p} = \text{star}(s^n(sgn_1, \dots, sgn_n))$ where

$$sgn_k = \begin{cases} (-1)^{\delta_J(k)} & \text{if } k \in \{i_0, \dots, i_p\} \\ 0 & \text{otherwise} \end{cases}$$

Since $J \neq \phi$, without loss of generality assume $1 \in J$ (we can always relabel coordinates). Note that for all $1 \leq i_0 < \dots < i_p \leq n$

$$F^\bullet(U_{i_0 \dots i_p}) = F^\bullet(U_{1 i_0 \dots i_p})$$

because F^\bullet a poset representation obtained from a legible diagram.

From the above Čech cover we get a Čech double complex

$$C^{p,q} = \bigoplus_{1 \leq i_0 < \dots < i_p \leq n} F^q(U_{i_0 \dots i_p})$$

where horizontal maps are Čech differentials(δ) and vertical maps are cochain differentials(d).

The mapping cone of

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c - \epsilon]); \mathcal{F}^\bullet)$$

is the total complex of the following double complex

$$C_{ext}^{p,q} = \begin{cases} C^{p,q} & \text{if } p \geq 0 \\ F^q(N_{x_{(I,J)}}) = F^q(\mathbb{R}^n) & \text{if } p = -1 \\ 0 & \text{otherwise} \end{cases}$$

where $\delta^{-1,q}$ is just a restriction map.

Now I will define a homotopy $h^{\bullet,\bullet} : C_{ext}^{\bullet,\bullet} \rightarrow C_{ext}^{\bullet-1,\bullet}$ on the Čech double complex that induces homotopy h on the total complex of $C_{ext}^{\bullet,\bullet}$. Let $c \in C^{p,q} := \bigoplus_{1 \leq i_0 < \dots < i_p \leq n} F^q(U_{i_0 \dots i_p})$, then we define $hc \in C^{p-1,q}$ so that

$$(hc)_{i_0 \dots i_p} = c_{1i_0 \dots i_p}$$

Here we are using the fact that if $1 \notin \{i_0, \dots, i_p\}$

$$F^q(U_{i_0 \dots i_p}) = F^q(U_{1i_0 \dots i_p})$$

Note that if one of i_0, \dots, i_p is 1, then $(hc)_{i_0 \dots i_p} = 0$. Also note that $h^{p,q-1}d^{p,q} = d^{p-1,q}h^{p,q}$. Now let's compute $hD_{Tot} + D_{Tot}h$ where D_{Tot} is the total differential. Suppose $c \in C_{ext}^{-1,q}$, then

- $(hD_{Tot}c) = (h(-d+\delta)c) = -(hdc) + (h\delta c)$. Because hdc lands on degree $p = -2$, $(hdc) = 0$ and $(hD_{Tot}c) = (h\delta c) = c|_{U_1} = c$. Here, we use the identity map between $F^q(N_{x_{(I,J)}}) = F^q(U_1)$ and think of $c|_{U_1}$ as an element of $F^q(N_{x_{(I,J)}})$ that uniquely extends $c|_{U_1}$, which is c itself.
- $(D_{Tot}hc) = D_{Tot}0 = 0$ because hc lands on degree $p = -2$.

Suppose $c \in C_{ext}^{p,q}$ ($p \geq 0$),

- $hD_{Tot} = h((-1)^p d + \delta) = (-1)^p hd + h\delta$

- $D_{Tot}h = ((-1)^{p-1}d + \delta)h = (-1)^{p-1}dh + \delta h$

Therefore, $hD_{Tot} + D_{Tot}h = h\delta + \delta h$ and

$$\begin{aligned}
 (h\delta c + \delta hc)_{i_0 \dots i_p} &= (h\delta c)_{i_0 \dots i_p} + (\delta hc)_{i_0 \dots i_p} \\
 &= (\delta c)_{1i_0 \dots i_p} + \sum_{j=0}^p (-1)^j (hc)_{i_0 \dots \hat{i}_j \dots i_p} \\
 &= [c_{i_0 \dots i_p} + \sum_{j=0}^p (-1)^{j+1} c_{1i_0 \dots \hat{i}_j \dots i_p}] + \sum_{j=0}^p (-1)^j (hc)_{i_0 \dots \hat{i}_j \dots i_p} \\
 &= c_{i_0 \dots i_p}
 \end{aligned}$$

Therefore, h is a homotopy between identity cochain map on $\text{Tot}(C_{ext}^{\bullet, \bullet})$ and a zero map. Therefore, we conclude that $\text{Tot}(C_{ext}^{\bullet, \bullet})$ is acyclic. (case2) $I^c \subset J^c$ i.e. $J = \phi$, then use the same Čech cover \mathcal{U} as in (case1). Then again the mapping cone of

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c - \epsilon]); \mathcal{F}^\bullet)$$

is the total complex of the following double complex up to a shift

$$F^\bullet(N_{x_{(I,J)}}) \rightarrow \bigoplus F^\bullet(U_{i_0}) \rightarrow \bigoplus F^\bullet(U_{i_0 i_1}) \rightarrow \dots$$

which is equal to

$$\begin{aligned}
 F^\bullet(star(s^n(0, \dots, 0))) &\rightarrow \bigoplus F^\bullet(star(s^n(sgn_{i_0} = +, \text{else } 0))) \rightarrow \\
 &\quad \bigoplus F^\bullet(star(s^n(sgn_{i_0} = sgn_{i_1} = +, \text{else } 0))) \rightarrow \dots
 \end{aligned}$$

which is equal to

$$\begin{aligned} F^\bullet(star(s^n(-, \dots, -))) &\rightarrow \oplus F^\bullet(star(s^n(sgn_{i_0} = +, \text{else } -))) \rightarrow \\ &\oplus F^\bullet(star(s^n(sgn_{i_0} = sgn_{i_1} = +, \text{else } -))) \rightarrow \dots \end{aligned}$$

whose total complex is the total complex of F^\bullet restricted to Q^n . Because $Q^n = Q^n_{s^n(-, \dots, -)}$ the proof is complete. \square

3.2 natural alternating diagram

Suppose we have a positive braid word ω , then we can draw the associated braid diagram $(i_1, \dots, i_{n-1} : [0, 1]_x \rightarrow [0, 1]_x \times (0, 1)_z)$ where i_k are smooth sections of the projection $[0, 1]_x \times (0, 1)_z \rightarrow [0, 1]_x$ on $[0, 1]_x \times (0, 1)_z$ and its cylindrical closure $S^1_x \times (0, 1)_z$. For example, if $\omega = s_1 s_1$, a braid word on 3 strand, then the cylindrical closure of the associated braid diagram is shown in the figure below.

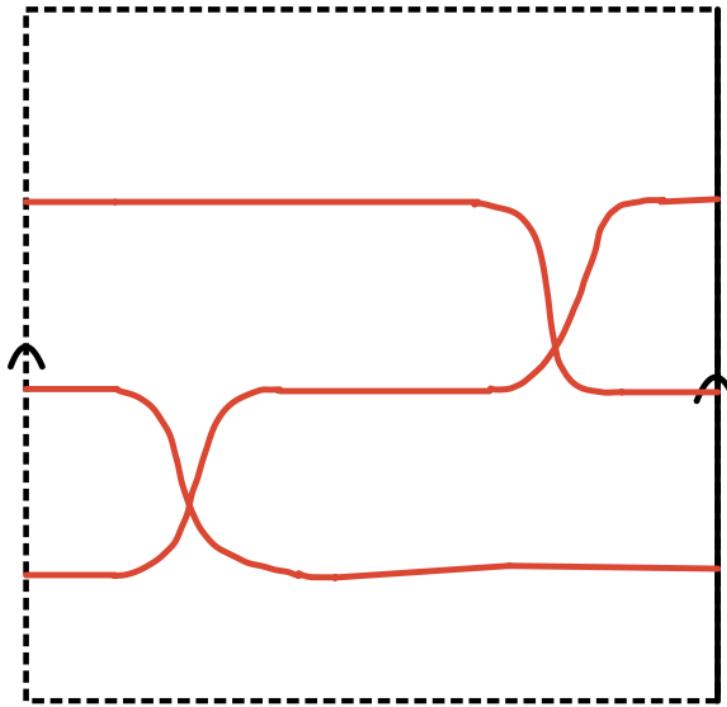


Figure 3.1: Your caption here

We will specify co-orientations of i_1, \dots, i_{n-1} so that we can think of the cylindrical closure of the braid word as the front projection of a Legendrian knot living inside the co-circle bundle of the cylindrical closure.

Let $x_0 \in [0, 1]$, we define the co-orientation at $i_k(x_0)$ to be $\xi = adx + cdz$ so that ξ vanishes at $\frac{di_k}{dt}|_{x=x_0}$, $\|(a, c)\| = 1$, and $c > 0$. This can be visually represented as hairs pointing upward(i.e. $c > 0$).

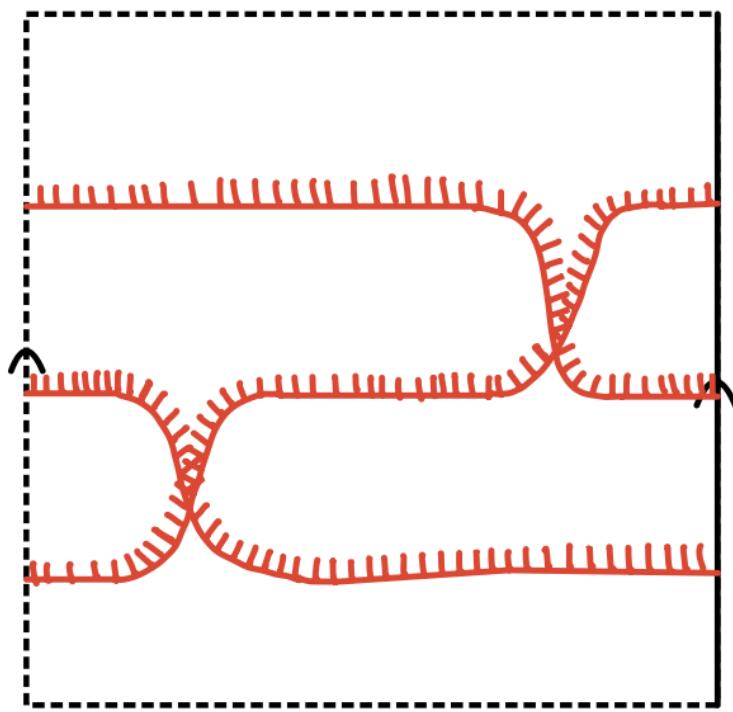


Figure 3.2: Your caption here

Suppose we have a Riemann sphere M with two punctures at 0 and ∞ . Topologically, M is homeomorphic to the boundaryless cylinder as shown in the figure below.

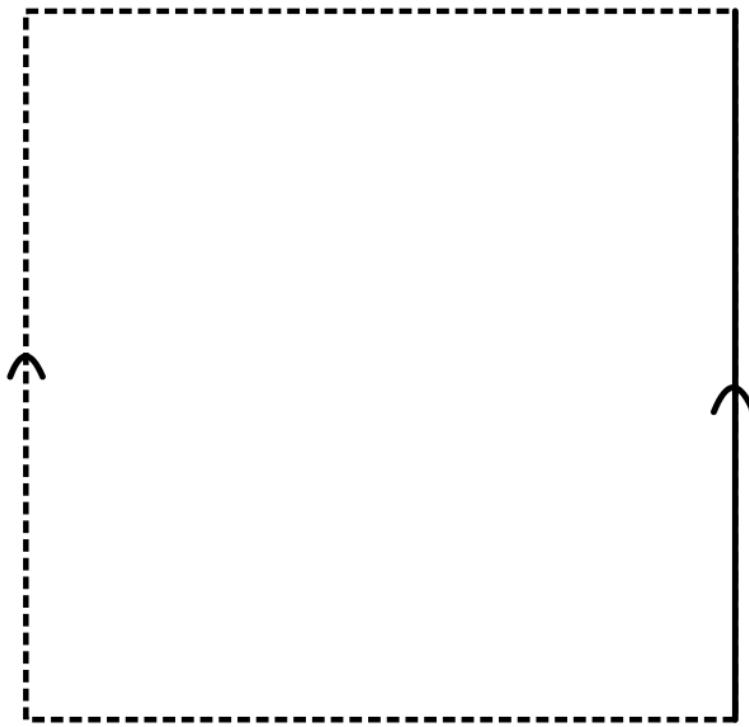


Figure 3.3: Your caption here

There are two distinguished ways of embedding the cylindrical closure of ω into M . We can embed the cylindrical closure onto the hemisphere containing 0(∞ resp.), i.e. the lower hemisphere(upper hemisphere resp.), in such a way that the embedding extends

- (i) to $S^1 \times \{0\}$) as an isomorphism onto the equator of M
- (ii) to $S^1 \times \{1\}$ as a constant map to 0(∞ resp.)

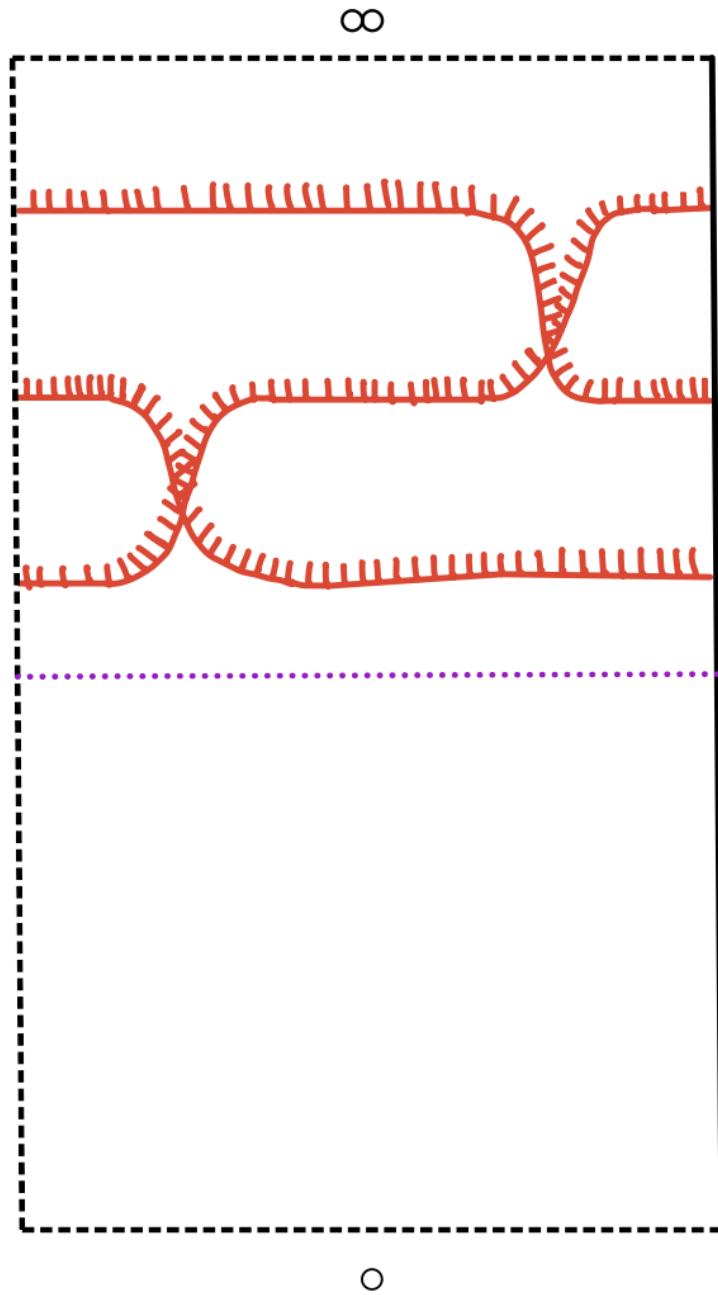


Figure 3.4: embedding of the cylindrical closure onto the upper hemisphere

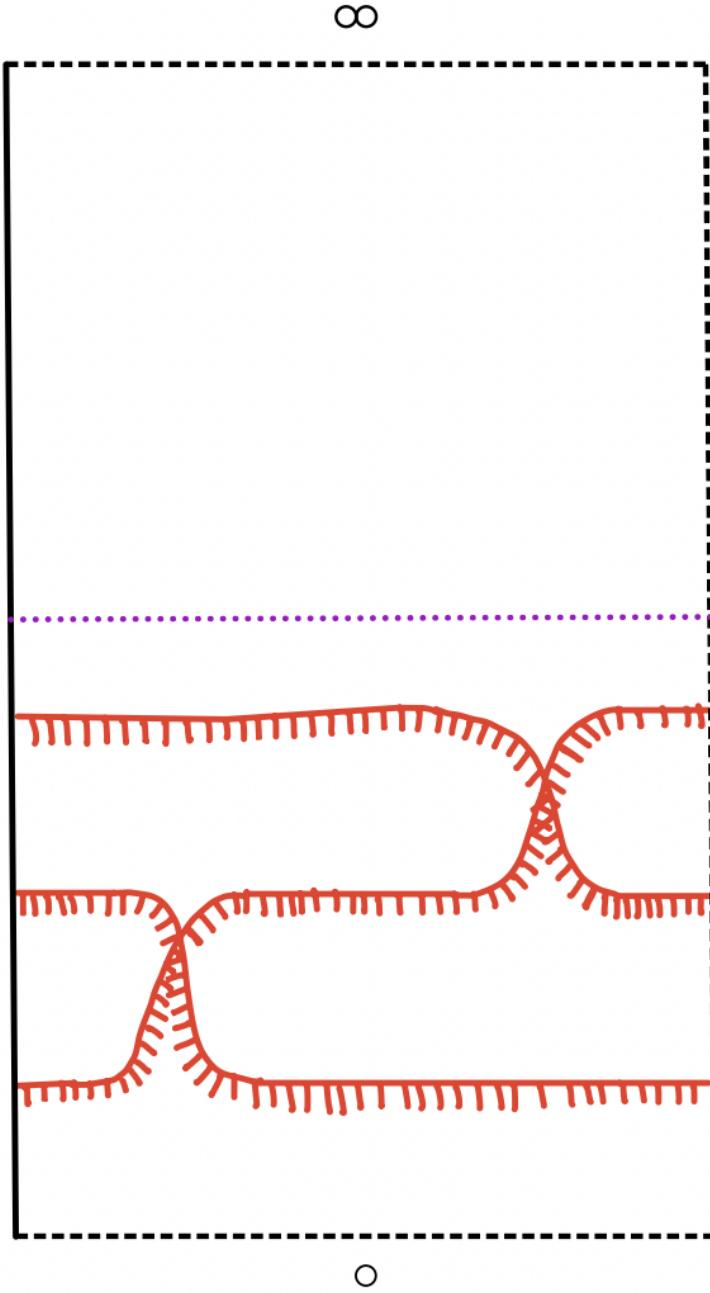


Figure 3.5: embedding of the cylindrical closure onto the lower hemisphere

Suppose we have a positive braid word ω on n strands, we have the following associated objects:

- M : A Riemann sphere with two punctures at 0 and ∞
- $\Phi_0 : \coprod_{i=1}^n S^1 \rightarrow M$: the front projection induced by the embedding of the

cylindrical closure of the trivial braid word onto the lower hemisphere

- ξ_0 : the co-orientation of Φ_0
- $\Phi_\infty : \coprod_{i=1}^m S^1 \rightarrow M$ (where $m \leq n$): the front projection given by the embedding of the cylindrical closure of the braid word ω onto the upper hemisphere
- ξ_∞ : the co-orientation of ι_∞

To simplify the notation, we will denote the pair $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$ resp.) as $\Lambda_0(\Lambda_\infty$ resp.). Also, we will abuse $\Lambda_0(\Lambda_\infty$ resp.) to denote the Legendrian associated to the pair $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$ resp.).

Now fix a positive braid word ω and the object $(M, \Lambda_0, \Lambda_\infty)$ associated with it which we call the separated diagram of ω . I will define a natural alternating braid diagram $(M, \Lambda'_0, \Lambda'_\infty)$ whose associated Legendrian is Legendrian isotopic to the Legendrian associated with $(M, \Lambda_0, \Lambda_\infty)$. I will construct an explicit Legendrian isotopy between them. Furthermore, I will construct cobordisms between constructible sheaves singular supported on $(M, \Lambda_0, \Lambda_\infty)$ and $(M, \Lambda'_0, \Lambda'_\infty)$ which will be the main result of this chapter. The isotopy will be only applied to Λ_0 , so the Λ_∞ will remain fixed i.e. $\Lambda_\infty = \Lambda'_\infty$.

First, let's draw Λ'_∞ as in red on M as follows :

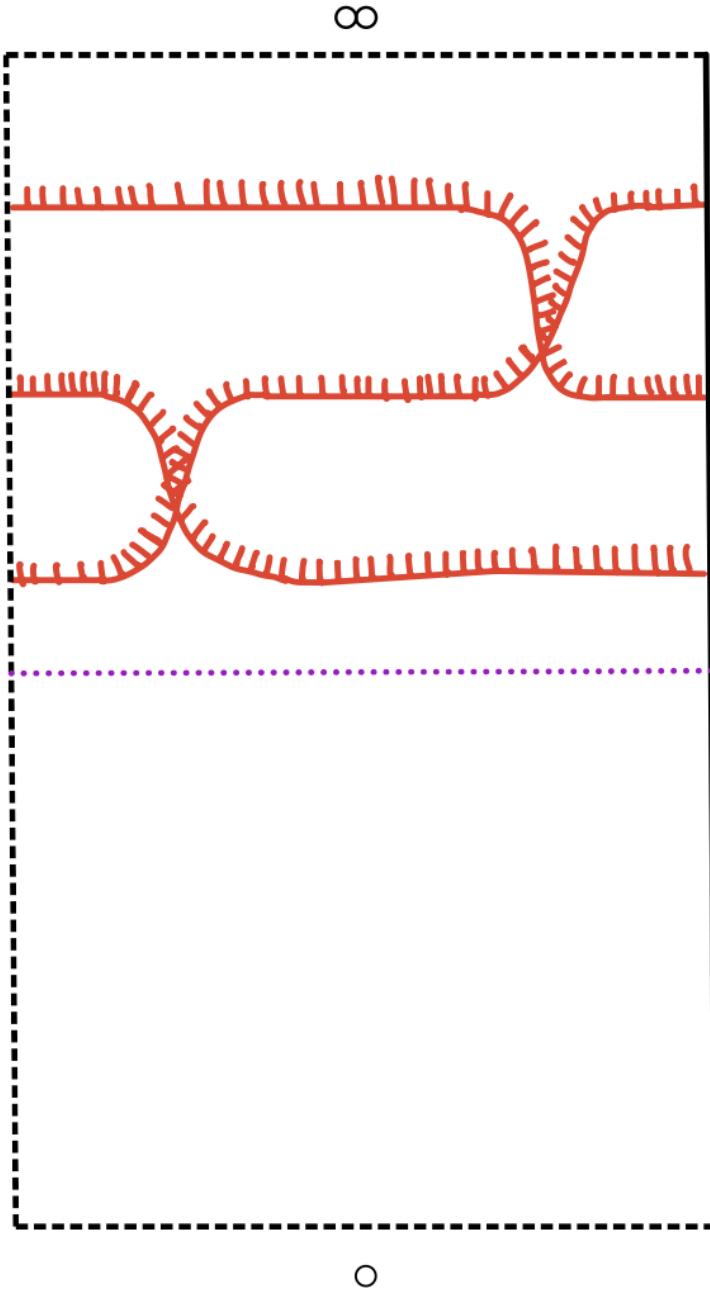


Figure 3.6: Your caption here

Now on the above diagram, let's draw Λ'_0 the part that is Legendrian isotopic to Λ_0 in blue. But before that, we need some definitions.

Definition 67. Suppose $\omega = s_{1_1}, \dots, s_{i_k}$, then the cylindrical closure can be parsed into a concatenation of k mutually disjoint regions where i^{th} region containing a part of the braid diagram corresponding to the generator s_{i_j} (figure below). We call

the region corresponding to s_{ij} as the j^{th} generator region(also its image under the embedding into M).

Below is the picture of the 1^{st} generator region of the cylindrical closure of $\omega = s_1 s_2$.

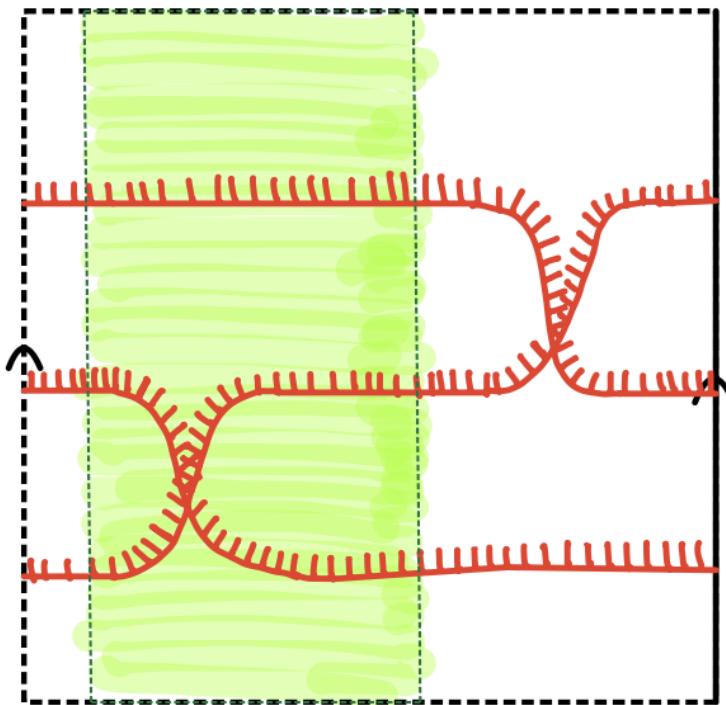


Figure 3.7: 1st generator region

Below is the picture of the 2^{nd} generator region of the cylindrical closure of $\omega = s_1 s_2$.

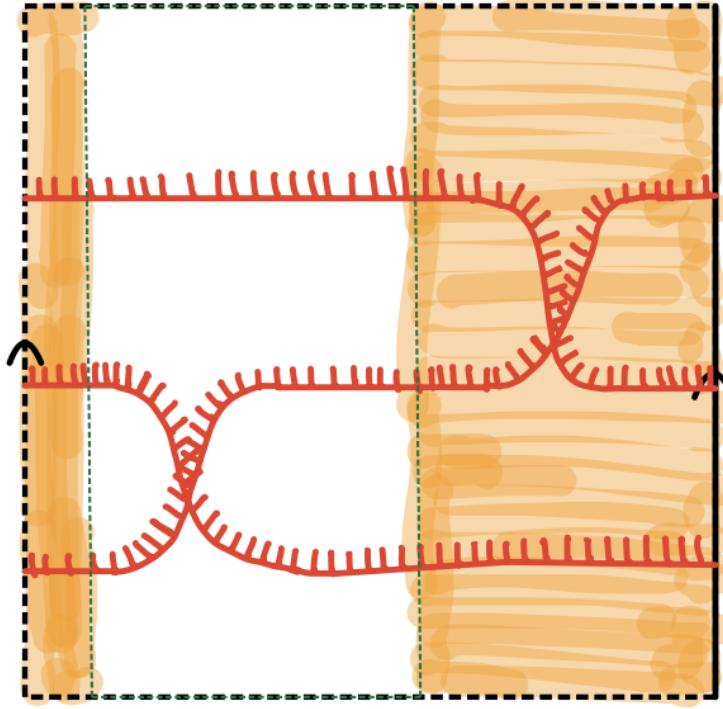


Figure 3.8: 2nd generator region

Definition 68. Suppose we set-theoretically subtract the union of all generator regions from the cylinder, we get k connected components. That is, for each $j = 1, \dots, k$, we have one component in between j^{th} and $(j + 1 \pmod k)^{th}$ regions. We call the neighborhood of this component inside the cylinder as j^{th} inter-generator region(also its image inside the cylinder under the embedding into M).

- inter-generator regions do not contain any crossing
- inter-generator regions are mutually disjoint
- j^{th} intergenerator region intersects with j^{th} and $j + 1^{th}$ (modulo k) generator region

Below is the picture of the 1^{st} inter-generator region of the cylindrical closure of $\omega = s_1 s_2$.

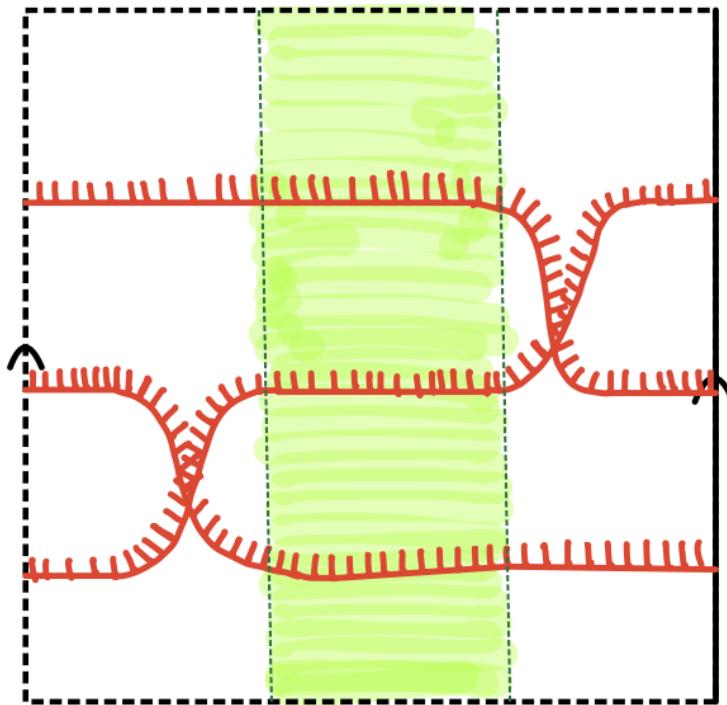


Figure 3.9: 1st inter-generator region

Below is the picture of the 2^{nd} generator region of the cylindrical closure of $\omega = s_1 s_2$.

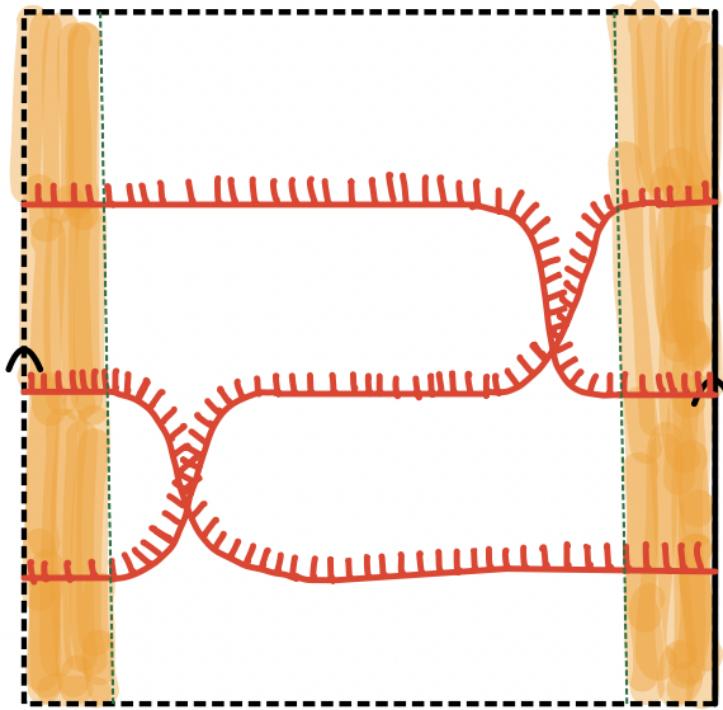


Figure 3.10: 2nd inter-generator region

Now I will draw Λ'_0 for each generator region so that they glue up to the whole Λ'_0 .

First, we restrict the diagram to j^{th} generator region, we have the following diagram: Note that i_j^{th} and $i_j + 1^{th}$ strands cross each other and all the other strands are horizontal.

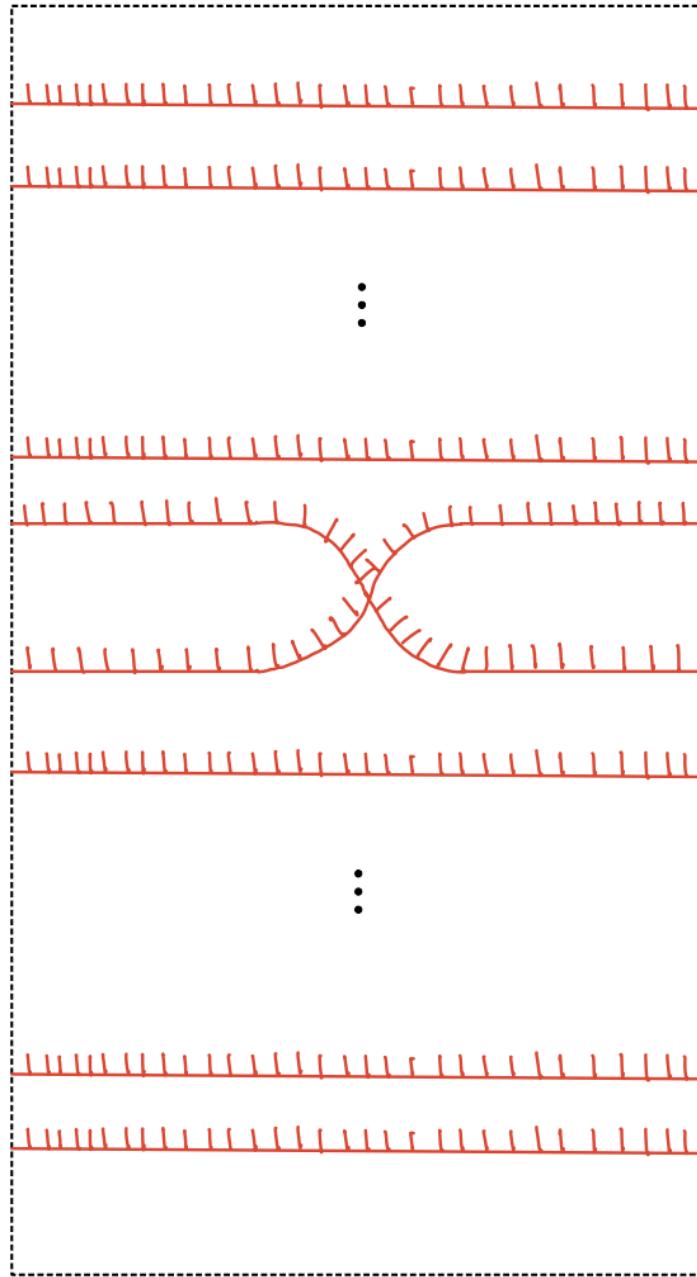


Figure 3.11: Your caption here

We label the strands from top to bottom using integers from 1 to n with reference to the left end points. This is the strand labelling scheme that I will use throughout this chapter.

I will draw Λ'_0 as blue strand on it as follows :

- l^{th} blue strand starts from the midpoint of the starting points of l^{th} and $l + 1^{th}$ red strands and ends at the midpoint of the end points of l^{th} and $l + 1^{th}$ red strands
- if $l \neq i_j$ and $i \neq i_j + 1$, then along the way the l^{th} blue strand crosses up and down once
- if $l = i_j$, l^{th} blue strand crosses l^{th} red strand up in the part before the crossing and then crosses $l + 1^{th}$ red strand down in the part after the crossing.
- if $l = i_j + 1$, l^{th} blue strand crosses $l + 1^{th}$ red strand up and down in the part before the crossing and then crosses l^{th} red strand up and down in the part after the crossing.

The picture below overlays Λ'_0 (drawn in blue strands with hairs pointing downward) on the previous diagram (use the precise numbering).

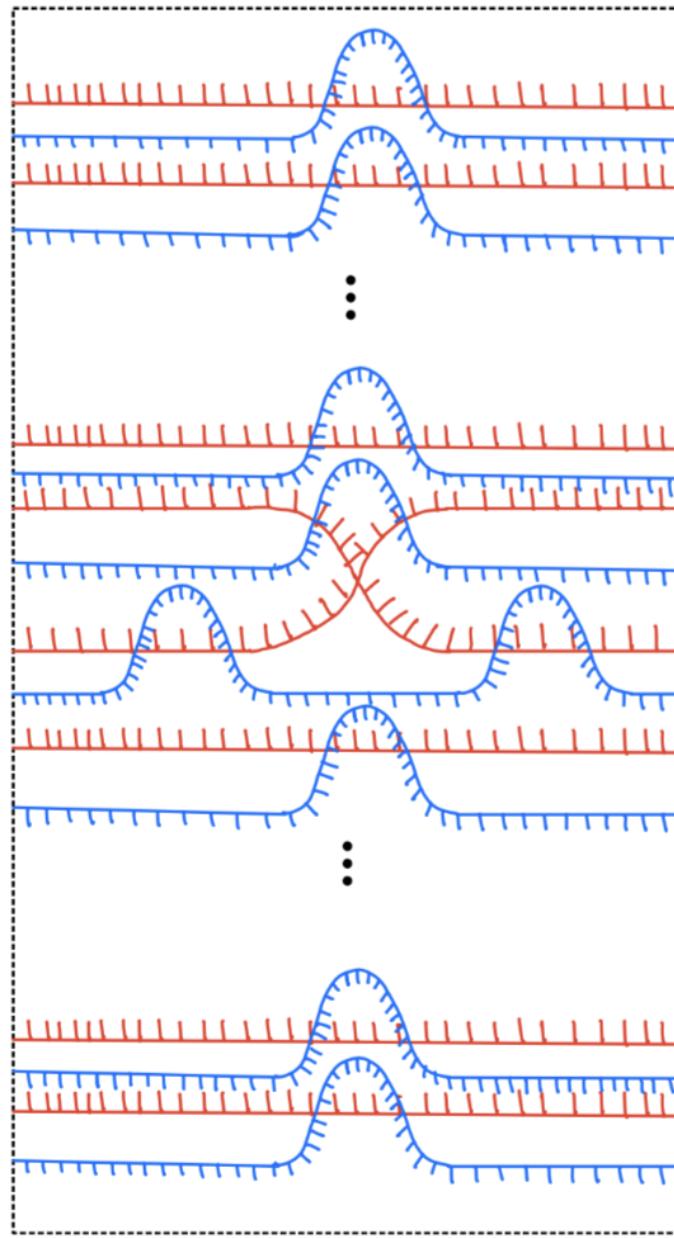


Figure 3.12: Your caption here

For the full alternating strand diagram, we take the closure of blue strands from the generator regions so that the end points from the bordering regions coincide.

The picture below shows how the global natural alternating strand diagram associated with $\omega = s_1 s_2$ looks like after gluing together local alternating strand diagrams of s_1 and s_2 .

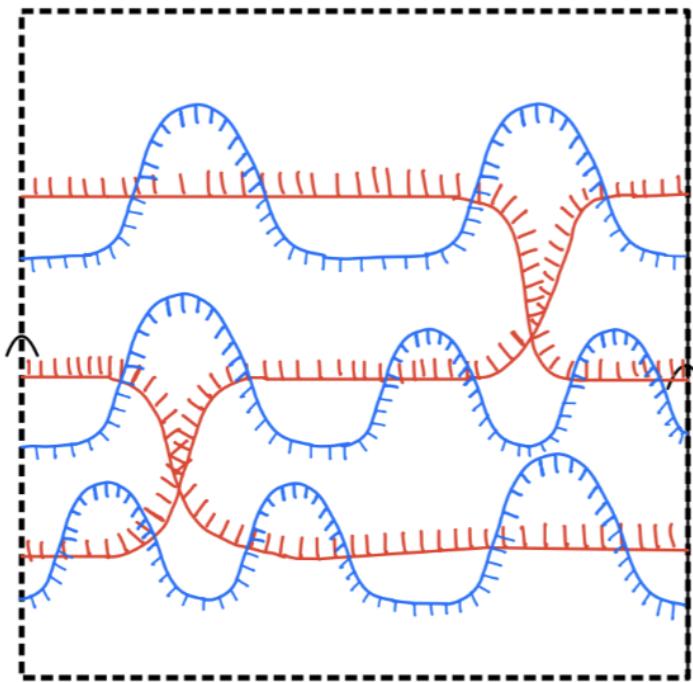


Figure 3.13: Your caption here

Theorem 69. The above defined strand diagram is alternating

Proof. we will denote

- the region with all the hairs pointing outward as \circ
- the region with all the hairs pointing inward as \triangle
- else with \times

for the generator region we have the following figure :

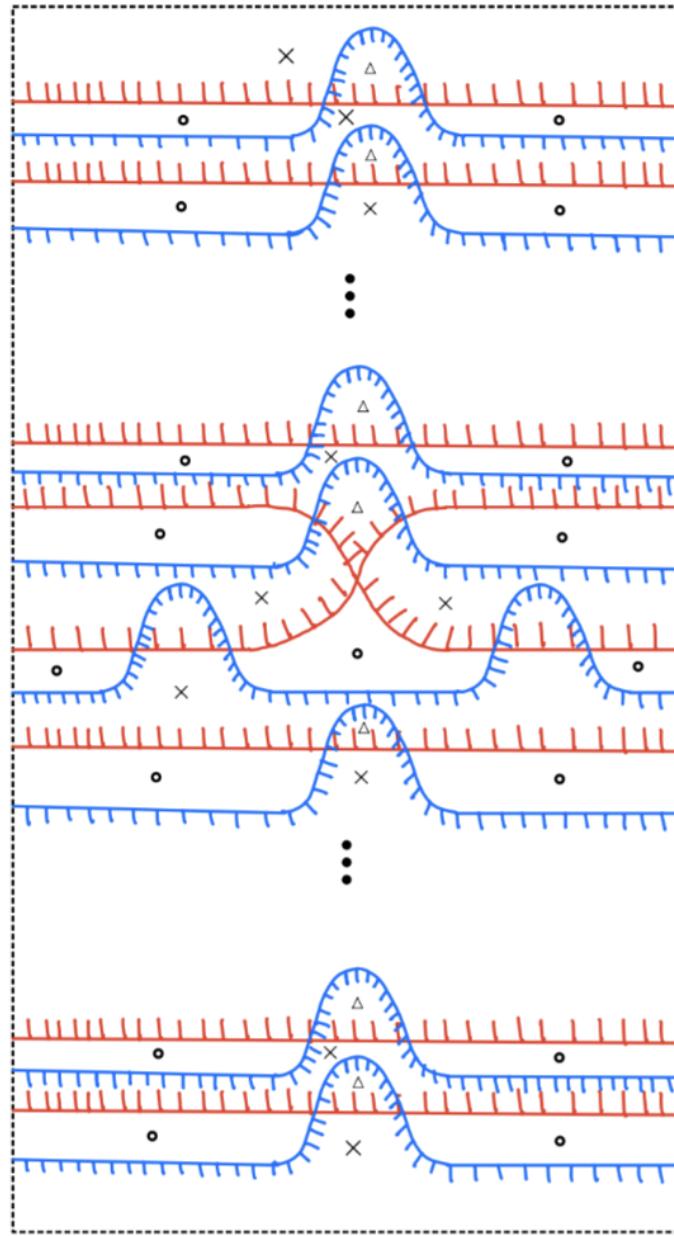


Figure 3.14: Your caption here

The above marking extends to the inter-generator region, we have the following figure:

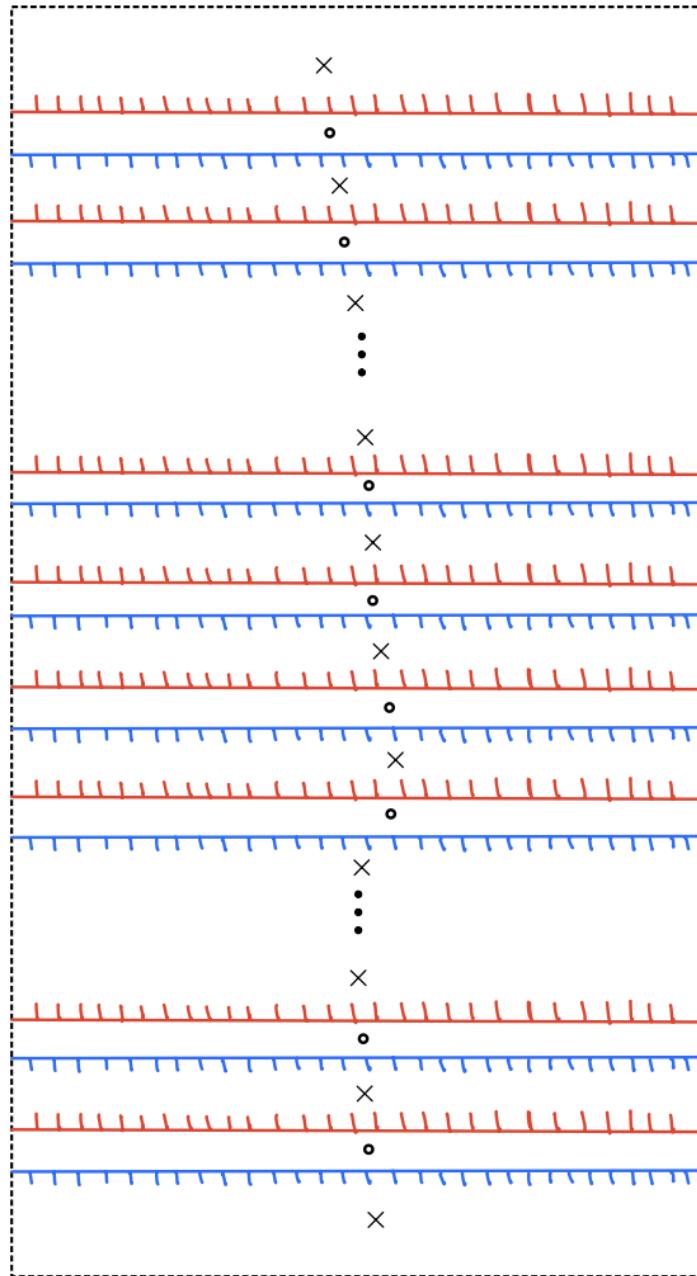


Figure 3.15: Your caption here

for each crossing, it satisfy the alternating condition. This diagram is indeed alternating. \square

3.3 local systems on natural alternating diagrams

Suppose we have a positive braid word ω then we have the associated natural alternating diagram $(M, \Lambda'_0, \Lambda'_{\infty})$ defined in the previous section.

We can associate a quiver Q to the alternating diagram in such a way that

- we have one vertex for regions where all hairs are pointing outward/inward
- for each crossing, we have an arrow from the vertex corresponding to the region where all hairs pointing outward to inward

For example, for each generator region of a natural alternating strand diagram:

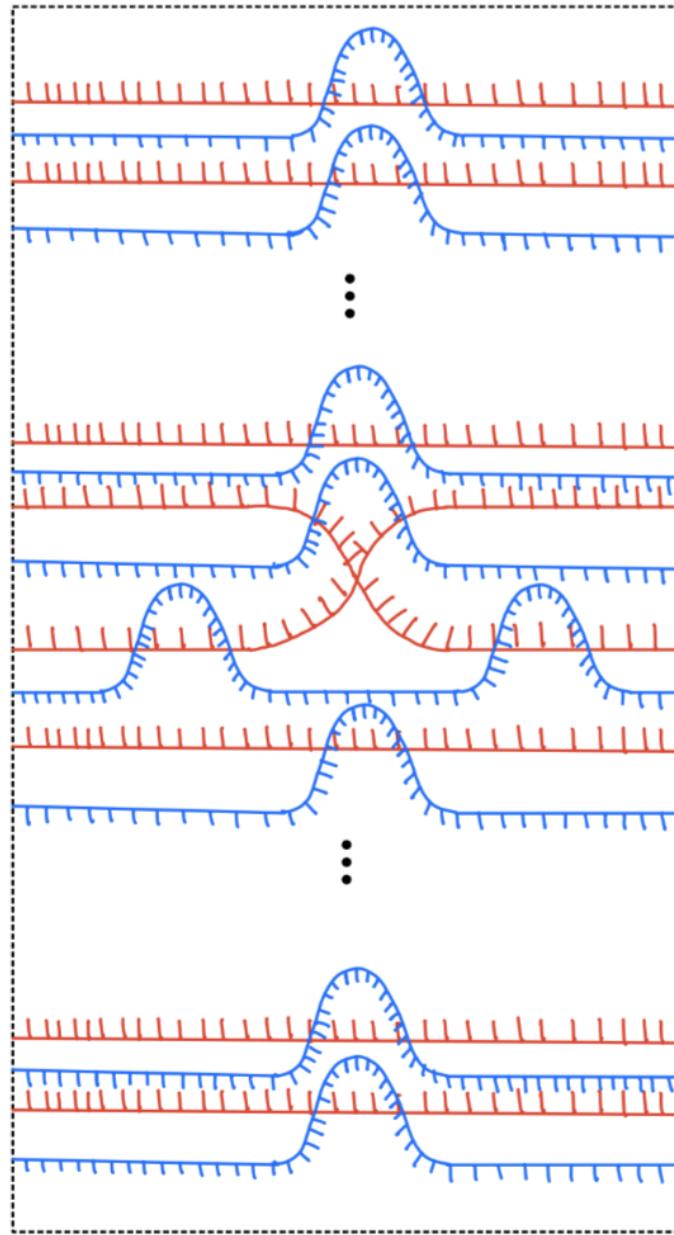


Figure 3.16: Your caption here

we have the following associated quiver:

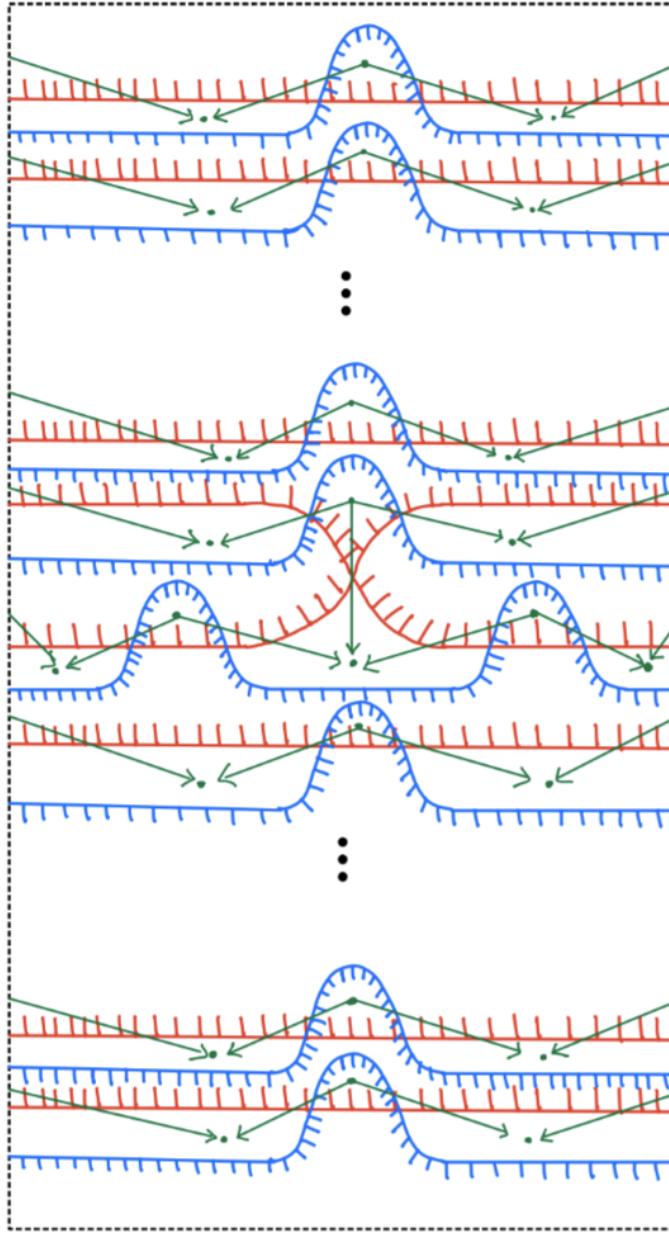


Figure 3.17: Your caption here

Once we have an alternating strand diagram, we have the associated conjugate surface S_{conj} . Furthermore, we can embed the underlying undirected graph of the quiver Q (i.e. the bipartite graph of the alternating coloring) in S_{conj} in such a way that S_{conj} deformation retracts to Q (with abuse of notation I will denote this underlying undirected graph as Q). Suppose we have a rank 1 local system on the conjugate

surface associated with $(M, \Lambda'_0, \Lambda'_{\infty})$, then restricting to Q , we get a local system on Q . Note that the pullback, induced by the restriction map, between the space of local systems $H^1(S_{conj}, \mathbb{C}^*) \rightarrow H^1(Q, \mathbb{C}^*)$ is an isomorphism.

$H^1(Q, \mathbb{C}^*)$ is isomorphic to $(\mathbb{C}^*)^{|Arr(Q)|} // (\mathbb{C}^*)^{|Vert(Q)|}$ here the group action is defined as the following : let $g_v \in (\mathbb{C}^*)^{|Vert(Q)|}$ (more precisely, $g_v := (g^{\delta_{w,v}})_{w \in Vert(Q)}$ where δ is the Kronecker delta), then $g_v \cdot (x_a)_{a \in Arr(Q)}$ is

- for entries with index a such that the source of a is v i.e. $s(a) = v$, we have

$$g_v \cdot x_a$$

- for entries with index a such that the target of a is v i.e. $t(a) = v$, we have

$$g_v^{-1} \cdot x_a$$

Now we define the associated alternating sheaf on some regular cell complex refinement of the natural alternating strand diagram associated with a rank 1 local systems on Q .

First, I will describe the special kind of regular cell complex associated with the alternating strand diagram called the regular cell complex refinement of the natural alternating strand diagram. I will define the refinement for each generator region and glue them to get the global regular cell complex.

Definition 70. Suppose we fix a generator region for the alternating strand diagram. Then we denote the j^{th} crossing(numbering starts from left to right) the i^{th} blue strand(numbering starts from top to bottom) crosses red strands as $c_{i,j}$. We will call the crossing between i^{th} and $i + 1^{th}$ red strand as c .

- (i) For each crossing $c_{i,j}$ we add

- when j is odd, locally near $c_{i,j}$ we have the following local diagram

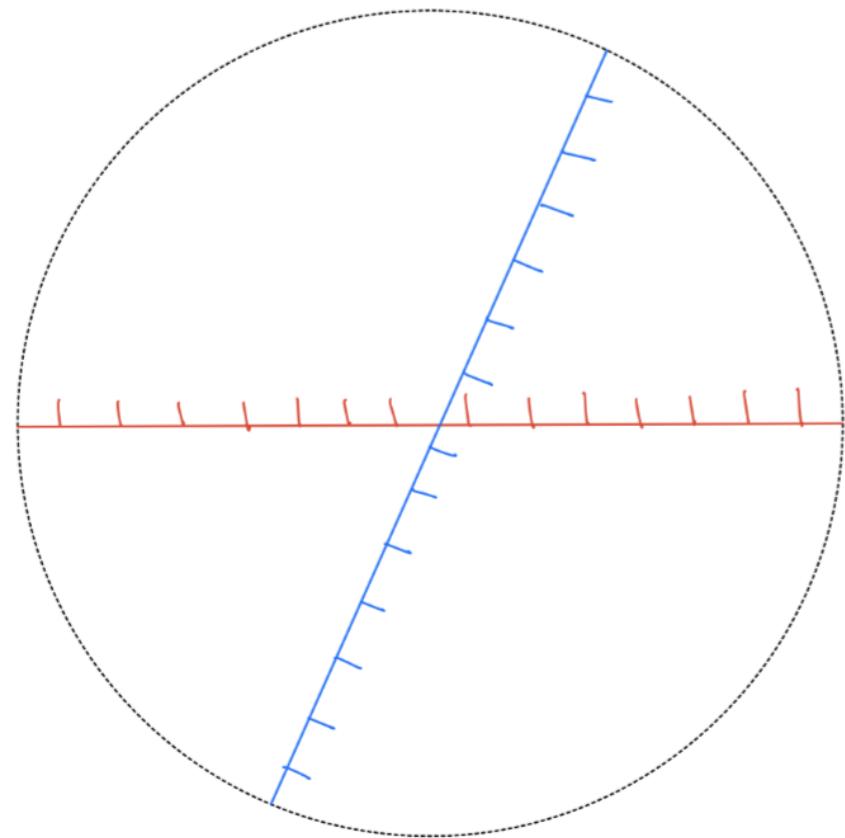


Figure 3.18: Your caption here

then we add squiggly lines with co-orientations and end points to get

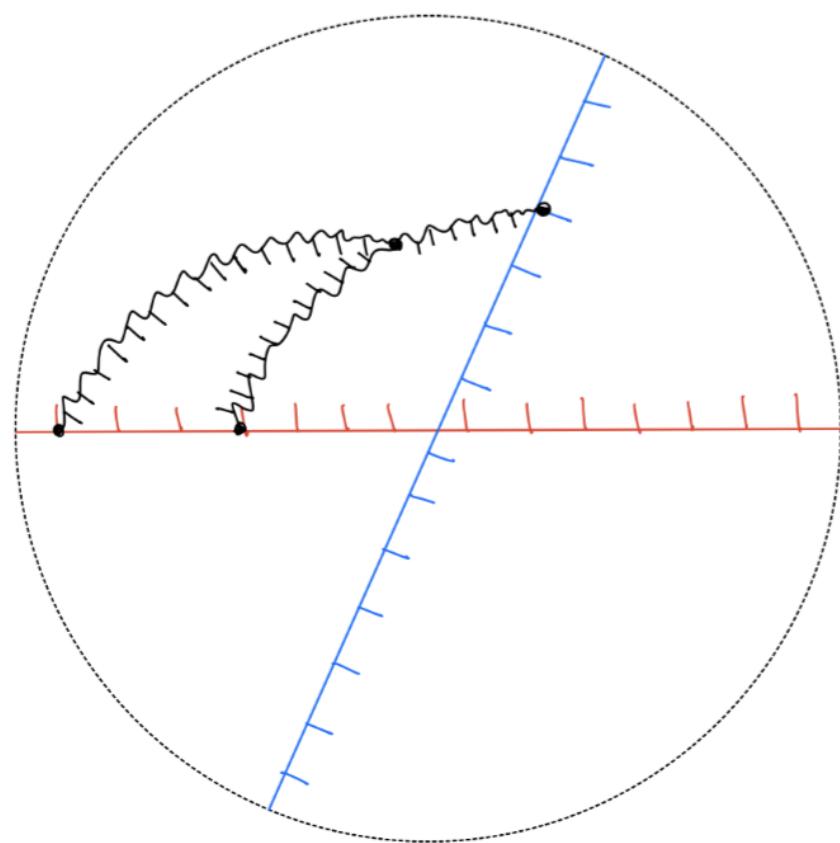


Figure 3.19: Your caption here

We call the region marked with * a crossing region.

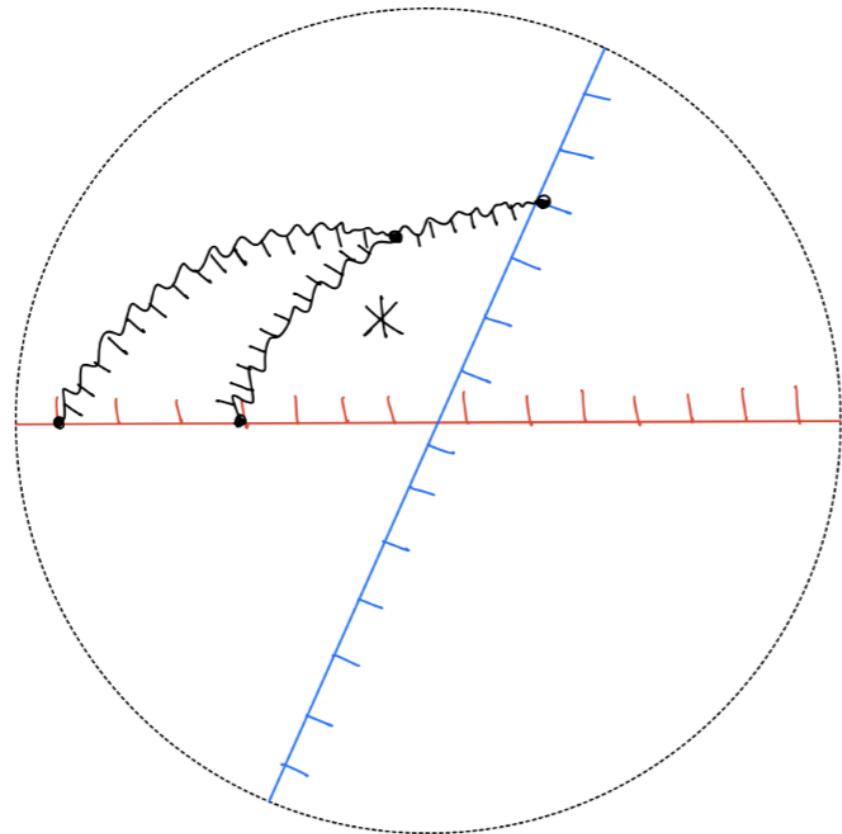


Figure 3.20: Your caption here

- when j is even, locally near $c_{i,j}$ we have the following local diagram

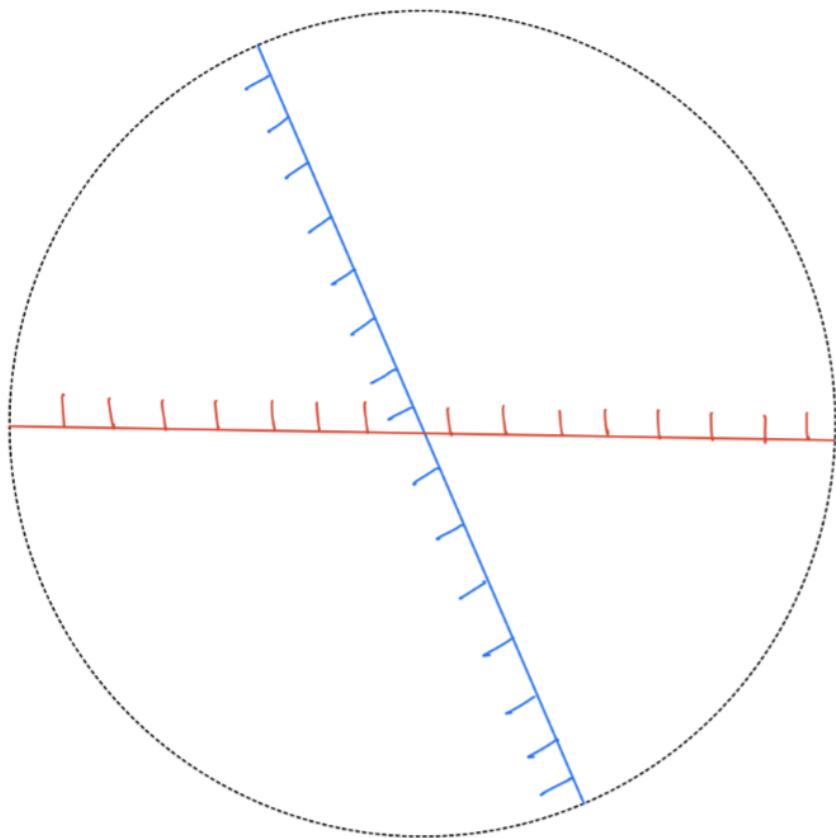


Figure 3.21: Your caption here

then we add squiggly lines with co-orientations and end points to get

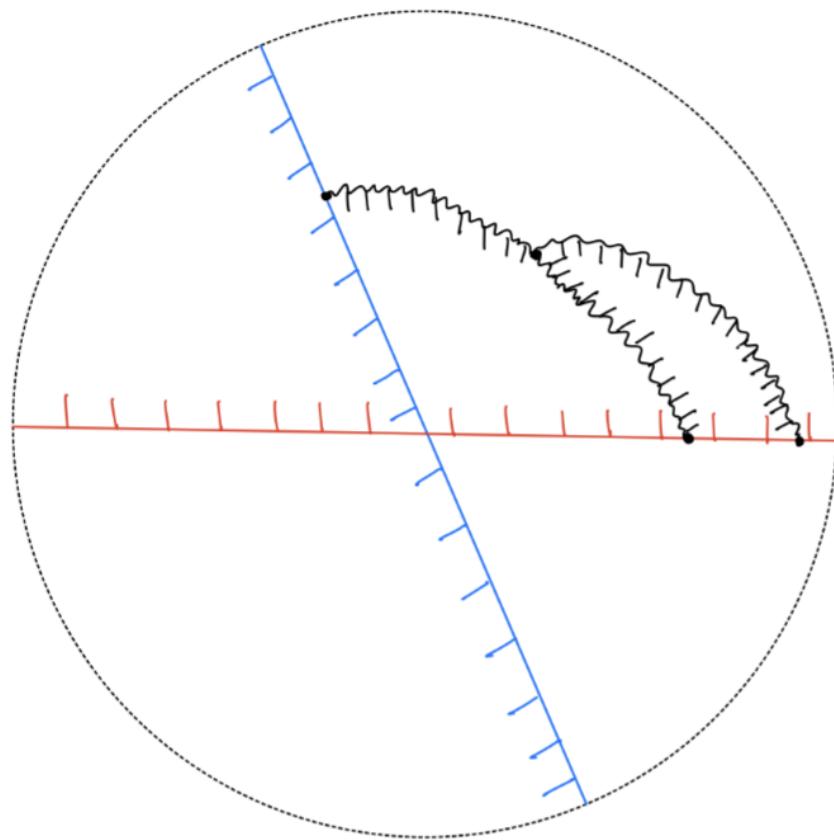


Figure 3.22: Your caption here

We call the region marked with * a crossing region.

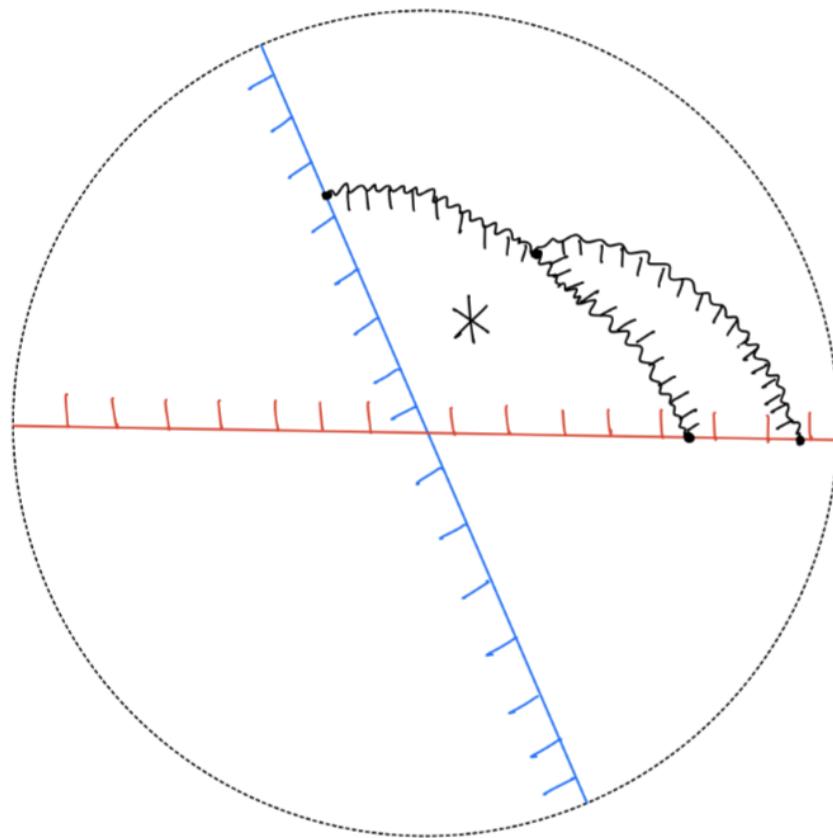


Figure 3.23: Your caption here

- (ii) For the crossing c , locally near the crossing, we have the following local diagram

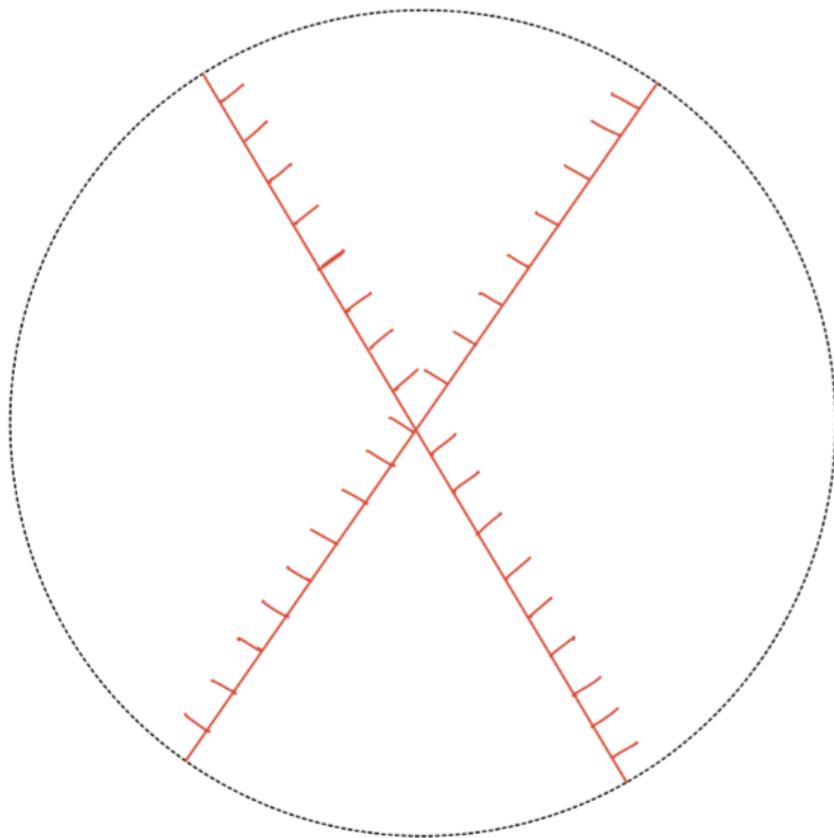


Figure 3.24: Your caption here

then we add squiggly lines with co-orientations and end points to get

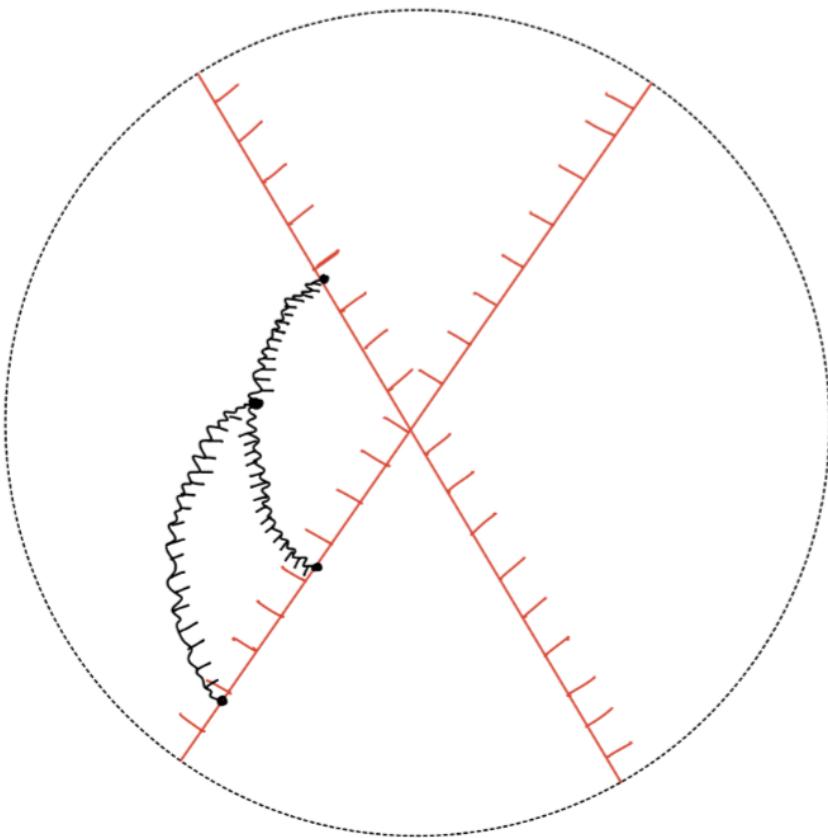


Figure 3.25: Your caption here

We call the region marked with * a crossing region.

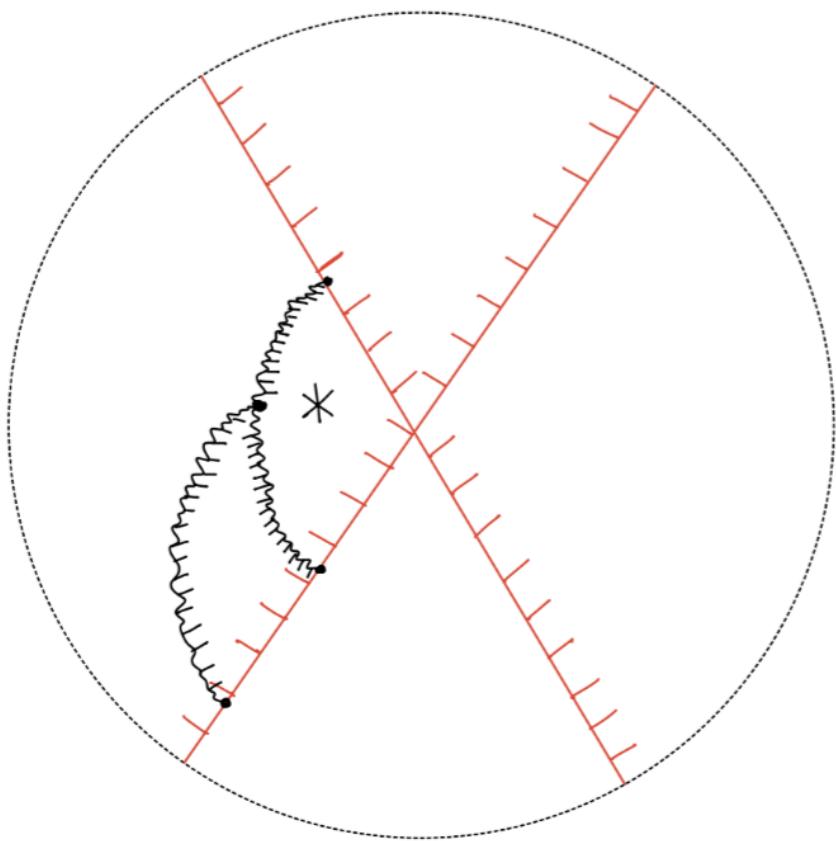


Figure 3.26: Your caption here

Below is the picture of a generator region of a natural alternating diagram:

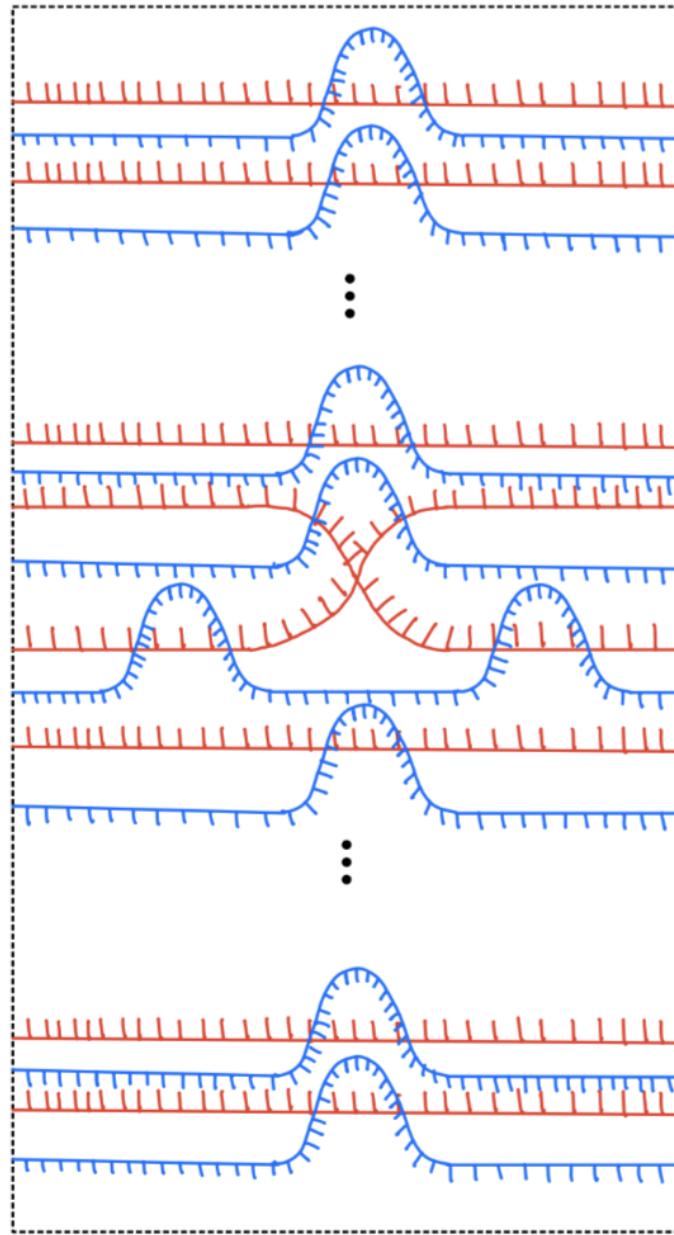


Figure 3.27: Your caption here

and below is the picture of the regular cell complex refinement in a generator region:

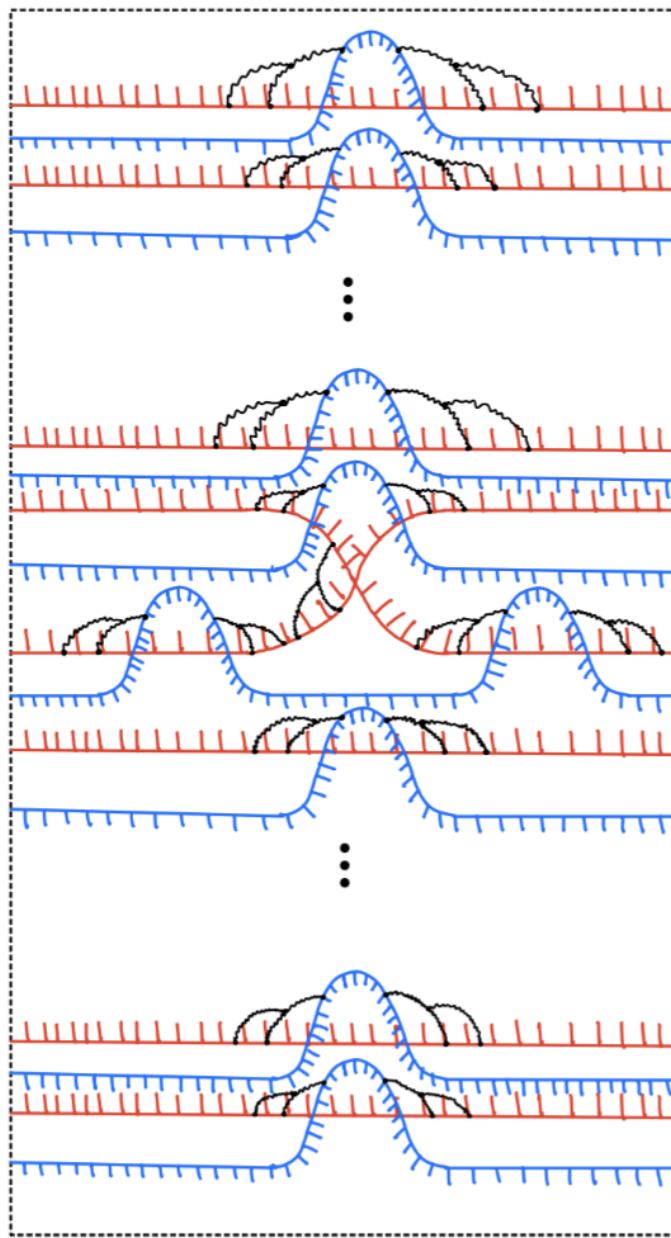


Figure 3.28: Your caption here

Next, we fix an inter-generator region

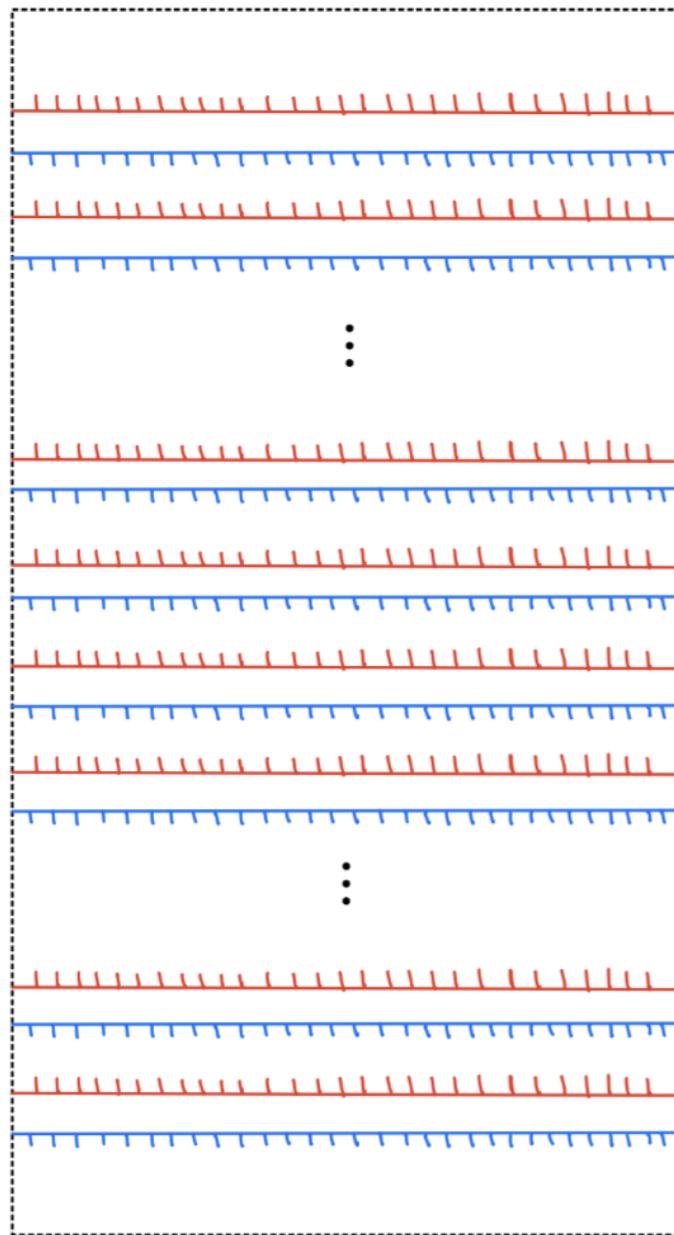


Figure 3.29: Your caption here

add a vertical squiggly line co-oriented towards the left

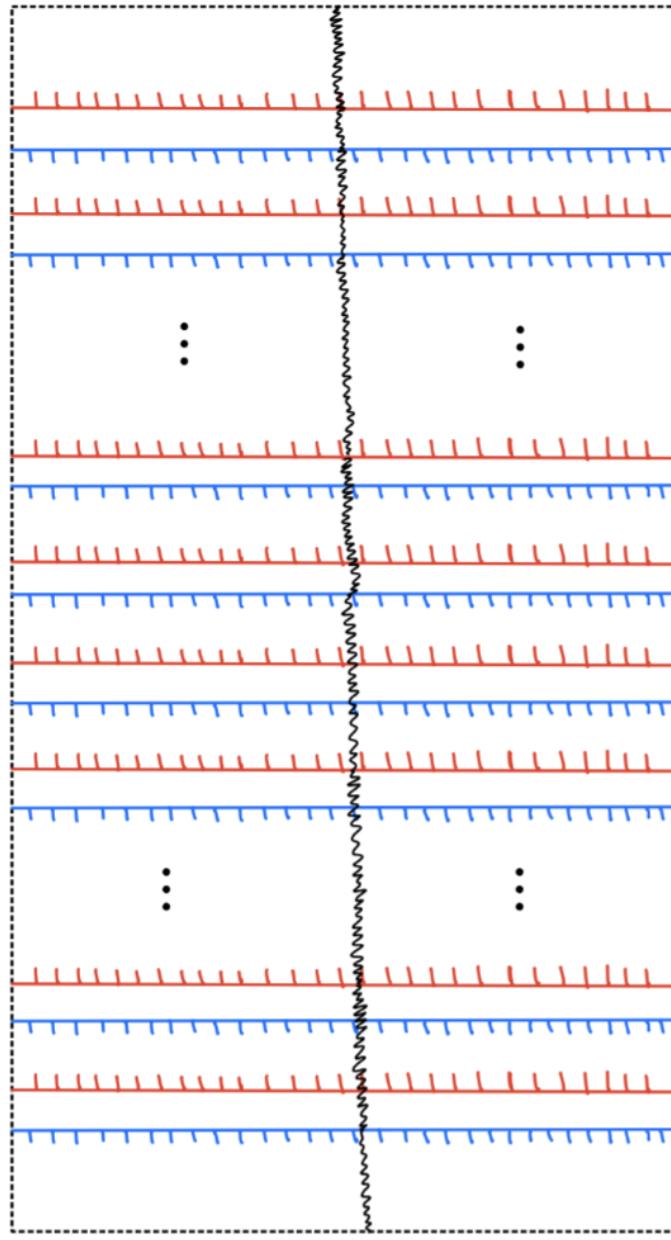


Figure 3.30: Your caption here

Now I will describe a way to specify a constructible sheaf on the above regular cell complex refinement associated with the local system on Q .

Definition 71. Suppose we have a local system on Q which can be represented as an element $(x_A)_{a \in Arr(Q)} (\mathbb{C}^*)^{|Arr(Q)|}$:

- (i) stalk of the region where all the hairs are pointing outward is $\mathbb{C}[-1]$

- (ii) stalk of the region where all the hairs(except the hairs on the squiggly lines) are pointing inward is \mathbb{C}
- (iii) stalk of the crossing regions is $\mathbb{C} \xrightarrow{\times x_a} \mathbb{C}$ where a is the arrow corresponding to the associated crossing.
- (iv) rest of the stalks are 0
- (v) the only nonzero genrization maps are from regions of type (iii) to (i), from (i) to (ii), or from (ii) to (ii)(the ones corresponding to vertical squiggly lines in the inter-generator regions)

The maps from (iii) to (i) are

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

The maps from (i) to (ii) are

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

The maps from (ii) to (ii) are identity maps.

Note that the group action maps a constructible sheaf to the isomorphic constructible sheaf. Therefore, we have a well-defined map $H^1(S_{conj}, \mathbb{C}^*) \rightarrow \mathcal{M}_1((M, \Lambda'_0, \Lambda'_{\infty}, \Lambda'_{squig}))$ where $\Lambda'_{squig} = (\Phi'_{squig}, \xi'_{squig})$ is the squiggly lines and it's co-orientations in the regular cell complex refinement of the natural alternating diagram.

3.4 1st Sheaf Cobordism

In this section, we define $cobord_1$, a compactly supported sheaf cobordism.

3.4.1 Notations

Definition 72. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 73. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_1$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_1$.

Definition 74. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_\bullet^{symbol} to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_{\bullet}^{symbol} a co-orientation of Φ_{\bullet}^{symbol} .
3. we denote the pair $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$ as $\Lambda_{\bullet}^{symbol}$. Later in the section, $\Lambda_{\bullet}^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₁.
4. we denote the triple $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$ as Λ_{\bullet} and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_{\bullet} will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₁.

Definition 75. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} -\frac{3}{4}e^{(\frac{x^2}{x^2-1})}t & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$ if $t \neq 0$
- $\{(1, 0), (-1, 0), (0, -\frac{3}{4}t)\} \subset Graph(\Psi_t)$

Definition 76. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$

$$(x, z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 77. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\} = \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_0^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig} : hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 78. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig} : hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 79. 1. We define the following subsets of $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -\frac{1}{2}\}$

- $\lambda_{\bullet}^{squig} := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$

2. We define co-orientations ξ_{\bullet}^{symbol} of $\lambda_{\bullet}^{symbol}$ as follows

- ξ_{\bullet}^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_{\bullet}^{∞} : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_{\bullet}^{squig} : hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 80. 1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .

2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_{\bullet} on $D_{r=2} \times [0, 1]$ induced by λ_{\bullet} i.e. strata are non-empty finite intersections of λ_{\bullet}^0 , $\lambda_{\bullet}^{\infty}$, and $\lambda_{\bullet}^{squig}$. Note that 2 dimensional strata has co-orientations inherited from λ_{\bullet} .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the following notations:

Definition 81. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 82. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

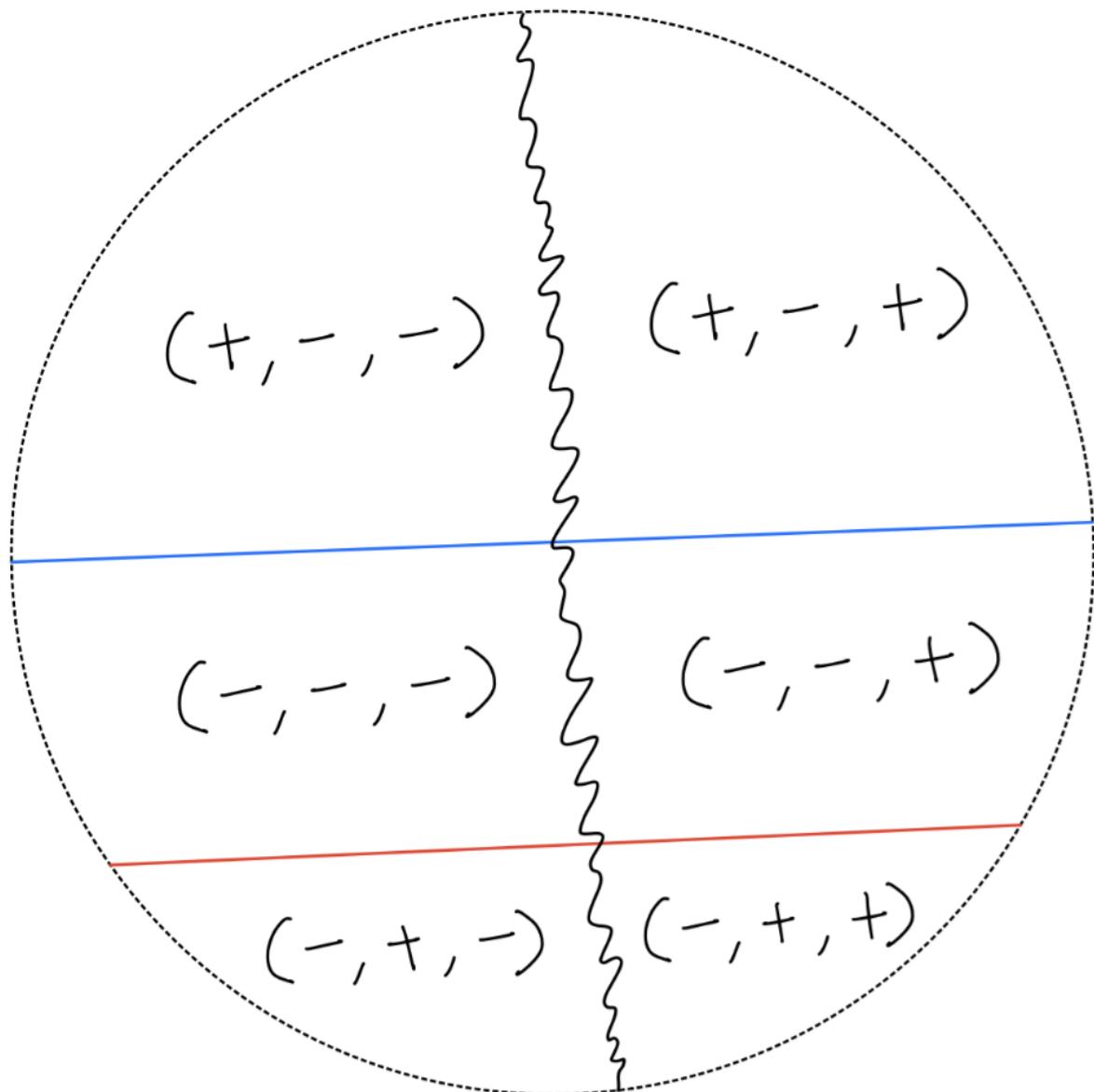
$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(-\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

Definition 83. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

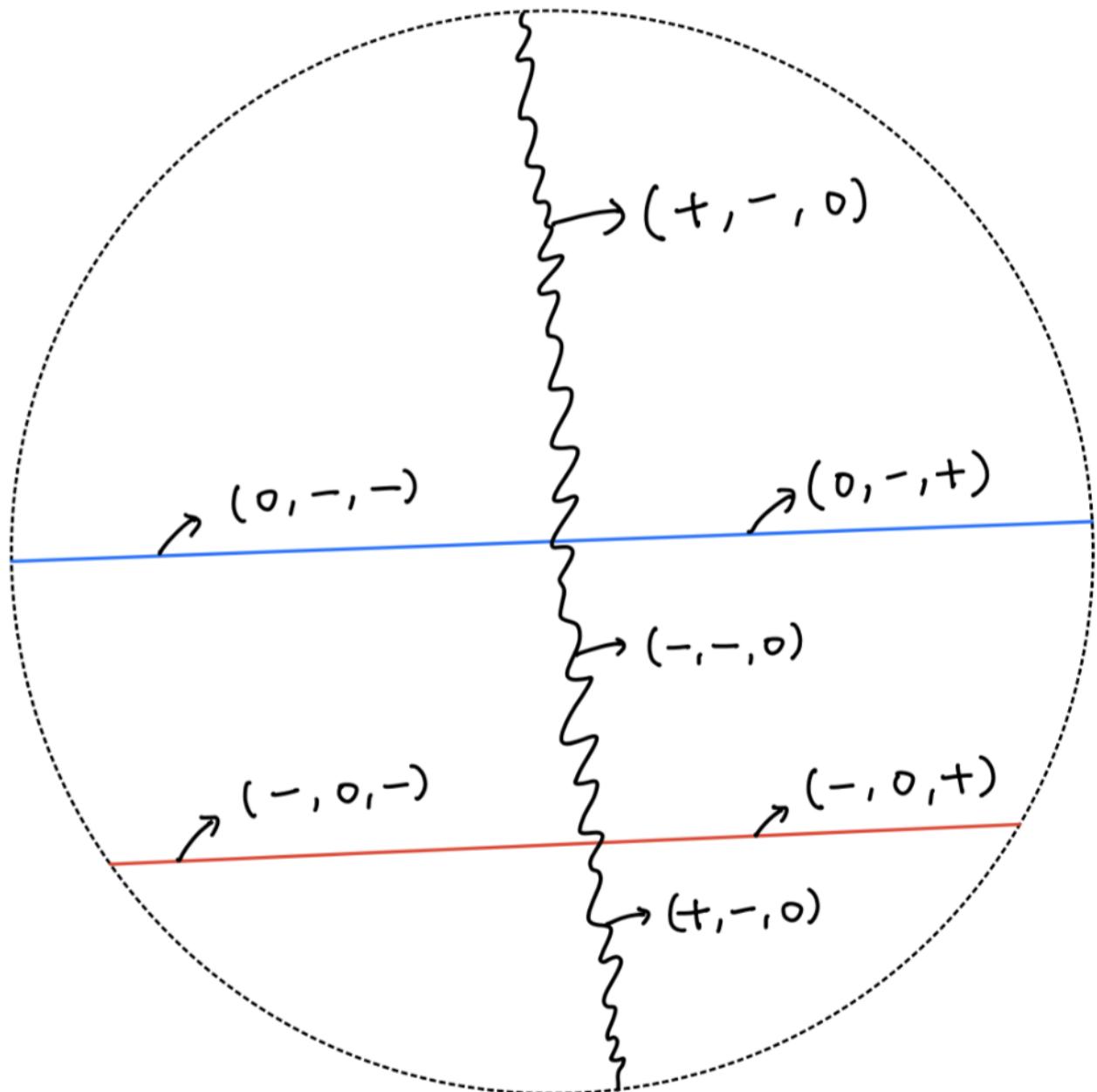
Definition 84. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the above notations:

1. \mathcal{S}_0 :

Figure 3.31: 2 dimensional strata of \mathcal{S}_0

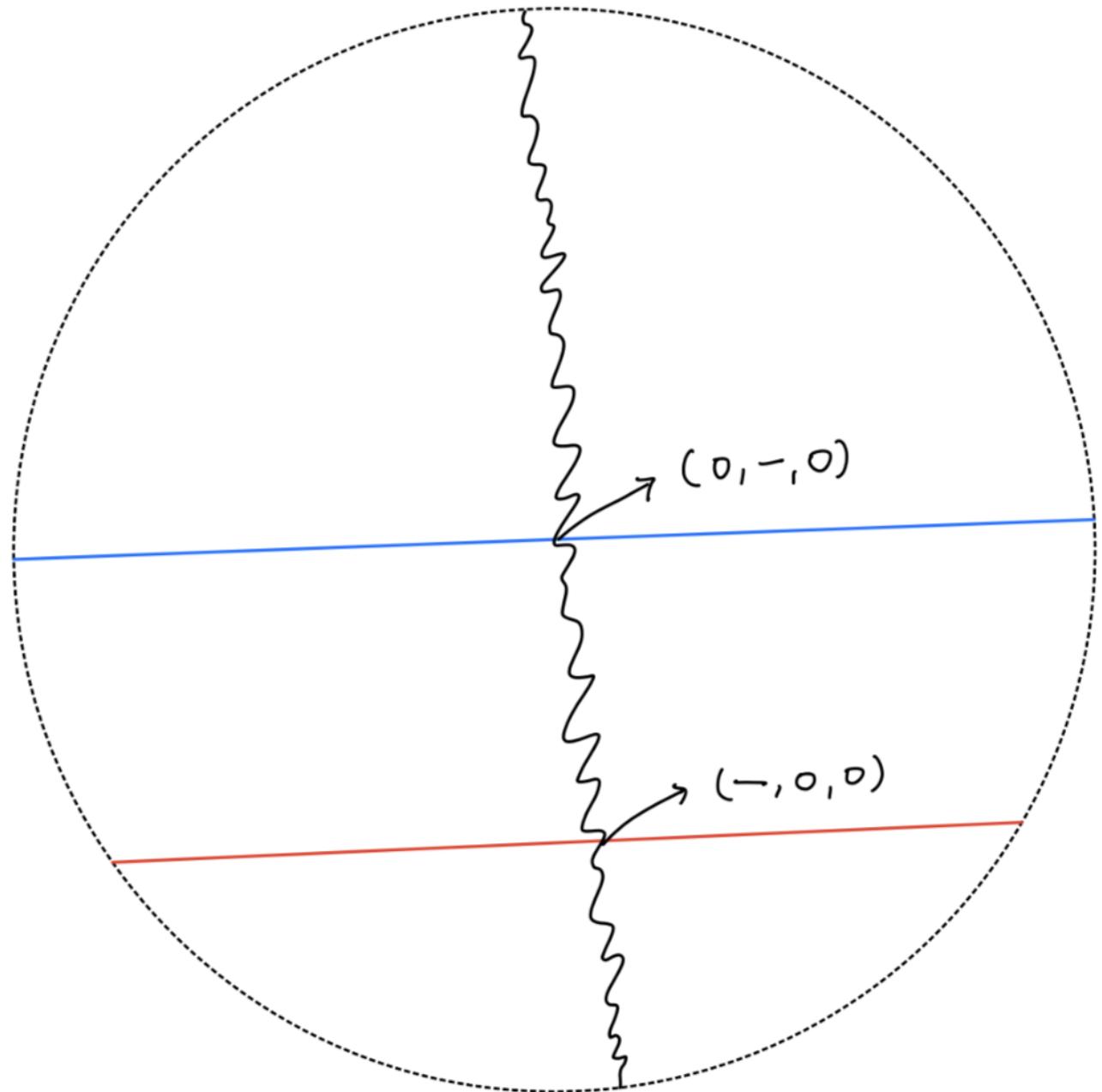
- 2 dimensional strata:

$$s_0(+, -, -), s_0(+, -, +), s_0(-, -, -), s_0(-, -, +), s_0(-, +, -), s_0(-, +, +)$$

Figure 3.32: 1 dimensional strata of \mathcal{S}_0

- 1 dimensional strata:

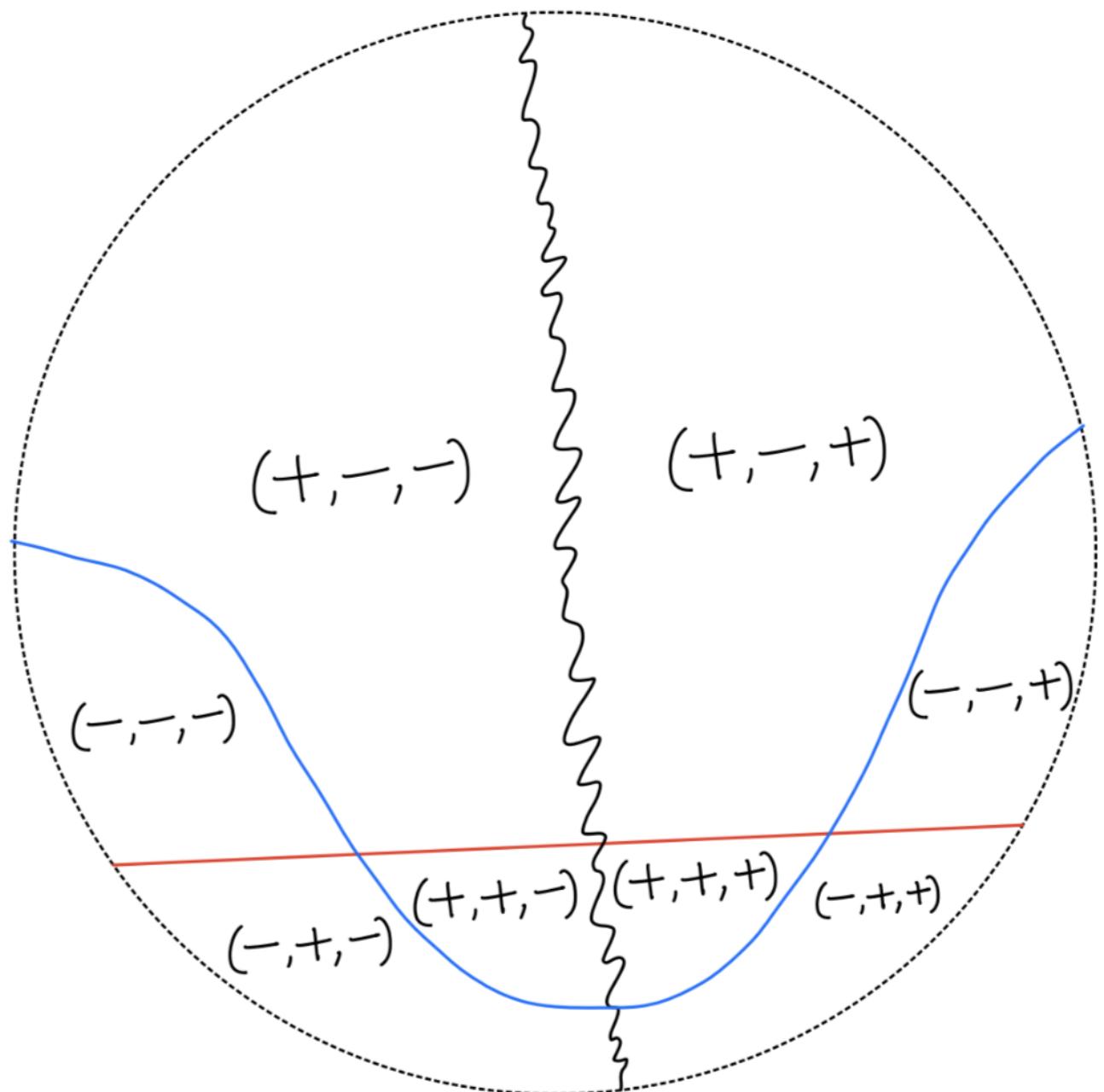
$$s_0(0, -, -), s_0(0, -, +), s_0(-, 0, -), s_0(-, 0, +), s_0(-, -, 0), s_0(-, +, 0), s_0(0, -, +)$$

Figure 3.33: 0 dimensional strata of \mathcal{S}_0

- 0 dimensional strata:

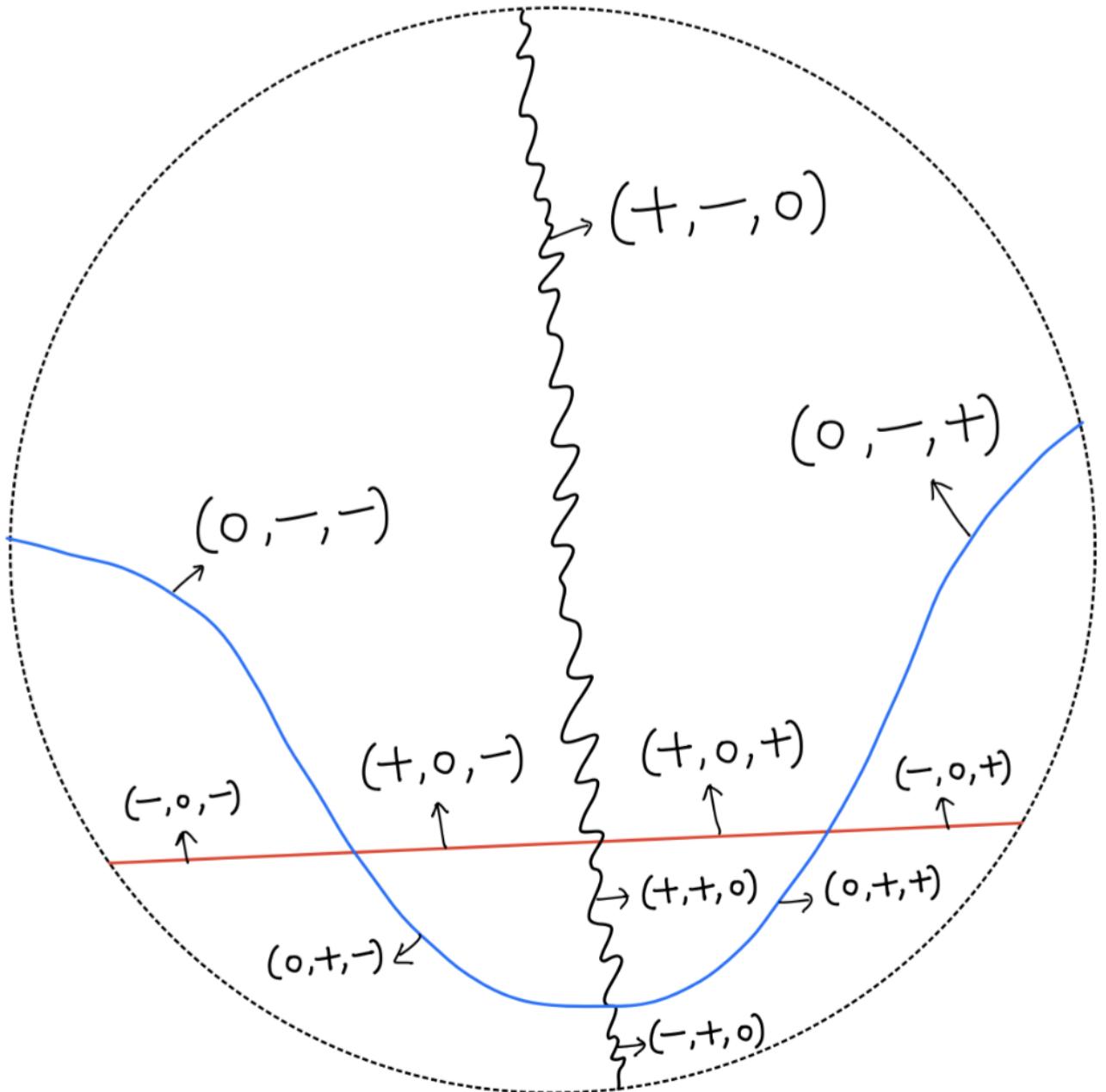
$$s_0(0, -, 0), s_0(-, 0, 0)$$

2. \mathcal{S}_1 :

Figure 3.34: 2 dimensional strata of \mathcal{S}_1

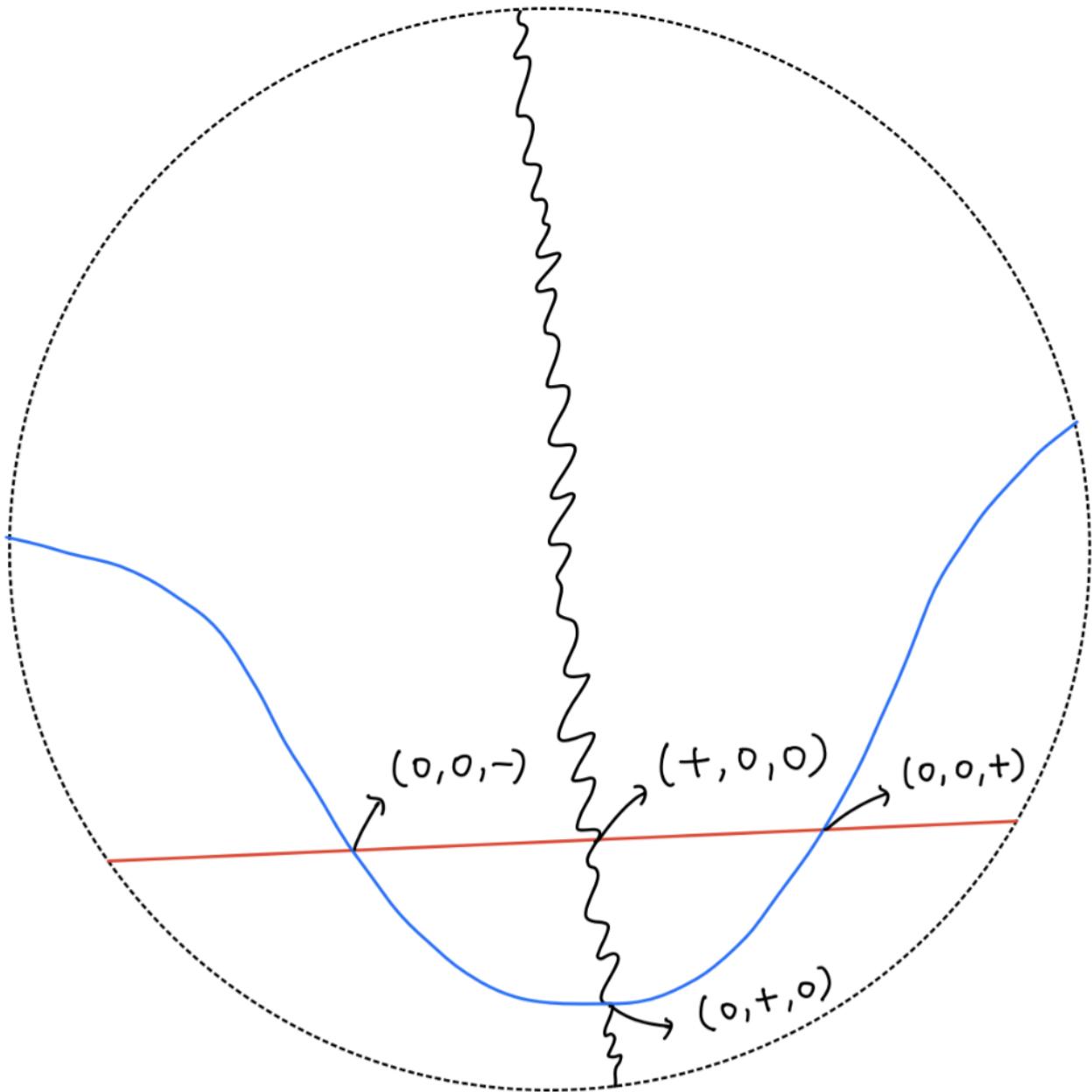
- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

Figure 3.35: 1 dimensional strata of \mathcal{S}_1

- 1 dimensional strata:

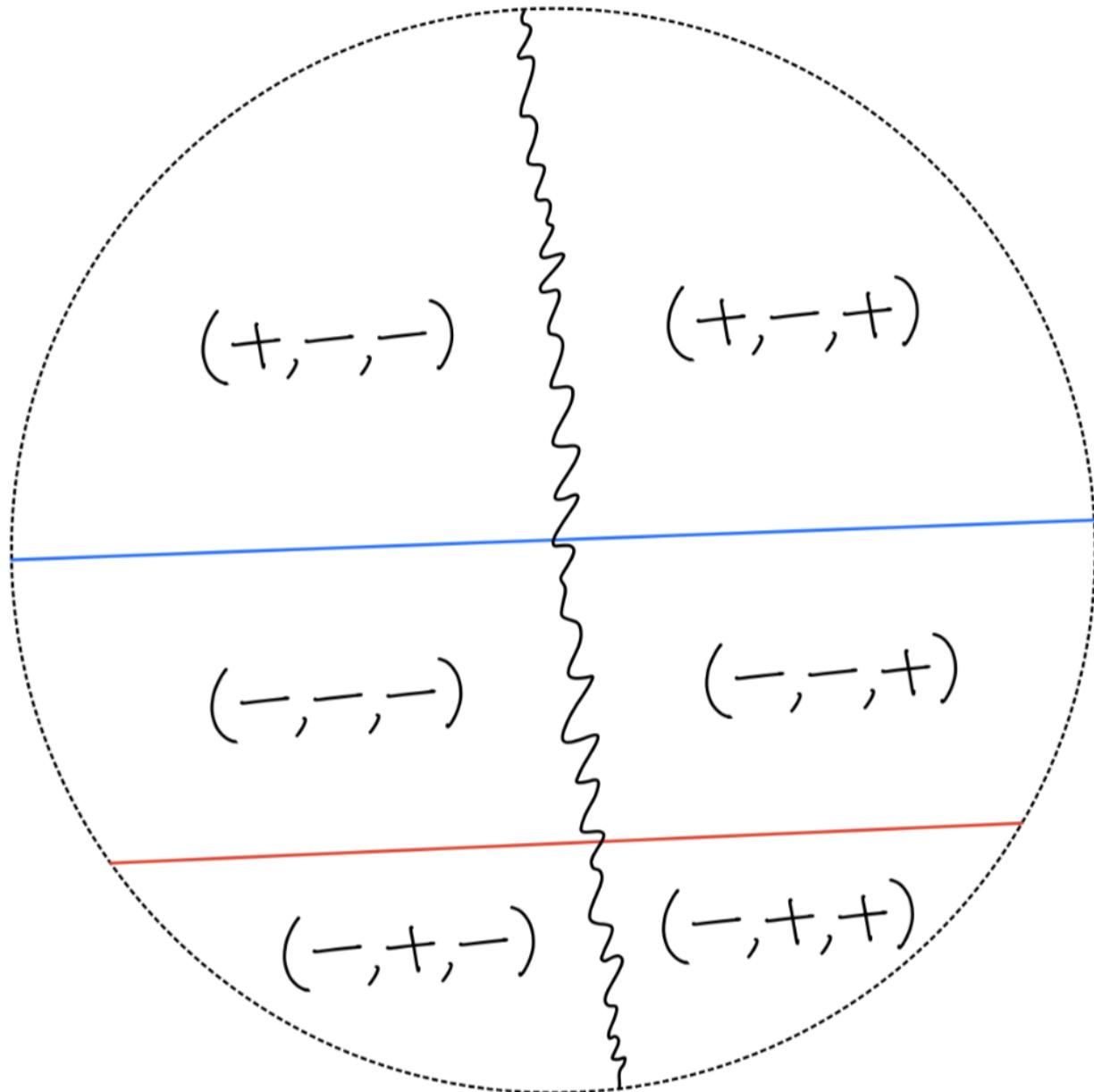
$\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3\} \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(-, -, 0)\}$

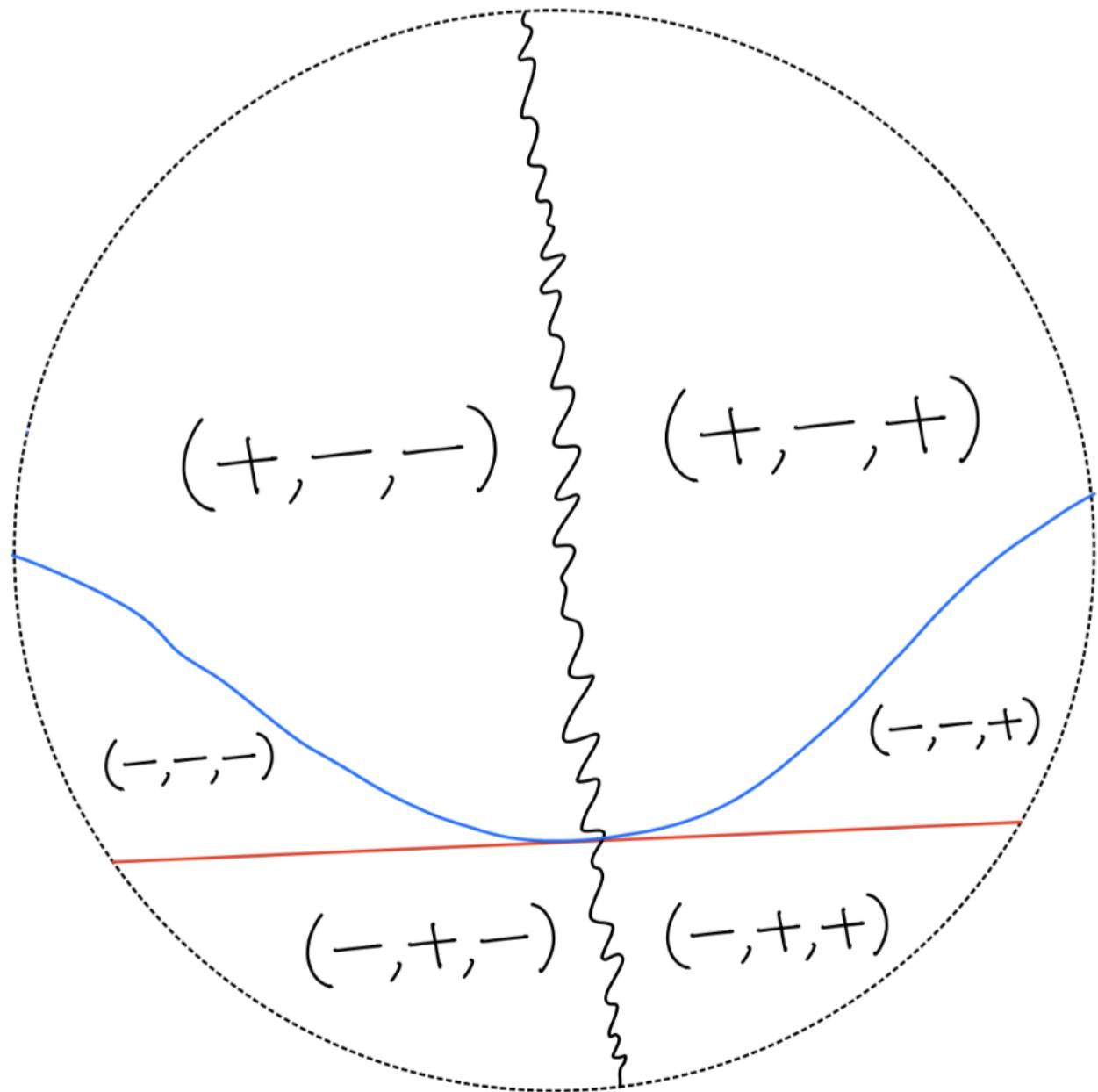
Figure 3.36: 0 dimensional strata of \mathcal{S}_1

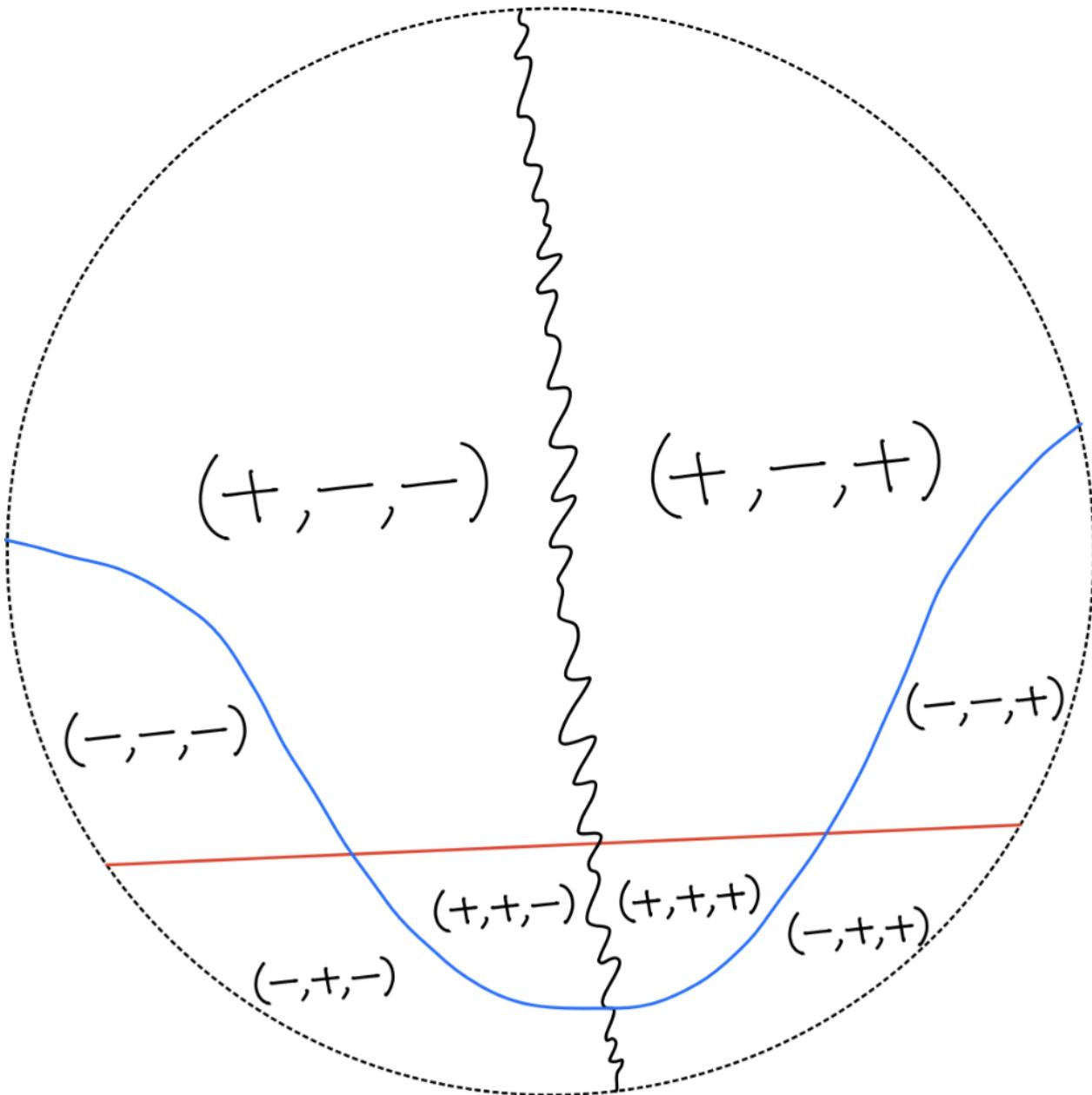
- 0 dimensional strata:

$$s_1(0,0,-), s_1(0,+0), s_1(0,0,+), s_1(+,0,0)$$

3. \mathcal{S}_\bullet :

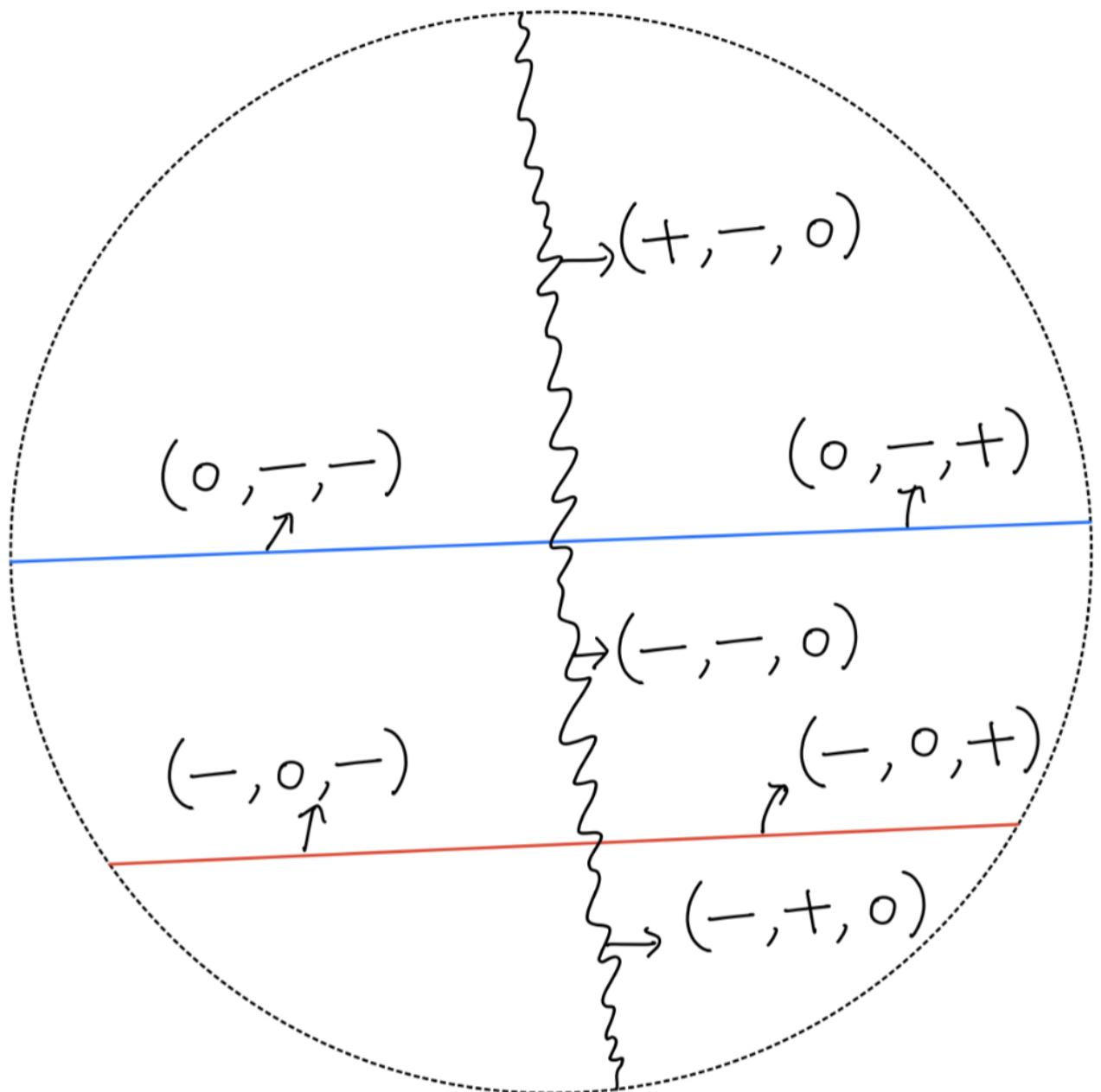
Figure 3.37: 3 dimensional strata of \mathcal{S}_\bullet at $t = 0$

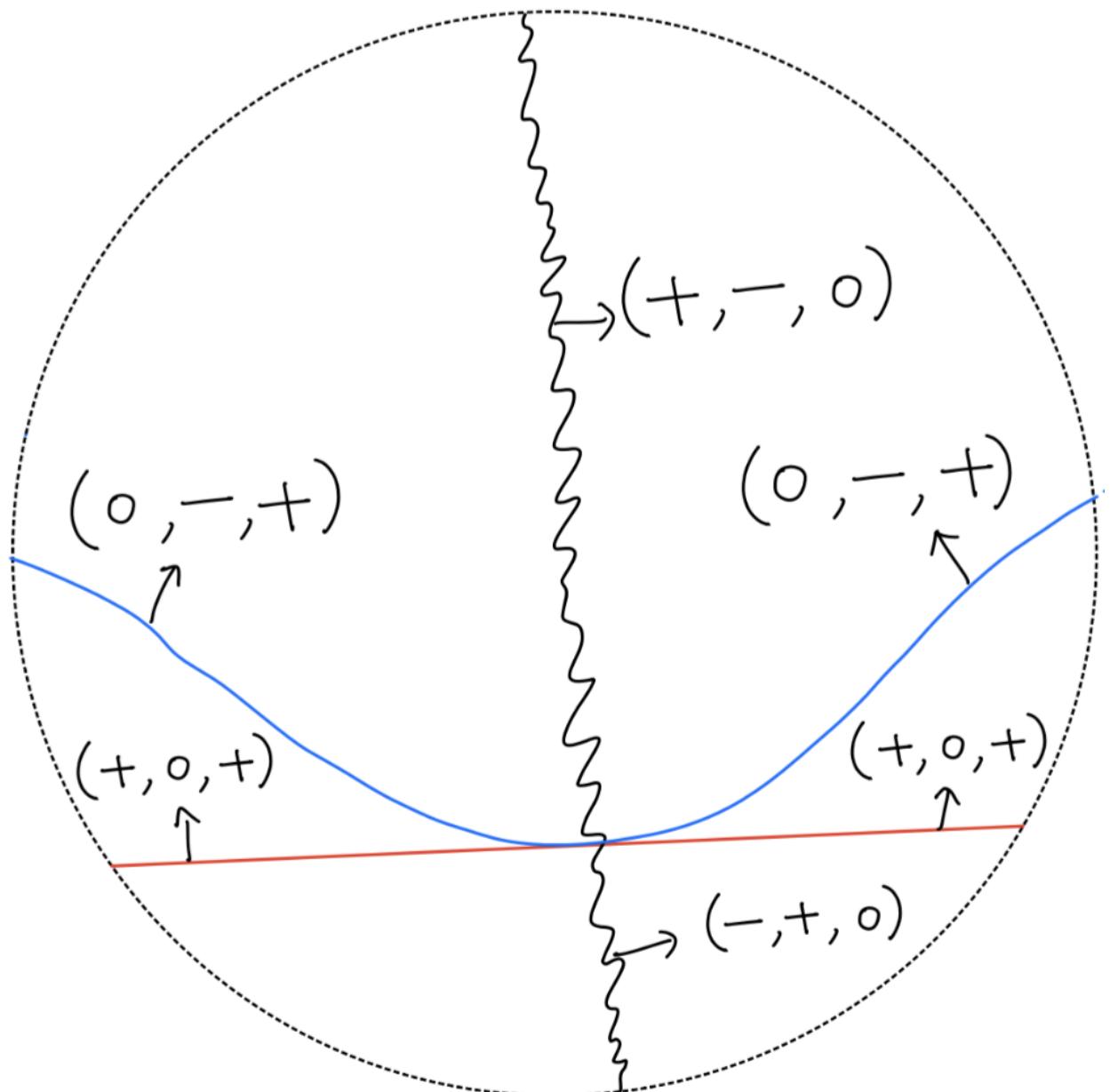
Figure 3.38: 3 dimensional strata of \mathcal{S}_\bullet at $t = \frac{1}{2}$

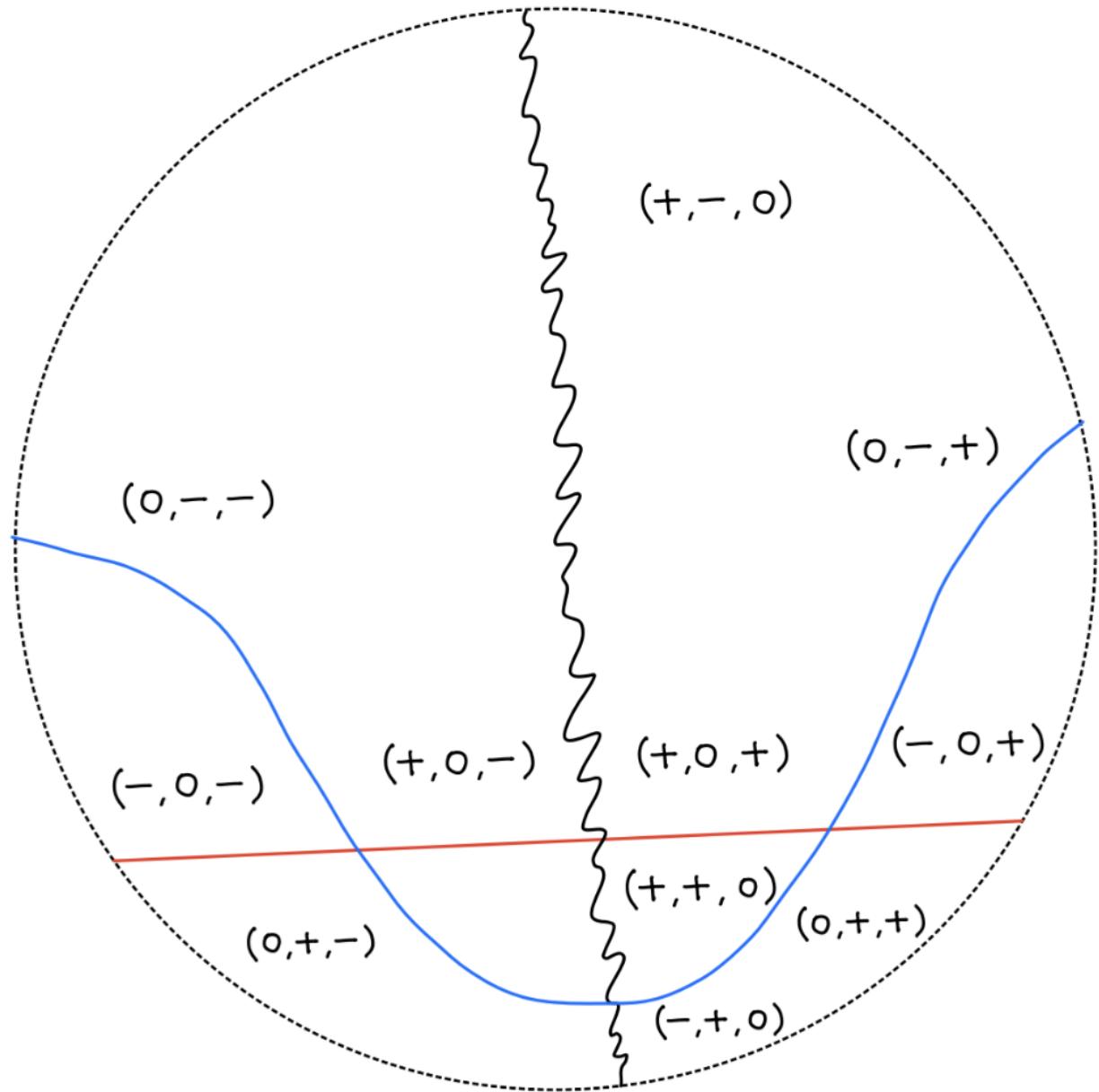
Figure 3.39: 3 dimensional strata of \mathcal{S}_\bullet at $t = 1$

- 3 dimensional strata:

$$\{s_\bullet(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

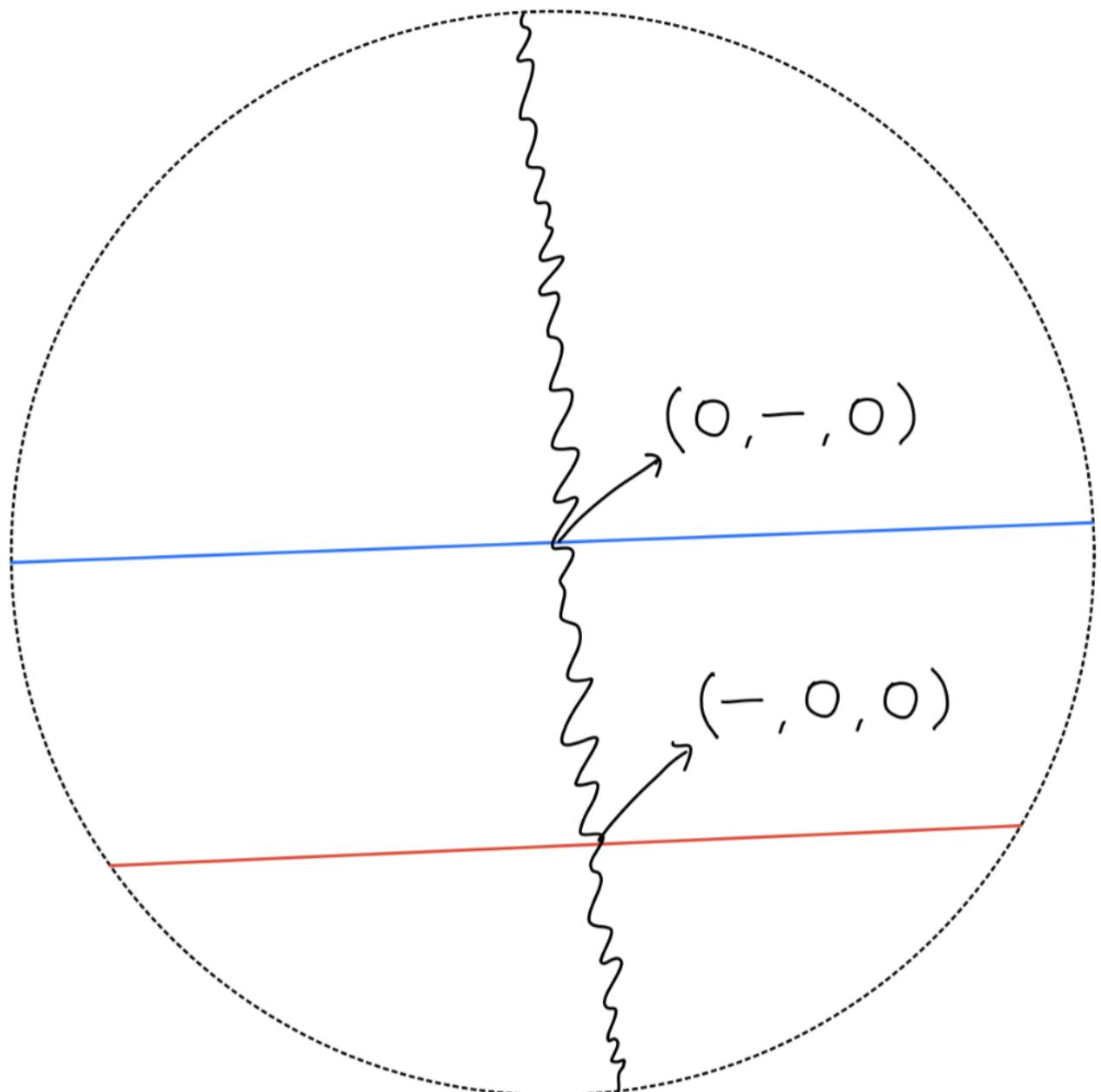
Figure 3.40: 2 dimensional strata of \mathcal{S}_\bullet at $t = 0$

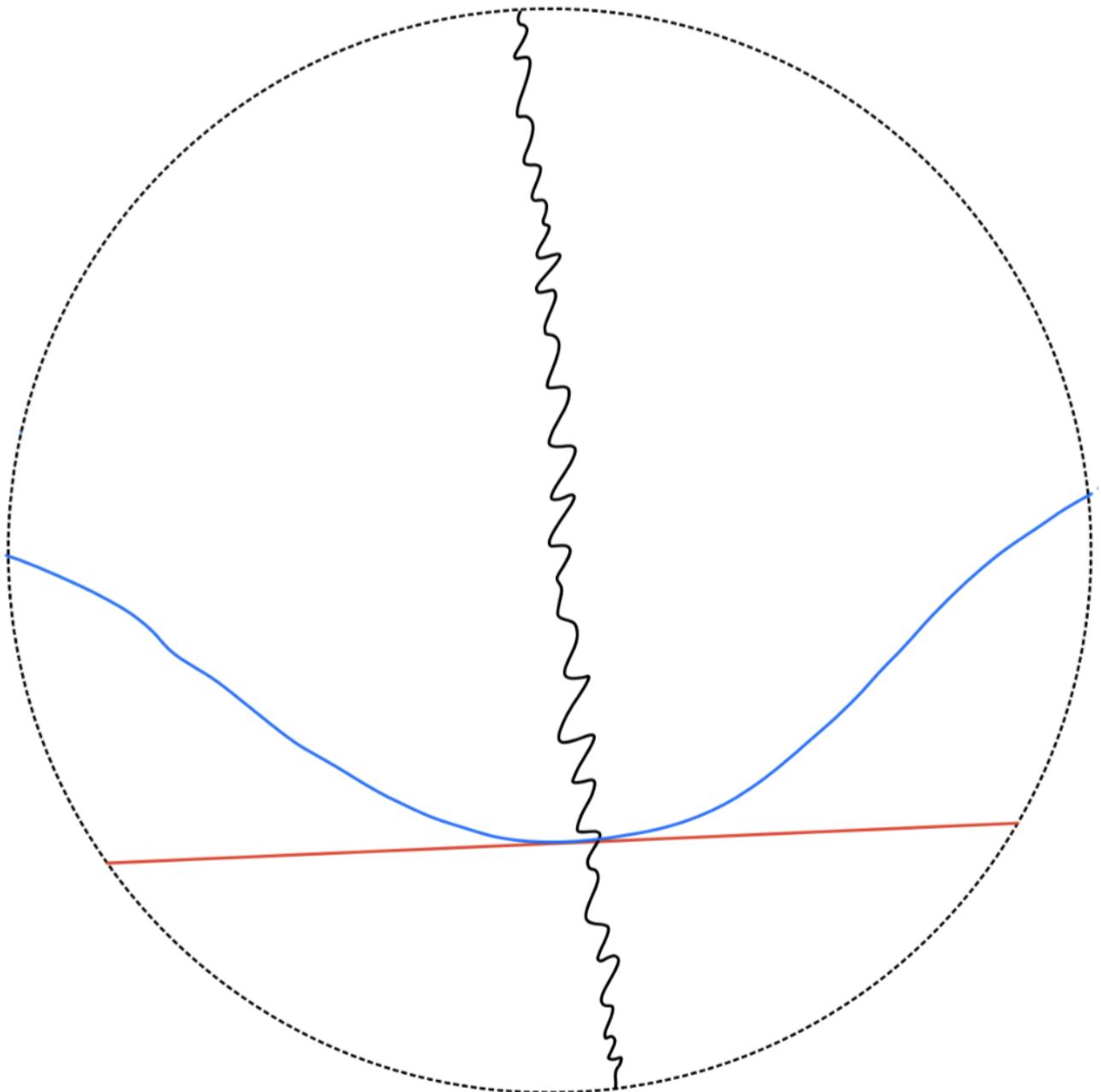
Figure 3.41: 2 dimensional strata of \mathcal{S}_\bullet at $t = \frac{1}{2}$

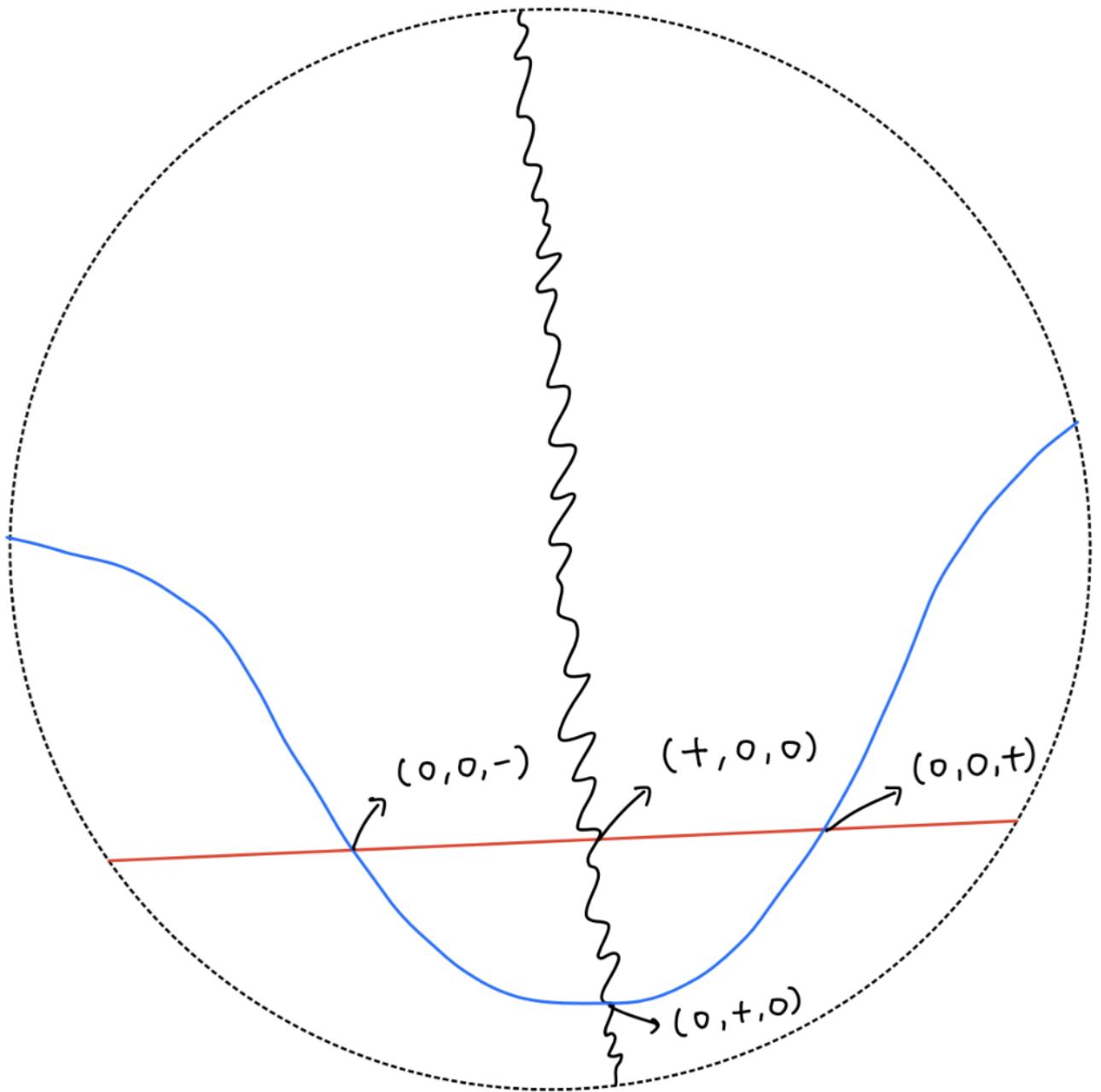
Figure 3.42: 2 dimensional strata of \mathcal{S}_\bullet at $t = 1$

- 2 dimensional strata:

$$\begin{aligned} & \{s_\bullet(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_\bullet(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_\bullet(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

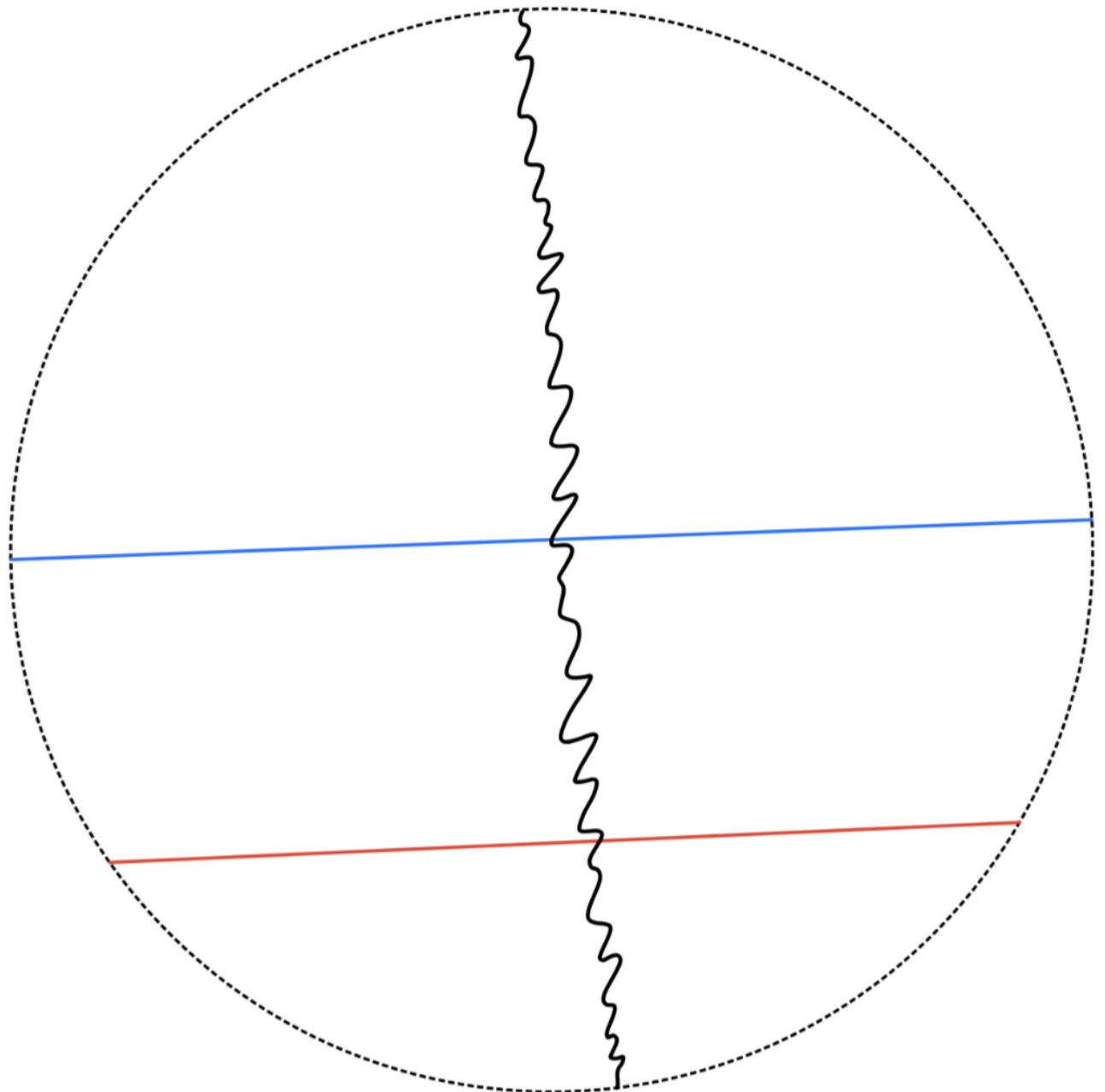
Figure 3.43: 1 dimensional strata of \mathcal{S}_\bullet at $t = 0$

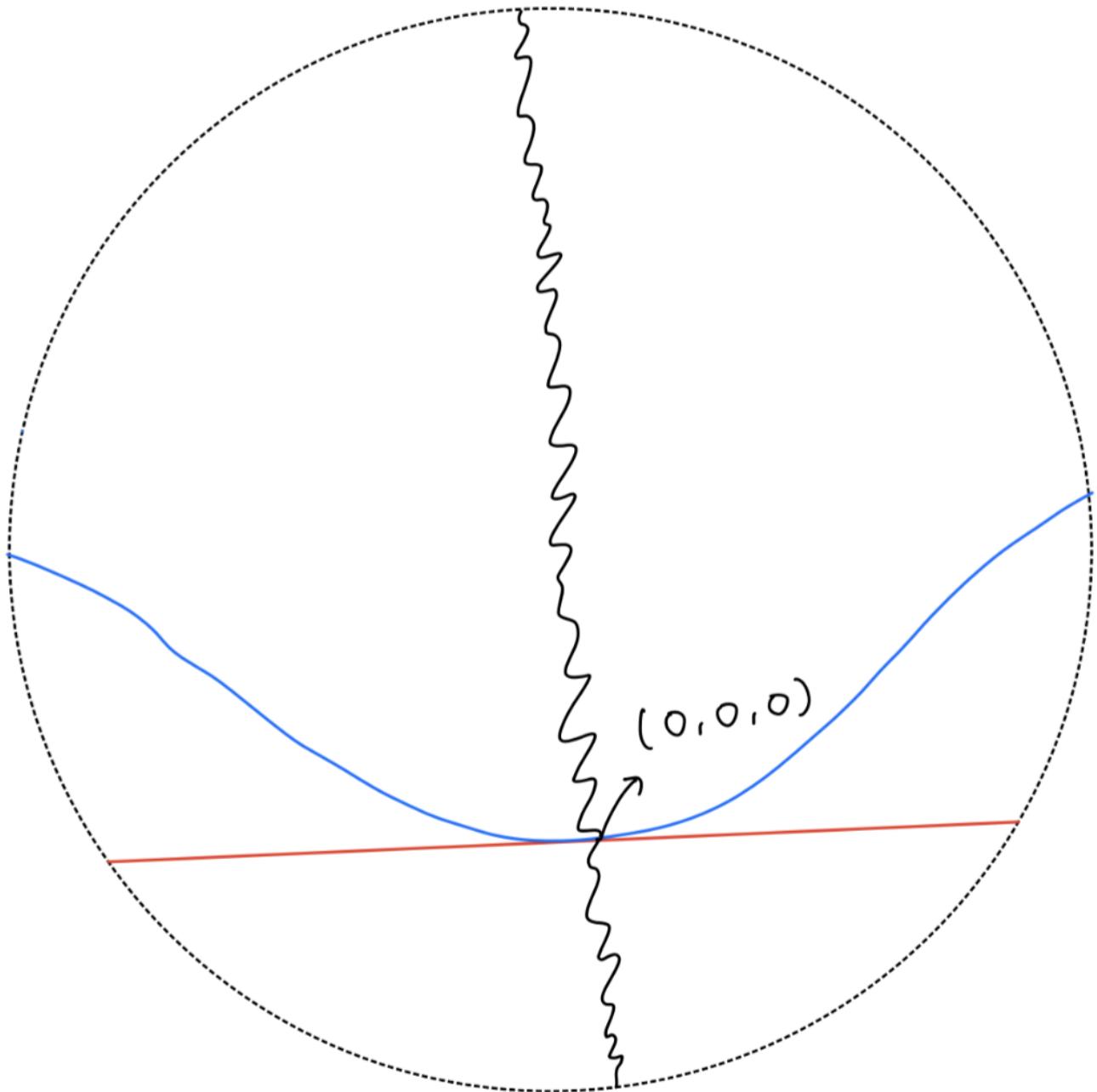
Figure 3.44: 1 dimensional strata of \mathcal{S}_\bullet at $t = \frac{1}{2}$

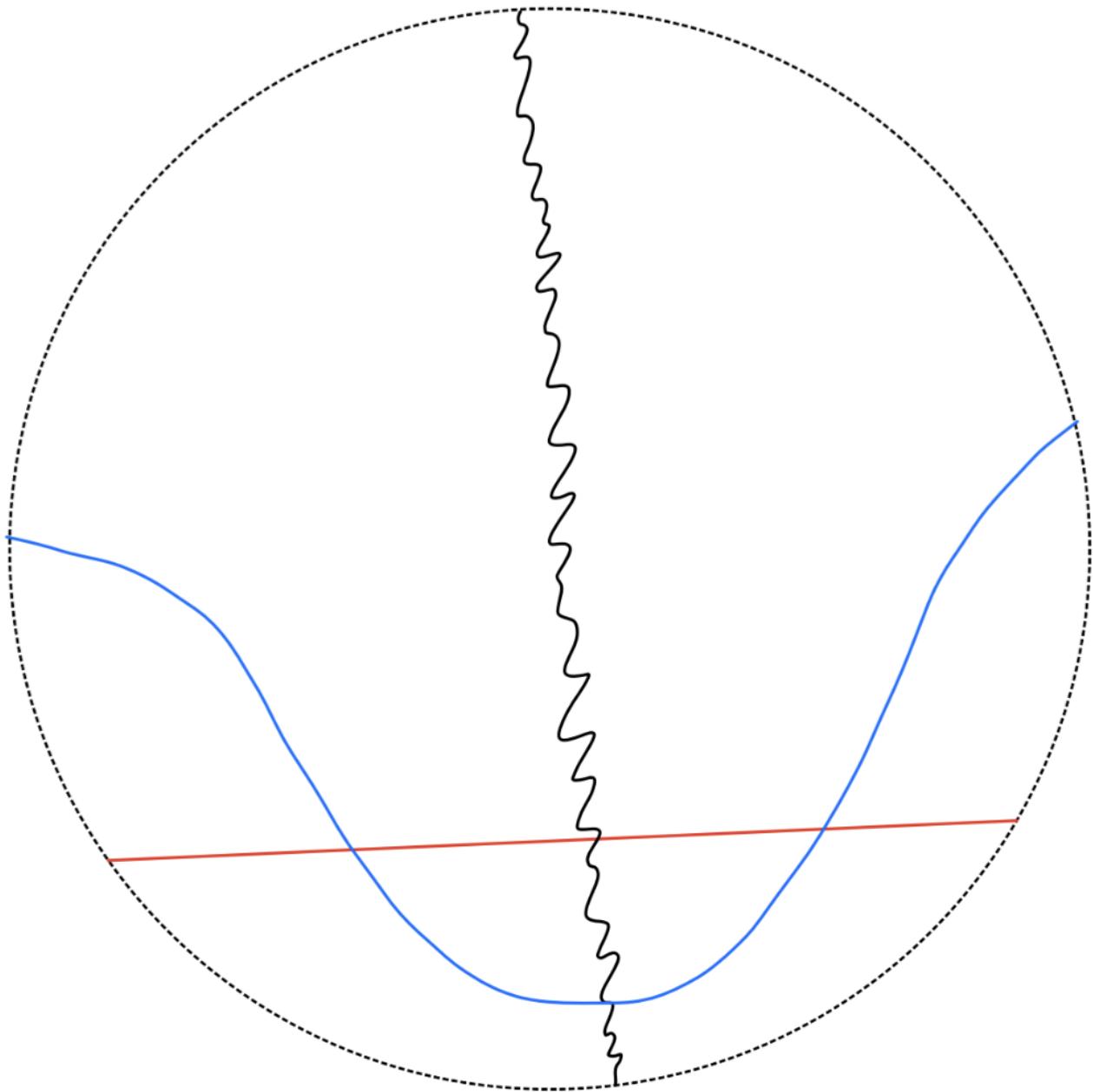
Figure 3.45: 1 dimensional strata of \mathcal{S}_\bullet at $t = 1$

- 1 dimensional strata:

$$\begin{aligned} & \{s_\bullet(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_\bullet(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_\bullet(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

Figure 3.46: 0 dimensional strata of \mathcal{S}_\bullet at $t = 0$

Figure 3.47: 0 dimensional strata of \mathcal{S}_\bullet at $t = \frac{1}{2}$

Figure 3.48: 0 dimensional strata of \mathcal{S}_\bullet at $t = 1$

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

Definition 85. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $\text{star}(s)$.

Definition 86. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s .

Definition 87. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 88. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 89. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 90. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_{\mathcal{S}}}(s_1 \rightarrow s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

This is well-defined because $F_{\mathcal{S}}$ is legible.

Definition 91. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.4.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.4.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

Stalks:

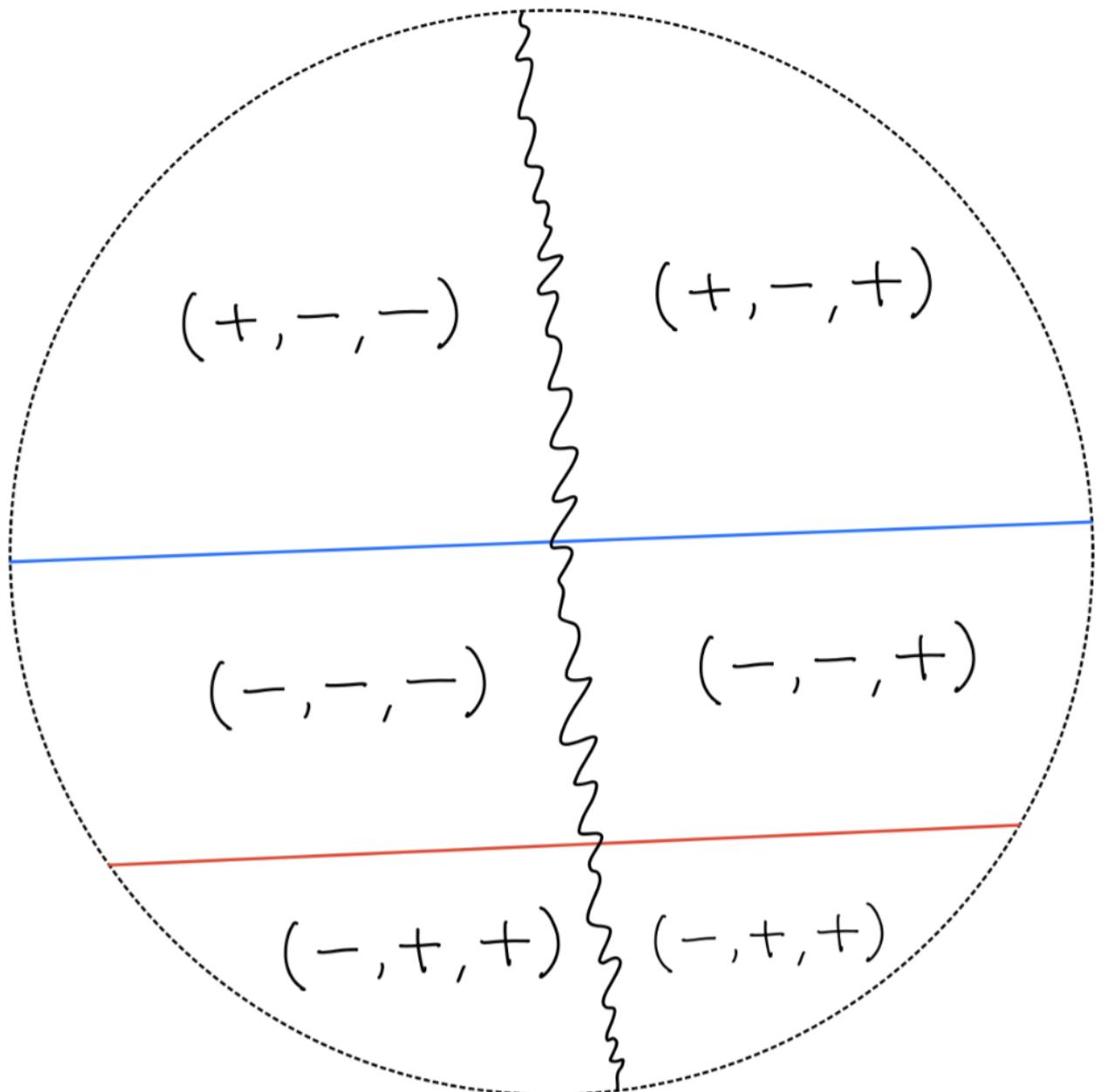


Figure 3.49

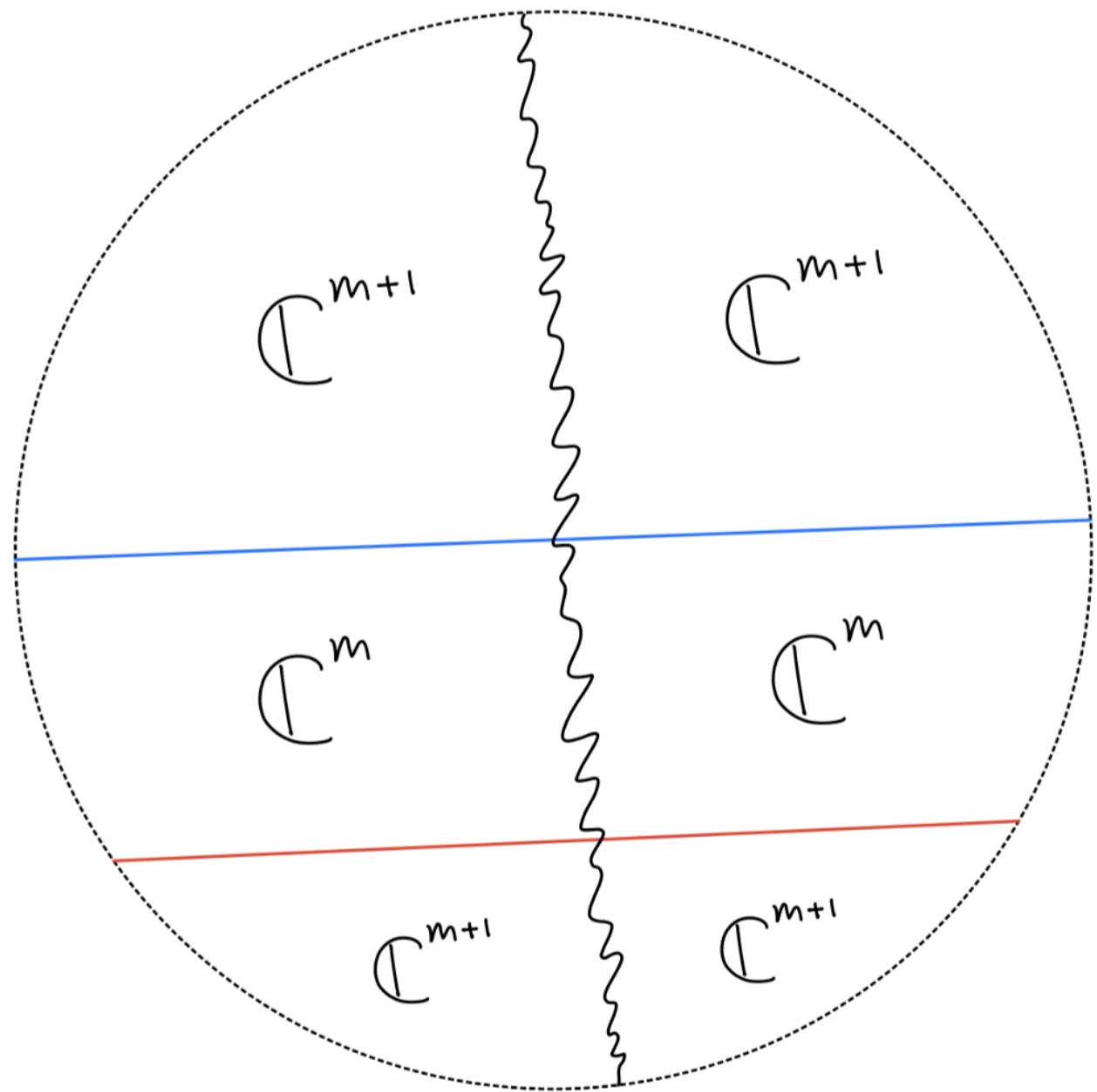


Figure 3.50

- $F_0(-, -, -) := \mathbb{C}^m$
- $F_0(-, -, +) := \mathbb{C}^m$
- $F_0(+, -, -) := \mathbb{C}^{m+1}$
- $F_0(+, -, +) := \mathbb{C}^{m+1}$

- $F_0(-, +, -) := \mathbb{C}^{m+1}$

- $F_0(-, +, +) := \mathbb{C}^{m+1}$

Generalization maps:

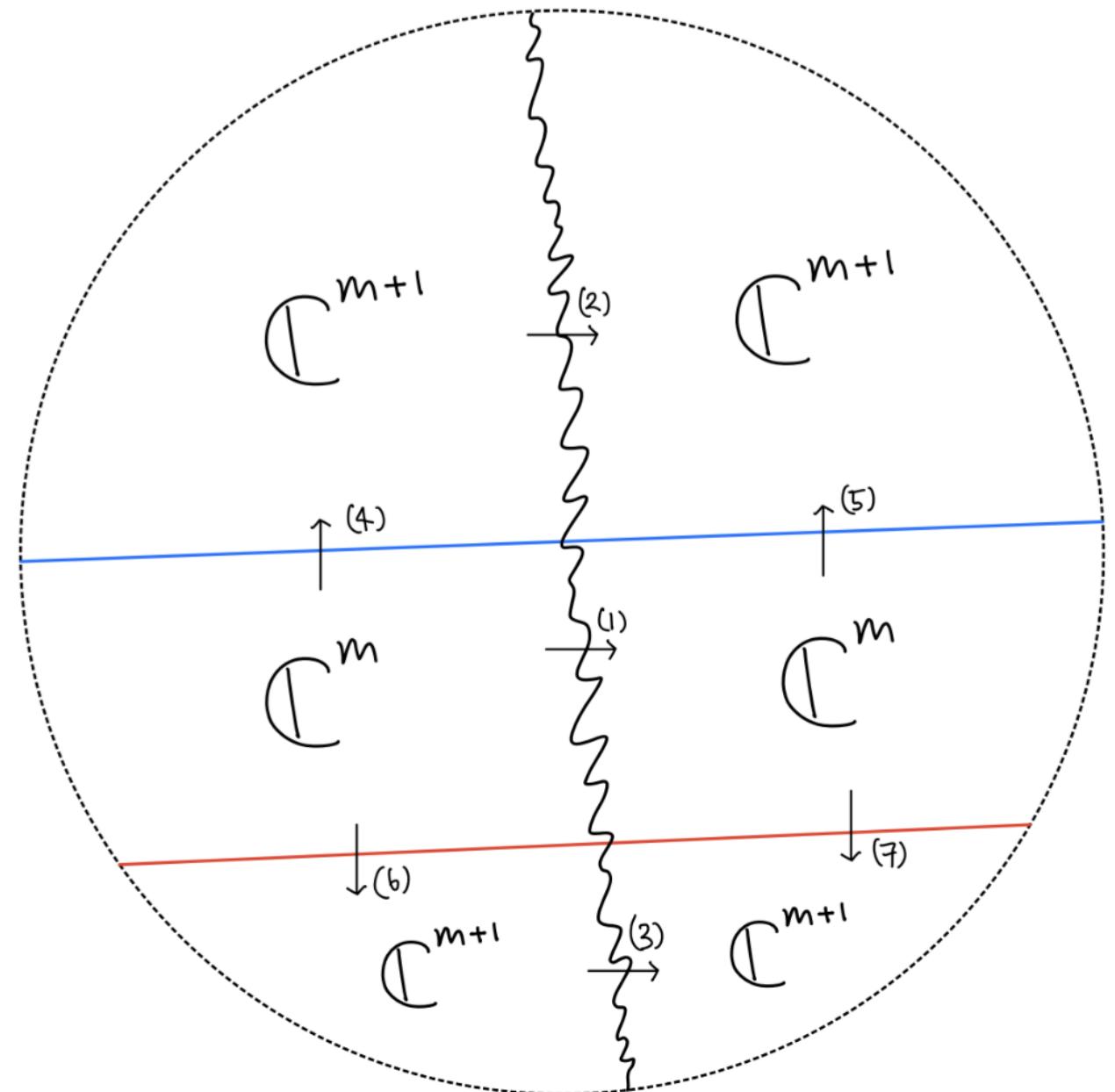


Figure 3.51

$$(1) \quad \mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

3.4.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say $cobord_2$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_{\bullet}(sgn_1, sgn_2, sgn_3) := F_{\bullet}(s_{\bullet}(sgn_1, sgn_2, sgn_3))$$

Stalks:

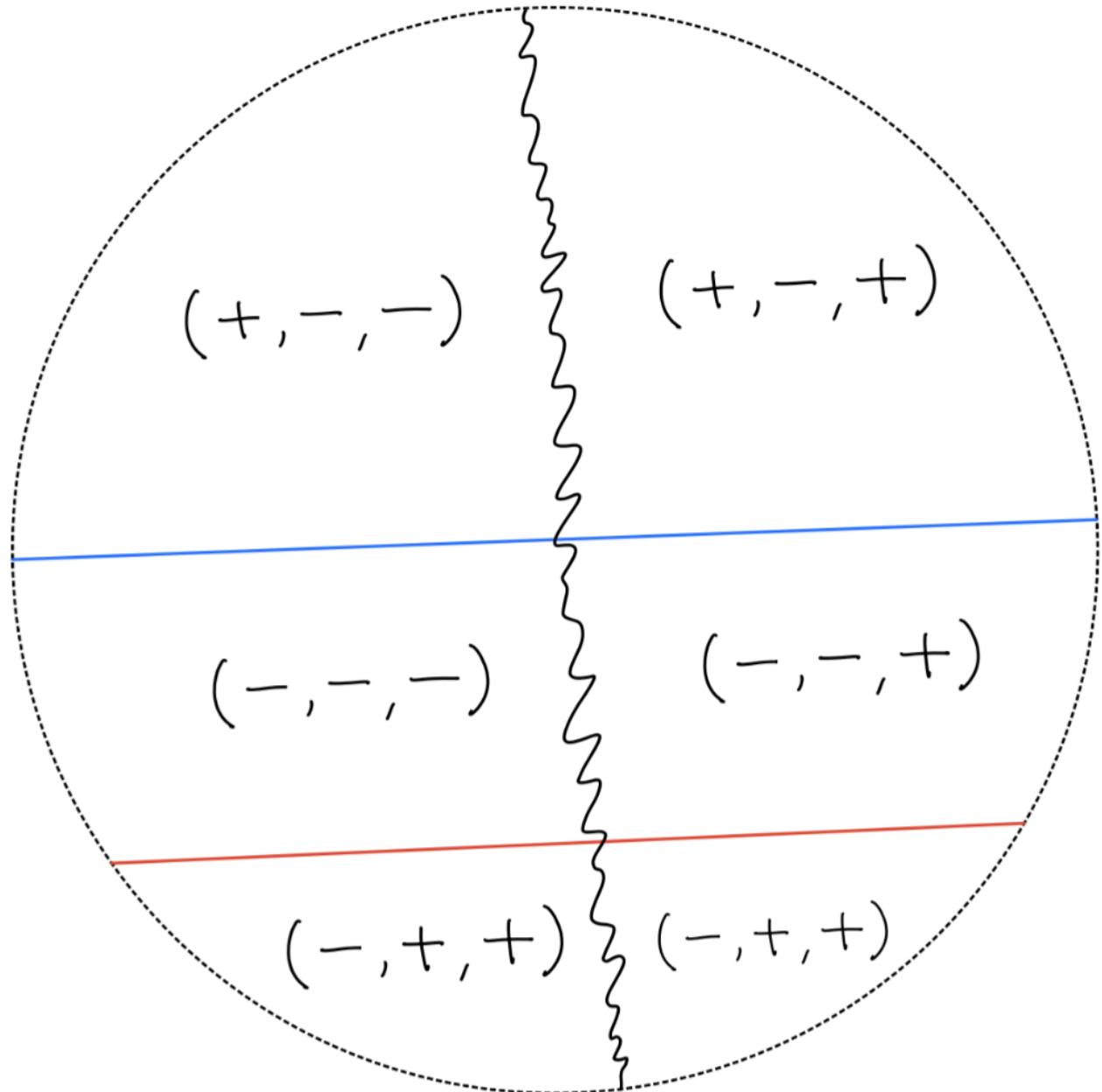


Figure 3.52

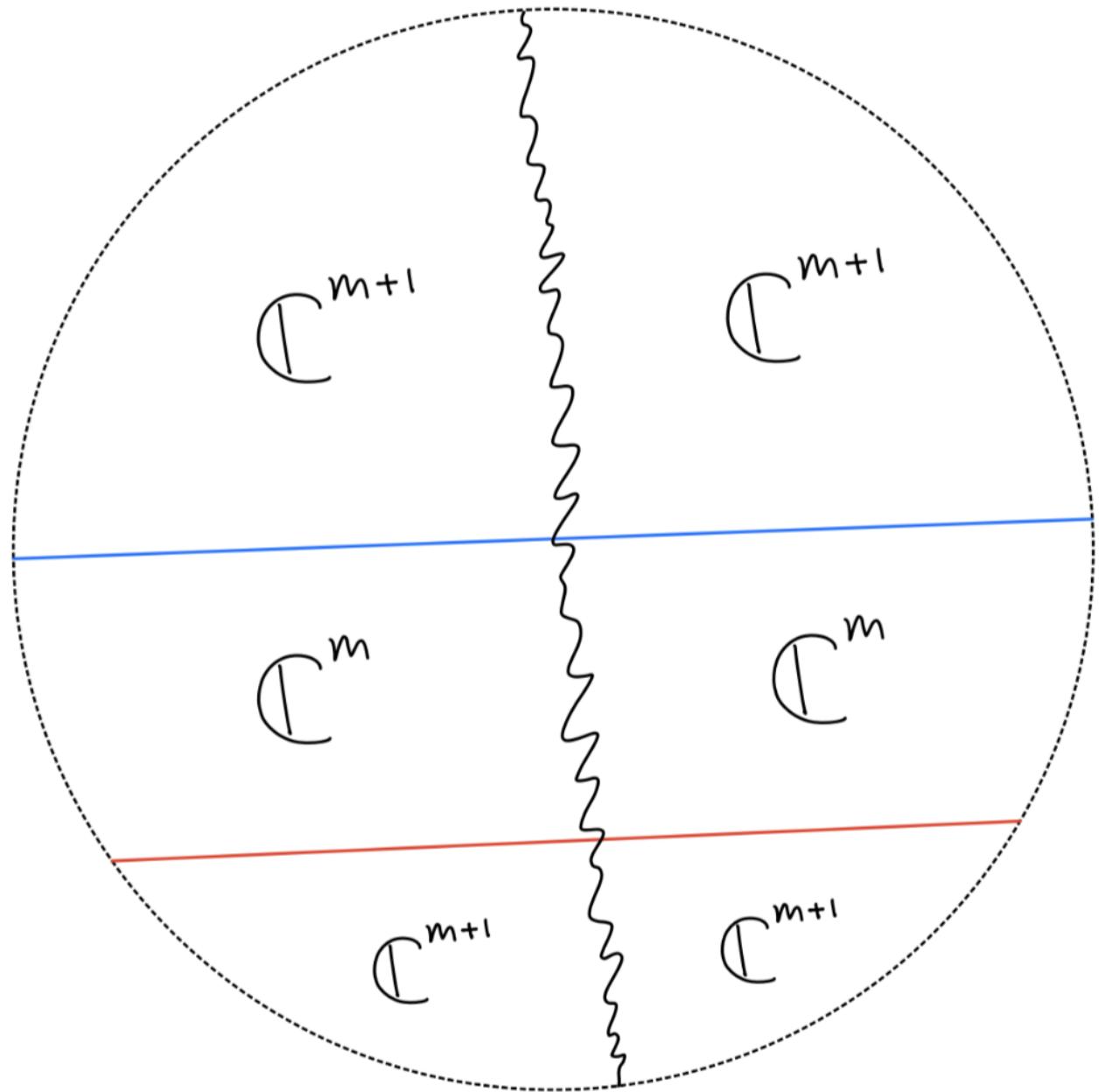


Figure 3.53

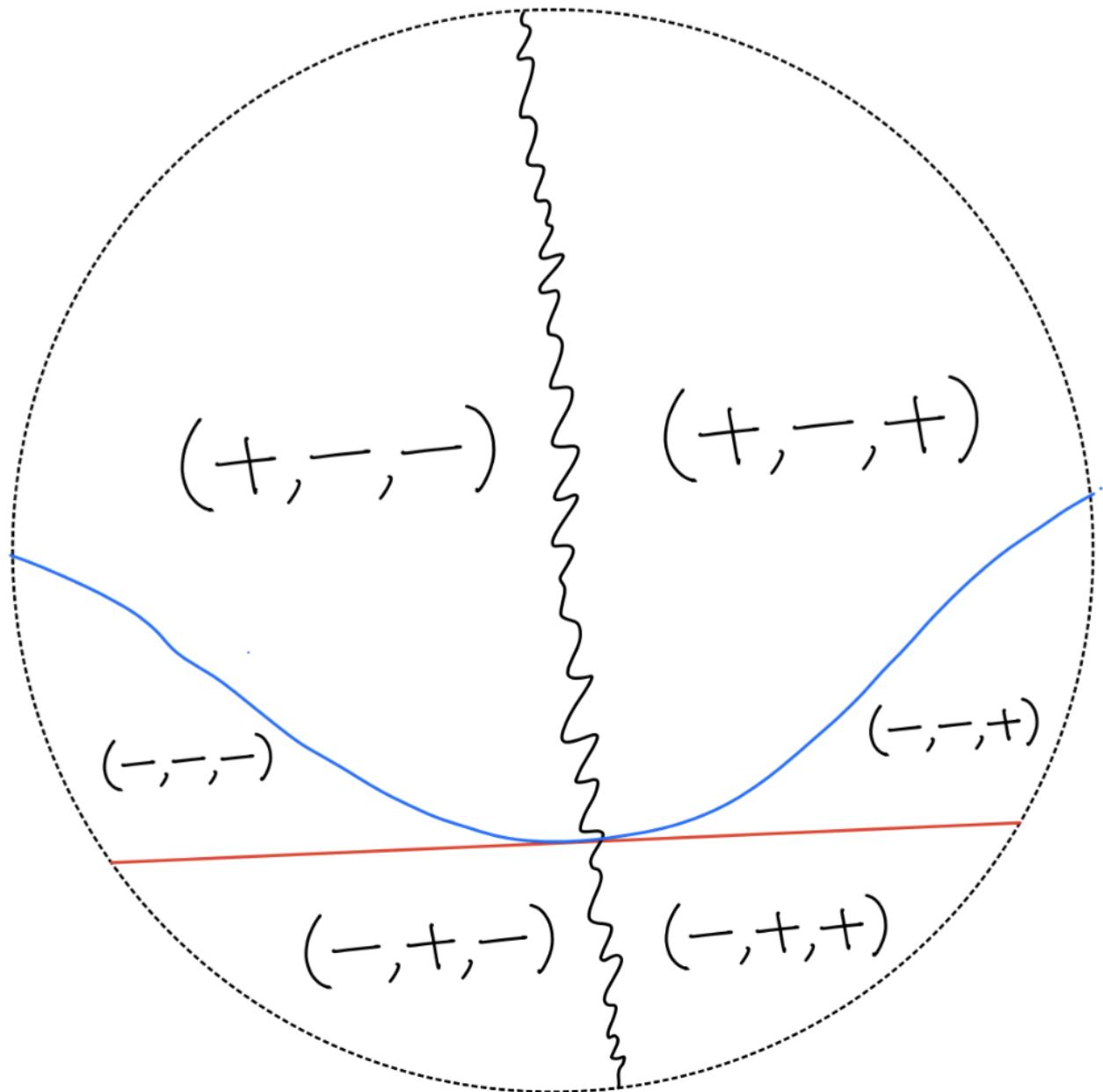


Figure 3.54

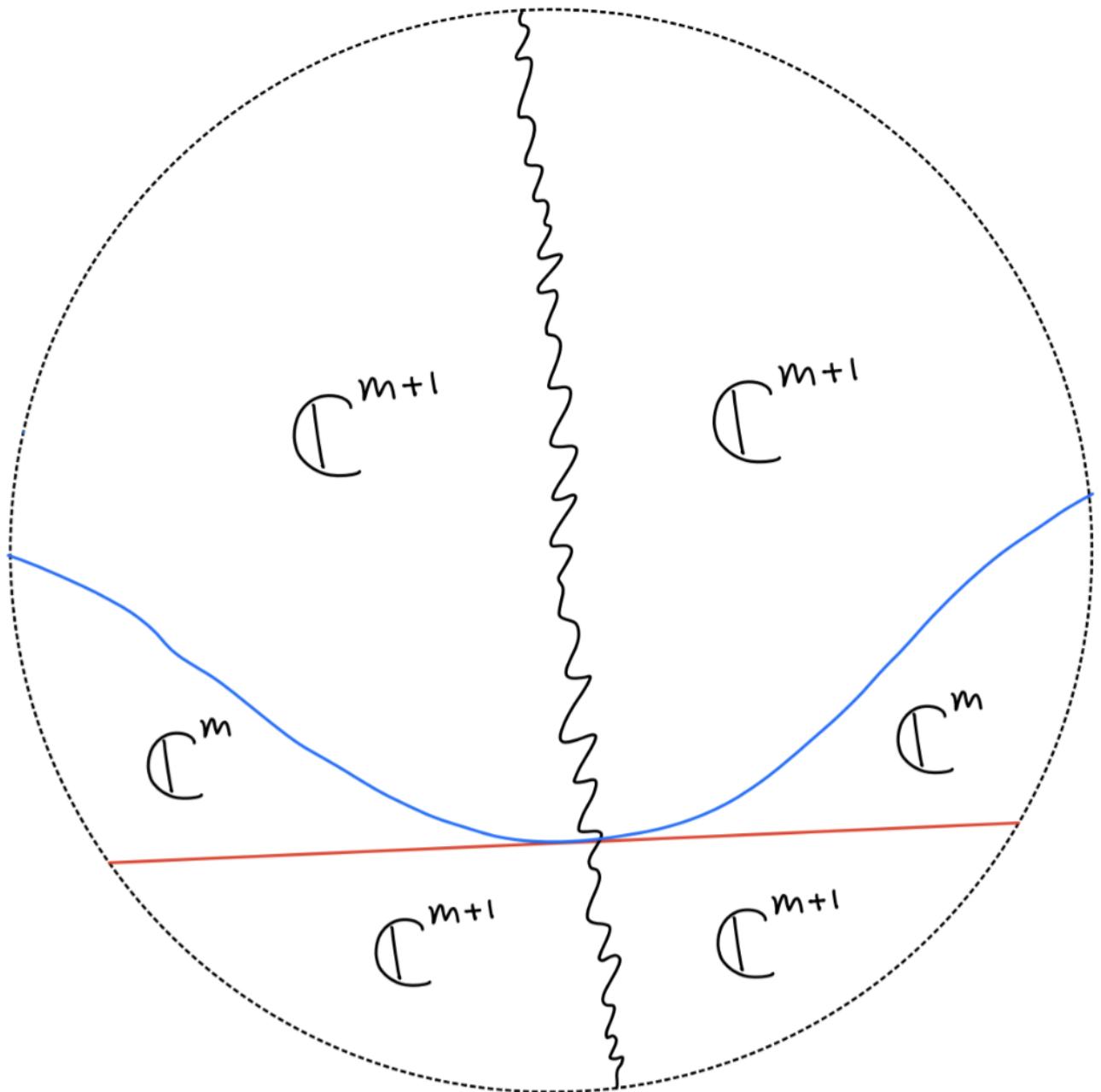


Figure 3.55

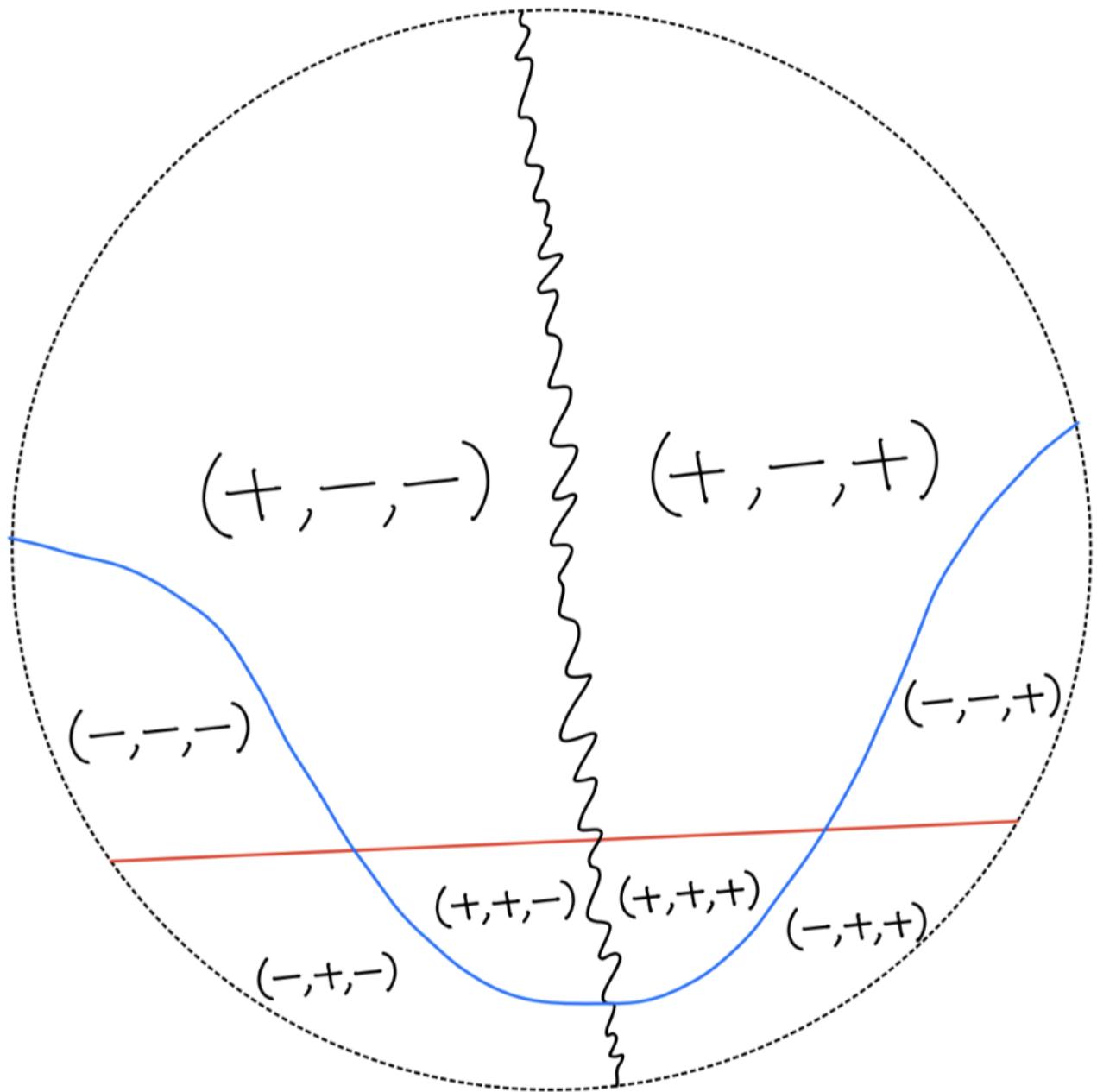


Figure 3.56

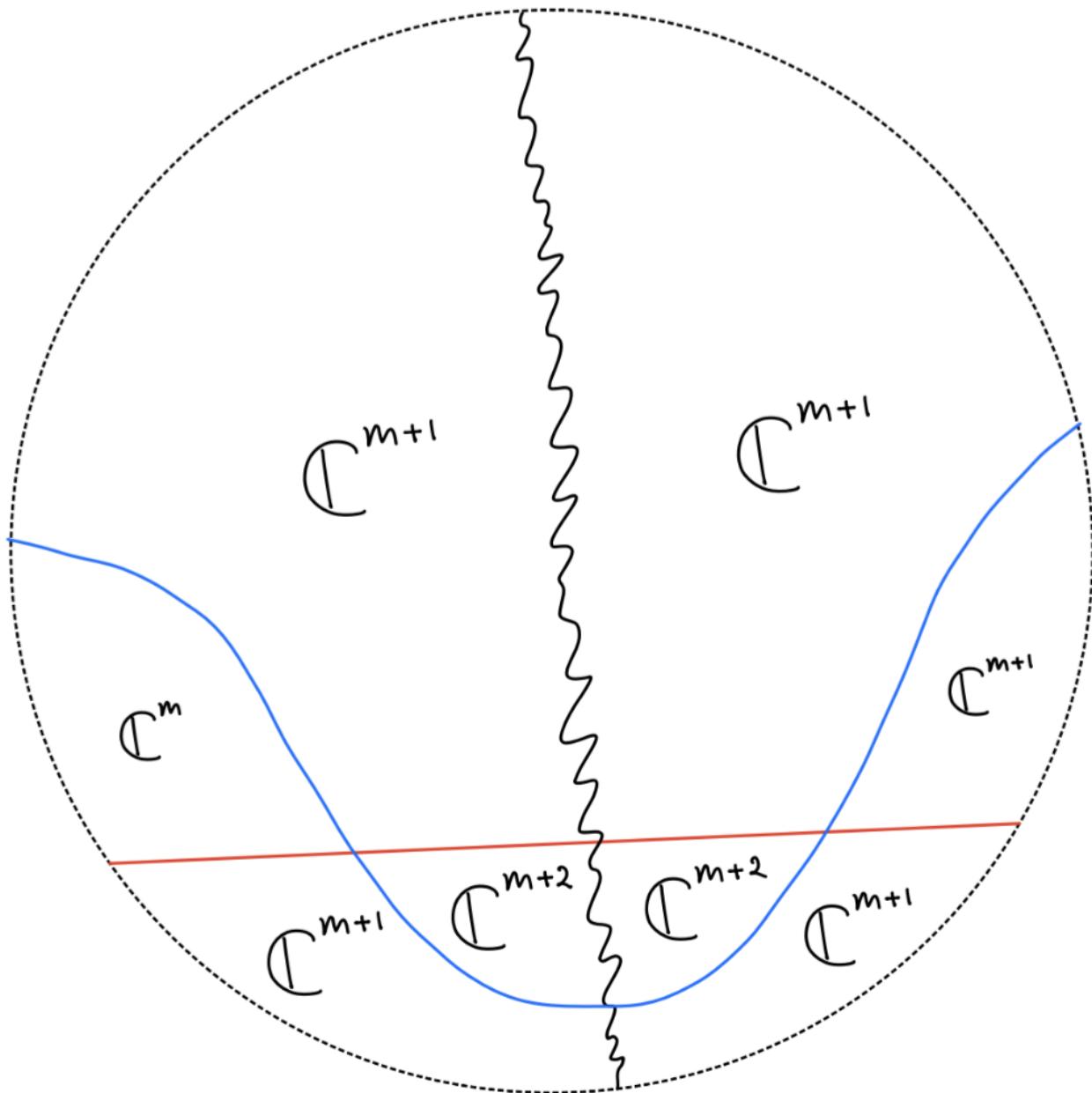


Figure 3.57

- $F_{\bullet}(-, -, -) := \mathbb{C}^m$
- $F_{\bullet}(-, -, +) := \mathbb{C}^m$
- $F_{\bullet}(+, -, -) := \mathbb{C}^{m+1}$
- $F_{\bullet}(+, -, +) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, -) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, +) := \mathbb{C}^{m+1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{m+2}$

- $F_\bullet(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

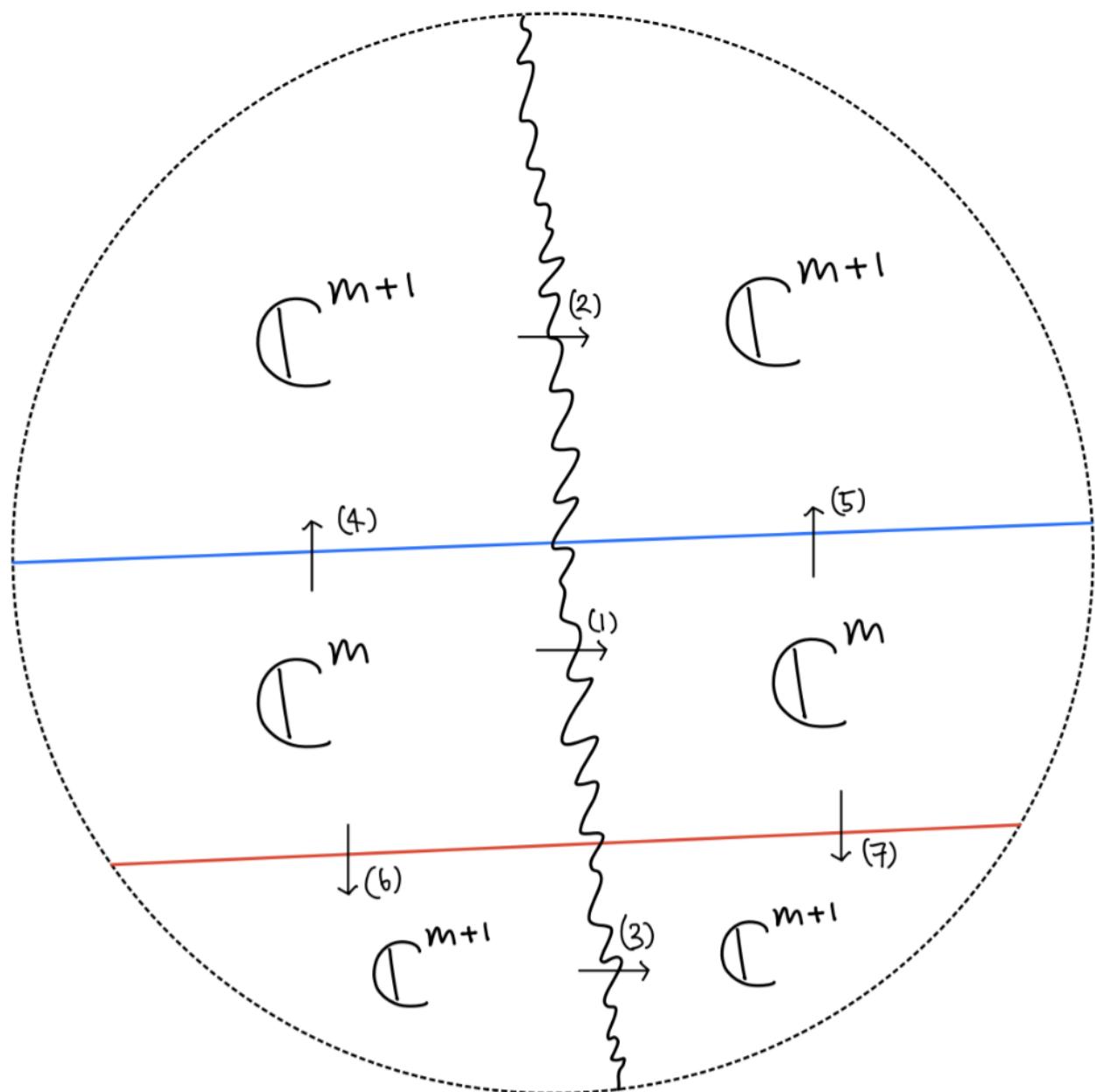


Figure 3.58

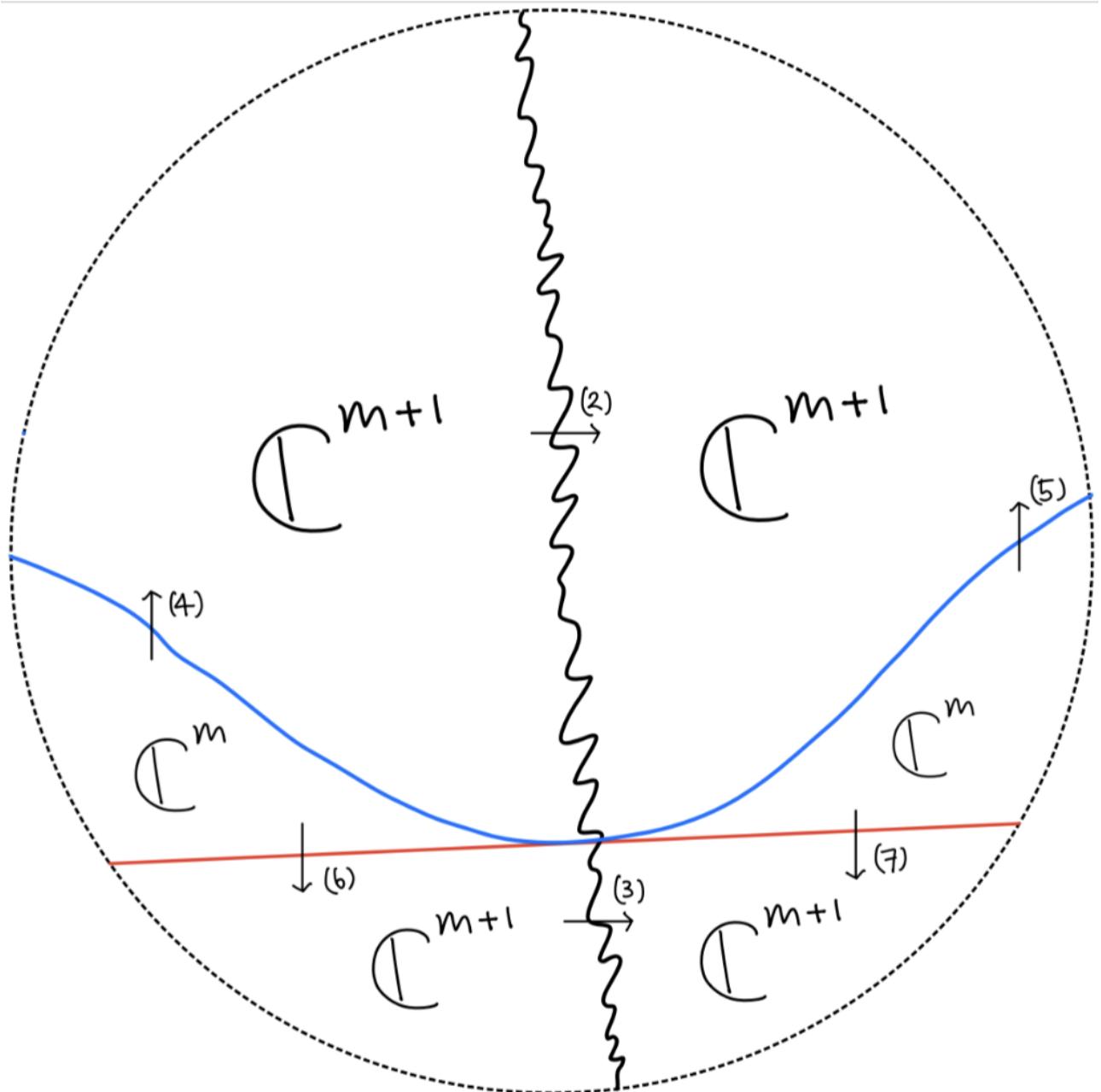


Figure 3.59

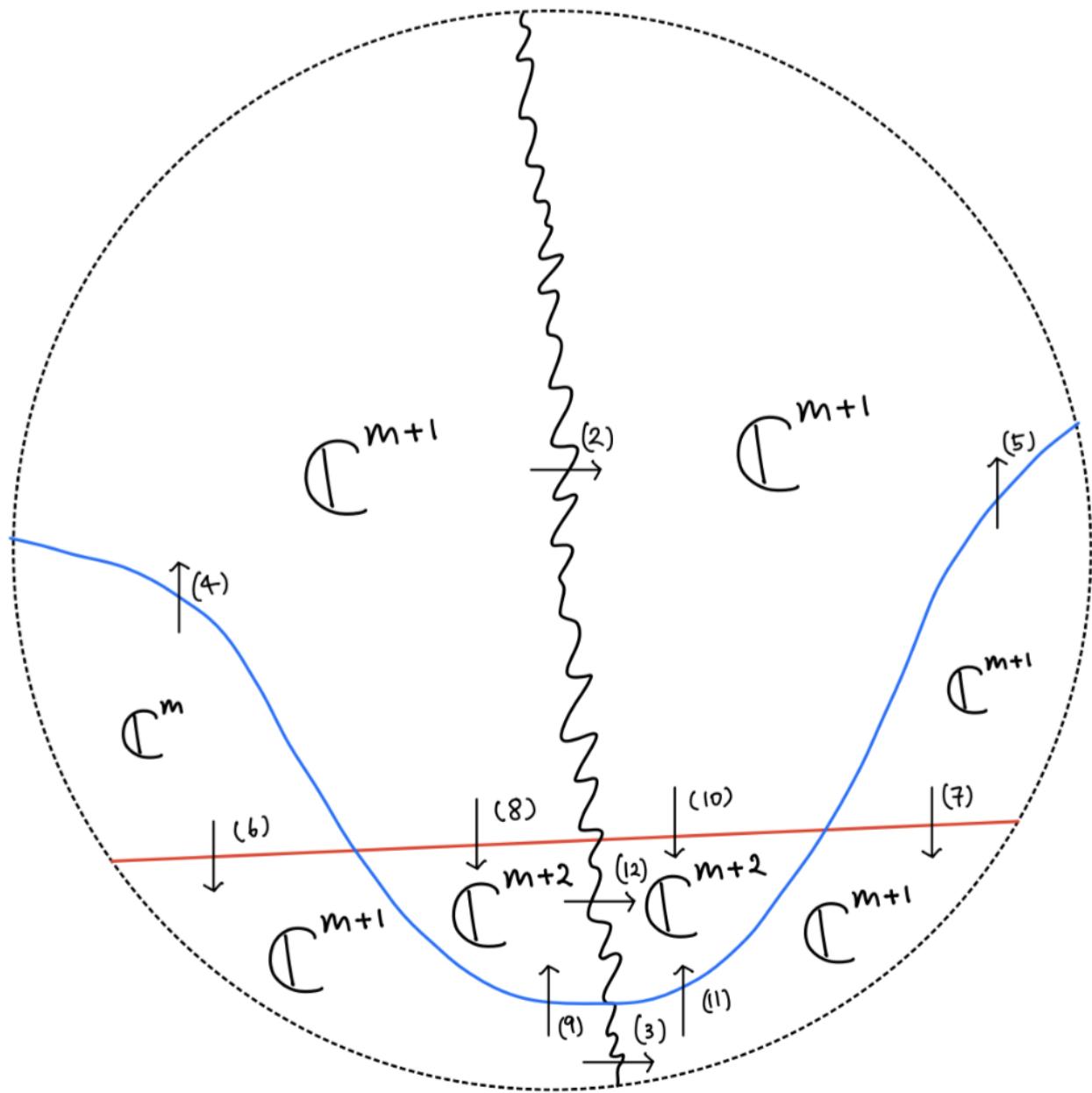


Figure 3.60

$$(1) \quad \mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(11) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(12) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 92. $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 93. we define γ_\bullet to be the composition

$$(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 94. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of M .

Note that there is a diffeomorphism between $D_{r=2} \times (0, 1)$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 .

To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}^m$
- $F^3(-, -, +) := \mathbb{C}^m$
- $F^3(+, -, -) := \mathbb{C}^{m+1}$
- $F^3(+, -, +) := \mathbb{C}^{m+1}$

- $F^3(-, +, -) := \mathbb{C}^{m+1}$

- $F^3(-, +, +) := \mathbb{C}^{m+1}$

- $F^3(+, +, -) := \mathbb{C}^{m+2}$

- $F^3(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

- $F^3(-, sgn_2, sgn_3) \rightarrow F^3(+, sgn_2, sgn_3) := \iota_1$

- $F^3(sgn_1, -, sgn_3) \rightarrow F^3(sgn_1, +, sgn_3) := \iota_0$

- $F^3(-, -, -) \rightarrow F^3(-, -, +) := T(2, m+1)$

- $F^3(+, -, -) \rightarrow F^3(+, -, +) := T(1, m+1)$

- $F^3(-, +, -) \rightarrow F^3(-, +, +) := T(2, m+2)$

- $F^3(+, +, -) \rightarrow F^3(+, +, +) := T$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^m \\
 \downarrow \iota_0 & & \downarrow \iota_0 \\
 \mathbb{C}^{m+1} & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1}
 \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-, +, -) & \xrightarrow{s(-,+0)} & s(-, +, +) & \\
 & \swarrow s(-,0,-) & & \searrow s(-,0,+)& \\
 s(-, -, -) & \xrightarrow{s(-,-0)} & s(-, -, +) & & s(0,+,+) \\
 & \downarrow s(0,-,-) & & \downarrow s(0,+,+) & \\
 & s(+, +, -) & \xrightarrow{s(+,+0)} & s(+, +, +) & \\
 & \swarrow s(+,0,-) & & \searrow s(+,0,+)& \\
 s(+, -, -) & \xrightarrow{s(+,-0)} & s(+, -, +) & &
 \end{array}$$

=

$$\begin{array}{ccccc}
 & & \mathbb{C}^{m+1} & & \\
 & \swarrow \iota_0 & \downarrow & \searrow \iota_0 & \\
 \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^m & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1} \\
 & \downarrow \iota_1 & & \downarrow \iota_1 & \\
 & & \mathbb{C}^{m+2} & & \\
 & \swarrow \iota_0 & \downarrow T_{\iota_1} & \searrow \iota_0 & \\
 \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+1} & \xrightarrow{T_{\iota_1}} & \mathbb{C}^{m+2}
 \end{array}$$

Note that $T(2, m+1), T(2, m+2), T(1, m+1), T$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

□

3.4.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the $cobord_2$. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

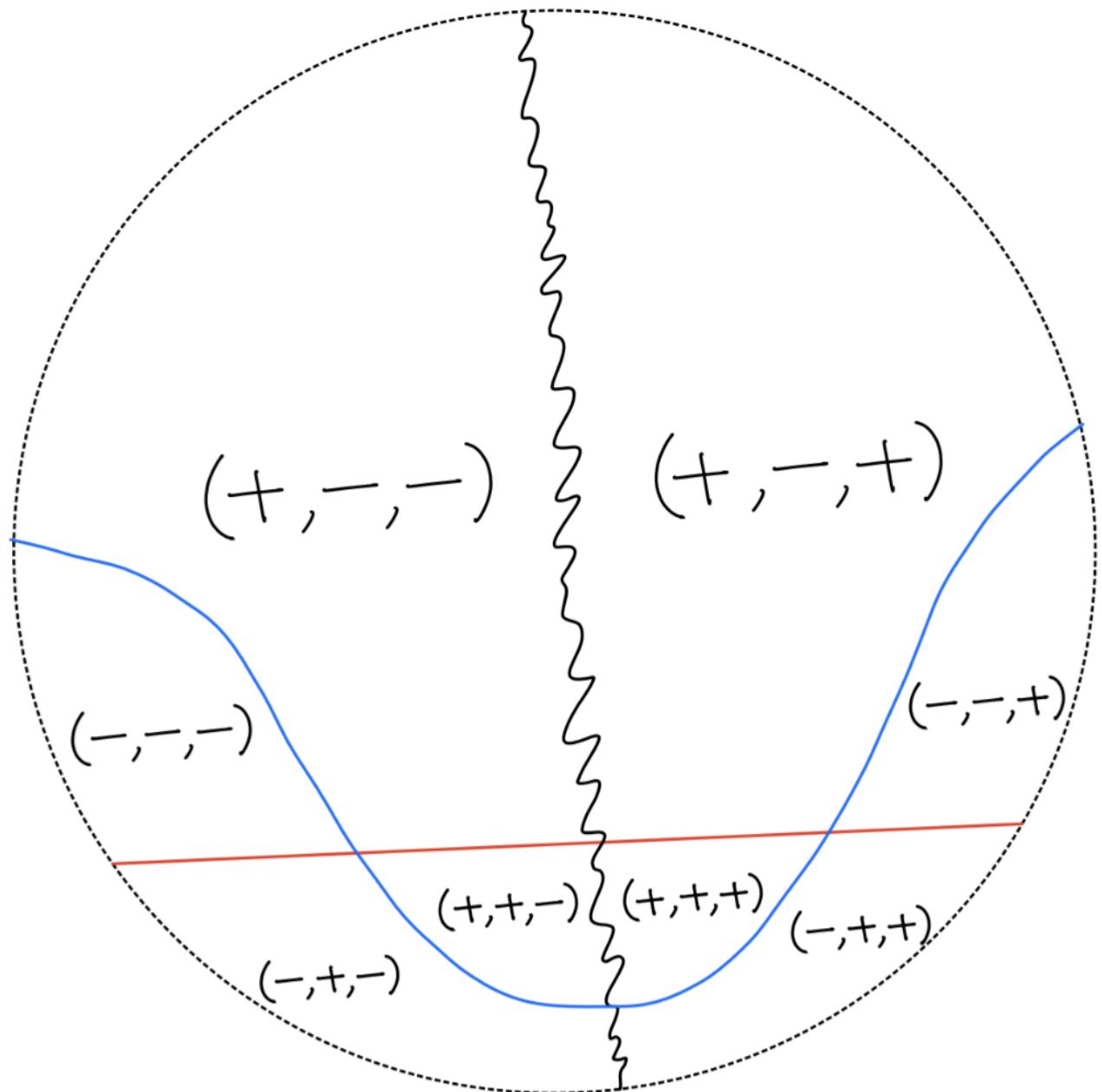


Figure 3.61

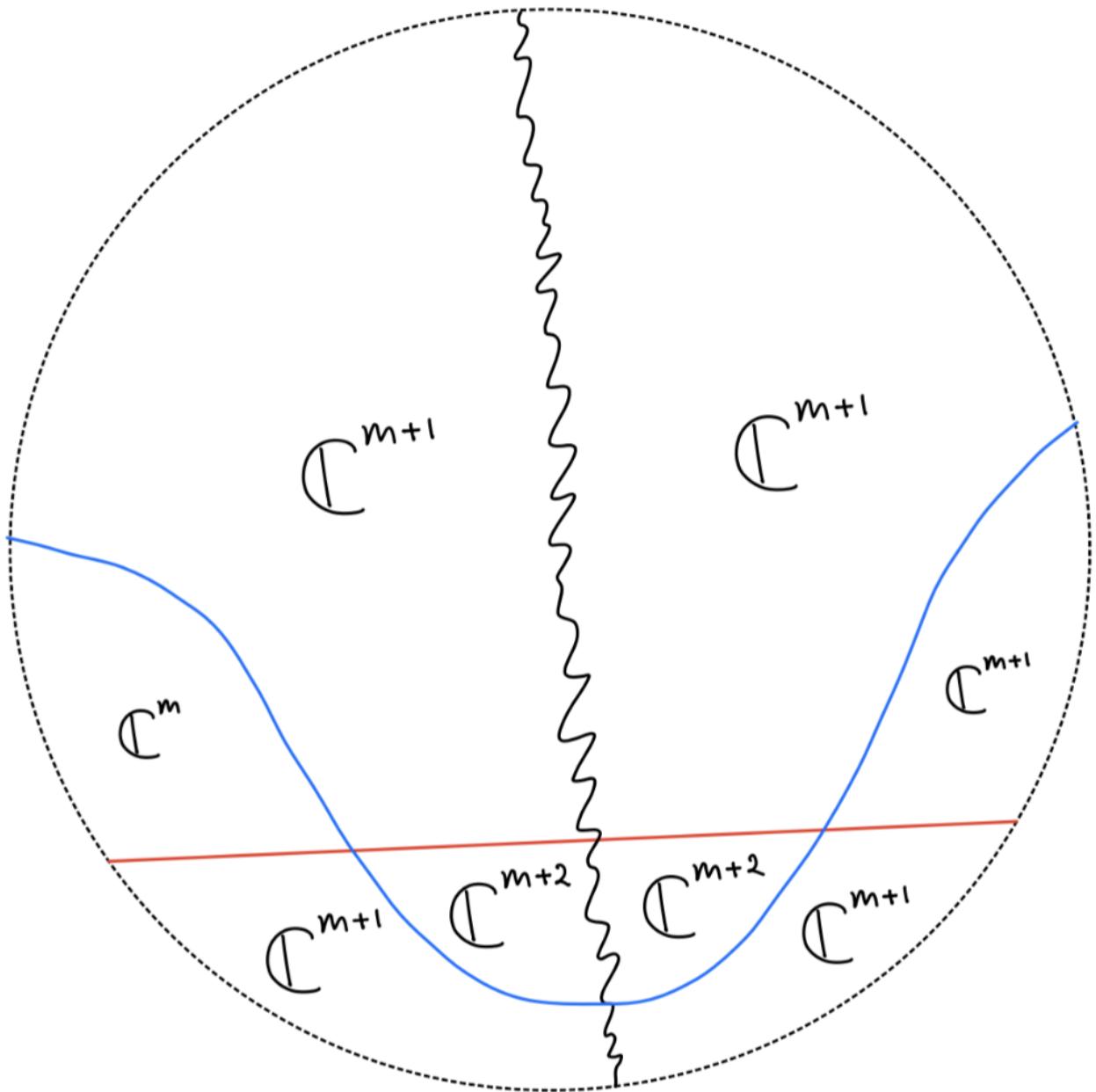


Figure 3.62

- $F_1(-, -, -) := \mathbb{C}^m$
- $F_1(-, -, +) := \mathbb{C}^m$
- $F_1(+, -, -) := \mathbb{C}^{m+1}$
- $F_1(+, -, +) := \mathbb{C}^{m+1}$

$$\bullet \ F_1(-, +, -) := \mathbb{C}^{m+1}$$

$$\bullet \ F_1(-, +, +) := \mathbb{C}^{m+1}$$

$$\bullet \ F_1(+, +, -) := \mathbb{C}^{m+2}$$

$$\bullet \ F_1(+, +, +) := \mathbb{C}^{m+2}$$

Generalization maps:

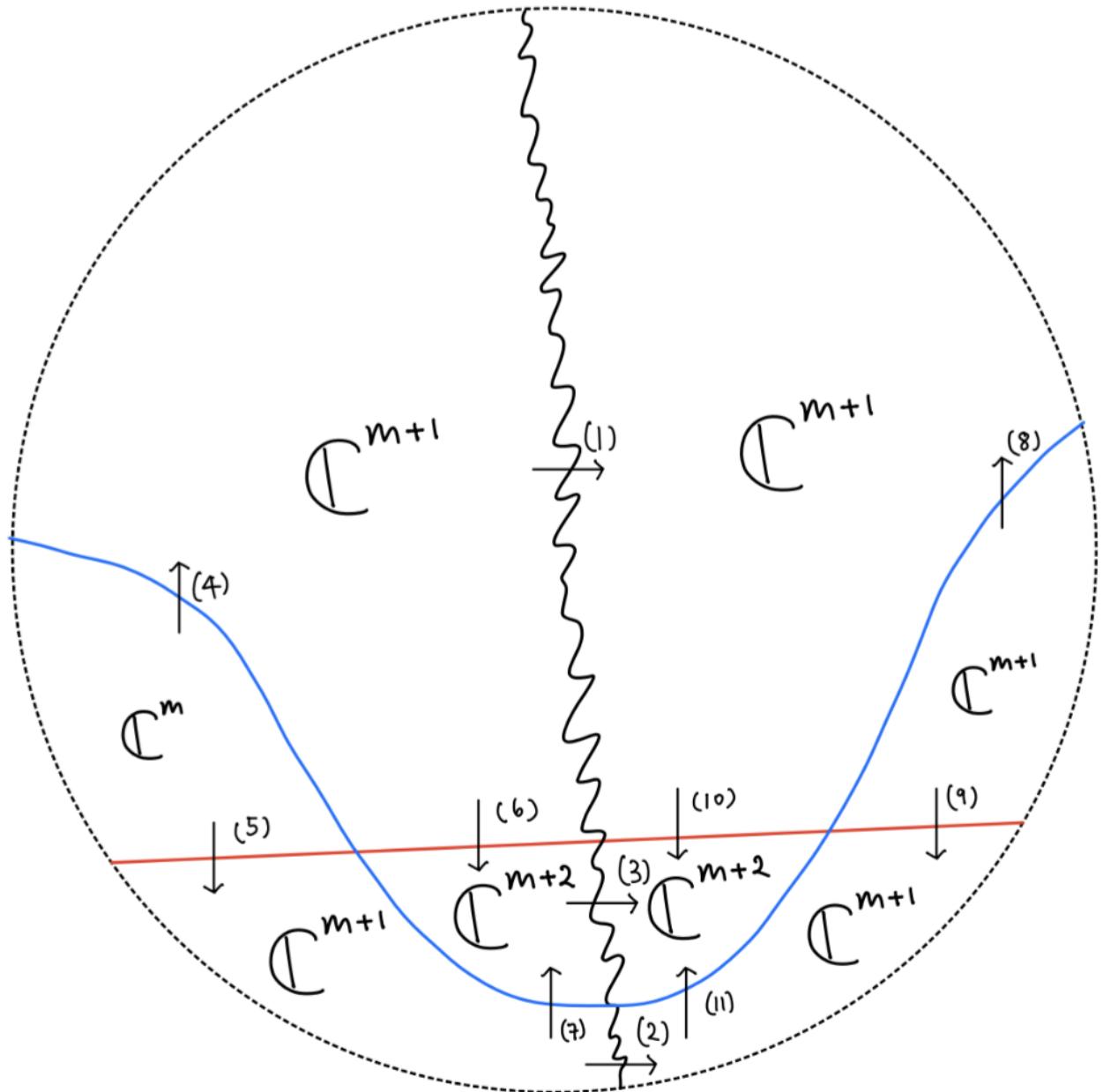


Figure 3.63

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(8) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(9) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(10) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(11) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 95. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0,1] \twoheadrightarrow (U \cap V)$$

3.5 2nd Sheaf Cobordism

In this section, we define $cobord_2$, a compactly supported sheaf cobordism.

3.5.1 Notations

Definition 96. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 97. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_2$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_2$.

Definition 98. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_{\bullet}^{symbol} to be smooth maps

$$\Phi_{\bullet}^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_{\bullet}^{symbol} a co-orientation of Φ_{\bullet}^{symbol} .

3. we denote the pair $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$ as $\Lambda_{\bullet}^{symbol}$. Later in the section, $\Lambda_{\bullet}^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₂.

4. we denote the triple $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$ as Λ_{\bullet} and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_{\bullet} will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₂.

Definition 99. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

Definition 100. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$

$$(x, z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 101. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_0^{squig} is the union of the following two components
 - (i) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
 - (ii) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig} : hairs are pointing downward direction i.e. coefficients of dz are negative.

Definition 102. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_1^{squig} is the union of the following three components
 - (i) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, -\frac{1}{2} \geq \Psi_1(x)\}$

$$(ii) \quad \{(0, z) \in D_{r=2} \mid -\frac{1}{2} < z < \frac{1}{2}\}$$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig} :
 - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
 - for (ii), hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 103. 1. We define the following subsets of $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = 0\}$
- λ_\bullet^{squig} is the union of the following two components
 - (i) $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0, 1] \mid \frac{1}{2} > \Psi_t(x)\}$
 - (ii) $\{(0, z, t) \in D_{r=2} \times [0, 1] \mid \Psi_t(0) < z < \frac{1}{2}\}$

2. We define co-orientations ξ_\bullet^{symbol} of λ_\bullet^{symbol} as follows

- ξ_\bullet^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_\bullet^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_\bullet^{squig} :
 - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.

- for (ii), hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 104.

1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .
2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_\bullet on $D_{r=2} \times [0, 1]$ induced by λ_\bullet i.e. strata are non-empty finite intersections of λ_\bullet^0 , λ_\bullet^∞ , and λ_\bullet^{squig} . Note that 2 dimensional strata has co-orientations inherited from λ_\bullet .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the following notations:

Definition 105. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 106. For $i = 1, 2, 3, 4$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned}
 s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\
 & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\
 & sgn(z - \frac{1}{2}) = sgn_3, \\
 & sgn(x) = sgn_4\}
 \end{aligned}$$

2. we use * as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{i^{th}}{\overset{*}{\uparrow}}, \cdots) := s_{t_0}(\cdots, \underset{i^{th}}{\overset{-}{\uparrow}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\overset{0}{\uparrow}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\overset{+}{\uparrow}}, \cdots)$$

3. we omit trailing *'s e.g. $s_0(+, -) = s_0(+, -, *, *)$

4. note that we do not omit *'s located in between $-$, 0 , $+$ e.g. $s_0(+, -, *, -) \neq s_0(+, -, -)$

Definition 107. For $i = 0, 1, 2, 3, 4$ and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned}
 s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\
 & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\
 & sgn(z - \frac{1}{2}) = sgn_3, \\
 & sgn(x) = sgn_4\}
 \end{aligned}$$

2. we use * as a wild card sign i.e.

$$s_{\bullet}(\cdots, \underset{i^{th}}{\overset{*}{\uparrow}}, \cdots) := s_{\bullet}(\cdots, \underset{i^{th}}{\overset{-}{\uparrow}}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{\overset{0}{\uparrow}}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{\overset{+}{\uparrow}}, \cdots)$$

3. we omit trailing *'s e.g. $s_\bullet(+, -) = s_\bullet(+, -, *, *)$
4. note that we do not omit *'s located in between $-$, 0 , $+$ e.g. $s_\bullet(+, -, *, -, -) \neq s_\bullet(+, -, -, -)$

Definition 108. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the above notations:

1. \mathcal{S}_0 :

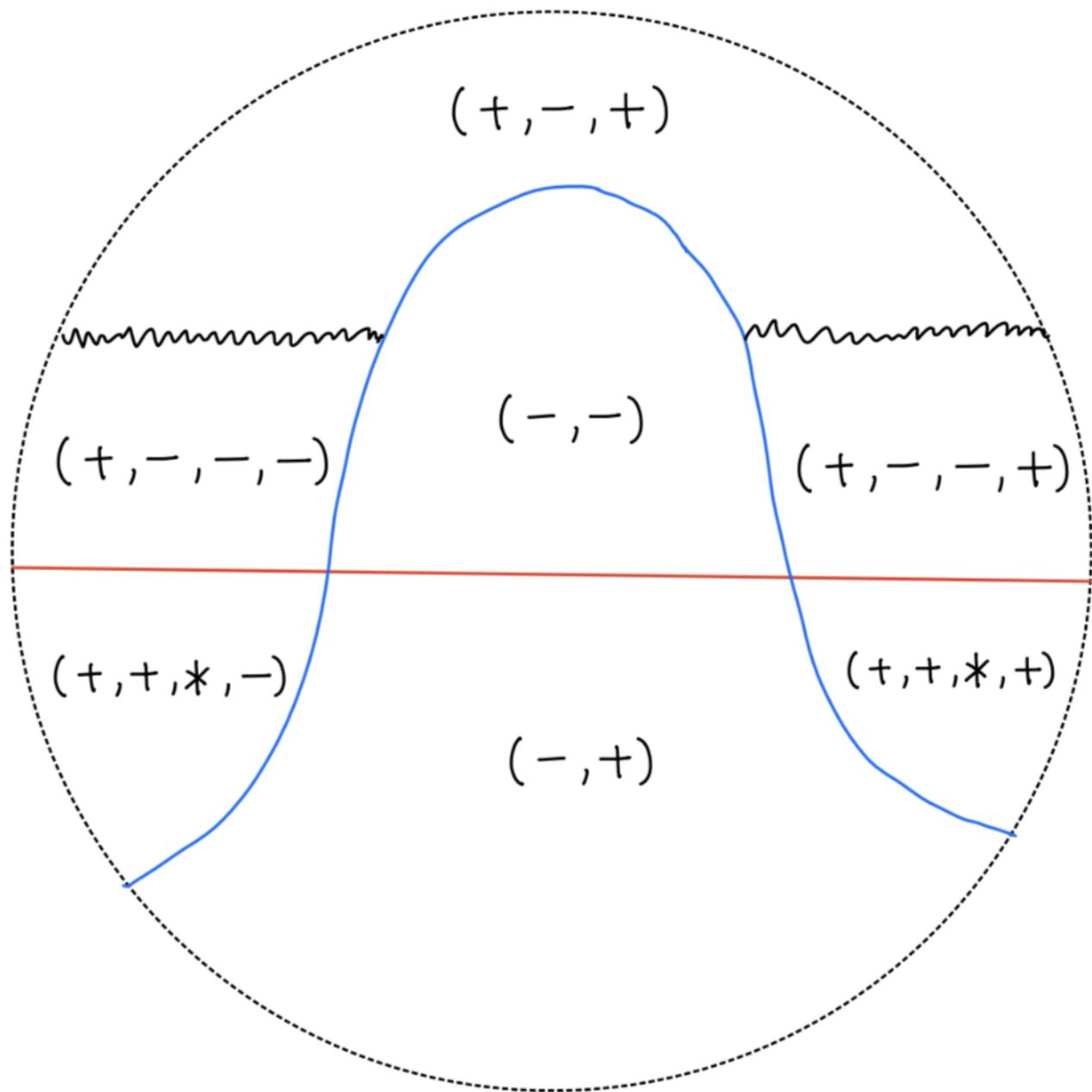


Figure 3.64

- 2 dimensional strata:

$$s_0(-,+), s_0(-,-), s_0(+,-,-,-), s_0(+,-,-,+), s_0(+,-,+), s_0(+,+,*,-), \\ s_0(+,+,*,-)$$

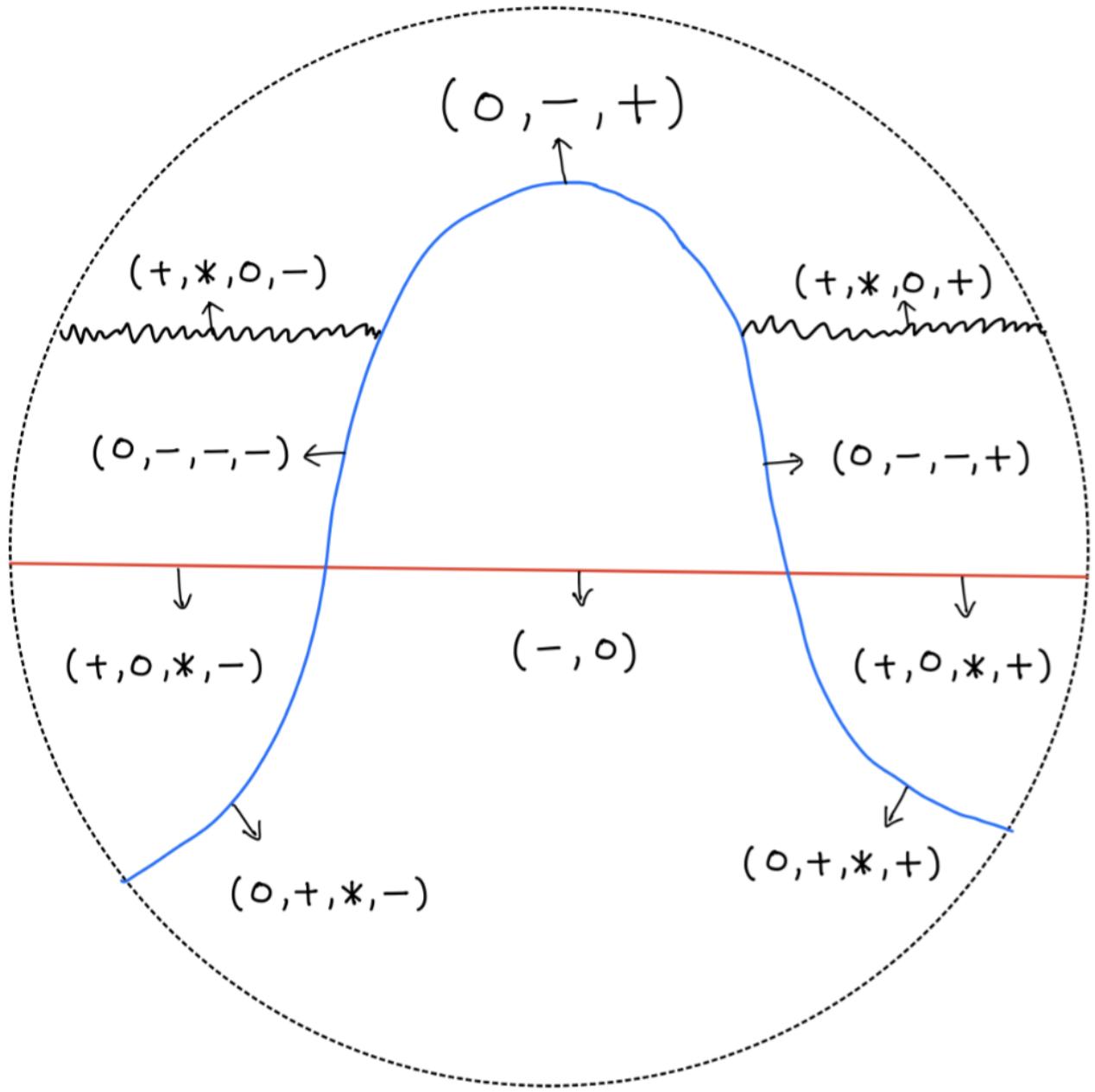


Figure 3.65

- 1 dimensional strata:

$$s_0(0, +, *, -), s_0(0, -, -, -), s_0(0, -, +), s_0(0, -, -, +), s_0(0, +, *, +), s_0(+, 0, *, -),$$

$$s_0(-, 0), s_0(+, 0, *, +), s_0(+, *, 0, -), s_0(+, *, 0, +)$$

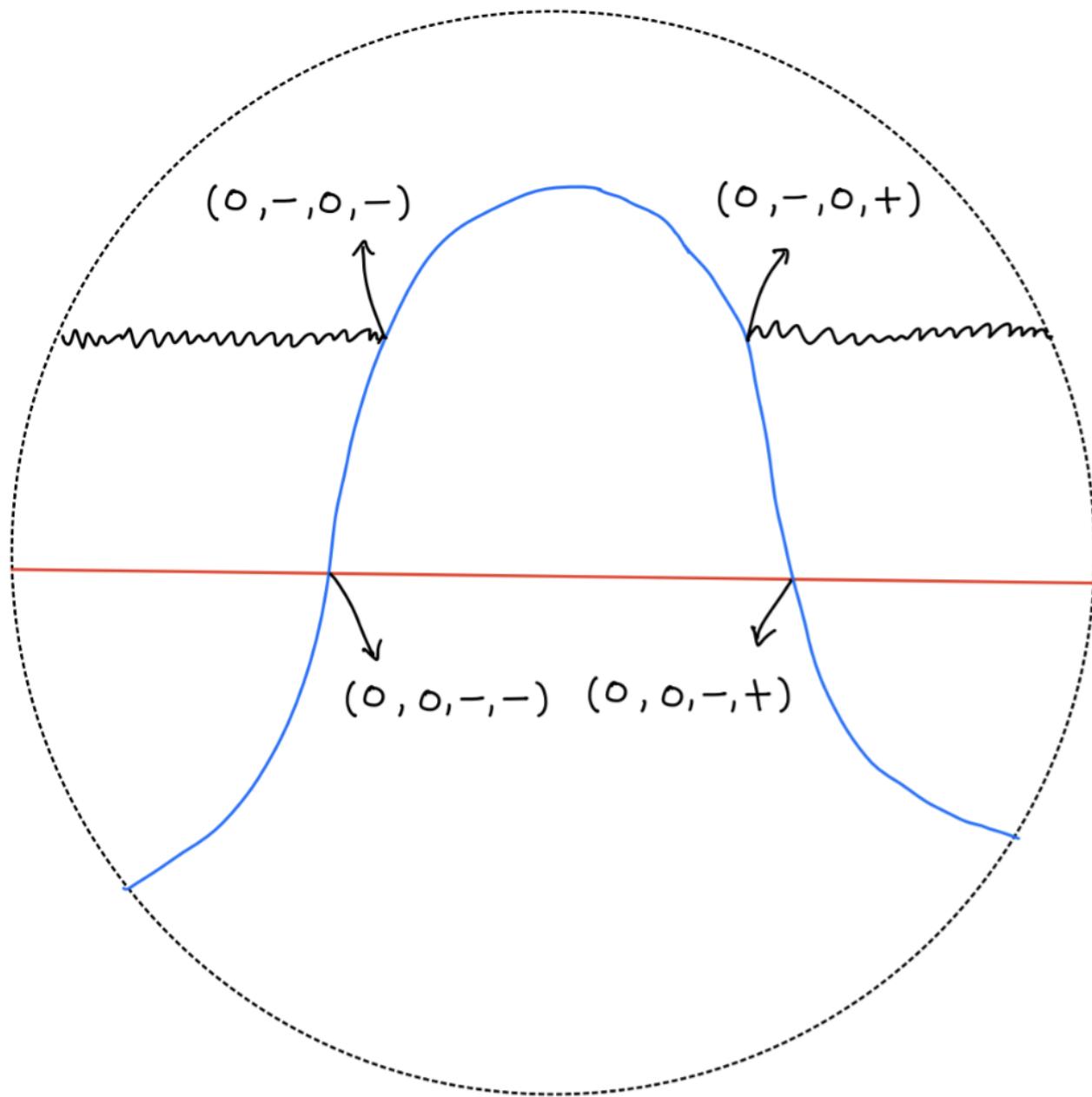


Figure 3.66

- 0 dimensional strata:

$$s_0(0, 0, -, -), s_0(0, -, 0, -), s_0(0, -, 0, +), s_0(0, 0, -, +)$$

2. \mathcal{S}_1 :

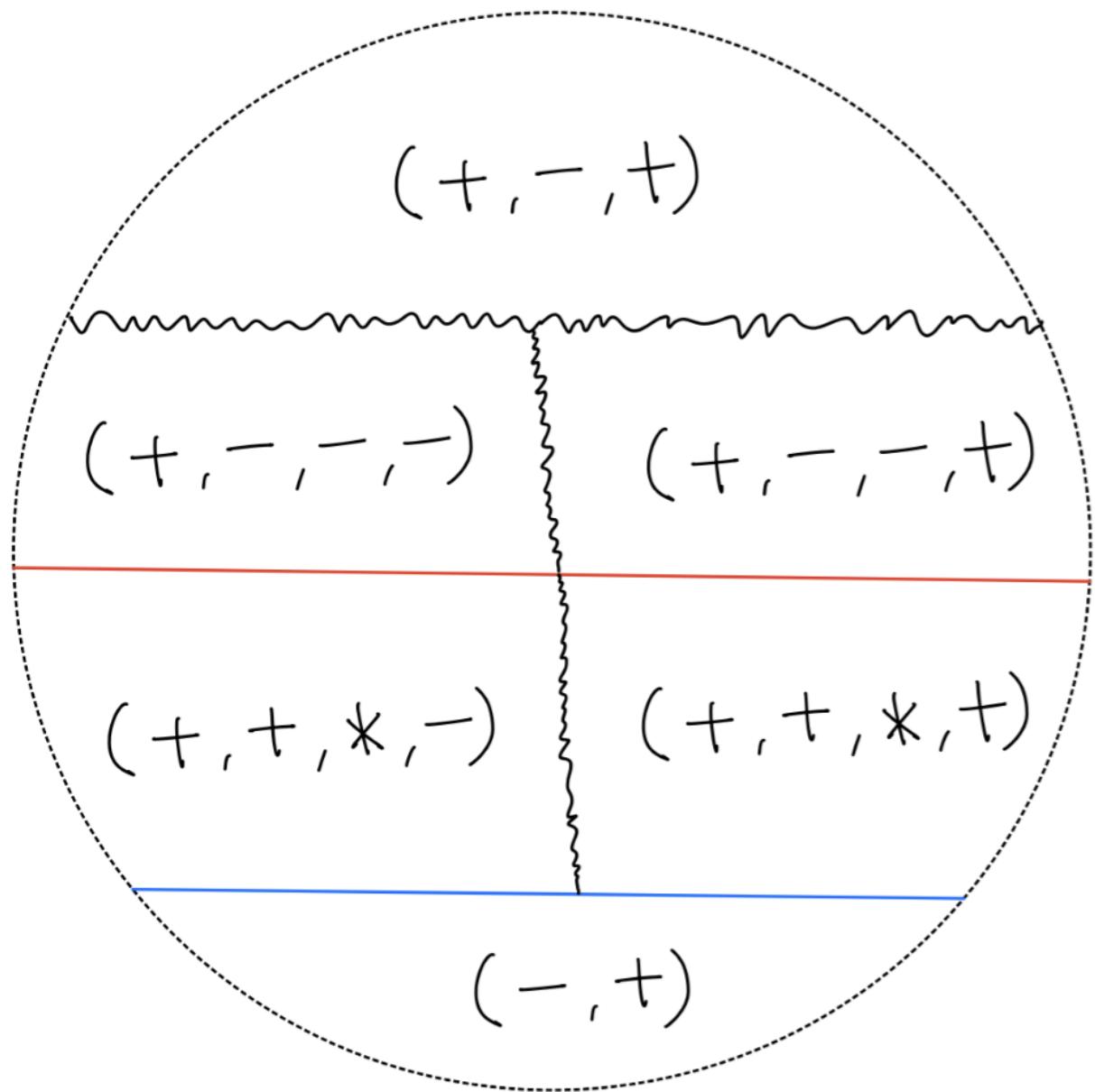


Figure 3.67

- 2 dimensional strata:

$$s_1(-, +), s_1(+, -, -, -), s_1(+, -, -, +), s_1(+, -, +), s_1(+, +, *, -), s_1(+, +, *, +)$$

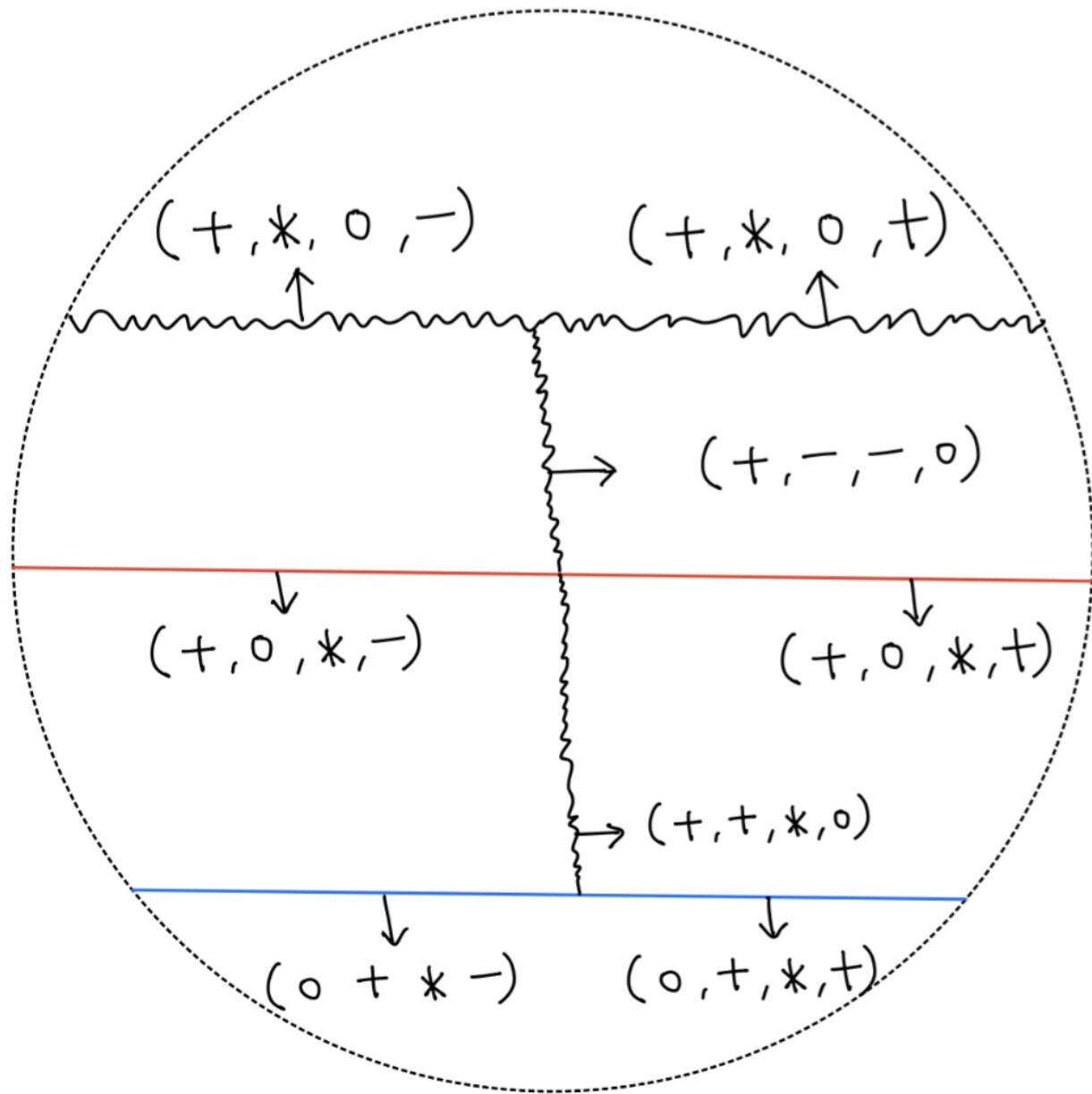


Figure 3.68

- 1 dimensional strata:

$$\begin{aligned}
 & s_1(0, +, *, -), s_1(0, +, *, +), s_1(+, 0, *, -), s_1(+, 0, *, +), s_1(+, *, 0, -), s_1(+, *, 0, +), \\
 & s_1(+, -, -, 0), s_1(+, +, *, 0)
 \end{aligned}$$

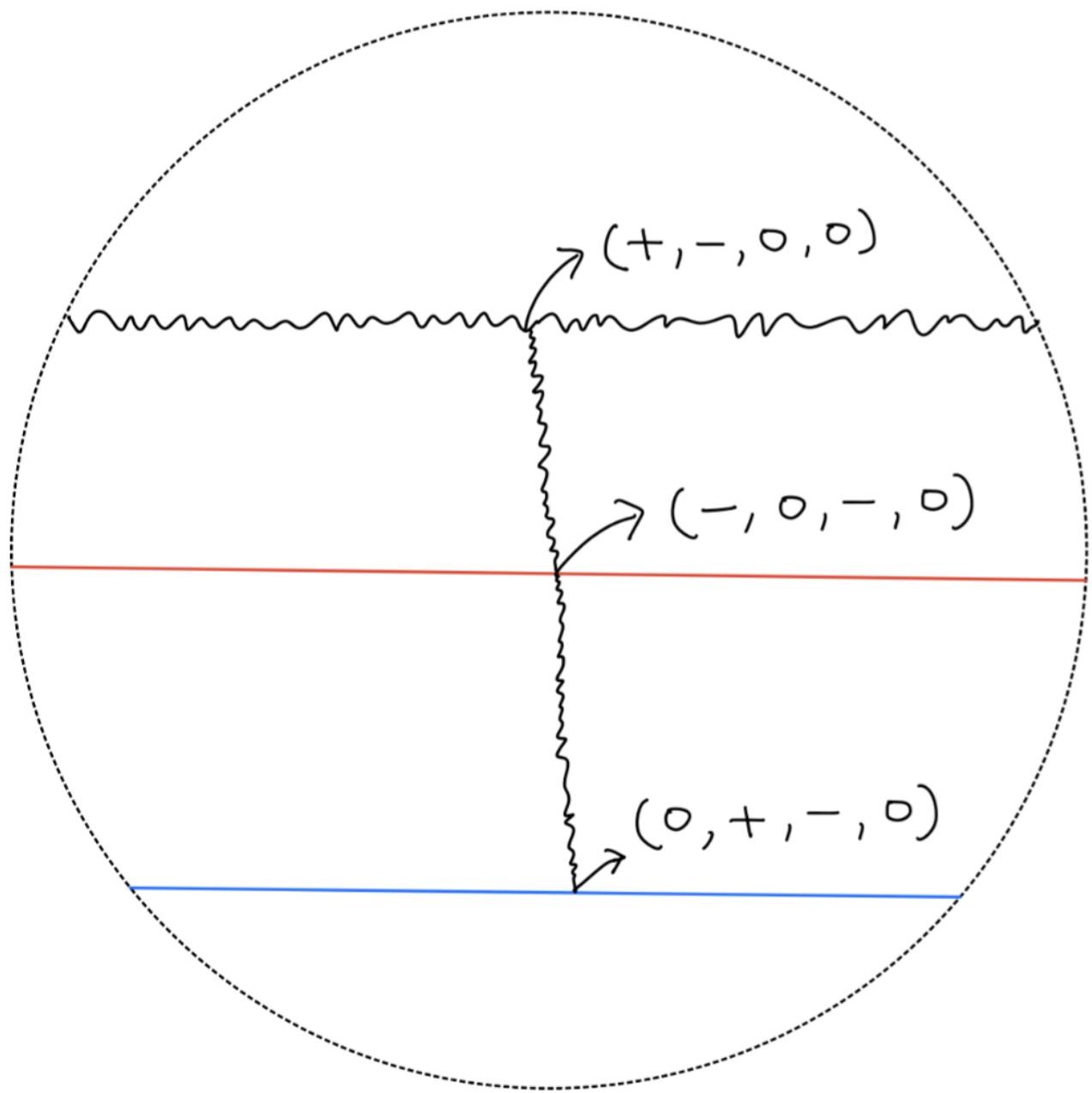


Figure 3.69

- 0 dimensional strata:

$$s_1(+, -, 0, 0), s_1(-, 0, -, 0), s_1(0, +, -, 0)$$

3. \mathcal{S}_\bullet :

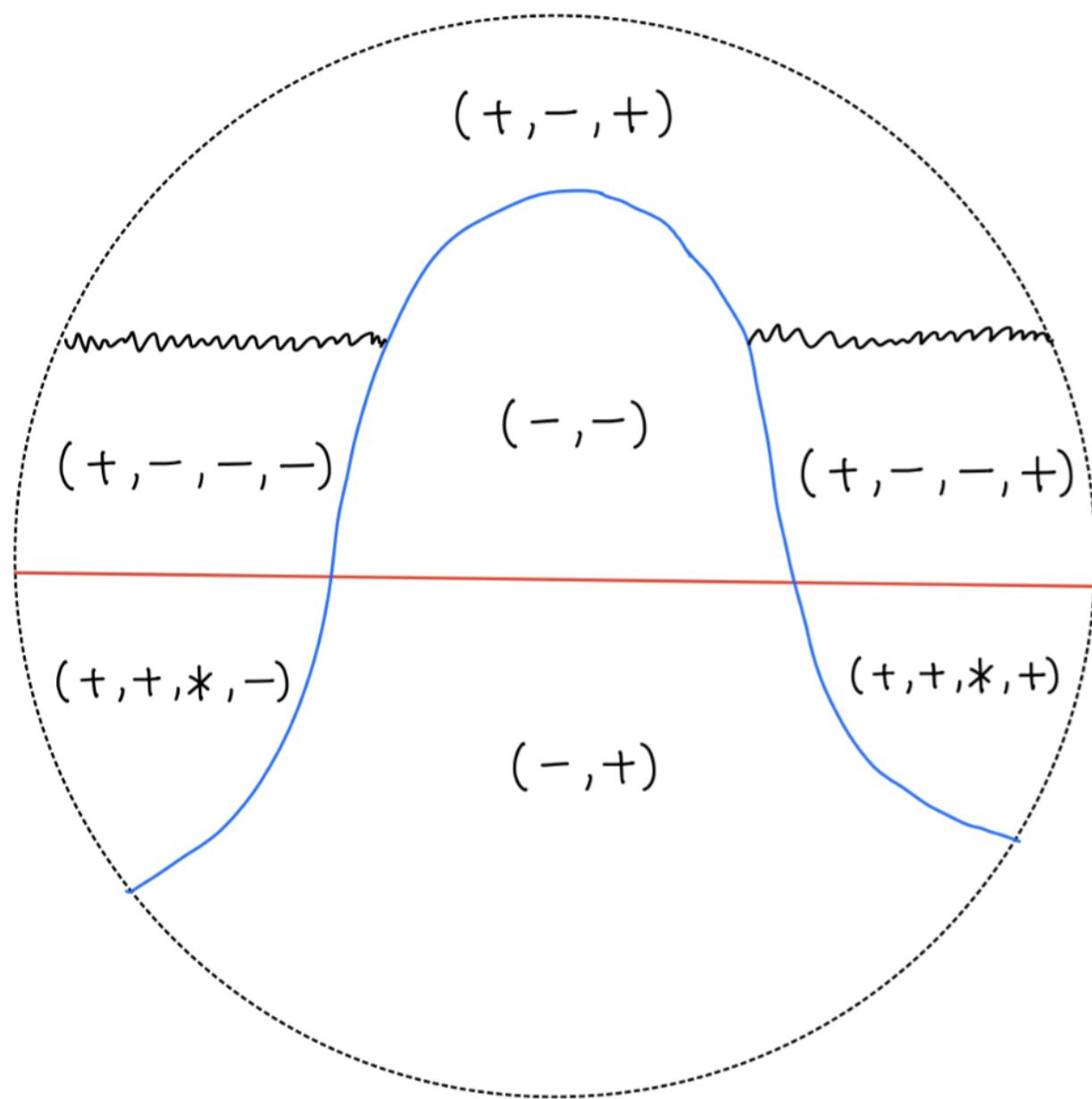


Figure 3.70

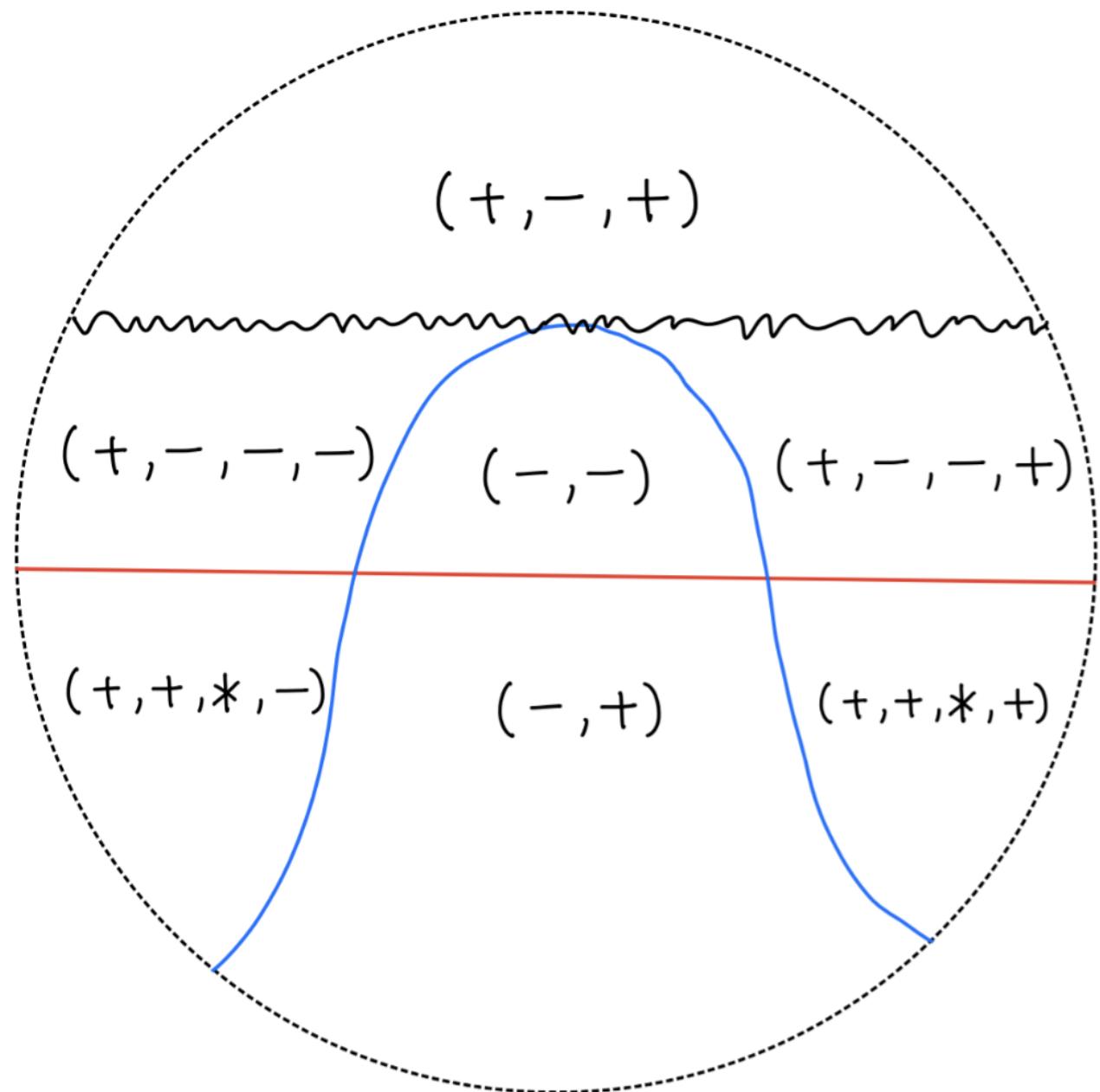


Figure 3.71

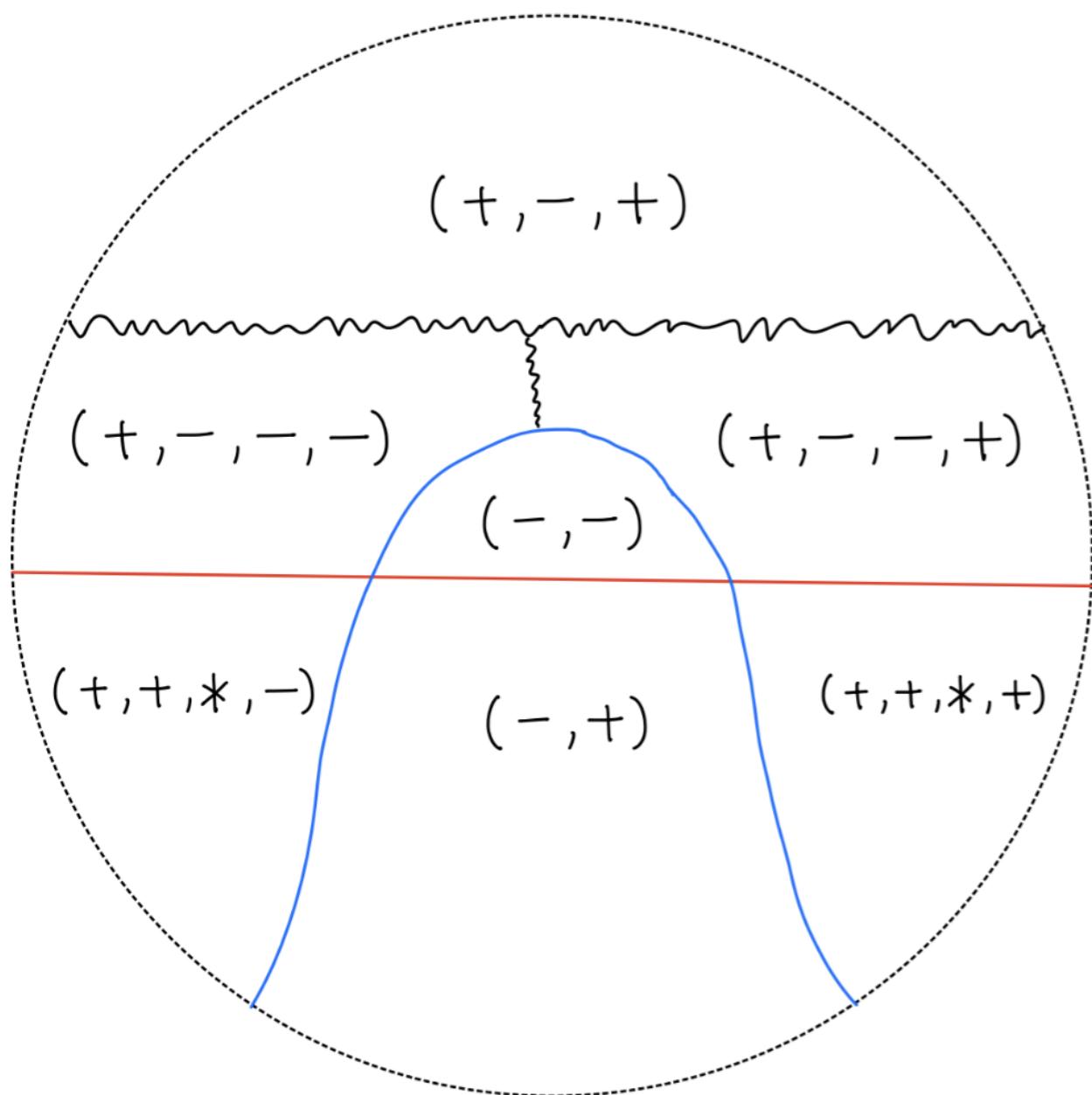


Figure 3.72

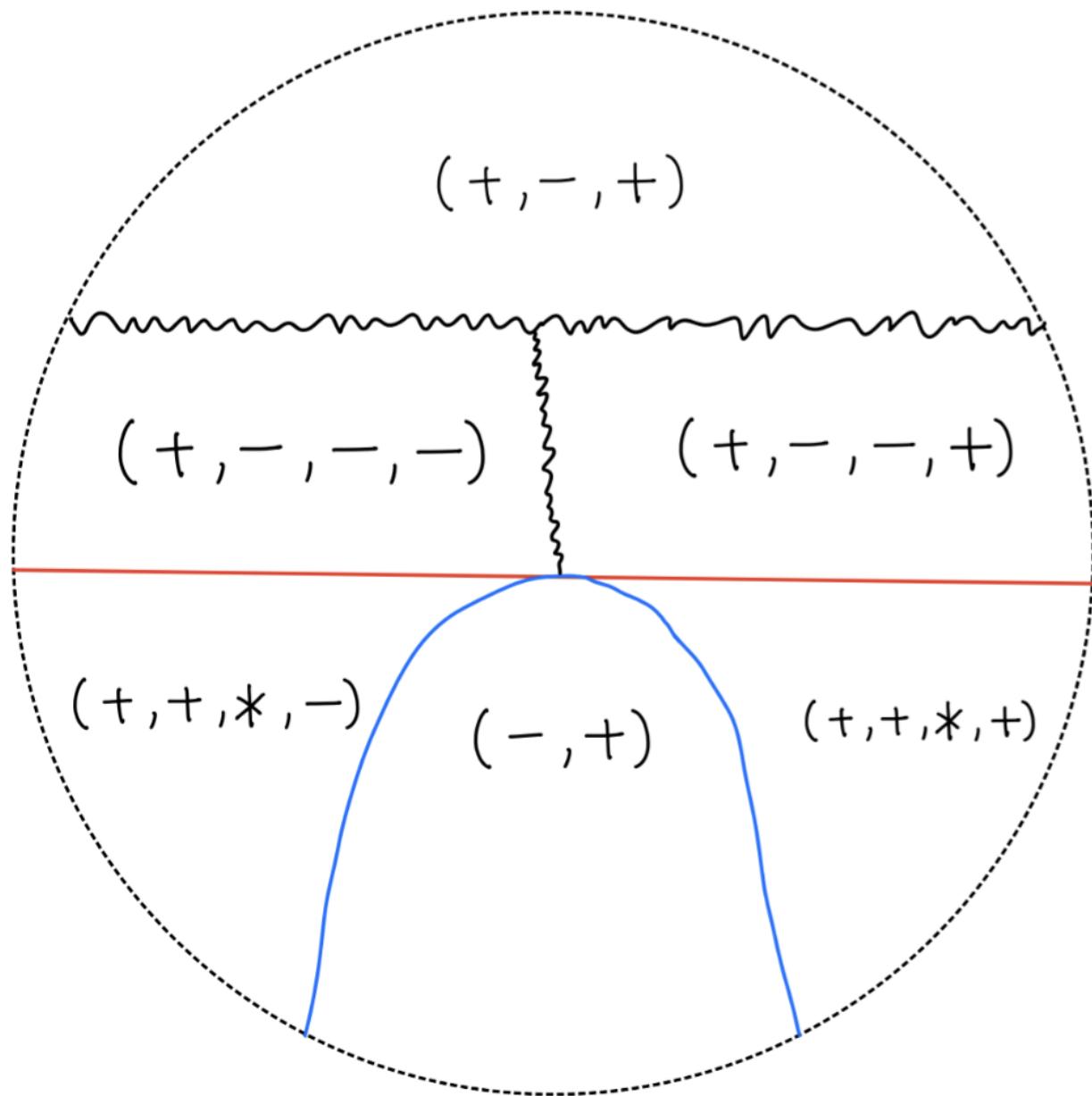


Figure 3.73

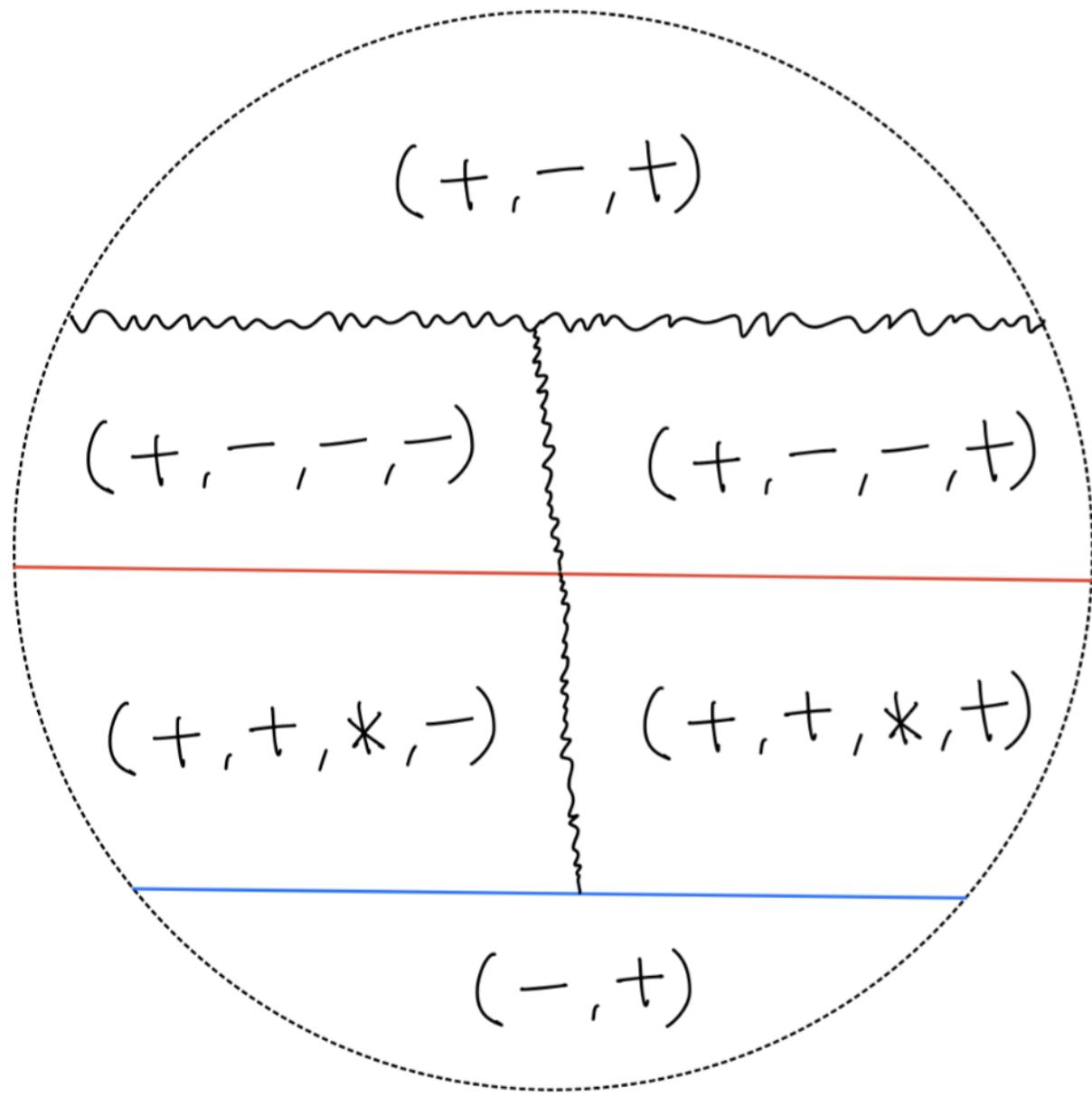


Figure 3.74

- 3 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(-,+), s_{\bullet}(-,-), s_{\bullet}(-,+), s_{\bullet}(+,-,-,-), s_{\bullet}(+,-,-,+), s_{\bullet}(+,-,+), \\
 & s_{\bullet}(+,*,-), s_{\bullet}(+,*,+)
 \end{aligned}$$

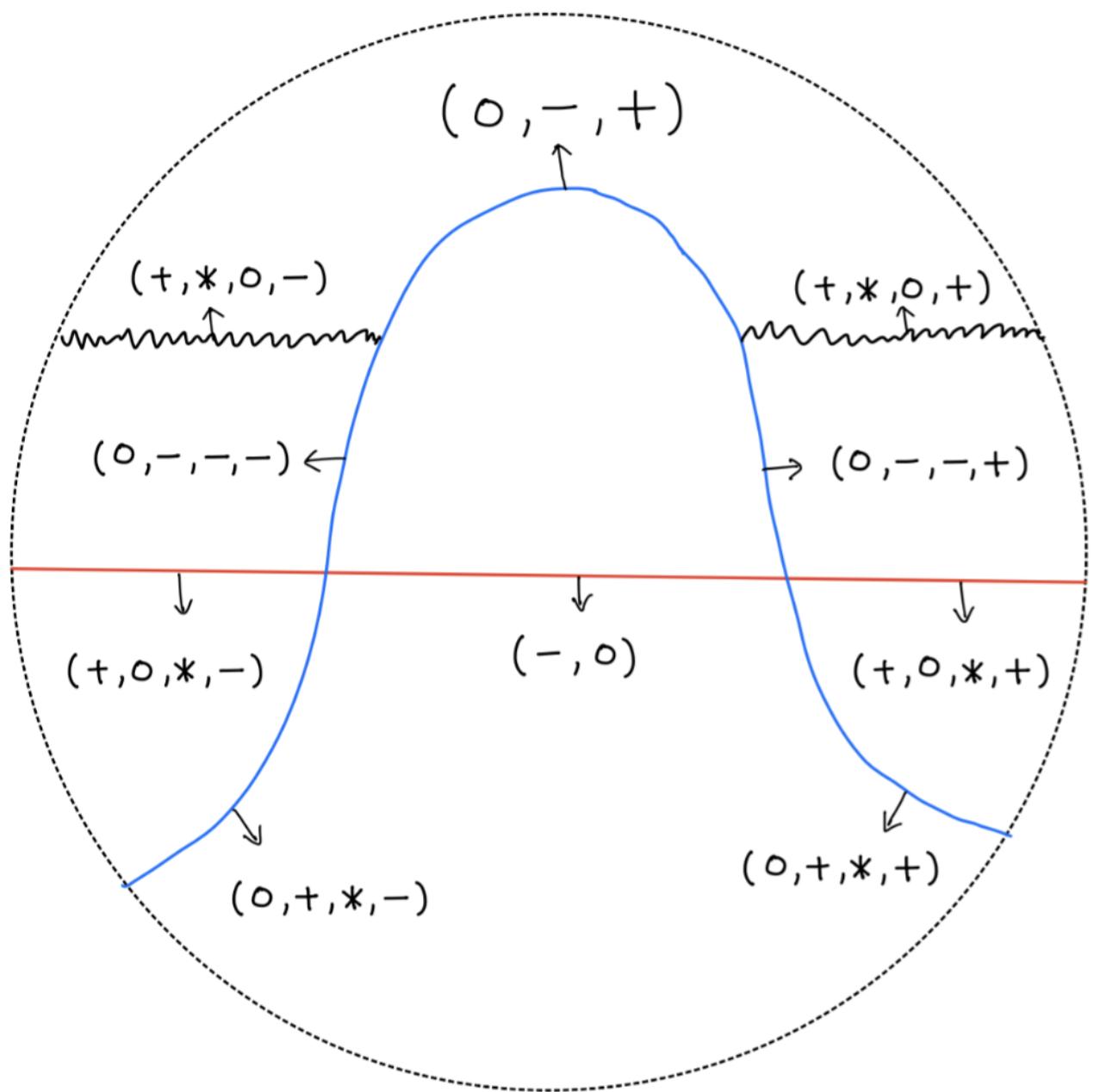


Figure 3.75

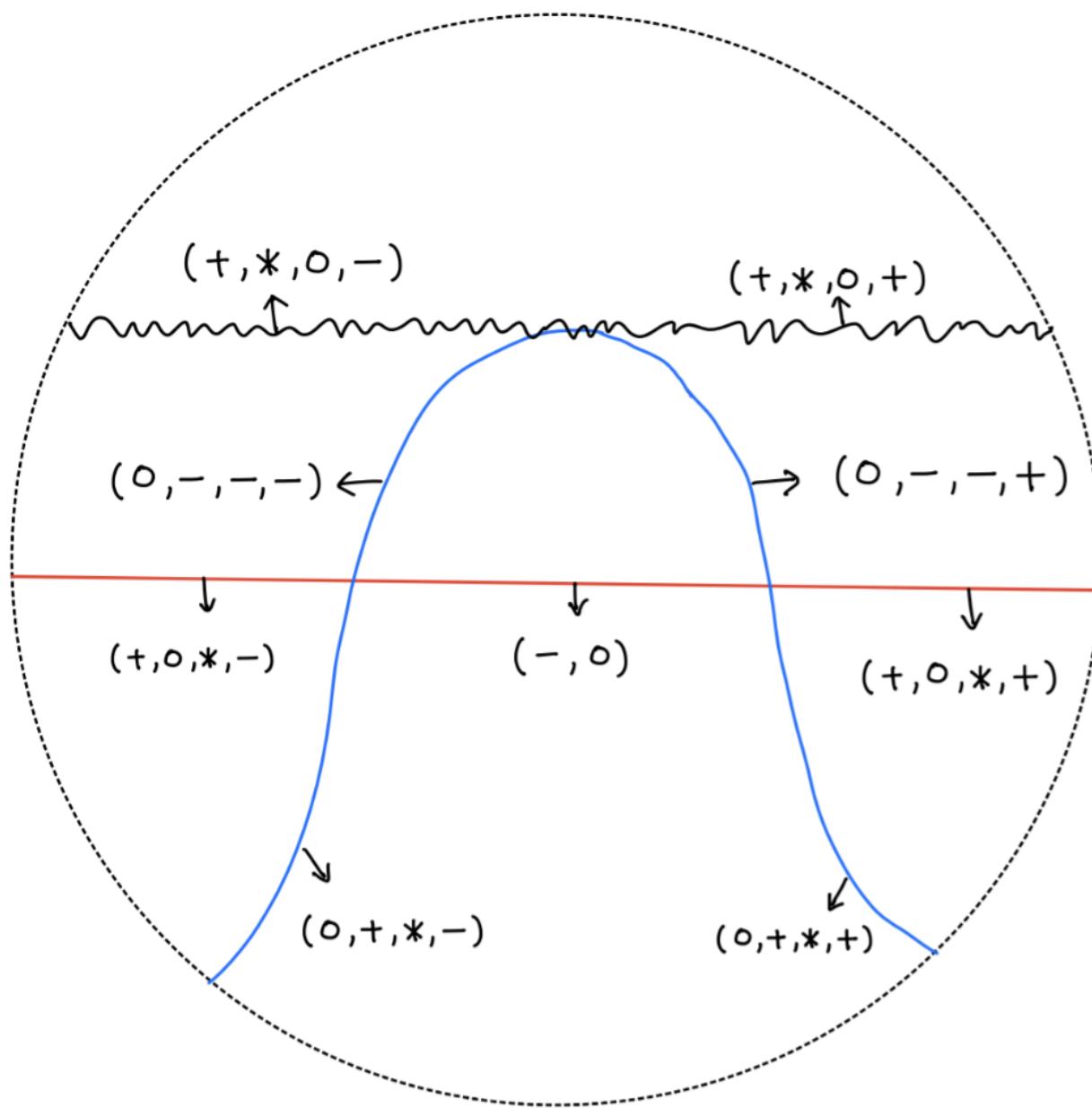


Figure 3.76

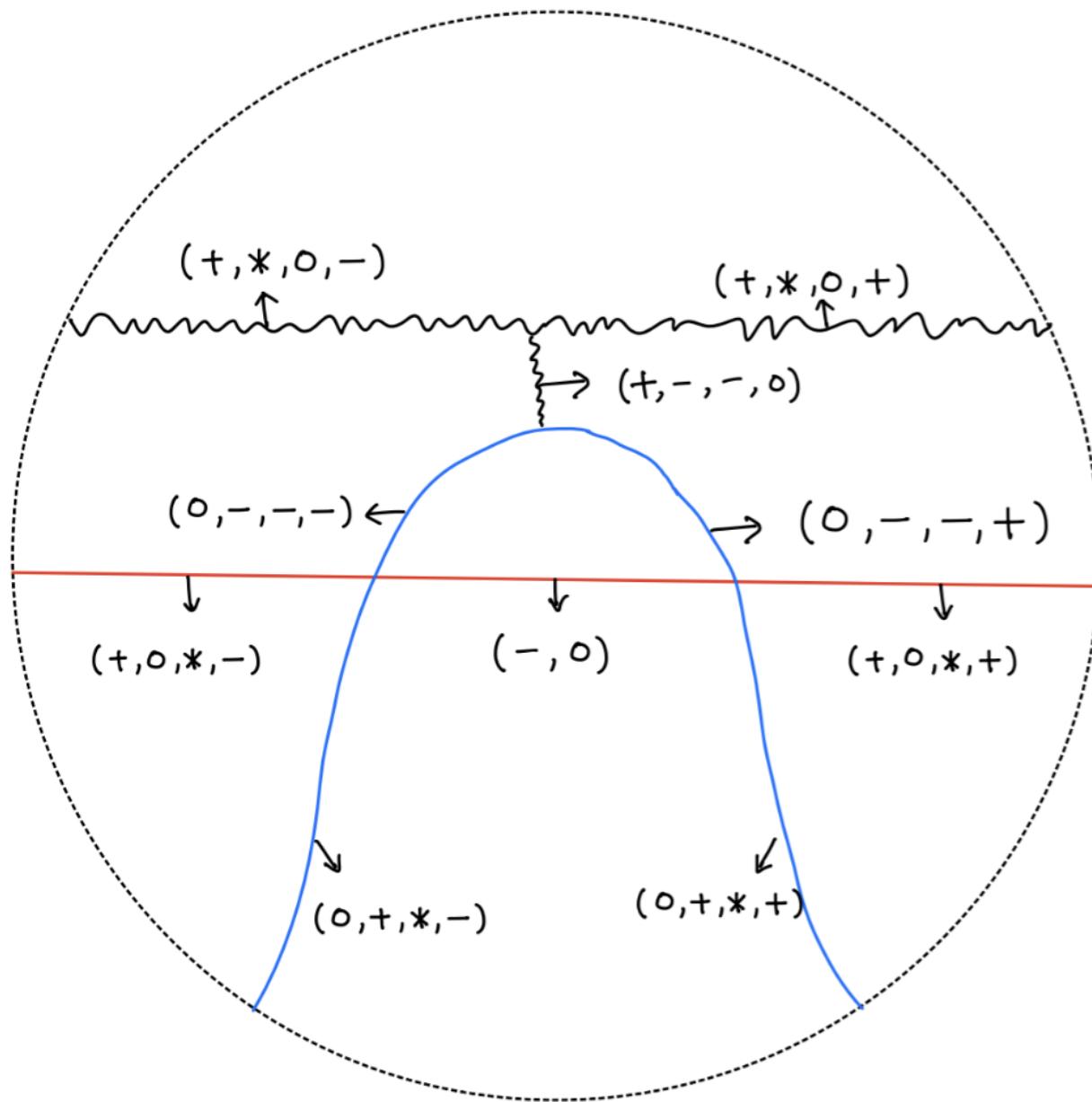


Figure 3.77

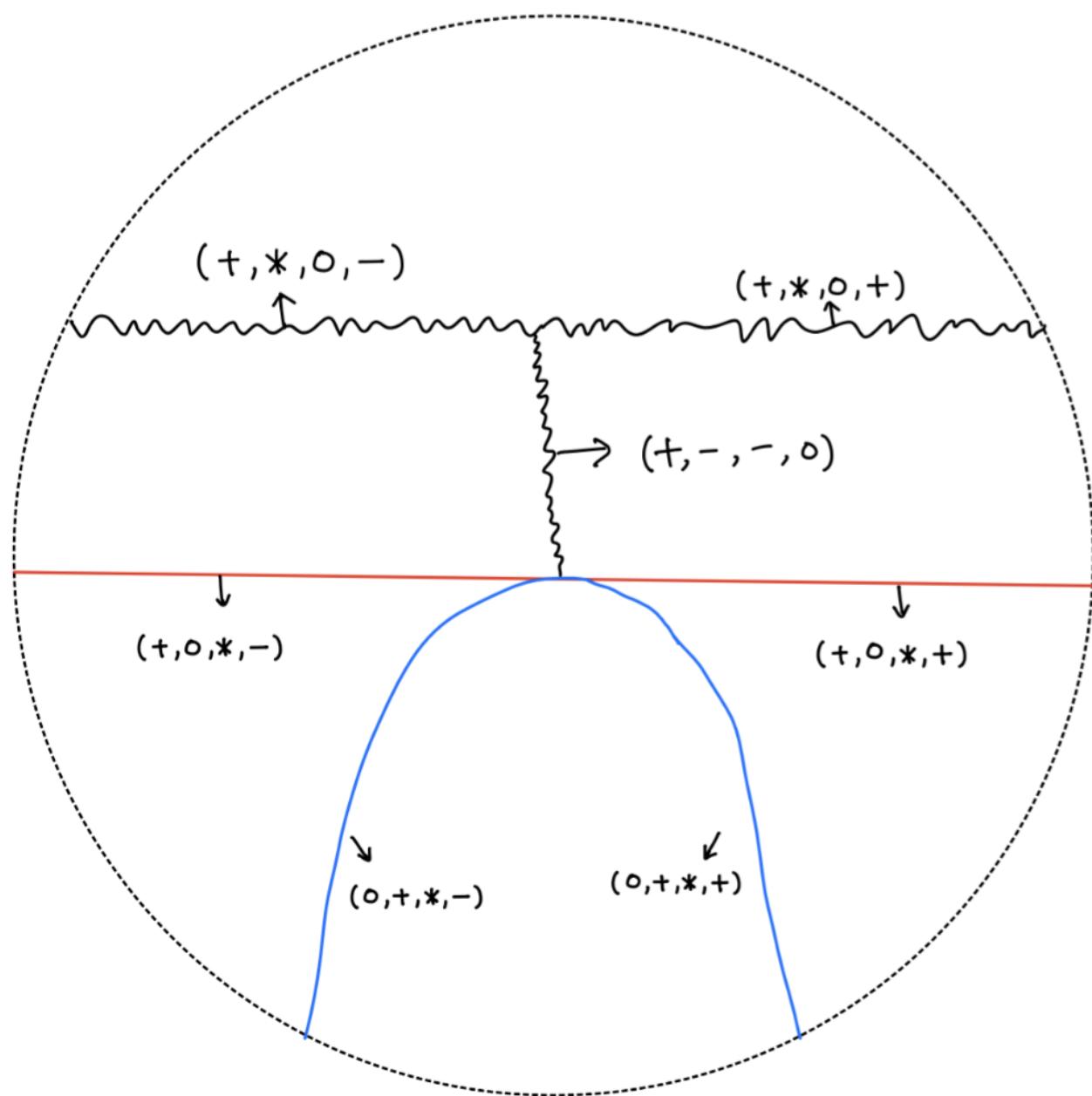


Figure 3.78

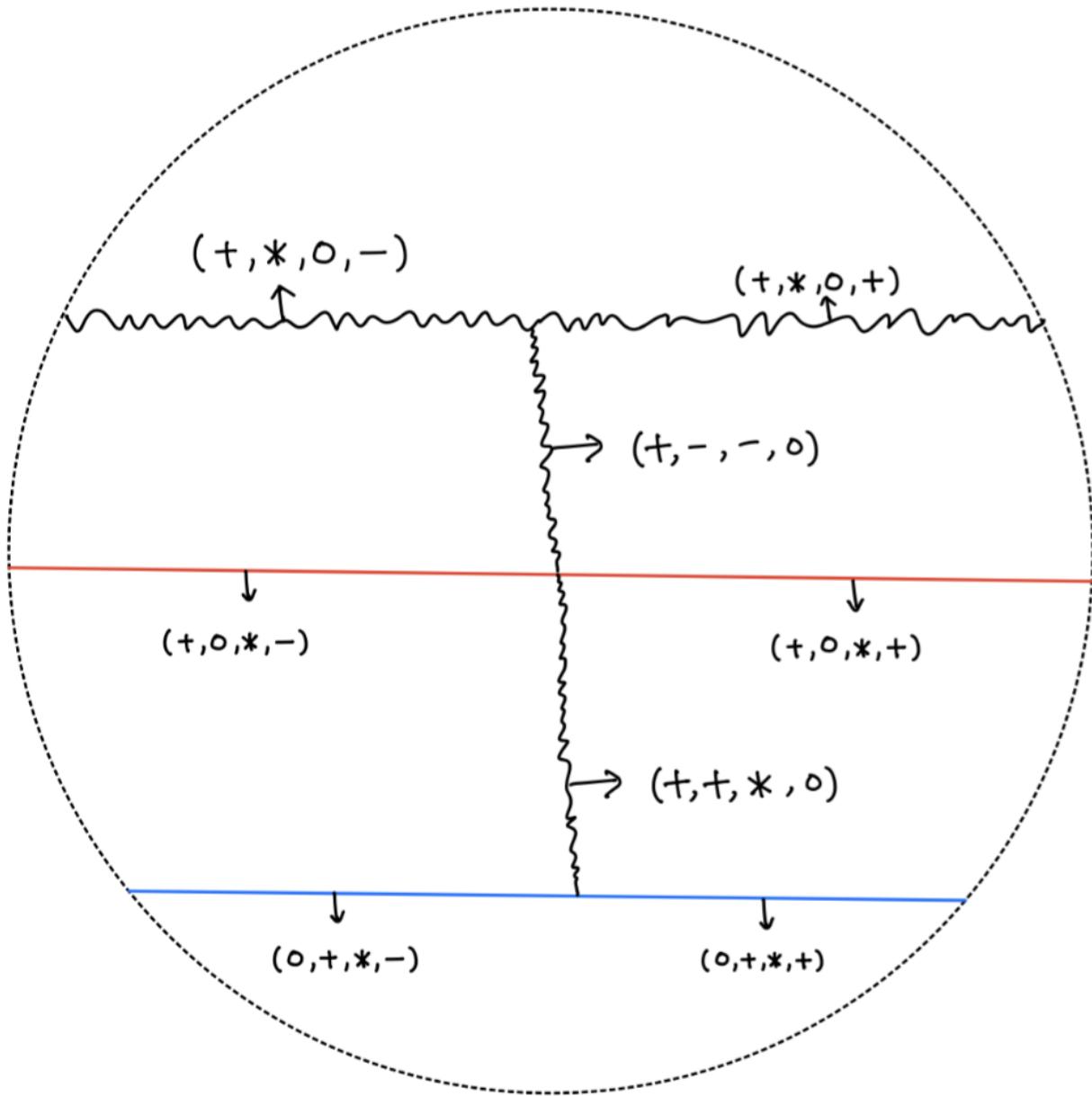


Figure 3.79

- 2 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, +, *, -), s_{\bullet}(0, -, -, -), s_{\bullet}(0, -, +), s_{\bullet}(0, -, -, +), s_{\bullet}(0, +, *, +), s_{\bullet}(+, 0, *, -), \\
 & s_{\bullet}(-, 0), s_{\bullet}(+, 0, *, +), s_{\bullet}(+, *, 0, -), s_{\bullet}(+, *, 0, +), s_{\bullet}(+, -, -, 0), s_{\bullet}(+, +, *, 0)
 \end{aligned}$$

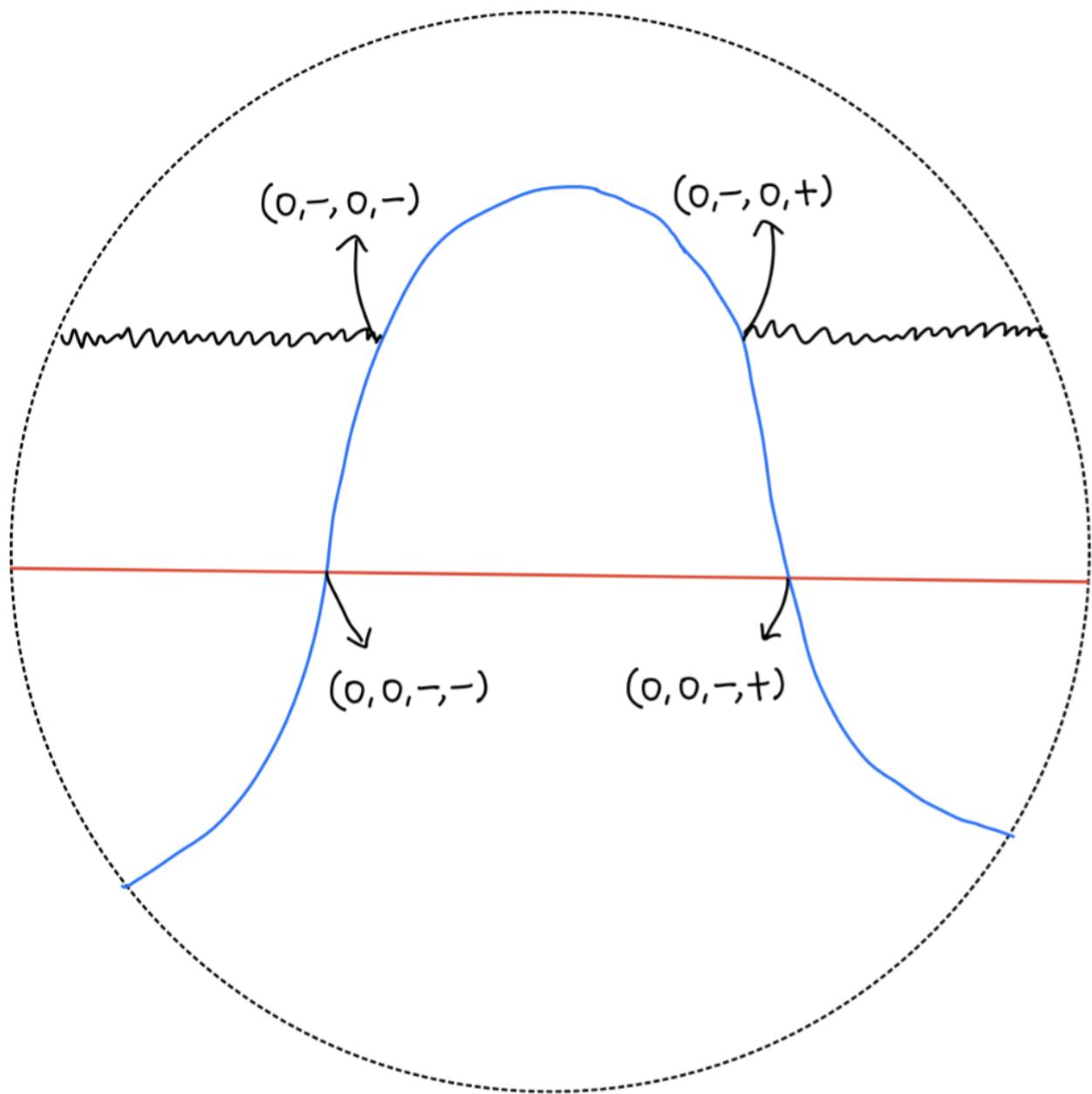


Figure 3.80

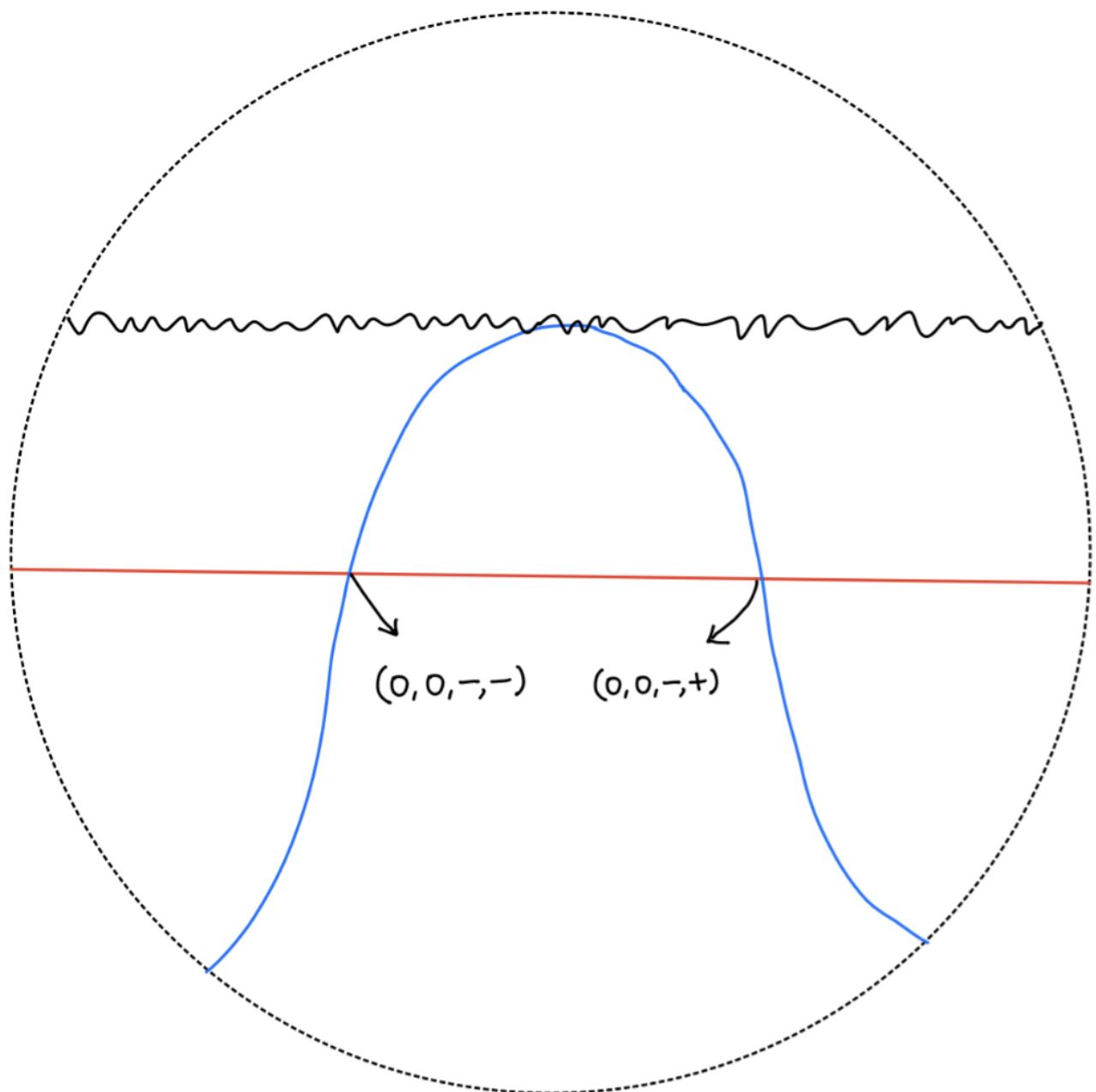


Figure 3.81

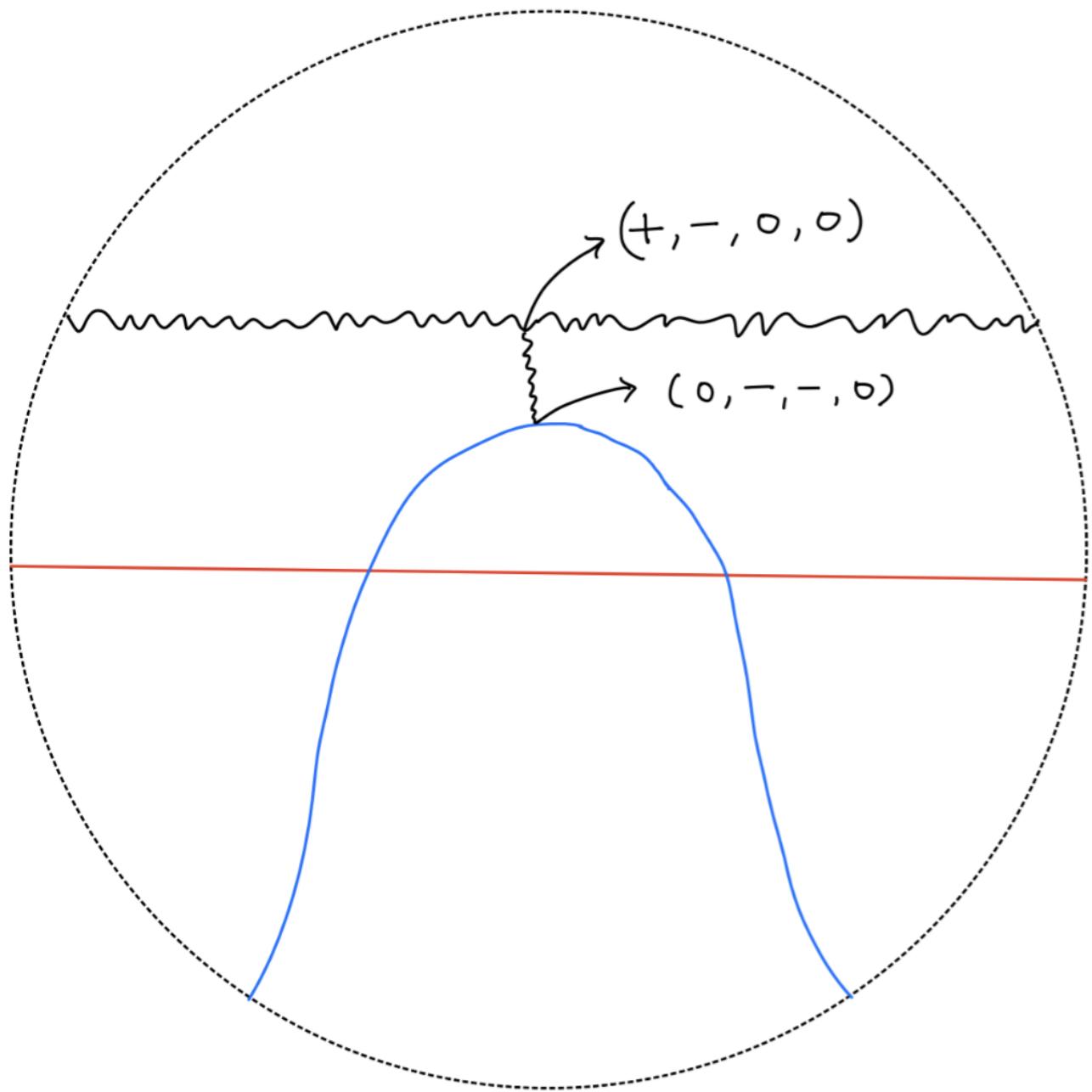


Figure 3.82

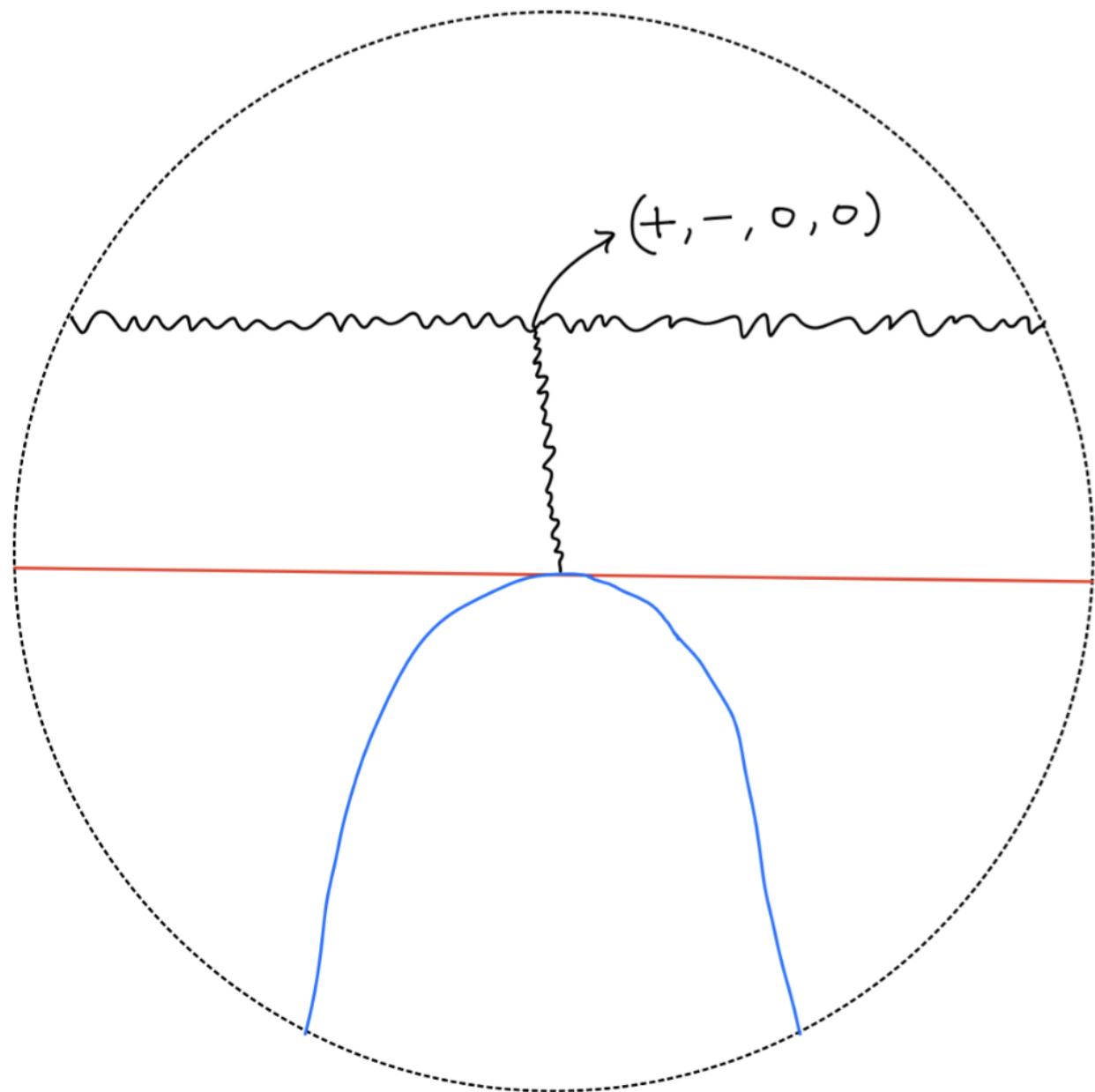


Figure 3.83

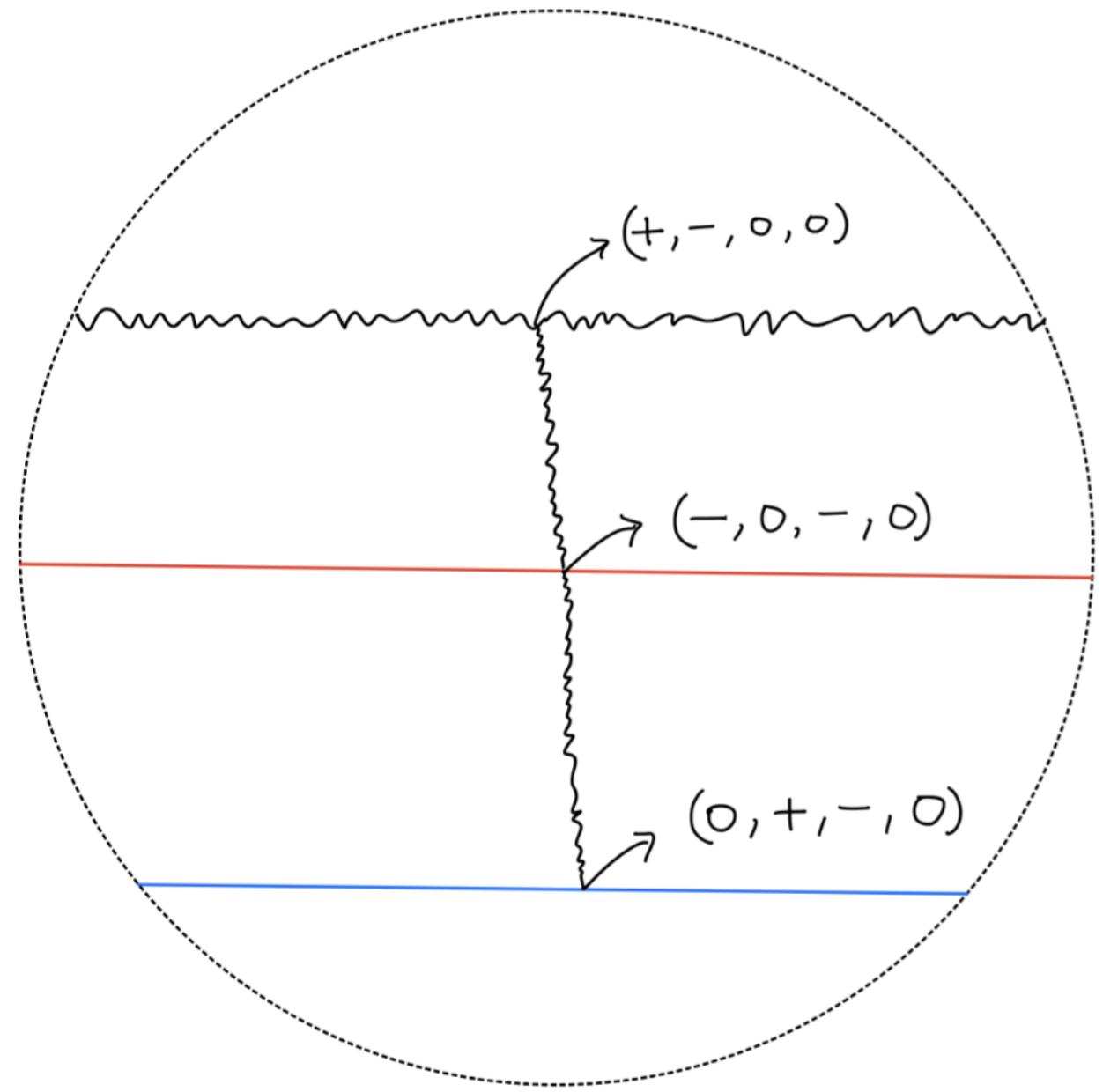


Figure 3.84

- 1 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, 0, -, -), s_{\bullet}(0, -, 0, -), s_{\bullet}(0, -, 0, +), s_{\bullet}(0, 0, -, +), s_{\bullet}(+, -, 0, 0), s_{\bullet}(0, -, -, 0), \\
 & s_{\bullet}(-, 0, -, 0), s_{\bullet}(0, +, -, 0)
 \end{aligned}$$

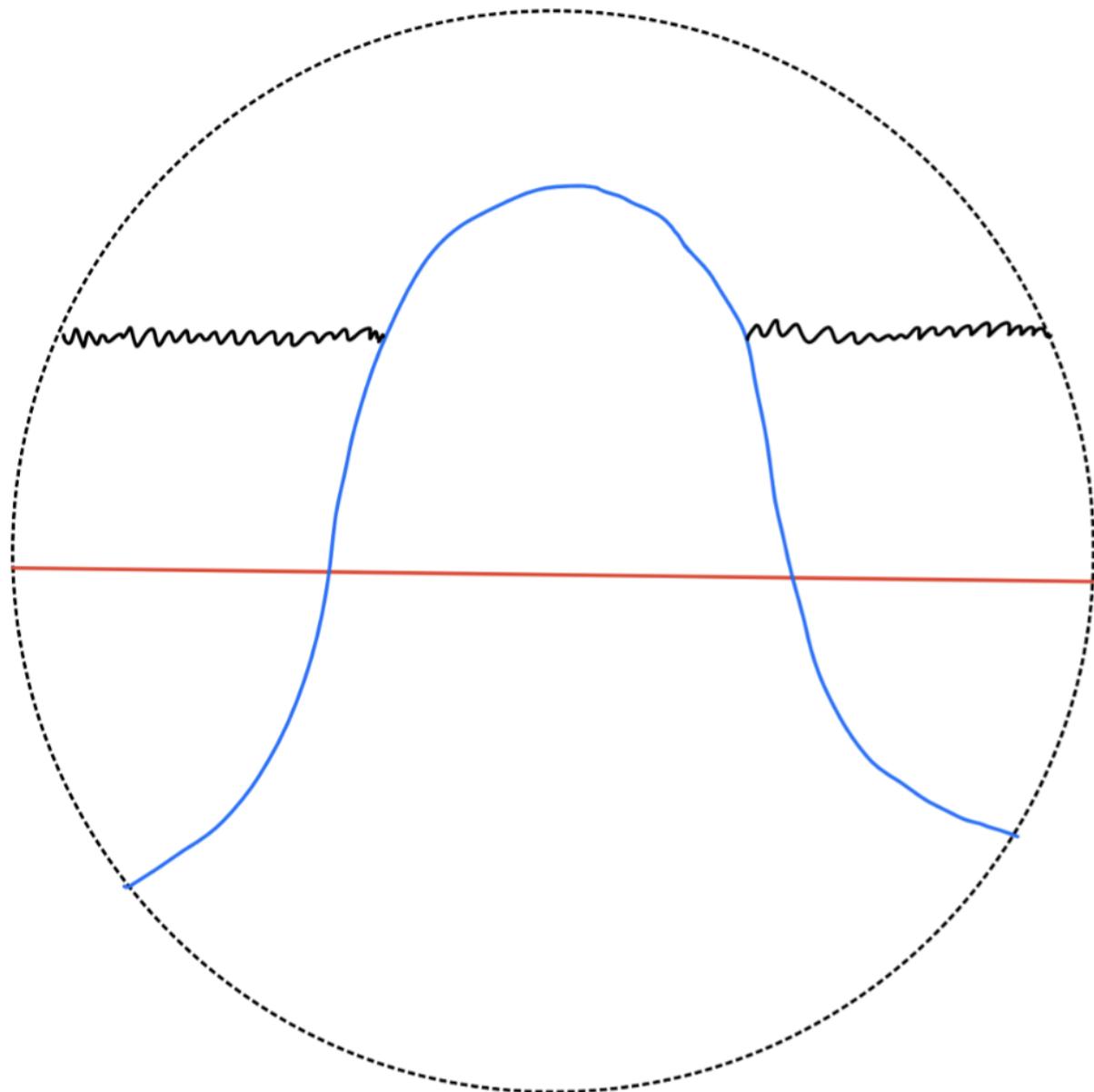


Figure 3.85

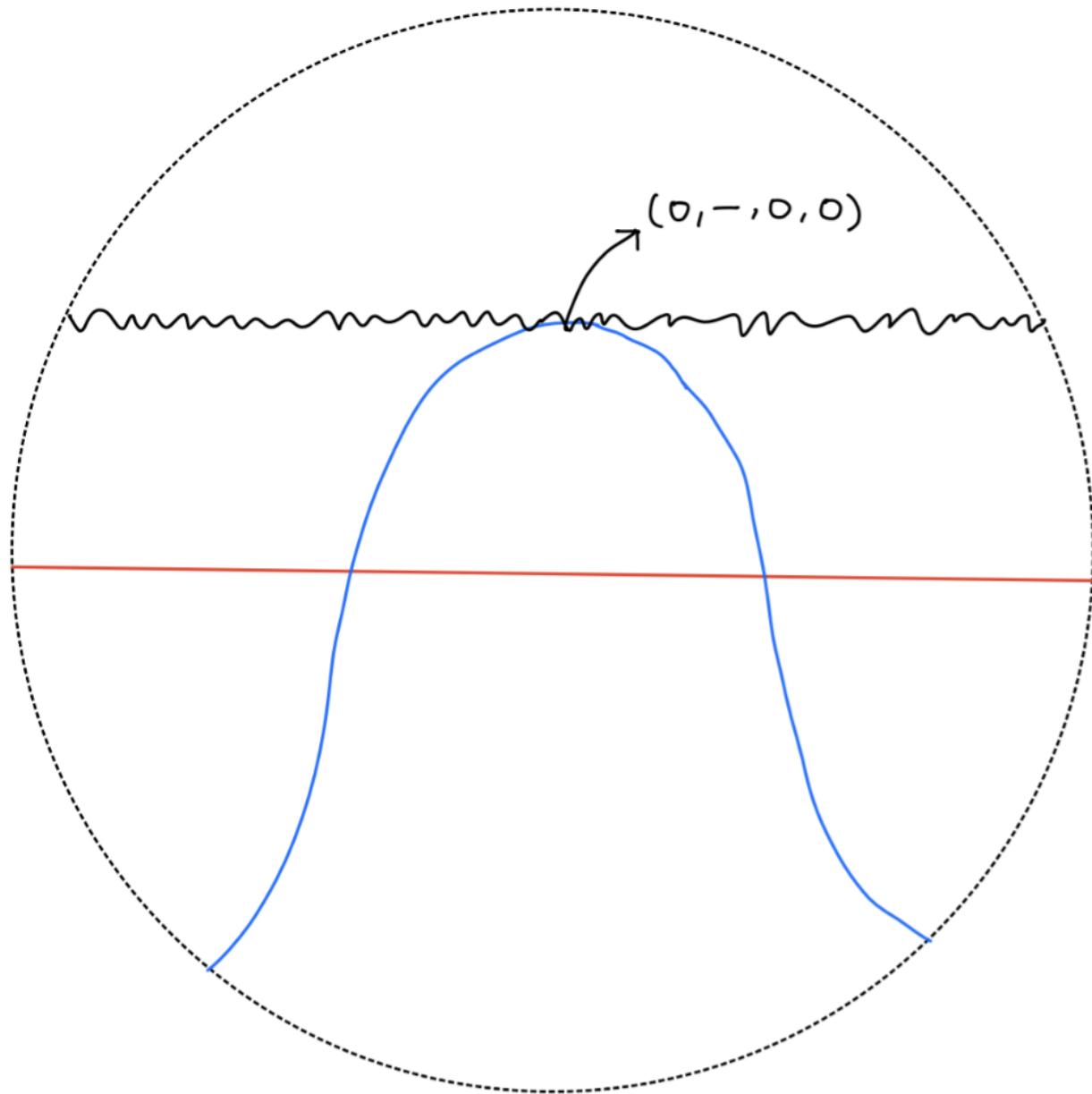


Figure 3.86

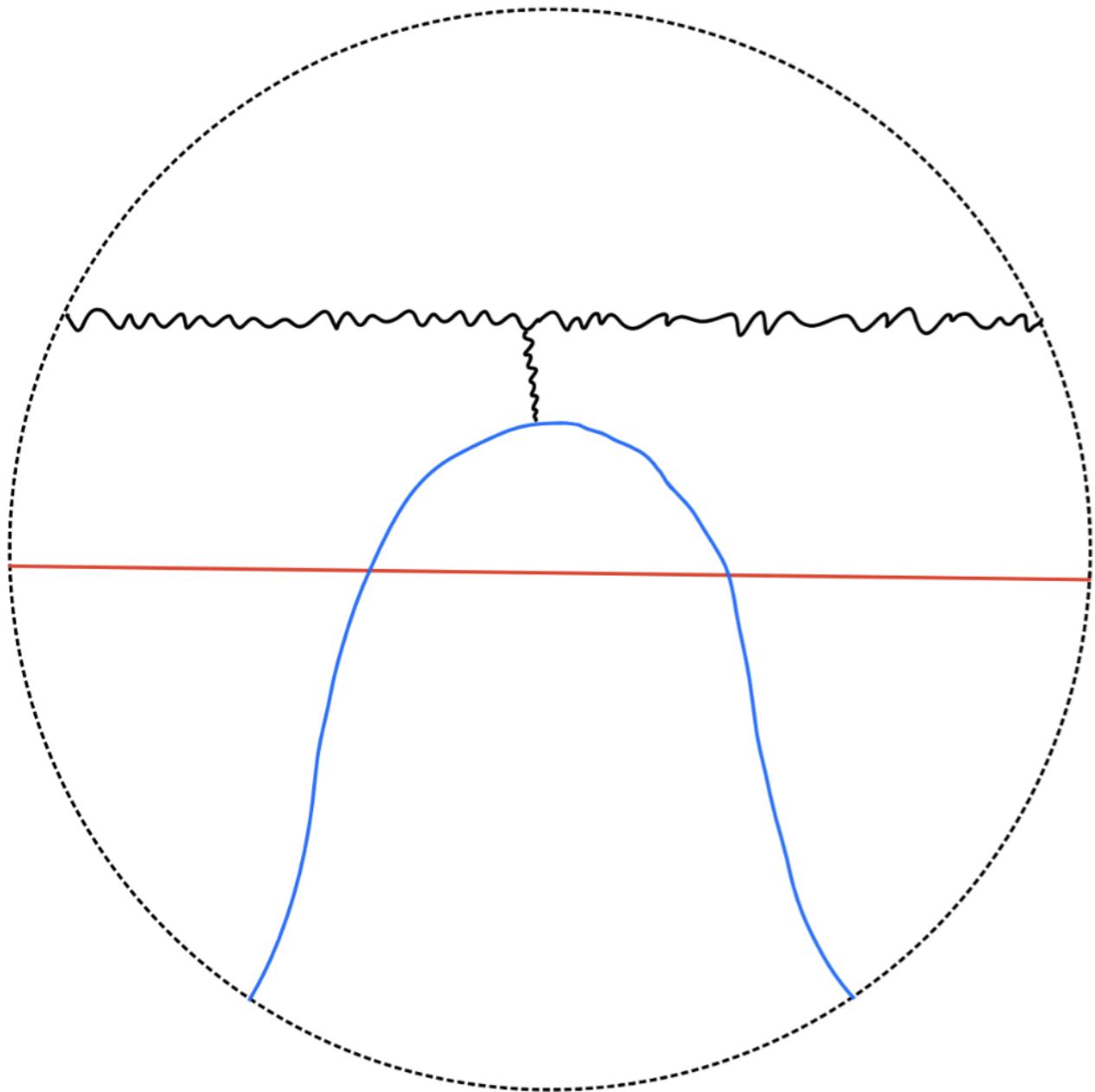


Figure 3.87

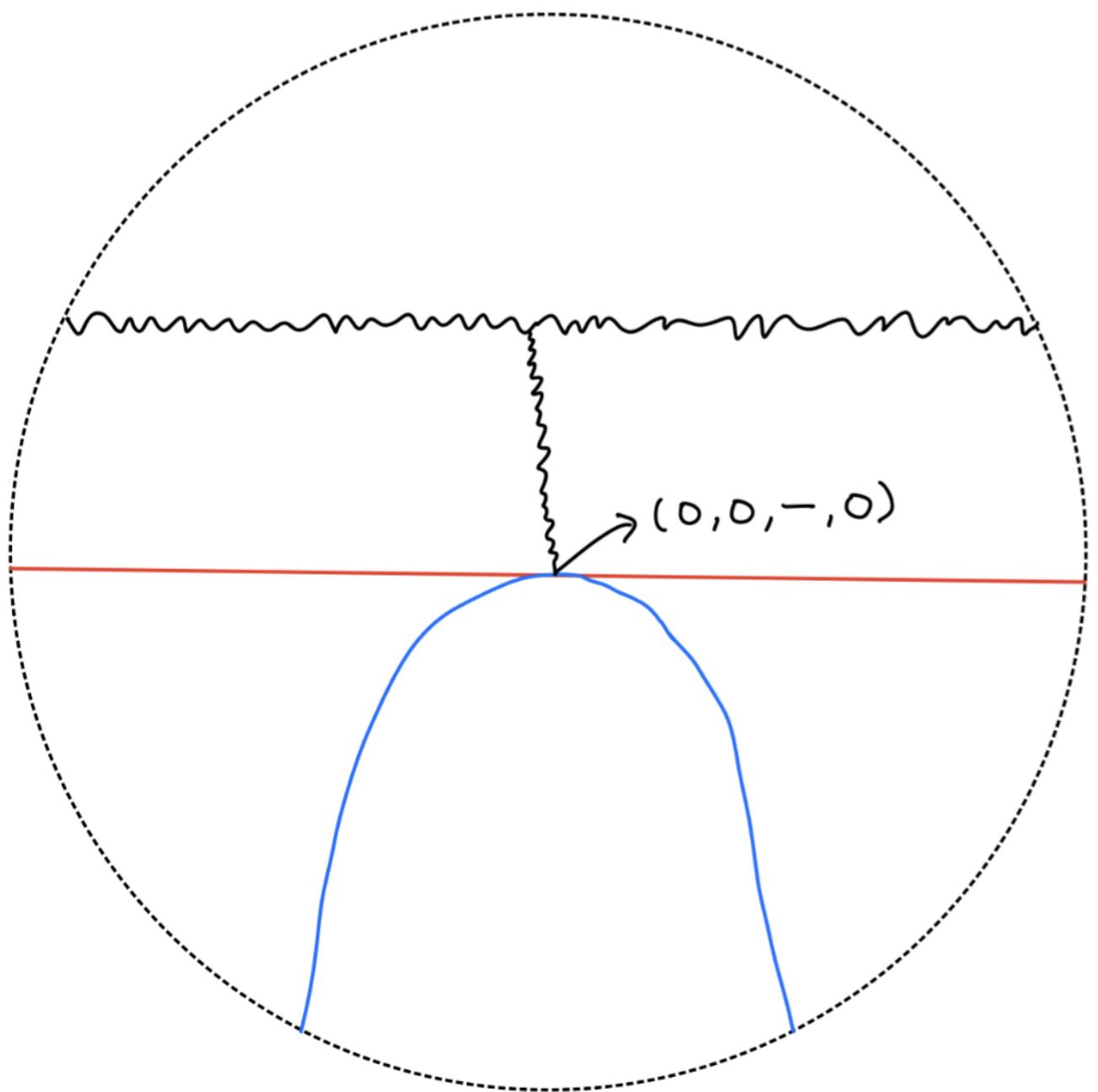


Figure 3.88

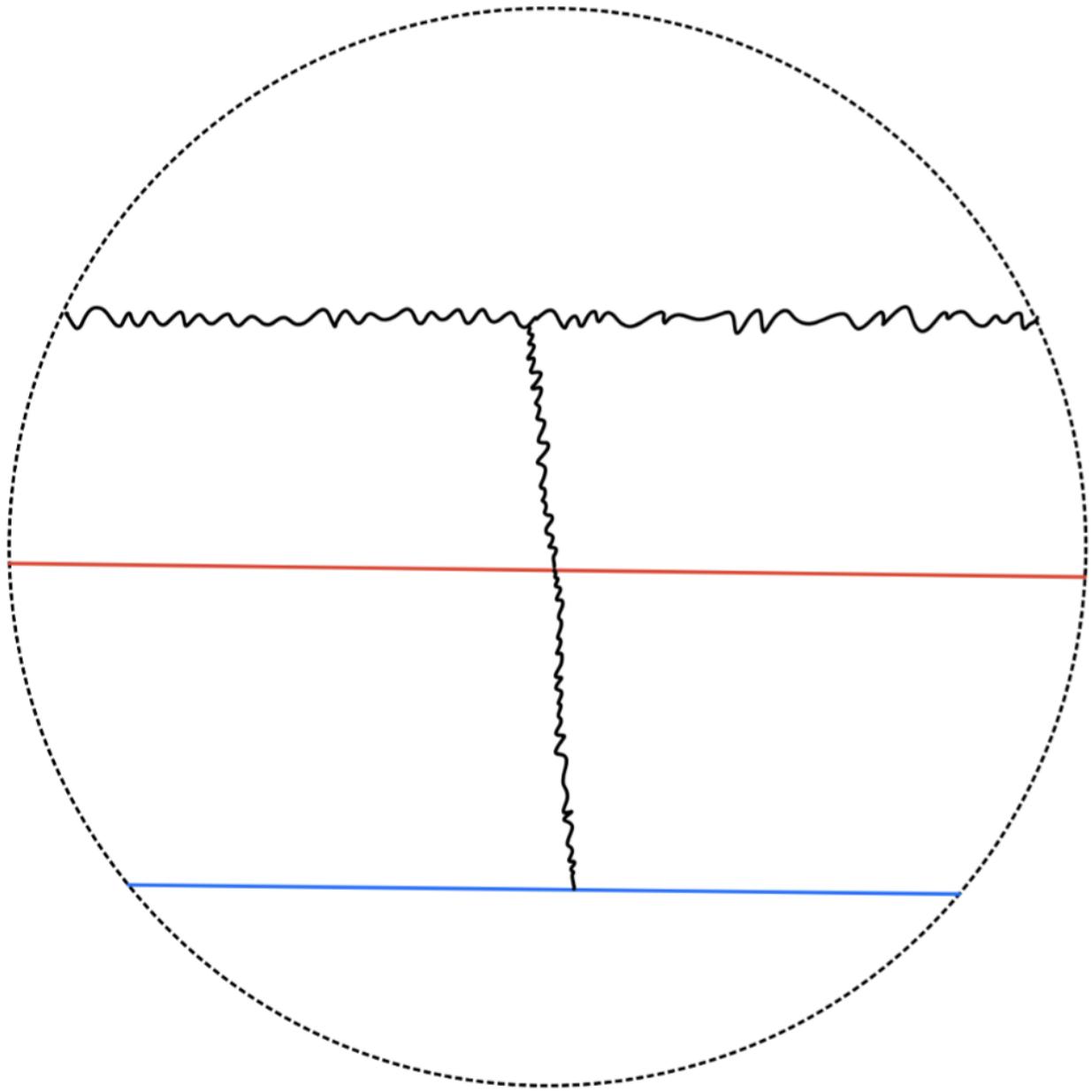


Figure 3.89

- 0 dimensional strata:

$$s_\bullet(0, -, 0, 0), s_\bullet(0, 0, -, 0)$$

Definition 109. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $\text{star}(s)$.

Definition 110. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of s .

Definition 111. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 112. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 113. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 114. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_{\mathcal{S}}}(s_1 \rightarrow s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

This is well-defined because $F_{\mathcal{S}}$ is legible.

Definition 115. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.5.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.5.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

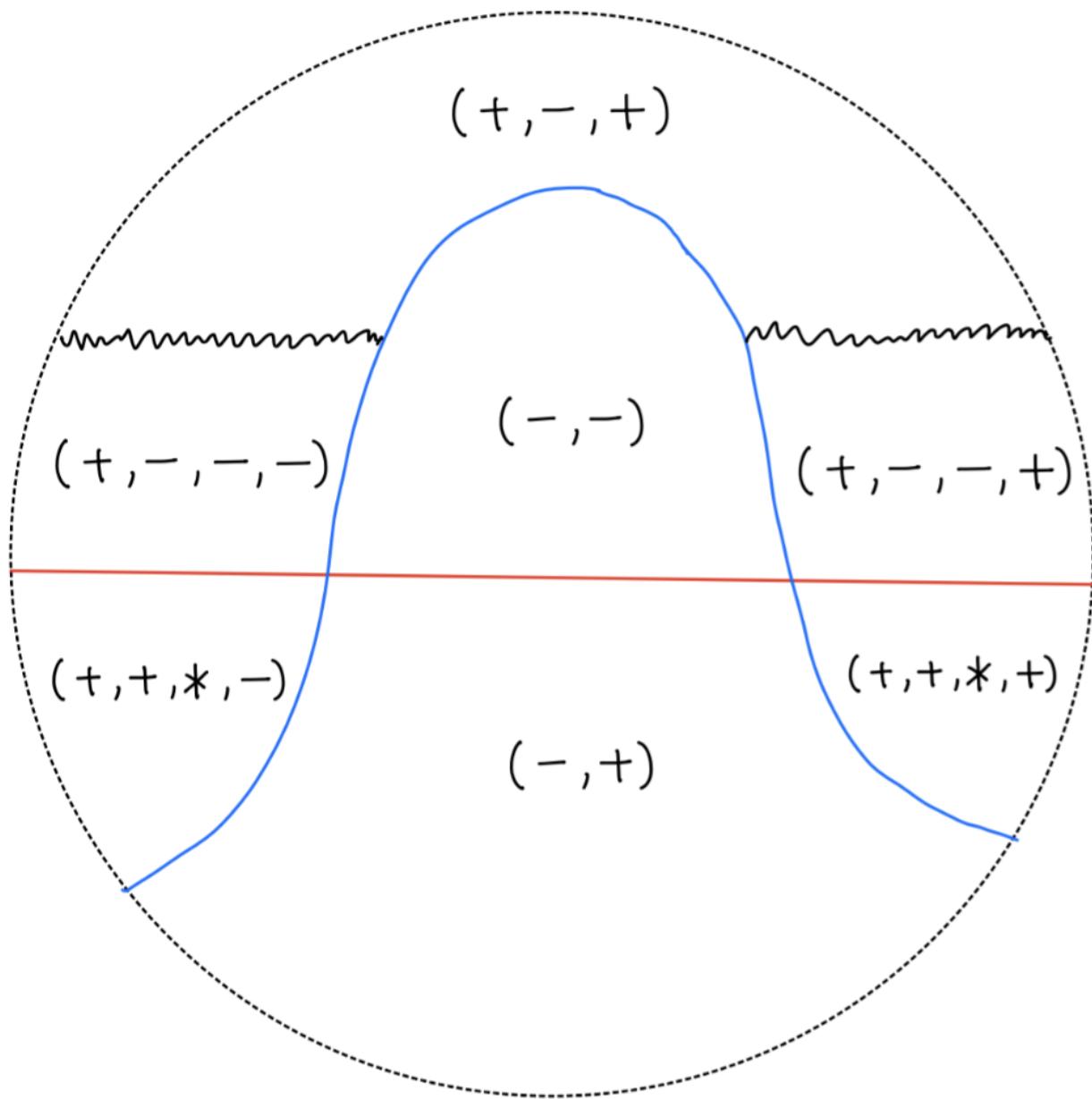


Figure 3.90

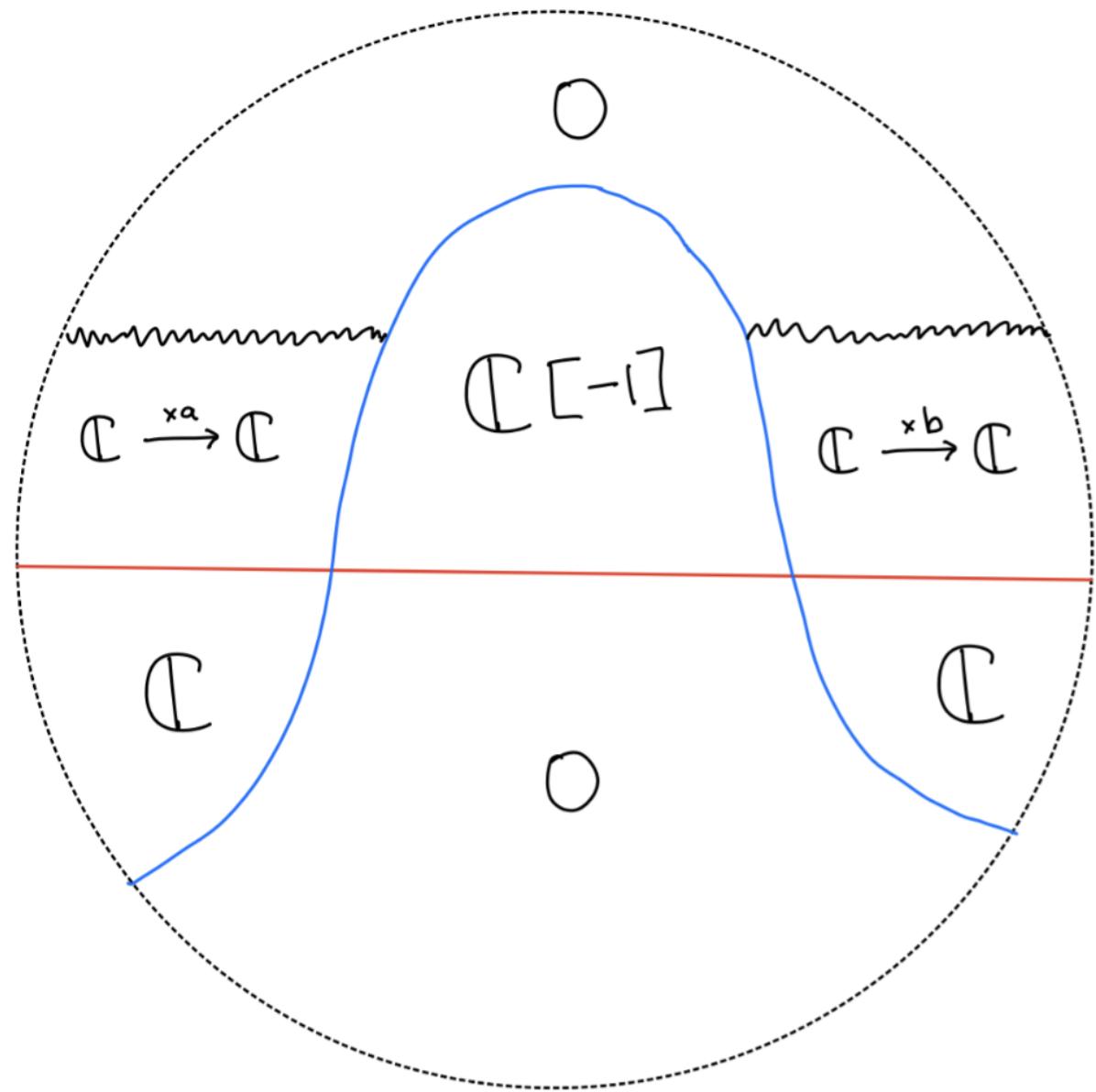


Figure 3.91

- $F_0(-, -) := \mathbb{C}[-1]$
- $F_0(-, +) := 0$
- $F_0(+, -, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(+, -, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_0(+, -, +) := 0$

- $F_0(+, +, *, -) := \mathbb{C}$

- $F_0(+, +, *, +) := \mathbb{C}$

Generalization maps:

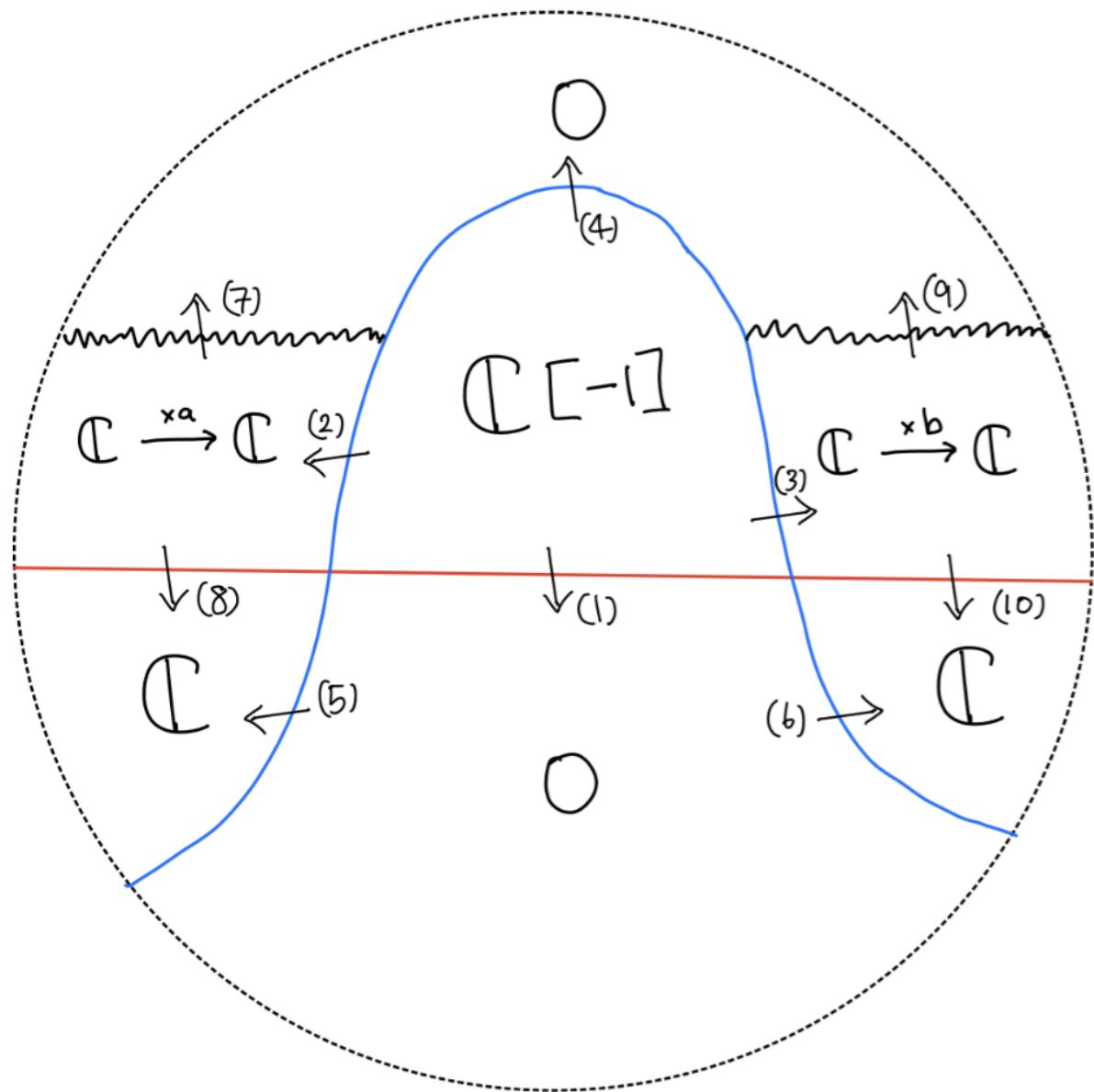


Figure 3.92

$$(1) \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

3.5.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say *cobord*₂, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

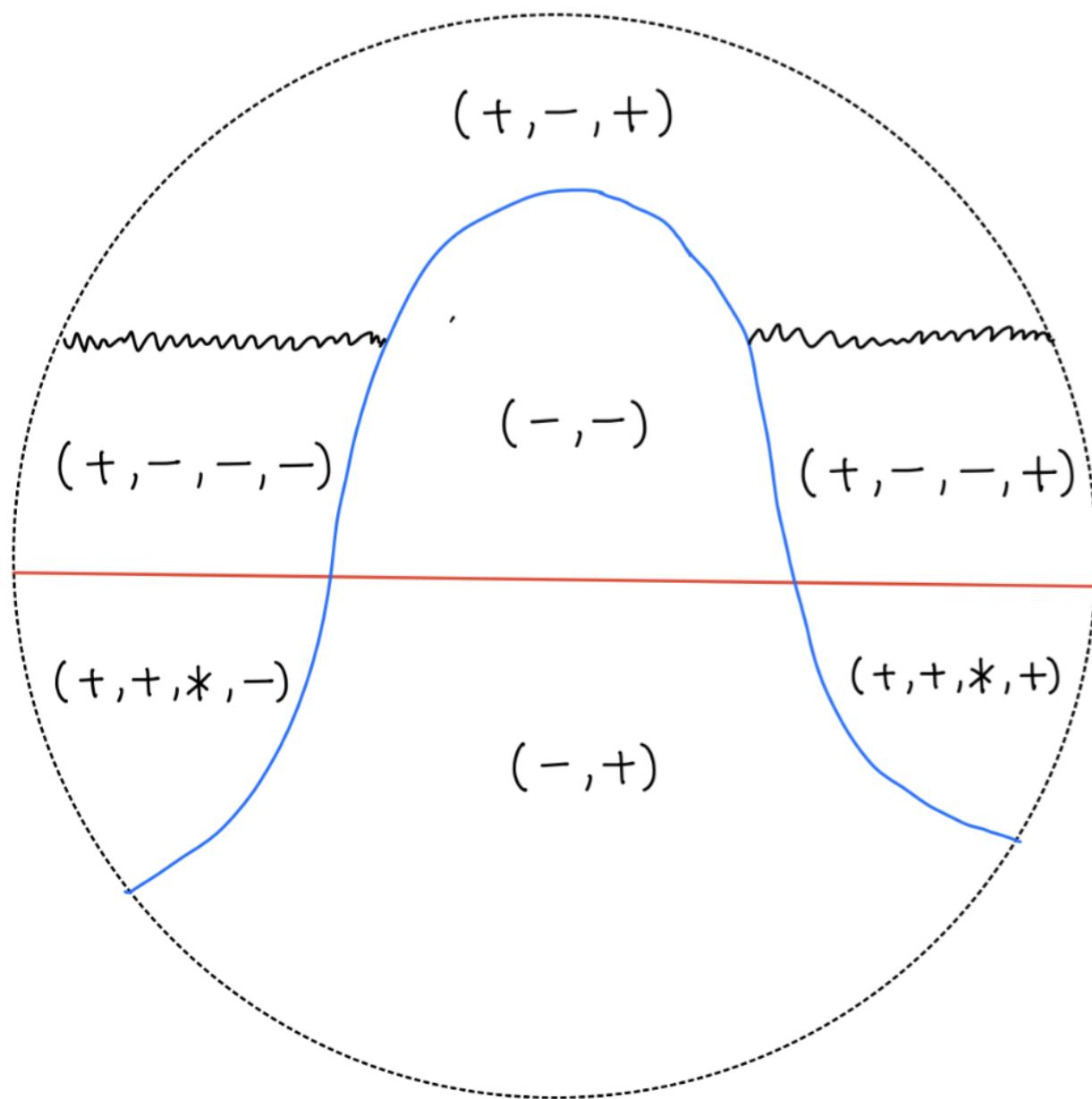


Figure 3.93

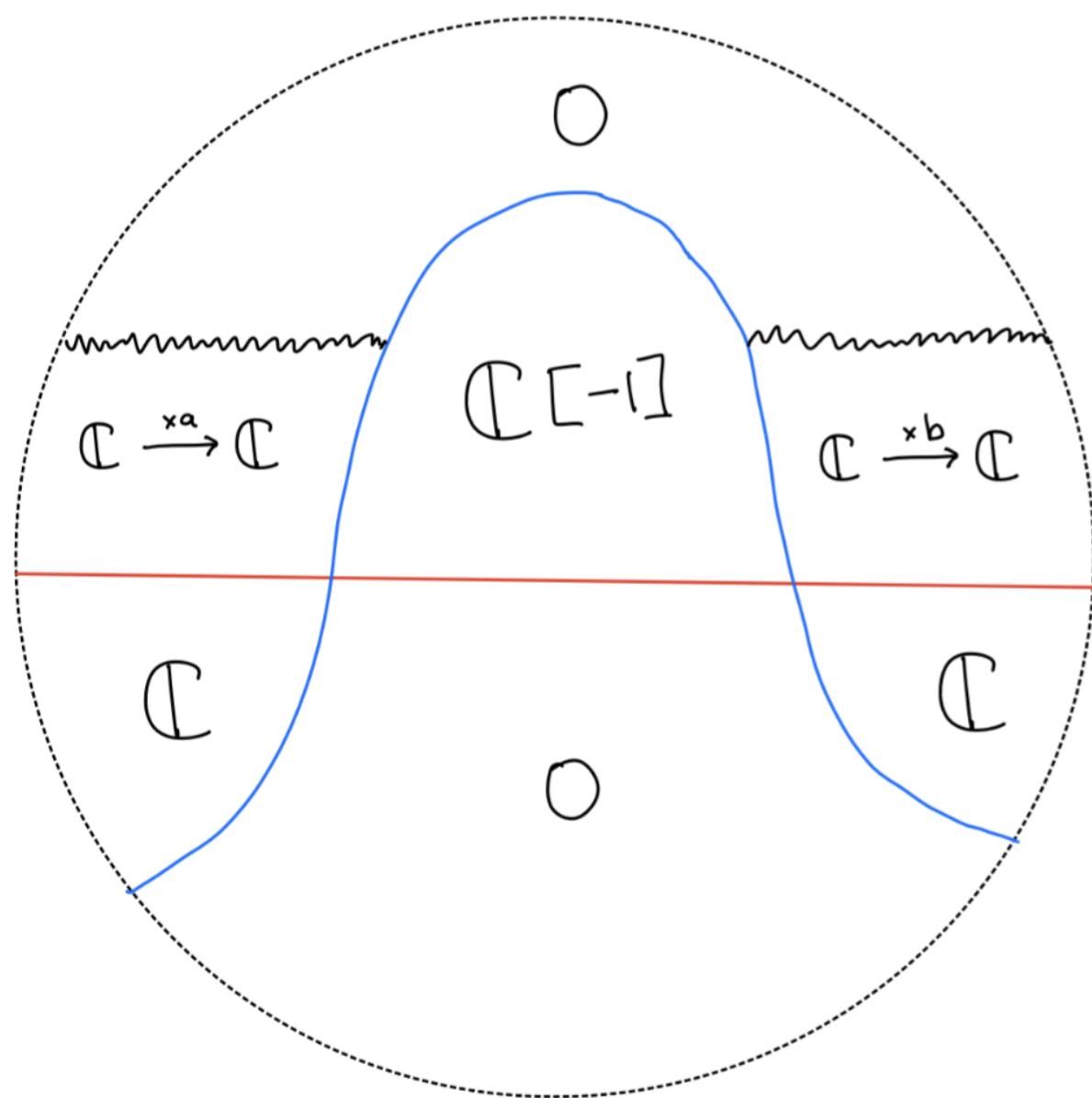


Figure 3.94

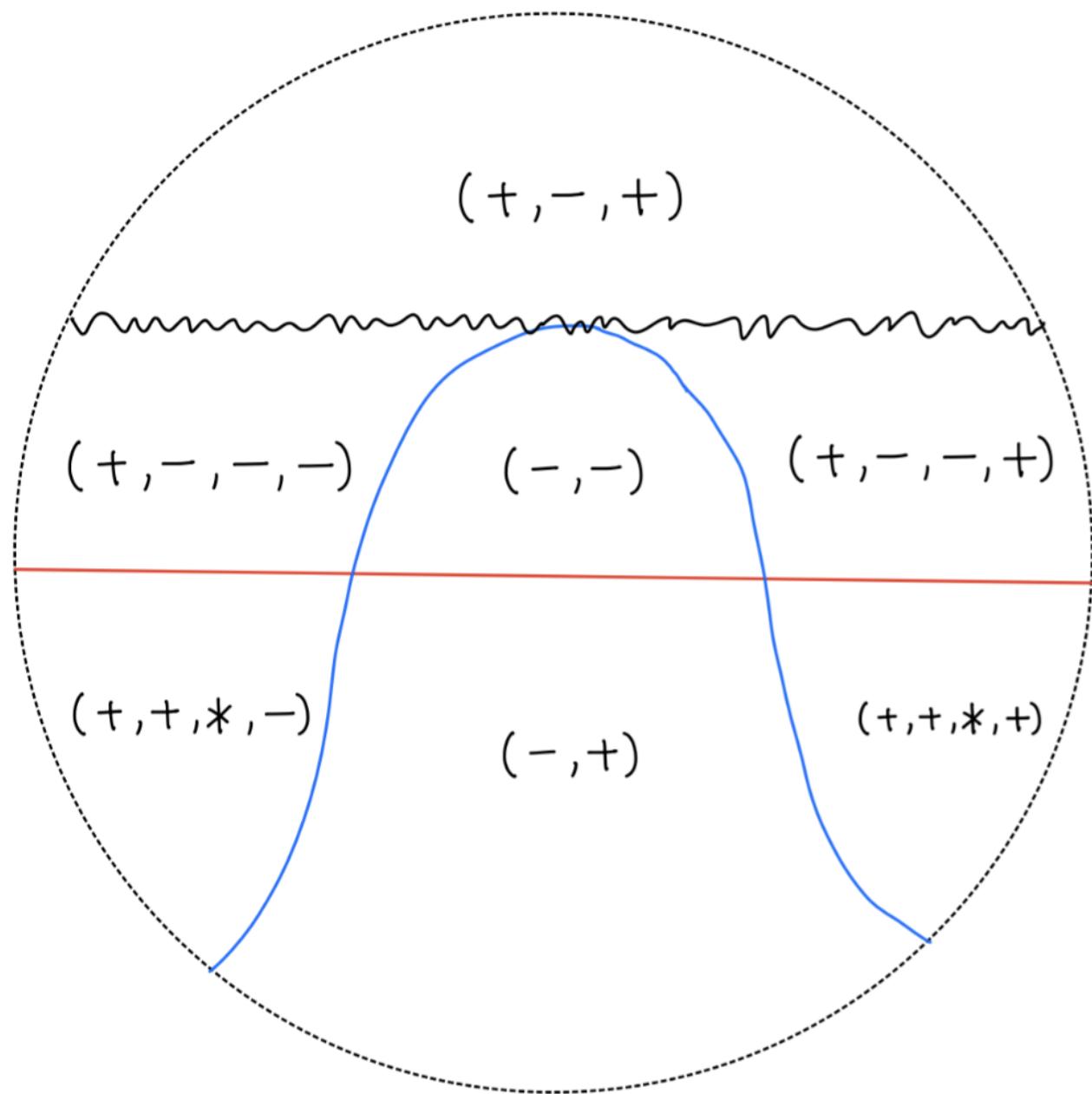


Figure 3.95

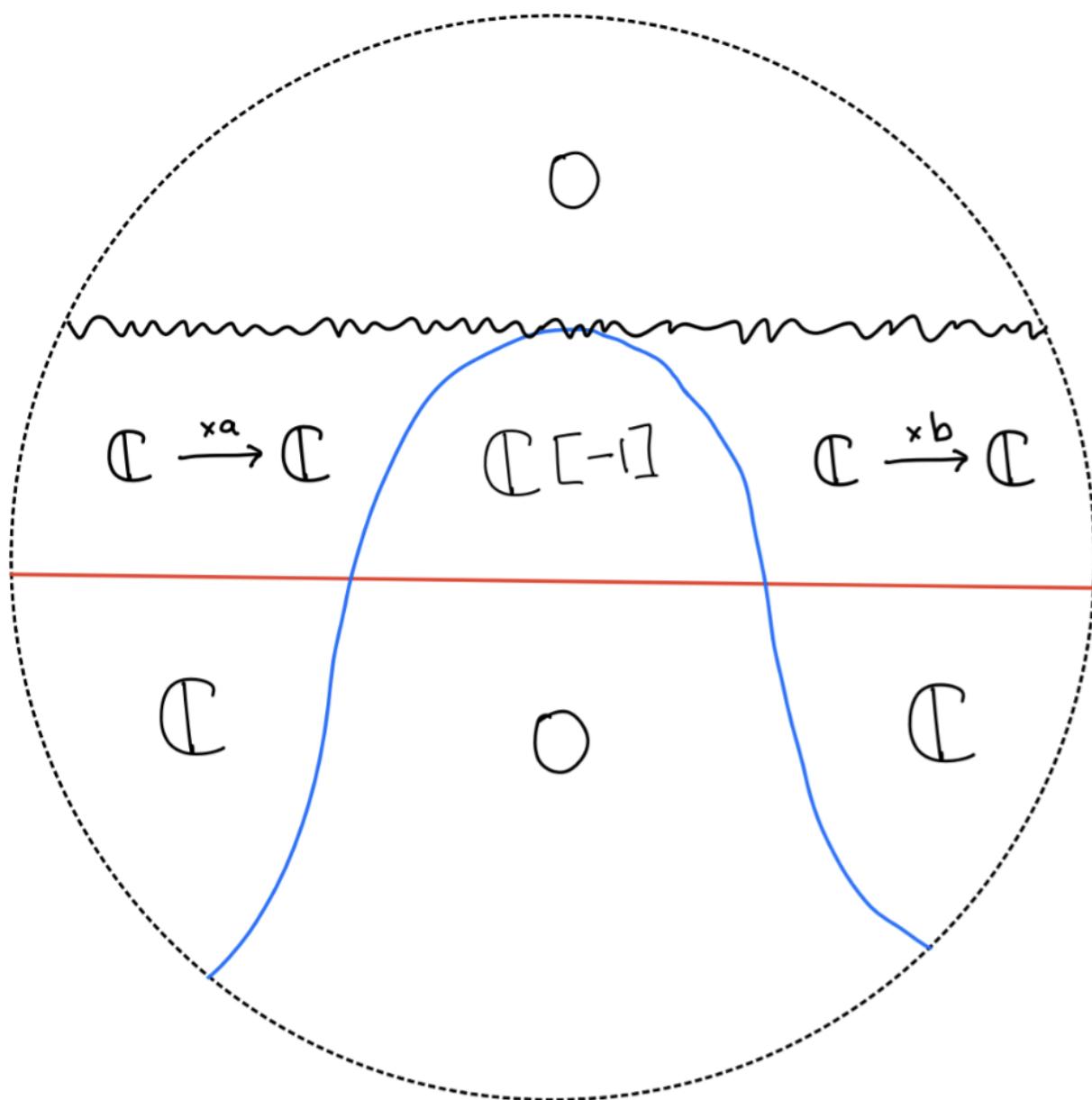


Figure 3.96

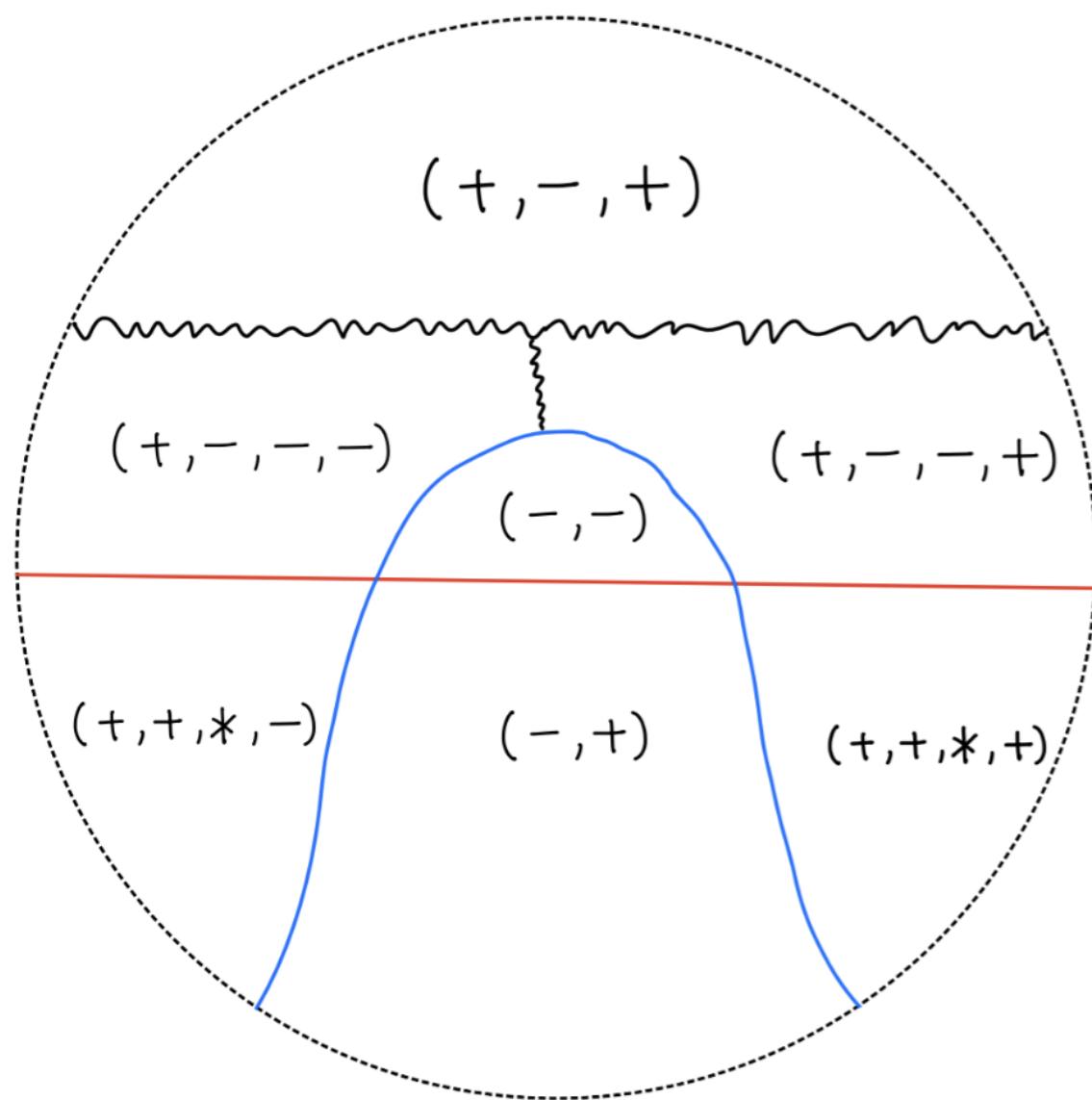


Figure 3.97

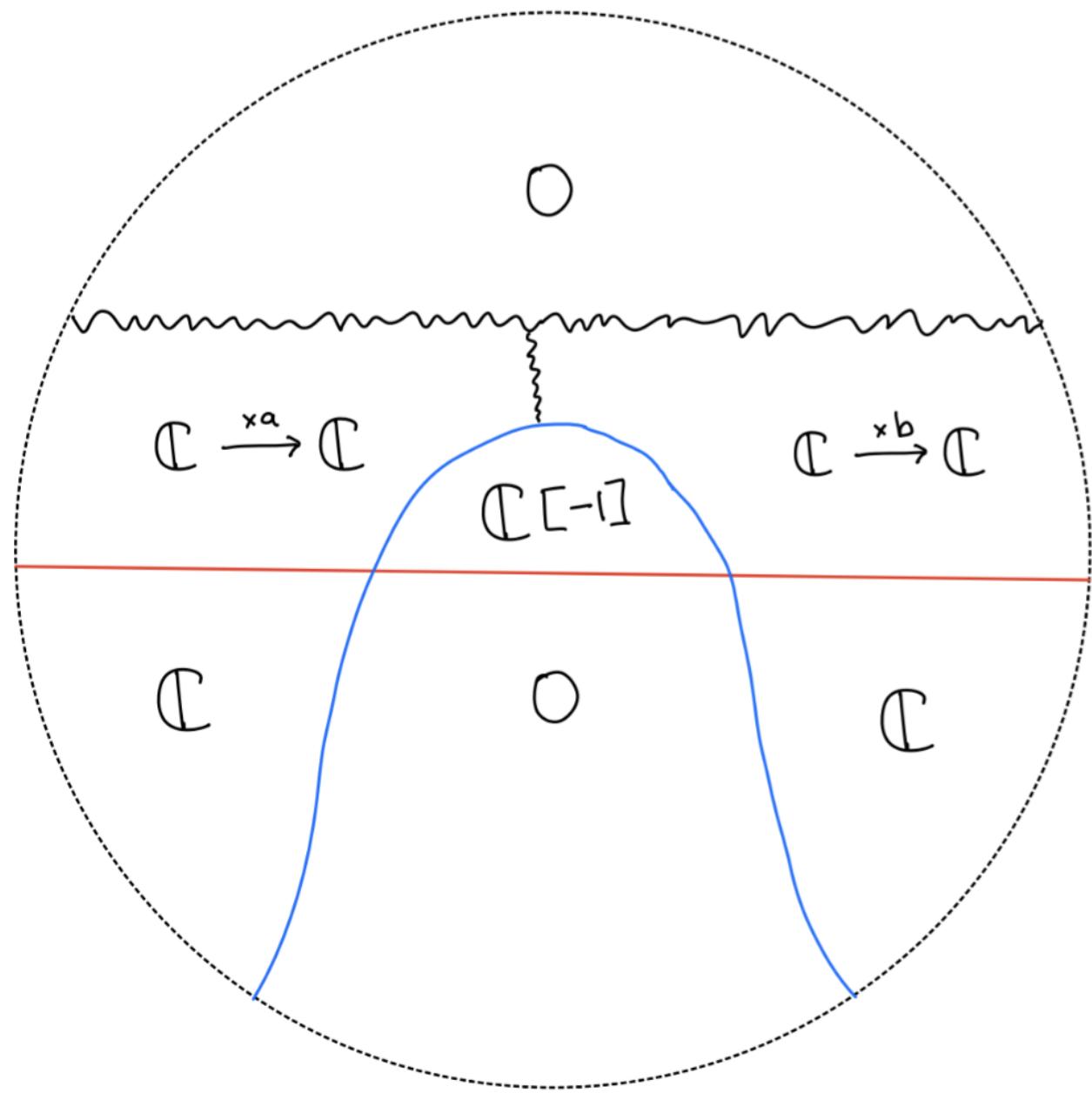


Figure 3.98

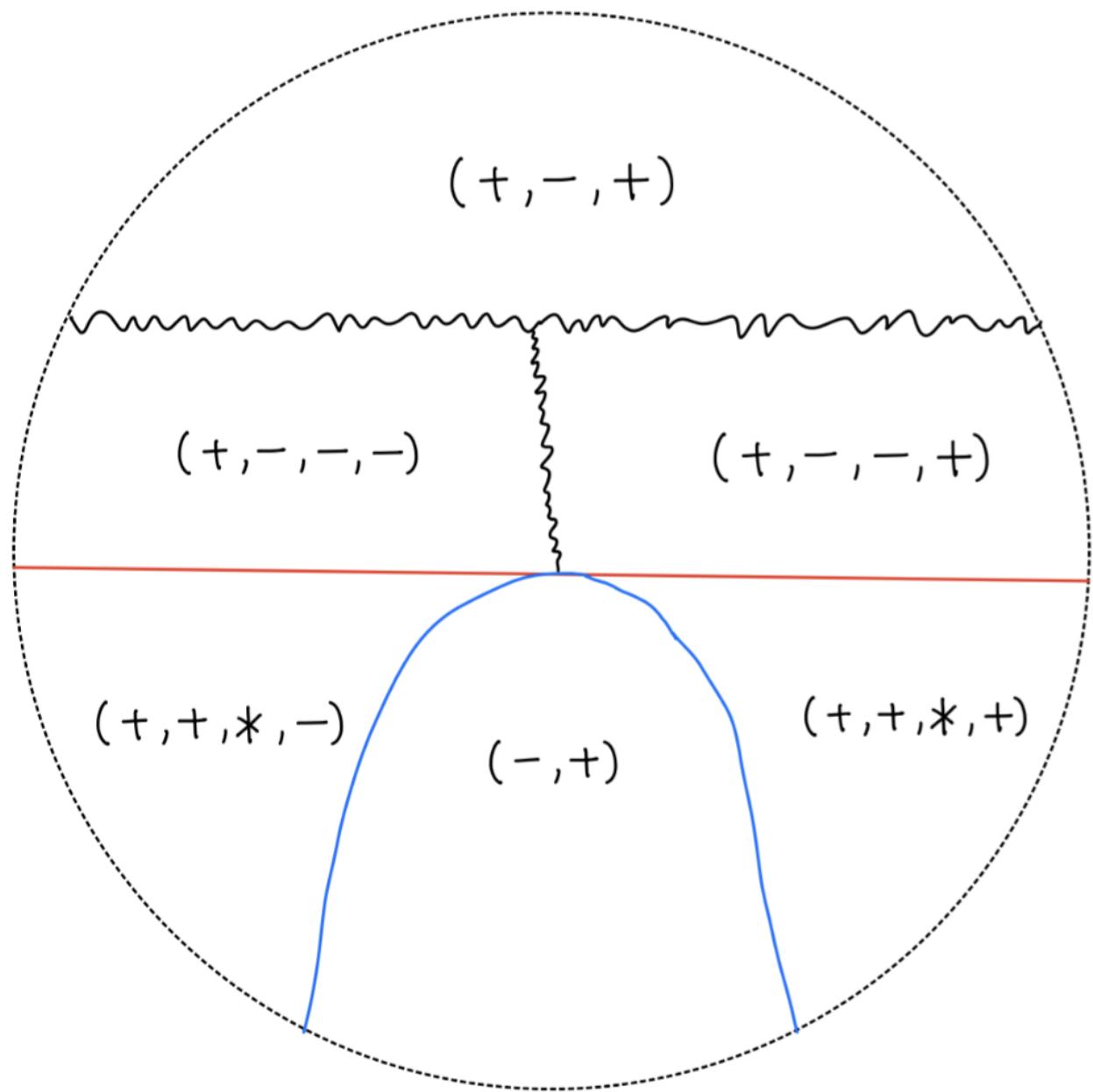


Figure 3.99

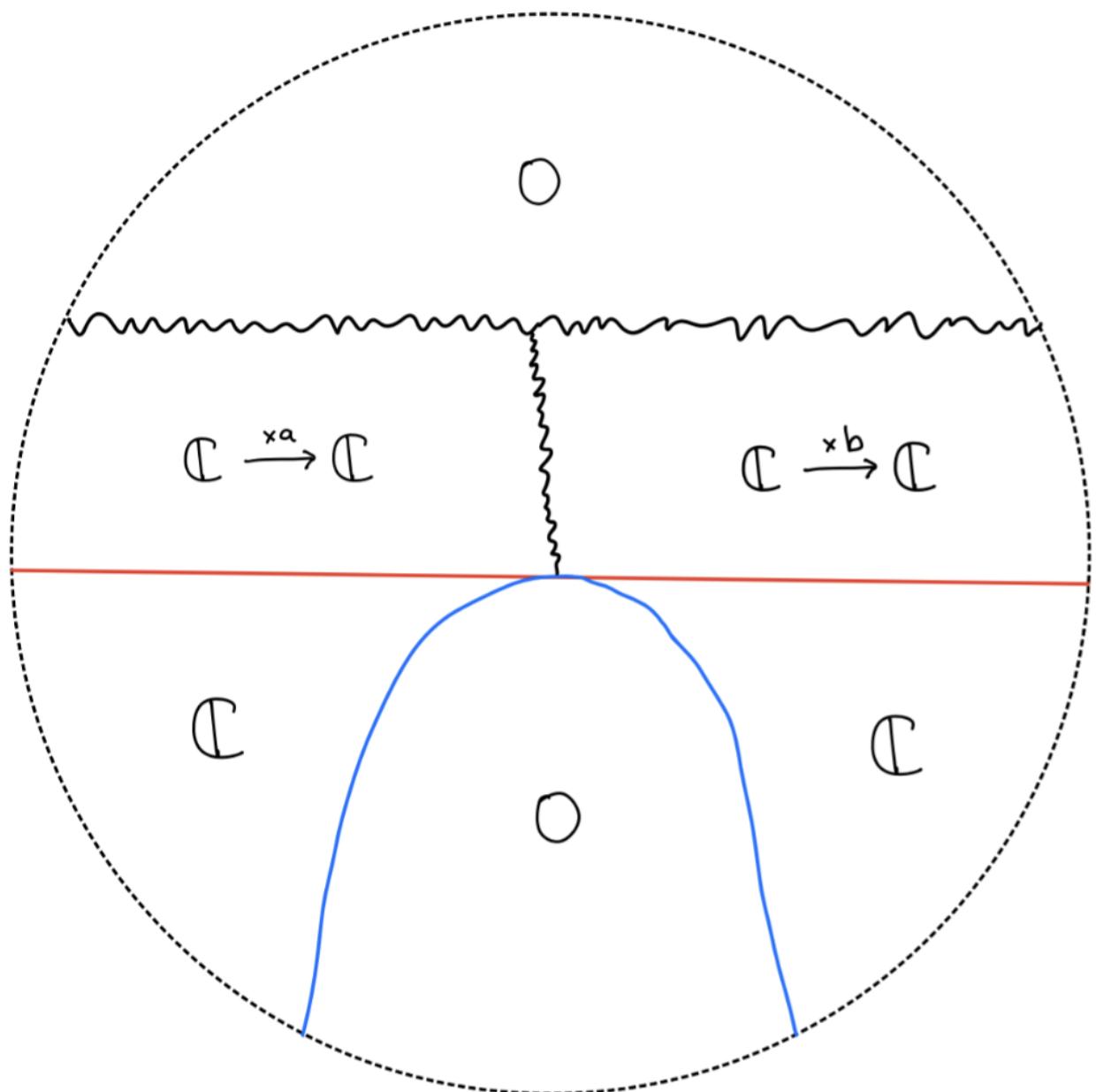


Figure 3.100

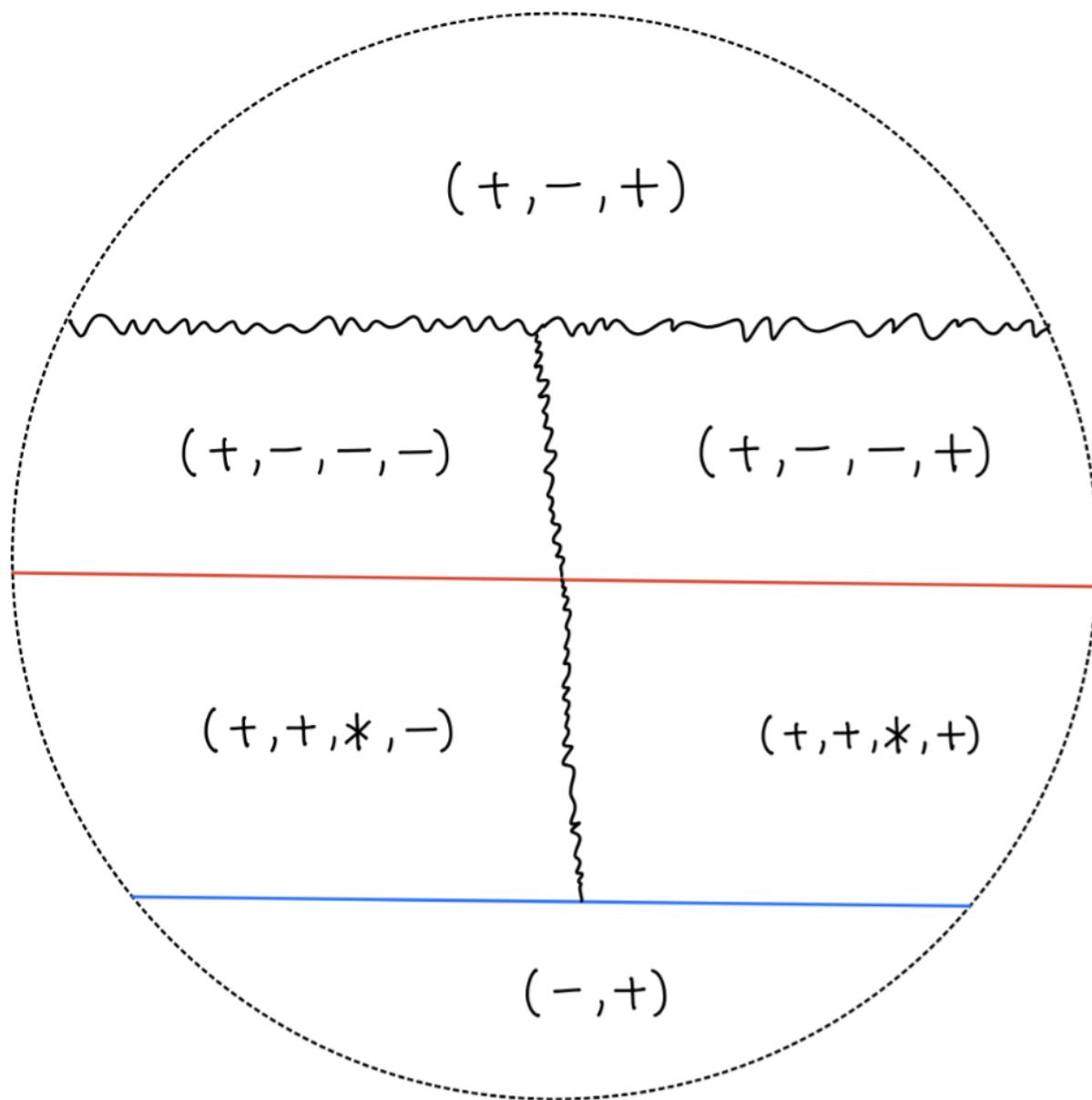


Figure 3.101

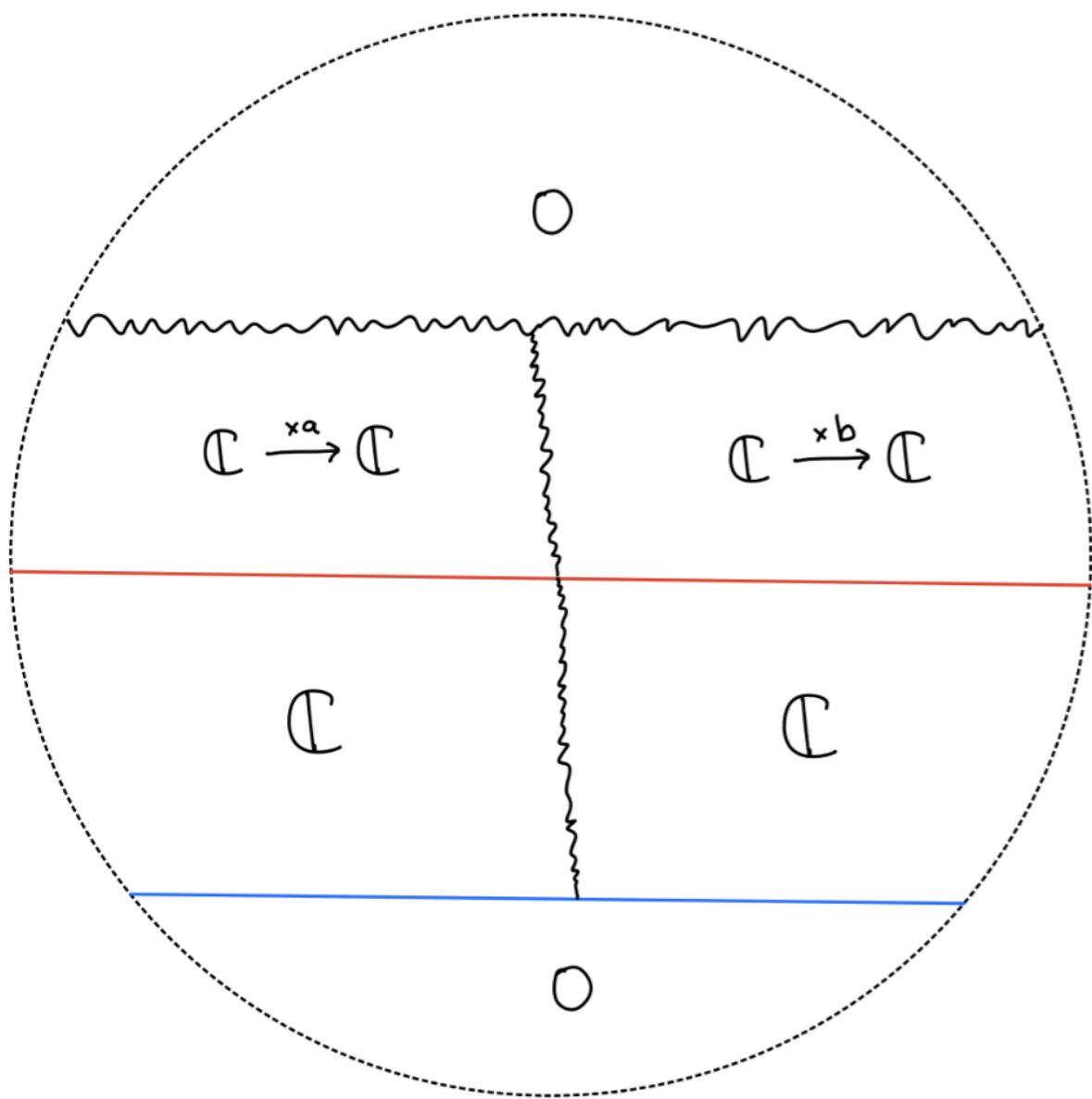


Figure 3.102

- $F_\bullet(-, -) := \mathbb{C}[-1]$
- $F_\bullet(-, +) := 0$
- $F_\bullet(+, -, -, -) := \mathbb{C} \xrightarrow{x^a} \mathbb{C}$
- $F_\bullet(+, -, -, +) := \mathbb{C} \xrightarrow{x^b} \mathbb{C}$
- $F_\bullet(+, -, +) := 0$

- $F_\bullet(+, +, *, -) := \mathbb{C}$

- $F_\bullet(+, +, *, +) := \mathbb{C}$

Generalization maps:

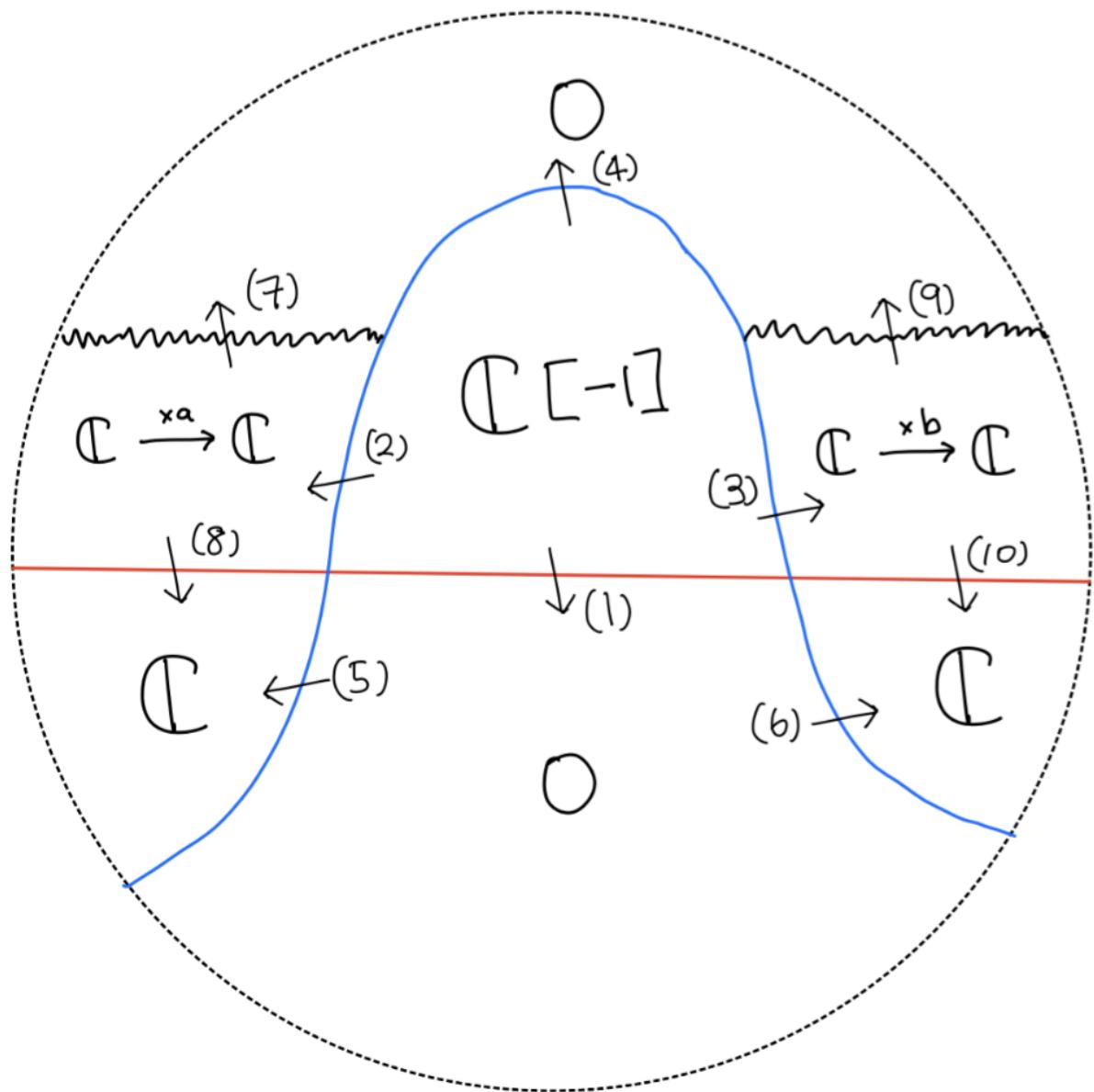


Figure 3.103

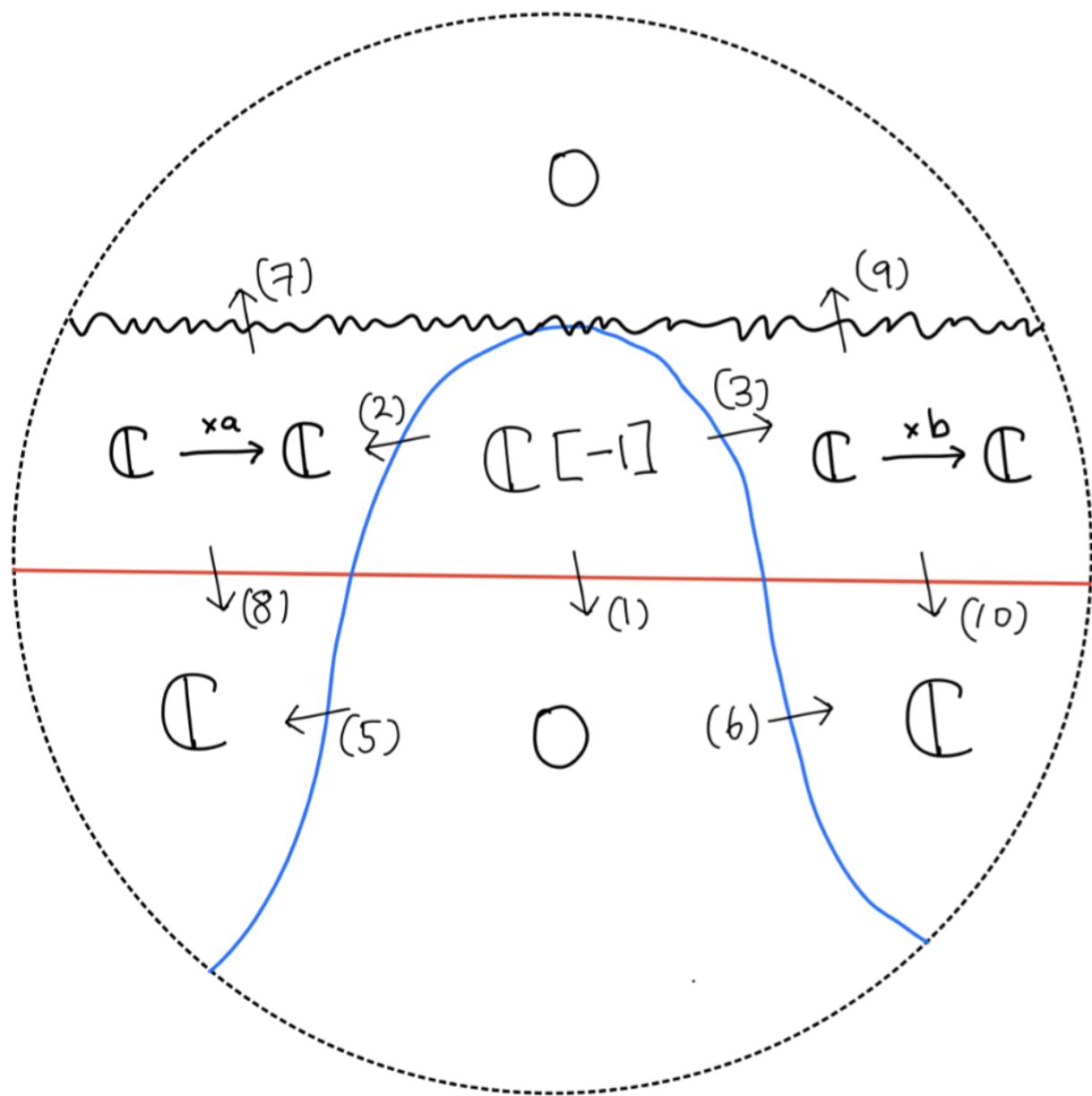


Figure 3.104

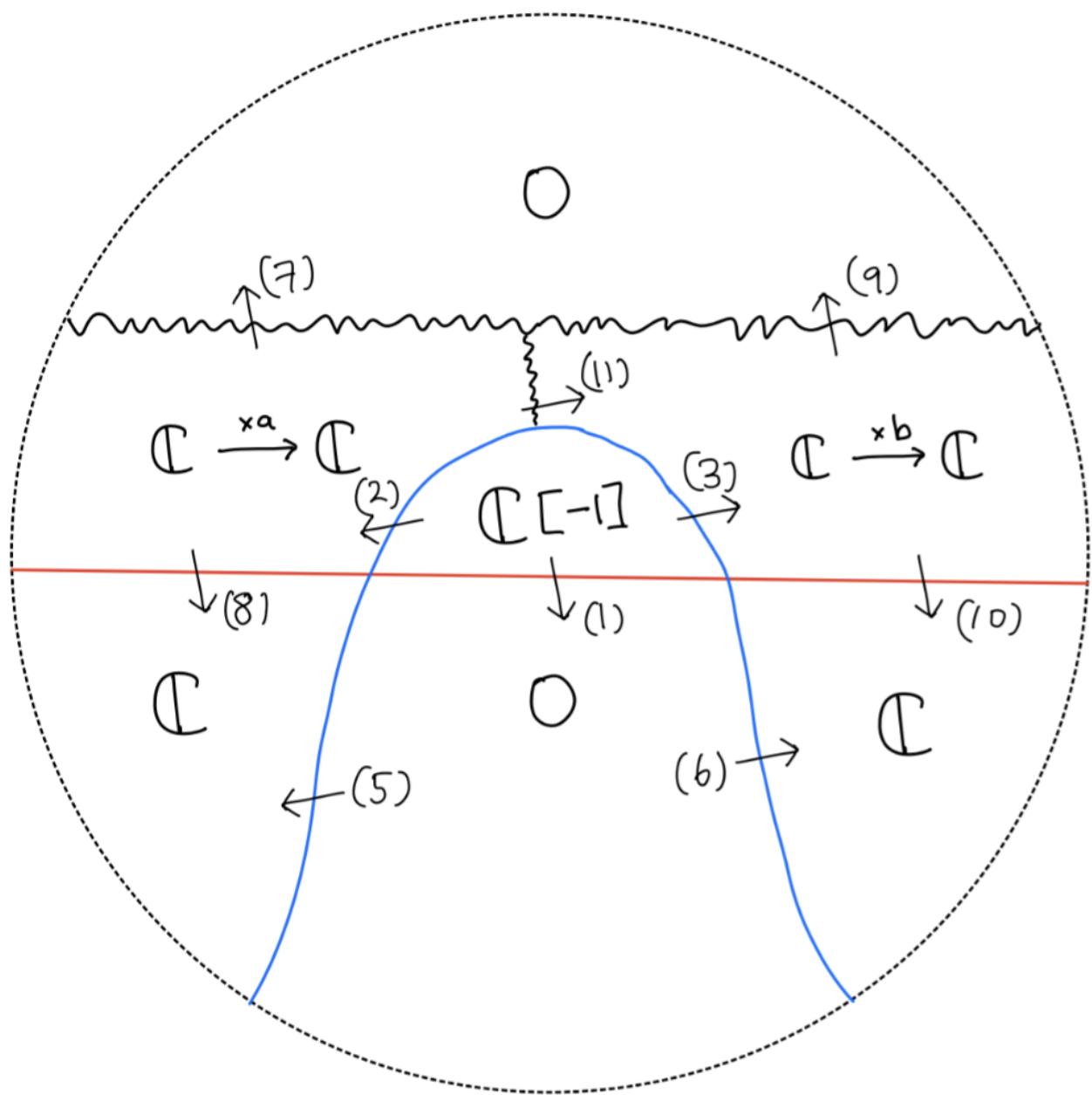


Figure 3.105

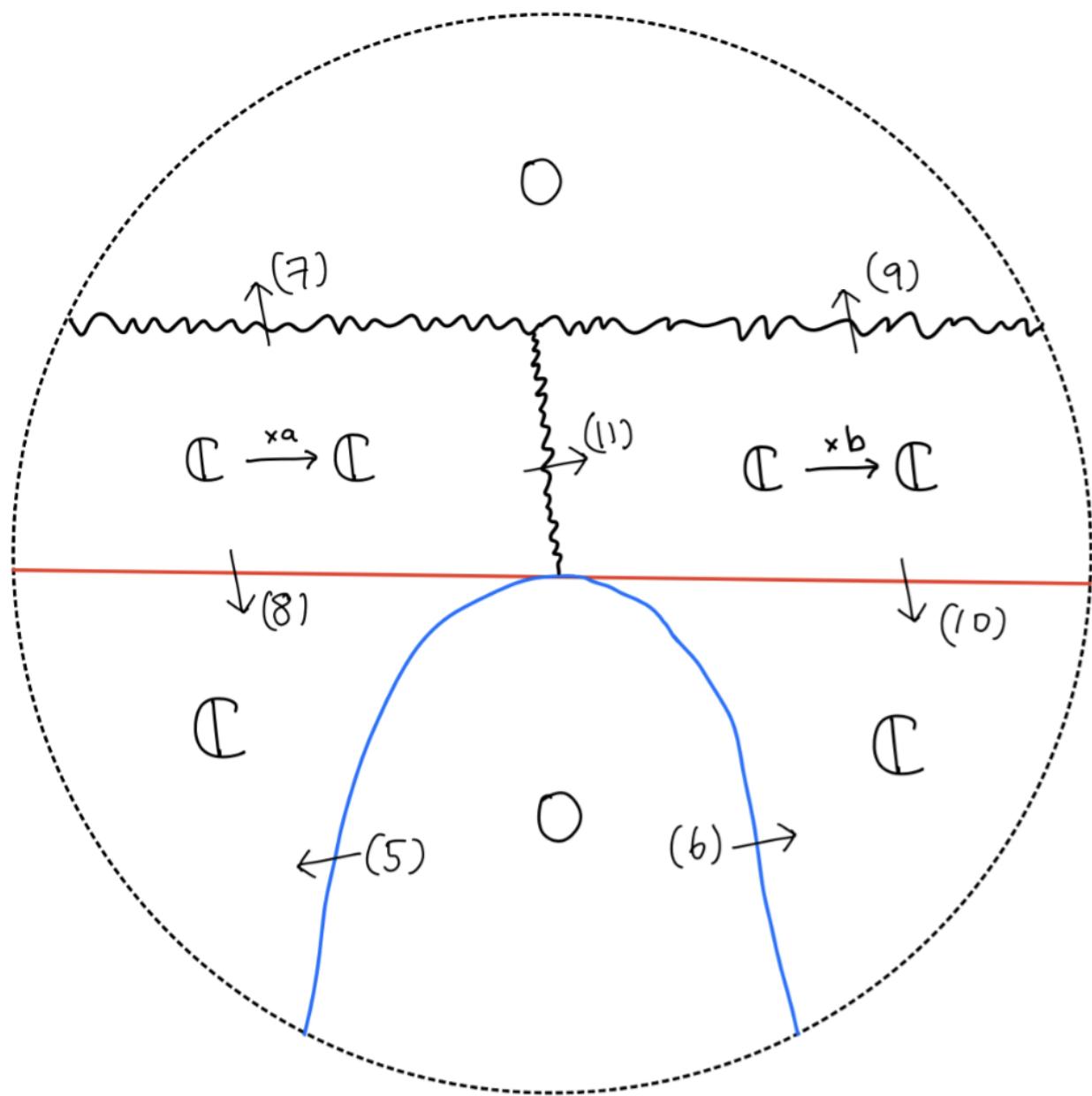


Figure 3.106

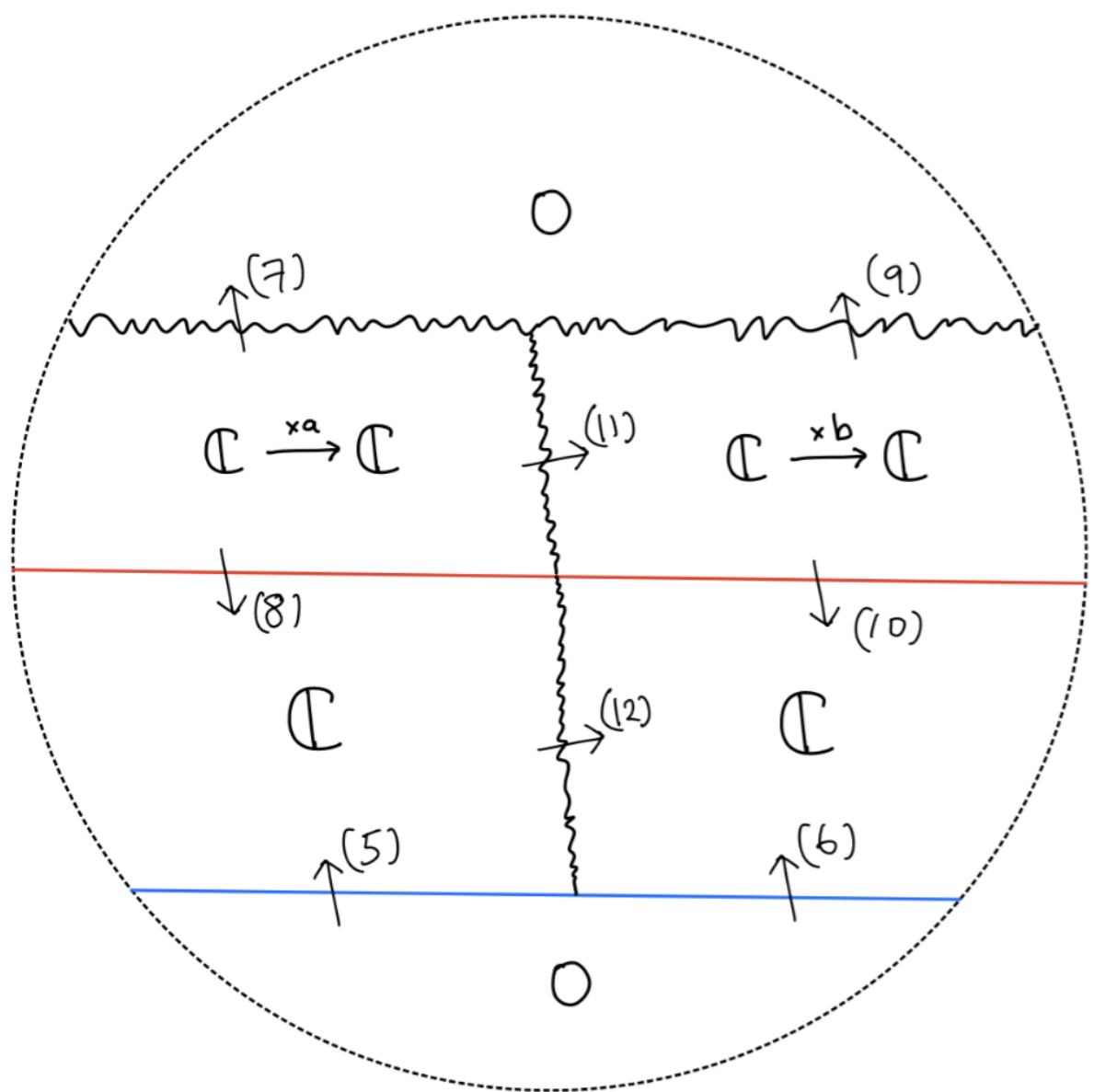


Figure 3.107

$$(1) \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \begin{array}{ccc} C & \xrightarrow{x^1} & C \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & C \end{array}$$

$$\quad \quad \quad \begin{array}{c} \uparrow \\ \times a \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 116. $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 117. we define γ_\bullet to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \longrightarrow & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \longleftarrow & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 118. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M \times [0,1], \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of $M \times [0,1]$.

Since \mathcal{F}^\bullet is constant along the time coordinate on $U^{tc} \times [0,1]$, it is enough to check for the points of $U \times [0,1] \cong D_{r=2} \times [0,1]$. Now consider the following open cover of $D_{r=2} \times [0,1]$

$$\{star(s_\bullet(0, -, 0, 0)), star(s_\bullet(0, 0, -, 0))\}$$

(1) First, let's show that the microlocal stalks of $\mathcal{F}^\bullet|_{star(s_\bullet(0, -, 0, 0))}$ vanishes. Note that there is a diffeomorphism between $star(s_\bullet(0, -, 0, 0))$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

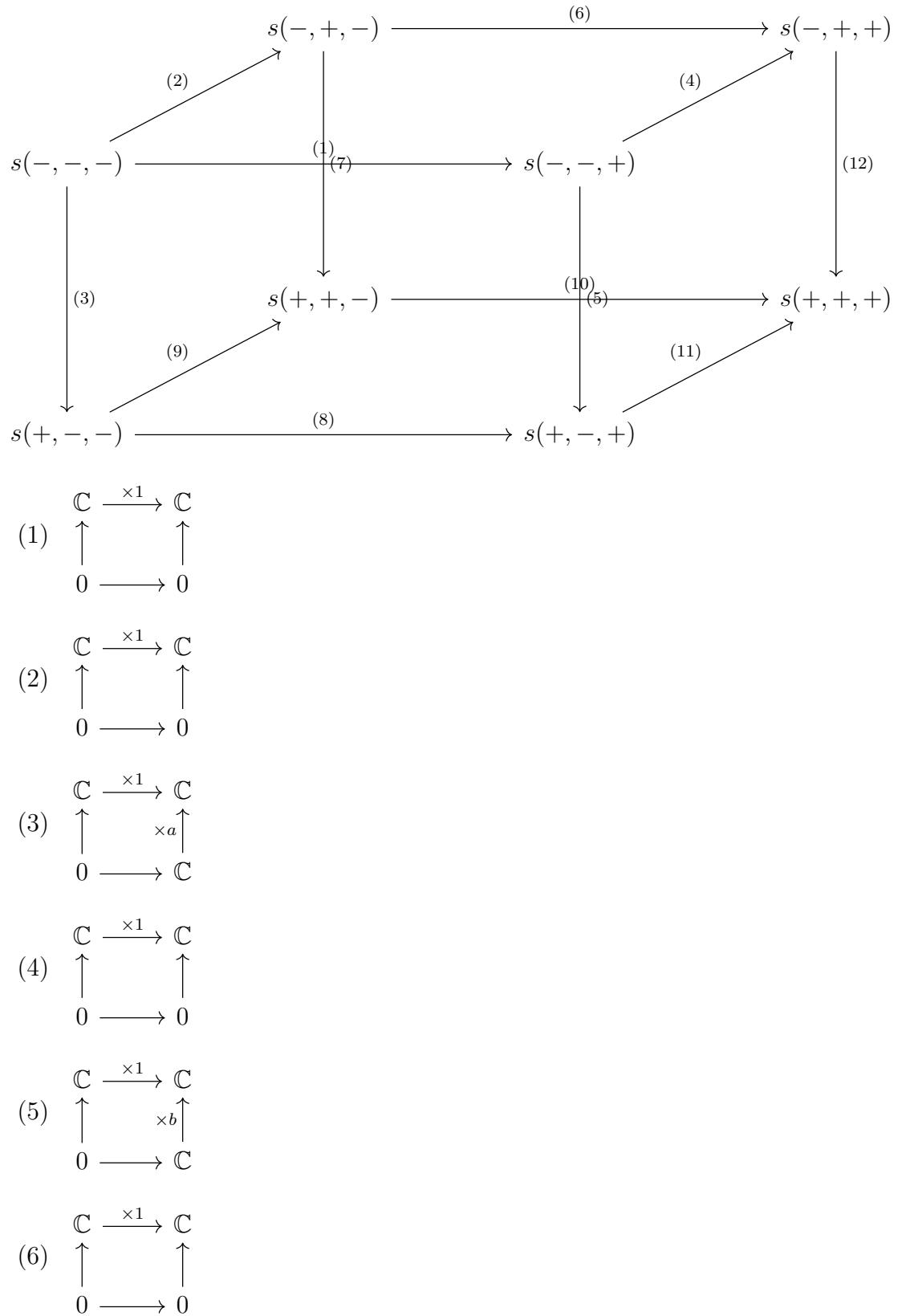
Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-, +, -) := \mathbb{C}[-1]$
- $F^3(-, +, +) := \mathbb{C}[-1]$
- $F^3(+, +, -) := 0$
- $F^3(+, +, +) := 0$

Generalization maps:



$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \end{array}$$

$$(iv) \quad \begin{array}{ccc} F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(v) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \end{array}$$

$$(vi) \quad \begin{array}{ccc} F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-, +, -) & \xrightarrow{s(-, +, 0)} & s(-, +, +) & \\
 & \nearrow s(-, 0, -) & \downarrow & \nearrow s(-, 0, +) & \downarrow s(0, +, +) \\
 s(-, -, -) & \xrightarrow[s(-, -, 0)]{s(0, +, -)} & s(-, -, +) & & \\
 & \downarrow s(0, -, -) & \downarrow & \downarrow s(0, +, +) & \\
 & s(+, +, -) & \xrightarrow[s(+, +, 0)]{s(0, +, +)} & s(+, +, +) & \\
 & \nearrow s(+, 0, -) & \downarrow & \nearrow s(+, 0, +) & \\
 s(+, -, -) & \xrightarrow{s(+, -, 0)} & s(+, -, +) & &
 \end{array}$$

$$\begin{array}{ccccc}
 & \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \\
 \swarrow & \downarrow & & \searrow & \downarrow \\
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & & 0 \\
 \downarrow & \downarrow & & \downarrow & \downarrow \\
 & 0 & \longrightarrow & 0 & \\
 \searrow & \downarrow & & \swarrow & \downarrow \\
 \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times b} \mathbb{C} & &
 \end{array}$$

Note that $\begin{array}{c} \mathbb{C} \xrightarrow{\times 1} \mathbb{C} \\ \uparrow \\ 0 \longrightarrow 0 \end{array}$, $\begin{array}{c} \mathbb{C} \xrightarrow{\times 1} \mathbb{C} \\ \uparrow \\ 0 \longrightarrow 0 \end{array}$, $\begin{array}{c} \mathbb{C} \xrightarrow{\times 1} \mathbb{C} \\ \uparrow \\ \mathbb{C} \xrightarrow{\times ab^{-1}} \mathbb{C} \end{array}$, $\begin{array}{c} 0 \longrightarrow 0 \\ \uparrow \\ 0 \longrightarrow 0 \end{array}$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of $\mathcal{F}^\bullet|_{\text{star}(s_\bullet(0,0,-,0))}$ vanishes. Note that there is a diffeomorphism between $\text{star}(s_\bullet(0,0,-,0))$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-, +, -) := 0$
- $F^3(-, +, +) := 0$
- $F^3(+, +, -) := \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}$

Generalization maps:

$$\begin{array}{ccccc}
 & s(-, +, -) & \xrightarrow{(6)} & s(-, +, +) & \\
 & \nearrow (2) & \downarrow (1) & \searrow (4) & \downarrow (12) \\
 s(-, -, -) & \xrightarrow{(7)} & s(-, -, +) & & \\
 & \downarrow (3) & & \downarrow (10) & \downarrow (5) \\
 & s(+, +, -) & \xrightarrow{(8)} & s(+, +, +) & \\
 & \nearrow (9) & & \searrow (11) & \\
 s(+, -, -) & \xrightarrow{(1)} & s(+, -, +) & &
 \end{array}$$

$\mathbb{C} \xrightarrow{\times 1} \mathbb{C}$
 (1) \uparrow \uparrow
 $0 \longrightarrow 0$

$\mathbb{C} \longrightarrow 0$
 (2) \uparrow \uparrow
 $0 \longrightarrow 0$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{ab^{-1}} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, +, -)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \longrightarrow \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, +) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \xrightarrow{\times b} \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{ab^{-1}} & \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & 0
 \end{array}$$

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \xrightarrow{(\times 1, \times ab^{-1})} \mathbb{C} \xrightarrow{\times b} \mathbb{C} \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \longrightarrow & \mathbb{C}
 \end{array}$$

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-,+,-) & \xrightarrow{s(-,+0)} & s(-,+,+) \\
 s(-,0,-) \nearrow & \downarrow & & \searrow s(-,0+) \\
 s(-,-,-) & \xrightarrow[s(-,-,0)]{s(0,+,-)} & s(-,-,+) & & \downarrow s(0,+,+) \\
 \downarrow s(0,-,-) & & & & \downarrow \\
 & s(+,+,-) & \xrightarrow[s(+,+0)]{s(0,+,-)} & s(+,+,+) \\
 s(+,0,-) \nearrow & \downarrow & \searrow s(+,0+) & & \\
 s(+,-,-) & \xrightarrow{s(+,-0)} & s(+,-,+) & &
 \end{array}
 =
 \begin{array}{ccccc}
 & 0 & \longrightarrow & 0 & \\
 \downarrow & \downarrow & & \downarrow & \\
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \longrightarrow & \mathbb{C} \\
 \downarrow & \downarrow & & \downarrow & \\
 & \mathbb{C} & \longrightarrow & \mathbb{C} & \\
 \downarrow & \downarrow & & \downarrow & \\
 \mathbb{C} \xrightarrow{a} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{b} \mathbb{C} & &
 \end{array}$$

Note that $\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} & , & 0 \longrightarrow 0 \\ \uparrow & & \uparrow & , & \uparrow \\ 0 & \longrightarrow & 0 & , & 0 \longrightarrow 0 \end{array}$, $\begin{array}{ccccc} 0 & \longrightarrow & 0 & , & \times a \uparrow \\ \uparrow & & \uparrow & , & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} & , & \mathbb{C} \xrightarrow{\times ab^{-1}} \mathbb{C} \end{array}$, $\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} & , & 0 \longrightarrow 0 \\ \uparrow & & \uparrow & , & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} & , & \mathbb{C} \xrightarrow{\times ab^{-1}} \mathbb{C} \end{array}$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete. \square

3.5.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the $cobord_2$. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphism $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

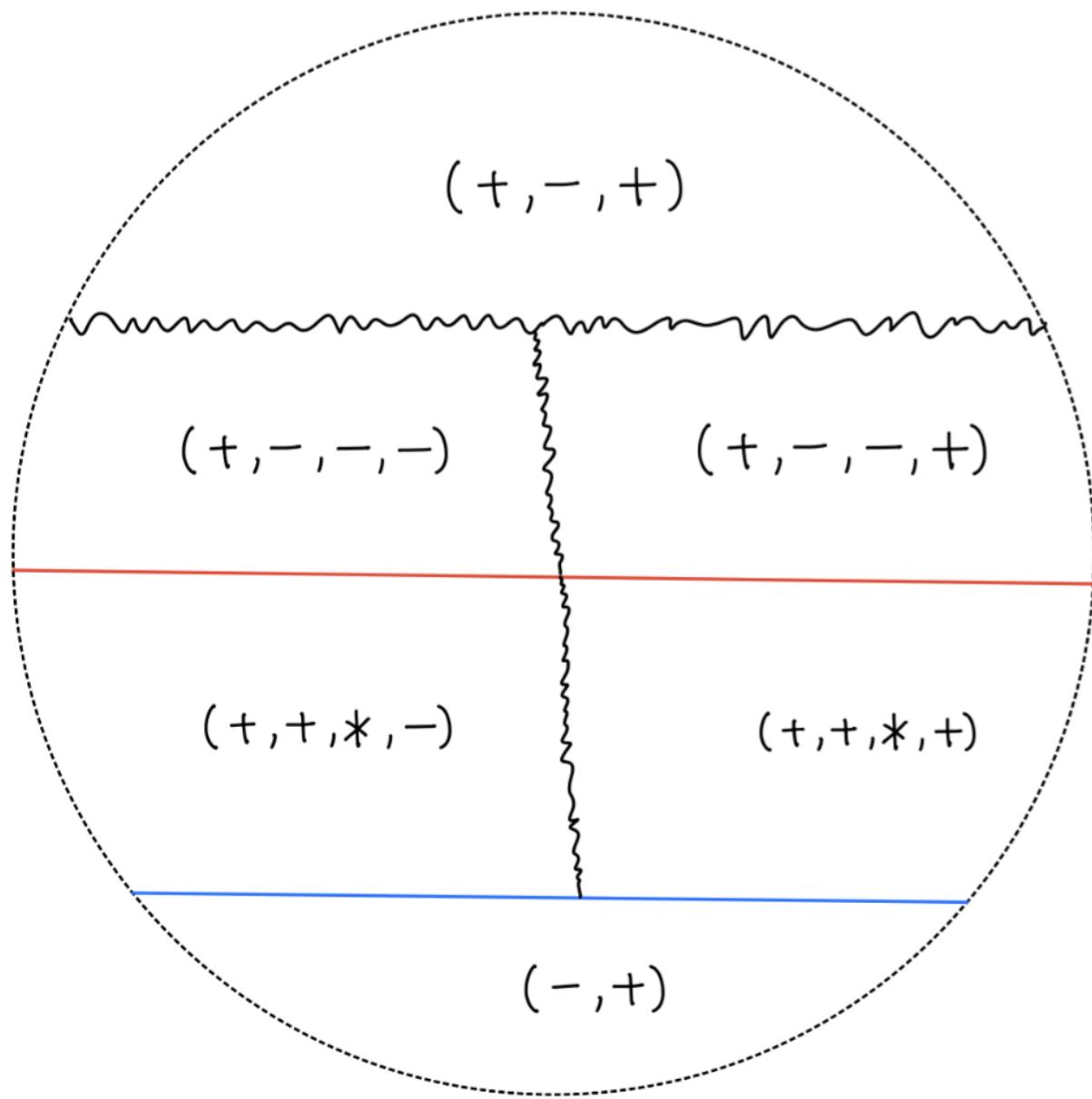


Figure 3.108

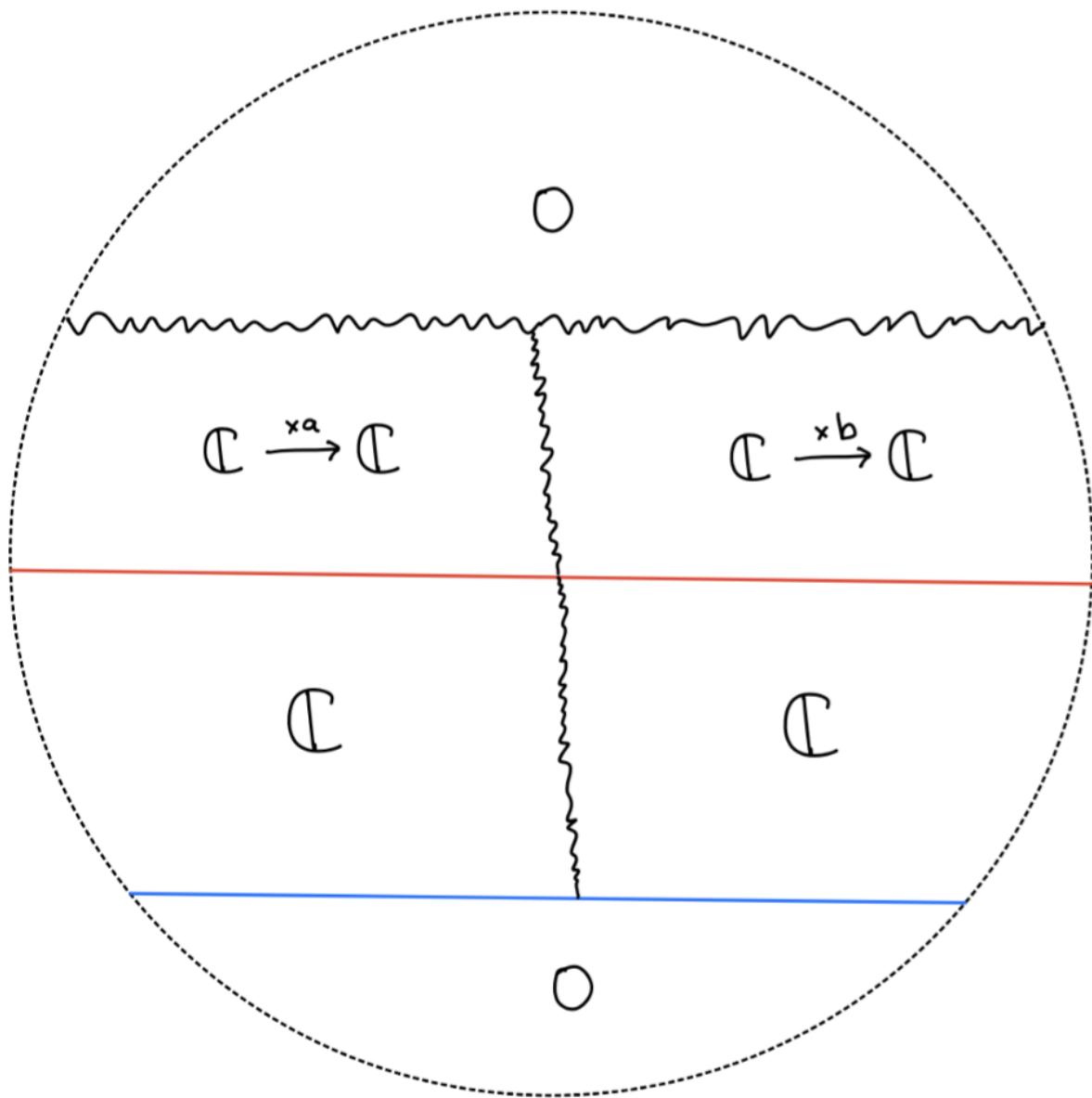


Figure 3.109

- $F_1(-, +) := 0$
- $F_1(+, -, -, -) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_1(+, -, -, +) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$
- $F_1(+, -, +) := 0$
- $F_1(+, +, *, -) := \mathbb{C}$

- $F_1(+, +, *, +) := \mathbb{C}$

Generalization maps:

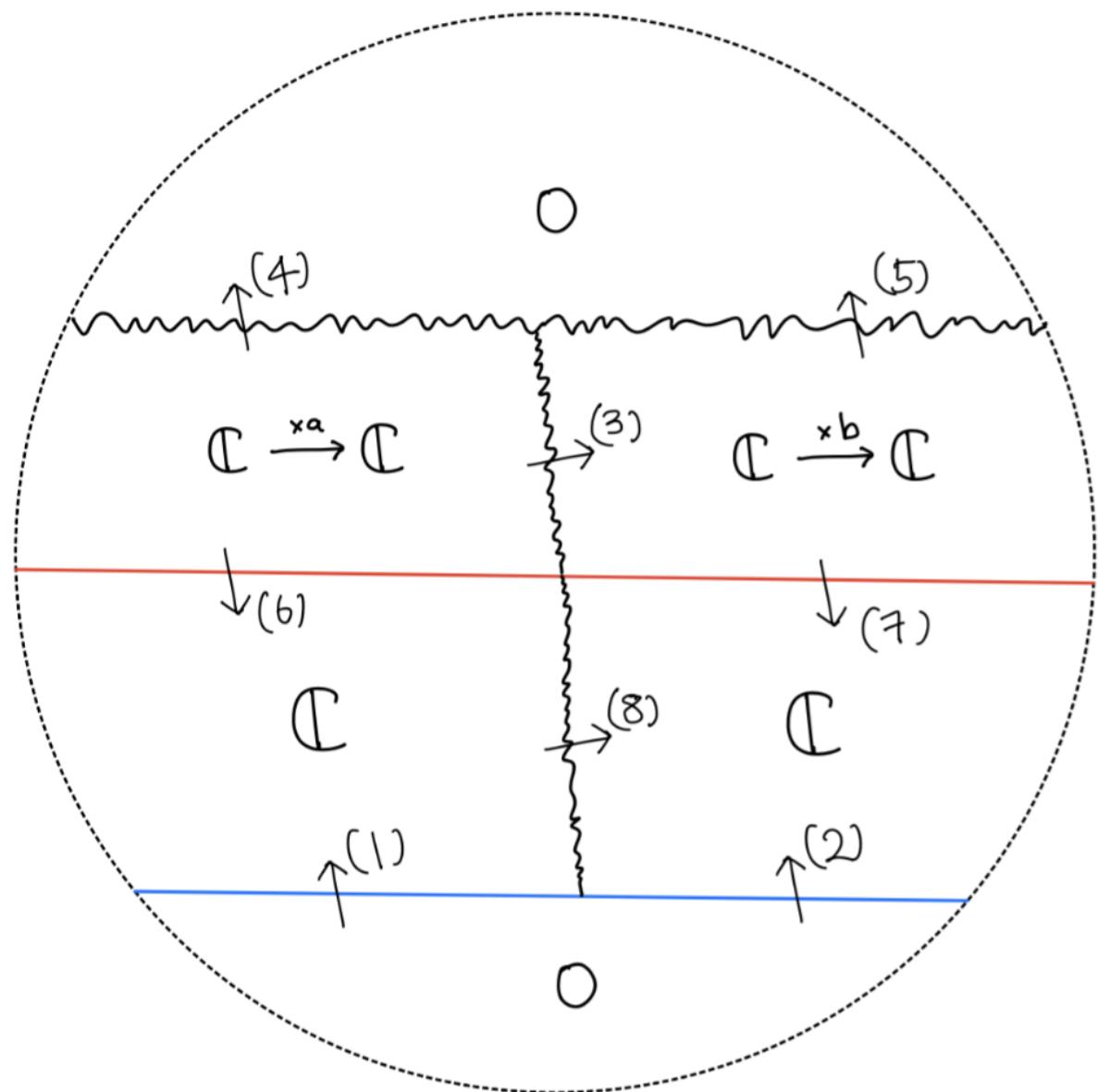


Figure 3.110

$$(1) \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \times b \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_{\bullet}|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 119. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

3.6 2nd Sheaf Cobordism'

In this section, we define $cobord'_2$, a compactly supported sheaf cobordism.

3.6.1 Notations

Definition 120. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 121. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying *cobord*₂ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying *cobord*₂.

Definition 122. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_\bullet^{symbol} to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript '*t*' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_\bullet^{symbol} a co-orientation of Φ_\bullet^{symbol} .

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as Λ_\bullet^{symbol} . Later in the section, Λ_\bullet^{symbol} will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₂.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as Λ_\bullet and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_\bullet will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₂.

Definition 123. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

Definition 124. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 125. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_0^{squia} is the union of the following three components
 - $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
 - $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$
 - $\{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.

- ξ_0^{squig} : the hairs of the components i, ii are pointing downward direction i.e. coefficients of dz are negative and the hairs of the component iii are pointing towards the right.

Definition 126. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_1^{squig} is the union of the following three components
 - (i) $\{(x, \frac{1}{2}) \in D_{r=2}\}$
 - (ii) $\{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig} :
 - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
 - for (ii), hairs are pointing towards the right i.e. coefficients of dx are positive.

Definition 127. 1. We define the following subsets of $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = 0\}$
- λ_\bullet^{squig} is the union of the following two components
 - (i) $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0, 1] \mid \frac{1}{2} > \Psi_t(x)\}$

$$(ii) \quad \{(0, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$$

2. We define co-orientations ξ_\bullet^{symbol} of λ_\bullet^{symbol} as follows

- ξ_\bullet^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_\bullet^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_\bullet^{squig} :
 - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
 - for (ii), hairs are pointing towards the right i.e. coefficients of dx are positive.

Definition 128. 1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .

2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_\bullet on $D_{r=2} \times [0, 1]$ induced by λ_\bullet i.e. strata are non-empty finite intersections of λ_\bullet^0 , λ_\bullet^∞ , and λ_\bullet^{squig} . Note that 2 dimensional strata has co-orientations inherited from λ_\bullet .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the following notations:

Definition 129. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 130. For $i = 1, 2, 3, 4$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := \{ & (x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\ & sgn(z - \frac{1}{2}) = sgn_3, \\ & sgn(-x) = sgn_4 \} \end{aligned}$$

2. we use * as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{i^{th}}{\ast}, \cdots) := s_{t_0}(\cdots, \underset{i^{th}}{-}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{0}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{+}, \cdots)$$

3. we omit trailing *'s e.g. $s_0(+, -) = s_0(+, -, *, *)$

4. note that we do not omit *'s located in between $-$, 0 , $+$ e.g. $s_0(+, -, *, -, -) \neq s_0(+, -, -, -)$

Definition 131. For $i = 0, 1, 2, 3, 4$ and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned}
 s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\
 & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\
 & sgn(z - \frac{1}{2}) = sgn_3, \\
 & sgn(-x) = sgn_4\}
 \end{aligned}$$

2. we use * as a wild card sign i.e.

$$s_{\bullet}(\cdots, \underset{i^{th}}{\ast}, \cdots) := s_{\bullet}(\cdots, \underset{i^{th}}{-}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{0}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{+}, \cdots)$$

3. we omit trailing *'s e.g. $s_{\bullet}(+, -) = s_{\bullet}(+, -, *, *)$

4. note that we do not omit *'s located in between $-, 0, +$ e.g. $s_{\bullet}(+, -, *, -, -) \neq s_{\bullet}(+, -, -, -)$

Definition 132. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the above notations:

1. \mathcal{S}_0 :

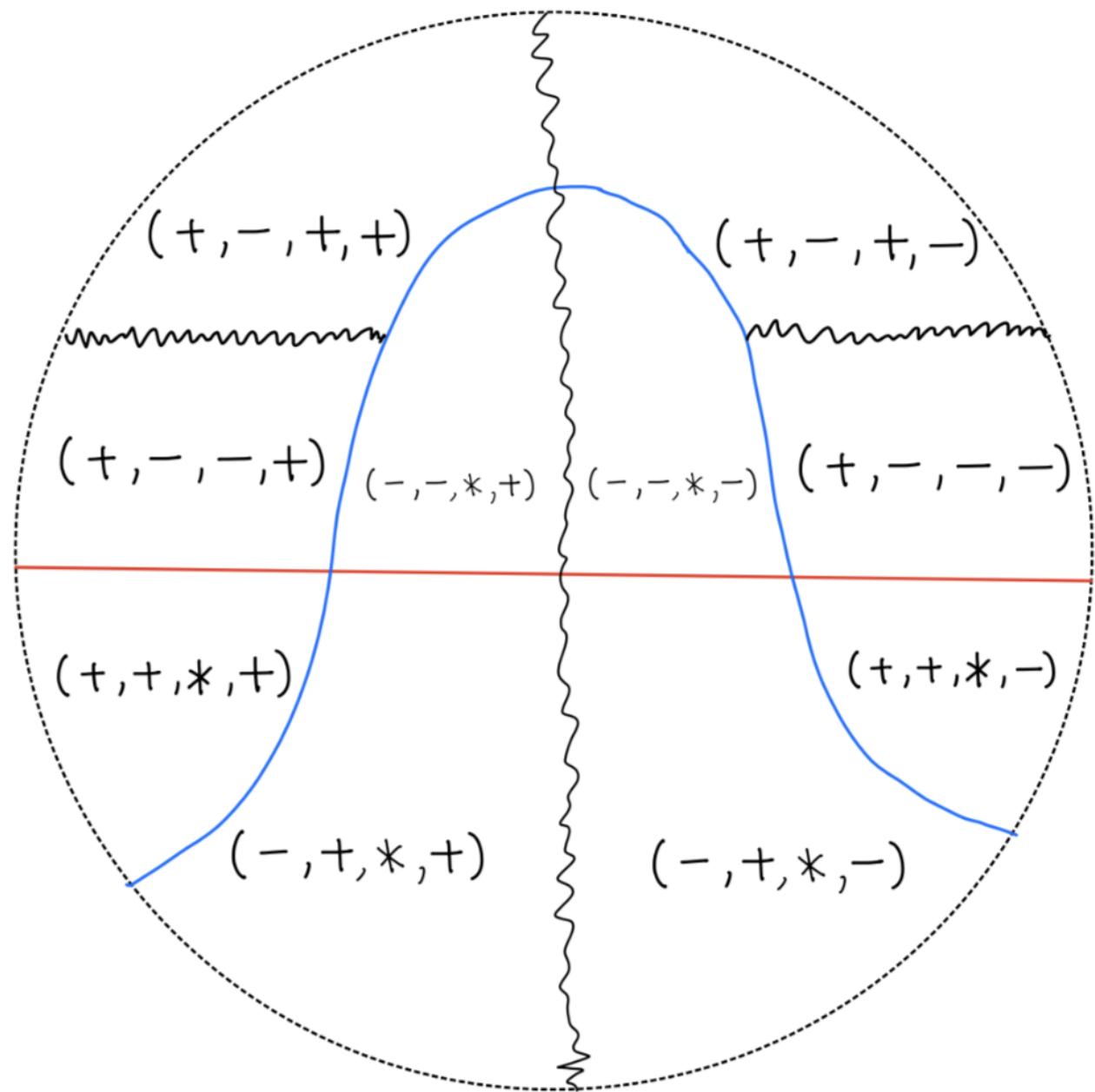


Figure 3.111

- 2 dimensional strata:

$s_0(-, -, *, -), s_0(-, -, *, +), s_0(+, -, -, -), s_0(+, -, -, +), s_0(+, -, +, -), s_0(+, -, +, +),$
 $s_0(+, +, *, -), s_0(+, +, *, -), s_0(-, +, *, -), s_0(-, +, *, +)$

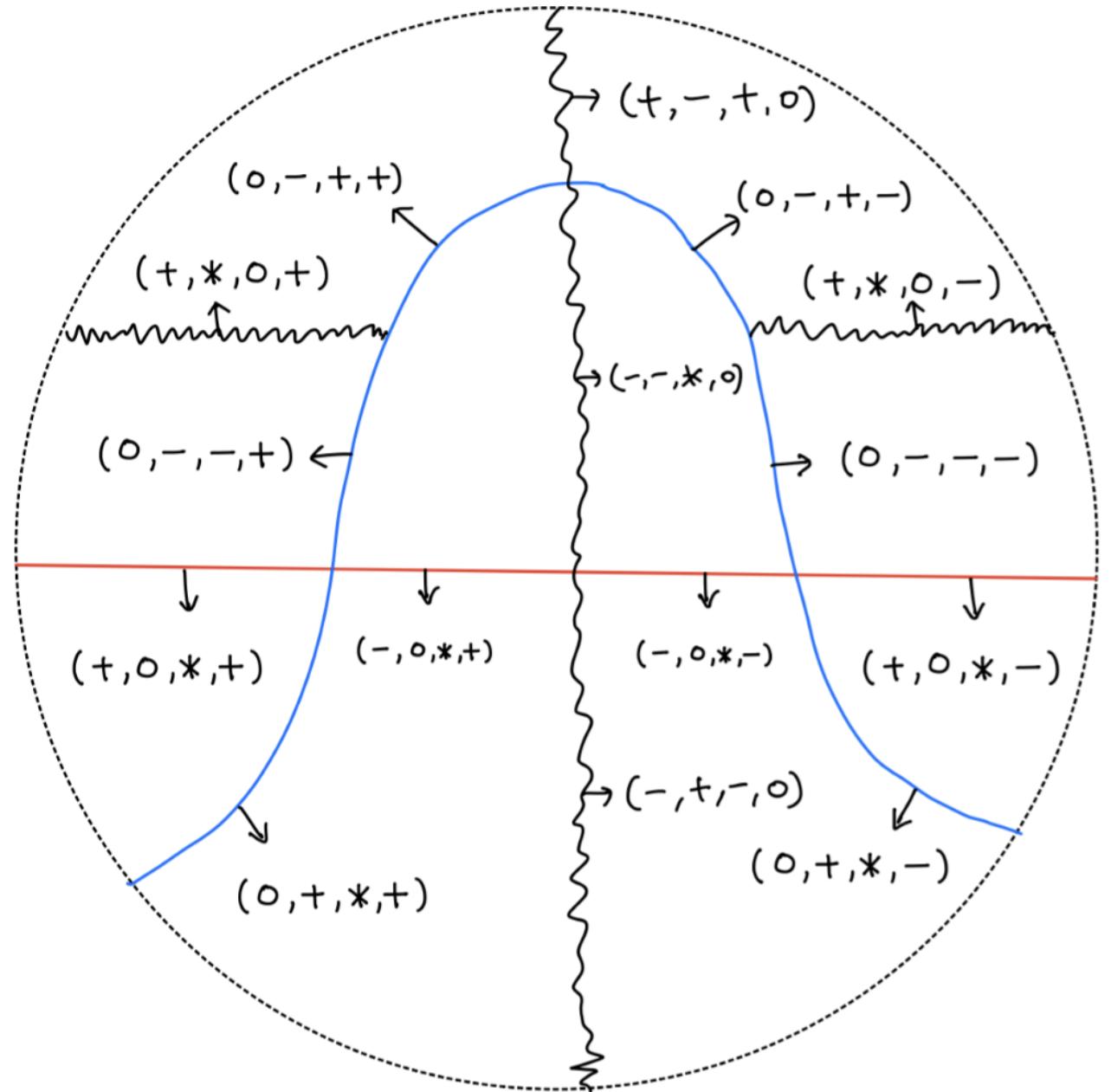


Figure 3.112

- 1 dimensional strata:

$s_0(0, +, *, -), s_0(0, -, -, -), s_0(0, -, +, -), s_0(0, -, +, +), s_0(0, -, -, +),$
 $s_0(0, +, *, +), s_0(+, 0, *, -), s_0(-, 0, *, -), s_0(-, 0, *, +), s_0(+, 0, *, +), s_0(+, *, 0, -),$
 $s_0(+, *, 0, +), s_0(+, -, +, 0), s_0(-, -, *, 0), s_0(-, +, -, 0)$

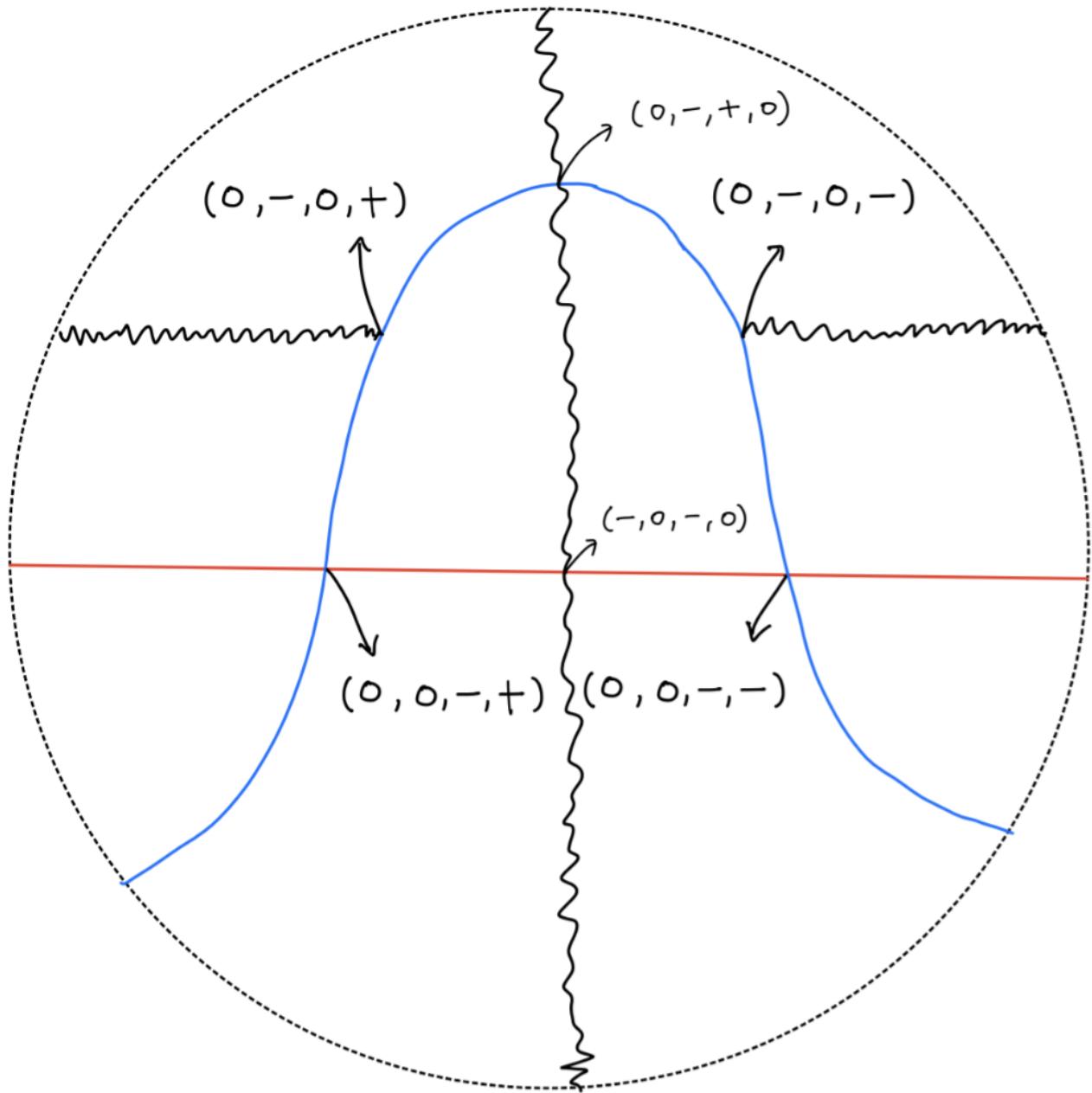


Figure 3.113

- 0 dimensional strata:

$$s_0(0, 0, -, -), s_0(0, -, 0, -), s_0(0, -, 0, +), s_0(0, 0, -, +), s_0(0, -, +, -), s_0(-, 0, -, 0)$$

2. \mathcal{S}_1 :

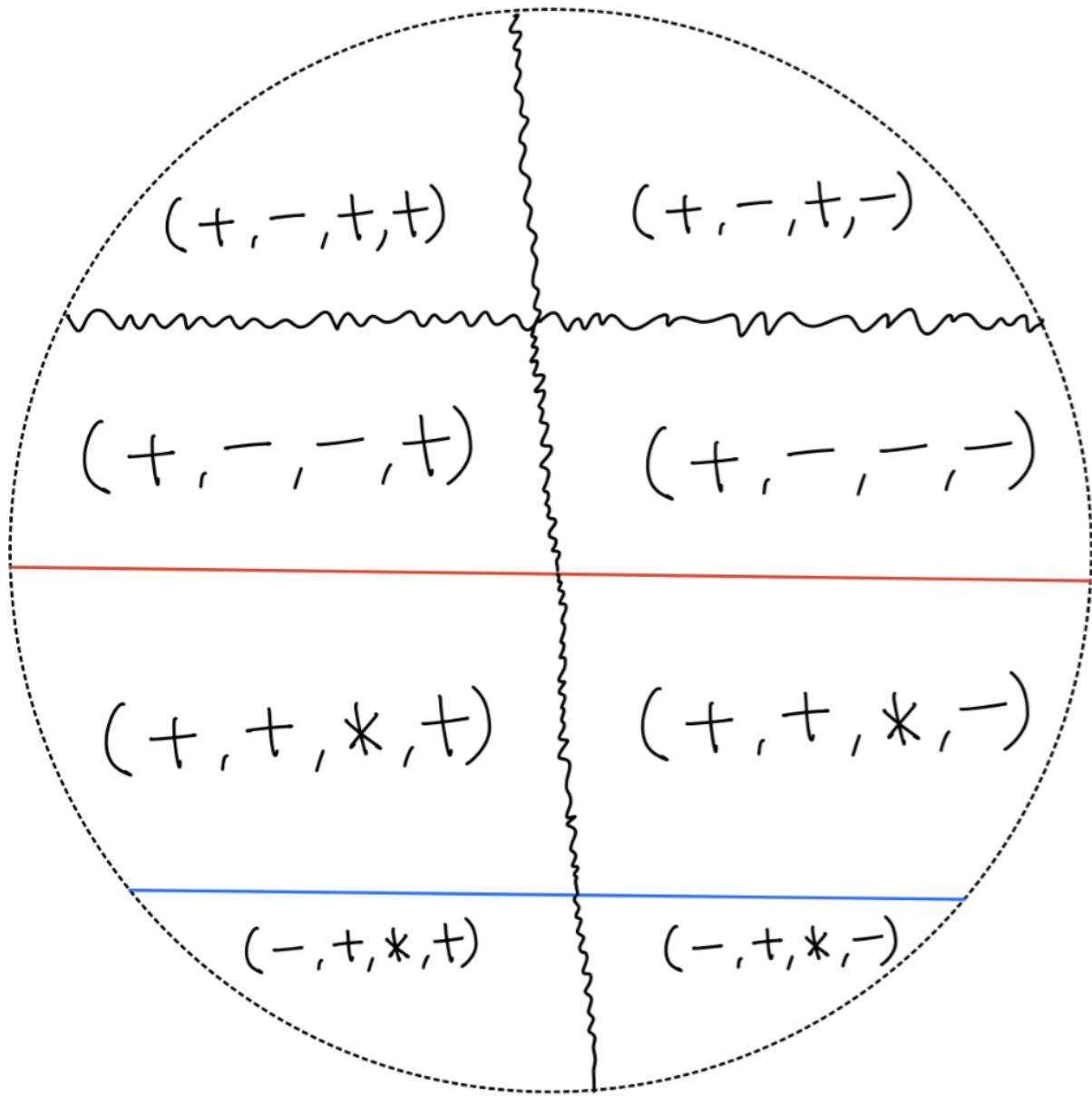


Figure 3.114

- 2 dimensional strata:

$$\begin{aligned}
 & s_1(-, +, *, -), s_1(-, +, *, +), s_1(+, -, -, -), s_1(+, -, -, +), s_1(+, -, +, -), \\
 & s_1(+, -, +, +), s_1(+, +, *, -), s_1(+, +, *, +)
 \end{aligned}$$

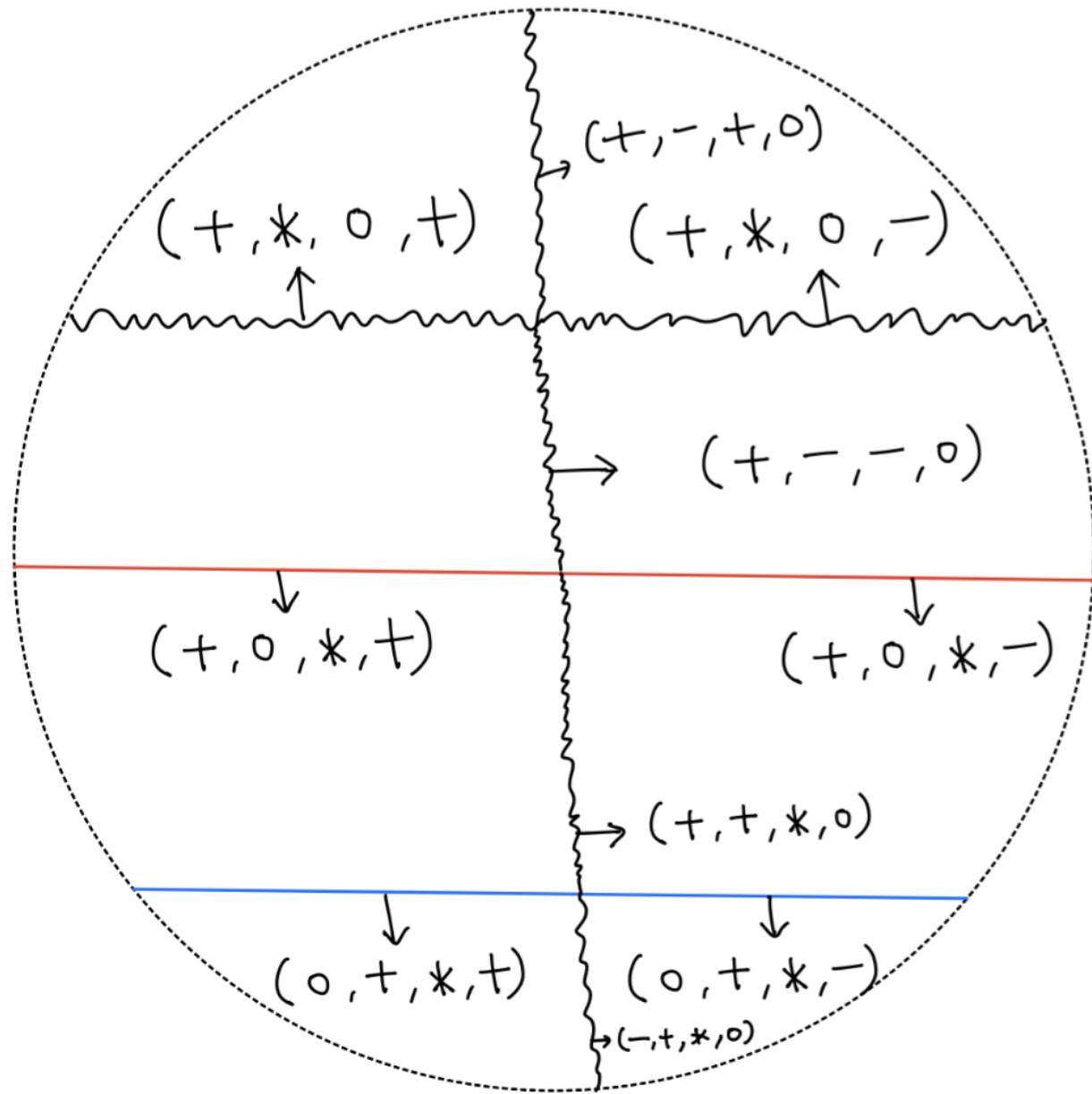


Figure 3.115

- 1 dimensional strata:

$$\begin{aligned}
 & s_1(0, +, *, -), s_1(0, +, *, +), s_1(+, 0, *, -), s_1(+, 0, *, +), s_1(+, *, 0, -), s_1(+, *, 0, +), \\
 & s_1(+, -, -, 0), s_1(+, +, *, 0), s_1(+, -, +, 0), s_1(-, +, *, 0)
 \end{aligned}$$

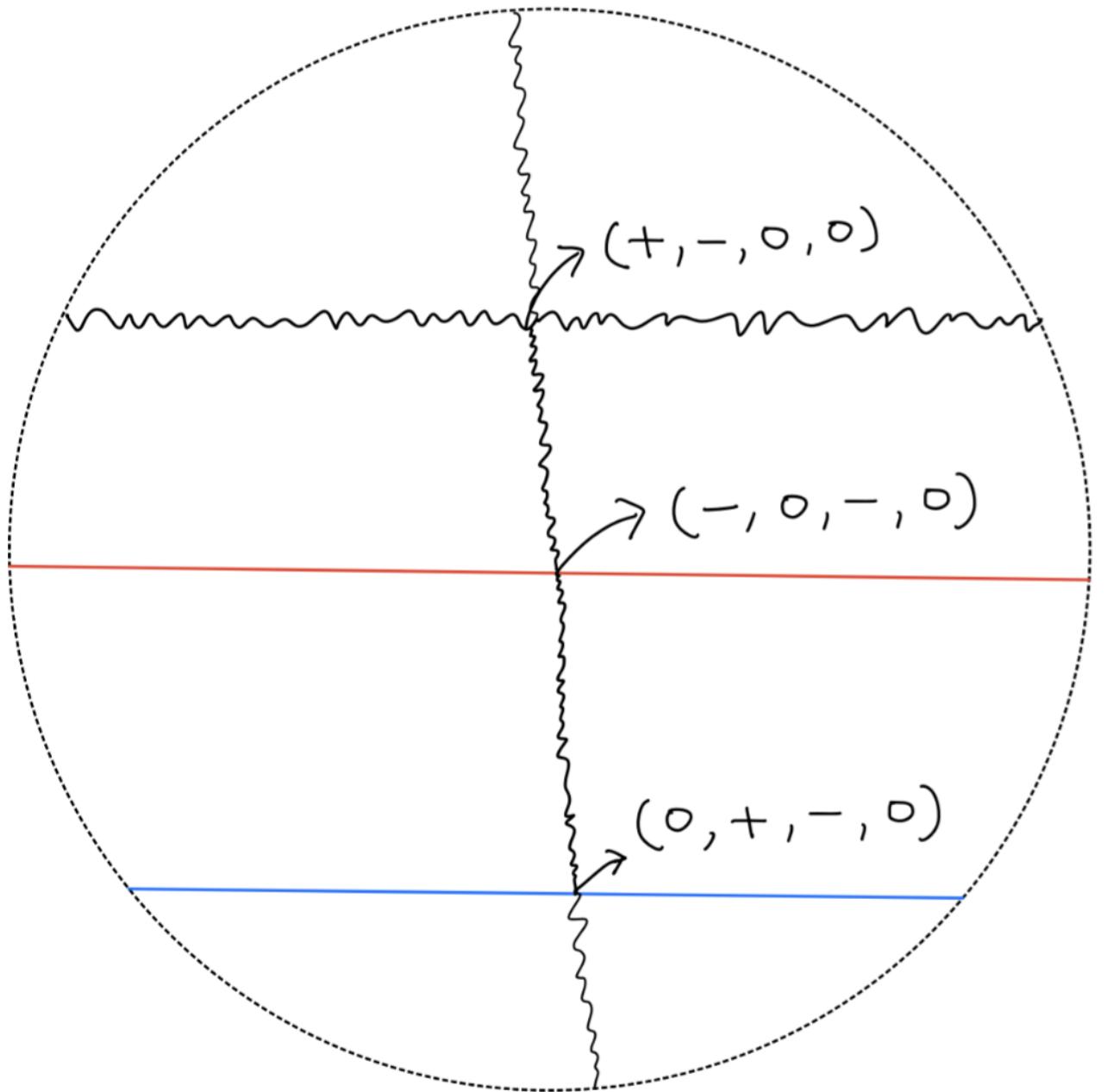


Figure 3.116

- 0 dimensional strata:

$$s_1(+, -, 0, 0), s_1(-, 0, -, 0), s_1(0, +, -, 0)$$

3. \mathcal{S}_\bullet :

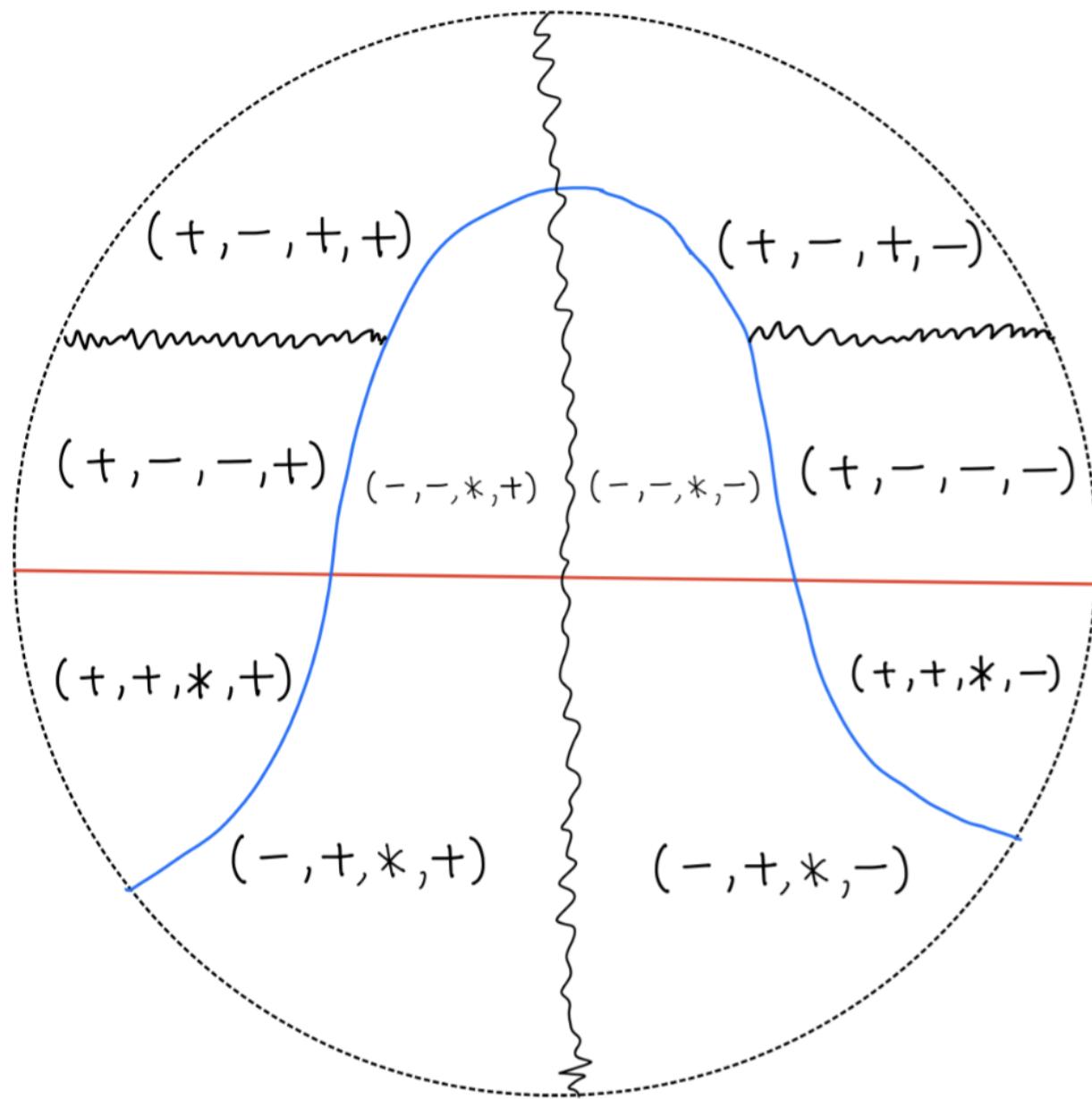


Figure 3.117

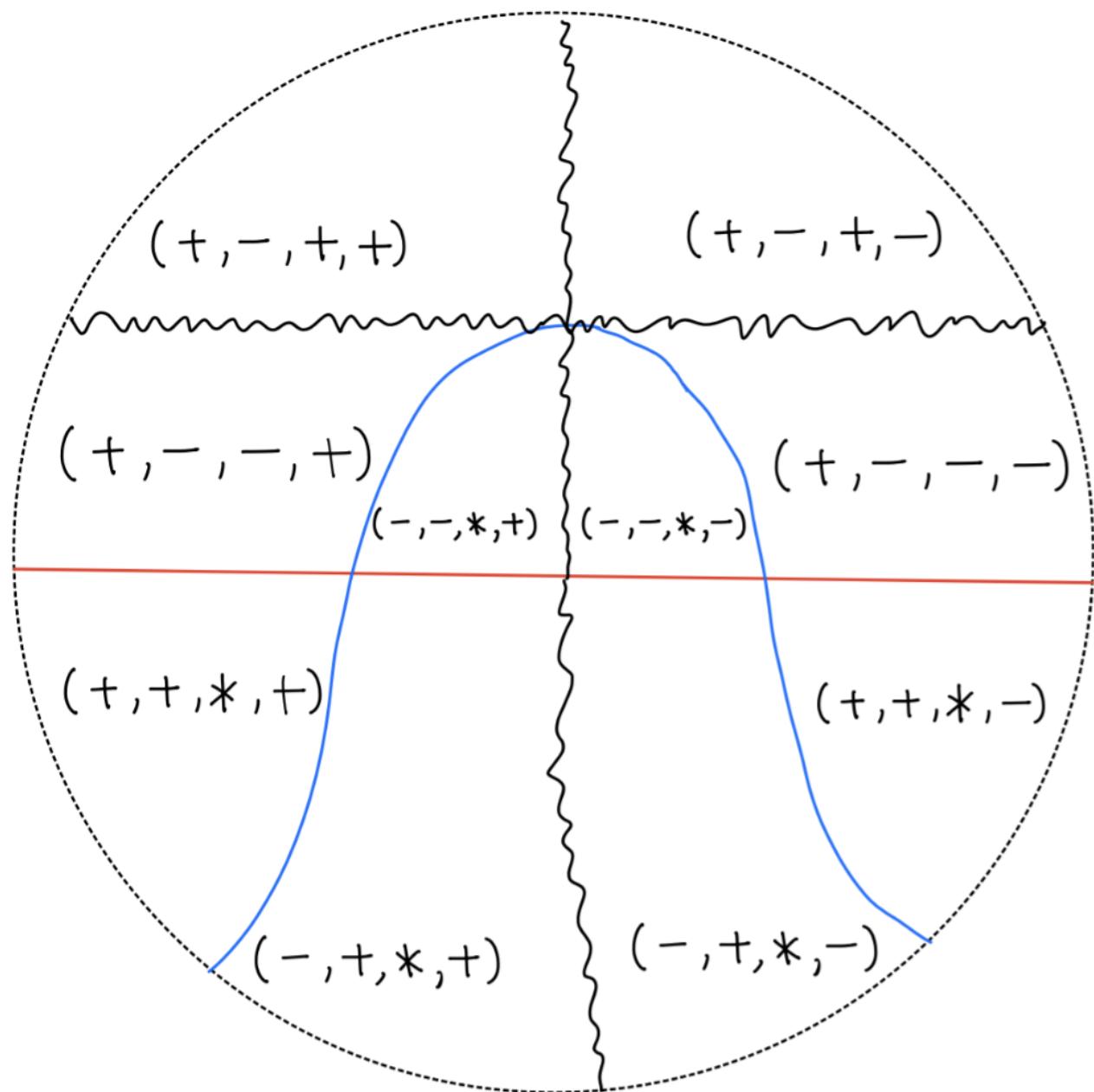


Figure 3.118

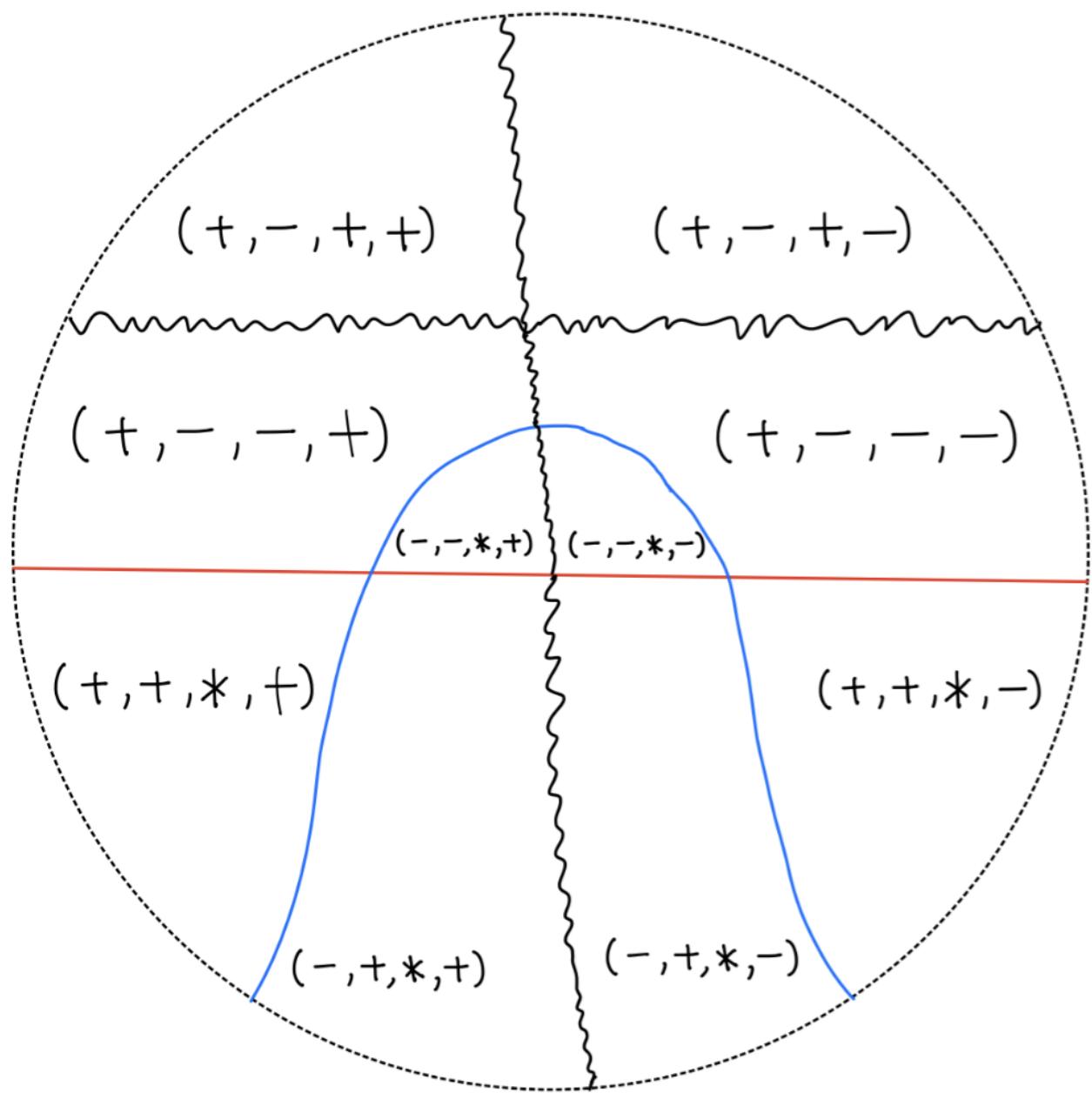


Figure 3.119

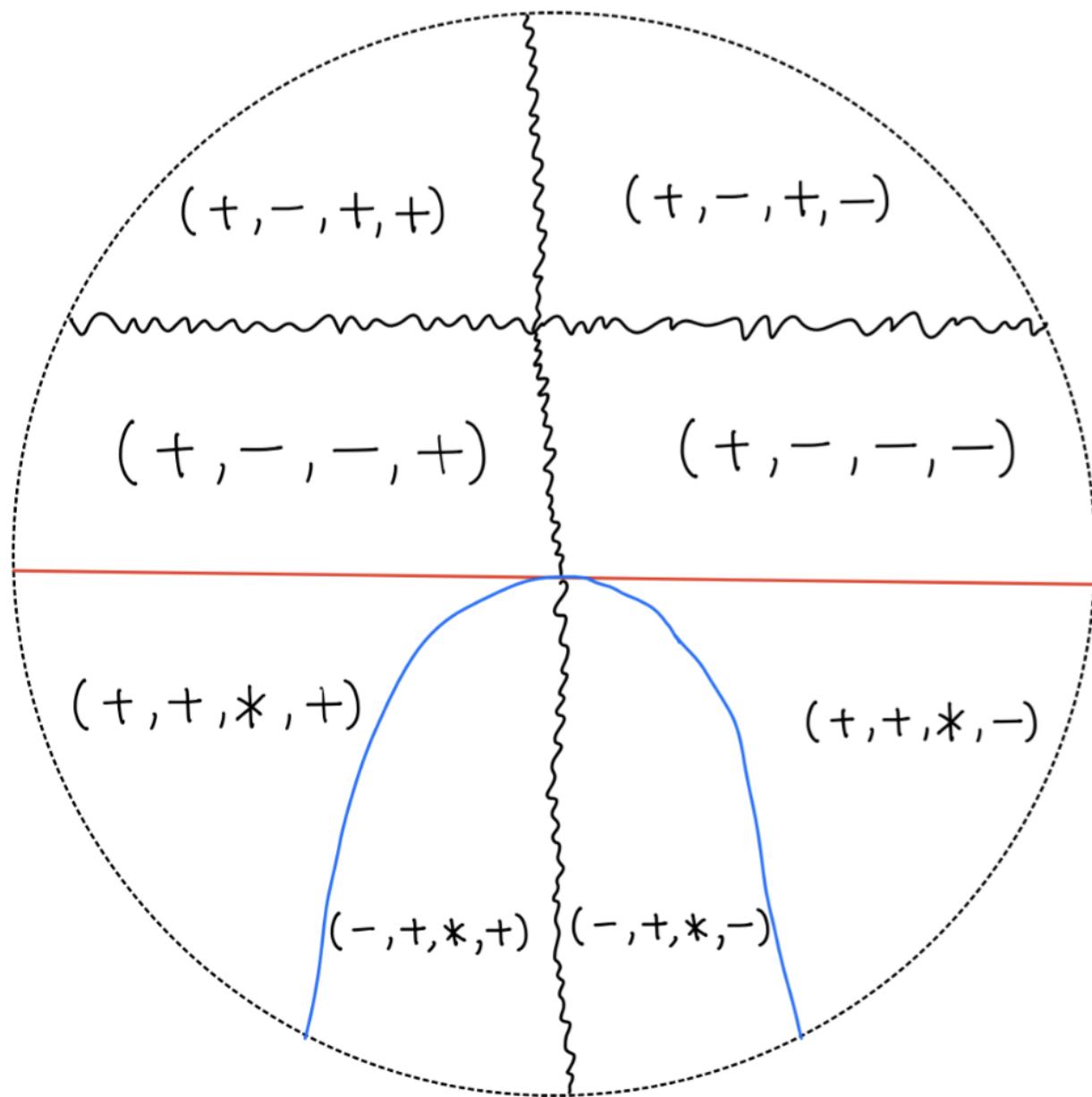


Figure 3.120

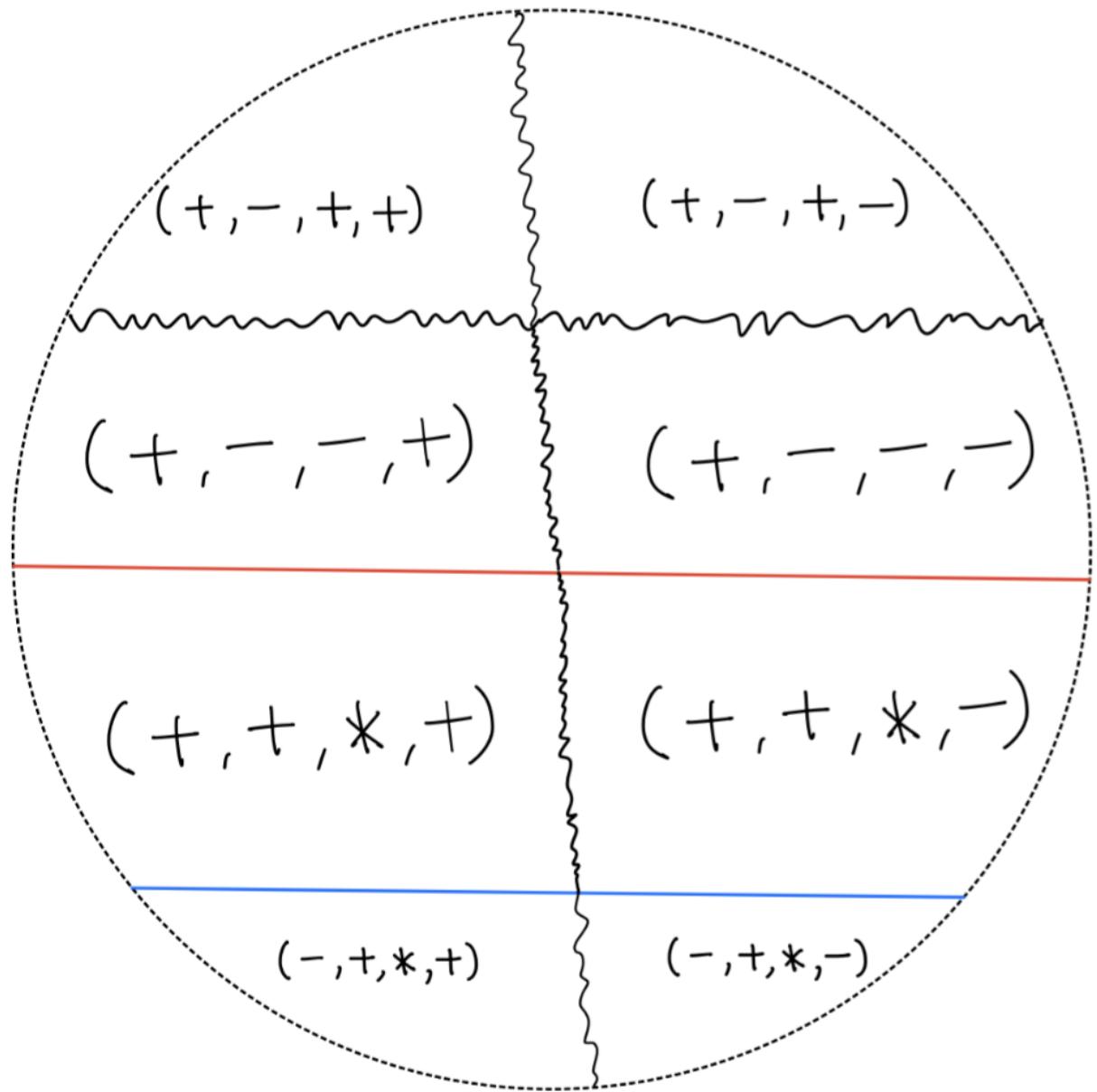


Figure 3.121

- 3 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(-,+,*,-), s_{\bullet}(-,+,*,+), s_{\bullet}(-,-,*,-), s_{\bullet}(-,-,*,+), s_{\bullet}(-,+), s_{\bullet}(+,-,-,-), \\
 & s_{\bullet}(+,-,-,+), s_{\bullet}(+,-,+,-), s_{\bullet}(+,-,+,+), s_{\bullet}(+,*,-), s_{\bullet}(+,*,+)
 \end{aligned}$$

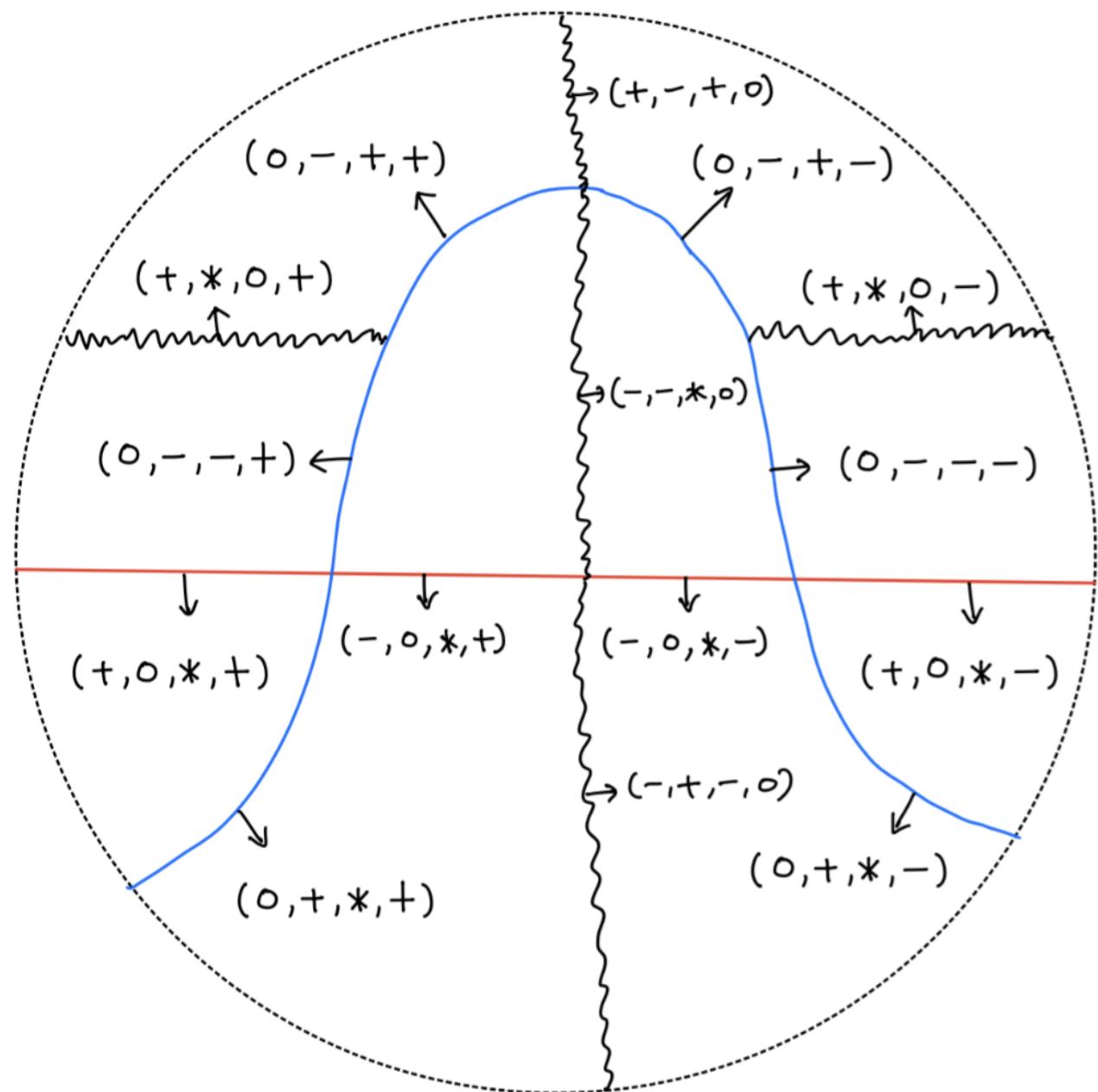


Figure 3.122

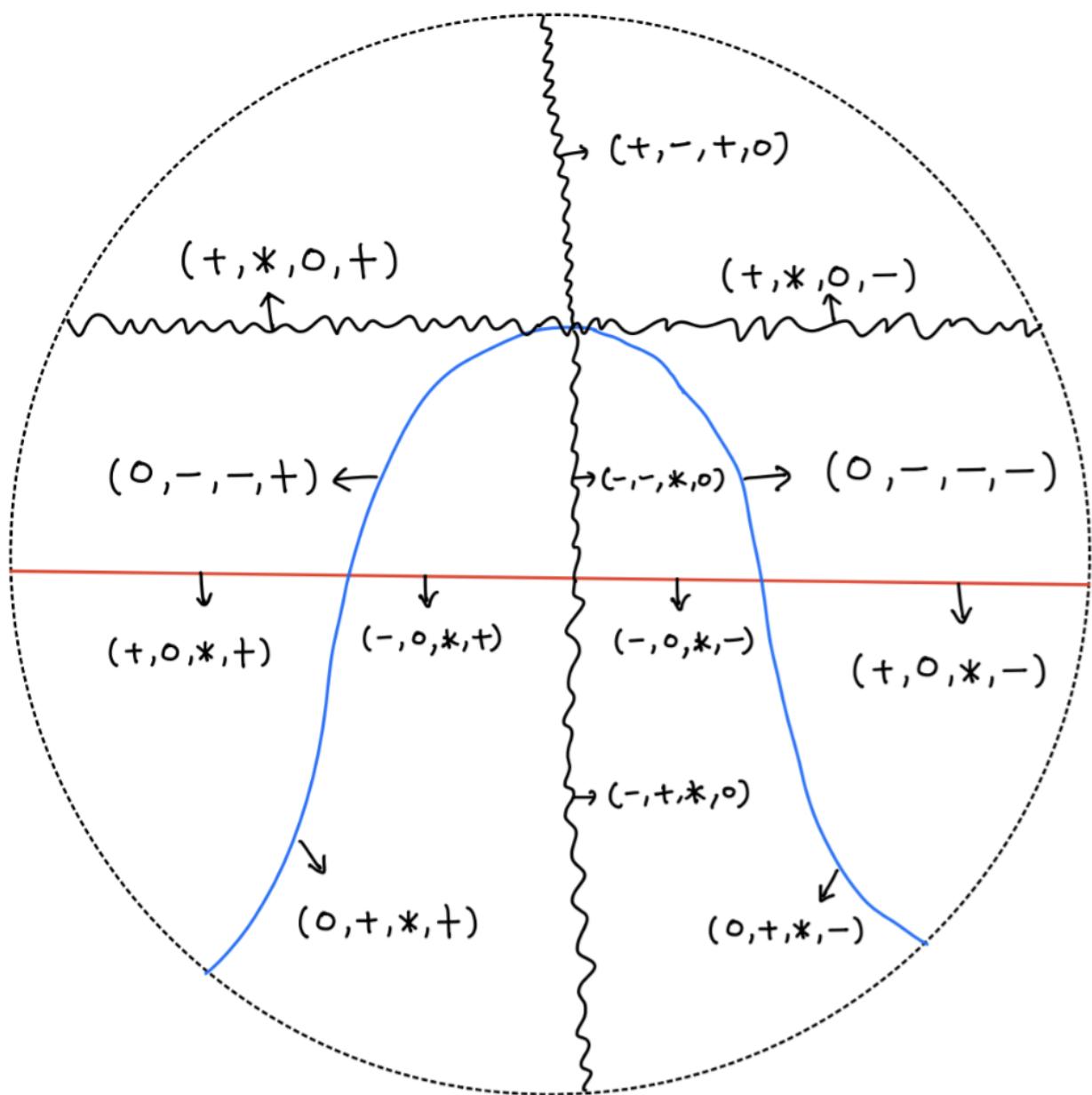


Figure 3.123

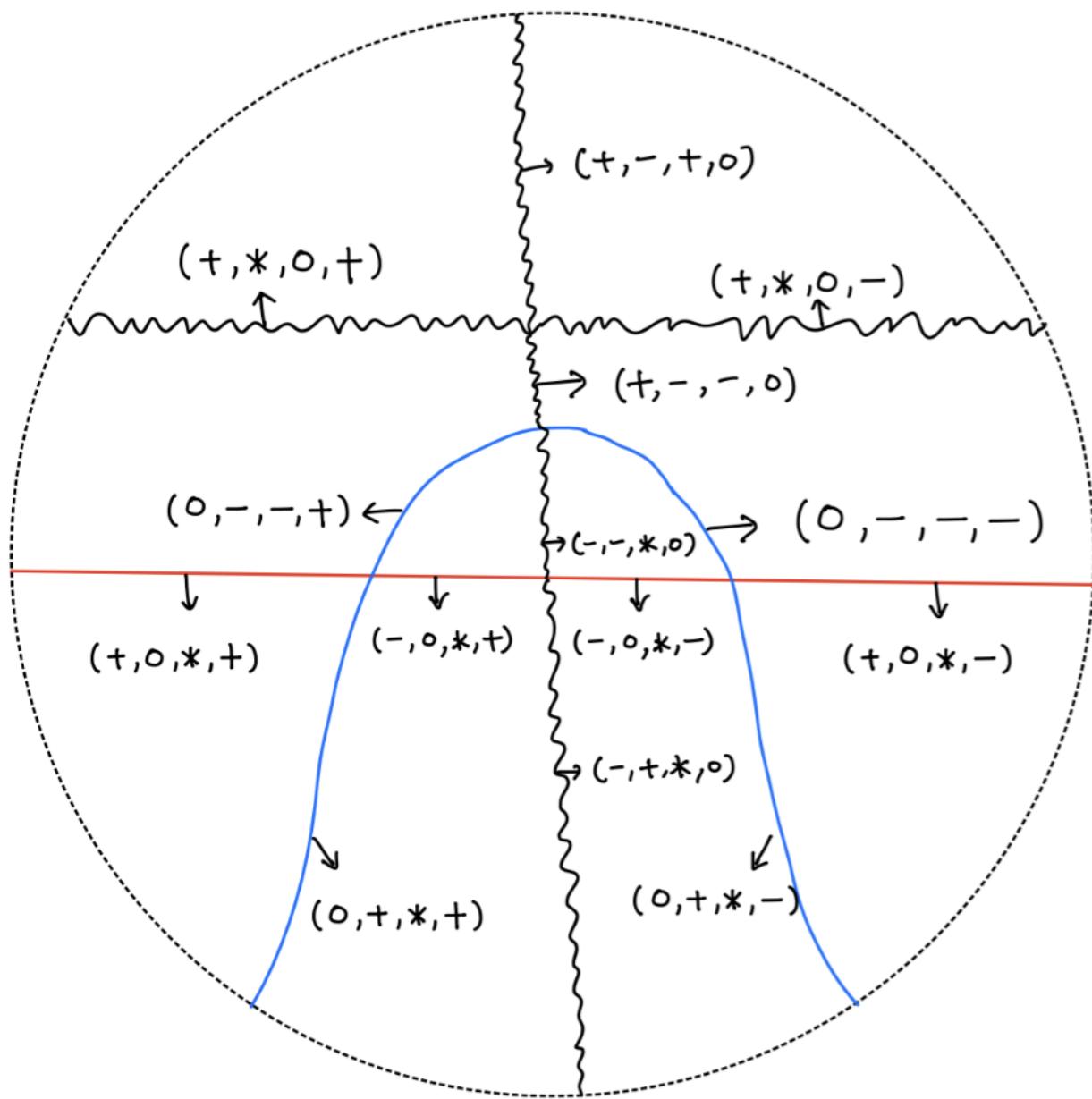


Figure 3.124

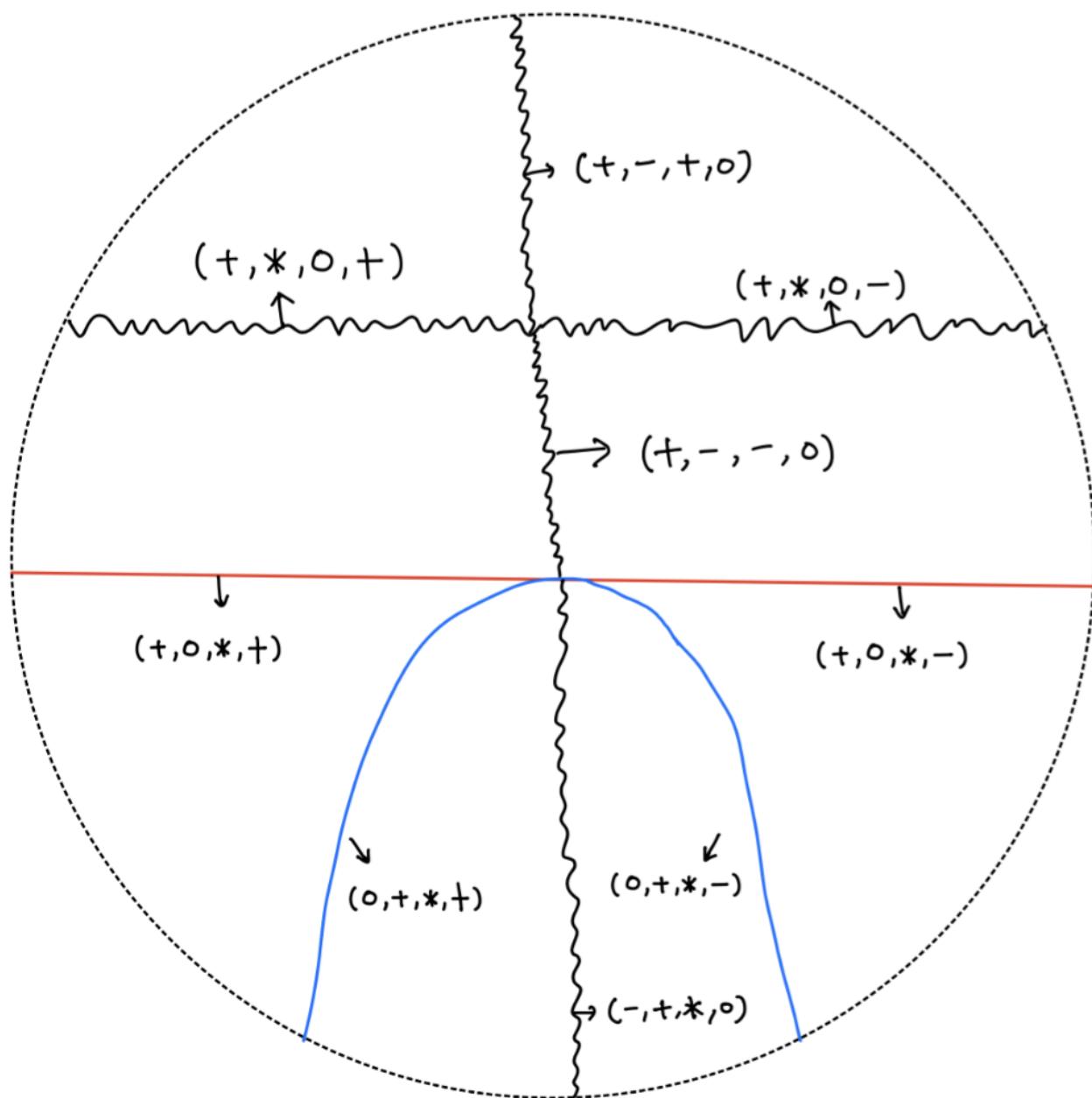


Figure 3.125

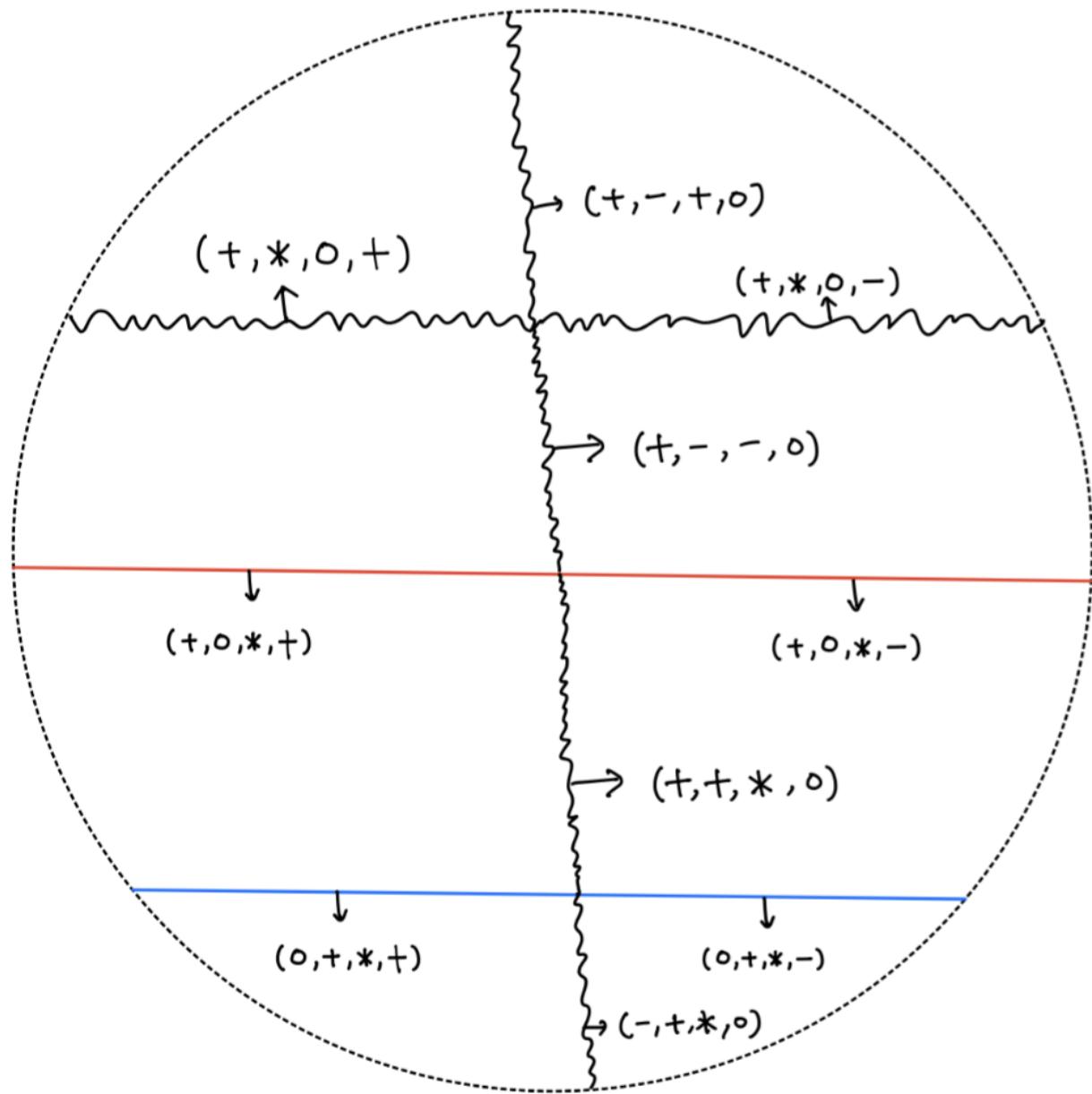


Figure 3.126

- 2 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, +, *, -), s_{\bullet}(0, -, -, -), s_{\bullet}(0, -, +), s_{\bullet}(0, -, +, -), s_{\bullet}(0, -, +, +), s_{\bullet}(0, -, -, +), \\
 & s_{\bullet}(0, +, *, +), s_{\bullet}(+, 0, *, -), s_{\bullet}(-, 0, *, -), s_{\bullet}(-, 0, *, +), s_{\bullet}(+, 0, *, +), s_{\bullet}(+, *, 0, -), \\
 & s_{\bullet}(+, *, 0, +), s_{\bullet}(+, -, -, 0), s_{\bullet}(+, +, *, 0), s_{\bullet}(+, -, +, 0), s_{\bullet}(-, +, *, 0), s_{\bullet}(-, -, *, 0)
 \end{aligned}$$

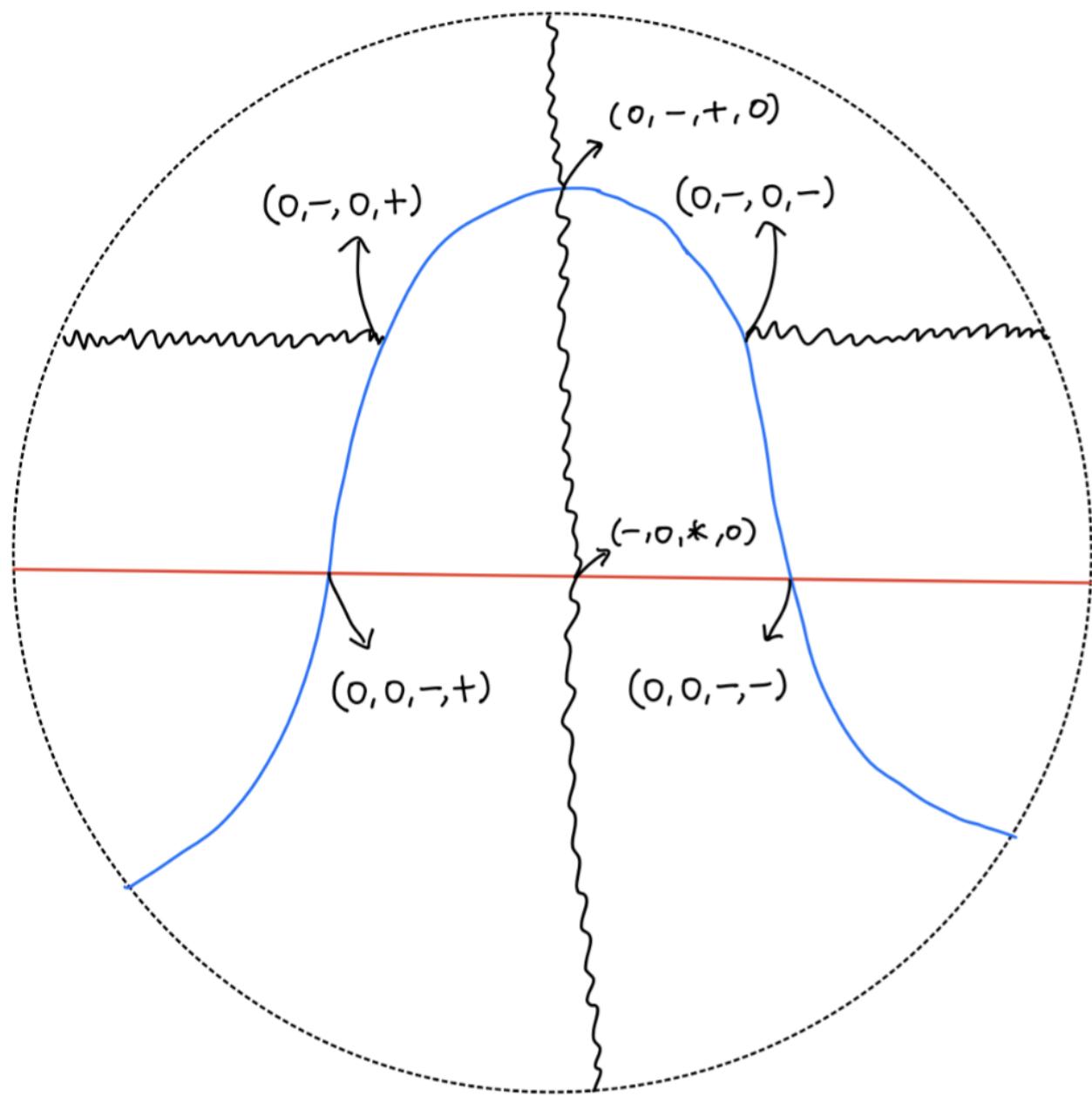


Figure 3.127

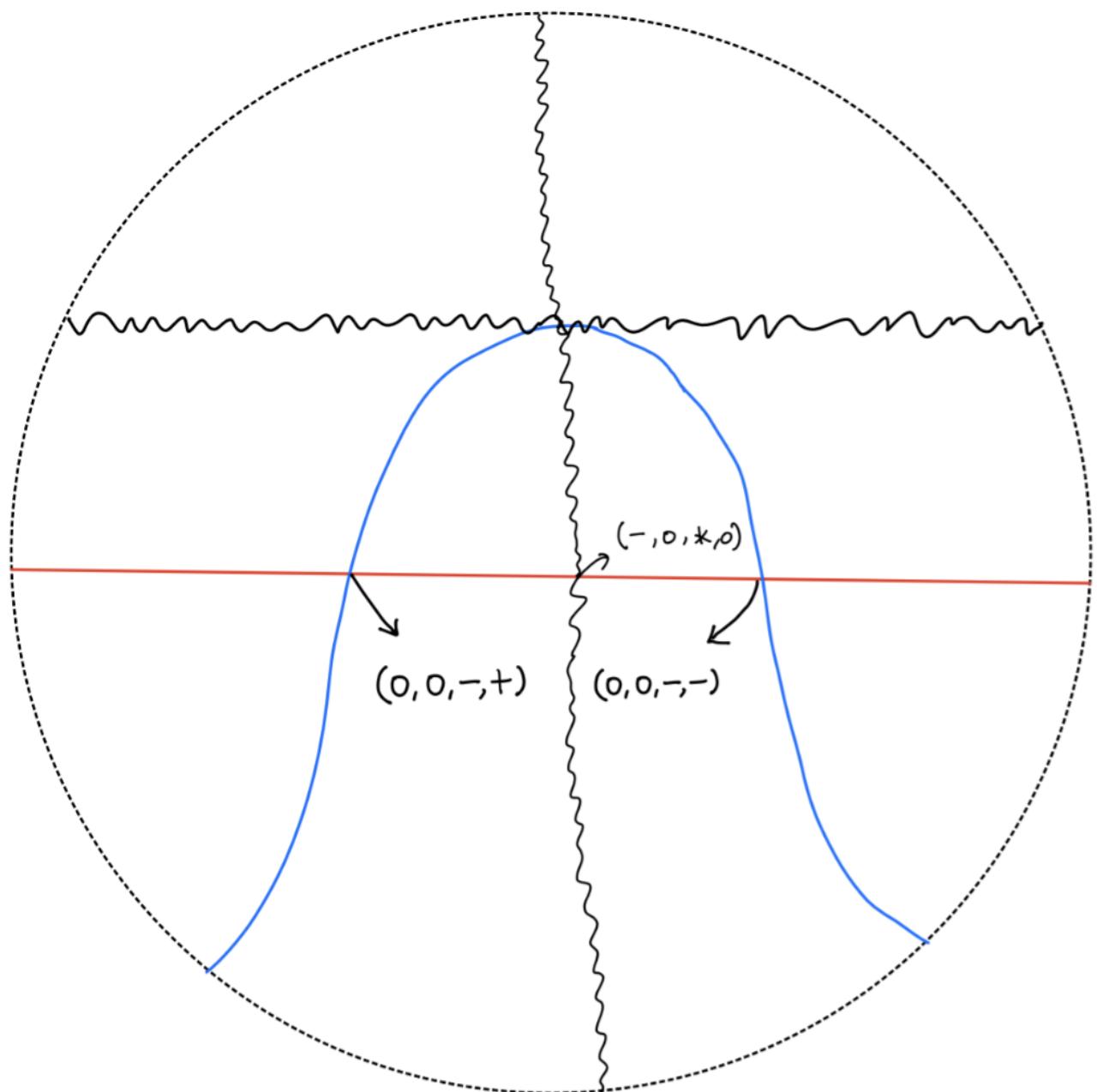


Figure 3.128

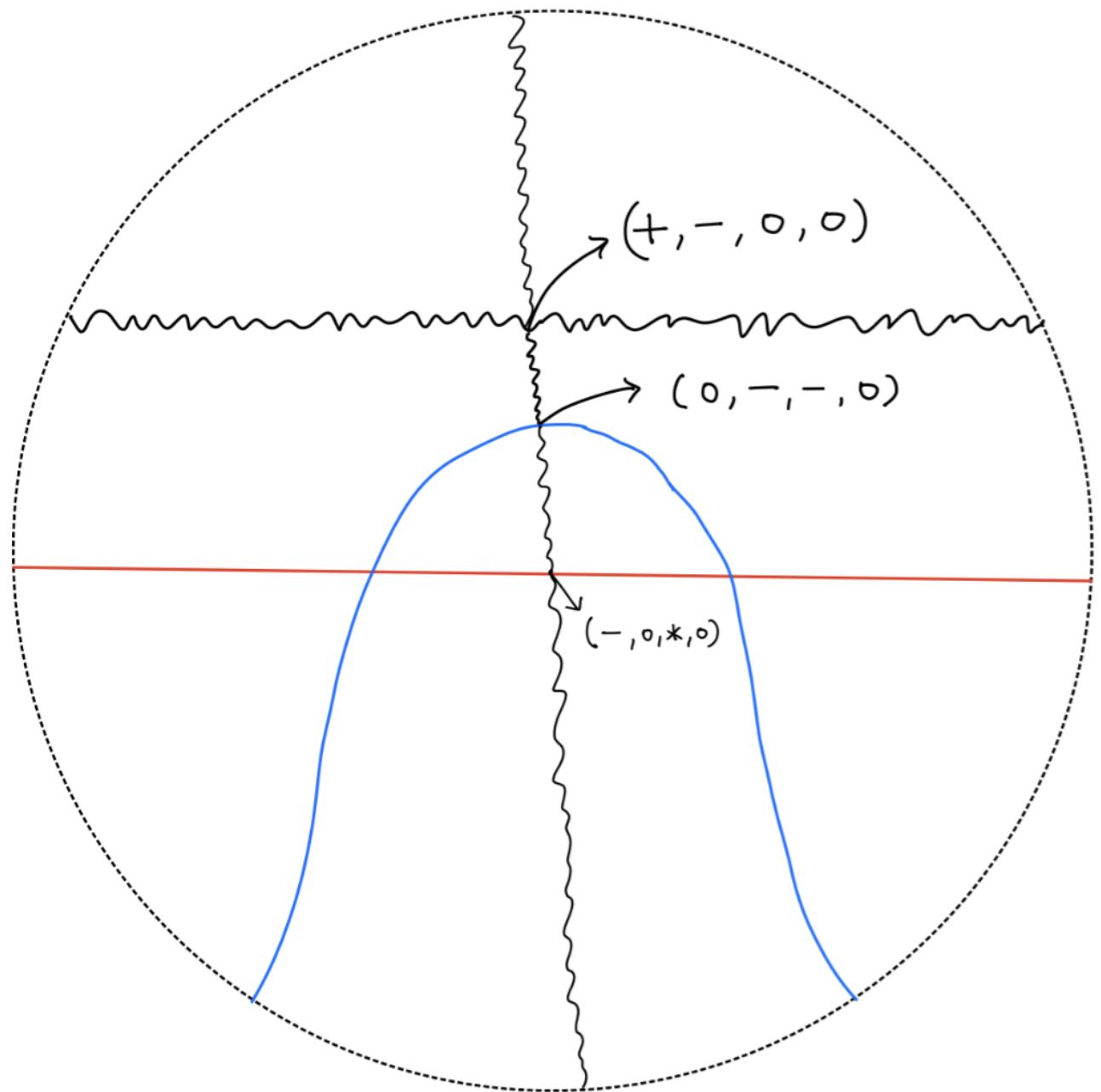


Figure 3.129

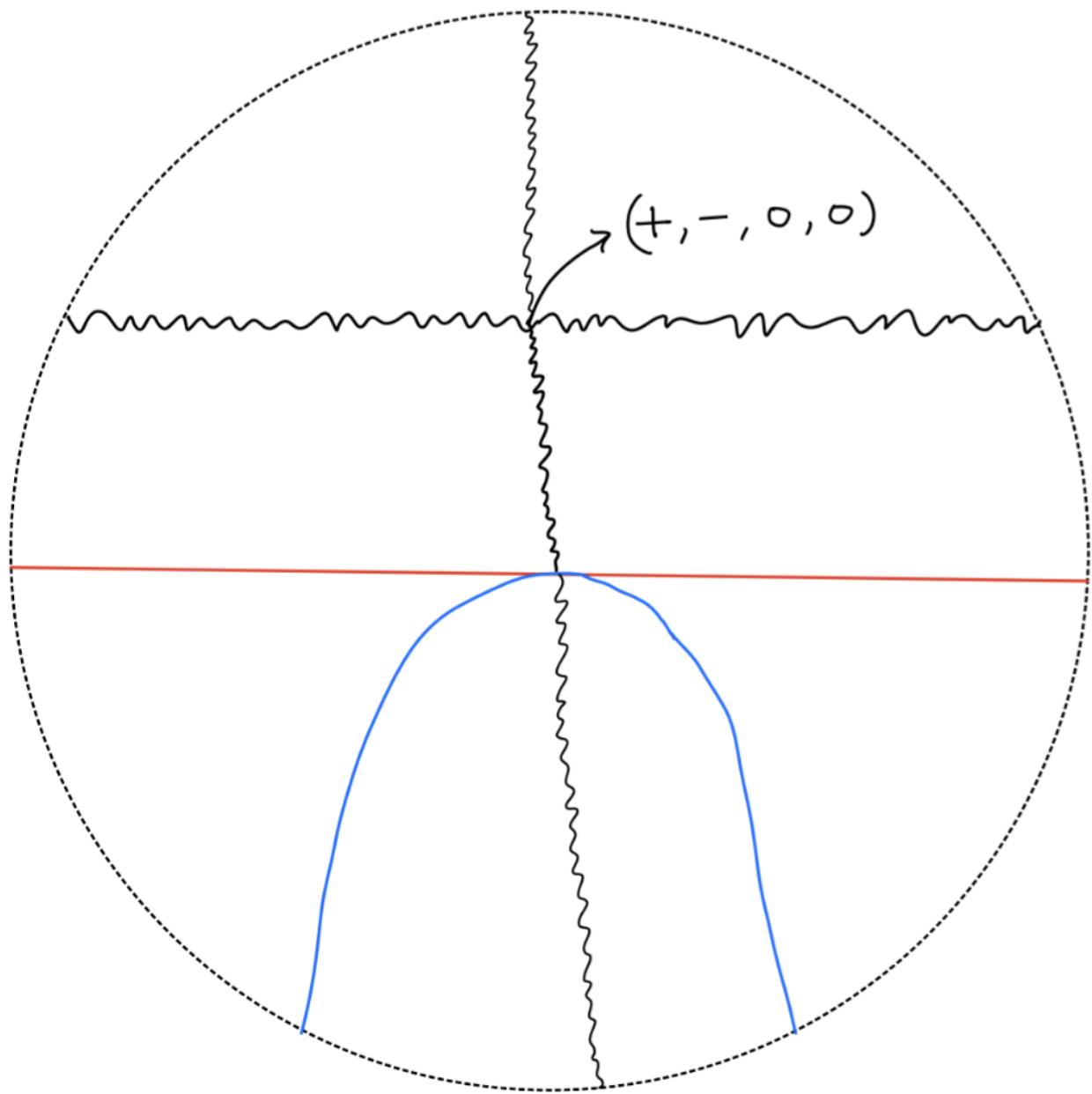


Figure 3.130

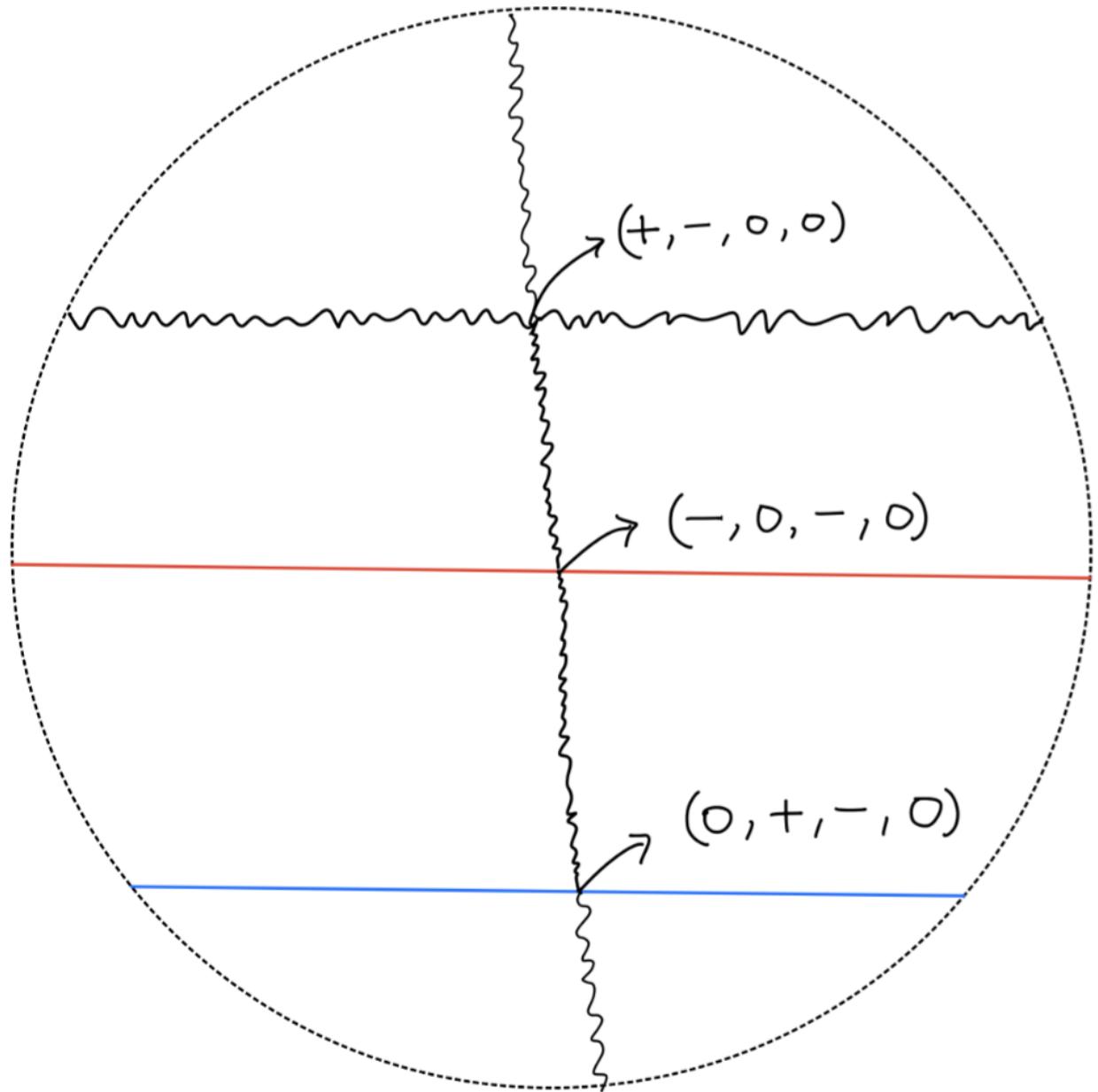


Figure 3.131

- 1 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, 0, -, -), s_{\bullet}(0, -, 0, -), s_{\bullet}(0, -, 0, +), s_{\bullet}(0, 0, -, +), s_{\bullet}(+, -, 0, 0), s_{\bullet}(0, -, -, 0), \\
 & s_{\bullet}(-, 0, -, 0), s_{\bullet}(0, +, -, 0), s_{\bullet}(0, -, +, 0), s_{\bullet}(-, 0, *, 0)
 \end{aligned}$$

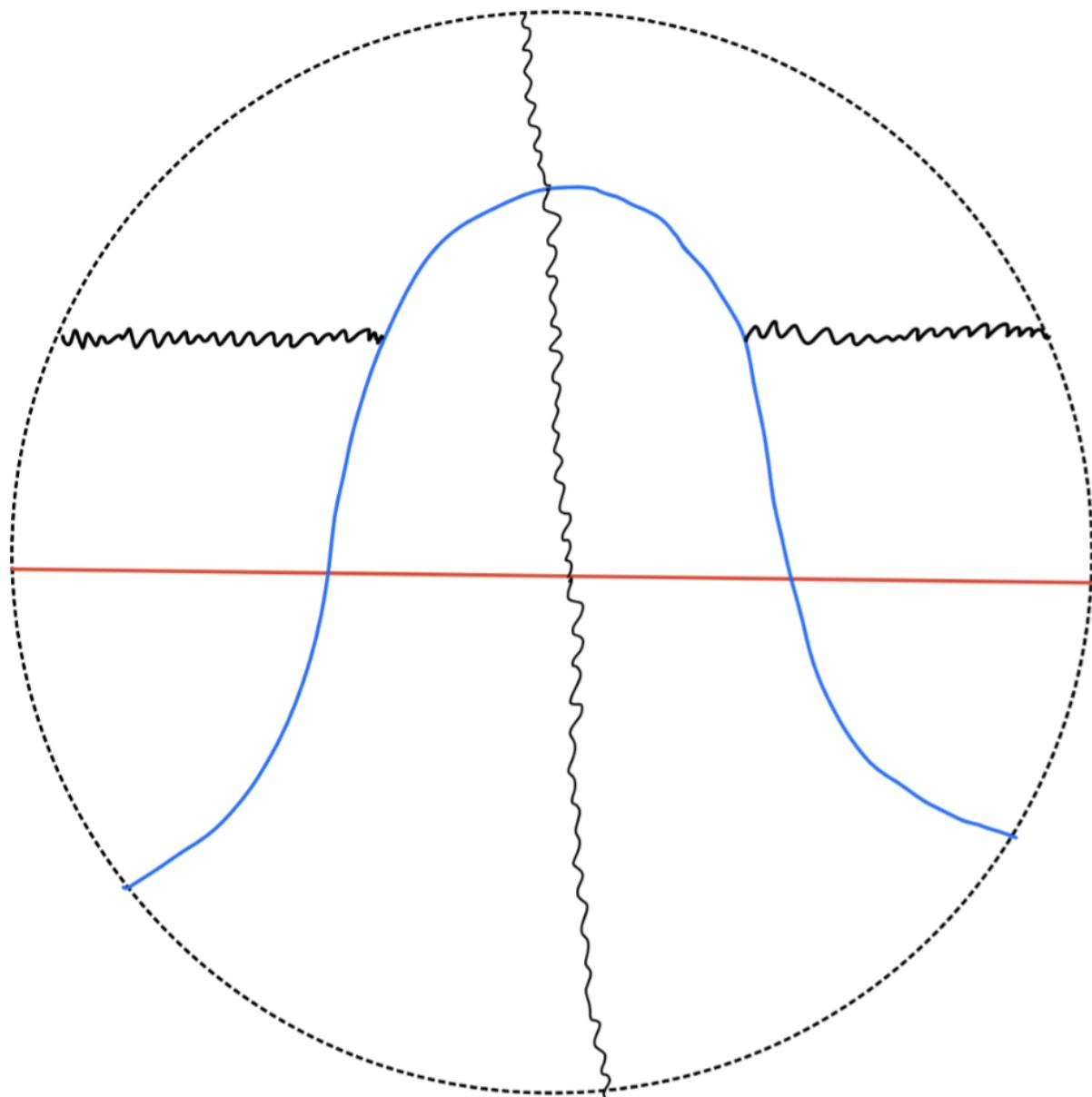


Figure 3.132

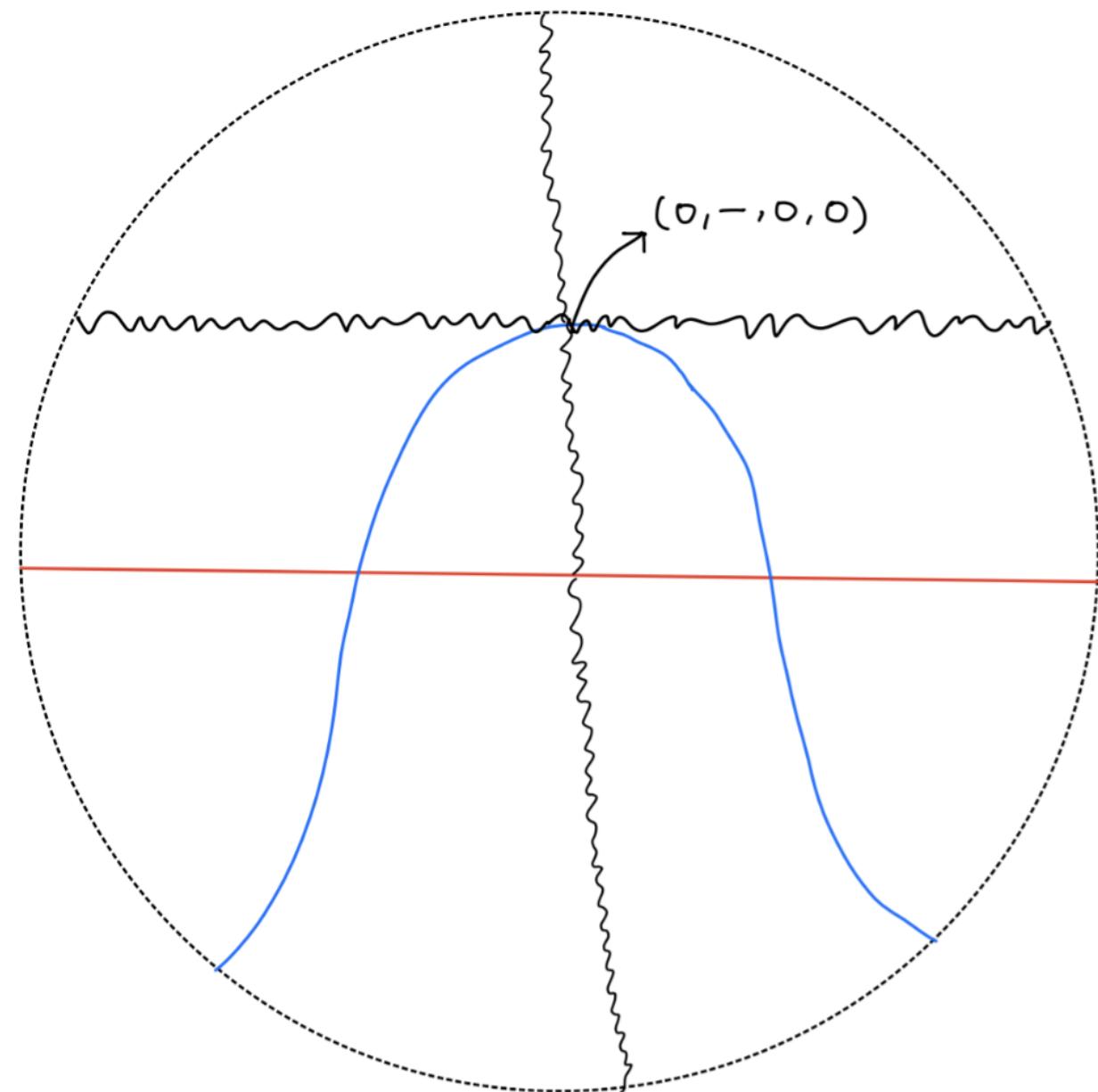


Figure 3.133

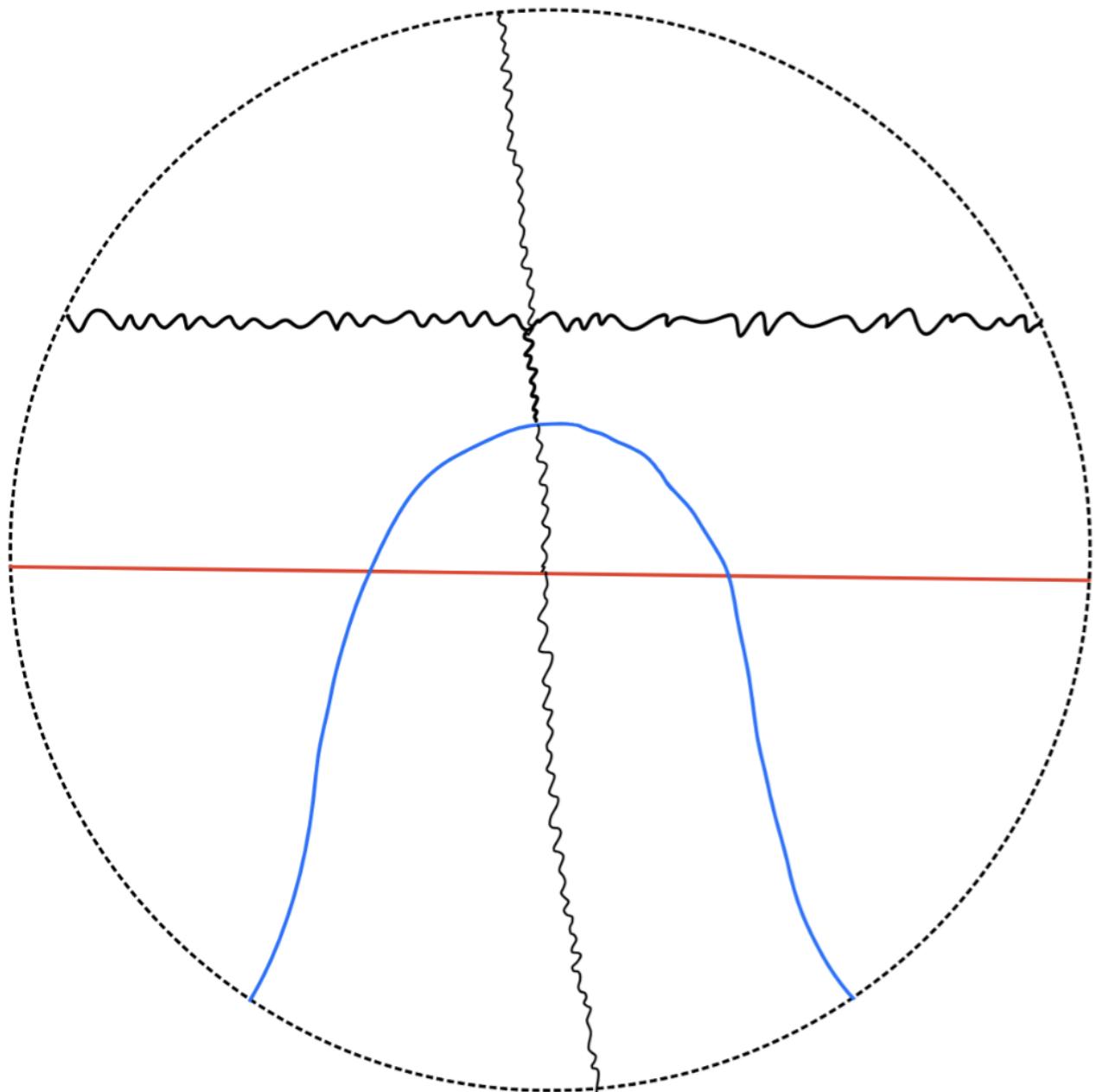


Figure 3.134

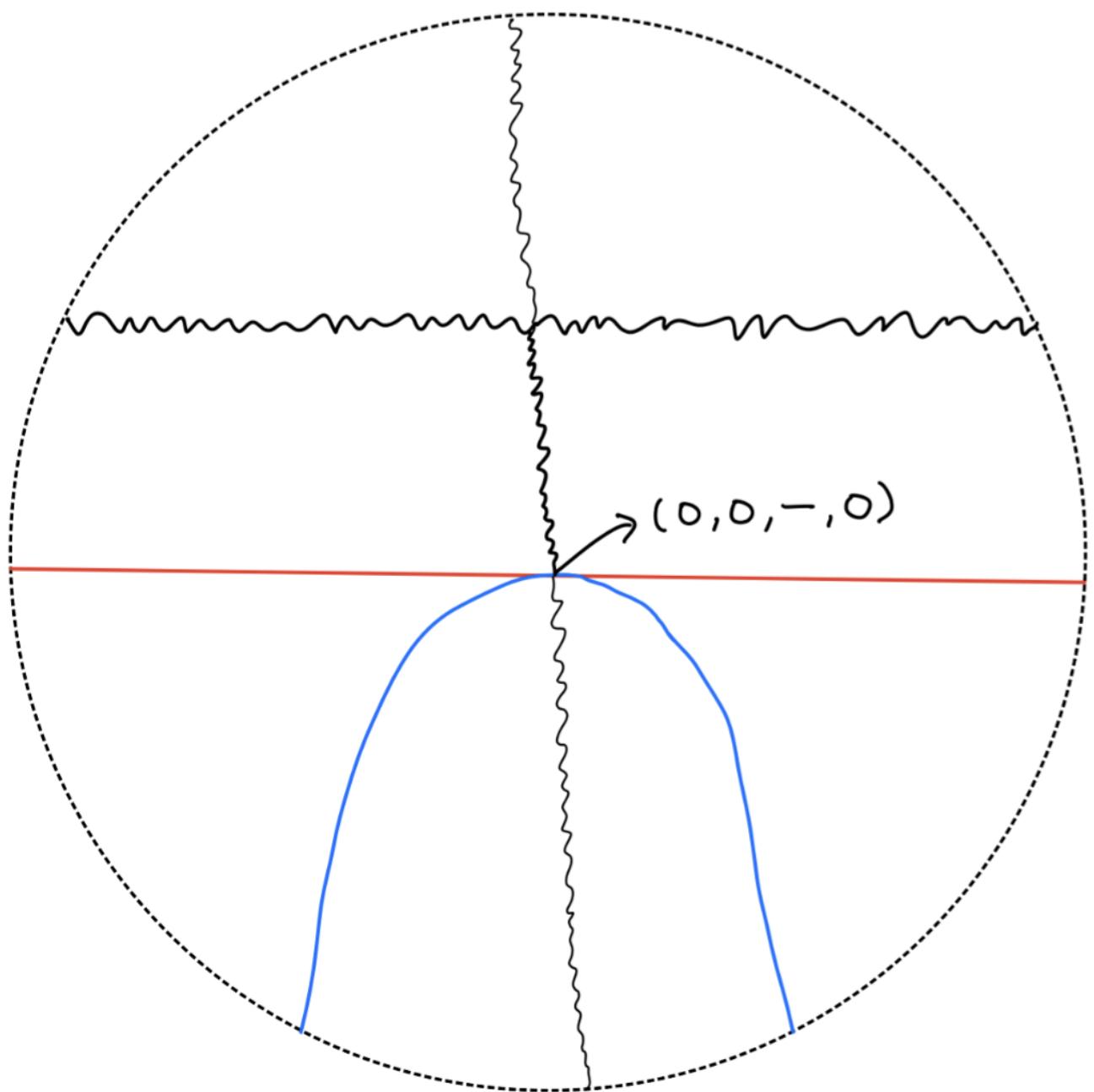


Figure 3.135

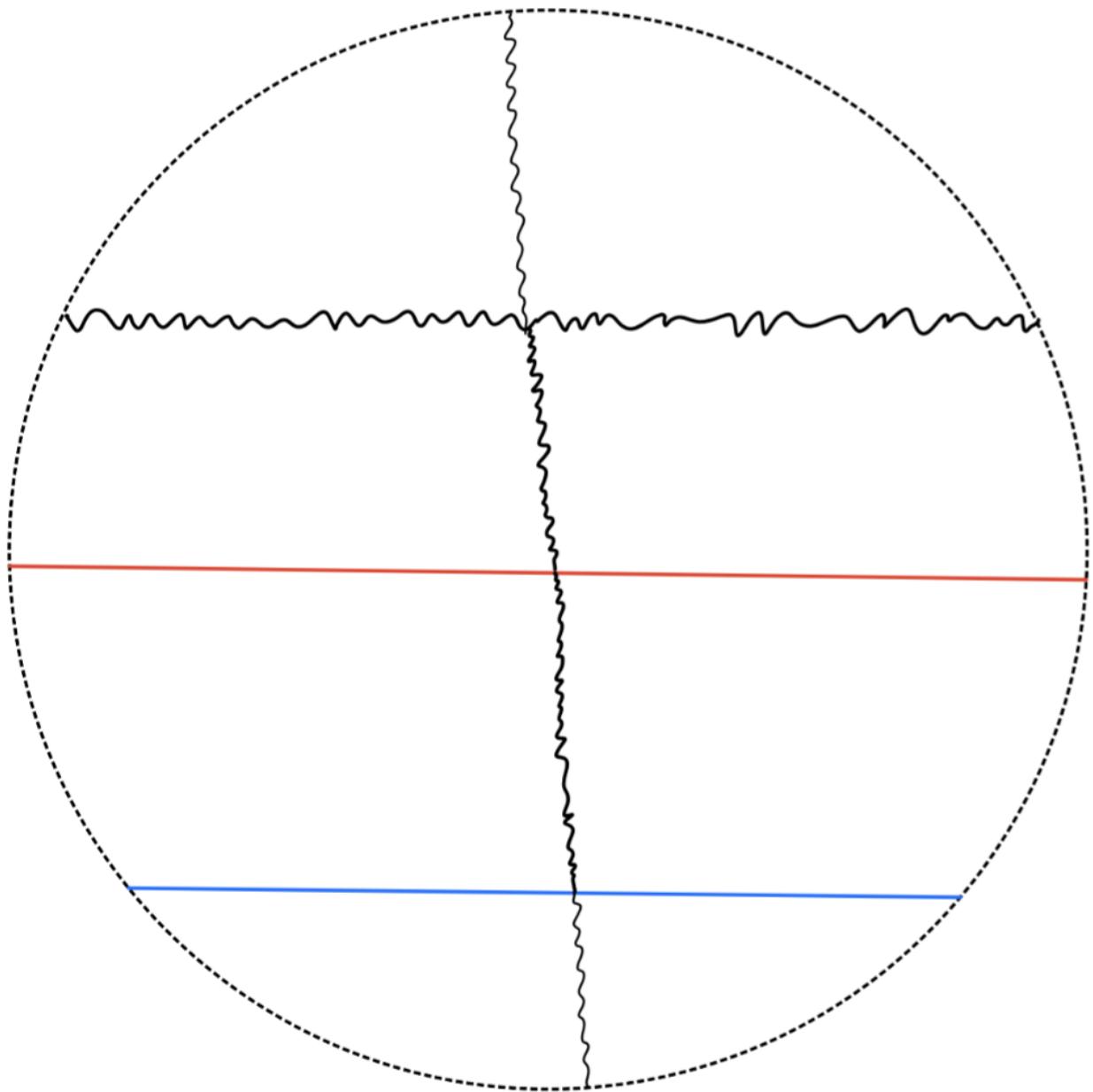


Figure 3.136

- 0 dimensional strata:

$$s_\bullet(0, -, 0, 0), s_\bullet(0, 0, -, 0)$$

Definition 133. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $\text{star}(s)$.

Definition 134. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of s .

Definition 135. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 136. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 137. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 138. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because F_S is legible.

Definition 139. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.6.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.6.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

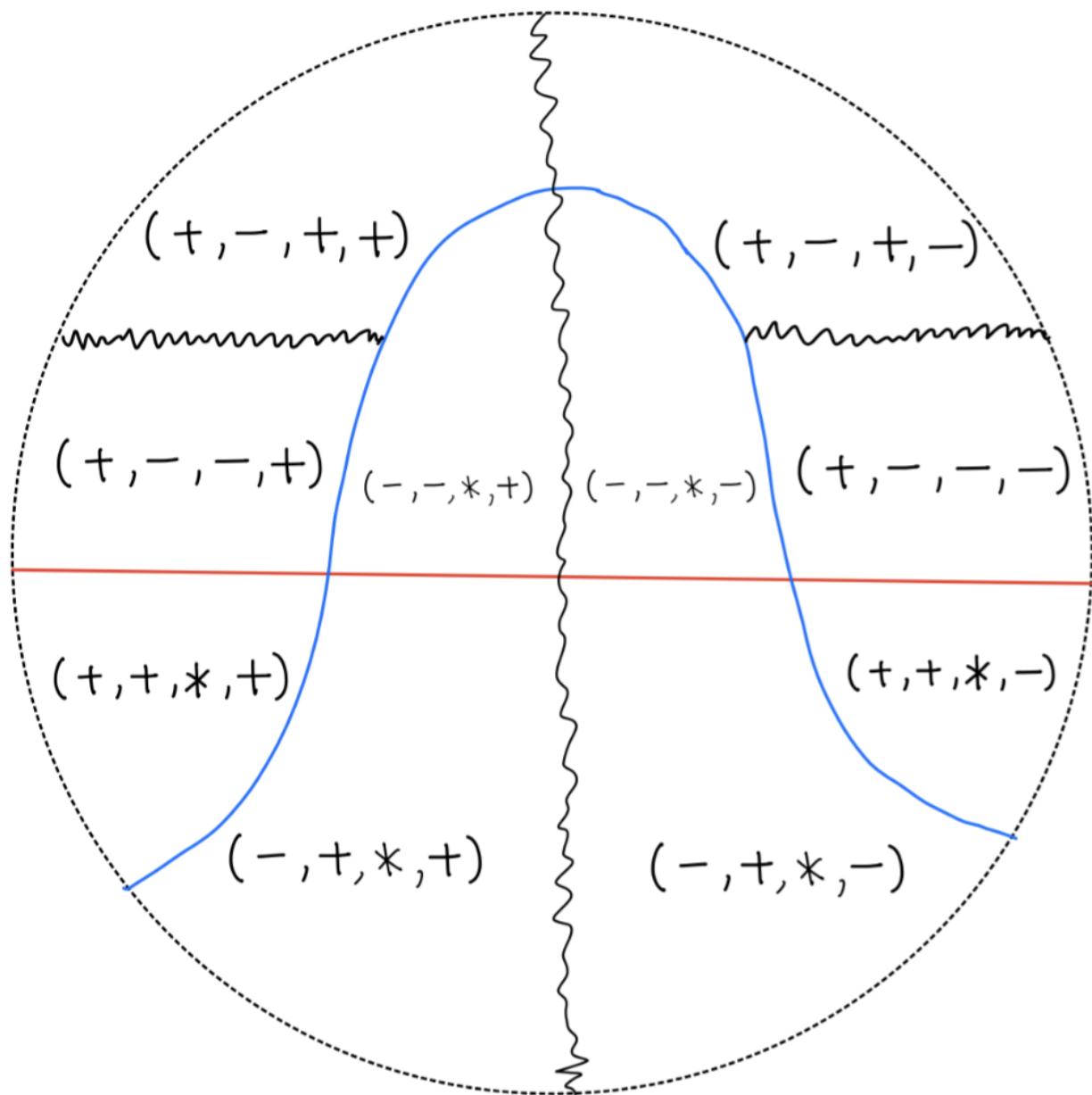


Figure 3.137

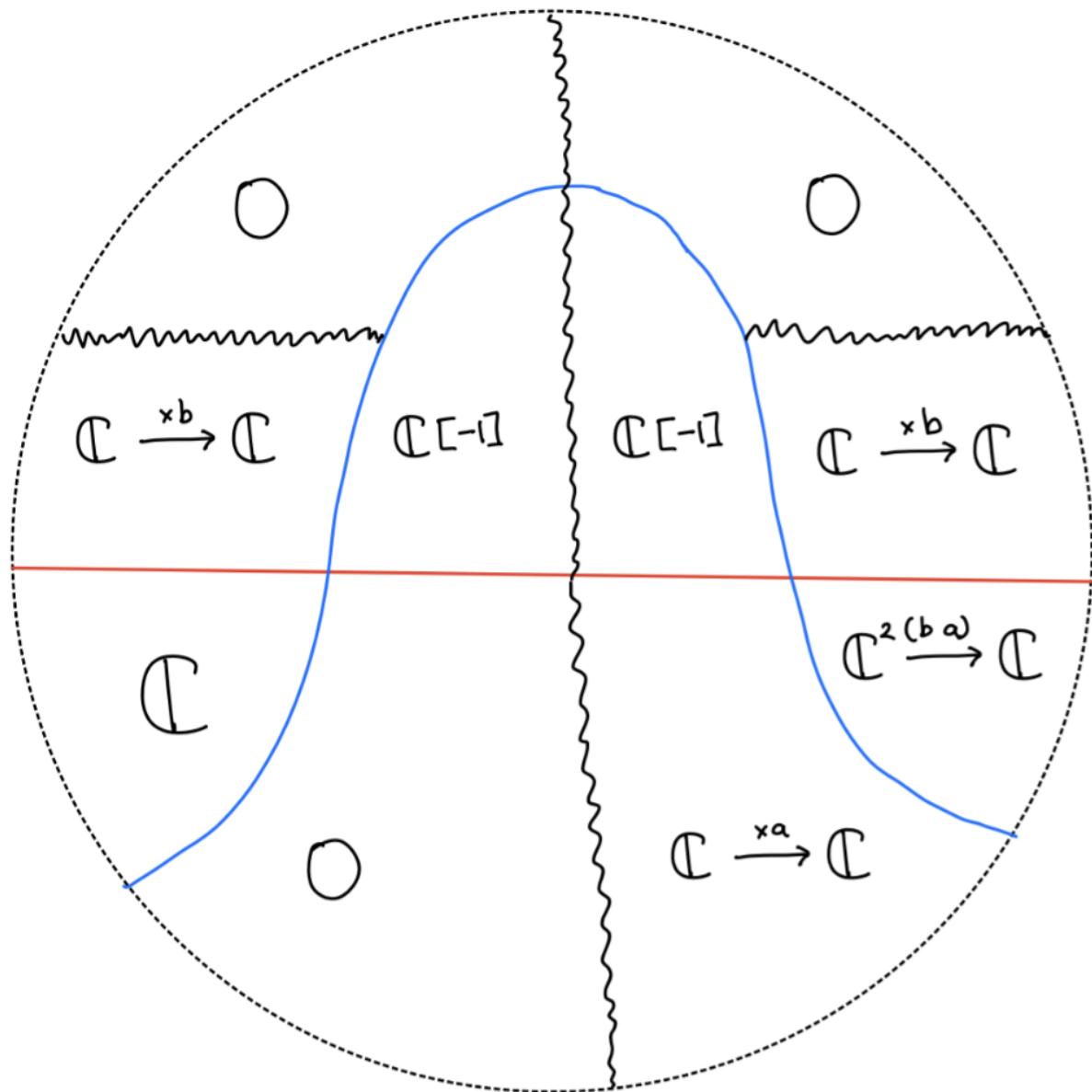


Figure 3.138

- $F_0(-, -, *, -) := \mathbb{C}[-1]$
- $F_0(-, -, *, +) := \mathbb{C}[-1]$
- $F_0(-, +, *, -) := \mathbb{C} \xrightarrow{x^a} \mathbb{C}$
- $F_0(-, +, *, +) := 0$
- $F_0(+, -, -, -) := \mathbb{C} \xrightarrow{x^b} \mathbb{C}$

- $F_0(+, -, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_0(+, -, +, -) := 0$
- $F_0(+, -, +, +) := 0$
- $F_0(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F_0(+, +, *, +) := \mathbb{C}$

Generalization maps:

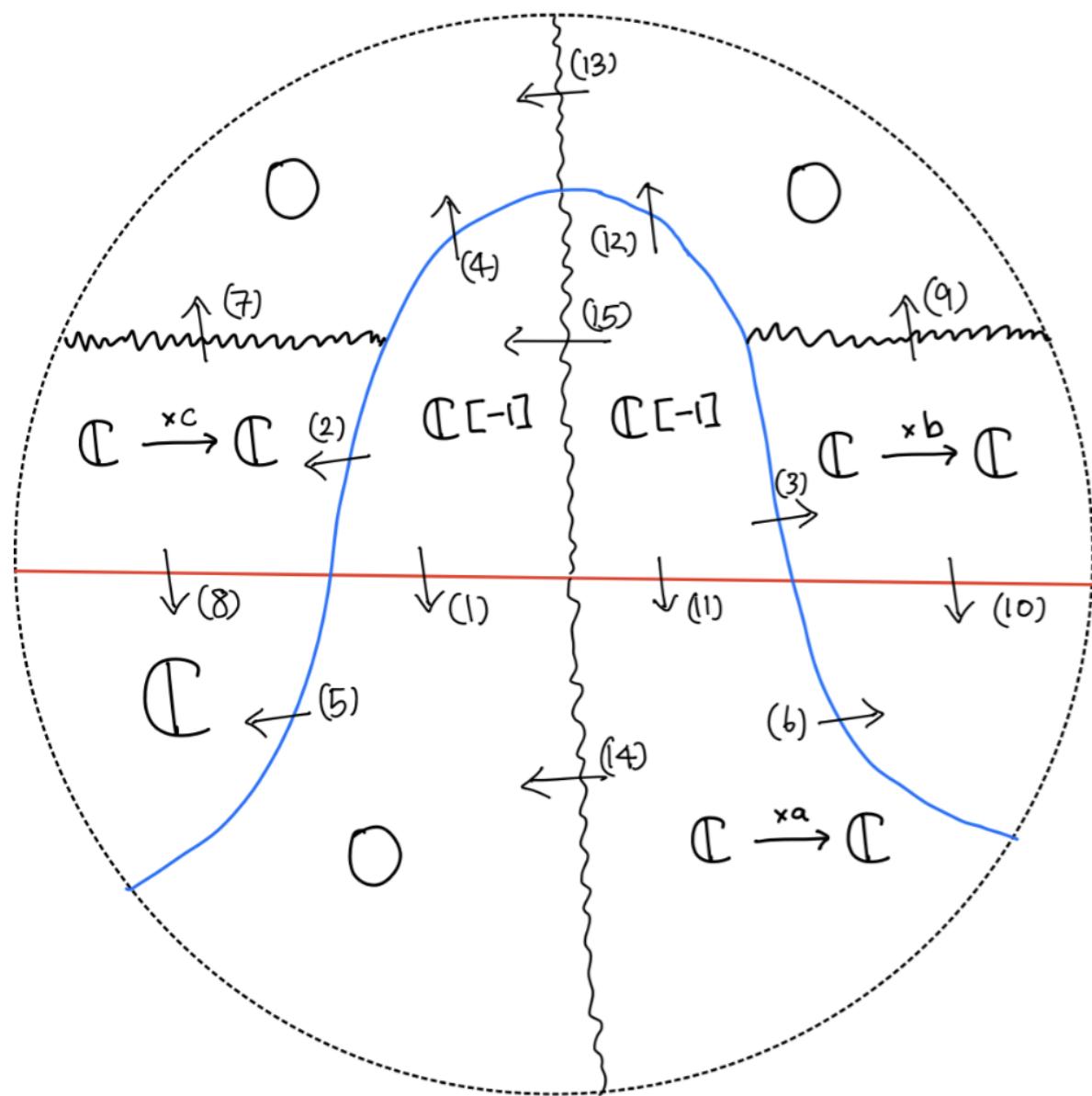


Figure 3.139

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{x_1} & \mathbb{C} \\ \uparrow & \uparrow x_c & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \times a \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(13) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(15) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

3.6.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say *cobord*₂, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := F_{\bullet}(s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

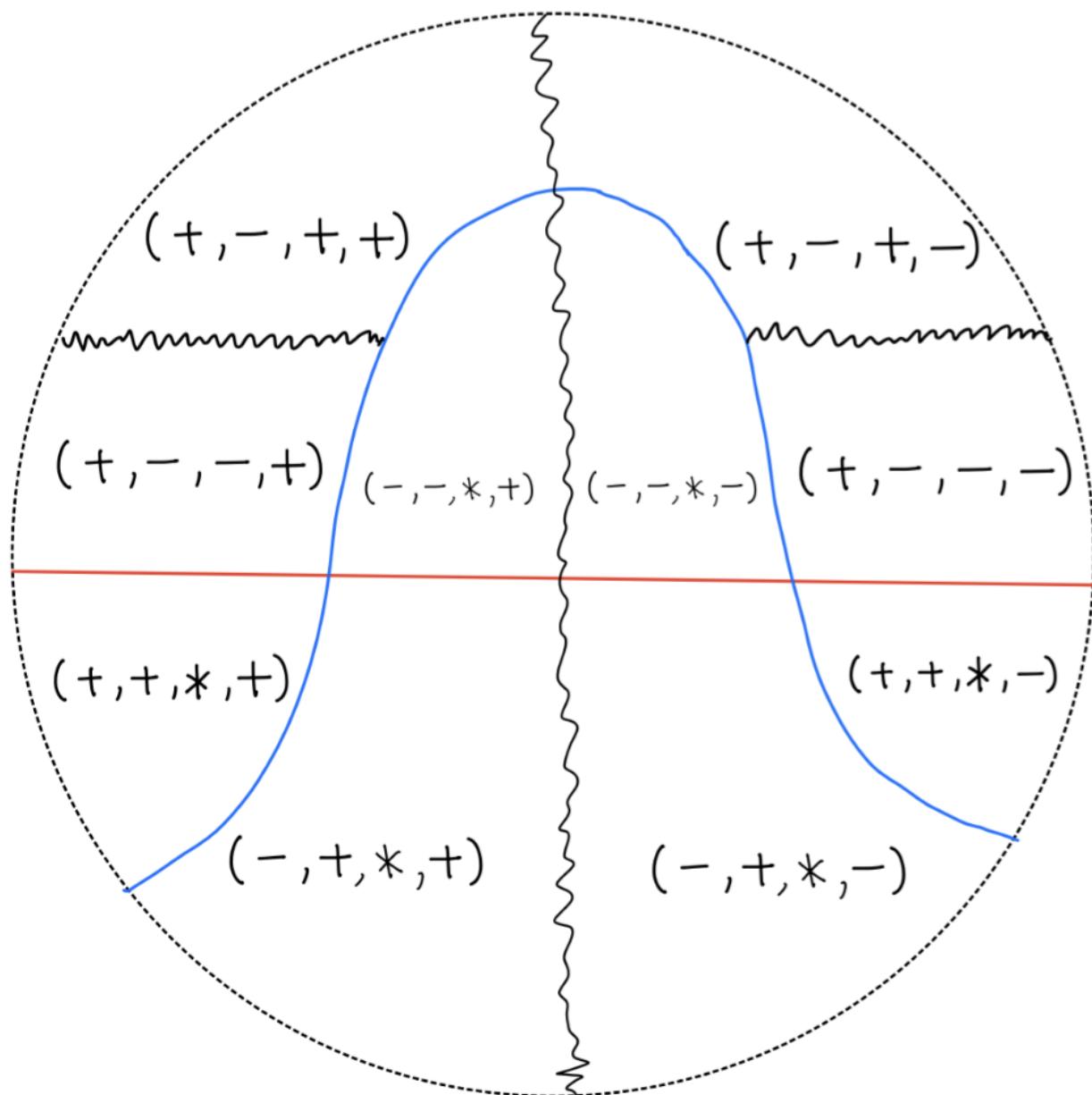


Figure 3.140

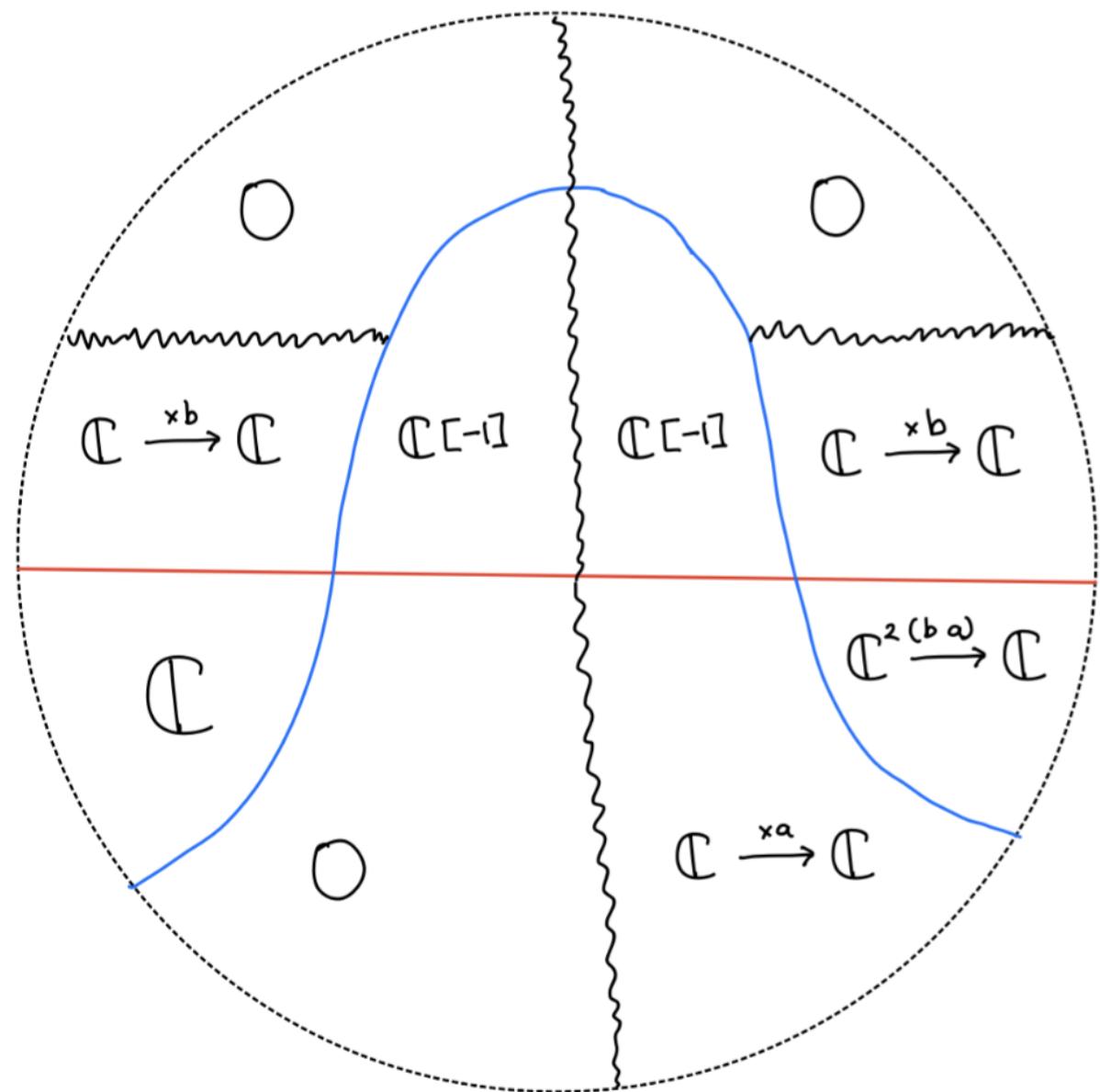


Figure 3.141

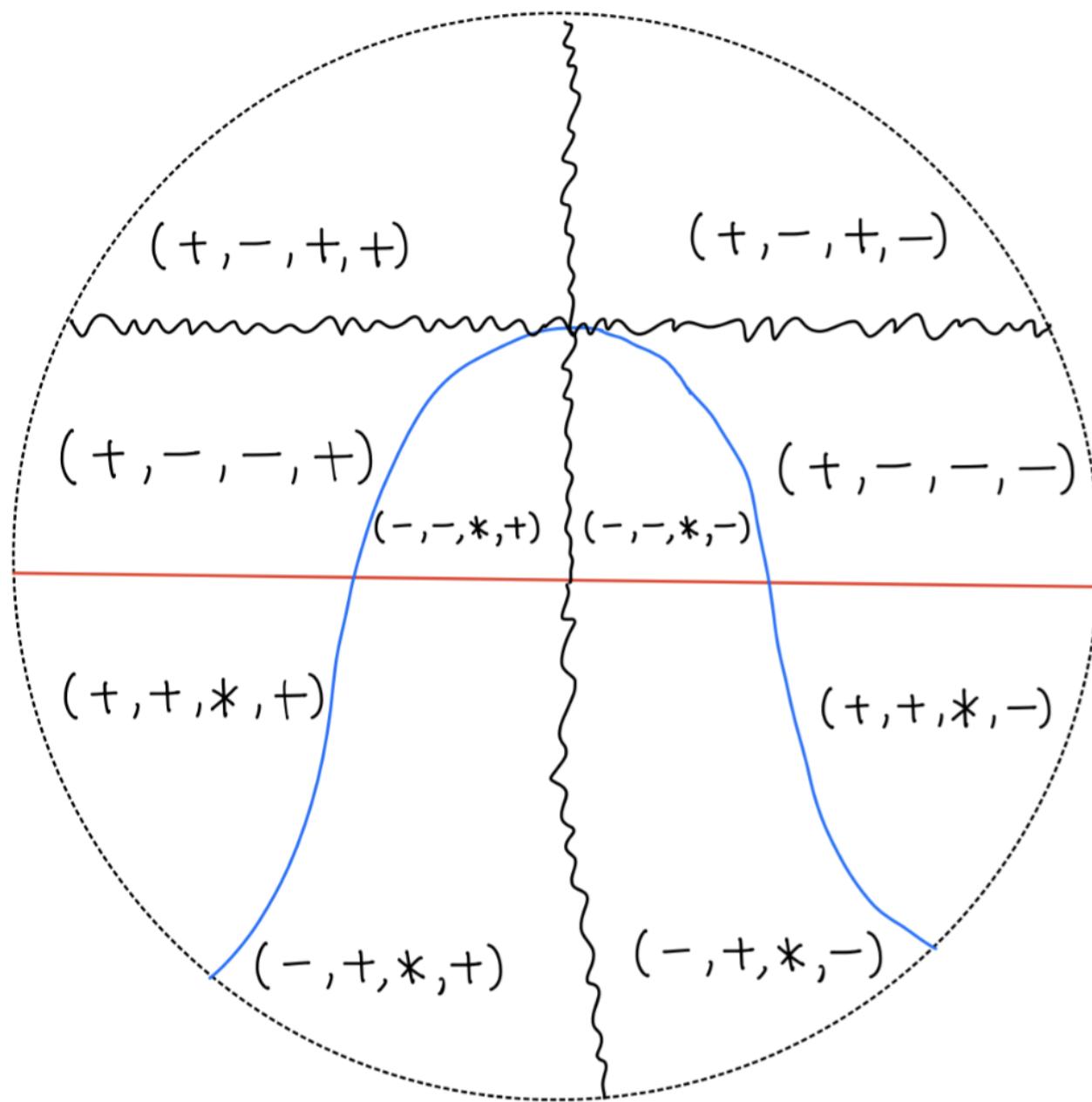


Figure 3.142

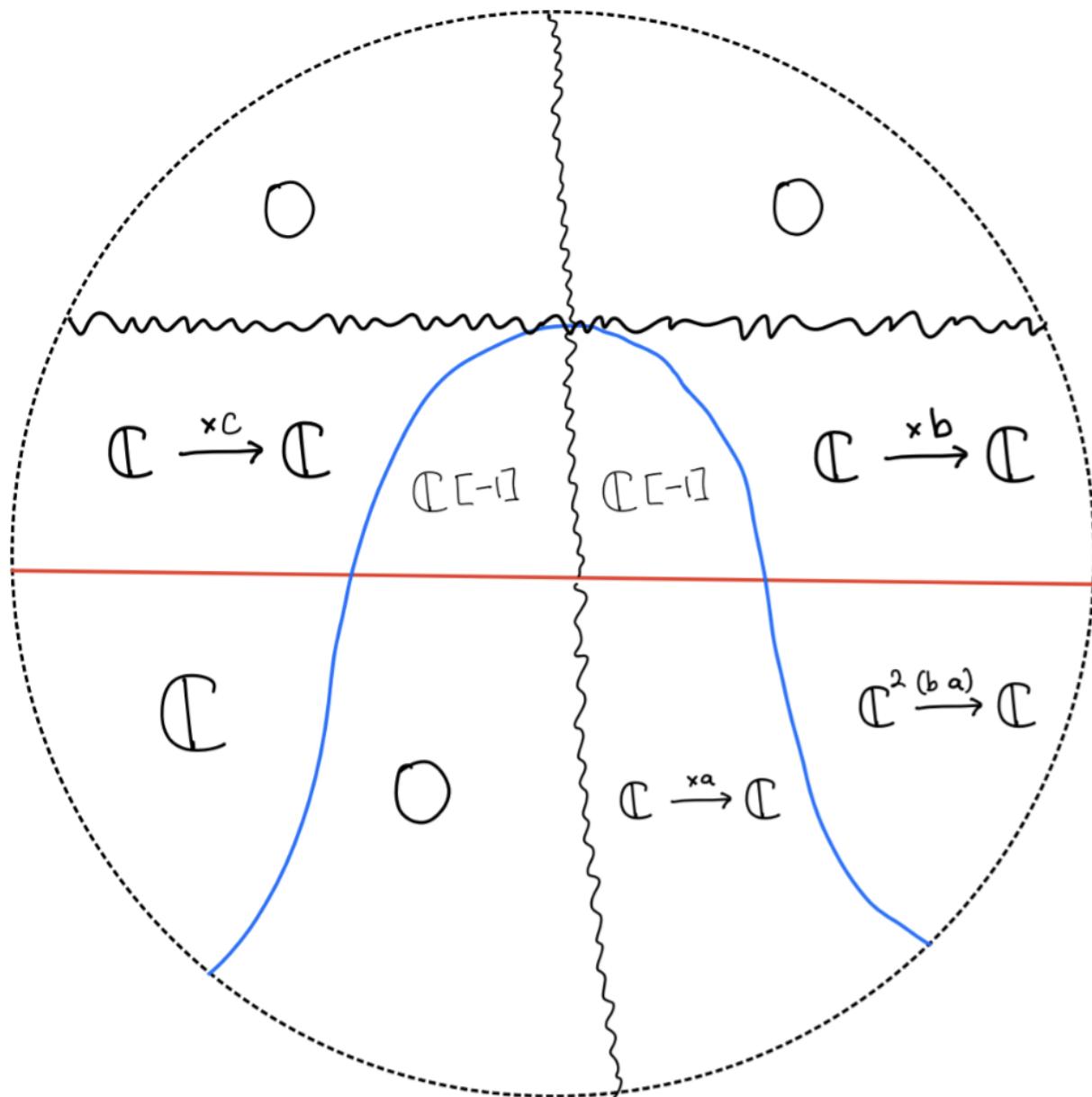


Figure 3.143

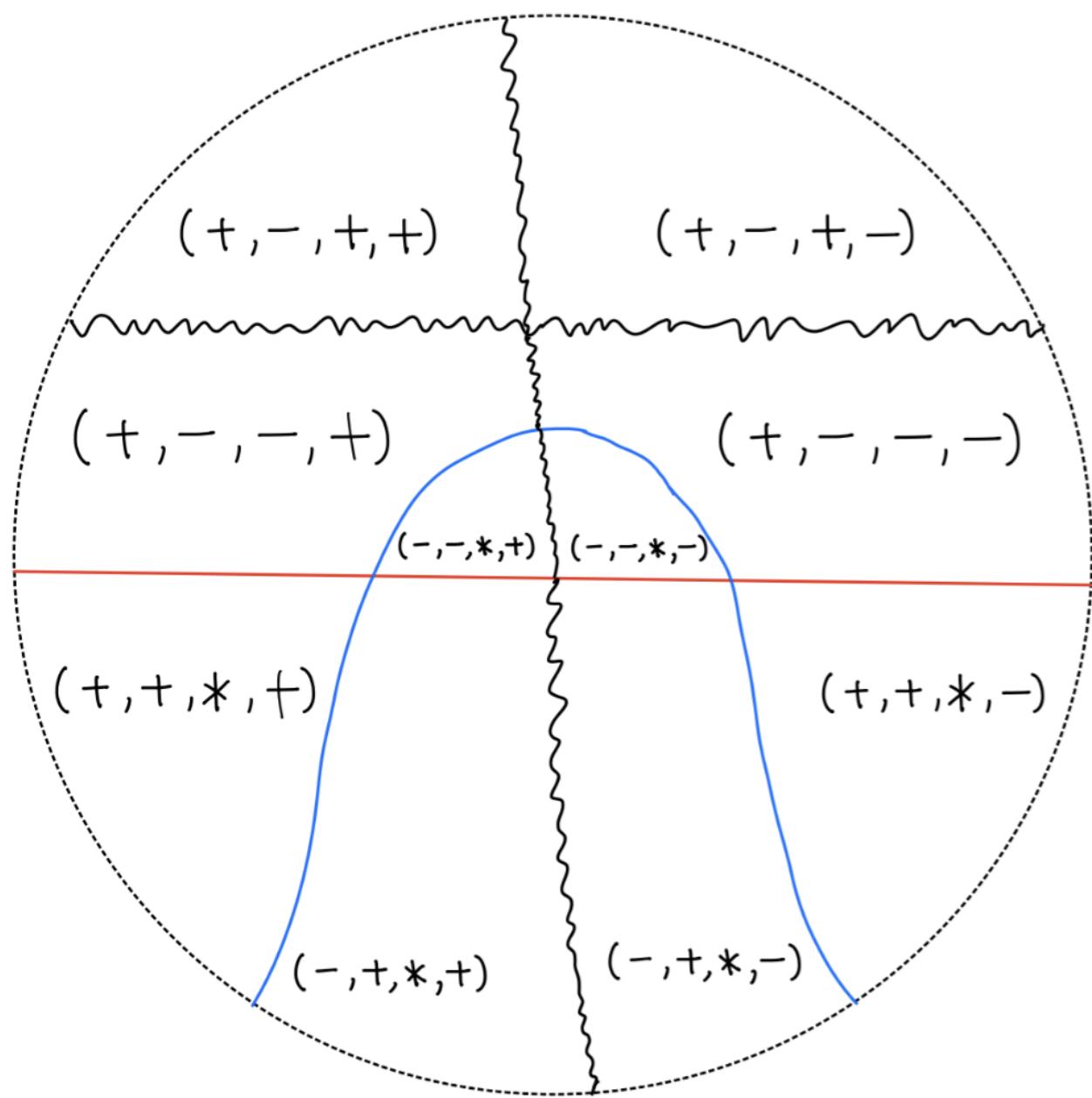


Figure 3.144

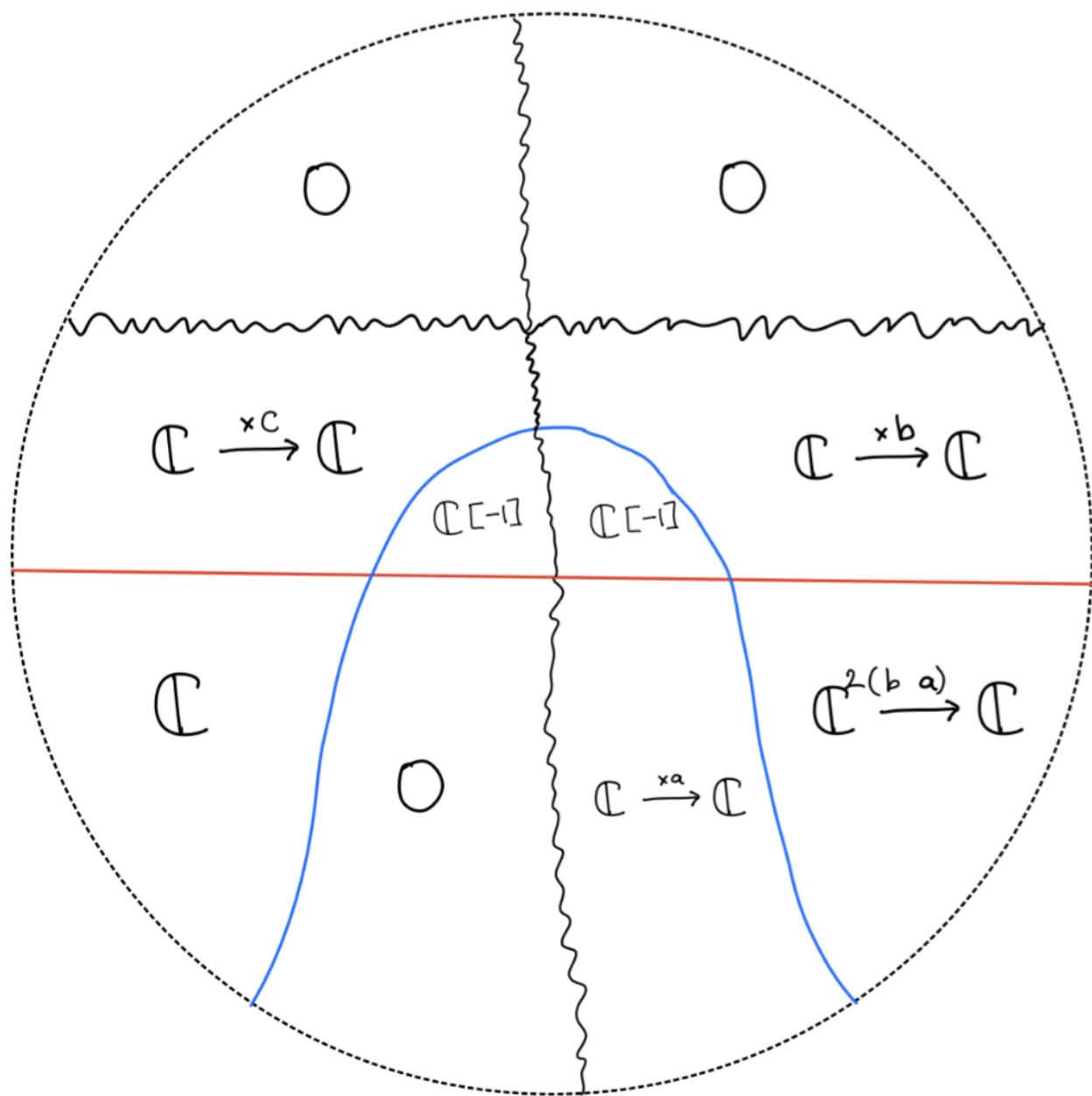


Figure 3.145

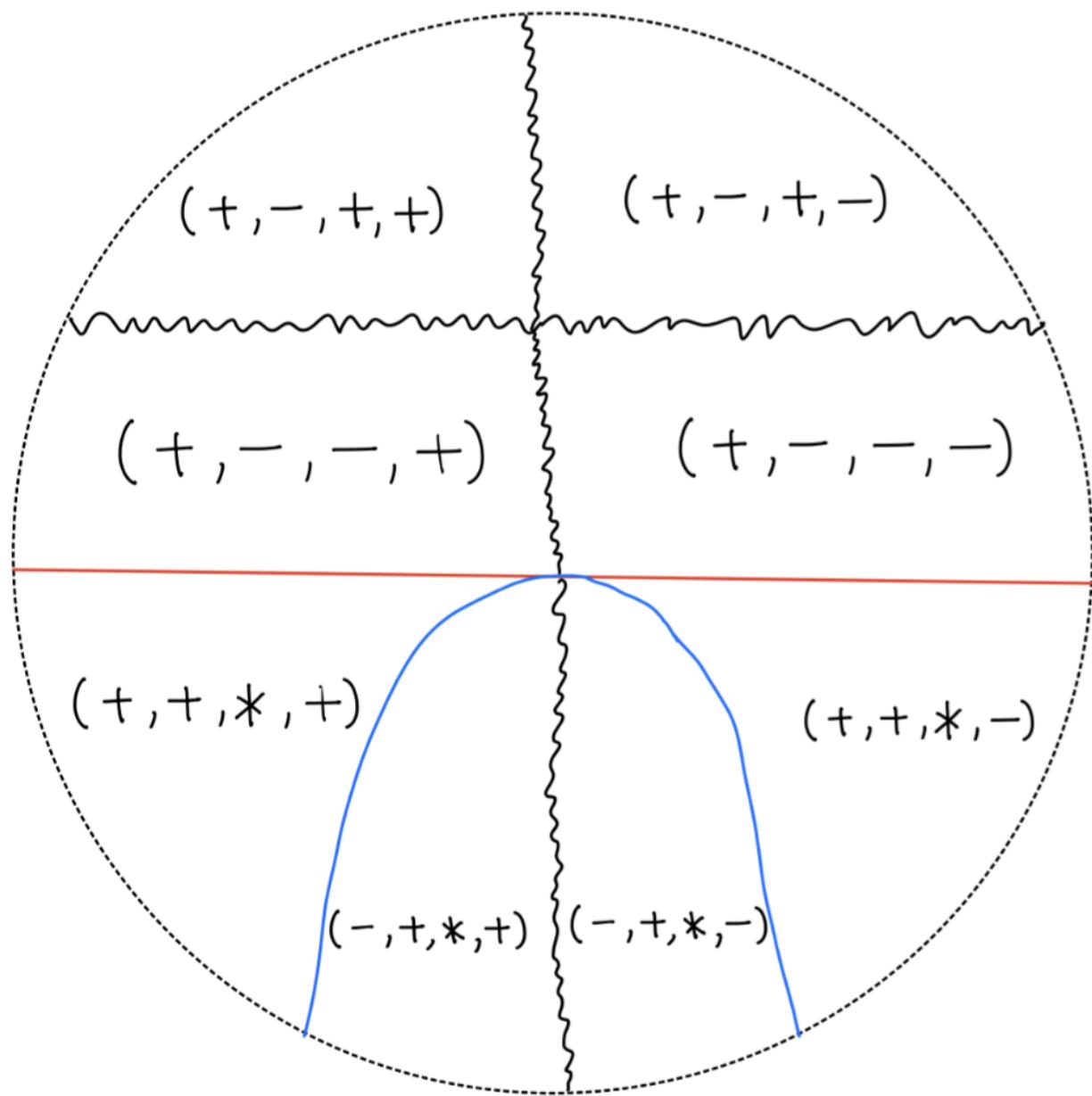


Figure 3.146

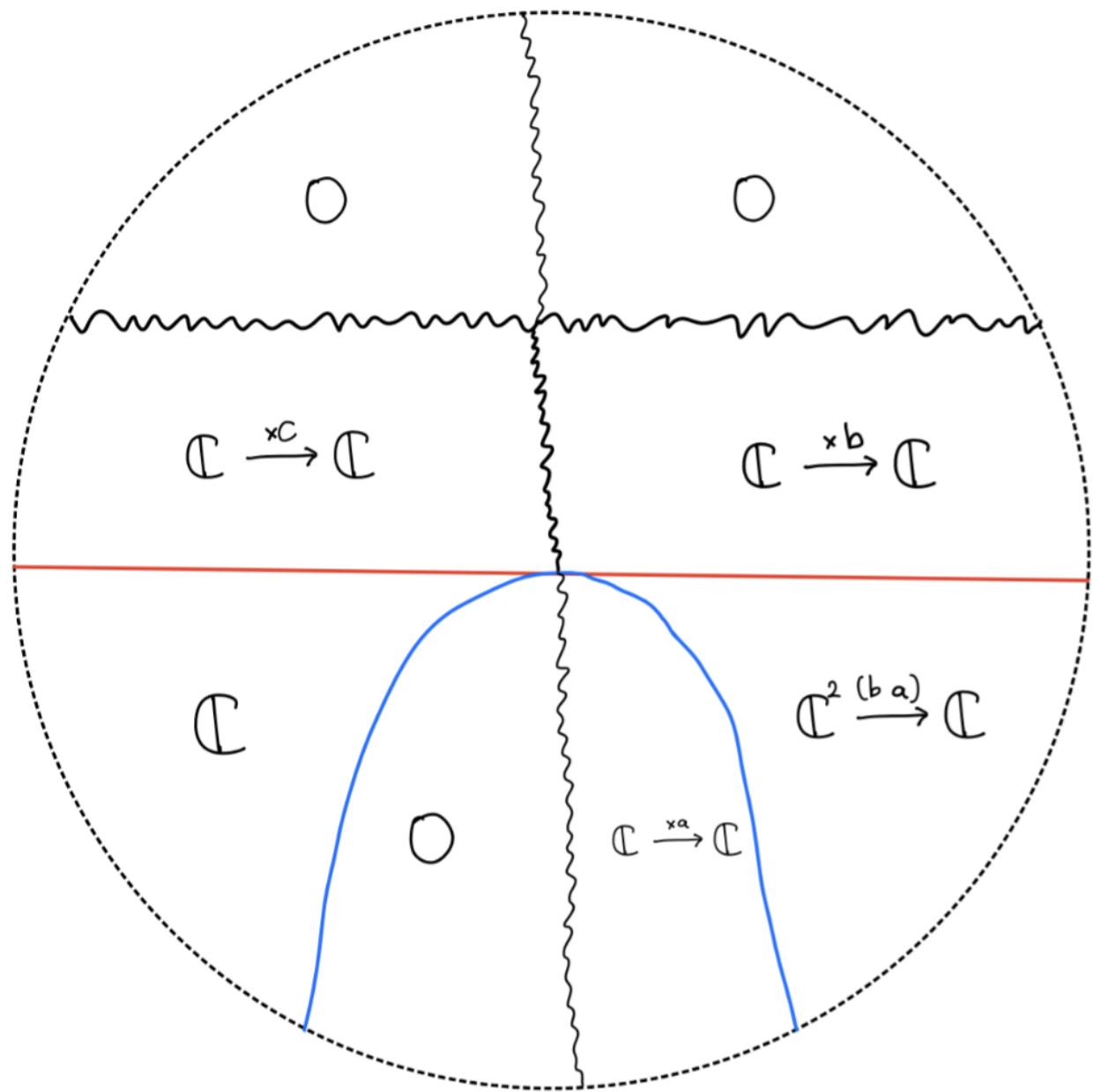


Figure 3.147

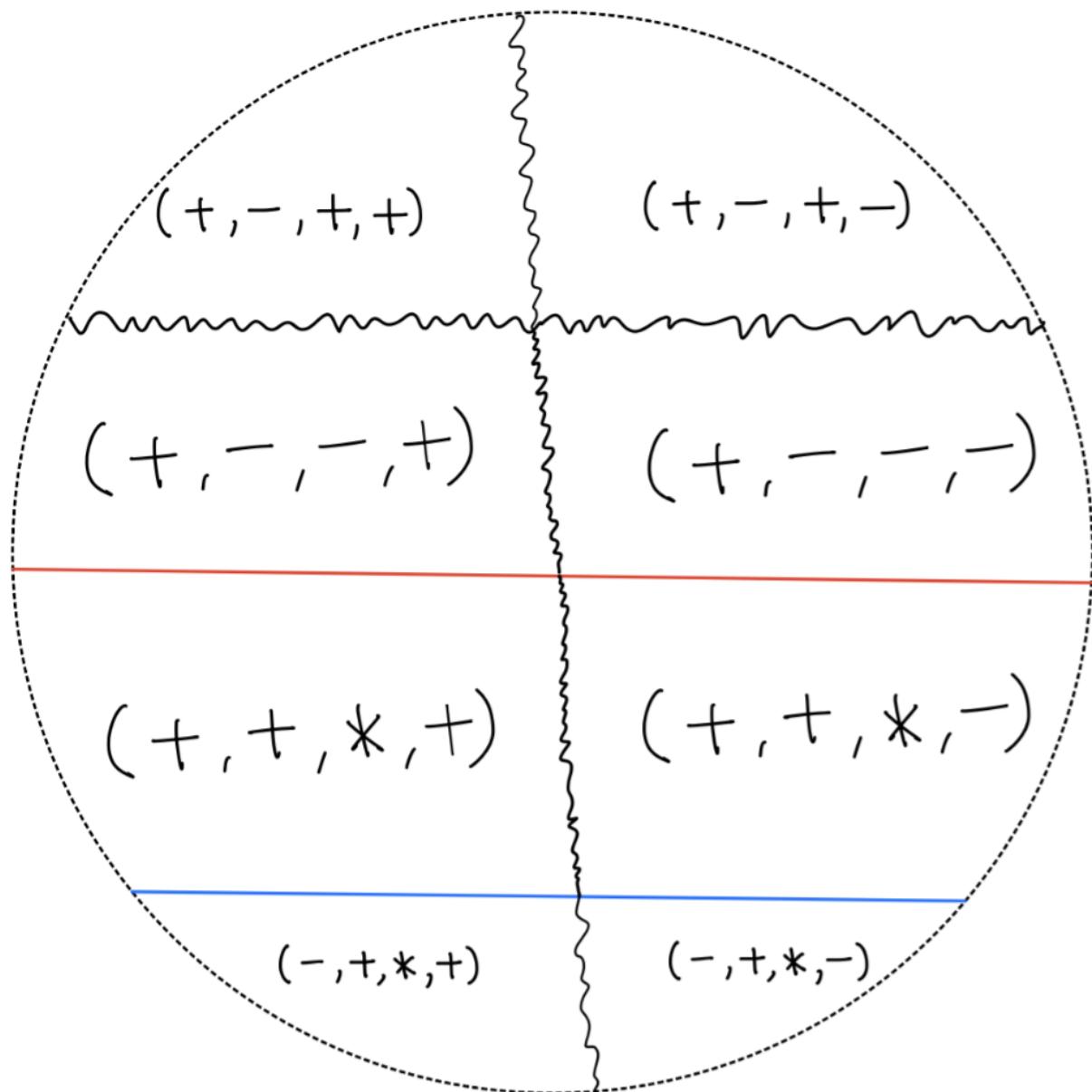


Figure 3.148

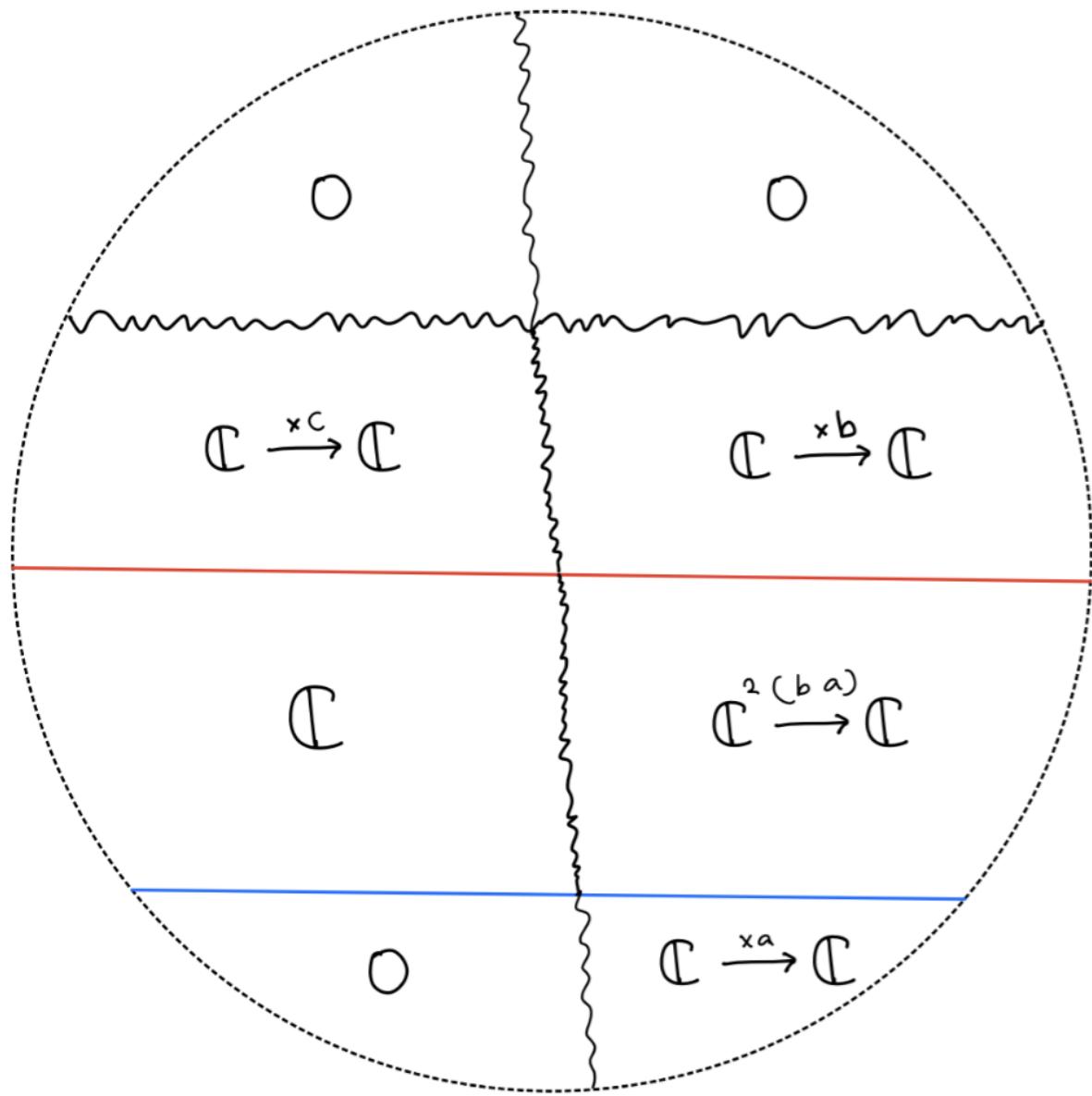


Figure 3.149

- $F_{\bullet}(-, -, *, -) := \mathbb{C}[-1]$
- $F_{\bullet}(-, -, *, +) := \mathbb{C}[-1]$
- $F_{\bullet}(-, +, *, -) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_{\bullet}(-, +, *, +) := 0$
- $F_{\bullet}(+, -, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$

- $F_\bullet(+, -, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_\bullet(+, -, +, -) := 0$
- $F_\bullet(+, -, +, +) := 0$
- $F_\bullet(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F_\bullet(+, +, *, +) := \mathbb{C}$

Generalization maps:

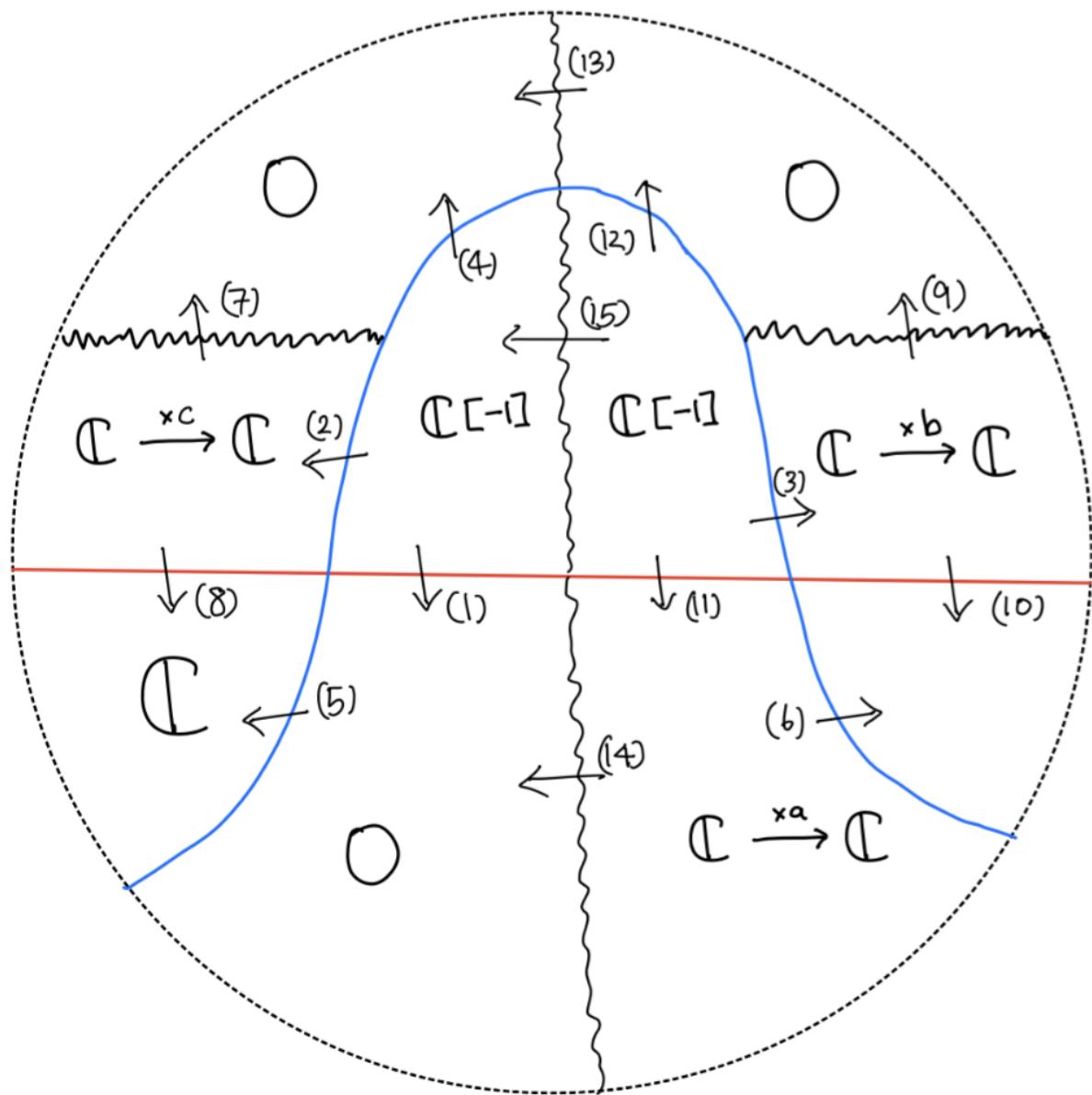


Figure 3.150

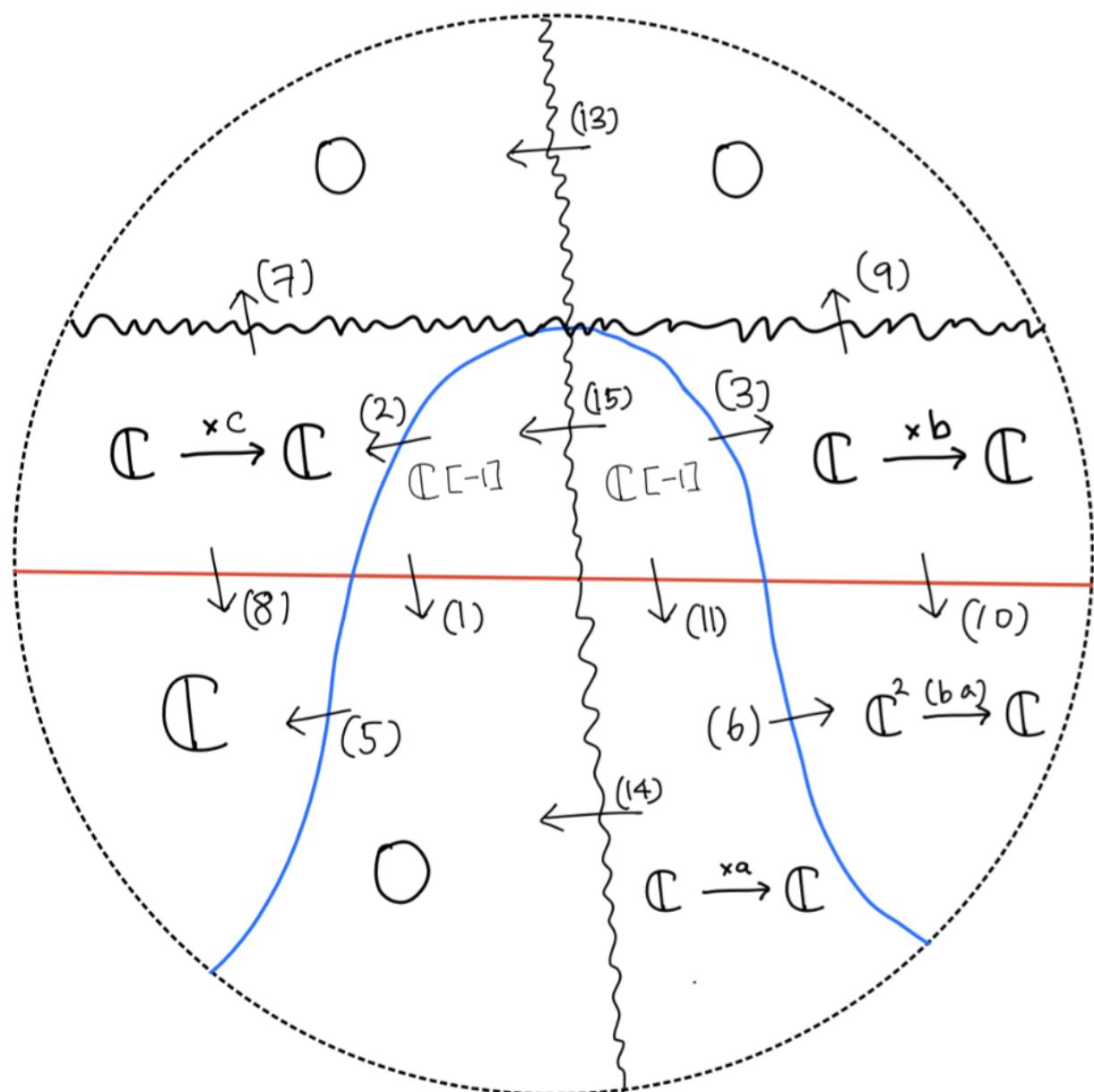


Figure 3.151

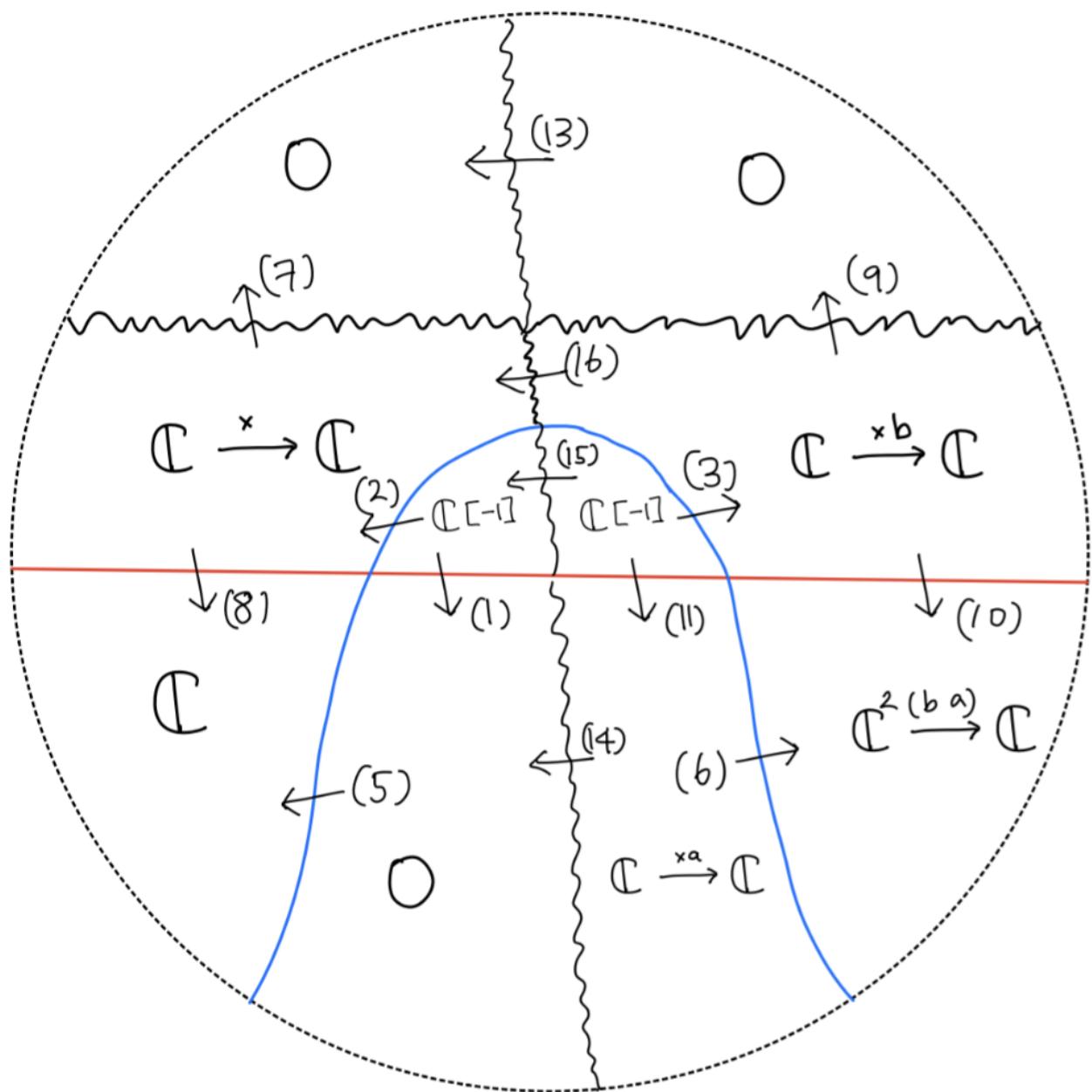


Figure 3.152

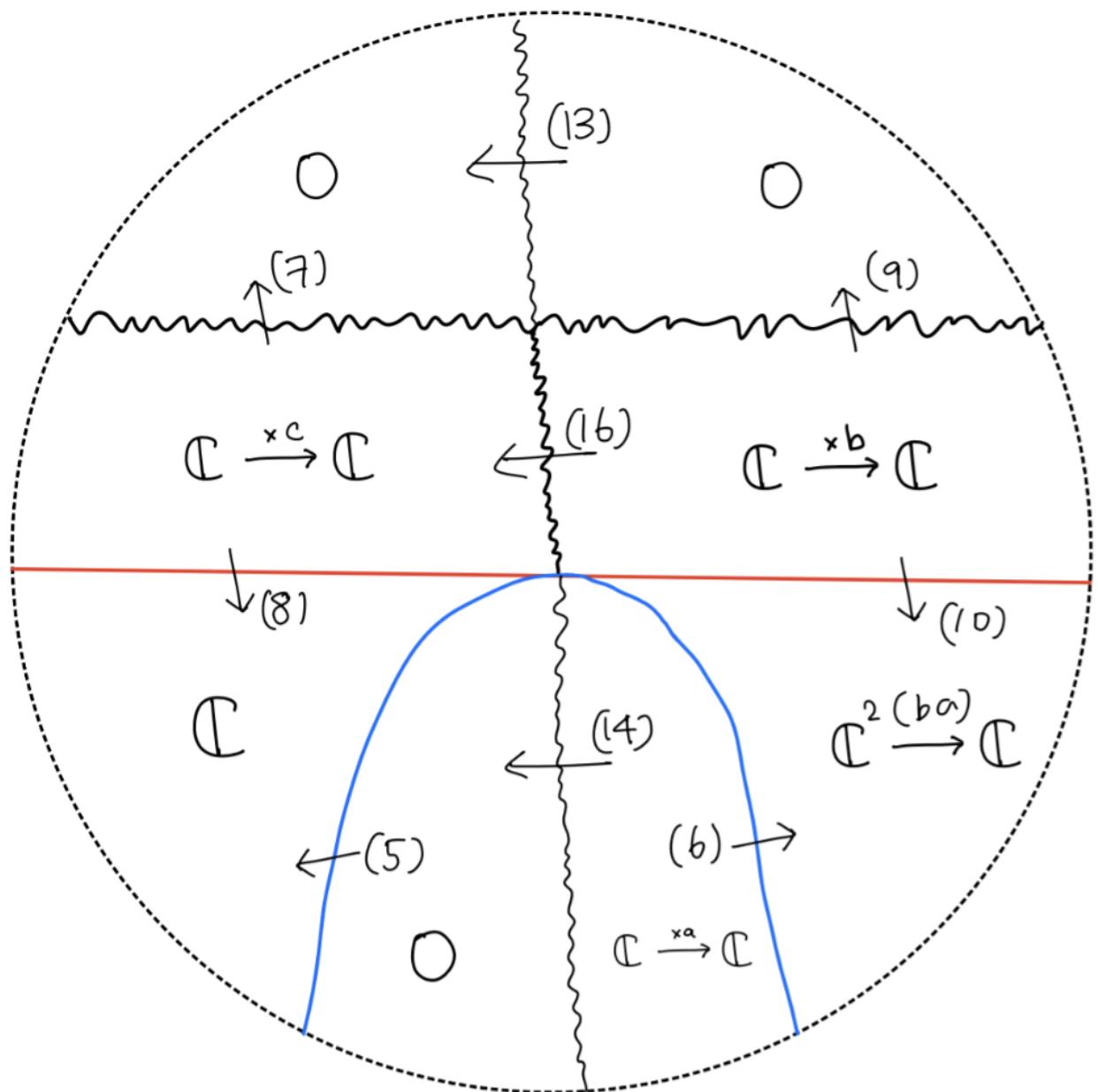


Figure 3.153

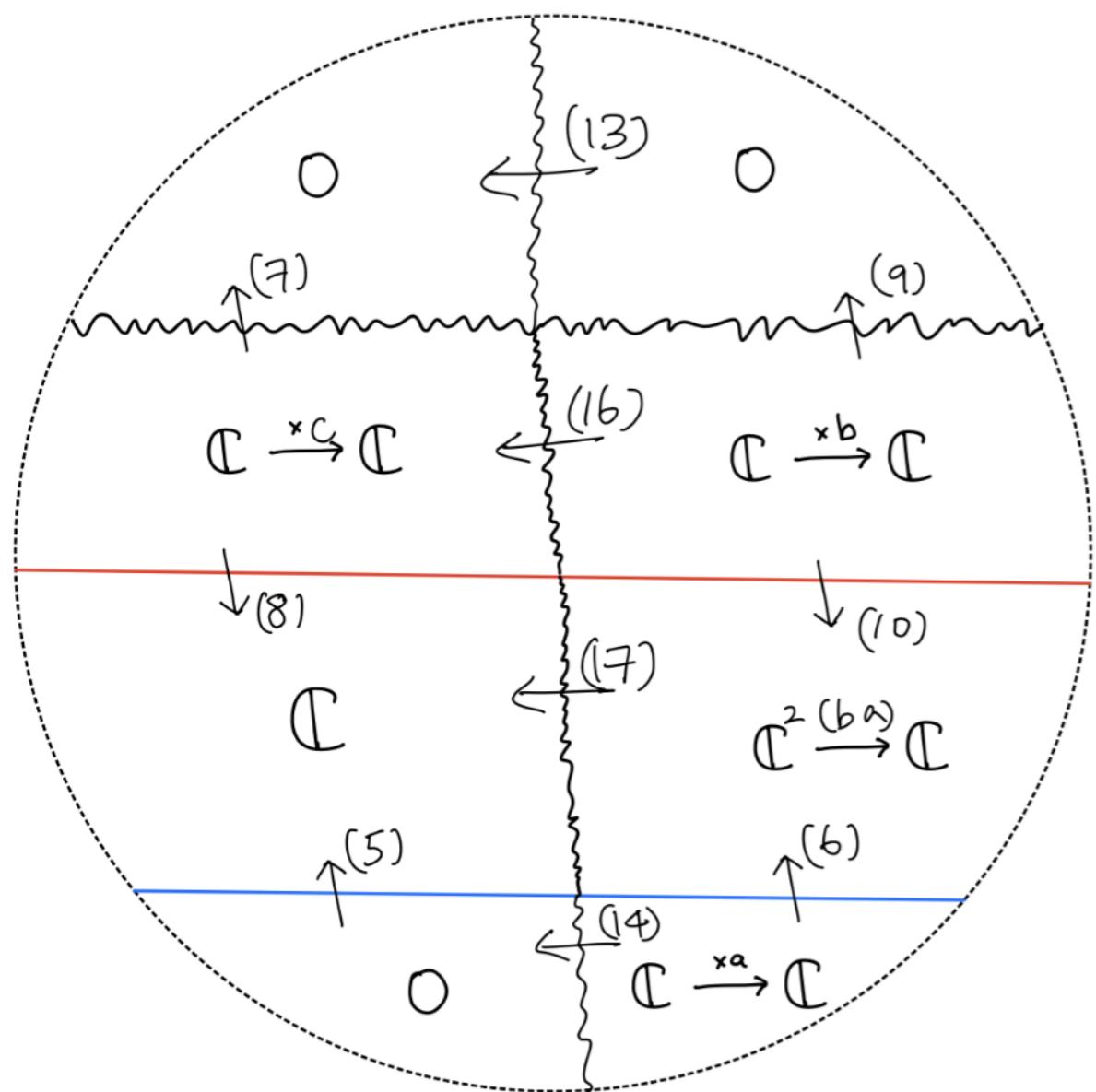


Figure 3.154

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} C & \xrightarrow{x_1} & C \\ \uparrow & \uparrow x_c & \\ 0 & \longrightarrow & C \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \times a \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(13) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times a} & 0 \end{array}$$

$$(15) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(16) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow_{\times b} & & \uparrow_{\times c} \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & 0 \end{array}$$

$$(17) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow_{(b \ a)} & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 140. $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 141. we define γ_\bullet to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 142. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M \times [0, 1], \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of $M \times [0, 1]$.

Since \mathcal{F}^\bullet is constant along the time coordinate on $U'^c \times [0, 1]$, it is enough to check for the points of $U \times [0, 1] \cong D_{r=2} \times [0, 1]$. Now consider the following open cover of $D_{r=2} \times [0, 1]$

$$\{star(s_\bullet(0, -, 0, 0)), star(s_\bullet(0, 0, -, 0))\}$$

(1) First, let's show that the microlocal stalks of $\mathcal{F}^\bullet|_{star(s_\bullet(0, -, 0, 0))}$ vanishes. Note that there is a diffeomorphism between $star(s_\bullet(0, -, 0, 0))$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 .

To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$

- $F^3(-, -, +) := \mathbb{C}[-1]$

- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$

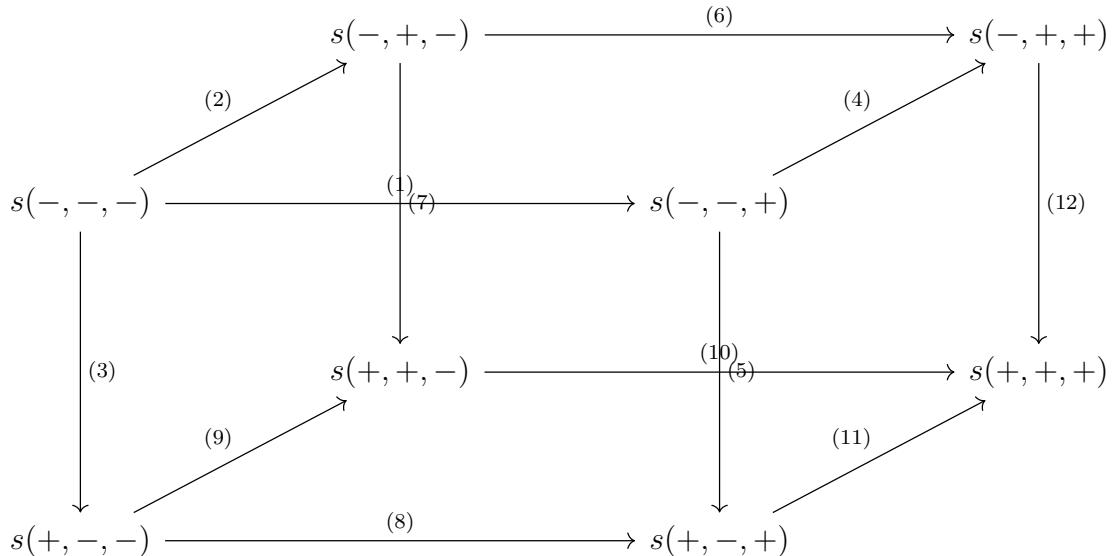
- $F^3(-, +, -) := \mathbb{C}[-1]$

- $F^3(-, +, +) := \mathbb{C}[-1]$

- $F^3(+, +, -) := 0$

- $F^3(+, +, +) := 0$

Generalization maps:



$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times c \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow \times c \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, +, -)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, +) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times c} & \mathbb{C} \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \xrightarrow{\times c} \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & 0
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1]
 \end{array}$$

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow \mathbb{C} \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & 0
 \end{array}$$

 (vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-,+,-) & \xrightarrow{s(-,+0)} & s(-,+,+) \\
 s(-,0,-) \nearrow & \downarrow & & \searrow s(-,0+) \\
 s(-,-,-) & \xrightarrow[s(-,-,0)]{s(0,+,-)} & s(-,-,+) & & \downarrow s(0,+,+) \\
 \downarrow s(0,-,-) & & & & \downarrow \\
 & s(+,+,-) & \xrightarrow[s(+,+0)]{s(0,+,-)} & s(+,+,+) \\
 s(+,0,-) \nearrow & \downarrow & \searrow s(+,0+) & & \\
 s(+,-,-) & \xrightarrow{s(+,-0)} & s(+,-,+) & & \\
 = & & & & \\
 & \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \\
 \mathbb{C}[-1] \nearrow & \downarrow & & \searrow & \downarrow \\
 & & \mathbb{C}[-1] & & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & 0 & \\
 \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times c} \mathbb{C} & & \\
 \mathbb{C} \xrightarrow{\times 1} \mathbb{C} & , & \mathbb{C} \xrightarrow{\times 1} \mathbb{C} & , & \mathbb{C} \xrightarrow{\times 1} \mathbb{C} \\
 \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 0 \longrightarrow 0 & & 0 \longrightarrow 0 & & \mathbb{C} \xrightarrow{\times bc^{-1}} \mathbb{C} \\
 & & & & \uparrow \\
 & & & & 0 \longrightarrow 0
 \end{array}$$

Note that the vertical arrows between the top row and the bottom row are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of $\mathcal{F}^\bullet|_{star(s_\bullet(0,0,-,0))}$ vanishes. Note that there is a diffeomorphism between $star(s_\bullet(0,0,-,0))$ and \mathbb{R}^3 that preserves

there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 .

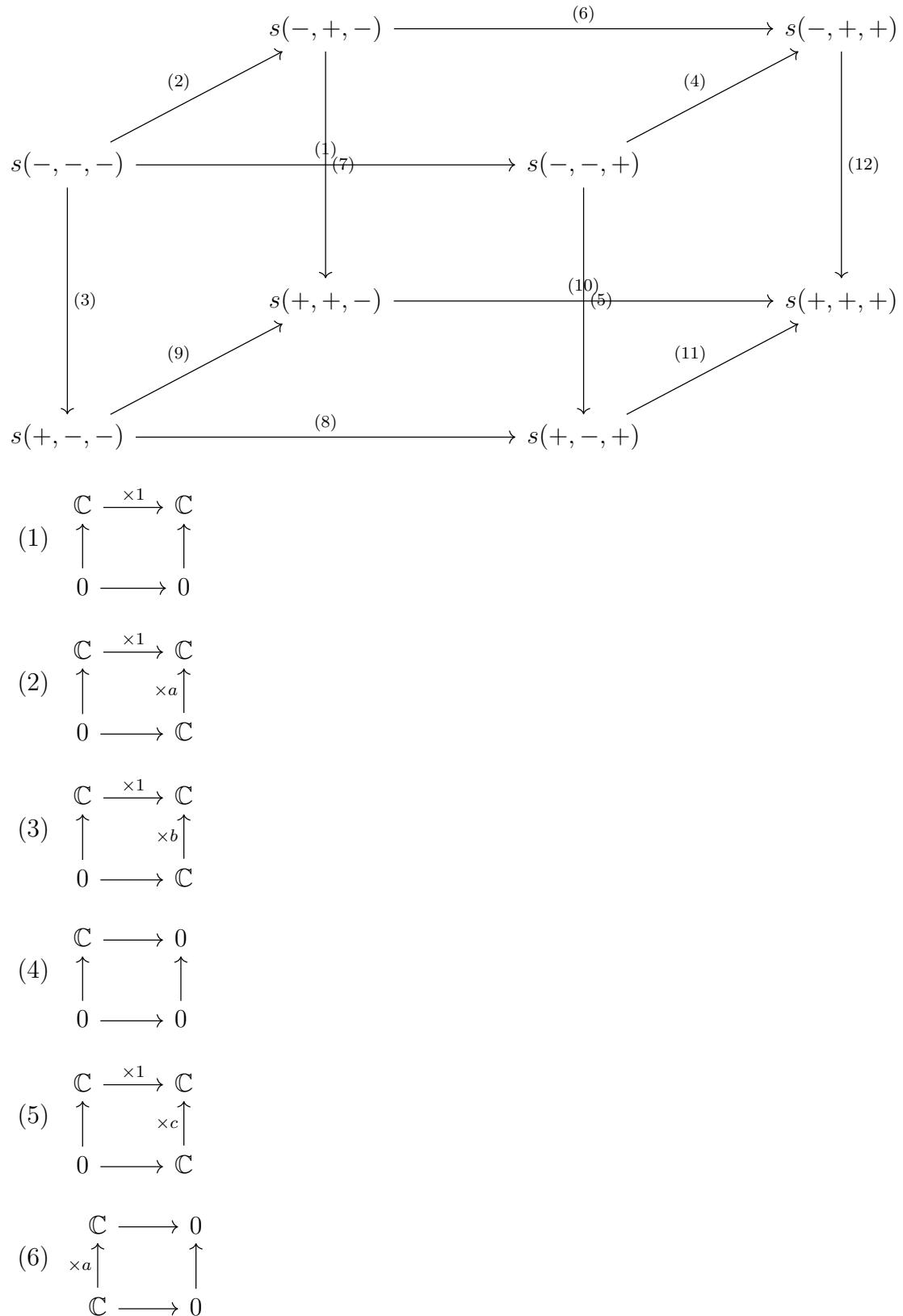
To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F^3(-, +, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(-, +, +) := 0$
- $F^3(+, +, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}$

Generalization maps:



$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \times_a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times_b \uparrow & & \times_c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times_b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \end{array}$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times c} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

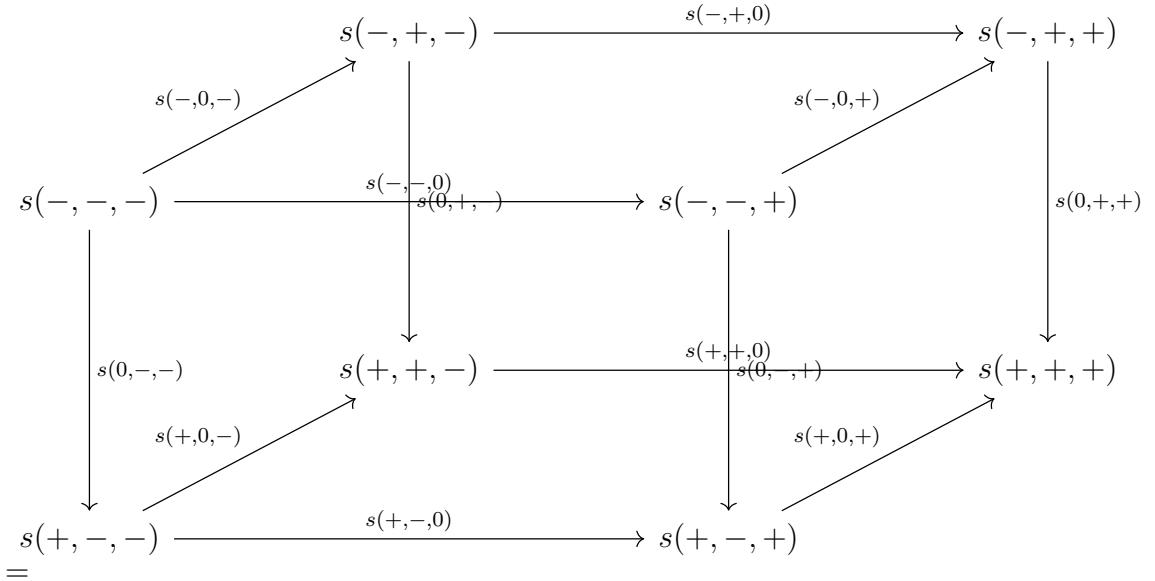
$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \end{array}$$

$$(iv) \quad \begin{array}{ccc} F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \longrightarrow 0 \\ \downarrow & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{(b \ a)} & \mathbb{C} \xrightarrow{(bc^{-1} \ 0)} \mathbb{C} \end{array}$$

$$(v) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \longrightarrow 0 \end{array}$$

$$(vi) \quad \begin{array}{ccc} F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow \mathbb{C} \xrightarrow{\times c} \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^2 & \xrightarrow{(b \ a)} & \mathbb{C} \longrightarrow \mathbb{C} \end{array}$$

(vii) the cubic diagram:



$$\begin{array}{ccccc}
 & \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & 0 \\
 & \downarrow & & & \downarrow \\
 \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C}[-1] & \xrightarrow{\quad} & \\
 & \downarrow & & \downarrow & \\
 & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} & \longrightarrow & \mathbb{C} & \\
 & \downarrow & & \downarrow & \\
 \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times c} \mathbb{C} & \longrightarrow & \\
 & \uparrow & , \uparrow & \uparrow & \\
 & \mathbb{C} \xrightarrow{\times 1} \mathbb{C} & \longrightarrow & 0 & \\
 & \uparrow & , \uparrow & \uparrow & \\
 & 0 \longrightarrow 0 & \longrightarrow & 0 & \\
 \text{Note that } & \mathbb{C} \longrightarrow 0 & , \mathbb{C} \xrightarrow{\times a} \mathbb{C} & , \mathbb{C} \xrightarrow{\times b} \mathbb{C} & , \mathbb{C} \xrightarrow{\times c} \mathbb{C} \\
 & \uparrow & , \uparrow & \uparrow & , \uparrow \\
 & \mathbb{C} \longrightarrow 0 & , \mathbb{C} \xrightarrow{\times bc^{-1}} \mathbb{C} & , \mathbb{C} \xrightarrow{\times bc^{-1}} \mathbb{C} & , \mathbb{C} \xrightarrow{\times bc^{-1}} \mathbb{C} \\
 & & , \uparrow & , \uparrow & \\
 & & \mathbb{C}^2 \xrightarrow{(bc^{-1} \ 0)} \mathbb{C} & \longrightarrow &
 \end{array}$$

are quasi-isomorphisms. Therefore, we can think of the cube diagram as quasi-isomorphism of two double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete. \square

3.6.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the $cobord_2$. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphism $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

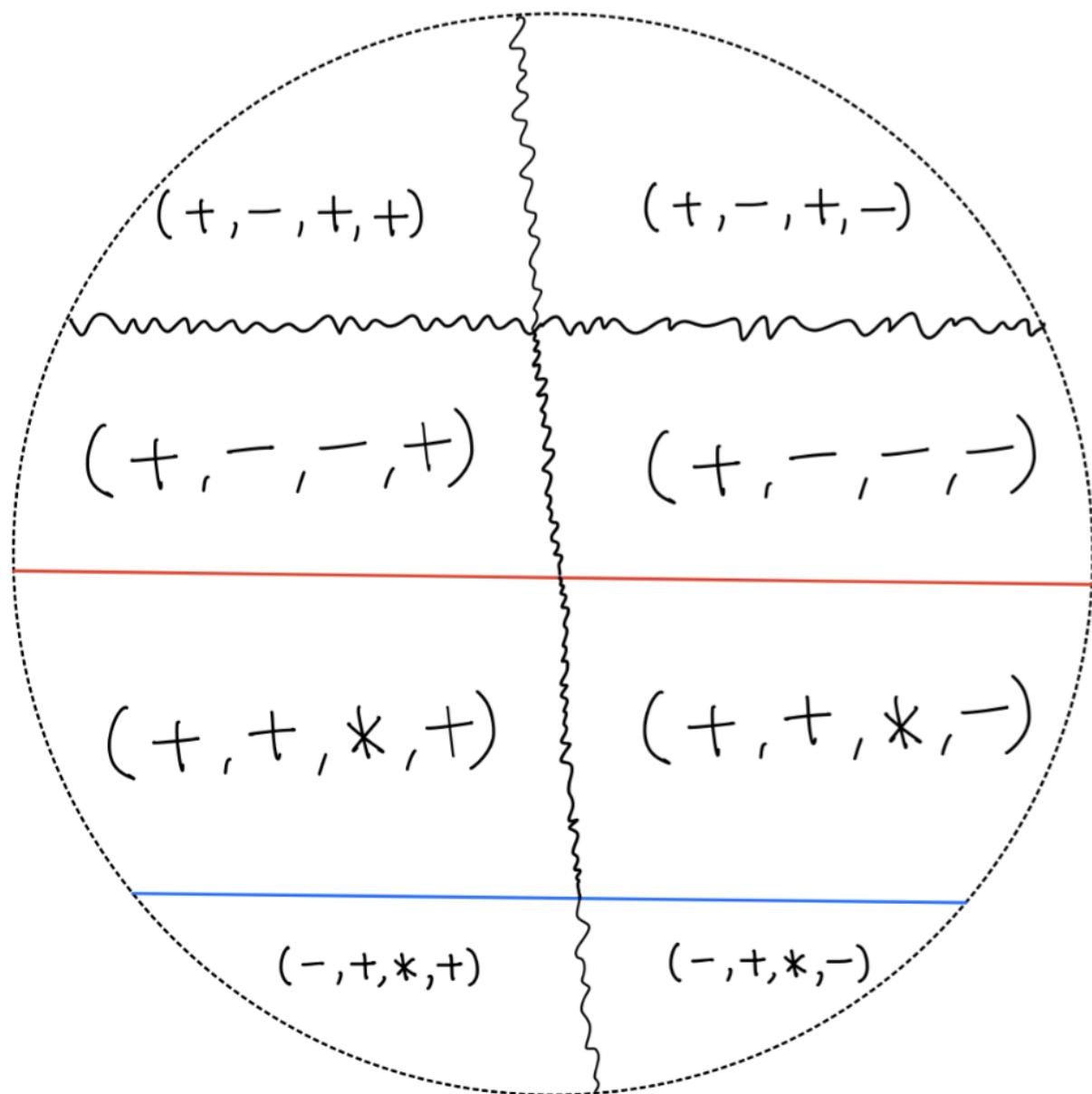


Figure 3.155

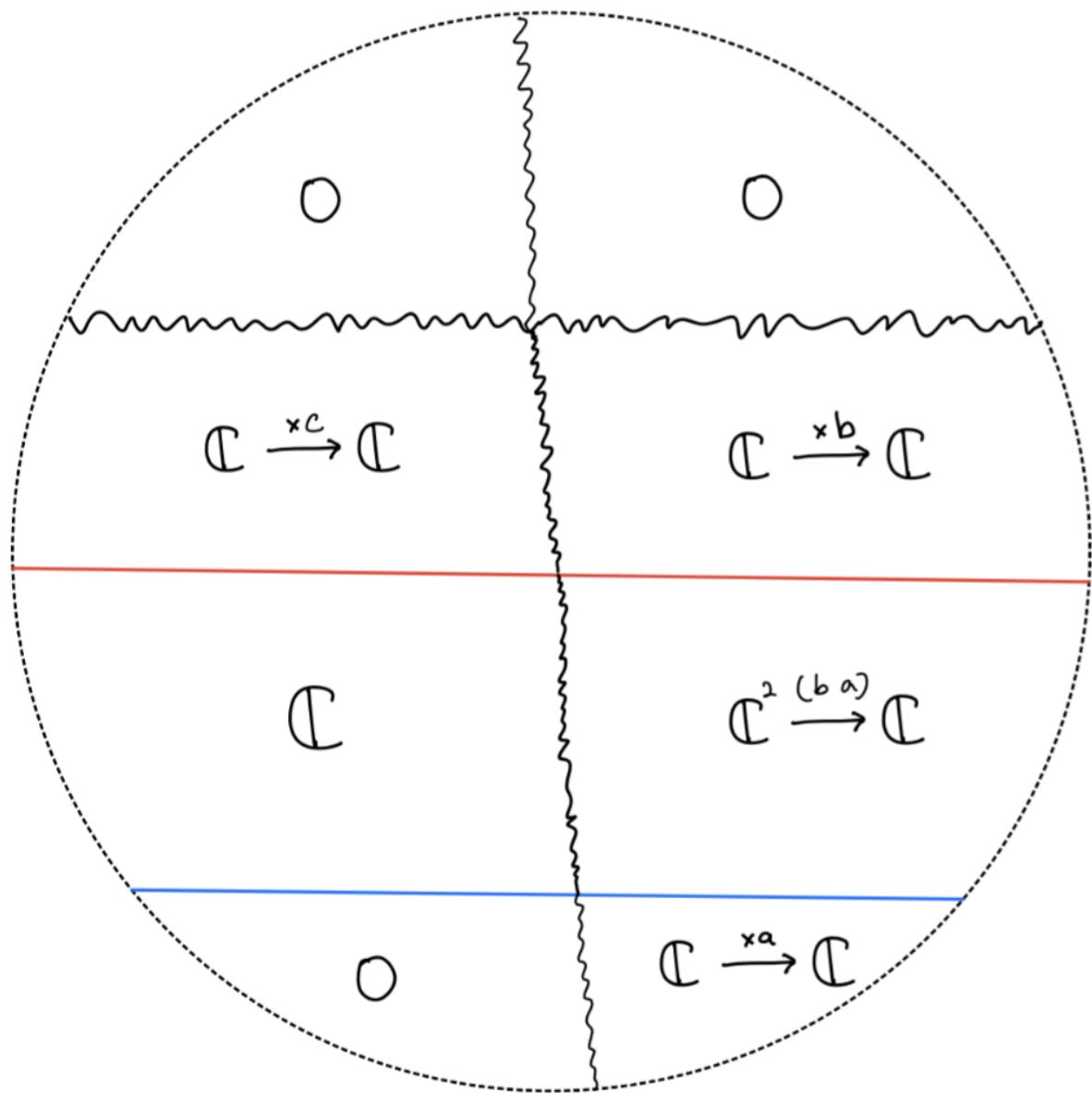


Figure 3.156

- $F_1(-, +, *, -) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$

- $F_1(-, +, *, +) := 0$

- $F_1(+, -, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$

- $F_1(+, -, -, +) := \mathbb{C} \xrightarrow{x_c} \mathbb{C}$

- $F_1(+, -, +, -) := 0$

- $F_1(+, -, +, +) := 0$
- $F_1(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F_1(+, +, *, +) := \mathbb{C}$

Generalization maps:

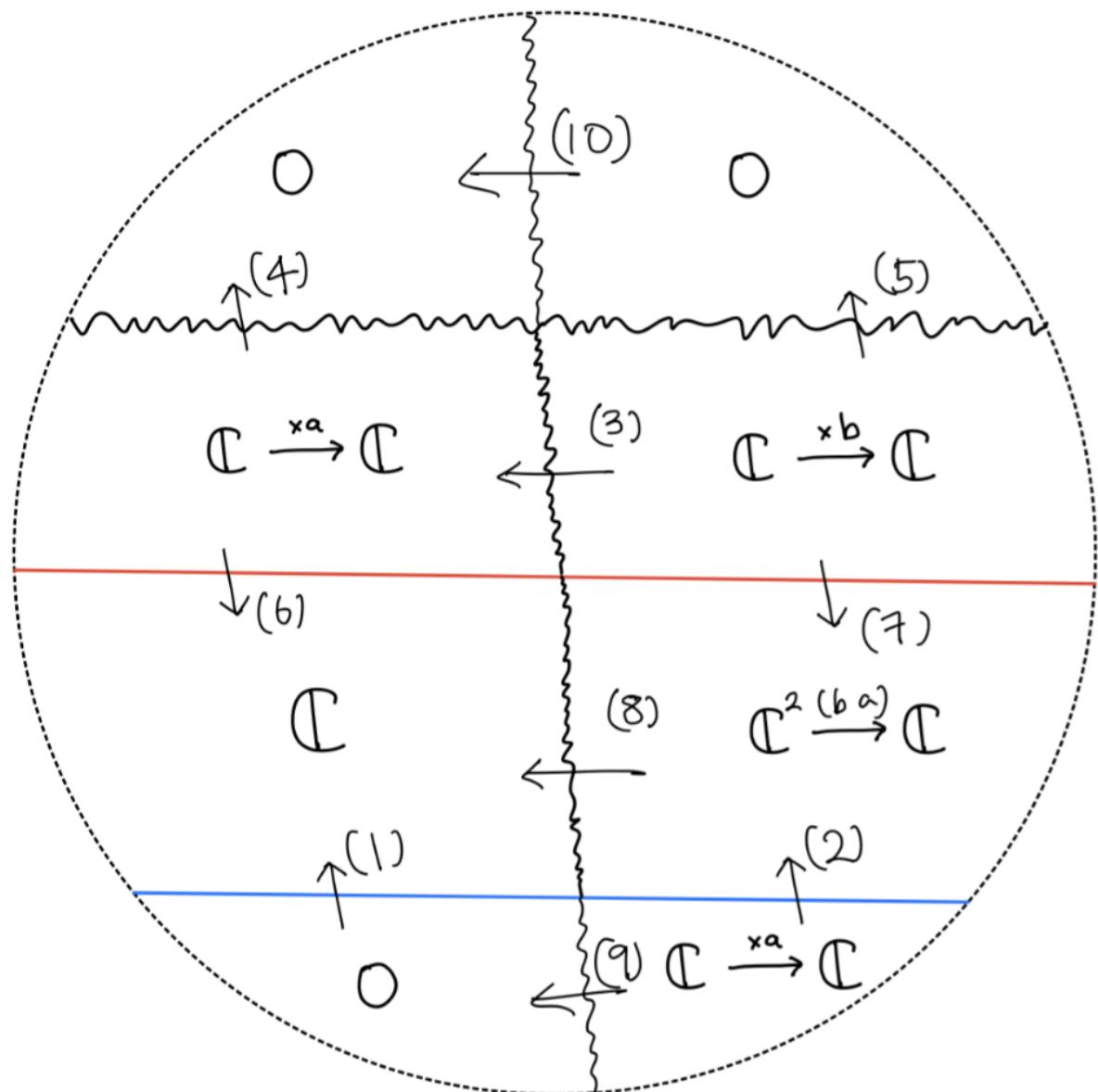


Figure 3.157

$$(1) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \times c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 143. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

3.7 3rd Sheaf Cobordism

In this section, we define *cobord*₃, a compactly supported sheaf cobordism.

3.7.1 Notations

Definition 144. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 145. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both
4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_3$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_3$.

Definition 146. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_\bullet^{symbol} to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript ' t ' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_\bullet^{symbol} a co-orientation of Φ_\bullet^{symbol} .

3. we denote the pair $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$ as $\Lambda_{\bullet}^{symbol}$. Later in the section, $\Lambda_{\bullet}^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₃.
4. we denote the triple $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$ as Λ_{\bullet} and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_{\bullet} will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₃.

Definition 147. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{3}{4}e^{(\frac{x^2}{x^2-1})}(1-t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$ if $t \neq 0$
- $\{(1, 0), (-1, 0), (0, \frac{3}{4}(1-t))\} \subset Graph(\Psi_t)$

Definition 148. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 149. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = \frac{1}{2}\}$
- $\lambda_0^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig} : hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 150. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = \frac{1}{2}\}$
- $\lambda_1^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig} : hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 151. 1. We define the following subsets of $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \frac{1}{2}\}$
- $\lambda_\bullet^{squig} := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$

2. We define co-orientations ξ_\bullet^{symbol} of λ_\bullet^{symbol} as follows

- ξ_\bullet^0 : hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_\bullet^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_\bullet^{squig} : hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 152.

1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .
2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_\bullet on $D_{r=2} \times [0, 1]$ induced by λ_\bullet i.e. strata are non-empty finite intersections of λ_\bullet^0 , λ_\bullet^∞ , and λ_\bullet^{squig} . Note that 2 dimensional strata has co-orientations inherited from λ_\bullet .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the following notations:

Definition 153. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 154. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

Definition 155. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

Definition 156. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the above notations:

1. \mathcal{S}_0 :

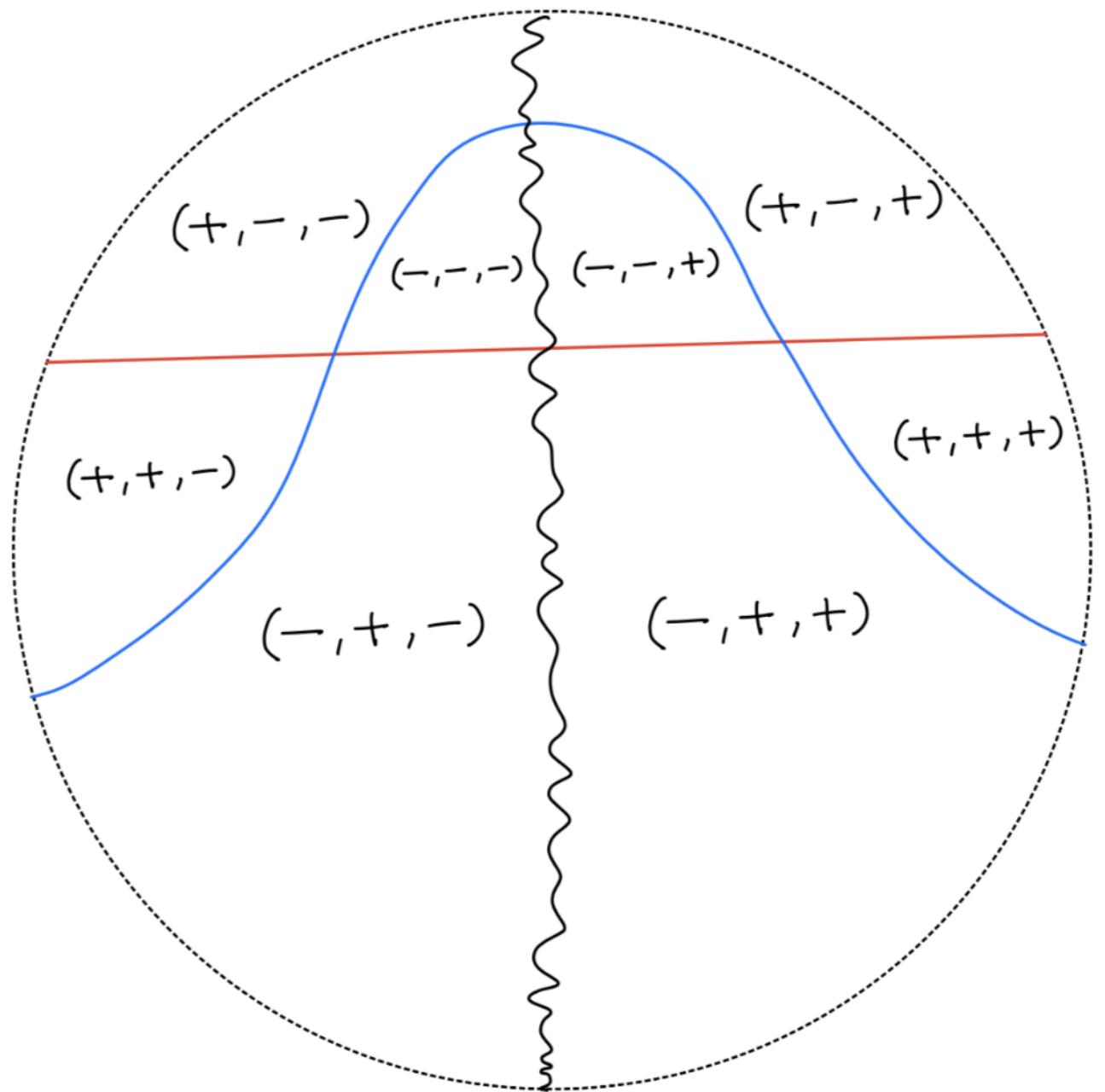


Figure 3.158

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

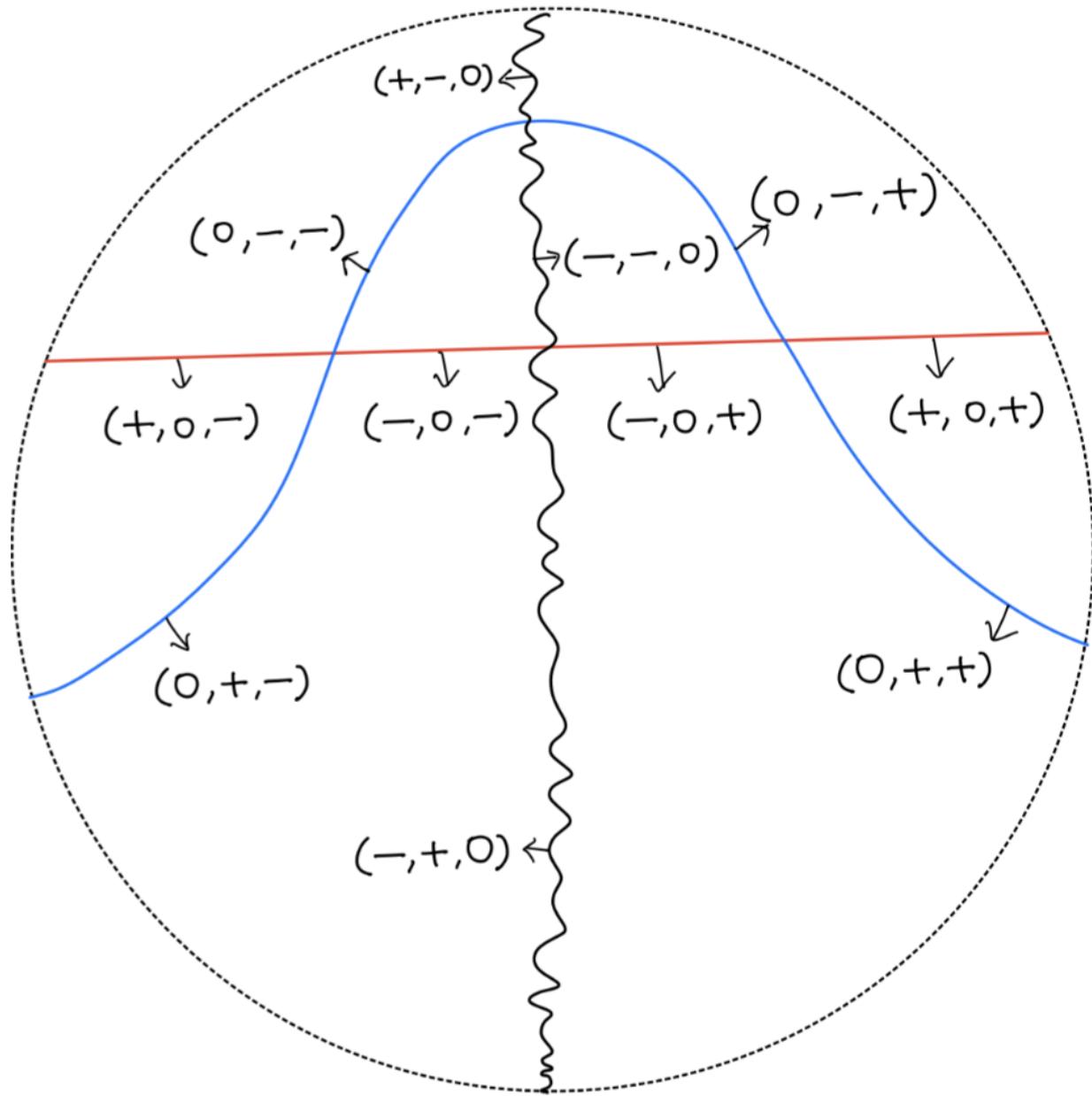


Figure 3.159

- 1 dimensional strata:

$$\begin{aligned} & \{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, +, 0)\} \end{aligned}$$

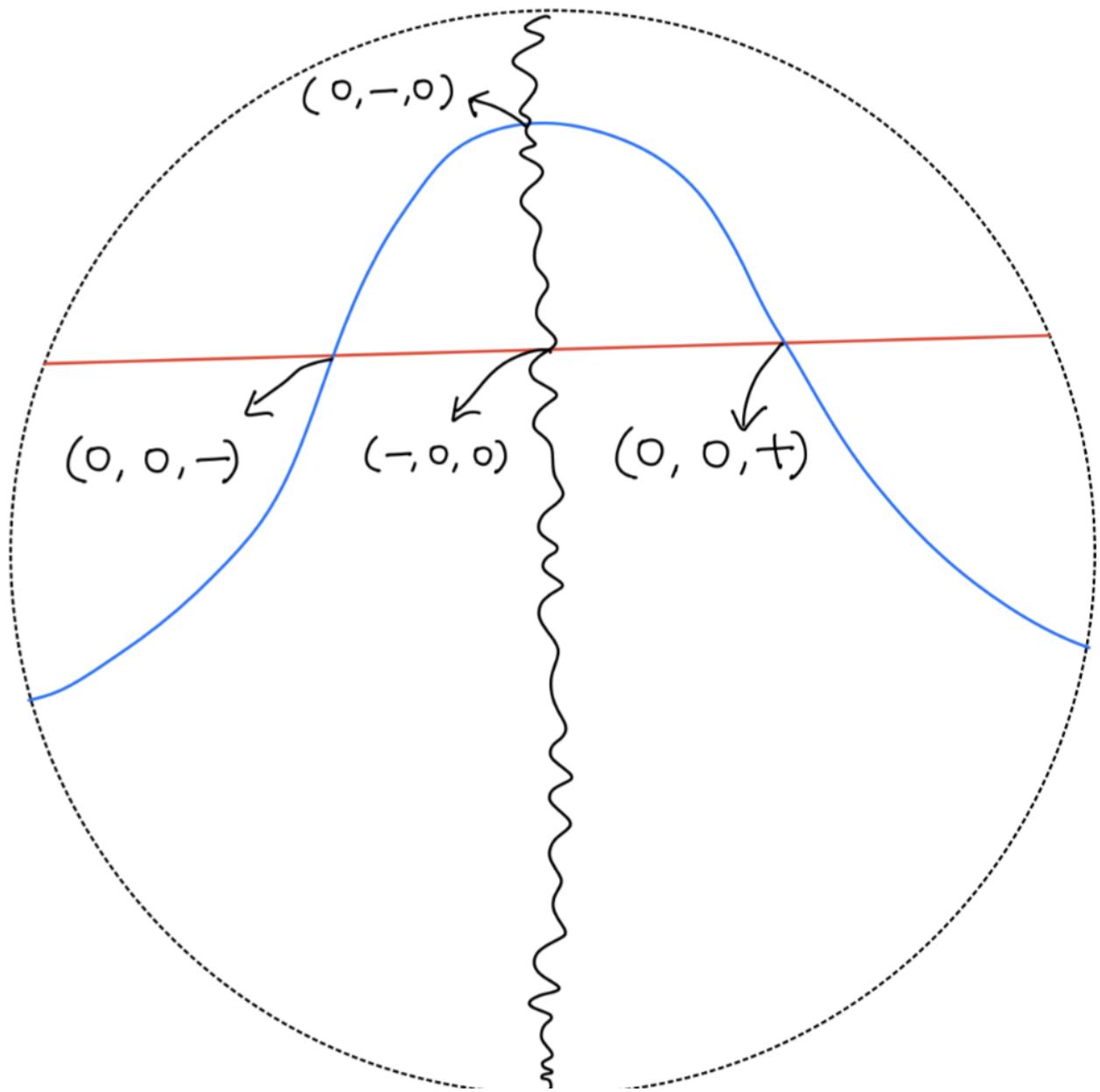


Figure 3.160

- 0 dimensional strata:

$$s_0(0, 0, +), s_0(-, 0, 0)$$

2. \mathcal{S}_1 :

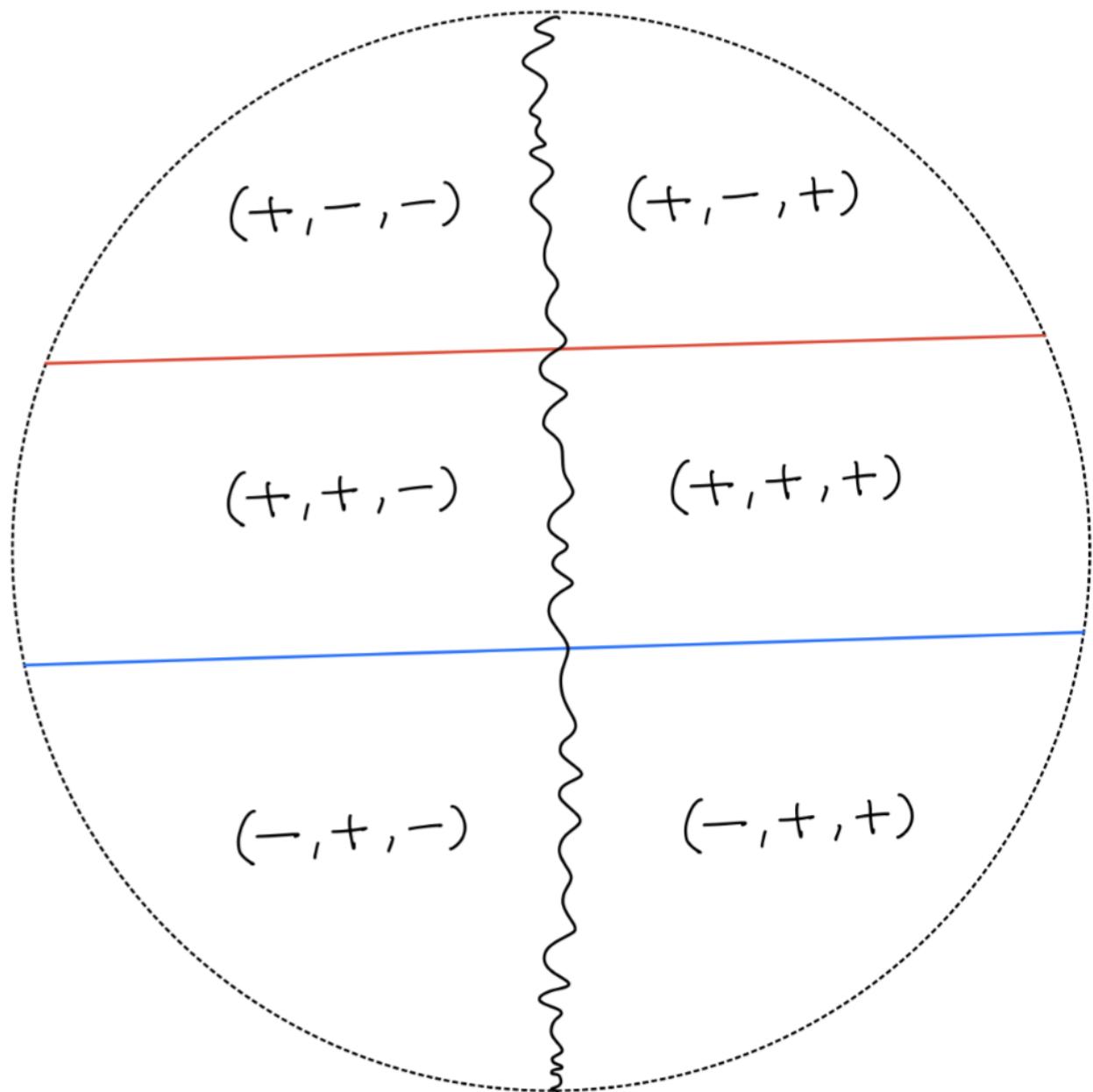


Figure 3.161

- 2 dimensional strata:

$$s_1(+, -, -), s_1(+, -, +), s_1(+, +, -), s_1(+, +, +), s_1(-, +, -), s_1(-, +, +)$$

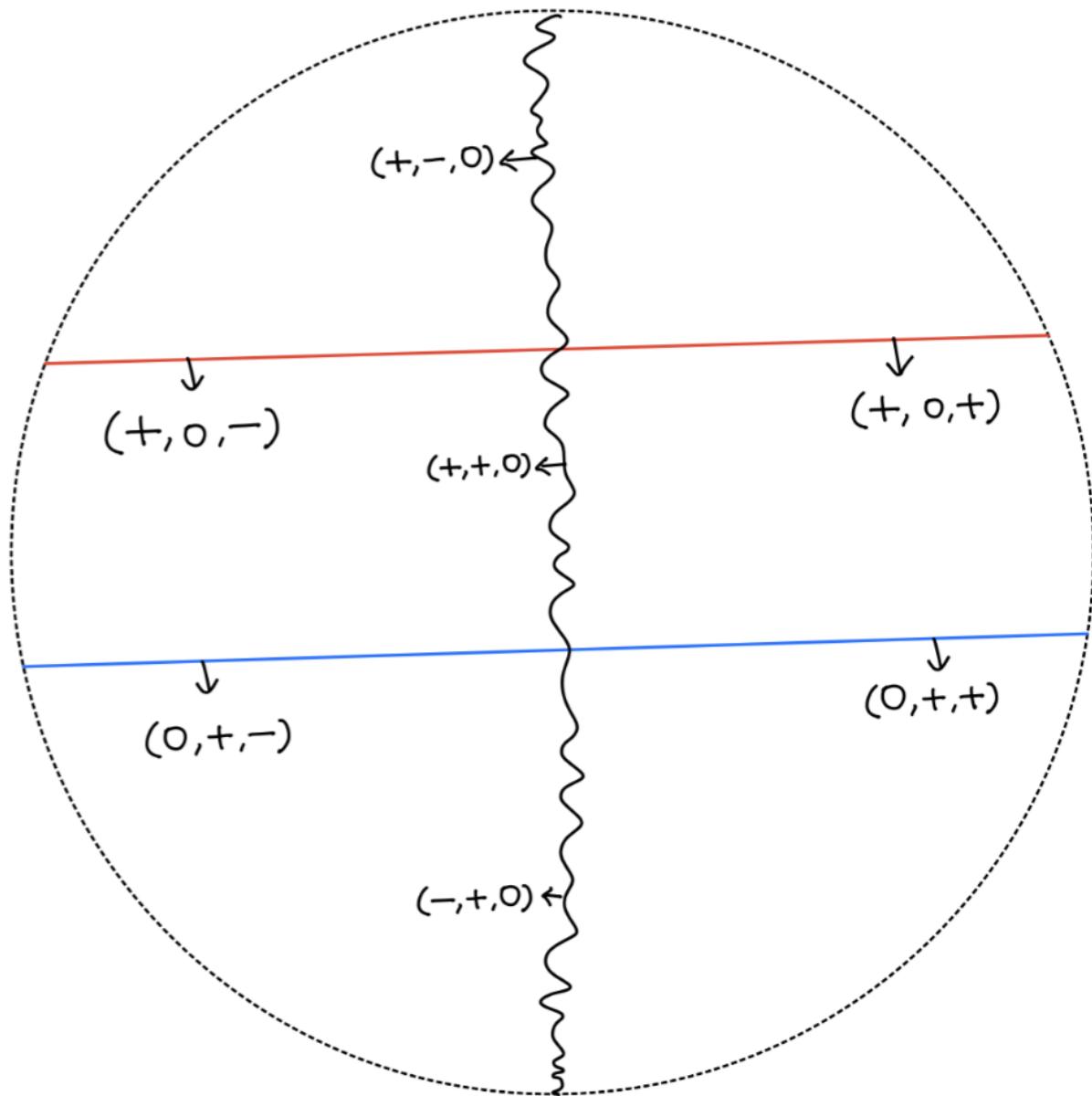


Figure 3.162

- 1 dimensional strata:

$$s_1(0, +, -), s_1(0, +, +), s_1(+, 0, -), s_1(+, 0, +), s_1(+, -, 0), s_1(+, +, 0), s_1(-, +, 0)$$

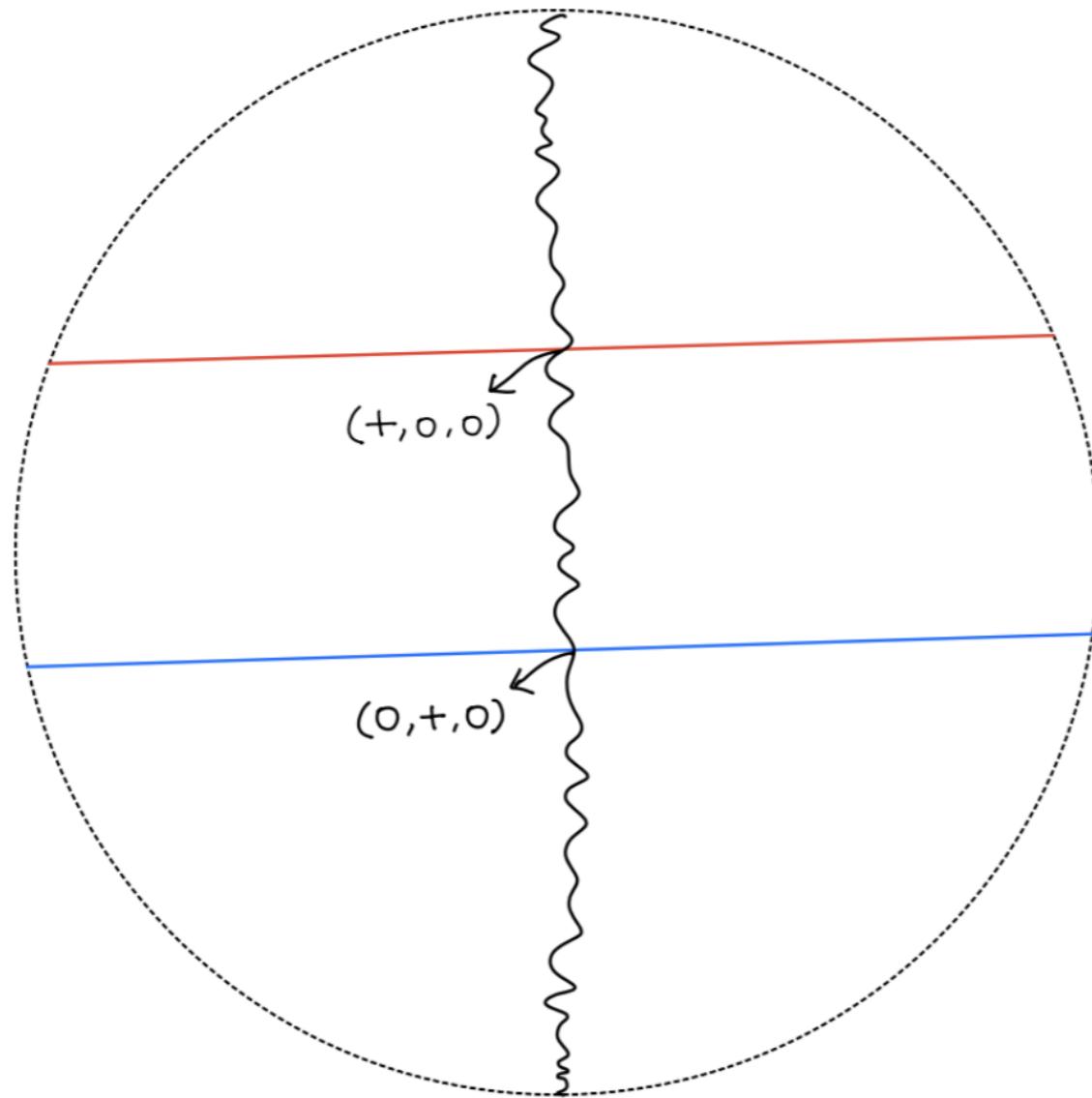


Figure 3.163

- 0 dimensional strata:

$$s_1(0, +, 0), s_1(+, 0, 0)$$

3. \mathcal{S}_\bullet :

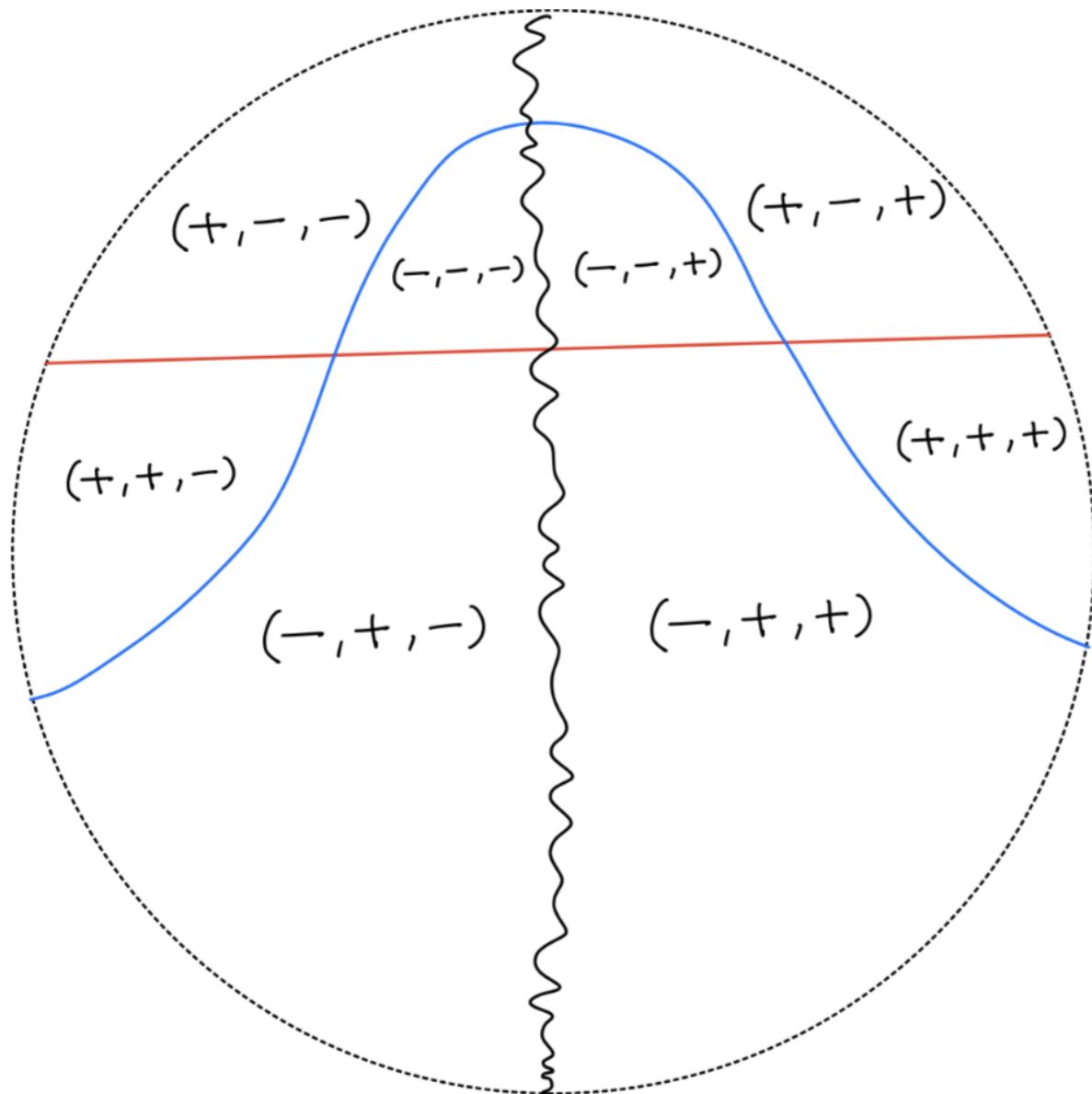


Figure 3.164

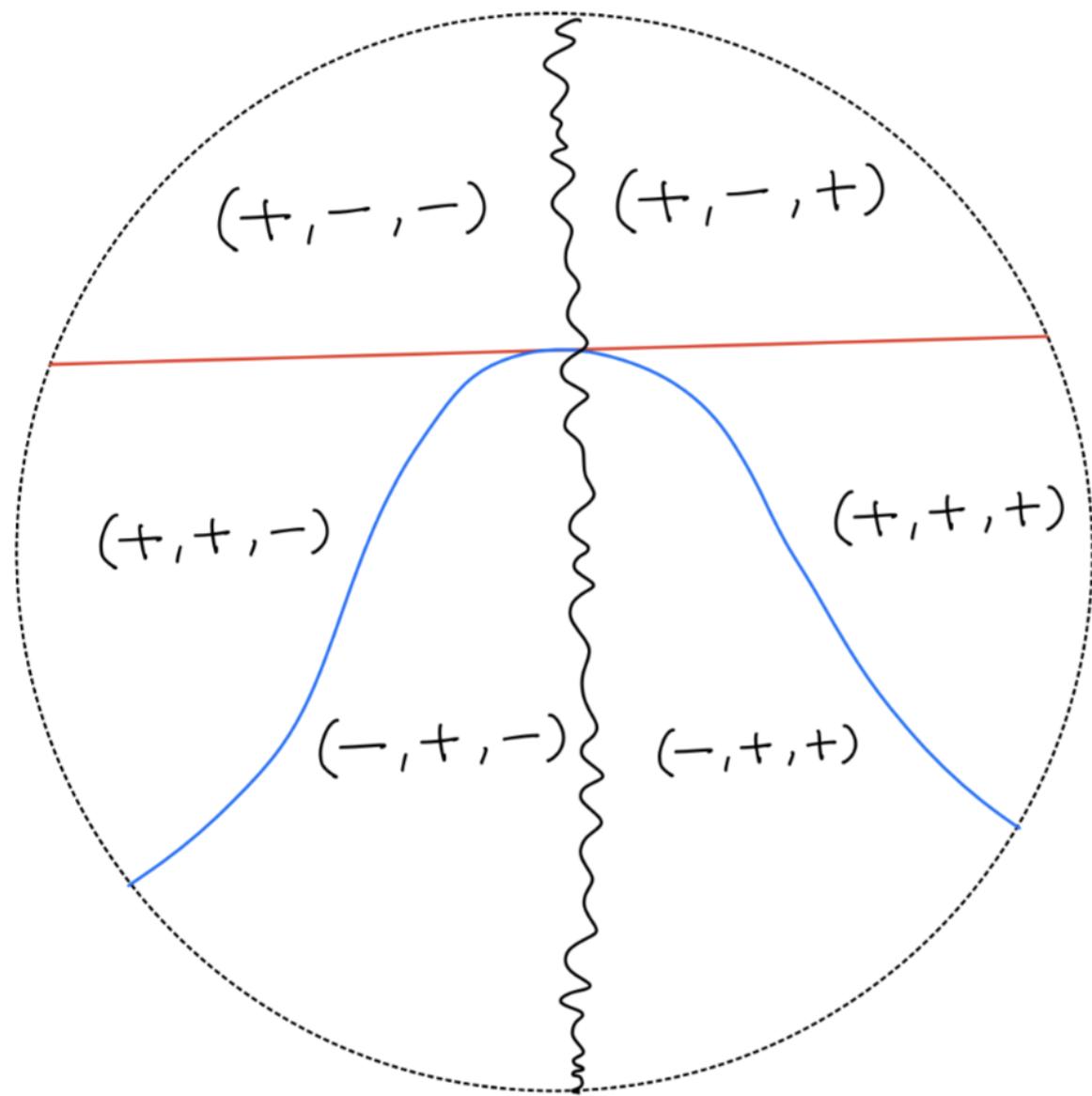


Figure 3.165

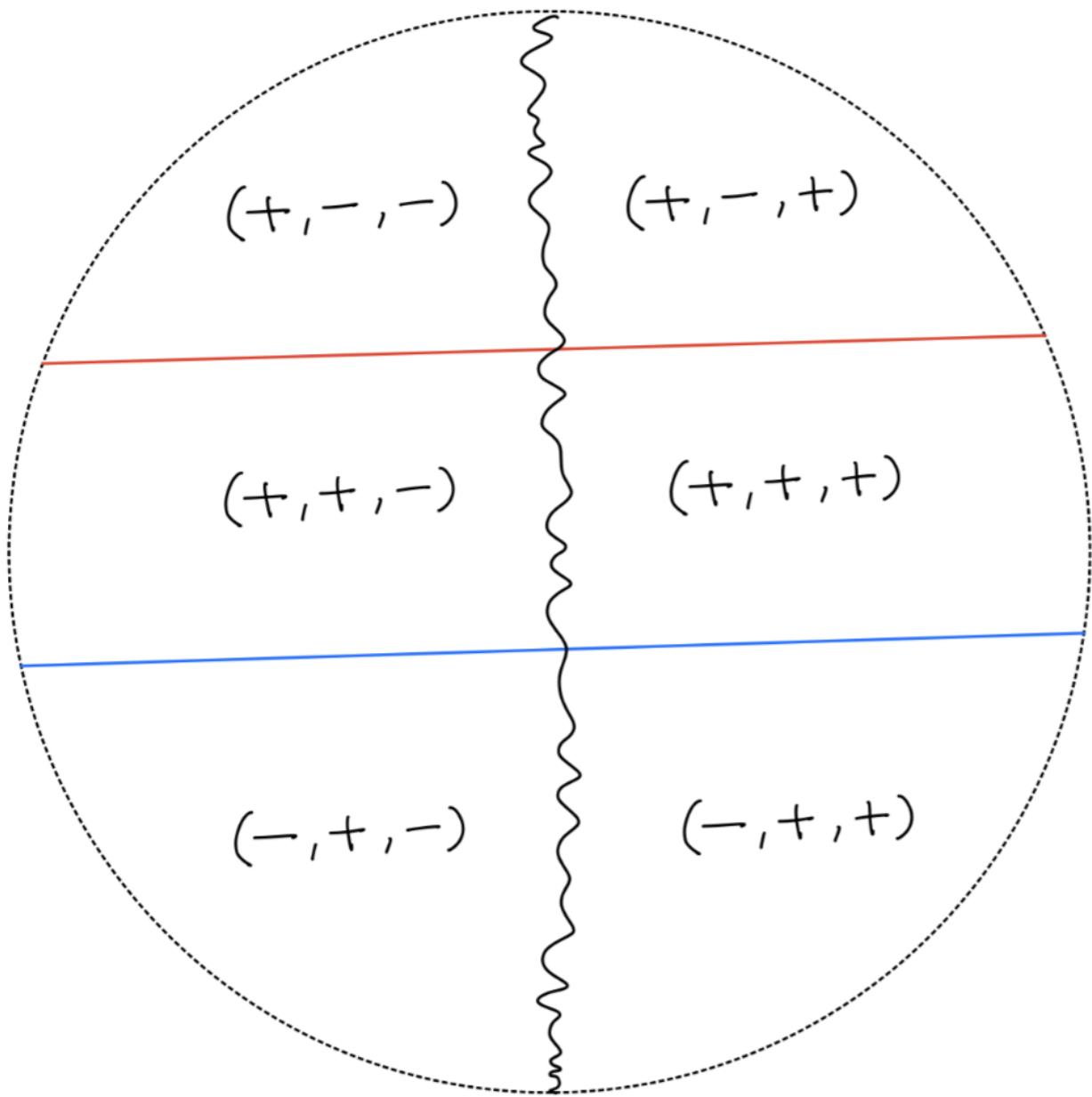


Figure 3.166

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

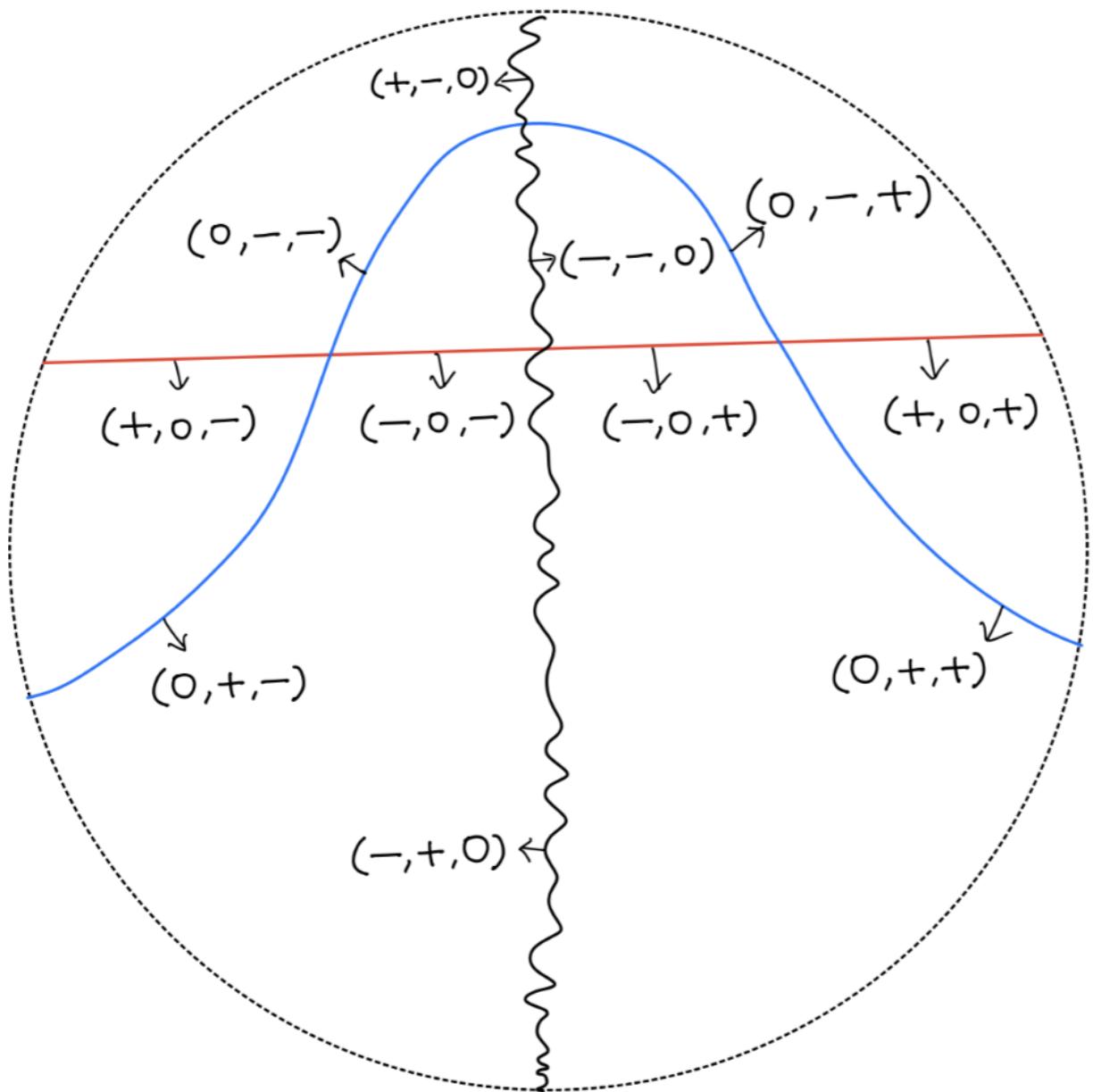


Figure 3.167

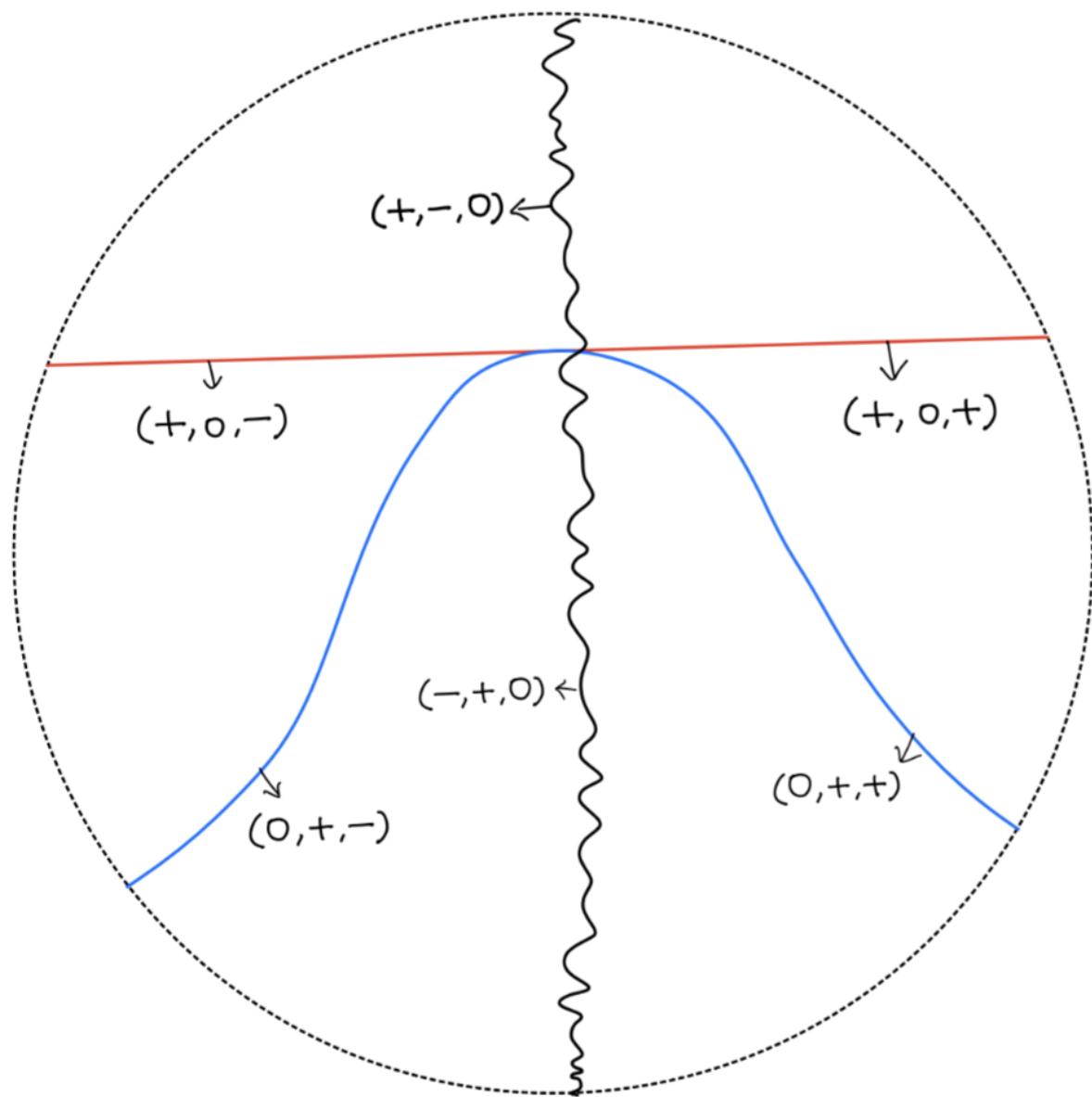


Figure 3.168

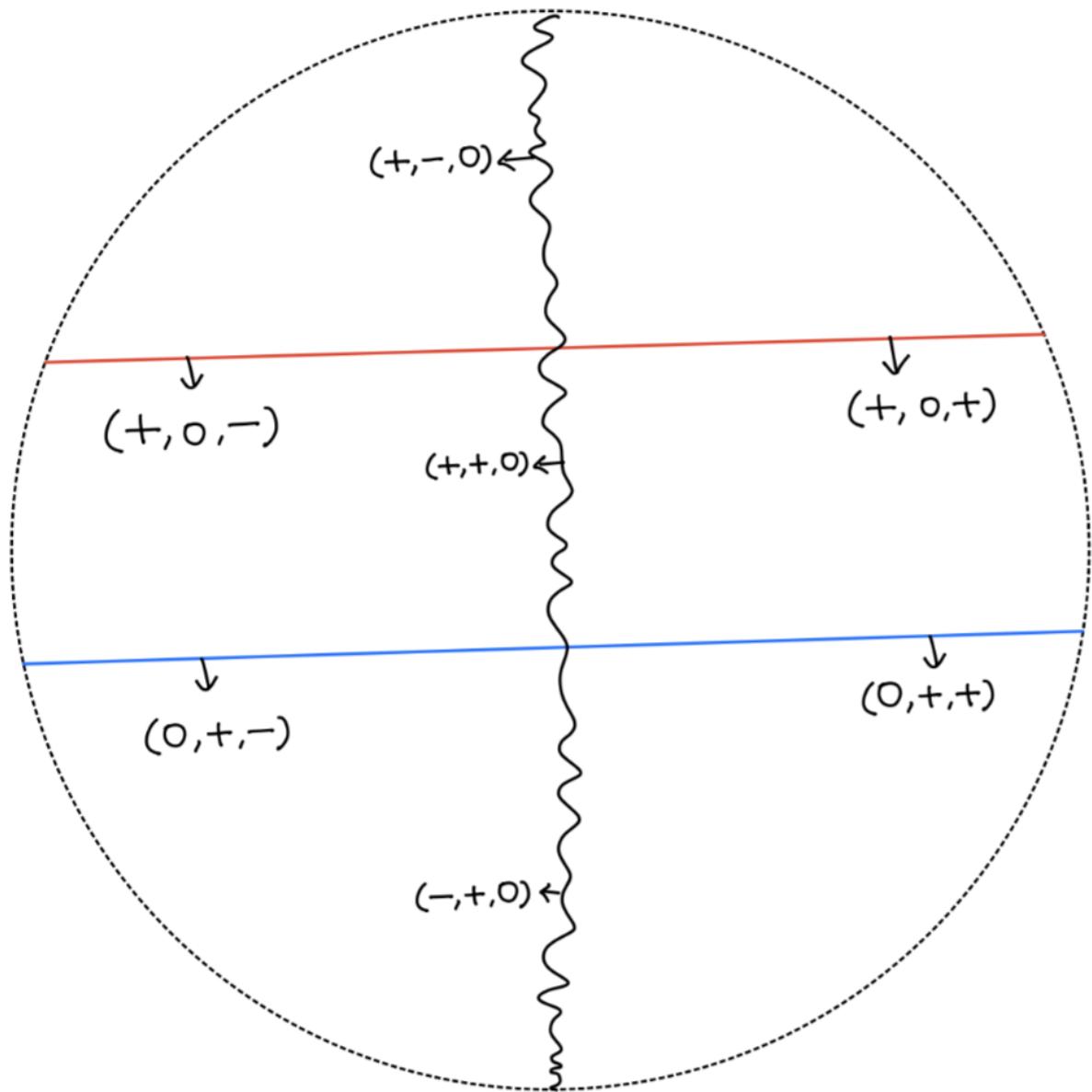


Figure 3.169

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

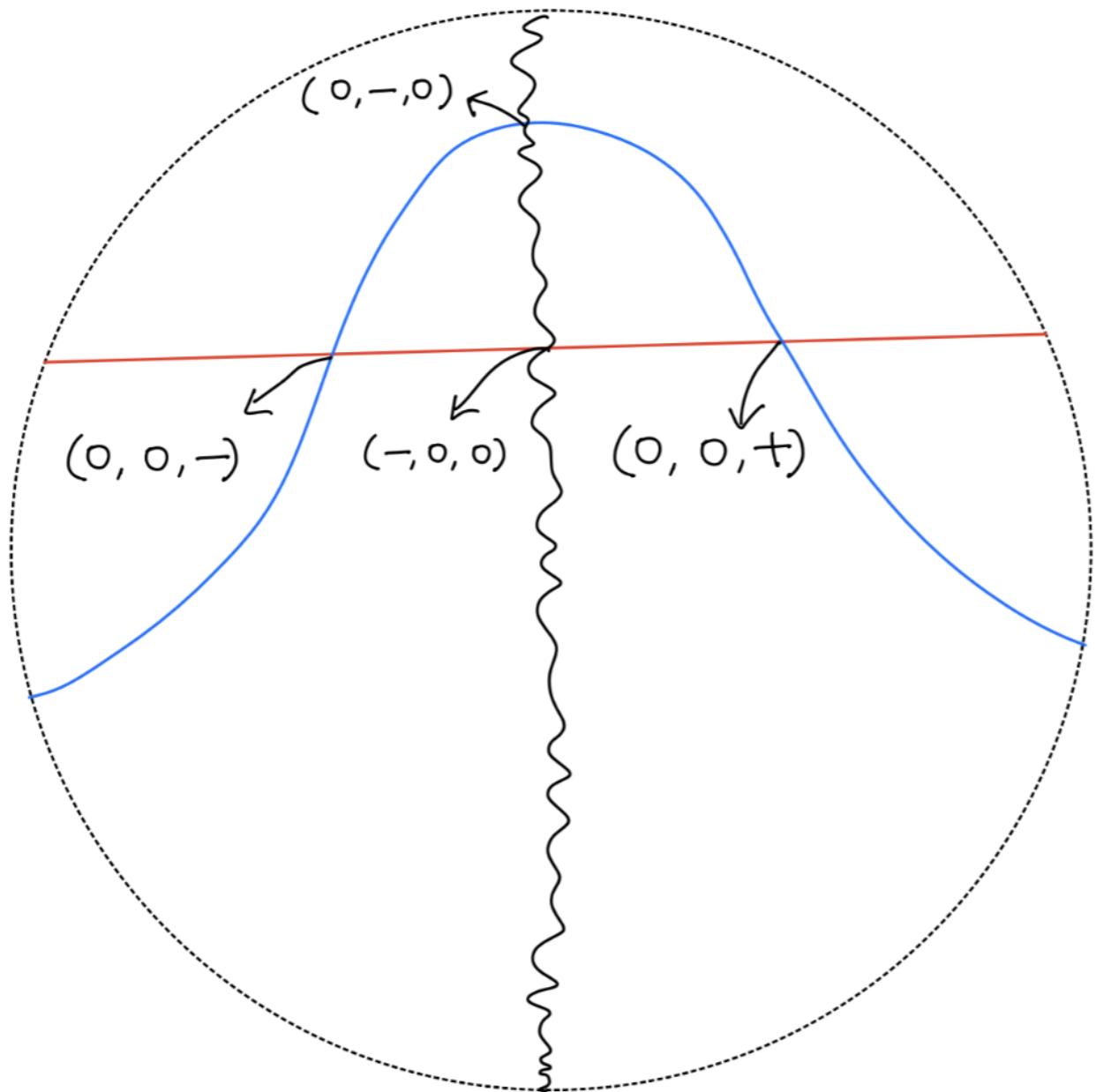


Figure 3.170

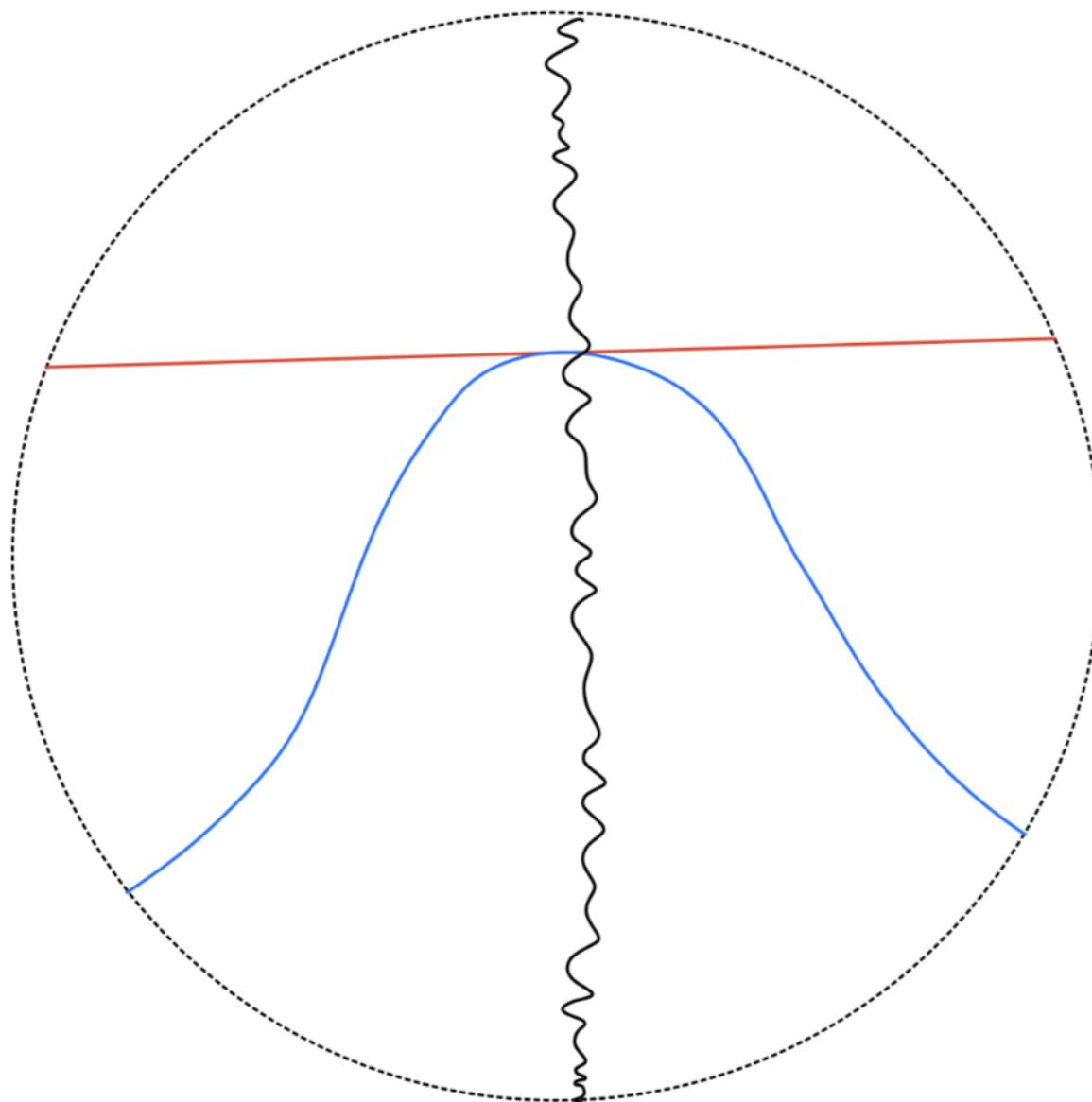


Figure 3.171

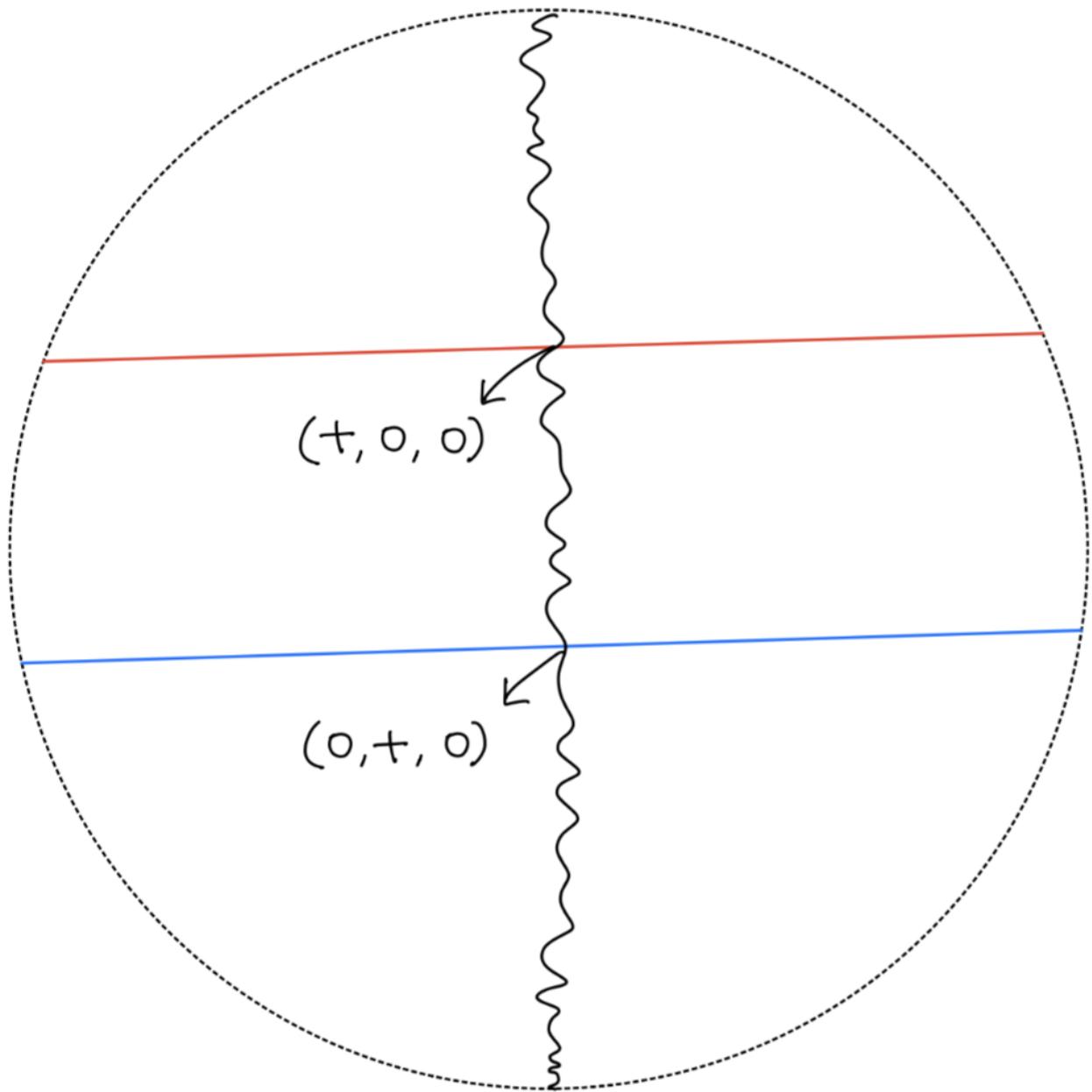


Figure 3.172

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

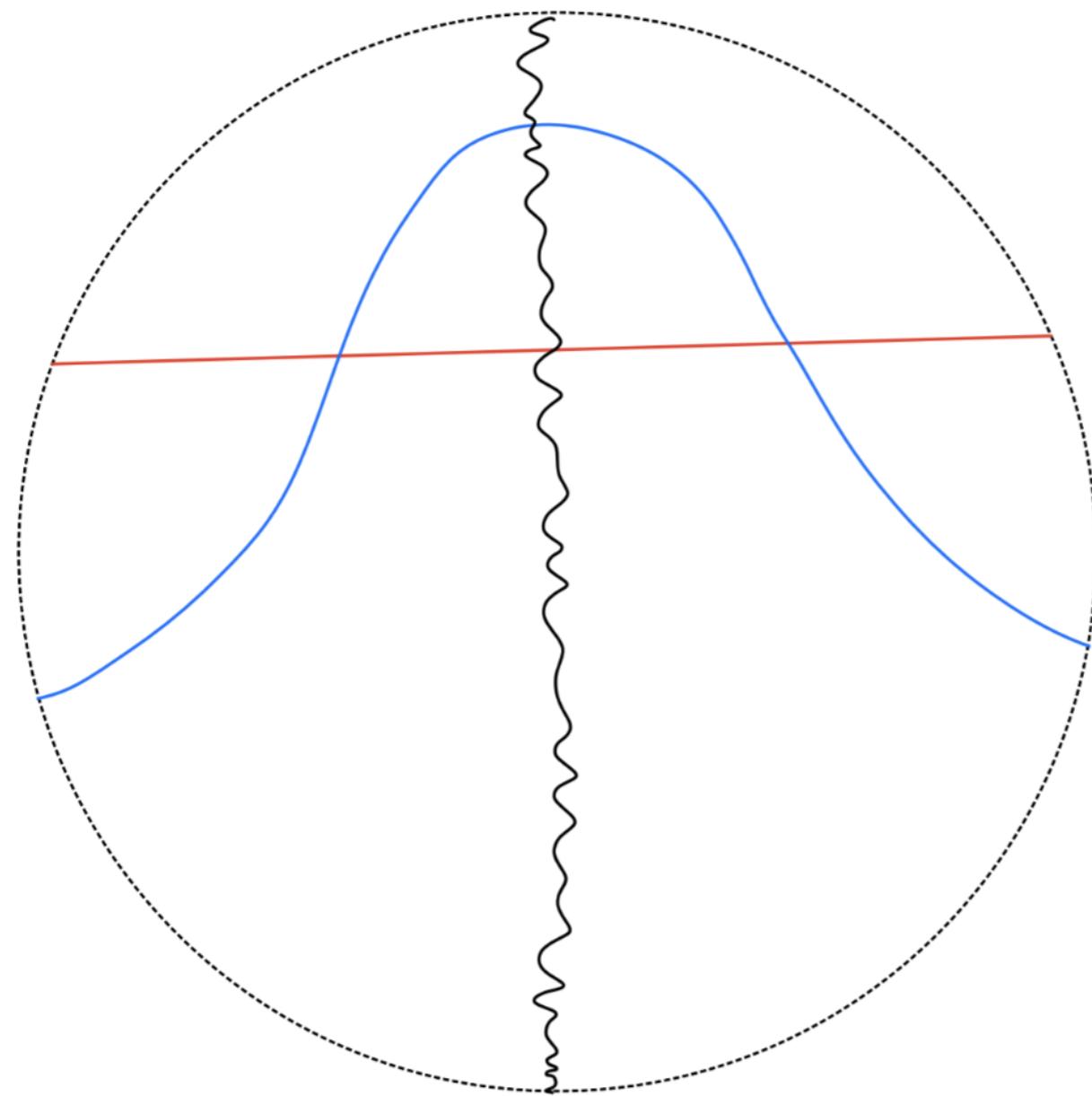


Figure 3.173

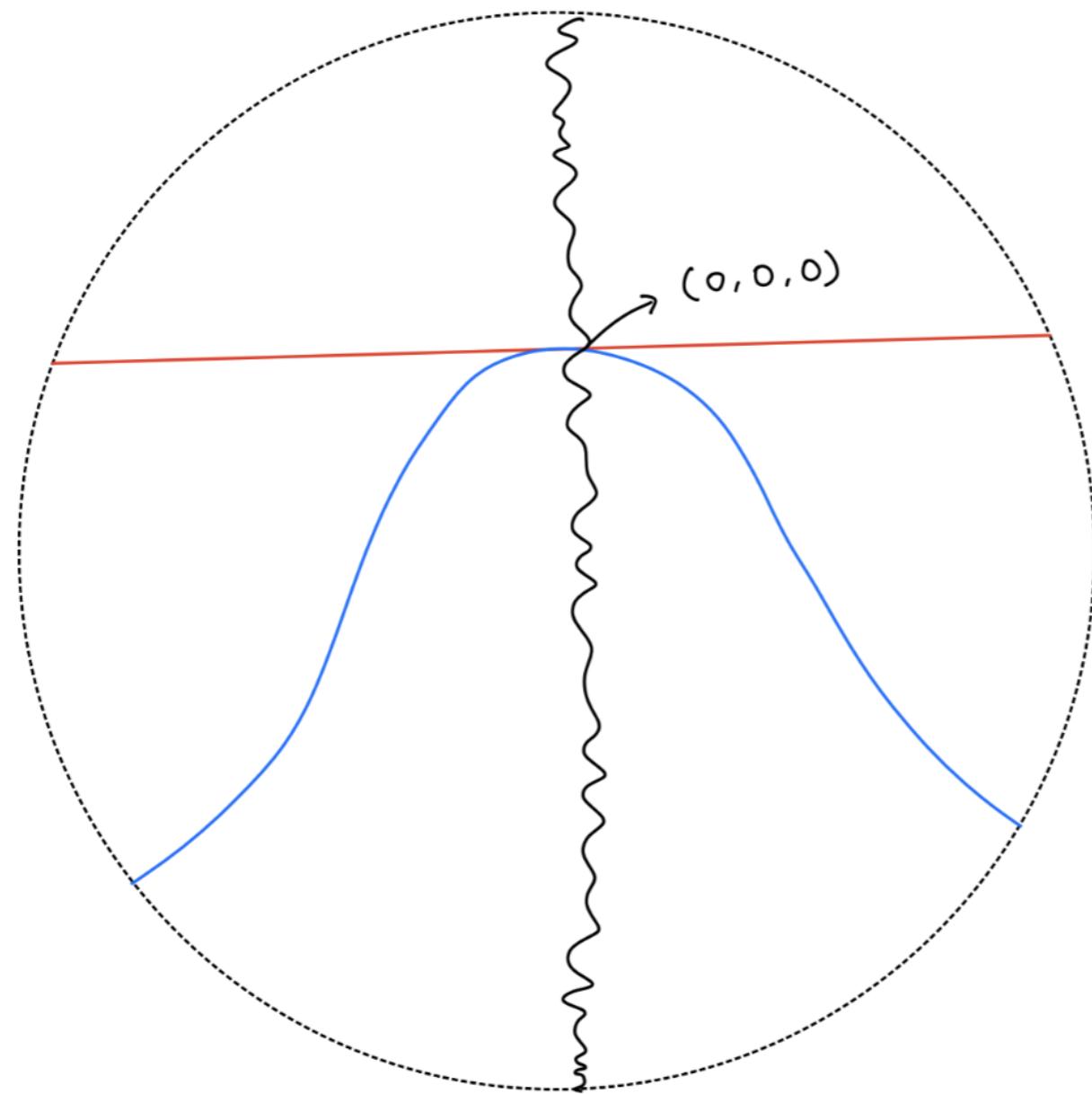


Figure 3.174

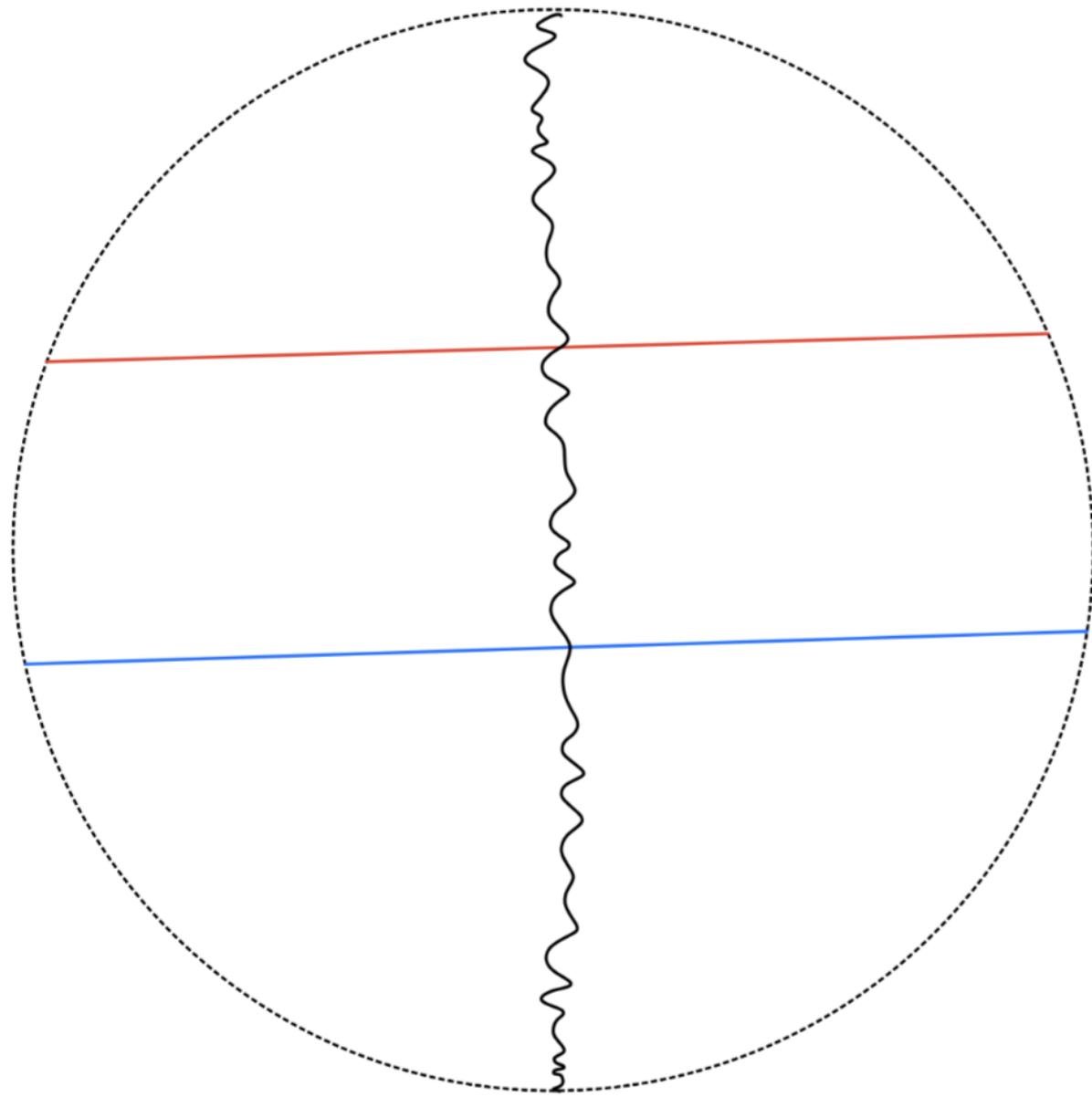


Figure 3.175

- 0 dimensional strata:

$$s_{\bullet}(0, 0, 0)$$

Definition 157. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $\text{star}(s)$.

Definition 158. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s .

Definition 159. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 160. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $\text{star}(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 161. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 162. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because F_S is legible.

Definition 163. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.7.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.7.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

Stalks:

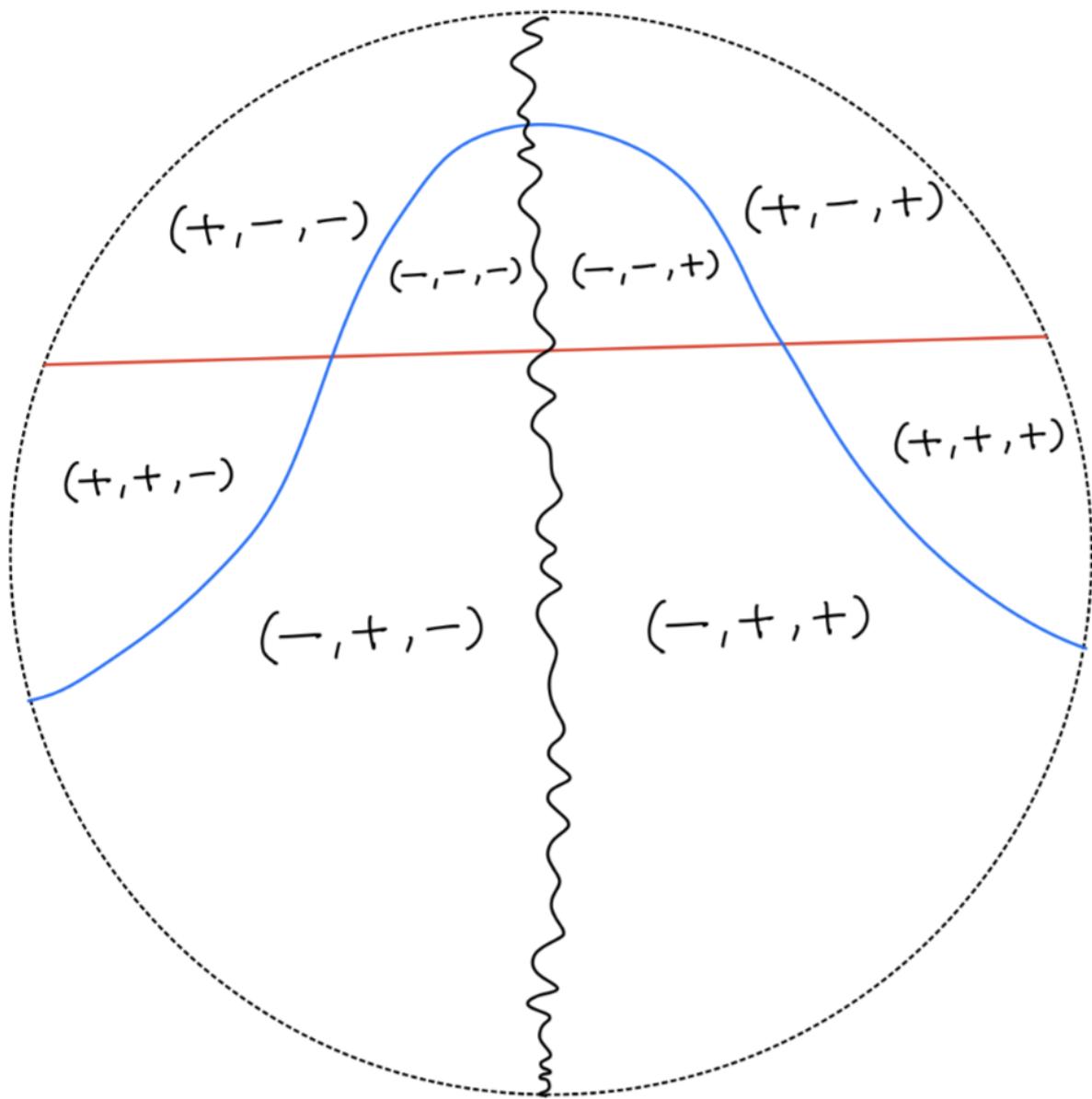


Figure 3.176

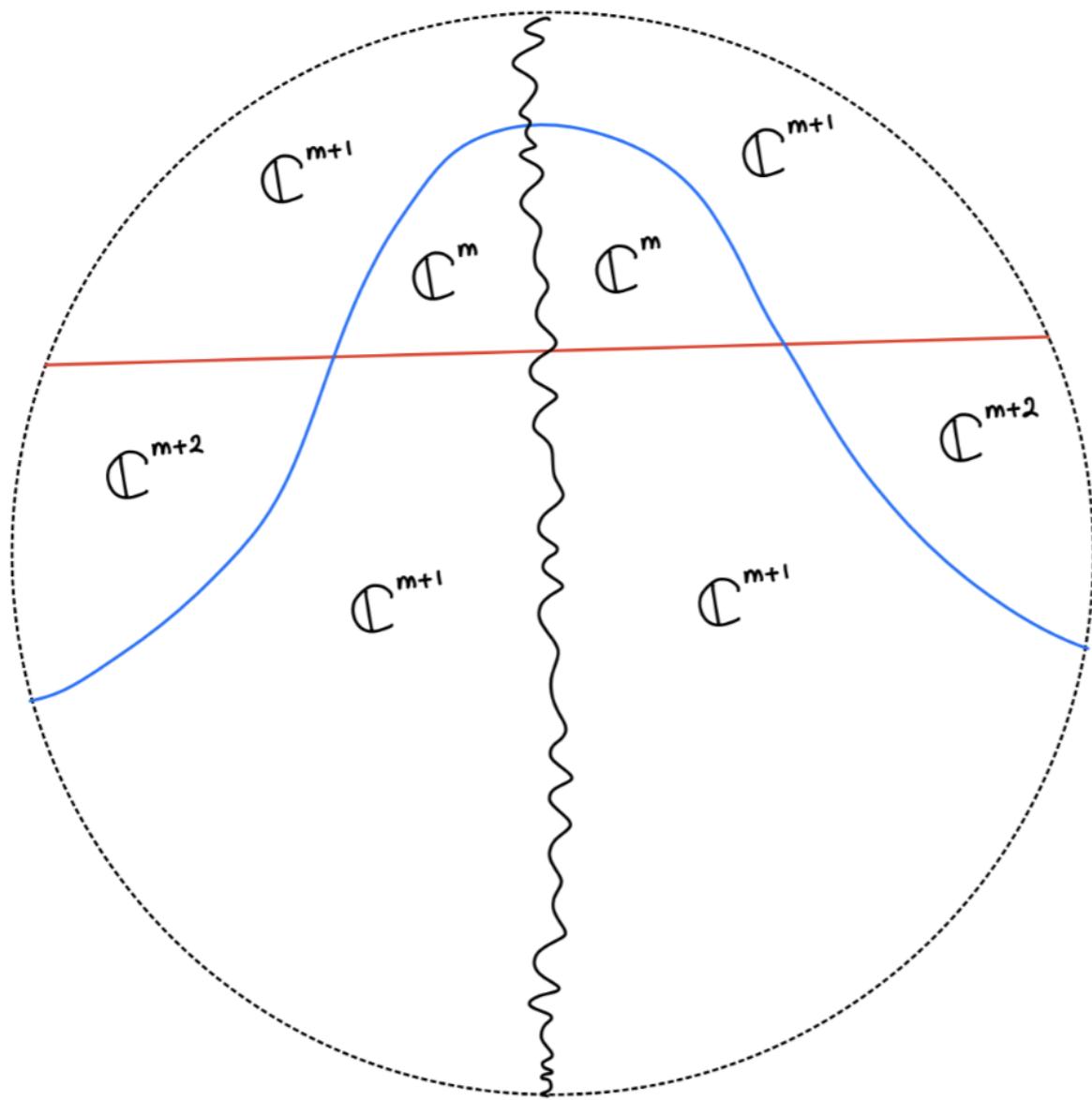


Figure 3.177

- $F_0(-, -, -) := \mathbb{C}^m$
- $F_0(-, -, +) := \mathbb{C}^m$
- $F_0(+, -, -) := \mathbb{C}^{m+1}$
- $F_0(+, -, +) := \mathbb{C}^{m+1}$
- $F_0(-, +, -) := \mathbb{C}^{m+1}$

- $F_0(-, +, +) := \mathbb{C}^{m+1}$

- $F_0(+, +, -) := \mathbb{C}^{m+2}$

- $F_0(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

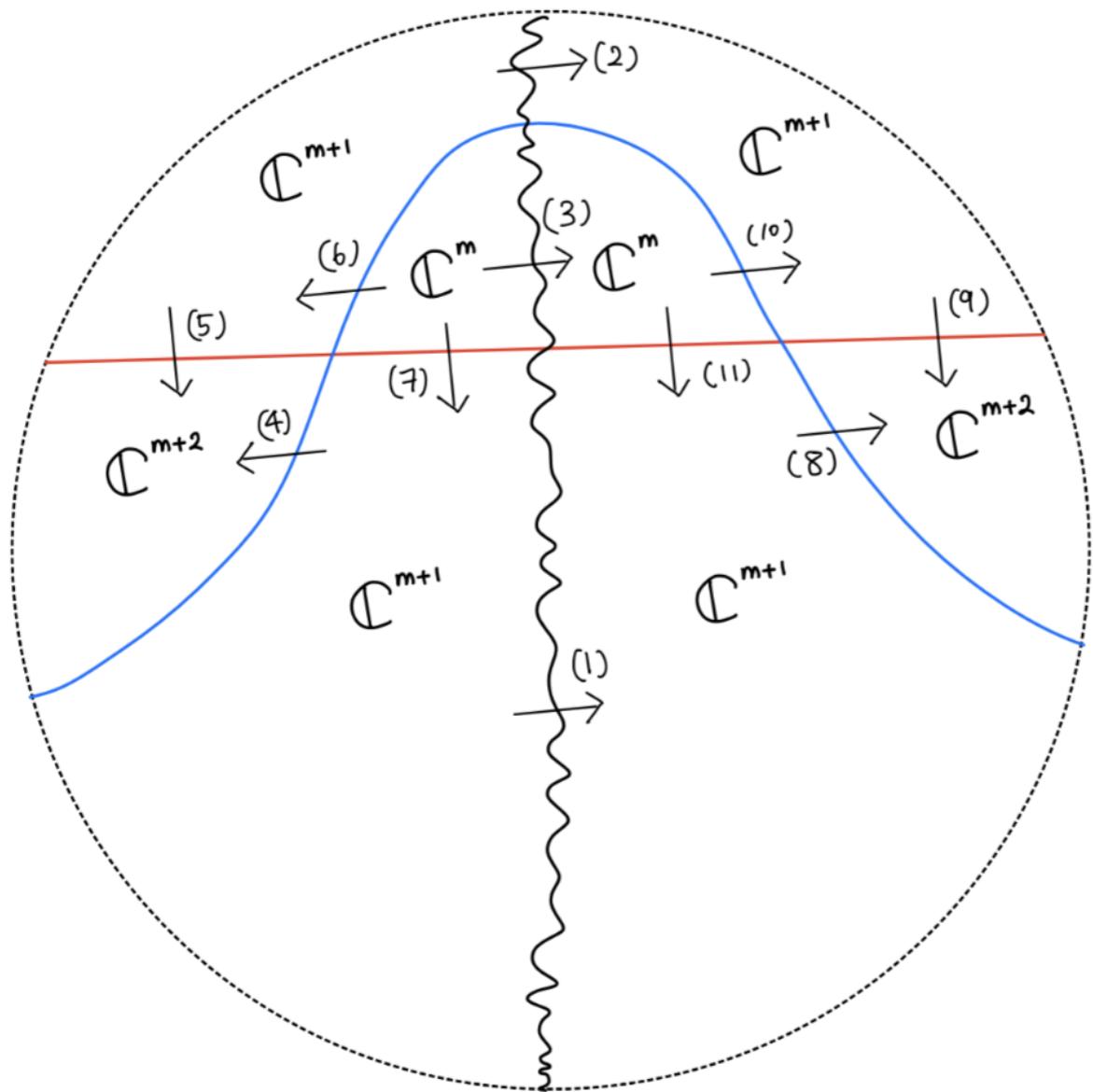


Figure 3.178

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^m \xrightarrow{T(\theta)} \mathbb{C}^m$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$$

$$(11) \quad \mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$$

3.7.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say *cobord*₂, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

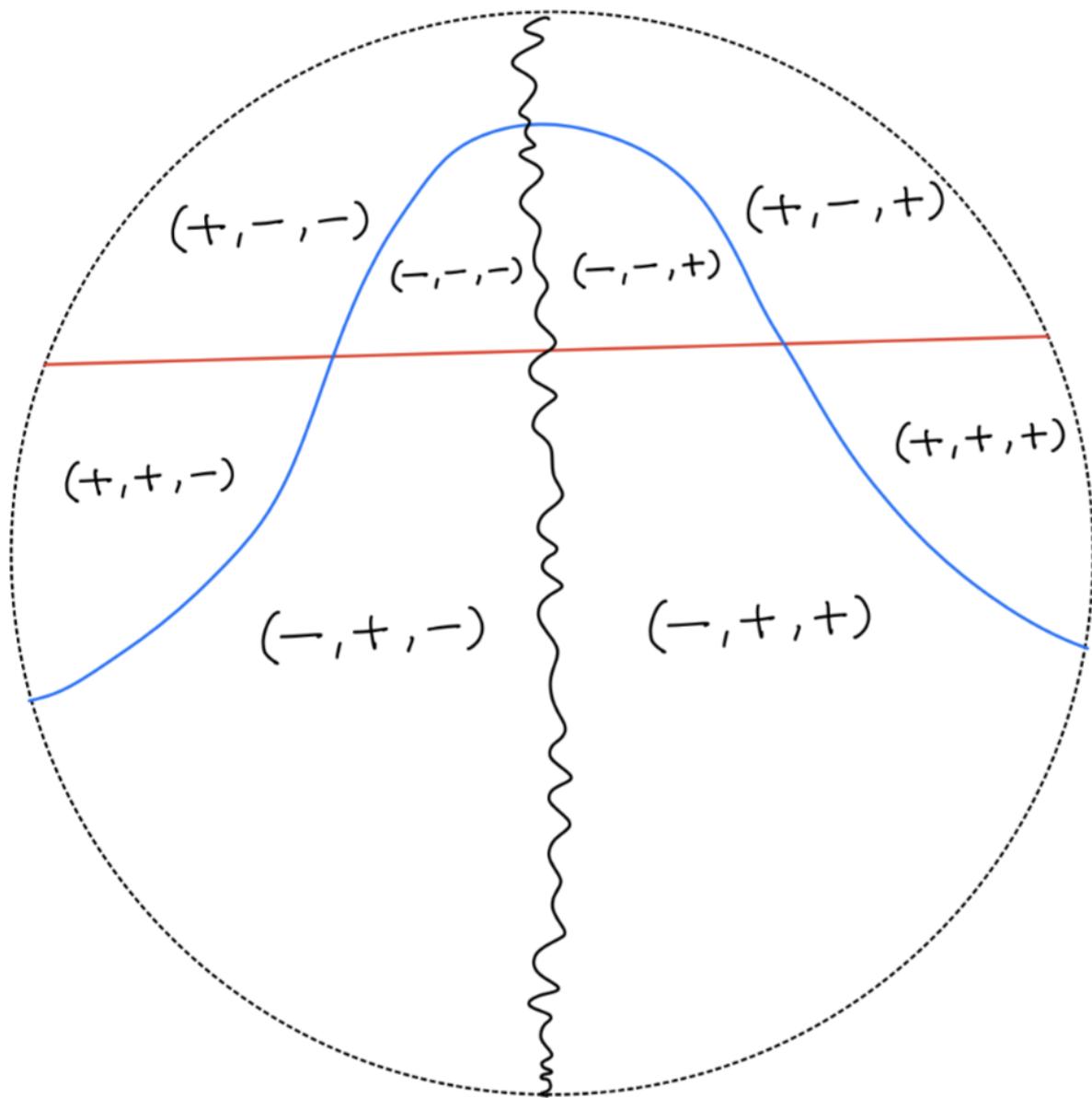


Figure 3.179

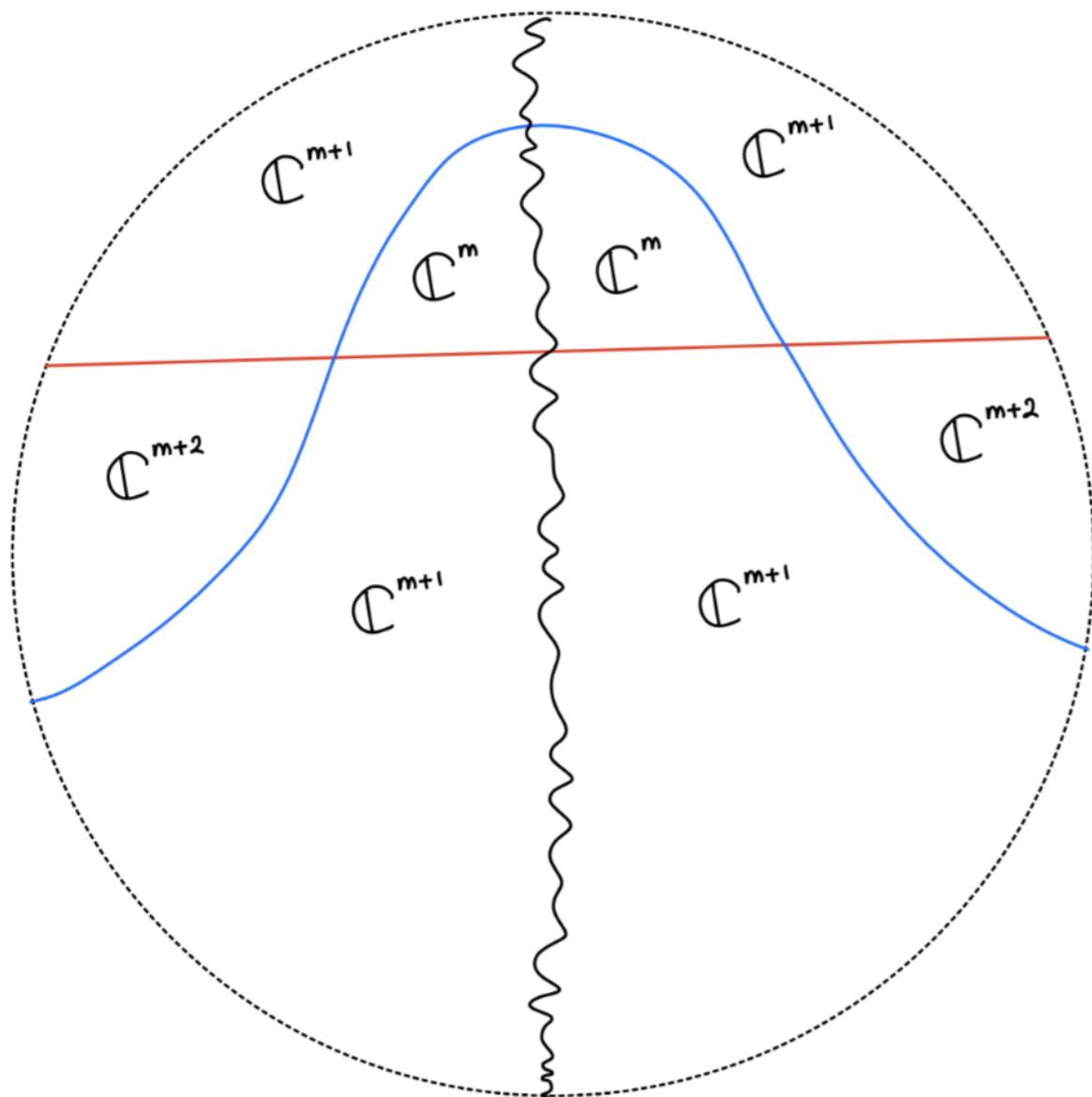


Figure 3.180

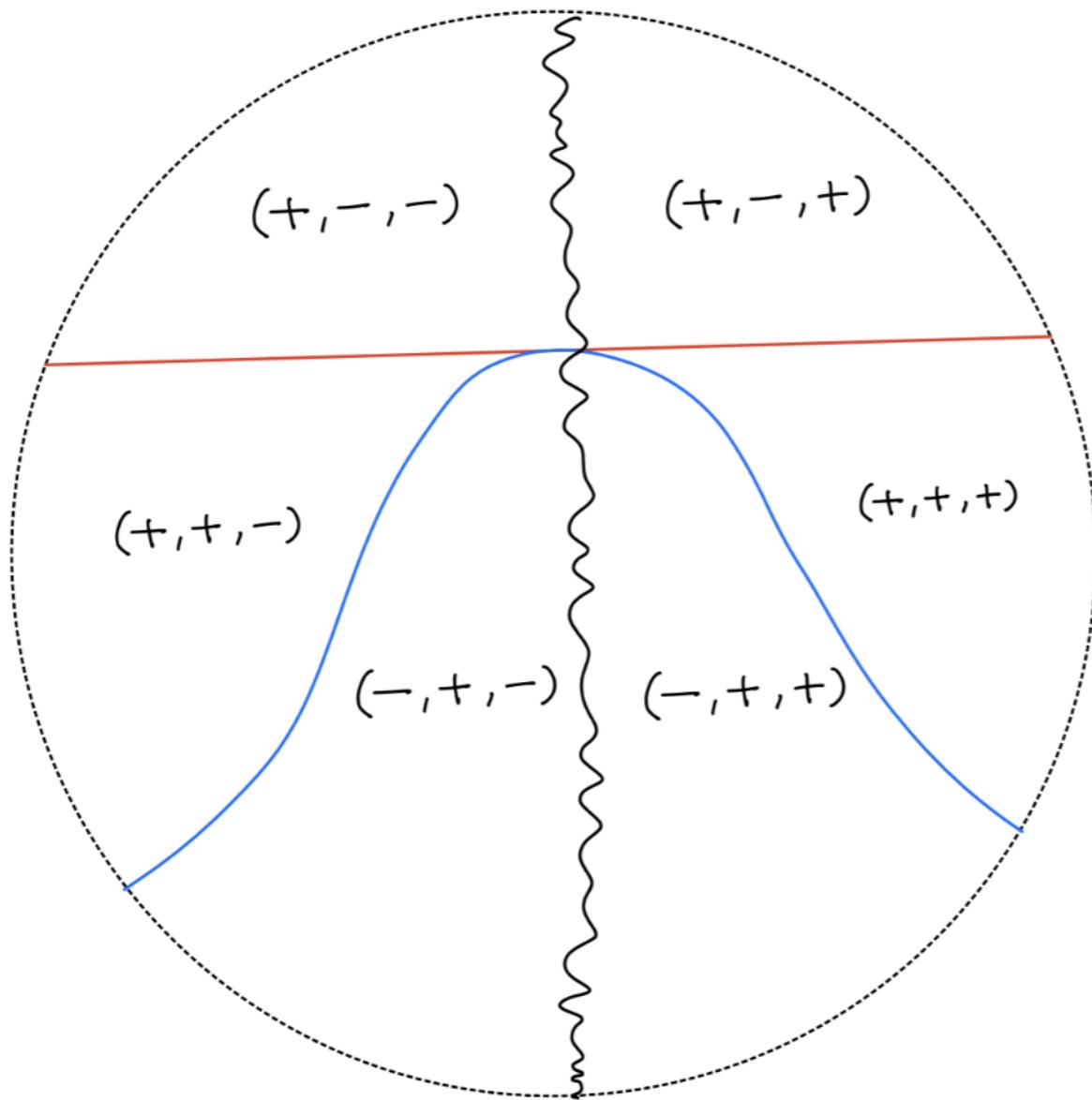


Figure 3.181

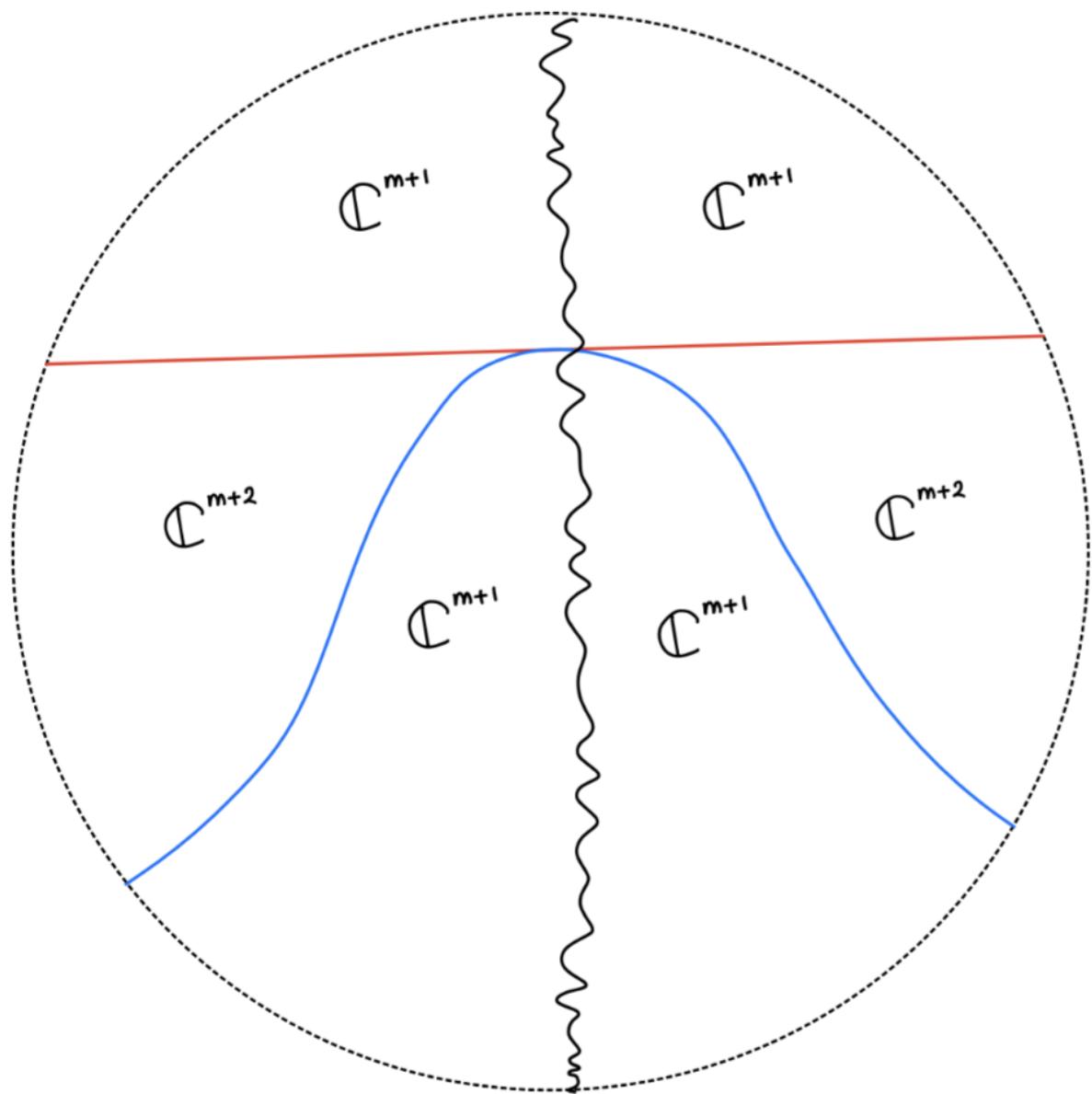


Figure 3.182

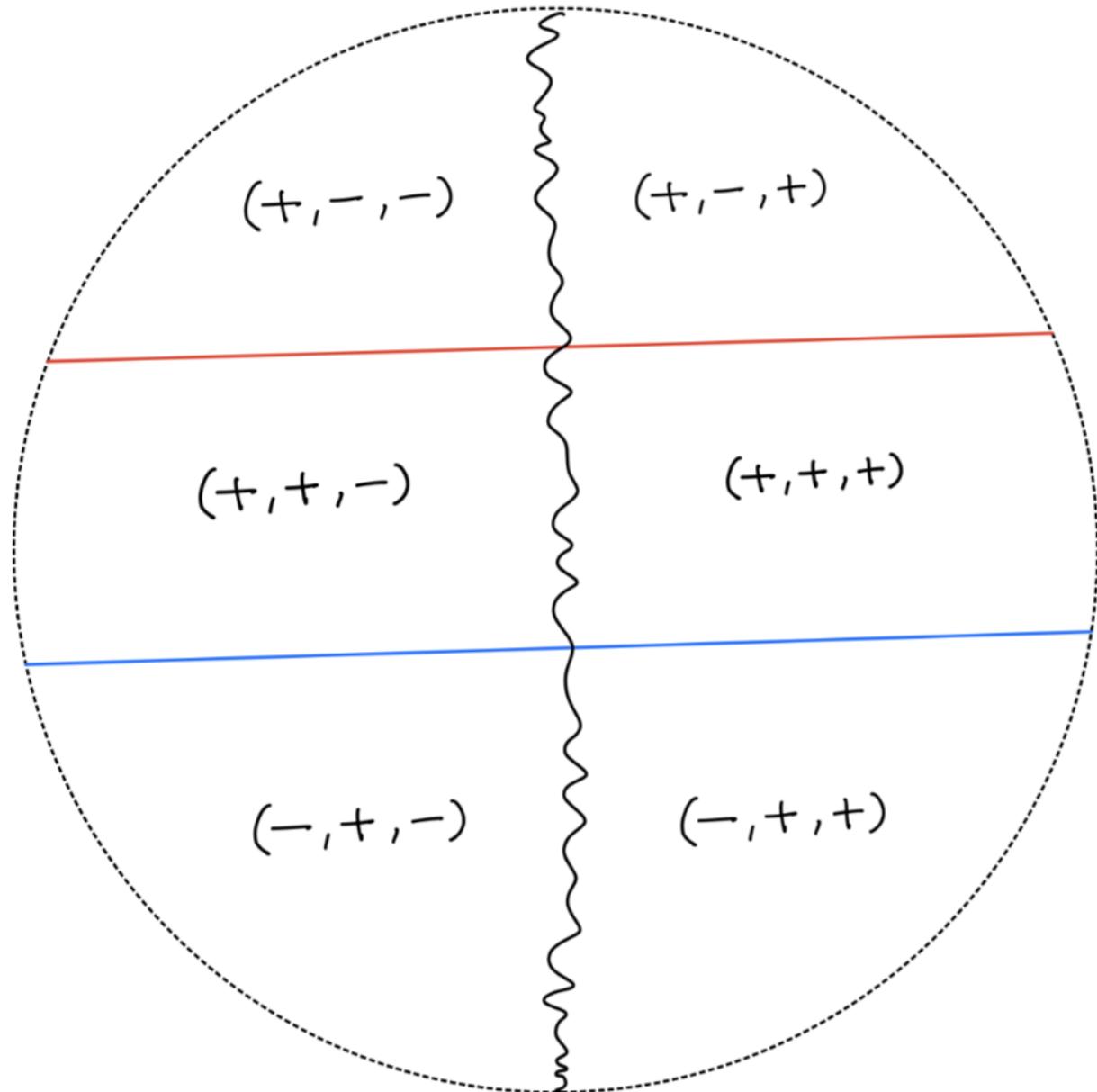


Figure 3.183

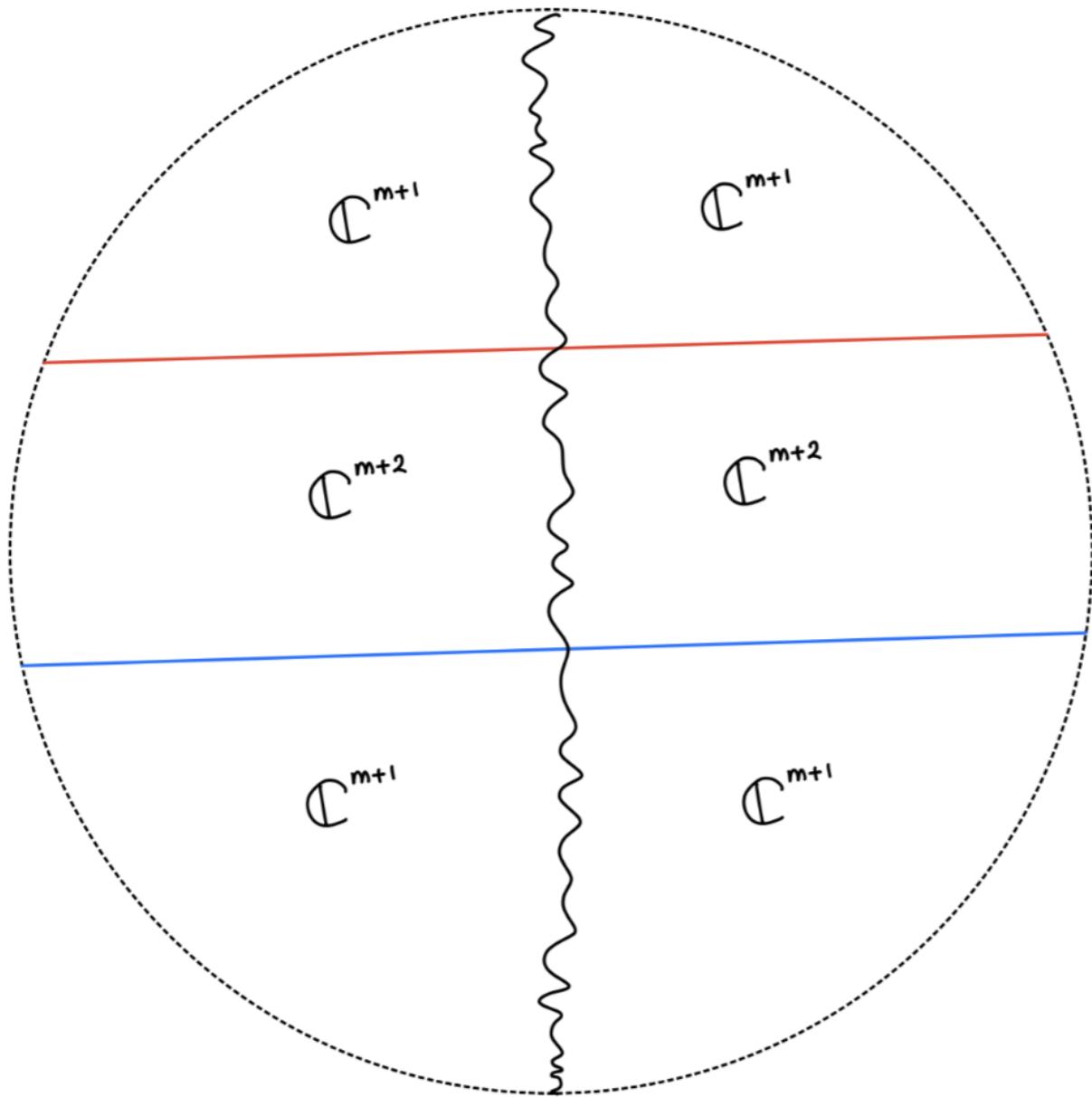


Figure 3.184

- $F_{\bullet}(-, -, -) := \mathbb{C}^m$
- $F_{\bullet}(-, -, +) := \mathbb{C}^m$
- $F_{\bullet}(+, -, -) := \mathbb{C}^{m+1}$
- $F_{\bullet}(+, -, +) := \mathbb{C}^{m+1}$
- $F_{\bullet}(-, +, -) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, +) := \mathbb{C}^{m+1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{m+2}$

- $F_\bullet(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

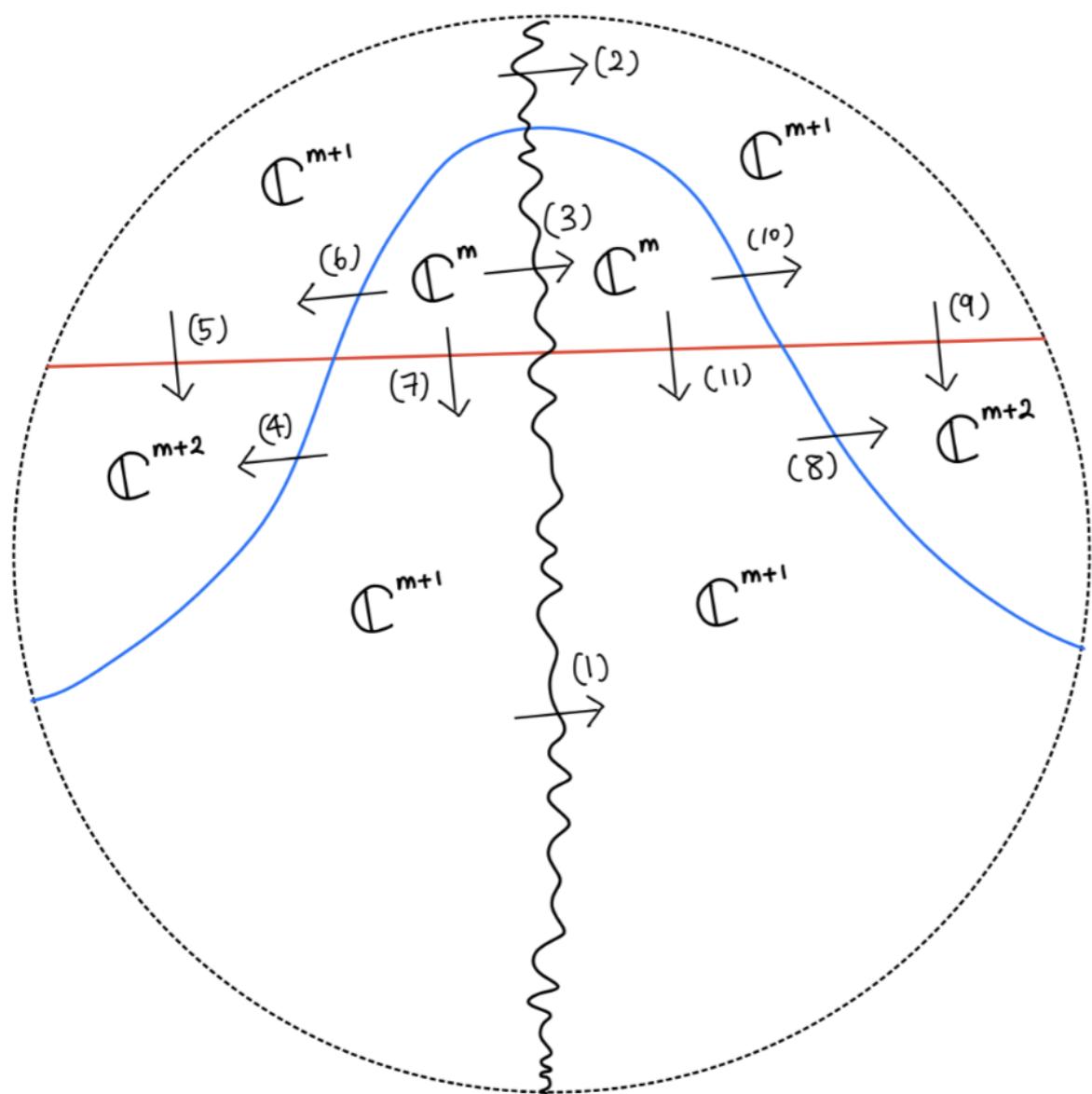


Figure 3.185

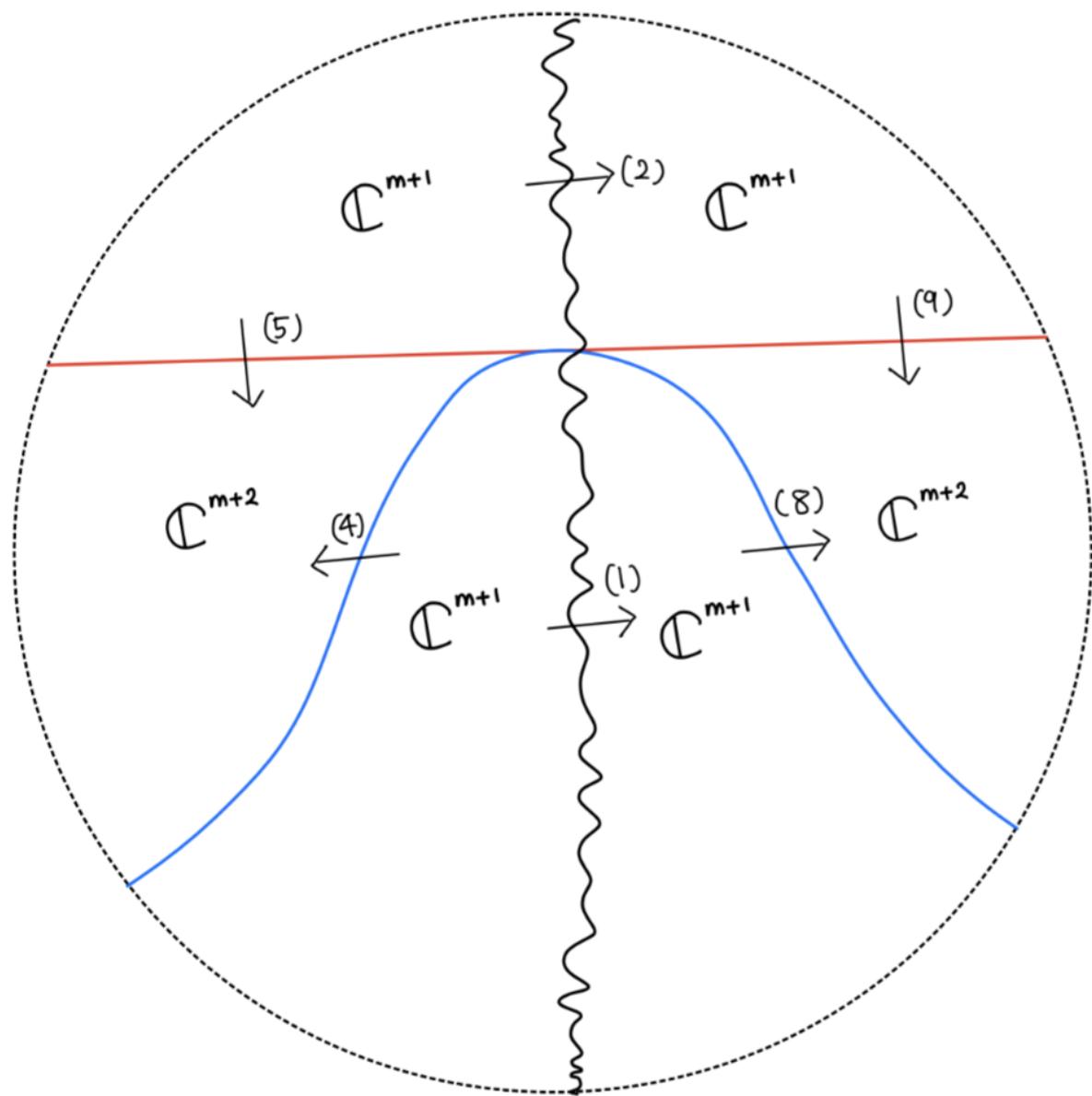


Figure 3.186

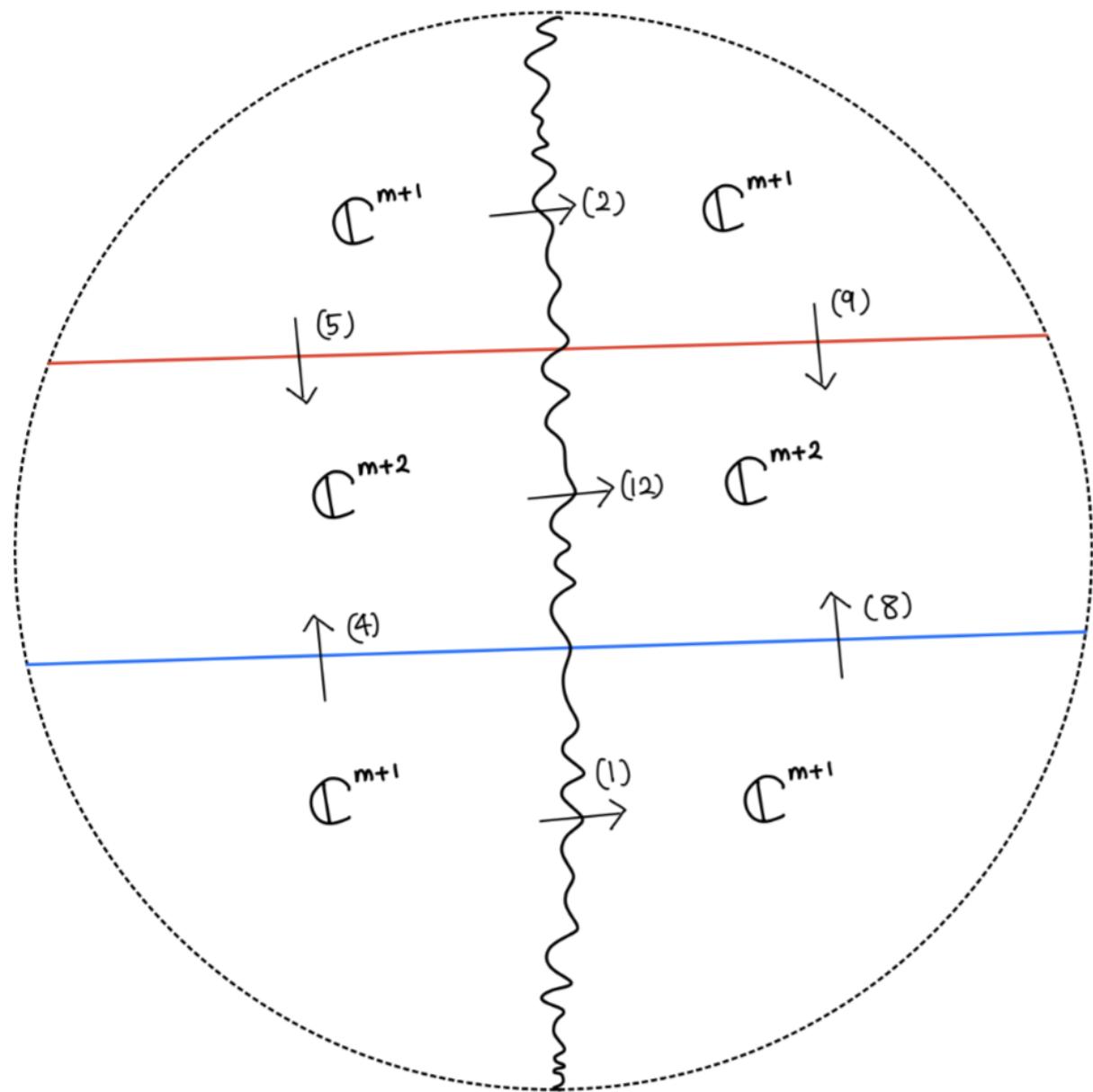


Figure 3.187

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^m \xrightarrow{\theta} \mathbb{C}^m$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$$

$$(11) \quad \mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$$

$$(12) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$

- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 164. $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 165. we define γ_\bullet to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 166. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of M .

Note that there is a diffeomorphism between $D_{r=2} \times (0, 1)$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}^m$
- $F^3(-, -, +) := \mathbb{C}^m$
- $F^3(+, -, -) := \mathbb{C}^{m+1}$
- $F^3(+, -, +) := \mathbb{C}^{m+1}$

- $F^3(-, +, -) := \mathbb{C}^{m+1}$

- $F^3(-, +, +) := \mathbb{C}^{m+1}$

- $F^3(+, +, -) := \mathbb{C}^{m+2}$

- $F^3(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

- $F^3(-, sgn_2, sgn_3) \rightarrow F^3(+, sgn_2, sgn_3) := \iota_1$ except $F(-, -, +) \rightarrow F^3(+, -, +) := \theta'$
and $F(-, +, +) \rightarrow F^3(+, +, +) := \mu'$

- $F^3(sgn_1, -, sgn_3) \rightarrow F^3(sgn_1, +, sgn_3) := \iota_0$ except $F(-, -, +) \rightarrow F^3(-, +, +) := \theta''$
and $F(+, -, +) \rightarrow F^3(+, +, +) := \nu'$

- $F^3(-, -, -) \rightarrow F^3(-, -, +) := \theta$

- $F^3(+, -, -) \rightarrow F^3(+, -, +) := \nu$

- $F^3(-, +, -) \rightarrow F^3(-, +, +) := \mu$

- $F^3(+, +, -) \rightarrow F^3(+, +, +) := T$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\theta''} & \mathbb{C}^{m+1} \\ \downarrow \theta' & & \downarrow \mu' \\ \mathbb{C}^{m+1} & \xrightarrow{\nu'} & \mathbb{C}^{m+2} \end{array}$$

: acyclicity follows from the crossing condition of F_0

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^m & \xrightarrow{\theta} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \theta' \\
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+2}
 \end{array}$$

: acyclicity follows from the crossing condition of F_0

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \mu' \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

: μ, T are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

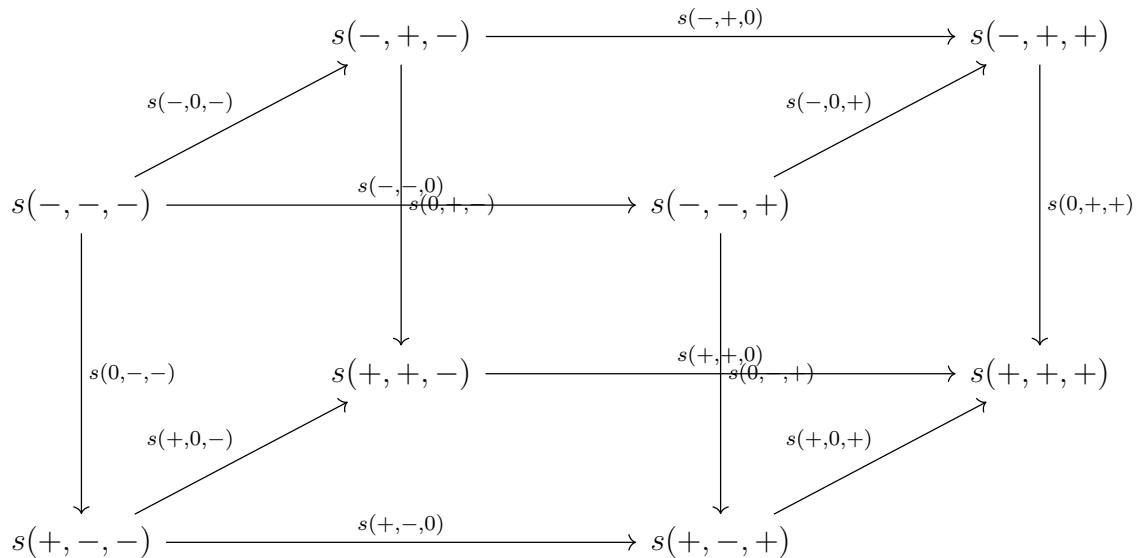
$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^m & \xrightarrow{\theta} & \mathbb{C}^m \\
 \downarrow \iota_0 & & \downarrow \theta'' \\
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1}
 \end{array}$$

: θ, μ are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{\nu} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \nu' \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

: ν, T are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

(vii) the cubic diagram:



$$\begin{array}{ccccc}
 & = & & & \\
 & & \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1} \\
 & \swarrow \iota_0 & \downarrow & & \searrow \theta'' \\
 \mathbb{C}^m & \xrightarrow{\theta_{\mu_1}} & \mathbb{C}^m & & \downarrow \mu' \\
 & \downarrow \iota_1 & & & \downarrow \\
 & & \mathbb{C}^{m+2} & \xrightarrow{T_{\theta'}} & \mathbb{C}^{m+2} \\
 & \swarrow \iota_0 & & & \searrow \nu' \\
 \mathbb{C}^{m+1} & \xrightarrow{\nu} & \mathbb{C}^{m+1} & &
 \end{array}$$

Note that θ, μ, ν, T are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

□

3.7.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the $cobord_2$. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphism $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

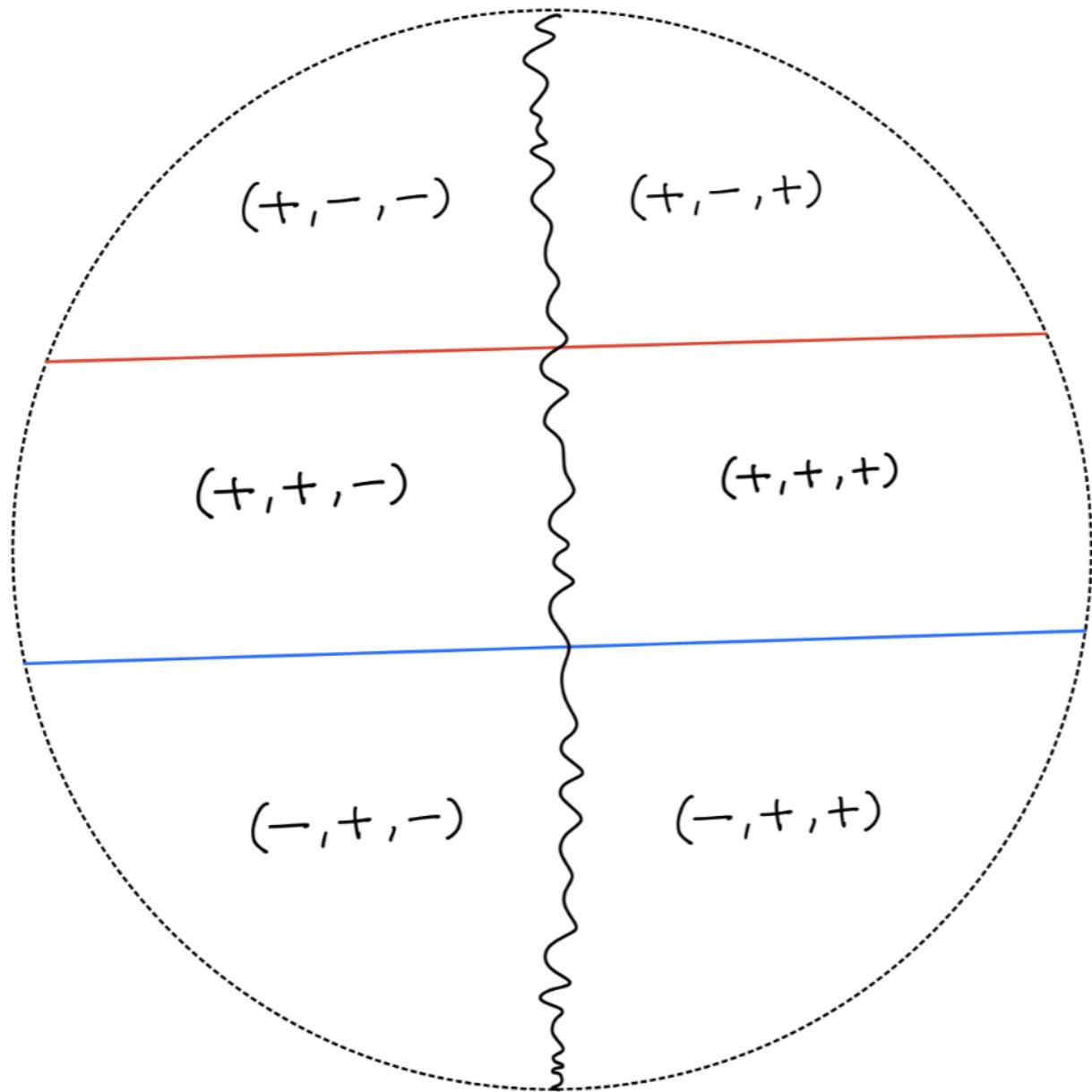


Figure 3.188

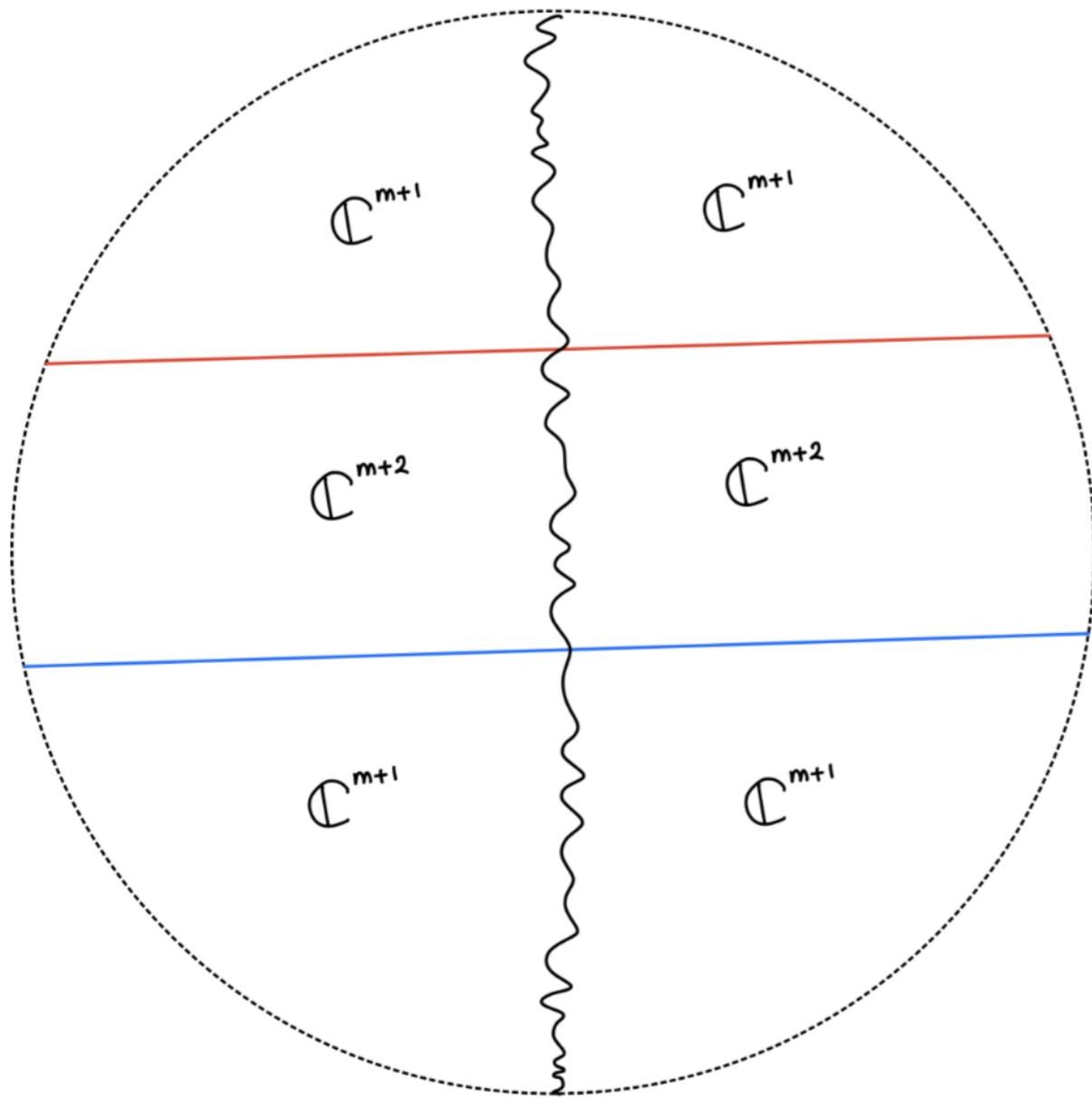


Figure 3.189

- $F_1(+, -, -) := \mathbb{C}^{m+1}$
- $F_1(+, -, +) := \mathbb{C}^{m+1}$
- $F_1(-, +, -) := \mathbb{C}^{m+1}$
- $F_1(-, +, +) := \mathbb{C}^{m+1}$
- $F_1(+, +, -) := \mathbb{C}^{m+2}$

- $F_1(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

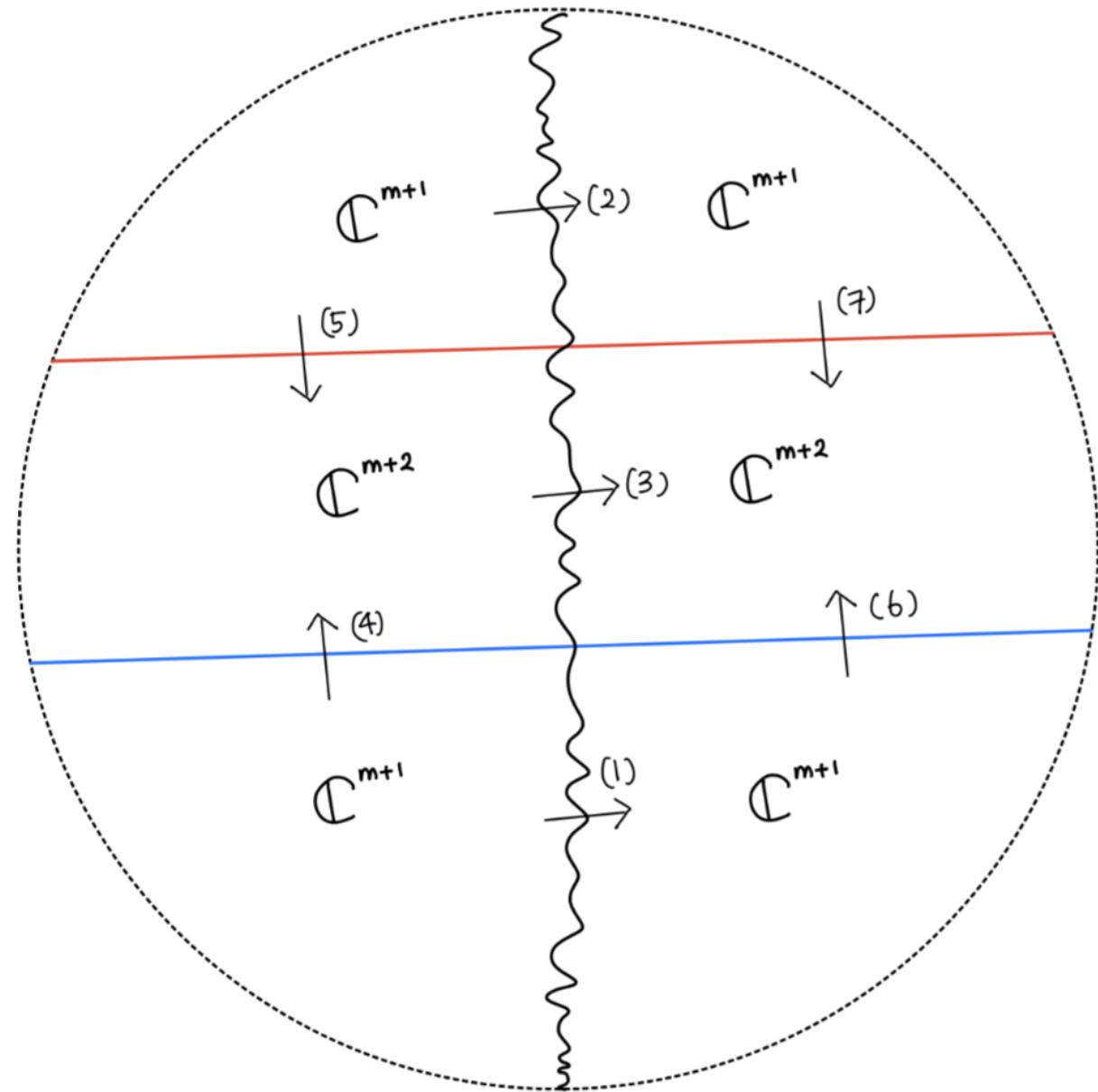


Figure 3.190

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$
- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 167. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

3.8 4th Sheaf Cobordism

In this section, we define $cobord_4$, a compactly supported sheaf cobordism.

3.8.1 Notations

Definition 168. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 169. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_4$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_4$.

Definition 170. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_{\bullet}^{symbol} to be smooth maps

$$\Phi_{\bullet}^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_{\bullet}^{symbol} a co-orientation of Φ_{\bullet}^{symbol} .

3. we denote the pair $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$ as $\Lambda_{\bullet}^{symbol}$. Later in the section, $\Lambda_{\bullet}^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₄.

4. we denote the triple $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$ as Λ_{\bullet} and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_{\bullet} will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₄.

Definition 171. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} e^{(\frac{x^2}{x^2-1})} (\frac{1}{2} - t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$ if $t \neq \frac{1}{2}$
- $\{(1, 0), (-1, 0), (0, \frac{1}{2} - t)\} \subset Graph(\Psi_t)$

Definition 172. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$

$$(x, z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 173. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_0(z)\}$
- λ_0^∞ is the union of the following two components
 - (i) $\{(x, z) \in D_{r=2} \mid z = x\}$
 - (ii) $\{(x, z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_0^∞ : hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 174. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_1(z)\}$
- λ_1^∞ is the union of the following two components
 - (i) $\{(x, z) \in D_{r=2} \mid z = x\}$
 - (ii) $\{(x, z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 175. 1. We define the following subsets of $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = \Psi_t(z)\}$
- λ_\bullet^∞ is the union of the following two components
 - (i) $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = x\}$
 - (ii) $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -x\}$

2. We define co-orientations ξ_\bullet^{symbol} of λ_\bullet^{symbol} as follows

- ξ_\bullet^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_\bullet^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 176. 1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .

2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_\bullet on $D_{r=2} \times [0, 1]$ induced by λ_\bullet i.e. strata are non-empty finite intersections of λ_\bullet^0 , λ_\bullet^∞ , and λ_\bullet^{squig} . Note that 2 dimensional strata has co-orientations inherited from λ_\bullet .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the following notations:

Definition 177. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 178. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(x - \Psi_{t_0}(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z)) = sgn_3\} \end{aligned}$$

Definition 179. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(x - \Psi_t(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z)) = sgn_3\} \end{aligned}$$

Definition 180. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the above notations:

1. \mathcal{S}_0 :

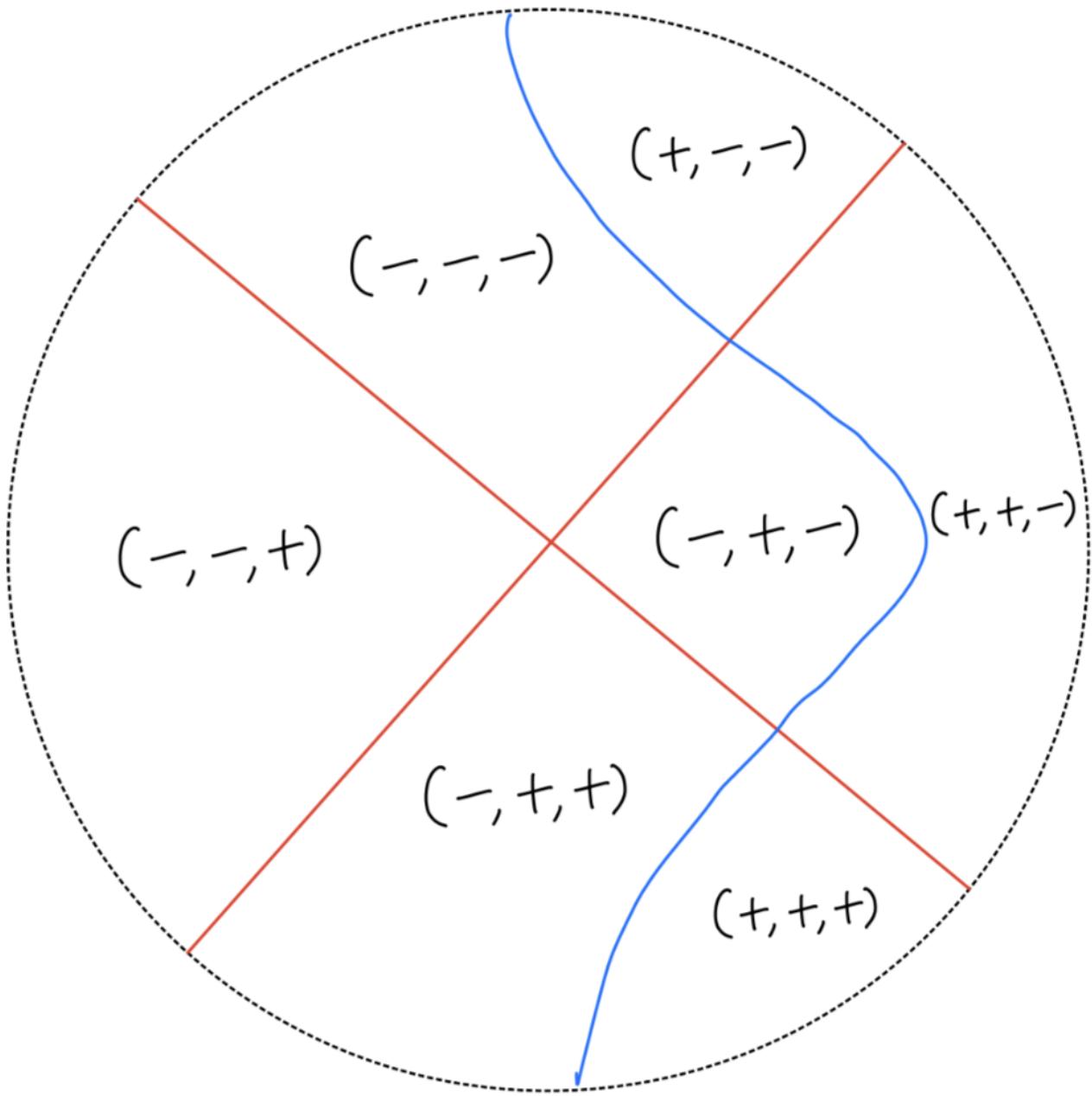


Figure 3.191

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_0(+, -, +)\}$$

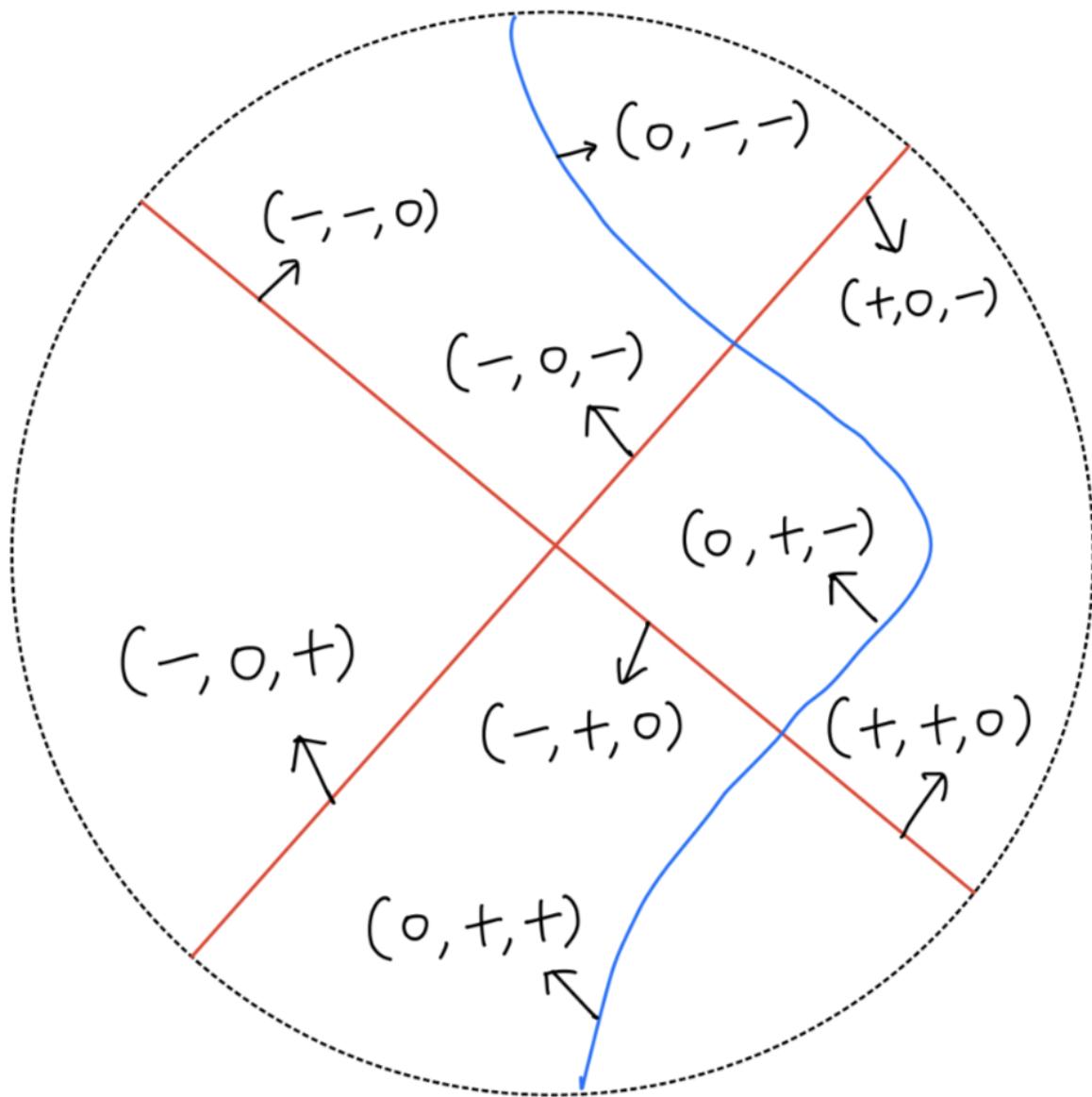


Figure 3.192

- 1 dimensional strata:

$\{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_0(0, -, +)\} \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_0(+, 0, +)\} \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, +, 0)\}$

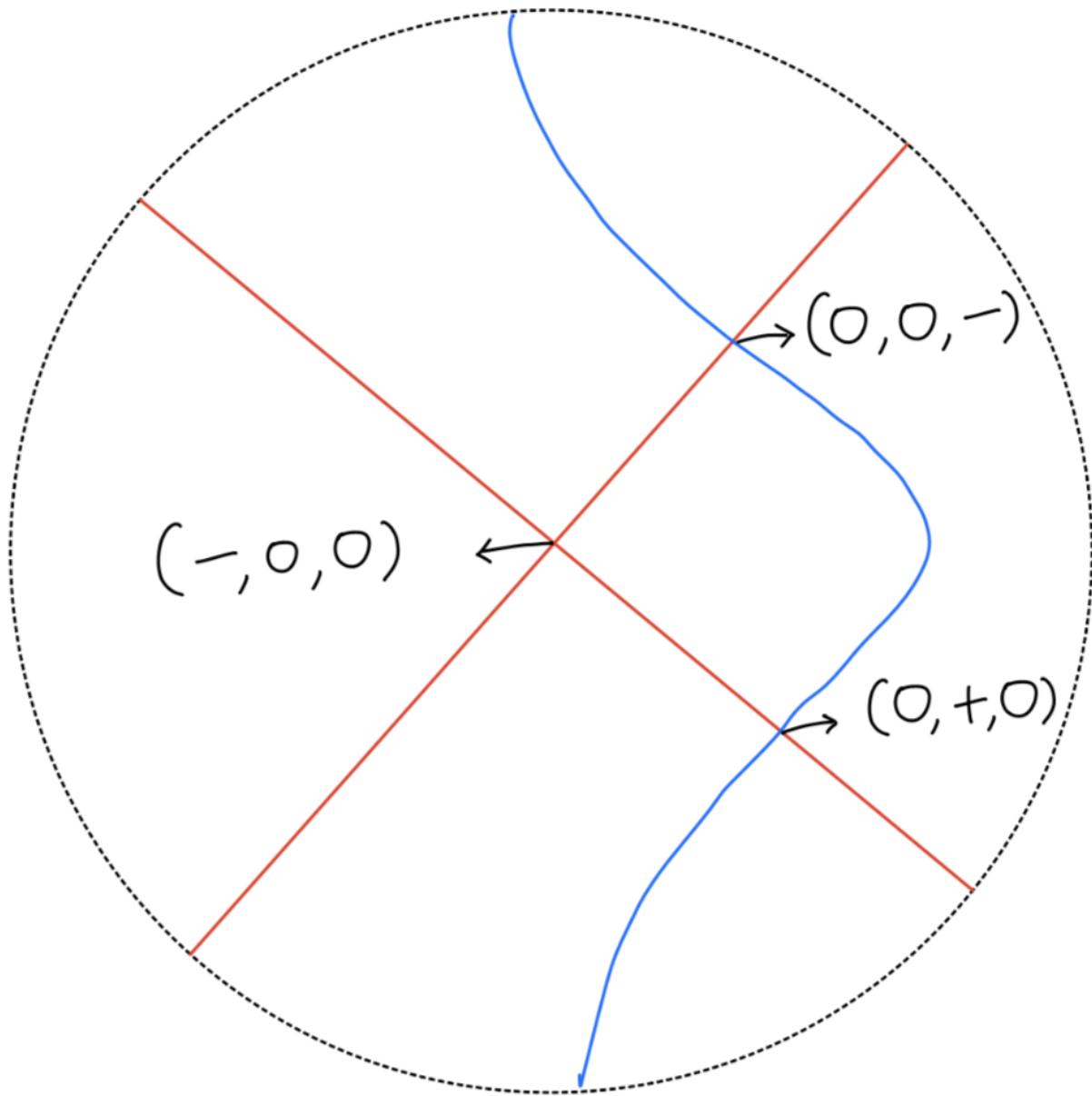


Figure 3.193

- 0 dimensional strata:

$$s_0(-, 0, 0), s_0(0, 0, -), s_0(0, +, 0)$$

2. \mathcal{S}_1 :

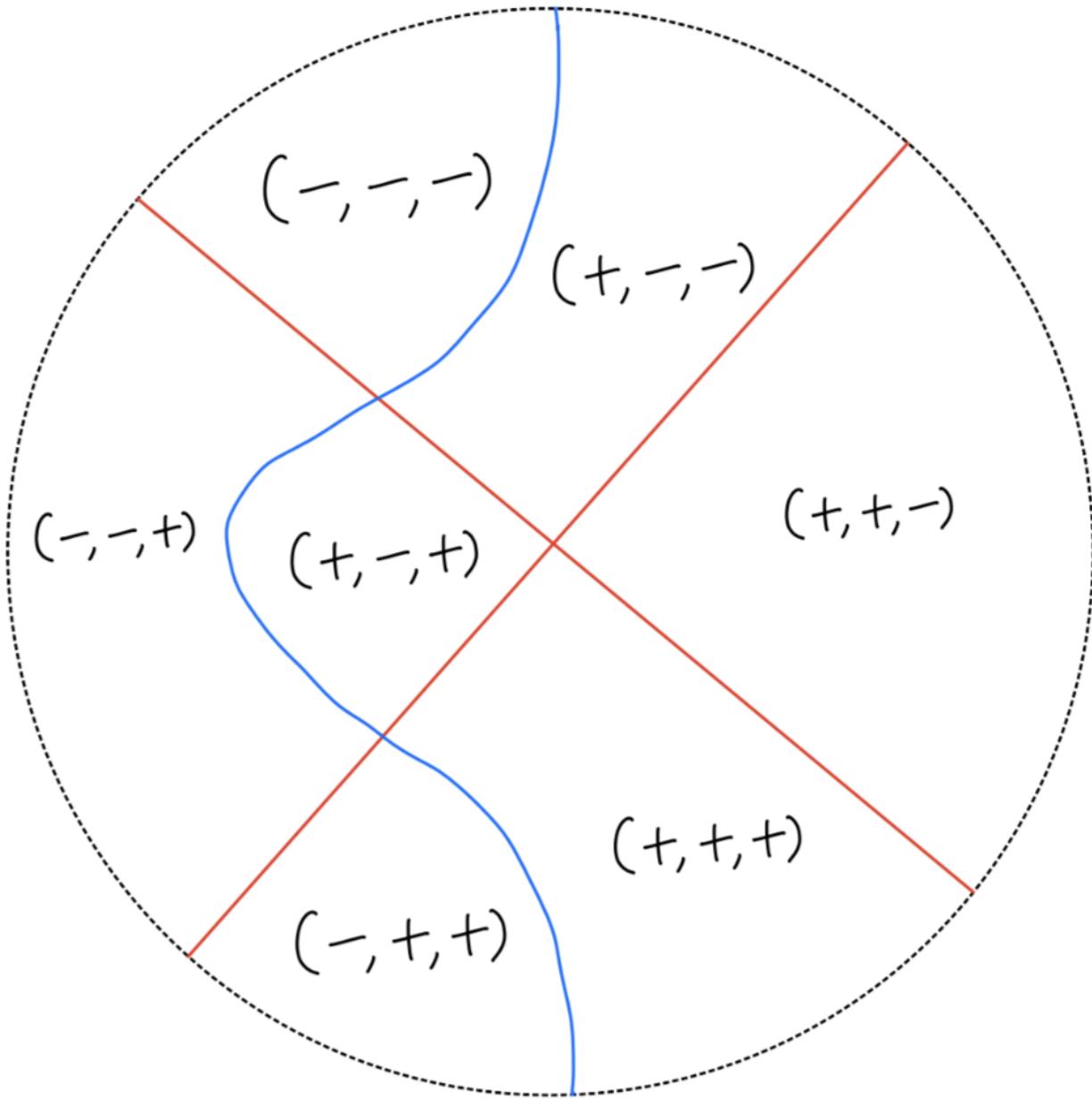


Figure 3.194

- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_1(-, +, -)\}$$

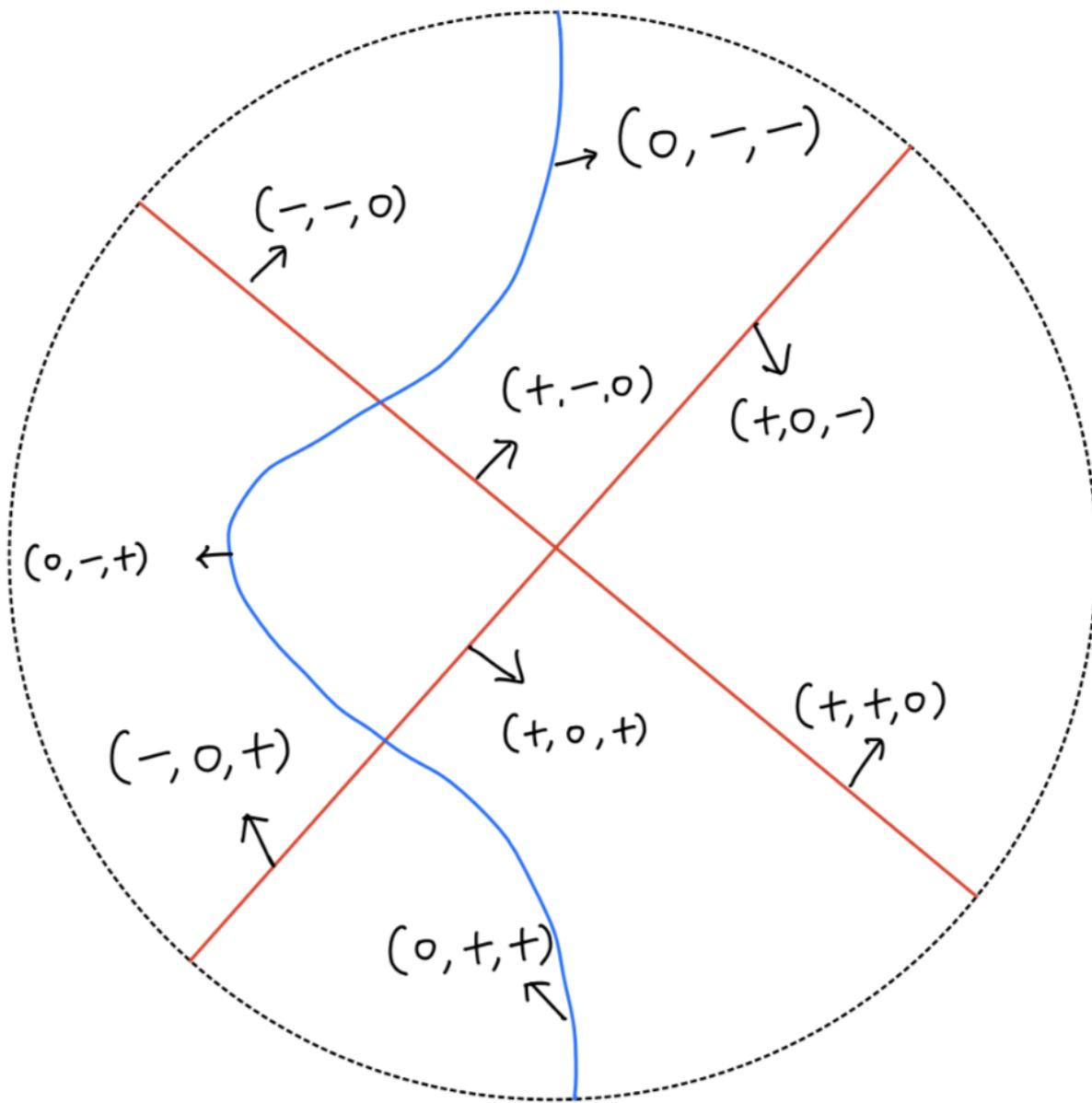


Figure 3.195

- 1 dimensional strata:

$\{s_1(0, \text{sgn}_2, \text{sgn}_3) \mid \text{sgn}_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_1(0, +, -)\} \cup \{s_1(\text{sgn}_1, 0, \text{sgn}_3) \mid \text{sgn}_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_1(-, 0, -)\} \cup \{s_1(\text{sgn}_1, \text{sgn}_2, 0) \mid \text{sgn}_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(0, 0, -)\}$

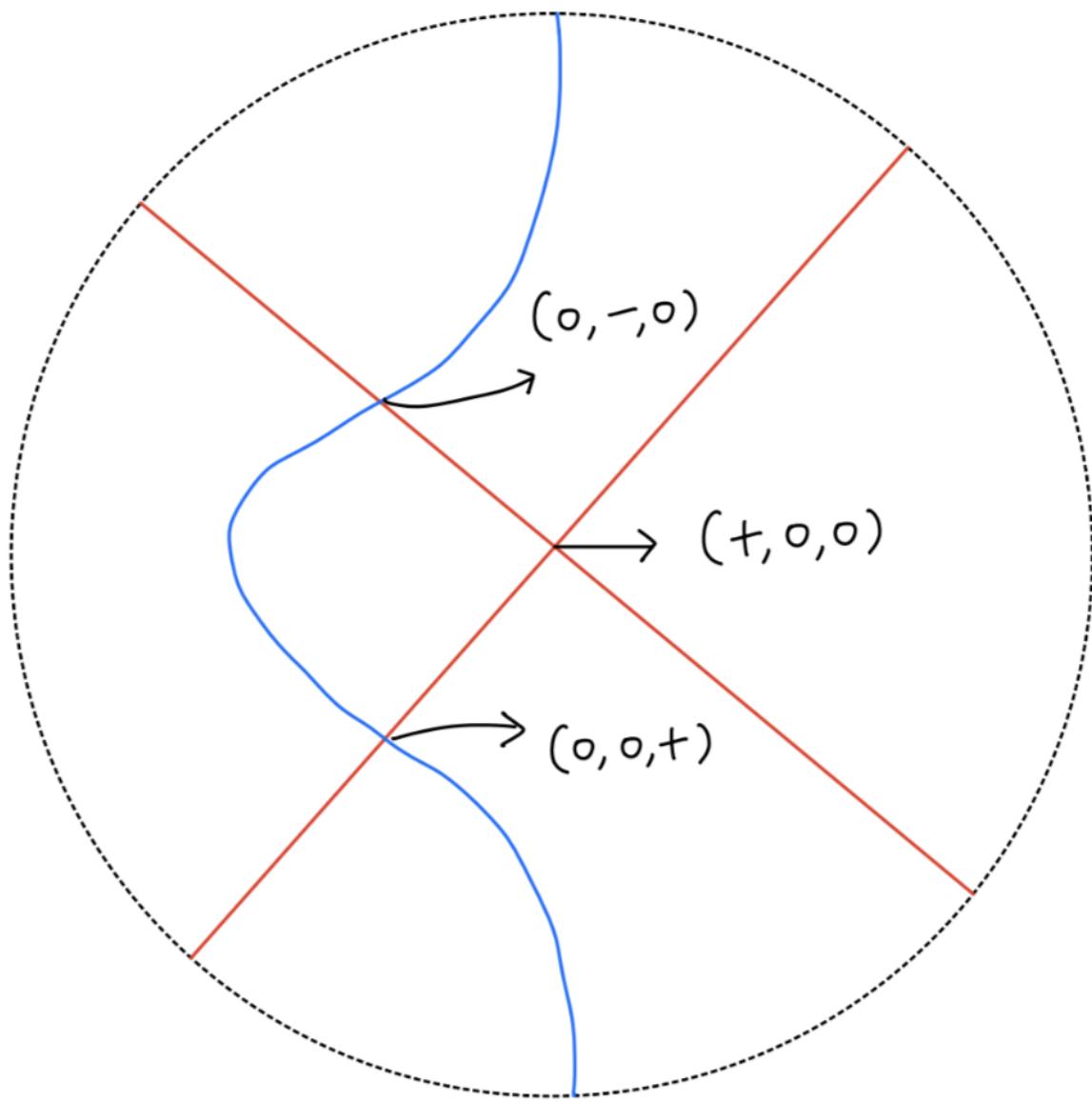


Figure 3.196

- 0 dimensional strata:

$$s_1(0, -, 0), s_1(+, 0, 0), s_1(0, 0, +)$$

3. \mathcal{S}_\bullet :

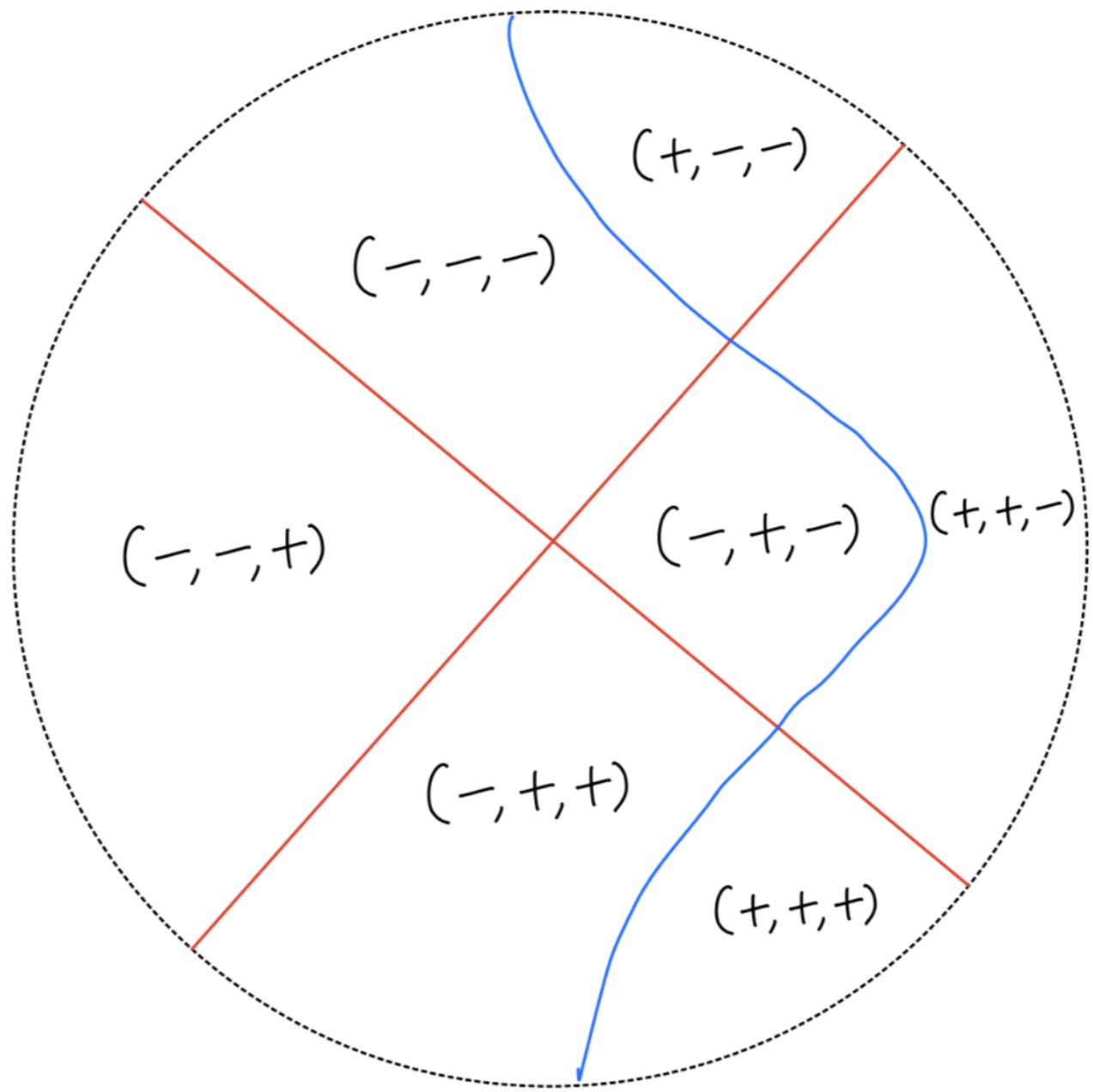


Figure 3.197

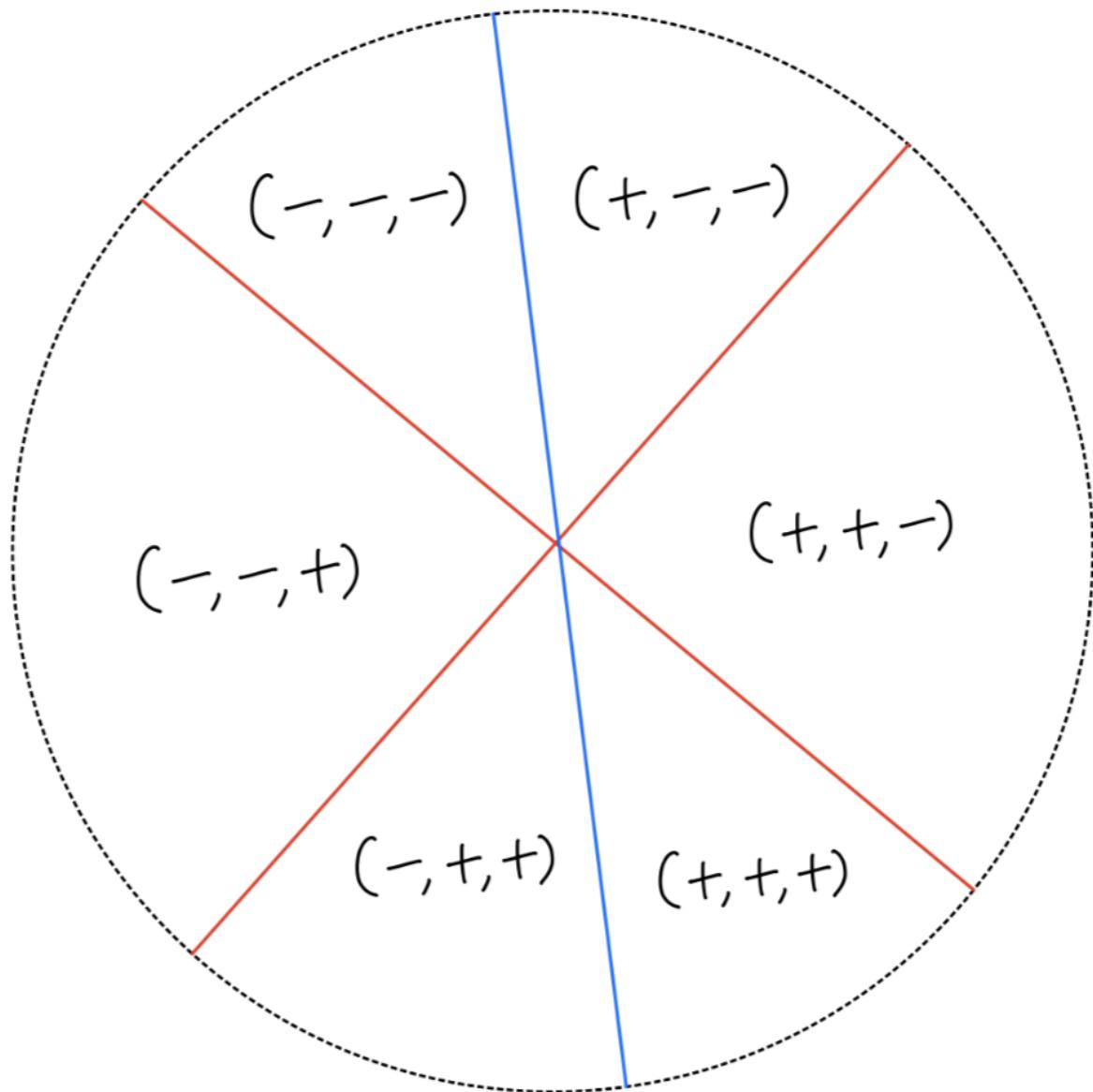


Figure 3.198

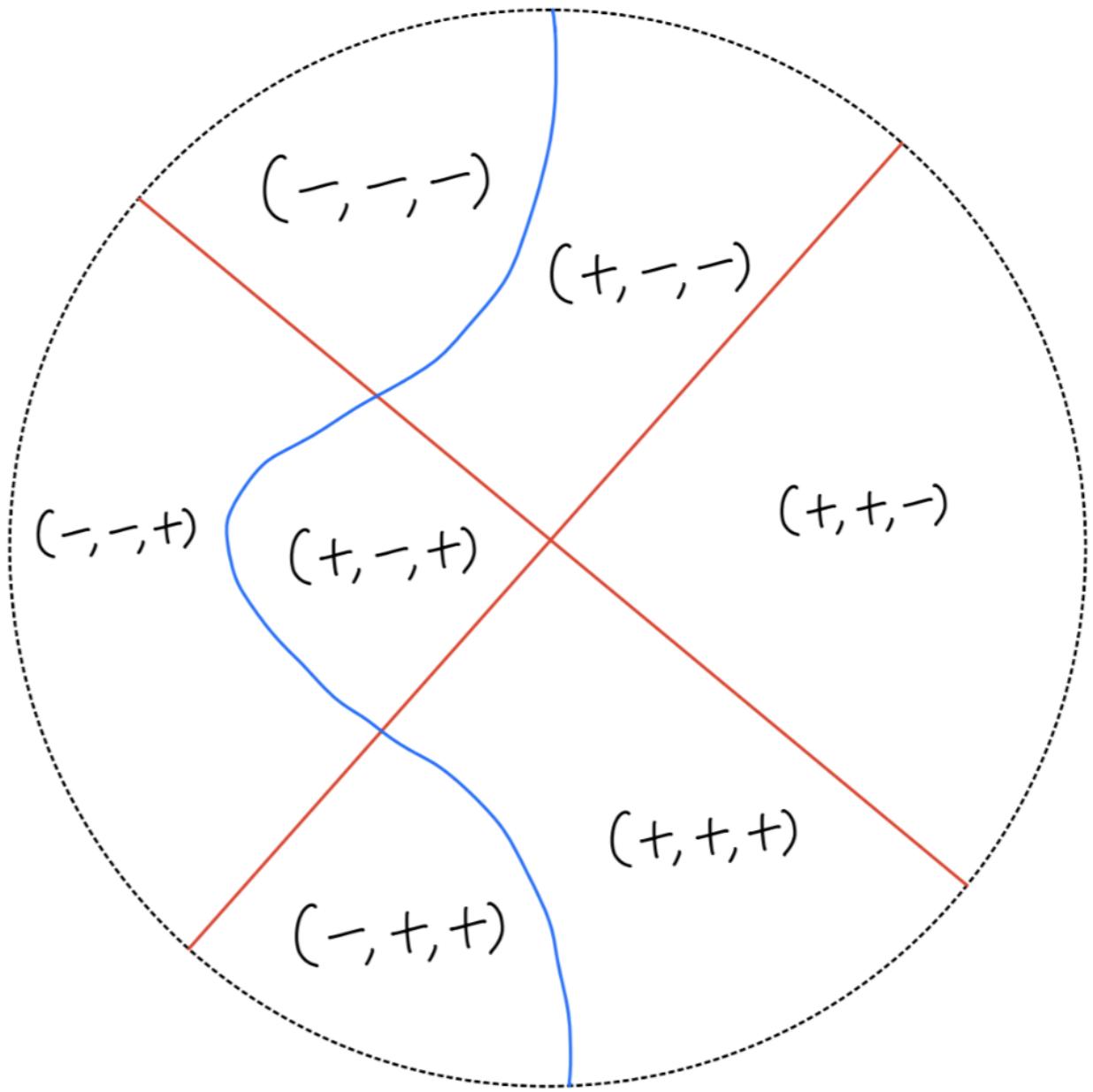


Figure 3.199

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

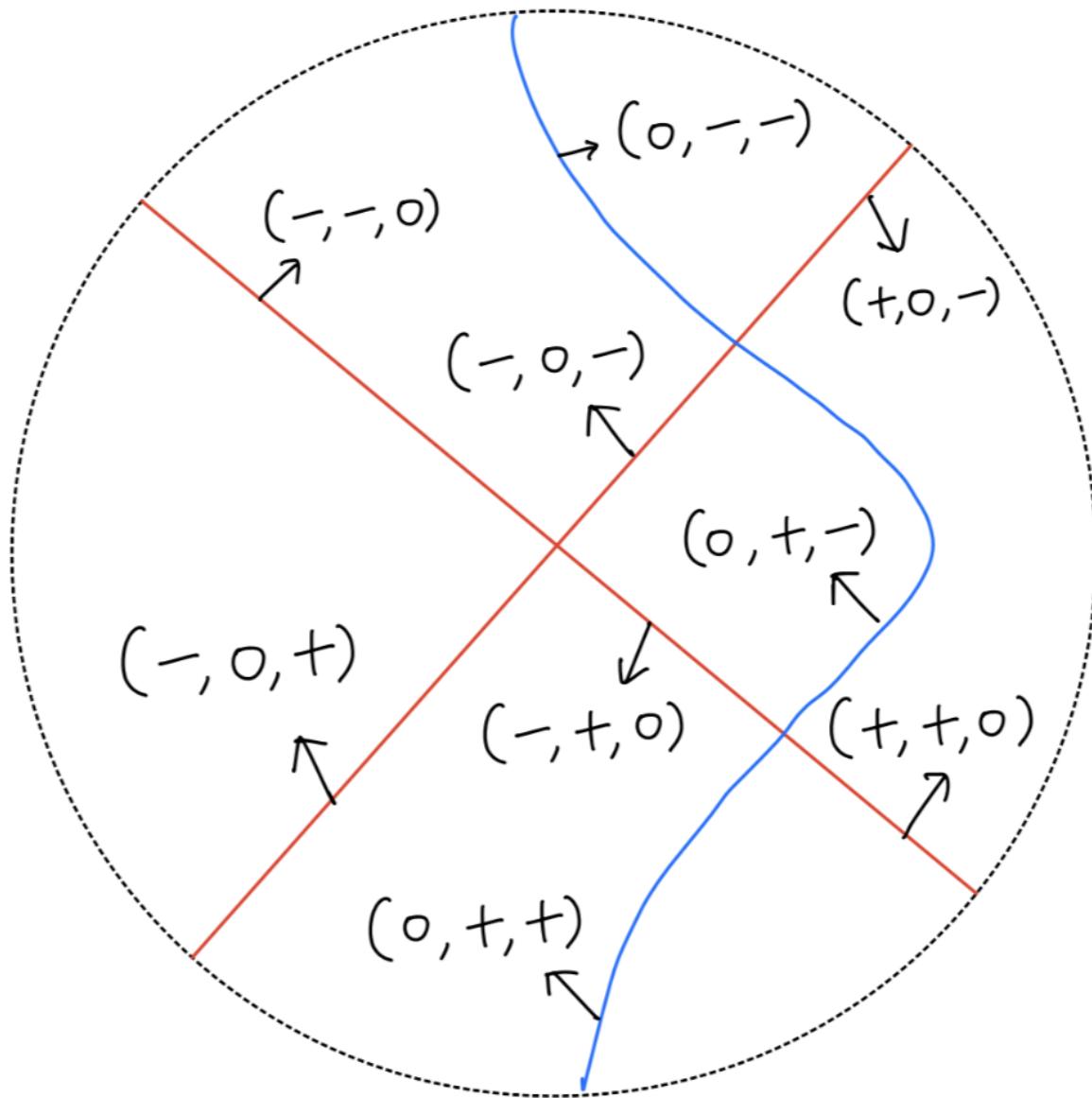


Figure 3.200

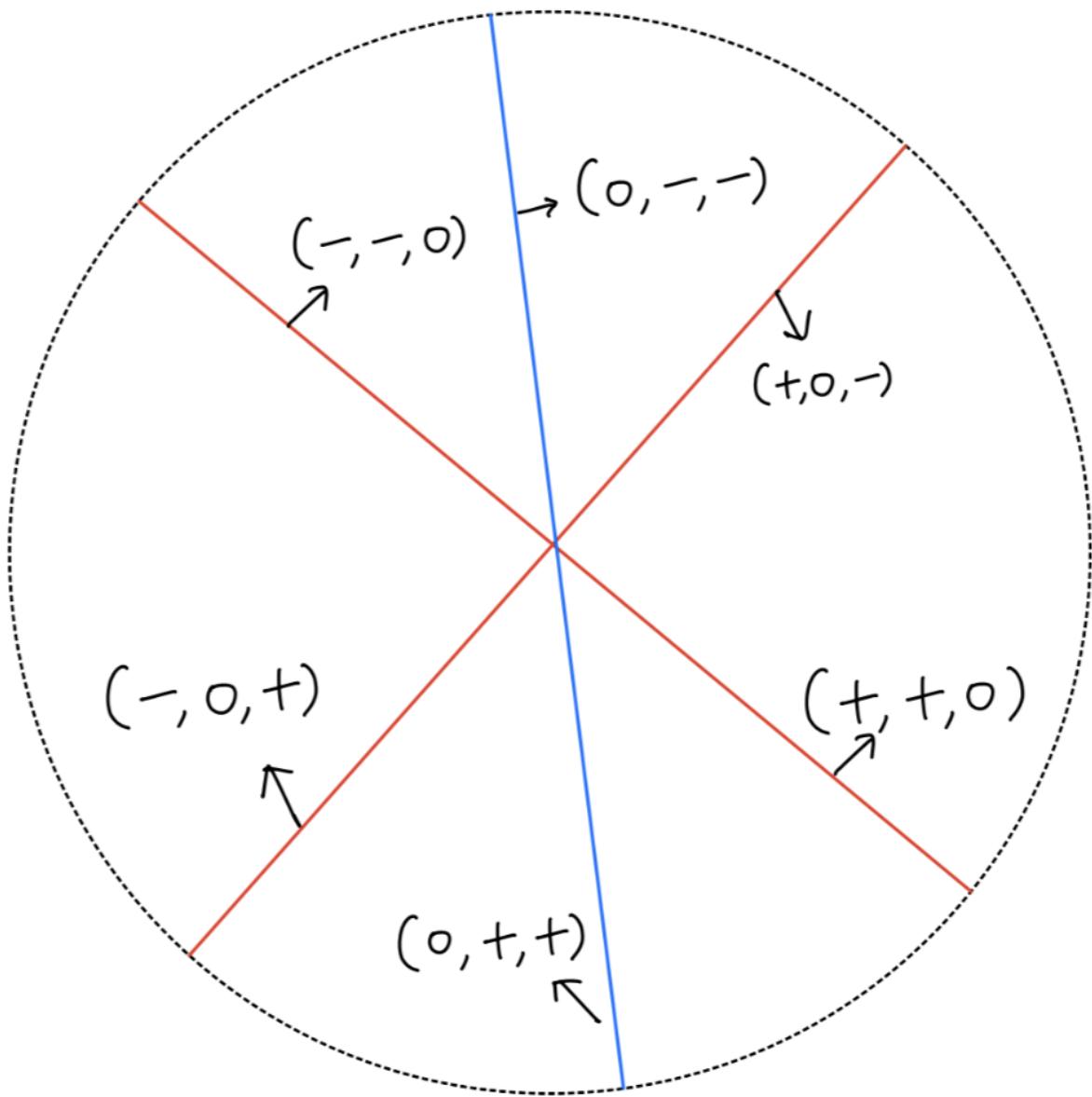


Figure 3.201

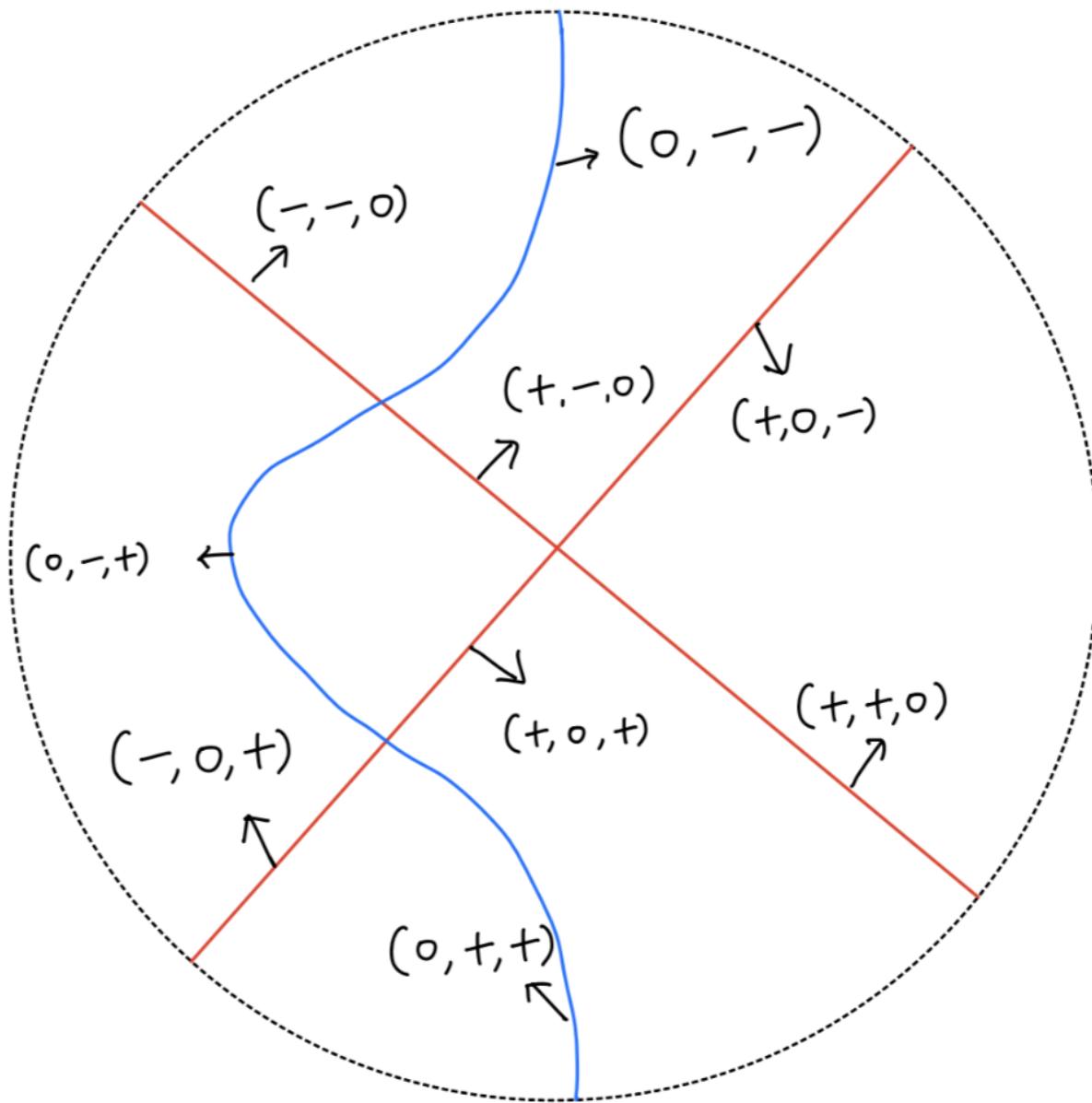


Figure 3.202

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

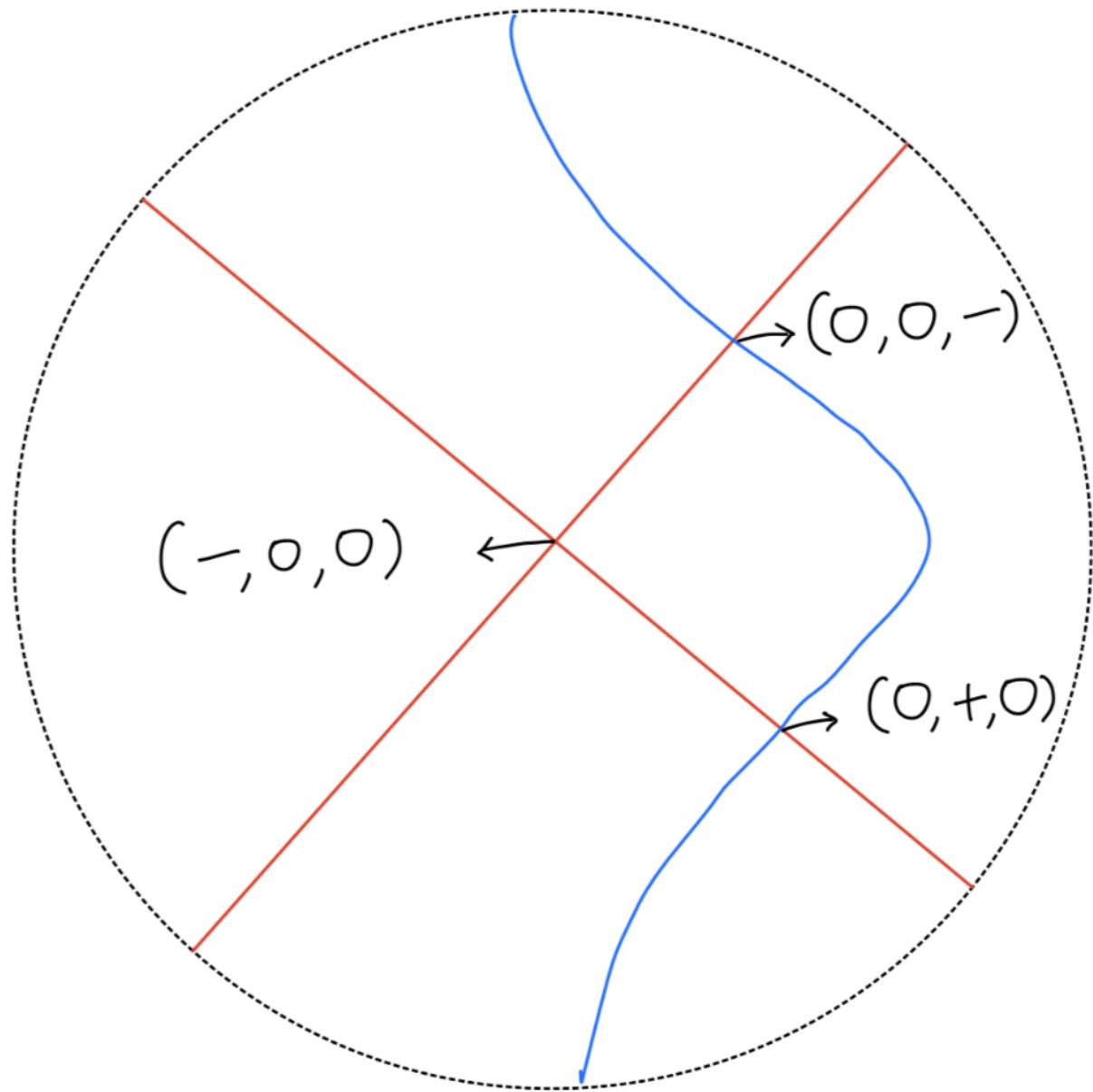


Figure 3.203

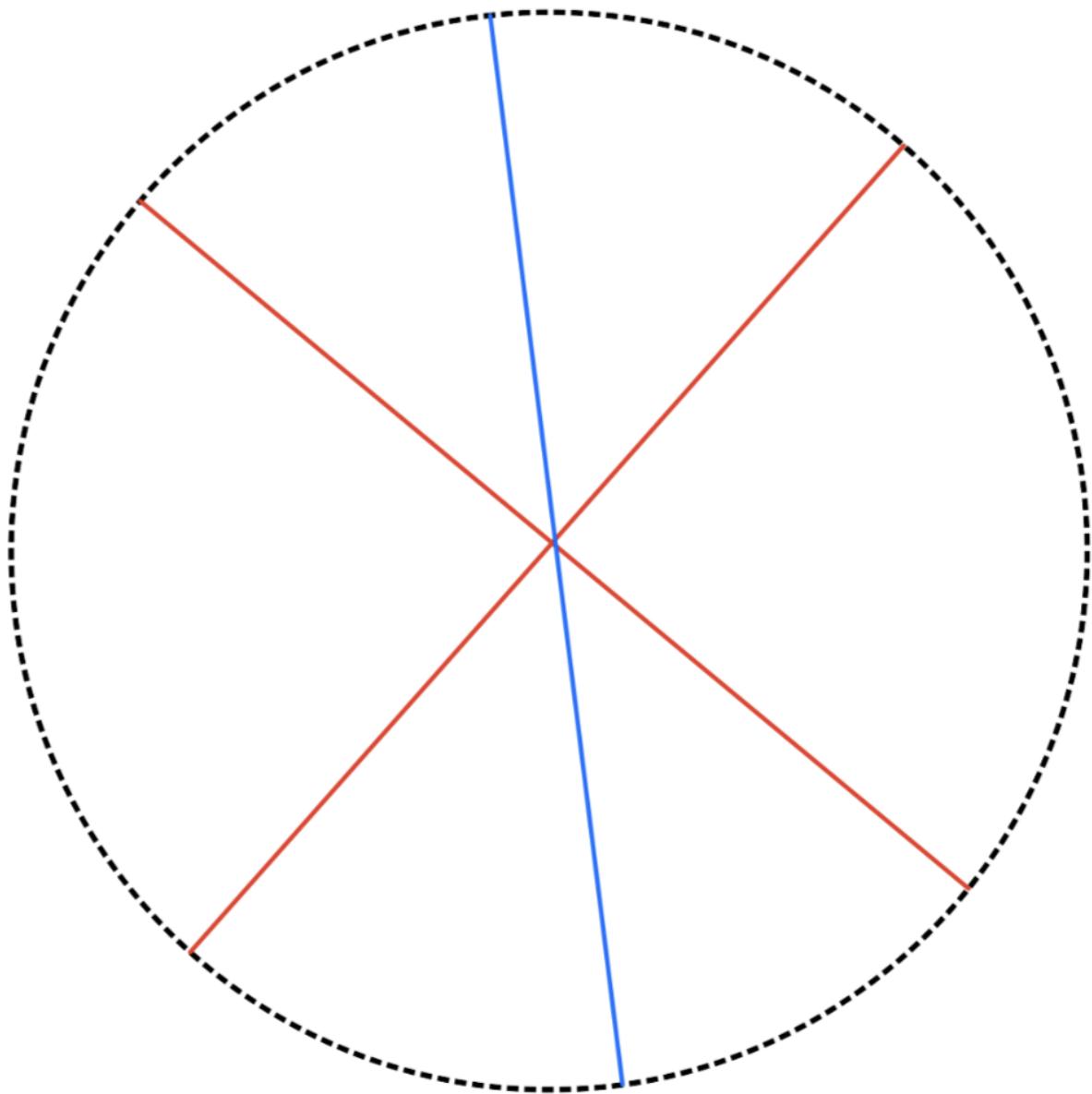


Figure 3.204

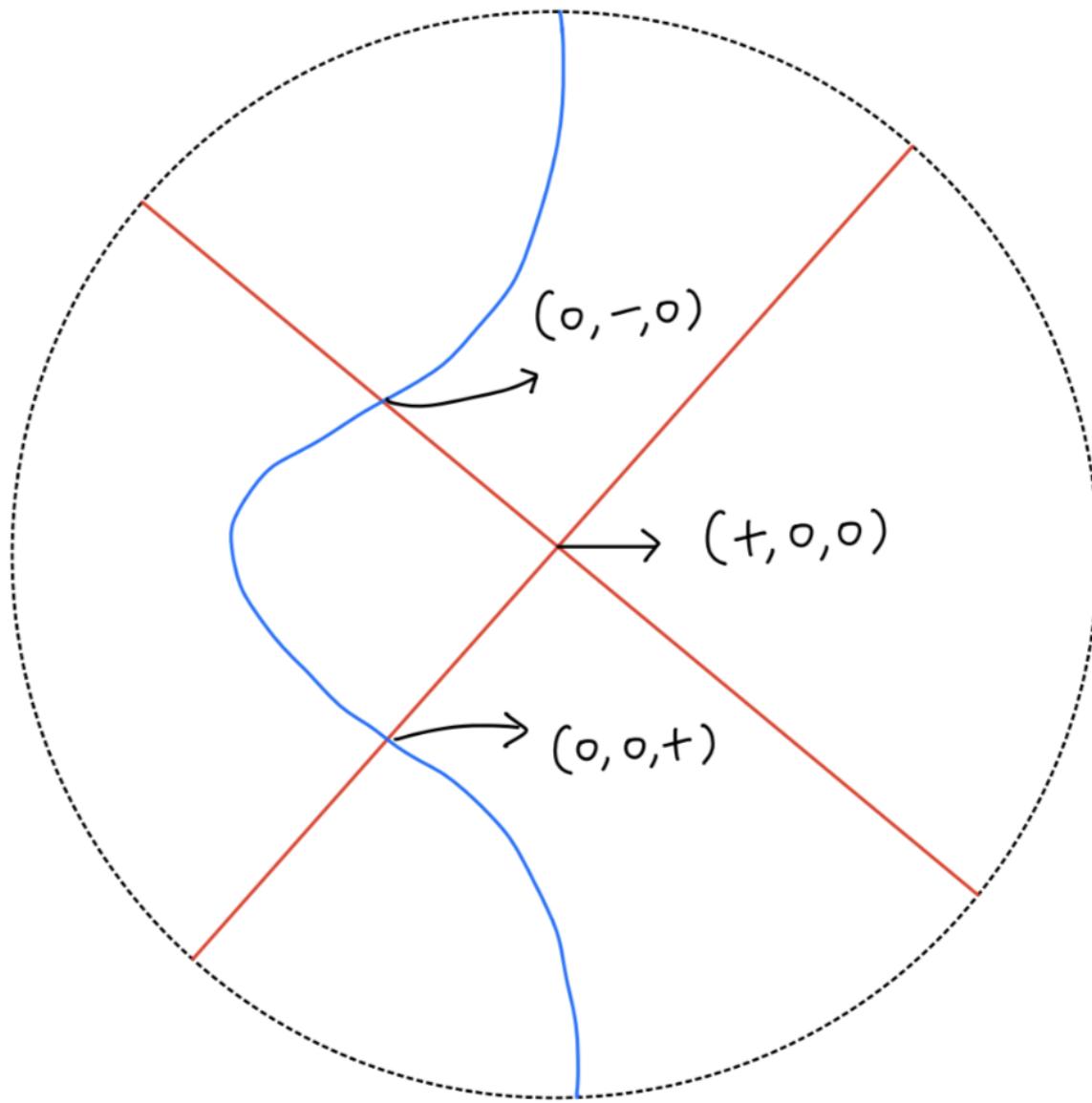


Figure 3.205

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

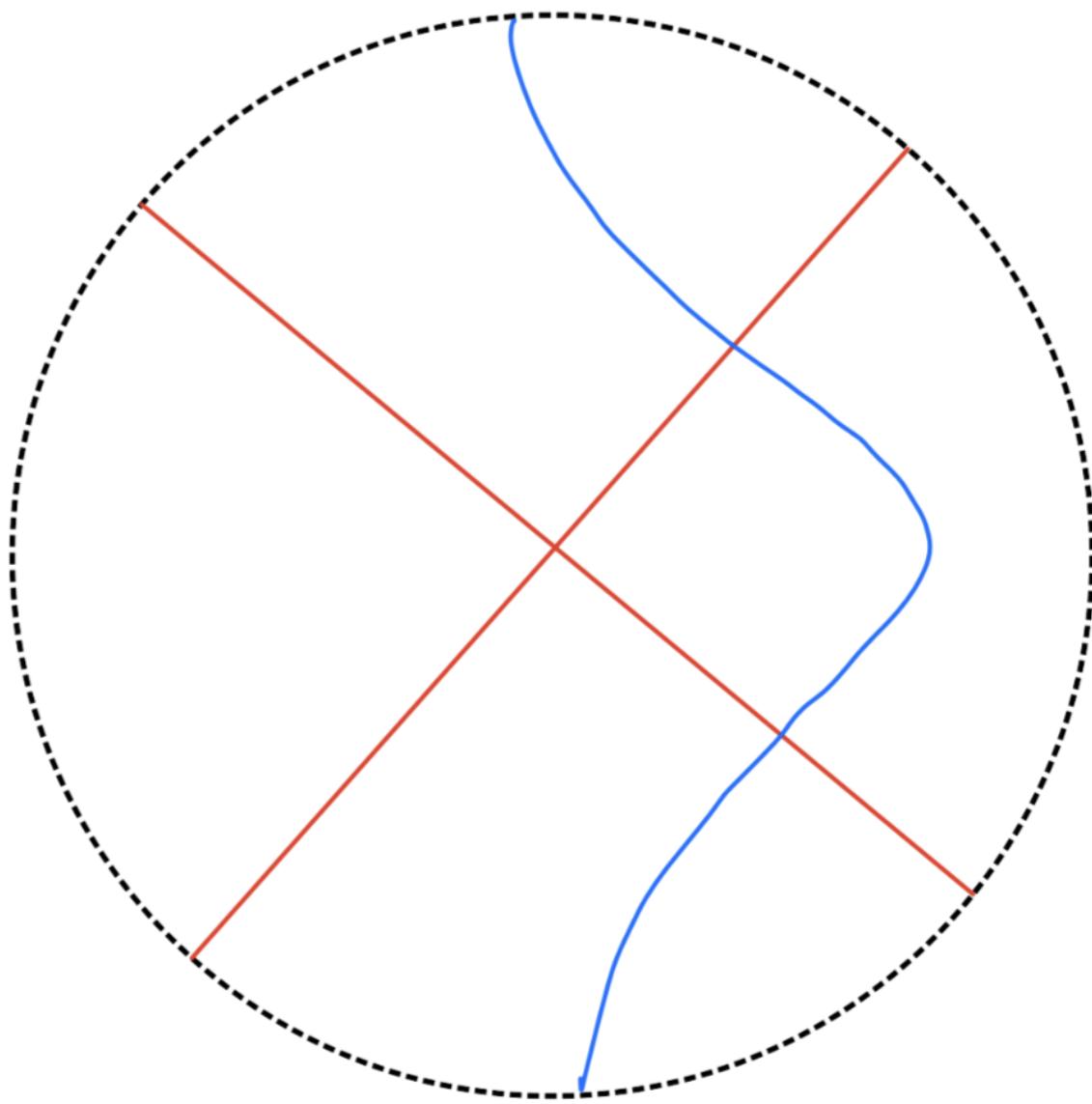


Figure 3.206

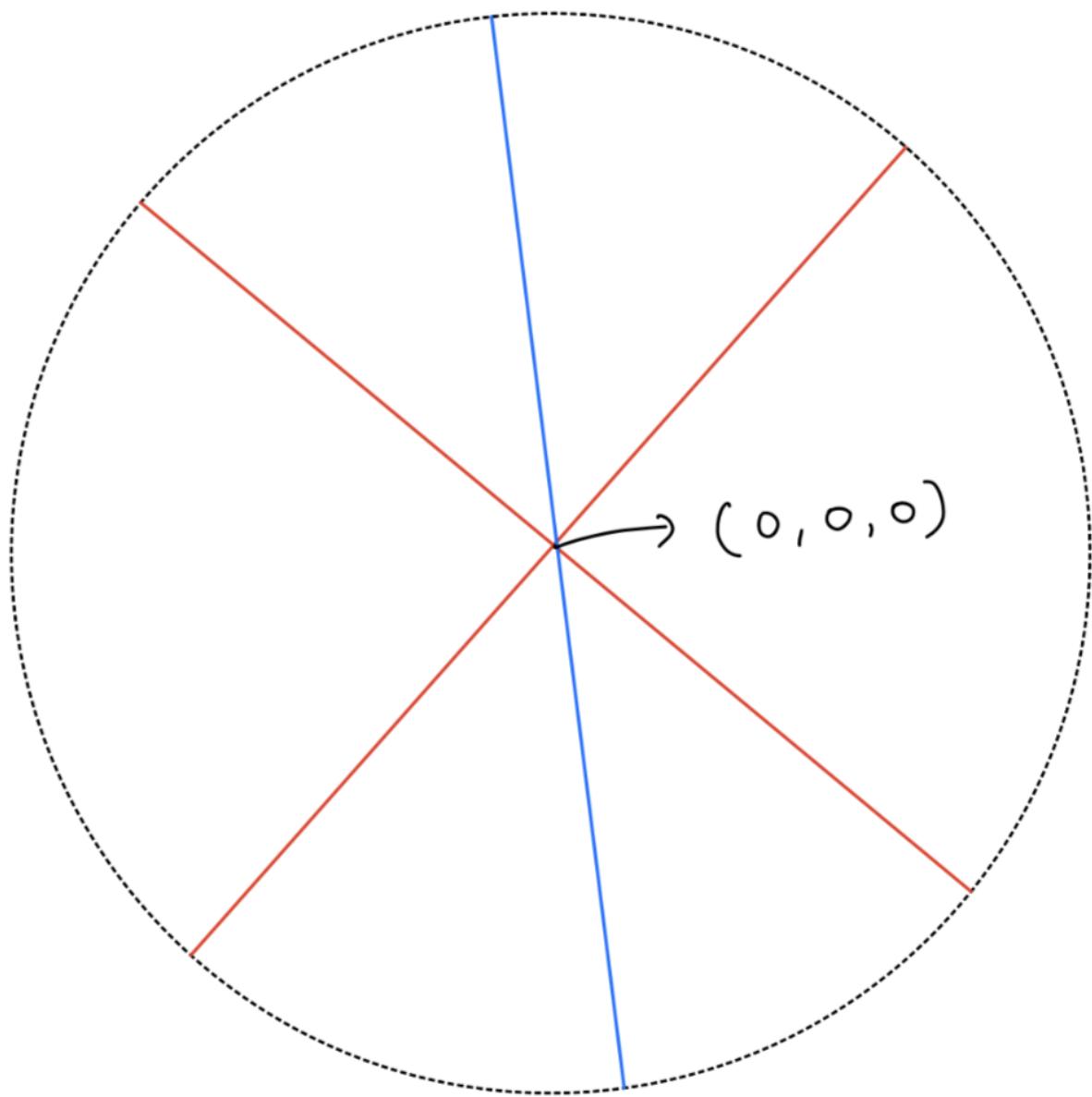


Figure 3.207

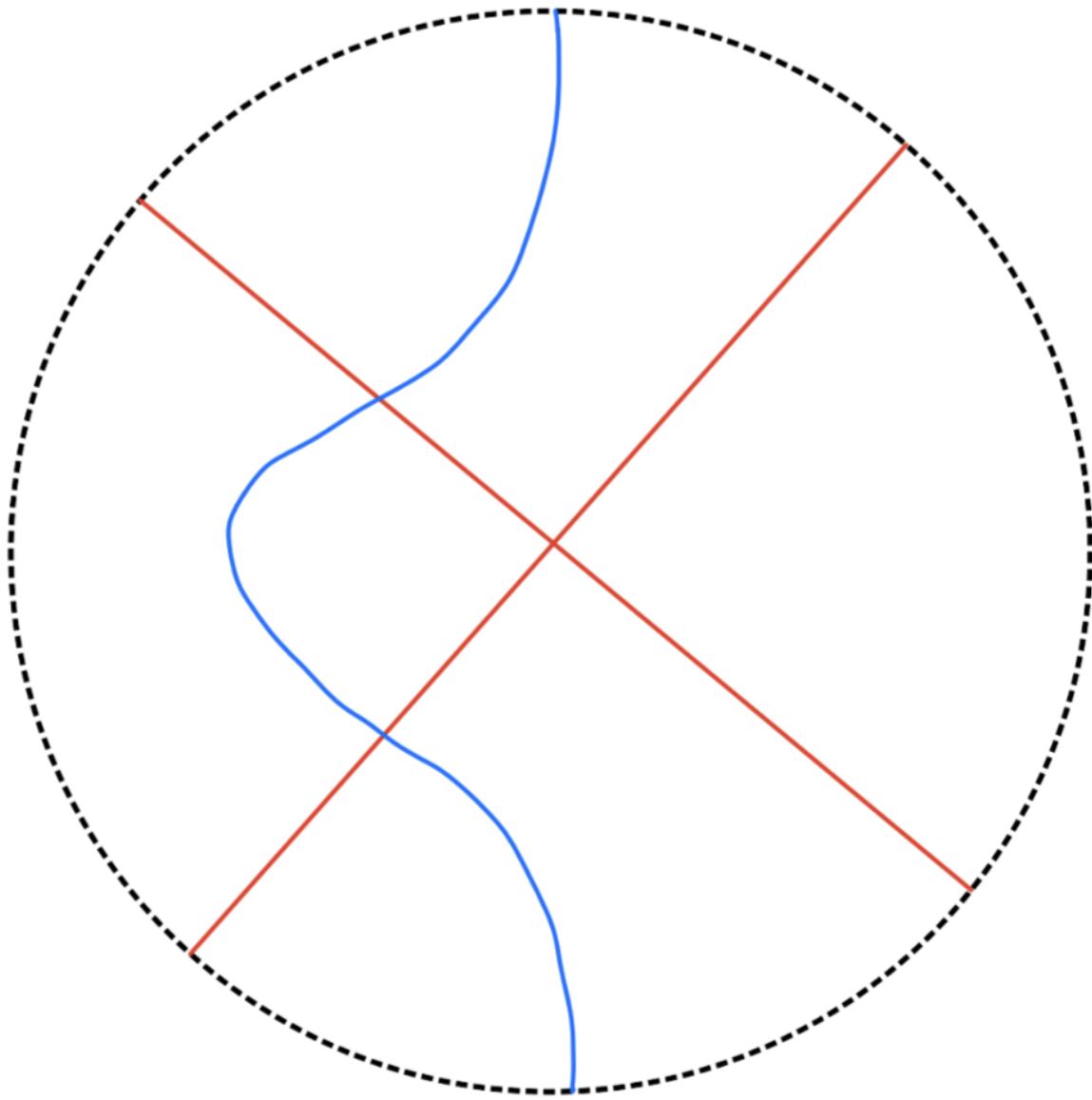


Figure 3.208

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

Definition 181. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $star(s)$.

Definition 182. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s .

Definition 183. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 184. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 185. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 186. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because F_S is legible.

Definition 187. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.8.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.8.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

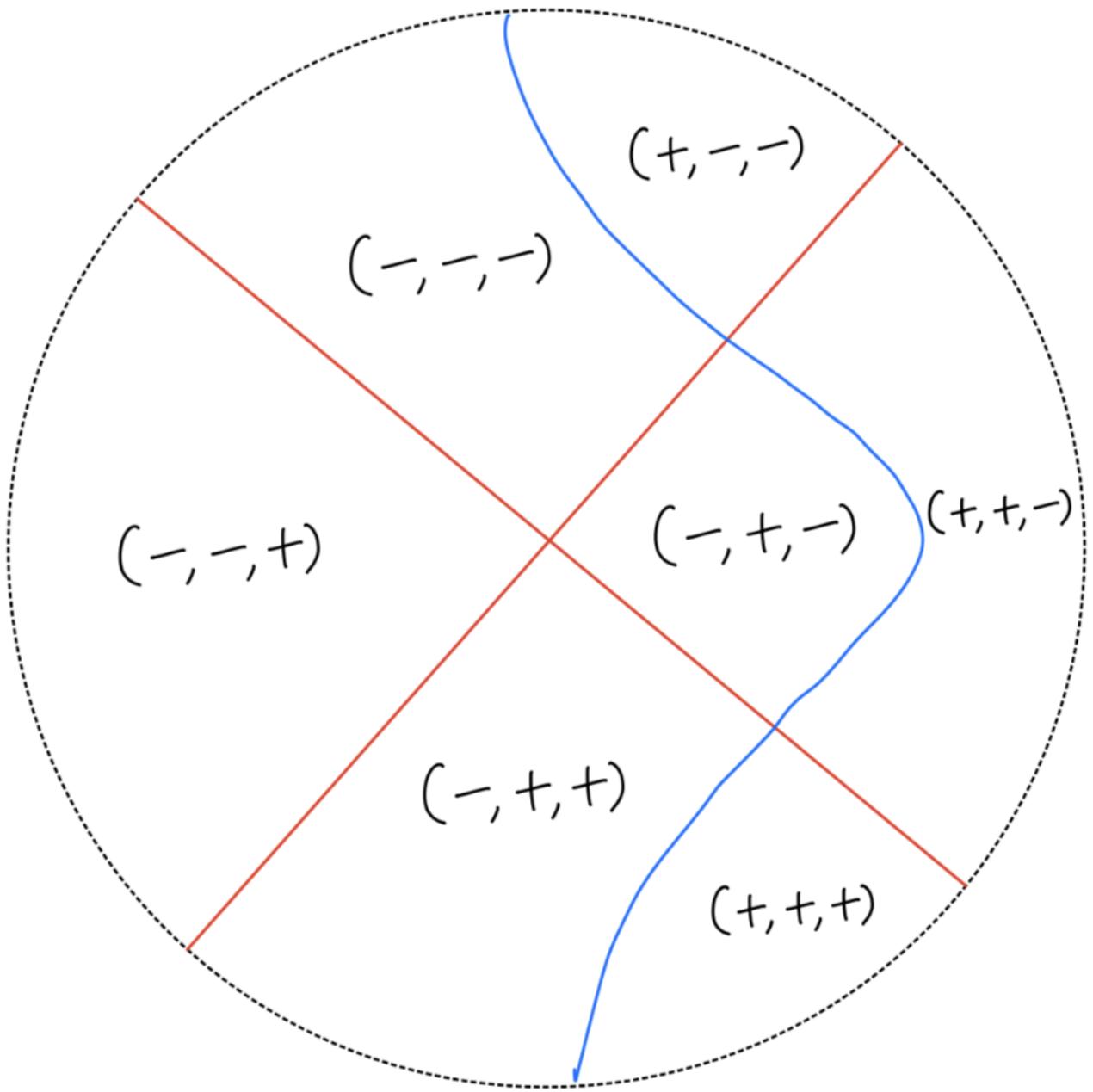


Figure 3.209

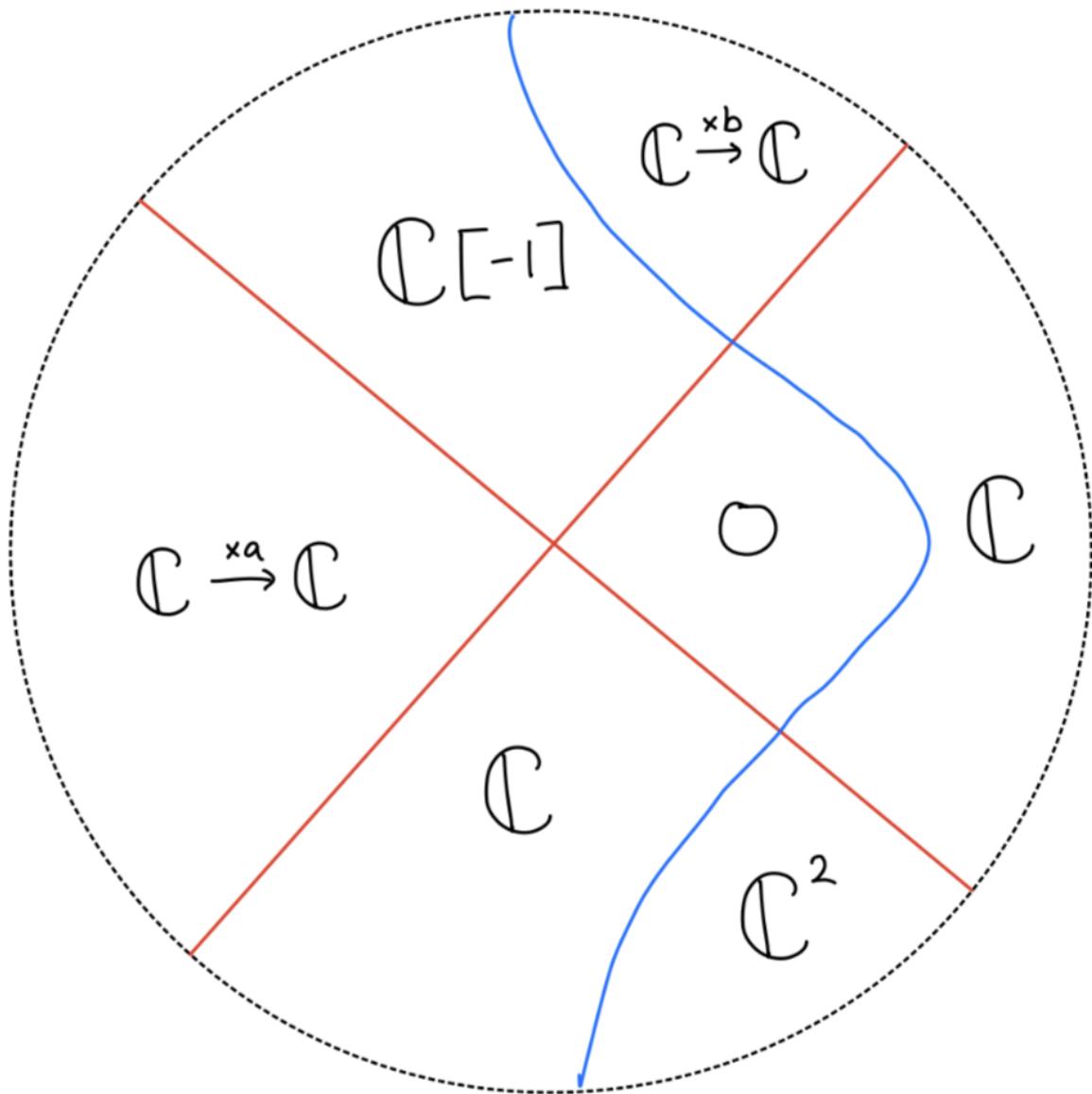


Figure 3.210

- $F_0(-, -, -) := \mathbb{C}[-1]$
- $F_0(+, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$
- $F_0(-, -, +) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_0(-, +, -) := 0$
- $F_0(+, +, -) := \mathbb{C}$

- $F_0(-, +, +) := \mathbb{C}$

- $F_0(+, +, +) := \mathbb{C}^2$

Generalization maps:

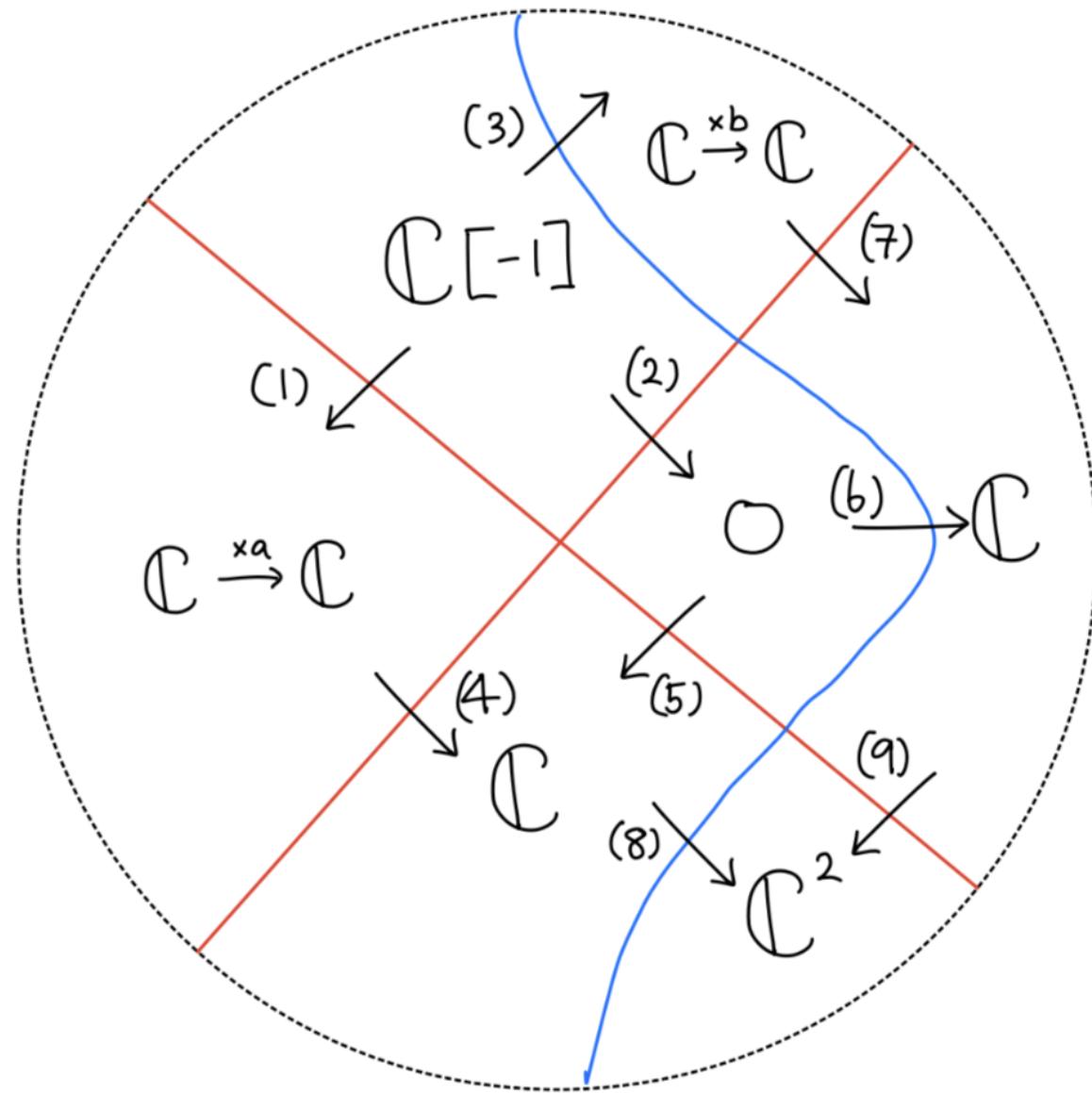


Figure 3.211

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & \uparrow \times a & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

3.8.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say $cobord_2$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphism, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

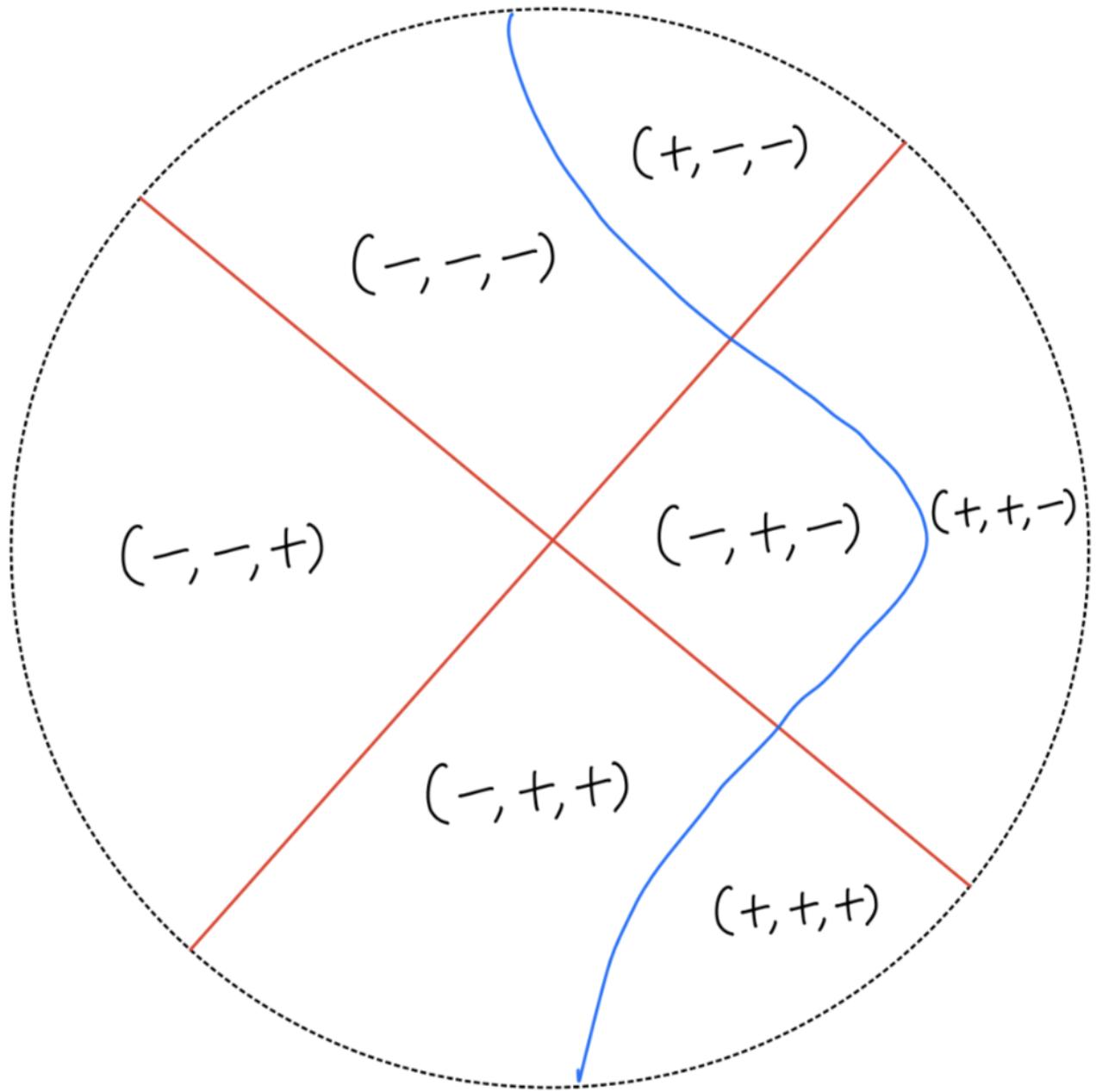


Figure 3.212

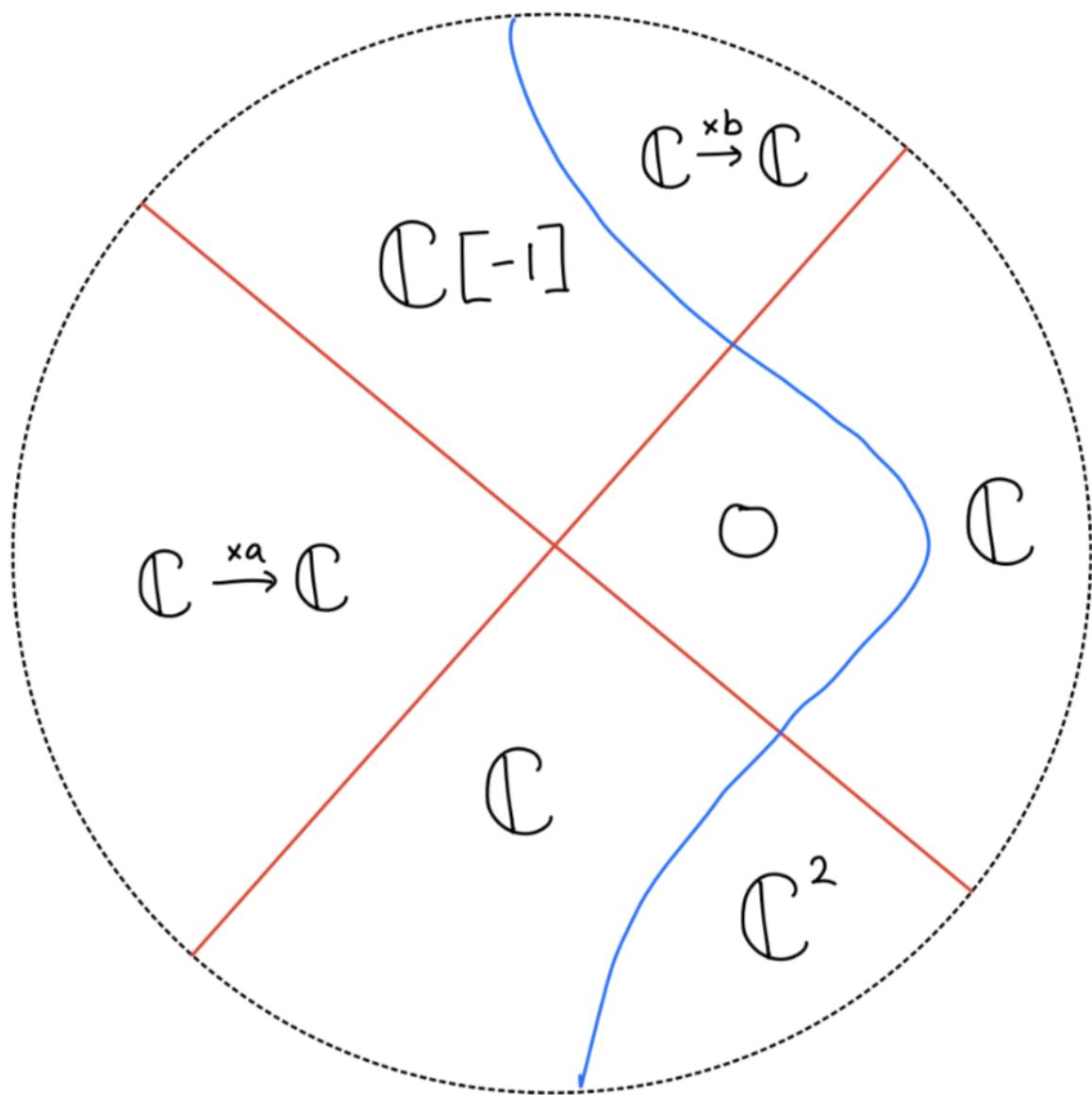


Figure 3.213

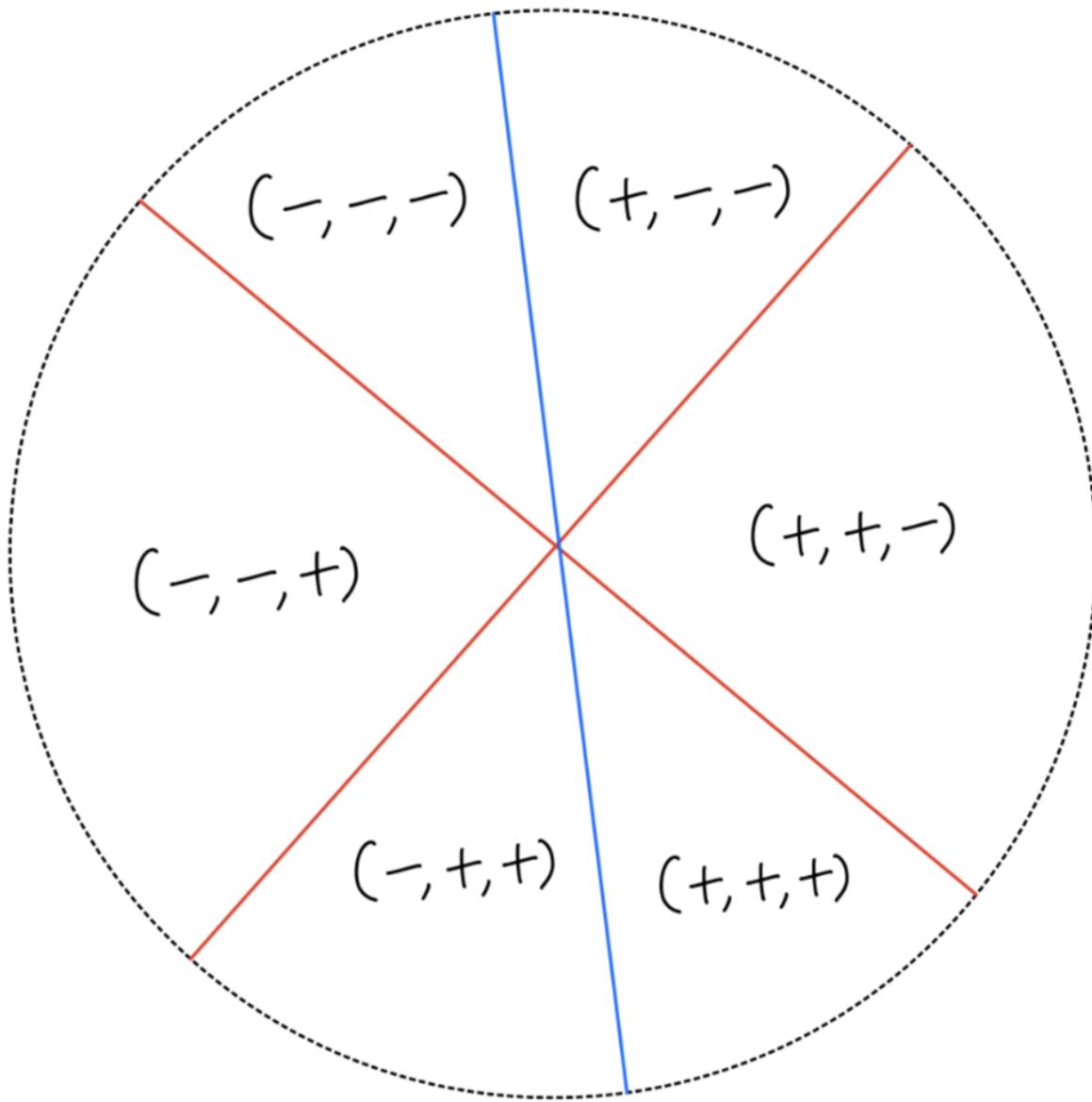


Figure 3.214

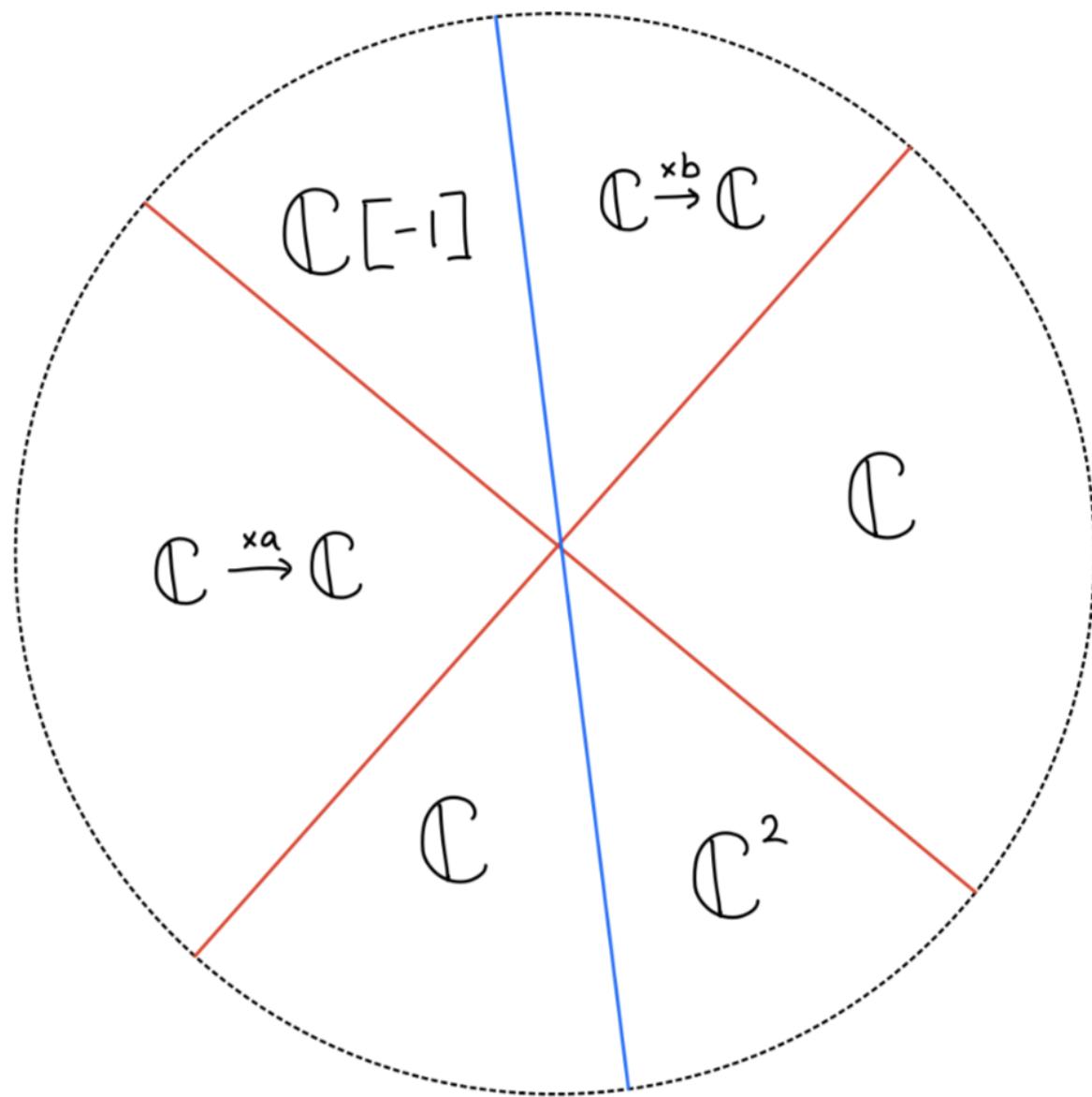


Figure 3.215

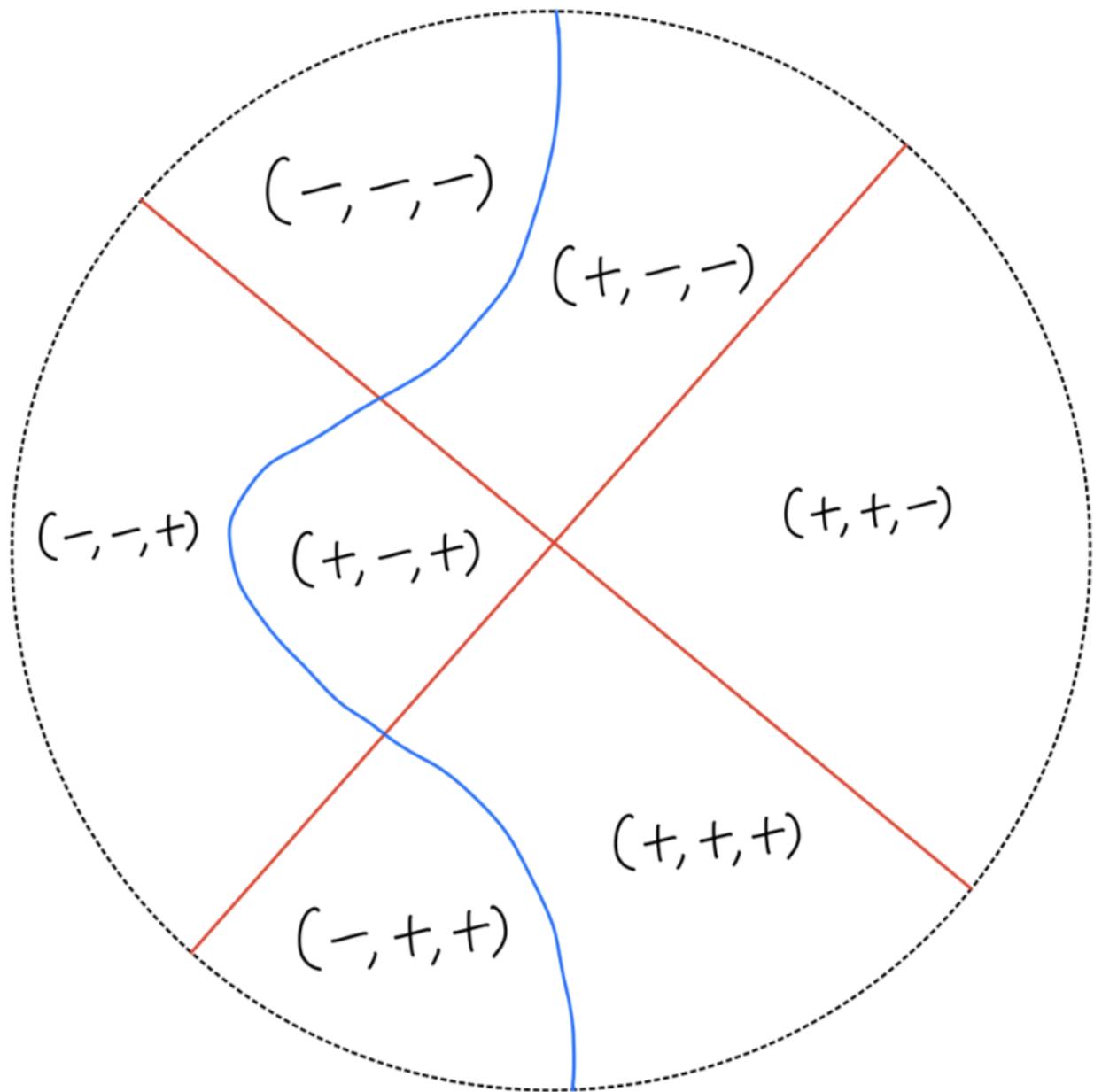


Figure 3.216

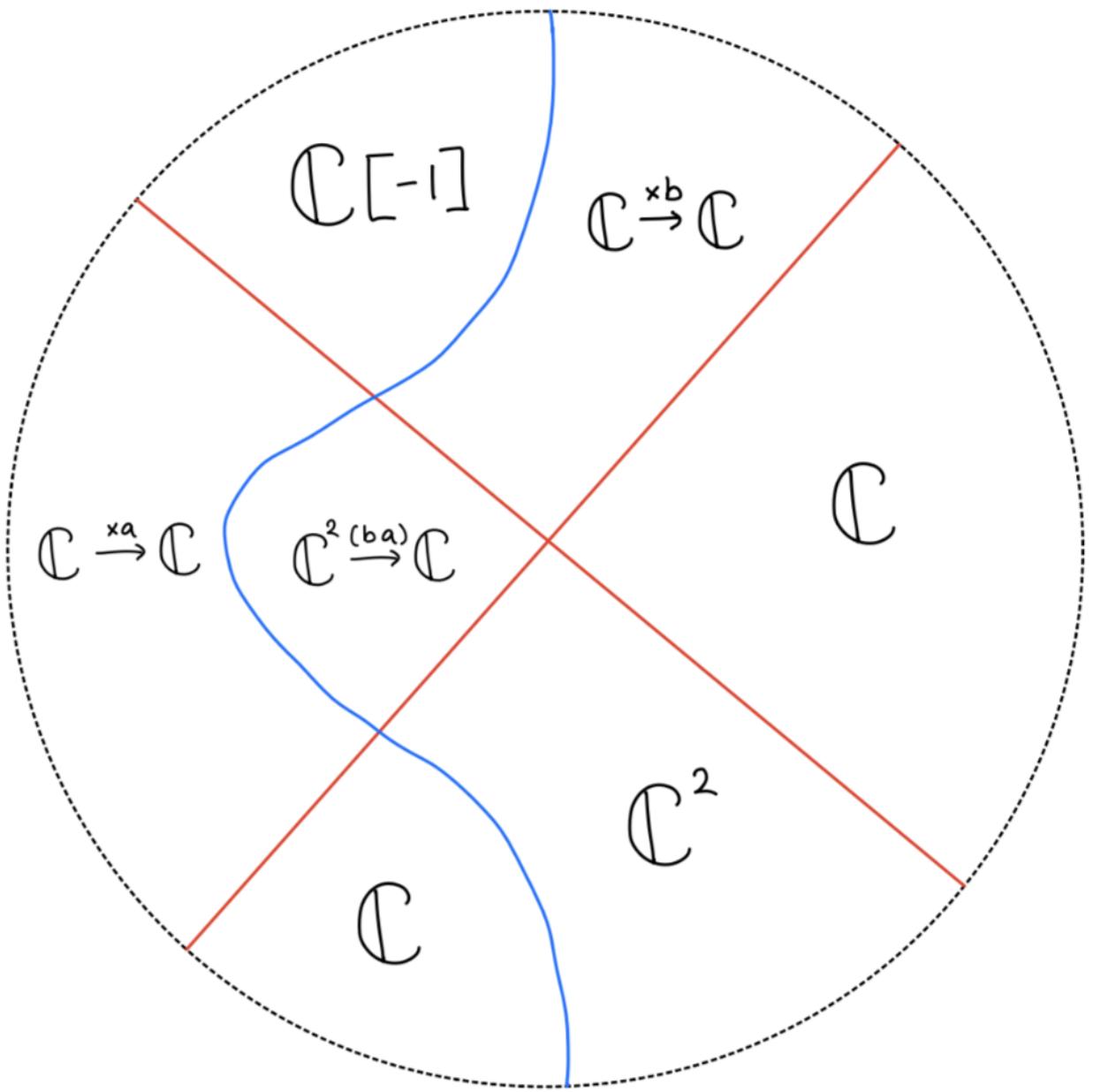


Figure 3.217

- $F_{\bullet}(-, -, -) := \mathbb{C}[-1]$
- $F_{\bullet}(-, -, +) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_{\bullet}(-, +, -) := 0$
- $F_{\bullet}(-, +, +) := \mathbb{C}$

- $F_\bullet(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_\bullet(+, -, +) := \mathbb{C}^2 \xrightarrow{\times(b\ a)} \mathbb{C}$
- $F_\bullet(+, +, -) := \mathbb{C}$
- $F_\bullet(+, +, +) := \mathbb{C}^2$

Generalization maps:

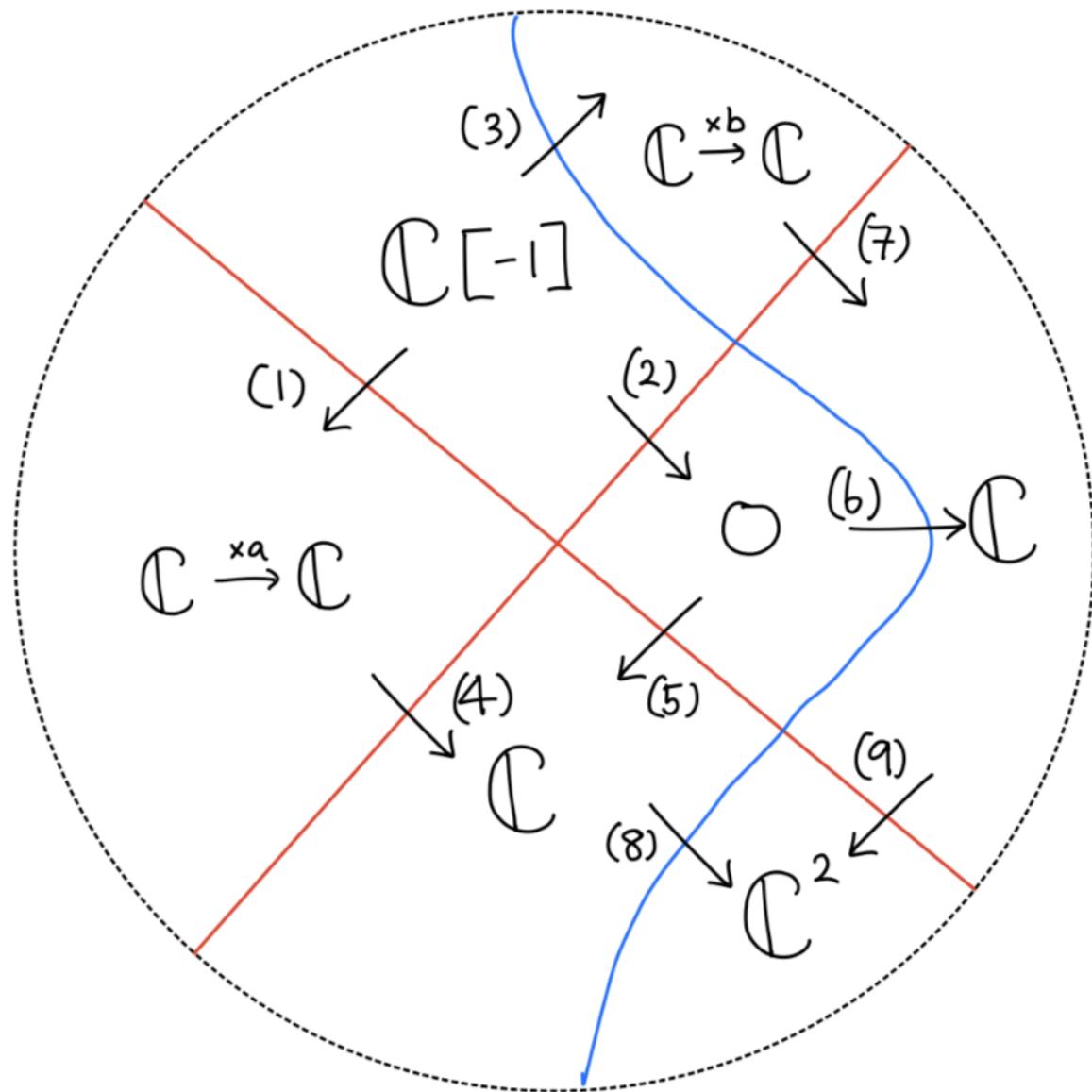


Figure 3.218

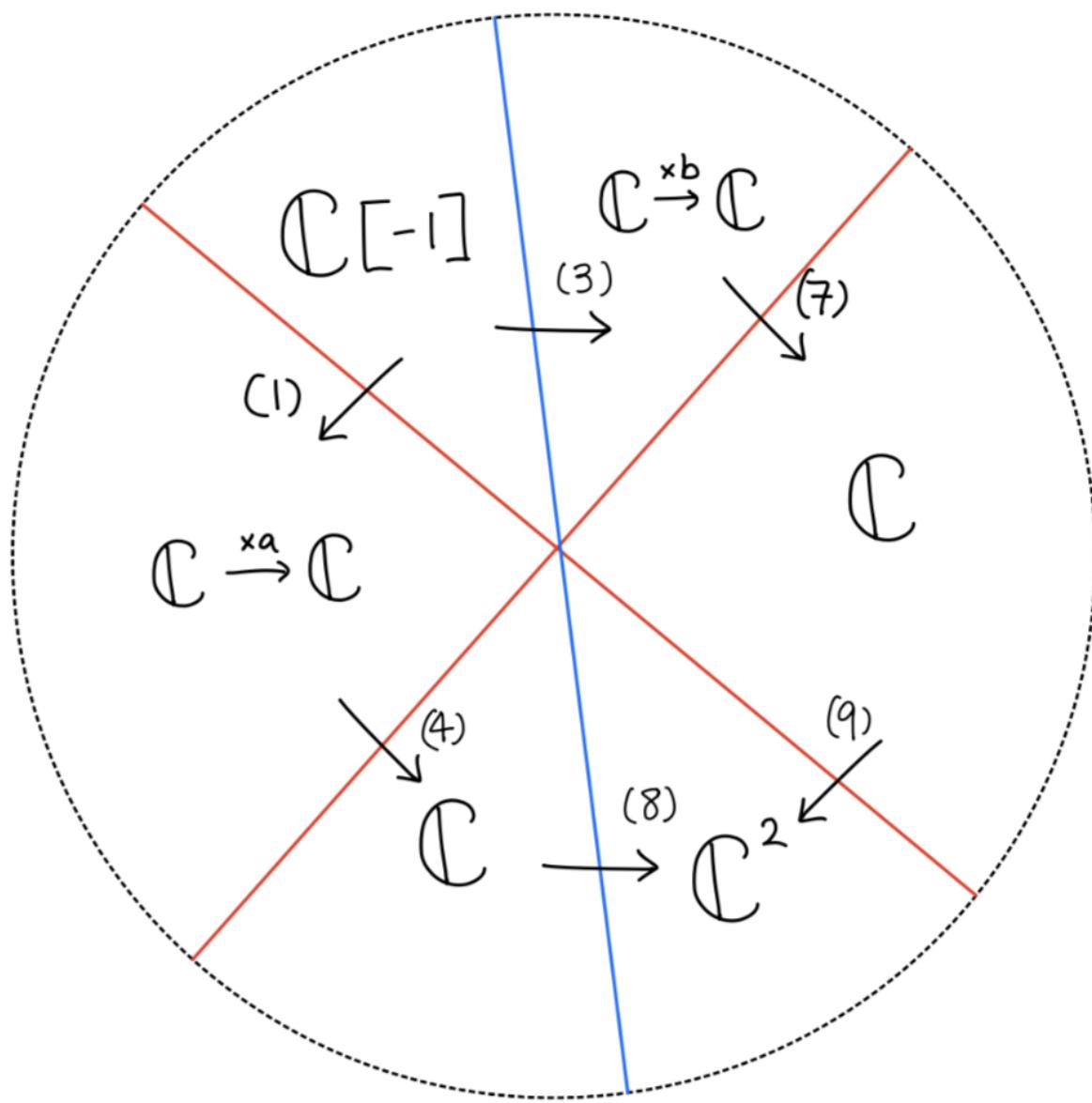


Figure 3.219

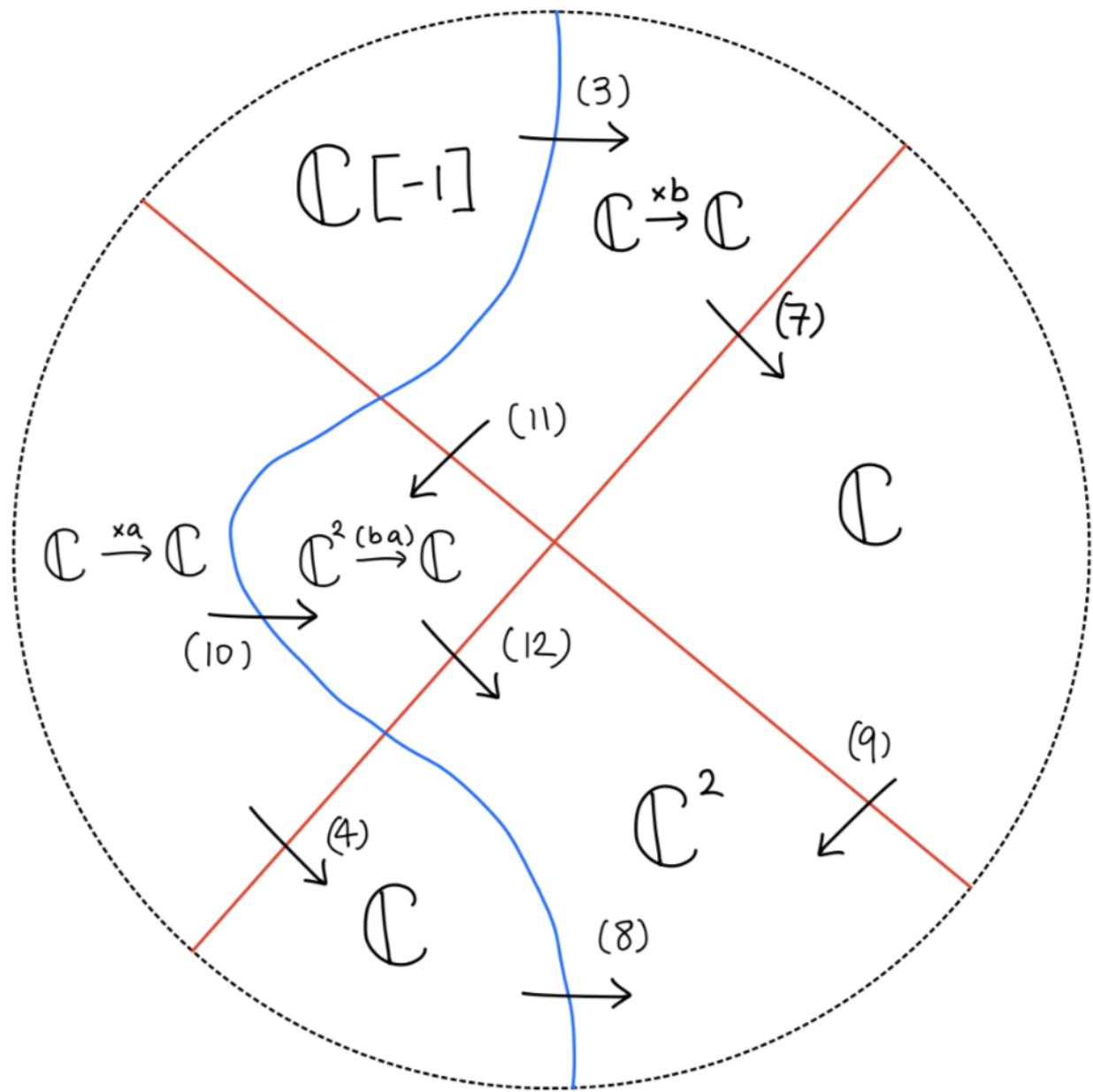


Figure 3.220

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & \uparrow \times a & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 188. $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 189. we define γ_\bullet to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \hookrightarrow & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \longrightarrow & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 190. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of M .

Note that there is a diffeomorphism between $D_{r=2} \times (0,1)$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 .

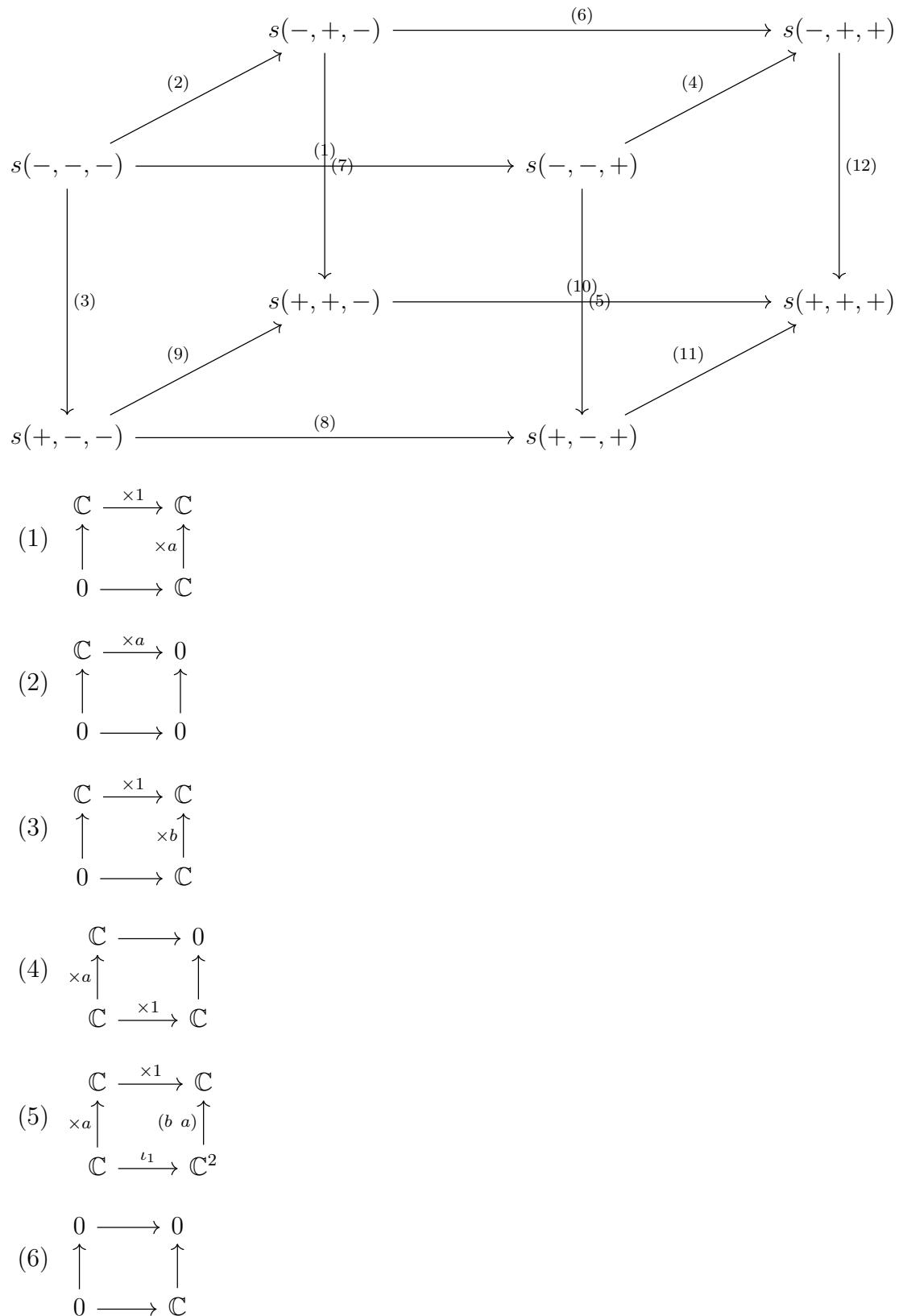
To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F^3(-, +, -) := 0$
- $F^3(-, +, +) := \mathbb{C}$
- $F^3(+, +, -) := \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}^2$

Generalization maps:



$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \xrightarrow{(\times 0, \times 1)} & \mathbb{C} \\ \downarrow (\times 1, \iota_1) & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{(0, id)} & \mathbb{C}^2 \end{array}$$

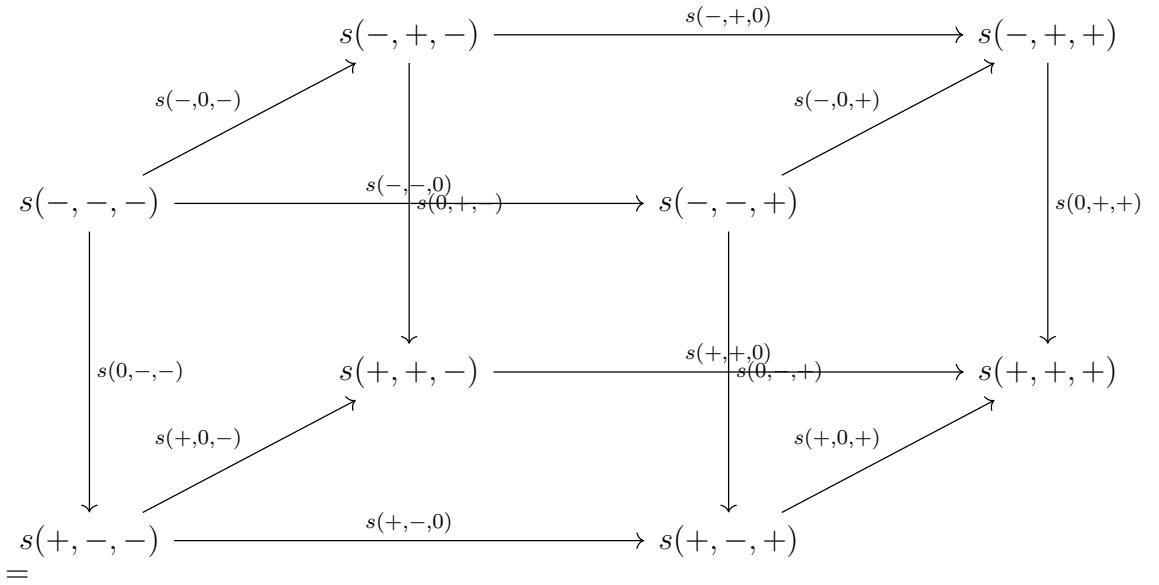
$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \xrightarrow{(\times 1, \times 0)} & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\ \downarrow (\times 1, \times 0) & & \downarrow (\times 1, \iota_1) \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \xrightarrow{(\times 1, \iota_0)} & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \end{array}$$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} =
 \begin{array}{ccc}
 0 & \xrightarrow{\iota_0} & \mathbb{C} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}[-1] & \xrightarrow{(\times 1, \times 0)} & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\
 \downarrow & & \downarrow (\times 0, \times 1) \\
 0 & \longrightarrow & \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \xrightarrow{(\times 1, \iota_0)} & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \\
 \downarrow (\times 0, \times 1) & & \downarrow (\times 0, id) \\
 \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2
 \end{array}$$

(vii) the cubic diagram:



$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \mathbb{C} \\
 & \nearrow 0 & \downarrow & & \searrow (\times 0, \times 1) & \downarrow & \\
 \mathbb{C}[-1] & \xrightarrow{(\times 1, \times 0)} & \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \nearrow (\times 1, \times 0) & \downarrow & & \searrow \iota_0(\times 1, \iota_1) & \downarrow & \mathbb{C}^2 \\
 & & \mathbb{C} & & & \downarrow & \\
 & \nearrow (\times 0, \times 1) & \downarrow & & \searrow (\times 0, id) & \downarrow & \\
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} & \xrightarrow{(\times 1, \iota_0)} & \mathbb{C}^2 & \xrightarrow{(b \ a)} & \mathbb{C}^2
 \end{array}$$

whose total complex is

$$\text{Tot}(0 \rightarrow \mathbb{C} \xrightarrow{(\times 1, \times 0)} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \xrightarrow{(\times 0, \times (-1))} \mathbb{C}^2 \xrightarrow{(0 \ b \ a)} \mathbb{C}^3 \xrightarrow{(\times 1, M_1)} \mathbb{C} \xrightarrow{(\times 1, M_3)} \mathbb{C} \xrightarrow{(\times 0, M_2)} \mathbb{C}^2)$$

where

$$M_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This is equal to

$$\text{Tot} \left(\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{M_1} & \mathbb{C}^2 & \xrightarrow{M_2} & \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & M_3 \uparrow & & M_4 \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C}^2 & \xrightarrow{M_5} & \mathbb{C} & \xrightarrow{M_6} & \mathbb{C}^2 \end{array} \right)$$

$$M_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & b & a & 0 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which is the following cochain complex

$$0 \rightarrow \mathbb{C}^3 \xrightarrow{M_1} \mathbb{C}^6 \xrightarrow{M_2} \mathbb{C}^3$$

$$M_1 = \begin{pmatrix} 1 & -b & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & b & a & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The final cochain complex is acyclic because of the following reasons

- $M_2 \circ M_1 = 0$
- M_1 is a monomorphism
- M_2 is an epimorphism

□

3.8.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the $cobord_2$. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

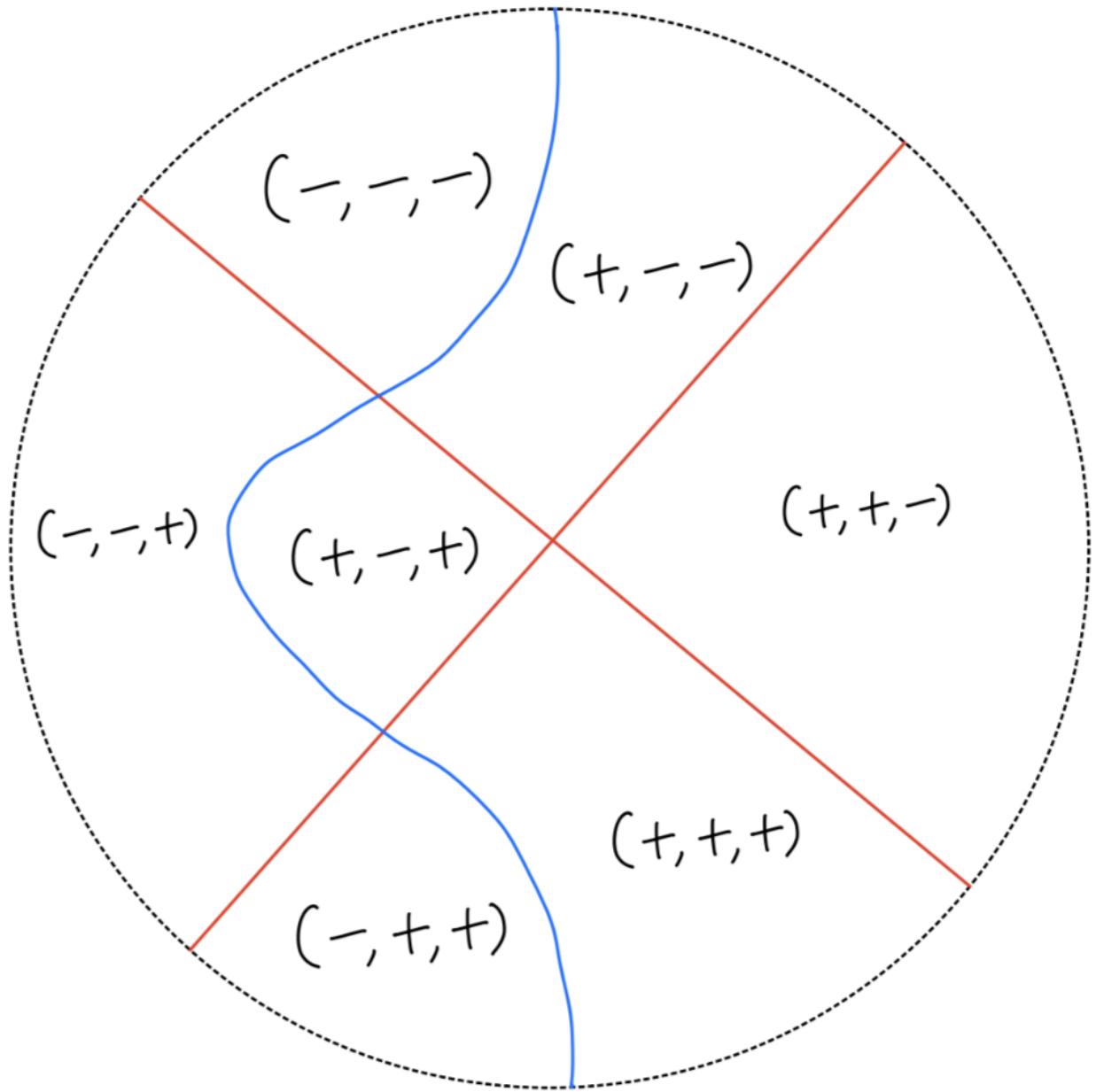


Figure 3.221

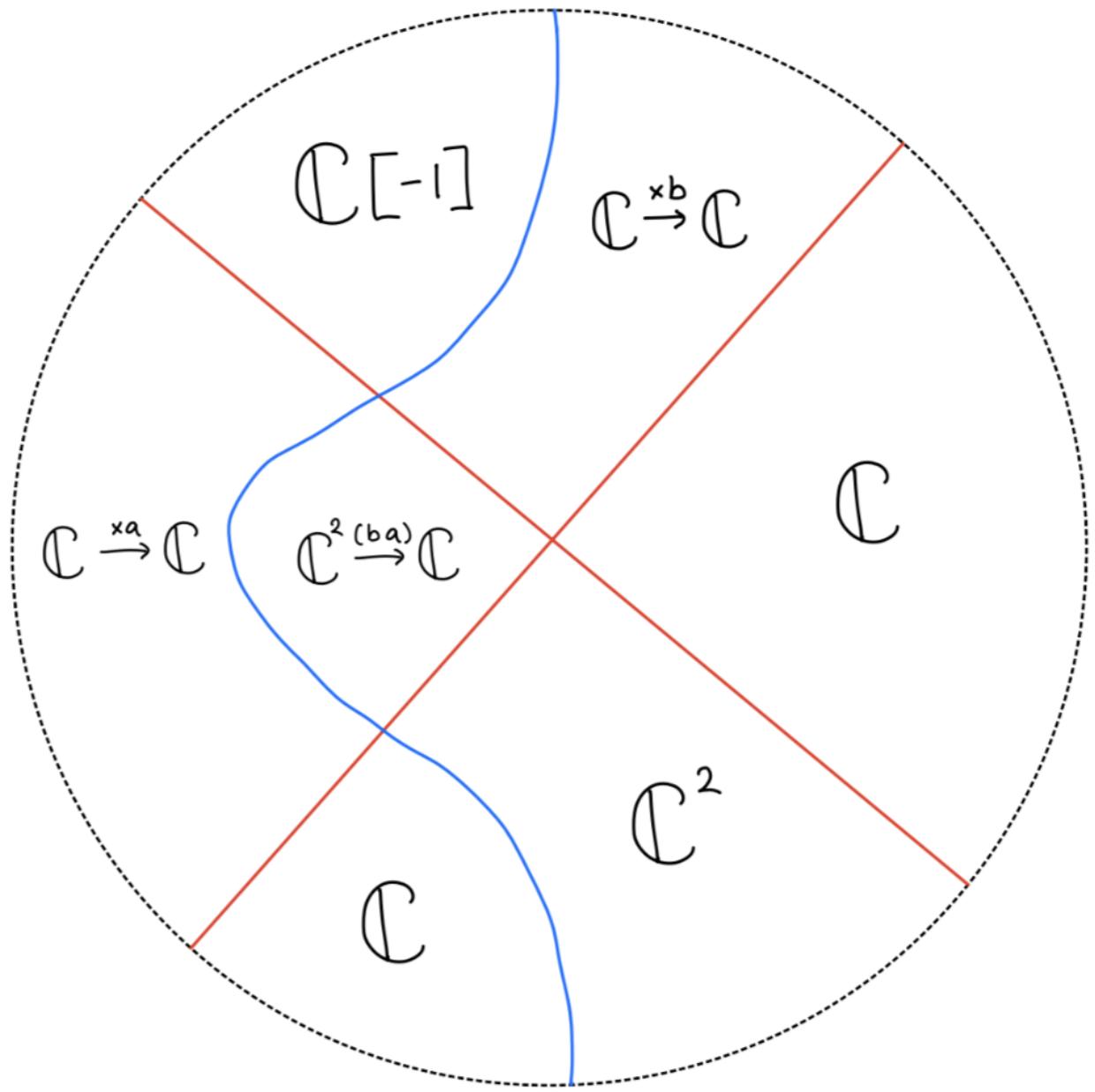


Figure 3.222

- $F_1(-, -, -) := \mathbb{C}[-1]$
- $F_1(+, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$
- $F_1(-, -, +) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_1(+, -, +) := \mathbb{C}^2 \xrightarrow{(b,a)} \mathbb{C}$

- $F_1(+, +, -) := \mathbb{C}$

- $F_1(-, +, +) := \mathbb{C}$

- $F_1(+, +, +) := \mathbb{C}^2$

Generalization maps:

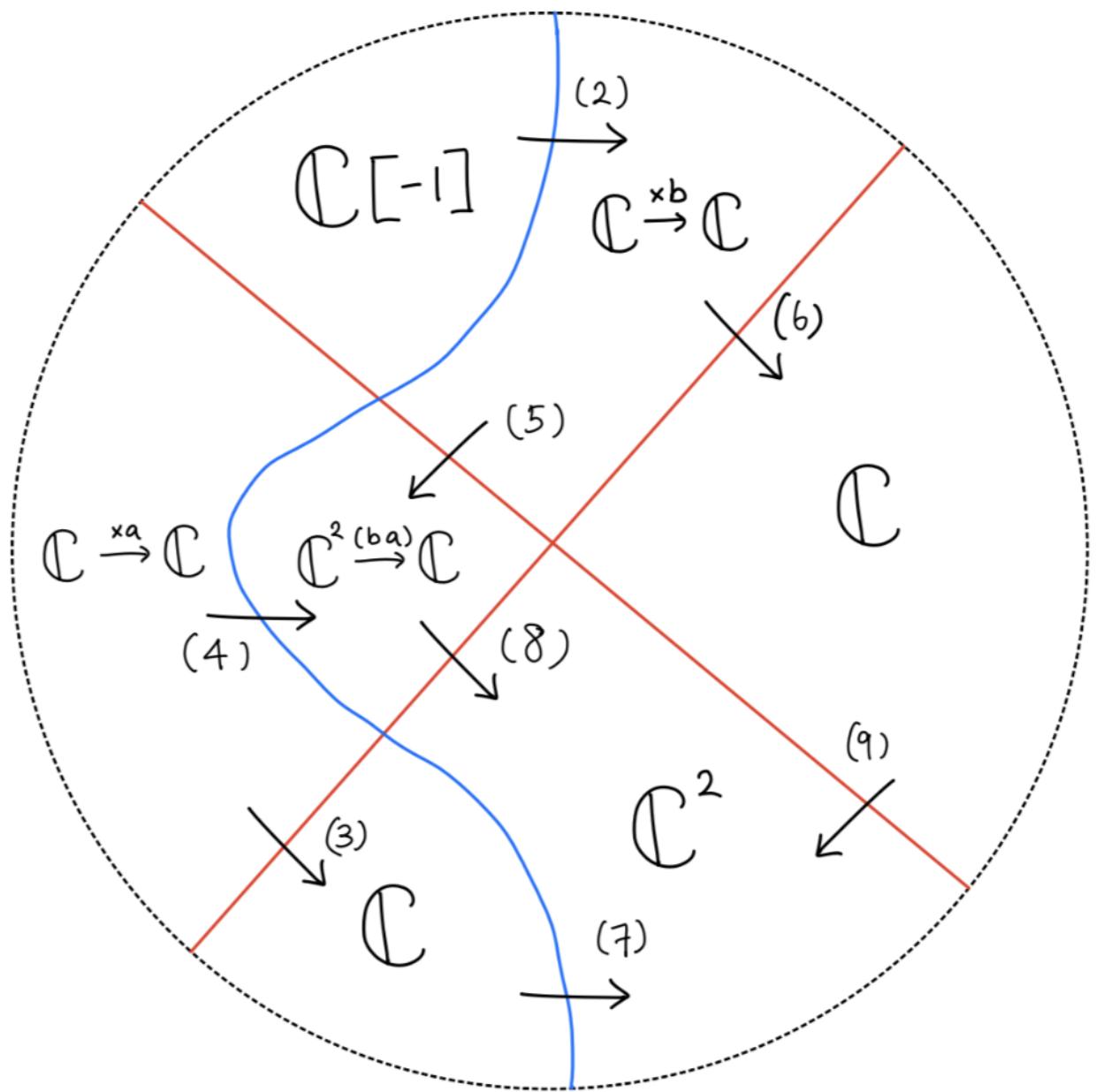


Figure 3.223

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times a & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times a & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times b & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times b & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 191. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

3.9 4th Sheaf Cobordism'

In this section, we define $cobord'_4$, a compactly supported sheaf cobordism.

3.9.1 Notations

Definition 192. M denotes a Riemann sphere with two punctures at 0 and ∞ . Topologically, M is homeomorphic to a cylinder.

Definition 193. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M , so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0 . Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_4$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_4$.

Definition 194. For $symbol \in \{0, \infty, squig\}$

1. we denote Φ_\bullet^{symbol} to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript ' t ' to distinguish the time coordinates from the space coordinates.

2. we denote Ξ_\bullet^{symbol} a co-orientation of Φ_\bullet^{symbol} .

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as Λ_\bullet^{symbol} . Later in the section, Λ_\bullet^{symbol} will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying *cobord*₄.
4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as Λ_\bullet and call it a squiggly isotopy from Λ_0 to Λ_1 . Later in the section, Λ_\bullet will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying *cobord*₄.

Definition 195. For $t \in [0, 1]$, we define $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} e^{(\frac{x^2}{x^2-1})}(\frac{1}{2} - t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$ if $t \neq \frac{1}{2}$
- $\{(1, 0), (-1, 0), (0, \frac{1}{2} - t)\} \subset Graph(\Psi_t)$

Definition 196. We denote the standard open disk in \mathbb{R}^2 of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0, 1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 197. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_0(z)\}$
 - λ_0^∞ is the union of the following two components
 - (i) $\{(x, z) \in D_{r=2} \mid z = x\}$
 - (ii) $\{(x, z) \in D_{r=2} \mid z = -x\}$
2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows
- ξ_0^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.
 - ξ_0^∞ : hairs are pointing towards the left i.e. coefficients of dx are positive.
- Definition 198.** 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$
- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_1(z)\}$
 - λ_1^∞ is the union of the following two components
 - (i) $\{(x, z) \in D_{r=2} \mid z = x\}$
 - (ii) $\{(x, z) \in D_{r=2} \mid z = -x\}$
2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows
- ξ_1^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.
 - ξ_1^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.
- Definition 199.** 1. We define the following subsets of $D_{r=2} \times [0, 1]$
- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = \Psi_t(z)\}$
 - λ_\bullet^∞ is the union of the following two components
 - (i) $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = x\}$
 - (ii) $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -x\}$

2. We define co-orientations ξ_\bullet^{symbol} of λ_\bullet^{symbol} as follows
- ξ_\bullet^0 : hairs are pointing towards the left i.e. coefficients of dx are negative.

- ξ_\bullet^∞ : hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 200. 1. Consider a stratification \mathcal{S}_0 on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0 .

2. Consider a stratification \mathcal{S}_1 on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1 .

Consider a stratification \mathcal{S}_\bullet on $D_{r=2} \times [0, 1]$ induced by λ_\bullet i.e. strata are non-empty finite intersections of λ_\bullet^0 , λ_\bullet^∞ , and λ_\bullet^{squig} . Note that 2 dimensional strata has co-orientations inherited from λ_\bullet .

Now let's list the strata of \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_\bullet using the following notations:

Definition 201. $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$ is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 202. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(x - \Psi_{t_0}(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z) = sgn_3\} \end{aligned}$$

Definition 203. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(x - \Psi_t(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z) = sgn_3\} \end{aligned}$$

Definition 204. Now I will describe \mathcal{S}_0 , \mathcal{S}_1 , and \mathcal{S}_{\bullet} using the above notations:

1. \mathcal{S}_0 :

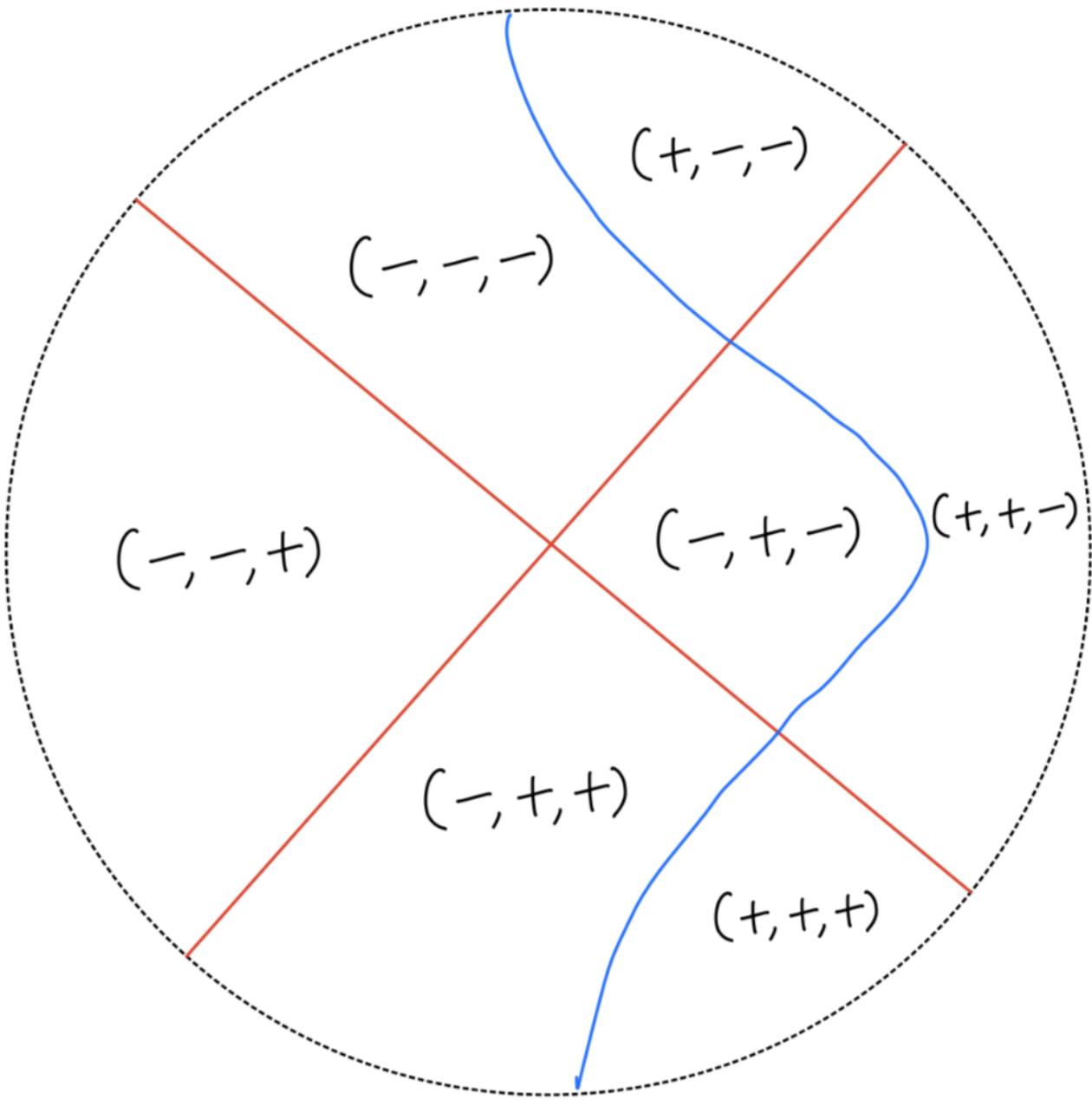


Figure 3.224

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_0(+, -, +)\}$$

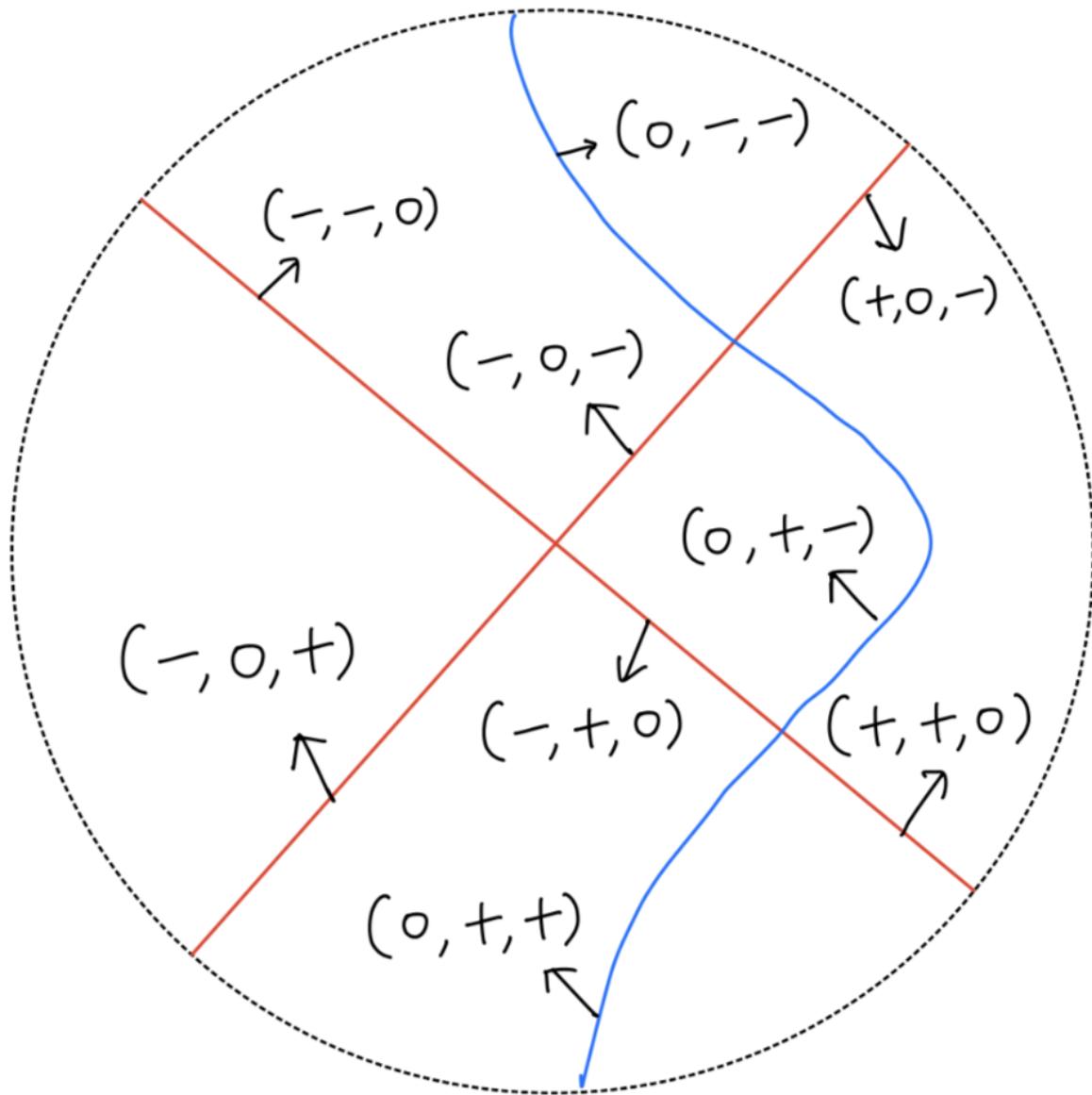


Figure 3.225

- 1 dimensional strata:

$\{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_0(0, -, +)\} \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_0(+, 0, +)\} \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, +, 0)\}$

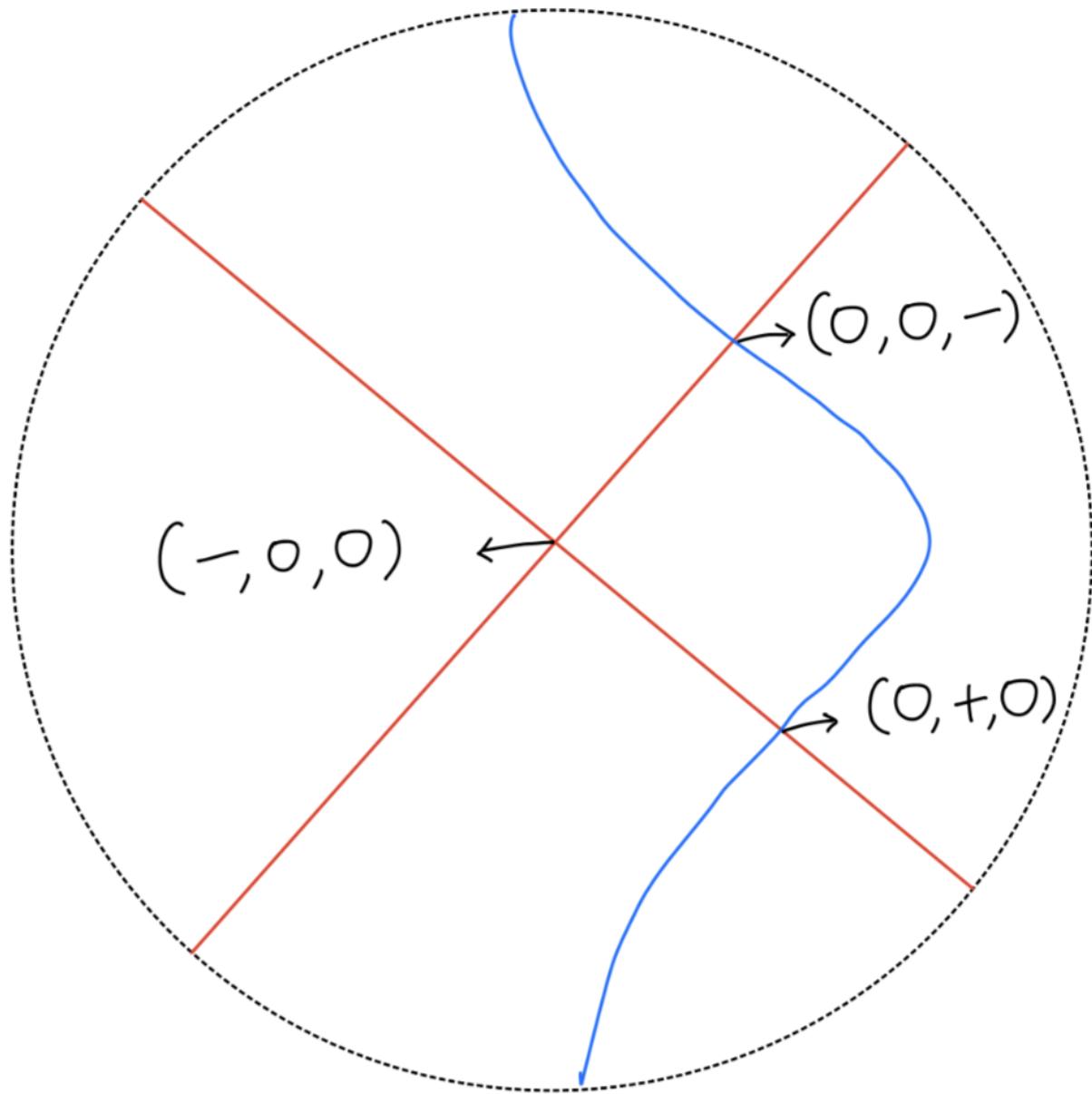


Figure 3.226

- 0 dimensional strata:

$$s_0(-, 0, 0), s_0(0, 0, -), s_0(0, +, 0)$$

2. \mathcal{S}_1 :

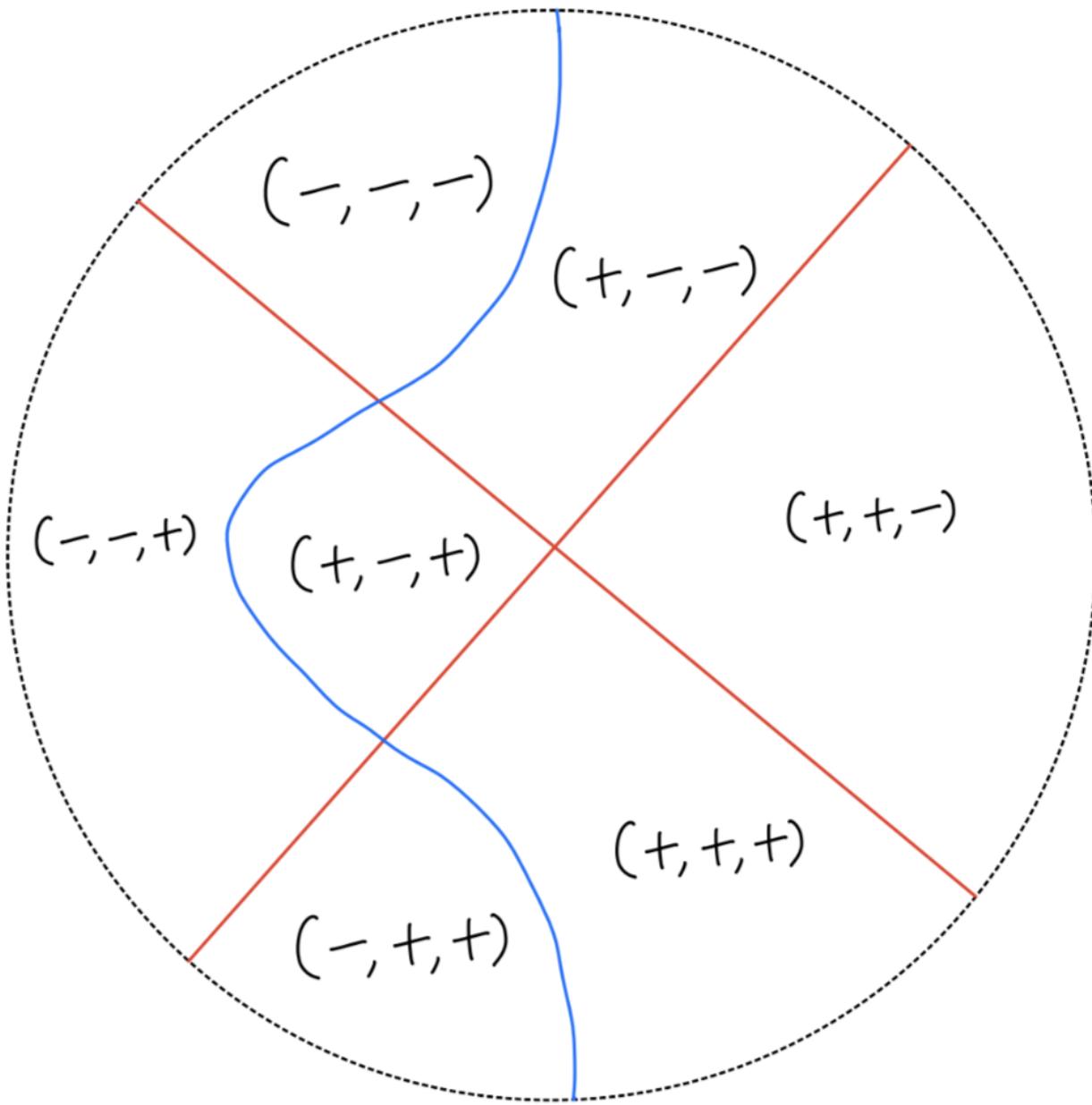


Figure 3.227

- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_1(-, +, -)\}$$

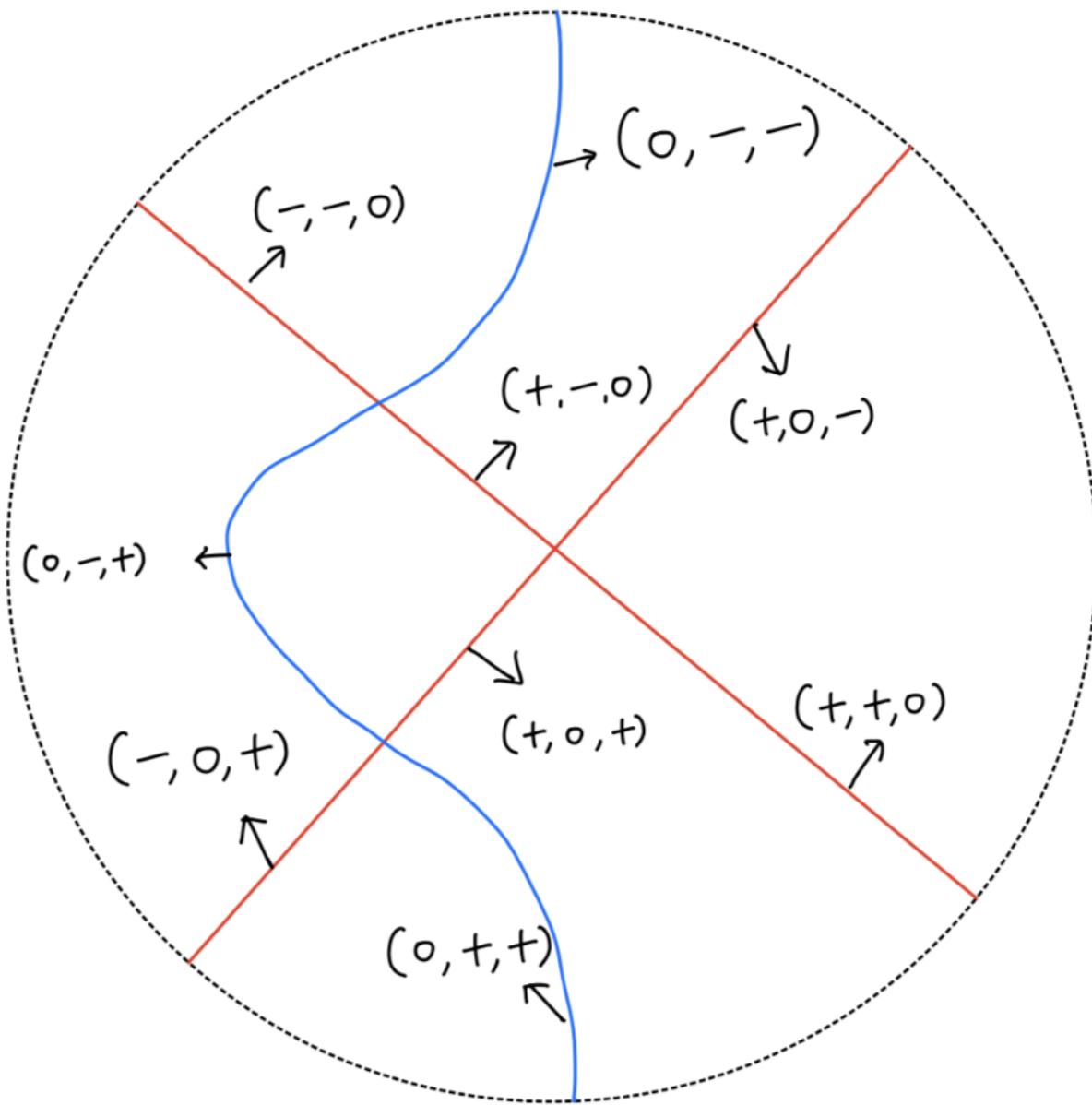


Figure 3.228

- 1 dimensional strata:

$\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_1(0, +, -)\} \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_1(-, 0, -)\} \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(-, -, 0)\}$

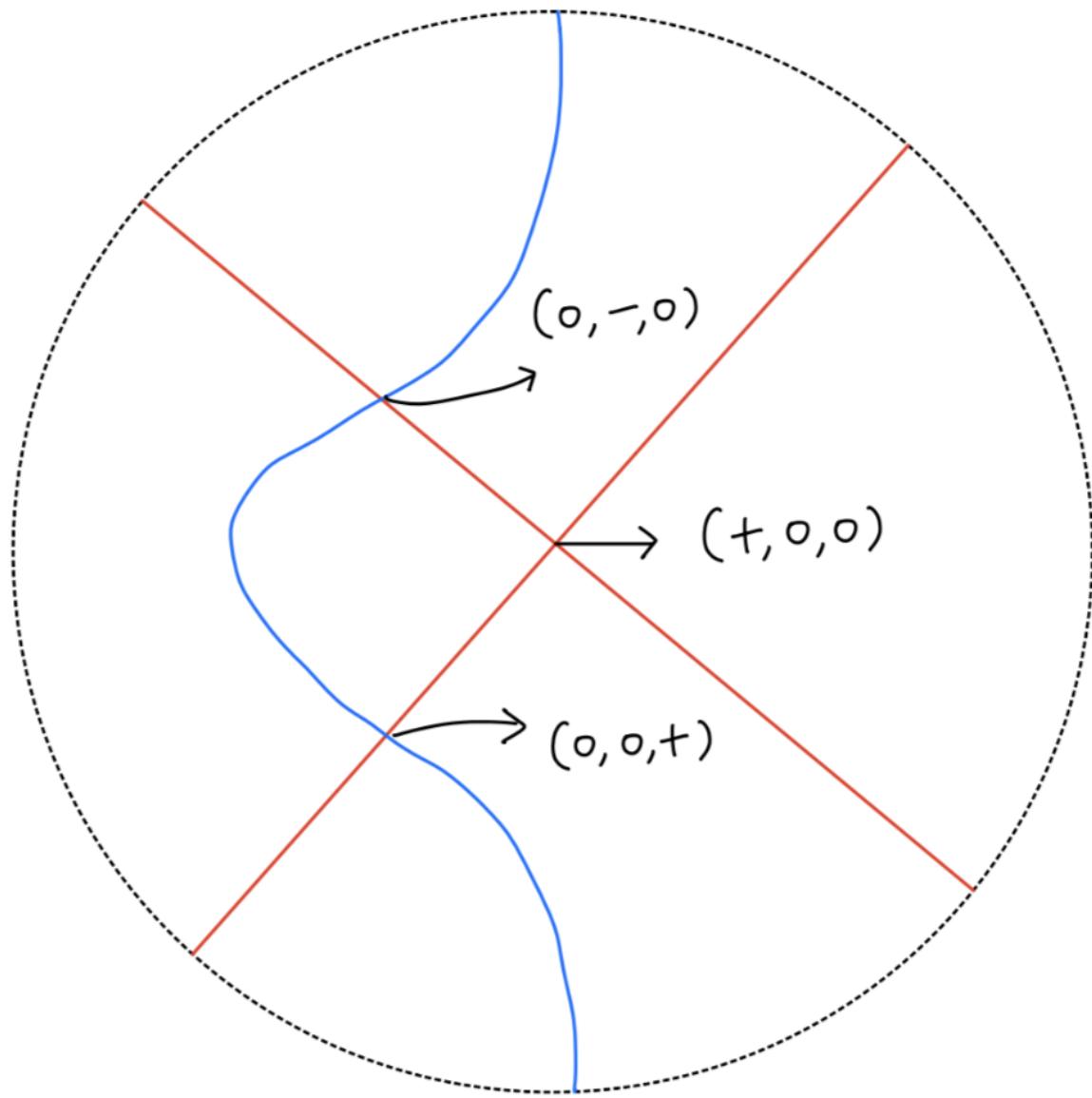


Figure 3.229

- 0 dimensional strata:

$$s_1(0, -, 0), s_1(+, 0, 0), s_1(0, 0, +)$$

3. \mathcal{S}_\bullet :

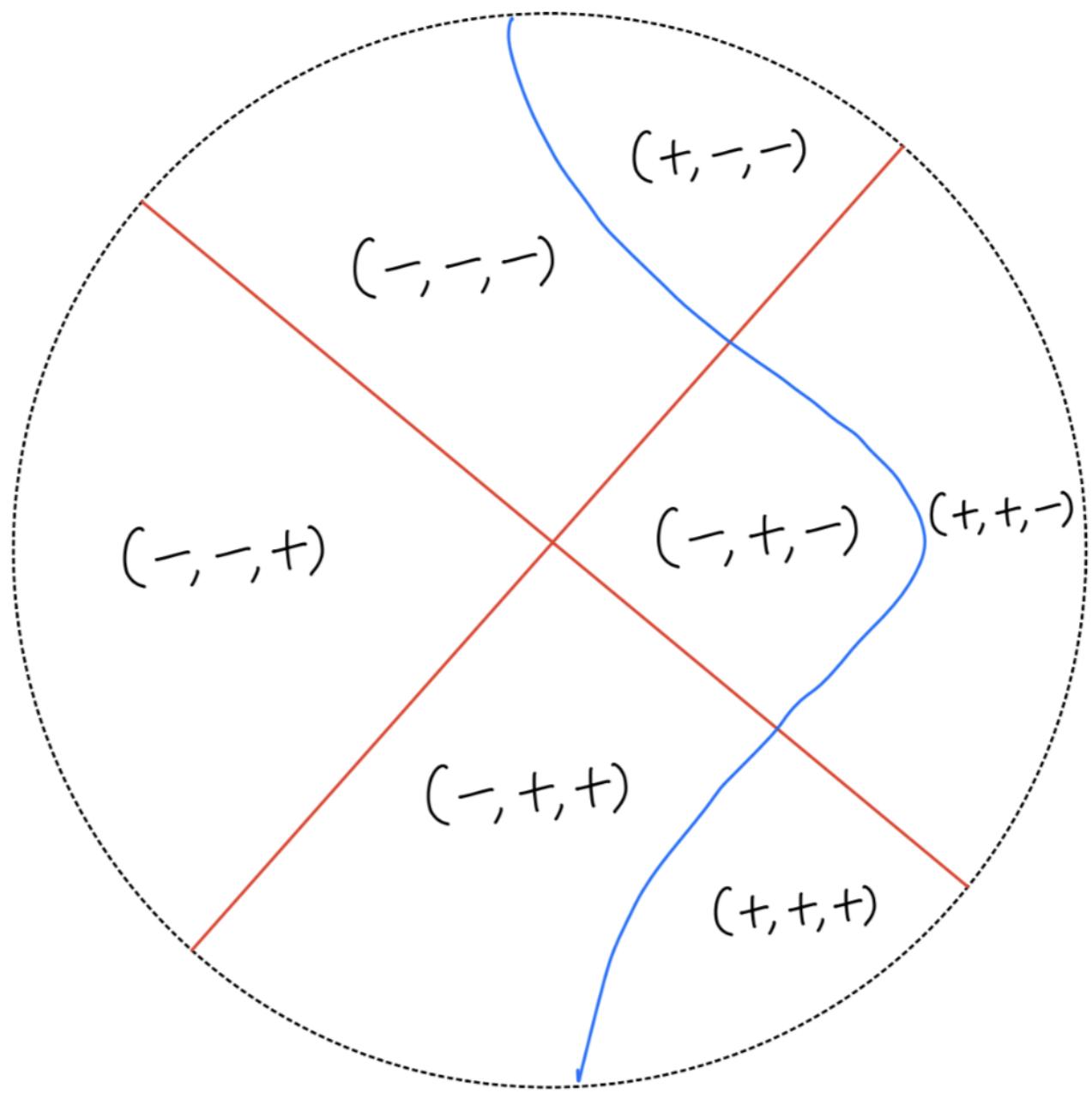


Figure 3.230

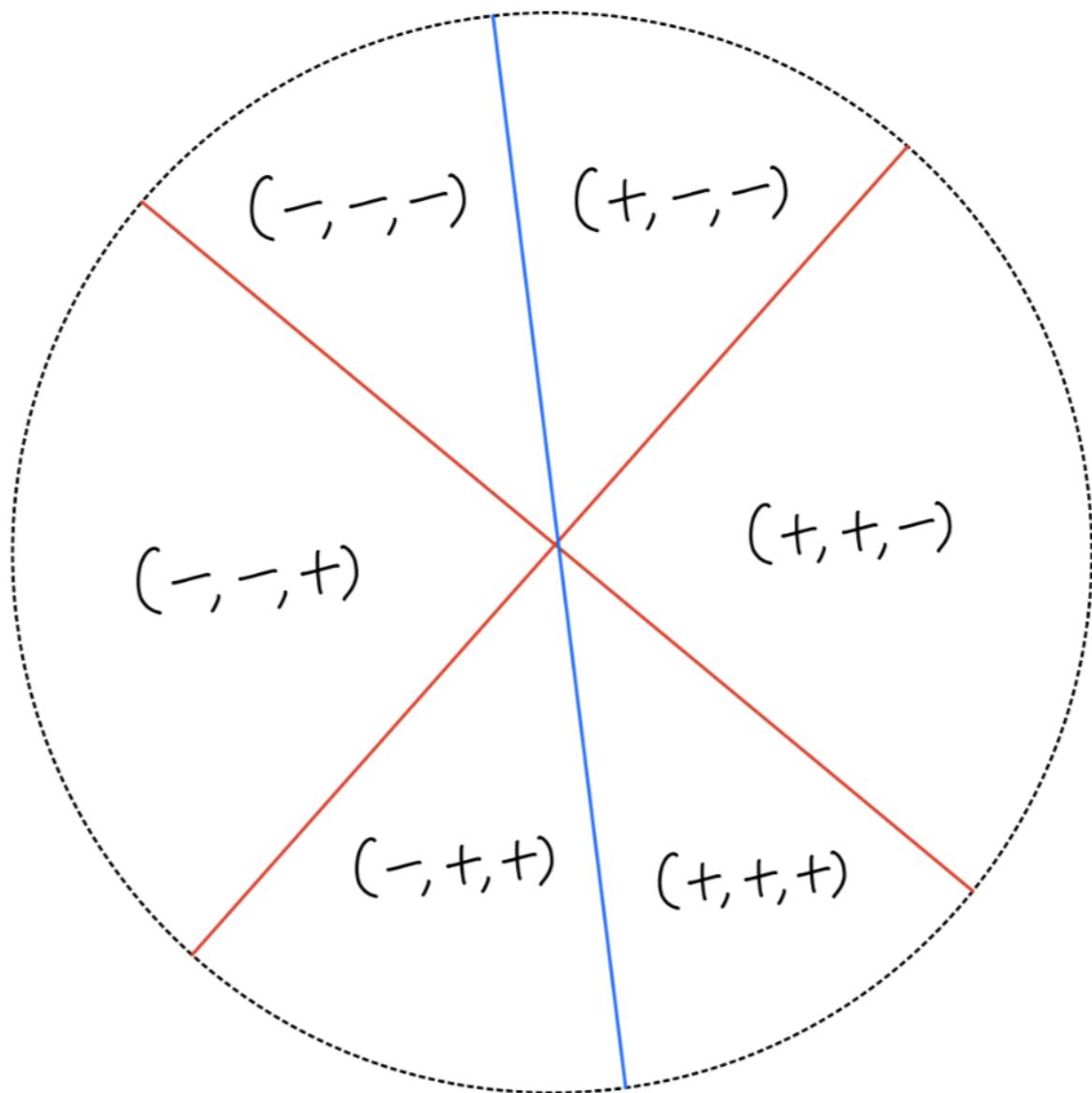


Figure 3.231

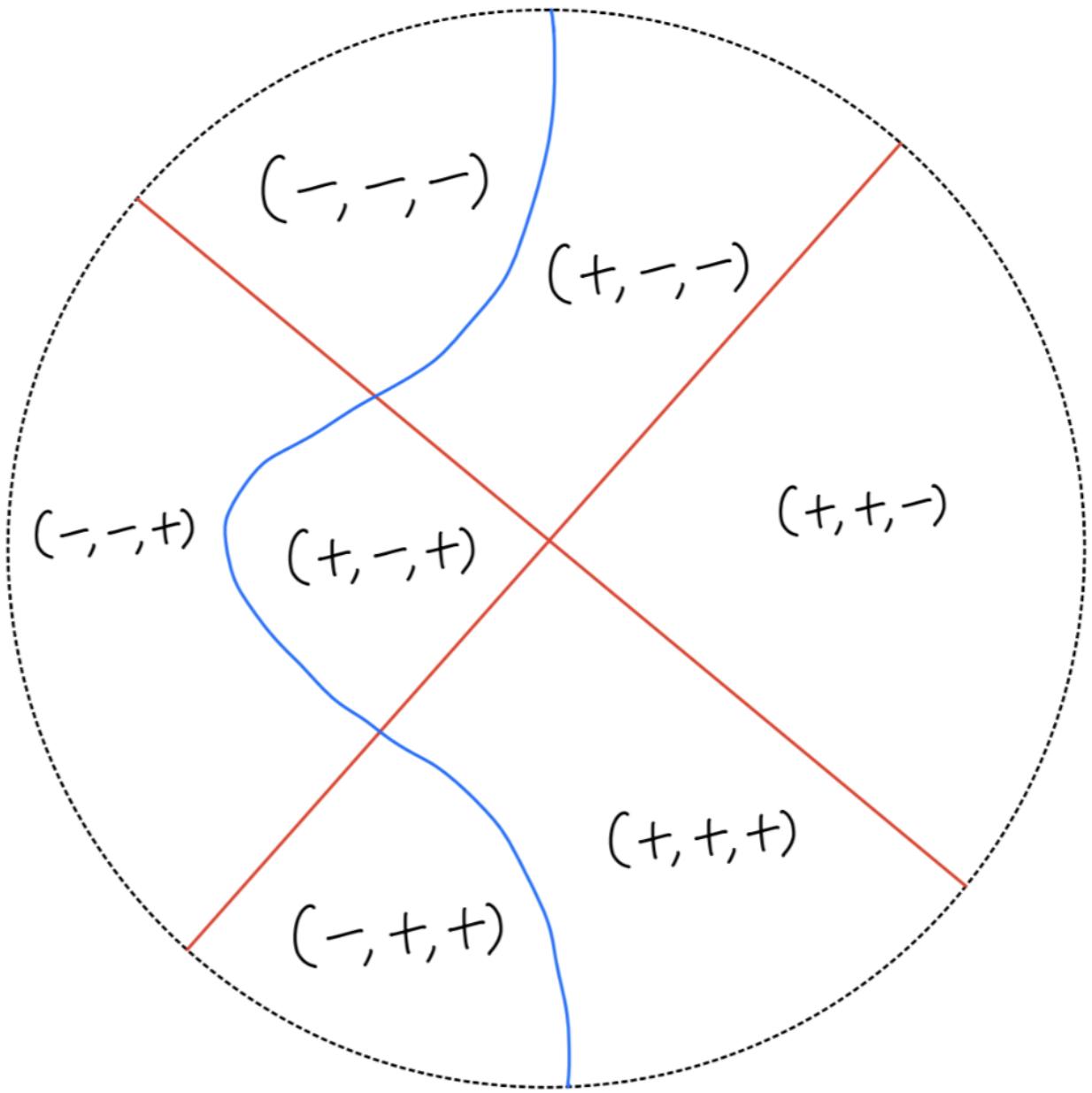


Figure 3.232

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

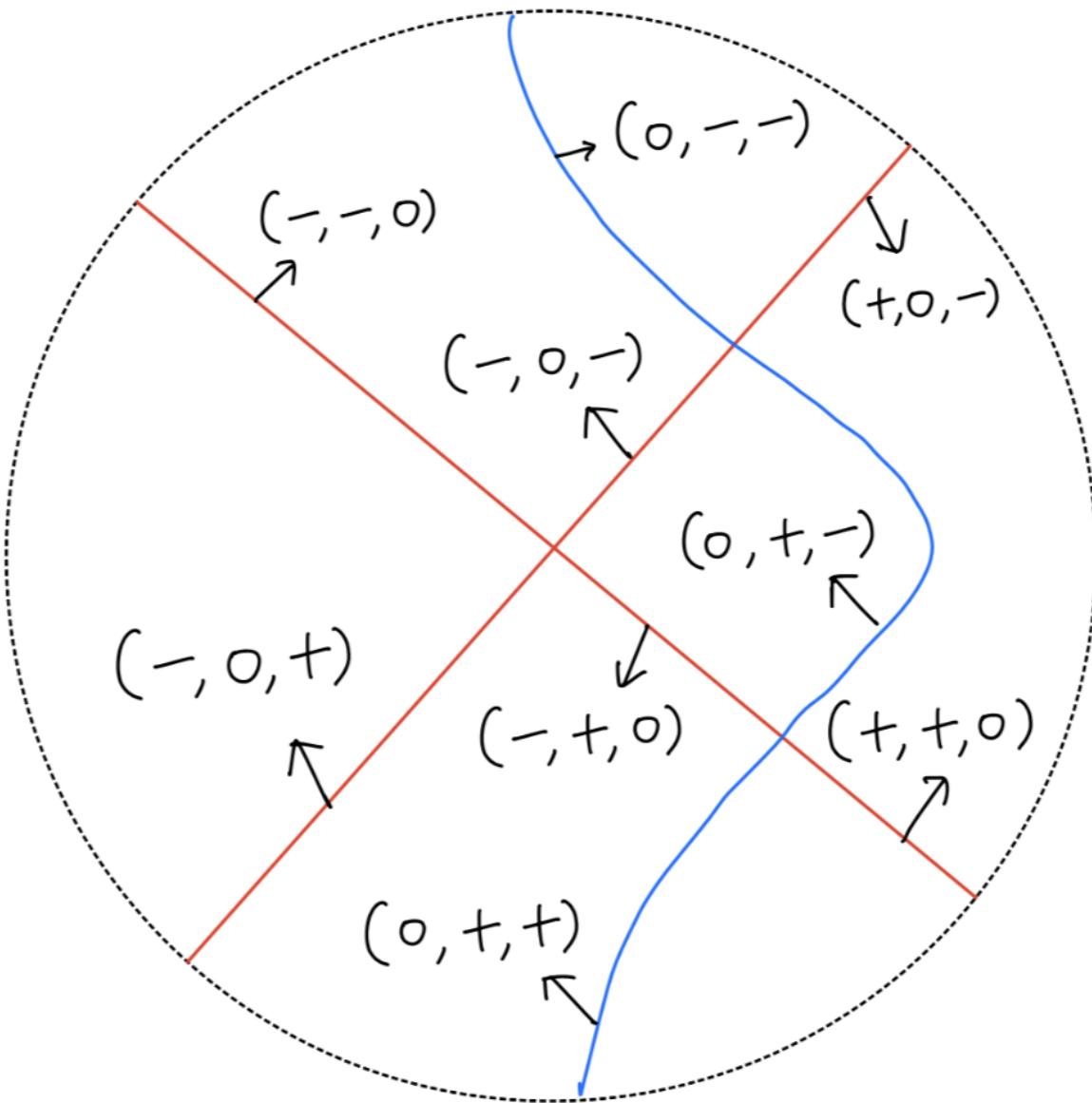


Figure 3.233

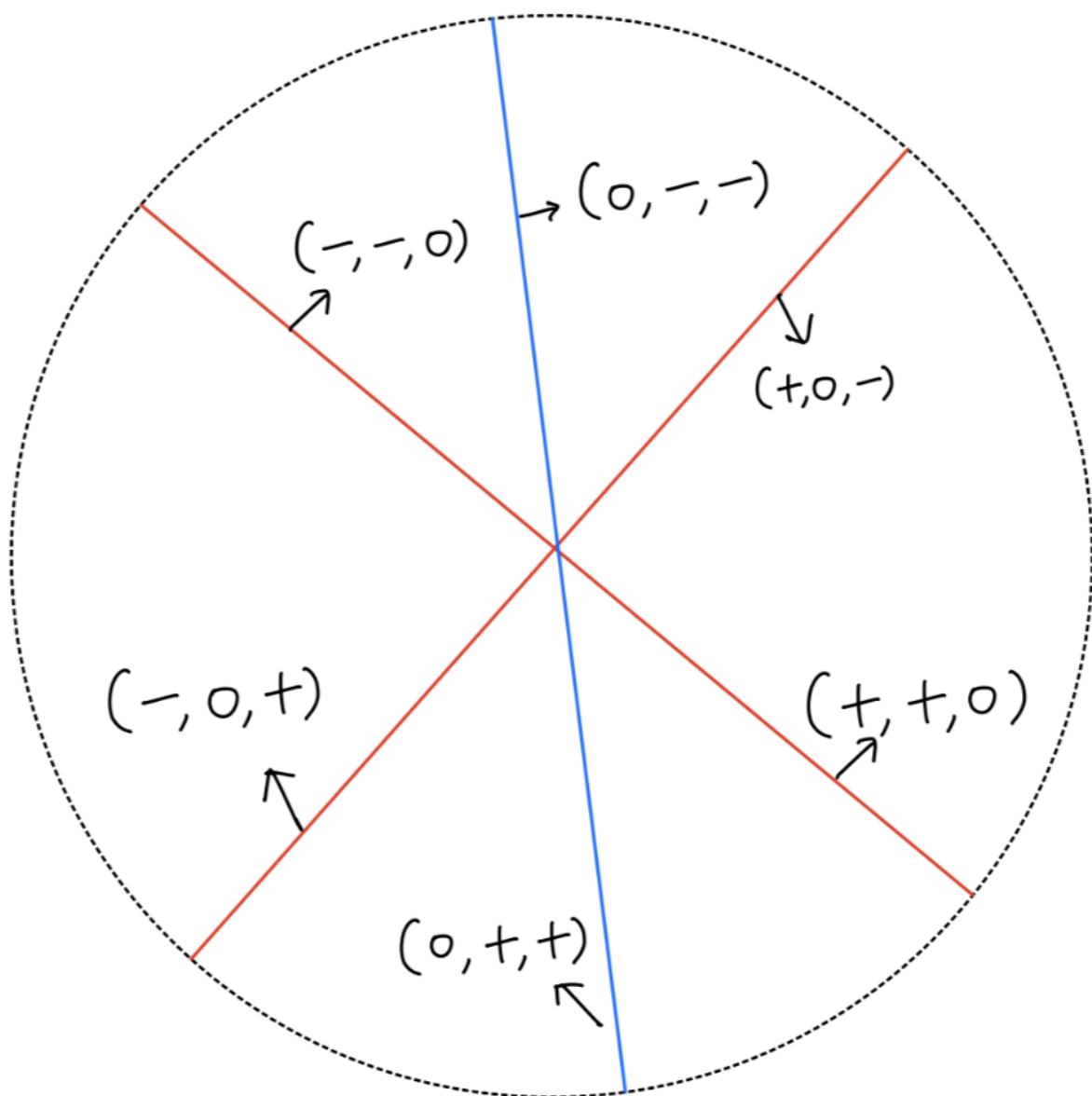


Figure 3.234

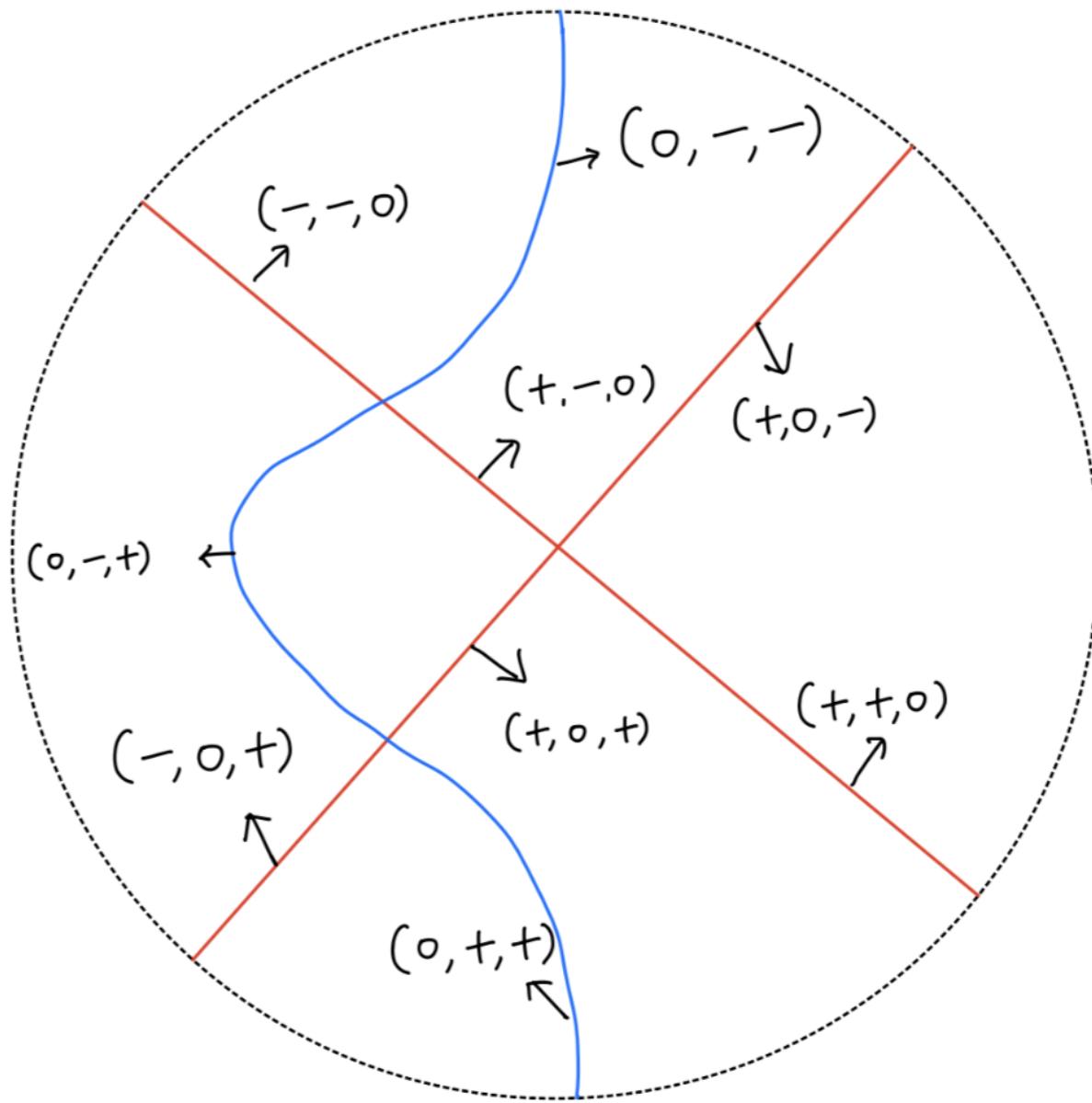


Figure 3.235

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

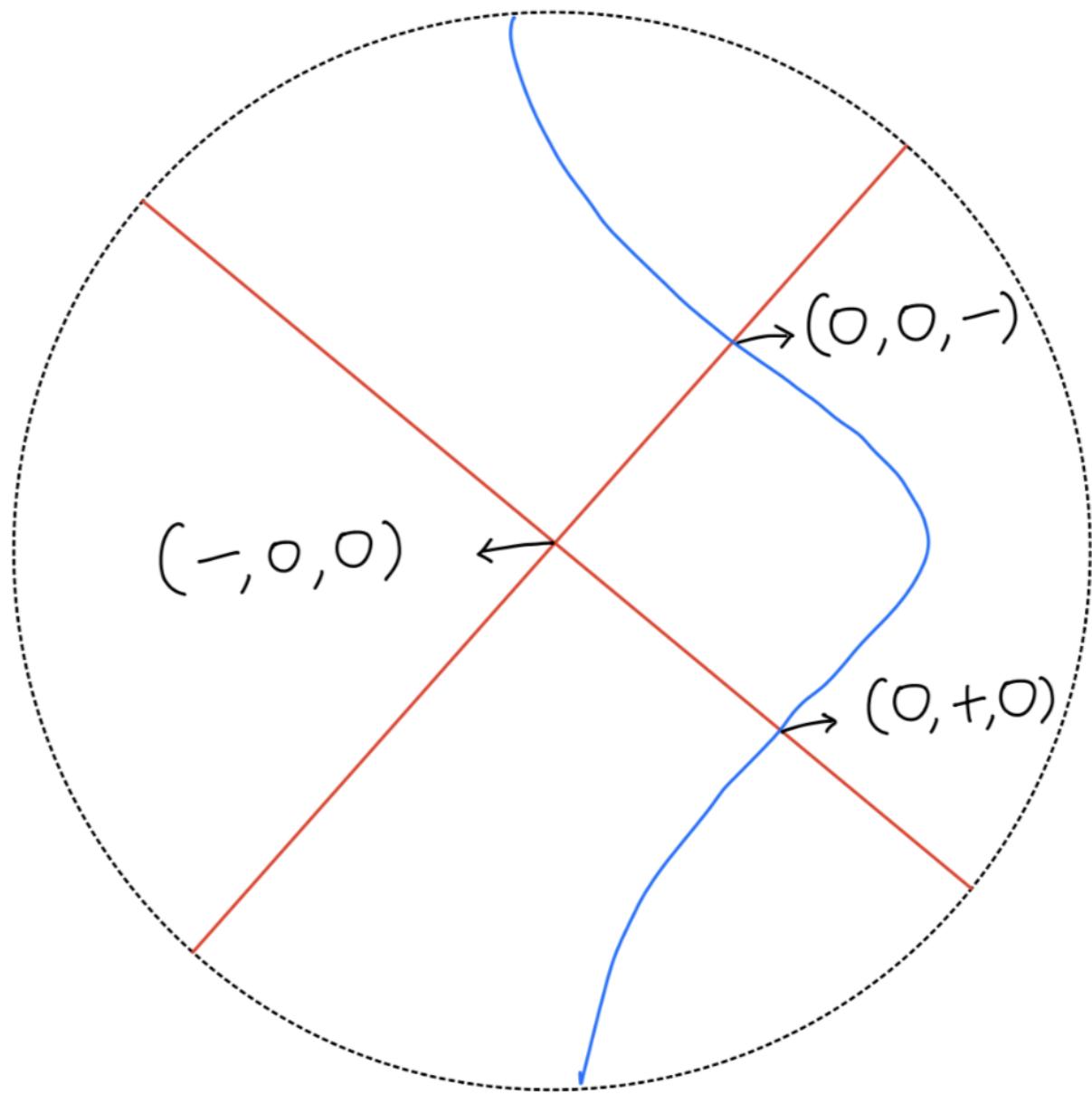


Figure 3.236

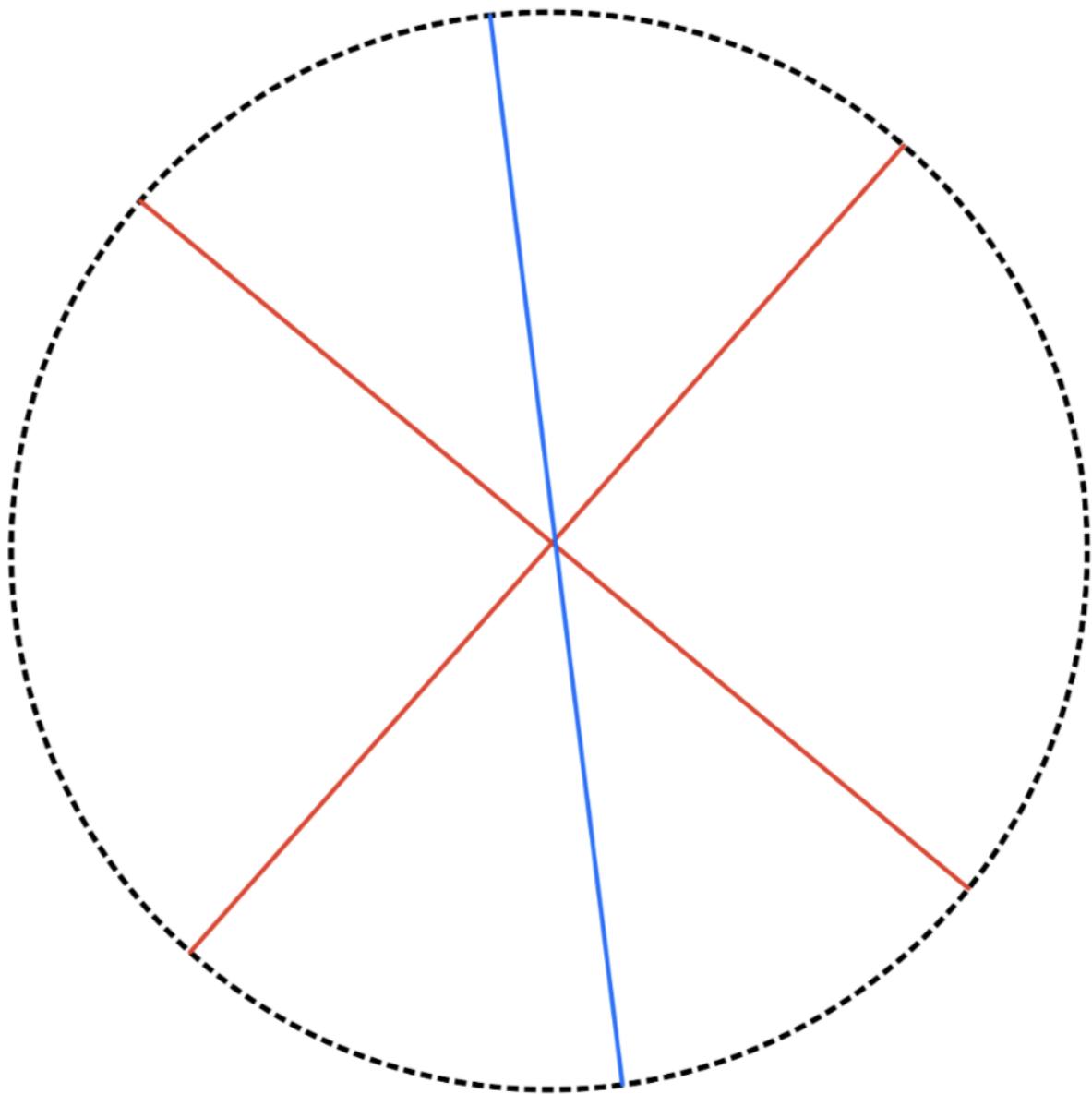


Figure 3.237

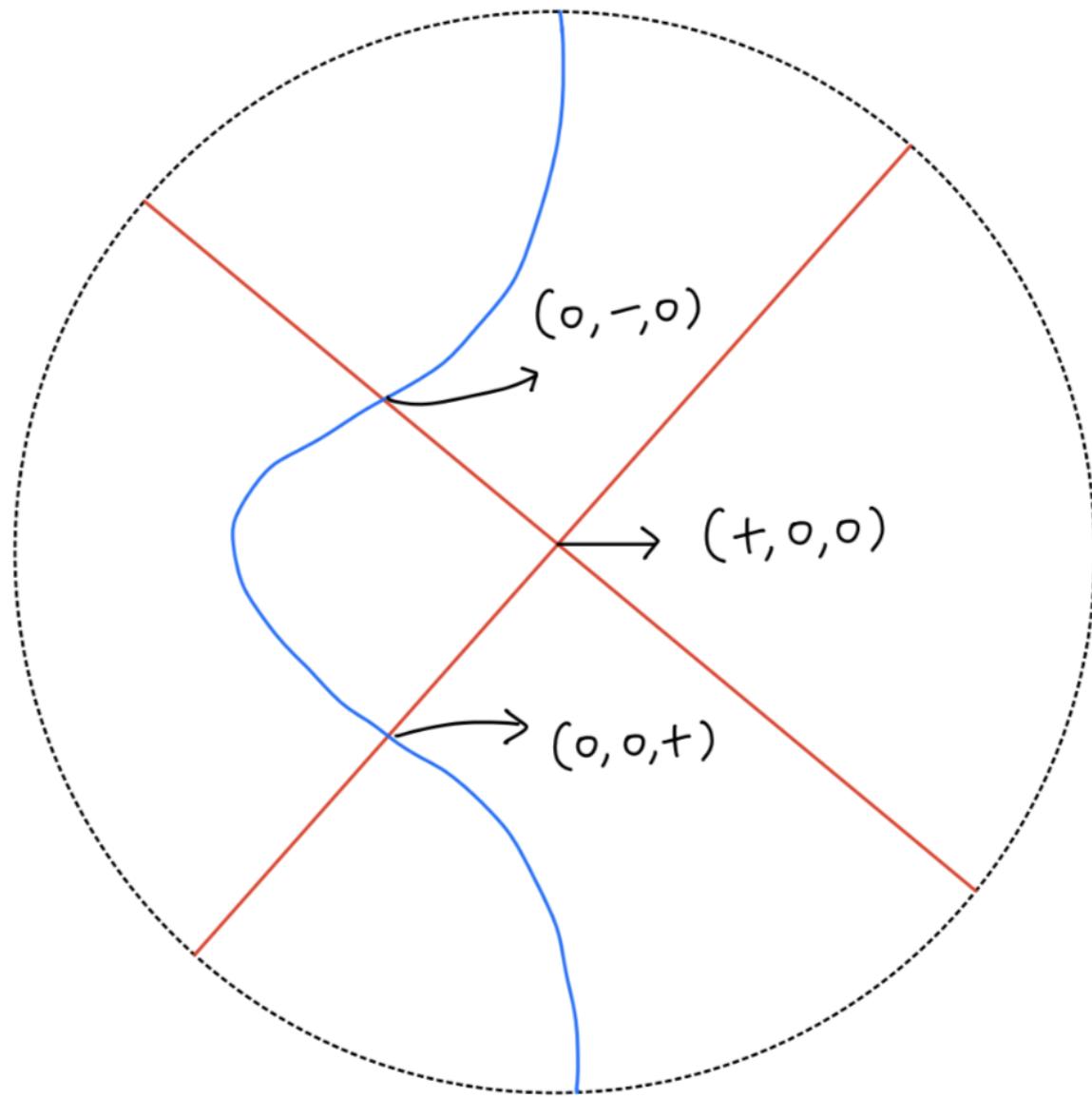


Figure 3.238

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

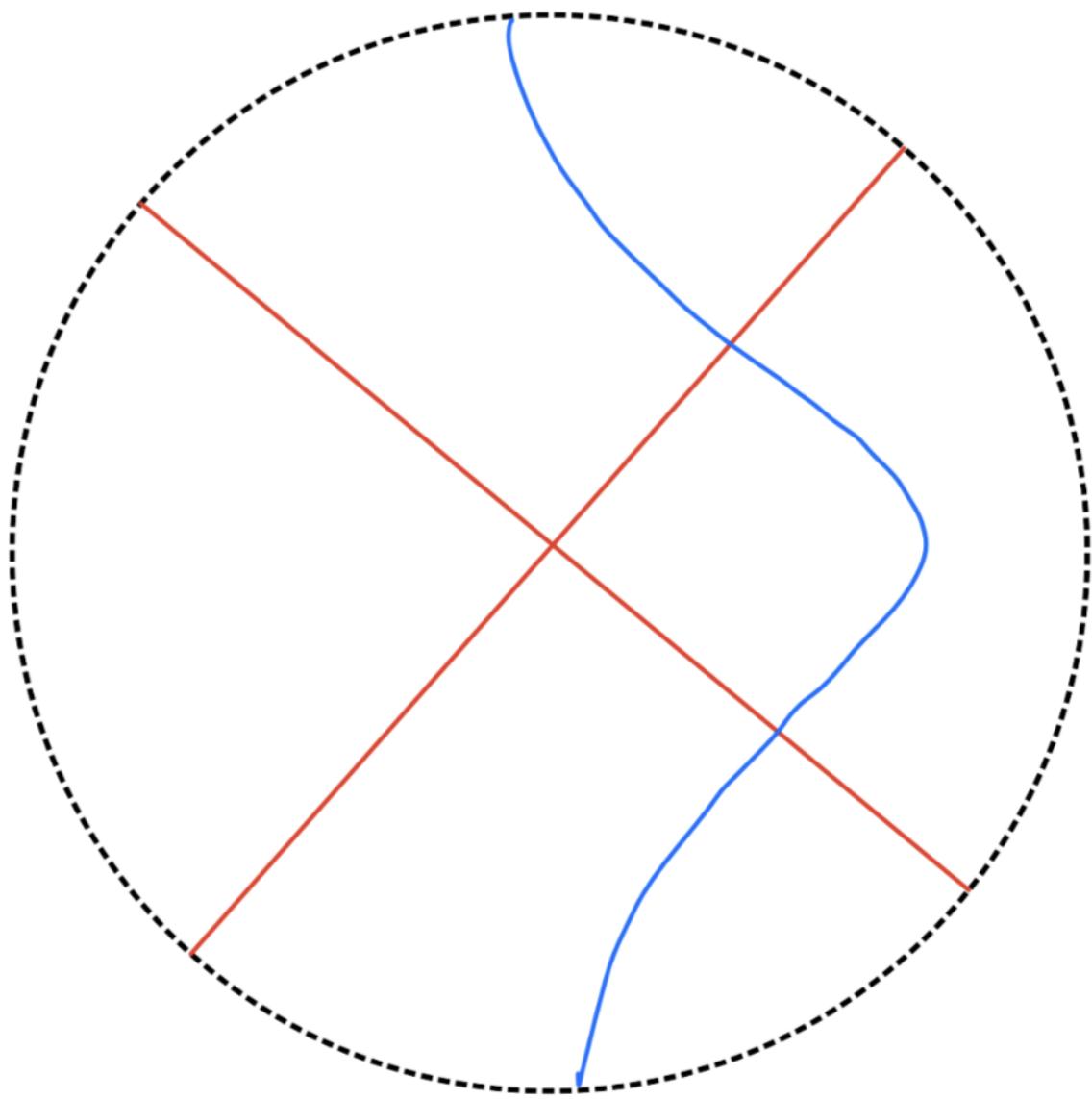


Figure 3.239

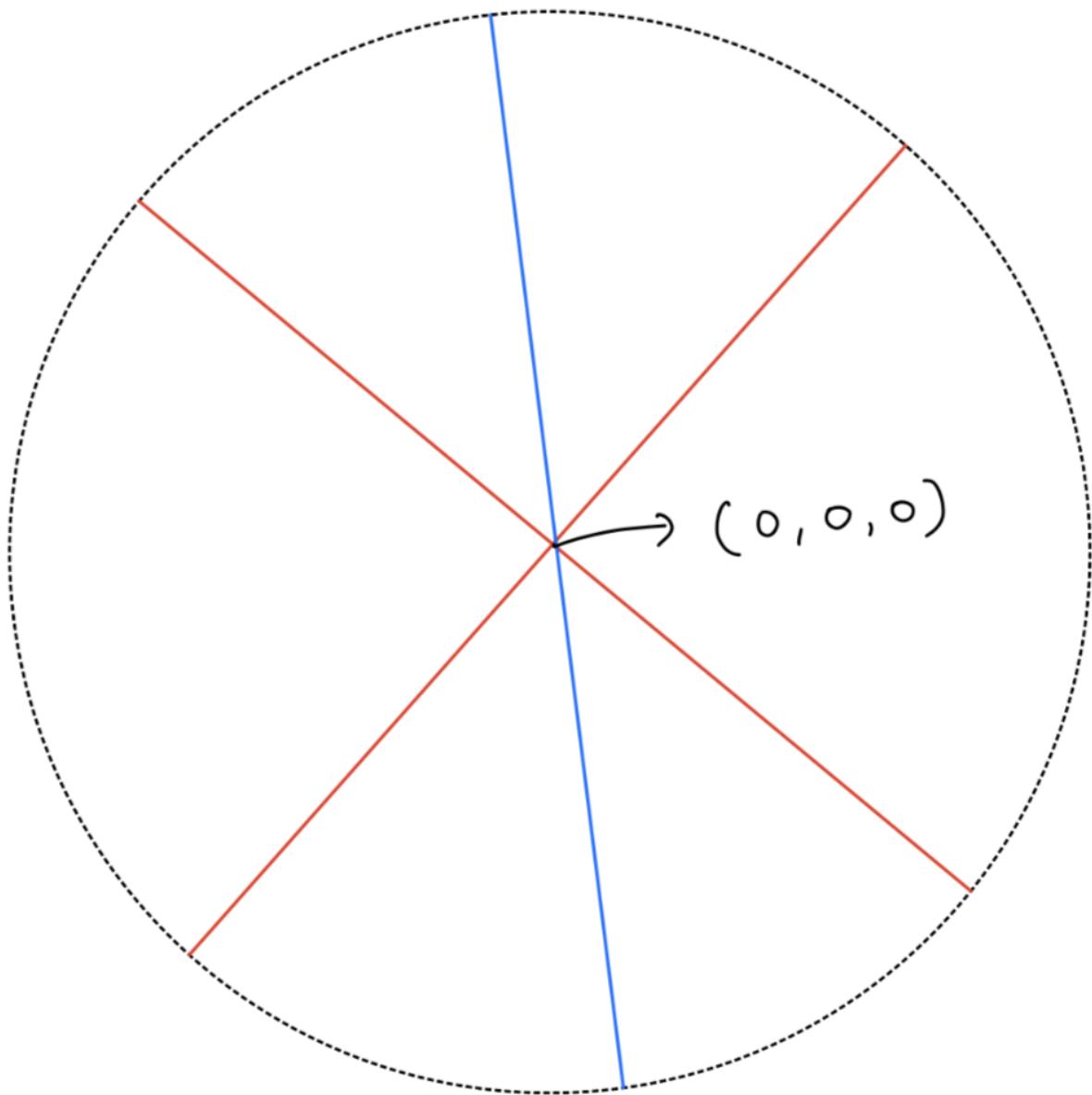


Figure 3.240

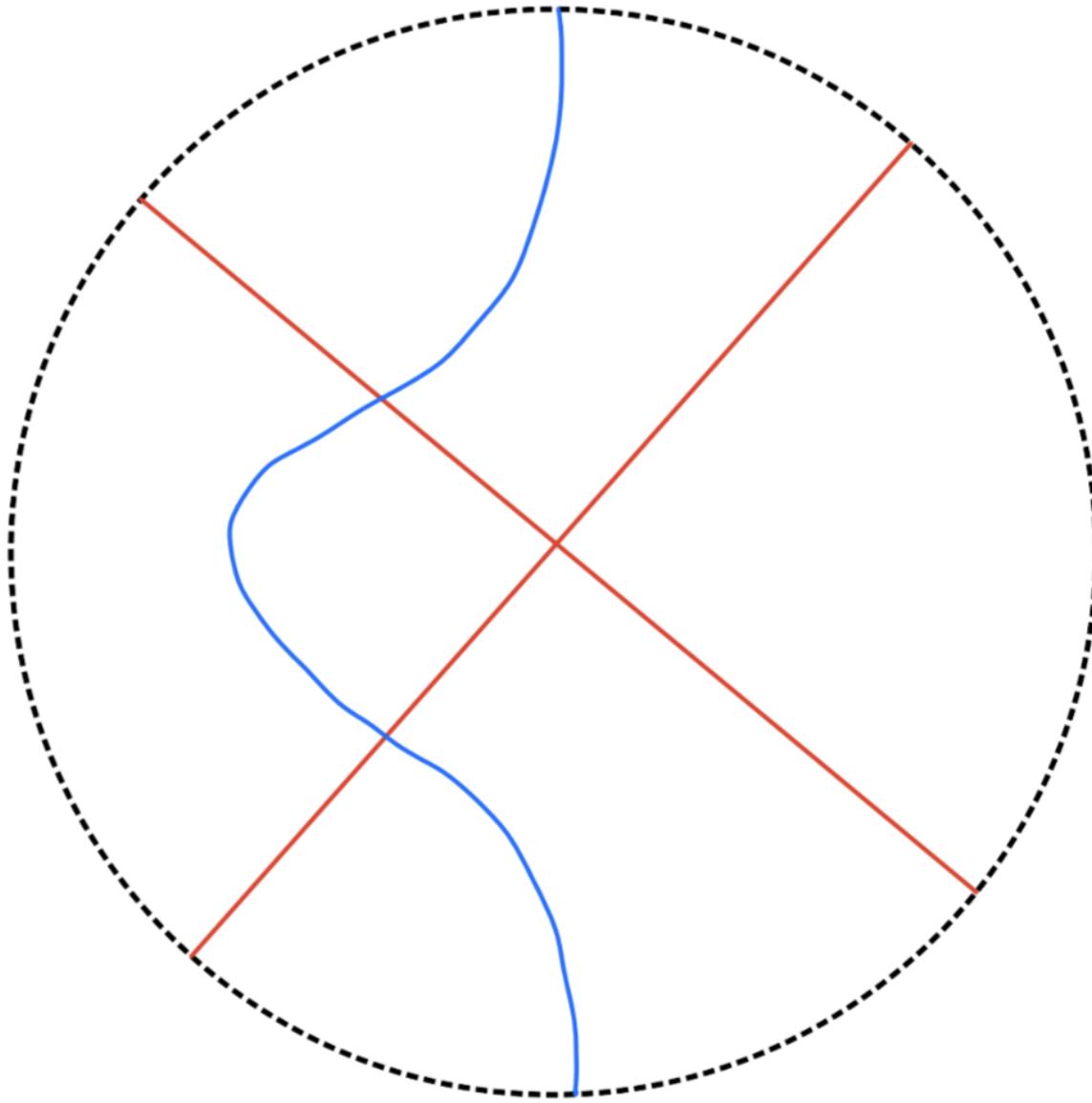


Figure 3.241

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

Definition 205. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $\text{star}(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to \mathcal{S} , say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of \mathcal{S} .
- whose arrows corresponds to codimension 1 strata of \mathcal{S} .
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the $star(s)$.

Definition 206. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s , say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s .

Definition 207. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

then \mathcal{S} is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to \mathcal{S} is legible if \mathcal{S} is.

Definition 208. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in $star(s)$.

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- \mathcal{S} is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$ from v to v' , $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 209. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Supoose \mathcal{S} is legible, then we define $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$ as

$\rho(s) :=$ the codimension 0 stratum corresponding to the initial vertex of $Q_{\mathcal{S},s}$

Definition 210. Suppose we have a manifold R with stratification \mathcal{S} such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions (= codimension 0 strata) contained in $star(s)$.
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset start(s_1)$, then $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say

(a_1, \dots, a_k) , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \dots \circ F_S(a_1)$$

This is well-defined because F_S is legible.

Definition 211. Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be the cochain complexes $(C^\bullet, \delta^\bullet)$ supported on degree 0 and 1 and ϕ^\bullet a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$, then

1. we denote $(C^\bullet, \delta^\bullet)$ as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote ϕ^\bullet as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

3.9.2 Setting

Suppose on M , we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M .
- a smooth chart from $D_{r=2}$, say $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

3.9.3 Sheaf at the Beginning

Suppose we have a sheaf \mathcal{F}_0 singular supported on Λ_0 such that $f^*\mathcal{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0 .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

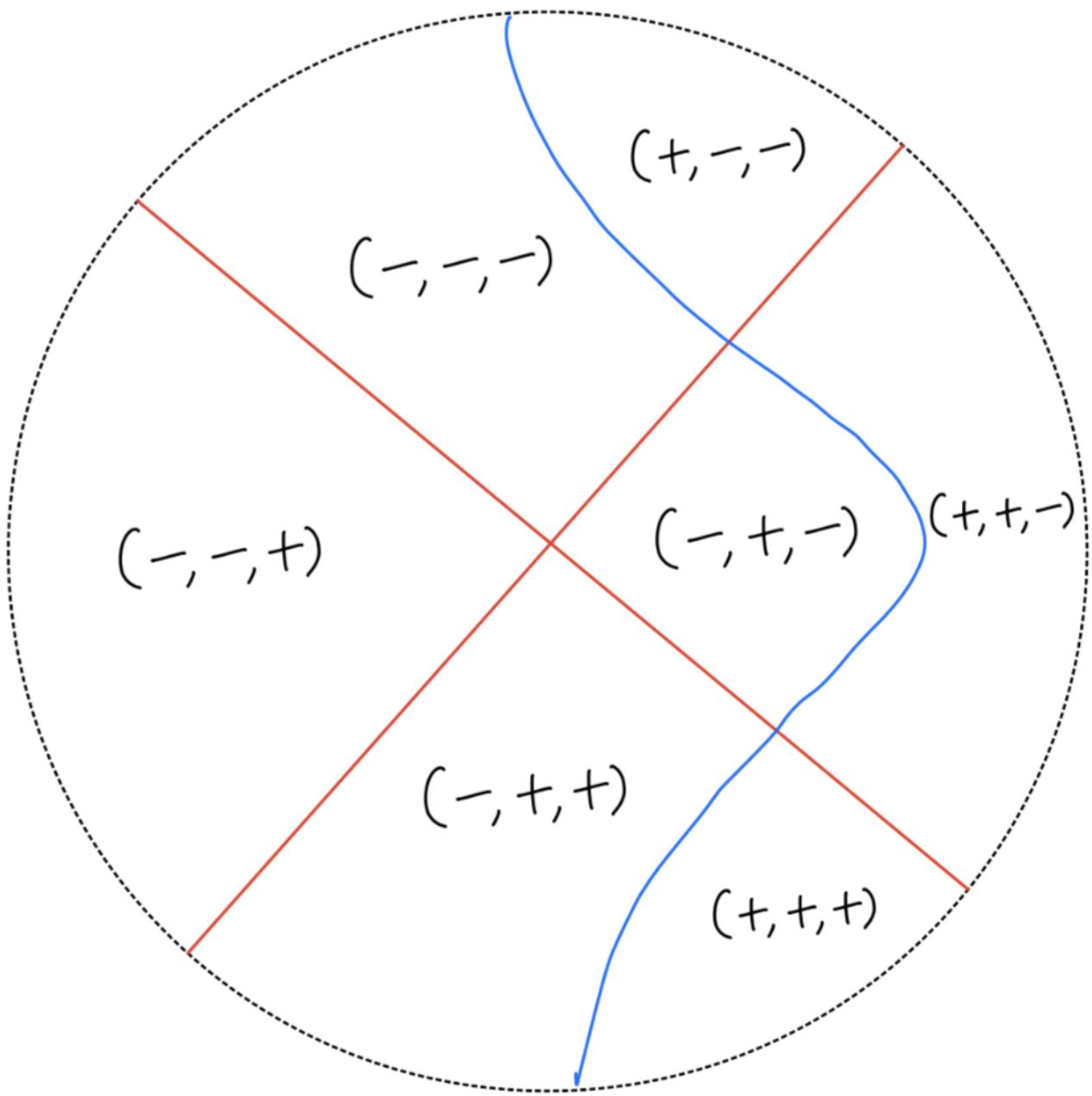


Figure 3.242

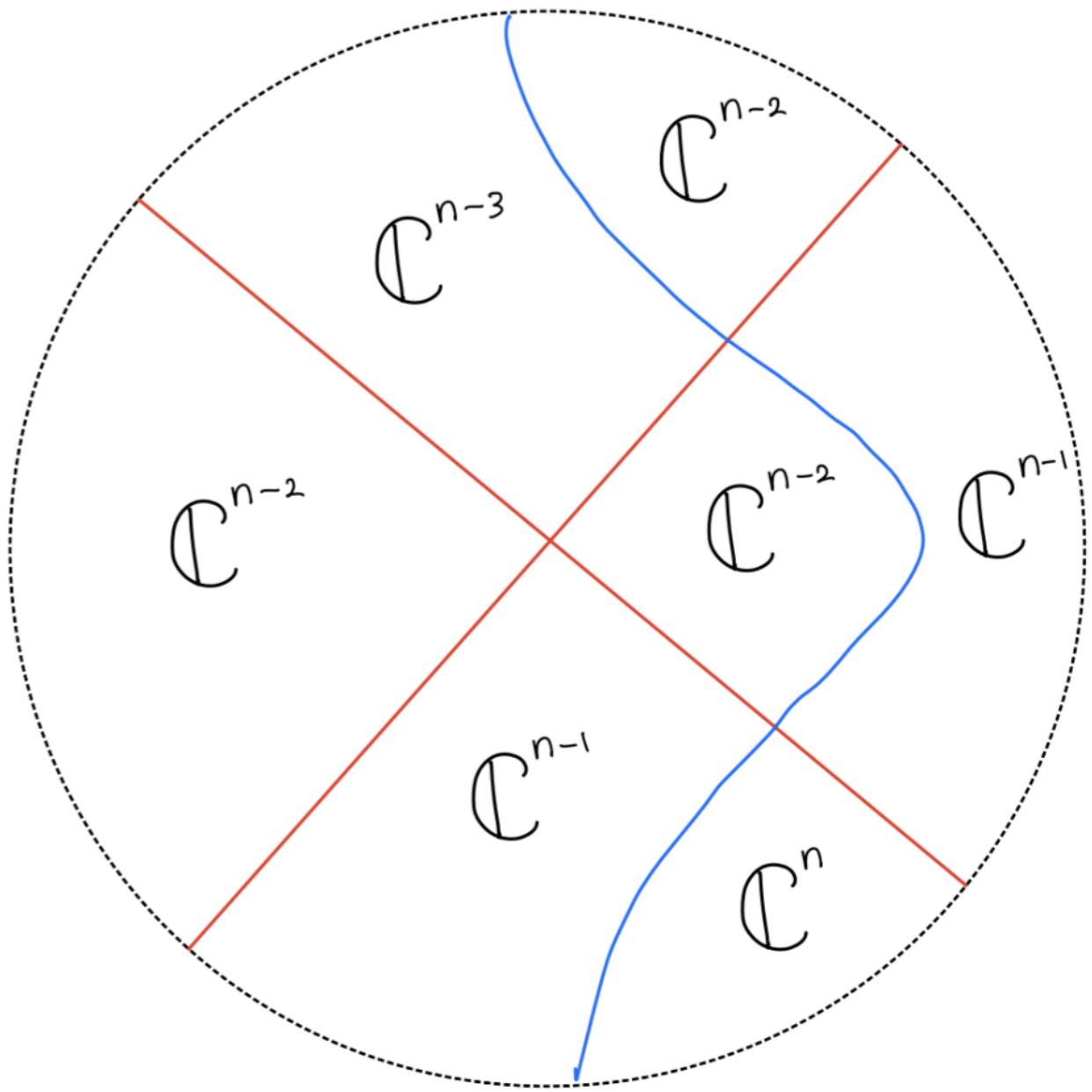


Figure 3.243

- $F_0(-, -, -) := \mathbb{C}^{n-3}$

- $F_0(+, -, -) := \mathbb{C}^{n-2}$

- $F_0(-, -, +) := \mathbb{C}^{n-2}$

- $F_0(-, +, -) := \mathbb{C}^{n-2}$

- $F_0(+, +, -) := \mathbb{C}^{n-1}$

- $F_0(-, +, +) := \mathbb{C}^{n-1}$

- $F_0(+, +, +) := \mathbb{C}^n$

Generalization maps:

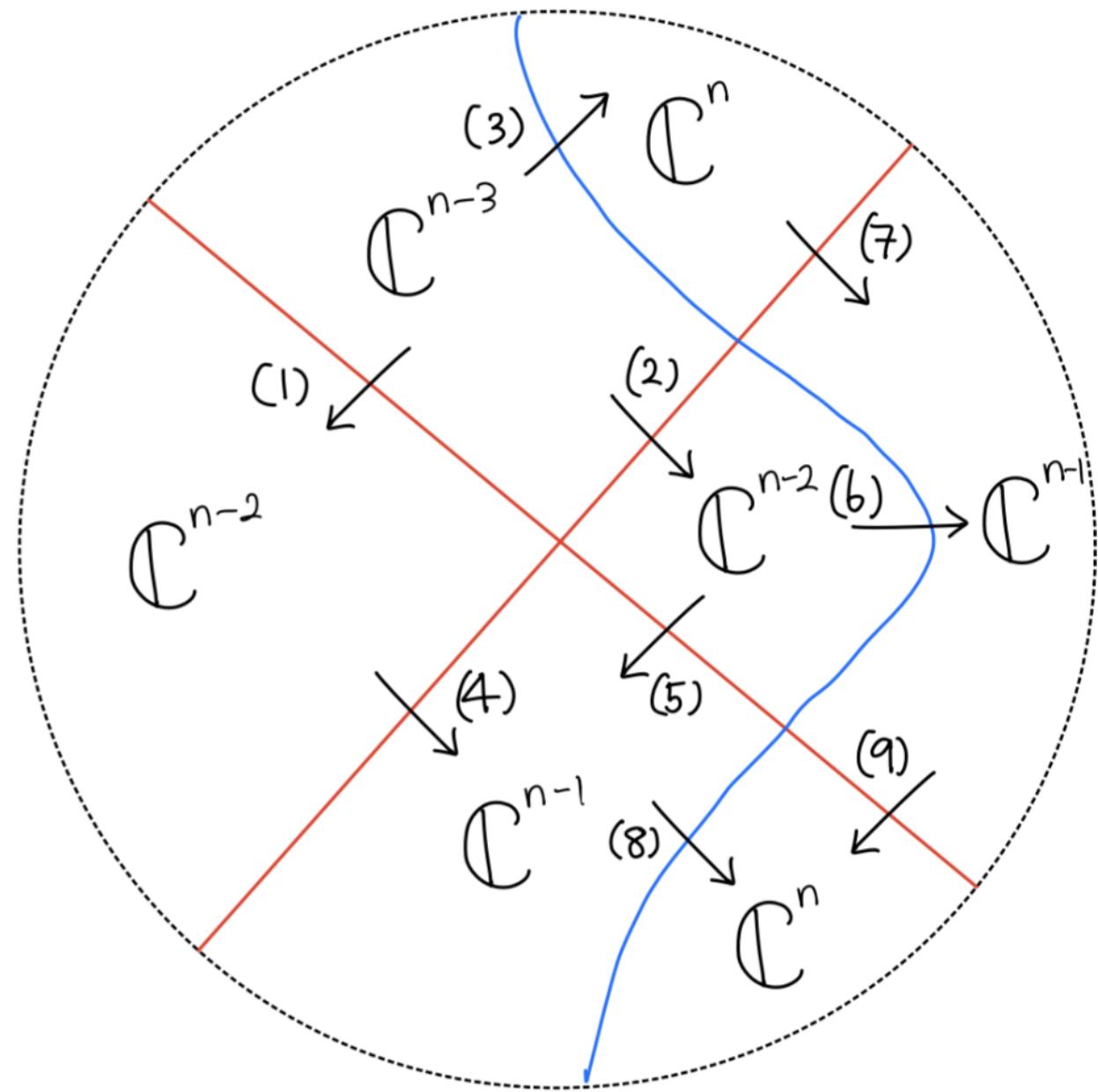


Figure 3.244

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(5) ι_0

(6) ι_1

(7) ι_0

(8) ι_1

(9) ι_0

3.9.4 Legendrian Cobordism

Then define a Legendrian cobordism \mathcal{F}_\bullet starting from \mathcal{F}_0 , say $cobord_2$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0, 1]$, say $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on $D_{r=2} \times [0, 1]$, say $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$.

A. Sheaf on $V \times [0, 1]$

First, I will define $\mathcal{F}_{V \times [0, 1]}$ to be $pr_1^*(\mathcal{F}_0|_V)$ where $pr_1 : V \times [0, 1] \rightarrow V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe $\mathcal{F}_{D_{r=2} \times [0, 1]}$ as $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from \mathcal{S}_\bullet to the category of perfect \mathbb{C} -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

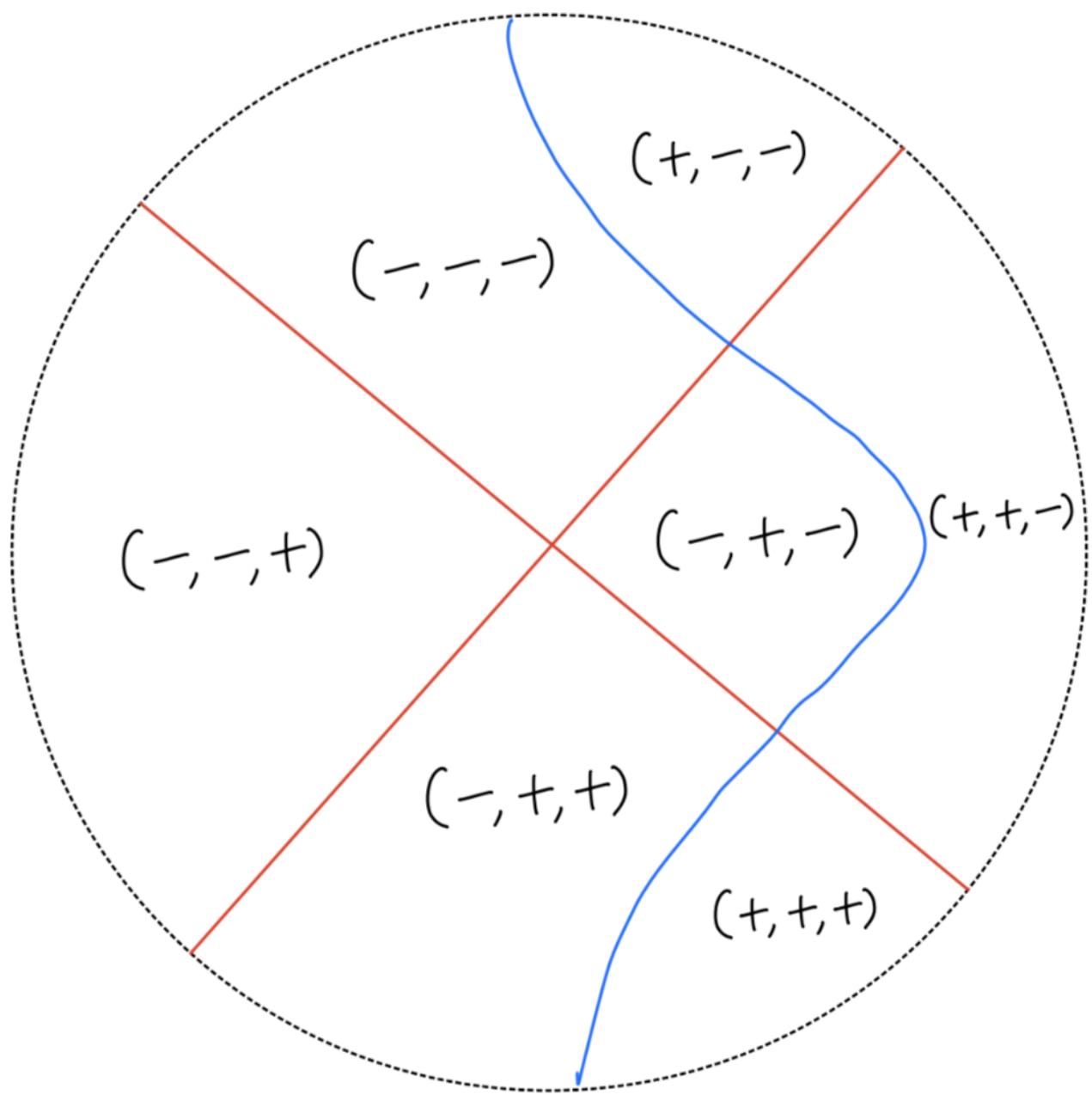


Figure 3.245

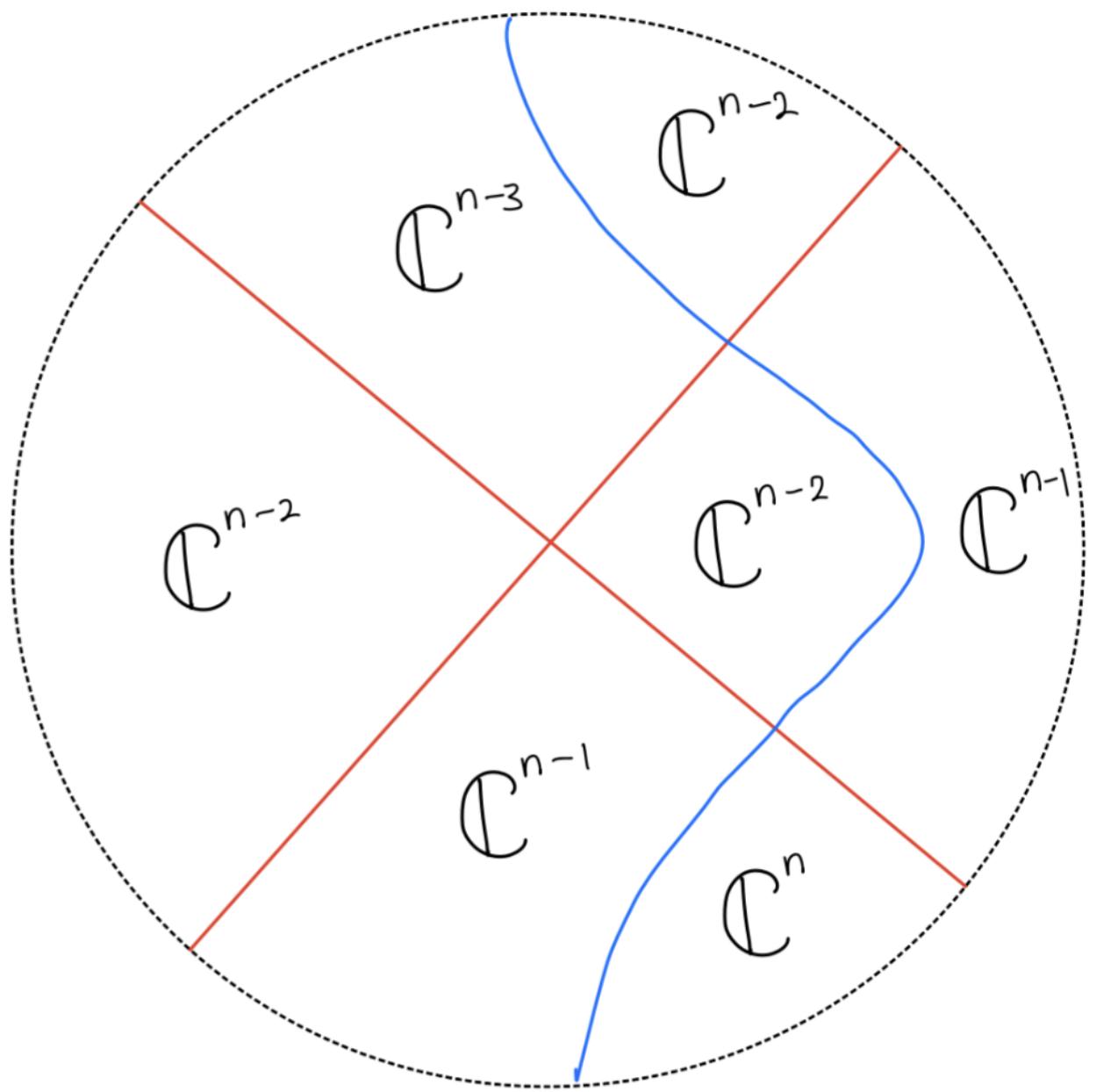


Figure 3.246

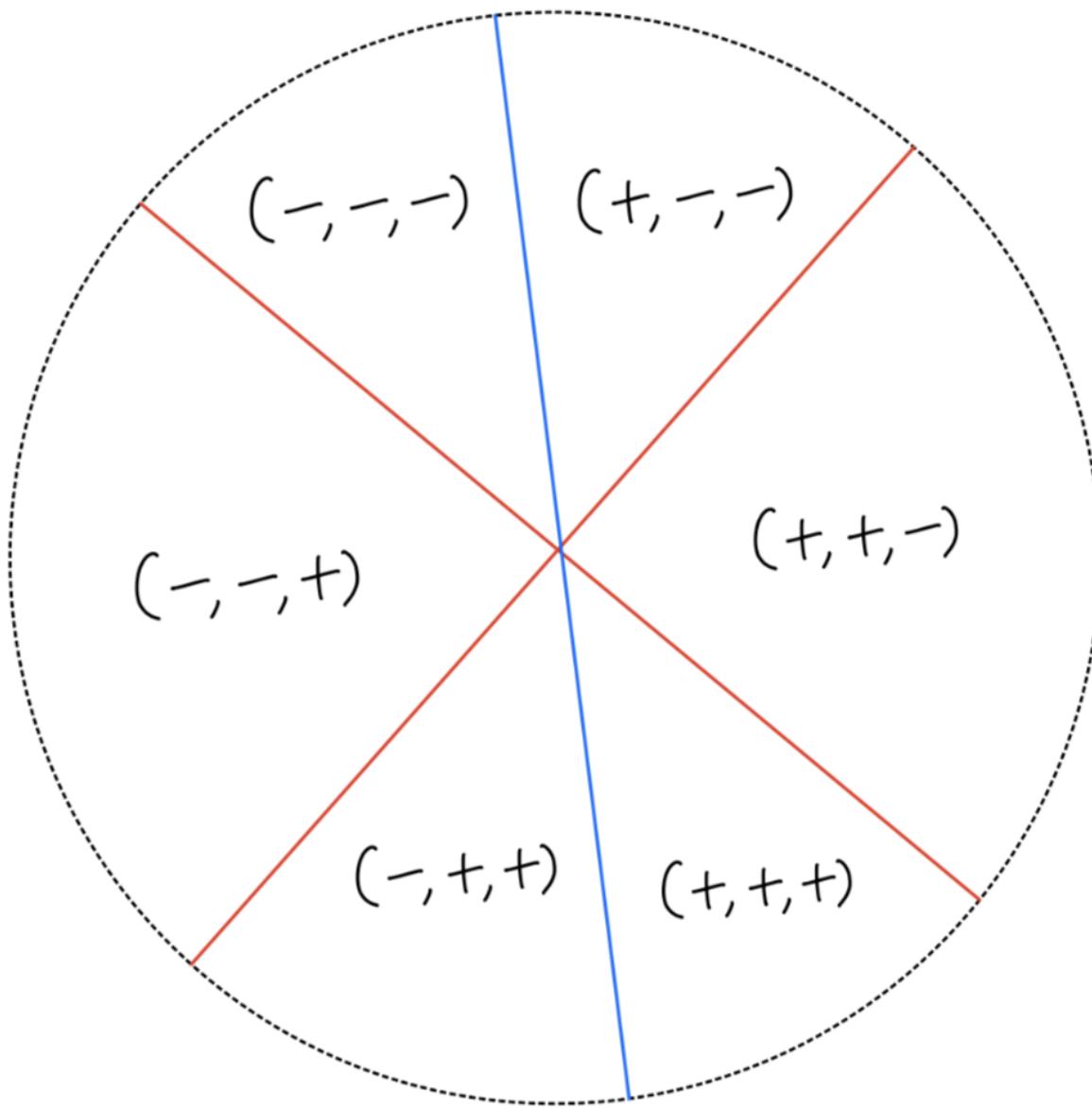


Figure 3.247

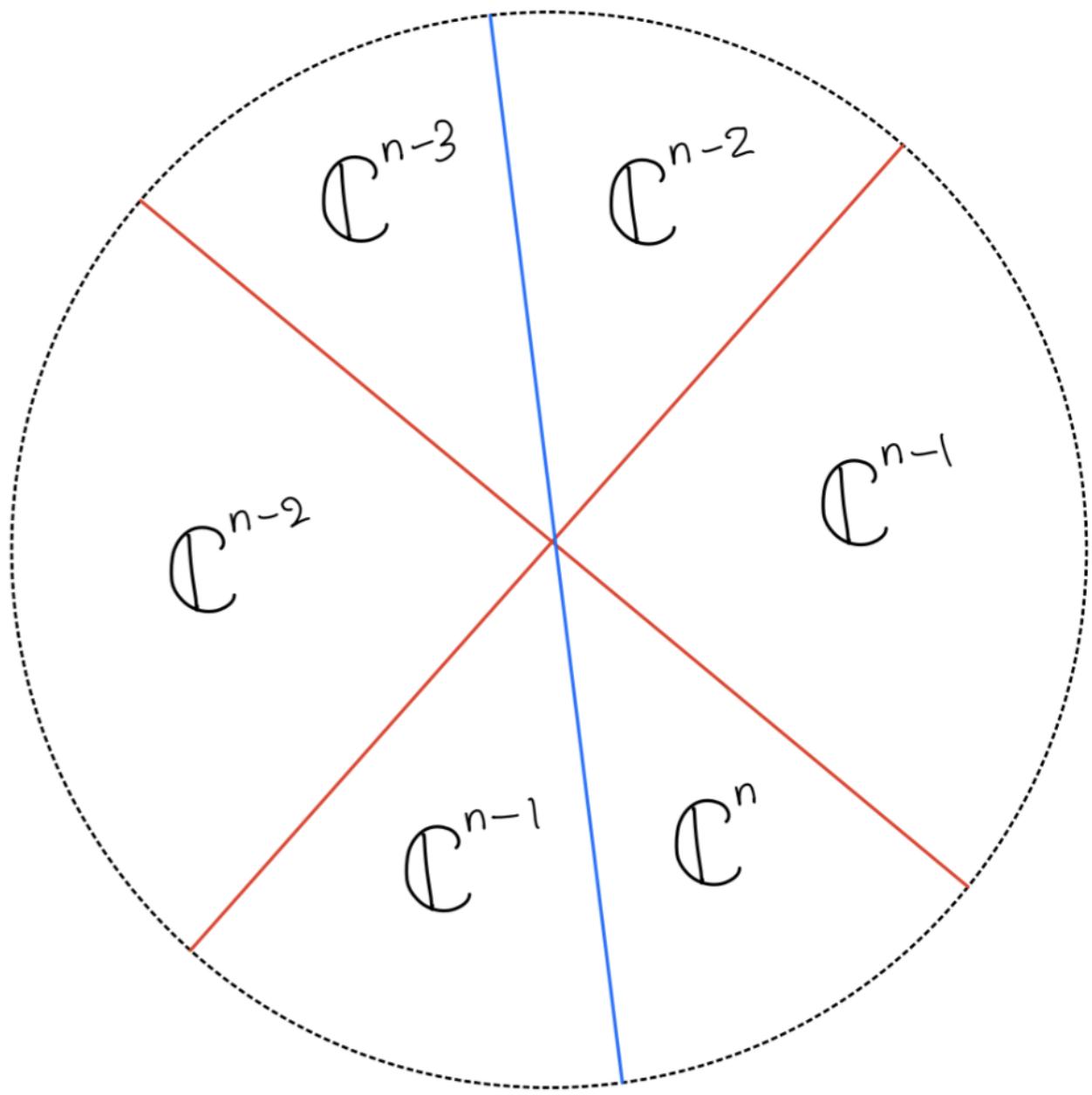


Figure 3.248

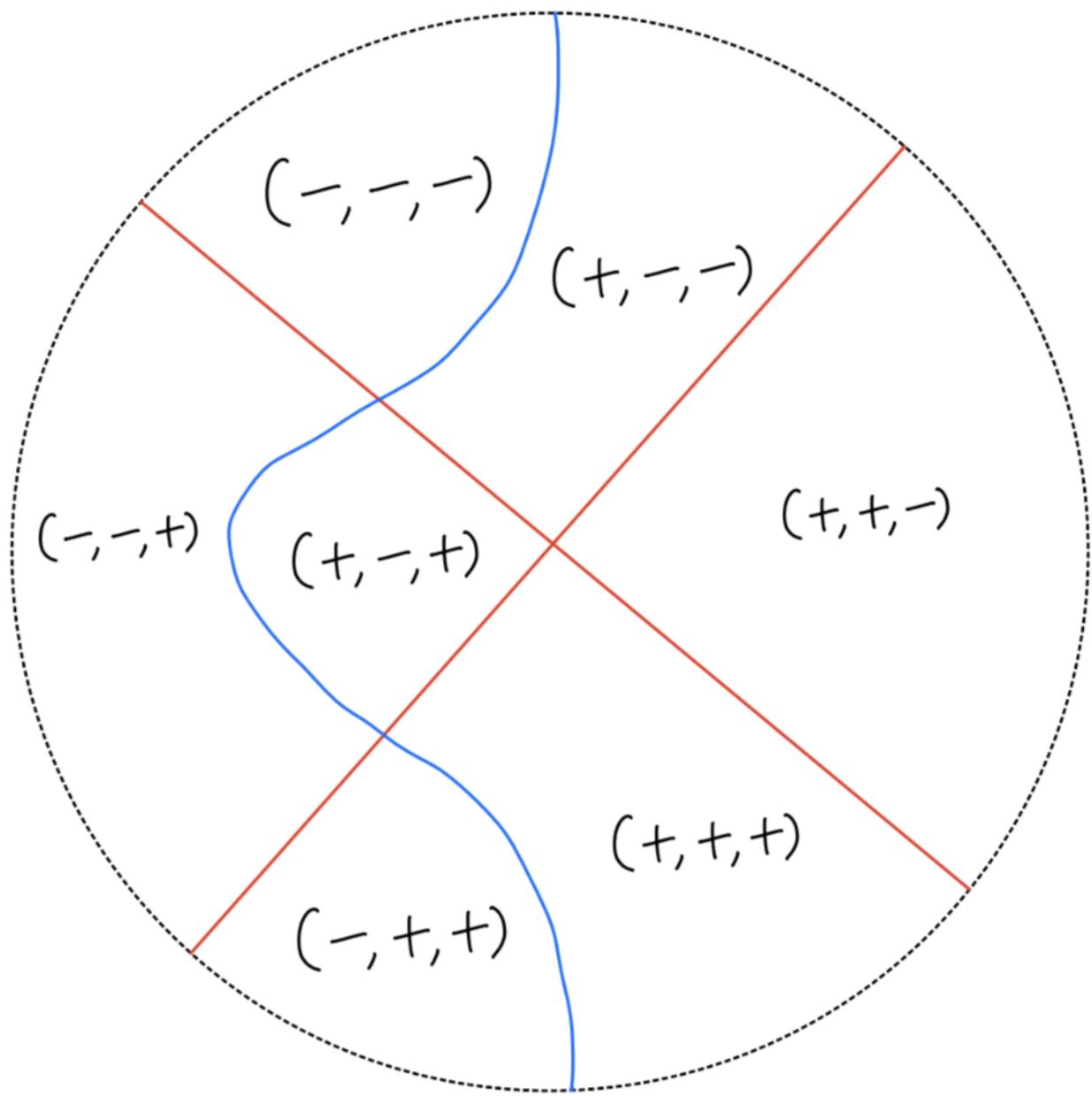


Figure 3.249

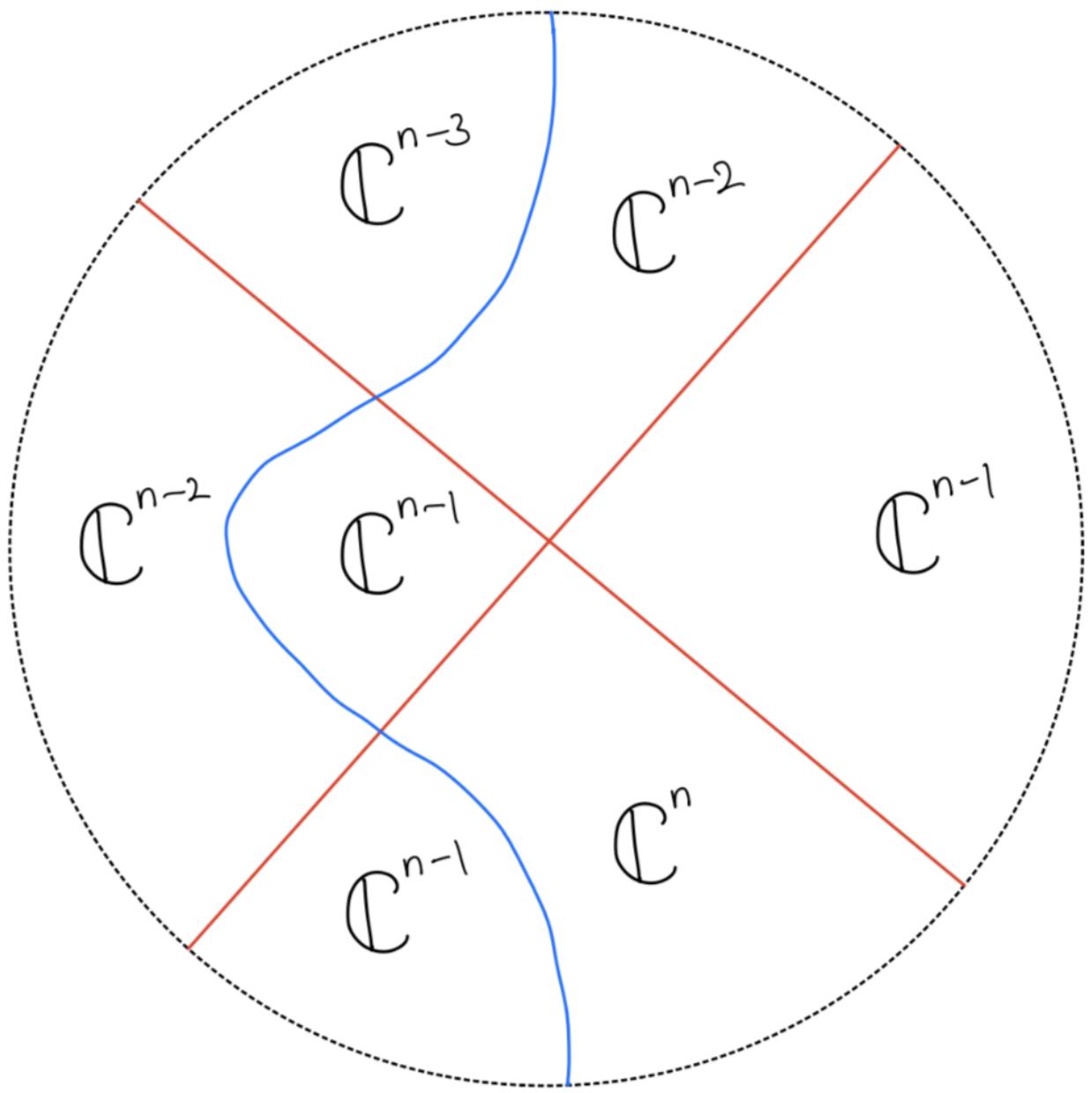


Figure 3.250

- $F_{\bullet}(-, -, -) := \mathbb{C}^{n-3}$

- $F_{\bullet}(-, -, +) := \mathbb{C}^{n-2}$

- $F_{\bullet}(-, +, -) := \mathbb{C}^{n-2}$

- $F_{\bullet}(-, +, +) := \mathbb{C}^{n-1}$

- $F_\bullet(+, -, -) := \mathbb{C}^{n-2}$

- $F_\bullet(+, -, +) := \mathbb{C}^{n-1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{n-1}$

- $F_\bullet(+, +, +) := \mathbb{C}^n$

Generalization maps:

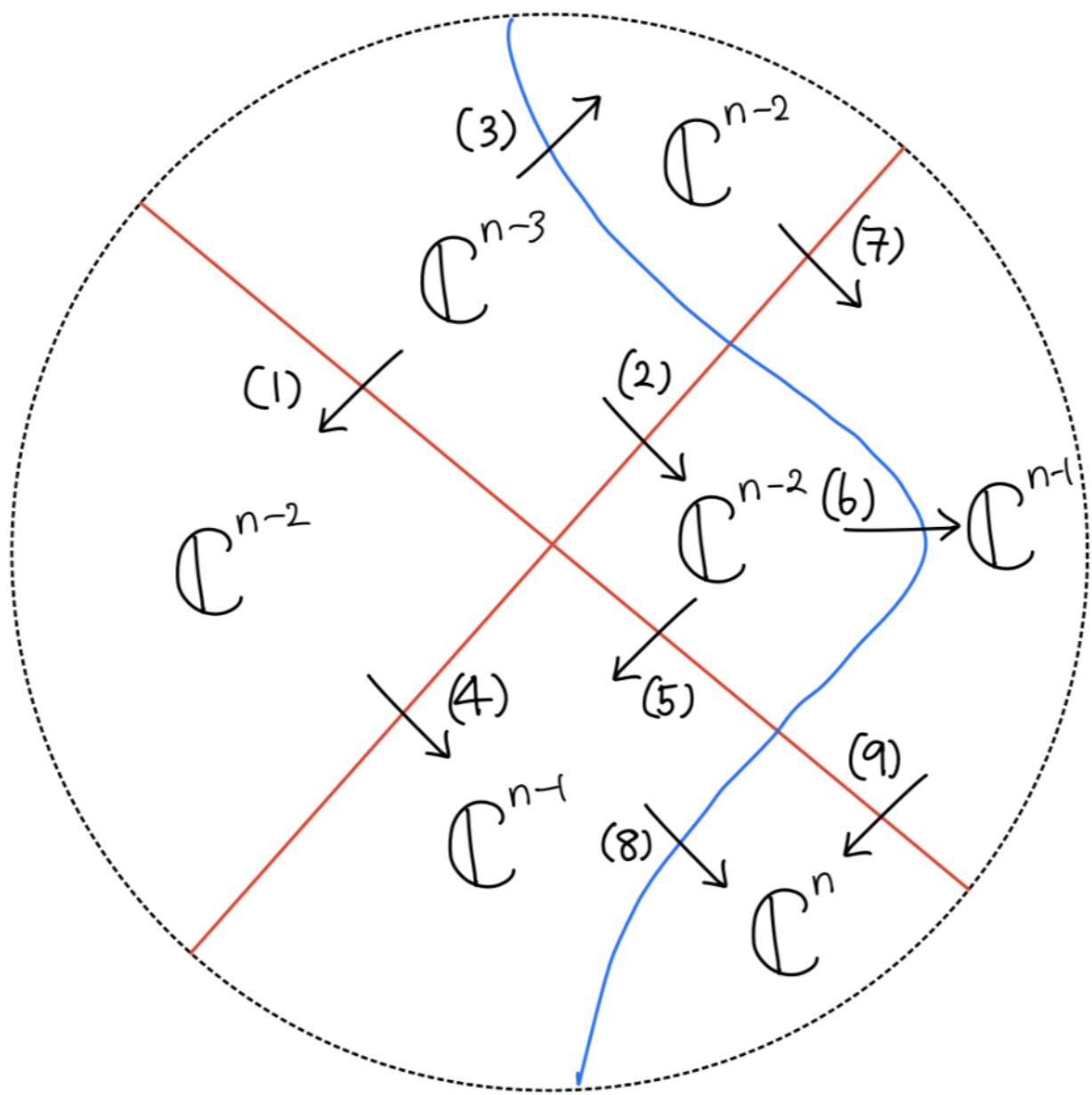


Figure 3.251

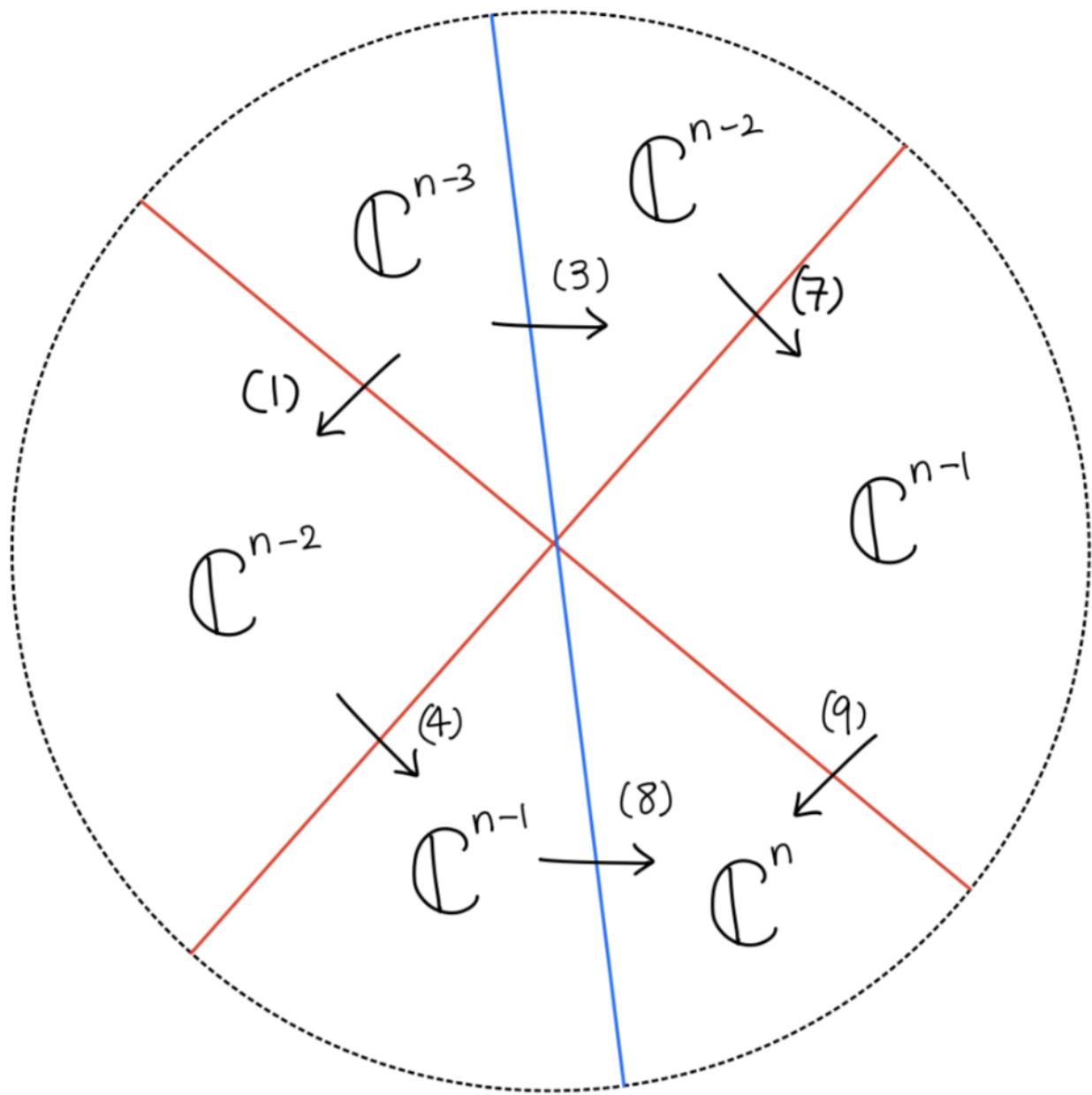


Figure 3.252

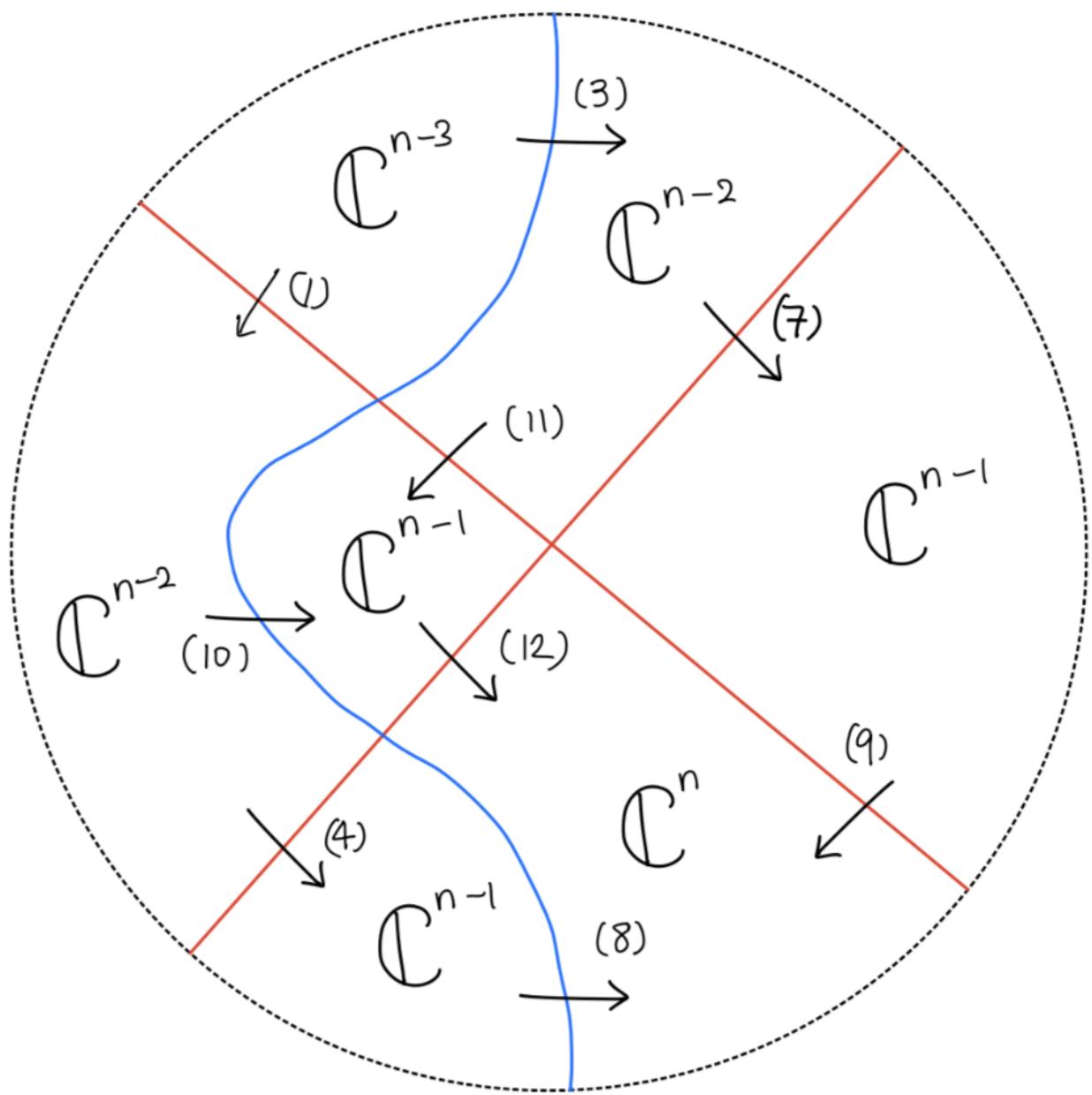


Figure 3.253

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I + n, n - 1$

(5) ι_0

(6) ι_1

(7) ι_0

(8) ι_1

(9) ι_0

(10) ι_1

(11) ι_0

(12) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the following fact.

Proposition 212. $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$ is the projection onto the first argument.

Proof. pass □

Definition 213. we define γ_\bullet to be the composition

$$(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism \mathcal{F}_\bullet , we show that this is a Legendrian cobordism.

Proposition 214. \mathcal{F}_\bullet is a Legendrian cobordism i.e. $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of \mathcal{F}_\bullet vanishes at every points on a contangent bundle of M .

Note that there is a diffeomorphism between $D_{r=2} \times (0, 1)$ and \mathbb{R}^3 that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of \mathcal{F}^\bullet along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of \mathcal{F}^\bullet along the diffeomorphism could be described using the following legible diagram, say F^3 . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-, -, -) := \mathbb{C}^{n-3}$

- $F^3(-, -, +) := \mathbb{C}^{n-2}$

- $F^3(+, -, -) := \mathbb{C}^{n-2}$

- $F^3(+, -, +) := \mathbb{C}^{n-2}$

- $F^3(-, +, -) := \mathbb{C}^{n-2}$

- $F^3(-, +, +) := \mathbb{C}^{n-1}$

- $F^3(+, +, -) := \mathbb{C}^{n-1}$

- $F^3(+, +, +) := \mathbb{C}^n$

Generalization maps:

$$\begin{array}{ccccc}
 & s(-, +, -) & \xrightarrow{(6)} & s(-, +, +) & \\
 \swarrow(2) & \downarrow(1)(7) & & \nearrow(4) & \downarrow(12) \\
 s(-, -, -) & \xrightarrow{(7)} & s(-, -, +) & & \\
 & \downarrow(3) & & \downarrow(5)(10) & \\
 & s(+, +, -) & \xrightarrow{(9)} & s(+, +, +) & \\
 \downarrow(8) & & & \downarrow(11) & \\
 s(+, -, -) & \xrightarrow{(8)} & s(+, -, +) & &
 \end{array}$$

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$

(5) ι_1

(6) ι_0

(7) ι_1

(8) ι_0

(9) ι_0

$$(10) \quad \iota_0$$

$$(11) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$$

$$(12) \quad \iota_1$$

To prove that microlocal stalk vanishes everywhere, by lemma??, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, +, -)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, +) & \longrightarrow & F^3(+, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{(*)} & \mathbb{C}^{n-1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-1} & \xrightarrow{(**)} & \mathbb{C}^n
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-1} & \xrightarrow{\iota_0} & \mathbb{C}^n
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_0 & & \downarrow (*) \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$

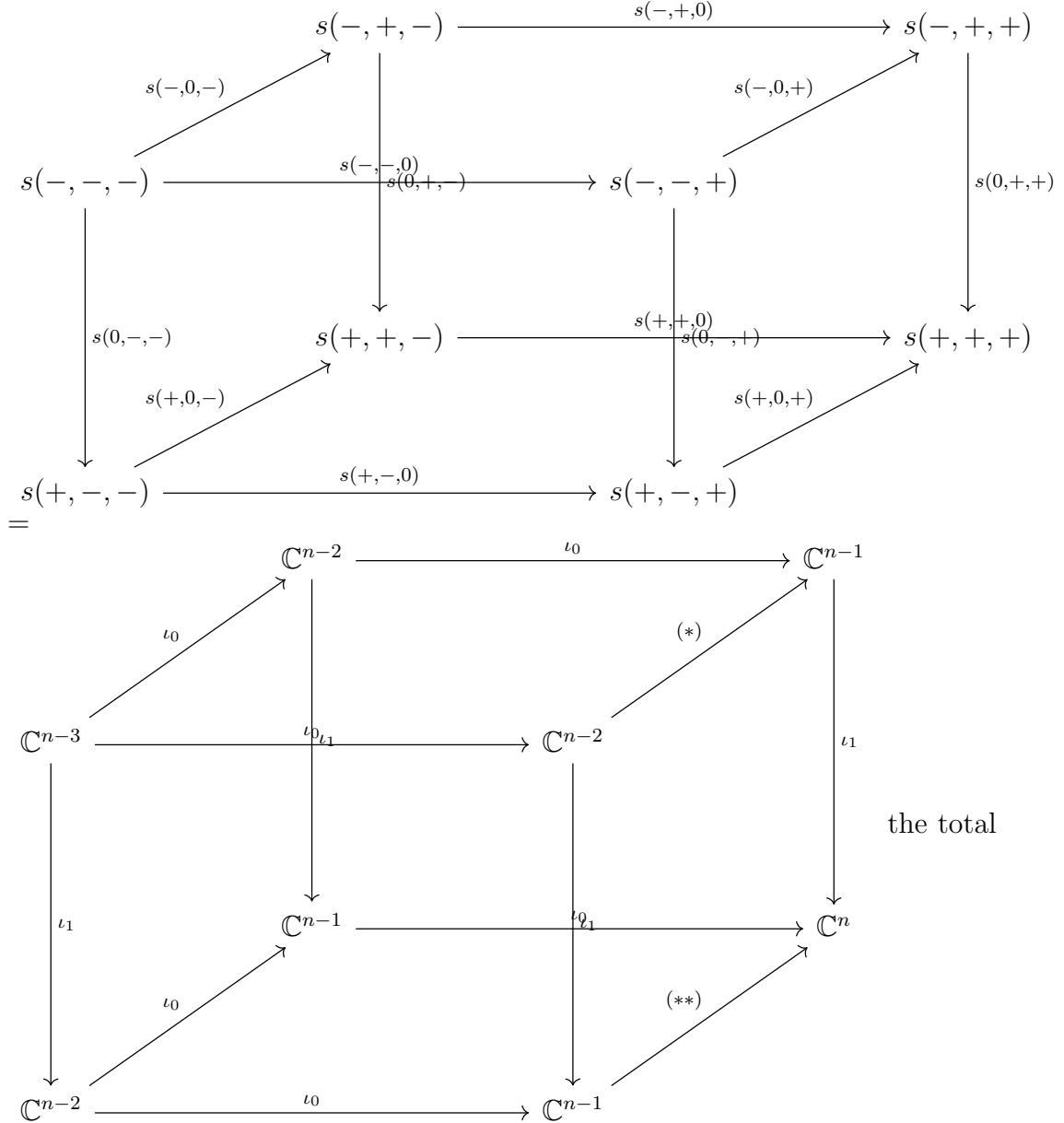
$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} =
 \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1} \\
 \downarrow \iota_0 & & \downarrow (**)
 \end{array} \text{ where}$$

$$\bullet \quad (*) = \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$$

$$\bullet \quad (**) = \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n, n-1}$$

The total complexes of $i - vi$ are acyclic because they are cartesian.

(vii) the cubic diagram:



complex is acyclic because by i, \dots, vi , the total complexes of the square diagrams are acyclic and the configuration of three planes

- $\mathbb{C}^{n-1} \xrightarrow{\text{iota}_0} \mathbb{C}$

- $\mathbb{C}^{n-1} \xrightarrow{\text{iota}_1} \mathbb{C}$

- $\mathbb{C}^{n-1} \xrightarrow{(**)} \mathbb{C}$

corresponding to $s(+, +, -,), s(-, +, +), s(+, -, +)$ are in general position.

□

3.9.5 Sheaf at the End

In this subsection, I will describe the sheaf \mathcal{F}_1 at the end of the *cobord*₂. By Mayer-Vietoris, $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V , say \mathcal{F}_V
- a sheaf on $D_{r=2}$, say $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathcal{F}_{V \times [0,1]}$ to $V \times \{1\}$, i.e. $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathcal{F}_{D_{r=2} \times [0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of F_\bullet defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

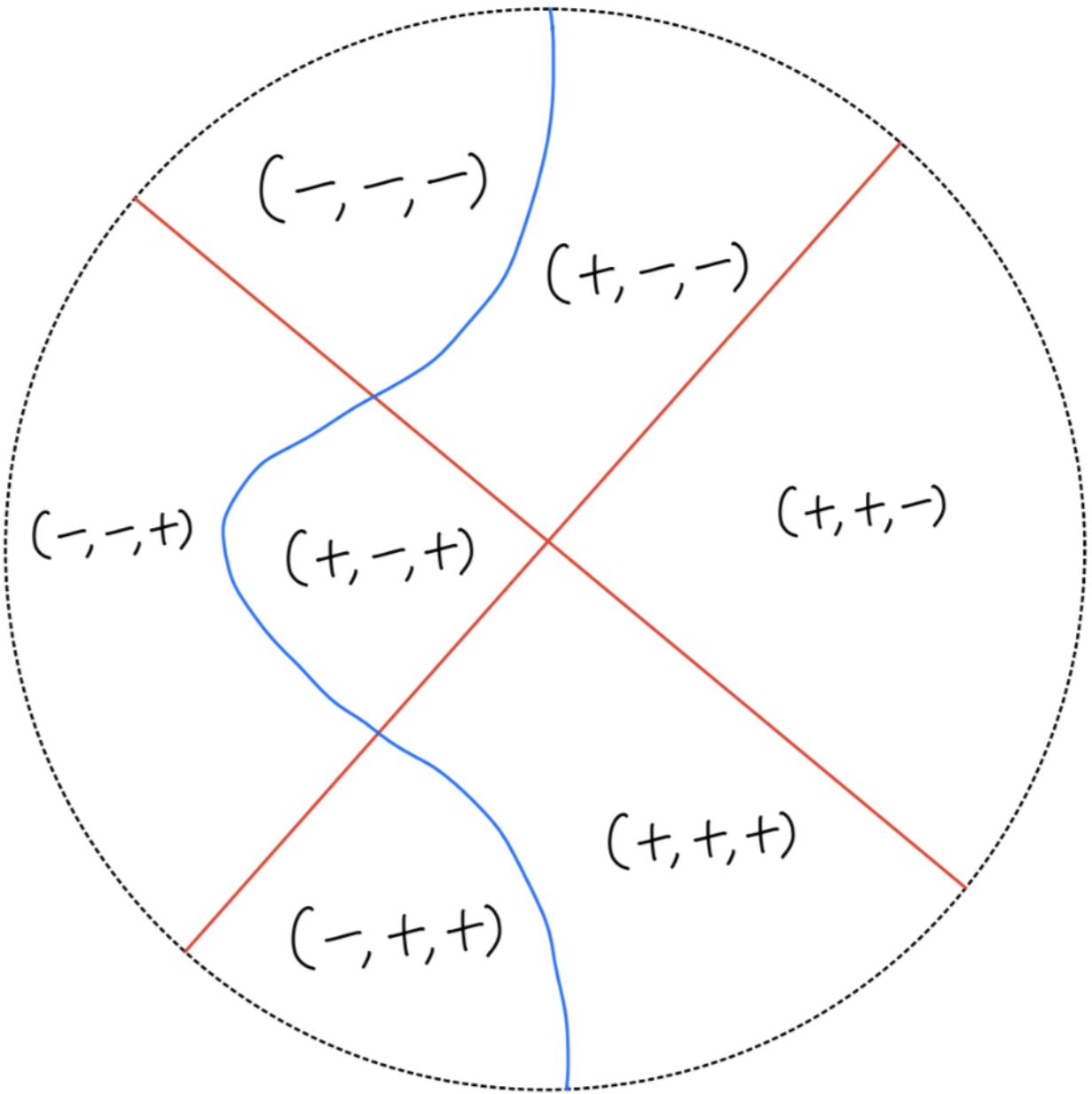


Figure 3.254

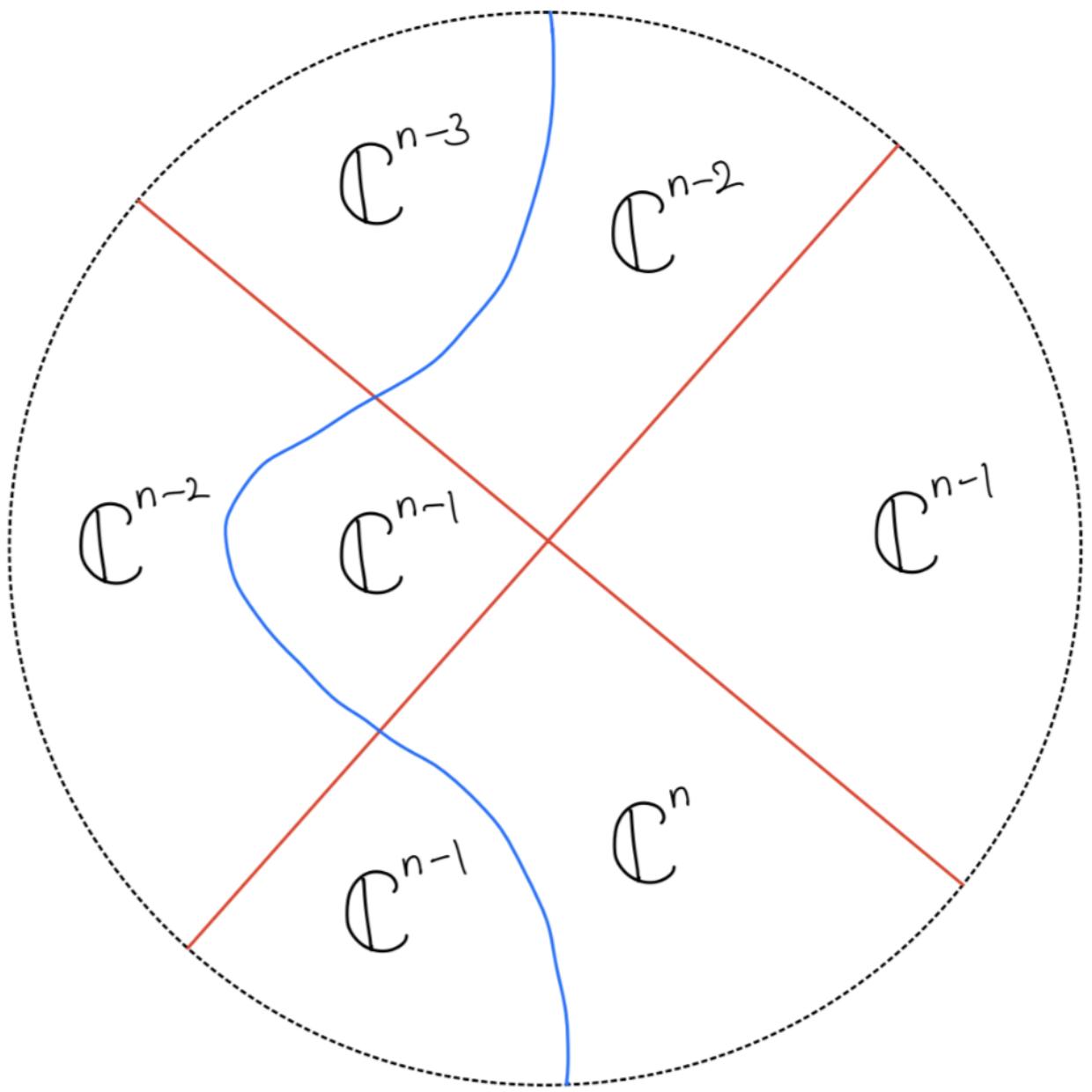


Figure 3.255

- $F_1(-, -, -) := \mathbb{C}^{n-3}$

- $F_1(+, -, -) := \mathbb{C}^{n-2}$

- $F_1(-, -, +) := \mathbb{C}^{n-2}$

- $F_1(+, -, +) := \mathbb{C}^{n-1}$

- $F_1(+, +, -) := \mathbb{C}^{n-1}$

- $F_1(-, +, +) := \mathbb{C}^{n-1}$

- $F_1(+, +, +) := \mathbb{C}^n$

Generalization maps:

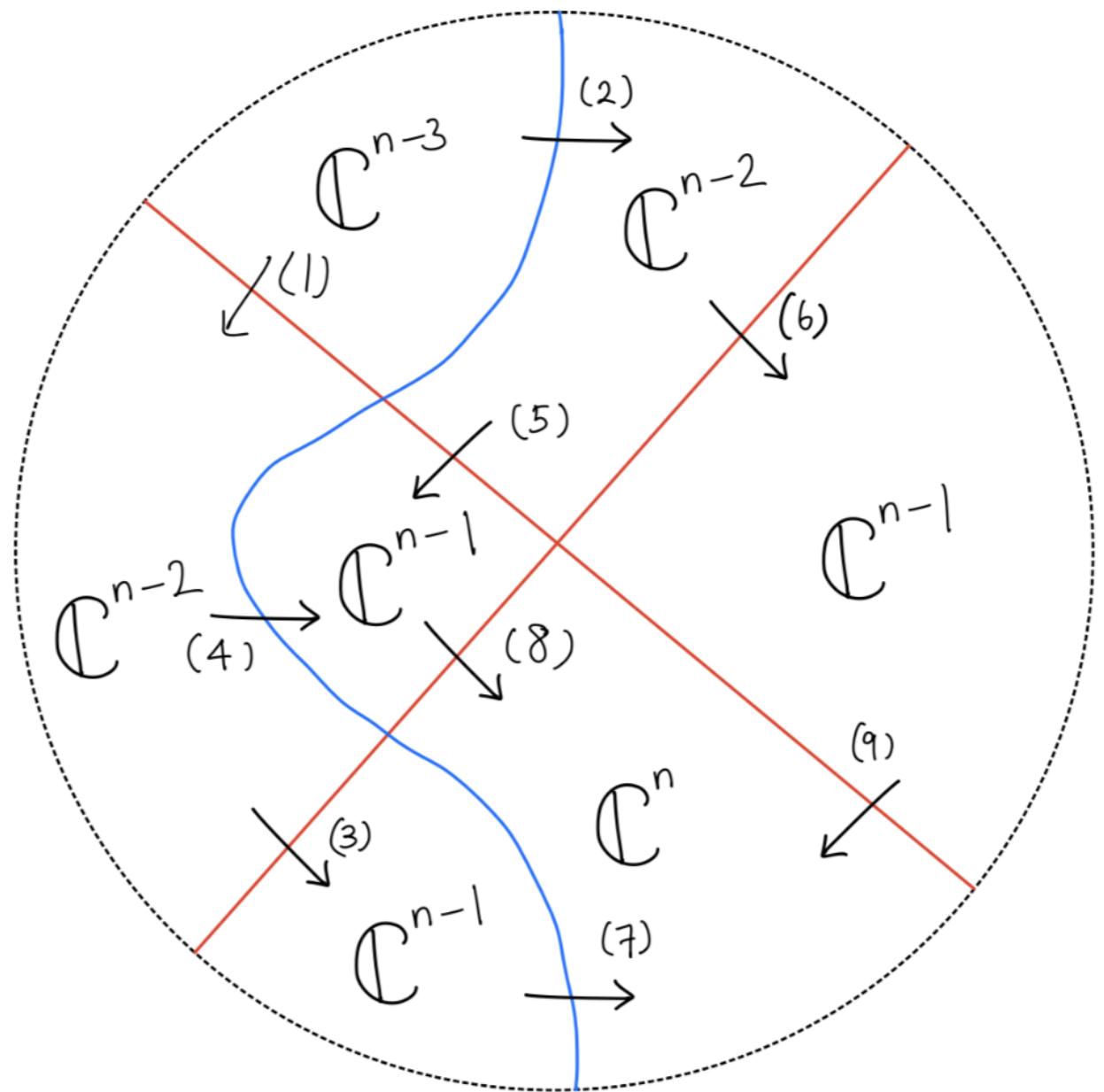


Figure 3.256

(1) ι_0

(2) ι_1

(3) ι_0

(4) ι_1

(5) ι_0

(6) ι_0

(7) ι_1

(8) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(9) ι_0

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$ is described as follows.

Definition 215. we define γ_1 to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathcal{F}_0|_{U \cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0,1] \twoheadrightarrow (U \cap V)$$

3.10 5th Sheaf Cobordism

Suppose we have a punctured Riemann sphere M and $\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{squig}$, a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-1, 1)_x \times (-n-1, 2)_z$ under f

- Λ_0^0 gets mapped to $\{(x, z) \in R \mid z = 1\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup_{k=1}^n \{(x, z) \in R \mid z = -k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x, z) \in R \mid x = 0\}$, co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

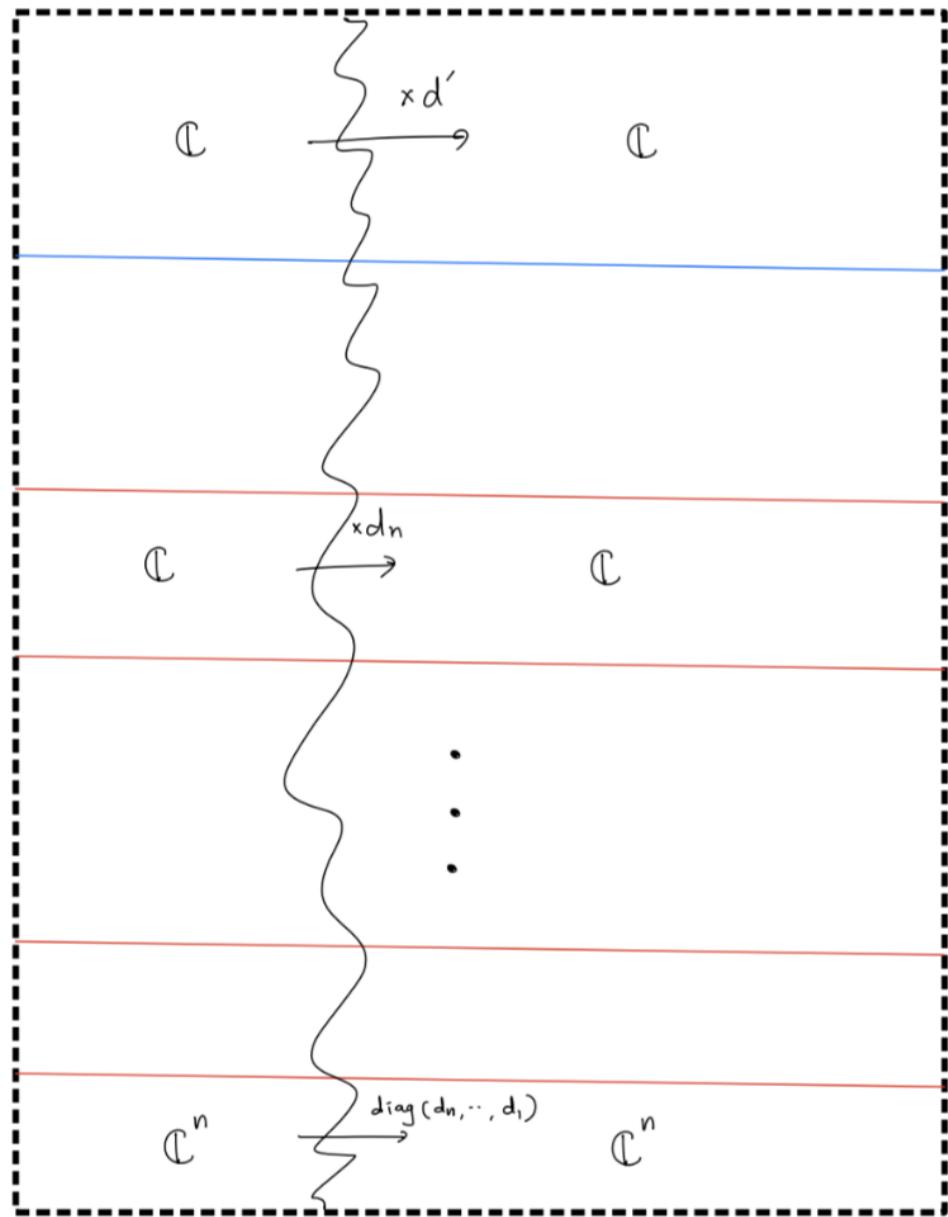


Figure 3.257

Then we define a cobordism starting from the above sheaf, say $cobord_5(n)$ supported on U . At the end of the cobordism, the sheaf, under the same chart f , is described as the following squiggly legible diagram.

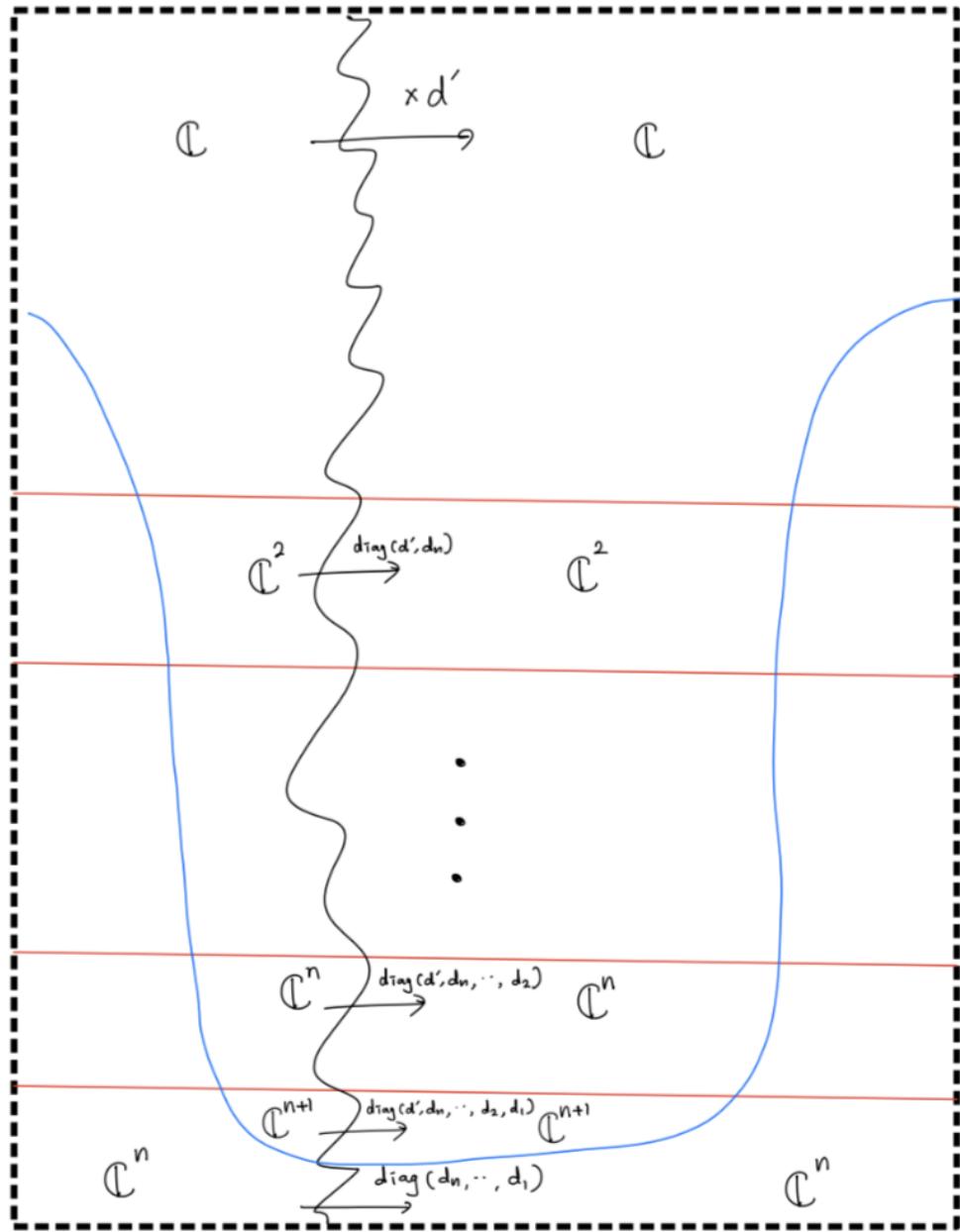


Figure 3.258

We define $cobord_5(n)$ inductively as follows.

- (i) For $n = 1$, we define $cobord_5(1)$ to be $cobord_1$ from

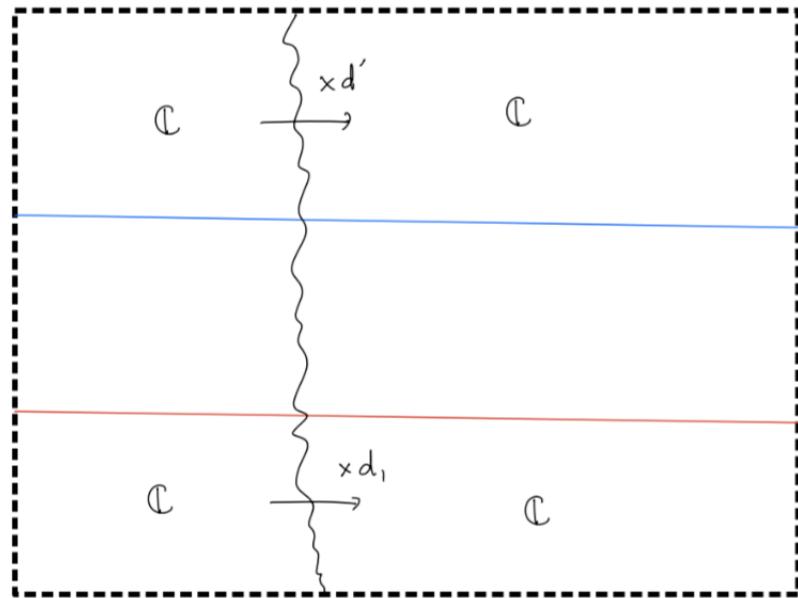


Figure 3.259

to

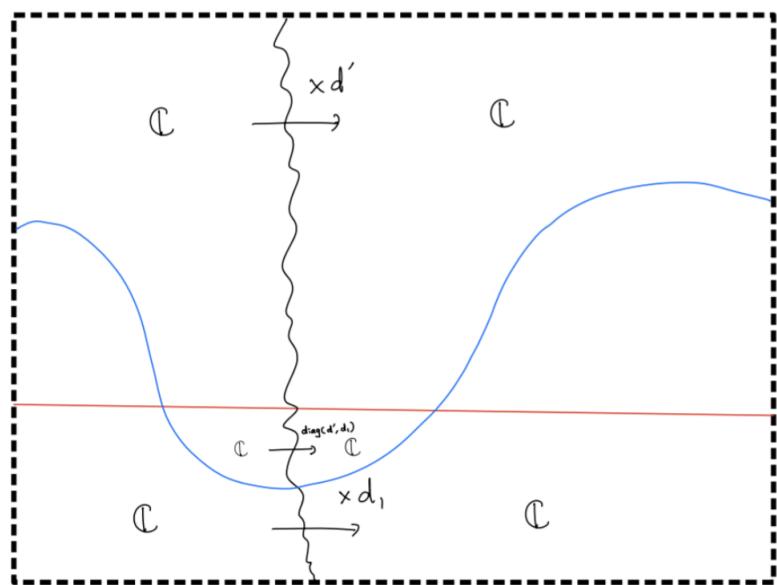


Figure 3.260

(ii) For $n > 0$,

(Step 1) we first apply $cobord_5(n - 1)$ to the square region surrounded by a purple

dotted line.

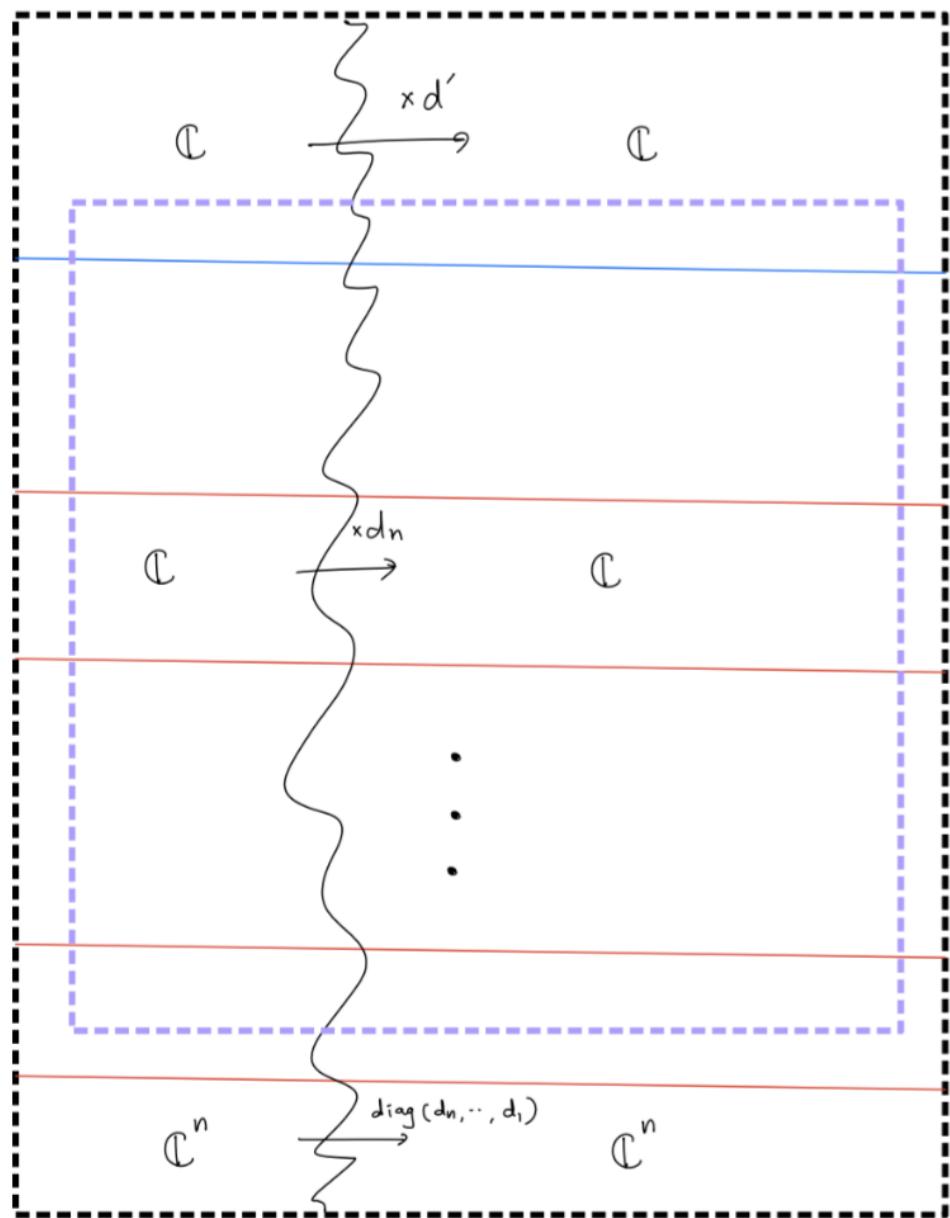


Figure 3.261

by induction hypothesis, we get

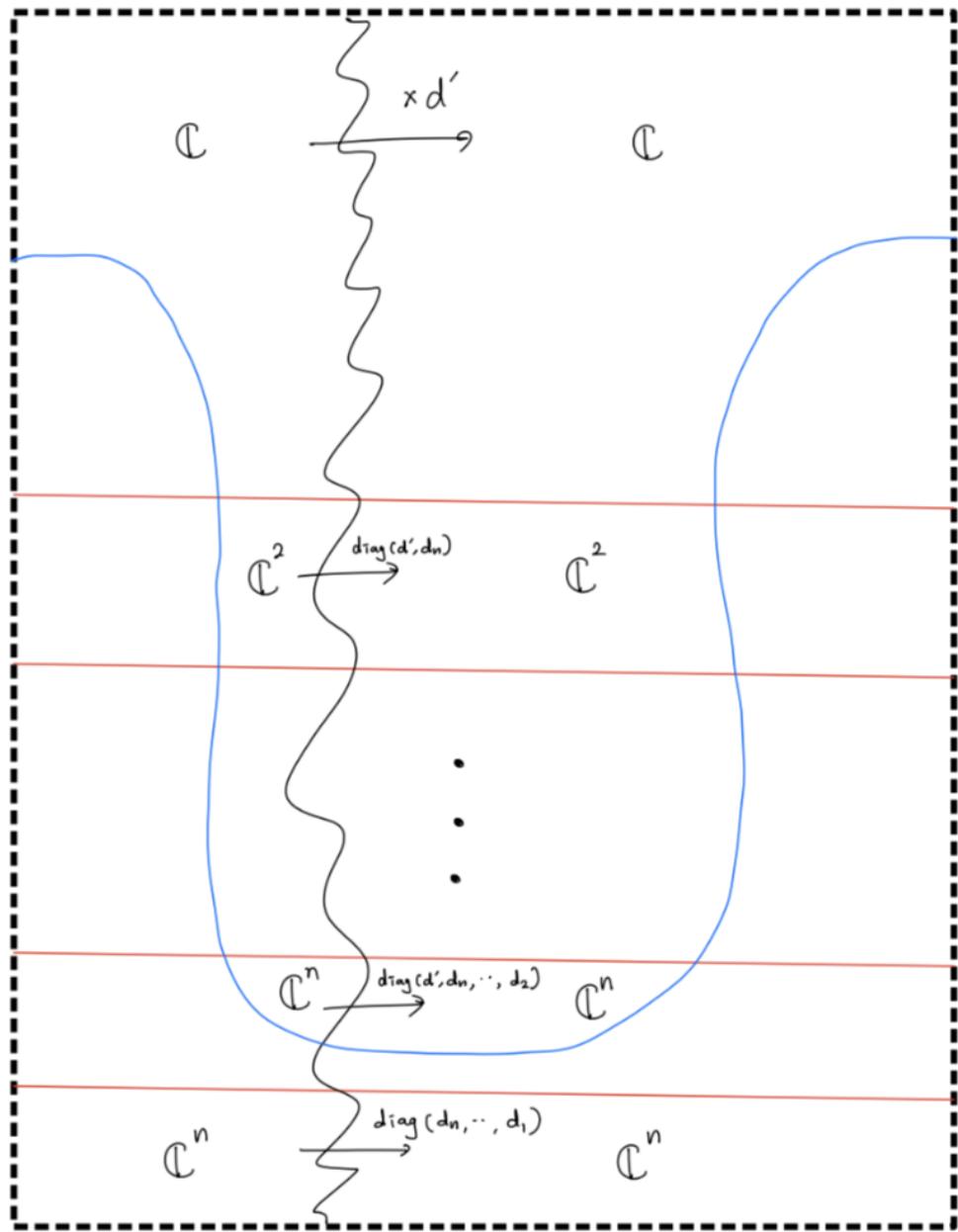


Figure 3.262

(Step 2) apply $cobord_1$ to the square region surrounded by purple dotted lines.

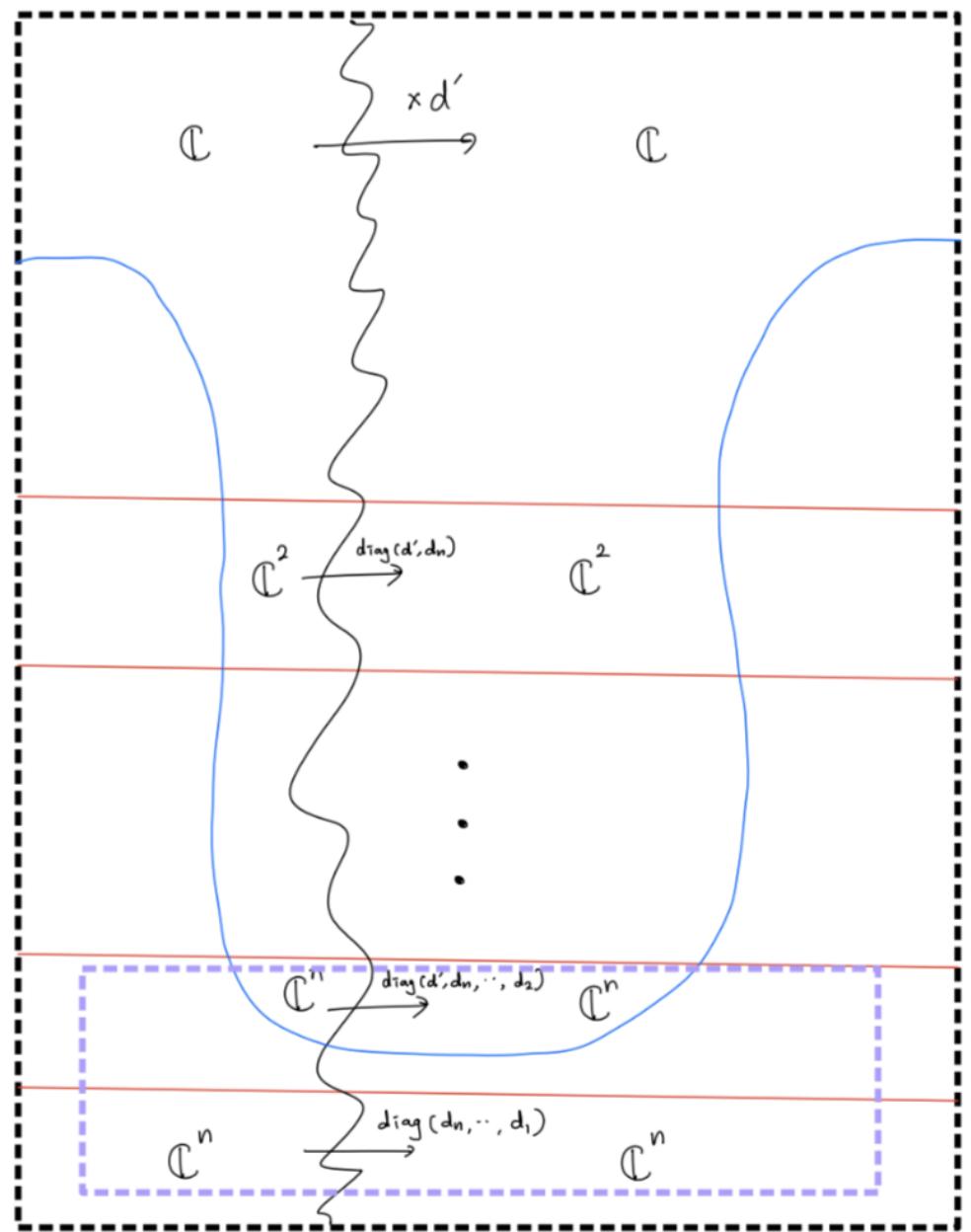


Figure 3.263

we get the final sheaf

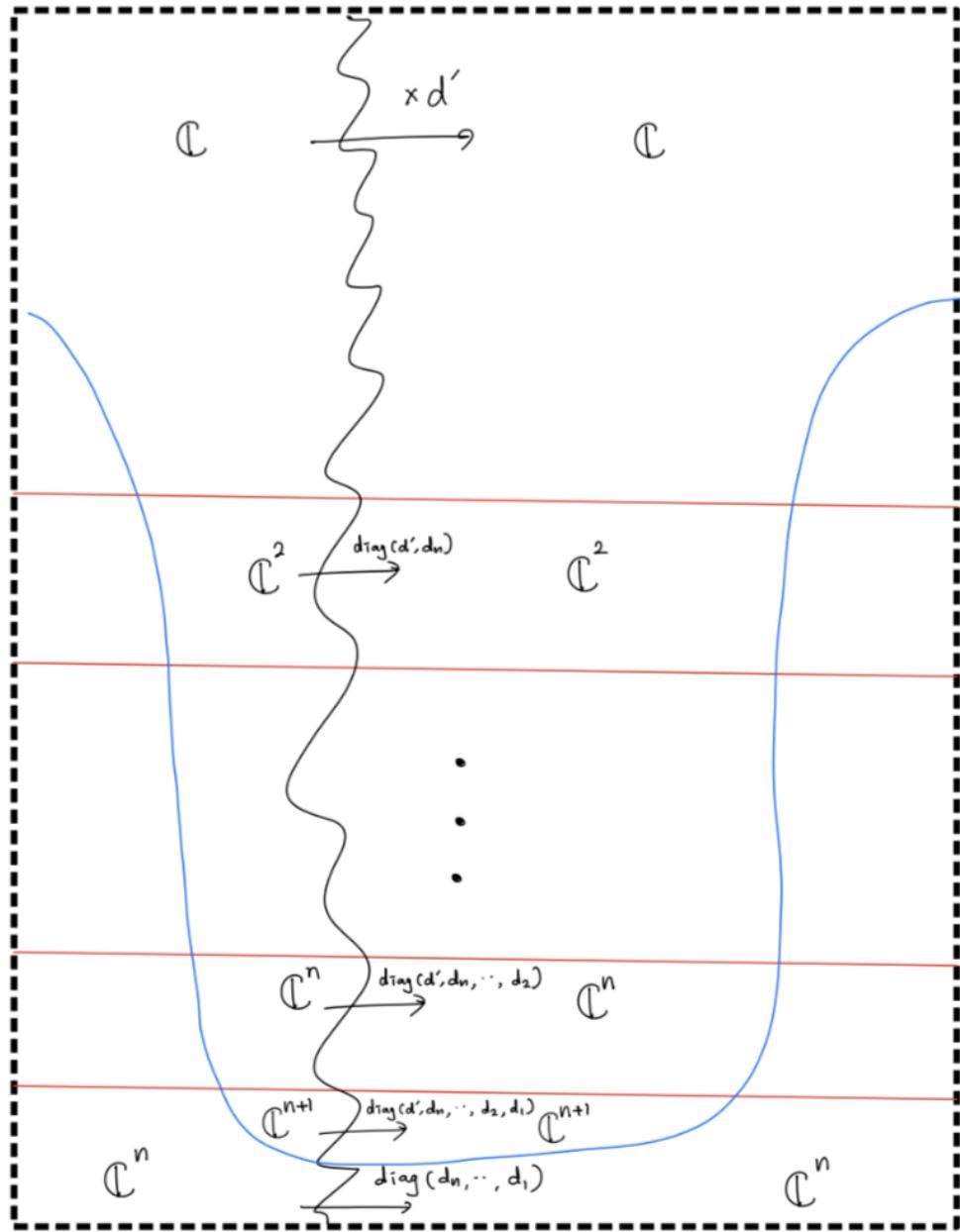


Figure 3.264

3.11 6th Sheaf Cobordism

Suppose we have a punctured Riemann sphere \$M\$ and \$\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{squig}\$, a nested regions \$U \subset U' \subset M\$, and a chart \$f : U \rightarrow \mathbb{R}^2\$ such that \$U'\$ maps to \$R := (-1, 1)_x \times (-2, n+1)_z\$ under \$f\$

- Λ_0^0 gets mapped to $\bigcup_{k=1}^n \{(x, z) \in R \mid z = k\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\{(x, z) \in R \mid z = -1\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x, z) \in R \mid x = 0\}$, co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

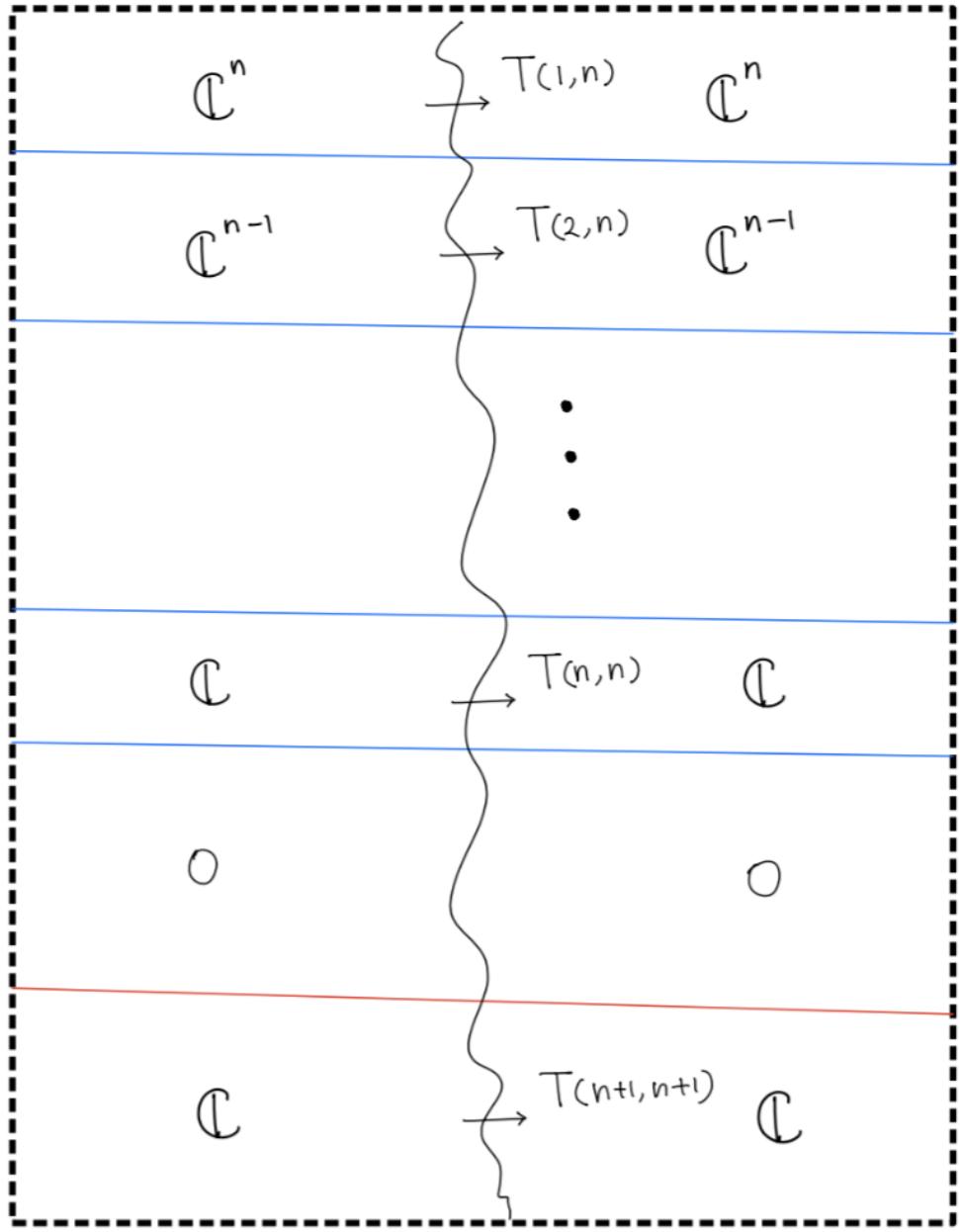


Figure 3.265

where $T \in M^{(n+1) \times (n+1)}$ such that preserves the following flags

- $\mathbb{C} \subset_{\iota_0} \cdots \subset_{\iota_0} \mathbb{C}^n$
- $\mathbb{C} \subset_{\iota_1} \mathbb{C}^n$

Then we define a cobordism starting from the above sheaf, say $cobord_6(n)$ supported on U , where n is the number of blue strands. At the end of the cobordism, the

sheaf, under the same chart f , is described as the following squiggly legible diagram.

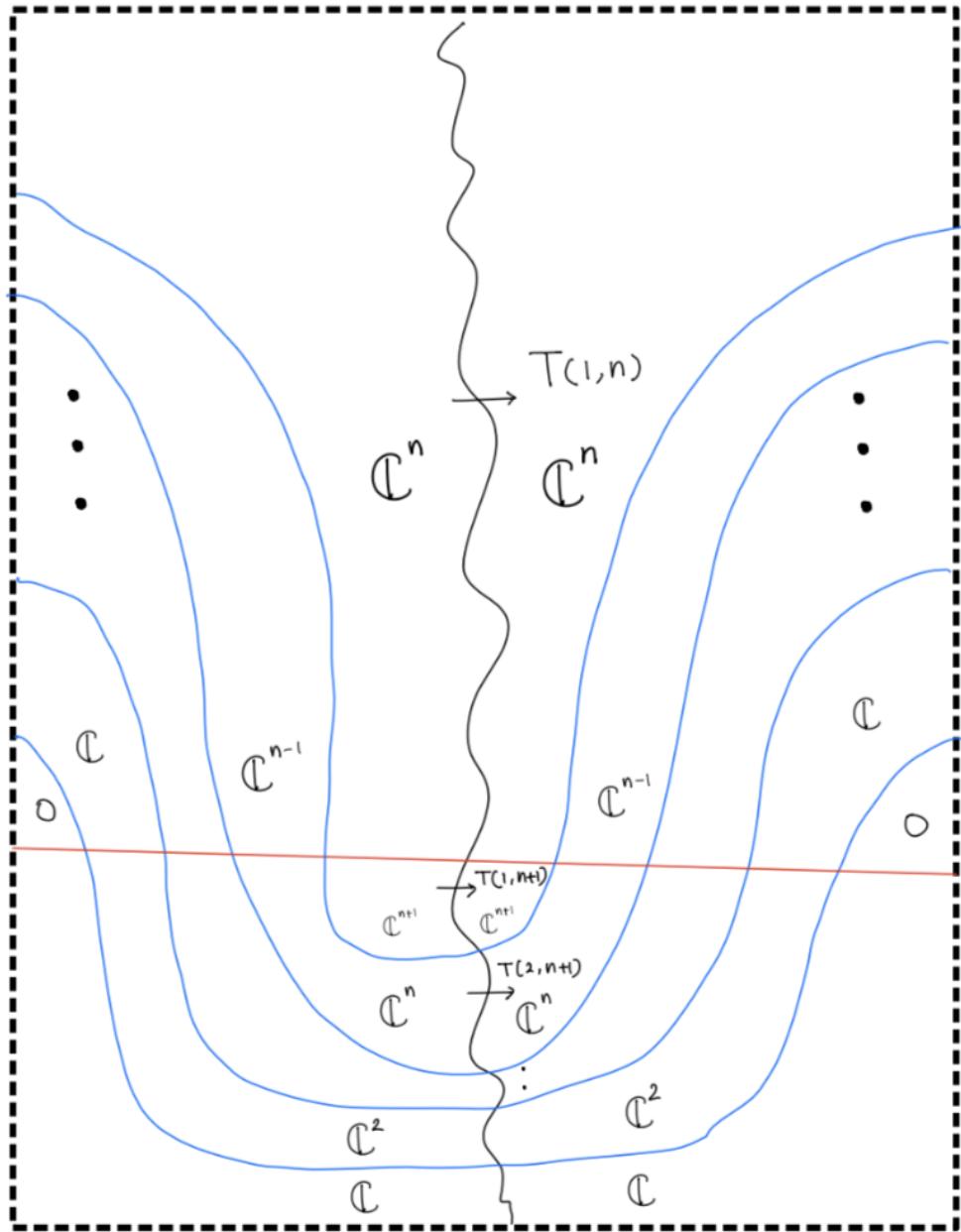


Figure 3.266

We define $cobord_6(n)$ inductively as follows.

- (i) For $n = 1$, we define $cobord_6(1)$ to be $cobord_1$ from

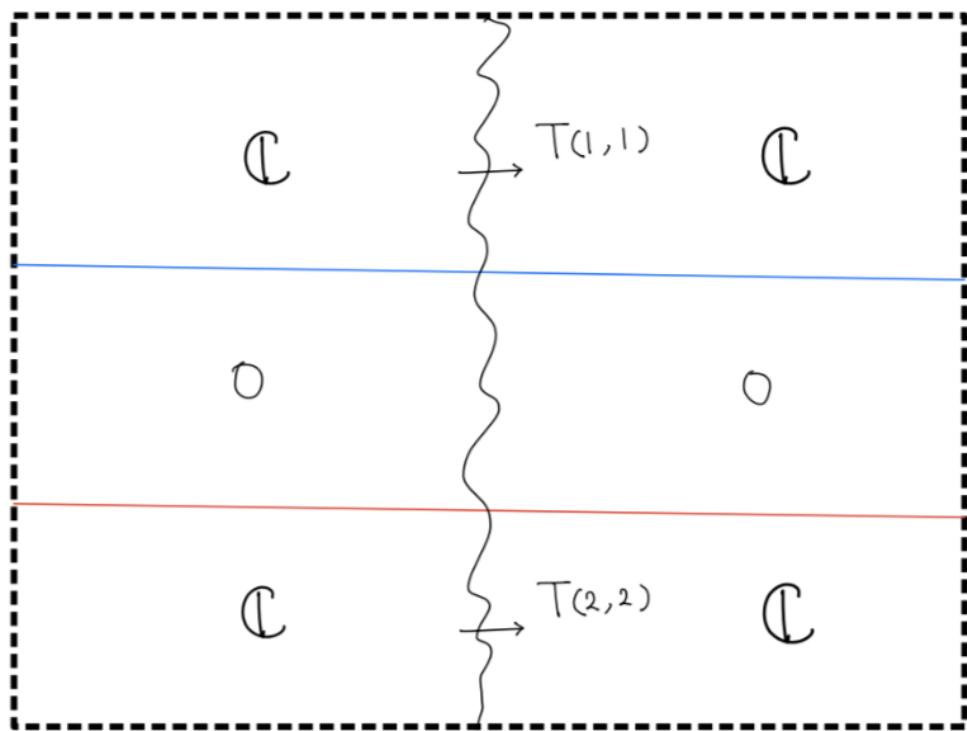


Figure 3.267

to

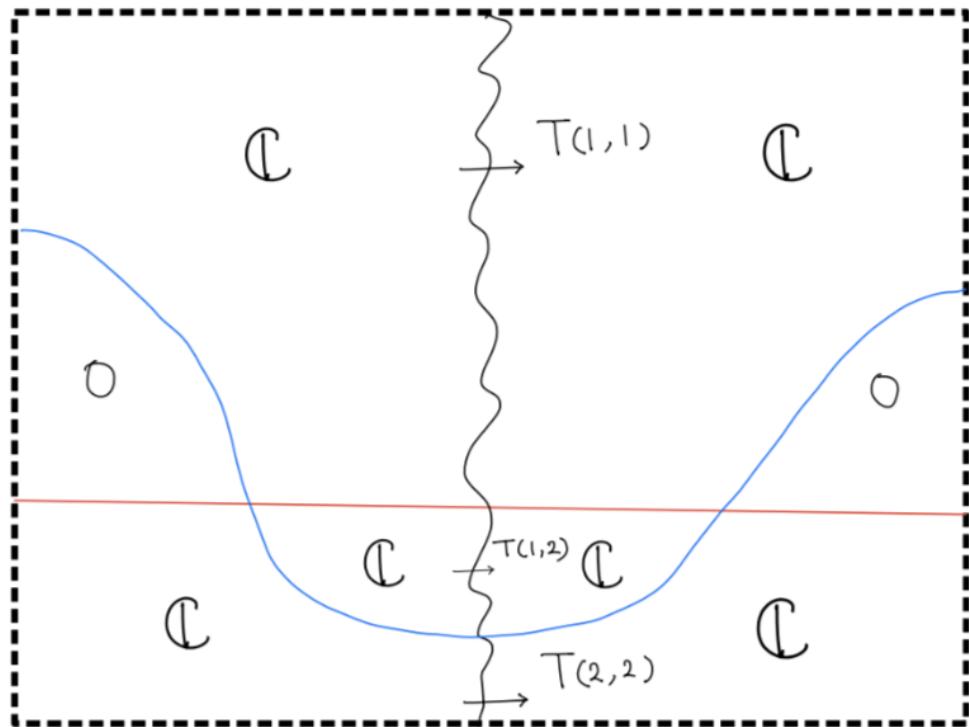


Figure 3.268

(ii) For $n > 0$,

(Step 1) we first apply $cobord_6(n - 1)$ to the square region surrounded by purple dotted lines.

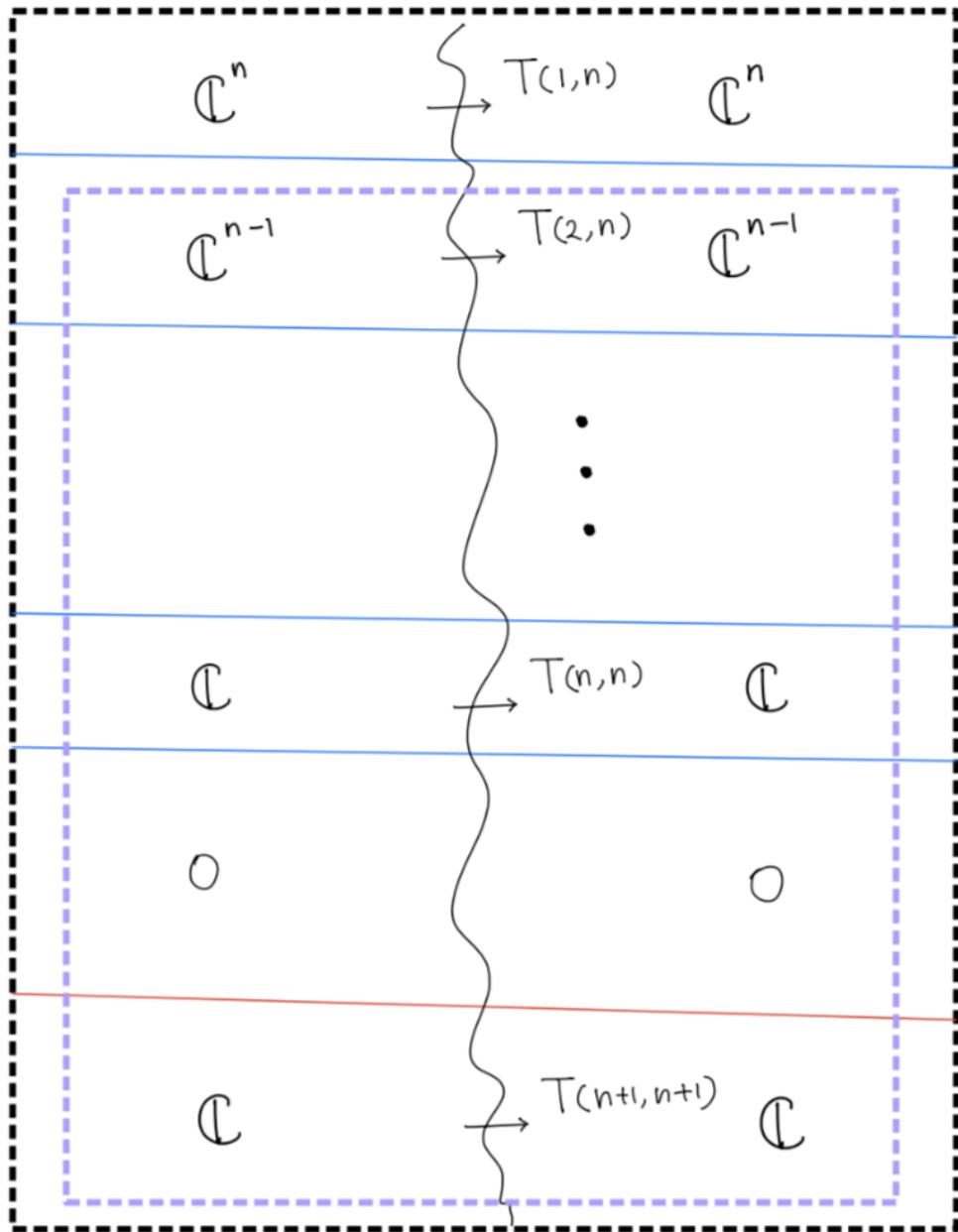


Figure 3.269

by induction hypothesis, we get

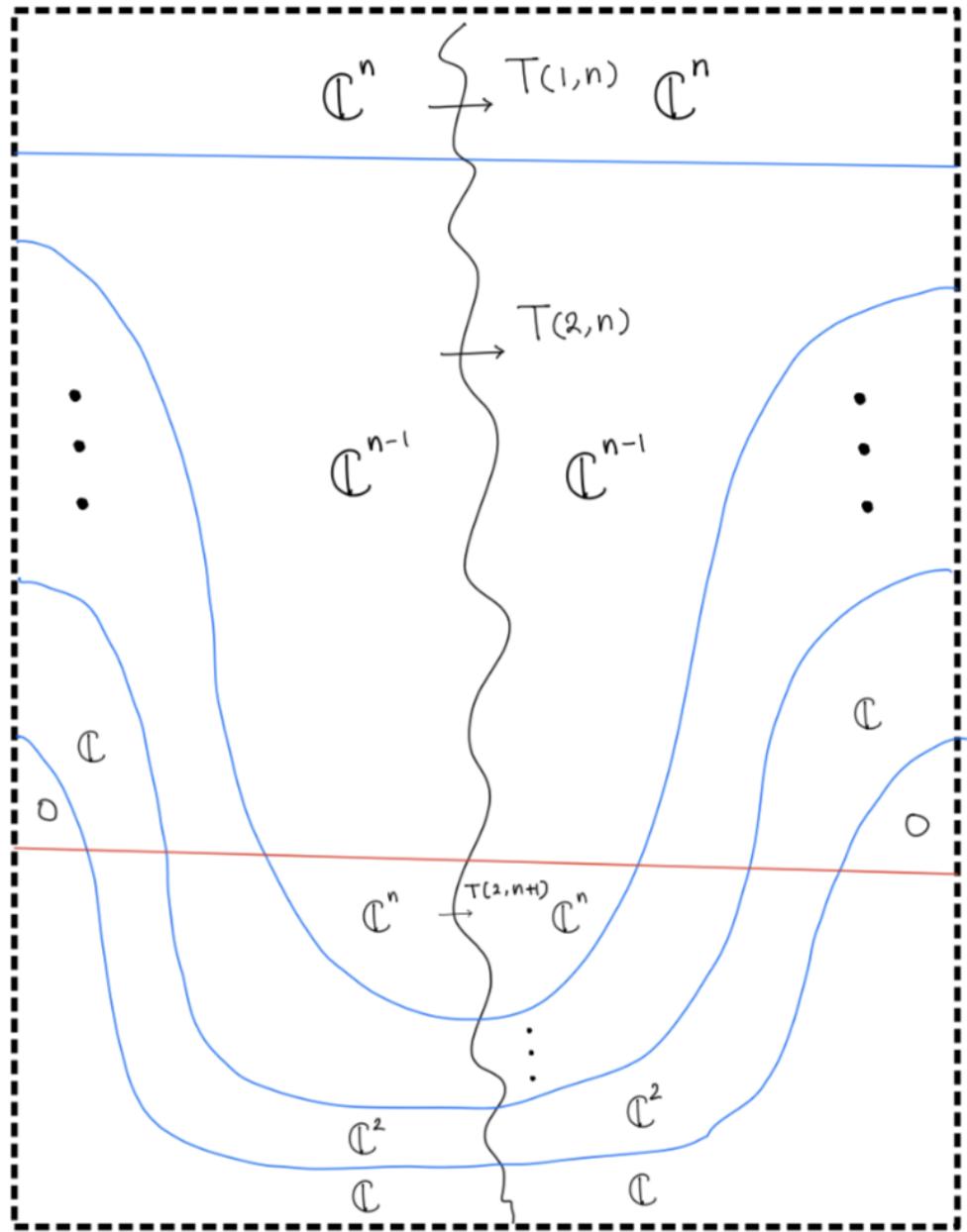


Figure 3.270

(Step 2) apply $cobord_1$ to the square region surrounded by purple dotted lines.

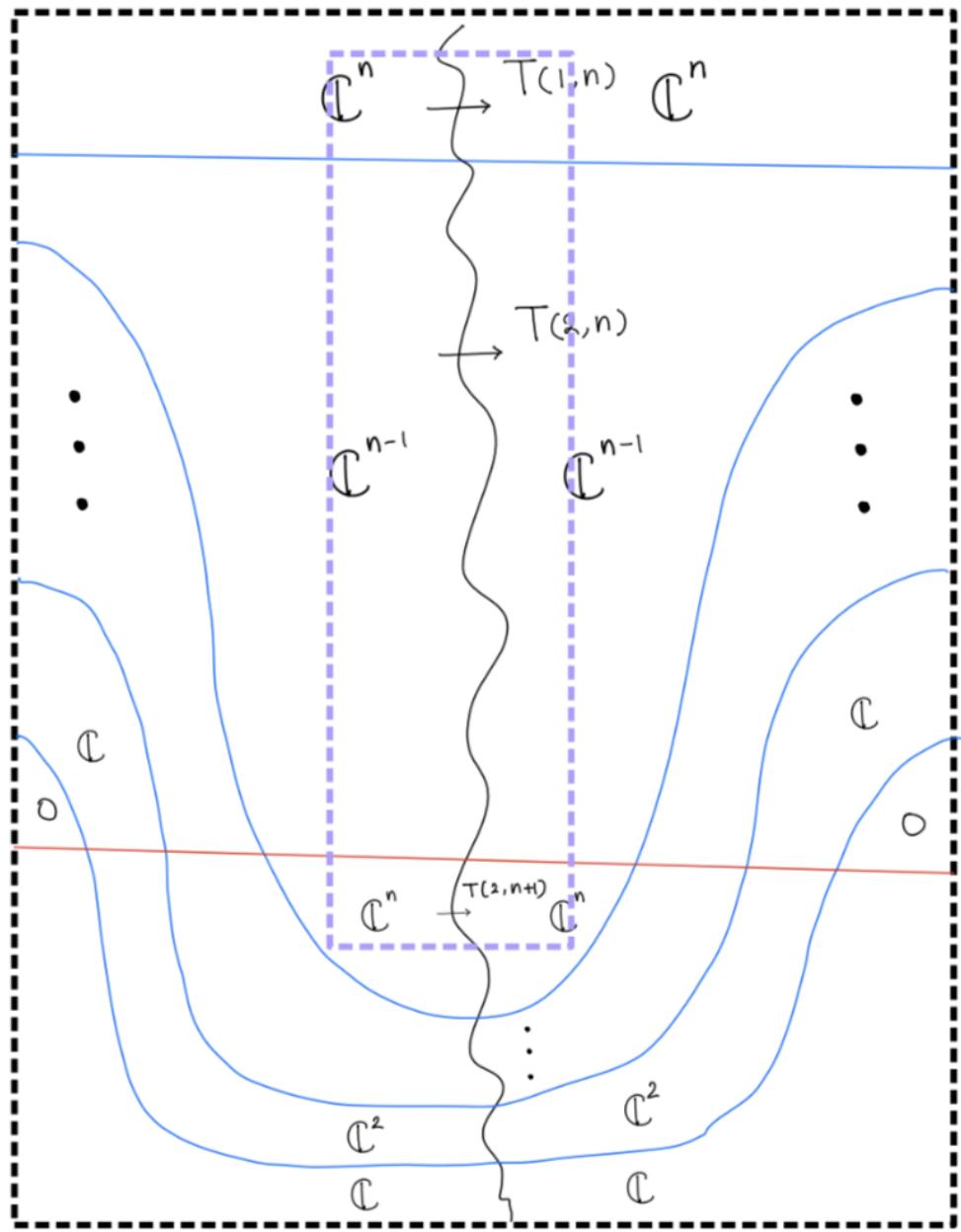


Figure 3.271

we get the final sheaf

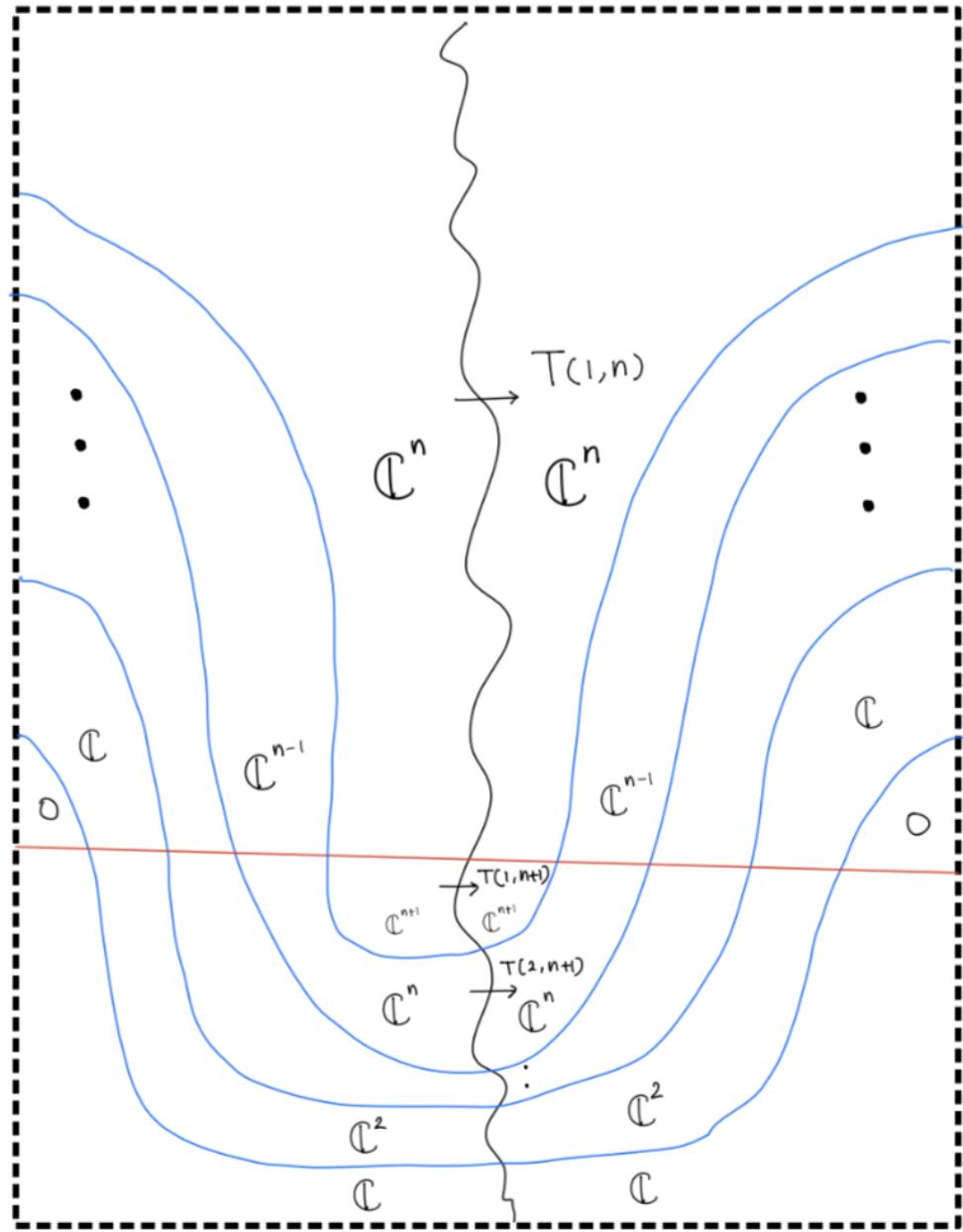


Figure 3.272

3.12 7th Sheaf Cobordism

Suppose we have a punctured Riemann sphere M and $\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{sqwig}$, a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-1, 1)_x \times (0, n+1)_z$ under f

- Λ_0^0 gets mapped to $\{(x, z) \in R \mid z = \Psi(\frac{1}{2}, n + \frac{1}{2})(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup_{k=1}^n \{(x, z) \in R \mid z = -k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to ϕ .

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

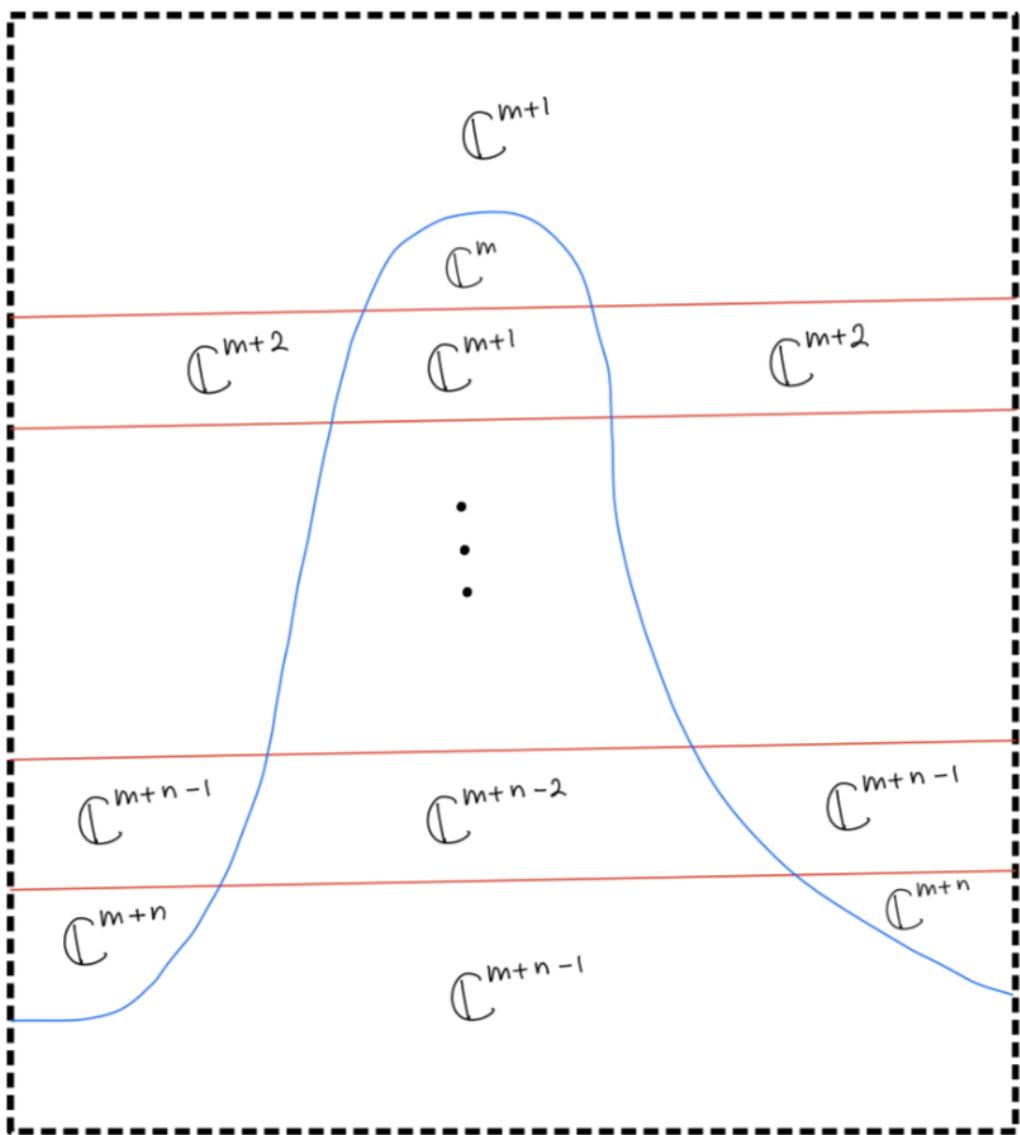


Figure 3.273

Then we define a cobordism starting from the above sheaf, say $cobord_5(n)$ supported on U , where n is the number of red strands. At the end of the cobordism , the sheaf, under the same chart f , is described as the following squiggly legible diagram.

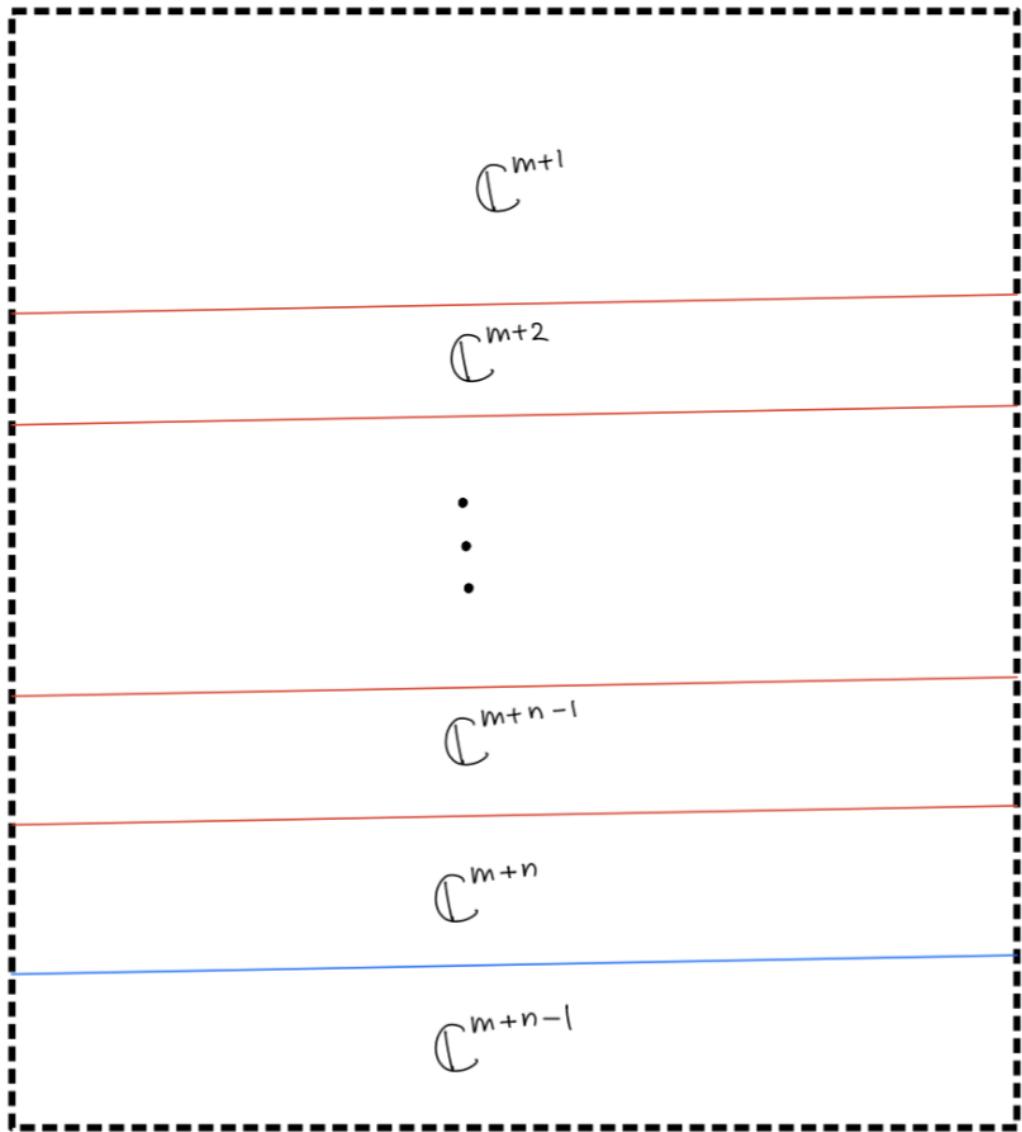


Figure 3.274

We define $cobord_7(n)$ inductively as follows.

- (i) For $n = 1$, we define $cobord_7(1)$ to be $cobord_3$ from

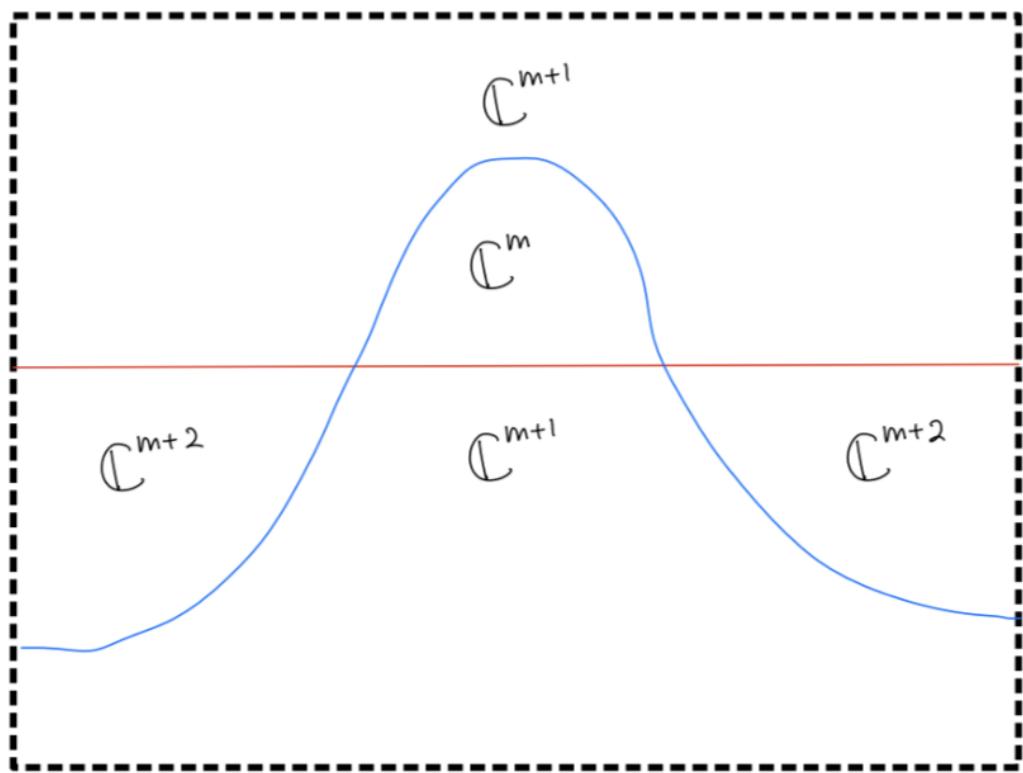


Figure 3.275

to

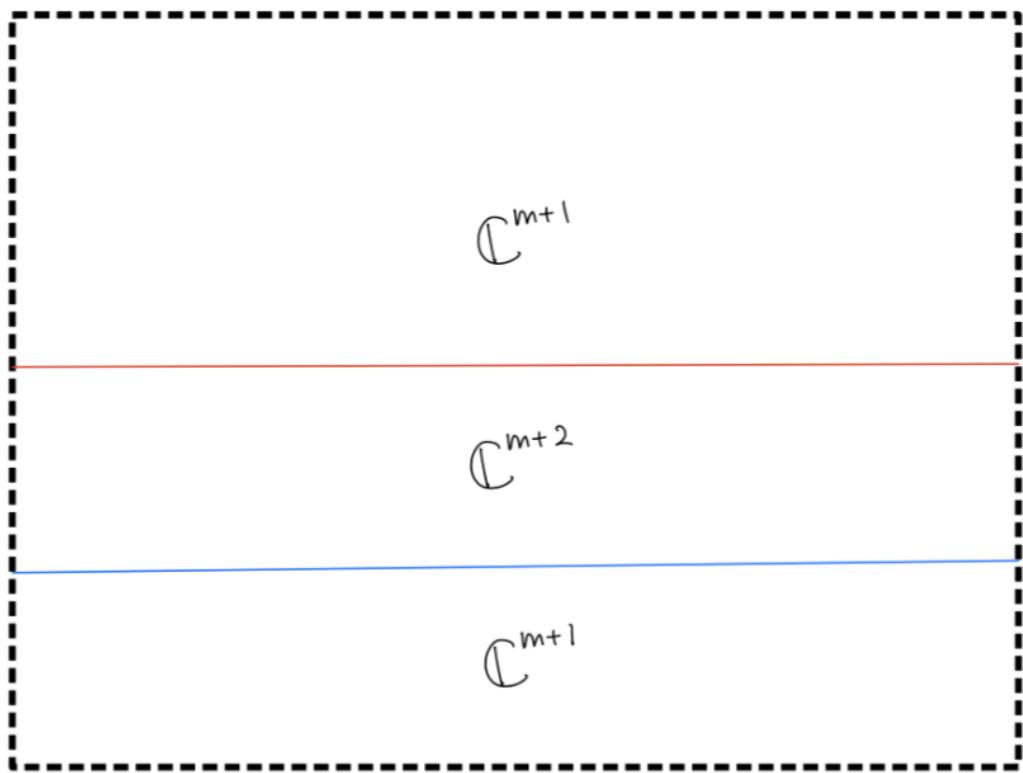


Figure 3.276

(ii) For $n > 0$,

(Step 1) we first apply $cobord_7(n - 1)$ to the square region surrounded by purple dotted lines.

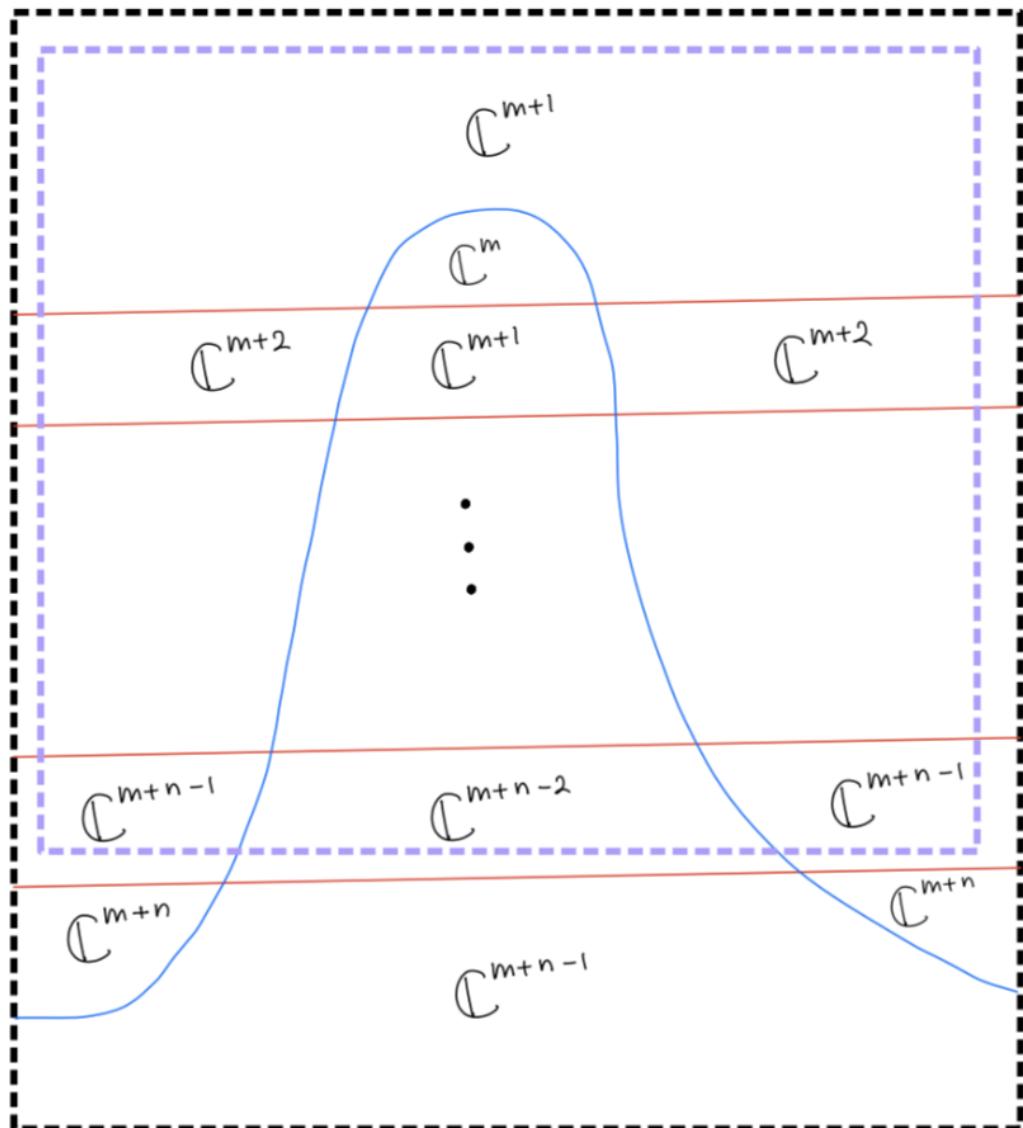


Figure 3.277

by induction hypothesis, we get

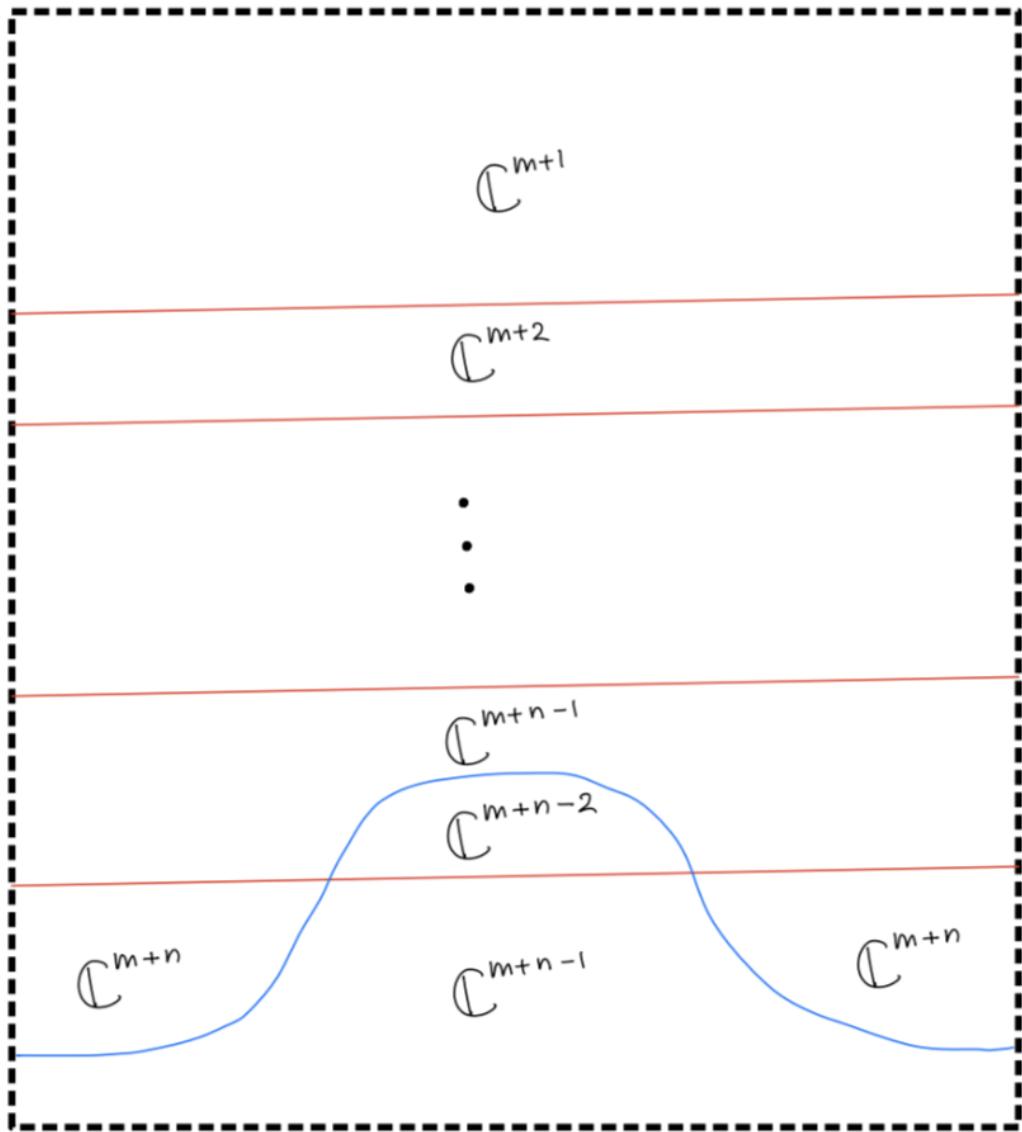


Figure 3.278

(Step 2) apply *cobord*₃ to the square region surrounded by purple dotted lines.

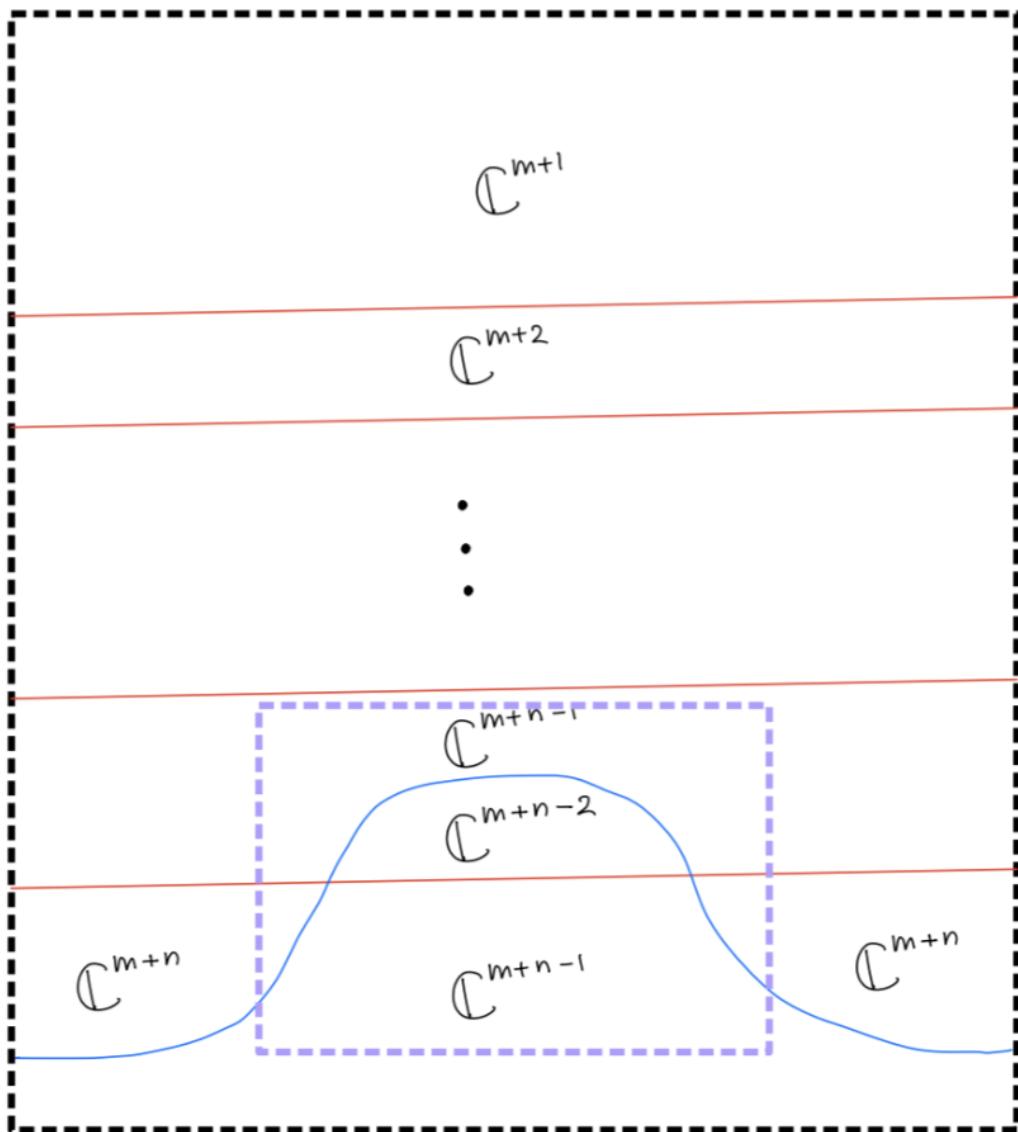


Figure 3.279

we get the final sheaf

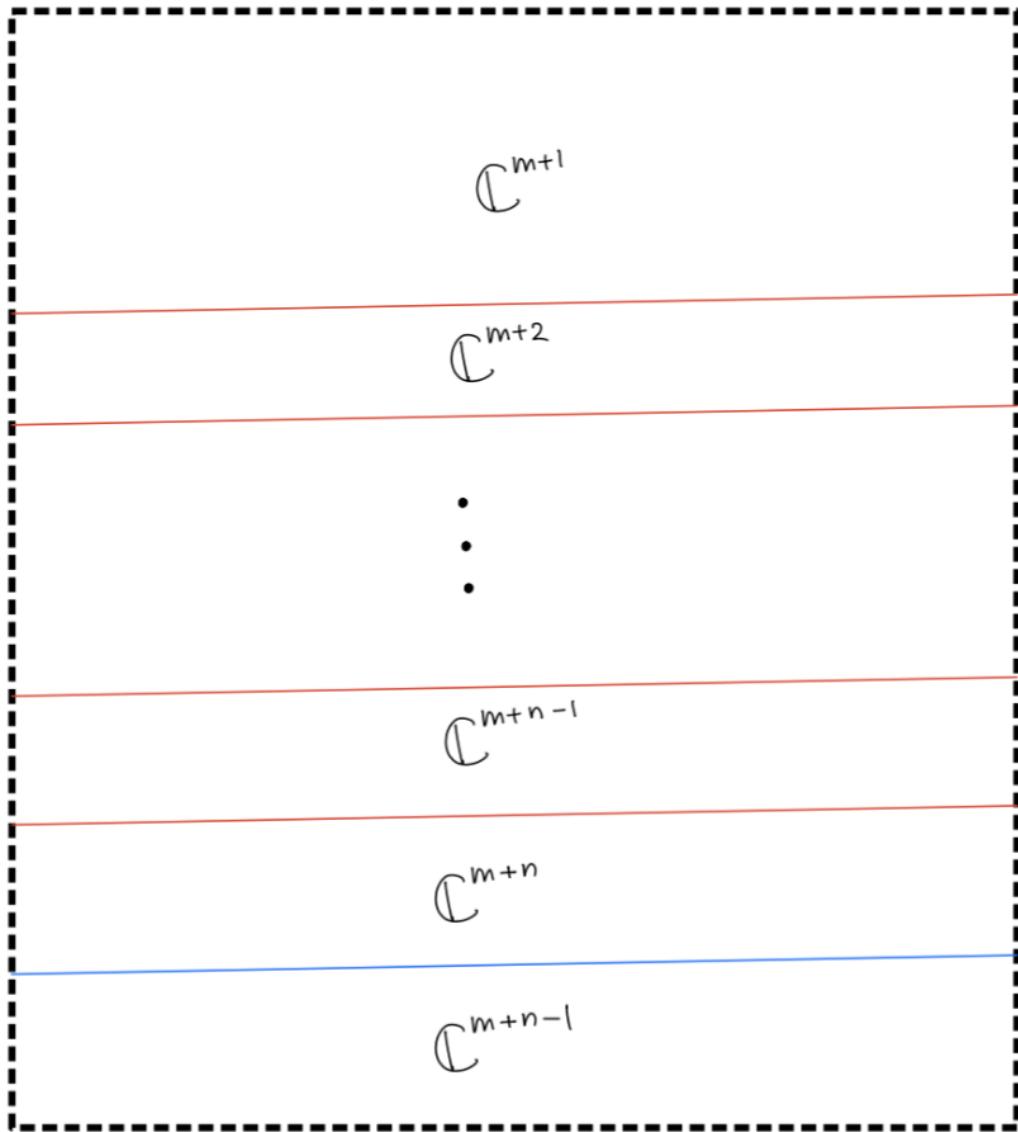


Figure 3.280

3.13 8th Sheaf Cobordism

Definition 216. (transition function) Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define a smooth transition function τ to be

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere M and Λ_0^0 , Λ_0^∞ , Λ_0^{squig} , a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-2, 2)_x \times (-1, 1)_z$ under f

- Λ_0^0 gets mapped to blue strands in the below figure, co-oriented upward.
- Λ_0^∞ gets mapped to red strands in the below figure, co-oriented downward.
- Λ_0^{squig} gets mapped to squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

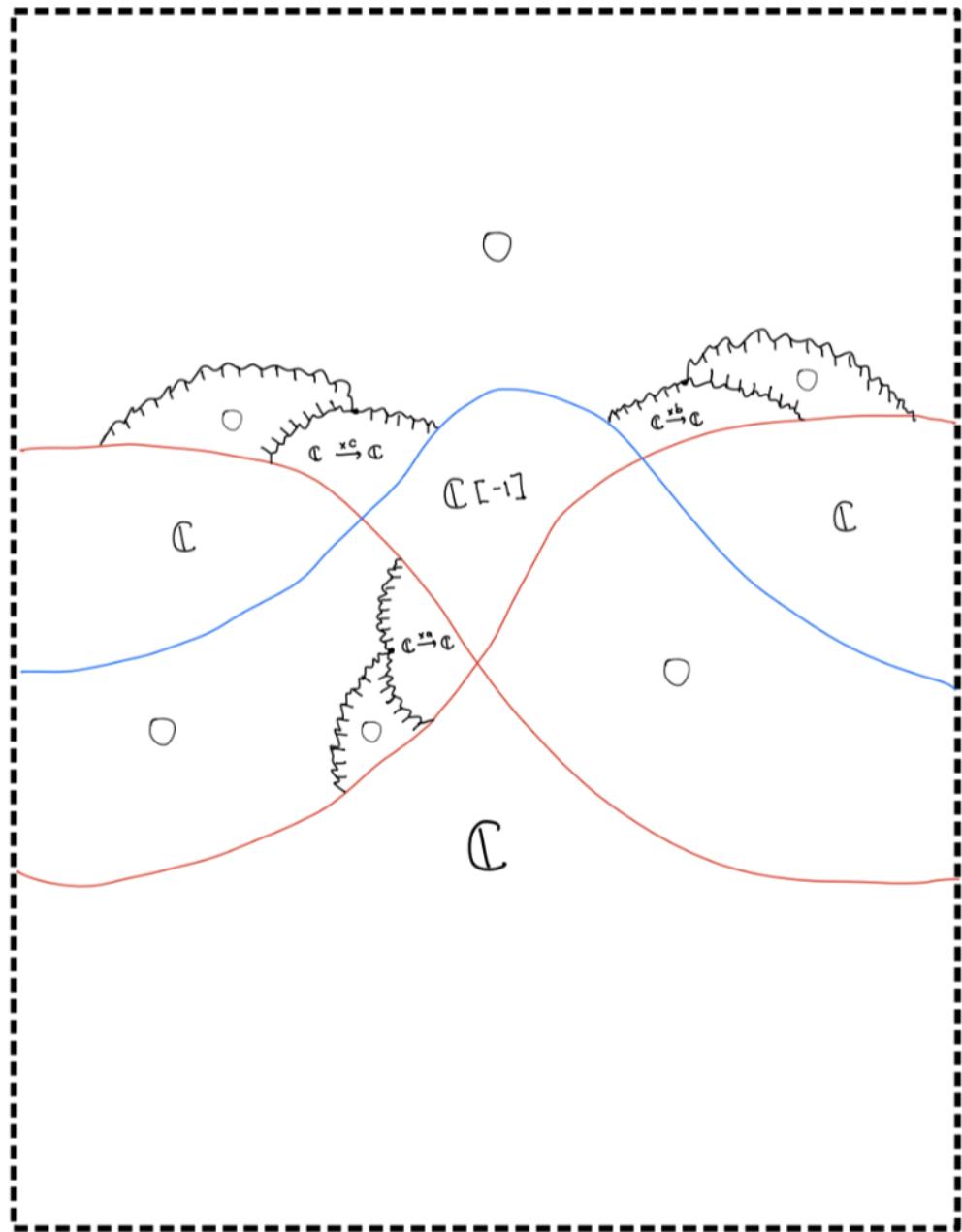


Figure 3.281

Then we define a cobordism starting from the above sheaf, say cobord_8 supported on U . At the end of the cobordism, the sheaf, under the same chart f , is described as the following squiggly legible diagram.

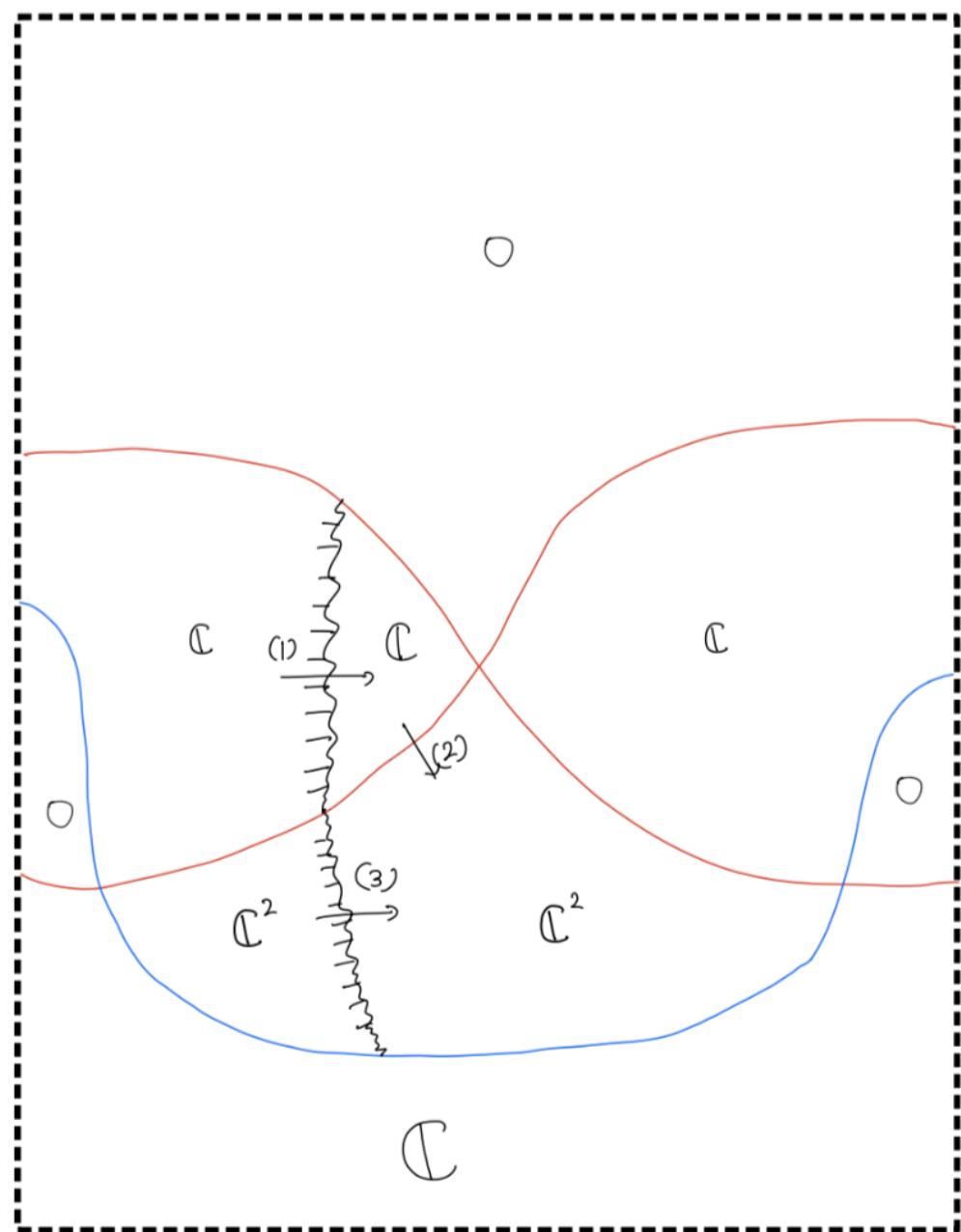


Figure 3.282

Generalizations maps:

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

$$(3) \begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$$

We define $cobord_8$ as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

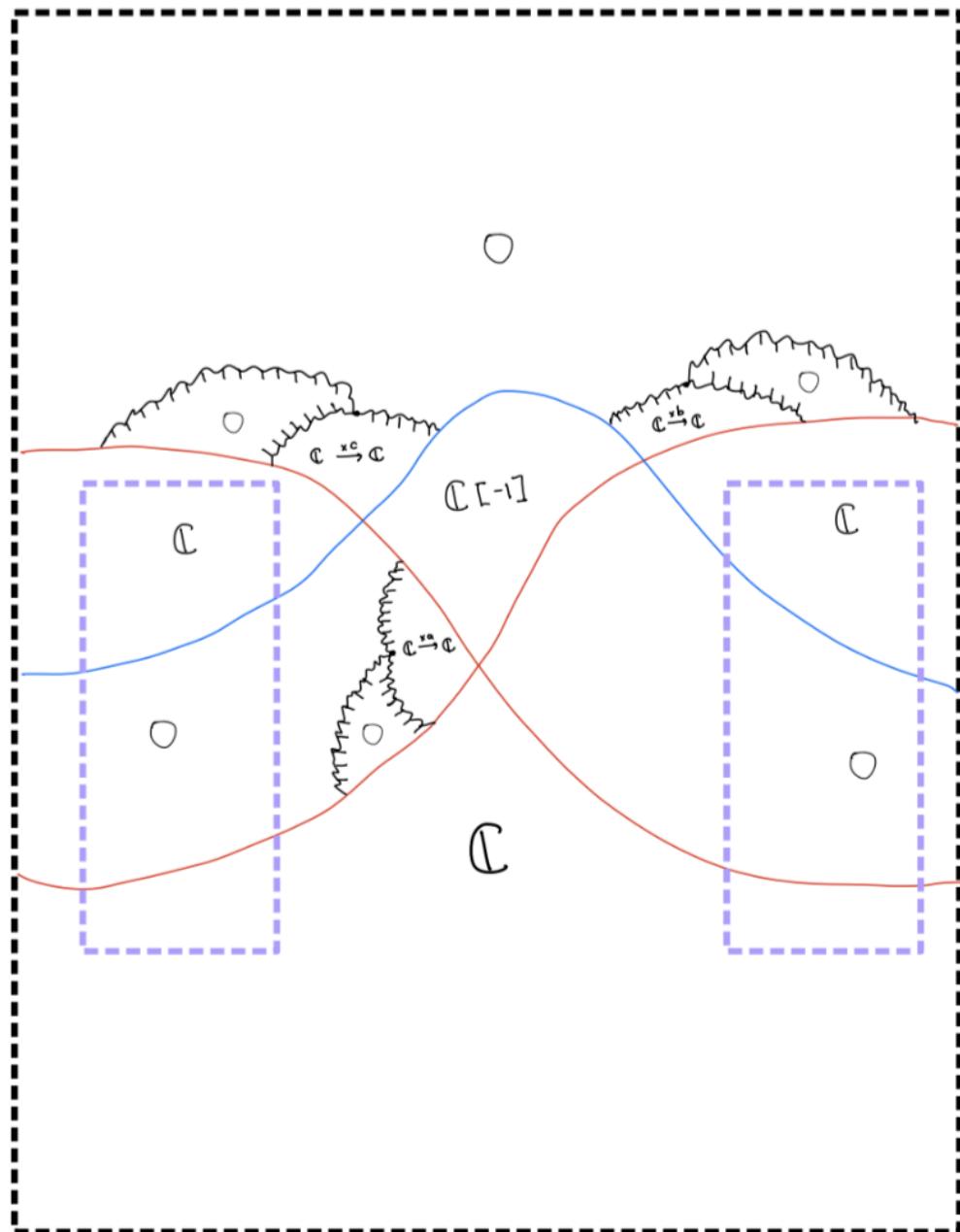


Figure 3.283

we get

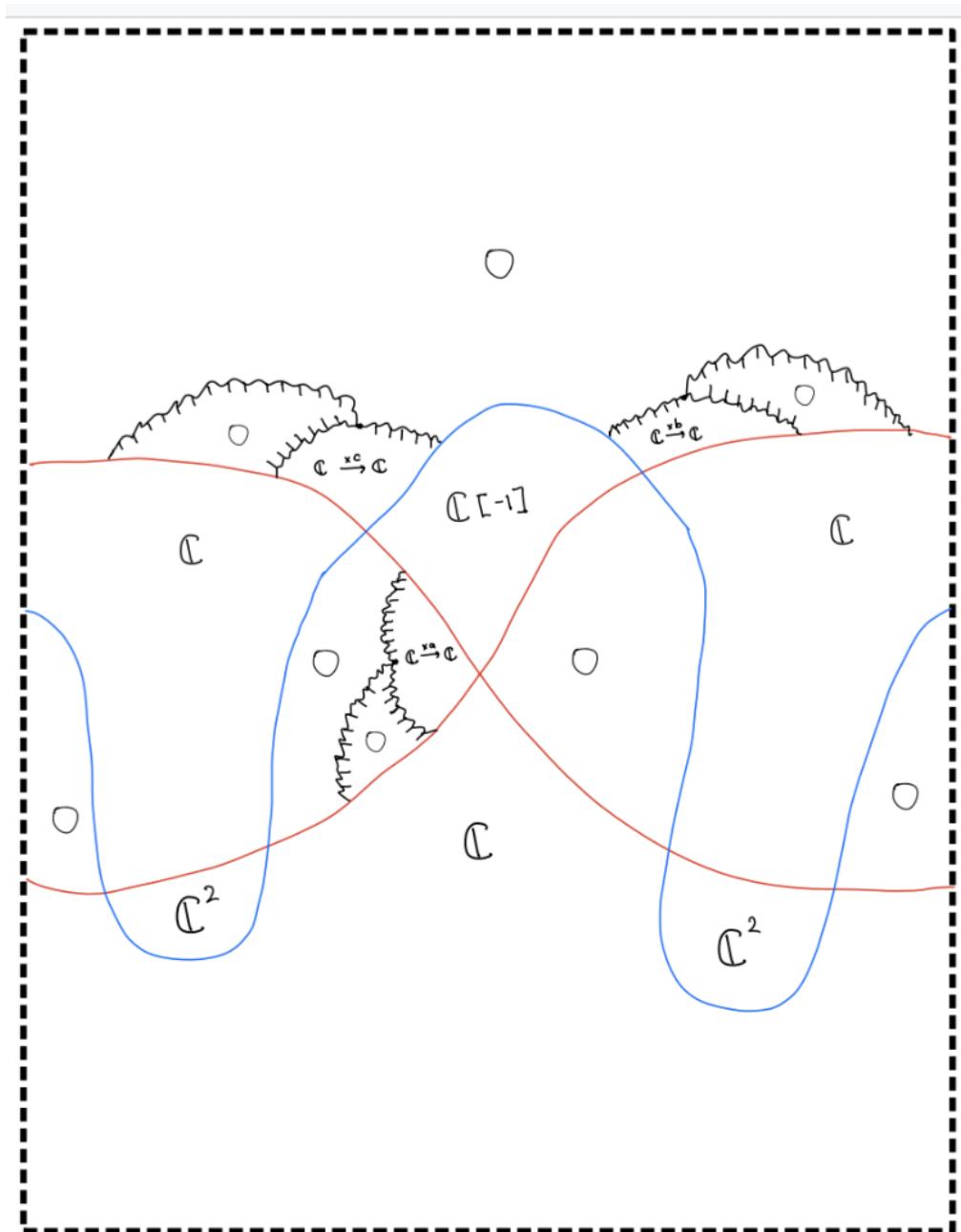


Figure 3.284

(Step 2) apply $cobord_4$ to the region surrounded by a purple dotted line.

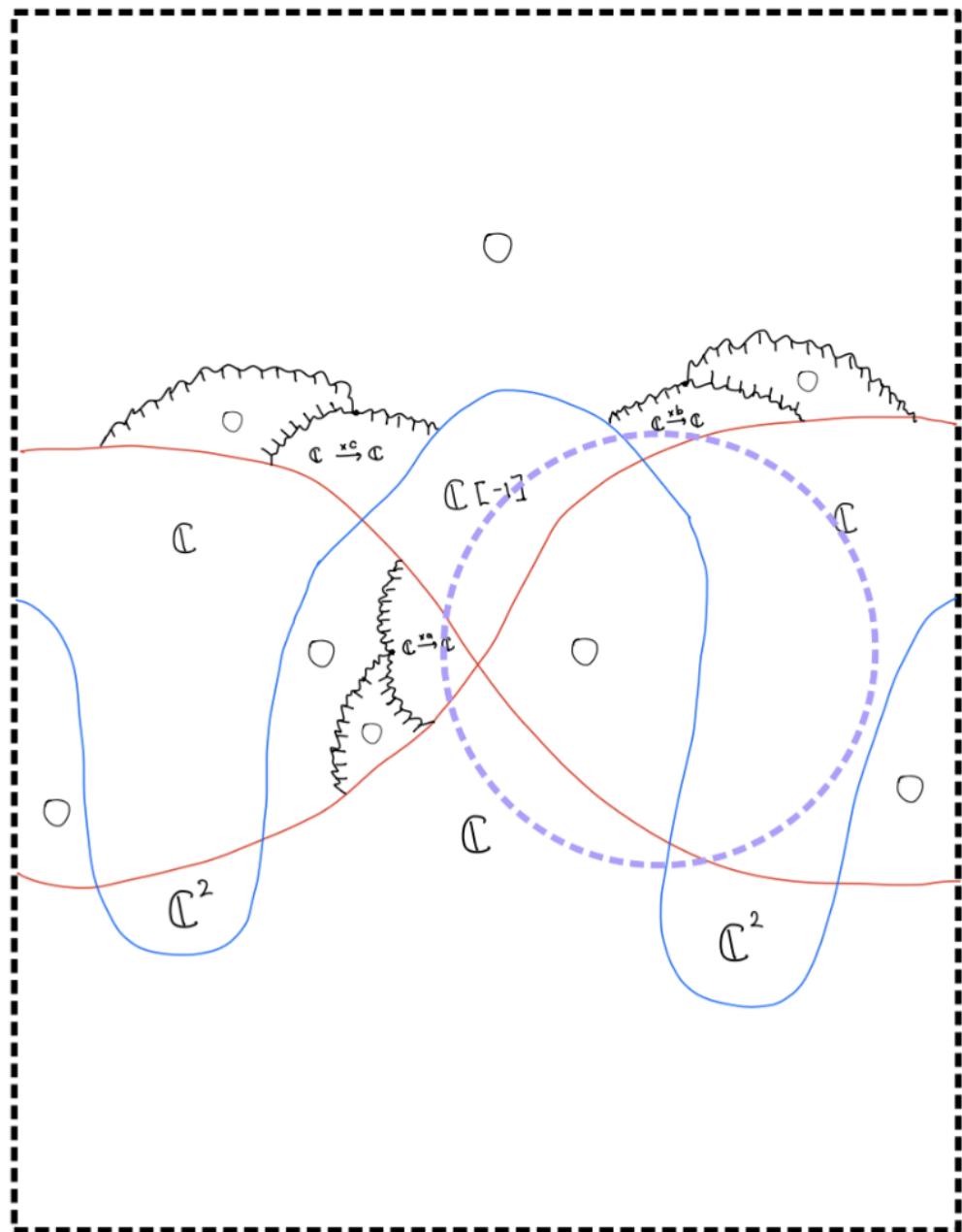


Figure 3.285

we get

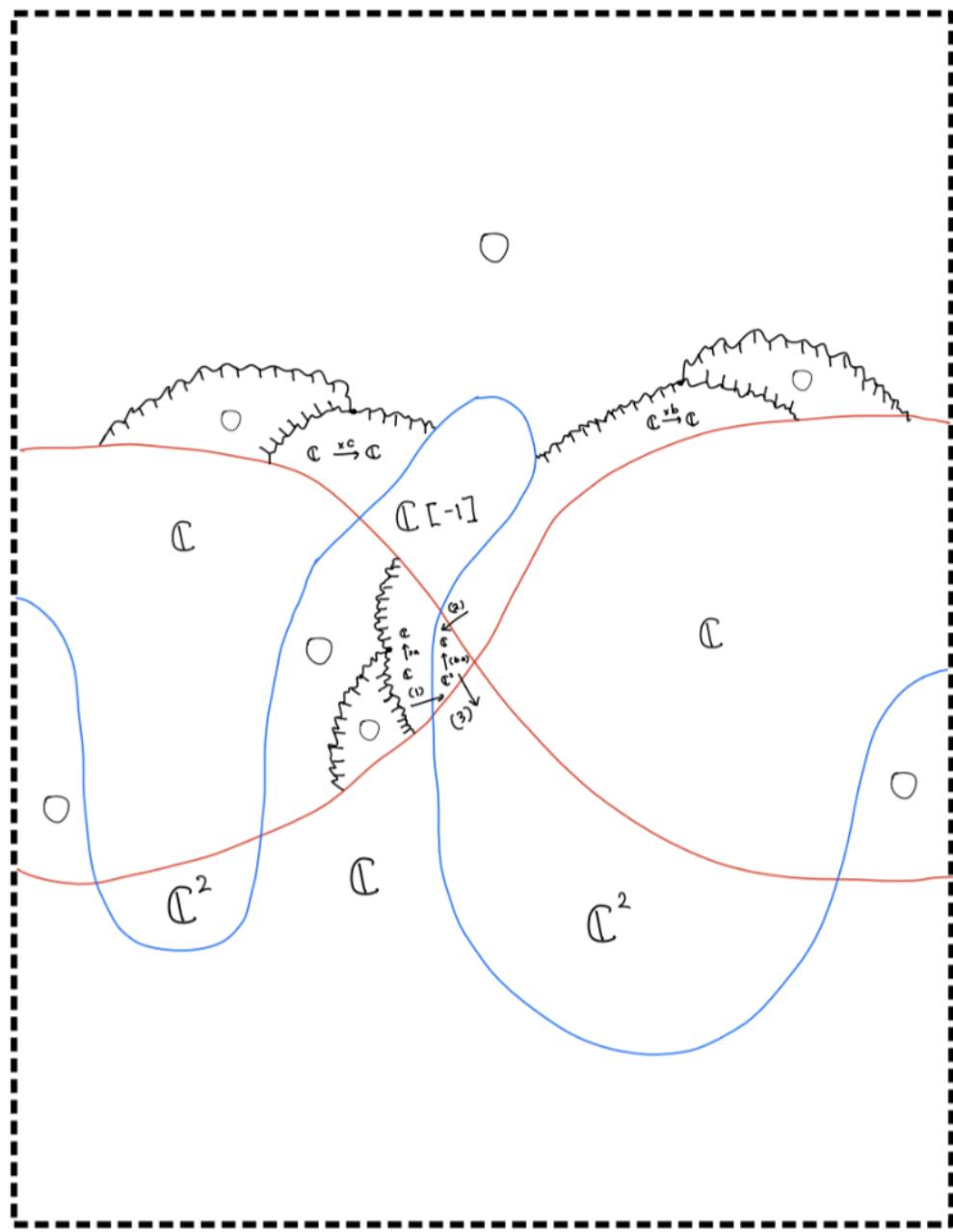


Figure 3.286

Generalizations maps:

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

(Step 3) apply $cobord'_2$ to the square region surrounded by purple dotted lines.

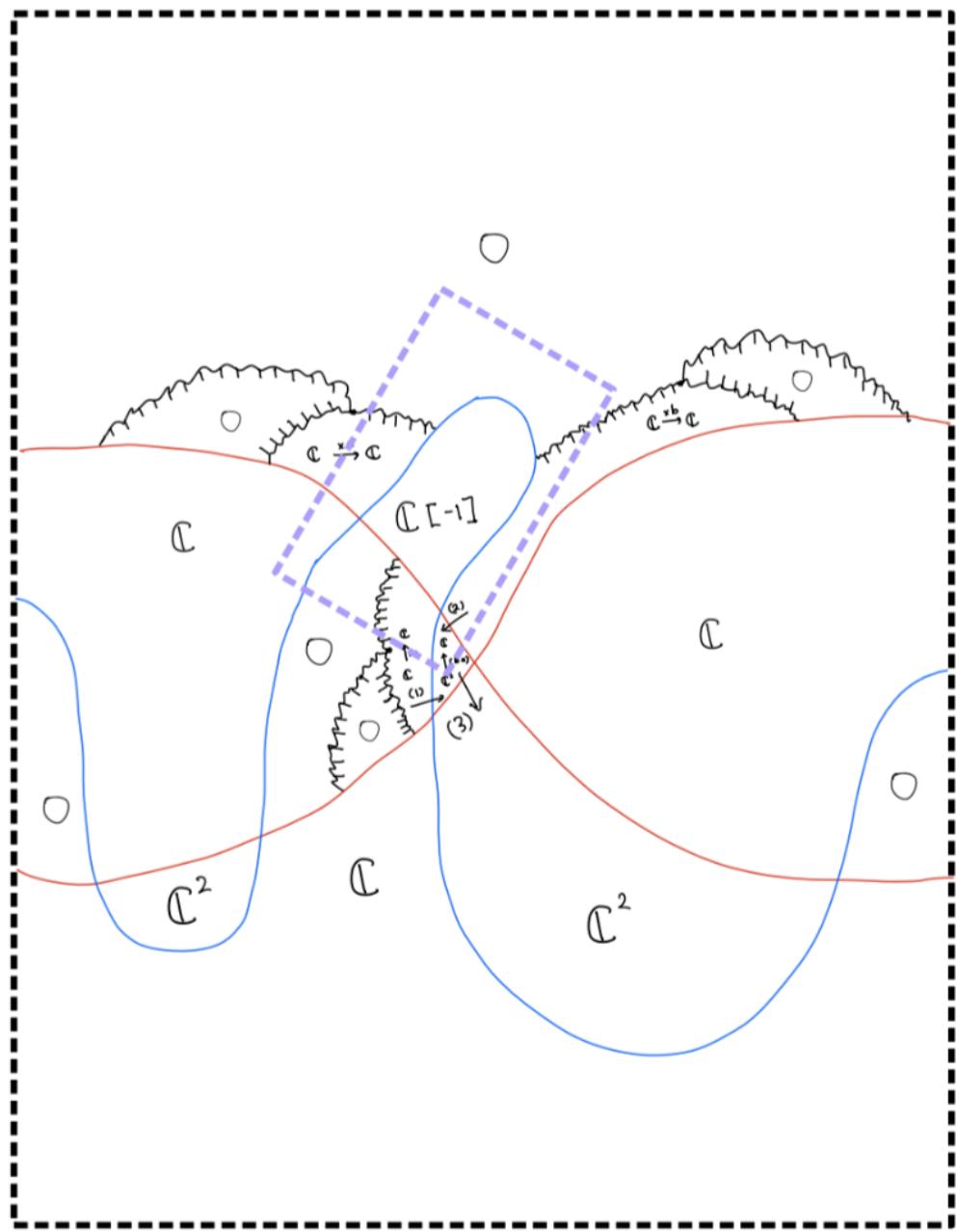


Figure 3.287

we get

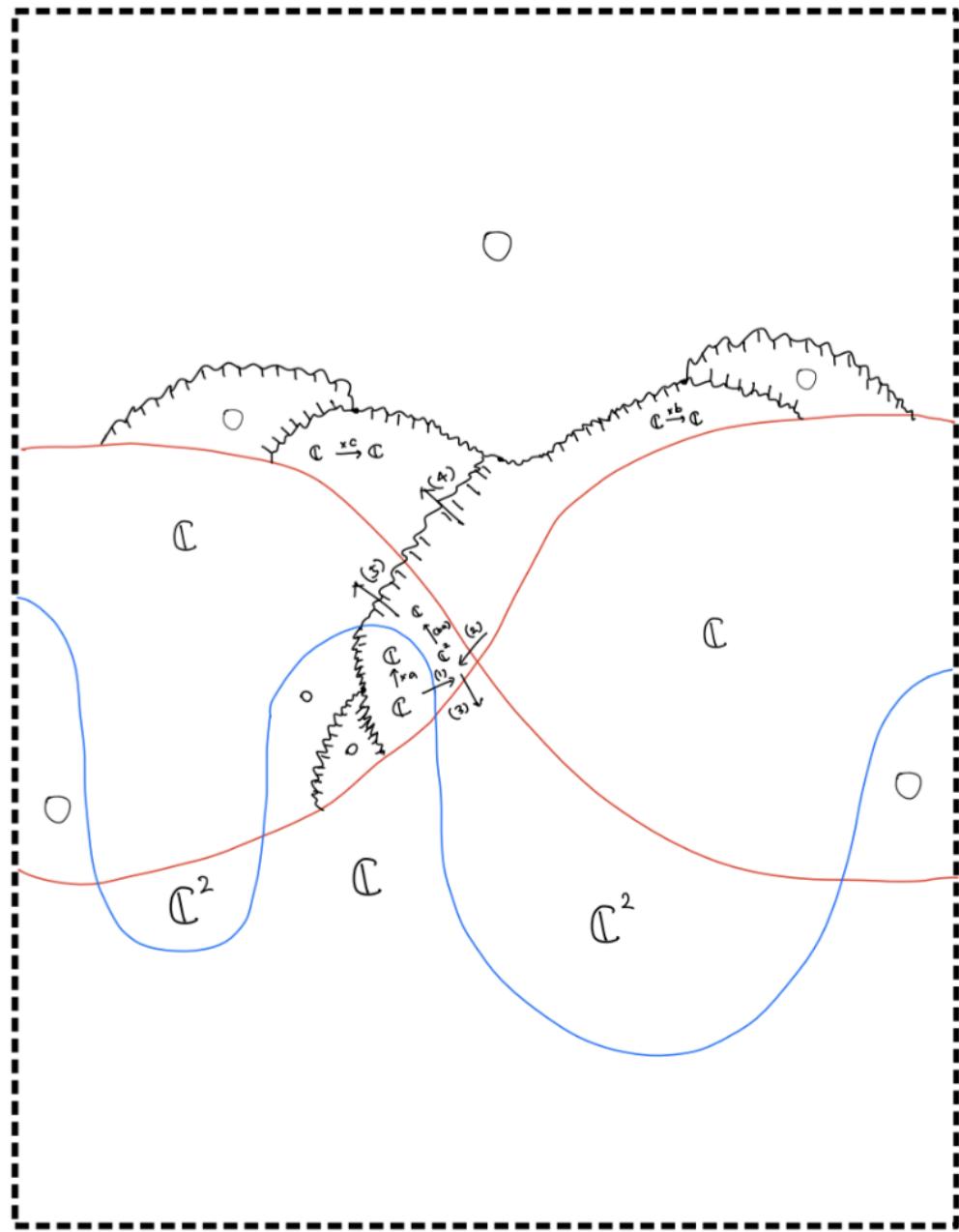


Figure 3.288

Generalizations maps:

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

After identifying

- \mathbb{C} with $\mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$ via $\begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$
- $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$, $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$, $\mathbb{C} \xrightarrow{\times c} \mathbb{C}$ with 0

the above sheaf is quasi-isomorphic to

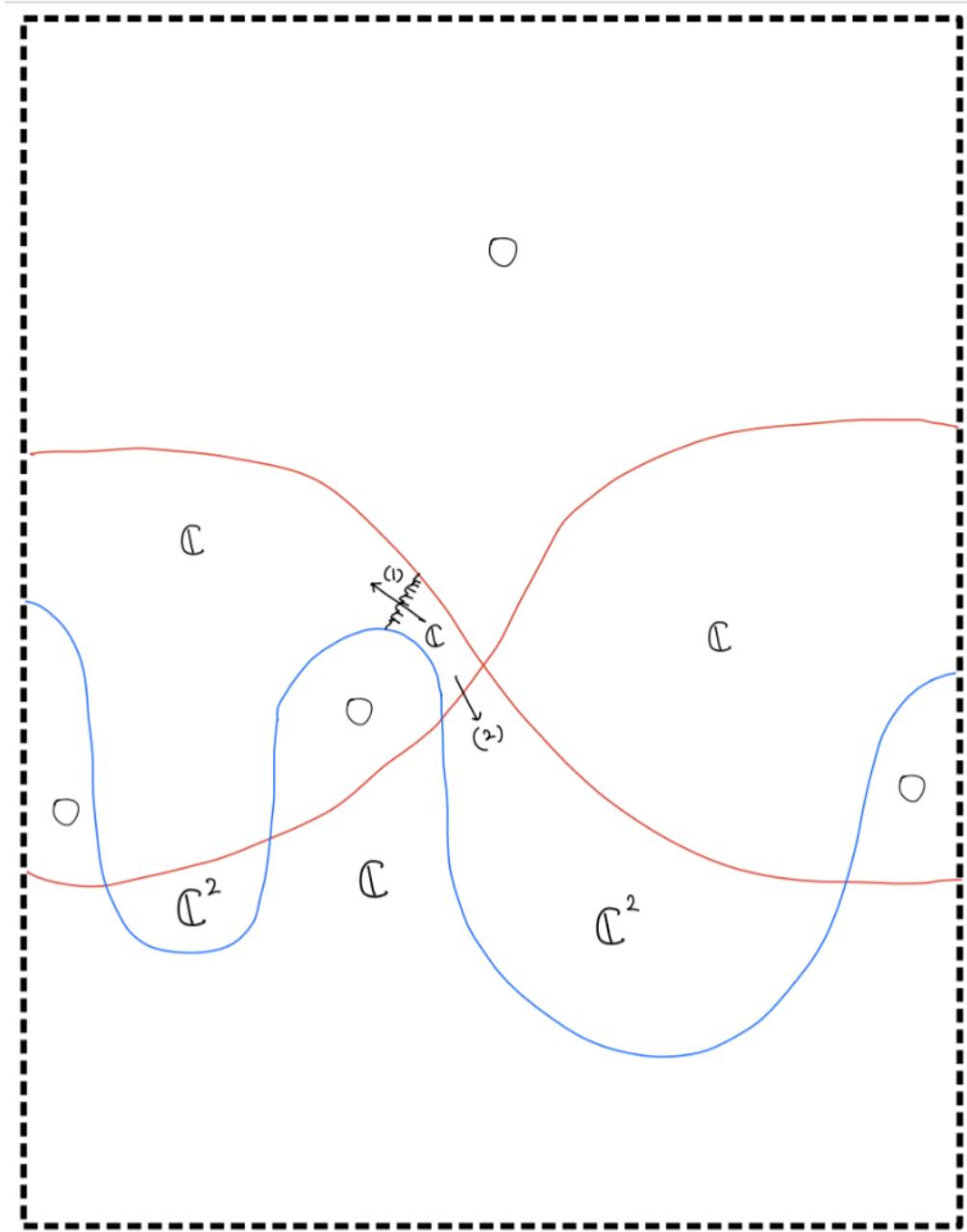


Figure 3.289

Generalizations maps:

$$(1) \times bc^{-1}$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

which is quasi-isomorphic to

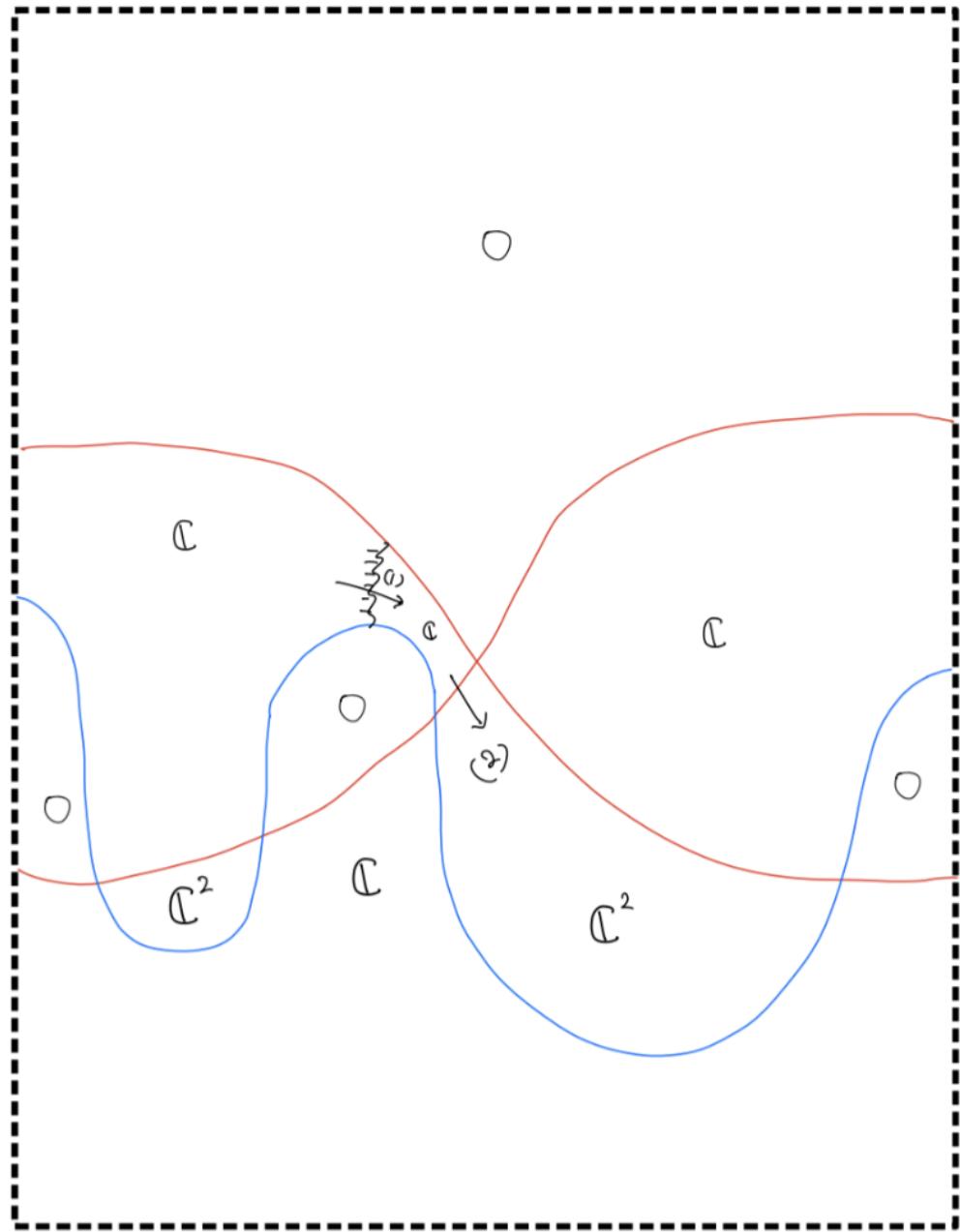


Figure 3.290

Generalizations maps:

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

(Step 4) apply $cobord_3$ to the region surrounded by a purple dotted line

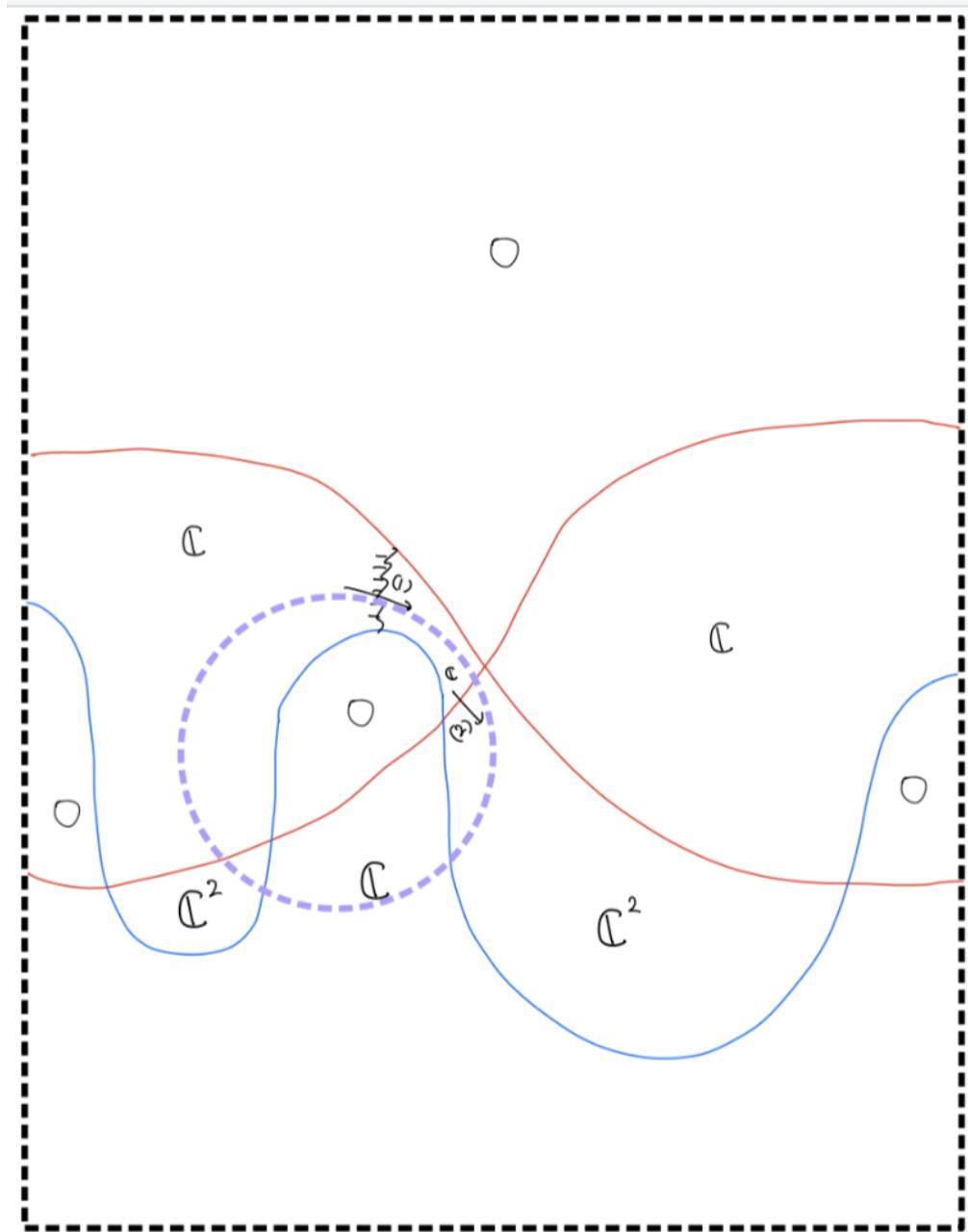


Figure 3.291

we get

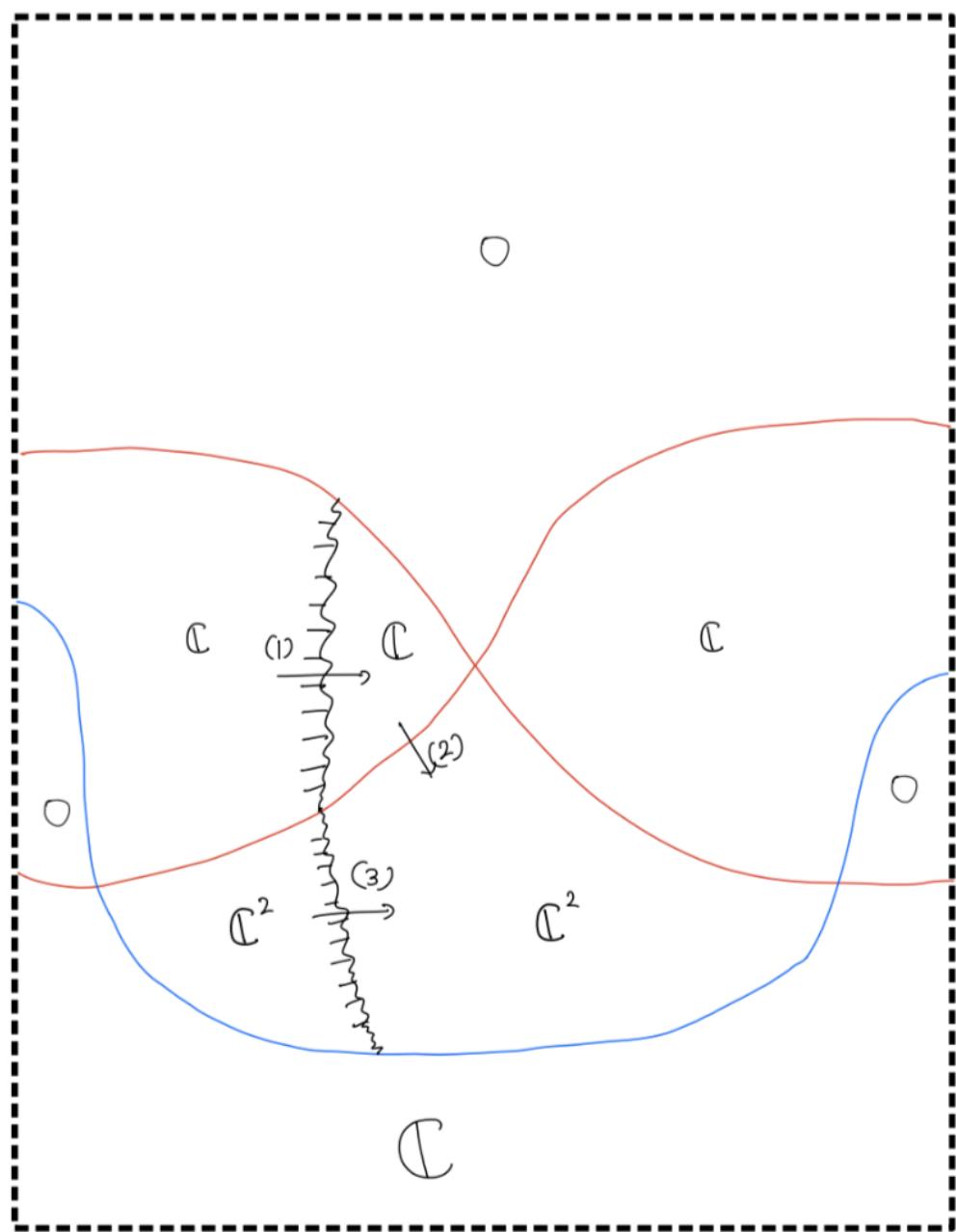


Figure 3.292

Generalizations maps:

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

$$(3) \begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$$

3.14 8th Sheaf Cobordism'

Definition 217. Smooth transition function consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere M and Λ_0^0 , Λ_0^∞ , Λ_0^{squig} , a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-2, 2)_x \times (-1, 1)_z$ under f

- Λ_0^0 gets mapped to $\{(x, z) \in R \mid z = \Psi(z_{lo} = 0, z_{hi} = 1)(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\{(x, z) \in R \mid z = \tau(\frac{x+1}{2}) - \frac{1}{2}\} \cup \{(x, z) \in R \mid z = -\tau(\frac{x+1}{2}) + \frac{1}{2}\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x, z) \in R \mid x = -c - \epsilon\}$ where $c > 0$, $\Psi(0, 1)(c) = \tau(\frac{c+1}{2}) - \frac{1}{2}$ and $\epsilon > 0$ is a small number such that $c > \epsilon$.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

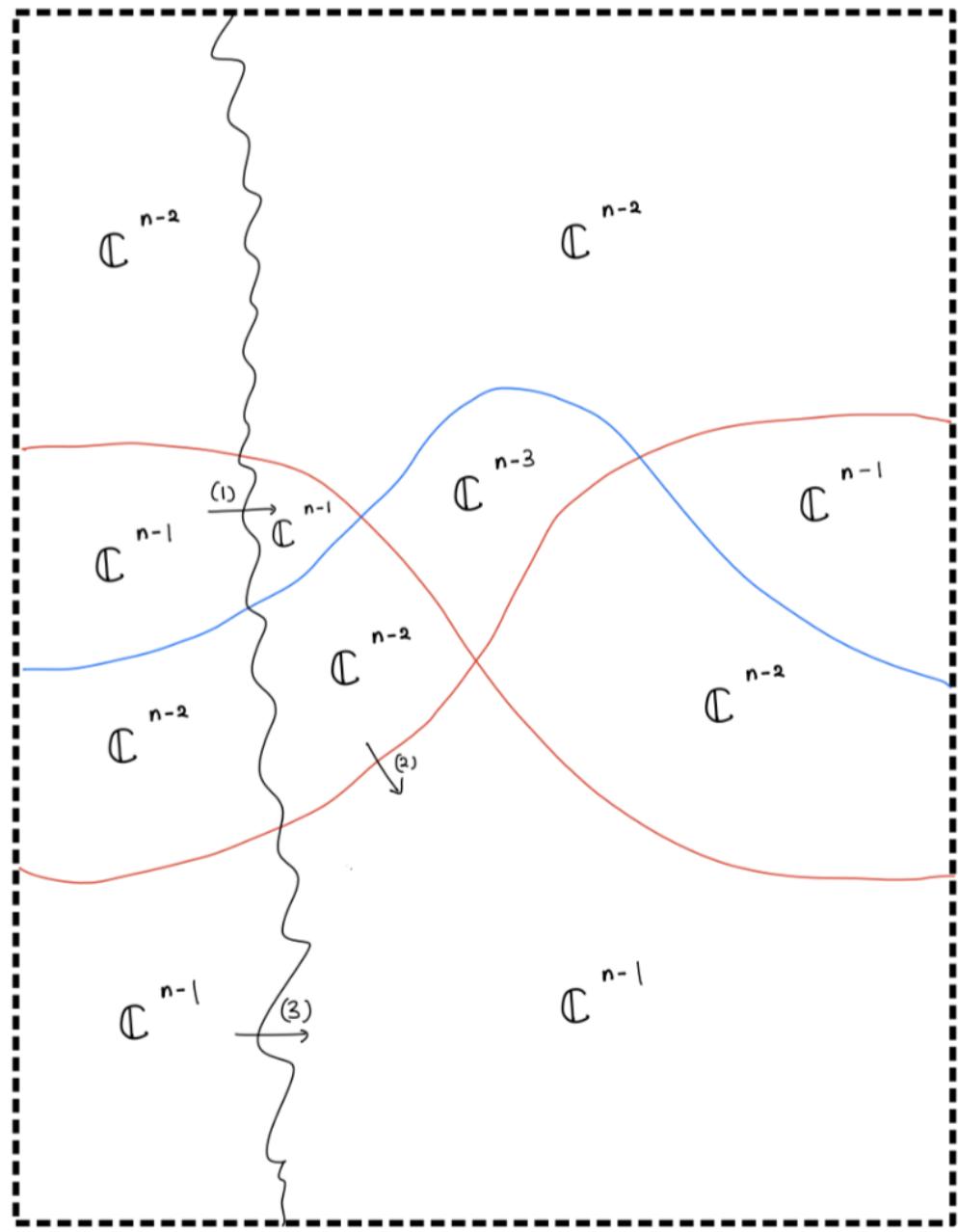


Figure 3.293

Generalizations maps:

- (1) $\text{diag}(d_n, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1, n-2}$
- (3) $\text{diag}(d_{n-1}, \dots, d_1) + eI_{n-1, n-2}$

where $e' = d_2^{-1}e$. Then we define a cobordism starting from the above sheaf, say $cobord'_8$ supported on U . At the end of the cobordism, the sheaf, under the same chart f , is described as the following squiggly legible diagram.

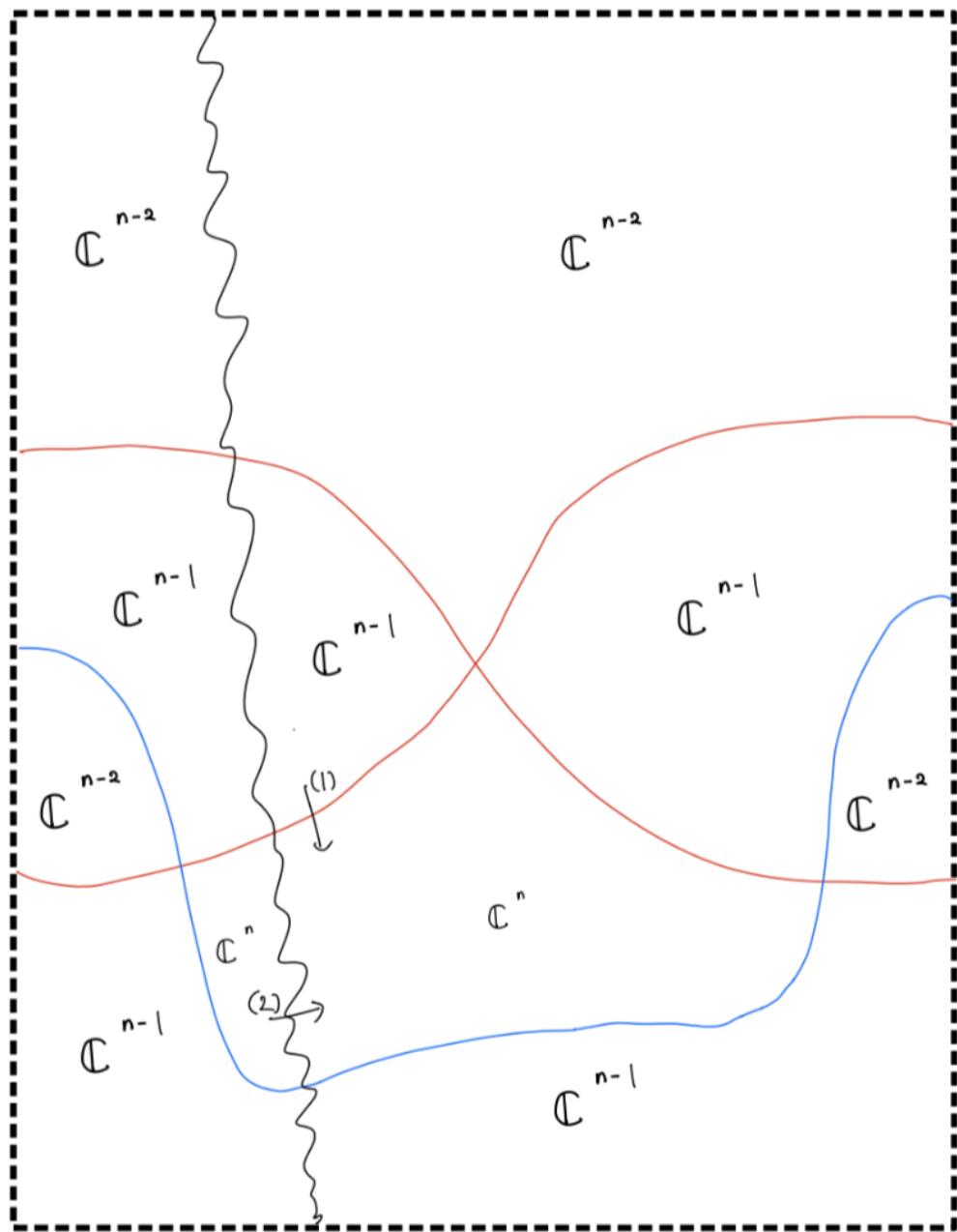


Figure 3.294

Generalizations maps:

$$(1) \quad \iota_0 \circ diag(1, \dots, 1) + e'I_{n,n-1}$$

$$(2) \ diag(d_n, \dots, d_1) + eI_{n,n-1}$$

We define $cobord'_8$ as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

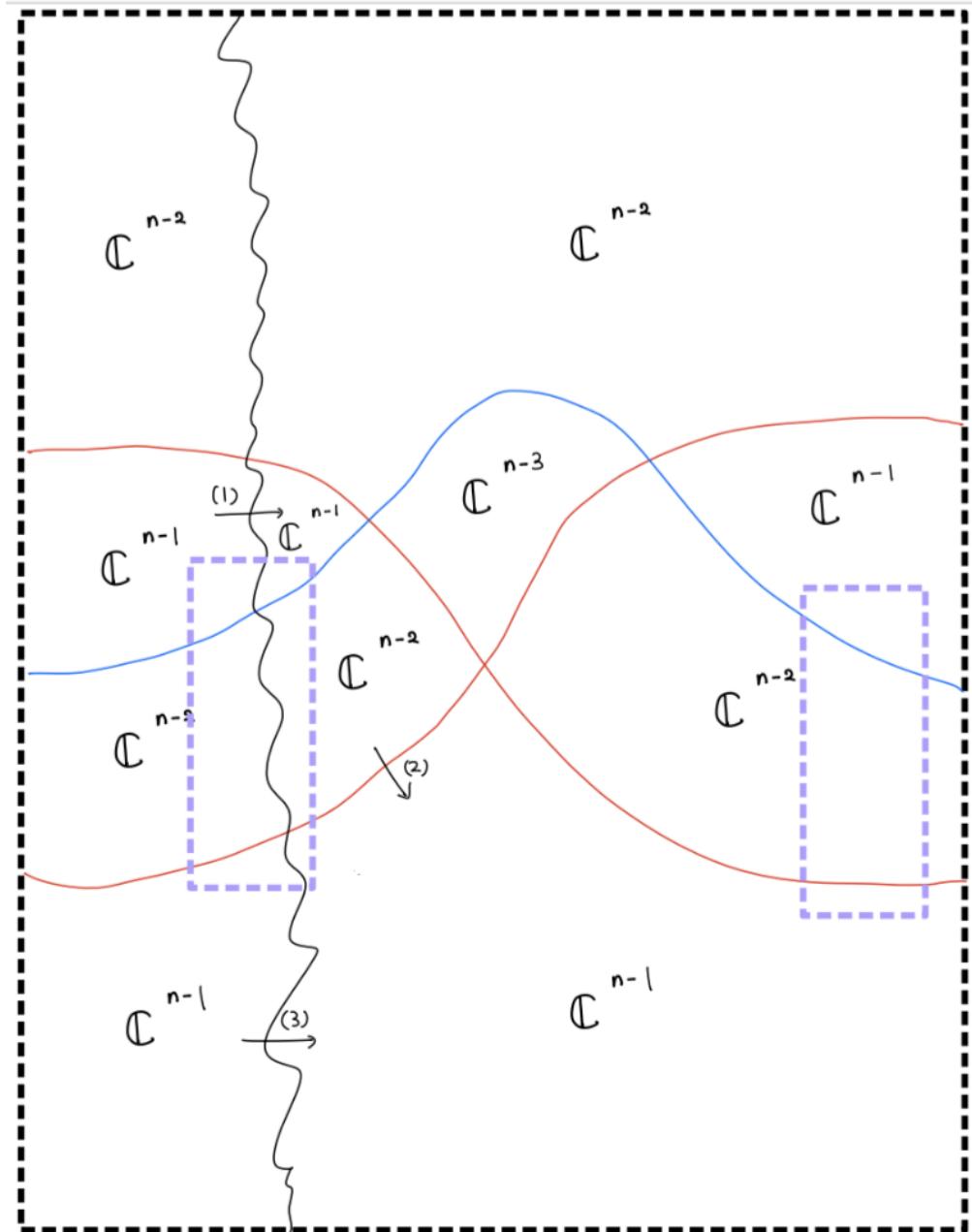


Figure 3.295

we get

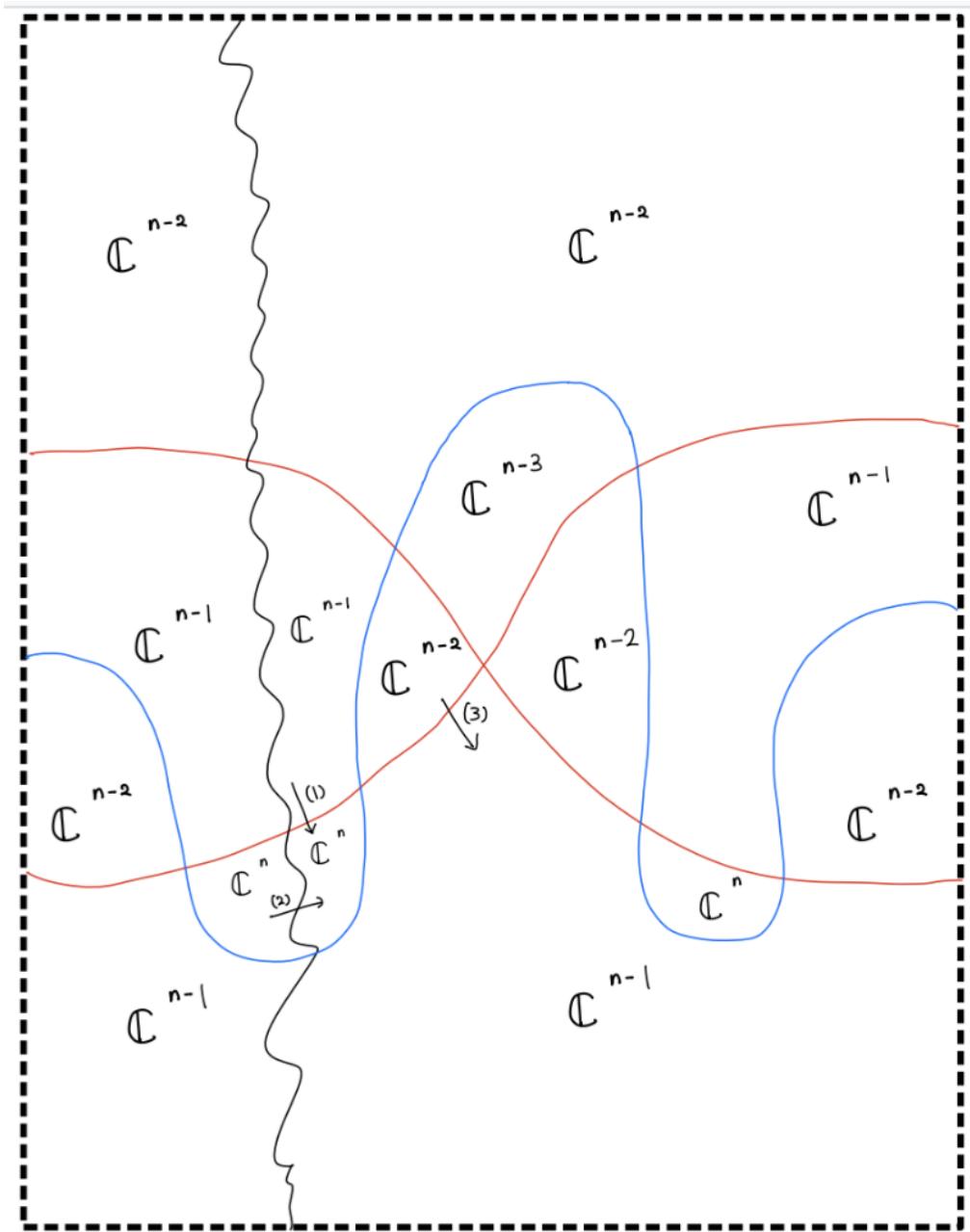


Figure 3.296

Generalizations maps:

$$(1) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$$

$$(2) \quad \text{diag}(d_n, \dots, d_1) + e I_{n,n-1}$$

$$(3) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1,n-2}$$

(Step 2) apply $cobord'_4$ to the region surrounded by a purple dotted line.

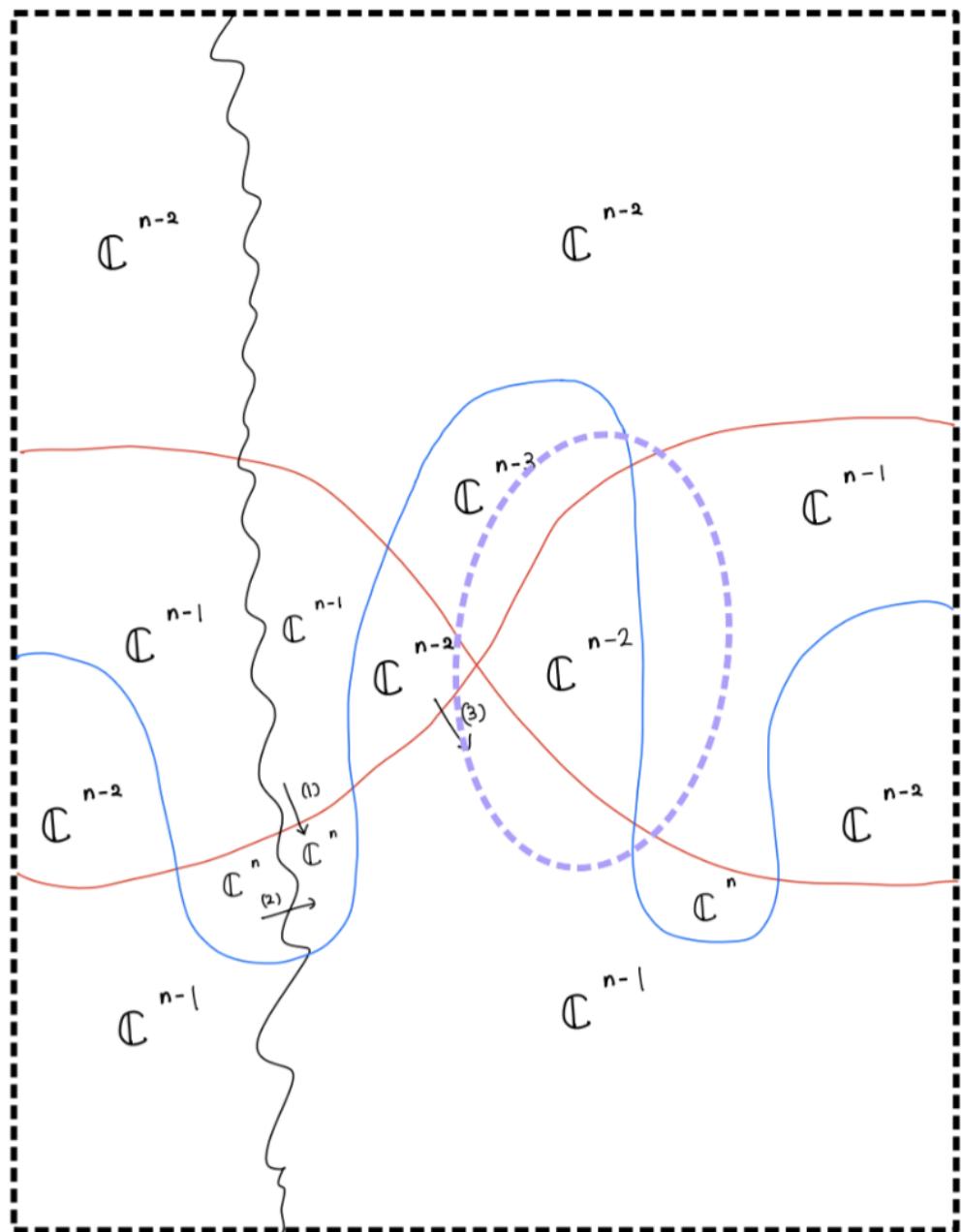


Figure 3.297

we get

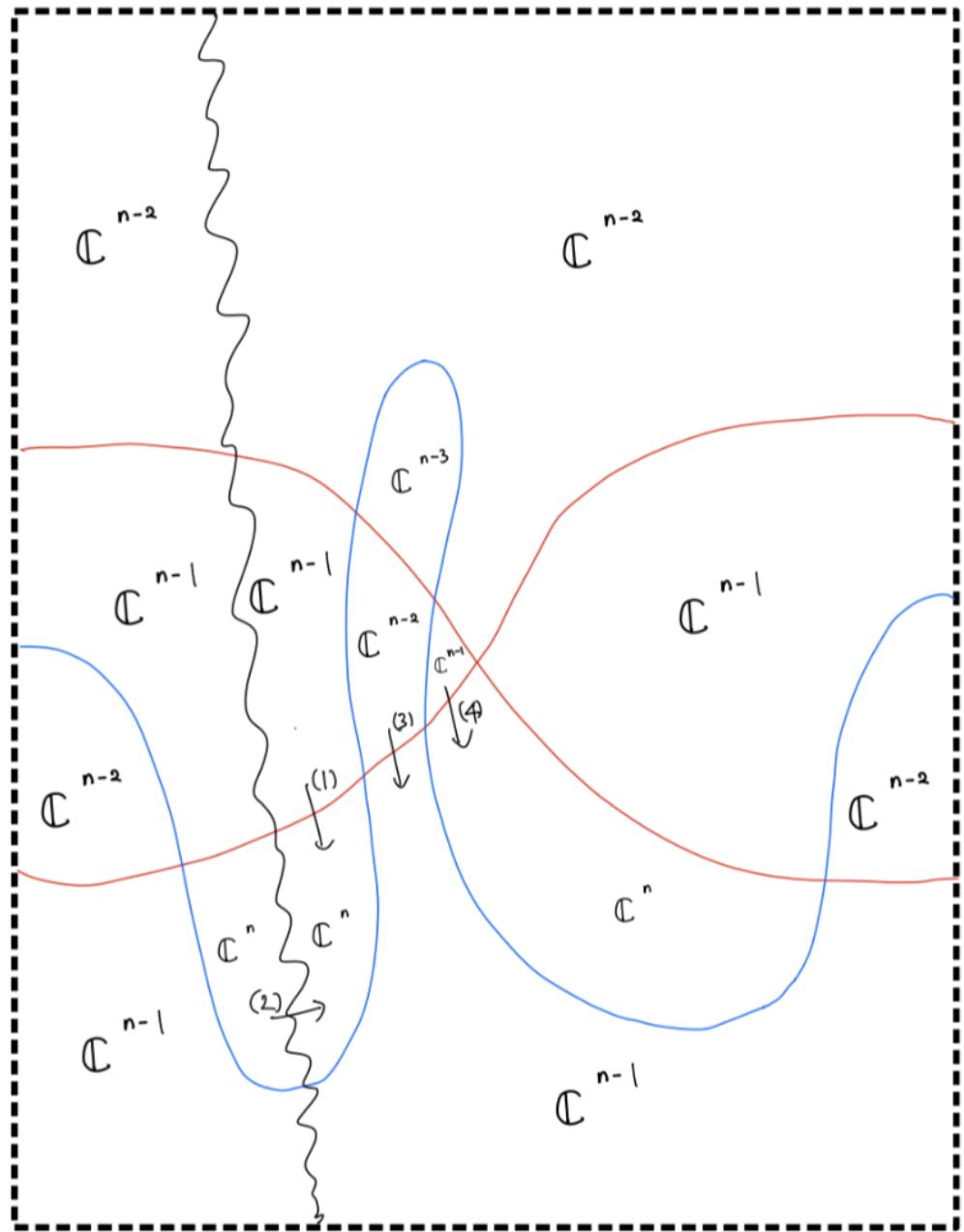


Figure 3.298

Generalizations maps:

- (1) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$
- (2) $\text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$
- (3) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1,n-2}$

$$(4) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$$

(Step 3) apply *cobord*₃ to the region surrounded by a purple dotted line.

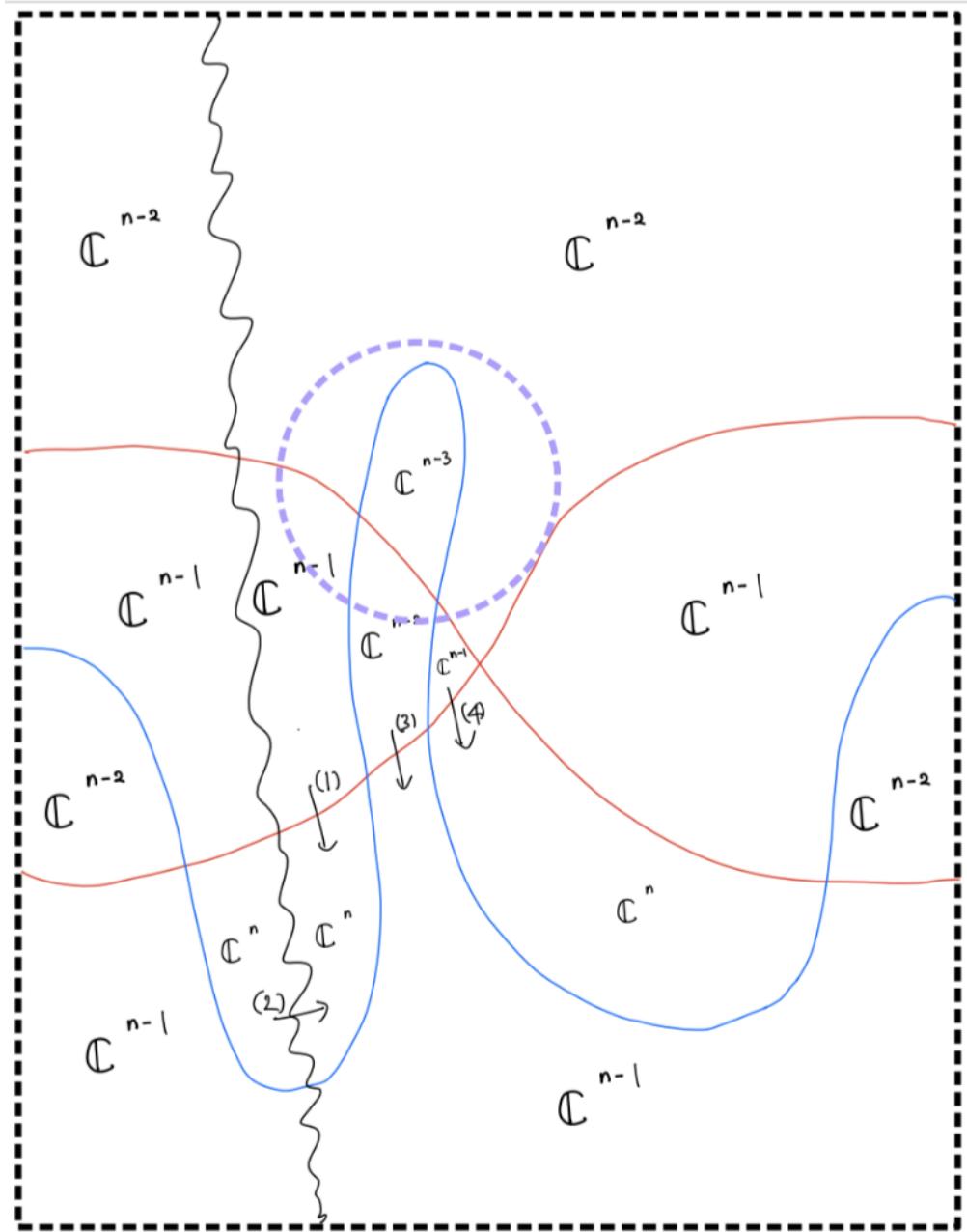


Figure 3.299

we get

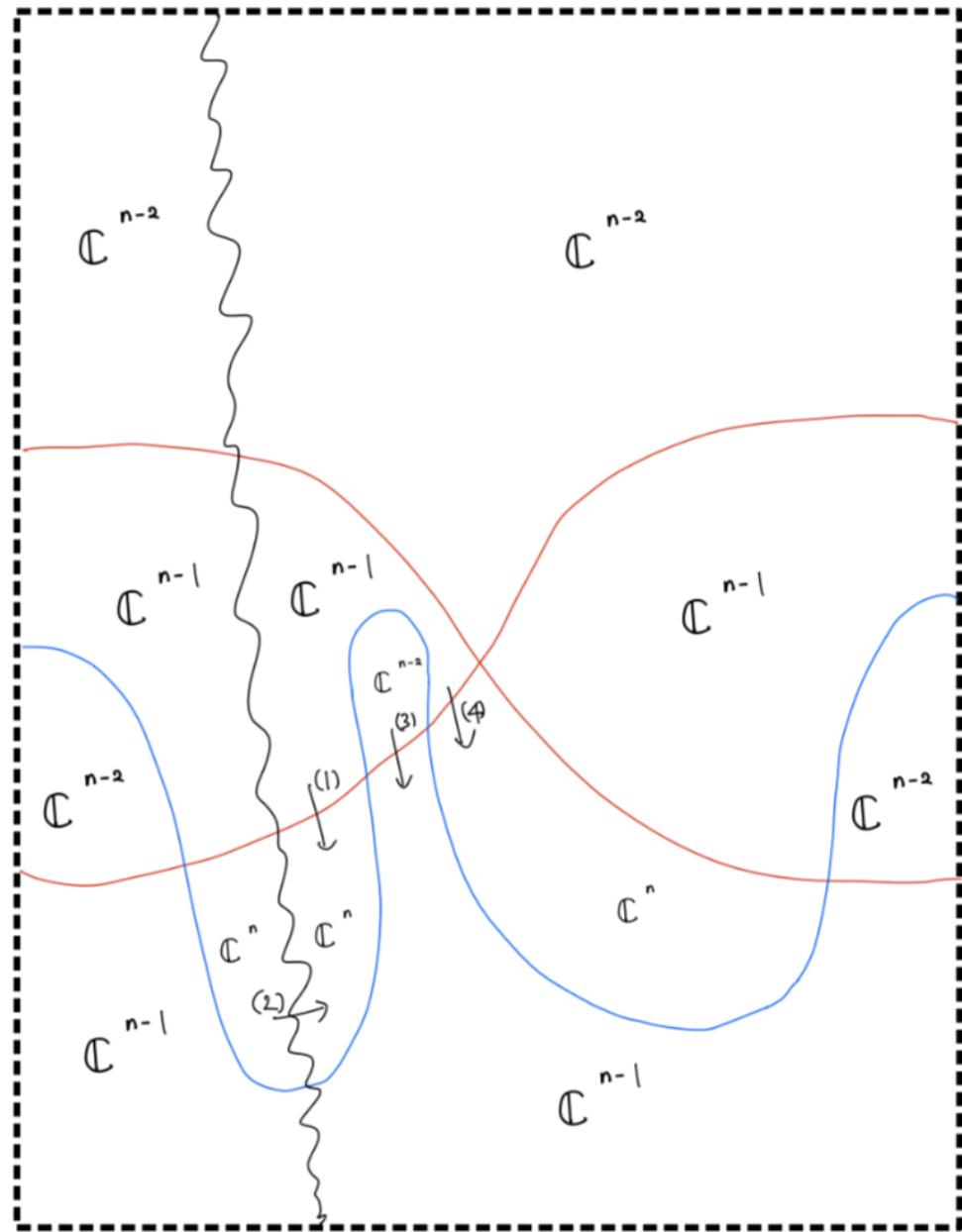


Figure 3.300

Generalizations maps:

- (1) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$
- (2) $\text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$
- (3) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1,n-2}$
- (4) $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(Step 4) apply $cobord_3$ to the region surrounded by a purple dotted line

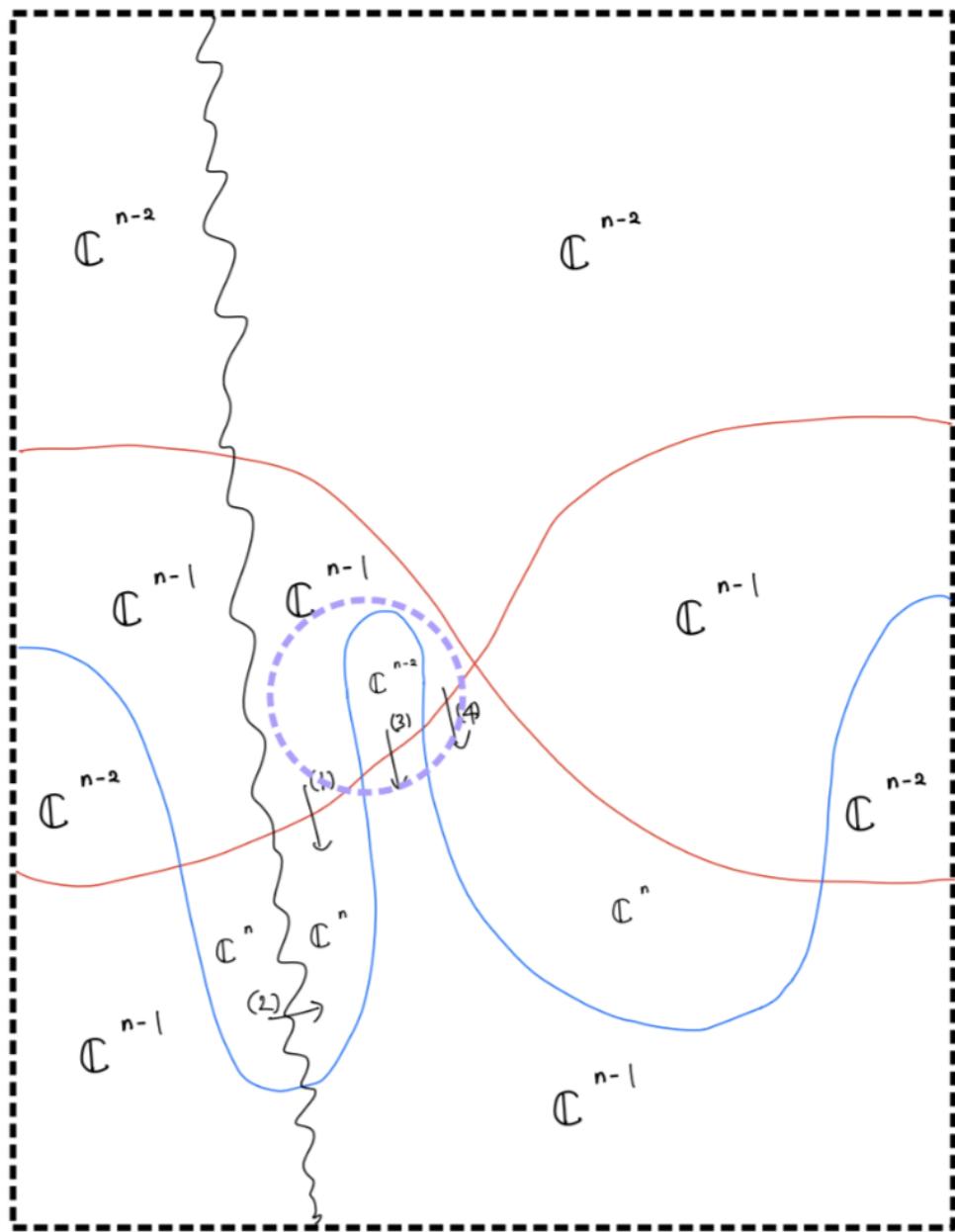


Figure 3.301

we get the final sheaf

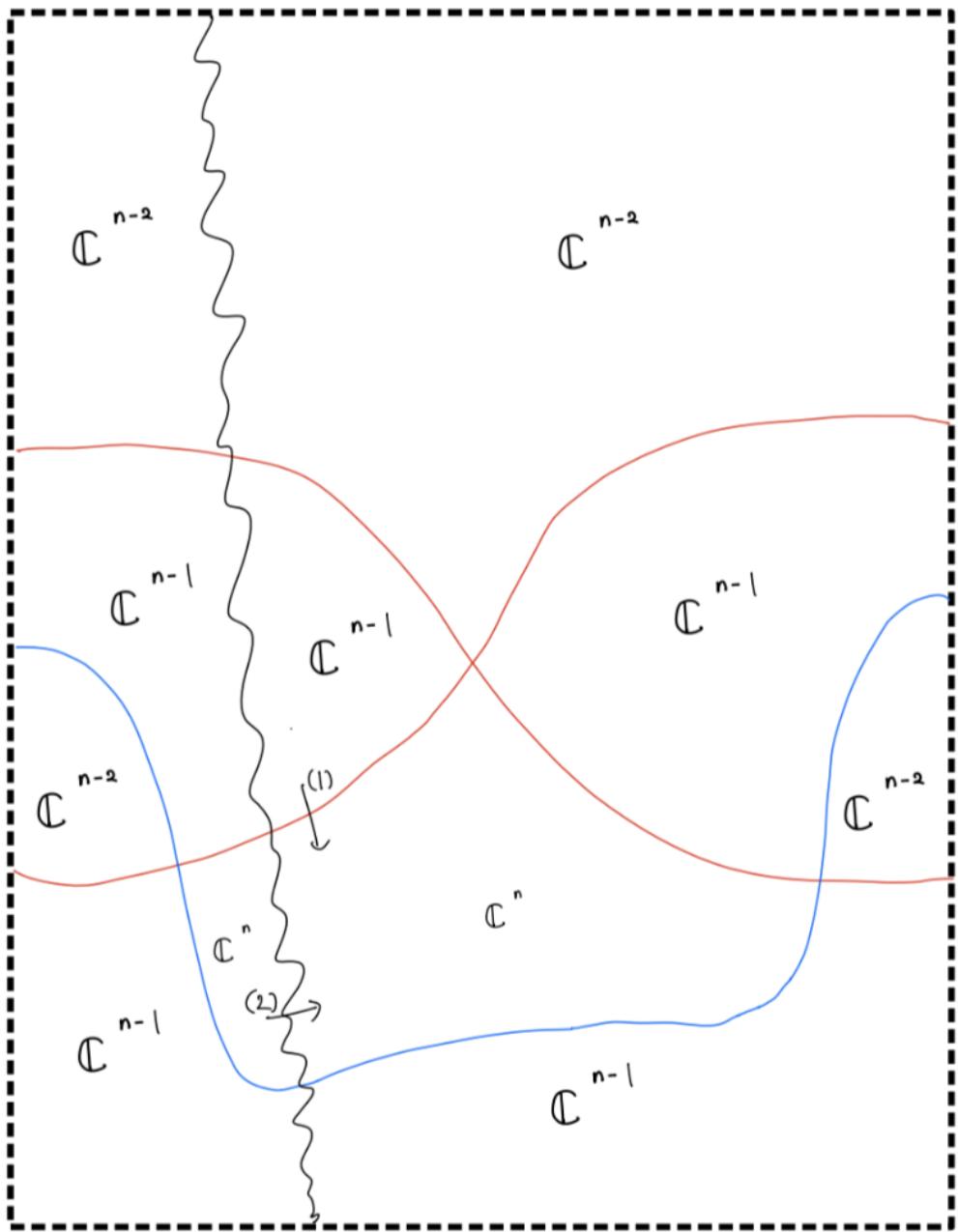


Figure 3.302

Generalizations maps:

$$(1) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$$

$$(2) \quad \text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$$

3.15 Sheaf Cobordism on Generator Regions

Definition 218. Smooth transition function consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere M and Λ_0^0 , Λ_0^∞ , Λ_0^{squig} , a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-1, 1)_x \times (-n-1, n+1)_z$ under f

- Λ_0^0 gets mapped to the red strands in the figure below, co-oriented upward.
- Λ_0^∞ gets mapped to the blue strands in the figure below, co-oriented downward.
- Λ_0^{squig} gets mapped to the squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

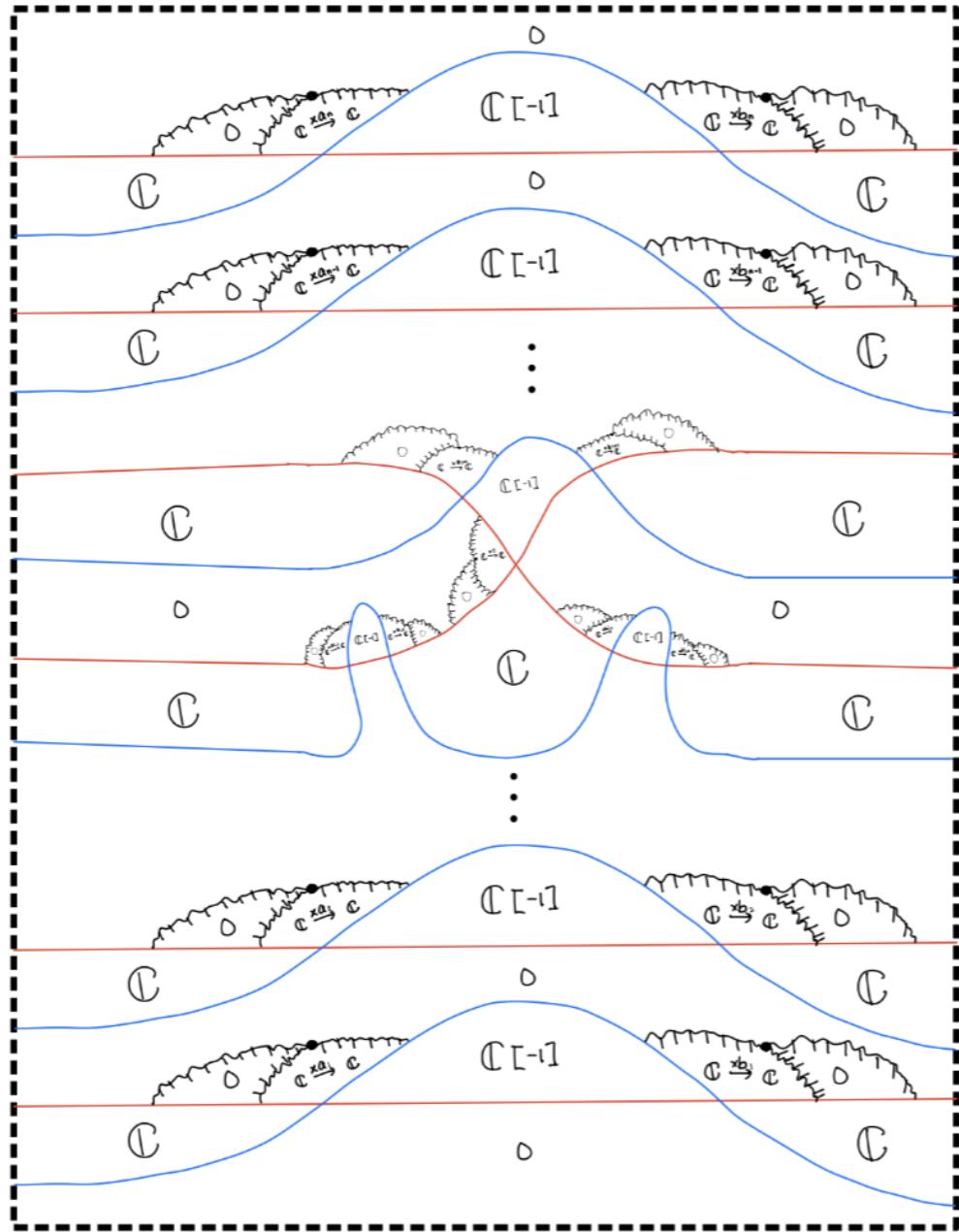


Figure 3.303

Then we define a cobordism starting from the above sheaf, say $cobord_{gen}(n)$ supported on U , where n is the number of blue strands (equivalently, red strands). At the end of the cobordism, the sheaf, under the same chart f , is described as the following squiggly legible diagram.

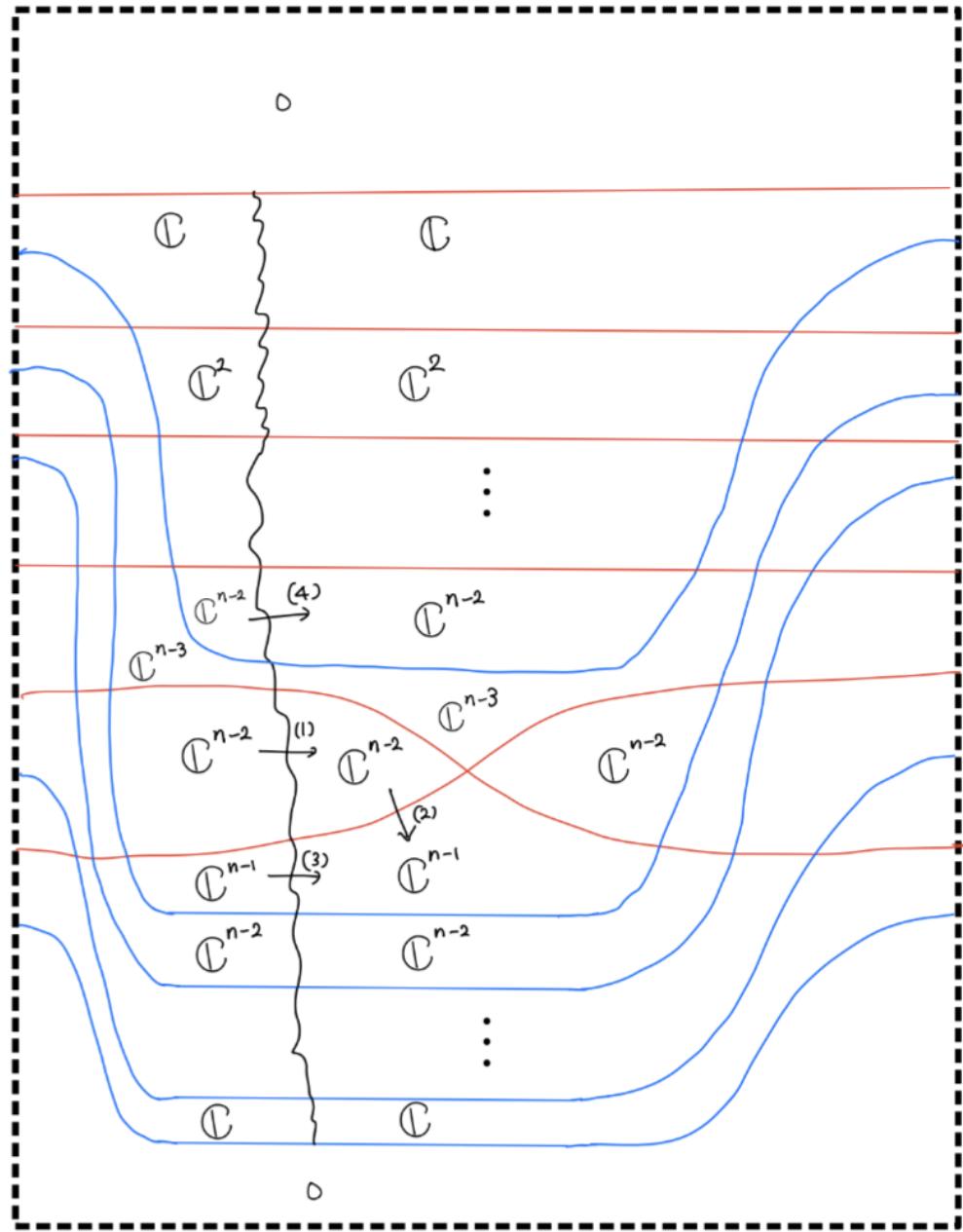


Figure 3.304

Generalization maps

- (1) $diag(d_n, \dots, d_{i+1})$
- (2) $\iota_0 \circ diag(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3) $diag(d_n, \dots, d_{i+1}) + e I_{n-i+1, n-i}$

where

- $d_r = a_r b_r^{-1}$
- $e = -a_{i+1} b_i^{-1} c$
- $e' = d_{i+1}^{-1} e$

We define $cobord_{gen}(n)$ inductively as follows.

- (i) For $n = 2$, we define $cobord_{gen}$ starting from the sheaf below

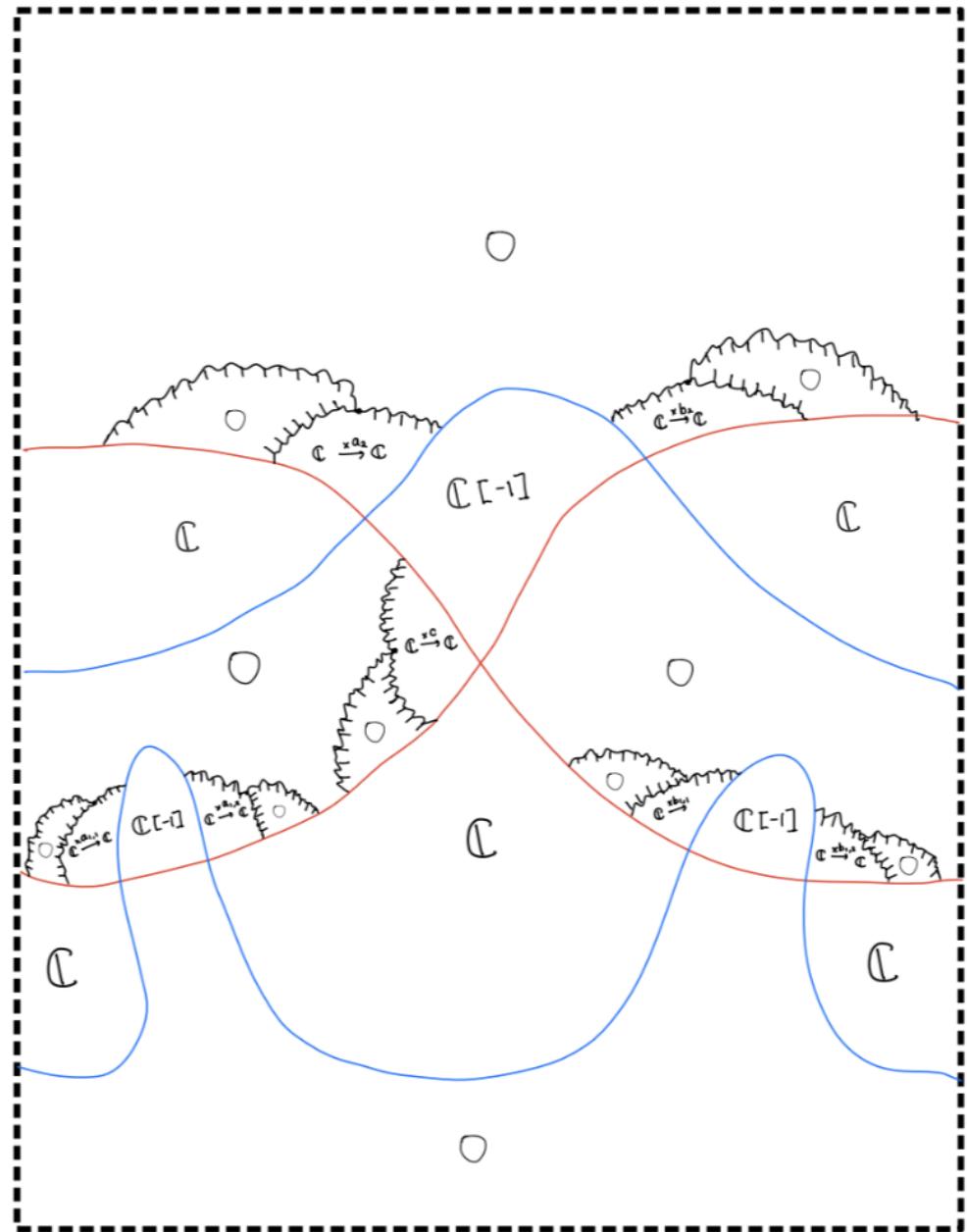


Figure 3.305

as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

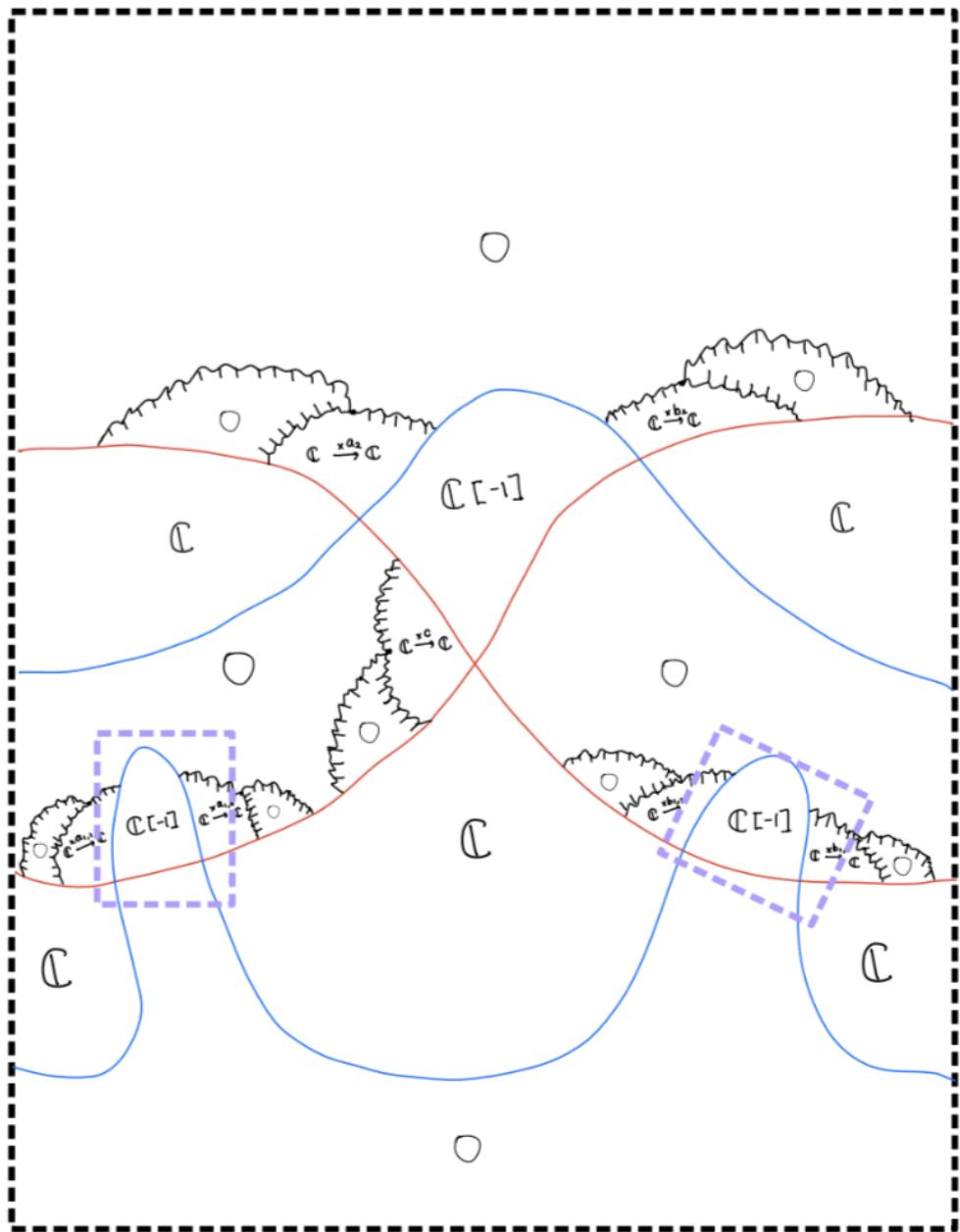


Figure 3.306

we get

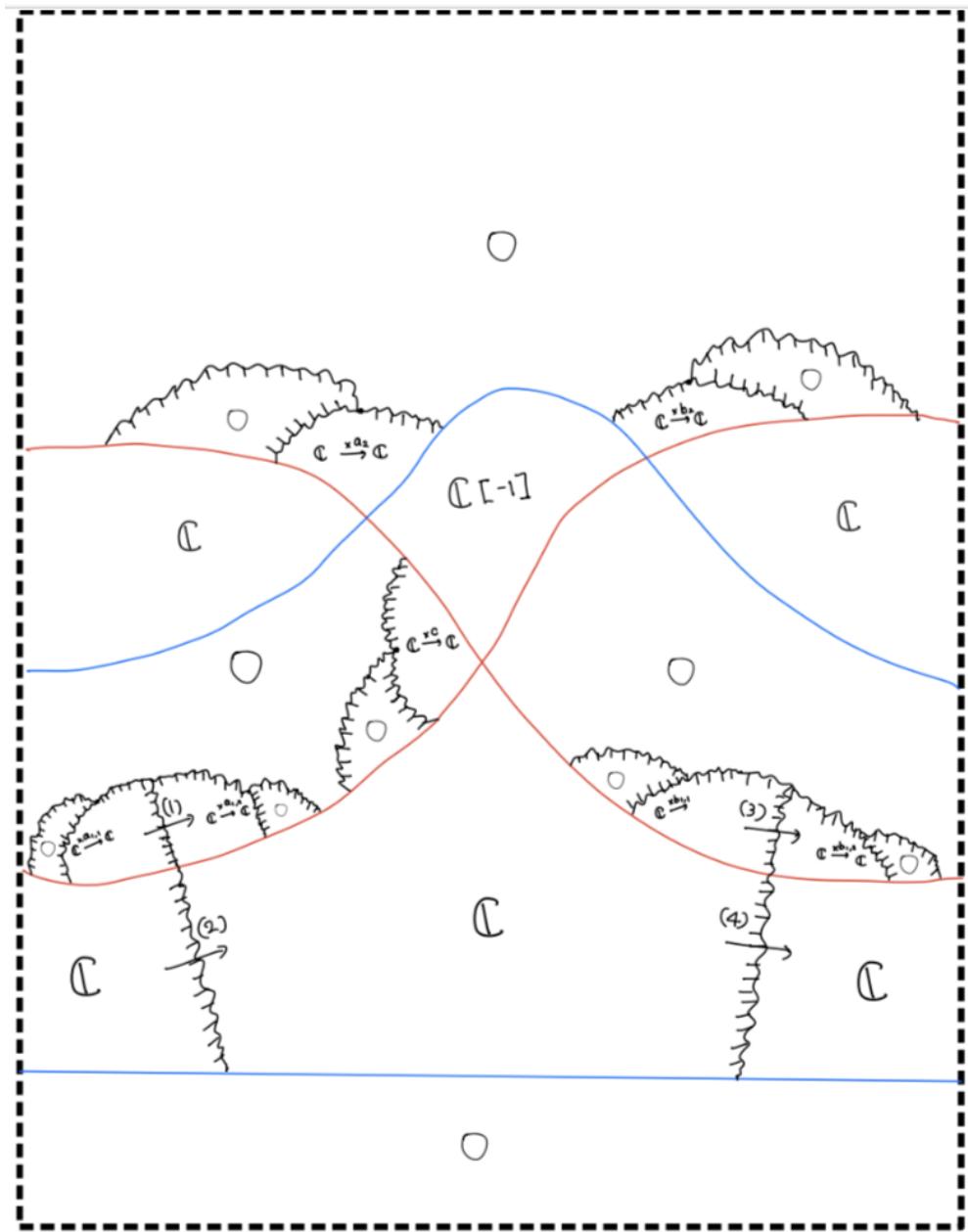


Figure 3.307

Generalization maps

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a_1 \uparrow & & \uparrow \times a_{1,2} \\ \mathbb{C} & \xrightarrow{\times a_{1,1}} & \mathbb{C} \end{array}$$

(2) $\times a_1$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b_1^{-1} \uparrow & & \uparrow \times b_{1,2} \\ \mathbb{C} & \xrightarrow{\times b_{1,1}} & \mathbb{C} \end{array}$$

$$(4) \quad \times b_1^{-1}$$

where

- $a_1 = a_{1,1}a_{1,2}^{-1}$
- $b_1 = b_{1,1}^{-1}b_{1,2}$

which is quasi-isomorphic to

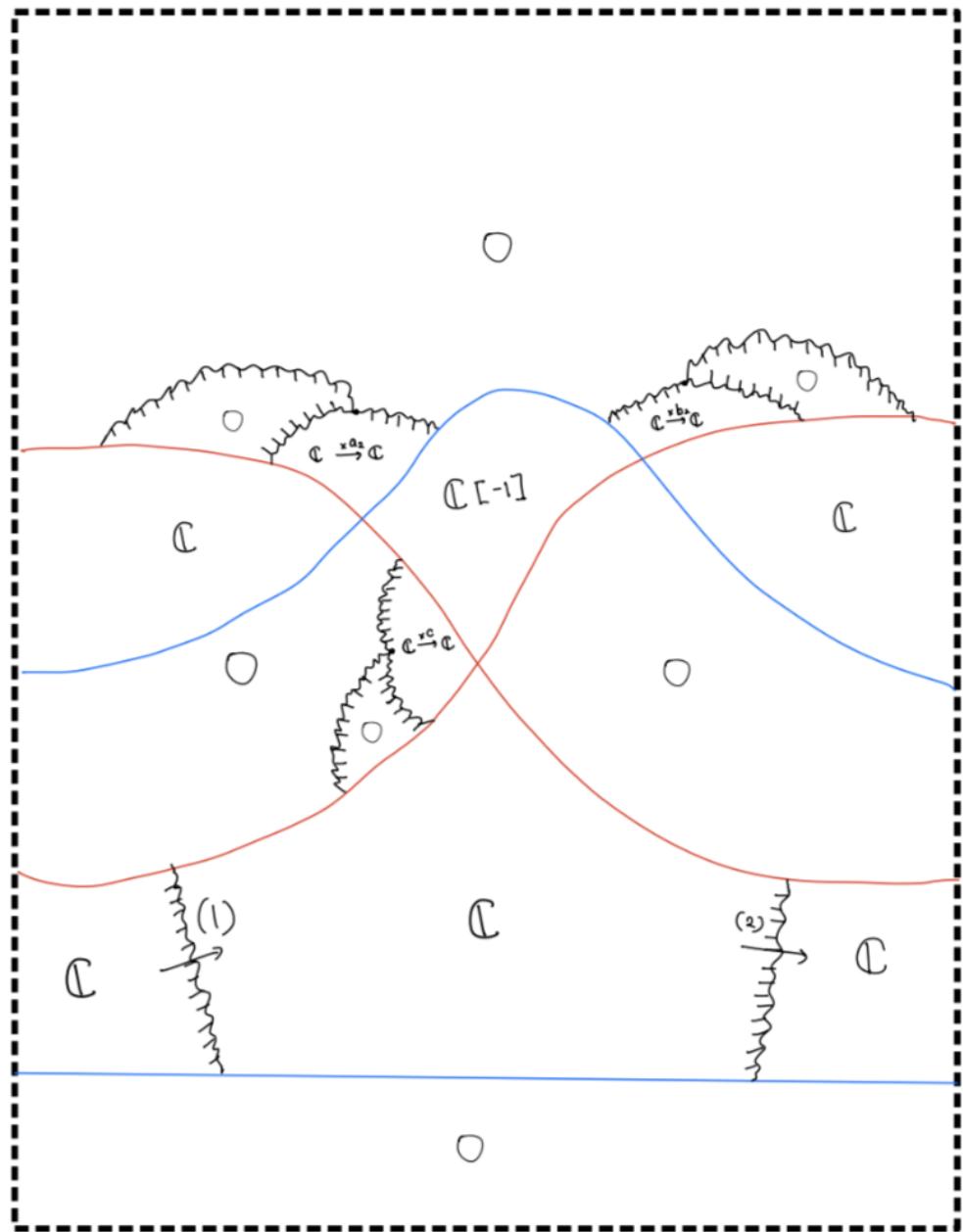


Figure 3.308

Generalization maps

(1) $\times a_1$

(2) $\times b_1^{-1}$

(Step 2) apply *cobord₈* to the region surrounded by a purple dotted line

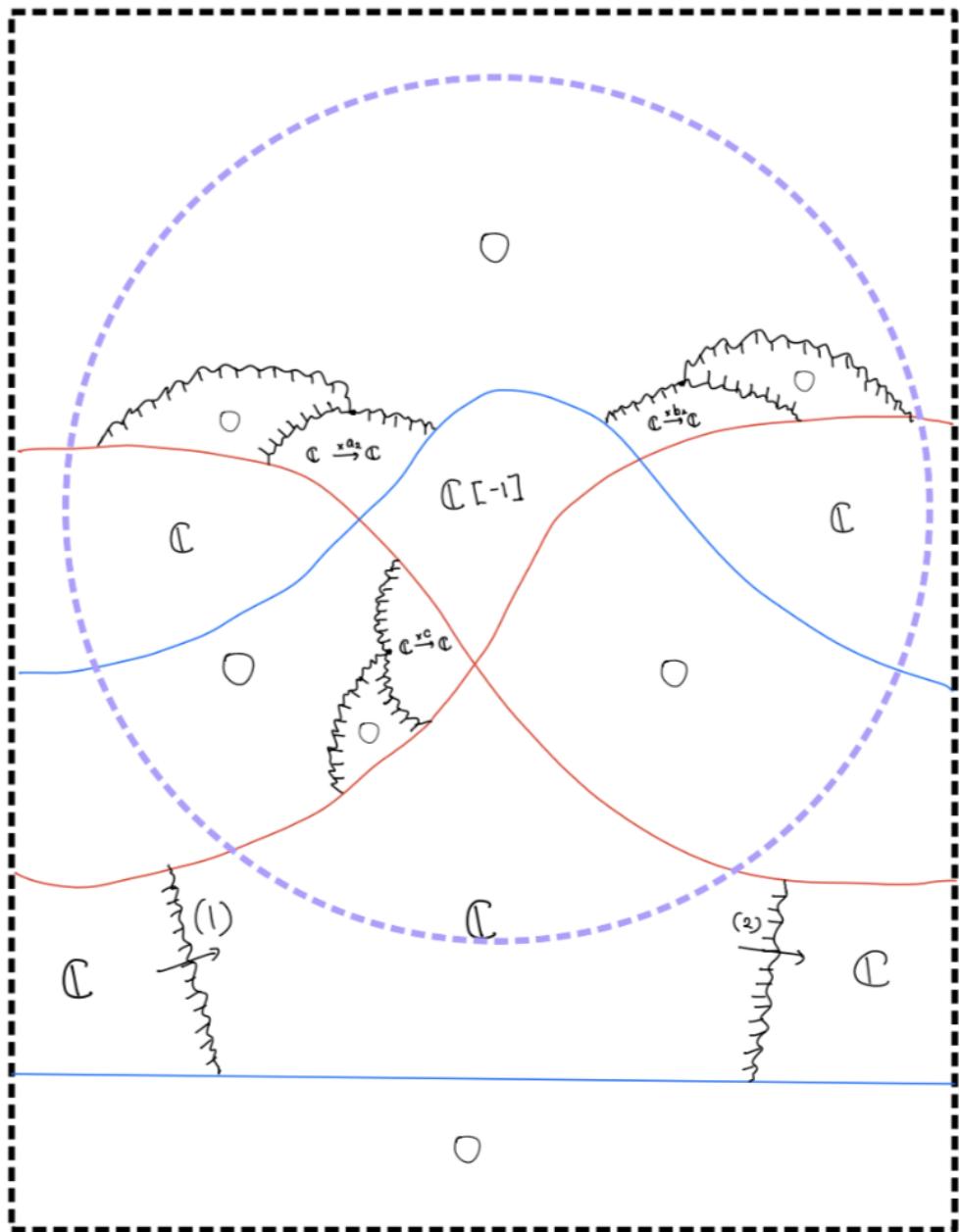


Figure 3.309

we get

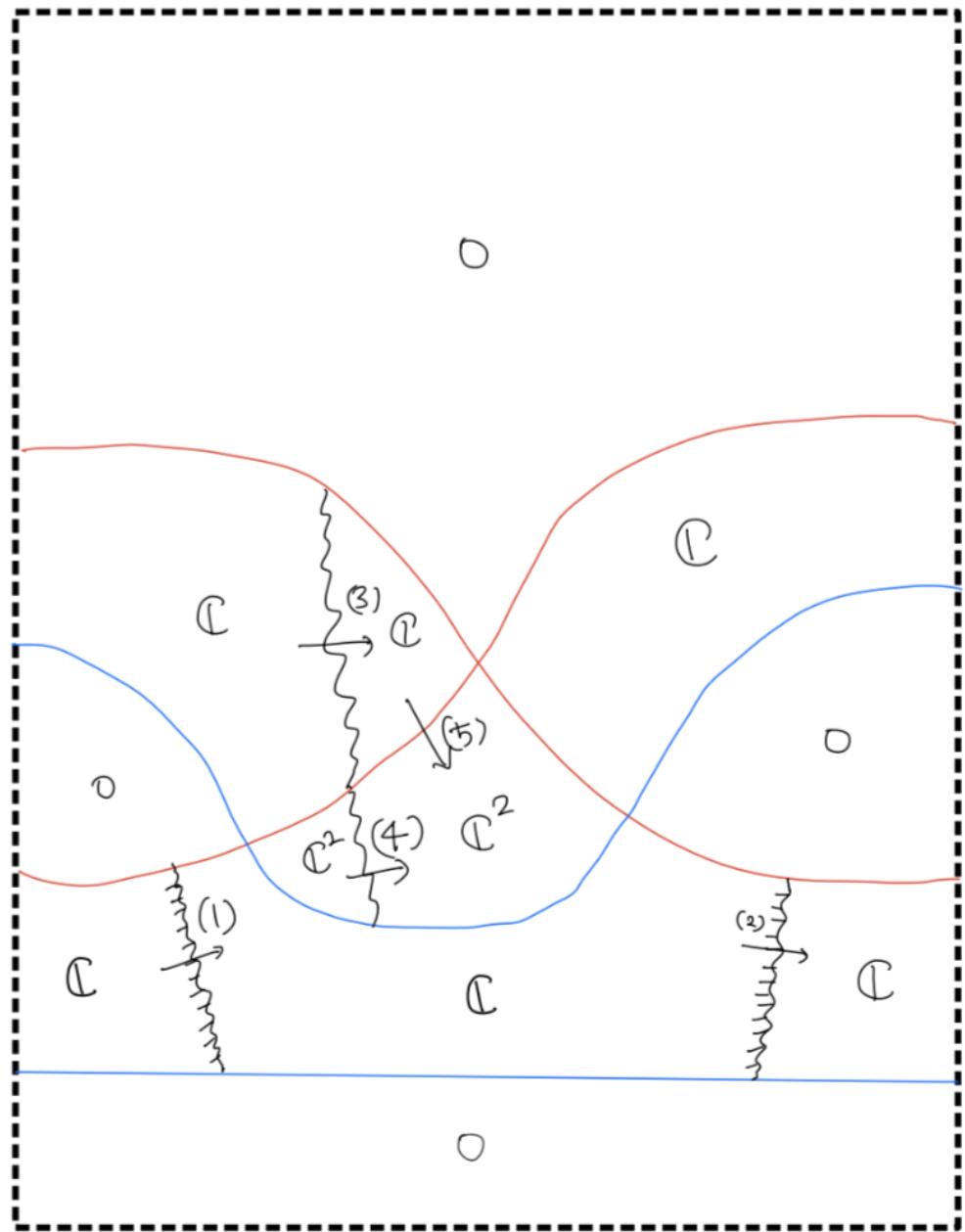


Figure 3.310

Generalization maps(1) $\times a_1$ (2) $\times b_1^{-1}$ (3) $\times a_2 b_2^{-1}$

$$(4) \begin{pmatrix} a_2 b_2^{-1} & 0 \\ -a_2 c^{-1} & 1 \end{pmatrix}$$

$$(5) \begin{pmatrix} 1 \\ -b_2 c^{-1} \end{pmatrix}$$

which is quasi-isomorphic to

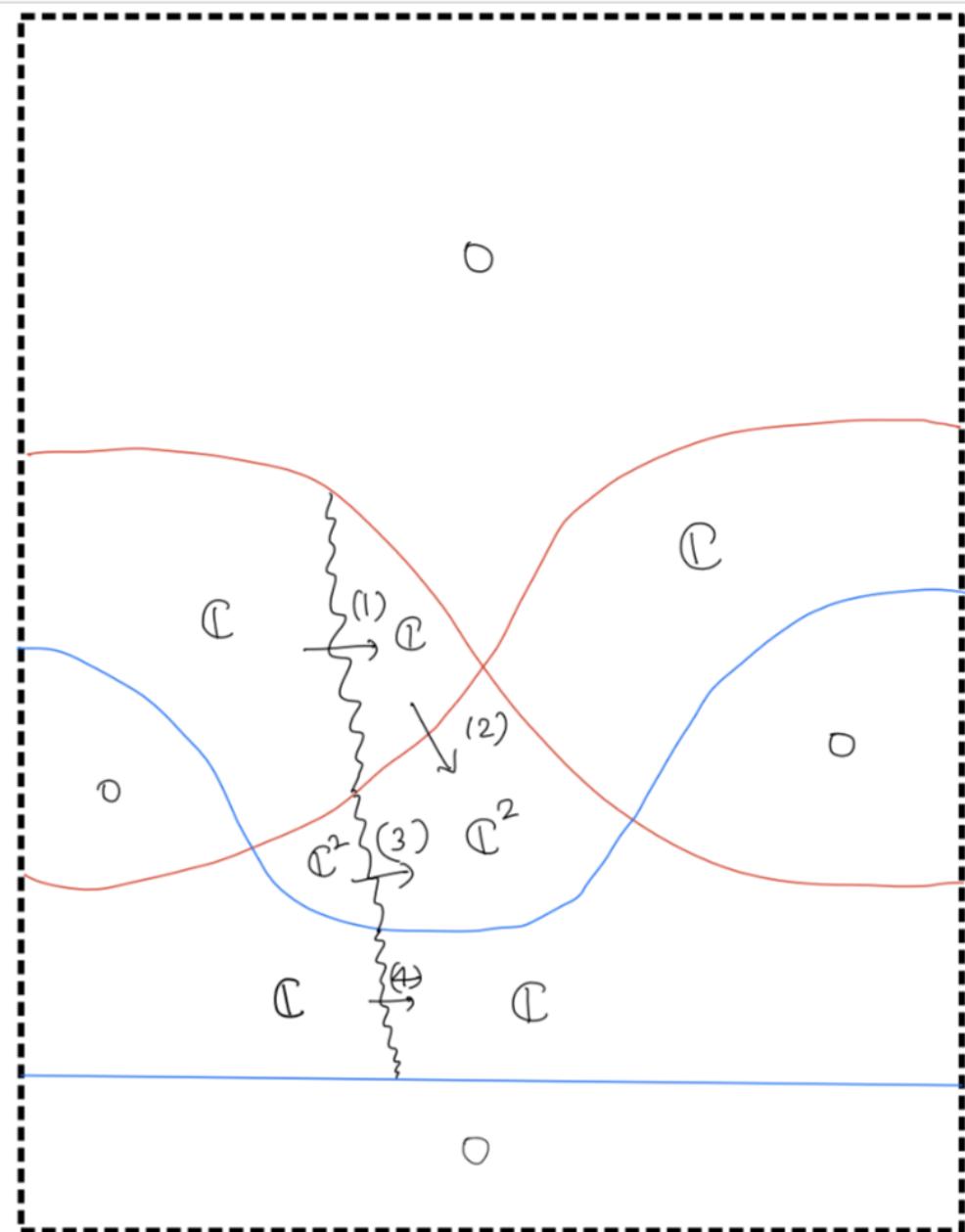


Figure 3.311

Generalization maps

$$\begin{aligned}
 (1) & \times d_2 \\
 (2) & \begin{pmatrix} 1 \\ e' \end{pmatrix} \\
 (3) & \begin{pmatrix} d_2 & 0 \\ e & d_1 \end{pmatrix} \\
 (4) & \times d_1
 \end{aligned}$$

where

- $d_r = a_r b_r^{-1}$
- $e = -a_2 b_1^{-1} c$
- $e' = d_2^{-1} e$

(ii) For $n > 2$,

(Case 1) if the generator s_i is $i \neq 1$,

(Step 1) we apply $cobord_{gen}(n-1)$ to the square region surrounded by purple dotted lines.

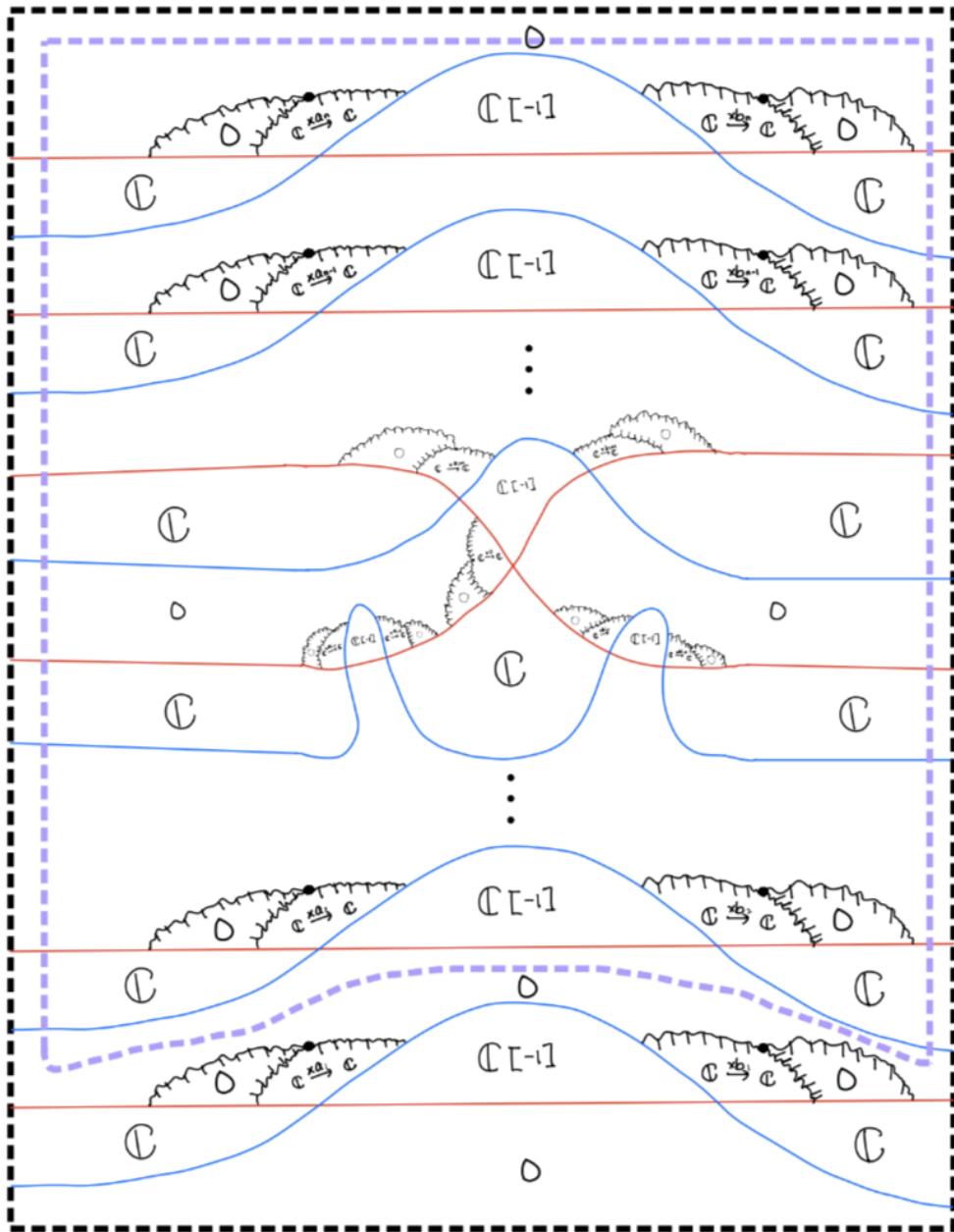


Figure 3.312

by induction hypothesis, we get

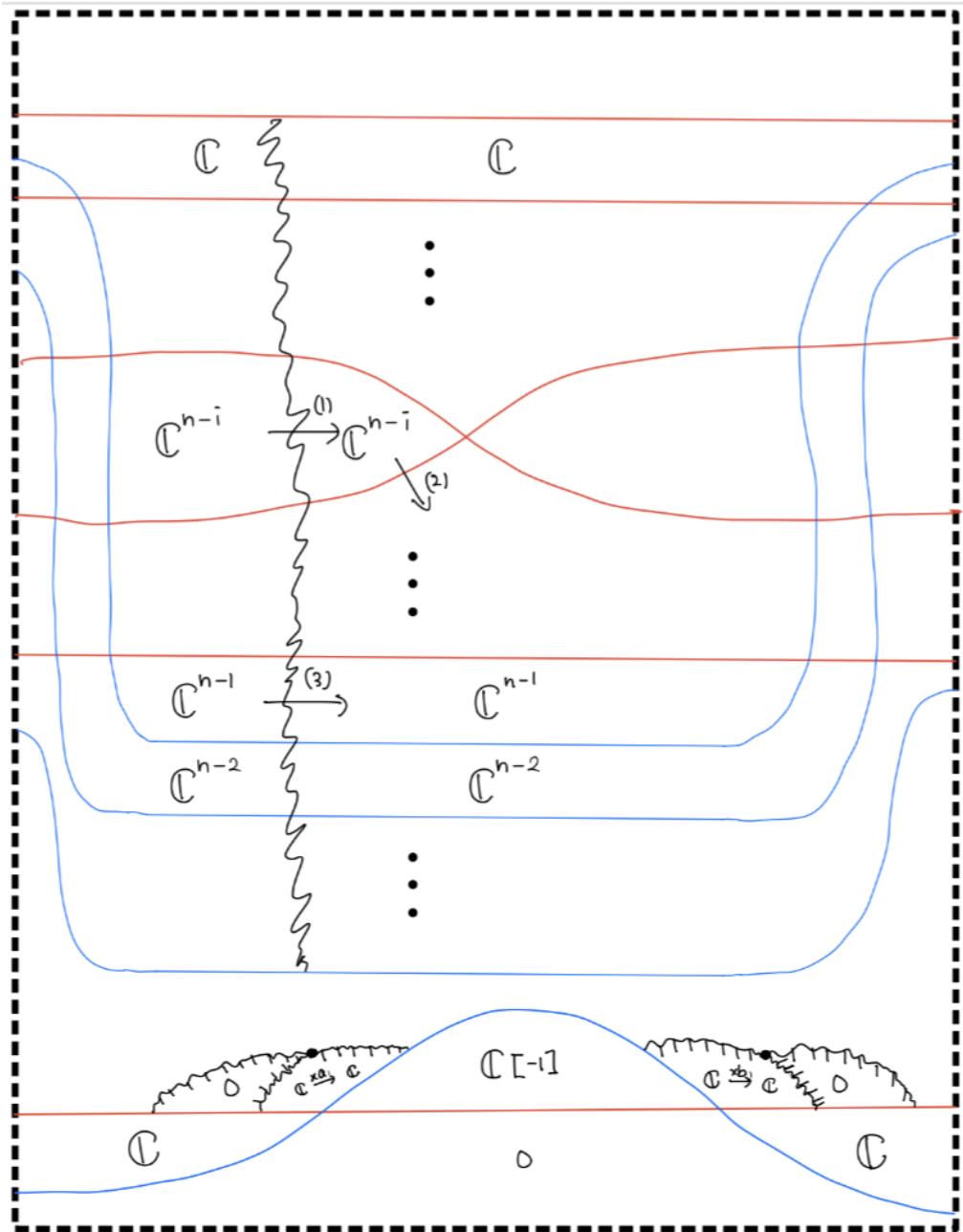


Figure 3.313

Generalization maps

- (1) $\text{diag}(d_n, \dots, d_{i+1})$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3) $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$

(Step 2) apply cobord_2 to the region surrounded by a purple dotted line.

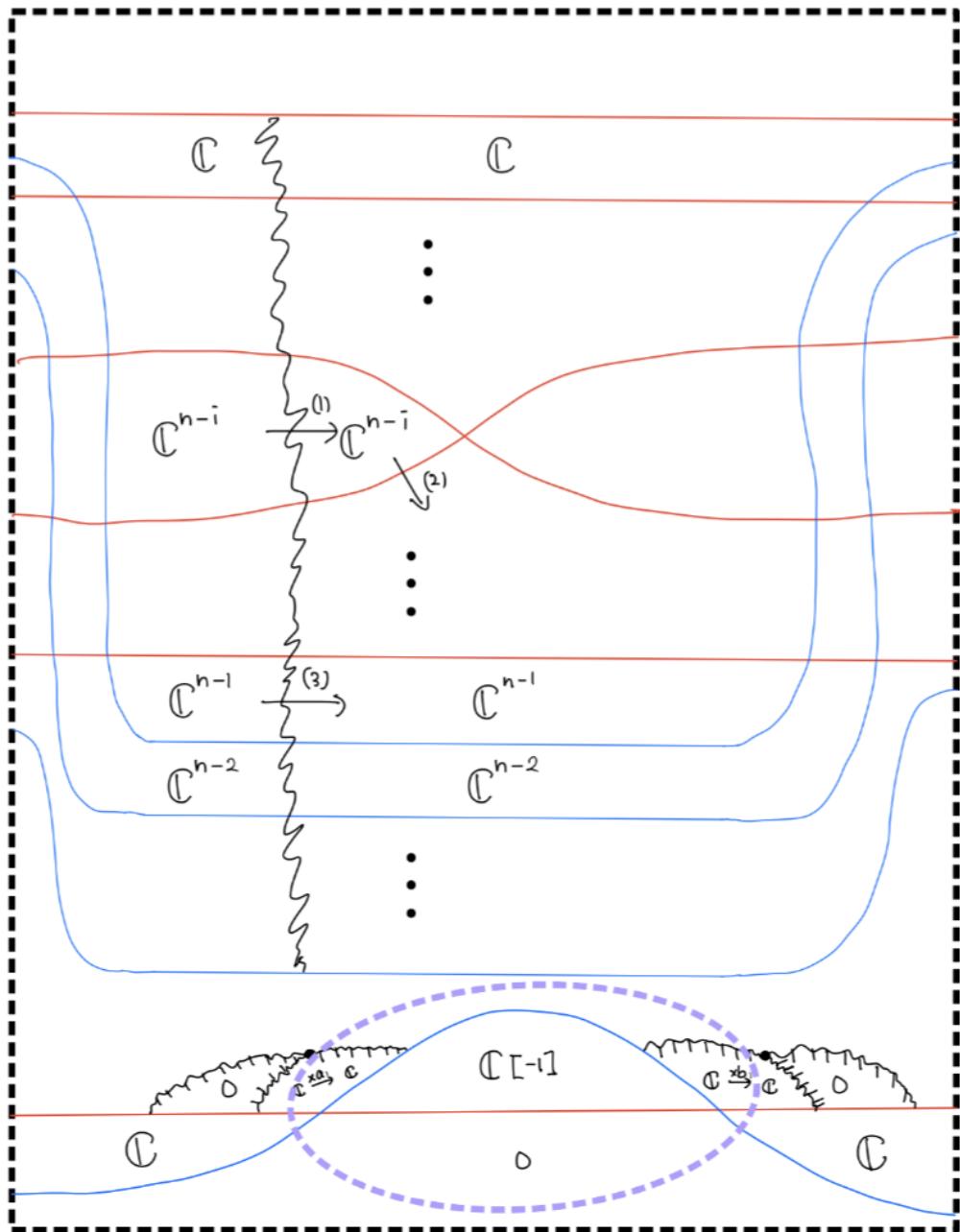


Figure 3.314

we get

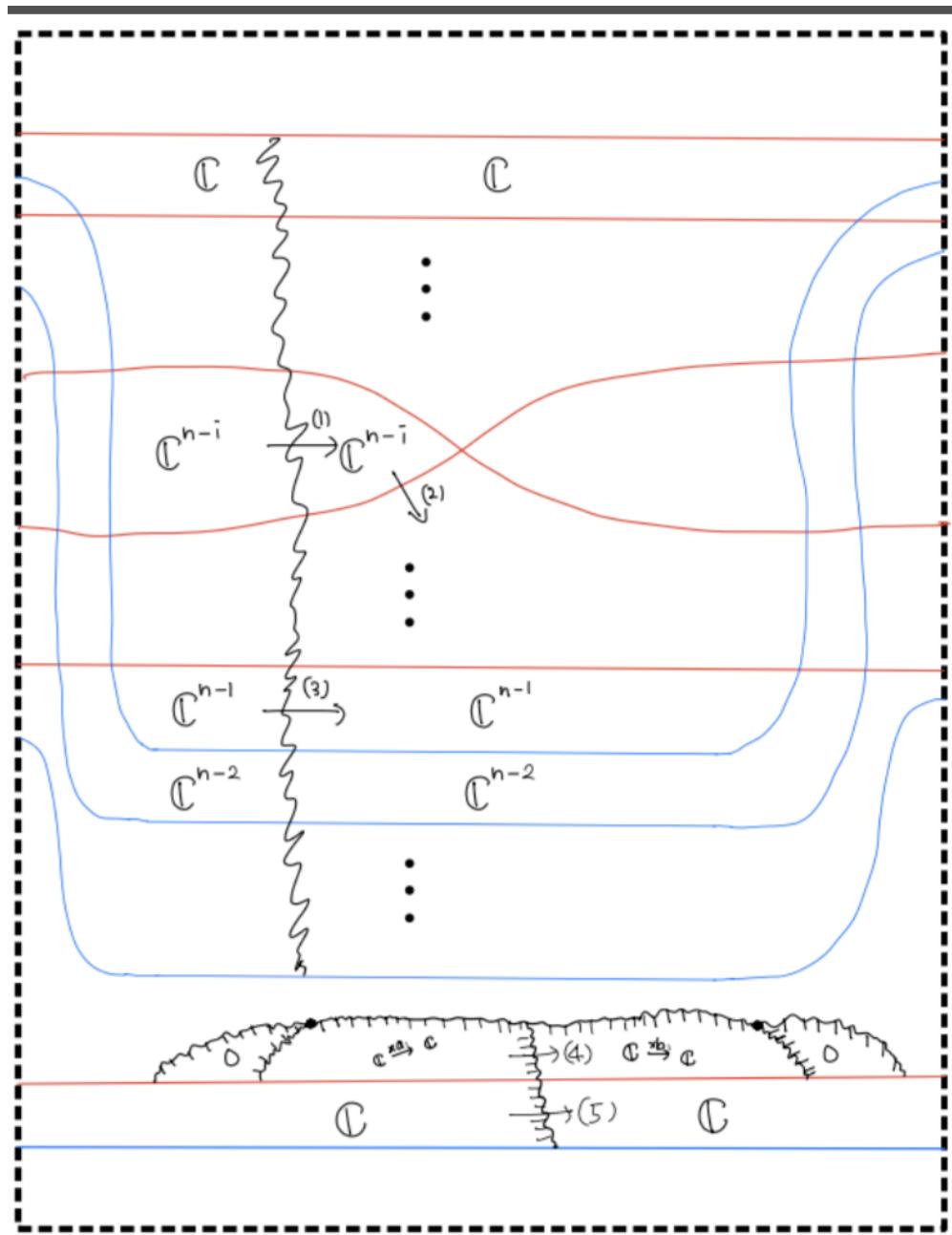


Figure 3.315

Generalization maps

- (1) $\text{diag}(d_n, \dots, d_{i+1})$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3) $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a_1 \uparrow & & \uparrow \times b_1 \\ \mathbb{C} & \xrightarrow{\times d_1} & \mathbb{C} \end{array}$$

(5) $\times d_1$

which is quasi-isomorphic to

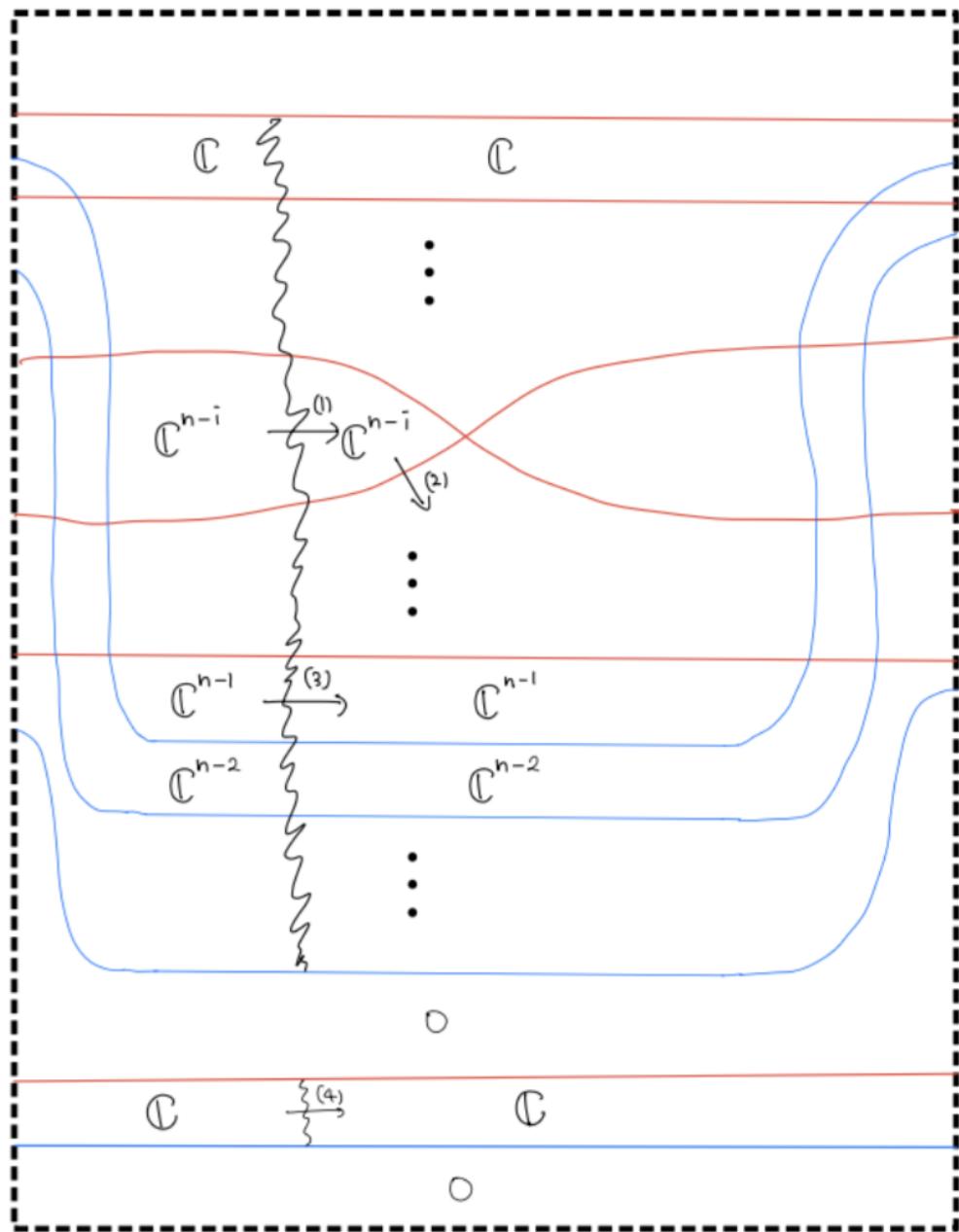


Figure 3.316

Generalization maps

- (1) $\text{diag}(d_n, \dots, d_{i+1})$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3) $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$
- (4) $\times d_1$

(Step 3) apply cobord_6 to the square region surrounded by purple dotted lines.

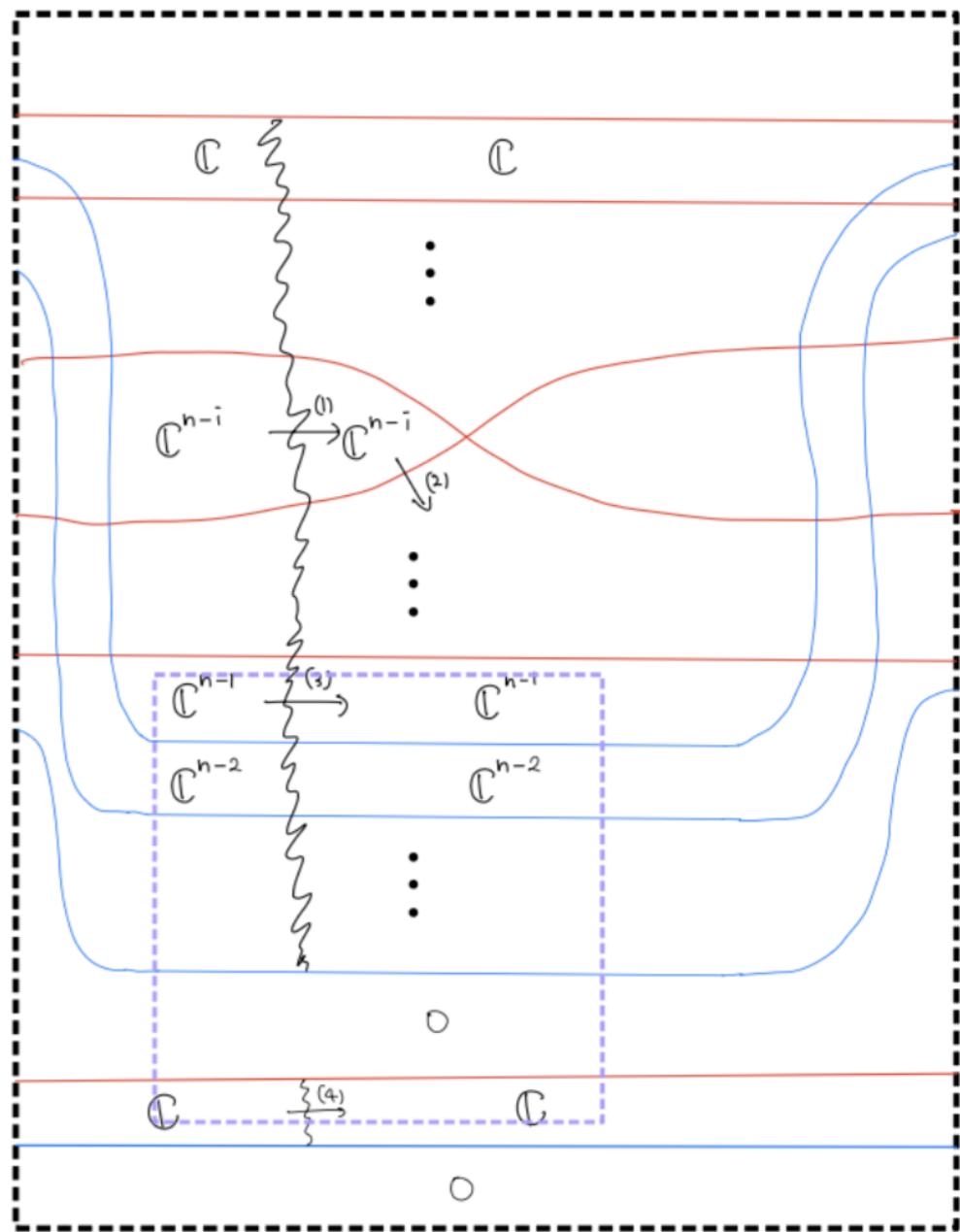


Figure 3.317

we get the final sheaf

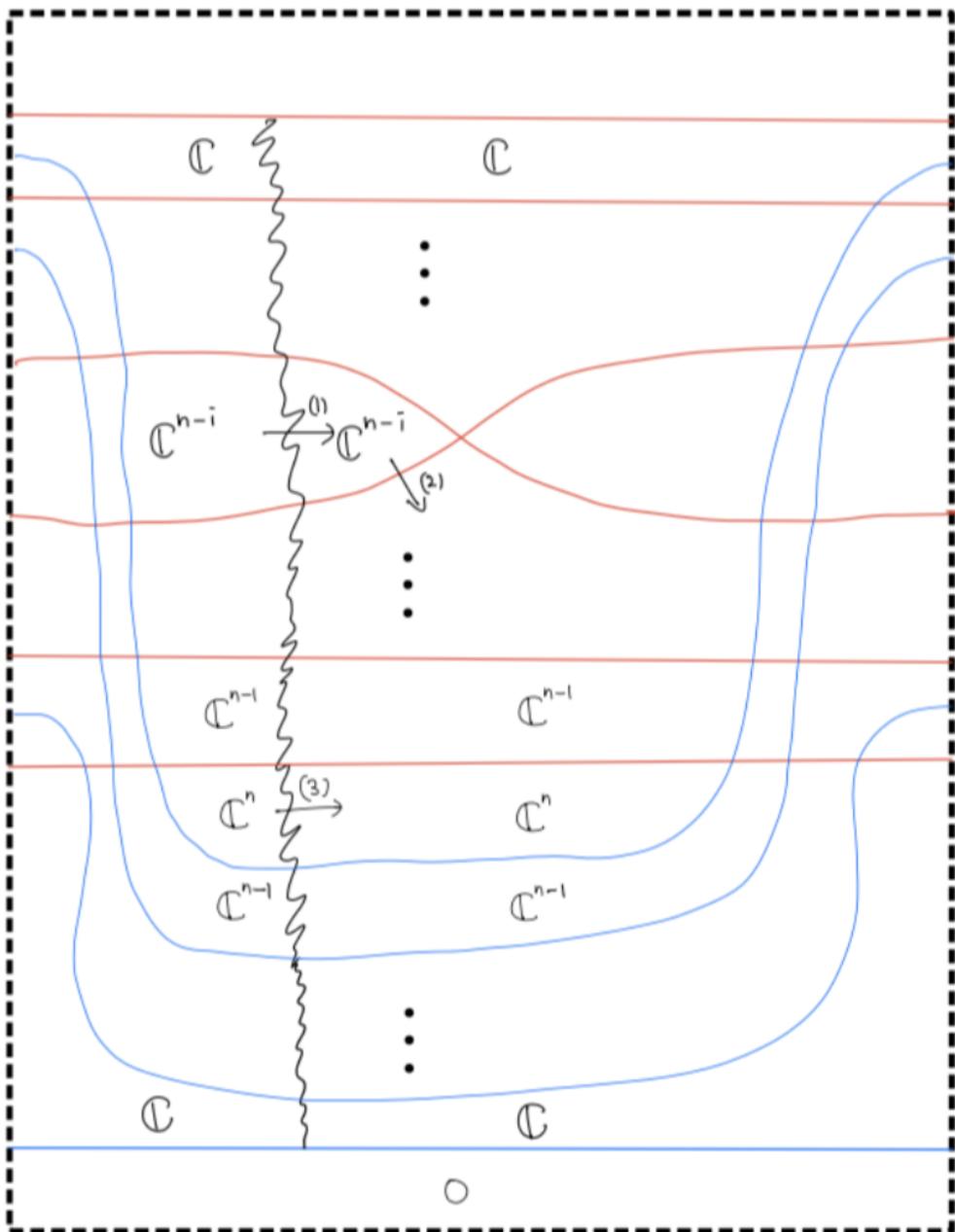


Figure 3.318

Generalization maps

- (1) $\text{diag}(d_n, \dots, d_{i+1})$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3) $\text{diag}(d_n, \dots, d_1) + e I_{n-i+1, n-i}$

(Case 2) If the generator s_i is $i = 1$,

(Step 1) apply $cobord_{gen}(n - 1)$ to the square region surrounded by purple dotted lines

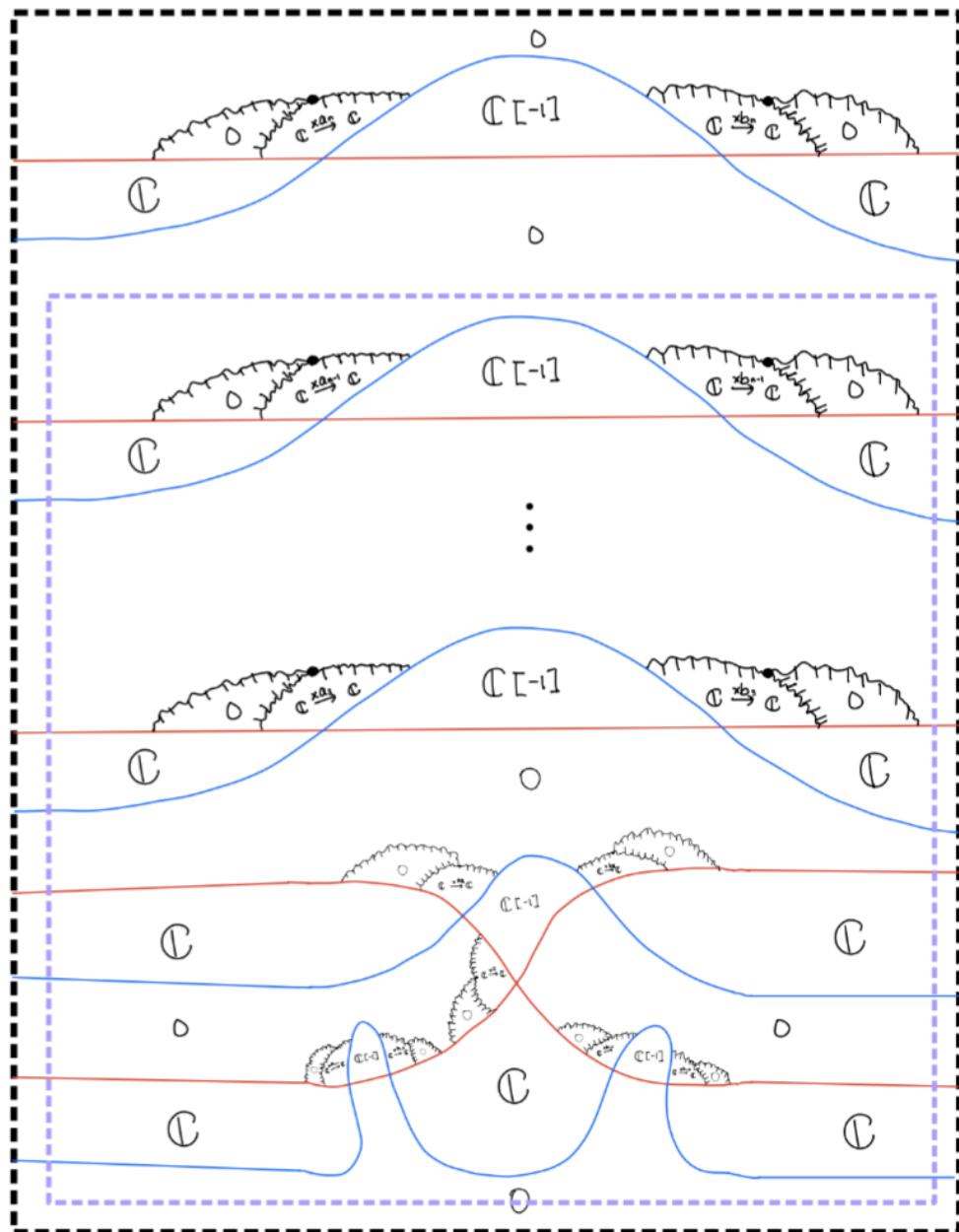


Figure 3.319

we get

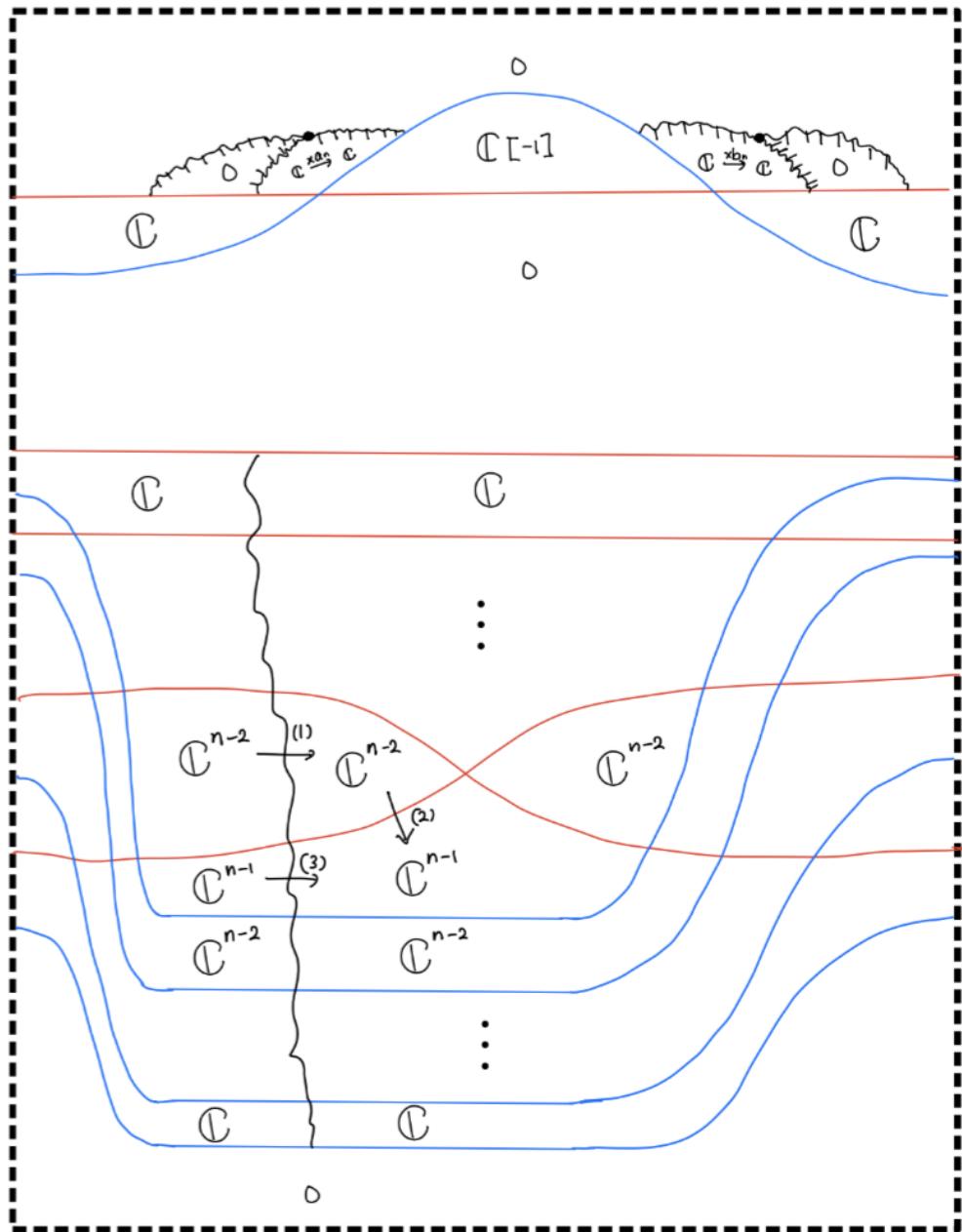


Figure 3.320

Generalization maps

- (1) $\text{diag}(d_{n-1}, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3) $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$

(Step 2) apply cobord_2 to the region surrounded by a purple dotted line

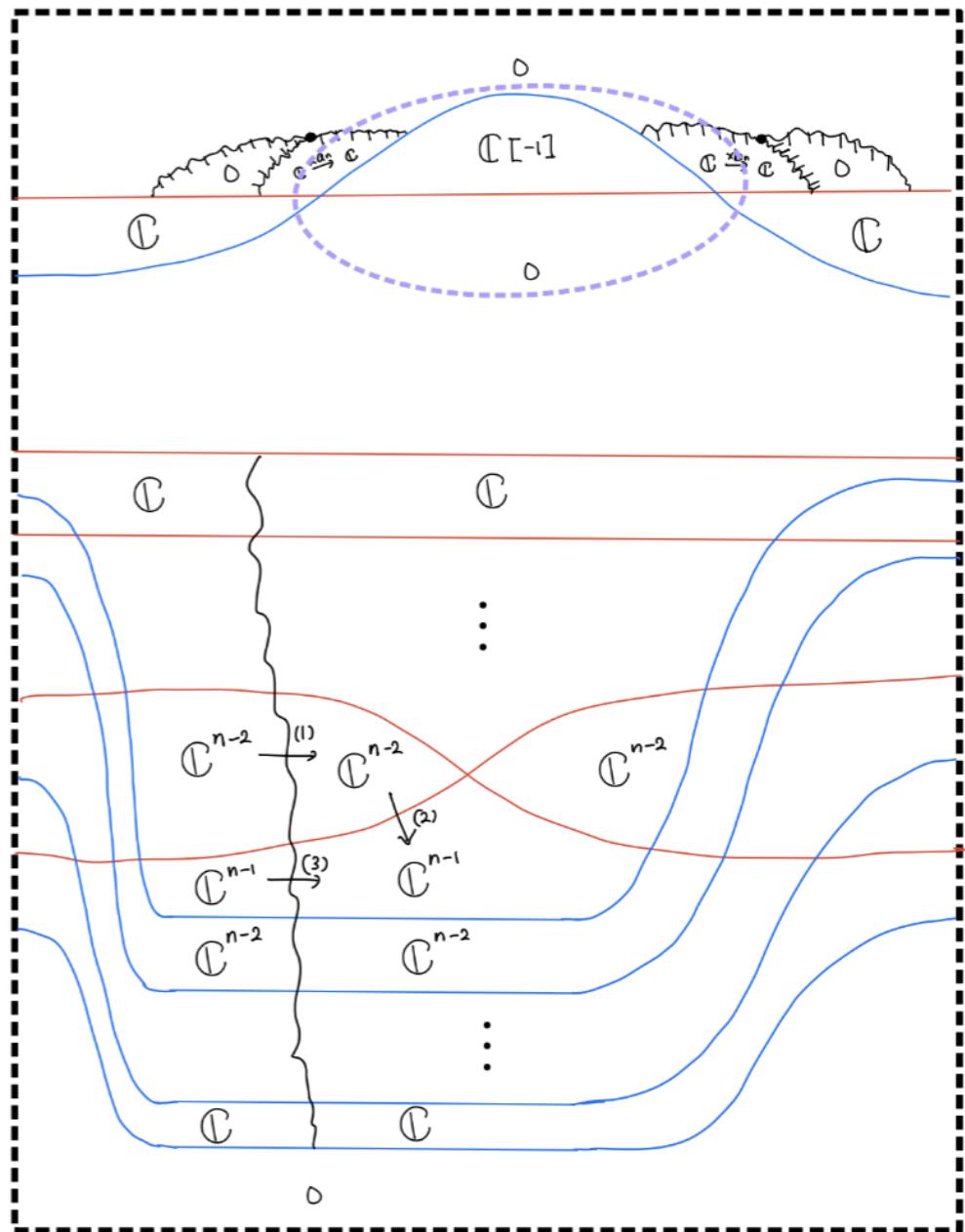


Figure 3.321

we get

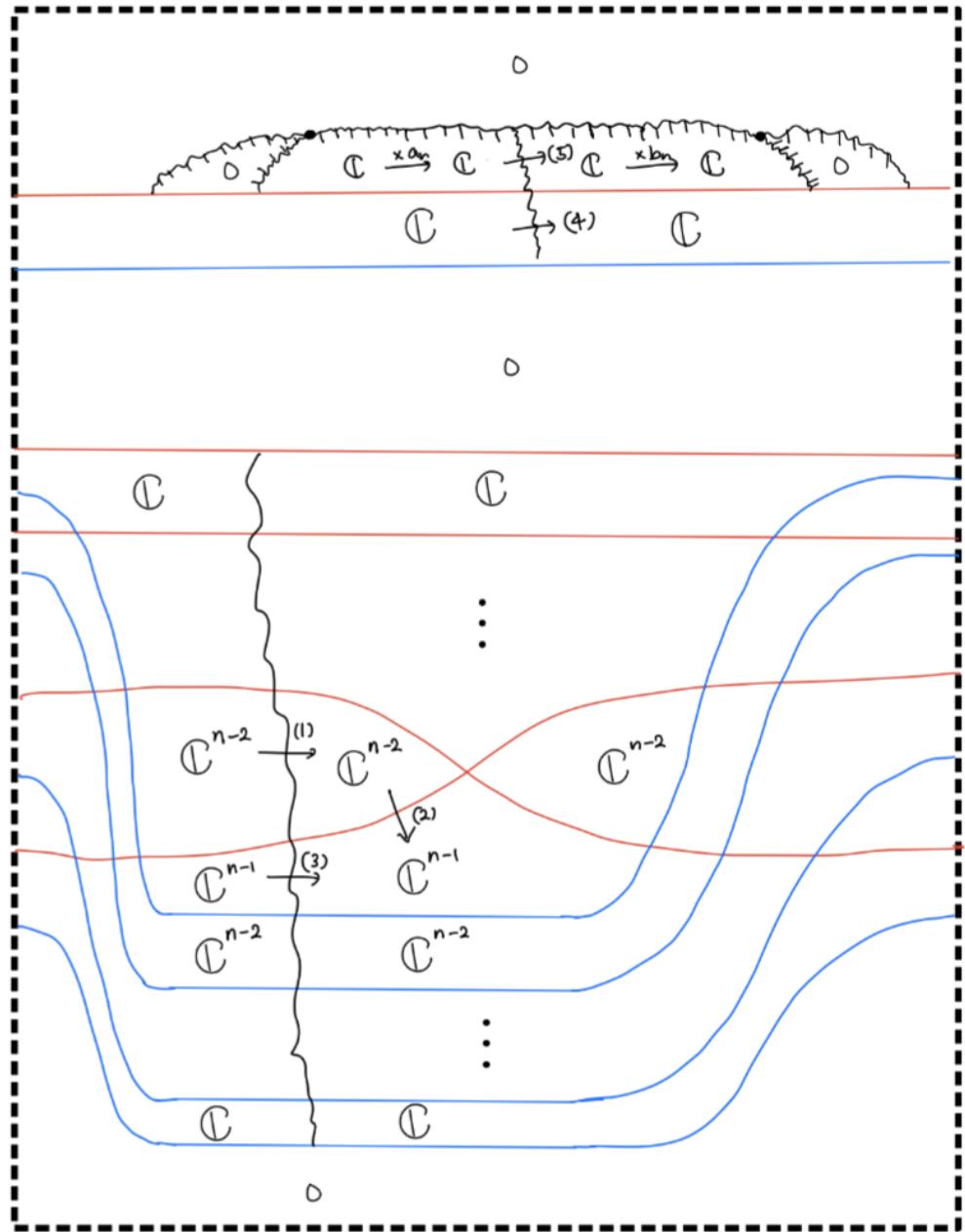


Figure 3.322

Generalization maps

- (1) $\text{diag}(d_{n-1}, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3) $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$
- (4) $\times d_n$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times d_n \uparrow & & \uparrow \times b_n \\ \mathbb{C} & \xrightarrow{\times d_n} & \mathbb{C} \end{array}$$

which is quasi-isomorphic to

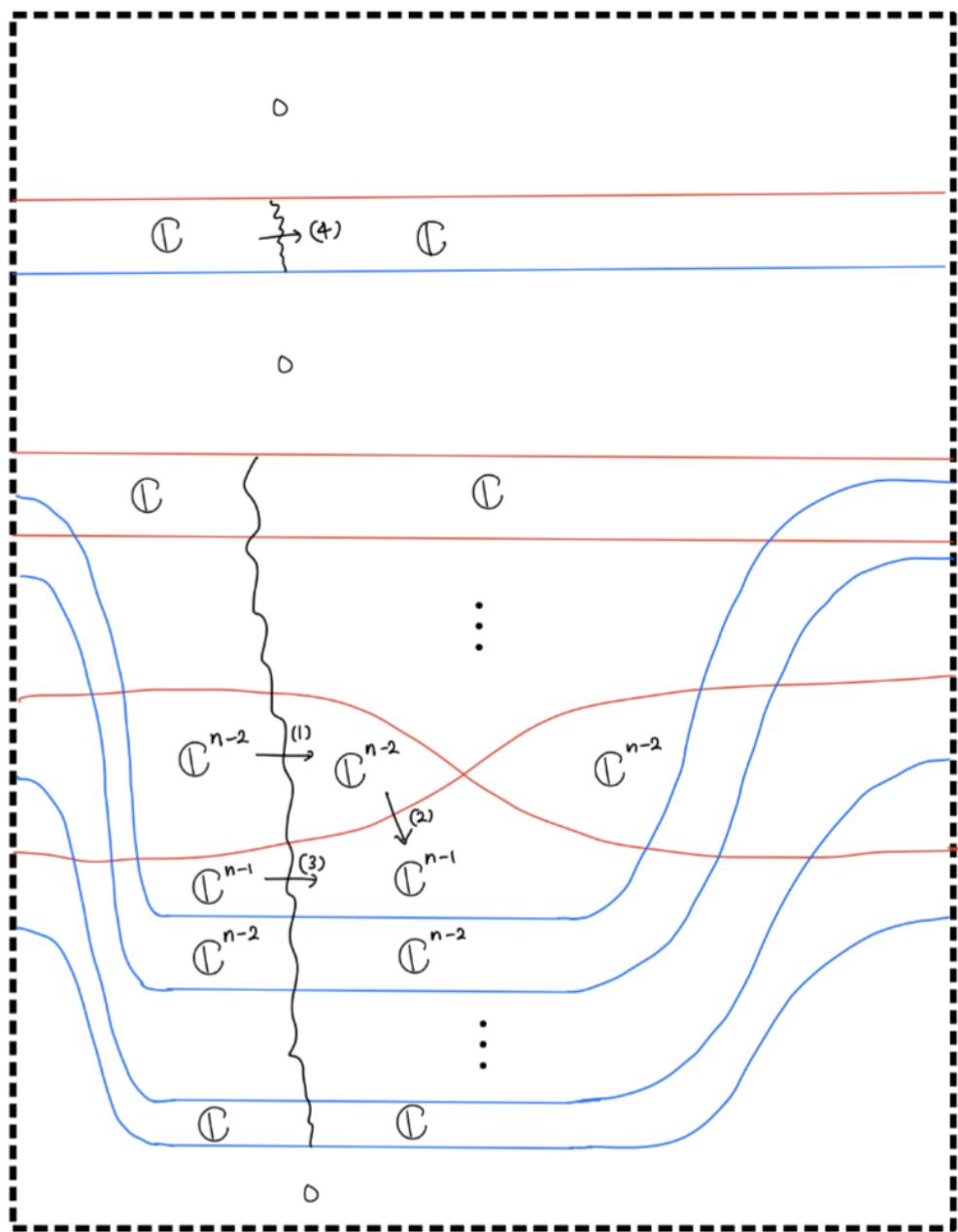


Figure 3.323

Generalization maps

(1) $\text{diag}(d_{n-1}, \dots, d_2)$

$$(2) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$$

$$(3) \quad \text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$$

$$(4) \quad \times d_n$$

(Step 3) apply cobord_5 to the square region surrounded by purple dotted lines

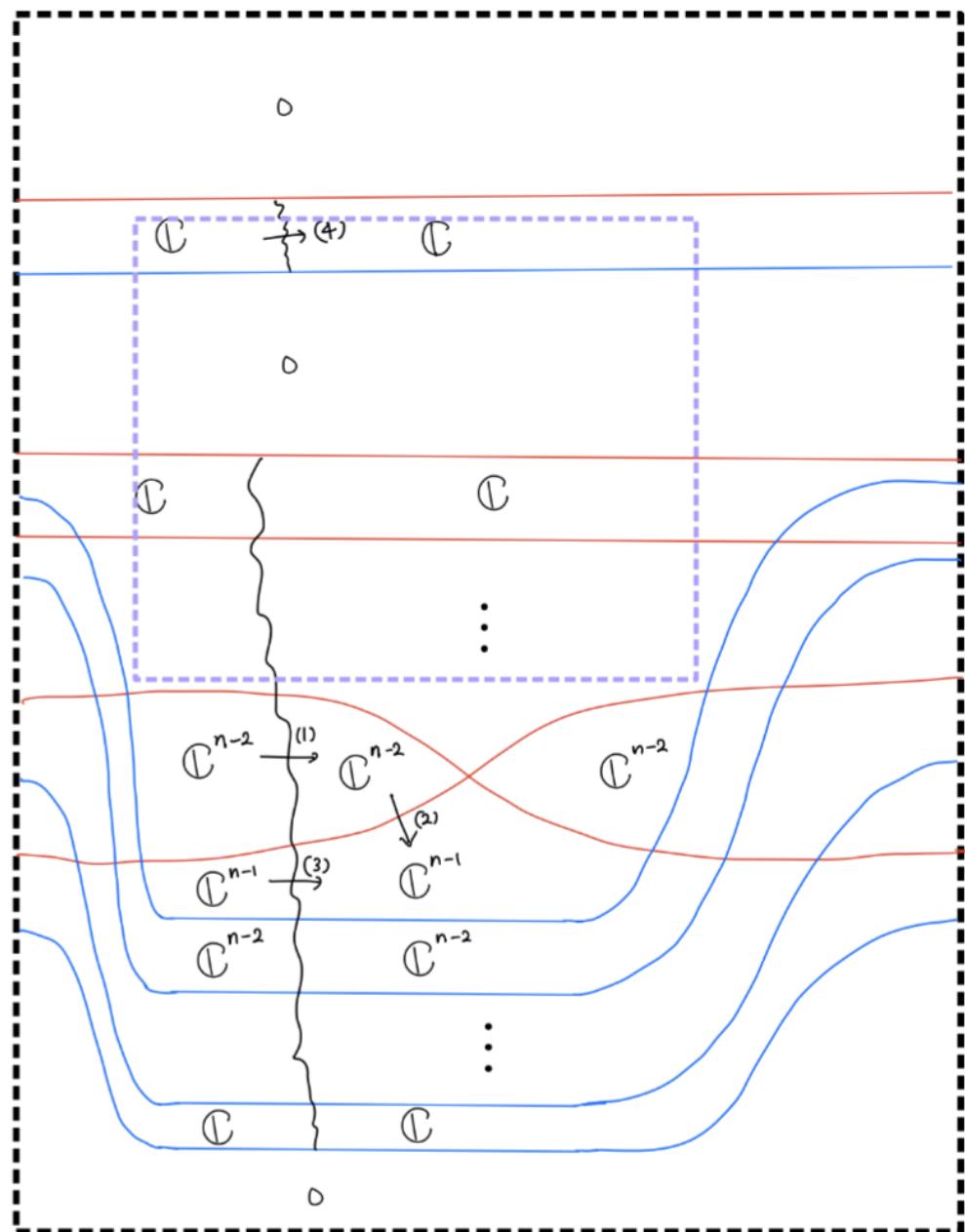


Figure 3.324

we get

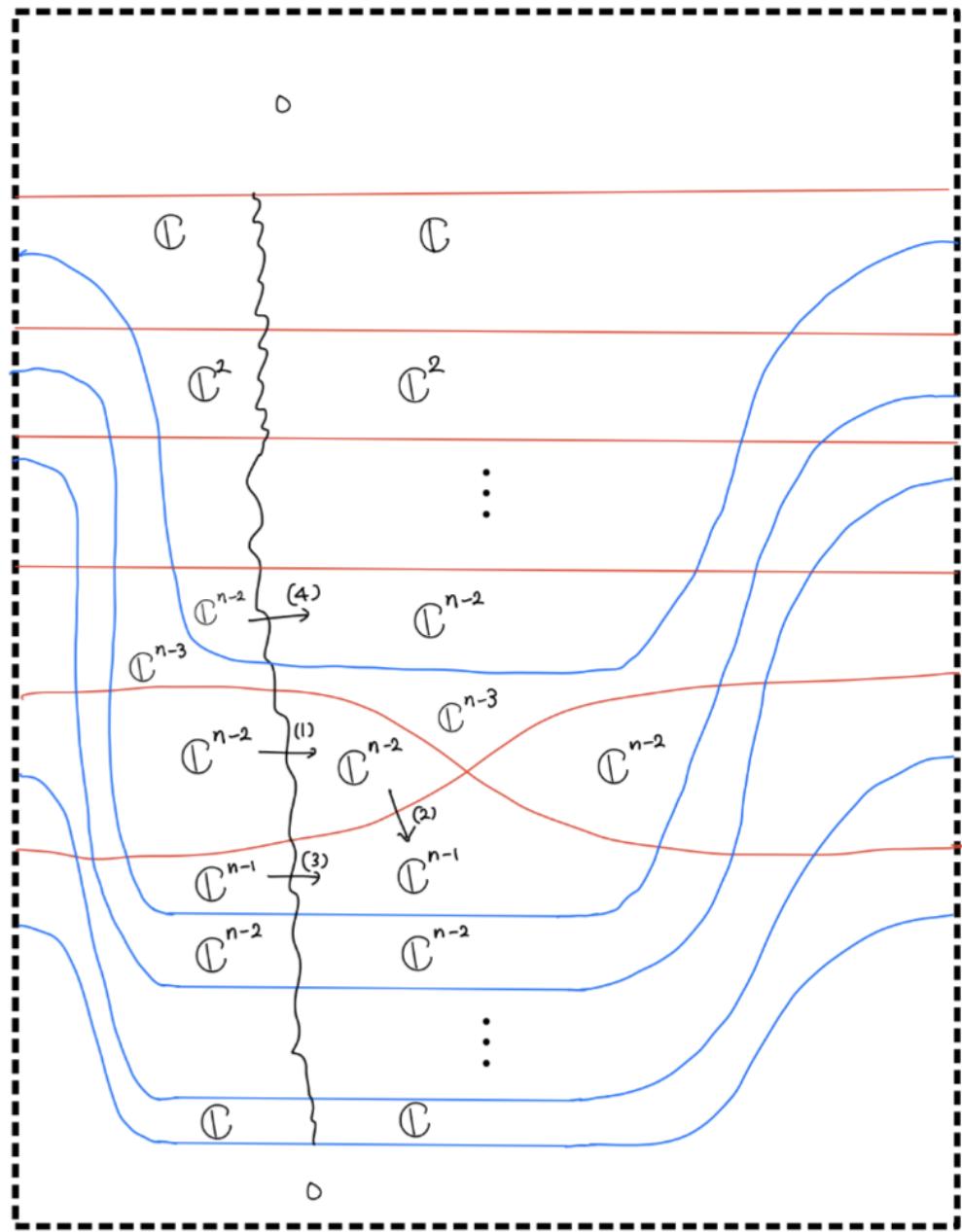


Figure 3.325

Generalization maps

- (1) $\text{diag}(d_{n-1}, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3) $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$
- (4) $\text{diag}(d_n, \dots, d_3)$

(Step 4) apply $cobord_1$ to the regions surrounded by purple dotted lines

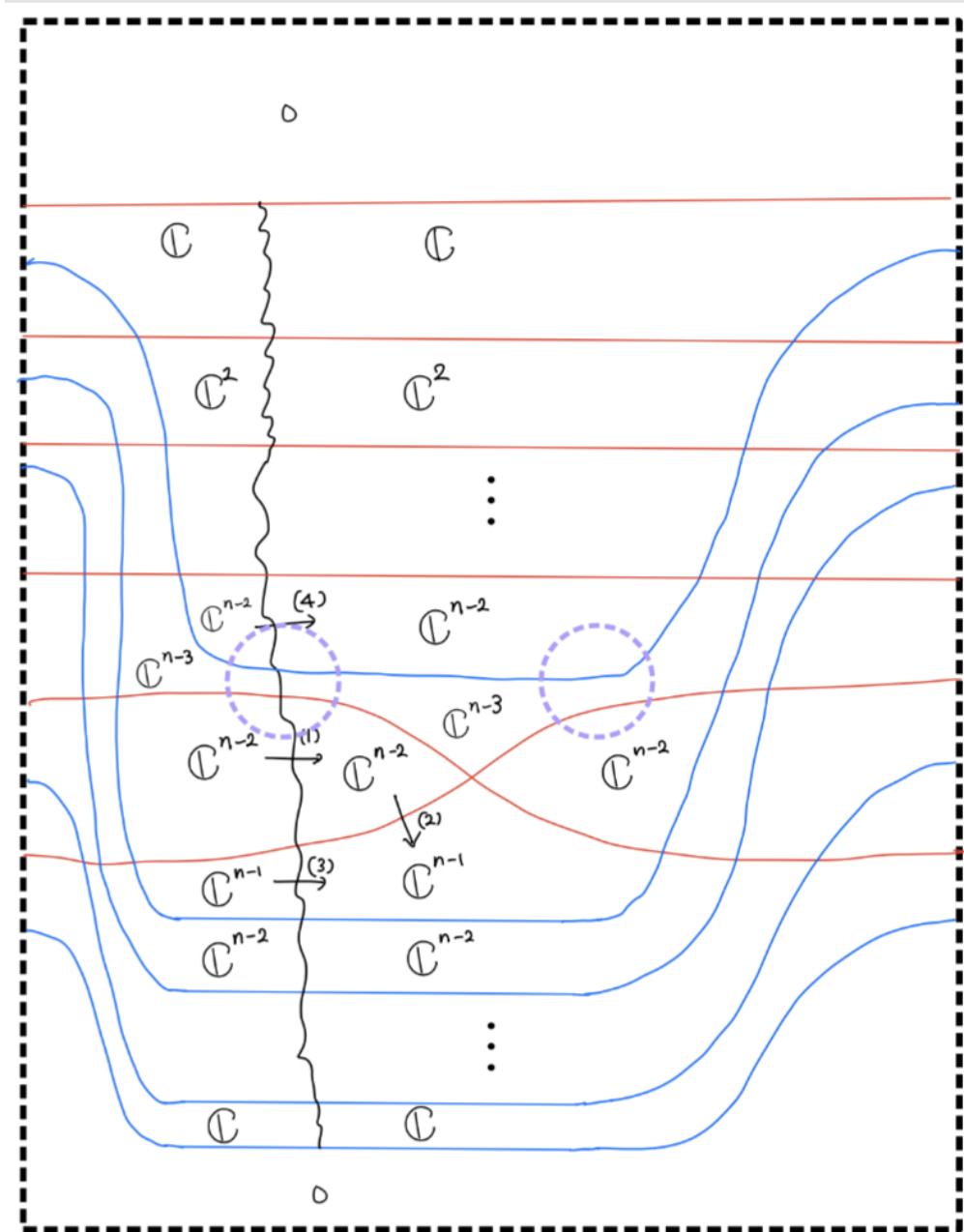


Figure 3.326

we get

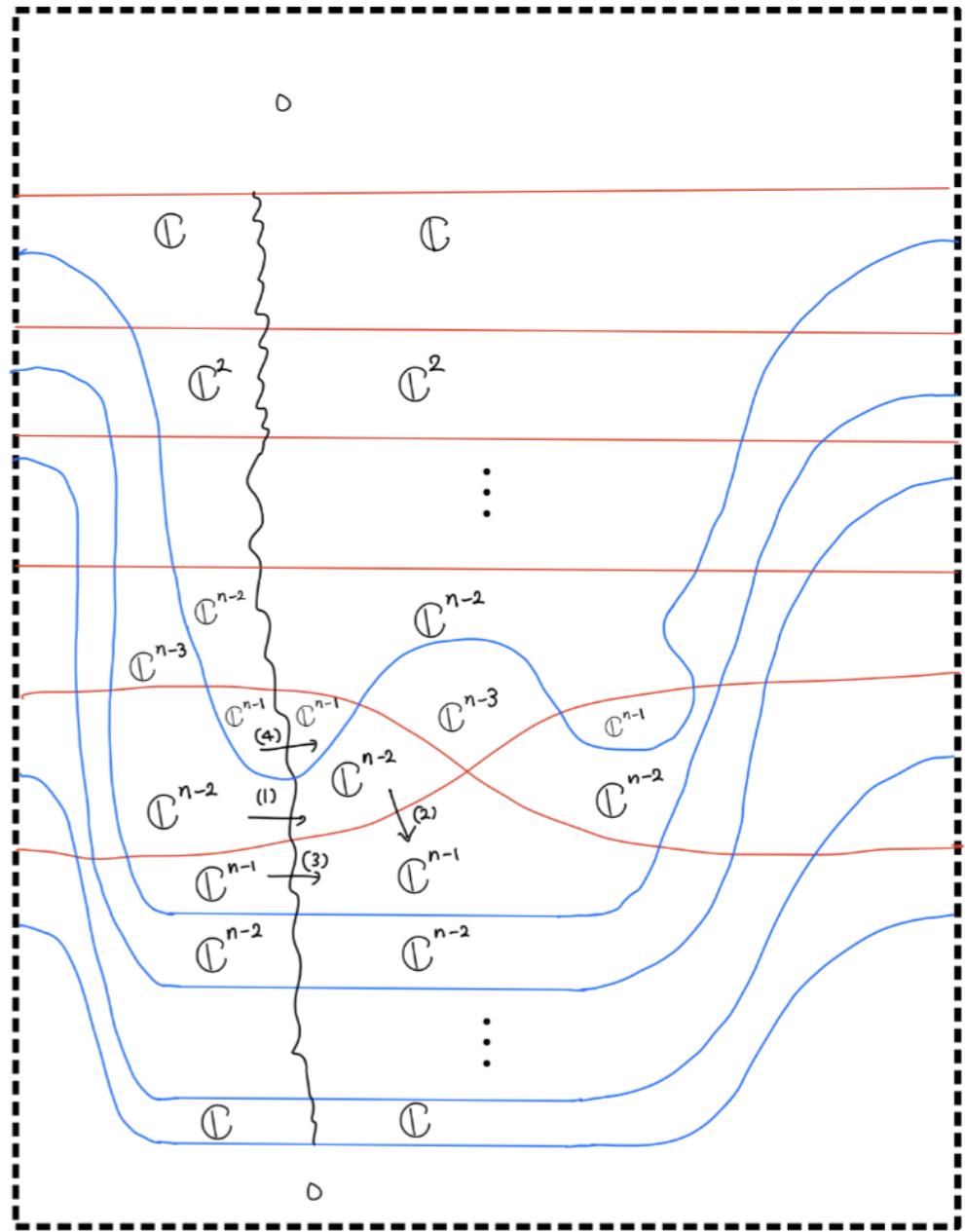


Figure 3.327

Generalization maps

- (1) $\text{diag}(d_{n-1}, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3) $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$
- (4) $\text{diag}(d_n, \dots, d_2)$

(Step 5) apply $cobord'_8$ to the region surrounded by purple dotted lines

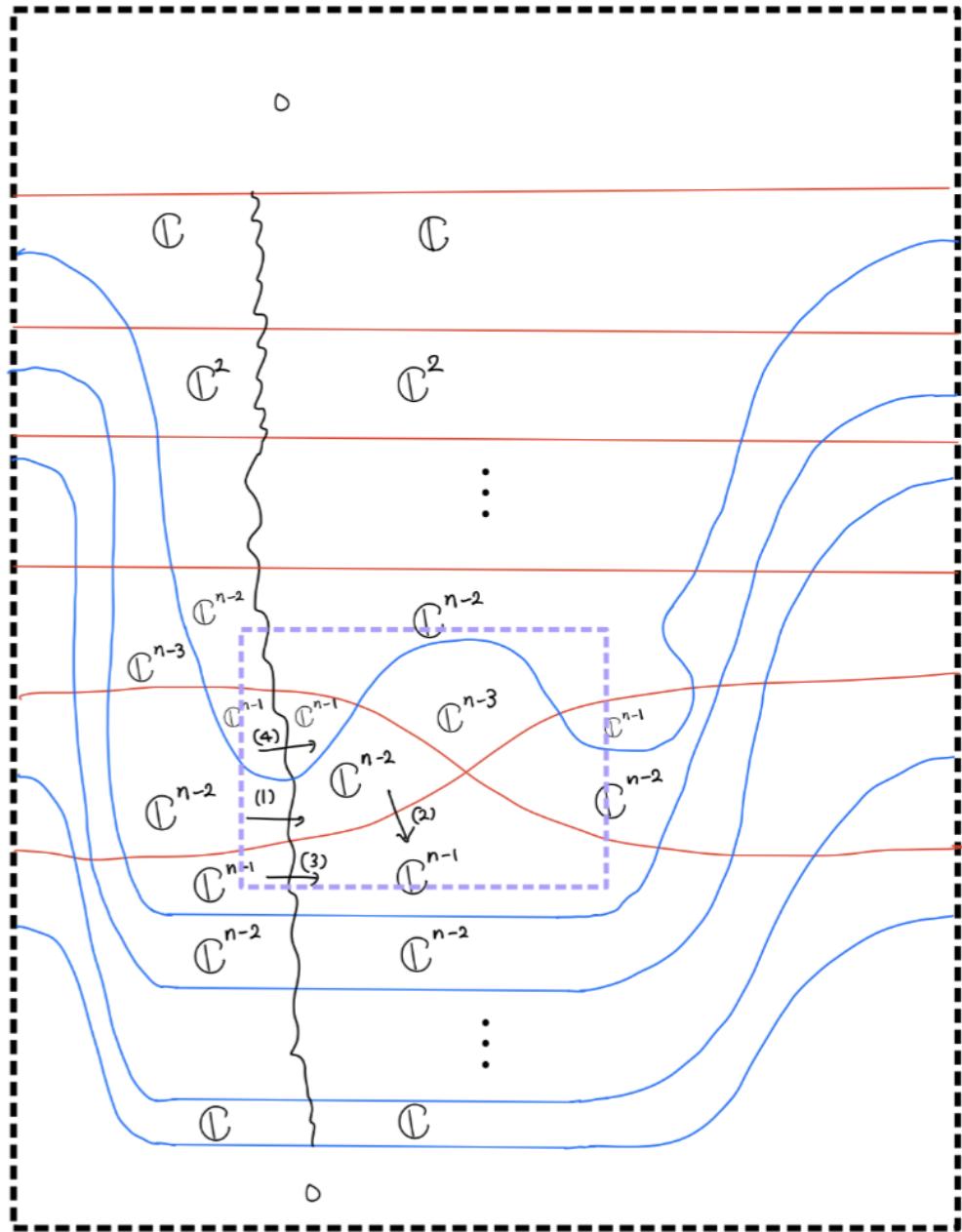


Figure 3.328

we get the final sheaf

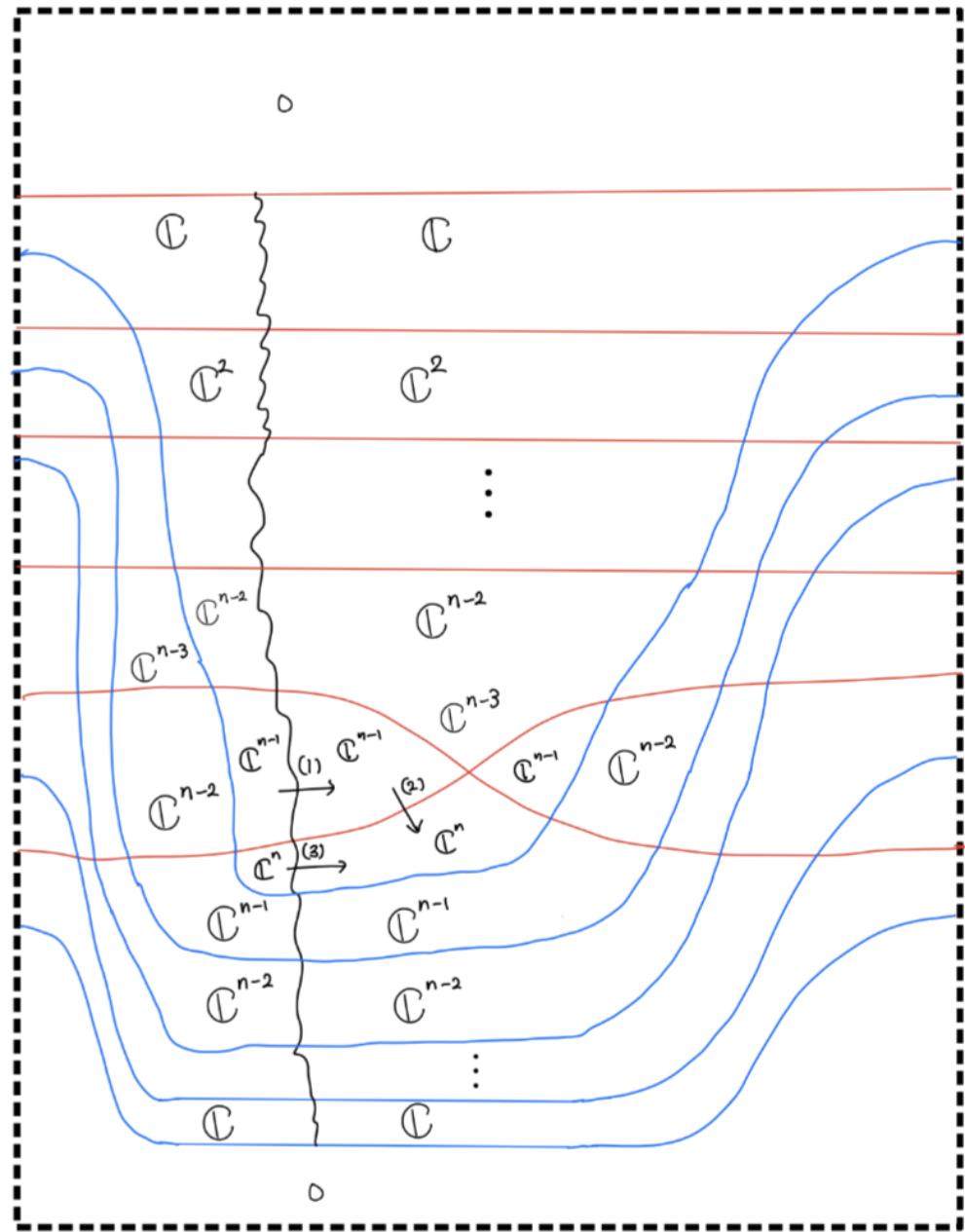


Figure 3.329

Generalization maps

- (1) $\text{diag}(d_n, \dots, d_2)$
- (2) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$
- (3) $\text{diag}(d_n, \dots, d_1) + e I_{n,n-1}$

3.16 Sheaf Cobordism on Inter-generator Regions

Definition 219. a bump function about $x = x_0$ of radius r_0 , with z range from z_{lo} to z_{hi} is a bump function

- supported on $[x_0 - r_0, x_0 + r_0]$
- peaks at (x_0, z_{hi})
- away from support, has value z_{lo}

We denote the bump function about $x = 0$ of radius 1 by $\Psi(z_{lo}, z_{hi})$

Suppose we have a punctured Riemann sphere M and Λ_0^0 , Λ_0^∞ , Λ_0^{squig} , a nested regions $U \subset U' \subset M$, and a chart $f : U \rightarrow \mathbb{R}^2$ such that U' maps to $R := (-1, 1)_x \times (-n-1, n+1)_z$ under f

- Λ_0^0 gets mapped to $\bigcup_{k=1}^n \{(x, z) \in R \mid z = \Psi(-k, n-k+\frac{1}{2})(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup_{k=1}^n \{(x, z) \in R \mid z = k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x, z) \in R \mid x = 0\}$, co-oriented toward the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

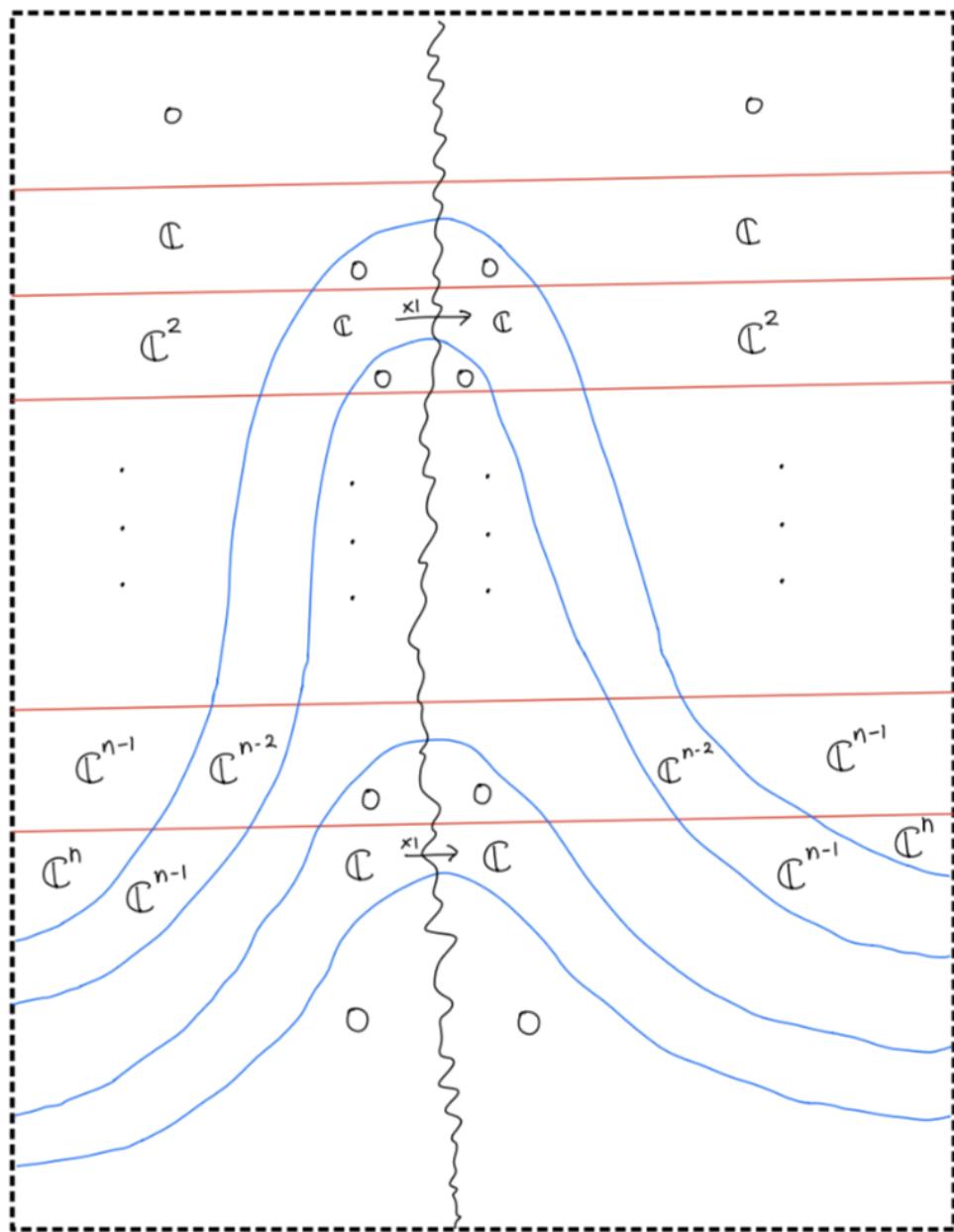


Figure 3.330

which is quasi-isomorphic to

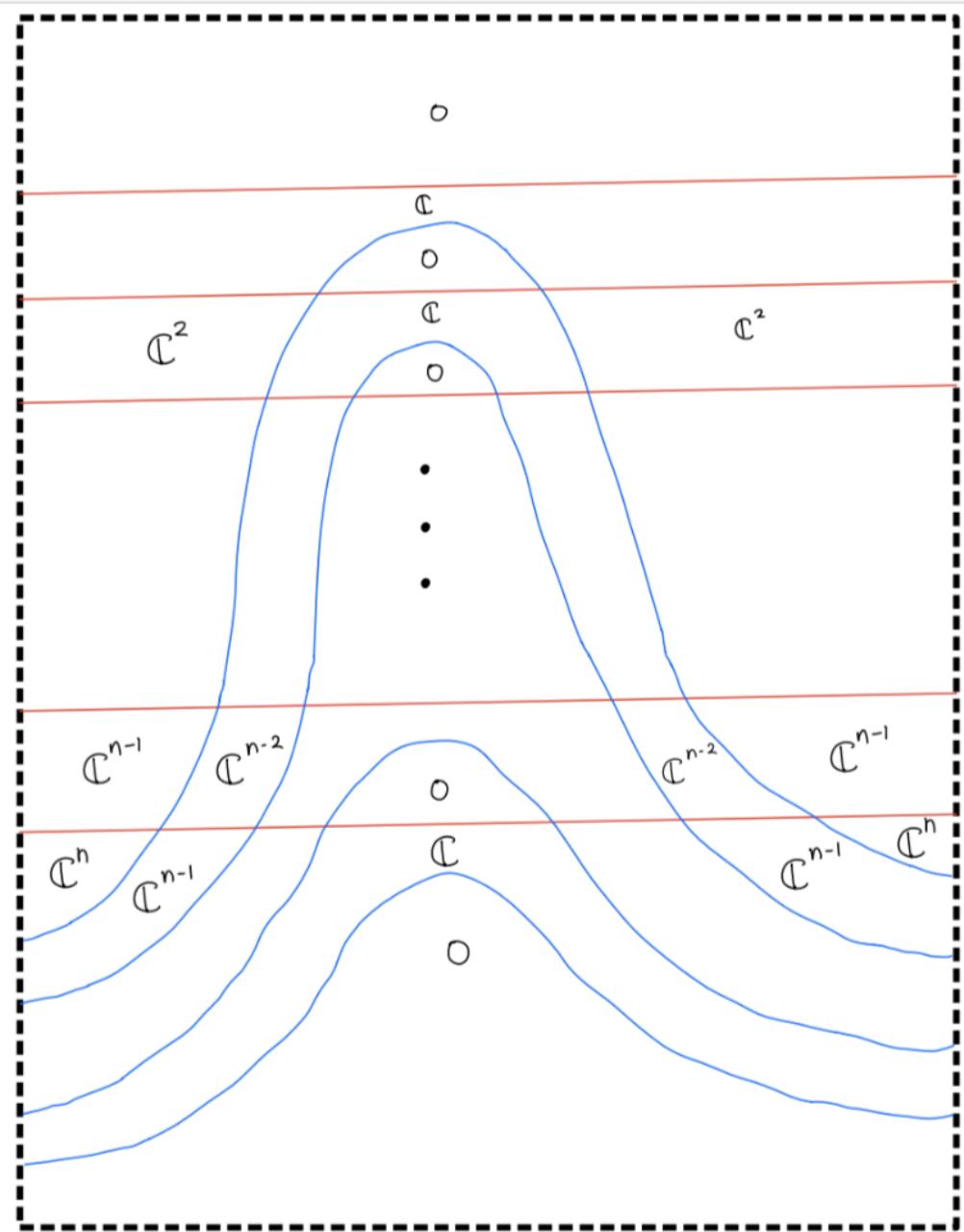


Figure 3.331

Then we define a cobordism starting from the above sheaf, say $cobord_{inter}(n)$ supported on U , where n is the number of blue strands (equivalently red strands). At the end of the cobordism, the sheaf, under the same chart f , is described as the following squiggly legible diagram.

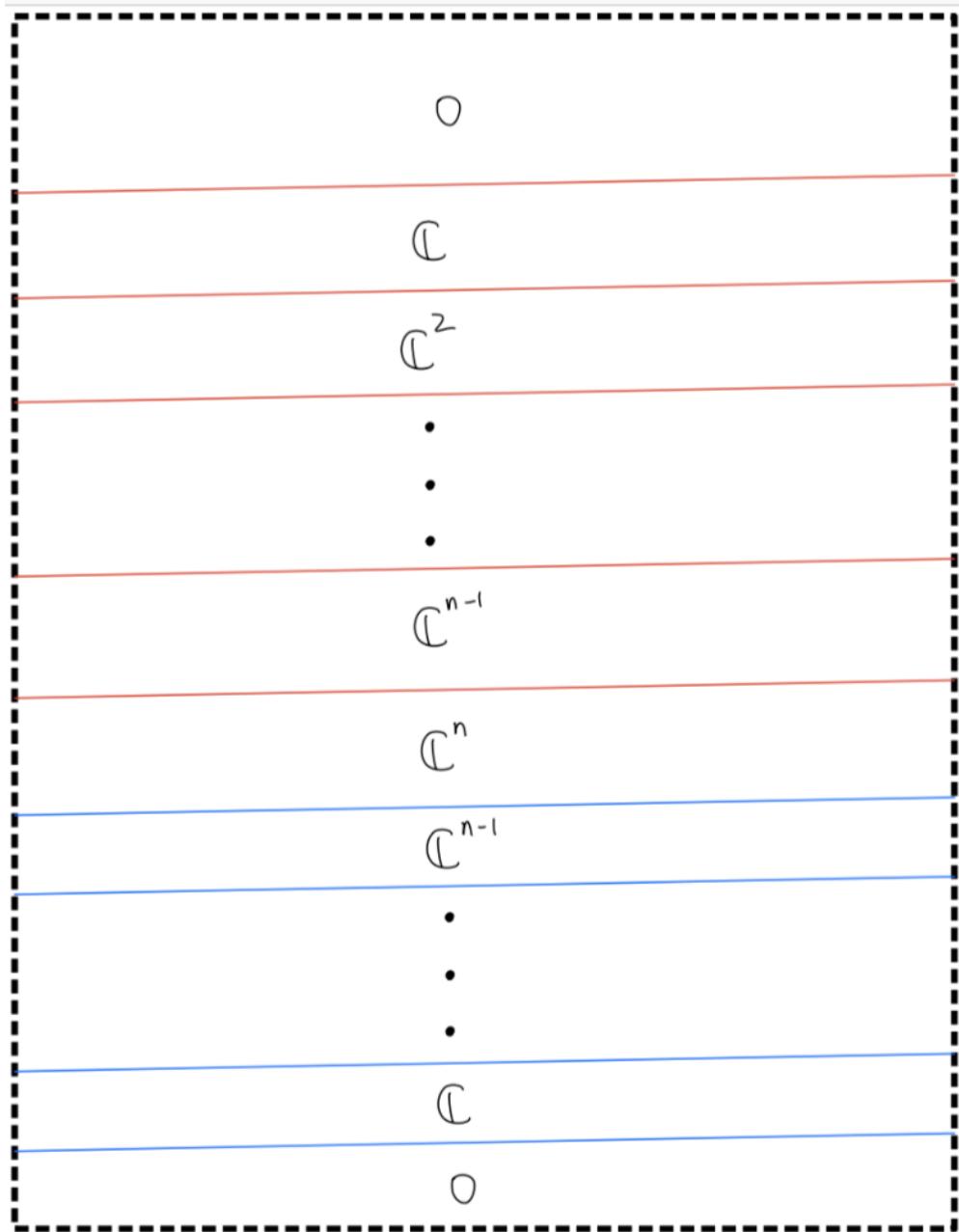


Figure 3.332

We define $cobord_{inter}(n)$ inductively as follows.

- (i) For $n = 1$, we define $cobord_{inter}(1)$ to be the null cobordism from

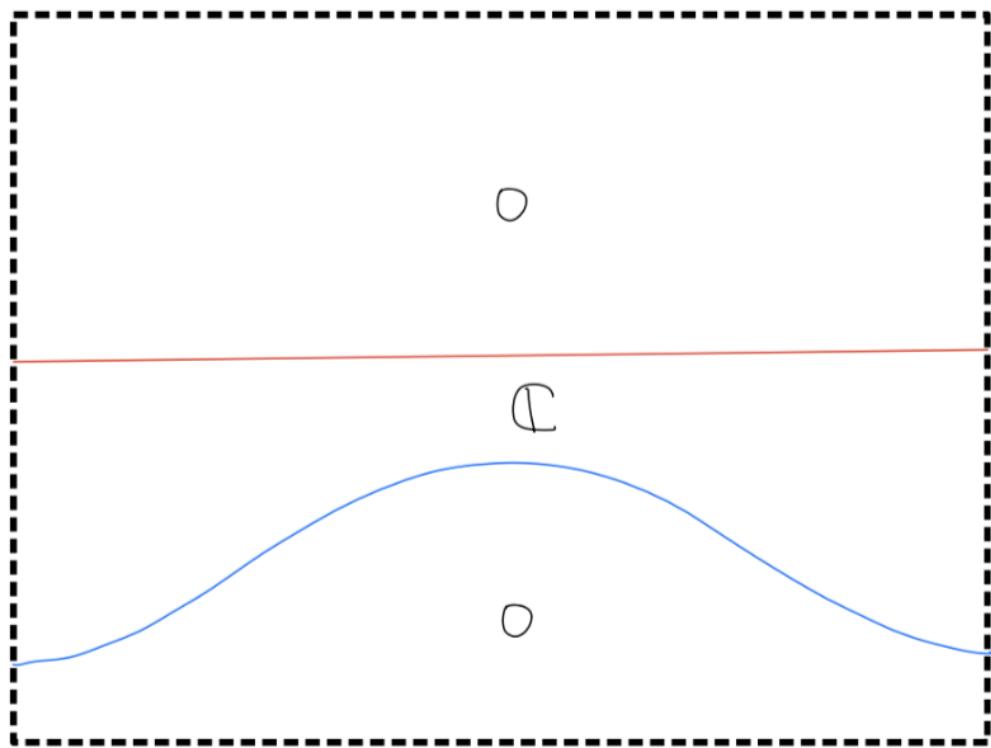


Figure 3.333

to

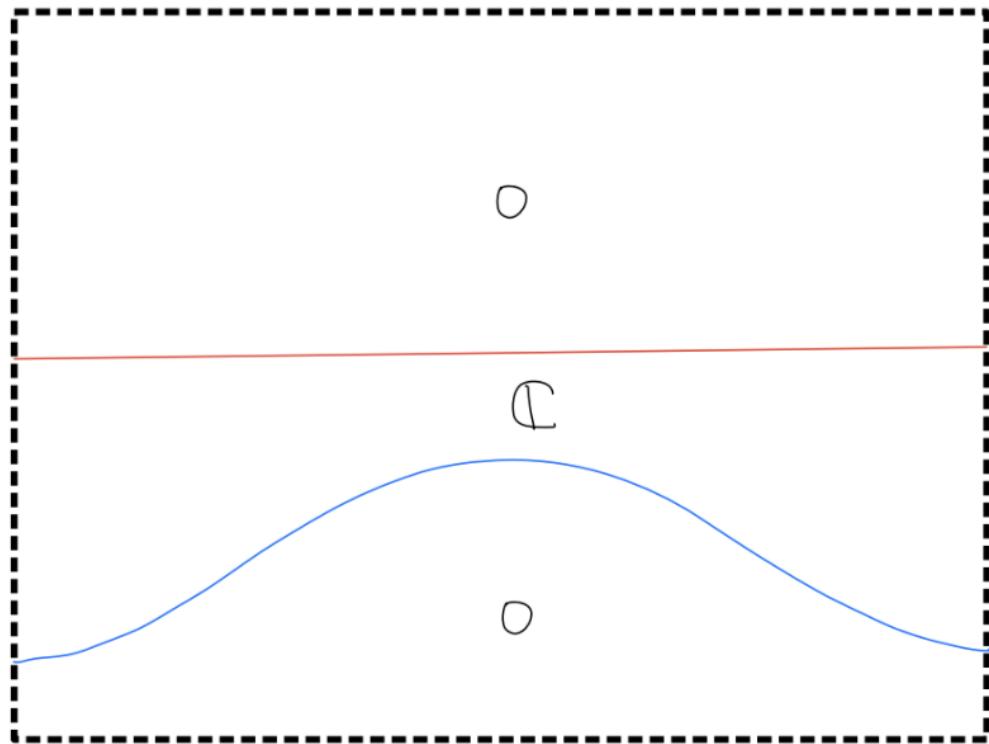


Figure 3.334

(ii) For $n > 1$,

(Step 1) we first apply $cobord_{inter}(n - 1)$ to the square region surrounded by purple dotted lines.

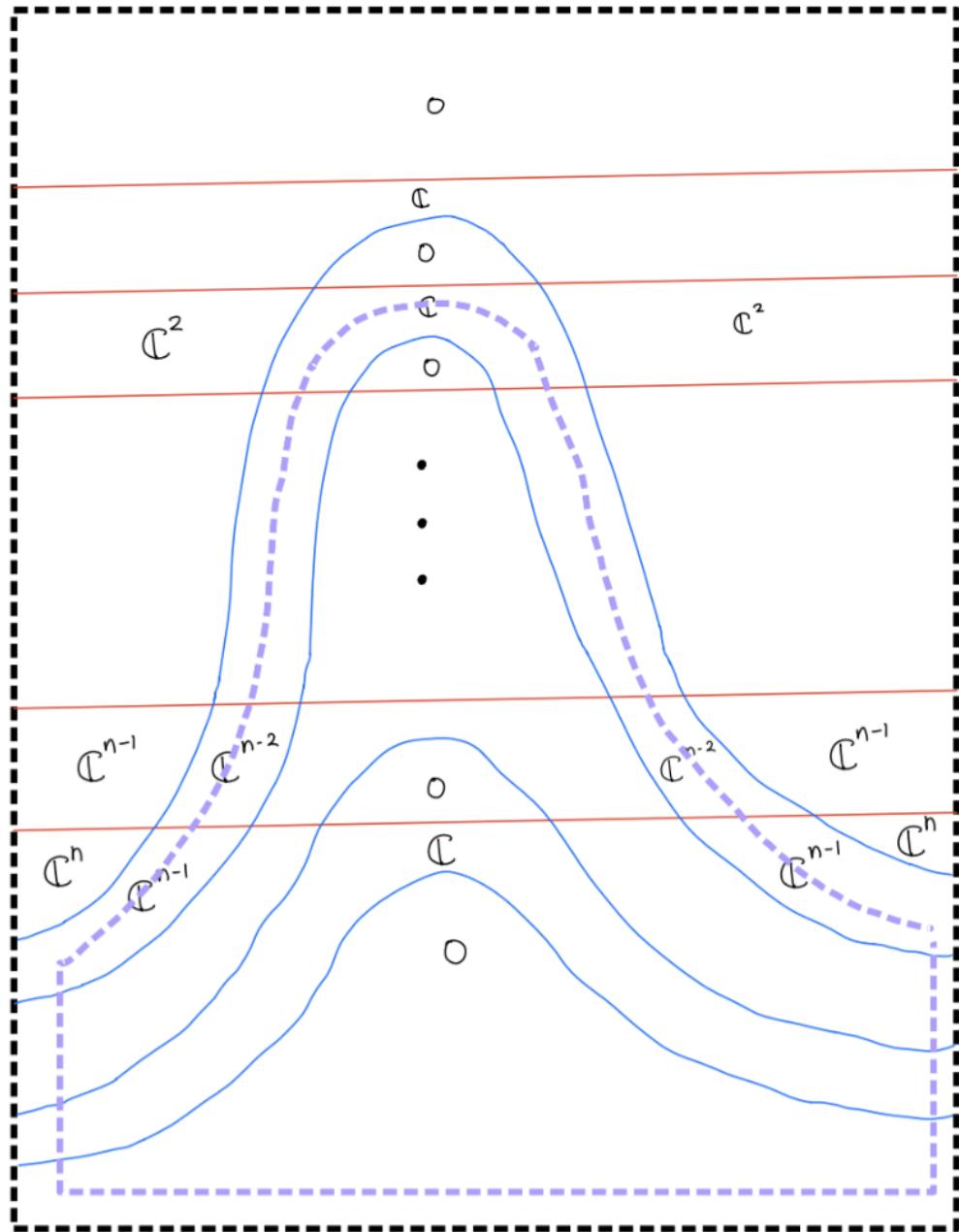


Figure 3.335

by induction hypothesis, we get

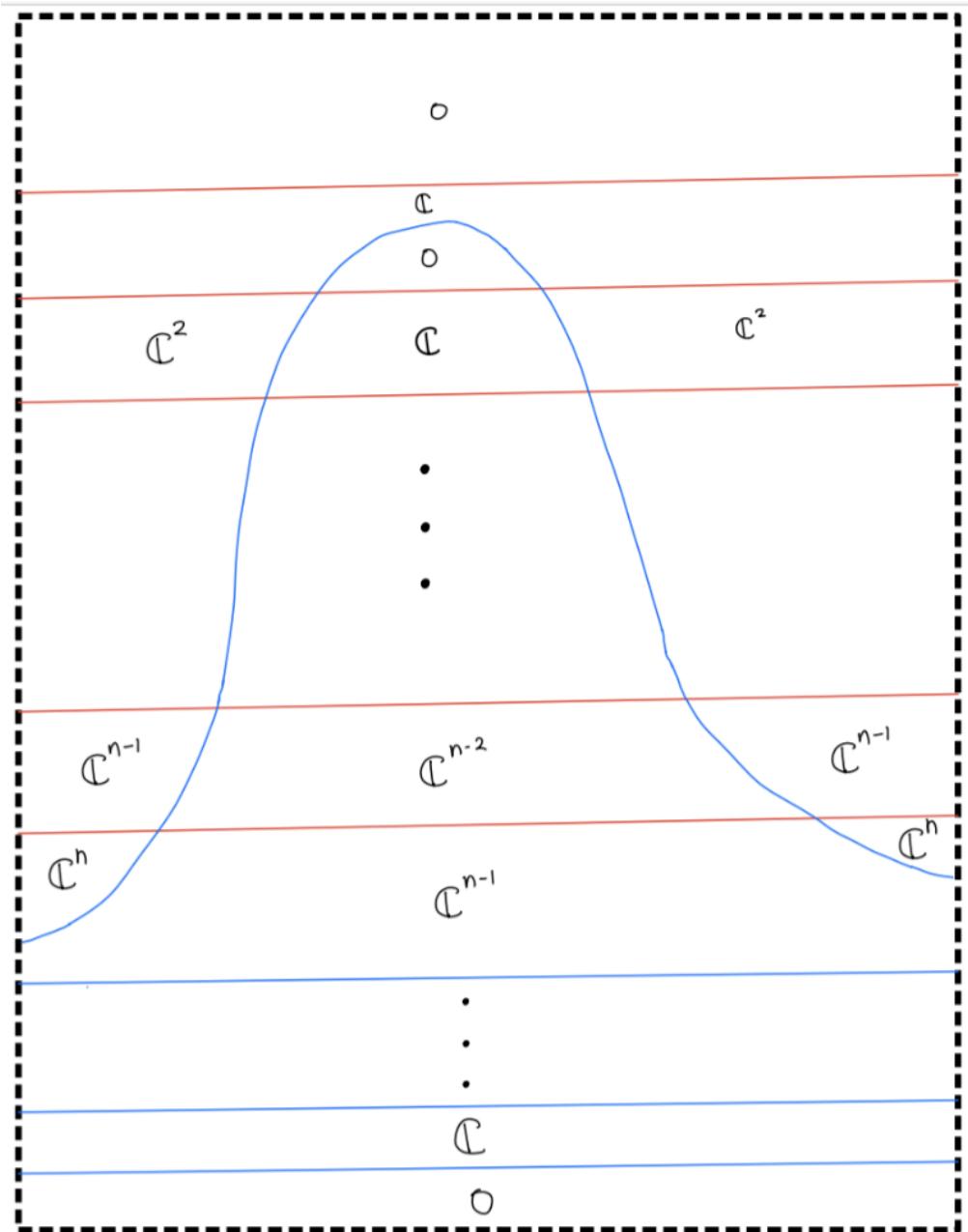


Figure 3.336

(Step 2) apply $cobord_7$ to the square region surrounded by purple dotted lines.

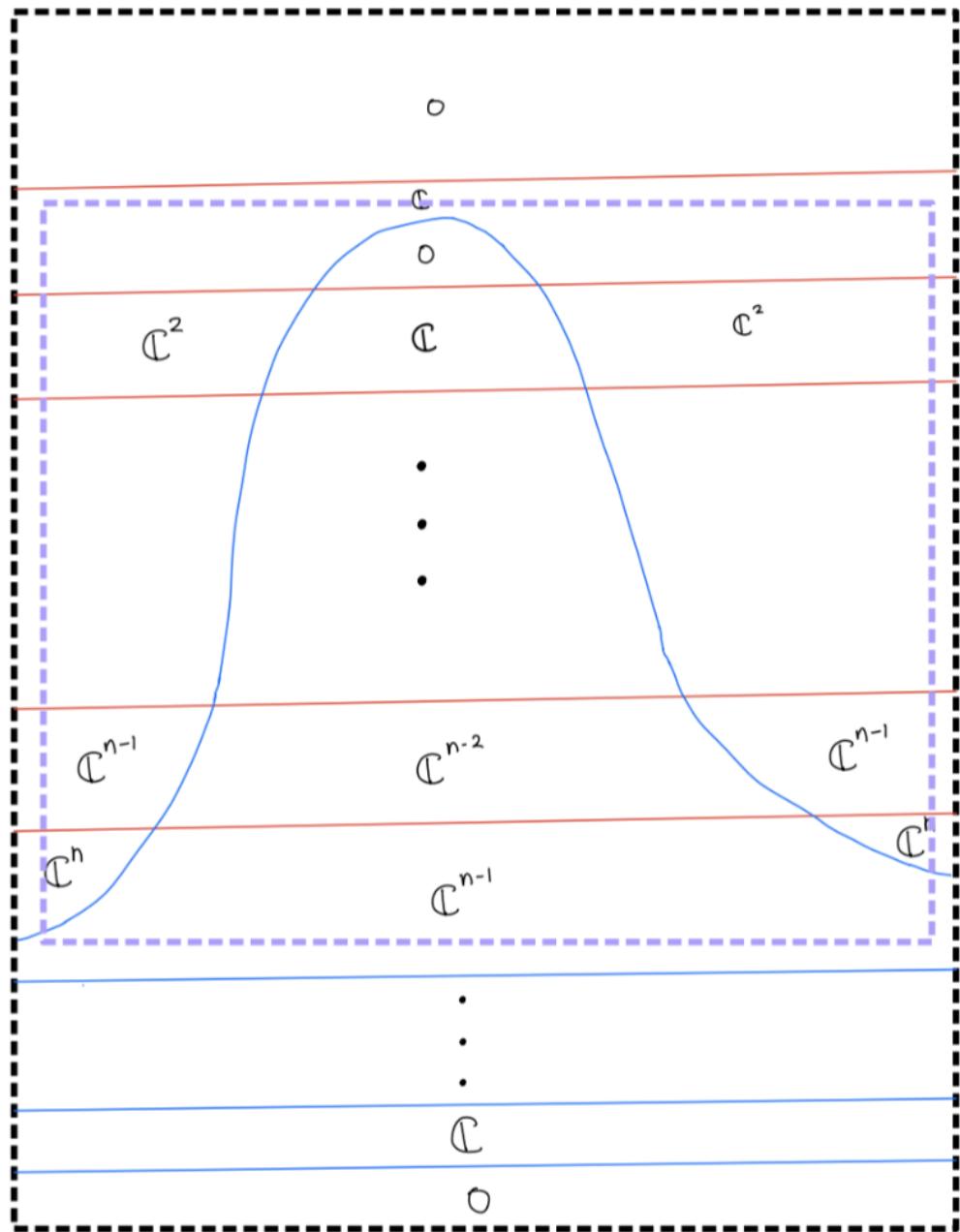


Figure 3.337

we get

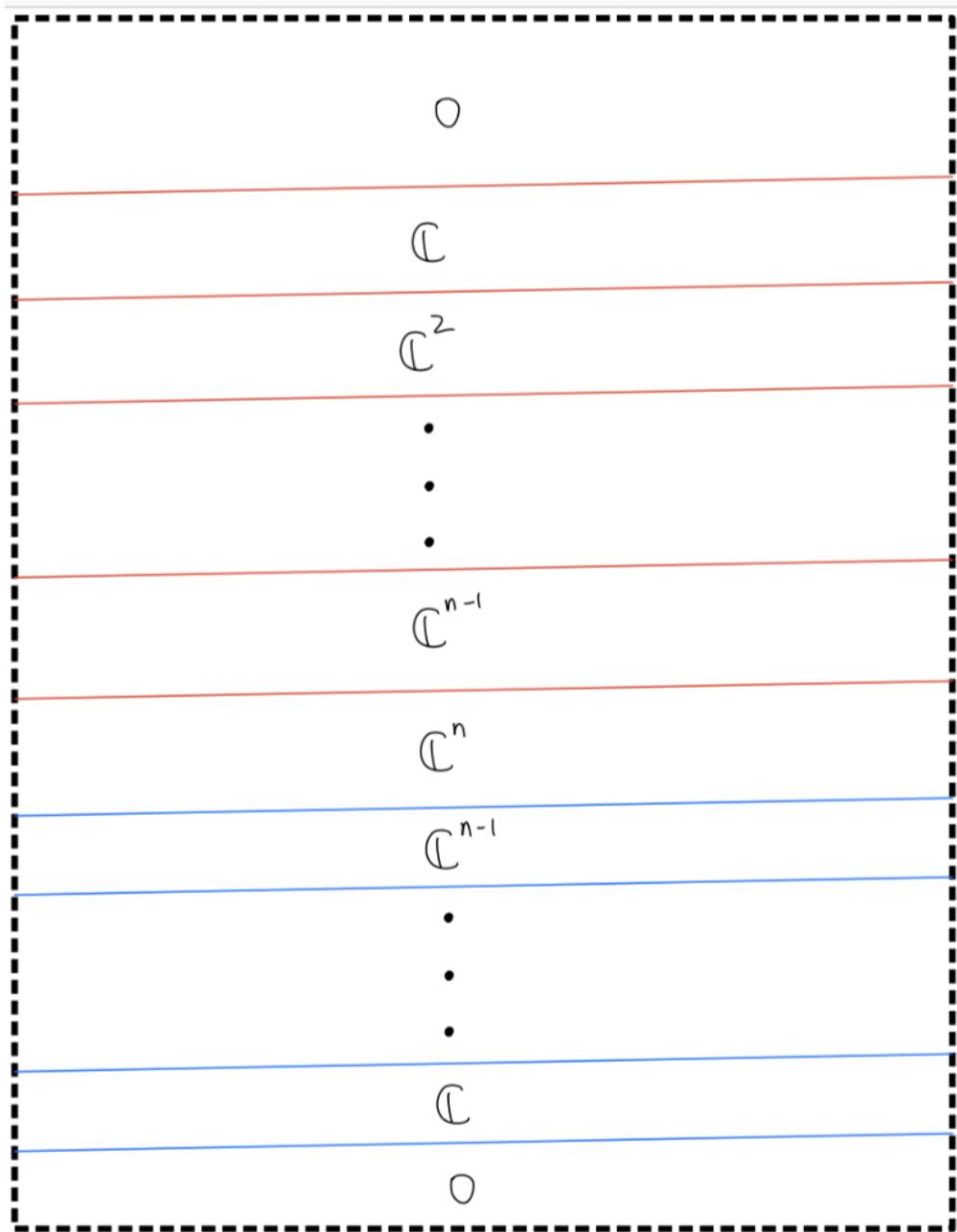


Figure 3.338

which is isomorphic to the final sheaf

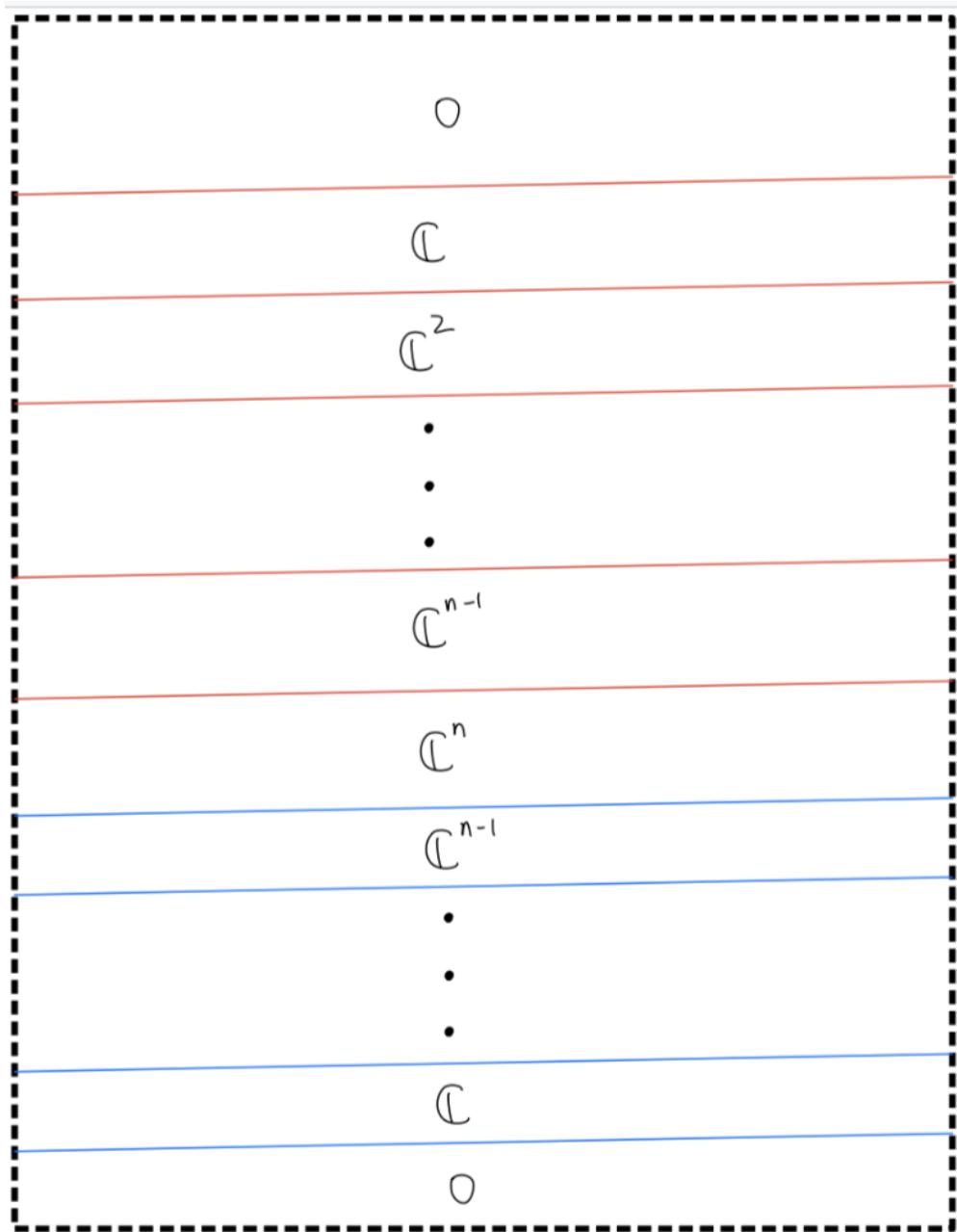


Figure 3.339

which is isomorphic to

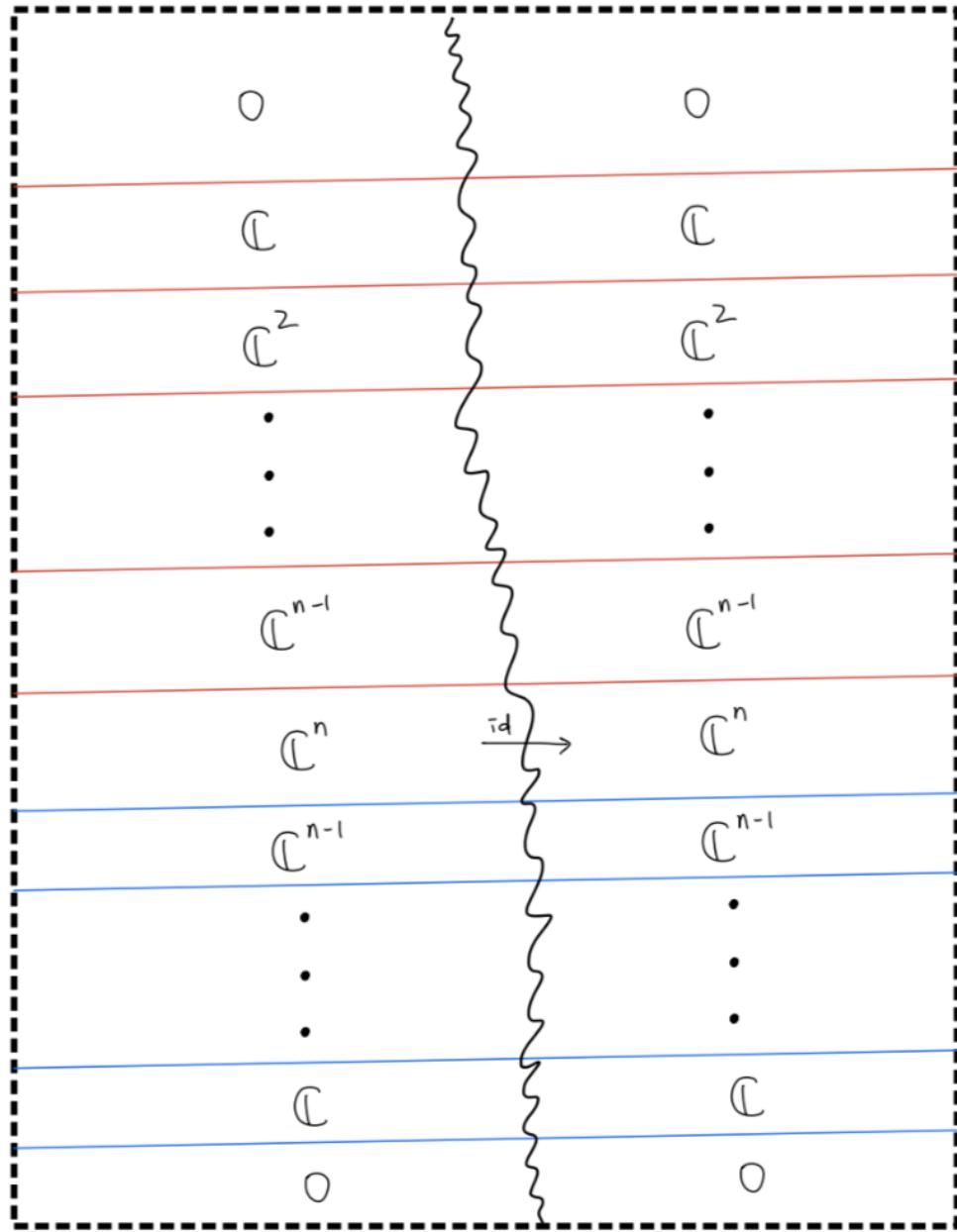


Figure 3.340

3.17 Full Sheaf Cobordism

Suppose we have a positive braid word $\omega = s_{i_1} \cdots s_{i_k}$ on n strands, then we have an embedding of the cylindrical closures of ω and the trivial braid ω_\emptyset in a Riemann sphere with punctures at $0, \infty$ described in section ???. Suppose we have an alternating

sheaf on the natural alternating diagram associated to $\omega \coprod \omega_\emptyset$ when restricted to the j^{th} generator region is described by the following legible diagram

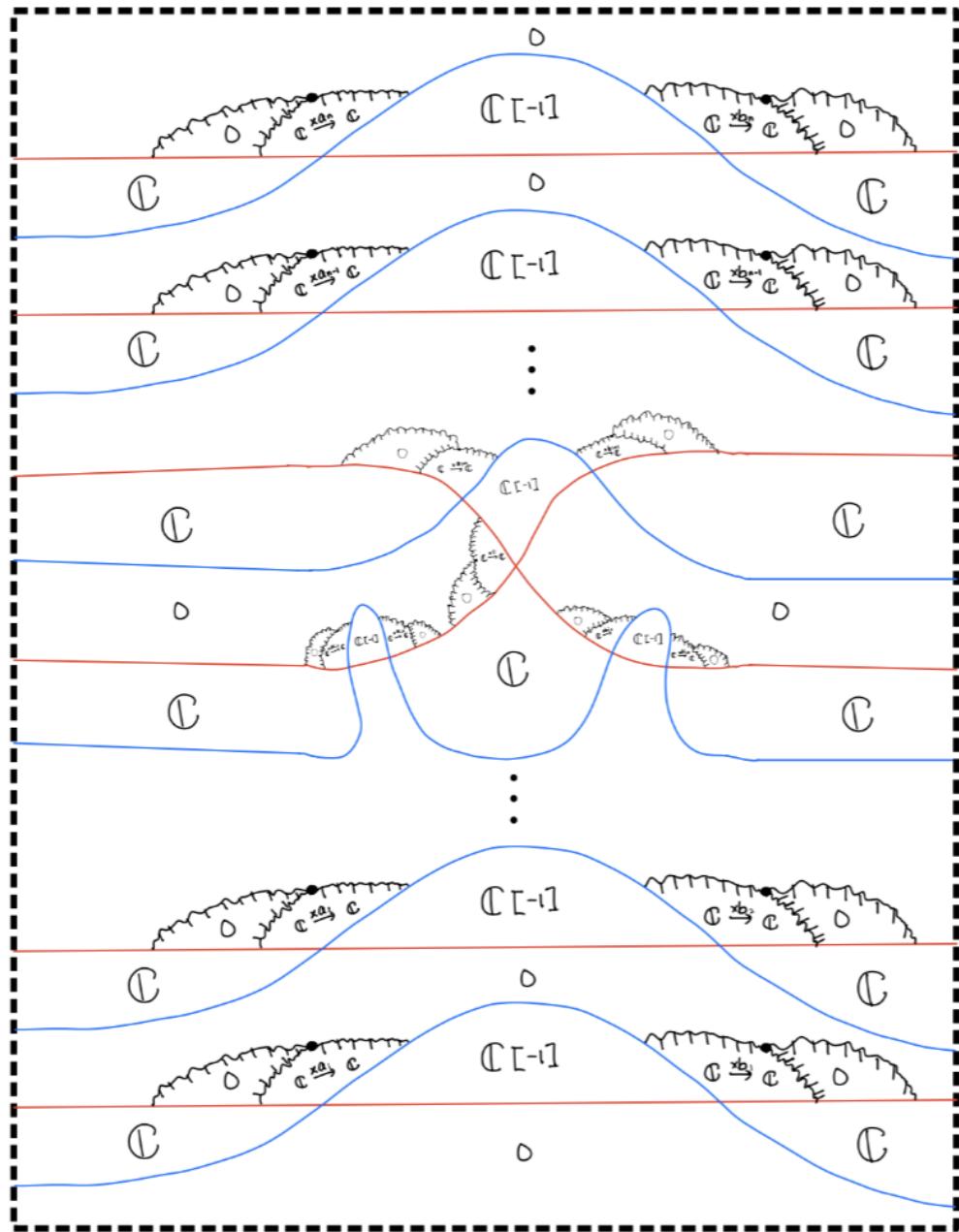


Figure 3.341

and when restricted to inter-generator regions is described by the following squiggly legible diagram

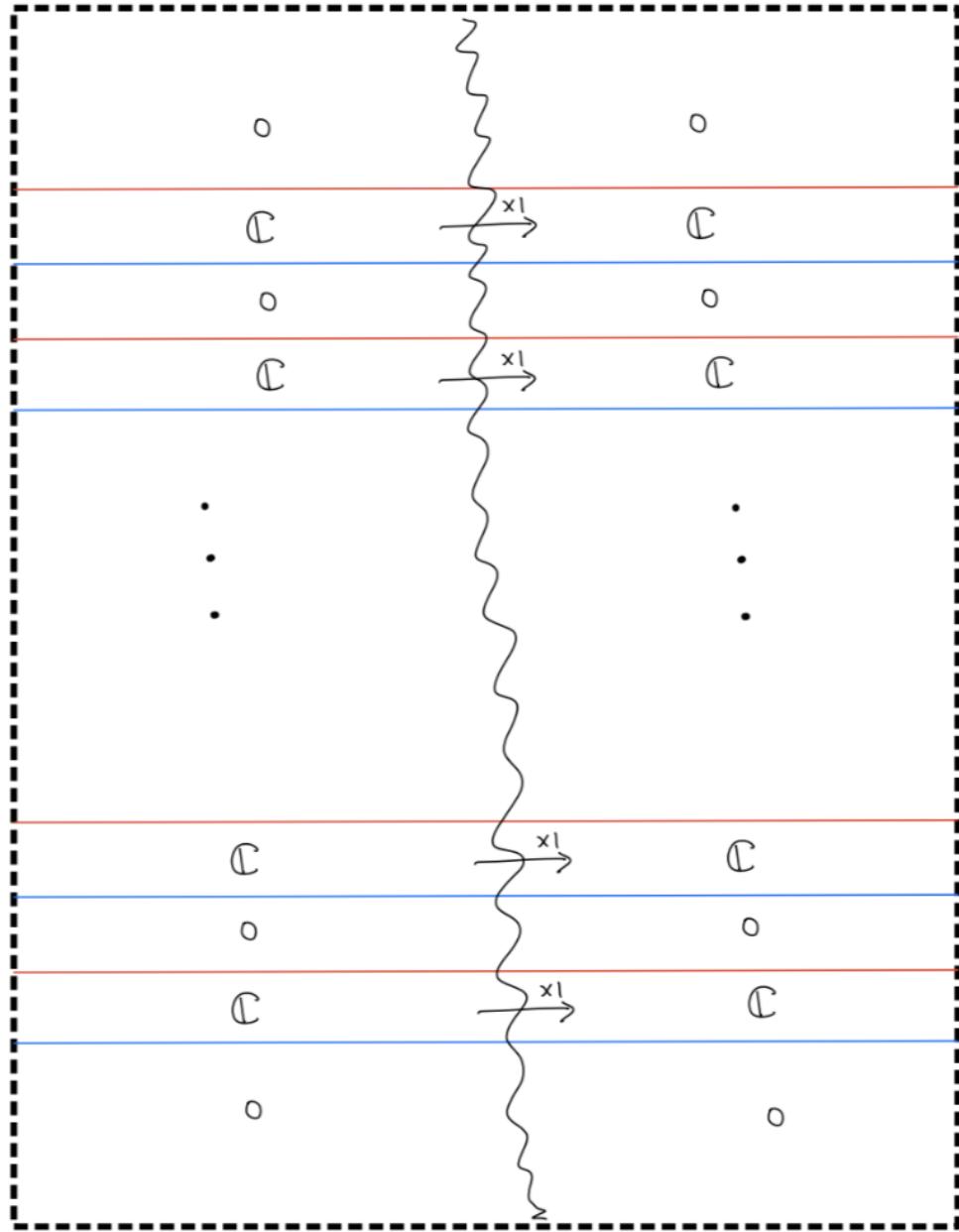


Figure 3.342

then we define a sheaf cobordism from the above alternating sheaf whose underlying Legendrian isotopy and at the separated diagram of $\omega \coprod \omega_\emptyset$, thereby describing a cluster coordinate.

(Step 1) to the j^{th} generator regions ($j = 1, \dots, k$), apply $cobord_{gen(n)}$, we get on the j^{th} generator region

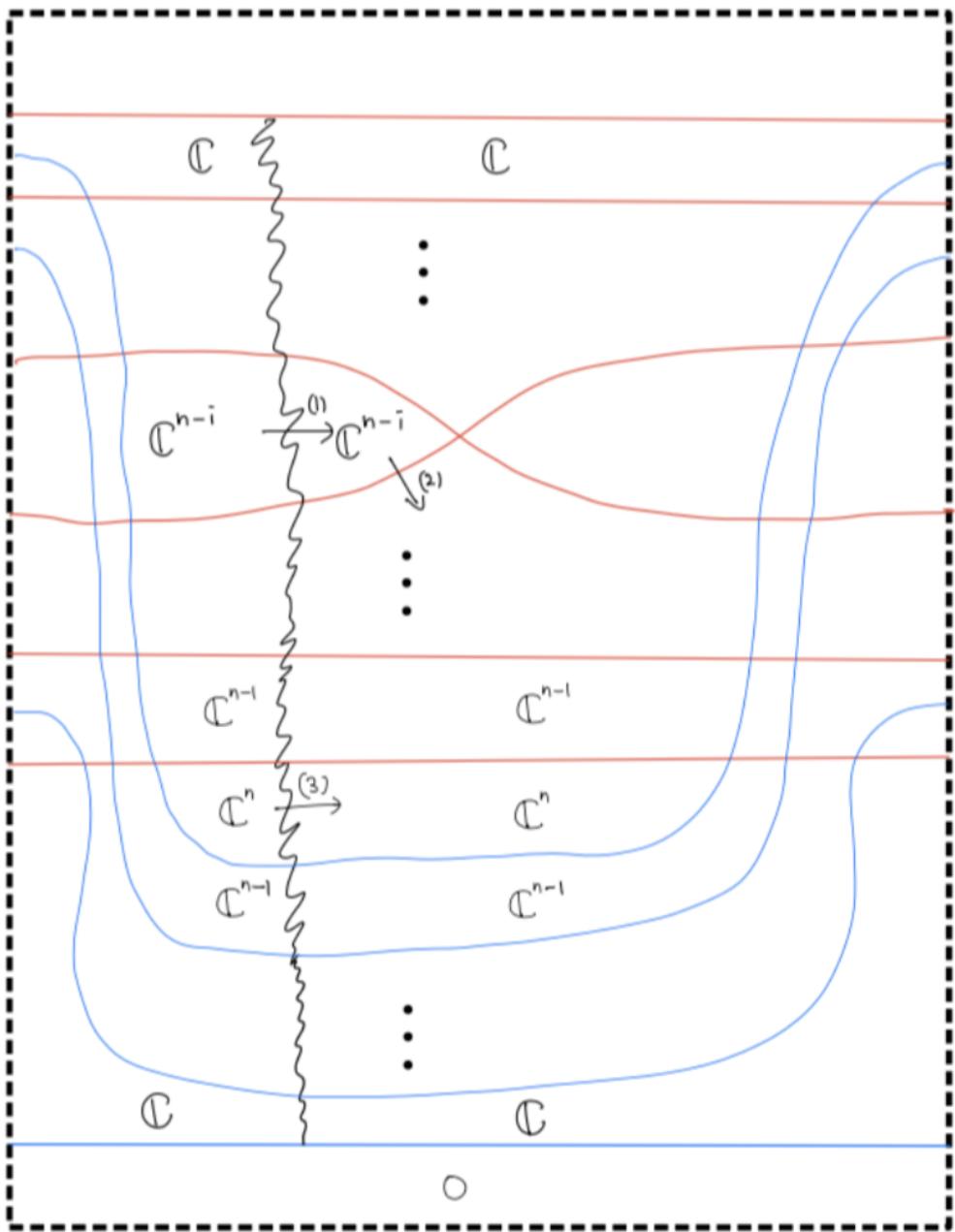


Figure 3.343

Generalization maps

- (a) $\text{diag}(d_n^j, \dots, d_{i+1}^j)$
- (b) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (c) $\text{diag}(d_n^j, \dots, d_1^j) + e I_{n-i+1, n-i}$

where

- $a_i = a_{i,1}a_{i,2}^{-1}$ and $b_i = b_{i,1}^{-1}b_{i,2}$
- $d_r = a_r b_r^{-1}$
- $e = -a_{i+1}b_i^{-1}c$
- $e' = d_{i+1}^{-1}e$

and on the inter-generator regions we get

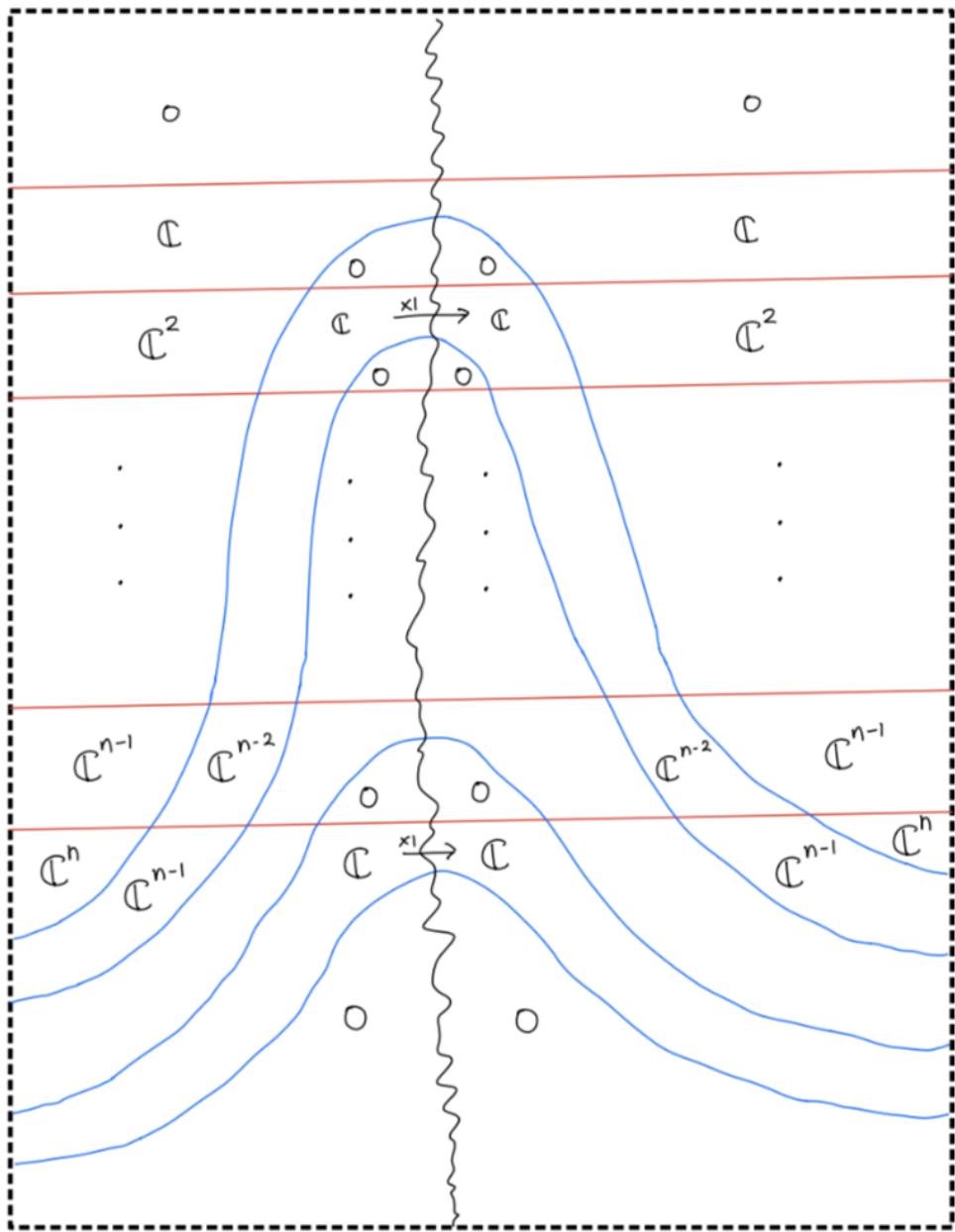


Figure 3.344

(Step 2) to the inter-generator regions, we apply $cobord_{inter}(n)$, on the j^{th} generator region, we get

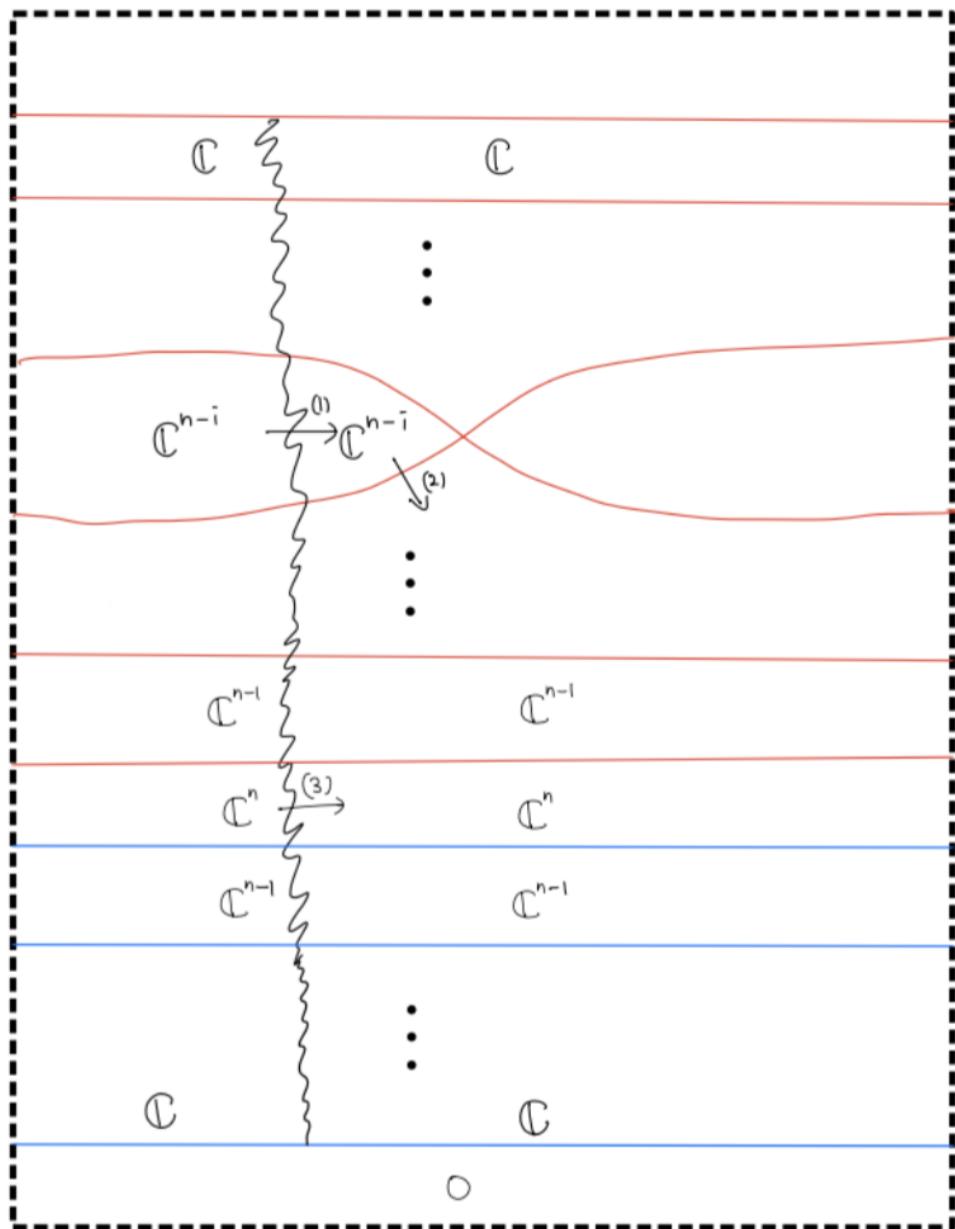


Figure 3.345

Generalization maps

- (a) $\text{diag}(d_n^j, \dots, d_{i+1}^j)$
- (b) $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (c) $\text{diag}(d_n^j, \dots, d_1^j) + e I_{n-i+1, n-i}$

and on the inter-generator regions, we get

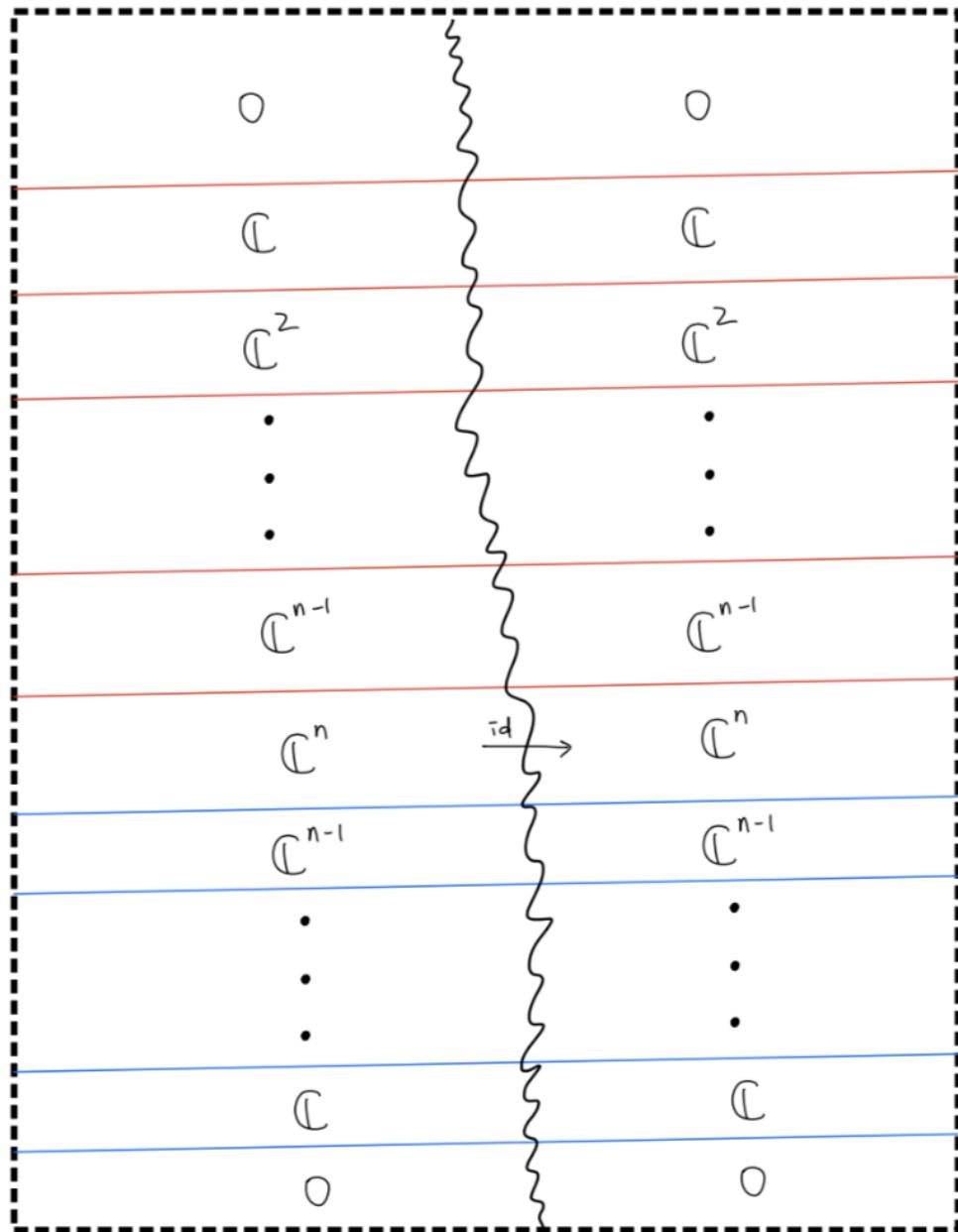


Figure 3.346

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