Moduli Spaces associated to Legendrian Links in the Cocircle Bundle of a Cylinder from Positive Braids

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Abstract

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Chapter 1

Introduction

1.1 Background and context

Contact Geometry

First, we review basics of contact geometry, see [Etn05][Gei08] for details. A contact structure on a $(2n-1)$-dimensional manifold X is a maximally nonintegrable distribution of $(2n-2)$-planes. A contact structure defined globally as a kernel for a chosen 1-form α is said to be co-oriented and α is said to be its co-orientation. An $(n-1)$-dimensional submanifold $\Lambda \subset X$ is said to be Legendrian if its tangent bundle is contained in the contact hyperplanes. Now let's look at several examples of contact manifolds.

(1) the contagent bundle T^*M of a manifold M carries a canonical 1-form which is in local coordinates (q_i, p_i), $\theta := \sum_i p_i dq_i$. This form is invariant under dialation in cotangent directions, therefore, θ descends to a well-defined 1-form(unique up to positive scalar function) on the cosphere bundle $T^\infty M := (T^*M - 0_M)/\mathbb{R}_+$ where 0_M is the zero section and $\mathbb{R}_+$-action is the dialation action along cotangent directions. Therefore, $T^\infty M$ is equipped with a natural contact structure.

(2) $\mathbb{R}^3_{x,y,z}$ with the global 1-form $\alpha = dz - ydx$ defines a co-oriented contact structure. We can embed $\mathbb{R}^3_{x,y,z}$ in $T^{\infty,-}\mathbb{R}^2_{x,z}$ as an open contact submanifold as follows:

$$\mathbb{R}^3_{x,y,z} \to \mathbb{R}^2_{x,z} \times S^1$$
$$(x, y, z) \mapsto (x, z; y, -1)$$

The image is the contact submanifold $T^{\infty,-}\mathbb{R}^2_{x,z}$ of "downward" covectors.

(3) Note that construction (2) is invariant under translation $x \mapsto x + 1$. Let $S^1_x := \mathbb{R}_x/\mathbb{Z}$, then standard 1-form $dz - ydx$ descends to $S^1_x \times \mathbb{R}_{y,z}$ and it embeds as the contact submanifold $T^{\infty,-}(S^1_x \times \mathbb{R}_z)$ in $T^\infty(S^1_x \times \mathbb{R}_z) \cong S^1_x \times \mathbb{R}_z \times S^1_y$. The"cocircle bundle" in the title of the thesis refers to $T^\infty(S^1_x \times \mathbb{R}_z)$.

Front Projection

Now let's restrict our attention to the case when the base manifold M is either $\mathbb{R}^2_{x,z}$ or $S^1_x \times \mathbb{R}_z$. We call the map $\pi : T^\infty M \to M$ the "front projection" and $\pi(\Lambda)$ the "front diagram".

- when $M = \mathbb{R}^2_{x,z}$, $\pi : \mathbb{R}^3_{x,y,z} \subset T^\infty\mathbb{R}^2_{x,z} \to \mathbb{R}^2_{x,z}$ is the projection onto x, z coordinates. We call $\mathbb{R}^2_{x,z}$ a front plane.

- similarly, when $M = S^1_x \times \mathbb{R}_z$, $\pi : S^1_x \times \mathbb{R}^2_{y,z} \subset T^\infty(S^1_x \times \mathbb{R}_z) \to S^1_x \times \mathbb{R}_z$ is the projection onto x, z coordinates. We call $S^1_x \times \mathbb{R}_z$ a front cylinder.

A Legendrian $\Lambda \subset T^\infty M$ at immersed points of $\pi(\Lambda)$can be recovered from $\pi(\Lambda)$ because Λ vanishes on the contact form $dz - ydx$, we have $y = \frac{dz}{dx}$ as long as it is in general position. Under Hamiltonian isotopy, Λ can be put in general position, i.e.

- $\pi|_\Lambda$ is locally injective.

- there are only finitely many points on M at which $\pi(\Lambda)$ is not an embedded submanifold. These are either
  - cusps where $\pi|_\Lambda$ is injective and $\frac{dz}{dx}$ has a well-defined limit of 0.
  - crossing where $\pi(\Lambda)$ is locally a transverse intersection of two smooth curves.

In this paper, we only allow $\pi(\Lambda)$ to have crossings. We won't consider $\pi(\Lambda)$ with cusps. There are no interesting examples in $\mathbb{R}^3_{x,y,z}$ but there are many interesting ones in the cocircle bundle of a cylinder. Suppose we have a smooth parametrized curve Φ in M

- which is an immersion
- which has finitely many self-intersections and they are transverse.

and a choice of preferred side, i.e. a co-orientation, diagramatically described using hairs normal to the Φ pointing the preferred direction. We say that the region on the preferred side lies beneath the the arc and the region on the other side lies above the arc. We can lift it to a Legendrian in $T^\infty M$. Therefore, the Legendrian in $T^\infty M$ amounts to the datum of the co-oriented front diagram.

Constructible sheaves

Let M be a manifold and $\mathcal{S}$ Whitney a stratification of M. A sheaf of $\mathbb{C}$-vector space $\mathscr{F}$ on M is $\mathcal{S}$-constructible if the restriction of $\mathscr{F}$ to each stratum of $s \in \mathcal{S}$, $\mathscr{F}|_s$, is a locally constant sheaf. $\mathscr{F}$ is called constructible if there exists a Whitney stratification $\mathcal{S}$ such that $\mathscr{F}$ is $\mathcal{S}$-constructible. $Sh^\bullet_{naive}(M;\mathbb{C})$ denotes the triangulated dg category whose objects are cochain complexes of sheaves of $\mathbb{C}$-vector spaces of bounded cohomology whose cohomology sheaves are constructible sheaves. If we localize $Sh^\bullet_{naive}(M;\mathbb{C})$ with respect to acyclic complexes, in the sense of [Dri04], we get

$Sh^\bullet(M;\mathbb{C})$. We denote $Sh^\bullet_{\mathcal{S}}(M;\mathbb{C})$ the full subcategory of $Sh^\bullet(M;\mathbb{C})$ of complexes whose cohomology sheaves are constructible with respect to $\mathcal{S}$. From now on we will call objects of $Sh^\bullet(M;\mathbb{C})$ simply as sheaves.

Singular Support

To each $\mathscr{F} \in Sh^\bullet(M;\mathbb{C})$, we assign a closed conical subset $SS(\mathscr{F}) \subset T^*M$, called the singular support of $\mathscr{F}$ following [GM83][Sch12][STZ17]. For more general treatment, see [KS13]. Fix a Riemannian metric on M and choose an ϵ-ball $B_\epsilon(x)$ around a point $x \in M$. The following constructions are independent of the choice of metrics. Let $\mathscr{F}$ be an $\mathcal{S}$-constructible sheaf on M. Fix a point $x \in M$ and a smooth function f on $B_\epsilon(x)$. For $\epsilon, \delta > 0$, we define the local Morse group to be

$$Mo_{x,f,\epsilon,\delta}(\mathscr{F}) := H^*(B_\epsilon(x) \cap f^{-1}((-\infty, f(x)+\delta)), B_\epsilon(x) \cap f^{-1}((-\infty, f(x)-\delta)); \mathscr{F})$$

For $\epsilon' < \epsilon$ and $\delta' < \delta$, there is a canonical restriction map

$$Mo_{x,f,\epsilon,\delta}(\mathscr{F}) \to Mo_{x,f,\epsilon',\delta'}(\mathscr{F})$$

The above restriction is an isomorphism if ϵ and δ are sufficiently small and f is stratified Morse at x. This allows us to define $Mo_{x,f}(\mathscr{F})$ for f suitably generic with respect to $\mathcal{S}$[KS13, Prop. 7.5.3]. In fact, $Mo_{x,f}(\mathscr{F})$ depends only on the Hessian of f at x. Let $\mathscr{F} \in Sh^\bullet(M;\mathbb{C})$, then $(x,\xi) \in T^*M$ is characteristic with respect to $\mathscr{F}$ if for some stratified Morse function f with $df_x = \xi$, the local Morse group $Mo_{x,f}(\mathscr{F})$ is nonzero. Then we define the singular support of $\mathscr{F}$ to be the closure of the set of characteristic covectors for $\mathscr{F}$ in T^*M. Singular support is a conic Lagrangian i.e. stable under $\mathbb{R}_+$-dialation along contangent fibers.

Fix a conic Lagrangian $L \subset T^*M$, then we define $Sh^\bullet_L(M;\mathbb{C})$ to be the full subcategory of $Sh^\bullet_L(M;\mathbb{C})$ whose objects are sheaves singular supported in L. If $\Lambda \subset T^\infty M$ is a Legendrian, then we define $Sh^\bullet_\Lambda(M;\mathbb{C}) := Sh^\bullet_{\mathbb{R}_+\Lambda\cup 0_M}(M;\mathbb{C})$ where $0_M \subset T^*M$ is the zero section.

Combinatorial Model of $Sh^\bullet_\Lambda(M;\mathbb{C})$

The first necessary condition for $\mathscr{F} \in Sh^\bullet(M;\mathbb{C})$ to belong to $Sh^\bullet_L(M;\mathbb{C})$ is that $\mathscr{F}$ is constructible with respect to the stratification on M induced by the front projection of Λ. That is the Whitney stratification whose zero dimensional strata are crossings, one dimensional strata are the arcs of $\pi(\Lambda)$ between crossings and two dimensional strata are the connected components of the complement of $\pi(\Lambda)$ in M.

Definition 1. Given a stratification $\mathcal{S}$, the star of a stratum $s \in \mathcal{S}$ is the union of strata that contain s in their closure. We view $\mathcal{S}$ as a poset category $s \leq t$ if and only if $s \subset \bar{t}$. We call the map from s to t as a generization map. We say $\mathcal{S}$ is a regular cell complex if every stratum is contractible and moreover the star of each stratum is contractible.

Definition 2. We define $Fun^\bullet_{naive}(\mathcal{S},\mathbb{C})$ to be the dg category of functors from $\mathcal{S}$ to the category whose objects are cochain complexes of $\mathbb{C}$-vector spaces and whose maps are cochain maps. We define $Fun^\bullet(\mathcal{S},\mathbb{C})$ the dg quotient [Dri04] of $Fun^\bullet_{naive}(\mathcal{S},\mathbb{C})$ by the thick subcategory of functors taking values in acyclic complexes.

Proposition 3. [Kas84], [She85], [Nad09, Lemma 2.3.2].Let $\mathcal{S}$ be a Whitney stratification of M. Consider the functor

$$\Gamma_{\mathcal{S}} : Sh^\bullet_{\mathcal{S}}(M;\mathbb{C}) \to Fun^\bullet(\mathcal{S},\mathbb{C})$$
$$\mathscr{F} \mapsto [s \mapsto \Gamma(star(s);\mathscr{F})]$$

If $\mathcal{S}$ is a regular cell complex, then $\Gamma_{\mathcal{S}}$ is a quasi-equivalence.

However, the stratification induced by the front diagram is not necessarily a regular cell complex. Therefore, we choose an $\mathcal{S}$ that refines of the stratification induced by $\pi(\Lambda)$. In figures, we draw the new line segments in squiggly lines.
The restriction of $\Gamma_{\mathcal{S}}$ to $Sh^{\bullet}_{\Lambda}(M;\mathbb{C})$ is quasi-fully faithful. We will describe the essential image of $Sh^{\bullet}_{\Lambda}(M;\mathbb{C})$ under $\Gamma_{\mathcal{S}}$.

Definition 4. Let $\mathcal{S}$ be a regular cell complex refining the stratification induced by the front diagram. we define $Fun^{\bullet}_{\Lambda}(\mathcal{S},\mathbb{C})$ to be the full subcategory of $Fun^{\bullet}(\mathcal{S},\mathbb{C})$ whose objects satisfy the following properties.

- every map from a zero dimensional stratum in $\mathcal{S}$ which is not a crossing is sent to quasi-isomorphism.
- every map from a one dimensional stratum which is not contained in an arc is sent to quasi-isomorphism.
- for each crossing $c \in \mathcal{S}$, Let N, E, S, W be the north, east, south, and west regions adjoining c, and let nw, ne, sw, se be the arcs separating them.

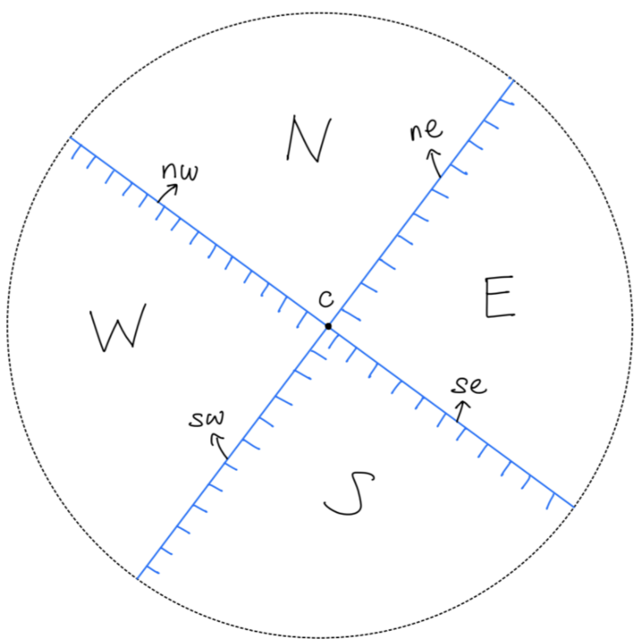

Figure 1.1

Then the following maps are sent to quasi-isomorphisms.

- maps between c, sw, se, and S.
- a map from nw to W.
- a map from ne to E.

- for each crossing $c \in \mathcal{S}$, we restrict the poset structure to the strata containing c in its closure, we get

$$\begin{tikzcd}
& & N & & \\
& nw \arrow[ur] \arrow[dl] & & ne \arrow[ul] \arrow[dr] & \\
W & & c \arrow[uu] \arrow[ul] \arrow[ur] \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dr] \arrow[dd] & & E \\
& sw \arrow[ul] \arrow[dr] & & se \arrow[ur] \arrow[dl] & \\
& & S & &
\end{tikzcd}$$

Then all triangles in the diagram should commute and the total complex of the following bicomplex

$$F(c) \to F(nw) \oplus F(ne) \to F(N)$$

should be acyclic.

A standard argument shows that the essential image of $Sh^{\bullet}_{\Lambda}(M, \mathbb{C})$ under $\Gamma_{\mathcal{S}}$ is $Fun^{\bullet}_{\Lambda}(\mathcal{S}, \mathbb{C})$[STZ17].

Legible Objects

Let M be a front surface, Φ a front diagram, $\Lambda \subset T^{\infty}M$ the associated Legendrian knot, and $\mathcal{S}$ a refinement of the stratification induced by Φ by adding squiggly lines. In this section, we define a "legible quiver" and a "squiggly legible diagram" to get a better handle on $Fun^{\bullet}_{\Lambda}(M;\mathbb{C})$.

Suppose we impose an auxiliary co-orientation on the squiggly lines as well, then this induces a quiver Q

Definition 5. Suppose we have a regular cell complex $\mathcal{S}$ that refines the stratification induced by Φ with auxiliary co-orientations on the added squiggly lines segments, then we define the associated quiver Q where

- the vertices of Q are 2 dimensional strata
- the arrows of Q are 1 dimensional strata where the sources and the targets are regions below and above the strata.

For a stratum $s \in \mathcal{S}$, we define Q_s to be the full sub-quiver of Q whose vertices are the regions incident with s.

Definition 6. We say that the quiver associated to $\mathcal{S}$ is *legible* if for every point stratum $p \in \mathcal{S}$, the sub-quiver Q_p is a poset with the smallest element.

Definition 7. A squiggly legible diagram on $\mathcal{S}$ is the representation $F^{\bullet}$ of the quiver Q associated to $\mathcal{S}$ valued in the cochain complex of $\mathbb{C}$-vector spaces subject to the condition that

- the maps corresponding to squiggly lines are quasi-isomorphisms.
- for fixed a source and a target vertices s and t, suppose we have to distinct paths in Q(a sequence of arrows) from s to t, say $(a_1, a_2, \cdots, a_k)$ and $(a'_1, a'_2, \cdots, a'_l)$, then $F^{\bullet}(a_k) \circ \cdots \circ F^{\bullet}(a_1) = F^{\bullet}(a'_l) \circ \cdots \circ F^{\bullet}(a'_1)$ i.e. the composition of cochain maps are path independent.

- at each crossing, we have surrounding region N, S, W, E

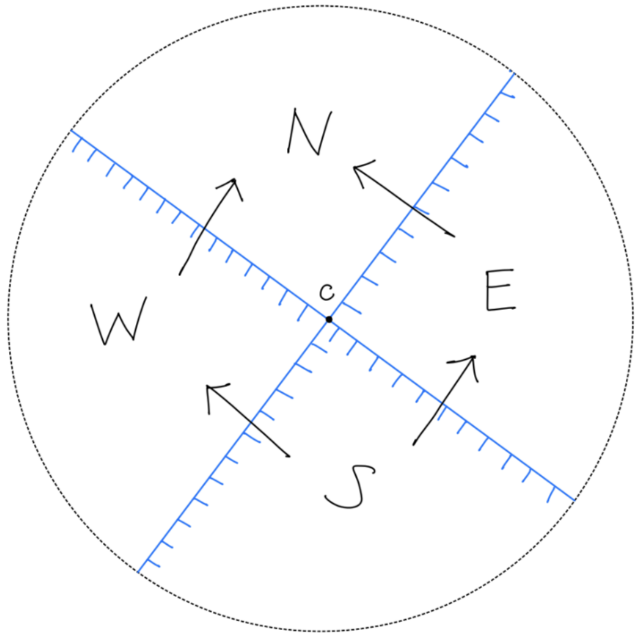

Figure 1.2

then the total complex of

$$F^{\bullet}(S) \to F^{\bullet}(W) \oplus F^{\bullet}(E) \to F^{\bullet}(N)$$

is acyclic.

Suppose we have $\mathcal{S}$ a refinement of the stratification induced by Φ and a squiggly legible diagram $F^{\bullet}$, then we can make an object $\overline{F}^{\bullet} \in Fun^{\bullet}_{\Lambda}(\mathcal{S}, \mathbb{C})$ out of it in the following way.

Definition 8. $\rho : \mathcal{S} \to \{R \in \mathcal{S} \mid dim_{\mathbb{R}}(R) = 2\}$ is a function that assigns $s \in \mathcal{S}$ to the smallest element in $\{R \in Vert(Q) \mid s \subset \overline{R}\}$.

Definition 9. Suppose $F^{\bullet}$ is a squiggly legible diagram on $\mathcal{S}$, then we define $\overline{F}^{\bullet} \in Fun^{\bullet}_{\Lambda}(\mathcal{S}, \mathbb{C})$ where

- for $s \in \mathcal{S}$, $\overline{F}^{\bullet}(s) := F^{\bullet}(\rho(s))$

- for $s_1, s_2 \in \mathcal{S}$ such that $s_2 \subset star(s_1)$, $\overline{F}^\bullet(s_1 \to s_2) := F^\bullet(a_k) \circ \cdots F^\bullet(a_1)$ where $(a_1, a_2, \cdots, a_k)$ is a path from $\rho(s_1)$ to $\rho(s_2)$ in the quiver associated to $\mathcal{S}$. This is well-defined because the composition of cochain maps are path independent by the definition of squiggly legible diagram and the existence of a path from $\rho(s_1)$ to $\rho(s_2)$ is guaranteed by the fact that Q_{s_2} is a subquiver of Q_{s_1} and $\rho(s_2), \rho(s_1)$ are the smallest elements from each of them.

In this paper, we only consider Φ, Λ, and $\mathcal{S}$ that objects of $Fun^\bullet(\mathcal{S}, \mathbb{C})$ arises from squiggly legible diagrams.A variant of [STZ17, Prop 3.22.] shows that every objects of $Sh^\bullet_\Lambda(M; \mathbb{C})$ is quasi-isomorphic to a squiggly legible diagram.

Microlocal Monodromy and Rank 1 Objects

In this section, we review the notion of microlocal monodromy of sheaves following[STZ17, Section 5.1]. For more general treatment, see [KS13, Ch. IV]. Supppose on M we have a front diagram Φ and a stratification $\mathcal{S}$ that refines the one induced by Φ by adding squiggly line segments. Suppose we are given a sheaf $F^\bullet$ in terms of squiggly legible diagram, we define the associated sheaf on Λ, called microlocal monodromy denoted $\mu mon(F^\bullet)$, by assigning cochain complexes of $\mathbb{C}$-vector spaces for each one dimensional stratum contained in Φ and cochain maps between them for each point stratum contained in Φ as follows:

- if s is a one dimensional stratum contained in Φ, $\mu mon(F^\bullet)(s) := Cone(F^\bullet(s))$
- if p is a point stratum in Φ, then p is either a crossing or the endpoint of a squiggly line. We define the cochain map for each case as follows:
  - when p is a crossing, locally near p the diagram looks like

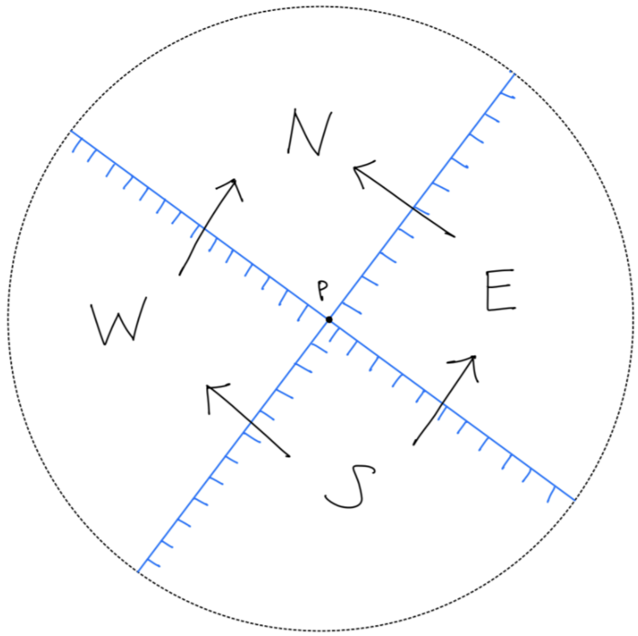

Figure 1.3

then p induces maps

$$Cone(S \to W) \to Cone(E \to W)$$
$$Cone(S \to E) \to Cone(W \to N)$$

– when p is the endpoint of a squiggly line segment, locally near p the diagram looks like one of the following diagrams

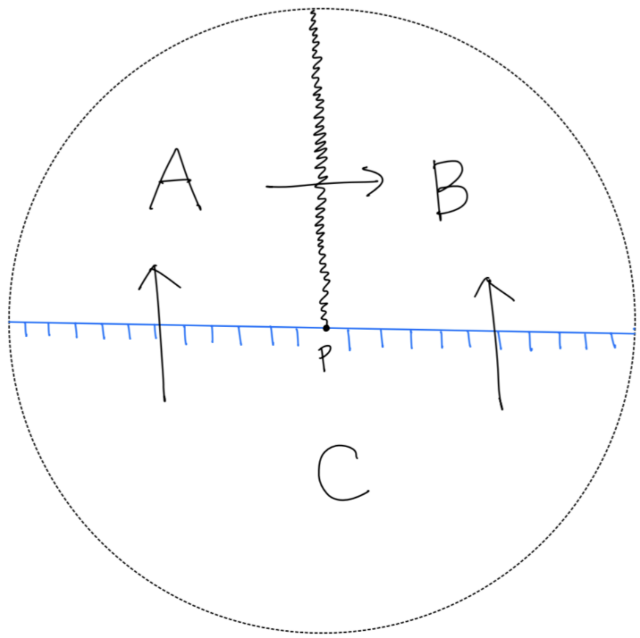

Figure 1.4

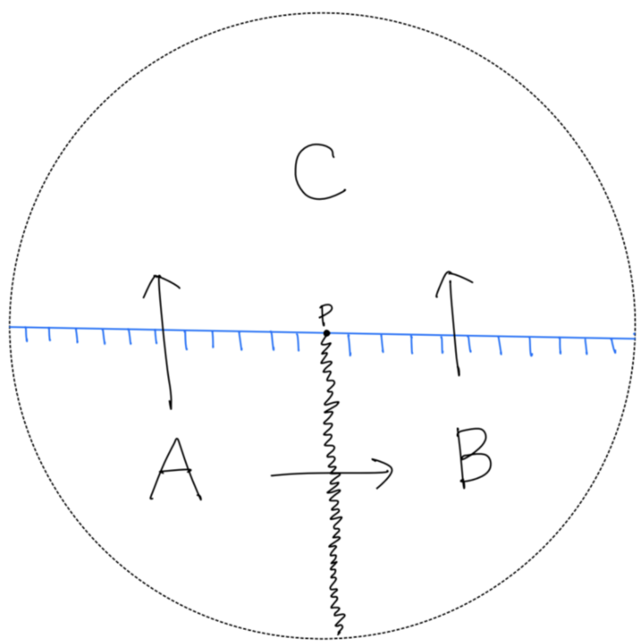

Figure 1.5

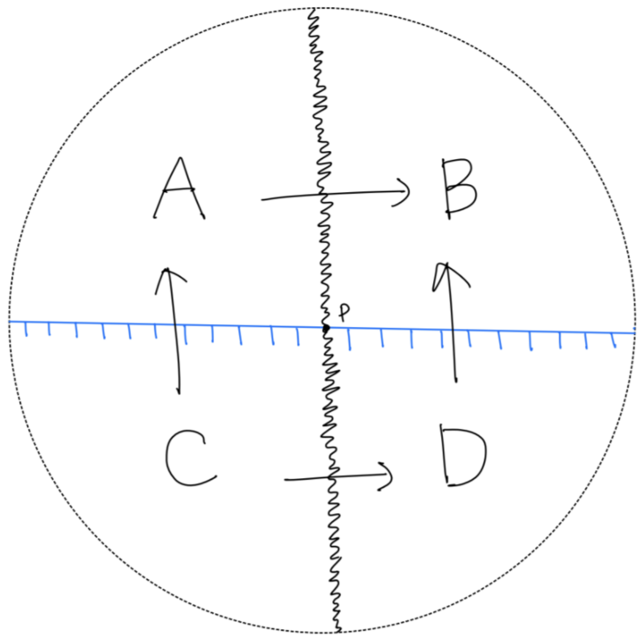

Figure 1.6

, then we have a map

$$Cone(C \to A) \to Cone(C \to B)$$

$$Cone(A \to C) \to Cone(B \to C)$$

$$Cone(C \to A) \to Cone(D \to B)$$

respectively.

The maps between cones are quasi-isomorphisms because of the crossing condition and the fact that the map corresponding to squiggly lines are quasi-isomorphism.

Definition 10. $F^\bullet$ is called *rank n object* if $\mu mon(F^\bullet)$ is a rank n local system concentrated in degree 0 on Λ.

Definition 11. The full subcategory of $Sh^\bullet_\Lambda(M;\mathbb{C})$ containing rank n objects is denoted $\mathcal{C}_n(M,\Lambda;\mathbb{C})$ and the moduli stack classifying such objects $\mathcal{M}_n(M,\Lambda;\mathbb{C})$. Additionally, let $\{\sigma_1, \cdots, \sigma_k\}$ be points in M that are contained in one of the regions separated by Φ. Then we define $\mathcal{C}_n(M, \Lambda, \{\sigma_1, \cdots, \sigma_k\}; \mathbb{C})$ to be the full subcategory of sheaves with vanishing stalks at $\{\sigma_1, \cdots, \sigma_k\}$, $\mathcal{M}(M, \Lambda, \{\sigma_1, \cdots, \sigma_k\}; \mathbb{C})$ the moduli space classifying such objects.

Moduli Spaces associated to Positive Braids

In this section, we define the moduli spaces associated to positive braids following[STWZ19, Section 3]. For background we refer to the survey [Toë14] and the foundational works [Lur04] [Toë09] [TV04] [TV05] [TV08]. Suppose we have a positive braid word ω of n strands(i.e. a word freely generated by $s_1, \cdots, s_{n-1}$), Then we can think of its cylindrical closure in $S^1_\theta \times (0,1)_r = (\mathbb{R}_\theta/\mathbb{Z}) \times (0,1)_r$. In this paper, we consider two kinds of moduli spaces associated to the braid which are the main objects of study in Chapter2 and Chapter3 respectively.

(1) First, we embedd the cylinder containing cylindrical closure of ω in $M = S^1_x \times \mathbb{R}_z$ via $(\theta, r) \mapsto (\theta, r-1)$. We can think of the cylindrical closure of ω in $S^1_x \times \mathbb{R}_z$ co-oriented downward($z < 0$) to be the front projection of a Legendrian knot Λ_ω living in $T^\infty M$. Let $\sigma_{z \ll 0}$ be a point in the non-compact region where $z \ll 0$. In Chapter2, we consider the moduli space $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C})$.

(2) Next, we embed the cylinder containing the cylindrical closure of ω in $M = S^1_x \times \mathbb{R}_z$ via $(\theta, r) \mapsto (\theta, 1-r)$ and embed the cylinder containing the cylindrical closure of the trivial braid $\omega_\emptyset$ via $(\theta, r) \mapsto (\theta, r-1)$. We get an embedding of the cylindrical closure of $\omega \coprod \omega_\emptyset$ in $S^1_x \times \mathbb{R}_z$ where the embedding of the cylindrical closure of ω is co-oriented upward and the embedding of the cylindrical closure of $\omega_\emptyset$ is co-oriented downward. We will call the embedding the separated diagram of ω and $\omega_\emptyset$ which we consider as the front projection of a Legendrian link $\Lambda_\infty \coprod \Lambda_0 \subset T^\infty M$. Let $\sigma_{z\ll 0}, \sigma_{z \gg 0}$ be points in the non-compact regions where

$z \ll 0$ and $z \gg 0$. In Chapter3, we consider the moduli space $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z\ll 0}, \sigma_{z\gg 0}\}; \mathbb{C})$.

Legendrian Isotopy and Sheaf Cobordism

Invariance of the category $\mathcal{C}_1(M, \Lambda, \{\sigma_1, \cdots, \sigma_k\}; \mathbb{C})$ under Legendrian isotopy follows from the main theorem of [GKS12].

Theorem 12. [GKS12]. Suppose $\Lambda_0 \subset T^\infty M$ is a Legendrian and $\Lambda_\bullet \subset T^\infty \times [0,1]$ a Legendrian isotopy where $\Lambda_\bullet|_{T^\infty \times \{0\}}$ is equal to Λ_0 under the natural identification $T^\infty \times \{0\} \cong T^\infty$, then the restriction functor

$$Sh^\bullet_{\Lambda_\bullet}(M \times [0,1]; \mathbb{C}) \to Sh^\bullet_{\Lambda_0}(M; \mathbb{C})$$

is a quasi-equivalence. Furthermore, this quasi-equivalence preserves rank n objects.

Definition 13. Suppose we have a Legendrian isotopy $\Lambda_\bullet \subset T^\infty \times [0,1]$ from $\Lambda_0 = \Lambda_\bullet|_{T^\infty \times \{0\}}$ to $\Lambda_1 = \Lambda_\bullet|_{T^\infty \times \{1\}}$ and a sheaf $\mathscr{F}_\bullet \in Sh^\bullet_{\Lambda_\bullet}(M \times [0,1]; \mathbb{C})$ that restricts to $\mathscr{F}_0$ and $\mathscr{F}_1$ at $M \times \{0\}$ and $M \times \{1\}$, then we say $\mathscr{F}_\bullet$ is a *sheaf cobordism* from $\mathscr{F}_0$ to $\mathscr{F}_1$.

Definition 14. Suppose $\Lambda_0, \Lambda_1 \subset T^\infty M$ are Legendrian links and there is a Legendrian isotopy $\Lambda_\bullet \subset T^{\infty,-} M \times [0,1]$ where $\Lambda_\bullet|_{T^{\infty,-} \times \{0\}}$ is equal to Λ_0 under the natural identification $T^{\infty,-} \times \{0\} \cong T^{\infty,-}$ and $\Lambda_\bullet|_{T^{\infty,-} \times \{1\}}$ is equal to Λ_0 under the natural identification $T^{\infty,-} \times \{1\} \cong T^{\infty,-}$, then we have quasi- equivalence

$$Sh^\bullet_{\Lambda_0}(M; \mathbb{C}) \xrightarrow{\sim} Sh^\bullet_{\Lambda_1}(M; \mathbb{C})$$

from the following correspondence

$$Sh^\bullet_{\Lambda_0}(M; \mathbb{C}) \xleftarrow{\sim} Sh^\bullet_{\Lambda_\bullet}(M \times [0,1]; \mathbb{C}) \xrightarrow{\sim} Sh^\bullet_{\Lambda_1}(M; \mathbb{C})$$

Furthermore, this quasi-equivalence restricts to

$$\mathcal{C}_n(M, \Lambda_0; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}(M, \Lambda_1; \mathbb{C})$$

When we have points $\{\sigma_1, \cdots, \sigma_k\} \subset M$ that are disjoint from the front projections of the Legendrian isotopy $\Lambda_\bullet$, we can further restrict the quasi-equivalence to

$$\mathcal{C}_n(M, \Lambda_0, \{\sigma_1, \cdots, \sigma_k\}; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}_n(M, \Lambda_1, \{\sigma_1, \cdots, \sigma_k\}; \mathbb{C})$$

Alternating Legendrians

Definition 15. Let M be a surface and $\Lambda \subset T^\infty M$ a Legendrian link. An *alternating coloring* for Λ is the datum of, for each region in the complement of the front diagram, a label of black, white, or null, subject to the following conditions

- the boundary of a black region is co-oriented inward.
- the boundary of a white region is co-oriented outward.
- the boundary of the null region have both inward and outward co-orientations.
- no black region shares a one dimensional border with white region and no null region shares a one dimensional border with another null region.

An *alternating Legendrian* is a Legendrian equipped with an alternating coloring and their front projection is called an *alternating strand diagram.* The *bipartite graph* of the alternating Legendrian is the graph whose vertices are black and white regions. Edges are connected if their closure intersect and are of distinct color.

Let $\hat{M}$ denote the real blow up of M at the crossings of the front projection of Λ. The blow down map $\hat{M} \to M$ is a diffeomorphism away from the crossing and the fiber above a crossing is the $\mathbb{RP}^1$ of lines tangent to the crossing. We define

$W \subset M$($B \subset M$ resp.) the union of the interiors of the white(black resp.) regions of the complement of the front projection.

Definition 16. $\overline{L}$ denote the closure of the preimage of $W \cup B$ in $\hat{M}$. It is a smooth surface with boundary and we refer to its interior L as the *conjuate surface* of Λ.

Definition 17. An *alternating sheaf* is an object of $Sh^{\bullet}_{\Lambda}(M;\mathbb{C})$ whose support is contained in the closure of the union of the white and black regions.

Theorem 18. [STWZ19, Thm. 4.16][STWZ19, Cor. 4.17]. The full subcategory of $Sh^{\bullet}_{\Lambda}(M;\mathbb{C})$ consisting of alternating sheaves is equivalent to the category of locally constant sheaves on L. Under this correspondence, rank 1 local systems of corresponds to rank 1 alternating sheaves.

Cluster Coordinates

In this section, we review a special type of coordinate system on the moduli space $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z\ll 0}, \sigma_{z\gg 0}\};\mathbb{C})$ called cluster coordinate.
Suppose we have an alternating Legendrian Λ_{alt} that is Legendrian isotopic to $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset}$ via a Legendrian isotopy $\Lambda_\bullet$ whose front projection does not touch $\{\sigma_{z\ll 0}, \sigma_{z\gg 0}\}$. Then the isotopy, by Theorem 12, induces an isomorphism between moduli spaces

$$\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z\ll 0}, \sigma_{z\gg 0}\};\mathbb{C}) \xrightarrow{\sim} \mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z\ll 0}, \sigma_{z\gg 0}\};\mathbb{C})$$

Furthermore, once we have alternating Legendrian, we can construct its conjugate surface L which deformation retracts to the bipartite graph $\Gamma_{\Lambda_{alt}}$ of alternating coloring on Λ_{alt}. Therefore, there is a sequence of isomorphisms

$$H^1(L, \mathbb{C}^*) \cong H^1(\Gamma_{\Lambda_{alt}}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})}$$

where $b_1(\Gamma_{\Lambda_{alt}})$ is the 1^{st} Betti number of $\Gamma_{\Lambda_{alt}}$.
Also, local systems on L corresponds to alternating sheaves in $M_1(S^1_x \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$ i.e. we have an inclusion

$$H^1(L, \mathbb{C}^*) \hookrightarrow M_1(S^1_x \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

In conclusion, we have an inclusion

$$(\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})} \hookrightarrow M_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

which is called a *cluster coordinate chart* of the moduli space. There is a rich theory of “cluster varieties”, covered by such cluster coordinate charts, but in this thesis we only construct one such chart.

1.2 Summary of results

Chapter 2

We define a class of braid words that we call *Sibuya braid words*. We show that the moduli spaces $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C})$ associated to Sibuya braid words ω parametrize configurations of points on a Projective space subject to certain non-degeneracy conditions. They are a generalization of the Sibuya space that Boalch studied in [Sib75][Boa15]. Their definition and a simple combinatorial characterization can be found in Definition 35 and Theorem 45. In this introduction, I will illustrate the concept with an example.
Consider a braid word $\omega = (s_1 s_2)^2$ on 3 strands, which is a power of the Coxeter braid. Then we have an embedding of ω into the fundamental domain of $S^1_x \times \mathbb{R}_z$ given in the below figure

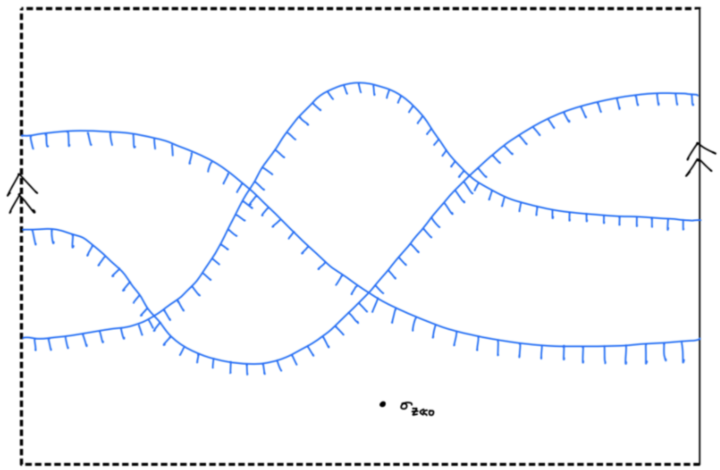

Figure 1.7

Since the stratification induced by ω is not a regular cell complex, for example non-compact regions at the top and the bottom are not contractible, we refine it by adding squiggly lines co-oriented towards left.

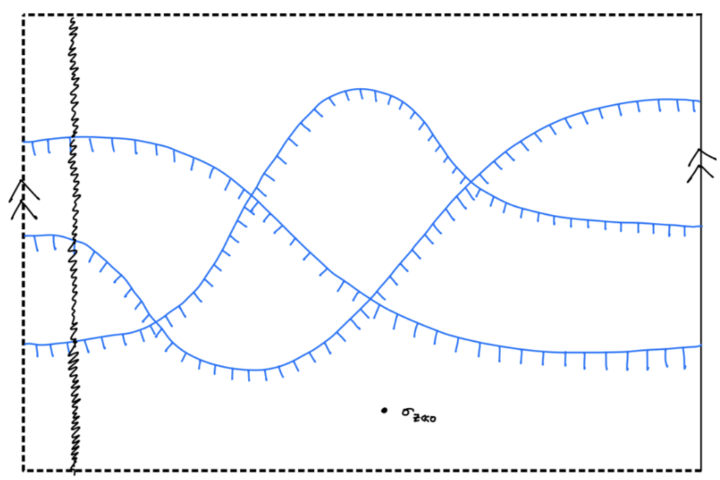

Figure 1.8

Consider an object of $C_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C})$. It can be described as a squiggly legible diagram by assigning cochain complexes of $\mathbb{C}$-vector spaces to regions and cochain maps to arcs subject to the conditions mentioned in (legible object subsection). Let's check how those conditions translates to our situation:

- the cochain complex assigned to the bottom region, which we call height 0 region, is acyclic because of the vanishing condition at $\sigma_{z\ll 0}$. We are considering cochain complexes of sheaves upto quasi-isomorphism, so we assign 0 stalk to the region.

- the stalk at the regions adjacent to the height 0 region that are not a height 0 region, which we call height 1 regions, should be quasi-isomorphic to dimension 1 vector space $\mathbb{C}$ because of the microlocal rank 1 condition on blue arcs separating height 0 region and height 1 regions i.e. the mapping cone of the cochain map corresponding to blue arcs should be rank 1 vector spaces concentrated in degree 0. Therefore, we assign $\mathbb{C}$ to these height 1 regions and incoming maps from the height 0 region zero maps.

- the stalk at the regions adjacent to the height 1 regions that are not height $0, 1$ regions, which we call height 2 regions, should be quasi-isomorphic to dimension 2 vector space $\mathbb{C}^2$ and incoming maps from the height 1 region to be monomorphisms because of the microlocal rank 1 condition again. Therefore, we assign $\mathbb{C}^2$ to the height 2 regions and incoming maps from the height 1 regions to be inclusion maps.

- Now, we are left with one non-compact region at the top which we call height 3 region. Again by microlocal rank 1 condition, the stalk should be $\mathbb{C}^3$ and the incoming maps from height 2 regions are inclusion maps.

- the maps corresponding to squiggly lines are isomorphisms of adjacent vector spaces.

- all diagrams should commute.

- for each crossing the crossing condition translates to

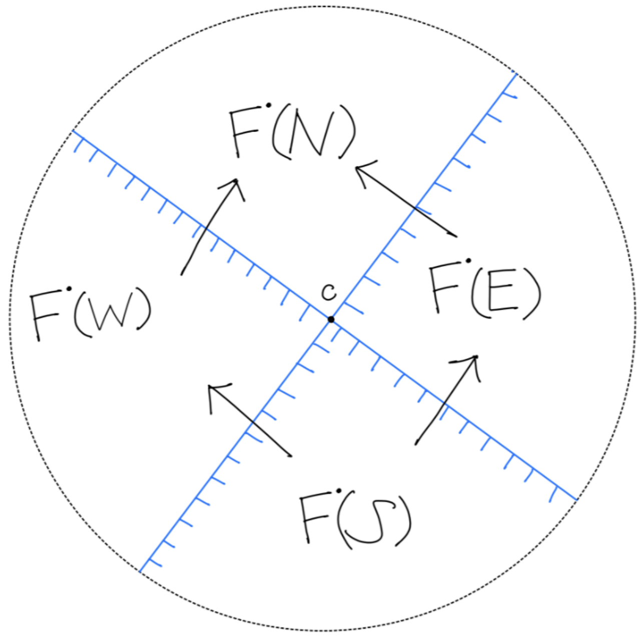

Figure 1.9

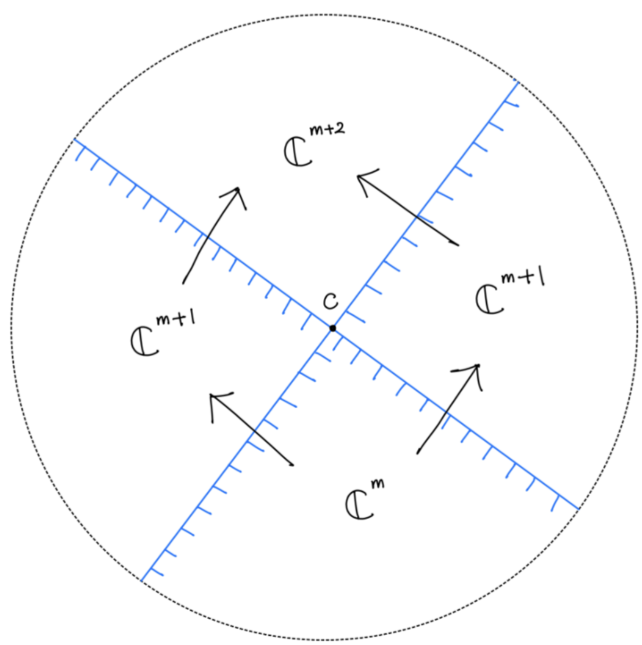

Figure 1.10

the inclusion of $F^\bullet(W)$ and $F^\bullet(E)$ in $F^\bullet(N)$ intersects transversely in $F^\bullet(S)$.

In summary, an object of $C_1(S^1_x \times \mathbb{R}_z, \Lambda, \{\sigma_{z\ll 0}\}; \mathbb{C})$ amounts to the data of assigning

- dimension i subspaces of $\mathbb{C}^3$ to height i regions subject to the condition that $F^\bullet(W) + F^\bullet(E) = F^\bullet(N)$ and $F^\bullet(W) \cap F^\bullet(E) = F^\bullet(S)$.

- an element $g \in GL_3(\mathbb{C})$ that maps the subspaces corresponding to the region to the left of squiggly lines to the subspaces corresponding to the region to the right of the squiggly lines.

One thing to note in the above example is that once we assign the dimension 1 subspaces to the height 1 regions, all the other subspaces are determined by the crossing conditions. More precisely, we can construct subspaces that are assigned to the regions of heights greater than 1 as follows: Suppose R is a region, then the subspace we assign to R is the subspace of $\mathbb{C}^3$ spanned by the 1 dimensional subspaces assigned to the height 1 regions that have paths to R. I will describe what this means in Chapter2. This fact that the sheaf is completely determined by its rank 1 stalks relies on following feature of the diagram Λ_ω:

> Every region of height bigger than 1 has more than or equal to 2 different regions of height 1 less adjacent to it.

We call a braid word with the above property as Sibuya braid word and a braid with Sibuya braid word representation as Sibuya braid.
But also note that not all assignments of 1 dimensional subspaces of $\mathbb{C}^3$ to height 1 regions gives rise to legitimate sheaves. In Chapter2, I will show how to restate the crossing conditions using only rank 1 stalks.
At the end of the Chapter, we will explicitly compute examples of the generalized Sibuya space.

Chapter 3

Chapter3 is the longest chapter in the thesis, but we can summarize it more briefly:

(i) we will systematically construct a natural alternating Legendrian(equivalently, alternating strand diagram) that are Legendrian isotopic to the separated Legendrian $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset} \subset T^\infty(S^1_x \times \mathbb{R}_z)$.

(ii) we will construct a Legendrian isotopy between the alternating Legendrians and the separated Legendrians.

(iii) we will explicitly describe the sheaf cobordism induced by the above isotopy.

Conventions and notation

- ω denotes positive braid words. In this thesis, all braid words are positive braid words.
- β denotes positive braids. In this thesis, all braids are positive braids.
- $\mathrm{Cross}(\omega)$ denotes the set of crossings in the braid word ω.
- L denotes Lagrangian surfaces in $T^*(S^1_x \times \mathbb{R}_z)$.
- Λ, λ denote Legendrian links in $T^\infty(S^1_x \times \mathbb{R}_z)$.
- Ξ, ξ denote co-orientations on Legendrian links.
- Φ denotes front projections of Legendrian links on $S^1_x \times \mathbb{R}_z$.
- $\mathcal{S}$ denotes refinements of the stratifications induced by front projections.
- $\mathscr{F}^\bullet$ denotes a cochain complex of sheaves whose cohomologies are constructible sheaves. We simply call these "sheaves" in this thesis.
- Q denotes quivers.
- $\mathrm{Vert}(Q)$ denotes the set of vertices of the quiver Q.

- Arr(Q) denotes the set of arrows of the quiver Q.

- Υ_0^k denotes the set of signature 0 height k vertices in a quiver.

- $I_k(v)$ denotes the set of height 1 vertices that have paths to vertex v in a quiver.

- $\mathrm{SS}(\mathscr{F}^\bullet)$ denotes the singular support of the sheaf $\mathscr{F}^\bullet$.

- $\mu mon(\mathscr{F}^\bullet)$ denotes the microlocal monodromy sheaf of $\mathscr{F}^\bullet$.

- $Sh^\bullet(M;\mathbb{C})$ denotes the dg category of sheaves on M.

- $Sh^\bullet_L(M;\mathbb{C})$ denotes the full subcategory of $Sh^\bullet(M;\mathbb{C})$ whose singular support is contained in the Lagrangian L.

- $Sh^\bullet_\Lambda(M;\mathbb{C})$ denotes the full subcategory of $Sh^\bullet(M;\mathbb{C})$ whose singular support is contained in the Lagrangian $\mathbb{R}_+\Lambda \cup 0_M$.

- $Fun^\bullet(\mathcal{S},\mathbb{C})$ denotes the dg category of functors from the poset category $\mathcal{S}$ to the category of cochain complexes of $\mathbb{C}$-vector spaces.

- $Fun^\bullet_\Lambda(\mathcal{S},\mathbb{C})$ denotes the full subcategory of $Fun^\bullet(\mathcal{S},\mathbb{C})$ whose objects under the quasi-equivalence $\Gamma_\mathcal{S}$ (see 3) correspond to sheaves singular supported in $\mathbb{R}_+\Lambda \cup 0_M$.

- $F^\bullet$ denotes quiver representations valued in the cochain complex of $\mathbb{C}$-vector spaces.

- $\overline{F}^\bullet$ denotes objects of $Fun^\bullet(\mathcal{S},\mathbb{C})$.

- $\mathcal{C}_n(M,\Lambda,\{\sigma_1,\cdots,\sigma_k\};\mathbb{C})$ denotes the full subcategory of $Sh^\bullet_\Lambda(M;\mathbb{C})$ consisting of rank n objects.

- $\mathcal{M}_n(M,\Lambda,\{\sigma_1,\cdots,\sigma_k\};\mathbb{C})$ denotes the moduli space parametrizing objects of $\mathcal{C}_n(M,\Lambda,\{\sigma_1,\cdots,\sigma_k\};\mathbb{C})$.

– $\mathcal{M}_\omega$ denotes the moduli space $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C})$.

– $\mathcal{M}^{fr}_\omega$ denotes the framed moduli space of $\mathcal{M}_\omega$, i.e., the moduli space parametrizing the objects of $\mathcal{M}_\omega$ with a choice of basis for the stalk at $\sigma_{z\gg 0}$, a fixed point in the non-compact region $z \gg 0$.

– ι_ω denotes the embedding of $\mathcal{M}^{fr}_\omega$ in the product of a general linear group with powers of projective spaces, i.e., $GL_n(\mathbb{C}) \times (\mathbb{P}^{n-1})^k$.

– L_{conj} denotes the conjugate surface of an alternating Legendrian.

– Γ_{bi} denotes the bipartite graph associated to an alternating Legendrian.

– $\Psi(x_0, r_0, z_{lo}, z_{hi})$ denotes the bump function about $x = x_0$ of radius r_0, with z range from z_{lo} to z_{hi} i.e. consider the standard bump function

$$\Psi_0(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

then

$$\Psi(x_0, r_0, z_{lo}, z_{hi}) := (z_{hi} - z_{lo})\Psi(\frac{x - x_0}{r_0}) + z_{lo}$$

We denote the bump function about $x = 0$ of radius 1 by $\Psi(z_{lo}, z_{hi})$.

– τ denotes the smooth transition function i.e. consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

then

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

– I denotes an identity matrix. We omit the size of the matrix when it can be inferred from context.

– I_{i_0,j_0} denotes the matrix whose (i,j)-entry is $\delta_{i,i_0} \cdot \delta_{j,j_0}$ where δ is the Kronecker delta.

– Let T be an n by m matrix, then $T(r_{start}, r_{end}, c_{start}, c_{end})$ denotes the truncation of the matrix T with row range from r_{start} to r_{end} and column range from c_{start} to c_{end}.

– $\mathrm{diag}(a_1, \cdots, a_k)$ denotes the diagonal matrix of size $k \times k$ whose (i,i)-entry is a_i.

– e_i denotes the i^{th} standard basis vector of $\mathbb{C}^m$.

– ι_k denotes the monomorphism from $\mathbb{C}^m$ to $\mathbb{C}^{m+1}$ that maps e_i to e_{i+k}. We omit the dimensions of the source and the target vectorspaces when it can be inferred from the context.

– $D_{r=r_0}$ denotes the standard disk in $\mathbb{R}^2$ of radius r_0 centered at the origin.

– Let $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$ be cochain complexes supported in degrees 0 and 1, and let $\phi^\bullet$ be a morphism between $(C^\bullet, \delta^\bullet)$ and $(C'^\bullet, \delta'^\bullet)$. Then:

  ∘ We denote $(C^\bullet, \delta^\bullet)$ as either:

    * $C^0 \xrightarrow{\delta^1} C^1$

    or

    * $$\begin{array}{c} C^1 \\ \uparrow{\scriptstyle \delta^1} \\ C^0 \end{array}$$

  ∘ We denote $\phi^\bullet$ as either:

- (ϕ^0, ϕ^1)

  or

$$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \uparrow^{\delta^1} & & \uparrow^{\delta'^1} \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

- We omit coboundary maps (δ^1) or cochain maps (ϕ^i for $i = 0, 1$) when either the source or the target is 0, in which case the map is obviously a zero map.

- When the source and target vector spaces of coboundary maps (δ^1) or cochain maps (ϕ^i for $i = 0, 1$) are identical and the map between them is omitted, it is understood to be an identity map.

– When drawing a squiggly legible diagram, we follow these conventions:

- All maps corresponding to red arcs are ι_0 unless otherwise specified.
- All maps corresponding to blue arcs are ι_1 unless otherwise specified.
- Stalks and generization maps are omitted when they can be inferred from the context.

Chapter 2

Multivalent Braids and Generalized Sibuya Spaces

2.1 Basic terminologies

Definition 19. Let ω be a braid word, then we have an embedding of the braid diagram in $[0,1]_x \times \mathbb{R}_z$, co-oriented below, as an *open front diagram* of ω.

Definition 20. Let ω be a braid word representing a braid $\beta \in Br_n^+$ i.e. $[\omega] = \beta$. Then we define Q_ω^o to be the quiver

- whose vertices are labeled by the regions of the open front projection
- whose arrows are labeled by pairs of vertices whose corresponding regions are adjacent(bordered by the front projection of the braid) subject to the condition that the arrows always go against the co-orientation(hairs).

There are two distinguished vertices of Q_ω^o. We denote the vertex corresponding to the region $z \to \infty$(resp. $z \to -\infty$) as U(resp. D).

Definition 21. Locally for each crossing c, there is a region all the hairs are pointing outward, we call this N_c(read north of c). Starting from N_c, as we move counterclockwise about the crossing, we call the corresponding regions N_c, E_c, S_c, W_c respectively(read north of, east of, south of, west of c).

Theorem 22. Suppose we have a fixed braid word ω and v is a vertex corresponding to a region given by the front projection of ω. Any path from d to a vertex v have same length.

Proof. we prove the statement by the induction on the length of the braid words. The statement is trivial for trivial braid word because there is only one path from d to any vertex. Suppose the statement is true for all quivers associated to braid words whose length is less than n. Suppose ω is a braid word of length n. Suppose there are two distinct paths p_1 and p_2 from d to v. Let $\omega = s_{i_1} \cdots s_{i_n}$. There are two cases to consider :

(case1) v is the region east of the new crossing generated by s_{i_n}. Then the paths to v must have passed the region right below the region of v because that's the only possible way to get to v under the constraint that the arrow always go against the co-orientation. If we remove the last edge and vertex from the paths p_1, p_2, we have paths p_1', p_2' ending at the same vertex(i.e. south of the crossing generated by s_{i_n}) and are entirely contained in the subquiver $Q^o_{\omega'}$ where $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Therefore, by the induction hypothesis the lengths of p_1' and p_2' are the same which immediately implies $\text{length}(p_1) = \text{length}(p_1') + 1 = \text{length}(p_2') + 1 = \text{length}(p_2)$.

(case2) Suppose v is not the region that is the east of the crossing generated by s_{i_n}. Without loss of generality, we can assume that two paths do not pass through the region east of the crossing, because we can always replace the part of the path $s_c \to e_c \to n_c$ with $S_c \to W_c \to N_c$ having the same length. Then once we know that two paths p_1 and p_2 do not pass through the region east of the crossing, we know that they are entirely contained in the subquiver $\omega' = s_{i_1} \cdots s_{i_{n-1}}$. Then the $\text{length}(p_1) = \text{length}(p_2)$ by the induction hypothesis. $\square$

Definition 23. The valency of the vertex of a quiver is the number of incoming arrows. The height of the vertex of a quiver is the length of a path starting from d ending at that vertex which is well-defined by the previous theorem. Note that for every crossing c, E_c and W_c have the same height.

Definition 24. We say two vertices are adjacent if there is a crossing c such that two vertices are E_c and W_c of this crossing. Let k be a positive integer, then we define a natural ordering on the set of all height k vertices generated by the relations, for each crossing $W_c \leq E_c$.

Theorem 25. The above ordering is well-defined and is a total ordering on the set of all height k vertices.

Proof. To prove the claim, we have to prove the following facts :

(i) For any two distinct points of height k, there is a chain of crossings connecting two points.

(ii) there is no chain of crossings starting at a point and ending at the same point.

We prove the claim by induction on the length of braid words. For the trivial braid, (i) holds because there is only one vertex for each height and (ii) holds because there is no arrow starting from that unique and ending at the point.

Now suppose the claim holds for all length $< n$ braids and suppose i does not hold for the braid word $\omega = s_{i_1} \cdots s_{i_n}$. Then one of the two vertices should be the vertex corresponding to E_c of the crossing generated by s_{i_n}. Let's call this vertex v and the other as v'. Then we know that by the induction hypothesis there is a chain of crossing connecting v' and W_c. Since W_c is connected by the crossing c to $E_c = v$ and v, v' are connected by a chain of crossings which is a contradiction.

Now suppose the claim holds for all length $< n$ braids and suppose ii. Suppose there is a point where there is a chain of crossing starting and ending at the same point. By the induction hypothesis, the chain of crossing generated by s_{i_n} along the way.

Without loss of generality, using cyclic shift we can assume the chain of crossings starts from e_c which is a contradiction because E_c is not the $W_{c'}$ of any crossing c'. □

Definition 26. There are two distinguished A_n sub-quivers of Q^o_ω. We denote R_ω(resp. L_ω) to be the full sub-quiver containing all the vertices corresponding to the rightmost(resp. leftmost) regions. Alternatively, the path following the largest(resp. smallest) arrows at each step. We denote the vertex of R_ω(resp. L_ω) of height k by R^k_ω(resp. L^k_ω).

Definition 27. Let $\mathcal{M}^{fr}_\omega$ be the framed moduli space classifying pairs (F, g) where F is the representation of Q^o_ω and $g \in \mathrm{GL}_n(\mathbb{C})$ subject to the following conditions:

- For each vertex the vectorspace associated to it is a subspace of $\mathbb{C}^n$ of dimension equal to its height.
- All maps are inclusion maps.
- $gF(R^k_\omega) = F(L^k_\omega)$ for all $k = 0, 1, \cdots, n$.
- For each crossing c, then the sequence $0 \to F(S_c) \to F(E_c) \oplus F(W_c) \to F(N_c) \to 0$ is a short exact sequence where maps are induced by the inclusion maps.

Remark 28. There is a natural left action of $x \in \mathrm{GL}_n(\mathbb{C})$ on $(F, g) \in \mathcal{M}^{fr}_\omega$, that is, $x \cdot (F, g) = (xF, xGx^{-1})$ where xF is left translation on quiver representation and xgx^{-1} is conjugation.

Theorem 29. [STZ17, Prop. 1.5] $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C})\backslash\mathcal{M}^{fr}_\omega$.

Now consider the following map

Definition 30. Suppose we have the braid word ω of a braid β. Let $\{v_1, v_2, \cdots, v_m\}$ be the complete list of all height 1 vertices of Q^o_ω such that $v_i < v_j$ if and only if $i < j$. We define ι_ω to be the forgetful map

$$\begin{aligned}\iota_\omega \;:\; \mathcal{M}^{fr}_\omega &\to (\mathbb{P}^{n-1})^m \times \mathrm{GL}_n(\mathbb{C})\\ (F, g) &\mapsto ([F(v_1)], \cdots, [F(v_m)], g)\end{aligned}$$

Definition 31. Let ω be a braid word. We denote the infinite cyclic copies of the original braid word ω by ω^∞. More precisely, suppose ω is a braid word given by a collection of sections $\{\sigma_i : [0,1]_x \to [0,1]_x \times \mathbb{R}^2_{y,z}\}_{i=1,\ldots,n}$, then ω^∞ is given by the collection $\{\overline{\sigma_i} : \mathbb{R}_x \to \mathbb{R}^3_{x,y,z} \mid \overline{\sigma}(x, y, z) = \sigma(x - [x], y, z)\}_{i=1,\ldots,n}$. Again, I will abuse the notation ω^∞ to denote its front projection onto $\mathbb{R}^2_{x,z}$.

Definition 32. Let ω be a braid word, then we have the quiver Q^o_ω associated to it. We define the quiver Q^∞_ω to be the quotient of $\prod_{i\in\mathbb{Z}} Q^o_\omega$ by the relations $\{R_{\omega,i} = L_{\omega,i+1}\}_{i\in\mathbb{Z}}$ where $R_{\omega,i}$(resp. $L_{\omega,i+1}$) is the subquiver R_ω(resp. L_ω) in the i^{th}(resp. $i + 1^{th}$) copy of Q^o_ω in $\prod_{i\in\mathbb{Z}} Q^o_\omega$. Therefore, we have the quotient map of quivers $q : \prod_{i\in\mathbb{Z}} Q^o_\omega \to Q^\infty_\omega$. Let the signature function $\sigma' : \mathrm{Vert}(\prod_{i\in\mathbb{Z}} Q_\omega) \to \mathbb{Z}$ be $\sigma'(v) = i$ if v is in the i^{th} copy of Q^o_ω in $\prod_{i\in\mathbb{Z}} Q_\omega$. Define $\sigma : Q^\infty_\omega \to \mathbb{Z}$ to be $\sigma(v) = \min_{w\in q^{-1}(v)} \sigma'(w)$ if $|q^{-1}(v)| < \infty$ and 0 otherwise. Note that for $v \in \mathrm{Vert}(Q^\infty_\omega)$, $q^{-1}(v)$ is infinite if and only if $R^{\mathrm{height}(v)}_\omega = L^{\mathrm{height}(v)}_\omega$ i.e. there is a unique vertex of height(v) in Q^o_ω. We can think of Q^o_ω as the full subquiver of Q^∞_ω spanned by the signature 0 vertices.

Definition 33. Suppose we have a quiver representation F_ω of Q^o_ω, then we define the induces quiver representation F^∞_ω of the quiver Q^∞_ω to be $F^\infty_\omega(v) := g^{\sigma(v)} \cdot F_\omega(v - \sigma(v))$.

Under the above correspondence, we get an isomorphism $\mathcal{M}^{fr}_\omega \cong \mathcal{M}^{fr,\infty}_\omega$.

Definition 34. Υ^k_0 is the set of all height k, signature zero vertices in Q^∞_ω. For each vertex v of Q^∞_ω, we define $I_k(v)$ to be the set of all the height k vertices that have paths to v.

2.2 Multivalent braid words

Definition 35. A braid word ω is multivalent if and only if all the vertices of height greater than 1 in Q_ω^∞ has valencies greater than 1.

Theorem 36. If the braid word ω is multivalent, then ι_ω is an embedding.

Proof. It is enough to prove that once we specify vectorspaces to height 1 vertices of Q_ω^o and $g \in \mathrm{GL}_n(\mathbb{C})$, the quiver representation of Q_ω^o extending the above datum is unique. Since $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$, it is enough to prove once we specify vectorspaces to signature 0 height 1 vertices of Q_ω^∞ and $g \in \mathrm{GL}_n(\mathbb{C})$, the quiver representation F_ω^∞ of Q_ω^∞ extending the above datum is unique. Since $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, for $v \in \mathrm{Vert}(Q_\omega^\infty)$, $F_\omega^\infty(v) = g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))$. If $\mathrm{height}(v) = 1$, then $v - \sigma(v)$ is height 1 signature 0 vertex. Therefore, $F_\omega^\infty(v)$ is uniquely determined.

Now we prove the statement by induction on the heights of vertices. Assume $\forall v \in \mathrm{Vert}(Q_\omega^\infty)$ with $\mathrm{height}(v) < h$, $F_\omega^\infty(v)$ are determined. Suppose $v \in \mathrm{Vert}(Q_\omega^\infty)$ such that $\mathrm{height}(v) = h > 1$, then there are at least two vertices of height $h-1$ that have arrows to v, say v' and v''. Without loss of generality, $v' \leq v''$. Then by $\omega^\infty, Q_\omega^\infty$-analogue of Theorem 25, there is a chain of crossings $c_1, \ldots, c_k$ and $v' = v_0, \ldots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i. Therefore, if we choose any c_i to be c, then $v = N_c$. By the induction hypothesis, $F_\omega^\infty(W_c)$, $F_\omega^\infty(E_c)$.and $F_\omega^\infty(S_c)$ have already been specified because heights of W_c,E_c,and S_c are $h-1$, $h-1$,and $h-2$ respectively. By the crossing condition,

$$0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$$

$F_\omega^\infty(N_c)$ is uniquely determined. $\square$

Theorem 37. If the braid word ω is multivalent, then the image of ι_ω is

$$\begin{aligned}
X_\omega = \{((z_v)_{v\in\Upsilon_0^1}, g) \in \prod_{v\in\Upsilon_0^1} \mathbb{P}_v^{n-1} \times \mathrm{GL}_n(\mathbb{C}) \mid \\
\forall u \in \mathrm{Vert}(Q_\omega^\infty),\ \dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(u), \\
\forall c \in \mathrm{Cross}(\omega^\infty),\ \dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(N_c)\}
\end{aligned}$$

where $\mathbb{P}_v^{n-1}$ is a copy of $\mathbb{P}^{n-1}$ labeled by $v \in \Upsilon_0^1$.

Proof. Instead of proving $\mathrm{Im}(\iota_\omega) = X_\omega$, I will prove that for ι', the map obtained by pre-composing the canonical isomorphism $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$ to ι_ω, $\mathrm{Im}(\iota') = X_\omega$.
First, let's prove $\mathrm{Im}(\iota') \subset X_\omega$. Recall that

$$\begin{aligned}
\mathcal{M}_\omega^{fr,\infty} = \{(F_\omega^\infty, g) \in \mathrm{Rep}(Q_\omega^\infty) \times \mathrm{GL}_n(\mathbb{C}) \mid \text{All the maps of } F_\omega^\infty \text{ are inclusion maps,} \\
\forall v \in \mathrm{Vert}(Q_\omega^\infty),\ \dim_{\mathbb{C}}(F_\omega^\infty(v)) = \mathrm{height}(v), \\
\forall v \in \mathrm{Vert}(Q_\omega^\infty),\ F_\omega^\infty(v+n) = g^n \cdot F_\omega^\infty(v), \\
\forall c \in \mathrm{Cross}(\omega^\infty),\ 0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0 \\
\text{are short exact sequences}\}
\end{aligned}$$

We need to show that for $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$, $\iota'_\omega((F_\omega^\infty, g)) = ((F_\omega^\infty(v))_{v\in\Upsilon_0^1}, g) \in X_\omega$ i.e.

- $\forall u \in \mathrm{Vert}(Q_\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = \mathrm{height}(u)$
- $\forall c \in \mathrm{Cross}(\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = \mathrm{height}(N_c)$

It is enough to prove that

- $\forall u \in \mathrm{Vert}(Q_\omega^\infty)$, $\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(u)$
- $\forall c \in \mathrm{Cross}(\omega^\infty)$, $\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(N_c)$

because of the condition $\forall u \in \mathrm{Vert}(Q_\omega^\infty),\ \dim_{\mathbb{C}}(F_\omega^\infty(u)) = \mathrm{height}(u)$ defining $\mathcal{M}_\omega^{fr,\infty}$. Moreover, $g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(v)$ by the definition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, we need to prove that,

(i) $\sum_{v \in I_1(u)} F_\omega^\infty(v) = F_\omega^\infty(u)$

(ii) $\sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) = F_\omega^\infty(N_c)$

Proof of (i) : For each $v \in I_1(u)$, there is a path from v to u, say $v = v_0 \to \cdots \to v_k = u$. Since $F_\omega^\infty(v_i) \subset F_\omega^\infty(v_{i+1})$, $F_\omega^\infty(v) \subset F_\omega^\infty(u)$. Therefore, $\sum_{v \in I_1(u)} F_\omega^\infty(v) \subset F_\omega^\infty(u)$. Conversely, we prove $F_\omega^\infty(u) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$ by induction on the height of u. If $\mathrm{height}(u) = 1$, the statement is trivial. Now suppose the statement holds for all $u \in \mathrm{Vert}(Q_\omega^\infty)$ of $\mathrm{height}(u) = h$. Since ω is multivalent, u has at least two vertices of height $h-1$ that have arrows to u, say v' and v''. Without loss of generality $v' \leq v''$. Then by $\omega^\infty, Q_\omega^\infty$-analogue of Theorem 25, there is a chain of crossings $c_1, \ldots, c_k$ and $v' = v_0, \ldots, v_k = v''$ where v_{i-1}, v_i are west and east of the crossing c_i. Therefore, if we choose any c_i to be c, then $u = N_c$ i.e. v is the north of the crossing c. Then by the crossing condition of $M_\omega^{fr,\infty}$

$$0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$$

we have $F_\omega^\infty(u) = F_\omega^\infty(N_c) = F_\omega^\infty(E_c) + F_\omega^\infty(W_c)$ by the induction hypothesis,

$$F_\omega^\infty(E_c) = \sum_{v \in I_1(E_c)} F_\omega^\infty(v),\ F_\omega^\infty(W_c) = \sum_{v \in I_1(W_c)} F_\omega^\infty(v)$$
$$\Rightarrow F_\omega^\infty(u) = \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v)$$

because E_c and W_c have arrows to $u = N_c \ \Rightarrow\ I_1(E_C) \cup I_1(W_c) \subset I_1(u)$. Therefore,

$$F_\omega^\infty(u) = \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$$

Proof of (ii) : By (i), we have

$$\begin{aligned}&\sum_{v\in I_1(E_c)\cup I_1(W_c)} F_\omega^\infty(v)\\ &= \sum_{v\in I_1(E_c)} F_\omega^\infty(v) \ + \sum_{v\in I_1(W_c)} F_\omega^\infty(v)\\ &= F_\omega^\infty(E_c) \ + \ F_\omega^\infty(W_c)\end{aligned}$$

This is equal to $F_\omega^\infty(N_c)$ by the crossing condition of $\mathcal{M}_\omega^{fr,\infty}$. Therefore, the proof of (ii) is complete.

Now let's prove $X_\omega \subset \mathrm{Im}(\iota'_\omega)$. Let $((z_v)_{v\in\Upsilon_0^1}, g)$ be an arbitrary point of X_ω. I will define a point (F_ω^∞, g) such that $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v\in\Upsilon_0^1}, g)$. We define a quiver representation F_ω^∞ to be

$$F_\omega^\infty(u) := \sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}$$

and all the arrows of F_ω^∞ are inclusion maps. The inclusion maps are well-defined because if there is an arrow from u to u', then $I_1(u) \subset I_1(u')$.

Note that if $u \in \Upsilon_0^1$ i.e. height$(u) = 1$ and $\sigma(u) = 0$, then

$$F_\omega^\infty(u) := \sum_{v\in\{u\}} g^{\sigma(v)} \cdot z_{v-\sigma(v)} = g^{\sigma(u)} \cdot z_{u-\sigma(u)}$$

If (F_ω^∞, g) is indeed a point of $\mathcal{M}_\omega^{fr,\infty}$, then $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v\in\Upsilon_0^1}, g)$. Thus, it is enough to prove that $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$ i.e.

(i) All maps of F_ω^∞ are inclusion maps

(ii) $\forall v \in \mathrm{Vert}(Q_\omega^\infty)$, $\dim_{\mathbb{C}}(F_\omega^\infty(v)) = \mathrm{height}(v)$

(iii) $\forall v \in \mathrm{Vert}(Q_\omega^\infty)$ and $n \in \mathbb{Z}$, $F_\omega^\infty(v + n) = F_\omega^\infty(v)$

(iv) $\forall c \in \mathrm{Cross}(\omega^\infty)$, $0 \to F_\omega^\infty(S_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$ are short exact sequences

We get (i) immediately from the definition of F_ω^∞.

To prove (ii), note that $\dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(u)$ because $((z_v)_{v\in\Upsilon_0^1}, g) \in X_\omega$. Therefore, $\dim_{\mathbb{C}}(F_\omega^\infty(u)) = \dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(u)$.

To prove (iii), note that $\forall u \in \mathrm{Vert}(Q_\omega^\infty)$ and $n \in \mathbb{Z}$, $I_1(u+n) = I_1(u)+n$. Therefore,

$$
\begin{aligned}
F_\omega^\infty(u+n) &= \sum_{v\in I_1(u+n)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\
&= \sum_{v\in I_1(u)+n} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\
&= \sum_{v-n\in I_1(u)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\
&= \sum_{v'\in I_1(u)} (g^{\sigma(v'+n)} \cdot z_{(v'+n)-\sigma(v'+n)}) \\
&= g^n \cdot (\sum_{v'\in I_1(u)} (g^{\sigma(v')} \cdot z_{v'-\sigma(v')})) \\
&= g^n \cdot F_\omega^\infty(u)
\end{aligned}
$$

Finally, let's prove (iv). Let $c \in \mathrm{Cross}(\omega^\infty)$, then

$$
\begin{aligned}
& \mathrm{height}(S_c)+2 = \mathrm{height}(E_c)+1 = \mathrm{height}(W_c)+1 = \mathrm{height}(N_c) \\
\Rightarrow\ & \dim_{\mathbb{C}}(F_\omega^\infty(S_c))+2 = \dim_{\mathbb{C}}(F_\omega^\infty(E_c))+1 = \dim_{\mathbb{C}}(F_\omega^\infty(W_c))+1 = \dim_{\mathbb{C}}(F_\omega^\infty(N_c))
\end{aligned}
$$

Since there are arrows from S_c to E_c, W_c and from E_c, W_c to N_c, $I_1(S_c) \subset I_1(E_c)$, $I_1(W_c) \subset I_1(N_c)$. Therefore, $F_\omega^\infty(S_c) \subset F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c)$ and $F_\omega^\infty(E_c) \cup F_\omega^\infty(W_c) \subset F_\omega^\infty(N_c)$.

By the condition that

$$\begin{aligned} & \dim_{\mathbb{C}}\Big(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}\Big) = \operatorname{height}(N_c) \\ \Rightarrow \ & \dim_{\mathbb{C}}(F_\omega^\infty(E_c) + F_\omega^\infty(W_c)) = \operatorname{height}(N_c) = \dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \\ \Rightarrow \ & F_\omega^\infty(E_c) + F_\omega^\infty(W_c) = F_\omega^\infty(N_c) \end{aligned}$$

we have surjection part of the short exact sequence i.e. $F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$ is exact. Thus we have a short exact sequence

$$0 \to F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c) \to F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \to F_\omega^\infty(N_c) \to 0$$

I claim that $F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c) = F_\omega^\infty(S_c)$. Since $F_\omega^\infty(E_c)$, $F_\omega^\infty(W_c)$,and $F_\omega^\infty(S_c)$ are codimension 1, 1,and 2 inside of $F_\omega^\infty(N_c)$ respectively, it is enough to show that $F_\omega^\infty(E_c)$ and $F_\omega^\infty(W_c)$ intersect transversely in $F_\omega^\infty(N_c)$ i.e. $F_\omega^\infty(E_c) \neq F_\omega^\infty(W_c)$. This is true because otherwise

$$\begin{aligned} \operatorname{height}(N_c) &= \dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \\ &= \dim_{\mathbb{C}}(F_\omega^\infty(E_c) + F_\omega^\infty(W_c)) \\ &= \dim_{\mathbb{C}}(F_\omega^\infty(E_c)) \\ &= \operatorname{height}(E_c) \end{aligned}$$

which is a contradiction.

Therefore, (F_ω^∞, g) satisfy all the condition (i)-(iv) i.e. $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$. $\square$

2.3 Bivalent braid words

Definition 38. A braid word ω is bivalent if it is multivalent and the valencies of all the vertices of height $1, \ldots, n-1$ in Q_ω^∞ are equal to 2 where n is the number of strands of ω.

Lemma 39. Let ω be a braid word and $v \in \mathrm{Vert}(Q_\omega(v \in \mathrm{Vert}(Q_\omega^\infty)$ resp.), then there is a path from v to the unique height n vertex U. The path passes through a vertex of height $n-1$.

Proof. If we prove the statement for Q_ω^o, then the proof for the Q_ω^∞ case follows immediately. Let's prove the claim by induction on the length of the braid word ω. If $\mathrm{length}(\omega) = 0$ i.e. trivial braid word, there is a path from D to U and along the way it passes through all the points. Therefore, restricting this path to start from the point that we are interested in gives the desired path. Now let's assume that the statement holds for all braid words of length less than k. Let $\omega = s_{i_1} \cdots s_{i_k}$, then define $\omega' = s_{i_1} \cdots s_{i_{k-1}}$. We get that $Q_{\omega'}^o$ is the subquiver of Q_ω^o where $\mathrm{Vert}(Q_\omega^o) - \mathrm{Vert}(Q_{\omega'}^o) = \{v\}$ is a singleton where v is the east of the crossing added by s_{i_k}, say c. Note that $v = E_c$ has an arrow to N_c and N_c has a path to U by the induction hypothesis because $N_c \in \mathrm{Vert}(Q_{\omega'}^o)$. Therefore, we can extend the path from N_c to U to start from E_c i.e. $E_c \to (N_c \to \cdots \to U)$. $\square$

Lemma 40. Let ω be a bivalent braid word and $u \in \mathrm{Vert}(Q_\omega^\infty)$ of $\mathrm{height}(u) < n$, then $|I_1(u)| = \mathrm{height}(u)$. In particular, if $c \in \mathrm{Cross}(\omega^\infty)$, then $I_1(S_c) = \mathrm{height}(S_c)$, $I_1(E_c) = \mathrm{height}(E_c)$, and $I_1(W_c) = \mathrm{height}(W_c)$.

Proof. We prove the statement by induction on the height of u. If $\mathrm{height}(u) = 1$, then $I_1(u) = \{u\} \Rightarrow |I_1(u)| = 1 = \mathrm{height}(u)$ holds. Now suppose the statement holds for vertices of heights less than h and $\mathrm{height}(u) = h$ where $h < n$. Since ω is bivalent, we have exactly two vertices of height $h-1$ and a crossing c, where those two vertices are the east and west of the crossing c. By the induction hypothesis, $|I_1(E_c)| = |I_1(W_c)| = h-1$, $|I_1(S_c)| = h-2$ and $I_1(S_c) \subset I_1(E_c), I_1(W_c)$. Let

$$\begin{aligned} I_1(E_c) - I_1(S_c) &= \{v_1\} \\ I_1(W_c) - I_1(S_c) &= \{v_2\} \end{aligned}$$

Since ω is bivalent,

$$\begin{aligned} I_1(N_c) &= I_1(E_c) \cup I_1(W_c) \\ &= \{v_1, v_2\} \cup I_1(S_c) \end{aligned}$$

If $v_1 \neq v_2 \Leftrightarrow I_1(E_c) \neq I_1(W_c)$, then $|I_1(N_c)| = 2 + (h-2) = h$. Therefore, it is enough to prove that $\forall c \in \mathrm{Cross}(\omega^\infty)$, $I_1(E_c) \neq I_1(W_c)$. This follows from Lemma 41 below. $\square$

Lemma 41. Let ω be a bivalent braid word and u, v be distinct vertices of Q_ω^∞ of the same height, then $I_1(u) \neq I_1(v)$. Note that the height of u, v cannot be n because there is only one vertex of height n, say U.

Proof. We prove the claim by the induction on the height of u and v. If $\mathrm{height}(u) = \mathrm{height}(v) = 1$, then the claim holds because $I_1(u) = \{u\}$ and $I_1(v) = \{v\}$.
Now suppose the claim holds for vertices of heights less than h. Then there are exactly two height $h-1$ vertices for each, say $\{u_1, u_2\}$, $\{v_1, v_2\}$, that have arrows to u and v. Note that there are crossings c, c' such that $\{u_1, u_2\} = \{E_c, W_c\}$, $\{v_1, v_2\} = \{E_{c'}, W_{c'}\}$. $\{u_1, u_2\} \neq \{v_1, v_2\}$, otherwise $c = c' \Rightarrow u = N_c = N_{c'} = v$ which is a contradiction. Therefore, by the induction hypothesis, $I_1(u) = I_1(u_1) \cup I_1(u_2) \neq I_1(v_1) \cup I_1(v_2) = I_1(v)$. $\square$

Lemma 42. Let ω be a bivalent braid word and $c \in \mathrm{Cross}(\omega^\infty)$, then $I_1(E_c) \cap I_1(W_c) = I_1(S_c)$ and $|I_1(E_c) \cup I_1(W_c)| = \mathrm{height}(N_c)$.

Proof. $I_1(S_c) \subset I_1(E_c) \cap I_1(W_c)$ because there are arrows from S_c to E_c, W_c. Since $E_c \neq W_c$, by Lemma 35, $I_1(E_c) \neq I_1(W_c)$. Therefore,

$$
\begin{aligned}
&|I_1(E_c) \cap I_1(W_c)| \leq |I_1(E_c)| - 1 = |I_1(S_c)| \\
\Rightarrow\ & I_1(E_c) \cap I_1(W_c) = I_1(S_c)
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
|I_1(E_c) \cup I_1(W_c)| &= |I_1(E_c)| + |I_1(W_c)| - |I_1(E_c) \cap I_1(W_c)| \\
&= |I_1(E_c)| + |I_1(W_c)| - |I_1(S_c)| \\
&= (h-1) + (h-1) - (h-2) = h \\
&= |I_1(N_c)| \\
&= \text{height}(N_c)
\end{aligned}
$$

$\square$

Definition 43. Suppose $L_1, L_2, \cdots, L_k \subset \mathbb{C}^n$ are lines(1 dimensional subspaces). We say $\{L_1, L_2, \cdots, L_k\}$ are linearly independent if and only if for nonzero $v_i \in L_i$, $\{v_1, v_2, \cdots, v_k\}$ are linearly independent. Note that the definition does not depend on the choice of v_i's.

Theorem 44. Let ω be a bivalent braid word, then ι_ω is an open embedding whose image is

$$
\begin{aligned}
X'_\omega = \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}^{n-1} \times \mathrm{GL}_n(\mathbb{C}) \mid \\
\forall c \in \mathrm{Cross}(\omega) \subset \mathrm{Cross}(\omega^\infty) \text{ with } \text{height}(N_c) = n, \\
\dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v - \sigma(v)}) = n\}
\end{aligned}
$$

Proof. Since bivalent braids are multivalent, by Theorem 37, the image of ι_ω is given by

$$\begin{aligned} X_\omega &= \{((z_v)_{v\in\Upsilon_0^1}, g) \in \prod_{v\in\Upsilon_0^1} \mathbb{P}_v^{n-1} \times \mathrm{GL}_n(\mathbb{C}) \mid \\ &\qquad \forall u \in \mathrm{Vert}(Q_\omega^\infty),\ \dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(u), \\ &\qquad \forall c \in \mathrm{Cross}(\omega^\infty),\ \dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(N_c)\} \end{aligned}$$

I claim that $X_\omega = X'_\omega$. Since the condition

$$\forall c \in \mathrm{Cross}(\omega) \subset \mathrm{Cross}(\omega^\infty) \text{ with } \mathrm{height}(N_c) = n,\ \dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n$$

defining X'_ω is subsumed in the condition

$$\forall c \in \mathrm{Cross}(\omega^\infty),\ \dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(N_c)$$

defining X_ω, $X_\omega \subset X'_\omega$.

Now let's prove $X'_\omega \subset X_\omega$ i.e. suppose $((z_v)_{v\in\Upsilon_0^1}, g)$ satisfy

(i) $\forall c \in \mathrm{Cross}(\omega) \subset \mathrm{Cross}(\omega^\infty)$ with $\mathrm{height}(N_c) = n$, $\dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = n$

then it also satisfy

(ii) $\forall u \in \mathrm{Vert}(Q_\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(u)$

(iii) $\forall c \in \mathrm{Cross}(\omega^\infty)$, $\dim_{\mathbb{C}}(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \mathrm{height}(N_c)$

Let $u \in \mathrm{Vert}(Q_\omega^\infty)$ and $c \in \mathrm{Cross}(\omega^\infty)$. First let's assume $\mathrm{height}(u) < n$. Since ω

is bivalent, by Lemma 40, $|I_1(u)| = \text{height}(u)$ and by Lemma 40 and Lemma 42, $|I_1(E_c) \cup I_1(W_c)| = |I_1(N_c)| = \text{height}(N_c)$. Therefore, the condition (i) is equivalent to saying that

$$\forall c \in \text{Cross}(\omega) \text{ with } \text{height}(N_c) = n,\ \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)} \text{ are linearly independent}$$

Likewise, the conditions (ii),(iii) are equivalent to saying that

$$\forall u \in \text{Vert}(Q_\omega^\infty),\ \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)} \text{ are linearly independent}$$
$$\forall c \in \text{Cross}(\omega^\infty),\ \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)} \text{ are linearly independent}$$

Let's prove (i) implies (ii) and (iii) using these paraphrased statements. Assume (i) holds for $((z_v)_{v \in \Upsilon_0^1}, g)$. Suppose for some $u \in \text{Vert}(Q_\omega^\infty)$, $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$ are linearly dependent. Let $u' = u - \sigma(u)$, then $I_1(u') = I_1(u) - \sigma(u)$. Therefore,

$$\begin{aligned} \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u')} &= \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v+\sigma(u) \in I_1(u)} \\ &= \{g^{\sigma(v')-\sigma(u)} \cdot z_{v'-\sigma(v')}\}_{v' \in I_1(u)} \\ &= g^{-\sigma(u)} \cdot \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)} \end{aligned}$$

are also linearly dependent. Therefore, without loss of generality, we can assume $\sigma(u) = 0$ i.e. $u \in \text{Vert}(Q_\omega) \subset \text{Vert}(Q_\omega^\infty)$. By Lemma 39, there is a path from u to a height $n-1$ vertex p. Thus, $I_1(u) \subset I_1(p)$. Since ω is bivalent, for some $c \in \text{Cross}(\omega)$, $p = E_c$ or W_c. By the condition (i), $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$ are linearly independent, thus its subset $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$ are linearly independent as well. Therefore, $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (ii).

Now let's show that $((z_v)_{v \in \Upsilon_0^1}, g)$ satisfy (iii). Suppose $c \in \text{Cross}(\omega^\infty)$. If $\text{height}(N_c) = n$, then the condition (i) is equal to the condition (iii), there is nothing to prove. Suppose $\text{height}(N_c) < n$, then $I_1(E_c) \cup I_1(W_c) = I_1(N_c)$. Thus, the condition (iii) translates to $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v\in I_1(N_c)}$ are linearly independent, which follows from (ii) that we already proved.

Now let's show that (iii) holds when height$(u) = n$ i.e. u is the unique height n point U. Since ω is bivalent, there is a crossing c such that $u = N_c$. Therefore,

$$\dim_{\mathbb{C}}\big(\sum_{v\in I_1(E_c)\cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}\big) = \text{height}(N_c) = n$$

Since $I_1(E_c) \cup I_1(W_c) \subset I_1(N_c) = I_1(u)$,

$$\dim_{\mathbb{C}}\big(\sum_{v\in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}\big) = n = \text{height}(u)$$

$\square$

Theorem 45. (Combinatorial characterization) A braid word is multivalent(bivalent resp.) if and only if in between s_i's there is at least one(exactly one resp.) s_{i-1} upto cyclic shifts for $2 \leq i < n$.

Proof. The statement follows from the fact that

- there is a one to one correspondence between the set of region of height k and s_k's in the braid word for $1 \leq k < n$.

- the valency of a region corresponding to a certain s_k is equal to the number of s_{k-1} in between that certain s_k and the next s_k.

$\square$

Definition 46. $\mathcal{M}_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z\ll 0}\}; \mathbb{C})$ is called *Generalized Sibuya space* if ω is a bivalent braid.

2.4 Examples : generalized Sibuya spaces

By nequation, I mean a formula that expresses the non-equality of two expressions. Whenever I denote capital X with subscript i.e. X_j, I mean an element of some projective space.

Lower case x's with subscripts are used to denote the homogeneous coordinates of X_j's.

For example $X_j = [x_{1,j} : \cdots : x_{n,j}]$.

I will also denote

$$(X_1, \cdots, X_{n-1}) = ([x_{1,1} : \cdots : x_{n,1}], [x_{1,2} : \cdots : x_{n,2}], \cdots, [x_{1,n-1} : \cdots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

We will use

$$\left(\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_n \end{array} \right) = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

to denote the matrix whose entries are $x_{i,j}$ which are sections of the line bundle $O_{\mathbb{P}^k}(1)$.

Thereby

$$det \left(\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_n \end{array} \right) = det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

is a section of the line bundle $\underbrace{O_{\mathbb{P}^k}(1) \boxtimes \cdots \boxtimes O_{\mathbb{P}^k}(1)}_{n-copies}$ on $\underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{n-copies}$.

Example 47. (**Empty Set**) In this section we compute 2 kinds of moduli spaces that are empty.

In the first case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than the number of strands with unipotent monodromy $u = I$

The framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \cdots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k-copies} \mid \det\left(\begin{array}{c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k} \end{array}\right) \neq 0,$$

$$\det\left(\begin{array}{c|c|c|c|c|c} X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k+1} \end{array}\right) \neq 0, \cdots,$$

$$\det\left(\begin{array}{c|c|c|c|c|c|c} X_k & X_1 & \cdots & X_k & X_1 & \cdots & X_{n-k-1} \end{array}\right) \neq 0\}$$

Take the first nequation and subtract 1^{st}column from the $(k+1)^{th}$column we get

$$\det\left(\begin{array}{c|c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & 0 & X_2 & \cdots & X_{n-k} \end{array}\right) \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the moduli space.

In the second case, we consider the moduli space of braids represented by a bivalent braid word that the number of s_1 in the expression is less than (the number of strands)-1 with unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The framed moduli space is

$$\begin{aligned} \mathcal{M}^{fr} = \{(X_1, \cdots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k-copies} \mid \det \left(\begin{array}{c|c|c|c|c|c|c} X_1 & X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k} \end{array} \right) \neq 0, \\ \det \left(\begin{array}{c|c|c|c|c|c} X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k+1} \end{array} \right) \neq 0, \cdots, \\ \det \left(\begin{array}{c|c|c|c|c|c|c} X_k & uX_1 & \cdots & uX_k & u^2X_1 & \cdots & u^2X_{n-k-1} \end{array} \right) \neq 0\} \end{aligned}$$

Take the first nequation and subtract 1^{st}(resp. 2^{nd}) column from the $(k+1)^{th}$(resp. $(k+2)^{th}$) column we get

$$\det \left(\begin{array}{c|c|c|c|c|c|c|c|c} & & & & 0 & 0 & & & \\ & & & & \vdots & \vdots & & & \\ X_1 & X_2 & \cdots & X_k & 0 & 0 & uX_3 & \cdots & uX_{n-k} \\ & & & & x_{n,1} & x_{n,2} & & & \\ & & & & 0 & 0 & & & \end{array} \right) \neq 0$$

which is never true. Therefore, the framed moduli space $\mathcal{M}^{fr} = \emptyset$ and so is the moduli space.

Example 48. (**Points**) In this section, we compute the moduli space of braids represented by the braid word

$$(s_1 s_2 \cdots s_{n-1})^{n-1}$$

and the unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

i.e. a unipotent matrix given by the partition $(\underbrace{1, 1, \cdots, 1}_{n-1}, 2)$. The framed moduli space associated to the above data is given as follows

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \left(\begin{array}{c|c|c|c|c} X_1 & X_2 & \cdots & X_{n-1} & uX_1 \end{array} \right) \neq 0,$$

$$\det \left(\begin{array}{c|c|c|c|c} X_2 & \cdots & X_n & uX_1 & uX_2 \end{array} \right) \neq 0, \cdots, \det \left(\begin{array}{c|c|c|c|c} X_{n-1} & uX_1 & \cdots & uX_{n-2} & uX_{n-1} \end{array} \right) \neq 0\}$$

Using the elementary column operation of subtracting the first column from the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_1 & X_2 & \cdots & X_{n-1} & 0 \\ & & & & x_{n,1} \\ & & & & 0 \end{array} \right) \neq 0,$$

$$\det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_2 & \cdots & X_{n-1} & uX_1 & 0 \\ & & & & x_{n,2} \\ & & & & 0 \end{array} \right) \neq 0, \cdots, \det \left(\begin{array}{c|c|c|c|c} & & & & 0 \\ & & & & \vdots \\ X_{n-1} & uX_1 & \cdots & uX_{n-2} & 0 \\ & & & & x_{n,n-1} \\ & & & & 0 \end{array} \right) \neq 0\}$$

Applying the cofactor expansion formula with respect to the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{n-copies} \mid x_{n,1} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0,$$

$$x_{n,2} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0, \cdots, x_{n,n-1} \cdot \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0\}$$

$$= \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{n-copies} \mid x_{n,i} \neq 0 \text{ for } i = 1, \cdots, n-1$$

$$, \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0\}$$

To get the moduli space out of the framed moduli space above, we have to quotient it out by the centralizer subgroup of u , that is,

$$\mathrm{C}_{\mathrm{GL}_n(\mathbb{C})}(u) = \{ \left(\begin{array}{ccc|cc} c_{1,1} & \cdots & c_{1,n-2} & 0 & c_{1,n} \\ \vdots & \ddots & \vdots & 0 & c_{1,n} \\ c_{n-2,1} & \cdots & c_{n-2,n-2} & 0 & c_{n-2,n} \\ \hline 0 & \cdots & 0 & 0 & c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-2} & c_{n,n-1} & c_{n,n} \end{array}\right) \in \mathrm{GL}_n(\mathbb{C}) |$$

$$\det \begin{pmatrix} c_{1,1} & \cdots & c_{1,n-2} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n-2} \end{pmatrix}, c_{n,n-1}, c_{n-1,n} \neq 0\}$$

It acts diagonally on $\underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$ where the action on each coordinate is given by left multiplication.

To simplify the notation, I will denote

$$(X_1, \cdots, X_{n-1}) = ([x_{1,1} : \cdots : x_{n,1}], [x_{1,2} : \cdots : x_{n,2}], \cdots, [x_{1,n-1} : \cdots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

I claim that for any $(X_1, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}}_{(n-1)-copies}$, there exists $A \in \mathrm{C}_{\mathrm{GL}_n(\mathbb{C})}(u)$ such that

$$A \cdot X = \left[\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline & & & \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

Let

$$A_1 = \left(\begin{array}{ccc|cc} & \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-2} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-2} \end{pmatrix}^{-1} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & & & \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_n \end{array}\right)$$

with $a_{n-1} \neq 0$ such that

$$\sum_{i=1}^{n} a_i \cdot (x_{i,1}, \cdots, x_{i,n}) = (1, \cdots, 1)$$

We can always find such because we know that

$$\det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0$$

Then we get

$$A_1 \cdot X = \left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ x_{n,1} & \cdots & x_{n,n-2} & x_{n,n-1} \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

Since we know that $x_{n,i} \neq 0$ for $i = 1, \cdots, n-1$, without loss of generality we put $x_{n,i} = 1$ for $i = 1, \cdots, n-1$. Then we get

$$A_1 \cdot X = \left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \end{array}\right]$$

such that $b_1 + \cdots + b_{n-1} + 1 \neq 0$. Now let

$$A_2 = \left(\begin{array}{ccc|cc} a_{1,1} & \cdots & a_{1,n-2} & 0 & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,1} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\ \hline & & & & \\ 0 & \cdots & 0 & & \\ & & & & I \\ 0 & \cdots & 0 & & \end{array}\right)$$

such that

$$a_{i,1}\cdot(1,0,\cdots,0,b_1)+a_{i,2}\cdot(0,1,0,\cdots,0,b_2)+\cdots+a_{i,n-2}\cdot(0,\cdots,0,1,b_{n-2})+a_{i,n}\cdot(1,\cdots,1)$$

$$=(0,\cdot,0,\overset{\mathrm{i}^{th}}{\overset{\downarrow}{1}},0,\cdots,0)$$

Again, we can find such because

$$\det\left[\begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ & & & b_{n-1} \\ \hline & & & \\ 1 & \cdots & 1 & 1 \end{array}\right]\neq 0$$

Then $A_2\cdot A_1$ is the desired A.

Example 49. (**Regular Unipotent Fibers**) Finite type integral schemes. Also, it has a stratification into rational varieties at most 1 stratum in each dimension. $(s_1)^3$, $(s_1s_2)^2$

Example1

In this section, we compute the moduli space associated to the braid word : $(s_1s_2)^2$ with regular unipotent monodromy. The framed moduli space is given by

$$\mathcal{M}^{fr}=\{([x:y:z],[a:b:c])\in\mathbb{P}^2\times\mathbb{P}^2 \mid \det\begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0\end{pmatrix}\neq 0,\ \det\begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0\end{pmatrix}\neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$C := \mathrm{C}_{\mathrm{GL}_3(\mathbb{C})}(u) = \{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \mid \alpha \neq 0 \}$$

We can cover $\mathcal{M}^{fr}$ with open subsets U_1, U_2 where

$$U_1 = \{([x:y:z],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \ \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, z \neq 0\}$$

$$U_2 = \{([x:y:z],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \ \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, c \neq 0\}$$

Therefore, we have a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

Quotienting out by the centralizer subgroup C, we get

$$\begin{tikzcd} \overline{U}_1 := C\backslash U_1 \arrow[r, hook] & \mathcal{M} = C\backslash\mathcal{M}^{fr} \\ \overline{U_1 \cap U}_2 := C\backslash U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & \overline{U}_2 := C\backslash U_2 \arrow[u, hook] \end{tikzcd}$$

Also note that, for the action of C, the centralizer subgroup of any element of $\mathcal{M}^{fr}$ is the set of scalar multiplication matrices.

First let's simplify, $\overline{U}_1$. Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{y}{z}$$
$$\gamma = \frac{y^2}{z^2} - \frac{x}{z}$$

This expression makes sense because $z \neq 0$ in U_1. If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\overline{U}_1 \cong \{([0:0:1],[a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} 0 & a & 0 \\ 0 & b & 1 \\ 1 & c & 0 \end{pmatrix} \neq 0, \ \det \begin{pmatrix} 0 & a & b \\ 0 & b & c \\ 1 & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\}$$

$$\cong \{[a:b:c] \in \mathbb{P}^2 \mid a \neq 0, \ \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0\}$$

$$\cong \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}$$

Under this identification,

$$\overline{U_1 \cap U_2} \cong \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

Now let's simplify, $\overline{U}_2$. Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{b}{c}$$
$$\gamma = \frac{b^2}{c^2} - \frac{a}{c}$$

This expression makes sense because $c \neq 0$ in U_1. If we take an element of U_2 with

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned} \overline{U}_2 &\cong \{([x:y:z],[0:0:1]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0,\ \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\ &\cong \{[x:y:z] \in \mathbb{P}^2 \mid x \neq 0,\ det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \neq 0\} \\ &\cong \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \end{aligned}$$

Under these identifications, the pushout square above becomes

$$\begin{tikzcd} \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \arrow[r, hook] & \mathcal{M} \\ \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \arrow[u, hook] \arrow[r, hook, "f"] & \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \arrow[u, hook] \end{tikzcd}$$

where

$$f : \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \to \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}$$
$$(b,c) \mapsto (-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c})$$

Now define a variety V to be

$$V := \{(Y,Z,W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0\} - \{(0,0,0)\}$$

I claim that V is isomorphic to $\mathcal{M}$. More precisely, I claim that

$$\begin{tikzcd}
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \arrow[rr, hook, "i"] & & V \\
\{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \arrow[u, hook, "g"] \arrow[r, hook, "f"] & \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \arrow[ru, hook, "\iota"] &
\end{tikzcd}$$

is a pushout square where

$$i : \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \to V$$
$$(b,c) \mapsto (-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}, b)$$

$$\iota : \{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \to V$$
$$(y,z) \mapsto (y, z, \frac{yz}{z - y^2})$$

It is easy to check the square commutes. f, g are inclusion maps by construction. ι is also an inclusion map because we can recover (y,z) from $\iota(y,z)$ by projecting onto the $1^{st}\&2^{nd}$ coordinates.

For i, we can recover b from $i(b,c)$ by projecting onto the 3^{rd} coordinate. We can recover c from $i(b,c)$ by multiplying $2^{nd}\&3^{rd}$ coordinates and dividing with -(the 1^{st}

coordinate).

Let's check that the images of i and ι form an open cover of V. The images of i and ι are

$$i(\{(b,c) \in \mathbb{A}^2 \mid c-b^2 \neq 0\}) = \{(Y,Z,W) \in \mathbb{A}^3 | Y^2W-ZW+YZ = 0, YW+Z+W^2 \neq 0\}$$

$$\begin{aligned}\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\}) &= \{(Y,Z,W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, Y^2 - Z \neq 0\} \\ &= \{(Y,Z,W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z-Y^2}, Y^2 - Z \neq 0\}\end{aligned}$$

Clearly, they are open subsets of V.

Let's check that if $(Y,Z,W) \in V$ and $Y^2 = Z$, then $YW + Z + W^2 \neq 0$. If $Y^2 = Z$, then the equation $Y^2W - ZW + YZ = 0$ becomes $YZ = 0$. Therefore, we get $Y = Z = 0$. Since $(0,0,0)$ is not contained in V, W can only take non-zero values. Therefore, $YW + Z + W^2 = W^2 \neq 0$. We conclude that the images of i and ι cover V.

Now let's check that

$$i^{-1}(\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\})) = \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

The image of ι is

$$\iota(\{(y,z) \in \mathbb{A}^2 \mid z-y^2 \neq 0\}) = \{(Y,Z,W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z-Y^2}, Y^2 - Z \neq 0\}$$

we have

$$\begin{aligned}
&i(b,c) \in \iota(\{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) \\
&\iff (-\frac{bc}{b^2-c})^2 \neq \frac{c^2}{b^2-c} \\
&\iff b^2c^2 \neq c^2(b^2-c) \\
&\iff c^3 \neq 0 \\
&\iff c \neq 0
\end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(y,z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b,c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

as desired.

Therefore, V is isomorphic to $\mathcal{M}$.

Example 50. In this section, we compute the moduli space associated to the braid word : s_1^3 with regular unipotent monodromy. The framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x:y],[z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det\begin{pmatrix} x & z \\ y & w \end{pmatrix} \neq 0, \\ \det\begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0,\ \det\begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is

$$C := \mathrm{C}_{\mathrm{GL}_2(\mathbb{C})}(u) = \{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \alpha \neq 0\}$$

From this point on, we will use the following notation

$$\infty := [1:0]$$
$$X := [x:y]$$
$$Z := [z:w]$$
$$A := [a:b]$$

We can cover $\mathcal{M}^{fr}$ with open subsets U_1, U_2 where

$$U_1 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid X \neq \infty\}$$
$$U_2 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid Z \neq \infty\}$$

Therefore, we have a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

Quotienting out by the centralizer subgroup C, we get

$$\begin{tikzcd} \overline{U}_1 := C\backslash U_1 \arrow[r, hook] & \mathcal{M} = C\backslash \mathcal{M}^{fr} \\ \overline{U_1 \cap U}_2 := C\backslash U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & \overline{U}_2 := C\backslash U_2 \arrow[u, hook] \end{tikzcd}$$

Also note that, for the action of C, the centralizer subgroup of any element of $\mathcal{M}^{fr}$

is the set of scalar multiplication matrices.

First let's simplify, $\overline{U}_1$. Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$
$$\beta = -\frac{x}{y}$$

This expression makes sense because $y \neq 0$ in U_1. If we take an element of U_1 with

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned}
\overline{U}_1 &\cong \{([0:1],[z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} 0 & z \\ 1 & w \end{pmatrix} \neq 0,\ \det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0,\ \det \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \neq 0\} \\
&\cong \{([z:w],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z \neq 0,\ bz - aw \neq 0,\ a - b \neq 0\} \\
&\cong \{(w,[a:b]) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid b - aw \neq 0,\ a - b \neq 0\}
\end{aligned}$$

Change variables $a' := a + b, b' := a - b$ we get

$$\begin{aligned}
\overline{U_1} &\cong \{(w, [a' : b']) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid \frac{a' - b'}{2} - \frac{(a' + b')w}{2} \neq 0,\ b' \neq 0\} \\
&\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a' - 1) - (a' + 1)w \neq 0\} \\
&\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a'' + 1)(1 - w) - 2 \neq 0\} \\
&\cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1\}
\end{aligned}$$

Under this identification,

$$\overline{U_1 \cap U_2} \cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1, 2w' \neq 1\}$$

Now let's simplify, $\overline{U}_2$. Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\begin{aligned}
\alpha &= 1 \\
\beta &= -\frac{z}{w}
\end{aligned}$$

This expression makes sense because $w \neq 0$ in U_2. If we take an element of U_2 with

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\begin{aligned}
\overline{U}_2 &\cong \{([x:y],[0:1],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \neq 0, \det \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0\} \\
&\cong \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x \neq 0,\ a \neq 0,\ ay - b(x+y) \neq 0\} \\
&\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y - b(1+y) \neq 0\} \\
&\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y(1-b) - b \neq 0\}
\end{aligned}$$

Change variables $b' := 1-b, y' := y+1$ we get

$$\overline{U_2} \cong \{(y',b') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid yb' \neq 1\}$$

Under these identifications, the pushout square above becomes

$$\begin{tikzcd}
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} \arrow[r, hook] & \mathcal{M} \\
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} \arrow[u, hook] \arrow[r, hook, "f"] & \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \arrow[u, hook]
\end{tikzcd}$$

where

$$\begin{aligned}
f : \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} &\to \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \\
(x,y) &\mapsto (2x, \frac{4x+y-4}{2xy-2})
\end{aligned}$$

Now define a variety V to be

$$V := \{(A, B, C) \in \mathbb{A}^3 \mid (AC-2)B = 2A + C - 4\} - \{(1, 1, 2)\}$$

I claim that V is isomorphic to $\mathcal{M}$. More precisely, I claim that

$$\begin{tikzcd}
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} \arrow[r, hook, "i"] & V \\
\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} \arrow[u, hook, "g"] \arrow[r, hook, "f"] & \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \arrow[u, hook, "\iota"']
\end{tikzcd}$$

is a pushout square where

$$\begin{aligned}
i : \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} &\to V \\
(x,y) &\mapsto (2x, \frac{4x+y-4}{2xy-2}, y) \\
\iota : \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} &\to V \\
(a,b) &\mapsto (a, b, \frac{2a+2b-4}{ab-1})
\end{aligned}$$

It is easy to see that the square commutes and f, g, i, ι are inclusion maps.
Let's check that the images of i and ι form an open cover of V.
The image of i and ι are

$$i(\{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\}) = \{(A, B, C) \in V | AC \neq 2\}$$

$$\iota(\{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V | AB \neq 1\}$$

Clearly, they are open subsets of V.
Let's check that if $(A, B, C) \in V$ and $AC = 2$, then $AB \neq 1$. If $AC = 2$, then the left hand side of the equation $(AC-2)B = 2A + C - 4$ becomes zero. Therefore, we get $AC = 2$ and $2A + C = 4$ which implies $2A + \frac{2}{A} = 4 \Leftrightarrow A^2 - 2A + 1 = 0$. Solving

the quadratic equation, we get $A = 1, C = 2$. Since $(1, 1, 2)$ is not contained in V, B can take any value except 1. Therefore, $AB \neq 1$. We conclude that the images of i and ι cover V.

Now let's check that

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\}$$

The image of ι is

$$\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V | AB \neq 1\}$$

we have

$$\begin{aligned}
&i(x, y) \in \iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) \\
&\iff 2x \cdot (\frac{4x + y - 4}{2xy - 2}) \neq 1 \\
&\iff 2x \cdot (4x + y - 4) \neq 2xy - 2 \\
&\iff 8x^2 - 8x + 2 \neq 0 \\
&\iff 2(2x - 1)^2 \neq 0 \\
&\iff 2x1 \neq 1
\end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \overline{U}_1 \mid 2x \neq 1\}$$

as desired.

Therefore, V is isomorphic to $\mathcal{M}$.

Example 51. In this section, we prove that the moduli space of a bivalent braid with regular unipotent monodromy is finite type integral scheme over $\mathbb{C}$ not necessarily separated. Suppose we have a bivalent braid word with n-strands and a unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

The framed moduli space is given as

$$\mathcal{M}^{fr} = \{X = (X^1, X^2, \cdots, X^k) \in (\mathbb{P}^{n-1})^k \mid f_1(X) \neq 0, \cdots f_m(X) \neq 0\}$$

where f_i 's are determinants with column vectors of the form $u^s \cdot X_j$. Note that the entries of the last row(i.e. the n^{th} row) is one of $x_{n,i}$ $(i = 1, \cdots, k)$. Thus $x_{n,i}$'s $(i = 1, \cdots, k)$ cannot be identically zero otherwise all of the f_r's will vanish. Therefore, we have an open cover of $\mathcal{M}^{fr}$, i.e. $\{U_i\}_{i=1,\cdots,k}$ where $U_i := \{X \in \mathcal{M}^{fr} \mid x_{n,i} \neq 0\}$. To get the moduli space, we quotient the framed moduli space with the centralizer subgroup of u in $\mathrm{GL}_n(\mathbb{C})$ i.e.

$$C := \mathrm{C}_{\mathrm{GL}_n(\mathbb{C})} = \{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \mid \alpha_1 \in \mathbb{C}^*,\ \alpha_i \in \mathbb{C}\ for\ i = 2, \cdots, n\}$$

Suppose we have an element

$$X = (X^1, \cdots, X^k) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \in U_i$$

Then there exists a

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \cdot \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

That is, recursively,

$$\alpha_1 = 1$$

$$\alpha_s = -\frac{1}{x_{n,i}}(x_{n-1,i}\alpha_{s-1} + x_{n-2,i}\alpha_{s-2} + \cdots x_{n-s+1,i}\alpha_1) = -\frac{1}{x_{n,i}}(\sum_{t=1}^{s-1} x_{n-t,i} \cdot \alpha_{s-t})$$

This expression makes sense because $x_{n,i} \neq 0$ in U_i. If we take an element of U_i with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each C-orbit, we see that

$$\overline{U}_i := C\backslash U_i \cong \{ \begin{array}{c} \begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix} \\ \begin{array}{c} \uparrow \\ \mathrm{i}^{th} column \end{array} \end{array} \in (\mathbb{P}^{n-1})^k \mid f_1(\begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix}) \neq 0, \cdots ,$$

$$f_m(\begin{bmatrix} x_{1,1} & & 0 & & x_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ x_{n-1,1} & & 0 & & x_{n-1,n} \\ x_{n,1} & & 1 & & x_{n,n} \end{bmatrix}) \neq 0\}$$

which are finite type scheme over $\mathbb{C}$. In summary, we have found an finite open cover of $\mathcal{M}$ i.e. $\{\overline{U}_i\}_{i=1,\cdots,k}$ such that each open is a finite type scheme over $\mathbb{C}$. Thus, $\mathcal{M}$ is also a finite type scheme over $\mathcal{C}$. Now we have a smooth surjective map $\pi : \mathcal{M}^{fr} \to \mathcal{M}$. $\mathcal{M}$ is irreducible because $\mathcal{M}^{fr}$ is irreducible and π is surjective. $\mathcal{M}$ is reduced because $\mathcal{M}^{fr}$ is reduced and π is smooth. Therefore, we conclude that the moduli space associated attached to bivalent braid with regular unipotent monodromy is finite type integral scheme over $\mathbb{C}$. But it may not be separated.

Example 52. In this section, I will provide an example of the moduli space associated to bivalent braid with regular unipotent monodromy that is non-separated. Consider a 2-strand braid given by the braid word s_1^2 and a regular unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The framed moduli space is given as follows

$$\begin{aligned}\mathcal{M}^{fr} &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], u \cdot [a:b] \neq [x:y]\} \\ &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], [a+b:b] \neq [x:y]\} \\ &= \{([x:y],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid ay \neq bx, (a+b)y \neq bx\}\end{aligned}$$

Then we have an open cover $\{U_i\}_{i=1,2}$ where

$$\begin{aligned}U_1 &:= \{([x:y],[a:b]) \in \mathcal{M}^{fr} | b \neq 0\} \\ U_2 &:= \{([x:y],[a:b]) \in \mathcal{M}^{fr} | y \neq 0\}\end{aligned}$$

We get a pushout square

$$\begin{tikzcd} U_1 \arrow[r, hook] & \mathcal{M}^{fr} \\ U_1 \cap U_2 \arrow[u, hook] \arrow[r, hook] & U_2 \arrow[u, hook] \end{tikzcd}$$

we take quotients of these opens with respect to the centralizer subgroup of u i.e.

$$C := \mathrm{C}_{\mathrm{GL}_2(\mathbb{C})} = \{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \alpha \neq 0\}$$

we get

$$\begin{aligned}
\overline{U}_1 := C\backslash U_1 &\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x\} \\
&\cong \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \\
\overline{U}_2 := C\backslash U_2 &\cong \{([0:1],[a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid a \neq 0, (a+b) \neq 0\} \\
&\cong \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \\
\overline{U}_1 \cap \overline{U}_2 := C\backslash U_1 &\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
&\cong \{([x:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
&\cong \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\}
\end{aligned}$$

and a pushout square

$$\begin{tikzcd}
\{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \arrow[r, hook] & \mathcal{M} \\
\{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \arrow[u, hook] \arrow[r, hook, "f"] & \cong \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \arrow[u, hook]
\end{tikzcd}$$

where

$$f : \{([1:y],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \to \{([0:1],[1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}$$

$$([1:y],[0:1]) \mapsto ([0:1],[1:-y])$$

Now consider the map

$$\begin{aligned}
g : \mathbb{A}^1 - \{0,1\} &\longrightarrow \overline{U}_1 \cap \overline{U}_2 \subseteq \mathcal{M} \\
y &\mapsto ([1:y],[0:1])
\end{aligned}$$

This map extends to $\mathbb{A}^1$ in two different ways i.e. we have two distinct h_1, h_2 that fit into the following commutative square

$$\begin{tikzcd} \mathbb{A}^1 - \{0,1\} \arrow[r, hook] \arrow[dr, "g"] & \mathbb{A}^1 - \{1\} \arrow[d, "h_i"] \\ & \mathcal{M} \end{tikzcd}$$

which are

$$h_1 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_1 \subseteq \mathcal{M}$$
$$y \mapsto ([1 : y], [0 : 1])$$

$$h_2 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_2 \subseteq \mathcal{M}$$
$$y \mapsto ([0 : 1], [1 : -y])$$

h_1 and h_2 are distinct because $h_1(0) \in U_1 - U2$. Therefore, by the valuative criterion for separatedness, $\mathcal{M}$ is non-separated.

Chapter 3

Natural Cluster Coordinates on Braid Moduli Spaces

3.1 Local Morse group computations

In this section, we compute the local Morse groups of a cochain complex of sheaves whose cohomologies are constructible with respect to the coordinate stratifications- See Definition 58.

Definitions and Notations

Definition 53. We define $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ such that

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Definition 54. We define $[n]$ to be a set of integers from 1 to n i.e.

$$[n] := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$$

Definition 55. Let X be a set, then we denote the power set of X as $\mathcal{P}(X)$.

Definition 56. We denote the i^{th} standard basis of $\mathbb{R}^n$ to be e_i^n.

Definition 57. Let

$$s^n(sgn_1, \cdots, sgn_n) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \text{for } k = 1, \cdots, n, \operatorname{sgn}(x_k) = sgn_k\}$$

To simplify the notation, we use $+$ instead of $+1$ and $-$ instead of -1. For example, $s^n(+, -, 0) := s^n(1, -1, 0)$.

Definition 58.

1. We define a *coordinate stratification* $\mathcal{S}^n$ on $\mathbb{R}^n$ to be

$$\{s^n(sgn_1, \cdots, sgn_n) \mid sgn_k \in \{-, 0, +\} \text{ for } k = 1, \cdots, n\}$$

2. There is a nautral poset structure of $\mathcal{S}^n$ such that $s^n(sgn_1, \cdots, sgn_n) \leq s^n(sgn_1', \cdots, sgn_n')$ if and only if $s^n(sgn_1', \cdots, sgn_n') \subset \operatorname{star}(s^n(sgn_1, \cdots, sgn_n))$. This is equivalent to saying that $|sgn_i| \leq |sgn_i'|$ for all $i \in [n]$.

3. There is also a natural poset structure on n-dimensional strata of $\mathcal{S}^n$, where $s^n(sgn_1, \cdots, sgn_n) \leq s^n(sgn_1', \cdots, sgn_n')$ if and only if $sgn_i \leq sgn_i'$ for all $i \in [n]$. Note that this is not the poset structure that is inherited from the poset structure mentioned above.

4. We give co-orientations to $(n-1)$-dimensional stratum as follows: Note that codimension 1 strata can be expressed as $s^n(sgn_1, \cdots, sgn_n)$ where only one of

arguments is zero, say sgn_i. Then we define the co-orientation of $s^n(sgn_1, \cdots, sgn_n)$ to be $-dx_i$ i.e. hairs are pointing the region $s^n(sgn_1, \cdots, \underset{\substack{\uparrow \\ i^{th}}}{-}, \cdots, sgn_n)$

5. Let $x \in \mathbb{R}^n$, then s^n_x is defined as the stratum in $\mathcal{S}^n$ containing x.

Suppose we have $\mathscr{F}^\bullet$ a complex of constructible sheaves on $\mathbb{R}^n$ constructible with respect to the coordinate stratification. In this section, we will compute local Morse groups of $\mathscr{F}^\bullet$ when given $(x, \xi) \in T^*\mathbb{R}^n$ i.e. we will compute the stalk of the microlocalization of $\mathscr{F}^\bullet(= \mu\mathscr{F}^\bullet)$ on $T^*\mathbb{R}^n$. Note that the singular support of $\mathscr{F}^\bullet$, $SS(\mathscr{F}^\bullet)$, is contained in the union of 2^n Lagrangian subspaces that are conormals of strata in $\mathcal{S}^n$ i.e. $\Lambda_{\mathcal{S}^n} = \bigcup_{s \in \mathcal{S}^n} N^*s \subset T^*\mathbb{R}^n$. These Lagrangians intersect with each other so union of them form a singular Lagrangian and it has smooth part in it, say $\Lambda^{smooth}_{\mathcal{S}^n} \subset \Lambda_{\mathcal{S}^n}$. Now we know that the singular support $SS(\mathscr{F}^\bullet) := supp(\mu\mathscr{F}^\bullet) \subset \Lambda^{smooth}_{\mathcal{S}^n}$ and $\mu\mathcal{F}^\bullet$ is constant on each component of $\Lambda^{smooth}_{\mathcal{S}^n}$, we define the local Morse group of a component of $\Lambda^{smooth}_{\mathcal{S}^n}$ with coefficient $\mathscr{F}^\bullet$ to be the local Morse group of a point in the component with coefficient $\mathscr{F}^\bullet$. We will see later that the components are contractible so this is well-defined upto unique isomorphism. Note that $\Lambda^{smooth}_{\mathcal{S}^n}$ has 2^{2n} components which are labelled by $\mathcal{P}([n]) \times \mathcal{P}([n])$. More precisely, for each $(I, J) \in \mathcal{P}([n]) \times \mathcal{P}([n])$, we have a component $comp^n_{(I,J)} \subset \Lambda^{smooth}_{\mathcal{S}^n}$ that is the image of

$$\iota^n_{(I,J)} : \mathbb{R}^n_{>0} \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

where $(x_1, \cdots, x_n)^T$ is mapped to

$$((\delta_I(1)\,\mathrm{sgn}_J(1)x_1, \cdots, \delta_I(n)\,\mathrm{sgn}_J(n)x_n)^T, ((1-\delta_I(1))\,\mathrm{sgn}_J(1)x_1, \cdots, (1-\delta_I(n))\,\mathrm{sgn}_J(n)x_n)^T)$$

where

- $\delta_I : [n] \to \{0,1\}$

$$\delta_I(k) = \begin{cases} 1 & \text{if } k \in I \\ 0 & \text{if } k \notin I \end{cases}$$

- $\mathrm{sgn}_J : [n] \to \{-1,1\}$

$$\mathrm{sgn}_J(k) = \begin{cases} 1 & \text{if } k \in J \\ -1 & \text{if } k \notin J \end{cases}$$

I will choose the representative of the $comp^n_{(I,J)}$ to be

$$(x_{(I,J)}, \xi_{(I,J)}) := \iota^n_{(I,J)}((1, \cdots, 1)^T)$$

Also, we define $G_{(I,J)}(\mathscr{F}^\bullet)$ to be the local Morse group of $comp^n_{(I,J)}$ with coefficient $\mathscr{F}^\bullet$ i.e.

$$H^*(N_x, N_x \cap \xi^{-1}((-\infty, c-\epsilon)); \mathscr{F}^\bullet)$$

where $(x,\xi) \in comp^n_{(I,J)}$, N_x a regular neighborhood of x, and ϵ a small positive number.

Definition 59. We define

- Q^n to be the quiver associated to $\mathcal{S}^n$ i.e. we have one vertex for each n-dimensional stratum and one arrow for each $(n-1)$-dimensional stratum where the direction goes against the co-orientation.

- $F^\bullet$ to be the legible diagram of $\mathscr{F}^\bullet$ i.e. the quiver representation of Q^n induced by $\mathscr{F}^\bullet$.

- Q^n_s to be the full suquiver of Q^n spanned by vertices whose corresponding strata are contained in the star of $s \in \mathcal{S}^n$.

Also, we simplify the notation as follows:

- $s^n_{(I,J)} := s^n_{x_{(I,J)}}$

- For $k \notin I$, $s^n_{(I,J)}(k) := s^n_{(x_{(I,J)} - \mathrm{sgn}_J(k)\cdot e^n_k)}$

Lemma 60. Suppose we have a cochain complex of $\mathcal{S}^n$-constructible sheaf $\mathscr{F}^\bullet$ on $\mathbb{R}^n$ that could be described by a legible diagram $F^\bullet$ on Q^n, then

$$G_{(I,J)}(\mathscr{F}^\bullet) = \begin{cases} 0 & \text{if } I^c \notin J \\ H^*(\mathrm{Tot}(F^\bullet|_{Q^n_{s_{(I,J)}}})) \text{ up to an overall shift in degree} & \text{if } k \in J \end{cases}$$

Proof. We prove the Lemma by induction on n.

(i) suppose I is not empty i.e. $x_{(I,J)} \neq 0$. We want to compute $G_{(I,J)}(\mathscr{F}^\bullet) = H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi^{-1}_{(I,J)}((-\infty, c-\epsilon]); \mathscr{F}^\bullet)$ where $N_{x_{(I,J)}}$ is a regular neighborhood of $x_{(I,J)}$, $c = \xi_{(I,J)}(x_{(I,J)})$, and ϵ a small positive real number. Once and for all fix $i \in I$, consider the following linear map

$$\mathrm{Lin}_i : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

where

$$\mathrm{Lin}_i(e_j) = \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } j \geq i \end{cases}$$

and let $\mathrm{Aff}_i := \mathrm{Lin}_i + e^n_i$ be the affine inclusion map. Define

- $I' = \{k \in I \mid k < i\} \cup \{k - 1 \mid k > i, k \in I\}$
- $J' = \{k \in J \mid k < i\} \cup \{k - 1 \mid k > i, k \in J\}$

and $(x_{(I',J')}, \xi_{(I',J')}) = \iota^{n-1}_{(I',J')}((1, \cdots, 1)^T)$, then $x_{(I,J)} \in \mathrm{Im}(\mathrm{Aff}_i)$ and $\alpha_i^{-1}(x_{(I,J)}) = x_{(I',J')}$ and $\xi_{(I,J)} \circ \mathrm{Aff}_i$. We let the regular neighborhood of $x_{(I,J)}$, $N_{x_{(I,J)}}$, to be $\mathrm{star}(s^{n-1}_{(I,J)})$ and the regular neighborhood of $x_{(I',J')}$, $N_{x_{(I',J')}}$, to be $\mathrm{star}(s^{n-1}_{(I',J')})$, $c = \xi_{(I,J)}(x_{(I,J)})$, and $c' = \xi_{(I',J')}(x_{(I',J')})$

(Claim) $H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi^{-1}_{(I,J)}((-\infty, c-\epsilon)); \mathscr{F}^\bullet) \cong H^*(N_{x_{(I',J')}}, N_{x_{(I',J')}} \cap \xi^{-1}_{(I',J')}((-\infty, c'-\epsilon)); \mathrm{Aff}_i^* \mathscr{F}^\bullet)$

(proof) Note that when we have a stratified spaces $(M, \mathcal{S}_M)$ and N, a subspace of M, we have an induced stratification $\mathcal{S}_N$ on N. Then we can identify poset $\mathcal{S}_N$ as the subposet of $\mathcal{S}_M$ via $(N \hookrightarrow M)_*$.

Now consider the following cartesian diagram of inclusion maps

$$\begin{tikzcd} N_{x_{(I,J)}} & N_{x_{(I,J)}} \cap \xi^{-1}_{(I,J)}((-\infty, c-\epsilon)) \arrow[l, hook'] \\ N_{x_{(I',J')}} \arrow[u, hook, "\mathrm{Aff}_i"] & N_{x_{(I',J')}} \cap \xi^{-1}_{(I',J')}((-\infty, c'-\epsilon)) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Note that Aff_i induces a poset isomorphism from the poset $\mathcal{S}_{N_{x_{(I',J')}}}$ to the poset $\mathcal{S}_{N_{x_{(I,J)}}}$ where via this isomorphism the subposet $\mathcal{S}_{N_{x_{(I',J')}} \cap \xi^{-1}_{(I',J')}((-\infty,c'-\epsilon))}$ gets identified with $\mathcal{S}_{N_{x_{(I,J)}} \cap \xi^{-1}_{(I,J)}((-\infty,c-\epsilon))}$. Furthermore, the representation of the poset $\mathcal{S}_{N_{x_{(I',J')}}}$ that gives rise to the sheaf $\mathrm{Aff}_i^* \mathscr{F}^\bullet$ is exactly the representation of the poset $\mathcal{S}_{N_{x_{(I,J)}}}$ that gives rise to the sheaf $\mathscr{F}^\bullet$ under the poset isomorphism $(\mathrm{Aff}_i)_*$ and the relative cohomology is completely determined by the poset, subpost structures and their representations. Therefore, the claim is proved.

Therefore, $G_{(I,J)}(\mathscr{F}^\bullet) \cong H^*(N_{x_{(I',J')}}, N_{x_{(I',J')}} \cap \xi^{-1}_{(I',J')}((-\infty, c'-\epsilon)); \mathrm{Aff}_i^* \mathscr{F}^\bullet)$. By the induction hypothesis if $I'^c \not\subset J'^c$, this is 0 and if $I'^c \subset J'^c$, this is equal to $\mathrm{Tot}(\mathrm{Aff}_i^*(F^\bullet|_{\mathcal{S}_{N_{x_{(I',J')}}}})) = \mathrm{Tot}(F^\bullet|_{\mathcal{S}_{N_{x_{(I,J)}}}}) = \mathrm{Tot}(F^\bullet|_{Q^n_{s_{(I,J)}}})$. Because $I^c \subset J^c$ if and only if $I'^c \subset J'^c$, we proved the case when $I \neq \phi$.

(ii) now consider the case where $I = \phi$.

(Case1) $I^c \not\subset J^c$ i.e. $J = \phi$, then we want to prove that $G_{(I,J)}(\mathscr{F}^\bullet) = 0$. Equivalently, we want to prove that the mapping cone of the following restriction map is acyclic.

$$R\Gamma(N_{x_{(I,J)}}; \mathscr{F}^\bullet) \to R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon)); \mathscr{F}^\bullet)$$

First, let's compute $R\Gamma(N_{x_{(I,J)}}; \mathscr{F}^\bullet)$. We can use a singleton Čech cover $\{N_{x_{(I,J)}} = \mathrm{star}(s^n_{(I,J)})\}$. Therefore, $R\Gamma(N_{x_{(I,J)}}; \mathscr{F}^\bullet) \cong \mathrm{Tot}(F^\bullet(N_{x_{(I,J)}}))$.

Next, let's compute $R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon]); \mathscr{F}^\bullet)$. We can use the following Čech cover

$$\mathcal{U} = \{U_k\}_{k \in [n]}$$

where

$$U_k := \text{star}(s^n_{x_{(I,J)}}(k)) = \text{star}(s^n(0, \cdots, 0, \underset{k^{th}}{\underset{\uparrow}{(-1)^{\delta_J(k)}}}, 0, \cdots, 0))$$

We define

$$U_{i_0 \cdots i_p} := N_{x_{(I,J)}} \cap U_{i_0} \cap \cdots \cap U_{i_p}$$

Note that

- if the index set in the subscript is empty, then $U = N_{x_{I,J}} = \mathbb{R}^n$
- $U_{i_0 \cdots i_p} = \text{star}(s^n(sgn_1, \cdots, sgn_n))$ where

$$sgn_k = \begin{cases} (-1)^{\delta_J(k)} & \text{if } k \in \{i_0, \cdots, i_p\} \\ 0 & \text{otherwise} \end{cases}$$

Since $J \neq \phi$, without loss of generality assume $1 \in J$(we can always relabel coordinates). Note that for all $1 \leq i_0 < \cdots < i_p \leq n$

$$F^\bullet(U_{i_0 \cdots i_p}) = F^\bullet(U_{1 i_0 \cdots i_p})$$

because $F^\bullet$ a poset representation obtained from a legible diagram.

From the above Čech cover we get a Čech double complex

$$C^{p,q} = \bigoplus_{1 \leq i_0 < \cdots < i_p \leq n} F^q(U_{i_0 \cdots i_p})$$

where horizontal maps are Čech differentials(δ) and vertical maps are cochain differentials(d).

The mapping cone of

$$R\Gamma(N_{x_{(I,J)}};\mathscr{F}^\bullet) \to R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon));\mathscr{F}^\bullet)$$

is the total complex of the following double complex

$$C^{p,q}_{ext} = \begin{cases} C^{p,q} & \text{if } p \geq 0 \\ F^q(N_{x_{(I,J)}}) = F^q(\mathbb{R}^n) & \text{if } p = -1 \\ 0 & \text{otherwise} \end{cases}$$

where $\delta^{-1,q}$ is just a restriction map.

Now I will define a homotopy $h^{\bullet,\bullet} : C^{\bullet,\bullet}_{ext} \to C^{\bullet-1,\bullet}_{ext}$ on the Čech double complex that induces homotopy h on the total complex of $C^{\bullet,\bullet}_{ext}$. Let $c \in C^{p,q} := \bigoplus_{1 \leq i_0 < \cdots < i_p \leq n} F^q(U_{i_0 \cdots i_p})$, then we define $hc \in C^{p-1,q}$ so that

$$(hc)_{i_0 \cdots i_p} = c_{1 i_0 \cdots i_p}$$

Here we are using the fact that if $1 \notin \{[i_0, \cdots, i_p\}$

$$F^q(U_{i_0 \cdots i_p}) = F^q(U_{1 i_0 \cdots i_p})$$

Note that if one of $i_0, \cdots, i_p$ is 1, then $(hc)_{i_0 \cdots i_p} = 0$. Also note that $h^{p,q-1}d^{p,q} = d^{p-1,q}h^{p,q}$. Now let's compute $hD_{\text{Tot}} + D_{\text{Tot}}h$ where D_{Tot} is the total differential. Suppose $c \in C^{-1,q}_{ext}$, then

- $(hD_{\text{Tot}}c) = (h(-d+\delta)c) = -(hdc) + (h\delta c)$. Because hdc lands on degree $p = -2$, $(hdc) = 0$ and $(hD_{\text{Tot}}c) = (h\delta c) = c|_{U_1} = c$. Here, we use the identity map between $F^q(N_{x_{(I,J)}}) = F^q(U_1)$ and think of $c|_{U_1}$ as an element of $F^q(N_{x_{(I,J)}})$ that uniquely extends $c|_{U_1}$, which is c itself.

- $(D_{\text{Tot}}hc) = D_{\text{Tot}}0 = 0$ because hc lands on degree $p = -2$.

Suppose $c \in C^{p,q}_{ext}$ $(p \geq 0)$,

- $hD_{\text{Tot}} = h((-1)^p d + \delta) = (-1)^p hd + h\delta$
- $D_{\text{Tot}}h = ((-1)^{p-1}d + \delta)h = (-1)^{p-1}dh + \delta h$

Therefore, $hD_{\text{Tot}} + D_{\text{Tot}}h = h\delta + \delta h$ and

$$\begin{aligned}
(h\delta c + \delta hc)_{i_0\cdots i_p} &= (h\delta c)_{i_0\cdots i_p} + (\delta hc)_{i_0\cdots i_p} \\
&= (\delta c)_{1 i_0\cdots i_p} + \sum_{j=0}^{p}(-1)^j (hc)_{i_0\cdots\hat{i}_j\cdots i_p} \\
&= [c_{i_0\cdots i_p} + \sum_{j=0}^{p}(-1)^{j+1} c_{1 i_0\cdots\hat{i}_j\cdots i_p}] + \sum_{j=0}^{p}(-1)^j (hc)_{i_0\cdots\hat{i}_j\cdots i_p} \\
&= c_{i_0\cdots i_p}
\end{aligned}$$

Therefore, h is a homotopy between identity cochain map on $\text{Tot}(C^{\bullet;\bullet}_{ext})$ and a zero map. Therefore, we conclude that $\text{Tot}(C^{\bullet;\bullet}_{ext})$ is acyclic.

(Case2) $I^c \subset J^c$ i.e. $J = \phi$, then use the same Čech cover $\mathcal{U}$ as in (case1). Then again the mapping cone of

$$R\Gamma(N_{x_{(I,J)}}; \mathscr{F}^\bullet) \to R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon)); \mathscr{F}^\bullet)$$

is the total complex of the following double complex up to a shift

$$F^\bullet(N_{x_{(I,J)}}) \to \oplus F^\bullet(U_{i_0}) \to \oplus F^\bullet(U_{i_0 i_1}) \to \cdots$$

which is equal to

$$F^\bullet(\mathrm{star}(s^n(0,\cdots,0))) \to \oplus F^\bullet(\mathrm{star}(s^n(sgn_{i_0}=+,\text{else }0))) \to$$
$$\oplus\, F^\bullet(\mathrm{star}(s^n(sgn_{i_0}=sgn_{i_1}=+,\text{else }0))) \to \cdots$$

which is equal to

$$F^\bullet(\mathrm{star}(s^n(-,\cdots,-))) \to \oplus F^\bullet(\mathrm{star}(s^n(sgn_{i_0}=+,\text{else }-))) \to$$
$$\oplus\, F^\bullet(\mathrm{star}(s^n(sgn_{i_0}=sgn_{i_1}=+,\text{else }-))) \to \cdots$$

whose total complex is the total complex of $F^\bullet$ restricted to Q^n. Because $Q^n = Q^n_{s^n(-,\cdots,-)}$ the proof is complete. □

3.2 Natural alternating diagrams

Suppose we have a positive braid word ω, then we can draw the associated braid diagram($i_1,\cdots,i_{n-1} : [0,1]_x \to [0,1]_x \times (0,1)_z$ where i_k are smooth sections of the projection $[0,1]_x \times (0,1)_z \to [0,1]_x$) on $[0,1]_x \times (0,1)_z$ and its cylindrical closure $S^1_x \times (0,1)_z$. For example, if $\omega = s_1 s_1$, a braid word on 3 strand, then the cylindrical closure of the associated braid diagram is shown in the figure below.

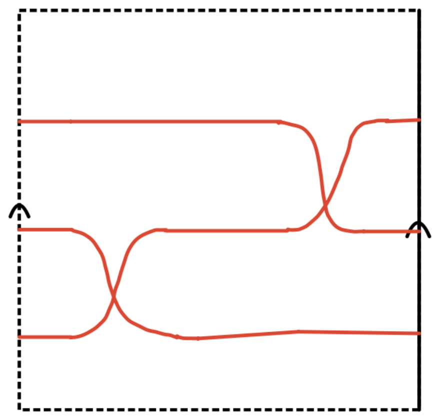

Figure 3.1

We will specify co-orientations of $i_1, \cdots, i_{n-1}$ so that we can think of the cylindrical closure of the braid word as the the front projection of a Legendrian knot living inside the co-circle bundle of the cylindrical closure.

Let $x_0 \in [0,1]$, we define the co-orientation at $i_k(x_0)$ to be $\xi = adx + cdz$ so that ξ vanishes at $\frac{di_k}{dt}|_{x=x_0}$, $||(a,c)|| = 1$, and $c > 0$. This can be visually represented as hairs pointing upward(i.e. $c > 0$).

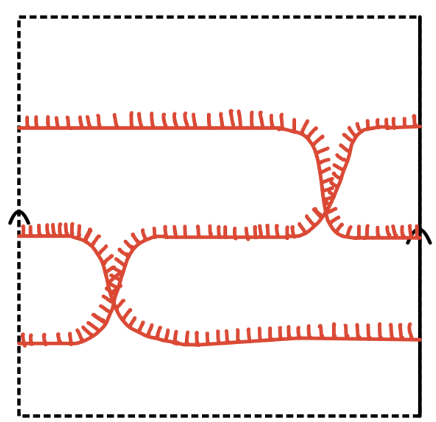

Figure 3.2

Suppose we have a Riemann sphere M with two punctures at 0 and ∞. M is diffeomorphic to the boundaryless cylinder as shown in the figure below.

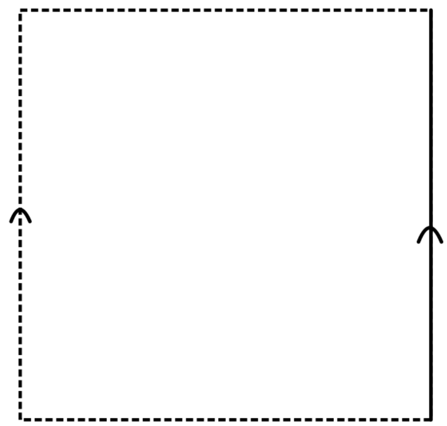

Figure 3.3

There are two distinguished ways of embedding the cylindrical closure of ω into M. We can embed the cylindrical closure onto the hemisphere containing $0(\infty$ resp.), i.e. the lower hemisphere(upper hemisphere resp.), in such a way that the embedding extends

(i) to $S^1 \times \{0\})$ as an isomophism onto the equator of M

(ii) to $S^1 \times \{1\}$ as a constant map to $0(\infty$ resp.)

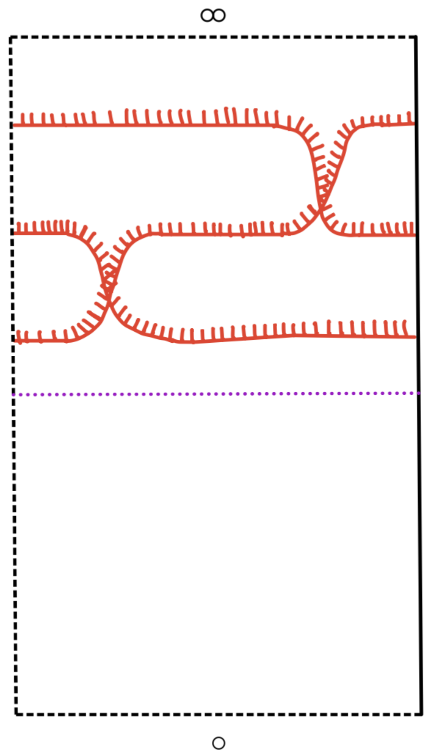

Figure 3.4

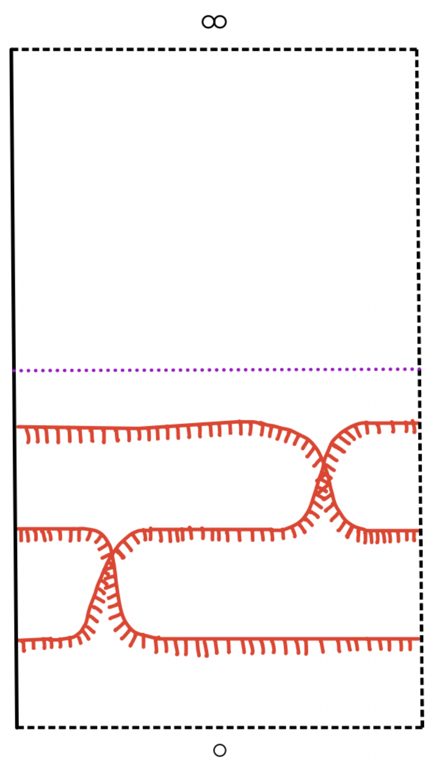

Figure 3.5

Suppose we have a positive braid word ω on n strands, we have the following associated objects:

- M: A Riemann sphere with two punctures at 0 and ∞
- $\Phi_0 : \coprod_{i=1}^{n} S^1 \to M$: the front projection induced by the embedding of the cylindrical closure of the trivial braid word onto the lower hemisphere
- ξ_0: the co-orientation of Φ_0
- $\Phi_\infty : \coprod_{i=1}^{m} S^1 \to M$ (where $m \leq n$): the front projection given by the embedding of the cylindrical closure of the braid word ω onto the upper hemisphere
- ξ_∞ : the co-orientation of ι_∞

To simplify the notation, we will denote the pair $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$ resp.) as $\Lambda_0(\Lambda_\infty$ resp.). Also, we will abuse $\Lambda_0(\Lambda_\infty$ resp.) to denote the Legendrian associated to the pair $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$ resp.).

Now fix a positive braid word ω and the object $(M, \Lambda_0, \Lambda_\infty)$ associated with it which we call the separated diagram of ω. I will define a natural alternating braid diagram $(M, \Lambda_0', \Lambda_\infty')$ whose associated Legendrian is Legendrian isotopic to the Legendrian associated with $(M, \Lambda_0, \Lambda_\infty)$. I will construct an explicit Legendrian isotopy between them. Furthermore, I will construct cobordisms between constructible sheaves singular supported on $(M, \Lambda_0, \Lambda_\infty)$ and $(M, \Lambda_0', \Lambda_\infty')$ which will be the main result of this chapter. The isotopy will be only applied to Λ_0, so the Λ_∞ will remain fixed i.e. $\Lambda_\infty = \Lambda_\infty'$.

First, let's draw Λ_∞' as in red on M as follows :

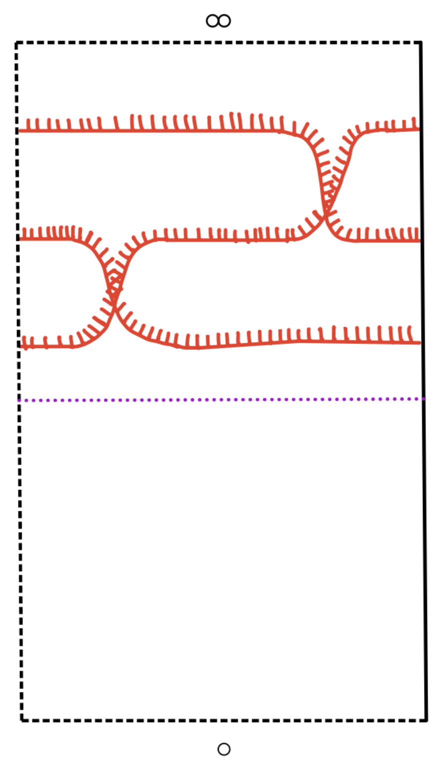

Figure 3.6

Now on the above diagram, let's draw Λ_0' the part that is Legendrian isotopic to Λ_0 in blue. But before that, we need some definitions.

Definition 61. Suppose $\omega = s_{1_1}, ..., s_{i_k}$, then the cylindrical closure can be parsed into a concatenation of k mutually disjoint regions where i^{th} region containing a part of the braid diagram corresponding to the generator s_{i_j} shown in the figure below. We call the region corresponding to s_{i_j} as the j^{th} generator region(also its image under the embedding into M).

Below is the picture of the 1^{st} generator region of the cylindrical closure of $\omega = s_1s_2$.

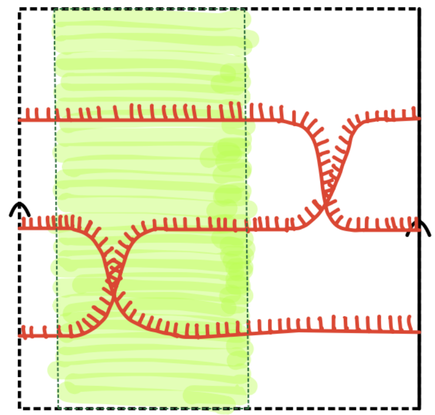

Figure 3.7

Below is the picture of the 2^{nd} generator region of the cylindrical closure of $\omega = s_1 s_2$.

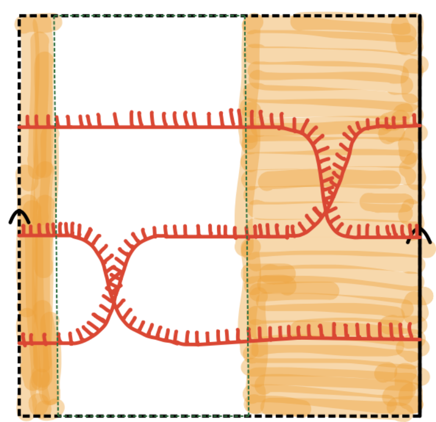

Figure 3.8

Definition 62. Suppose we set-theoretically subtract the union of all generator regions from the cylinder, we get k connected components. That is, for each $j = 1, ..., k$, we have one component in between j^{th} and $(j + 1 \pmod{k})^{th}$ regions. We call the neighborhood of this component inside the cylinder as j^{th} inter-generator region(also its image inside the cylinder under the embedding into M).

- inter-generator regions do not contain any crossing
- inter-generator regions are mutually disjoint

- j^{th} intergenerator region intersects with j^{th} and $j+1^{th}$(modulo k) generator region

Below is the picture of the 1^{st} inter-generator region of the cylindrical closure of $\omega = s_1 s_2$.

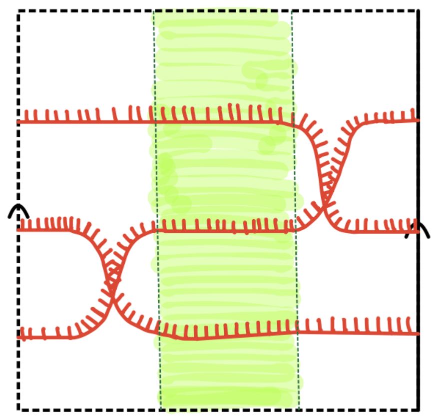

Figure 3.9

Below is the picture of the 2^{nd} generator region of the cylindrical closure of $\omega = s_1 s_2$.

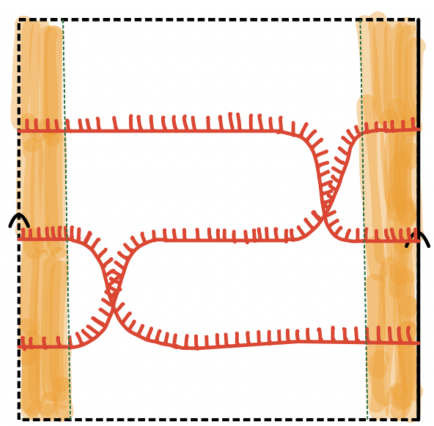

Figure 3.10

Now I will draw Λ_0' for each generator region so that they glue up to the whole Λ_0'.

First, we restrict the diagram to j^{th} generator region, we have the following diagram: Note that i_j^{th} and $i_j + 1^{th}$ strands cross each other and all the other strands are horizontal.

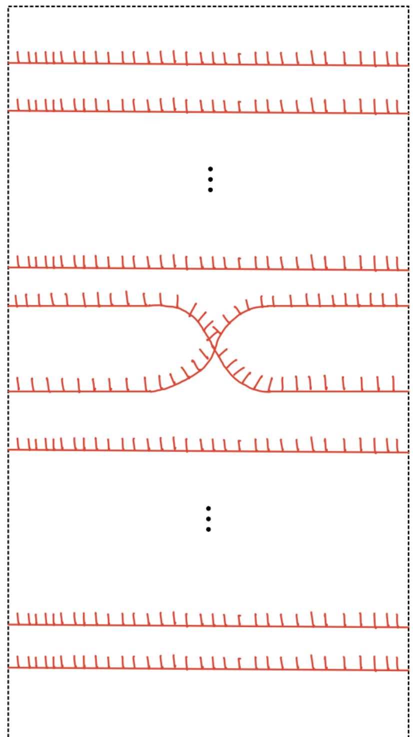

Figure 3.11

We label the strands from bottom to top using integers from 1 to n with reference to the left end points. This is the strand labelling scheme that I will use throughout this chapter.

I will draw Λ_0' as blue strand on it as follows :

- l^{th} blue strand starts from the midpoint of the starting points of $l-1^{th}$ and l^{th} red strands and ends at the midpoint of the end points of $l-1^{th}$ and l^{th} red strands

- if $l \neq i_j$ and $i \neq i_j + 1$, then along the way the l^{th} blue strand crosses up and down once

- if $l = i_j + 1$, l^{th} blue strand crosses l^{th} red strand up in the part before the crossing and then crosses $l - 1^{th}$ red strand down in the part after the crossing.

- if $l = i_j$, l^{th} blue strand crosses l^{th} red strand up and down in the part before the crossing and then crosses $l + 1^{th}$ red strand up and down in the part after the crossing.

The picture below overlays Λ_0', drawn in blue strands with hairs pointing downward, on the previous diagram.

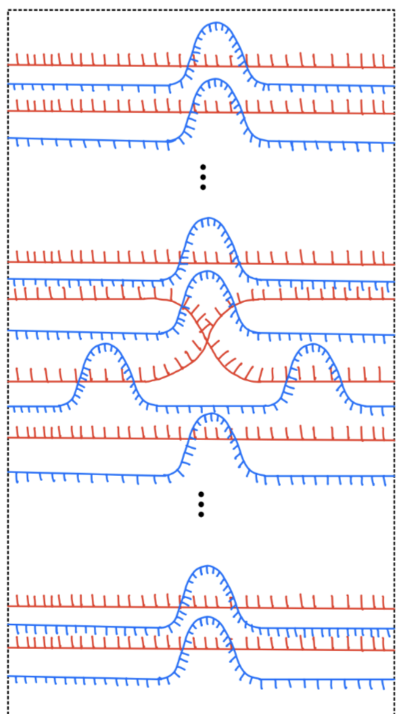

Figure 3.12

For the full alternating strand diagram, we take the closure of blue strands from the generator regions so that the end points from the bordering regions coincide.

The picture below shows how the global natural alternating strand diagram associated with $\omega = s_1 s_2$ looks like after gluing together local alternating strand diagrams of s_1 and s_2.

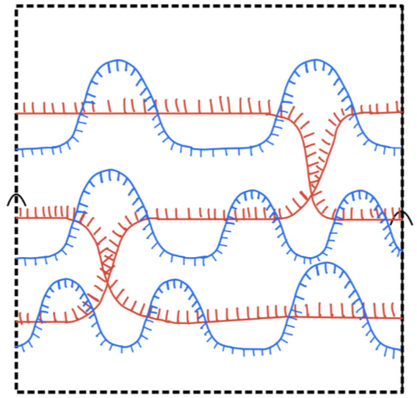

Figure 3.13

Theorem 63. The above defined strand diagram is alternating

Proof. we will denote

- the region with all the hairs pointing outward as $\circ$
- the region with all the hairs pointing inward as $\triangle$
- else with $\times$

for the generator region we have the following figure :

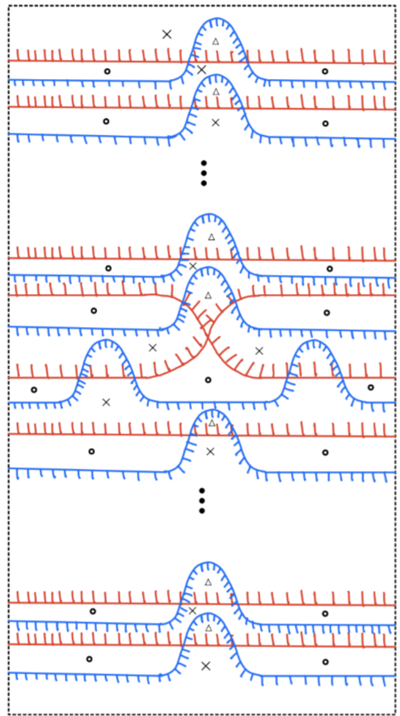

Figure 3.14

The above marking extends to the inter-generator region, we have the following figure:

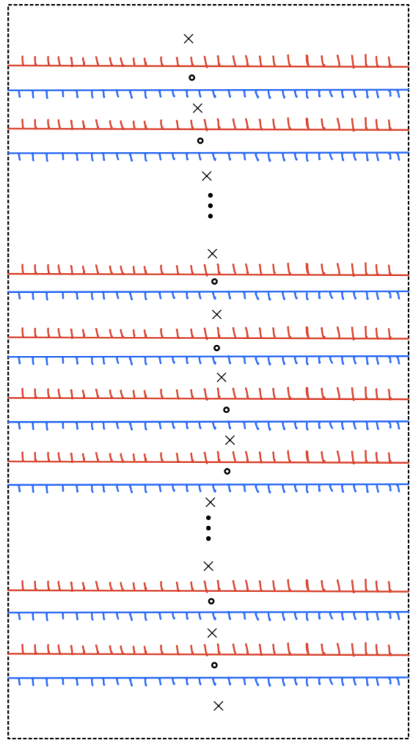

Figure 3.15

for each crossing, it satisfy the alternating condition. This diagram is indeed alternating. □

3.3 Local systems on natural alternating diagrams

Suppose we have a positive braid word ω then we have the associated natural alternating diagram $(M, \Lambda'_0, \Lambda'_\infty)$ defined in the previous section.

We can associate a quiver Q to the alternating diagram in such a way that

- we have one vertex for regions where all hairs are pointing outward/inward
- for each crossing, we have an arrow from the vertex corresponding to the region where all hairs pointing outward to inward

For example, for each generator region of a natural alternating strand diagram:

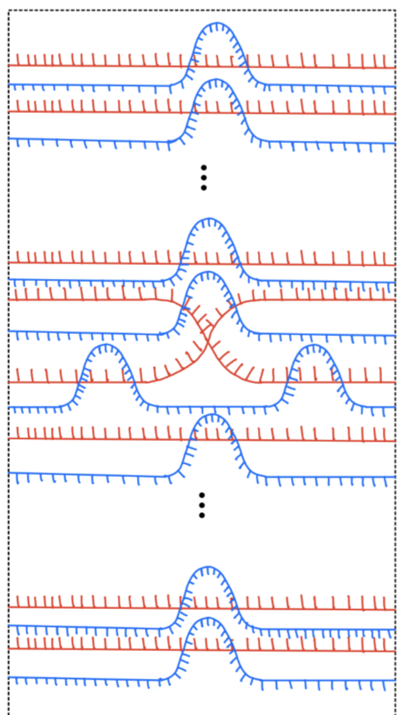

Figure 3.16

we have the following associated quiver:

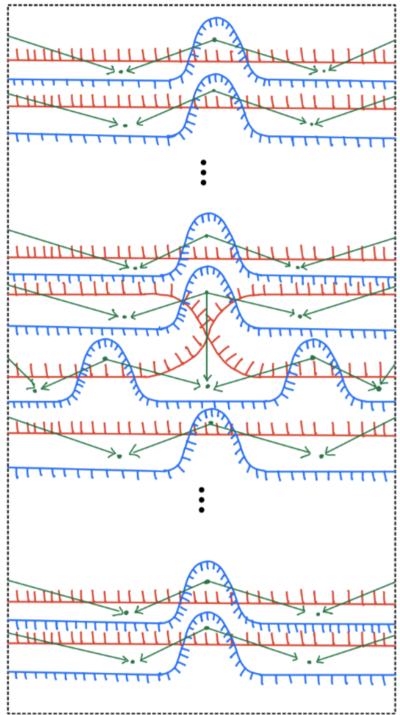

Figure 3.17

Once we have an alternating strand diagram, we have the associated conjugate surface L_{conj}. Furthermore, we can embed the underlying undirected graph Γ_{bi} of the quiver Q(i.e. the bipartite graph of the alternating coloring) in L_{conj} in such a way that L_{conj} deformation retracts to Γ_{bi}. Suppose we have a rank 1 local system on the conjugate surface associated with $(M, \Lambda'_0, \Lambda'_\infty)$, then restricting to Q, we get a local system on Q. Note that the pullback, induced by the restriction map, between the space of local systems $H^1(L_{conj}, \mathbb{C}^*) \to H^1(\Gamma_{bi}, \mathbb{C}^*)$ is an isomorphism.

$H^1(\Gamma_{bi}, \mathbb{C}^*)$ is isomorphic to $(\mathbb{C}^*)^{|Arr(Q)|}//(\mathbb{C}^*)^{|Vert(Q)|}$ here the group action is defined as the following : let $g_v \in (\mathbb{C}^*)^{|Vert(Q)|}$ (more precisely, $g_v := (g^{\delta_{w,v}})_{w \in Vert(Q)}$ where δ is the Kronecker delta), then $g_v \cdot (x_a)_{a \in Arr(Q)}$ is

- for entries with index a such that the source of a is v i.e. $s(a) = v$, we have $g_v \cdot x_a$
- for entries with index a such that the target of a is v i.e. $t(a) = v$, we have

$g_v^{-1} \cdot x_a$

Now we define the associated alternating sheaf on some regular cell complex refinement of the natural alternating strand diagram associated with a rank 1 local systems on Q.

First, I will describe the special kind of regular cell complex associated with the alternating strand diagram called the regular cell complex refinement of the natural alternating strand diagram. I will define the refinement for each generator region and glue them to get the global regular cell complex.

Definition 64. Suppose we fix a generator region for the alternating strand diagram. Then we denote the j^{th} crossing(numbering starts from left to right) the i^{th} blue strand(numbering starts from top to bottom) crosses red strands as $c_{i,j}$. We will call the crossing between i^{th} and $i+1^{th}$ red strand as c.

(i) For each crossing $c_{i,j}$ we add

- when j is odd, locally near $c_{i,j}$ we have the following local diagram

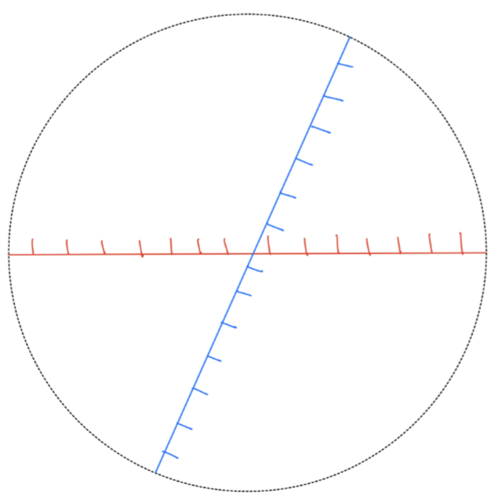

Figure 3.18

then we add squiggly lines with co-orientations and end points to get

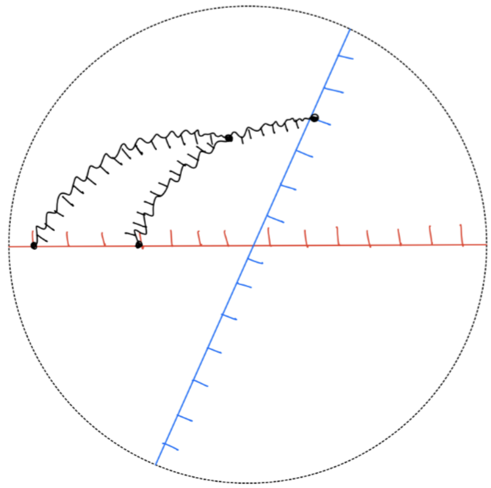

Figure 3.19

We call the region marked with $*$ a crossing region.

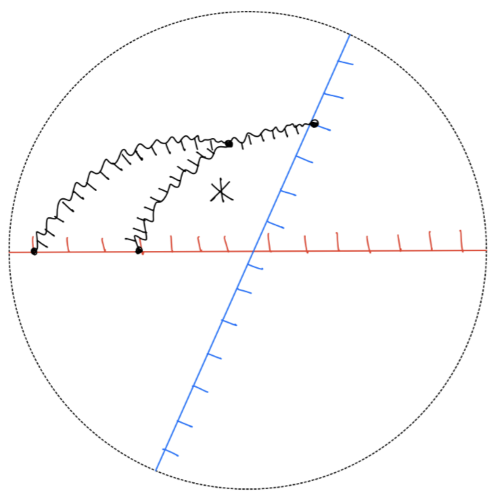

Figure 3.20

- when j is even, locally near $c_{i,j}$ we have the following local diagram

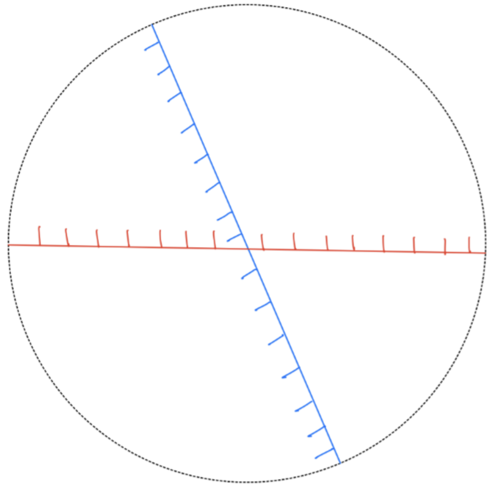

Figure 3.21

then we add squiggly lines with co-orientations and end points to get

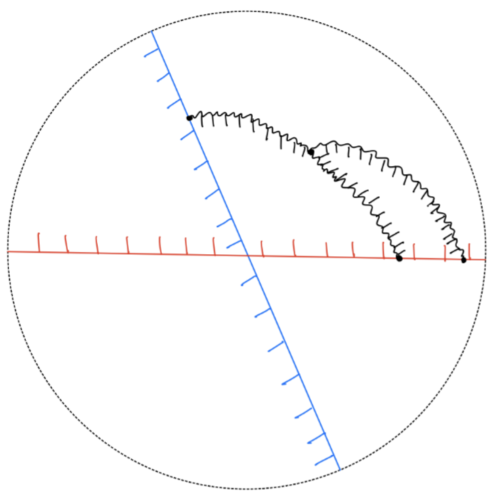

Figure 3.22

We call the region marked with $*$ a crossing region.

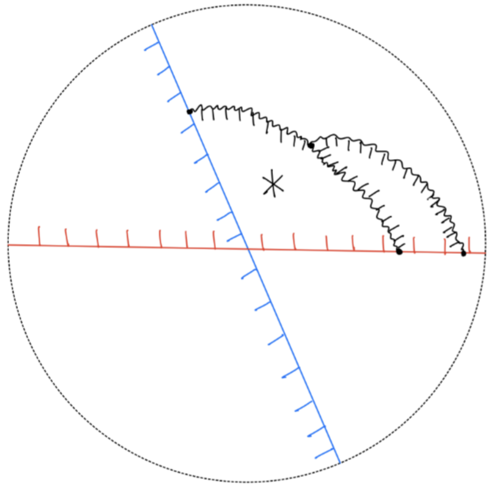

Figure 3.23

(ii) For the crossing c, locally near the crossing, we have the following local diagram

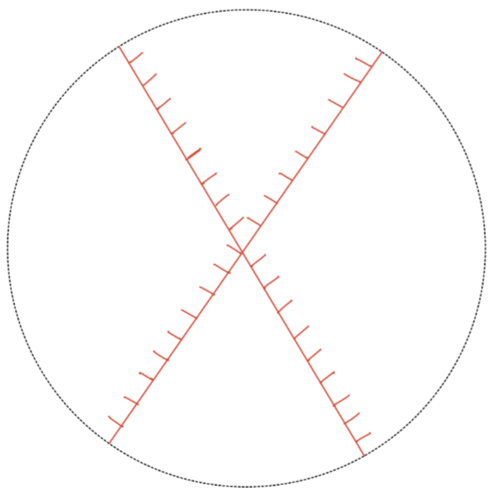

Figure 3.24

then we add squiggly lines with co-orientations and end points to get

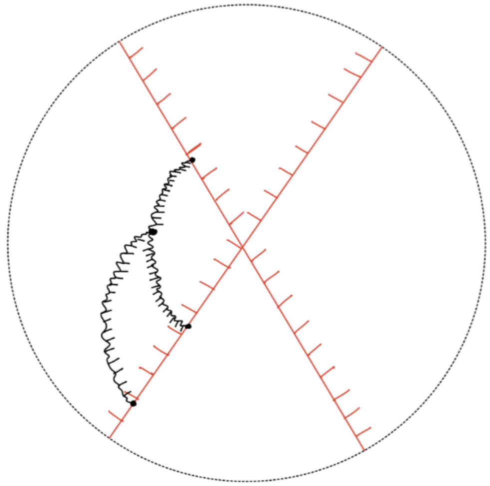

Figure 3.25

We call the region marked with $*$ a crossing region.

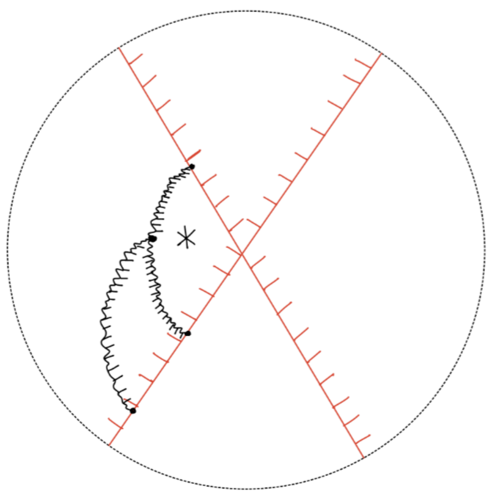

Figure 3.26

Below is the picture of a generator region of a natural alternating diagram:

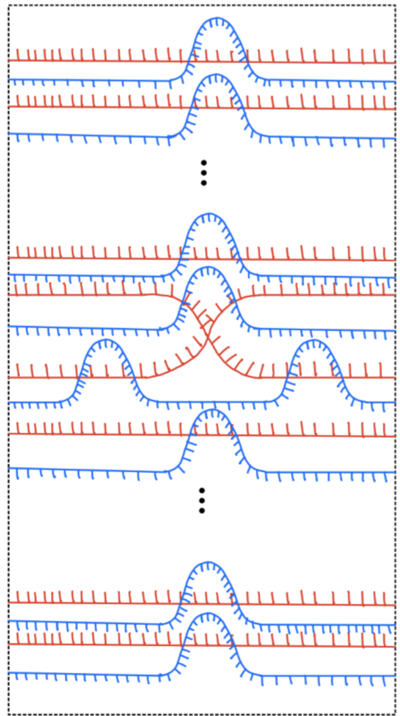

Figure 3.27

and below is the picture of the regular cell complex refinement in a generator region:

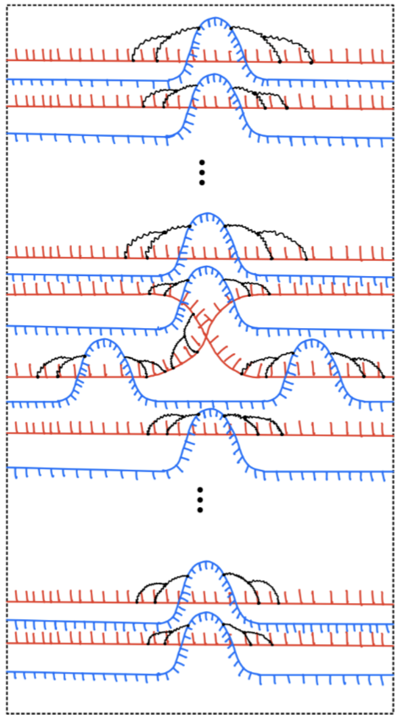

Figure 3.28

Next, we fix an inter-generator region

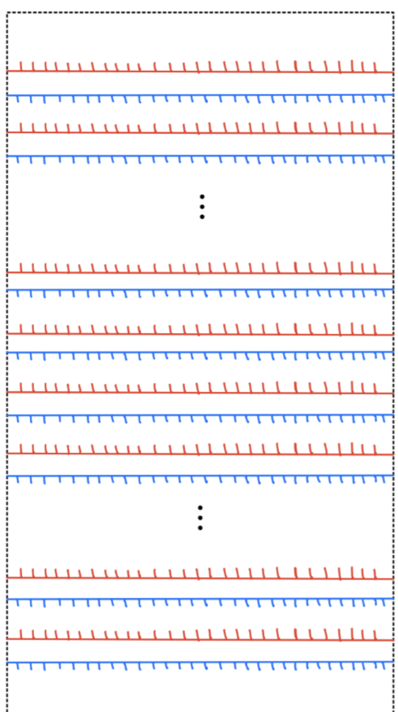

Figure 3.29

add a vertical squiggly line co-oriented towards the left

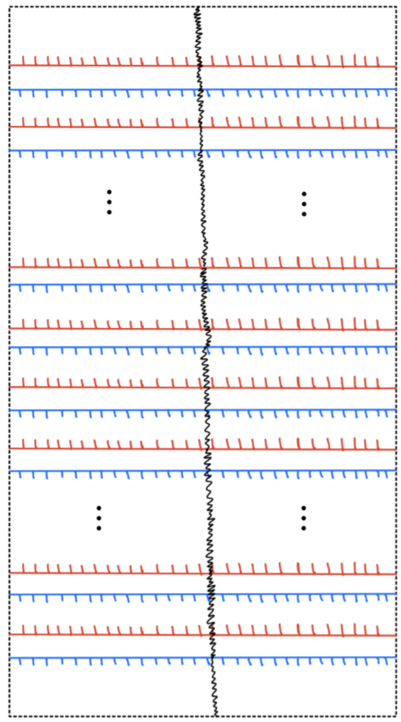

Figure 3.30

Now I will describe a way to specify a constructible sheaf on the above regular cell complex refinement associated with the local system on Γ_{bi}.

Definition 65. Suppose we have a local system on Q which can be represented as an element $(x_A)_{a \in Arr(Q)}(\mathbb{C}^*)^{|Arr(Q)|}$:

(i) stalk of the region where all the hairs are pointing outward is $\mathbb{C}[-1]$

(ii) stalk of the region where all the hairs(except the hairs on the squiggly lines) are pointing inward is $\mathbb{C}$

(iii) stalk of the crossing regions is $\mathbb{C} \xrightarrow{\times x_a} \mathbb{C}$ where a is the arrow corresponding to the associated crossing.

(iv) rest of the stalks are 0

(v) the only nonzero genrization maps are from regions of type (iii) to (i), from (i) to (ii), or from (ii) to (ii)(the ones corresponding to vertical squiggly lines in the inter-generator regions)

The maps from (iii) to (i) are

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

The maps from (i) to (ii) are

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

The maps from (ii) to (ii) are identity maps.

Note that the group action maps a constructible sheaf to the isomorphic constructible sheaf. Therefore, we have a well-defined map $H^1(L_{conj}, \mathbb{C}^*) \to \mathcal{M}_1(M, \Lambda', \{\sigma_{z\ll 0}, \sigma_{z\gg 0}\}; \mathbb{C})$ where $\Lambda' = \Lambda'_0 \coprod \Lambda'_\infty$, $\sigma_{z\ll 0}$ is a point in the region containing 0, and $z \gg 0$ is a point in the region containing ∞.

3.4 1st sheaf cobordism

In this section, we define *cobord*$_1$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

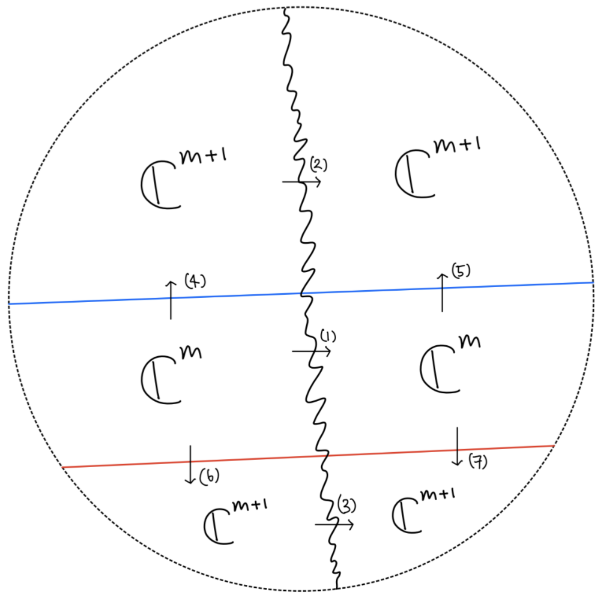

Figure 3.31

Generization maps:

(1) $\mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$

(2) $\mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$

(4) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(5) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(6) $\mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

to

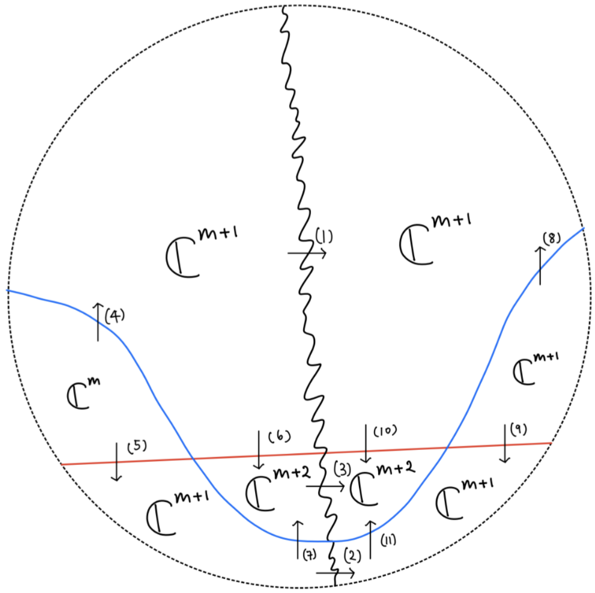

Figure 3.32

Generization maps:

(1) $\mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

(4) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(5) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(6) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(7) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(8) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(9) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(10) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(11) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

Notations

Definition 66. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 67. For $t_0 \in \{0,1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$
$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$
$$\Phi_{t_0}^{squig} : [0,1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_1$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_1$.

Definition 68. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_\bullet^{symbol}$ to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_\bullet^\infty : (S^1)^m \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_\bullet^{squig} : \coprod_{1 \le i \le k} ([0,1] \times [a_i, b_i]_t) \to M \times [0,1]_t$$

   where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_\bullet^{symbol}$ a co-orientation of $\Phi_\bullet^{symbol}$.

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as $\Lambda_\bullet^{symbol}$. Later in the section, $\Lambda_\bullet^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord_1$.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as $\Lambda_\bullet$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_\bullet$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord_1$.

Definition 69. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} -\frac{3}{4} e^{(\frac{x^2}{x^2-1})} t & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1,1]$ if $t \neq 0$
- $\{(1,0),\ (-1,0), (0, -\frac{3}{4}t)\} \subset Graph(\Psi_t)$

Definition 70. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$
$$(x,z) \mapsto (x,z,t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 71. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_0(x)\} = \{(x,z) \in D_{r=2} \mid z = 0\}$
- $\lambda_0^\infty := \{(x,z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_0^{squig} := \{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig}: hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 72. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_1(x)\}$
- $\lambda_1^\infty := \{(x,z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^{squig} := \{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing downward direction i.e. coefficients of dz are negative.

- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig}: hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 73. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda_\bullet^0 := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = -\frac{1}{2}\}$
- $\lambda_\bullet^{squig} := \{(x,z,t) \in D_{r=2} \times [0,1] \mid x = 0\}$

2. We define co-orientations $\xi_\bullet^{symbol}$ of $\lambda_\bullet^{symbol}$ as follows

- $\xi_\bullet^0$: hairs are pointing downward direction i.e. coefficients of dz are negative.
- $\xi_\bullet^\infty$: hairs are pointing upward direction i.e. coefficients of dz are positive.
- $\xi_\bullet^{squig}$: hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 74. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda^0_\bullet$, $\lambda^\infty_\bullet$, and $\lambda^{squig}_\bullet$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 75. $\operatorname{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\operatorname{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 76. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := \ & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \operatorname{sgn}(z - \Psi_{t_0}(x)) = sgn_1, \ \operatorname{sgn}(-\frac{1}{2} - z) = sgn_2, \\ & \operatorname{sgn}((x) = sgn_3\} \end{aligned}$$

Definition 77. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3) := \ & \{(x, z, t) \in D_{r=2} \times [0,1] \mid \\ & \operatorname{sgn}(z - \Psi_t(x)) = sgn_1, \ \operatorname{sgn}(\frac{1}{2} - z) = sgn_2, \\ & \operatorname{sgn}((x) = sgn_3\} \end{aligned}$$

Definition 78. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the above notations:

1. $\mathcal{S}_0$:

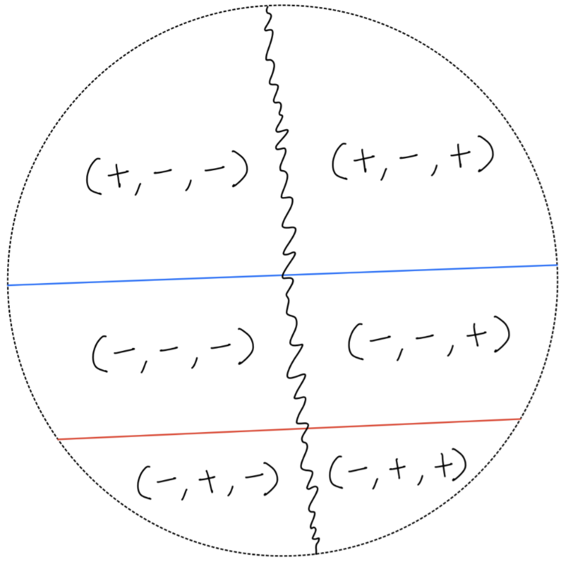

Figure 3.33

- 2 dimensional strata:

  $s_0(+,-,-)$,$s_0(+,-,+)$,$s_0(-,-,-)$,$s_0(-,-,+)$,$s_0(-,+,-)$,$s_0(-,+,+)$

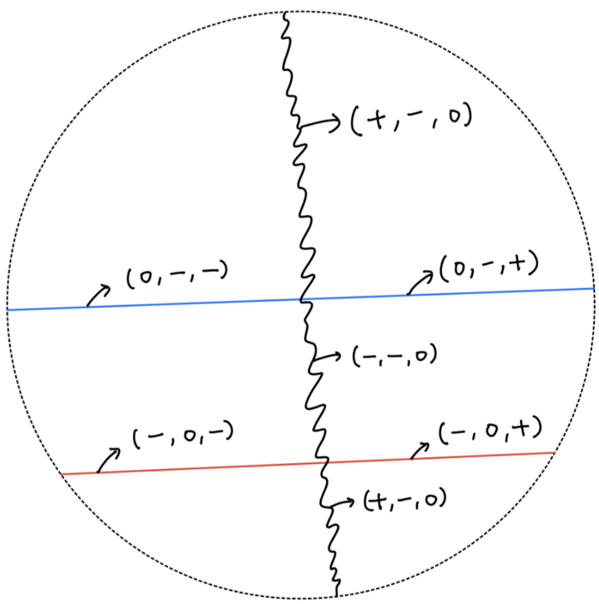

Figure 3.34

- 1 dimensional strata:

  $s_0(0,-,-)$,$s_0(0,-,+)$,$s_0(-,0,-)$,$s_0(-,0,+)$,$s_0(-,-,0)$,$s_0(-,+,0)$,$s_0(0,-,+)$

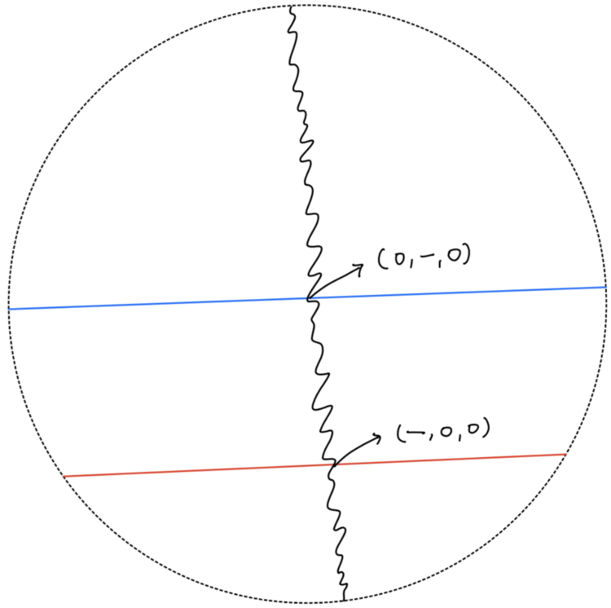

Figure 3.35

- 0 dimensional strata:

  $s_0(0, -, 0)$, $s_0(-, 0, 0)$

2. $\mathcal{S}_1$:

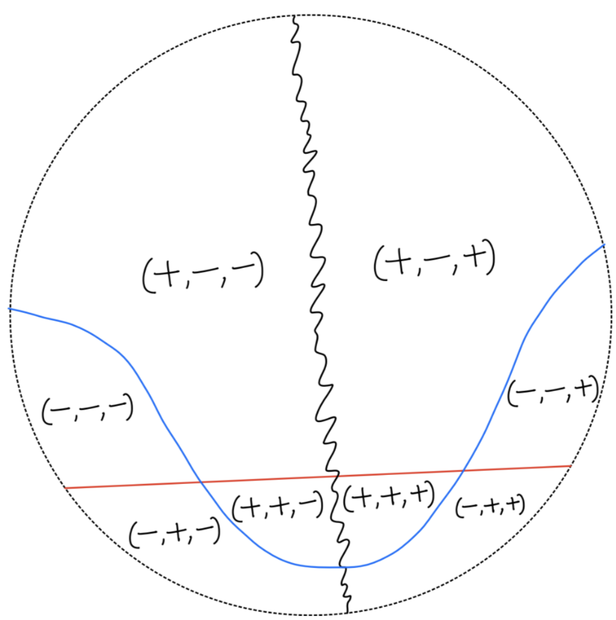

Figure 3.36

- 2 dimensional strata:

$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3$\}$

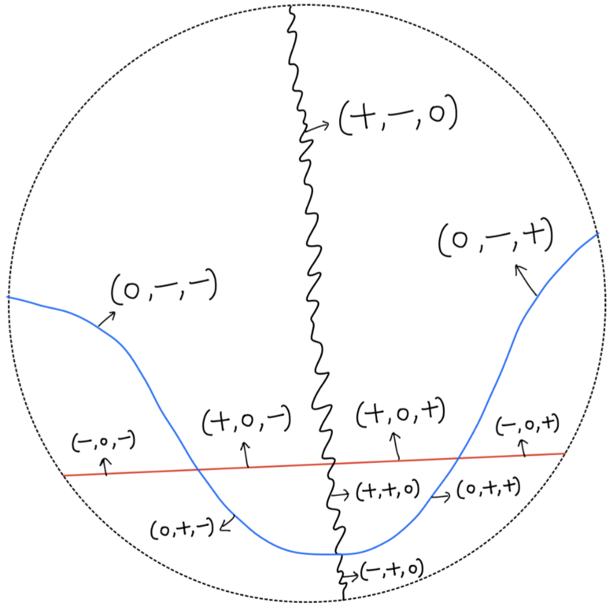

Figure 3.37

- 1 dimensional strata:

  $\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3$\} \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_1(-,-,0)\}$

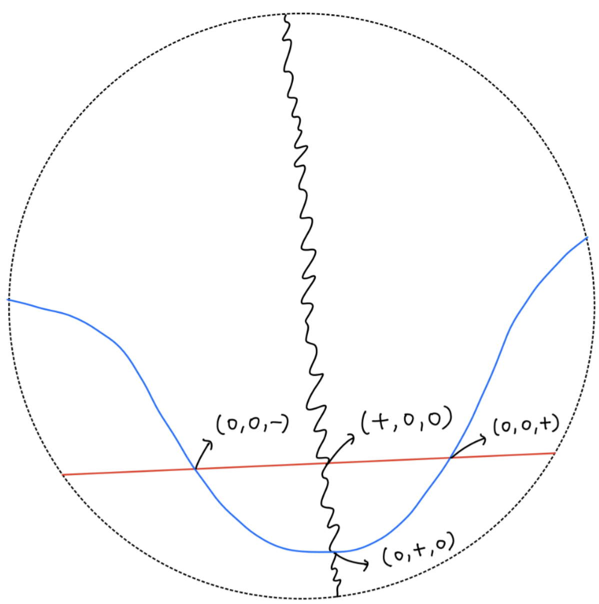

Figure 3.38

- 0 dimensional strata:

  $s_1(0,0,-)$, $s_1(0,+,0)$, $s_1(0,0,+)$, $s_1(+,0,0)$

3. $\mathcal{S}_\bullet$:

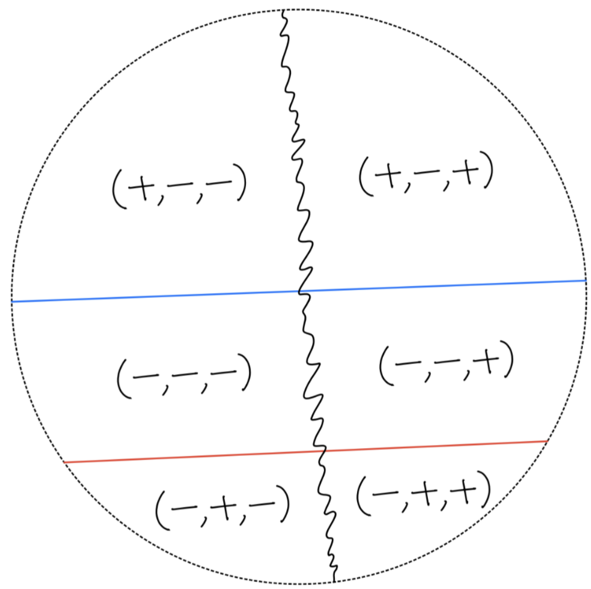

Figure 3.39

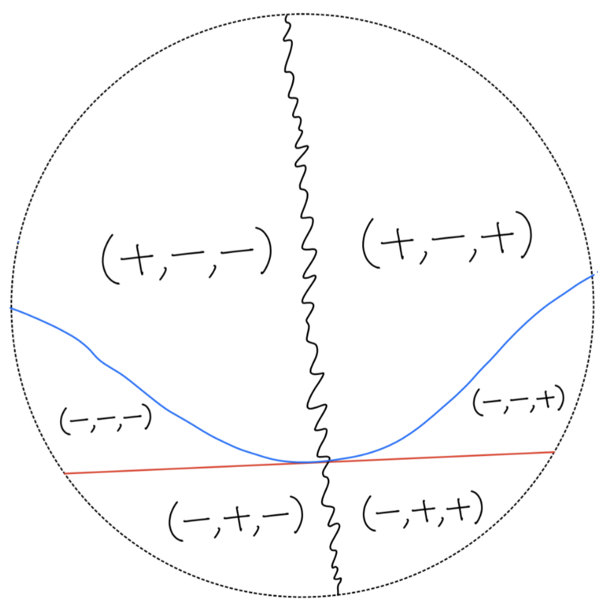

Figure 3.40

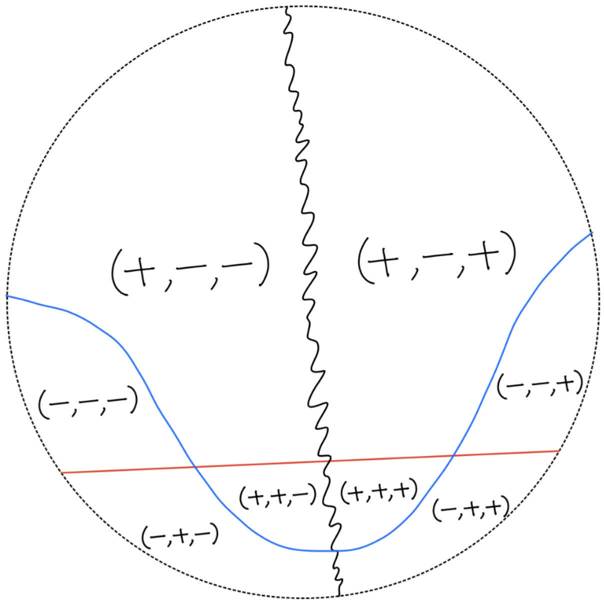

Figure 3.41

- 3 dimensional strata:

  $\{s_\bullet(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3$\}$

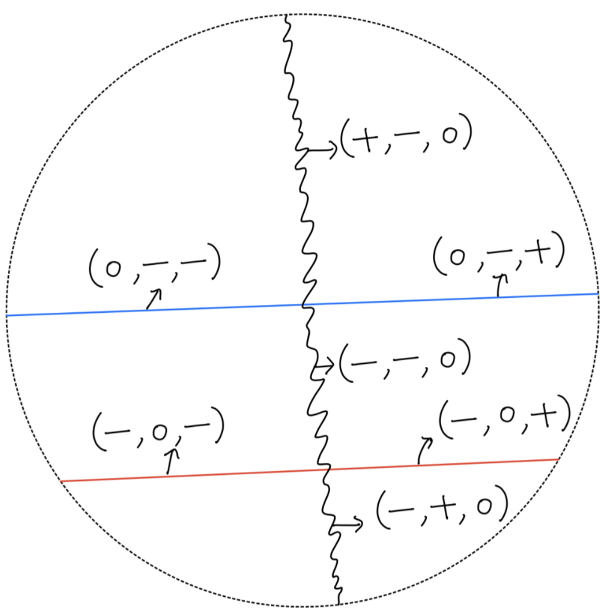

Figure 3.42

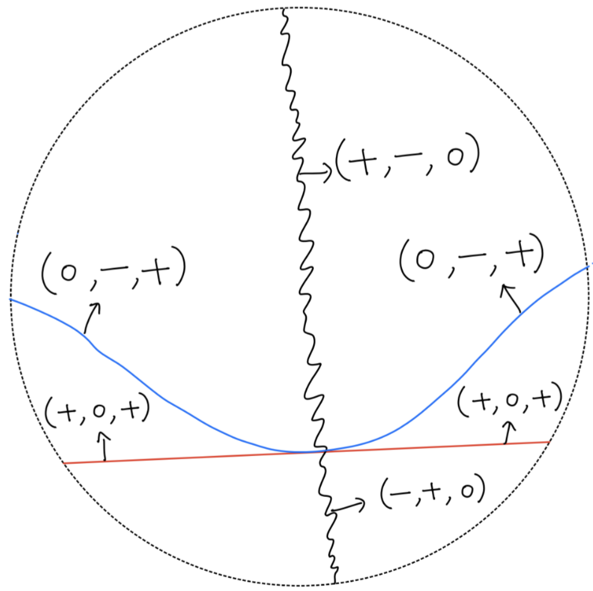

Figure 3.43

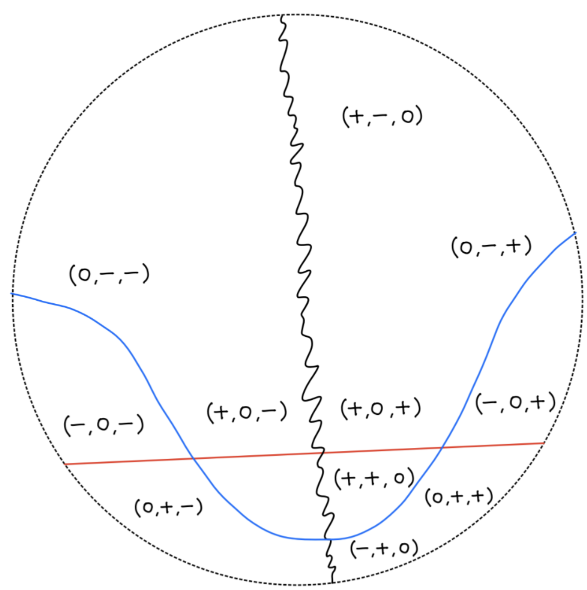

Figure 3.44

- 2 dimensional strata:

  $\{s_\bullet(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_\bullet(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3$\} \cup \{s_\bullet(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2$\}$

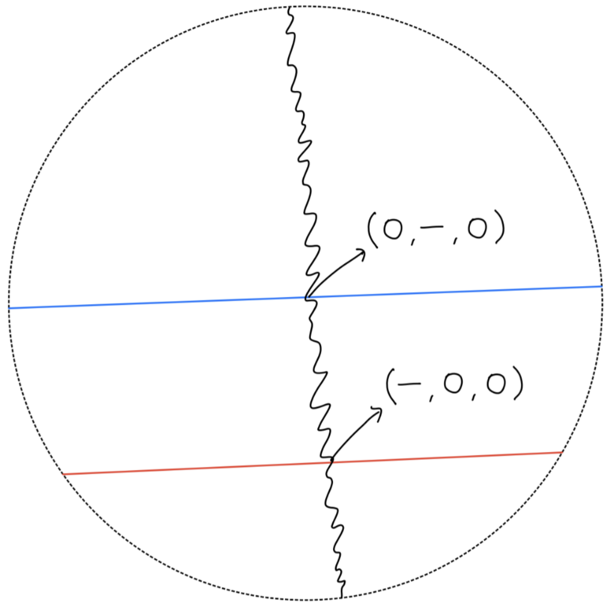

Figure 3.45

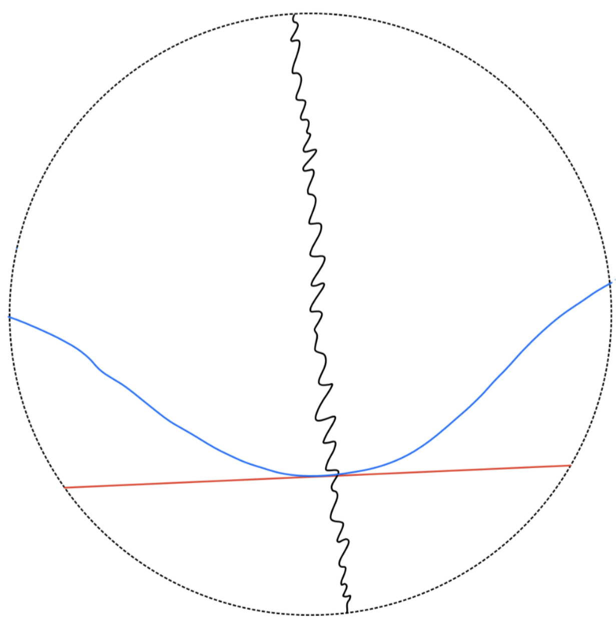

Figure 3.46

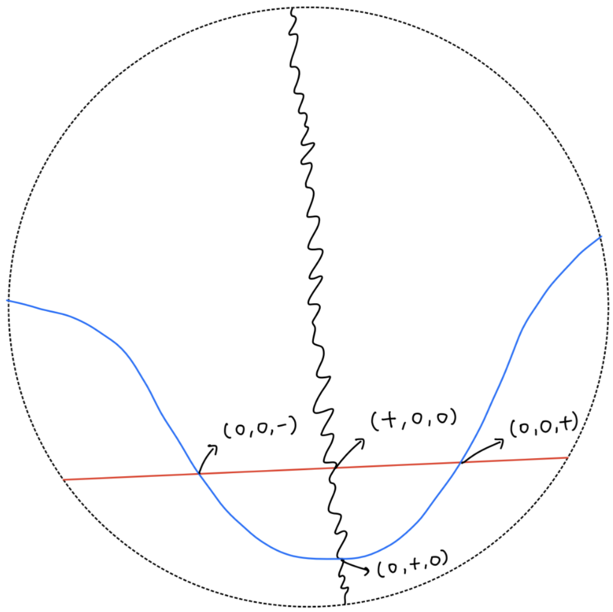

Figure 3.47

- 1 dimensional strata:

  $\{s_\bullet(sgn_1, 0, 0) \mid sgn_1 \in \{-,+\}\} \cup \{s_\bullet(0, sgn_2, 0) \mid sgn_2 \in \{-,+\}\} \cup \{s_\bullet(0, 0, sgn_3) \mid sgn_3 \in \{-,+\}\}$

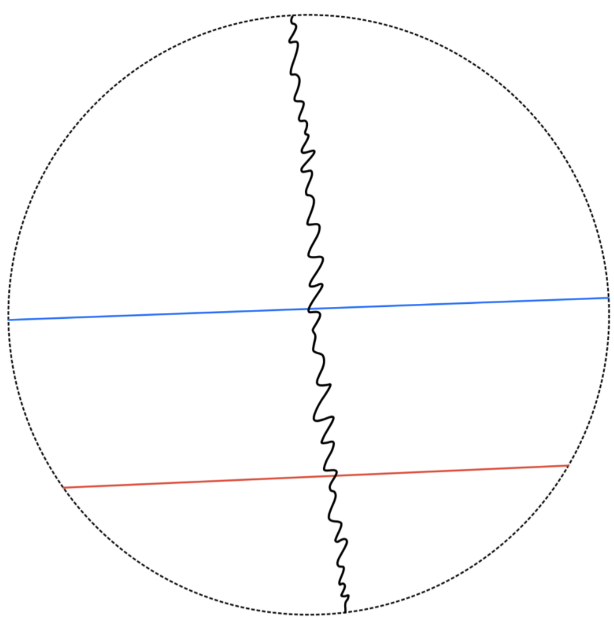

Figure 3.48

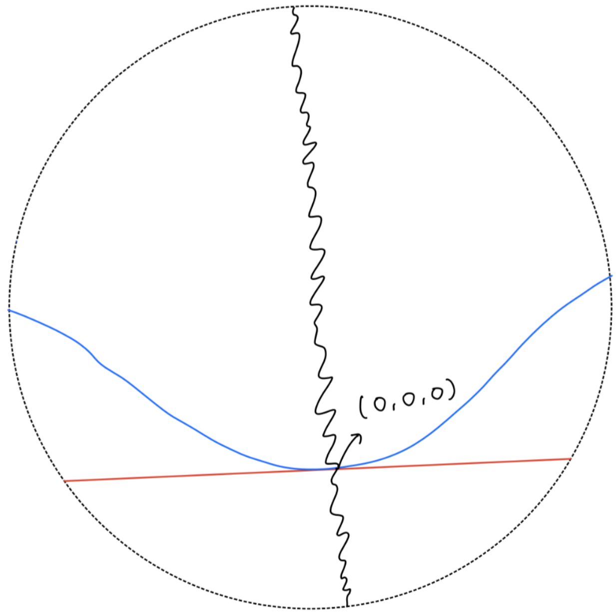

Figure 3.49

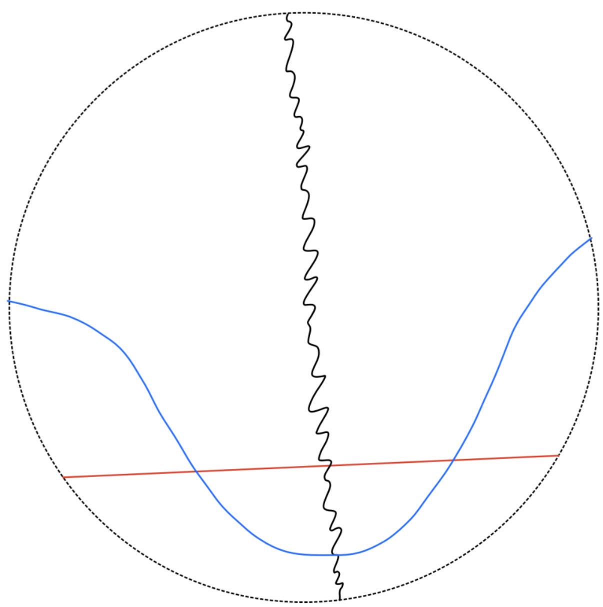

Figure 3.50

- 0 dimensional strata:

  $s_\bullet(0, 0, 0)$

Definition 79. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 80. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s.

Definition 81. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 82. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 83. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \text{ codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 84. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.

- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \text{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$, then

$$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

  This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M

- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.

- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'

- λ_0^0 is mapped to $\Lambda_0^0|_U$

- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$

- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

Stalks:

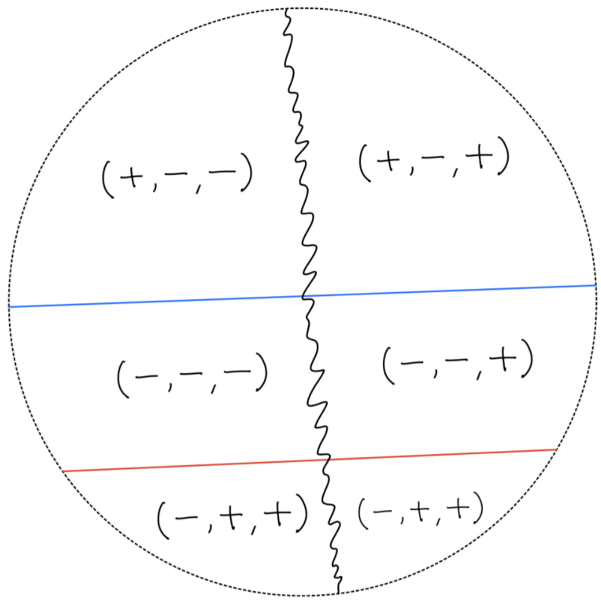

Figure 3.51

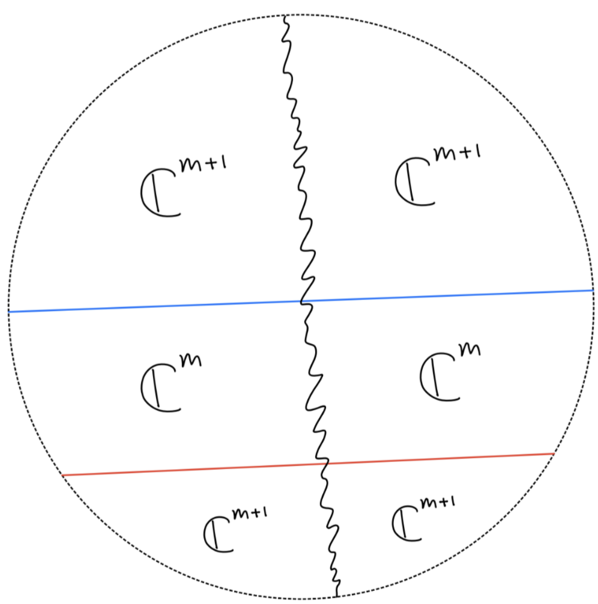

Figure 3.52

- $F_0(-,-,-) := \mathbb{C}^m$
- $F_0(-,-,+) := \mathbb{C}^m$
- $F_0(+,-,-) := \mathbb{C}^{m+1}$
- $F_0(+,-,+) := \mathbb{C}^{m+1}$
- $F_0(-,+,-) := \mathbb{C}^{m+1}$
- $F_0(-,+,+) := \mathbb{C}^{m+1}$

Generization maps:

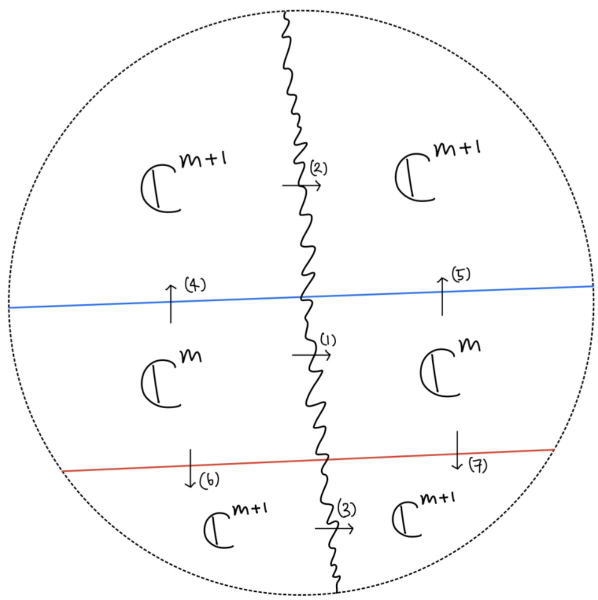

Figure 3.53

(1) $\mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$

(2) $\mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$

(4) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(5) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(6) $\mathbb{C}^m \overset{\iota_0}{\to} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^m \overset{\iota_0}{\to} \mathbb{C}^{m+1}$

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord_1$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

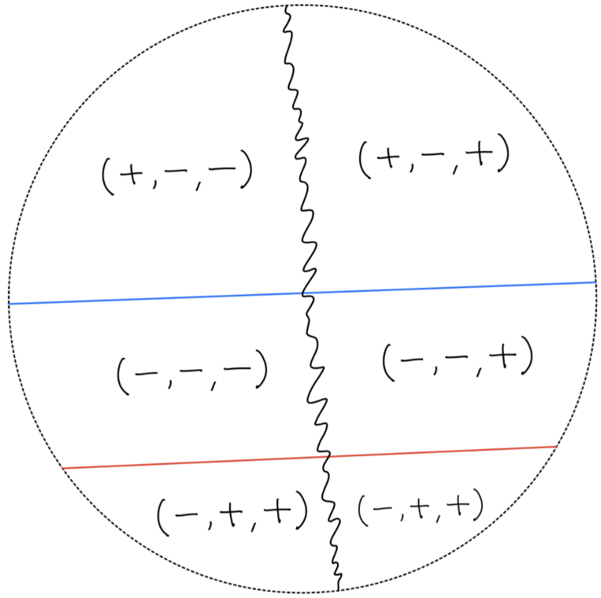

Figure 3.54

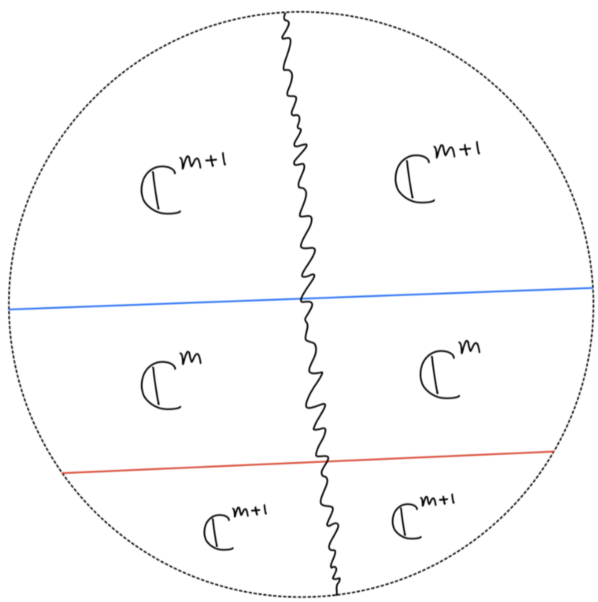

Figure 3.55

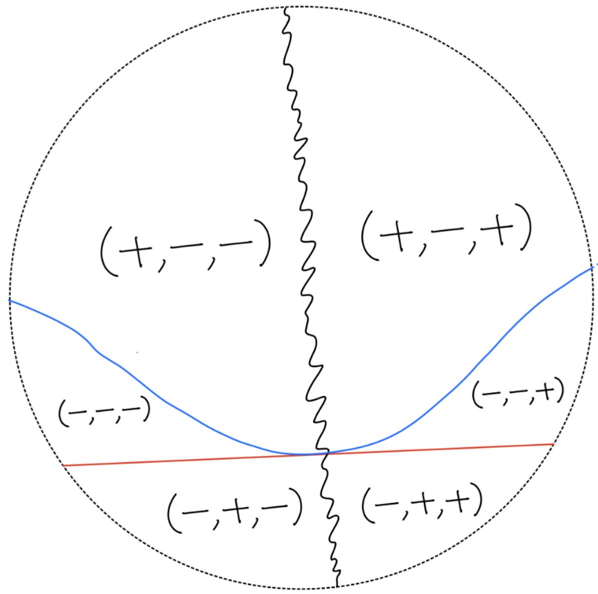

Figure 3.56

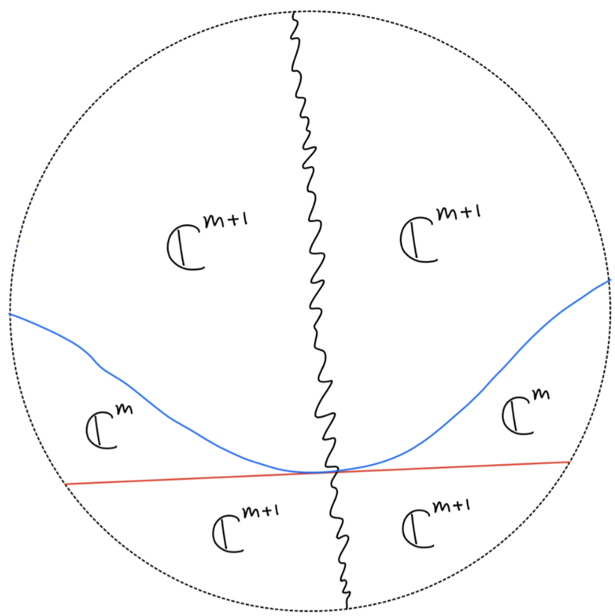

Figure 3.57

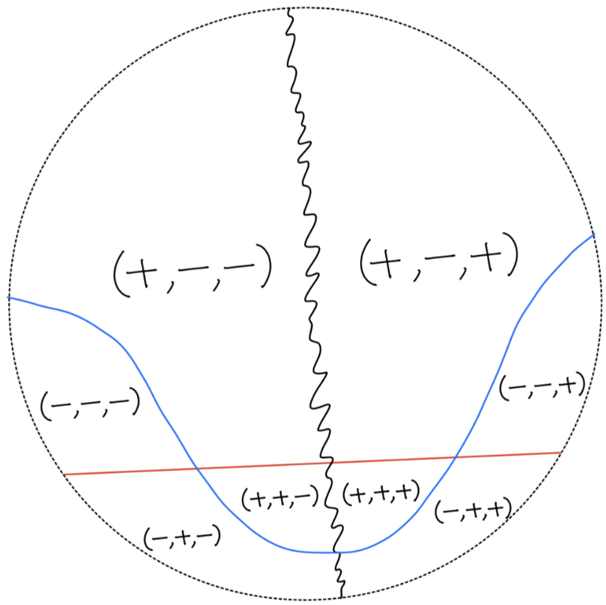

Figure 3.58

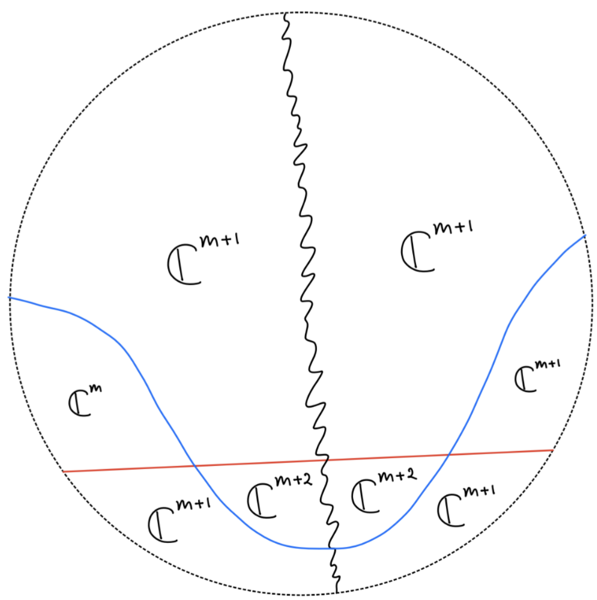

Figure 3.59

- $F_\bullet(-,-,-) := \mathbb{C}^m$
- $F_\bullet(-,-,+) := \mathbb{C}^m$
- $F_\bullet(+,-,-) := \mathbb{C}^{m+1}$

- $F_\bullet(+,-,+) := \mathbb{C}^{m+1}$
- $F_\bullet(-,+,-) := \mathbb{C}^{m+1}$
- $F_\bullet(-,+,+) := \mathbb{C}^{m+1}$
- $F_\bullet(+,+,-) := \mathbb{C}^{m+2}$
- $F_\bullet(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

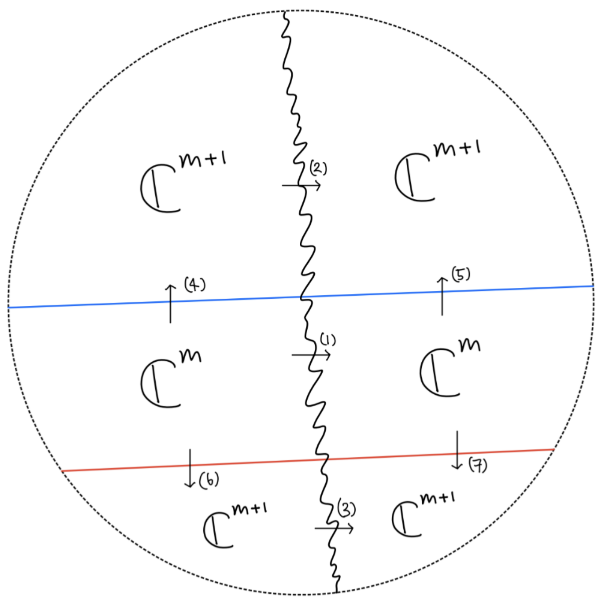

Figure 3.60

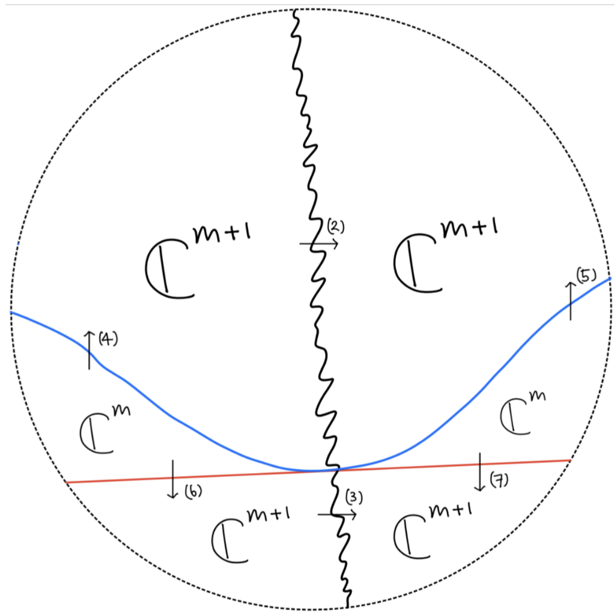

Figure 3.61

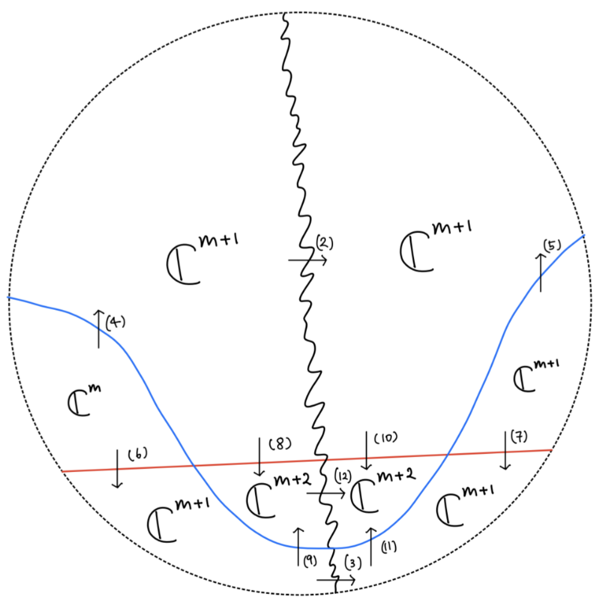

Figure 3.62

(1) $\mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$

(2) $\mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$

(4) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(5) $\mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(6) $\mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(8) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(9) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(10) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(11) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(12) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$ using the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$ where $pr_1 : (U\cap V)\times[0,1] \to (U\cap V)$ is the projection onto the first argument.

Definition 85. we define $\gamma_\bullet$ to be the composition

$$(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U\cap V)\times[0,1]} = \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd}
(U\cap V)\times[0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V\times[0,1] \arrow[d, "pr_1"] \\
(U\cap V) \arrow[r, hook] & V
\end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 86. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of M.
Note that there is a diffeomorphism beteween $D_{r=2} \times (0,1)$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}^m$
- $F^3(-,-,+) := \mathbb{C}^m$
- $F^3(+,-,-) := \mathbb{C}^{m+1}$
- $F^3(+,-,+) := \mathbb{C}^{m+1}$
- $F^3(-,+,-) := \mathbb{C}^{m+1}$
- $F^3(-,+,+) := \mathbb{C}^{m+1}$
- $F^3(+,+,-) := \mathbb{C}^{m+2}$

- $F^3(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

- $F^3(-, sgn_2, sgn_3) \to F^3(+, sgn_2, sgn_3) := \iota_1$
- $F^3(sgn_1, -, sgn_3) \to F^3(sgn_1, +, sgn_3) := \iota_0$
- $F^3(-,-,-) \to F^3(-,-,+) := T(2, m+1)$
- $F^3(+,-.-) \to F^3(+,-,+) := T(1, m+1)$
- $F^3(-,+,-) \to F^3(-,+,+) := T(2, m+2)$
- $F^3(+,+,-) \to F^3(+,+,+) := T$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

(i)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^m \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{m+1} \arrow[r, "\iota_0"] & \mathbb{C}^{m+2} \end{tikzcd}$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^m \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{m+1} \arrow[r, "\iota_0"] & \mathbb{C}^{m+2} \end{tikzcd}$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^m \arrow[r, "\mathrm{T(2,m+1)}"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{m+1} \arrow[r, "\mathrm{T(1,m+1)}"] & \mathbb{C}^{m+2} \end{tikzcd}$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{m+1} \arrow[r, "\mathrm{T(2,m+2)}"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{m+2} \arrow[r, "\mathrm{T}"] & \mathbb{C}^{m+2} \end{tikzcd}$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(v)
$$
\begin{tikzcd}
F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\
F^3(-,+,-) \arrow[r] & F^3(-,+,+)
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C}^m \arrow[r, "\mathrm{T(2,m+1)}"] \arrow[d, "\iota_0"] & \mathbb{C}^m \arrow[d, "\iota_0"] \\
\mathbb{C}^{m+1} \arrow[r, "\mathrm{T(2,m+2)}"] & \mathbb{C}^{m+1}
\end{tikzcd}
$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(vi)
$$
\begin{tikzcd}
F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\
F^3(+,+,-) \arrow[r] & F^3(+,+,+)
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{C}^{m+1} \arrow[r, "\mathrm{T(1,m+1)}"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\iota_1"] \\
\mathbb{C}^{m+2} \arrow[r, "\mathrm{T}"] & \mathbb{C}^{m+2}
\end{tikzcd}
$$
: This is a cartesian diagram, therefore, the total complex is acyclic

(vii) the cubic diagram:

$$
\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\
s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\
 & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] &
\end{tikzcd}
$$

=

$$\begin{tikzcd}
& \mathbb{C}^{m+1} \arrow[rr, "T(2{,}m+2)"] \arrow[dd, "\iota_1"] & & \mathbb{C}^{m+1} \arrow[dd, "\iota_1"] \\
\mathbb{C}^{m} \arrow[ur, "\iota_0"] \arrow[rr, "T(2{,}m+1)"] \arrow[dd, "\iota_1"'] & & \mathbb{C}^{m} \arrow[ur, "\iota_0"'] \arrow[dd] & \\
& \mathbb{C}^{m+2} \arrow[rr, "T"] & & \mathbb{C}^{m+2} \\
\mathbb{C}^{m+1} \arrow[ur, "\iota_0"] \arrow[rr, "T(1{,}m+1)"] & & \mathbb{C}^{m+1} \arrow[ur, "\iota_0"] &
\end{tikzcd}$$

Note that $T(2, m+1), T(2, m+2), T(1, m+1), T$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

□

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the $cobord_1$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_*\mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V\times\{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V\times\{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

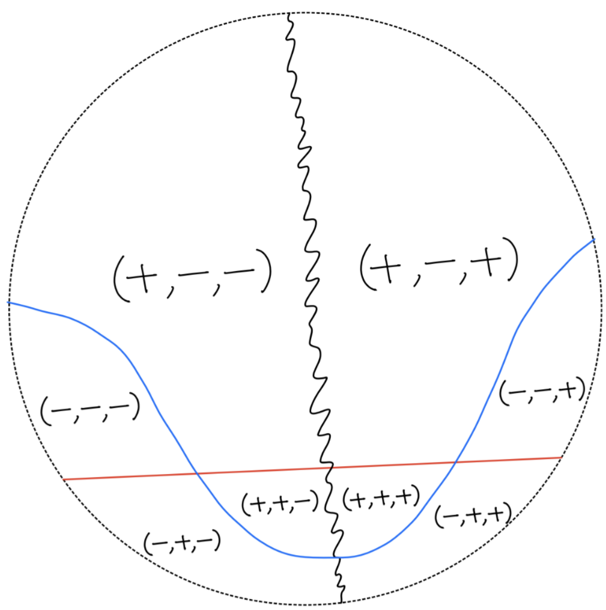

Figure 3.63

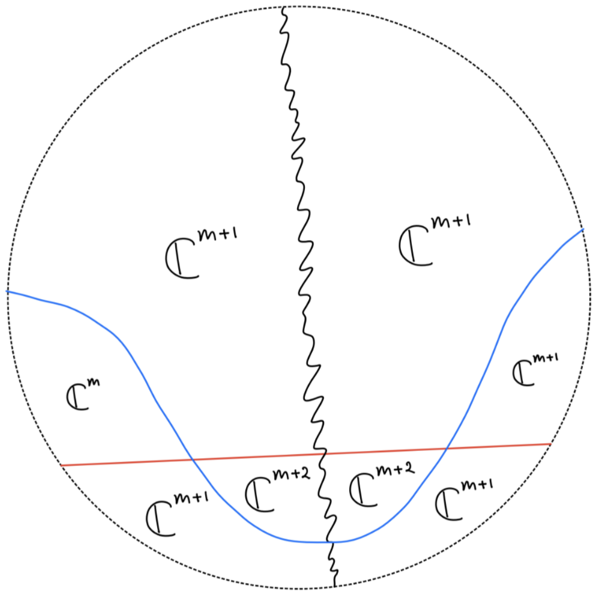

Figure 3.64

- $F_1(-,-,-) := \mathbb{C}^m$
- $F_1(-,-,+) := \mathbb{C}^m$
- $F_1(+,-,-) := \mathbb{C}^{m+1}$
- $F_1(+,-,+) := \mathbb{C}^{m+1}$
- $F_1(-,+,-) := \mathbb{C}^{m+1}$
- $F_1(-,+,+) := \mathbb{C}^{m+1}$
- $F_1(+,+,-) := \mathbb{C}^{m+2}$
- $F_1(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

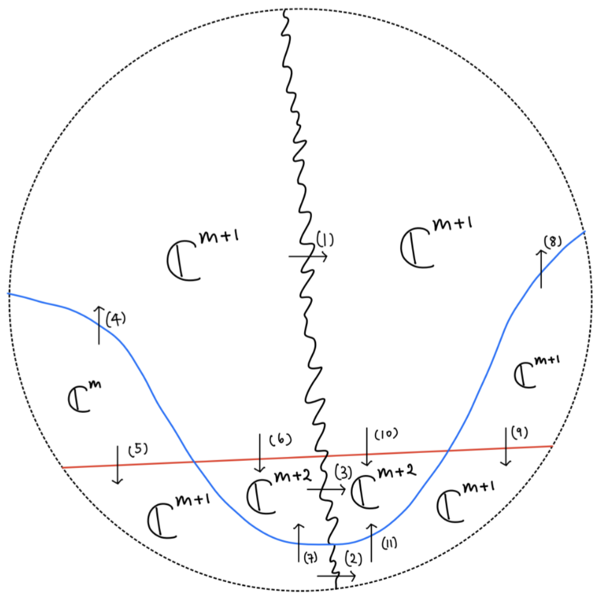

Figure 3.65

(1) $\mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

(4) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(5) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(6) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(7) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(8) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(9) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(10) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(11) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 87. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.5 2nd sheaf cobordism

In this section, we define $cobord_2$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism:

from

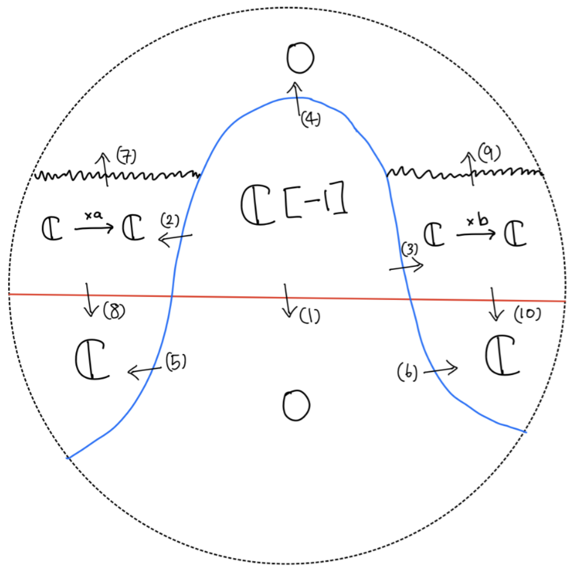

Figure 3.66

Generization maps:

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

to

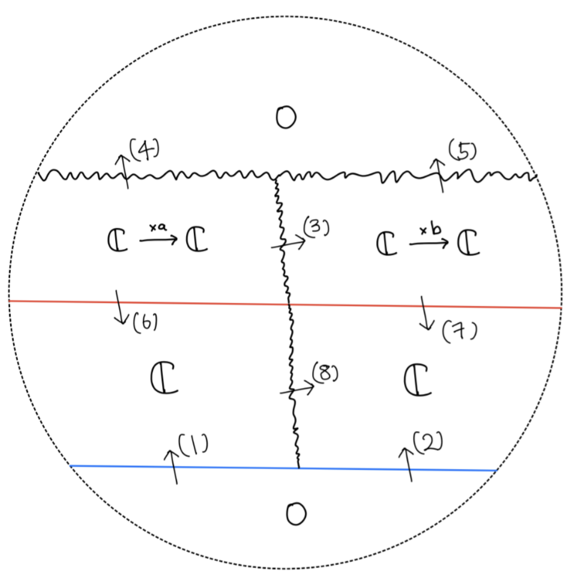

Figure 3.67

Generization maps:

(1)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(2)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(5)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(6)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

Notations

Definition 88. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 89. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$
$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$
$$\Phi_{t_0}^{squig} : [0,1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_2$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_2$.

Definition 90. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{\bullet}^{symbol}$ to be smooth maps

$$\Phi_{\bullet}^{0} : (S^1)^n \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_{\bullet}^{squig} : \coprod_{1 \le i \le k} ([0,1] \times [a_i, b_i]_t) \to M \times [0,1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_{\bullet}^{symbol}$ a co-orientation of $\Phi_{\bullet}^{symbol}$.

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as $\Lambda_\bullet^{symbol}$. Later in the section, $\Lambda_\bullet^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord_2$.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as $\Lambda_\bullet$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_\bullet$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord_2$.

Definition 91. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

Definition 92. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x,z) &\mapsto (x,z,t_0) \end{aligned}$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 93. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_0^{squig} is the union of the following two components
  (i) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  (ii) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig}: hairs are pointing downward direction i.e. coefficients of dz are negative.

Definition 94. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- λ_1^{squig} is the union of the following three components
  (i) $\{(x, \frac{1}{2}) \in D_{r=2}\}$
  (ii) $\{(0, z) \in D_{r=2} \mid -\frac{1}{2} < z < \frac{1}{2}\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig}:

- for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
- for (ii), hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 95. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda^0_\bullet := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = \Psi_t(x)\}$
- $\lambda^\infty_\bullet := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = 0\}$
- $\lambda^{squig}_\bullet$ is the union of the following two components

  (i) $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0,1] \mid \frac{1}{2} > \Psi_t(x)\}$

  (ii) $\{(0,z,t) \in D_{r=2} \times [0,1] \mid \Psi_t(0) < z < \frac{1}{2}\}$

2. We define co-orientations $\xi^{symbol}_\bullet$ of $\lambda^{symbol}_\bullet$ as follows

- $\xi^0_\bullet$: hairs are pointing downward direction i.e. coefficients of dz are negative.
- $\xi^\infty_\bullet$: hairs are pointing upward direction i.e. coefficients of dz are positive.
- $\xi^{squig}_\bullet$:
  - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
  - for (ii), hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 96. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda_\bullet^0$, $\lambda_\bullet^\infty$, and $\lambda_\bullet^{squig}$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 97. $\mathrm{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 98. For $i = 1, 2, 3, 4$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := \ & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \mathrm{sgn}(z - \Psi_{t_0}(x)) = sgn_1, \ \mathrm{sgn}(-z) = sgn_2, \\ & \mathrm{sgn}(z - \frac{1}{2}) = sgn_3, \\ & \mathrm{sgn}(x) = sgn_4\} \end{aligned}$$

2. we use $*$ as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{*}, \cdots) := s_{t_0}(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{-}, \cdots) \cup s_{t_0}(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{0}, \cdots) \cup s_{t_0}(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{+}, \cdots)$$

3. we omit trailing $*$'s e.g. $s_0(+,-) = s_0(+,-,*,*)$

4. note that we do not oit $*$'s located in between $-,0,+$ e.g. $s_0(+,-,*,-) \neq s_0(+,-,-)$

Definition 99. For $i = 0,1,2,3,4$ and $sgn_i \in \{-,0,+\}$,

1. we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := \ & \{(x,z,t) \in D_{r=2} \times [0,1] \mid \\ & \operatorname{sgn}(z - \Psi_t(x)) = sgn_1, \ \operatorname{sgn}(-z) = sgn_2, \\ & \operatorname{sgn}(z - \frac{1}{2}) = sgn_3, \\ & \operatorname{sgn}(x) = sgn_4\} \end{aligned}$$

2. we use $*$ as a wild card sign i.e.

$$s_\bullet(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{*}, \cdots) := s_\bullet(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{-}, \cdots) \cup s_\bullet(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{0}, \cdots) \cup s_\bullet(\cdots, \underset{\substack{\uparrow \\ i^{th}}}{+}, \cdots)$$

3. we omit trailing $*$'s e.g. $s_\bullet(+,-) = s_\bullet(+,-,*,*)$

4. note that we do not oit $*$'s located in between $-,0,+$ e.g. $s_\bullet(+,-,*,-) \neq s_\bullet(+,-,-)$

Definition 100. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the above notations:

1. $\mathcal{S}_0$:

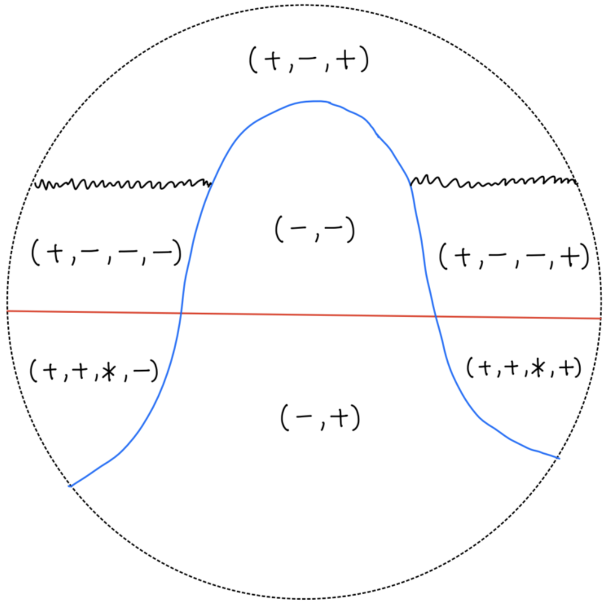

Figure 3.68

- 2 dimensional strata:

  $s_0(-,+)$,$s_0(-,-)$, $s_0(+,-,-,-)$, $s_0(+,-,-,+)$, $s_0(+,-,+)$, $s_0(+,+,*,-)$, $s_0(+,+,*,-)$

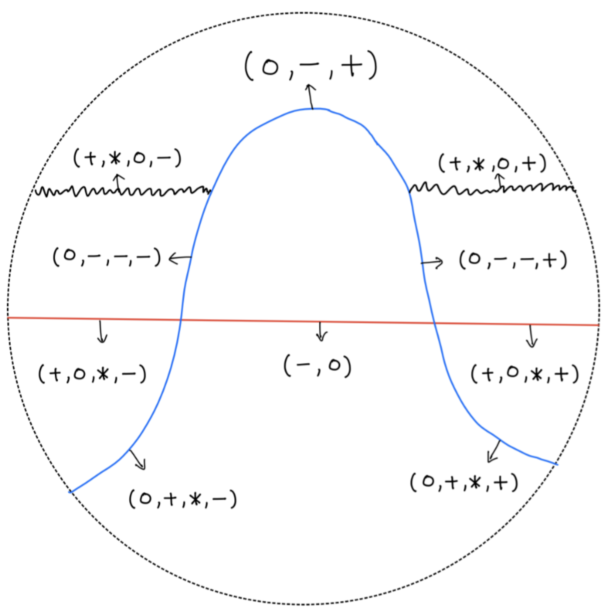

Figure 3.69

- 1 dimensional strata:

$s_0(0,+,*,-)$, $s_0(0,-,-,-)$, $s_0(0,-,+)$, $s_0(0,-,-,+)$, $s_0(0,+,*,+)$, $s_0(+,0,*,-)$, $s_0(-,0)$, $s_0(+,0,*,+)$, $s_0(+,*,0,-)$, $s_0(+,*,0,+)$

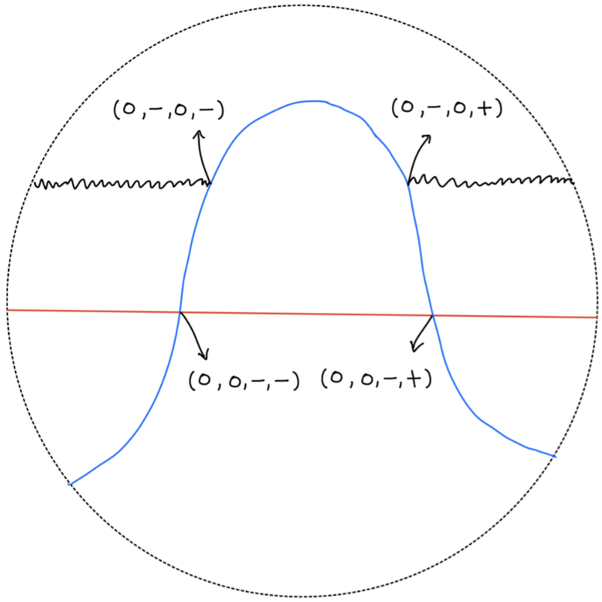

Figure 3.70

- 0 dimensional strata:

  $s_0(0,0,-,-)$, $s_0(0,-,0,-)$, $s_0(0,-,0,+)$, $s_0(0,0,-,+)$

2. $\mathcal{S}_1$:

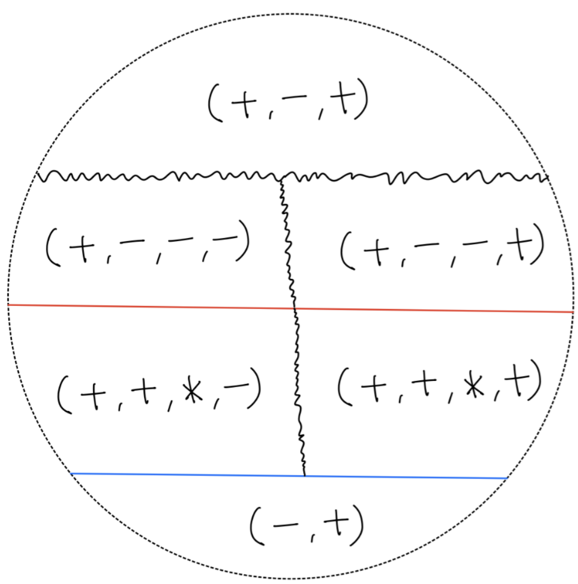

Figure 3.71

- 2 dimensional strata:

  $s_1(-,+)$, $s_1(+,-,-,-)$, $s_1(+,-,-,+)$, $s_1(+,-,+)$, $s_1(+,+,*,-)$, $s_1(+,+,*,+)$

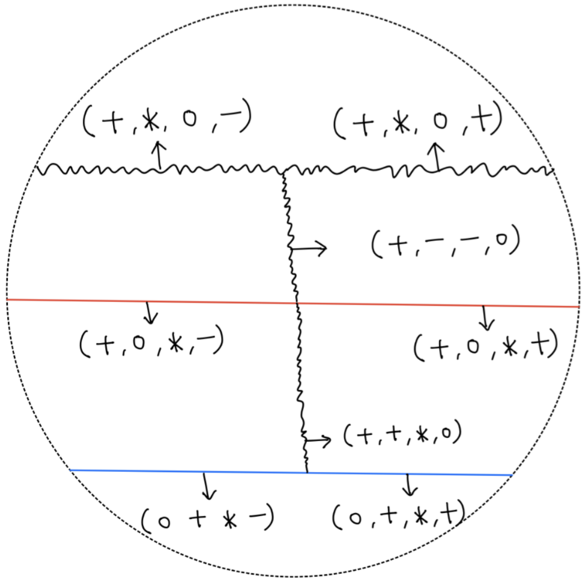

Figure 3.72

- 1 dimensional strata:

  $s_1(0,+,*,-)$, $s_1(0,+,*,+)$, $s_1(+,0,*,-)$, $s_1(+,0,*,+)$, $s_1(+,*,0,-)$, $s_1(+,*,0,+)$, $s_1(+,-,-,0)$, $s_1(+,+,*,0)$

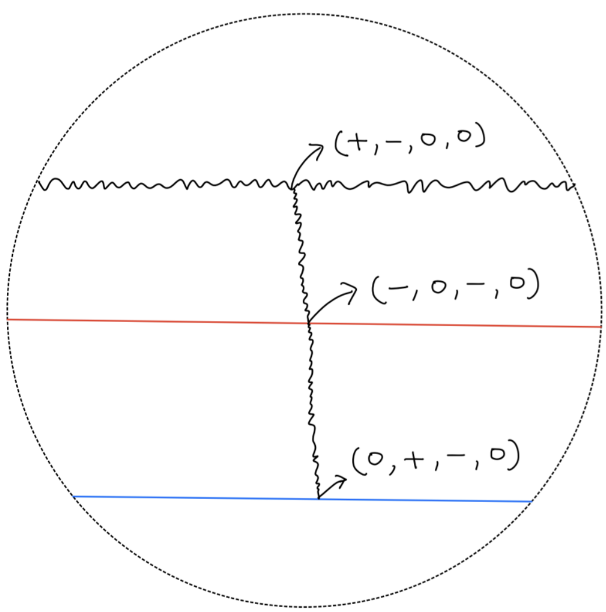

Figure 3.73

- 0 dimensional strata:

  $s_1(+,-,0,0)$, $s_1(-,0,-,0)$, $s_1(0,+,-,0)$

3. $\mathcal{S}_\bullet$:

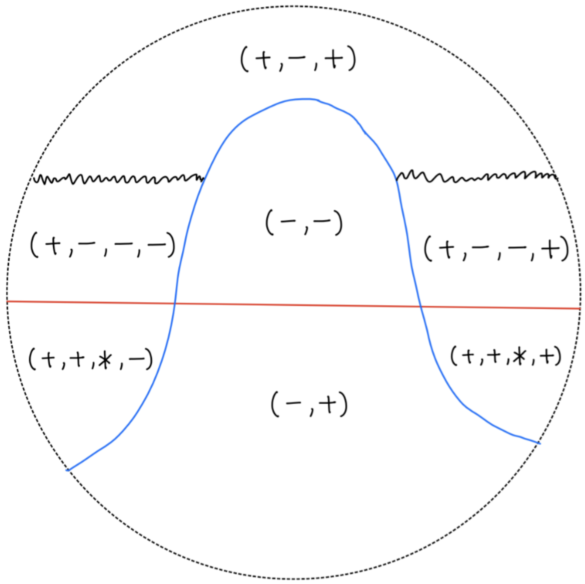

Figure 3.74

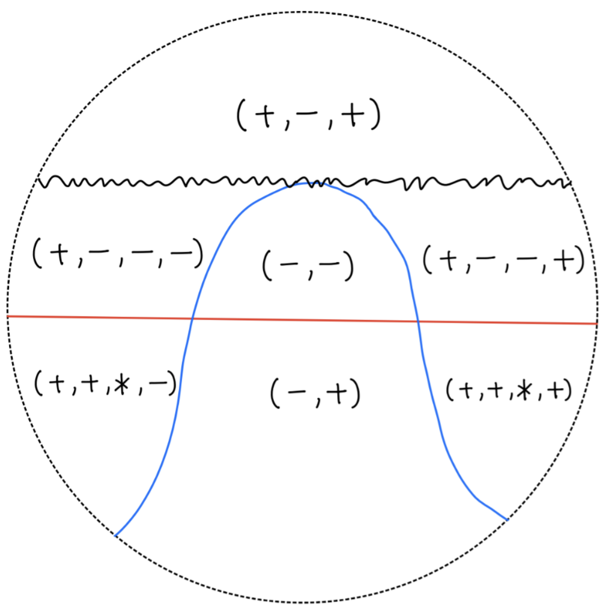

Figure 3.75

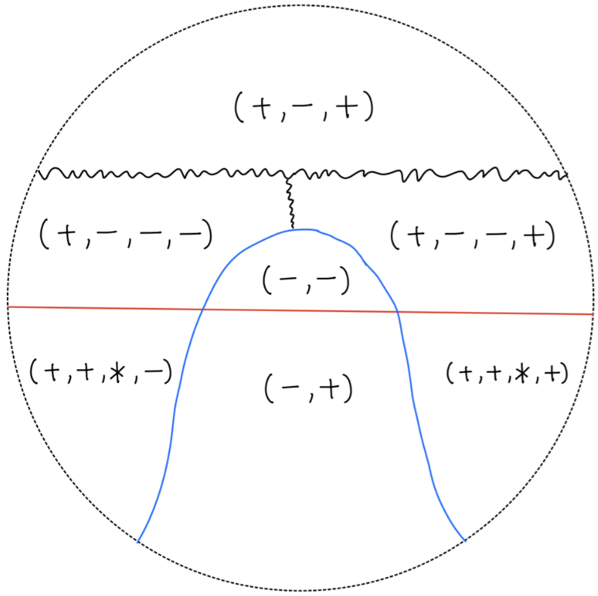

Figure 3.76

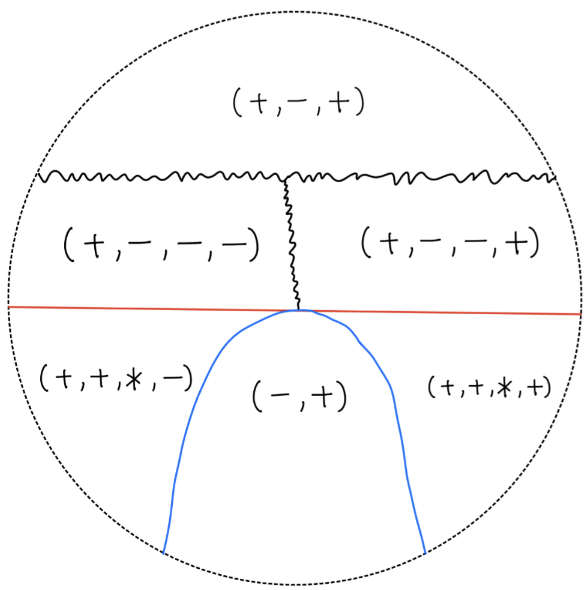

Figure 3.77

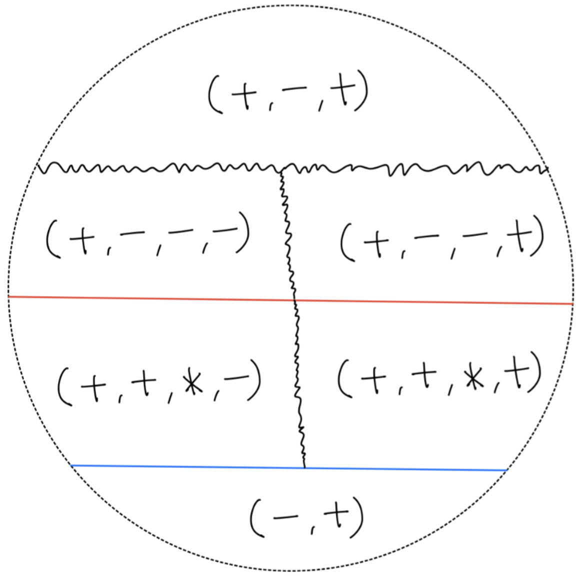

Figure 3.78

- 3 dimensional strata:

  $s_\bullet(-,+),\ s_\bullet(-,-),\ s_\bullet(-,+),\ s_\bullet(+,-,-,-),\ s_\bullet(+,-,-,+),\ s_\bullet(+,-,+),$
  $s_\bullet(+,+,*,-),\ s_\bullet(+,+,*,+)$

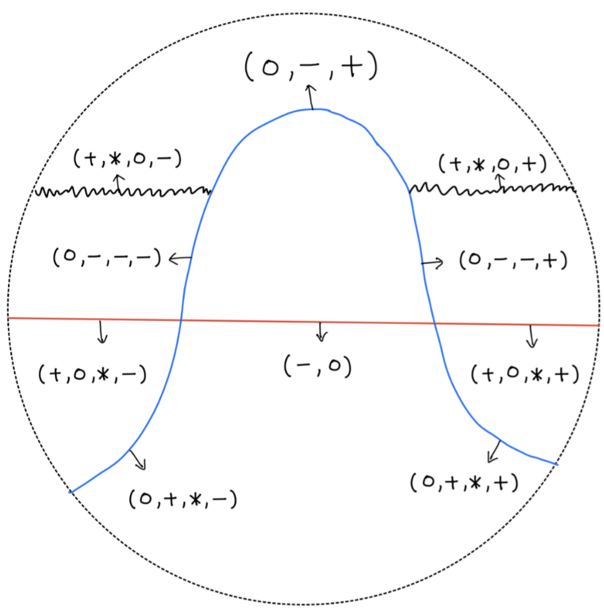

Figure 3.79

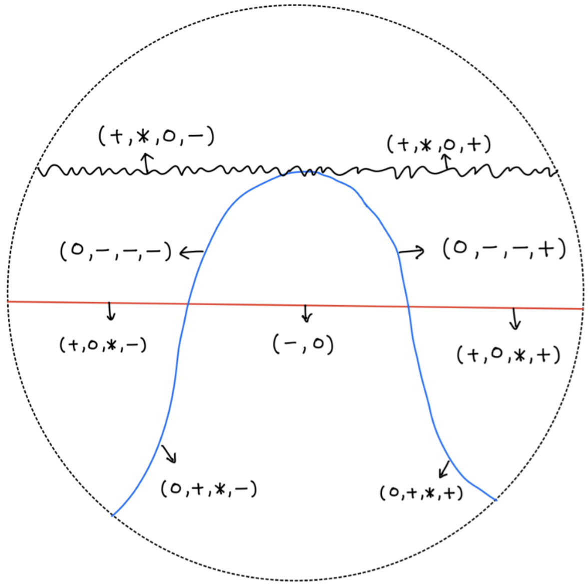

Figure 3.80

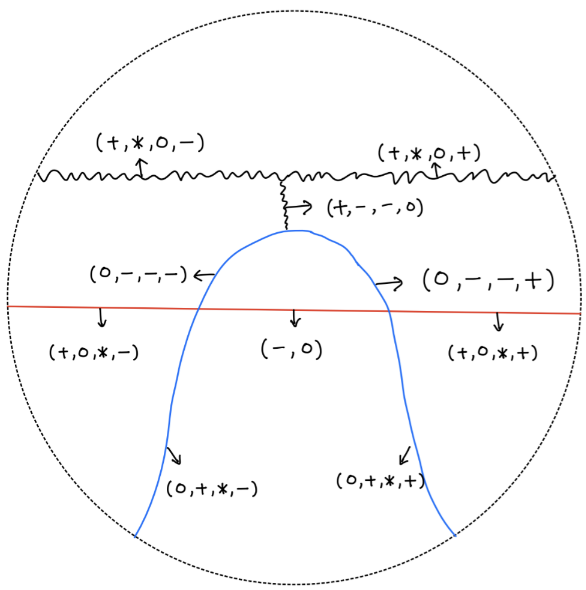

Figure 3.81

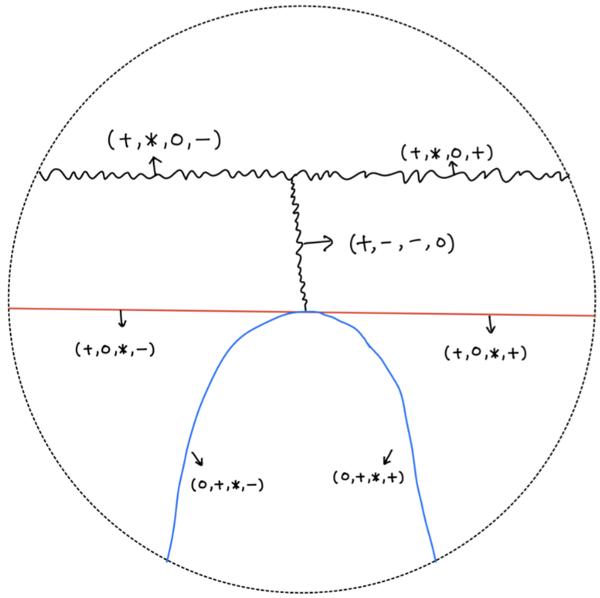

Figure 3.82

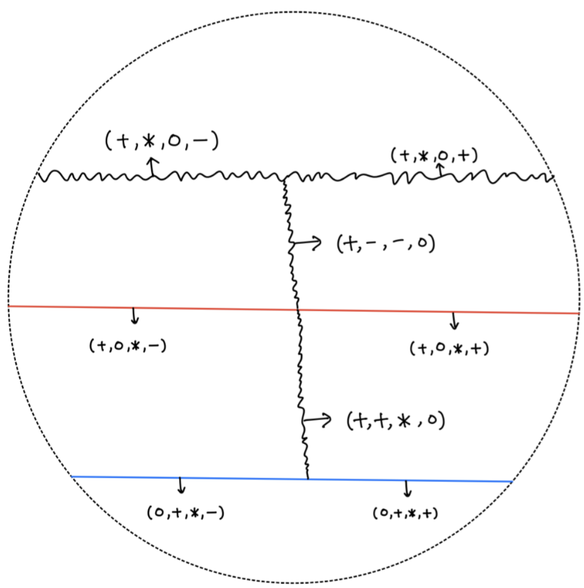

Figure 3.83

- 2 dimensional strata:

  $s_\bullet(0,+,*,-)$, $s_\bullet(0,-,-,-)$, $s_\bullet(0,-,+)$, $s_\bullet(0,-,-,+)$, $s_\bullet(0,+,*,+)$, $s_\bullet(+,0,*,-)$, $s_\bullet(-,0)$, $s_\bullet(+,0,*,+)$, $s_\bullet(+,*,0,-)$, $s_\bullet(+,*,0,+)$, $s_\bullet(+,-,-,0)$, $s_\bullet(+,+,*,0)$

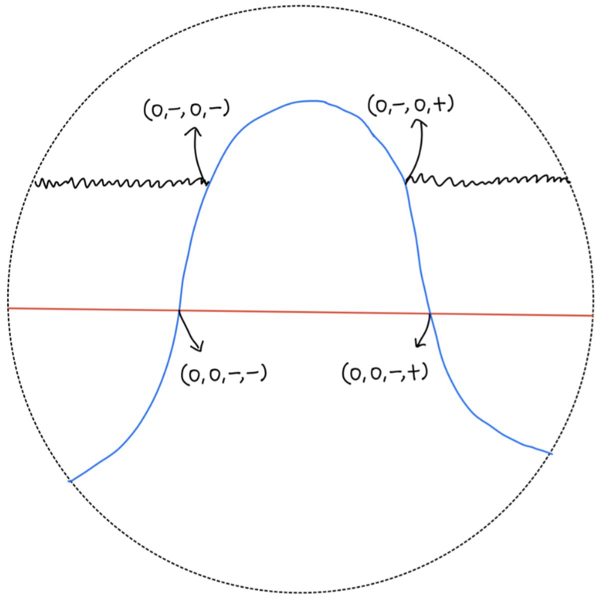

Figure 3.84

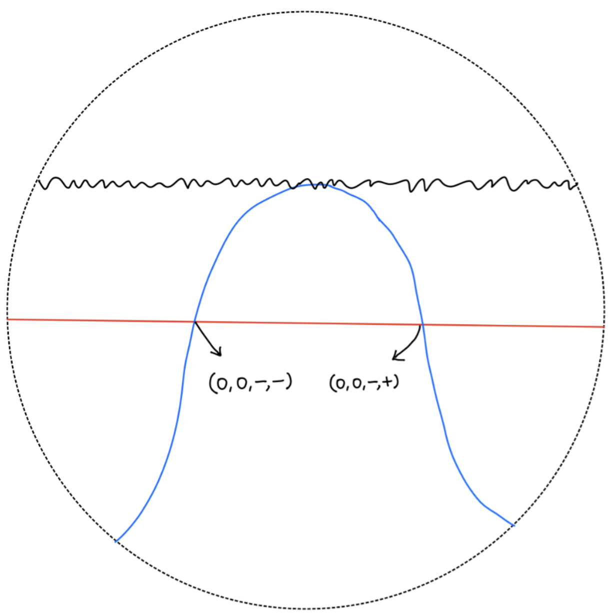

Figure 3.85

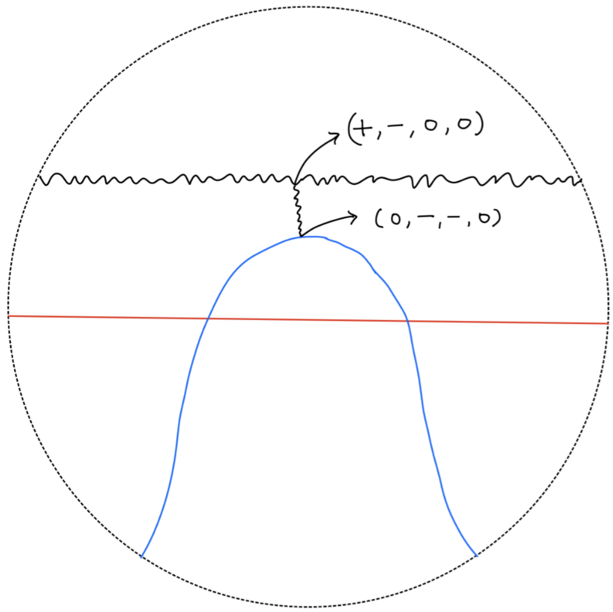

Figure 3.86

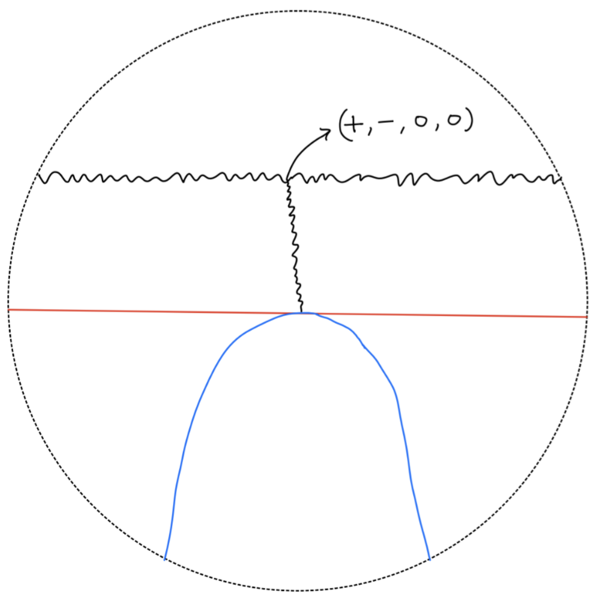

Figure 3.87

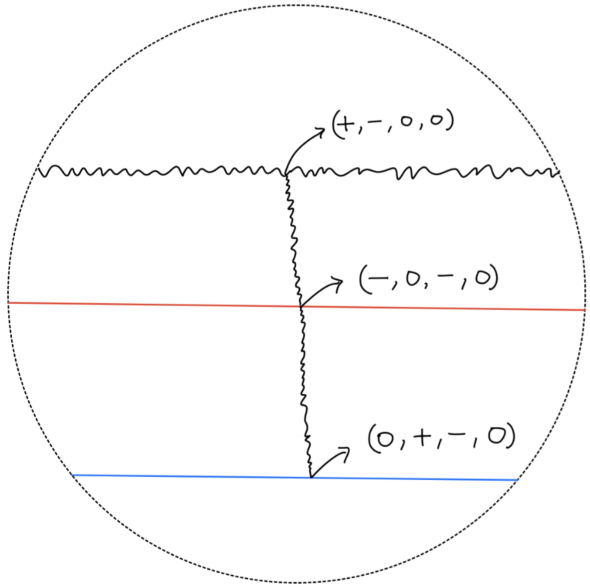

Figure 3.88

- 1 dimensional strata:

  $s_\bullet(0,0,-,-)$, $s_\bullet(0,-,0,-)$, $s_\bullet(0,-,0,+)$, $s_\bullet(0,0,-,+)$, $s_\bullet(+,-,0,0)$, $s_\bullet(0,-,-,0)$, $s_\bullet(-,0,-,0)$, $s_\bullet(0,+,-,0)$

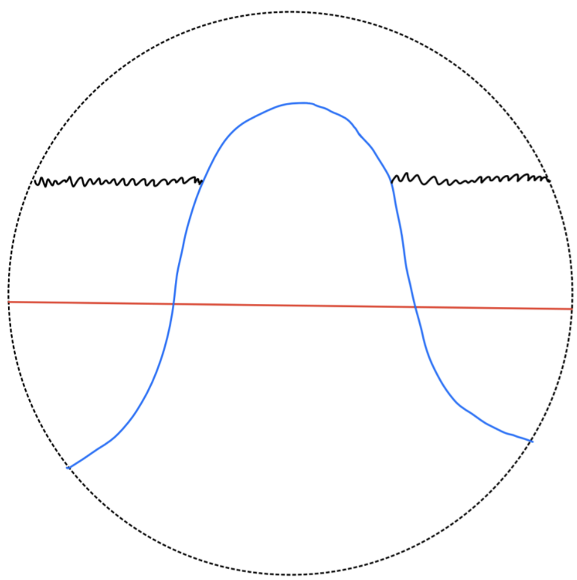

Figure 3.89

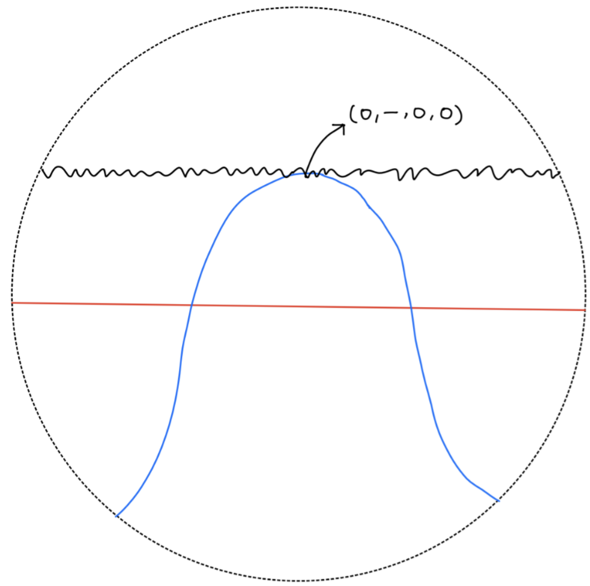

Figure 3.90

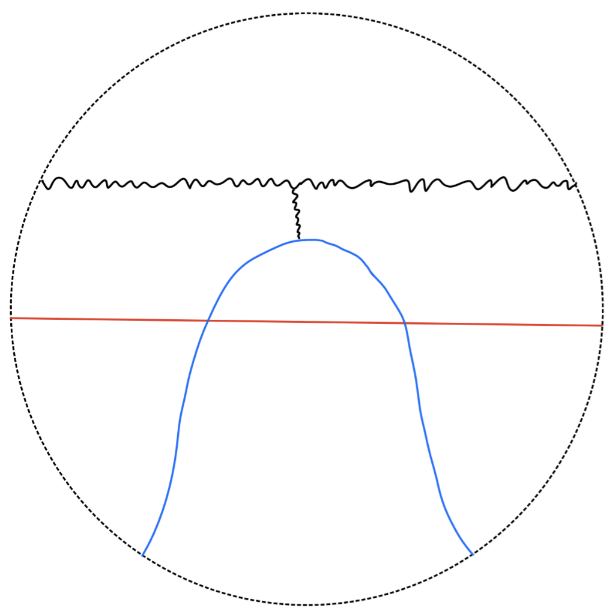

Figure 3.91

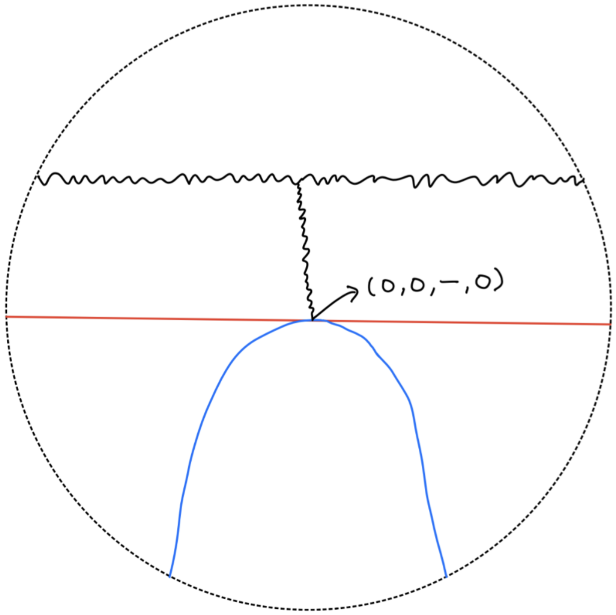

Figure 3.92

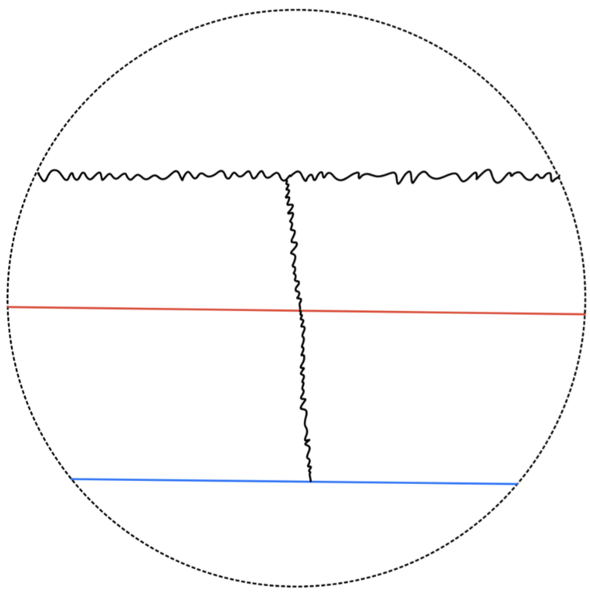

Figure 3.93

- 0 dimensional strata:

  $s_\bullet(0,-,0,0)$, $s_\bullet(0,0,-,0)$

Definition 101. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 102. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of s.

Definition 103. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 104. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 105. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \operatorname{codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 106. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \text{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$, then

$$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

  This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.
- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

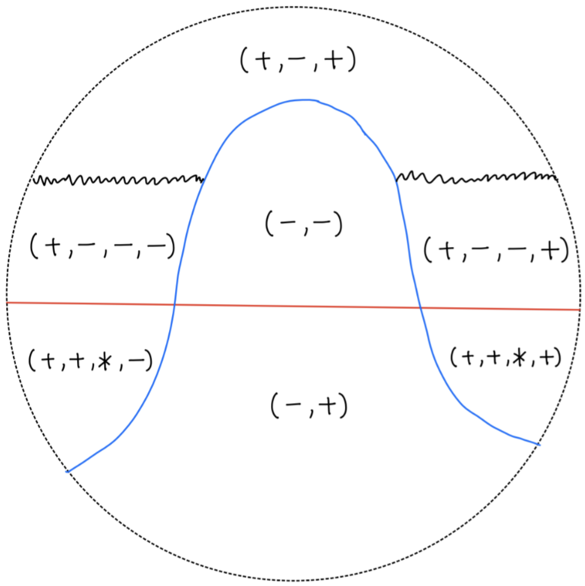

Figure 3.94

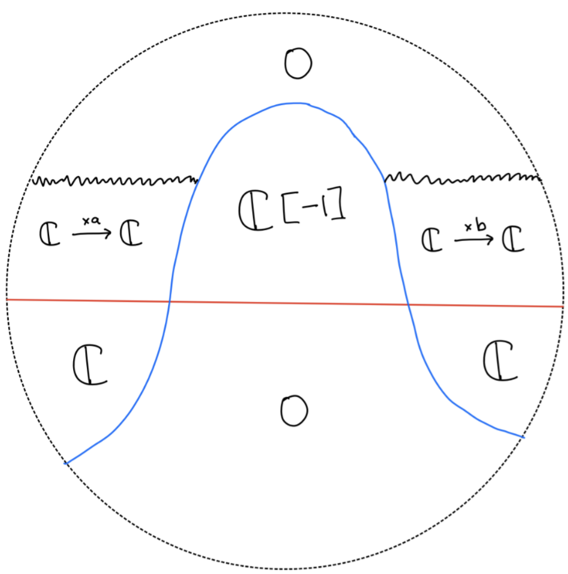

Figure 3.95

- $F_0(-,-) := \mathbb{C}[-1]$
- $F_0(-,+) := 0$
- $F_0(+,-,-,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(+,-,-,+) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_0(+,-,+) := 0$
- $F_0(+,+,*,-) := \mathbb{C}$
- $F_0(+,+,*,+) := \mathbb{C}$

Generization maps:

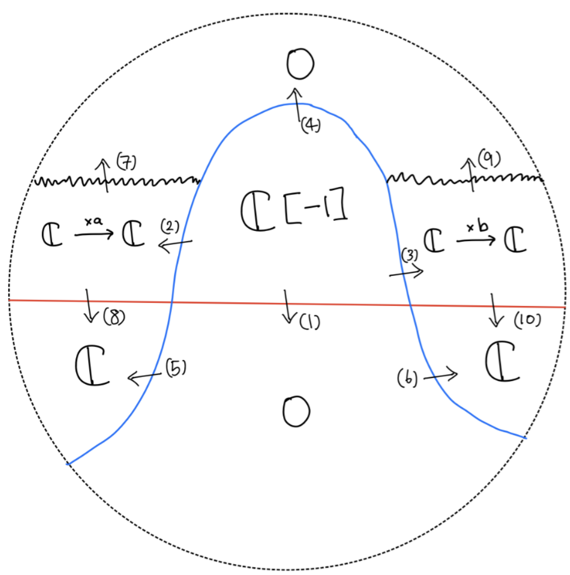

Figure 3.96

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\
\uparrow & & \uparrow{\scriptstyle \times b} \\
0 & \longrightarrow & \mathbb{C}
\end{array} \tag{3}
$$

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array} \tag{4}
$$

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \mathbb{C}
\end{array} \tag{5}
$$

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \mathbb{C}
\end{array} \tag{6}
$$

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & 0 \\
{\scriptstyle \times a}\uparrow & & \uparrow \\
\mathbb{C} & \longrightarrow & 0
\end{array} \tag{7}
$$

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & 0 \\
{\scriptstyle \times a}\uparrow & & \uparrow \\
\mathbb{C} & \xrightarrow{\times 1} & \mathbb{C}
\end{array} \tag{8}
$$

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & 0 \\
{\scriptstyle \times b}\uparrow & & \uparrow \\
\mathbb{C} & \longrightarrow & 0
\end{array} \tag{9}
$$

$$
\begin{array}{ccc}
\mathbb{C} & \longrightarrow & 0 \\
{\scriptstyle \times b}\uparrow & & \uparrow \\
\mathbb{C} & \xrightarrow{\times 1} & \mathbb{C}
\end{array} \tag{10}
$$

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord_2$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

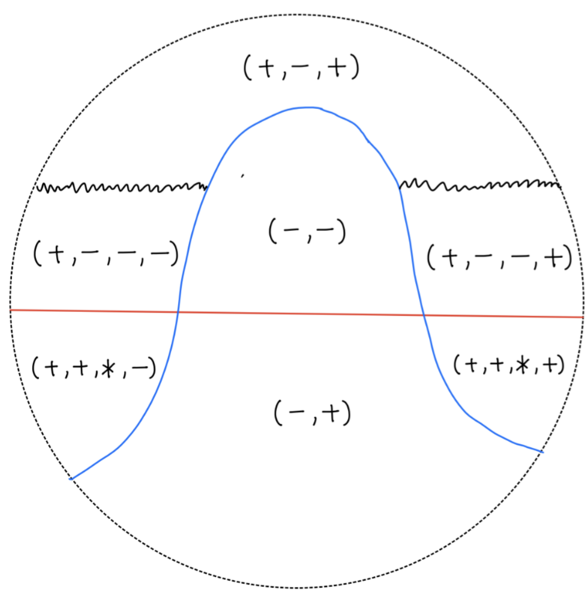

Figure 3.97

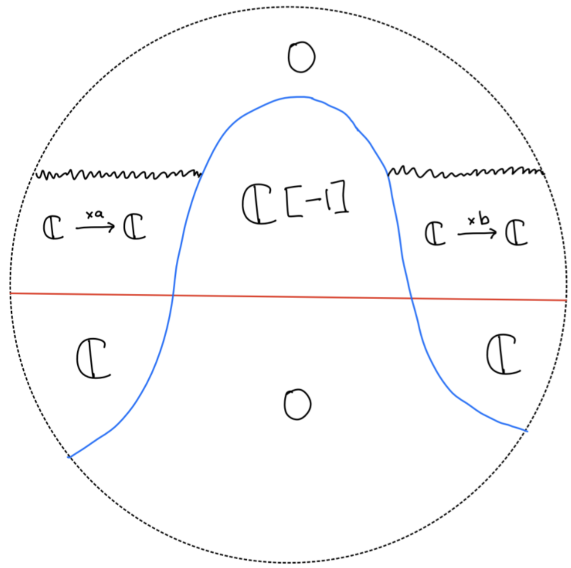

Figure 3.98

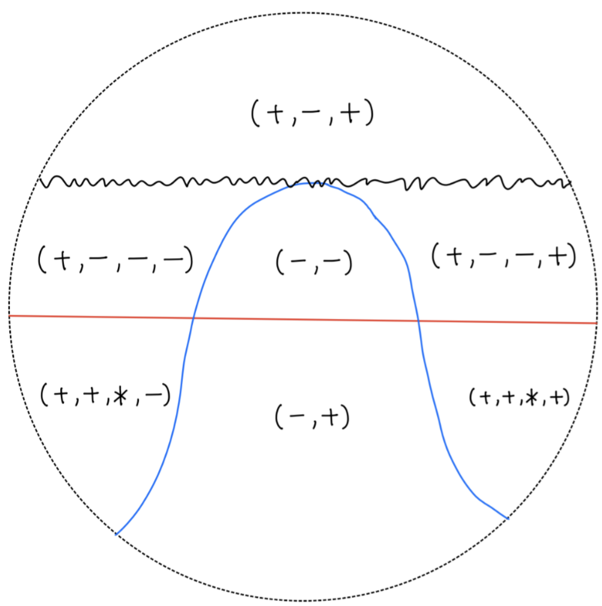

Figure 3.99

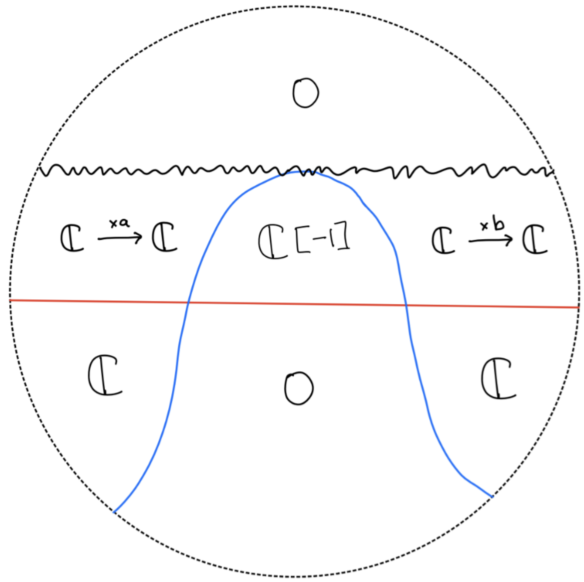

Figure 3.100

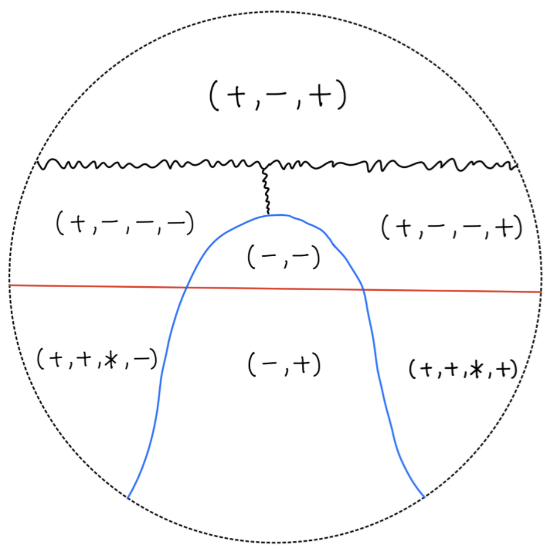

Figure 3.101

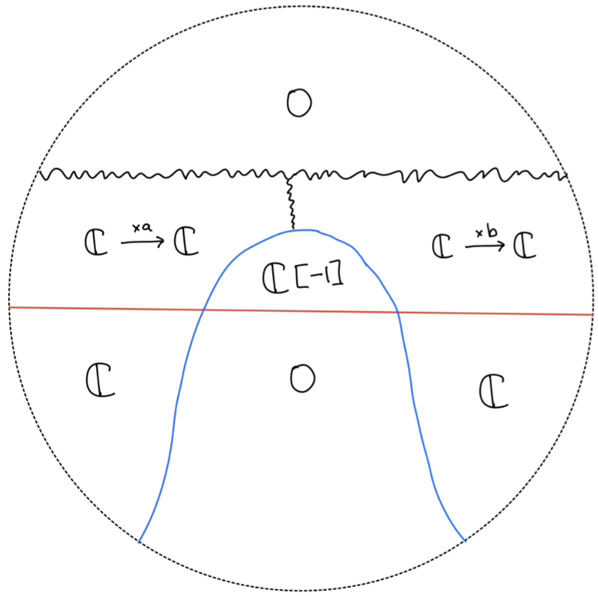

Figure 3.102

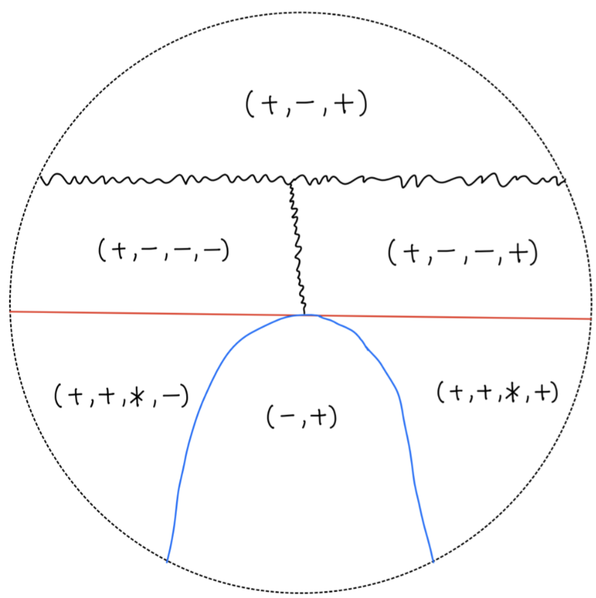

Figure 3.103

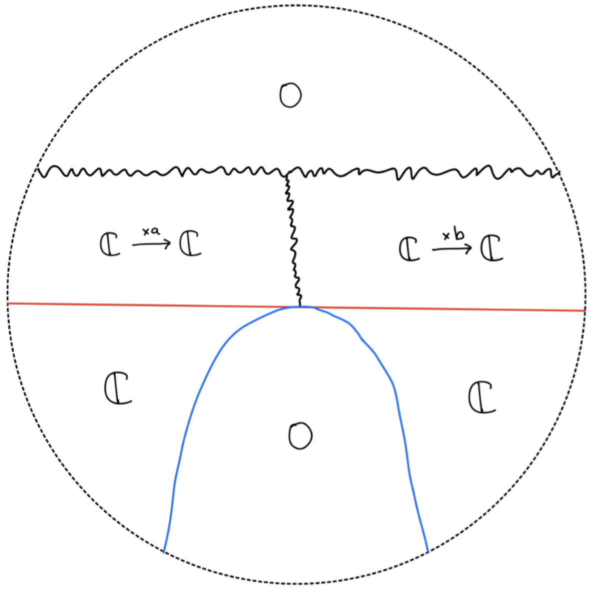

Figure 3.104

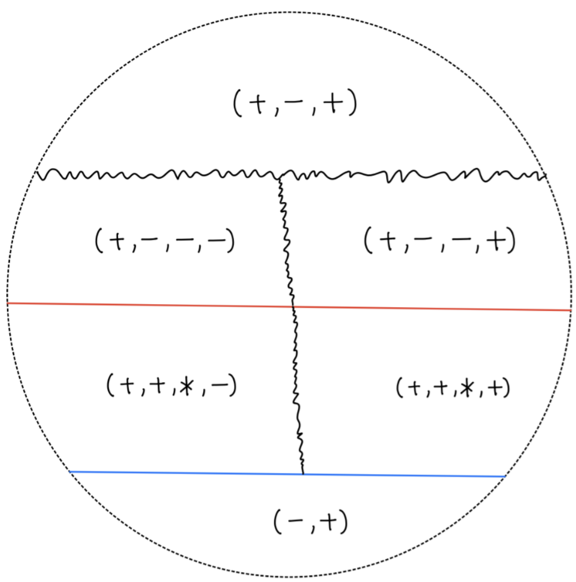

Figure 3.105

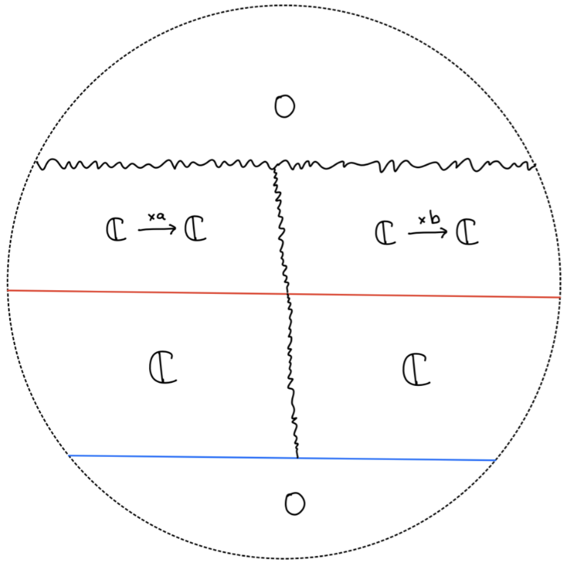

Figure 3.106

- $F_\bullet(-,-) := \mathbb{C}[-1]$
- $F_\bullet(-,+) := 0$
- $F_\bullet(+,-,-,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_\bullet(+,-,-,+) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_\bullet(+,-,+) := 0$
- $F_\bullet(+,+,*,-) := \mathbb{C}$
- $F_\bullet(+,+,*,+) := \mathbb{C}$

Generization maps:

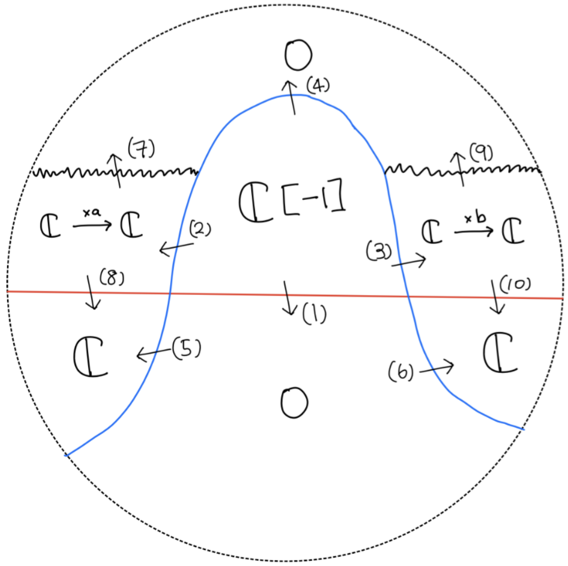

Figure 3.107

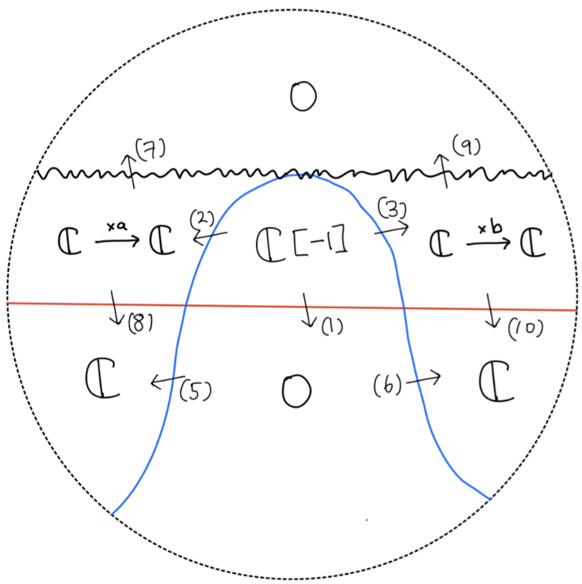

Figure 3.108

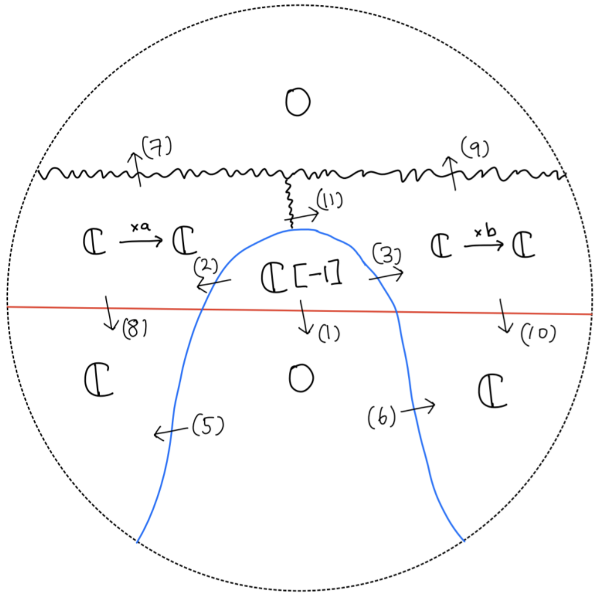

Figure 3.109

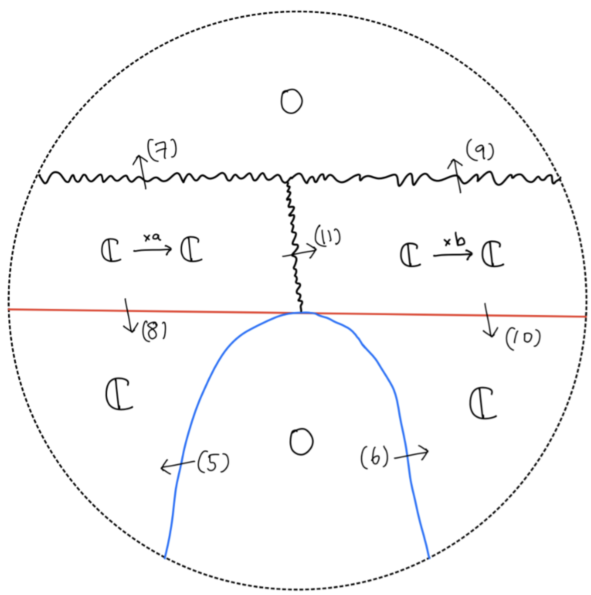

Figure 3.110

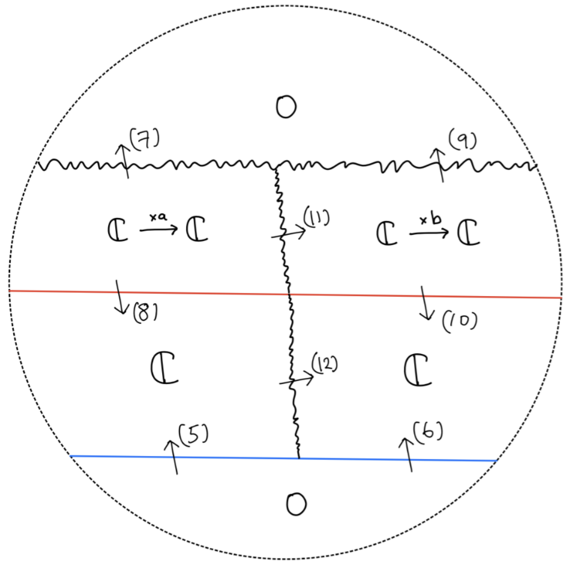

Figure 3.111

(1) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(2) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(3) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(5) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(6) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(9)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(10)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(11)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(12)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$ using the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$ where $pr_1 : (U\cap V)\times[0,1] \to (U\cap V)$ is the projection onto the first argument.

Definition 107. we define $\gamma_\bullet$ to be the composition

$$(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U\cap V)\times[0,1]} = \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times [0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V \times [0,1] \arrow[d, "pr_1"] \\
(U \cap V) \arrow[r, hook] & V
\end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 108. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M \times [0,1], \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of $M \times [0,1]$.
Since $\mathscr{F}^\bullet$ is constant along the time coordinate on $U'^c \times [0,1]$, it is enough to check for the points of $U \times [0,1] \cong D_{r=2} \times [0,1]$. Now consider the following open cover of $D_{r=2} \times [0,1]$

$$\{\mathrm{star}(s_\bullet(0,-,0,0)), \mathrm{star}(s_\bullet(0,0,-,0))\}$$

(1) First, let's show that the microlocal stalks of $\mathscr{F}^\bullet|_{\mathrm{star}(s_\bullet(0,-,0,0))}$ vanishes. Note that there is a diffeomorphism beteween $\mathrm{star}(s_\bullet(0,-,0,0))$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}[-1]$
- $F^3(-,-,+) := \mathbb{C}[-1]$
- $F^3(+,-,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+,-,+) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-,+,-) := \mathbb{C}[-1]$
- $F^3(-,+,+) := \mathbb{C}[-1]$
- $F^3(+,+,-) := 0$
- $F^3(+,+,+) := 0$

Generization maps:

$$\begin{tikzcd}
& s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
& s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}$$

(1) $$\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

(2) $$\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

$$(3)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

$$(4)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(11)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(12)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$\text{(i)}\quad \begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

$$\text{(ii)}\quad \begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

$$\text{(iii)}\quad \begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \end{tikzcd}$$

$$\text{(iv)}\quad \begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

$$\text{(v)}\quad \begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C}[-1] \arrow[r] & \mathbb{C}[-1] \end{tikzcd}$$

$$\text{(vi)}\quad \begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

(vii) the cubic diagram:

$$\begin{tikzcd}
& s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)" near start] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\
s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)" near start] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)" near start] & \\
& s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] &
\end{tikzcd}$$

$=$

$$\begin{tikzcd}
& \mathbb{C}[-1] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[dd] \\
\mathbb{C}[-1] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[ur] \arrow[dd] & \\
& 0 \arrow[rr] & & 0 \\
\mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[ur] \arrow[rr] & & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

For i-vi, horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For vii, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of $\mathscr{F}^\bullet|_{\mathrm{star}(s_\bullet(0,0,-,0))}$ vanishes. Note

that there is a diffeomorphism beteween star($s_\bullet(0,0,-,0)$) and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}[-1]$
- $F^3(-,-,+) := \mathbb{C}[-1]$
- $F^3(+,-,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+,-,+) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-,+,-) := 0$
- $F^3(-,+,+) := 0$
- $F^3(+,+,-) := \mathbb{C}$
- $F^3(+,+,+) := \mathbb{C}$

Generization maps:

$$
\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
 & s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}
$$

$$
(1) \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

$$
(2) \quad
\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

$$
(3) \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"']
\end{tikzcd}
$$

$$
(4) \quad
\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

$$
(5) \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"']
\end{tikzcd}
$$

$$
(6) \quad
\begin{tikzcd}
0 \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

(7)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(9)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(10)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "ab^{-1}"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(11)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(12)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

(i)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \end{tikzcd}$$

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \arrow[r, "ab^{-1}"] & \mathbb{C} \end{tikzcd}$$

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r, "(\times ab^{-1}, \times 1)"] \arrow[d] & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(vii) the cubic diagram:

$$\begin{tikzcd} & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\ s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\ & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\ s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] & \end{tikzcd}$$

=

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
\mathbb{C}[-1] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[ur] \arrow[dd] & \\
 & \mathbb{C} \arrow[rr] & & \mathbb{C} \\
\mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[ur] \arrow[rr] & & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

For iii-vi, horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For iand ii, straightforward calculation shows that the total complexes are acyclic.

For vii, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u, "\times b"] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[r, "\times 1"] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u] \end{tikzcd}$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete. □

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the $cobord_2$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_*\mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V \times \{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

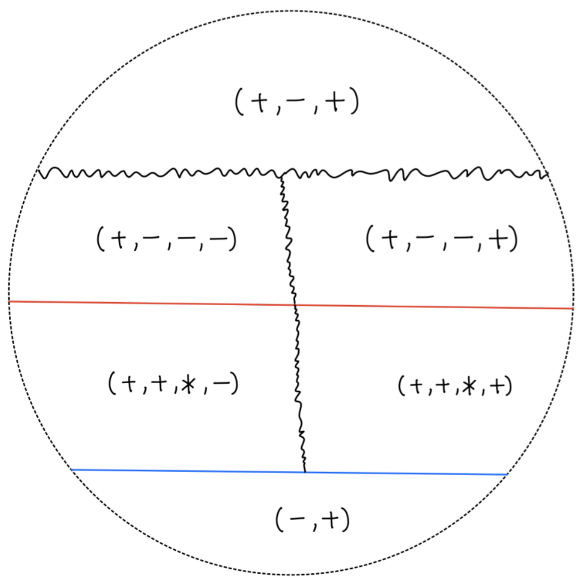

Figure 3.112

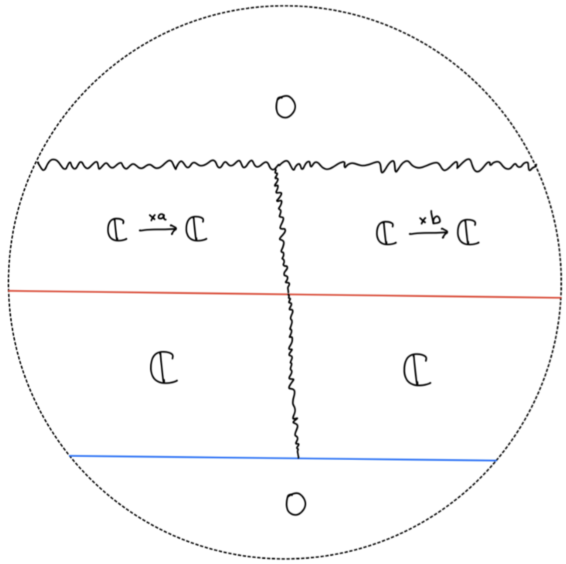

Figure 3.113

- $F_1(-,+) := 0$
- $F_1(+,-,-,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_1(+,-,-,+) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_1(+,-,+) := 0$
- $F_1(+,+,*,-) := \mathbb{C}$
- $F_1(+,+,*,+) := \mathbb{C}$

Generization maps:

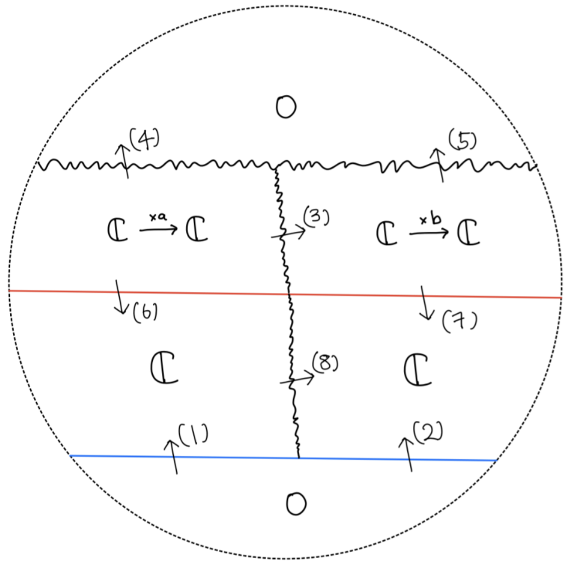

Figure 3.114

$$(1)\quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2)\quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3)\quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ {\scriptstyle \times a}\uparrow & & {\scriptstyle \times b}\uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(4)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times a}\uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times b}\uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times a}\uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\times ab^{-1}"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 109. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.6 2nd sheaf cobordism’

In this section, we define $cobord_2'$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

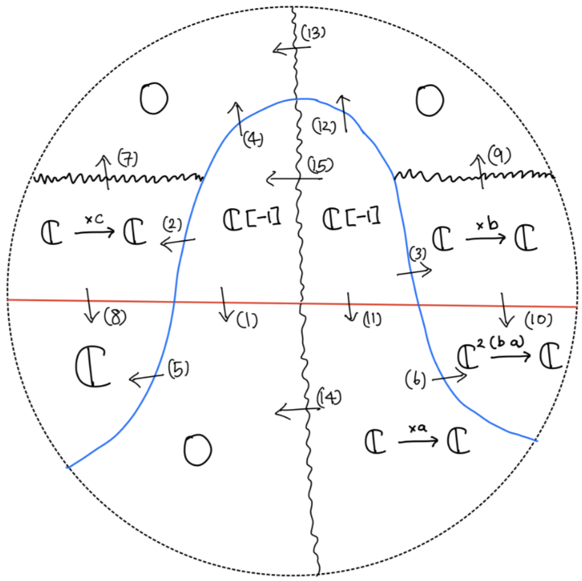

Figure 3.115

Generization maps:

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[r, "\iota_1"] \arrow[u, "\times a"] & \mathbb{C}^2 \arrow[u, "(b\ a)"'] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r] \arrow[u, "\times c"] & 0 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r, "\times 1"] \arrow[u, "\times c"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r] \arrow[u, "\times b"] & 0 \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[r, "\iota_0"] \arrow[u, "\times b"] & \mathbb{C}^2 \arrow[u, "(b\ a)"'] \end{tikzcd}$$

$$(11)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

$$(12)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

$$(13)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

$$(14)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r] \arrow[u, "\times a"] & 0 \arrow[u] \end{tikzcd}$$

(15)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

to

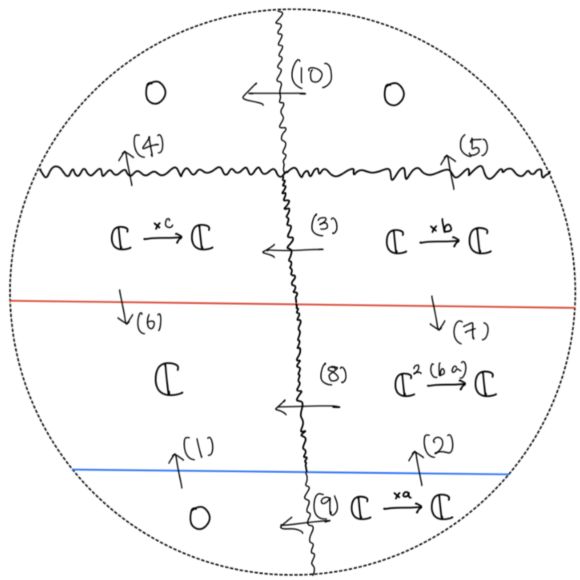

Figure 3.116

Generization maps:

(1)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times b}\uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times c}\uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7)\quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ {\scriptstyle \times b}\uparrow & & \uparrow{\scriptstyle (b\ \ a)} \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(8)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle (b\ \ a)}\uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1}\ \ 0)} & \mathbb{C} \end{array}$$

$$(9)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle \times a}\uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

Notations

Definition 110. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 111. For $t_0 \in \{0,1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$
$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$
$$\Phi_{t_0}^{squig} : [0,1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_2'$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_2'$.

Definition 112. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{\bullet}^{symbol}$ to be smooth maps

$$\begin{aligned}
&\Phi_{\bullet}^{0} : (S^1)^n \times [0,1]_t \to M \times [0,1]_t \\
&\Phi_{\bullet}^{\infty} : (S^1)^m \times [0,1]_t \to M \times [0,1]_t \\
&\Phi_{\bullet}^{squig} : \coprod_{1 \le i \le k} ([0,1] \times [a_i, b_i]_t) \to M \times [0,1]_t
\end{aligned}$$

   where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_{\bullet}^{symbol}$ a co-orientation of $\Phi_{\bullet}^{symbol}$.

3. we denote the pair $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$ as $\Lambda_{\bullet}^{symbol}$. Later in the section, $\Lambda_{\bullet}^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord_2'$.

4. we denote the triple $(\Lambda_{\bullet}^{0}, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$ as $\Lambda_{\bullet}$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_{\bullet}$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord_2'$.

Definition 113. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

Definition 114. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$
$$(x,z) \mapsto (x,z,t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 115. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x,z) \in D_{r=2} \mid z = 0\}$
- λ_0^{squig} is the union of the following three components
  - (i) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  - (ii) $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  - (iii) $\{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig}: the hairs of the components i, ii are pointing downward direction i.e. coefficients of dz are negative and the hairs of the component iii are pointing towards the right.

Definition 116. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x,z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x,z) \in D_{r=2} \mid z = 0\}$
- λ_1^{squig} is the union of the following three components

  (i) $\{(x, \frac{1}{2}) \in D_{r=2}\}$

  (ii) $\{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig}:
  - for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
  - for (ii), hairs are pointing towards the right i.e. coefficients of dx are positive.

Definition 117. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda_\bullet^0 := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = 0\}$
- $\lambda_\bullet^{squig}$ is the union of the following two components
  (i) $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0,1] \mid \frac{1}{2} > \Psi_t(x)\}$
  (ii) $\{(0, z, t) \in D_{r=2} \times [0,1] \mid x = 0\}$

2. We define co-orientations $\xi_\bullet^{symbol}$ of $\lambda_\bullet^{symbol}$ as follows
   - $\xi_\bullet^0$: hairs are pointing downward direction i.e. coefficients of dz are negative.
   - $\xi_\bullet^\infty$: hairs are pointing upward direction i.e. coefficients of dz are positive.
   - $\xi_\bullet^{squig}$:
     – for (i), hairs are pointing downward direction i.e. coefficients of dz are negative.
     – for (ii), hairs are pointing towards the right i.e. coefficients of dx are positive.

Definition 118. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the

connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda_\bullet^0$, $\lambda_\bullet^\infty$, and $\lambda_\bullet^{squig}$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 119. $\mathrm{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 120. For $i = 1, 2, 3, 4$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := \; & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \mathrm{sgn}(z - \Psi_{t_0}(x)) = sgn_1, \;\; \mathrm{sgn}(-z) = sgn_2, \\ & \mathrm{sgn}(z - \frac{1}{2}) = sgn_3, \\ & \mathrm{sgn}(-x) = sgn_4\} \end{aligned}$$

2. we use $*$ as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{i^{th}}{\underset{\uparrow}{*}}, \cdots) := s_{t_0}(\cdots, \underset{i^{th}}{\underset{\uparrow}{-}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\underset{\uparrow}{0}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\underset{\uparrow}{+}}, \cdots)$$

3. we omit trailing $*$'s e.g. $s_0(+, -) = s_0(+, -, *, *)$

4. note that we do not oit $*$'s located in between $-, 0, +$ e.g. $s_0(+, -, *, -) \neq s_0(+, -, -)$

Definition 121. For $i = 0, 1, 2, 3, 4$ and $sgn_i \in \{-, 0, +\}$,

1. we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := \; & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & \operatorname{sgn}(z - \Psi_t(x)) = sgn_1,\ \operatorname{sgn}(-z) = sgn_2, \\ & \operatorname{sgn}(z - \frac{1}{2}) = sgn_3, \\ & \operatorname{sgn}(-x) = sgn_4\} \end{aligned}$$

2. we use $*$ as a wild card sign i.e.

$$s_\bullet(\cdots, \underset{i^{th}}{\underset{\uparrow}{*}}, \cdots) := s_\bullet(\cdots, \underset{i^{th}}{\underset{\uparrow}{-}}, \cdots) \cup s_\bullet(\cdots, \underset{i^{th}}{\underset{\uparrow}{0}}, \cdots) \cup s_\bullet(\cdots, \underset{i^{th}}{\underset{\uparrow}{+}}, \cdots)$$

3. we omit trailing $*$'s e.g. $s_\bullet(+, -) = s_\bullet(+, -, *, *)$

4. note that we do not oit $*$'s located in between $-, 0, +$ e.g. $s_\bullet(+, -, *, -) \neq s_\bullet(+, -, -)$

Definition 122. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the above notations:

1. $\mathcal{S}_0$:

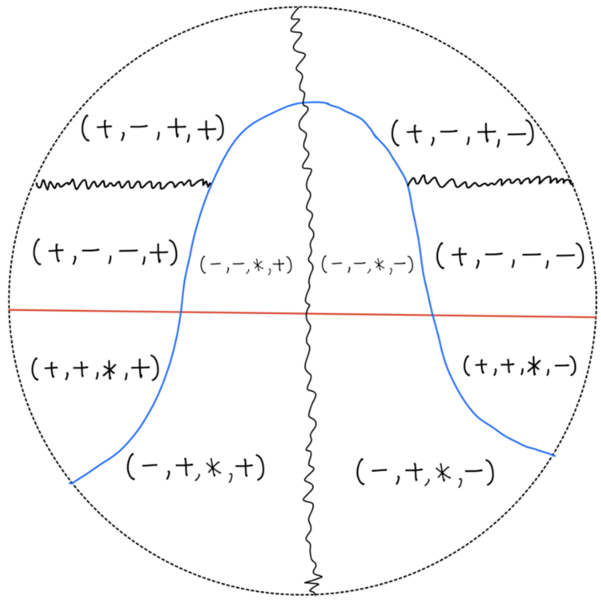

Figure 3.117

- 2 dimensional strata:

  $s_0(-,-,*,-)$,$s_0(-,-,*,+)$, $s_0(+,-,-,-)$, $s_0(+,-,-,+)$, $s_0(+,-,+,-)$,$s_0(+,-,+,+)$, $s_0(+,+,*,-)$, $s_0(+,+,*,-)$,$s_0(-,+,*,-)$,$s_0(-,+,*,+)$

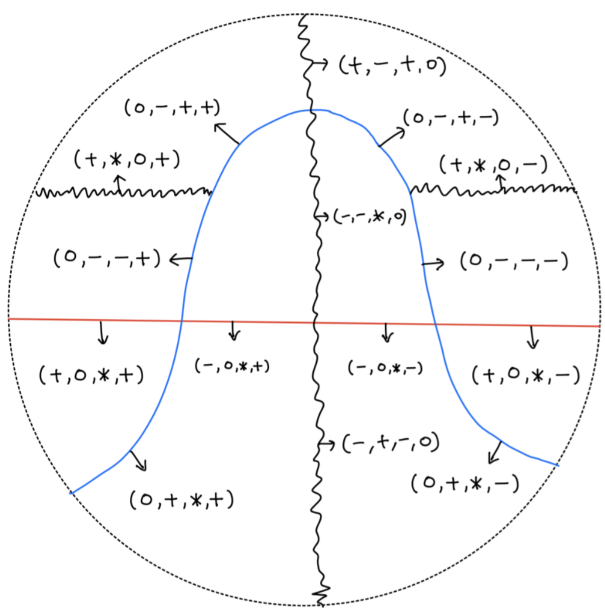

Figure 3.118

- 1 dimensional strata:

$s_0(0,+,*,-)$, $s_0(0,-,-,-)$, $s_0(0,-,+,-)$, $s_0(0,-,+,+)$, $s_0(0,-,-,+)$, $s_0(0,+,*,+)$, $s_0(+,0,*,-)$, $s_0(-,0,*,-)$, $s_0(-,0,*,+)$, $s_0(+,0,*,+)$, $s_0(+,*,0,-)$, $s_0(+,*,0,+)$, $s_0(+,-,+,0)$, $s_0(-,-,*,0)$, $s_0(-,+,-,0)$

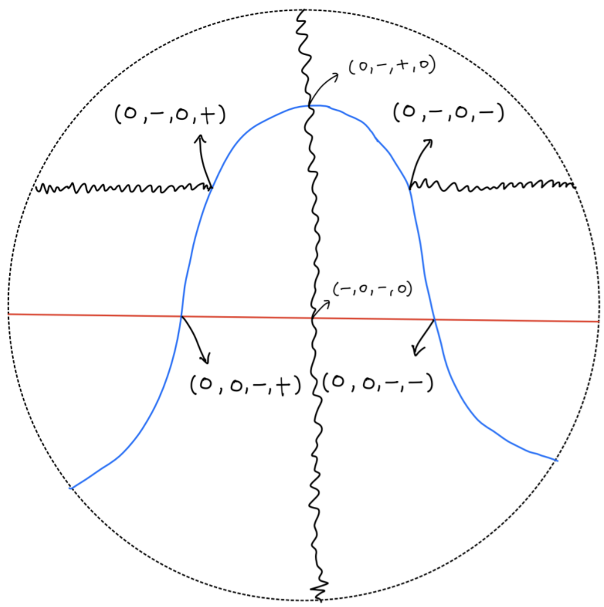

Figure 3.119

- 0 dimensional strata:

  $s_0(0,0,-,-)$, $s_0(0,-,0,-)$, $s_0(0,-,0,+)$, $s_0(0,0,-,+)$, $s_0(0,-,+,-)$, $s_0(-,0,-,0)$

2. $\mathcal{S}_1$:

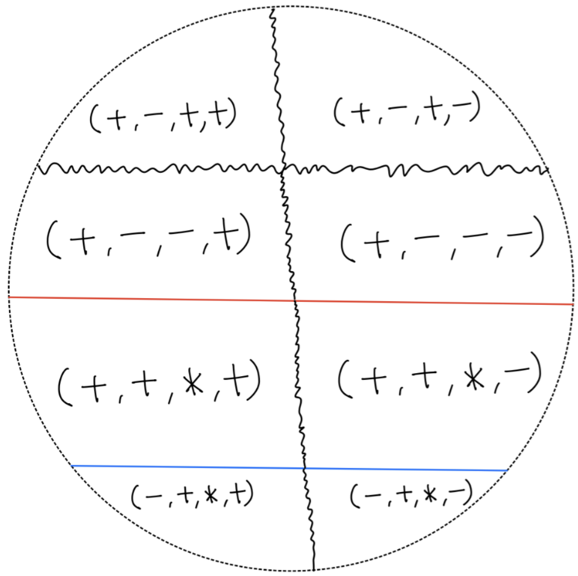

Figure 3.120

- 2 dimensional strata:

  $s_1(-,+,*,-)$, $s_1(-,+,*,+)$, $s_1(+,-,-,-)$, $s_1(+,-,-,+)$, $s_1(+,-,+,-)$, $s_1(+,-,+,+)$, $s_1(+,+,*,-)$, $s_1(+,+,*,+)$

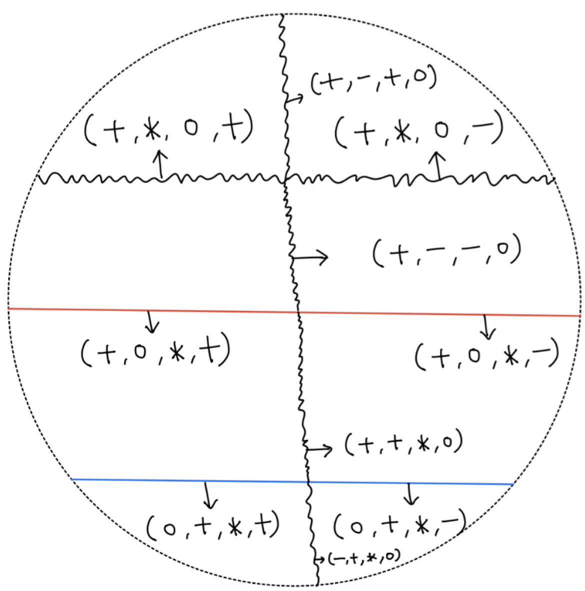

Figure 3.121

- 1 dimensional strata:

  $s_1(0,+,*,-)$, $s_1(0,+,*,+)$, $s_1(+,0,*,-)$, $s_1(+,0,*,+)$, $s_1(+,*,0,-)$, $s_1(+,*,0,+)$,

$s_1(+,-,-,0)$, $s_1(+,+,*,0)$, $s_1(+,-,+,0)$, $s_1(-,+,*,0)$

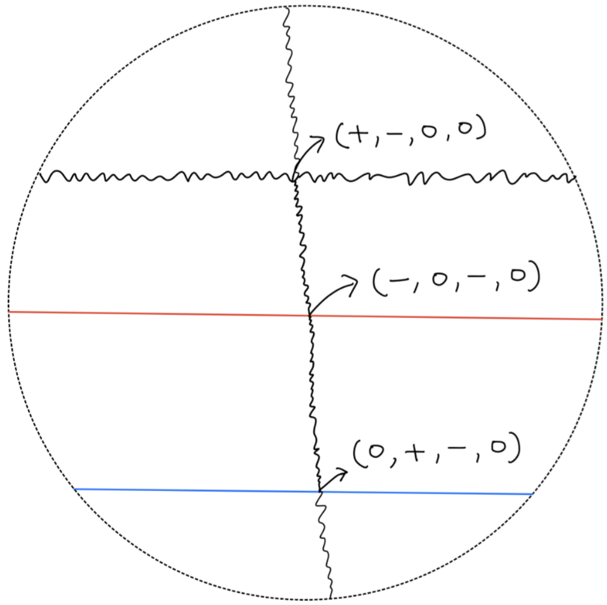

Figure 3.122

- 0 dimensional strata:

  $s_1(+,-,0,0)$, $s_1(-,0,-,0)$, $s_1(0,+,-,0)$

3. $\mathcal{S}_\bullet$:

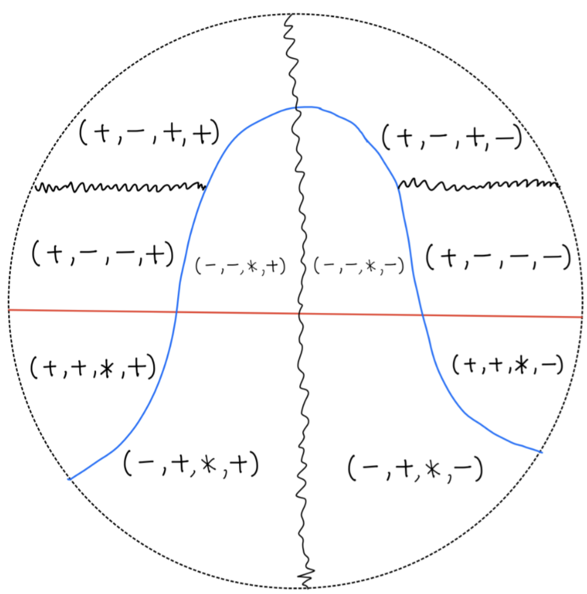

Figure 3.123

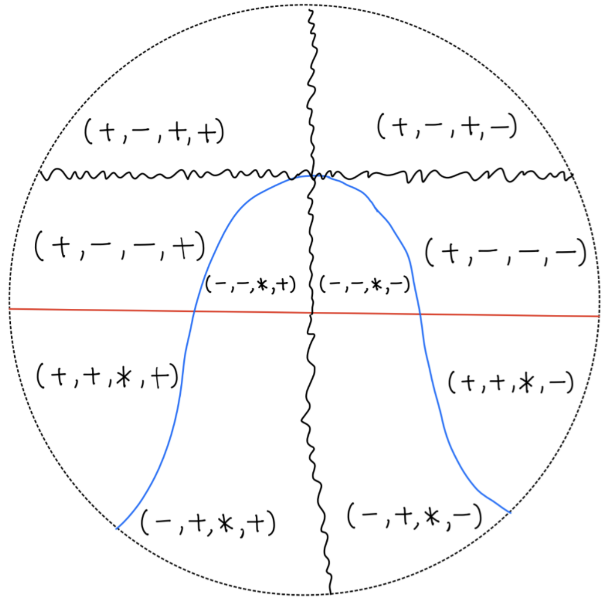

Figure 3.124

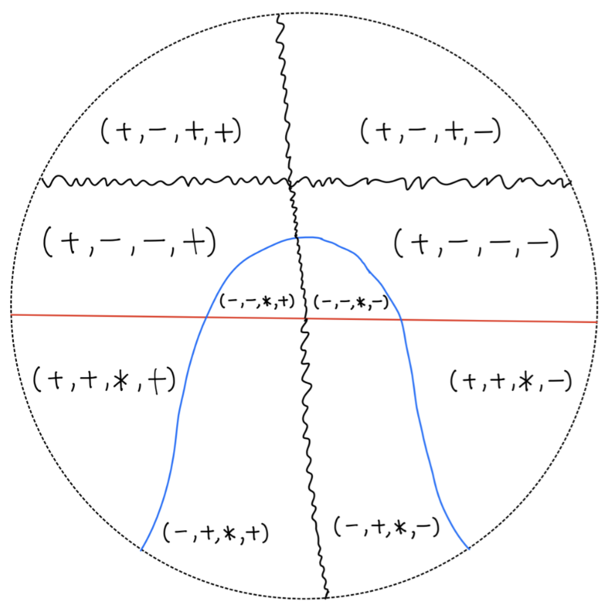

Figure 3.125

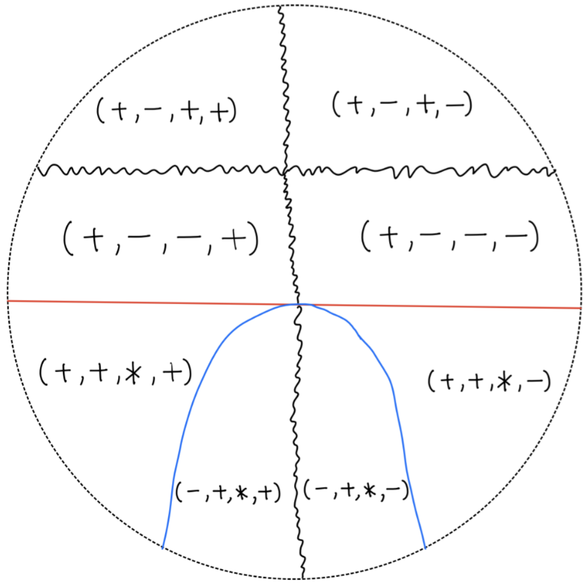

Figure 3.126

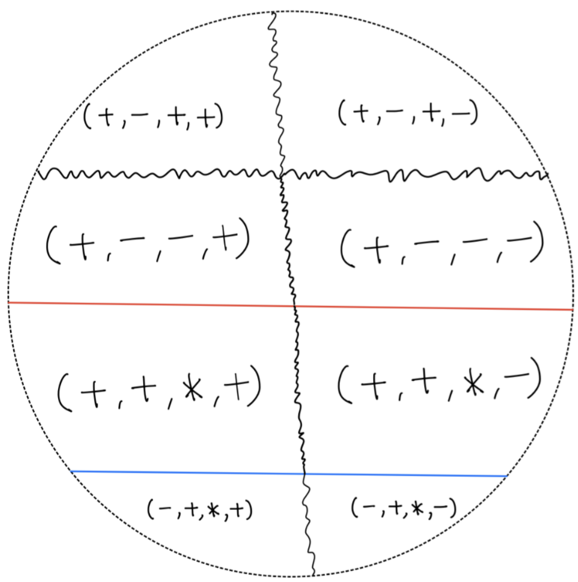

Figure 3.127

- 3 dimensional strata:

  $s_\bullet(-,+,*,-)$, $s_\bullet(-,+,*,+)$, $s_\bullet(-,-,*,-)$, $s_\bullet(-,-,*,+)$, $s_\bullet(-,+)$, $s_\bullet(+,-,-,-)$, $s_\bullet(+,-,-,+)$, $s_\bullet(+,-,+,-)$, $s_\bullet(+,-,+,+)$, $s_\bullet(+,+,*,-)$, $s_\bullet(+,+,*,+)$

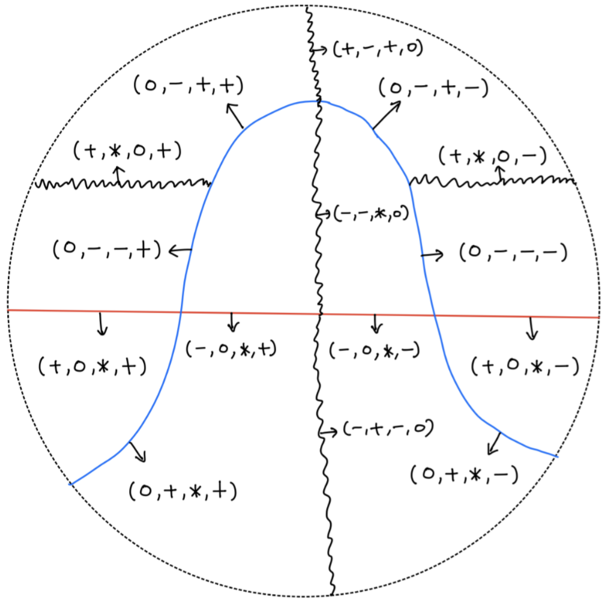

Figure 3.128

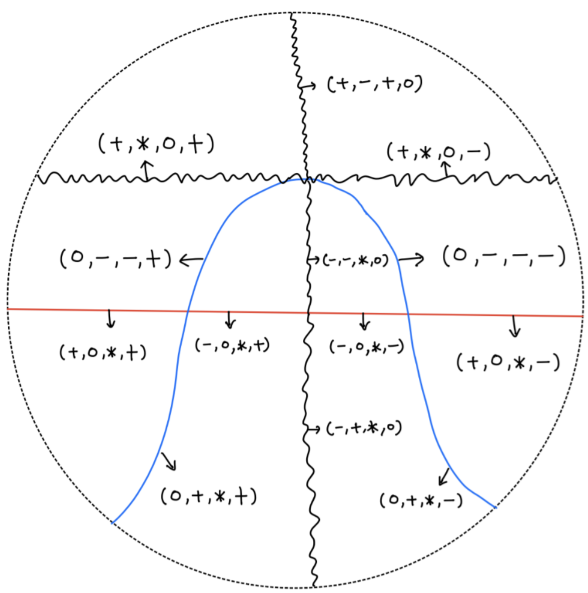

Figure 3.129

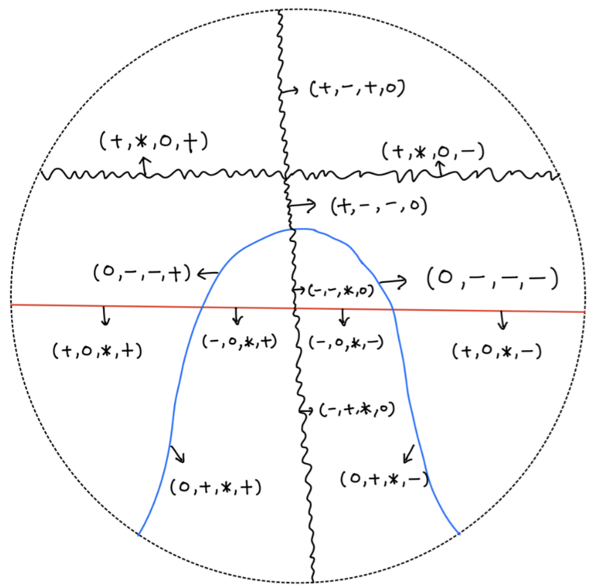

Figure 3.130

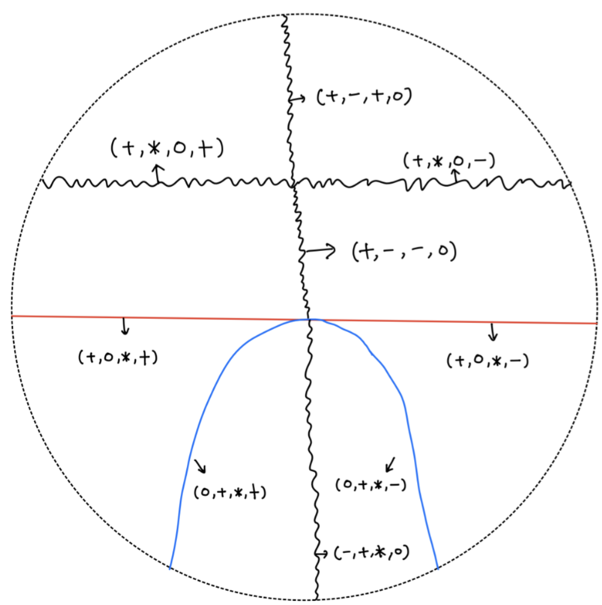

Figure 3.131

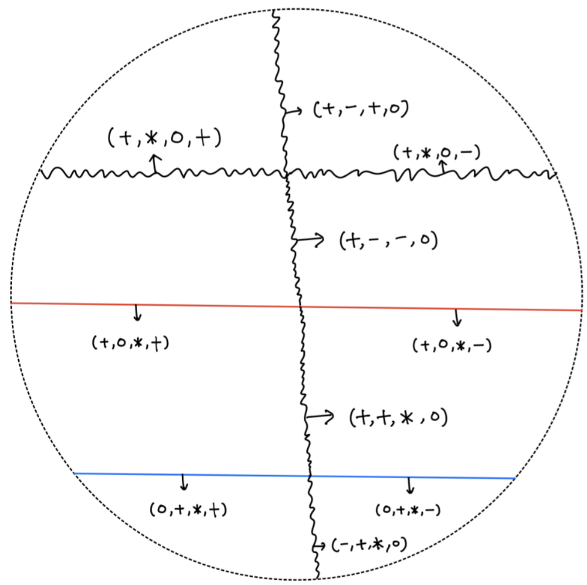

Figure 3.132

- 2 dimensional strata:

  $s_\bullet(0,+,*,-)$, $s_\bullet(0,-,-,-)$, $s_\bullet(0,-,+)$, $s_\bullet(0,-,+,-)$, $s_\bullet(0,-,+,+)$, $s_\bullet(0,-,-,+)$, $s_\bullet(0,+,*,+)$, $s_\bullet(+,0,*,-)$, $s_\bullet(-,0,*,-)$, $s_\bullet(-,0,*,+)$, $s_\bullet(+,0,*,+)$, $s_\bullet(+,*,0,-)$, $s_\bullet(+,*,0,+)$, $s_\bullet(+,-,-,0)$, $s_\bullet(+,+,*,0)$, $s_\bullet(+,-,+,0)$, $s_\bullet(-,+,*,0)$, $s_\bullet(-,-,*,0)$

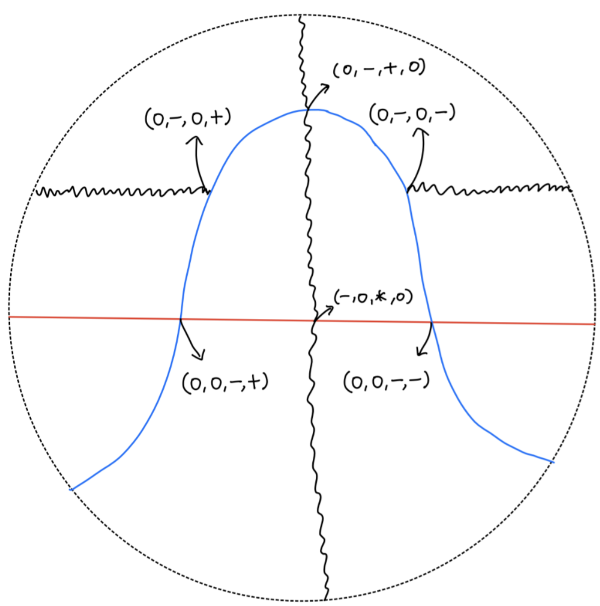

Figure 3.133

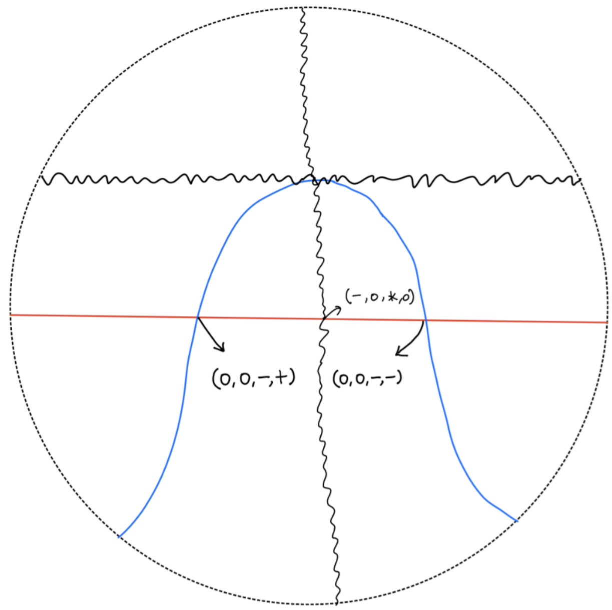

Figure 3.134

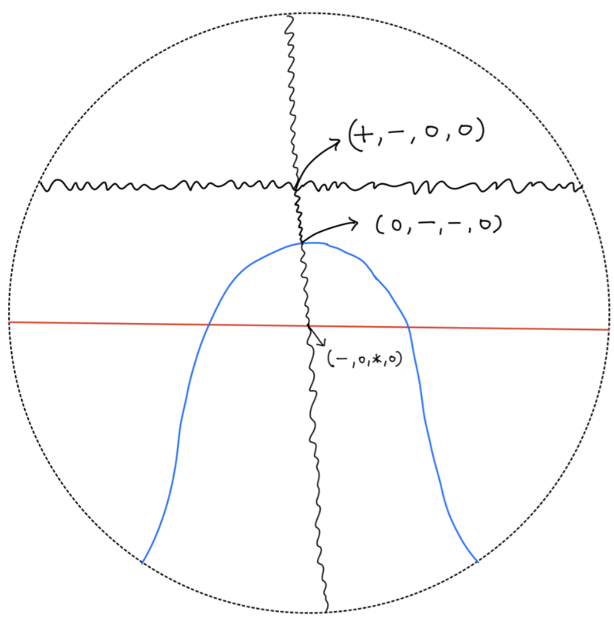

Figure 3.135

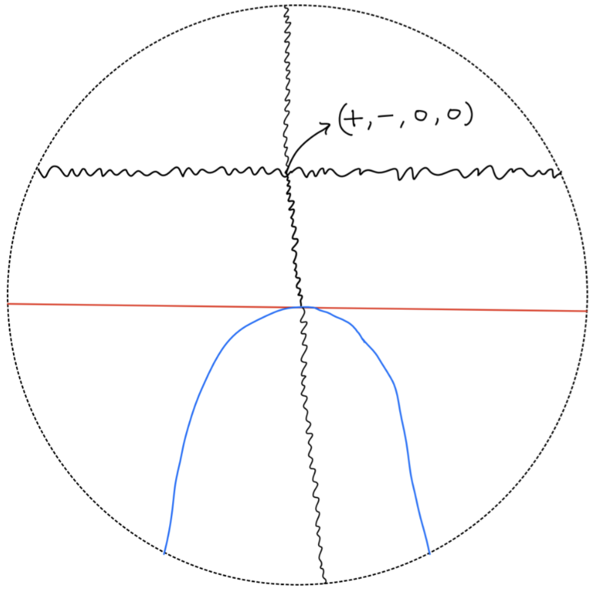

Figure 3.136

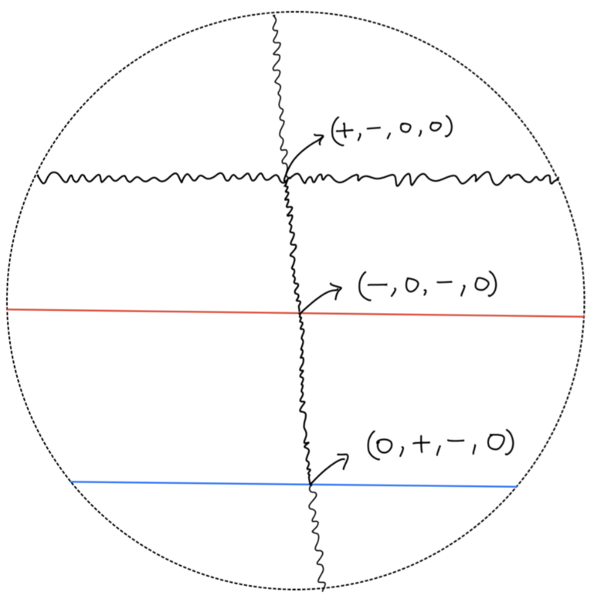

Figure 3.137

- 1 dimensional strata:

  $s_\bullet(0,0,-,-)$, $s_\bullet(0,-,0,-)$, $s_\bullet(0,-,0,+)$, $s_\bullet(0,0,-,+)$, $s_\bullet(+,-,0,0)$, $s_\bullet(0,-,-,0)$, $s_\bullet(-,0,-,0)$, $s_\bullet(0,+,-,0)$, $s_\bullet(0,-,+,0)$, $s_\bullet(-,0,*,0)$

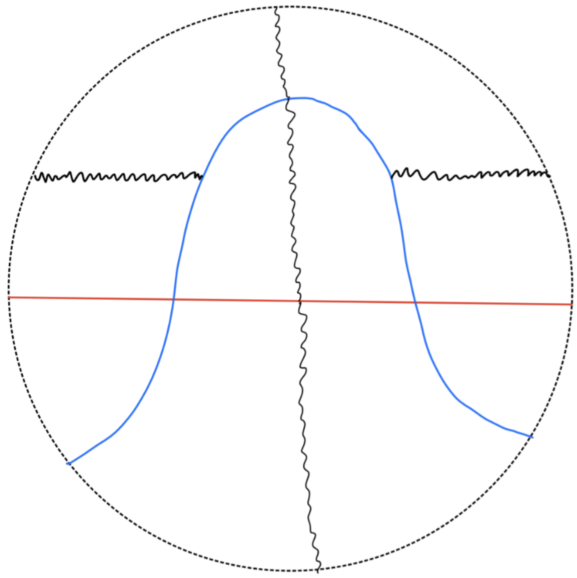

Figure 3.138

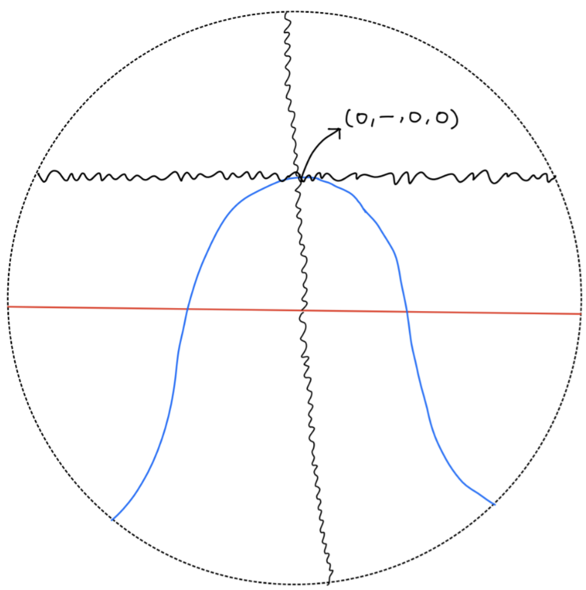

Figure 3.139

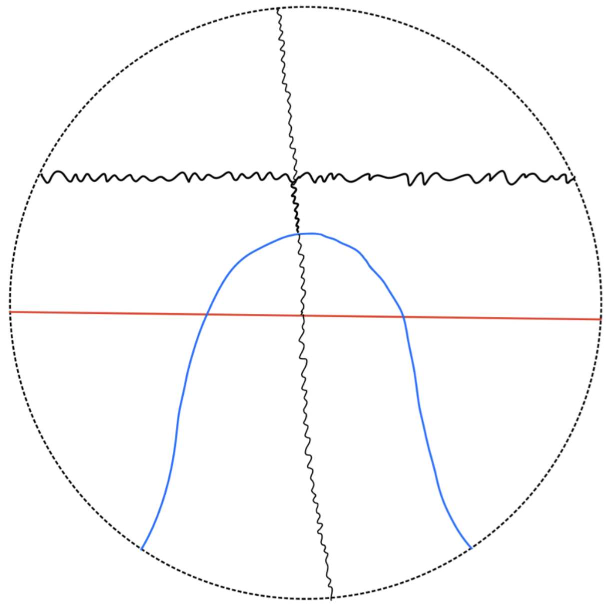

Figure 3.140

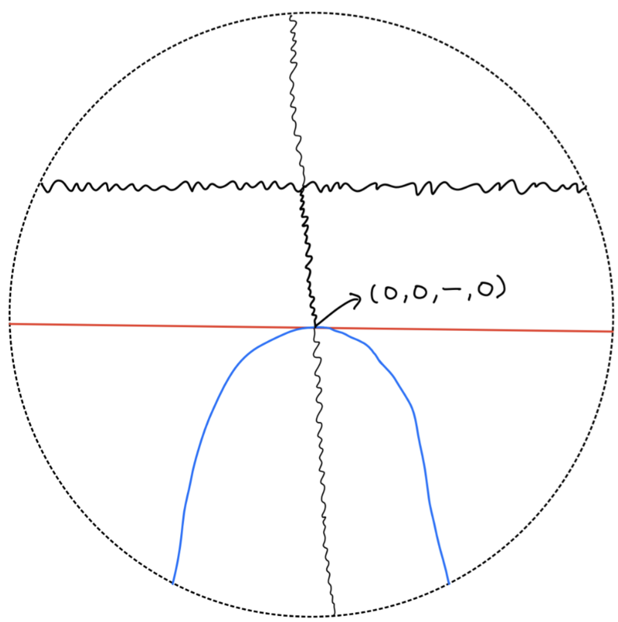

Figure 3.141

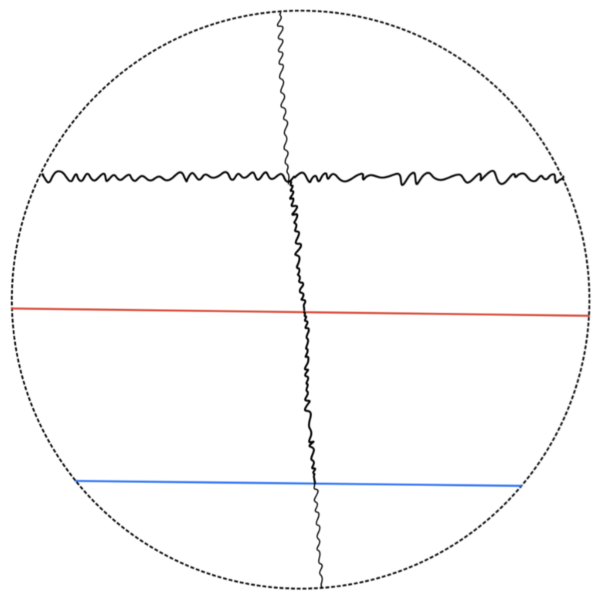

Figure 3.142

- 0 dimensional strata:

  $s_\bullet(0,-,0,0)$, $s_\bullet(0,0,-,0)$

Definition 123. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 124. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of s.

Definition 125. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 126. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 127. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \text{ codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 128. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \text{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$, then
$$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$
This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M

- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.
- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

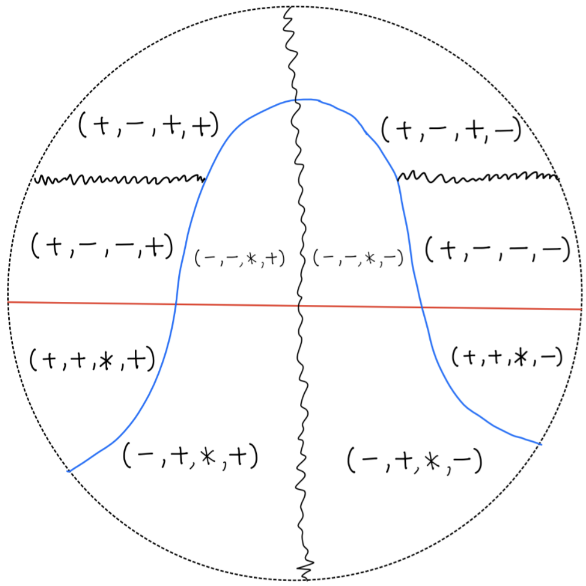

Figure 3.143

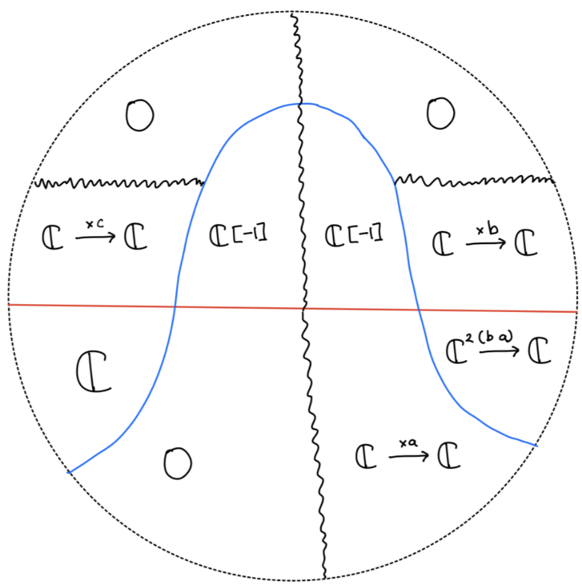

Figure 3.144

- $F_0(-,-,*,-) := \mathbb{C}[-1]$
- $F_0(-,-,*,+) := \mathbb{C}[-1]$
- $F_0(-,+,*,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(-,+,*,+) := 0$

- $F_0(+,-,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_0(+,-,-,+) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_0(+,-,+,-) := 0$
- $F_0(+,-,+,+) := 0$
- $F_0(+,+,*,-) := \mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C}$
- $F_0(+,+,*,+) := \mathbb{C}$

Generization maps:

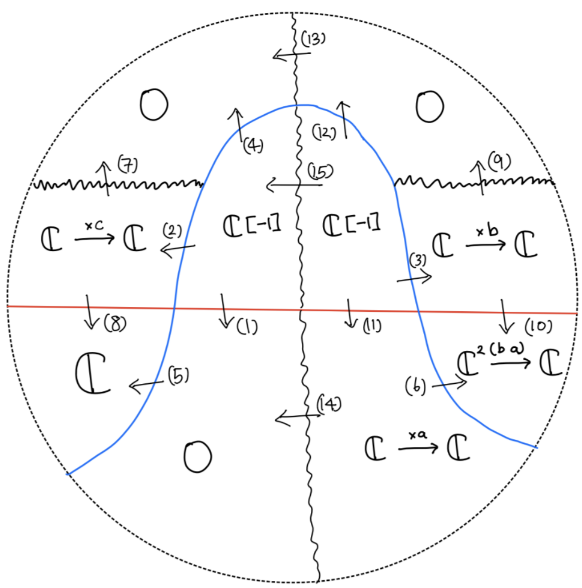

Figure 3.145

(1)
$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

(2)
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow {\scriptstyle \times c} \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd} \tag{5}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd} \tag{8}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{9}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd} \tag{10}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd} \tag{11}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{12}$$

$$(13)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(14)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(15)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord'_2$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

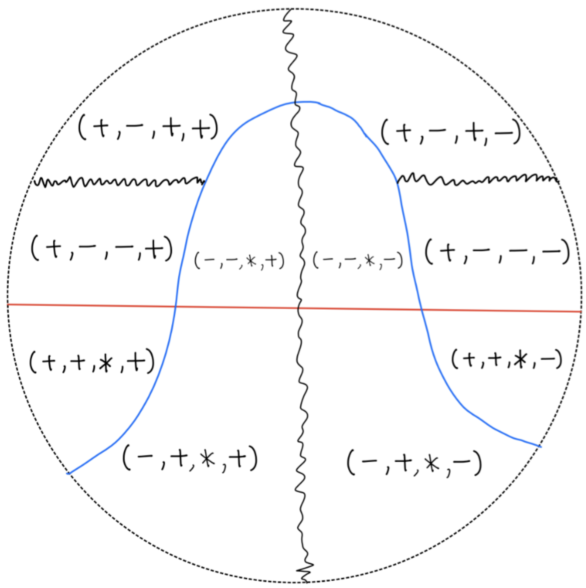

Figure 3.146

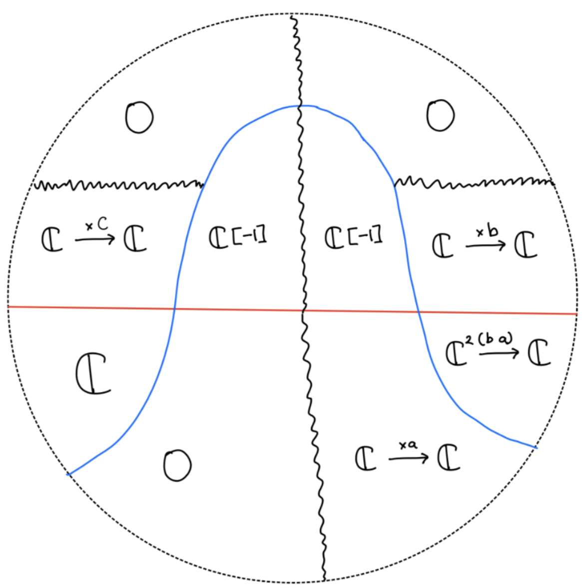

Figure 3.147

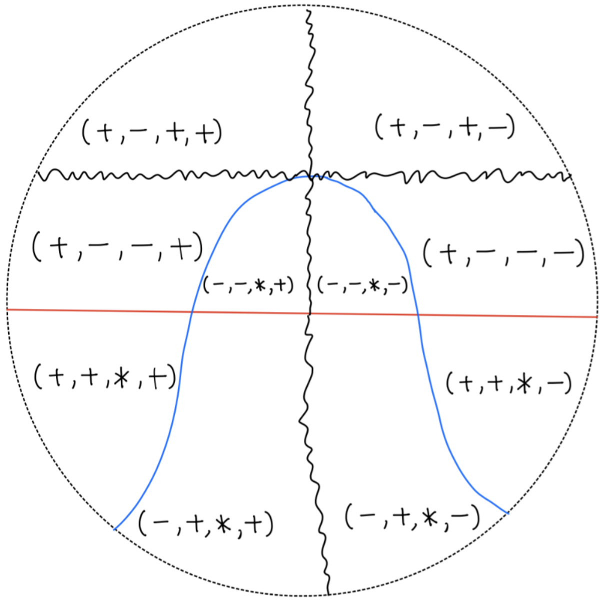

Figure 3.148

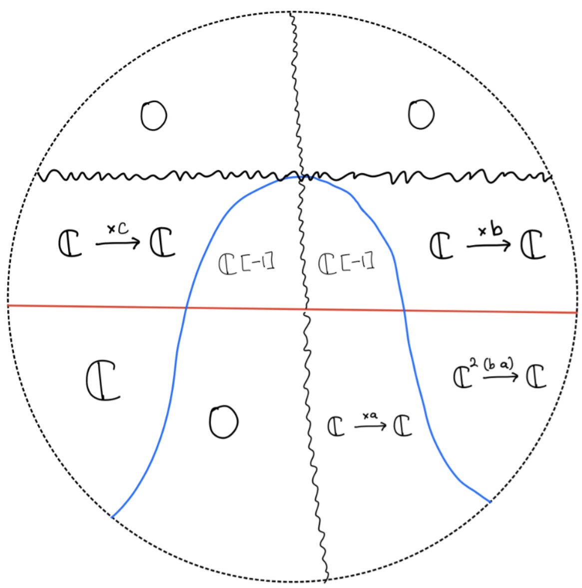

Figure 3.149

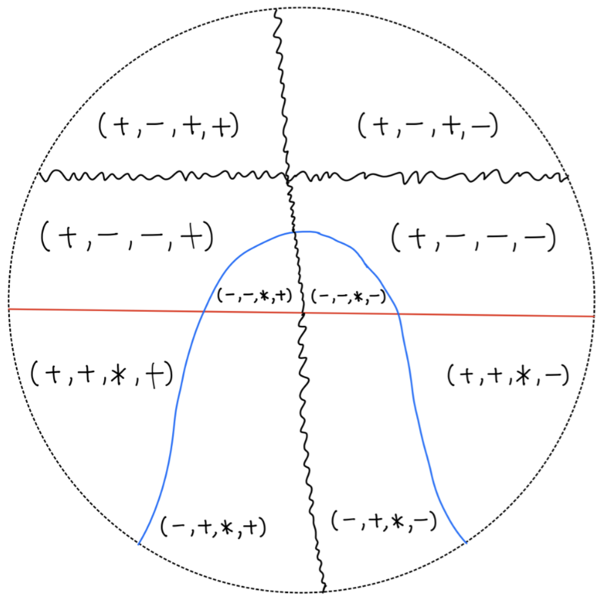

Figure 3.150

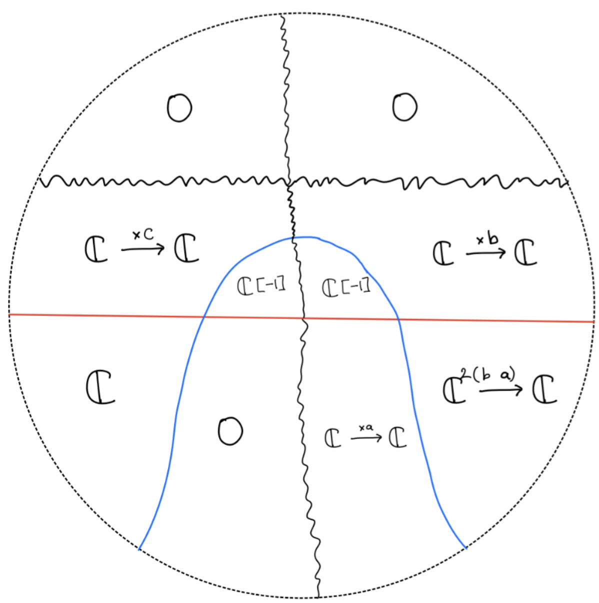

Figure 3.151

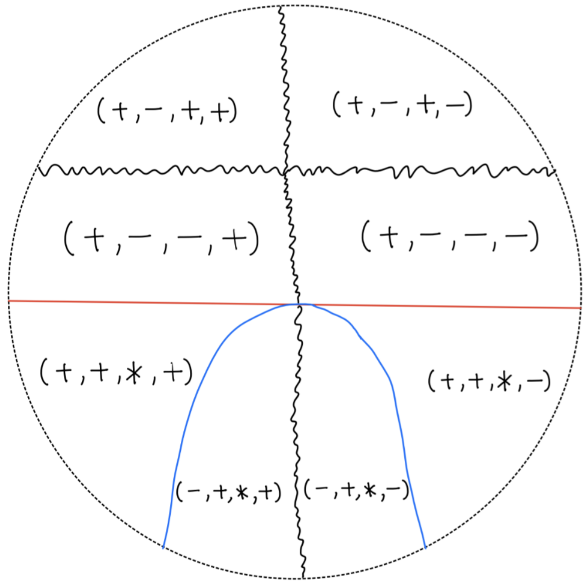

Figure 3.152

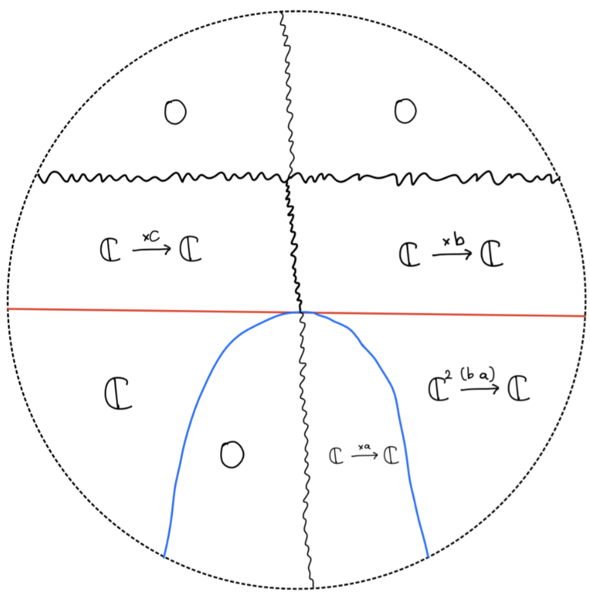

Figure 3.153

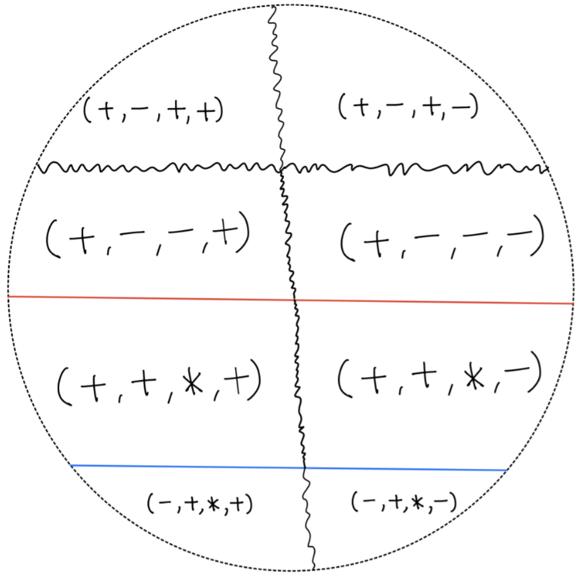

Figure 3.154

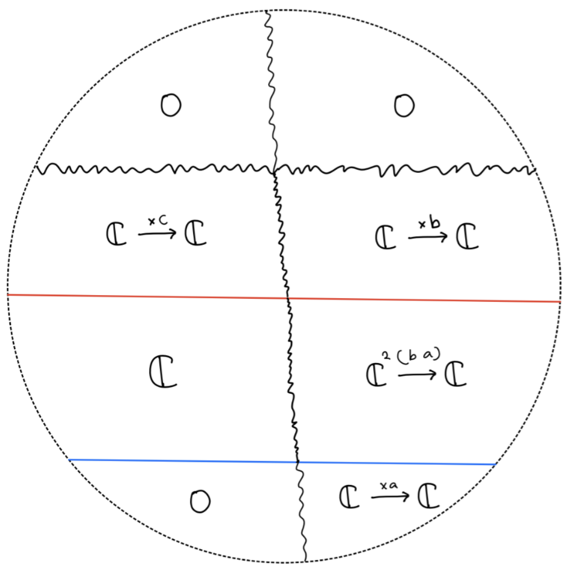

Figure 3.155

- $F_\bullet(-,-,*,-) := \mathbb{C}[-1]$
- $F_\bullet(-,-,*,+) := \mathbb{C}[-1]$
- $F_\bullet(-,+,*,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_\bullet(-,+,*,+) := 0$

- $F_\bullet(+,-,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_\bullet(+,-,-,+) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_\bullet(+,-,+,-) := 0$
- $F_\bullet(+,-,+,+) := 0$
- $F_\bullet(+,+,*,-) := \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C}$
- $F_\bullet(+,+,*,+) := \mathbb{C}$

Generization maps:

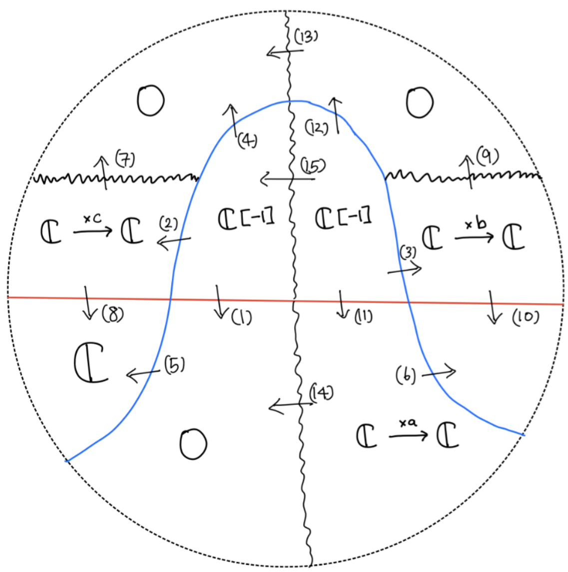

Figure 3.156

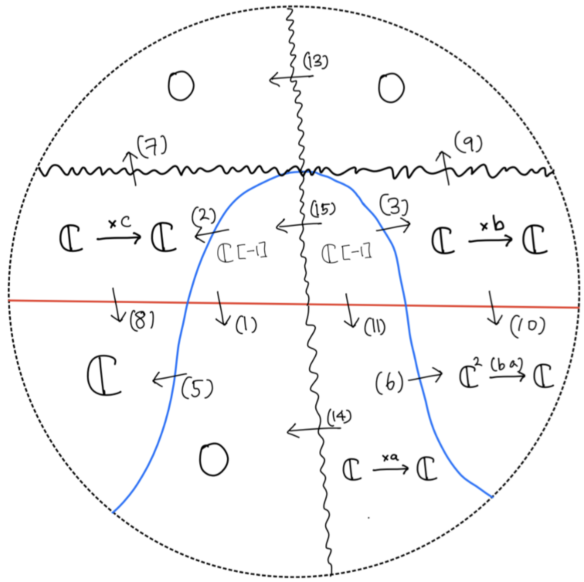

Figure 3.157

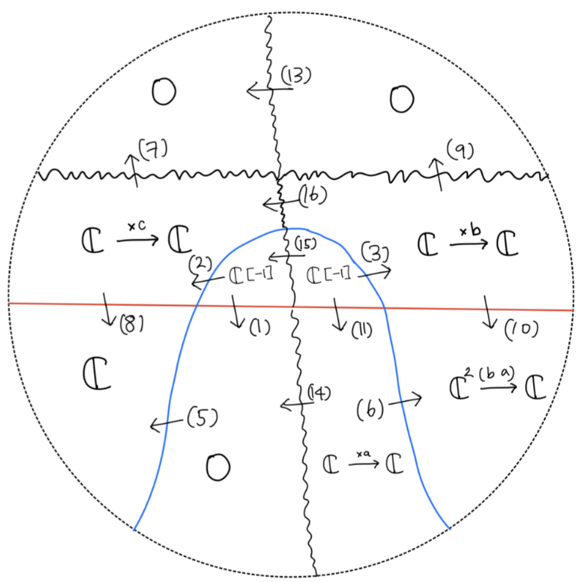

Figure 3.158

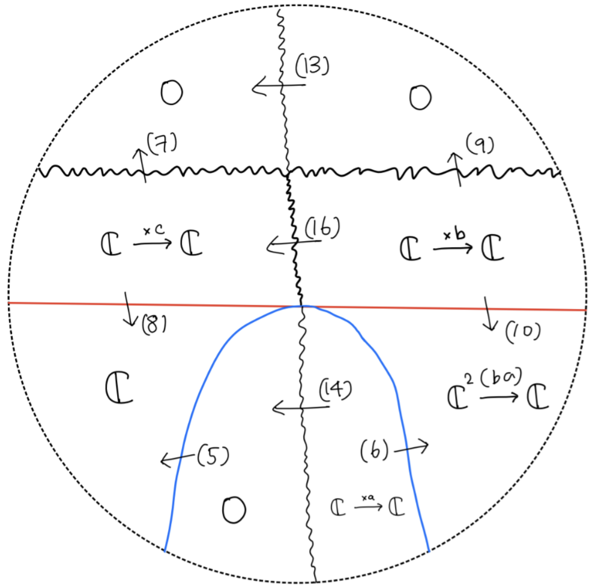

Figure 3.159

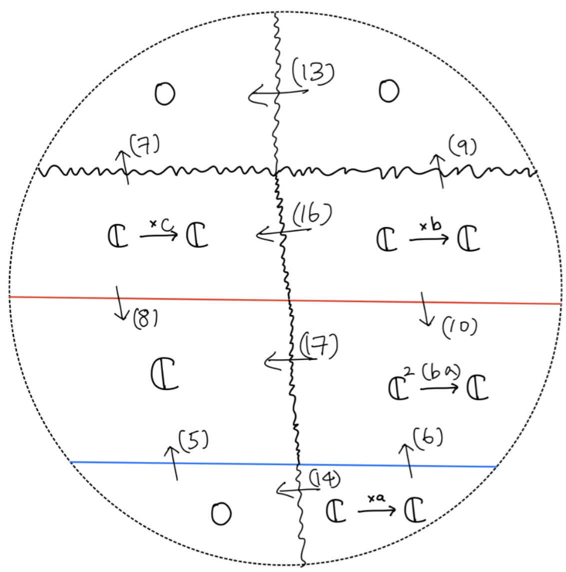

Figure 3.160

(1)
$$\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c"']
\end{tikzcd}$$

$$(3)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

$$(4)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[r, "\iota_1"] \arrow[u, "\times a"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r] \arrow[u, "\times c"] & 0 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r, "\times 1"] \arrow[u, "\times c"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[r] \arrow[u, "\times b"] & 0 \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[r, "\iota_0"] \arrow[u, "\times b"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

$$(11)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[r] \arrow[u] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

$$(12)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

$$(13)\quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times a} & 0 \end{array}$$

$$(15)\quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(16)\quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ {\scriptstyle \times b}\uparrow & & {\scriptstyle \times c}\uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & 0 \end{array}$$

$$(17)\quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ {\scriptstyle (b\ \ a)}\uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1}\ \ 0)} & \mathbb{C} \end{array}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$ using the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$ where $pr_1 : (U\cap V)\times[0,1] \to (U\cap V)$ is the projection onto the first argument.

Definition 129. we define $\gamma_\bullet$ to be the composition

$$(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U\cap V)\times[0,1]} = \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd} (U \cap V) \times [0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V \times [0,1] \arrow[d, "pr_1"] \\ (U \cap V) \arrow[r, hook] & V \end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 130. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M \times [0,1], \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of $M \times [0,1]$.
Since $\mathscr{F}^\bullet$ is constant along the time coordinate on $U'^c \times [0,1]$, it is enough to check for the points of $U \times [0,1] \cong D_{r=2} \times [0,1]$. Now consider the following open cover of $D_{r=2} \times [0,1]$

$$\{\mathrm{star}(s_\bullet(0,-,0,0)), \mathrm{star}(s_\bullet(0,0,-,0))\}$$

(1) First, let's show that the microlocal stalks of $\mathscr{F}^\bullet|_{\mathrm{star}(s_\bullet(0,-,0,0))}$ vanishes. Note that there is a diffeomorphism beteween $\mathrm{star}(s_\bullet(0,-,0,0))$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}[-1]$
- $F^3(-,-,+) := \mathbb{C}[-1]$
- $F^3(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+,-,+) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F^3(-,+,-) := \mathbb{C}[-1]$
- $F^3(-,+,+) := \mathbb{C}[-1]$
- $F^3(+,+,-) := 0$
- $F^3(+,+,+) := 0$

Generization maps:

$$\begin{tikzcd}
& s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
& s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}$$

(1) $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$

(2) $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$

$$(3)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

$$(4)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(11)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$(12)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

(i)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \end{tikzcd}$$

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C}[-1] \arrow[r] & \mathbb{C}[-1] \end{tikzcd}$$

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

(vii) the cubic diagram:

$$\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\
s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\
 & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] &
\end{tikzcd}$$

$=$

$$\begin{tikzcd}
 & \mathbb{C}[-1] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[dd] \\
\mathbb{C}[-1] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[ur] \arrow[dd] & \\
 & 0 \arrow[rr] & & 0 \\
\mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[ur] \arrow[rr] & & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

For i-vi, horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For vii, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$, $\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$ are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of $\mathscr{F}^\bullet|_{\mathrm{star}(s_\bullet(0,0,-,0))}$ vanishes. Note

that there is a diffeomorphism beteween star($s_\bullet(0,0,-,0)$) and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}[-1]$
- $F^3(-,-,+) := \mathbb{C}[-1]$
- $F^3(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+,-,+) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F^3(-,+,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(-,+,+) := 0$
- $F^3(+,+,-) := \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C}$
- $F^3(+,+,+) := \mathbb{C}$

Generization maps:

$$
\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
 & s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}
$$

(1) $$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

(2) $$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"']
\end{tikzcd}
$$

(3) $$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"']
\end{tikzcd}
$$

(4) $$
\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

(5) $$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times c"']
\end{tikzcd}
$$

(6) $$
\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
\mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(8)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$$

(9)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(10)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "(bc^{-1}\ \ 0)"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(11)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times c"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(12)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

(i)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & \mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C} \end{tikzcd}$$

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \end{tikzcd}$$

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C} \arrow[r, "(bc^{-1}\ 0)"] & \mathbb{C} \end{tikzcd}$$

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & \mathbb{C}[-1] \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[d] \\ \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(vii) the cubic diagram:

$$\begin{tikzcd} & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\ s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\ & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\ s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] & \end{tikzcd}$$

$=$

$$\begin{tikzcd}
& \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
\mathbb{C}[-1] \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{C}[-1] \arrow[ur] \arrow[dd] & \\
& \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C} \arrow[rr] & & \mathbb{C} \\
\mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[ur] \arrow[rr] & & \mathbb{C} \xrightarrow{\times c} \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

For iii-vi, the horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For iand ii, straightforward calculation shows that the total complexes are acyclic.

For vii, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"'] \end{tikzcd}$, $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ a)"] \arrow[r, "(bc^{-1}\ 0)"] & \mathbb{C} \arrow[u] \end{tikzcd}$ are quasi-isomorphisms. Therefore, we can think of the cube diagram as quasi-isomorphism of two double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete. □

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the *cobord*$'_2$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$

- a gluing isomorphsim $\gamma_1 : f_* \mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V \times \{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

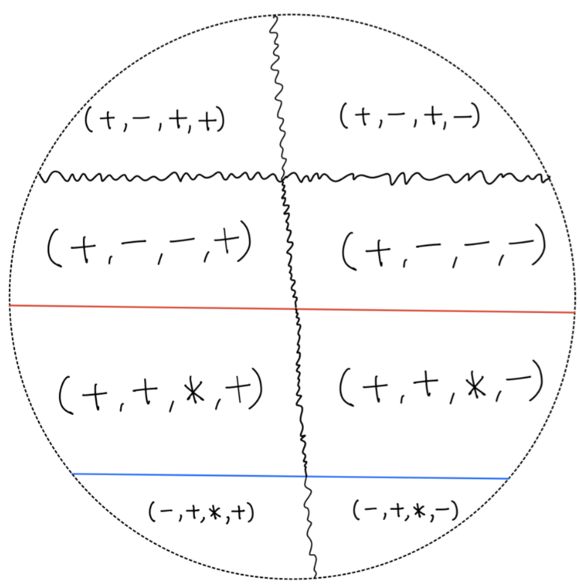

Figure 3.161

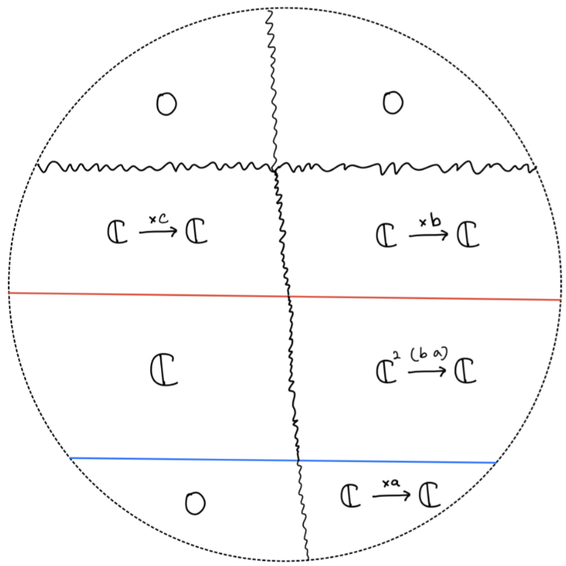

Figure 3.162

- $F_1(-,+,*,-) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_1(-,+,*,+) := 0$
- $F_1(+,-,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_1(+,-,-,+) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_1(+,-,+,-) := 0$
- $F_1(+,-,+,+) := 0$
- $F_1(+,+,*,-) := \mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C}$
- $F_1(+,+,*,+) := \mathbb{C}$

Generization maps:

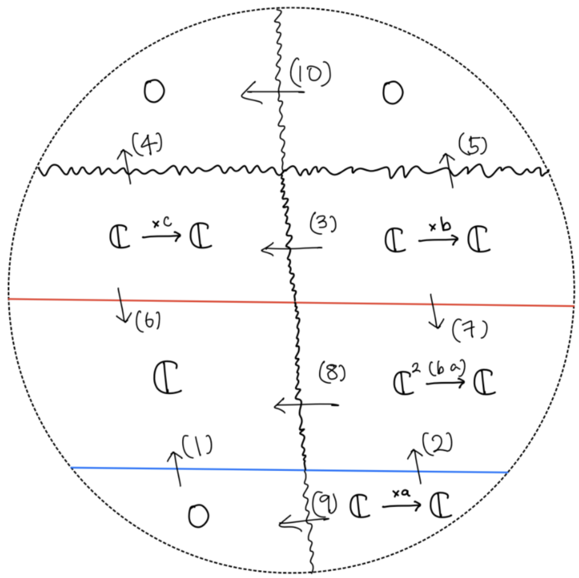

Figure 3.163

(1)
$$\begin{tikzcd}
0 \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u]
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
\mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"']
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\
\mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "\times c"']
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
\mathbb{C} \arrow[u, "\times c"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

(5)
$$\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
\mathbb{C} \arrow[u, "\times b"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

(6)
$$\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 \\
\mathbb{C} \arrow[u, "\times c"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u]
\end{tikzcd}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(8)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "(bc^{-1}\ \ 0)"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(9)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 131. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.7 3rd sheaf cobordism

In this section, we define $cobord_3$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

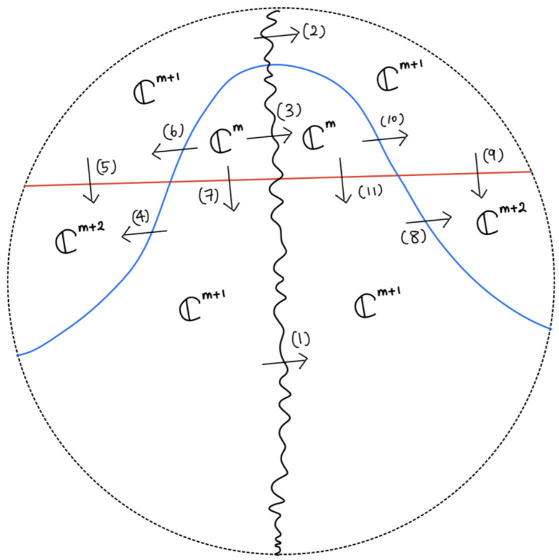

Figure 3.164

Generization maps:

(1) $\mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m} \xrightarrow{\theta} \mathbb{C}^{m}$

(4) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(5) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(6) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(8) $\mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$

(9) $\mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$

(10) $\mathbb{C}^{m} \xrightarrow{\theta'} \mathbb{C}^{m+1}$

(11) $\mathbb{C}^{m} \xrightarrow{\theta''} \mathbb{C}^{m+1}$

to

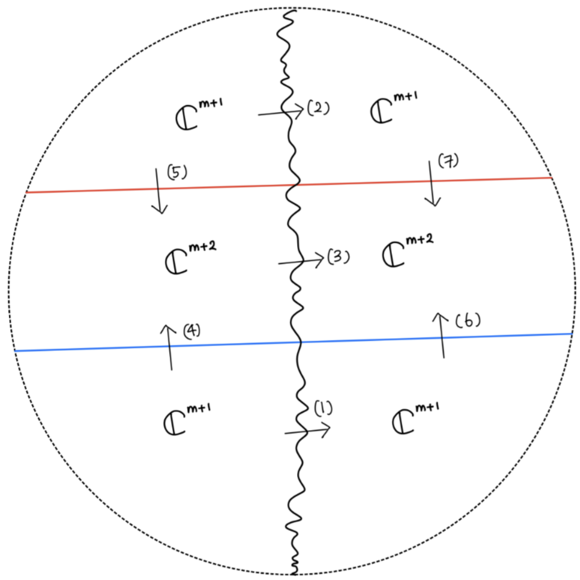

Figure 3.165

Generization maps:

(1) $\mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

(4) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(5) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(6) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(7) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$
- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

Notations

Definition 132. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 133. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$

$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_3$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_3$.

Definition 134. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_\bullet^{symbol}$ to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_\bullet^\infty : (S^1)^m \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_\bullet^{squig} : \coprod_{1\leq i\leq k} ([0,1] \times [a_i, b_i]_t) \to M \times [0,1]_t$$

   where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_\bullet^{symbol}$ a co-orientation of $\Phi_\bullet^{symbol}$.

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as $\Lambda_\bullet^{symbol}$. Later in the section, $\Lambda_\bullet^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord_3$.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as $\Lambda_\bullet$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_\bullet$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord_3$.

Definition 135. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} \frac{3}{4} e^{\left(\frac{x^2}{x^2-1}\right)}(1-t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1,1]$ if $t \neq 0$
- $\{(1,0),\ (-1,0), (0, \frac{3}{4}(1-t))\} \subset Graph(\Psi_t)$

Definition 136. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$
$$(x,z) \mapsto (x,z,t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 137. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x,z) \in D_{r=2} \mid z = \frac{1}{2}\}$
- $\lambda_0^{squig} := \{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing downward direction i.e. coefficients of dz are negative.
- ξ_0^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_0^{squig}: hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 138. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x,z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x,z) \in D_{r=2} \mid z = 0\}$
- $\lambda_1^\infty := \{(x,z) \in D_{r=2} \mid z = \frac{1}{2}\}$
- $\lambda_1^{squig} := \{(x,z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing downward direction i.e. coefficients of dz are negative.

- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.
- ξ_1^{squig}: hairs are pointing towards the left i.e. coefficients of dx are negative.

Definition 139. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda_\bullet^0 := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x,z,t) \in D_{r=2} \times [0,1] \mid z = \frac{1}{2}\}$
- $\lambda_\bullet^{squig} := \{(x,z,t) \in D_{r=2} \times [0,1] \mid x = 0\}$

2. We define co-orientations $\xi_\bullet^{symbol}$ of $\lambda_\bullet^{symbol}$ as follows

- $\xi_\bullet^0$: hairs are pointing downward direction i.e. coefficients of dz are negative.
- $\xi_\bullet^\infty$: hairs are pointing upward direction i.e. coefficients of dz are positive.
- $\xi_\bullet^{squig}$: hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 140. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda_\bullet^0$, $\lambda_\bullet^\infty$, and $\lambda_\bullet^{squig}$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 141. $\text{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\text{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 142. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := \ & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \text{sgn}(z - \Psi_{t_0}(x)) = sgn_1, \ \ \text{sgn}(\frac{1}{2} - z) = sgn_2, \\ & \text{sgn}((x) = sgn_3\} \end{aligned}$$

Definition 143. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3) := \ & \{(x, z, t) \in D_{r=2} \times [0,1] \mid \\ & \text{sgn}(z - \Psi_t(x)) = sgn_1, \ \ \text{sgn}(\frac{1}{2} - z) = sgn_2, \\ & \text{sgn}((x) = sgn_3\} \end{aligned}$$

Definition 144. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the above notations:

1. $\mathcal{S}_0$:

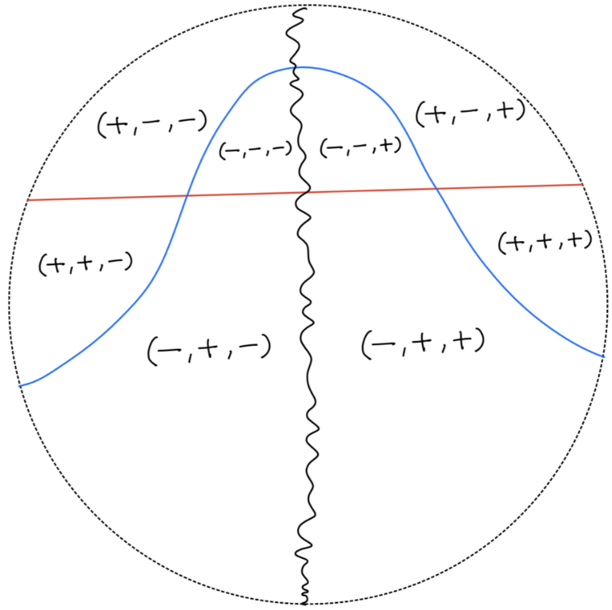

Figure 3.166

- 2 dimensional strata:

  $\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3$\}$

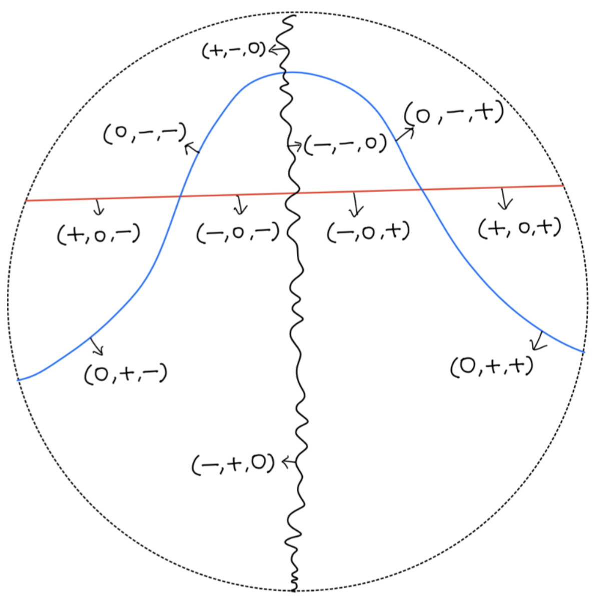

Figure 3.167

- 1 dimensional strata:

  $\{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in$

$\{-,+\}$ for i=1,3$\}\cup\{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_0(+,+,0)\}$

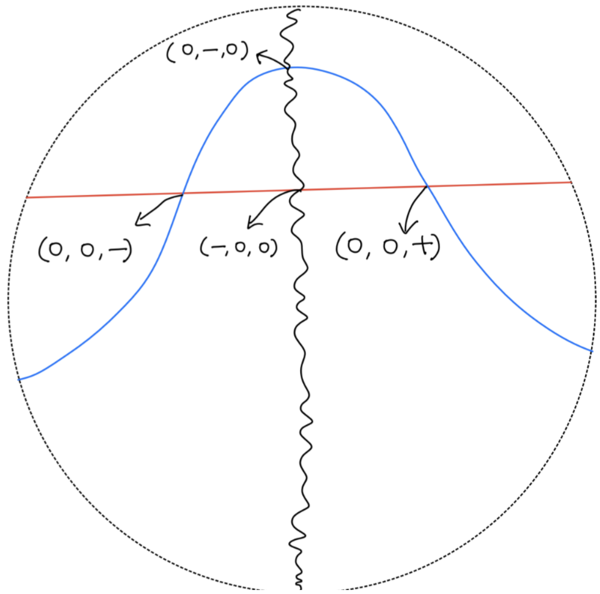

Figure 3.168

- 0 dimensional strata:

  $s_0(0, 0, +)$, $s_0(-, 0, 0)$

2. $\mathcal{S}_1$:

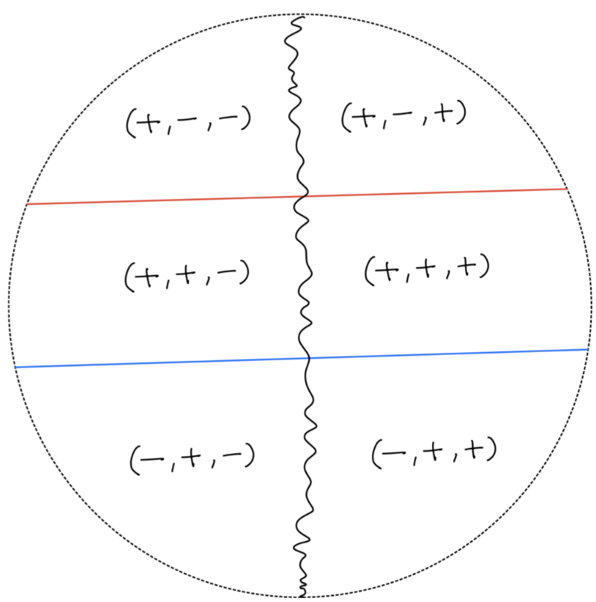

Figure 3.169

- 2 dimensional strata:

  $s_1(+,-,-)$,$s_1(+,-,+)$,$s_1(+,+,-)$,$s_1(+,+,+)$,$s_1(-,+,-)$,$s_1(-,+,+)$

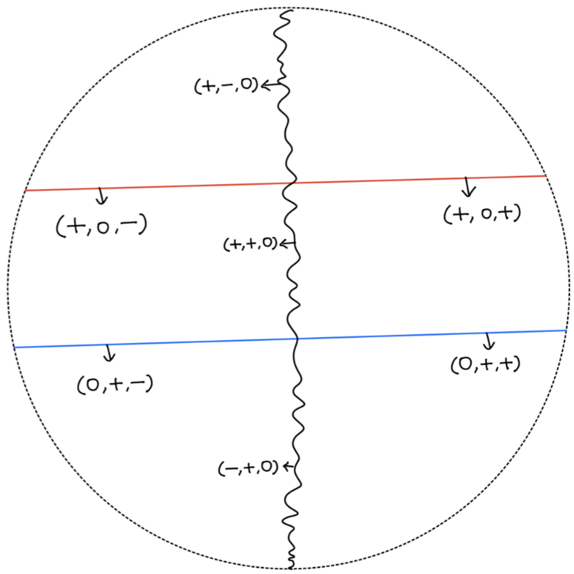

Figure 3.170

- 1 dimensional strata:

  $s_1(0,+,-)$,$s_1(0,+,+)$,$s_1(+,0,-)$,$s_1(+,0,+)$,$s_1(+,-,0)$,$s_1(+,+,0)$,$s_1(-,+,0)$

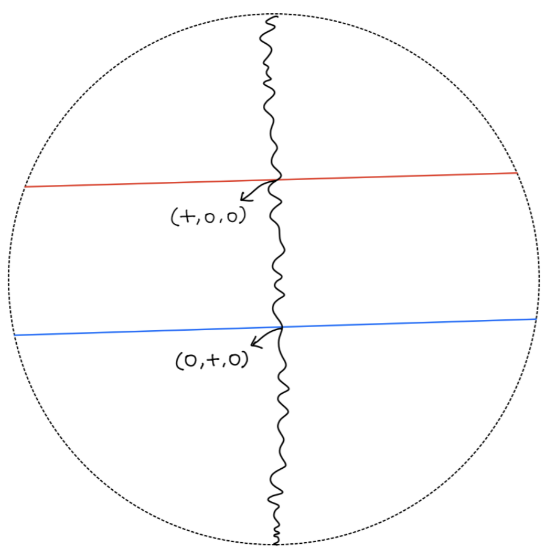

Figure 3.171

- 0 dimensional strata:

$s_1(0,+,0),\ s_1(+,0,0)$

3. $\mathcal{S}_\bullet$:

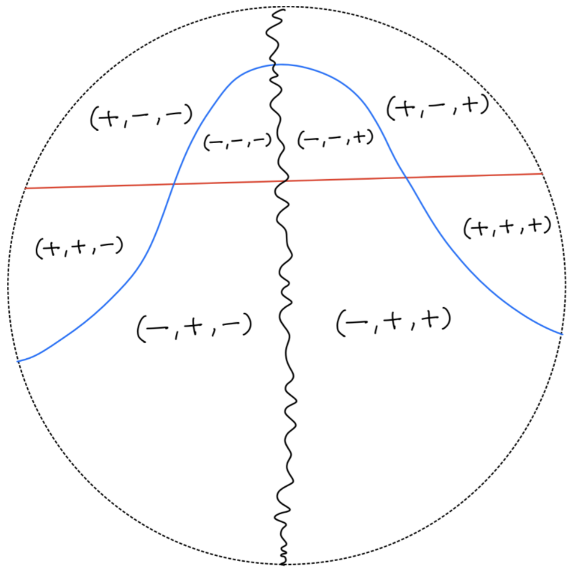

Figure 3.172

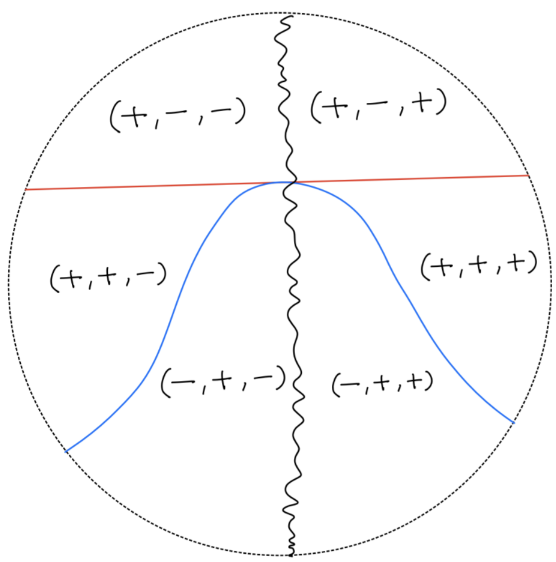

Figure 3.173

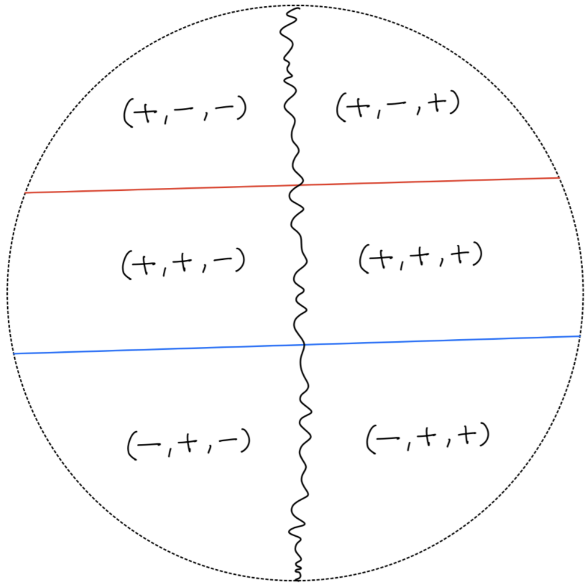

Figure 3.174

- 3 dimensional strata:

  $\{s_\bullet(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3$\}$

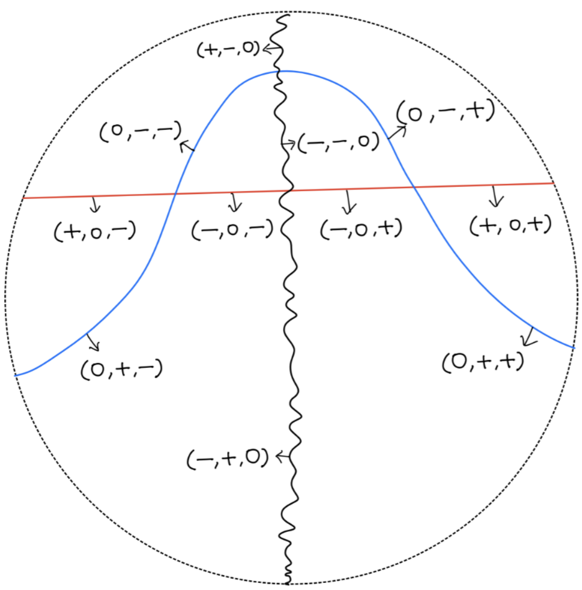

Figure 3.175

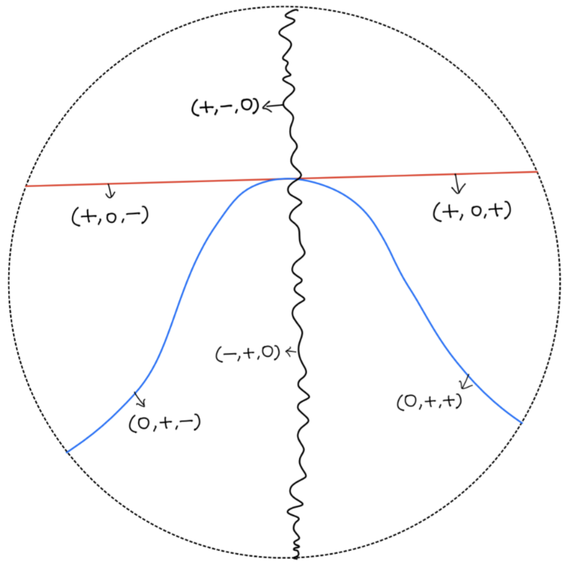

Figure 3.176

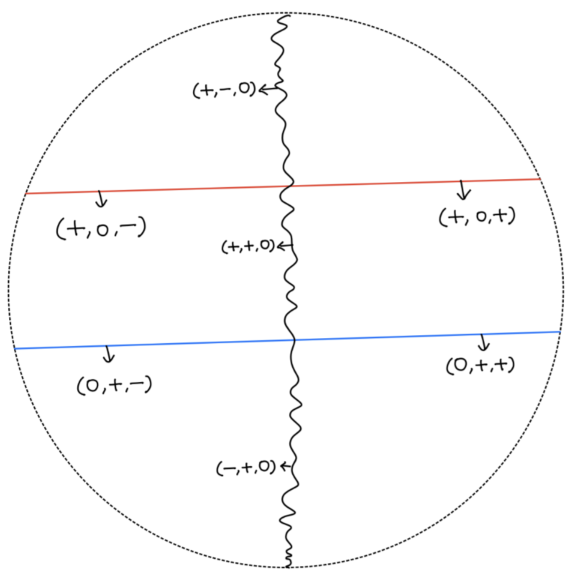

Figure 3.177

- 2 dimensional strata:

  $\{s_\bullet(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_\bullet(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3$\} \cup \{s_\bullet(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2$\}$

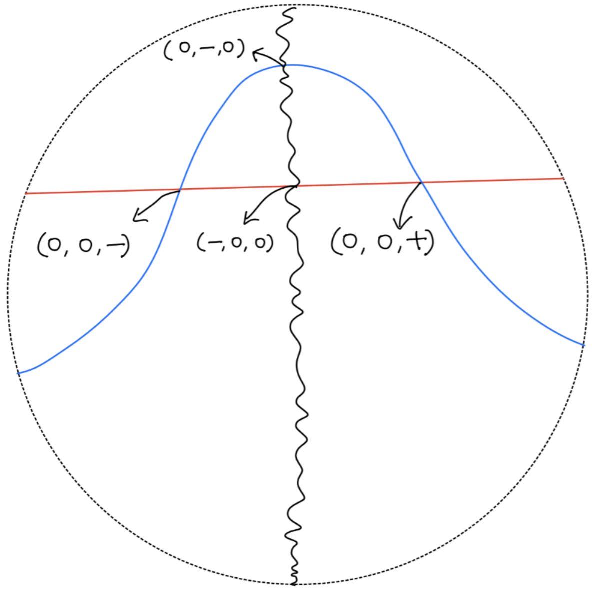

Figure 3.178

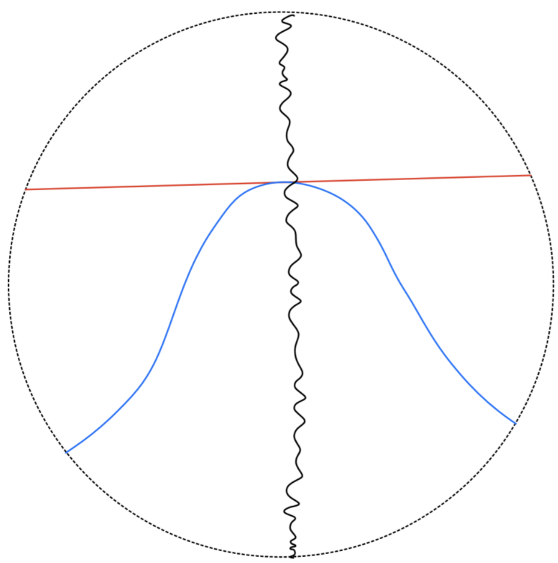

Figure 3.179

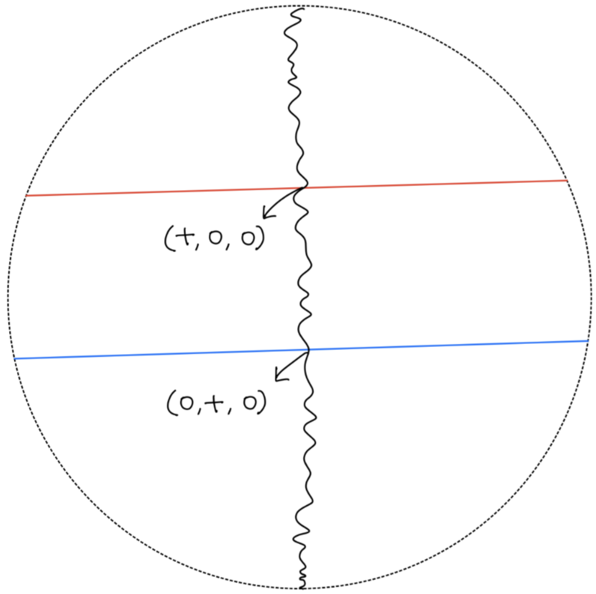

Figure 3.180

- 1 dimensional strata:

  $\{s_\bullet(sgn_1, 0, 0) \mid sgn_1 \in \{-,+\}\} \cup \{s_\bullet(0, sgn_2, 0) \mid sgn_2 \in \{-,+\}\} \cup \{s_\bullet(0, 0, sgn_3) \mid sgn_3 \in \{-,+\}\}$

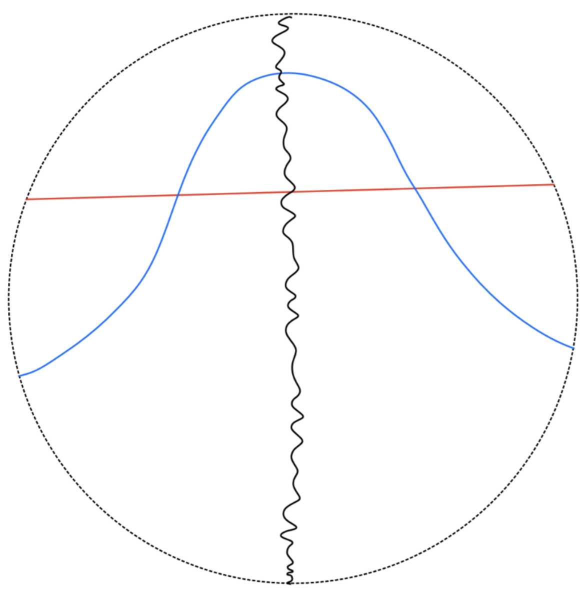

Figure 3.181

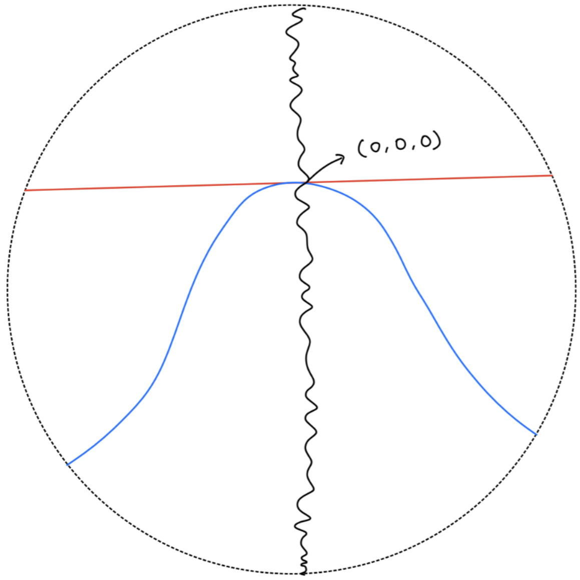

Figure 3.182

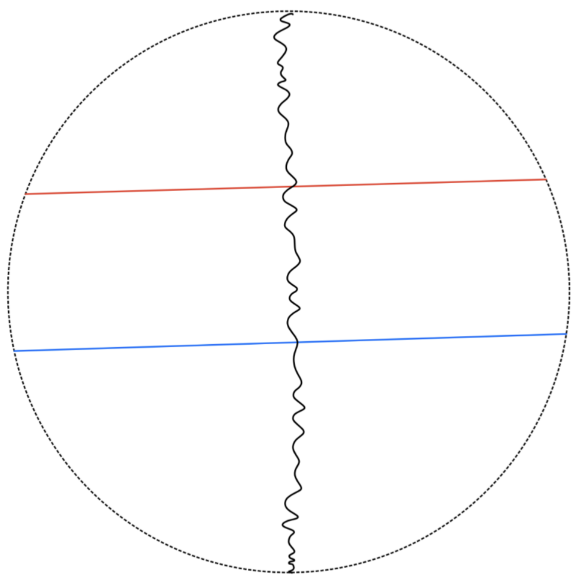

Figure 3.183

- 0 dimensional strata:

  $s_\bullet(0,0,0)$

Definition 145. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 146. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s.

Definition 147. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 148. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 149. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \text{ codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 150. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.

- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \mathrm{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$, then

  $$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

  This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.
- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

Stalks:

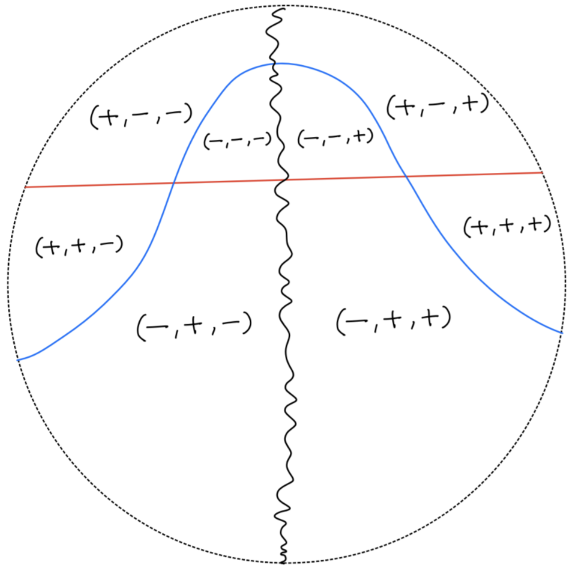

Figure 3.184

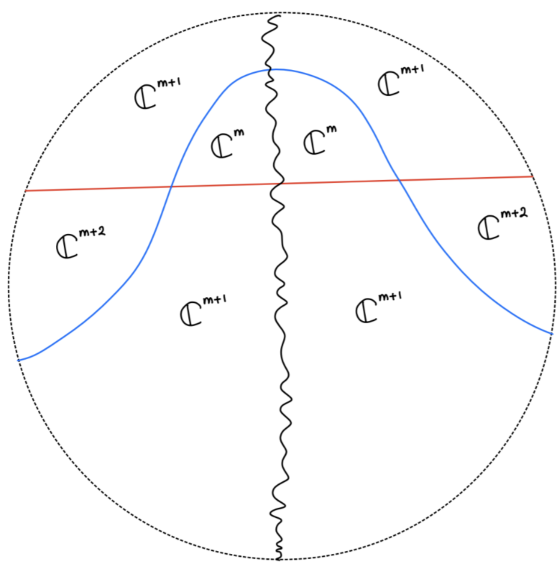

Figure 3.185

- $F_0(-,-,-) := \mathbb{C}^m$
- $F_0(-,-,+) := \mathbb{C}^m$
- $F_0(+,-,-) := \mathbb{C}^{m+1}$

- $F_0(+,-,+) := \mathbb{C}^{m+1}$
- $F_0(-,+,-) := \mathbb{C}^{m+1}$
- $F_0(-,+,+) := \mathbb{C}^{m+1}$
- $F_0(+,+,-) := \mathbb{C}^{m+2}$
- $F_0(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

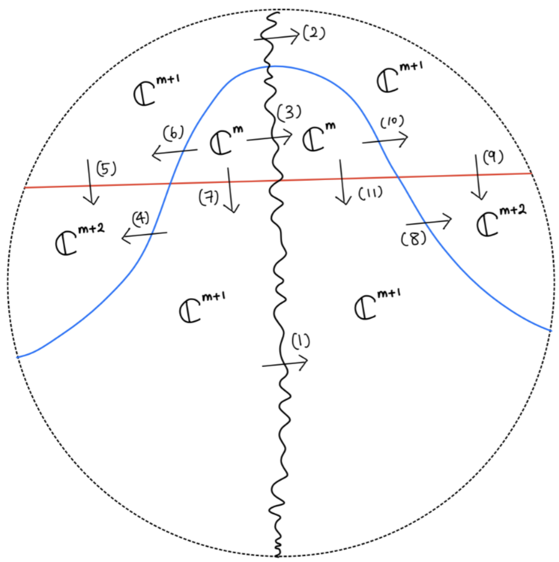

Figure 3.186

(1) $\mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m} \xrightarrow{\theta} \mathbb{C}^{m}$

(4) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(5) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(6) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(8) $\mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$

(9) $\mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$

(10) $\mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$

(11) $\mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord_3$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

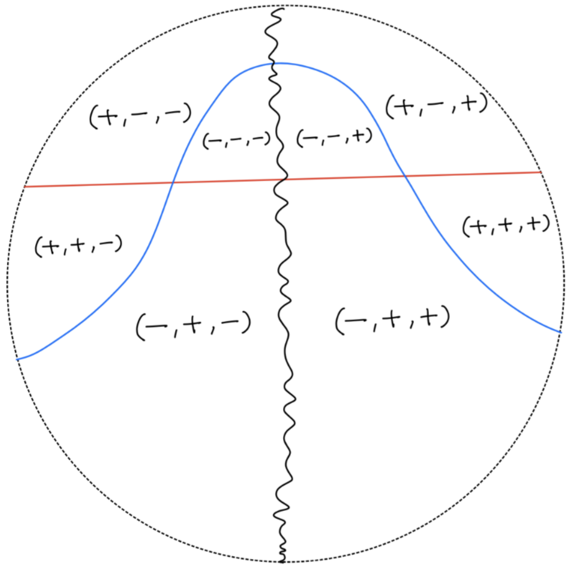

Figure 3.187

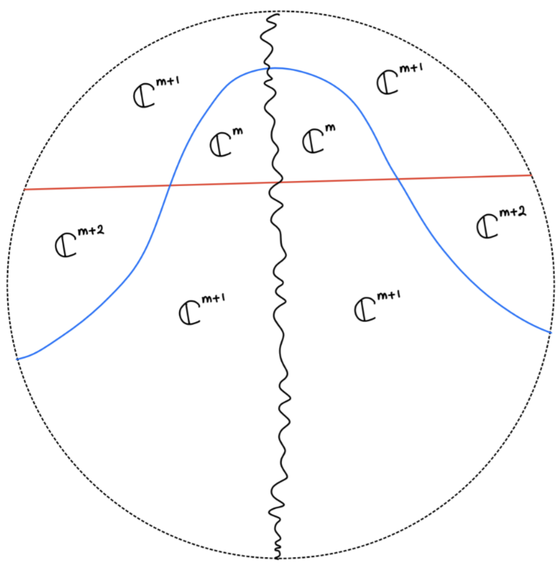

Figure 3.188

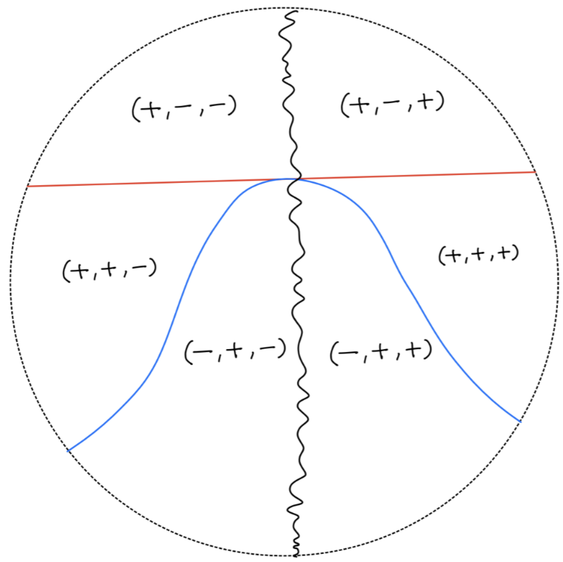

Figure 3.189

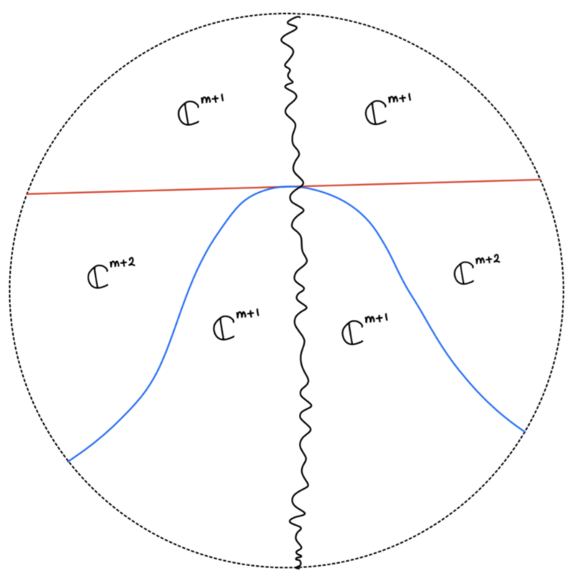

Figure 3.190

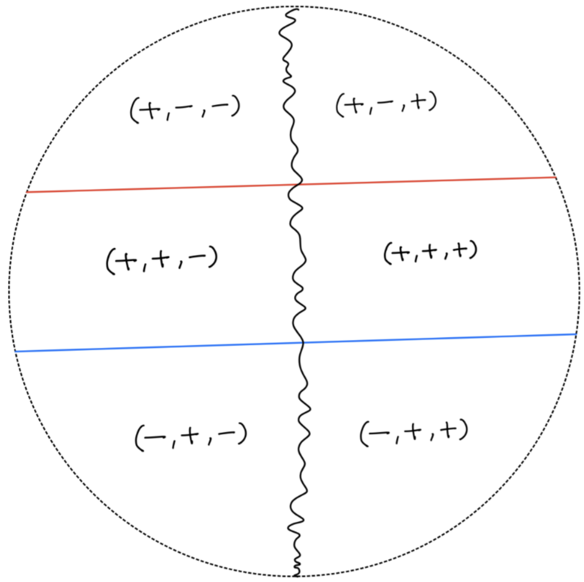

Figure 3.191

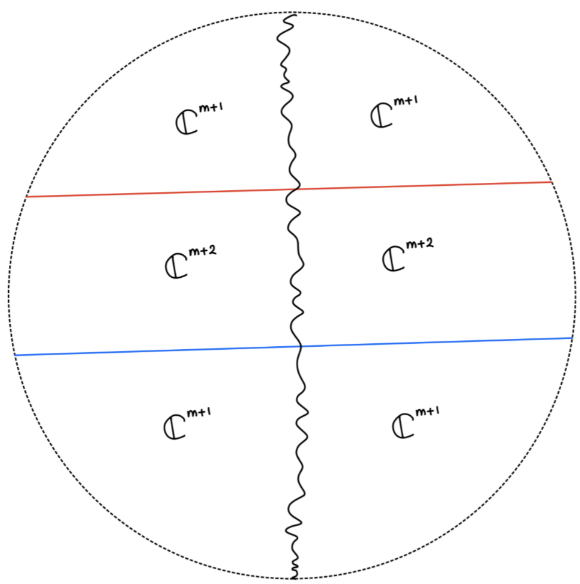

Figure 3.192

- $F_\bullet(-,-,-) := \mathbb{C}^m$
- $F_\bullet(-,-,+) := \mathbb{C}^m$
- $F_\bullet(+,-,-) := \mathbb{C}^{m+1}$
- $F_\bullet(+,-,+) := \mathbb{C}^{m+1}$

- $F_\bullet(-,+,-) := \mathbb{C}^{m+1}$
- $F_\bullet(-,+,+) := \mathbb{C}^{m+1}$
- $F_\bullet(+,+,-) := \mathbb{C}^{m+2}$
- $F_\bullet(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

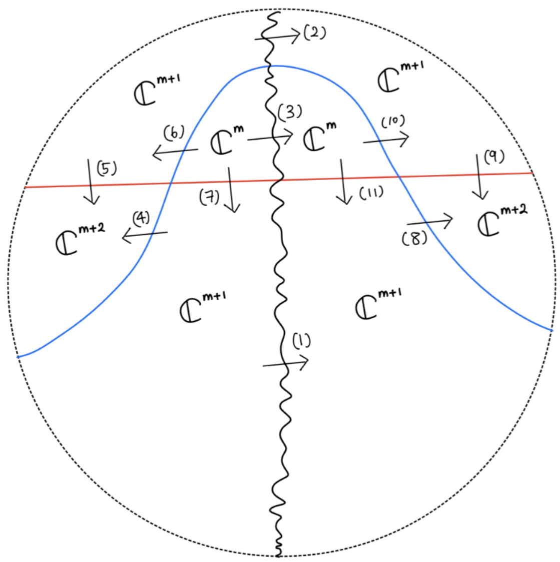

Figure 3.193

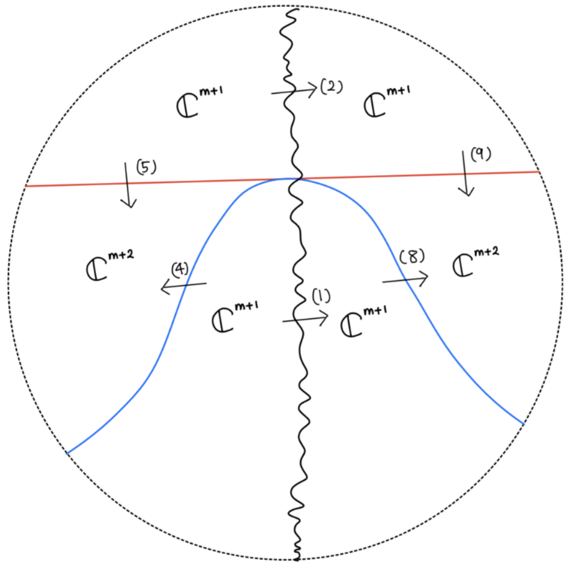

Figure 3.194

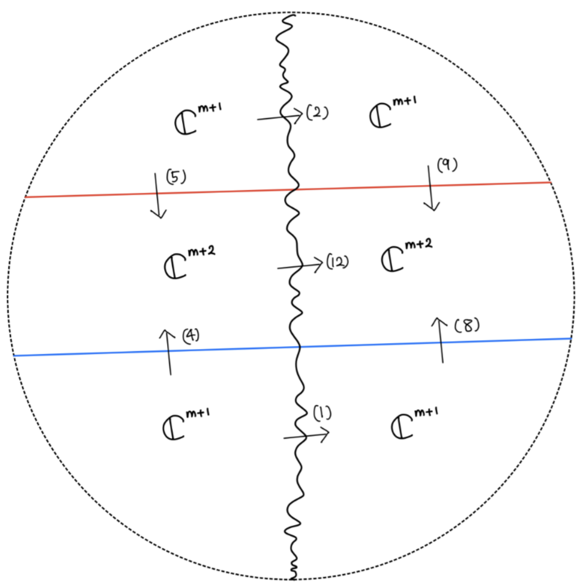

Figure 3.195

(1) $\mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m} \xrightarrow{\theta} \mathbb{C}^{m}$

(4) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(5) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(6) $\mathbb{C}^{m} \xrightarrow{\iota_1} \mathbb{C}^{m+1}$

(7) $\mathbb{C}^{m} \xrightarrow{\iota_0} \mathbb{C}^{m+1}$

(8) $\mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$

(9) $\mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$

(10) $\mathbb{C}^{m} \xrightarrow{\theta'} \mathbb{C}^{m+1}$

(11) $\mathbb{C}^{m} \xrightarrow{\theta''} \mathbb{C}^{m+1}$

(12) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

where

- $T(1,1,m+2,m+1) = \nu' \circ \nu$
- $T(1,2,m+2,m+2) = \mu' \circ \mu$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$ using the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$ where $pr_1 : (U\cap V)\times[0,1] \to (U\cap V)$ is the projection onto the first argument.

Definition 151. we define $\gamma_\bullet$ to be the composition

$$(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U\cap V)\times[0,1]} = \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times [0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V \times [0,1] \arrow[d, "pr_1"] \\
(U \cap V) \arrow[r, hook] & V
\end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 152. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of M.
Note that there is a diffeomorphism beteween $D_{r=2} \times (0,1)$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}^m$
- $F^3(-,-,+) := \mathbb{C}^m$
- $F^3(+,-,-) := \mathbb{C}^{m+1}$
- $F^3(+,-,+) := \mathbb{C}^{m+1}$

- $F^3(-,+,-) := \mathbb{C}^{m+1}$
- $F^3(-,+,+) := \mathbb{C}^{m+1}$
- $F^3(+,+,-) := \mathbb{C}^{m+2}$
- $F^3(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

- $F^3(-,sgn_2,sgn_3) \to F^3(+,sgn_2,sgn_3) := \iota_1$ except $F^(-,-,+) \to F^3(+,-,+):=\theta'$ and $F^(-,+,+) \to F^3(+,+,+):=\mu'$
- $F^3(sgn_1,-,sgn_3) \to F^3(sgn_1,+,sgn_3) := \iota_0$ except $F^(-,-,+) \to F^3(-,+,+):=\theta''$ and $F^(+,-,+) \to F^3(+,+,+):=\nu'$
- $F^3(-,-,-) \to F^3(-,-,+) := \theta$
- $F^3(+,-.-) \to F^3(+,-,+) := \nu$
- $F^3(-,+,-) \to F^3(-,+,+) := \mu$
- $F^3(+,+,-) \to F^3(+,+,+) := T$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$\text{(i)} \quad \begin{array}{ccc} F^3(-,-,-) & \longrightarrow & F^3(-,+,-) \\ \downarrow & & \downarrow \\ F^3(+,-,-) & \longrightarrow & F^3(+,+,-) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_1} \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$\text{(ii)} \quad \begin{array}{ccc} F^3(-,-,+) & \longrightarrow & F^3(-,+,+) \\ \downarrow & & \downarrow \\ F^3(+,-,+) & \longrightarrow & F^3(+,+,+) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\theta''} & \mathbb{C}^{m+1} \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \mu'} \\ \mathbb{C}^{m+1} & \xrightarrow{\nu'} & \mathbb{C}^{m+2} \end{array}$$

: acyclicity follows from the crossing condition of F_0

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^m \arrow[r, "\theta"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\theta'"] \\ \mathbb{C}^{m+1} \arrow[r, "\mu"] & \mathbb{C}^{m+2} \end{tikzcd}$$

: acyclicity follows from the crossing condition of F_0

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{m+1} \arrow[r, "\mu"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\mu'"] \\ \mathbb{C}^{m+2} \arrow[r, "\mathrm{T}"] & \mathbb{C}^{m+2} \end{tikzcd}$$

: μ, T are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^m \arrow[r, "\theta"] \arrow[d, "\iota_0"] & \mathbb{C}^{m} \arrow[d, "\theta''"] \\ \mathbb{C}^{m+1} \arrow[r, "\mu"] & \mathbb{C}^{m+1} \end{tikzcd}$$

: θ, μ are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{m+1} \arrow[r, "\nu"] \arrow[d, "\iota_1"] & \mathbb{C}^{m+1} \arrow[d, "\nu'"] \\ \mathbb{C}^{m+2} \arrow[r, "\mathrm{T}"] & \mathbb{C}^{m+2} \end{tikzcd}$$

: ν, T are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

(vii) the cubic diagram:

$$\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "{s(-,+,0)}"] \arrow[dd, "{s(0,+,-)}" near start] & & s(-,+,+) \arrow[dd, "{s(0,+,+)}"] \\
s(-,-,-) \arrow[ur, "{s(-,0,-)}"] \arrow[rr, "{s(-,-,0)}" near start] \arrow[dd, "{s(0,-,-)}"] & & s(-,-,+) \arrow[ur, "{s(-,0,+)}"] \arrow[dd, "{s(0,-,+)}" near start] & \\
 & s(+,+,-) \arrow[rr, "{s(+,+,0)}" near start] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "{s(+,0,-)}"] \arrow[rr, "{s(+,-,0)}"] & & s(+,-,+) \arrow[ur, "{s(+,0,+)}"] &
\end{tikzcd}$$

$$
=
\begin{tikzcd}
 & \mathbb{C}^{m+1} \arrow[rr, "\mu"] \arrow[dd, "\iota_1"] & & \mathbb{C}^{m+1} \arrow[dd, "\mu'"] \\
\mathbb{C}^{m} \arrow[ur, "\iota_0"] \arrow[rr, "\theta"] \arrow[dd, "\iota_1"] & & \mathbb{C}^{m} \arrow[ur, "\theta''"] \arrow[dd, "T"] & \\
 & \mathbb{C}^{m+2} \arrow[rr, "\theta'"] & & \mathbb{C}^{m+2} \\
\mathbb{C}^{m+1} \arrow[ur, "\iota_0"] \arrow[rr, "\nu"] & & \mathbb{C}^{m+1} \arrow[ur, "\nu'"] &
\end{tikzcd}
$$

Note that θ, μ, ν, T are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

$\square$

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the $cobord_3$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_*\mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V\times\{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V\times\{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

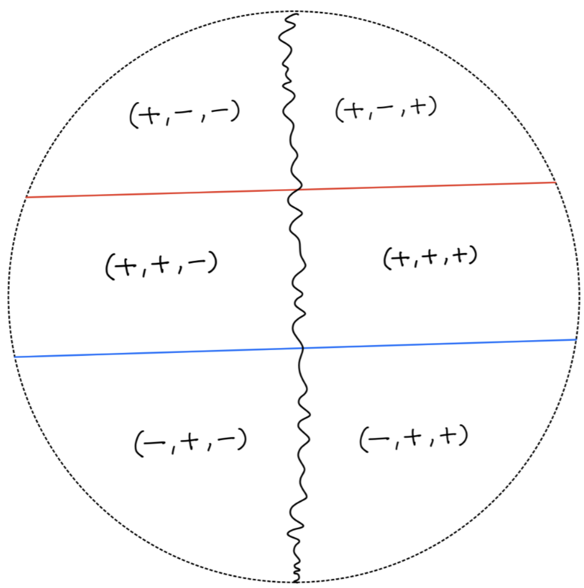

Figure 3.196

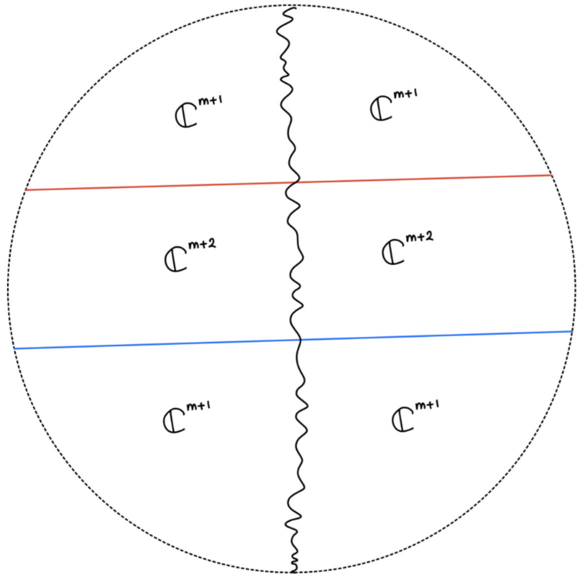

Figure 3.197

- $F_1(+,-,-) := \mathbb{C}^{m+1}$
- $F_1(+,-,+) := \mathbb{C}^{m+1}$
- $F_1(-,+,-) := \mathbb{C}^{m+1}$
- $F_1(-,+,+) := \mathbb{C}^{m+1}$
- $F_1(+,+,-) := \mathbb{C}^{m+2}$
- $F_1(+,+,+) := \mathbb{C}^{m+2}$

Generization maps:

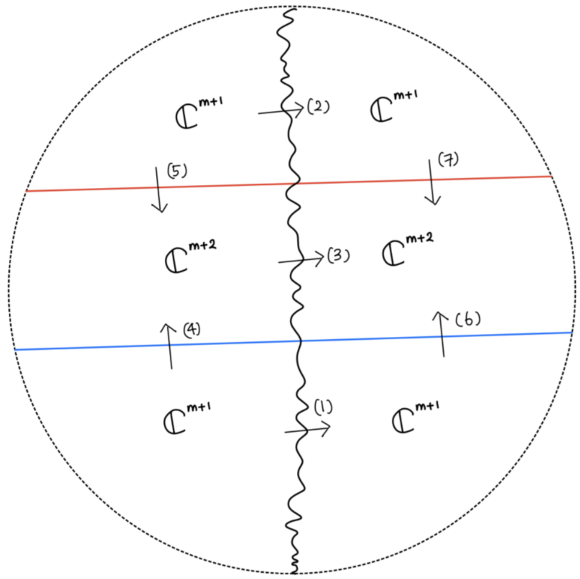

Figure 3.198

(1) $\mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$

(2) $\mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$

(3) $\mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$

(4) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(5) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

(6) $\mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$

(7) $\mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$
- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 153. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.8 4th sheaf cobordism

In this section, we define $cobord_4$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

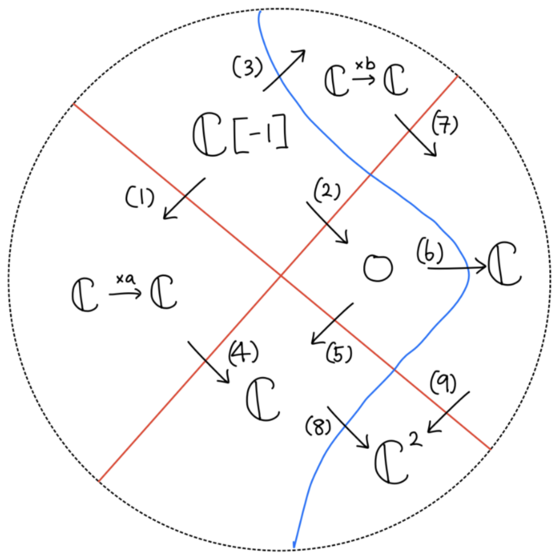

Figure 3.199

Generization maps:

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(5)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

to

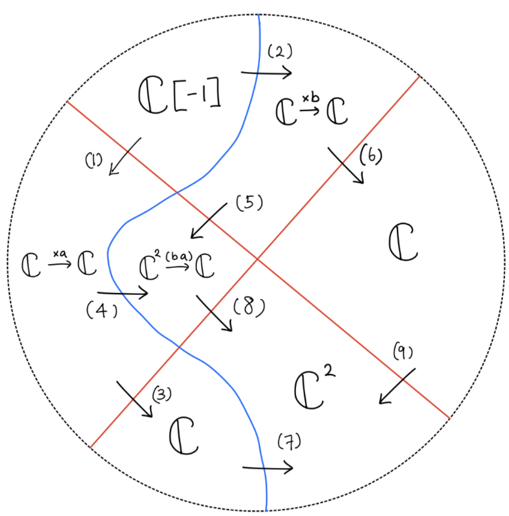

Figure 3.200

Generization maps:

$$(1)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(2) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(3) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(4) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(5) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(6) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(7) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(8) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(9) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

Notations

Definition 154. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 155. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$
$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$
$$\Phi_{t_0}^{squig} : [0,1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_4$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_4$.

Definition 156. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{\bullet}^{symbol}$ to be smooth maps

$$\Phi_{\bullet}^{0} : (S^1)^n \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0,1]_t \to M \times [0,1]_t$$
$$\Phi_{\bullet}^{squig} : \coprod_{1 \le i \le k} ([0,1] \times [a_i, b_i]_t) \to M \times [0,1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_{\bullet}^{symbol}$ a co-orientation of $\Phi_{\bullet}^{symbol}$.

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as $\Lambda_\bullet^{symbol}$. Later in the section, $\Lambda_\bullet^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord_4$.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as $\Lambda_\bullet$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_\bullet$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord_4$.

Definition 157. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} e^{(\frac{x^2}{x^2-1})}(\frac{1}{2} - t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$ if $t \neq \frac{1}{2}$
- $\{(1,0),\ (-1,0), (0, \frac{1}{2} - t)\} \subset Graph(\Psi_t)$

Definition 158. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$
$$(x,z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 159. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x,z) \in D_{r=2} \mid x = \Psi_0(z)\}$
- λ_0^∞ is the union of the following two components
  (i) $\{(x,z) \in D_{r=2} \mid z = x\}$
  (ii) $\{(x,z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_0^∞: hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 160. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x,z) \in D_{r=2} \mid x = \Psi_1(z)\}$
- λ_1^∞ is the union of the following two components
  (i) $\{(x,z) \in D_{r=2} \mid z = x\}$
  (ii) $\{(x,z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 161. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda_\bullet^0 := \{(x,z,t) \in D_{r=2} \times [0,1] \mid x = \Psi_t(z)\}$
- $\lambda_\bullet^\infty$ is the union of the following two components
  (i) $\{(x,z,t) \in D_{r=2} \times [0,1] \mid z = x\}$
  (ii) $\{(x,z,t) \in D_{r=2} \times [0,1] \mid z = -x\}$

2. We define co-orientations $\xi_\bullet^{symbol}$ of $\lambda_\bullet^{symbol}$ as follows

- $\xi_\bullet^0$: hairs are pointing towards the left i.e. coefficients of dx are negative.

- $\xi_\bullet^\infty$: hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 162. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda_\bullet^0$, $\lambda_\bullet^\infty$, and $\lambda_\bullet^{squig}$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 163. $\mathrm{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 164. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := \ & \{(x,z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \mathrm{sgn}(x - \Psi_{t_0}(z)) = sgn_1, \ \ \mathrm{sgn}(x - z) = sgn_2, \\ & \mathrm{sgn}((-x - z) = sgn_3\} \end{aligned}$$

Definition 165. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := \ & \{(x,z,t) \in D_{r=2} \times [0,1] \mid \\ & \mathrm{sgn}(x - \Psi_{t}(z)) = sgn_1, \ \ \mathrm{sgn}(x - z) = sgn_2, \\ & \mathrm{sgn}((-x - z) = sgn_3\} \end{aligned}$$

Definition 166. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the above notations:

1. $\mathcal{S}_0$:

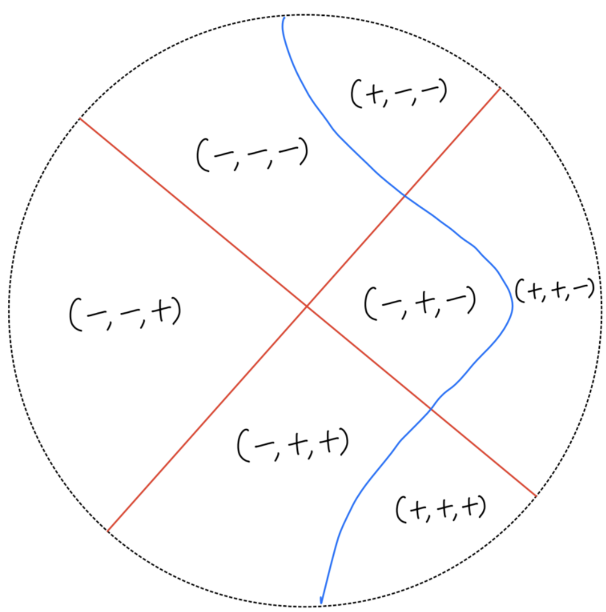

Figure 3.201

- 2 dimensional strata:

  $\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\}$ for i=1,2,3, except $s_0(+,-,+)\}$

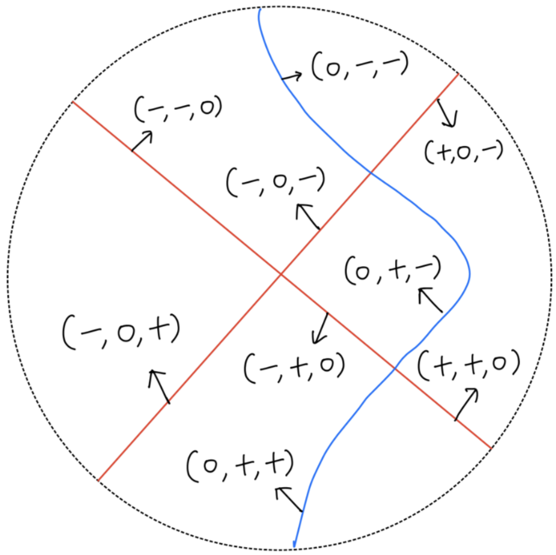

Figure 3.202

- 1 dimensional strata:

  $\{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3, except $s_0(0,-,+)\}$

  $\cup\, \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3, except $s_0(+,0,+)\}$

  $\cup\, \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_0(+,-,0)\}$

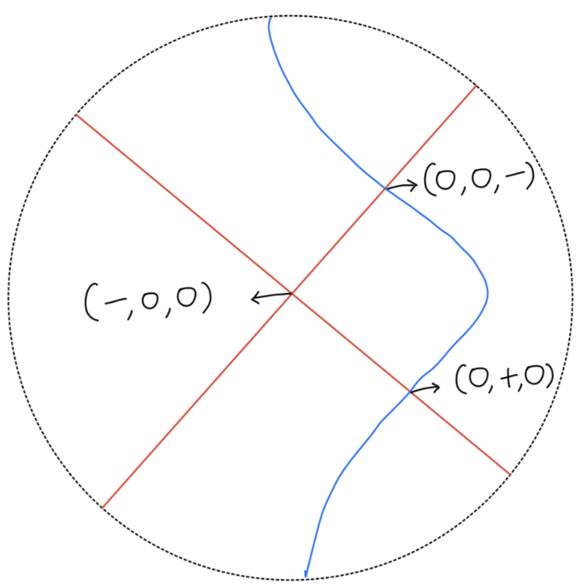

Figure 3.203

- 0 dimensional strata:

$s_0(-,0,0),\ s_0(0,0,-),\ s_0(0,+,0)$

2. $\mathcal{S}_1$:

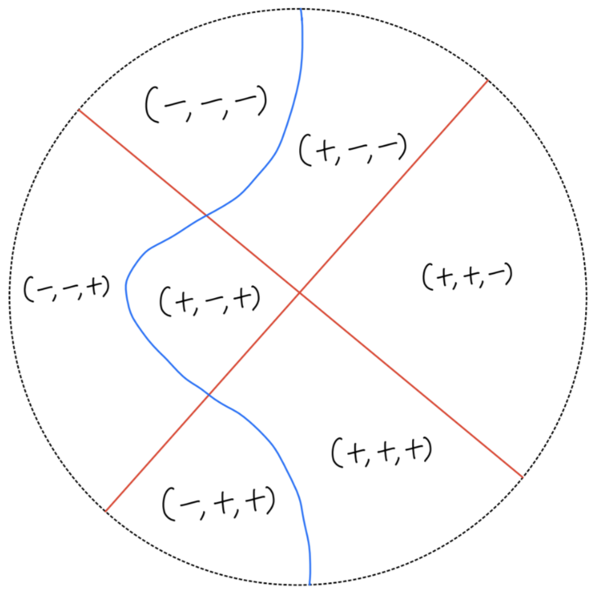

Figure 3.204

- 2 dimensional strata:

  $\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3, except $s_1(-,+,-)\}$

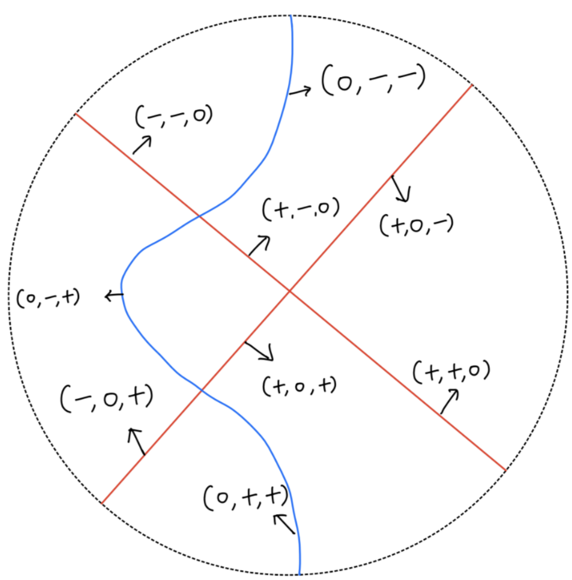

Figure 3.205

- 1 dimensional strata:

  $\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3, except $s_1(0,+,-)\}$

  $\cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3, except $s_1(-,0,-)\}$

  $\cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_1(-,+,0)\}$

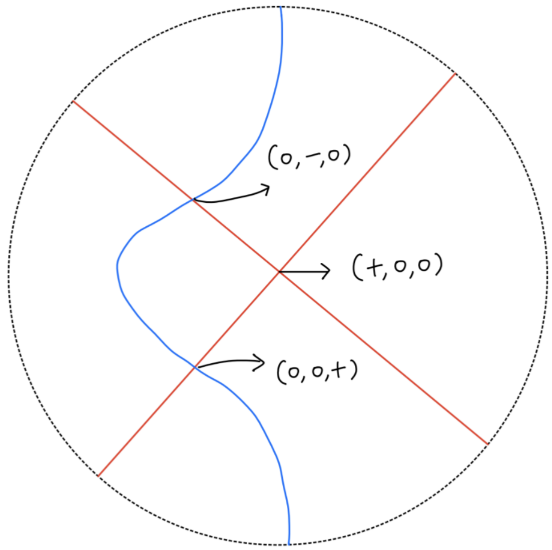

Figure 3.206

- 0 dimensional strata:

  $s_1(0,-,0),\ s_1(+,0,0),\ s_1(0,0,+)$

3. $\mathcal{S}_\bullet$:

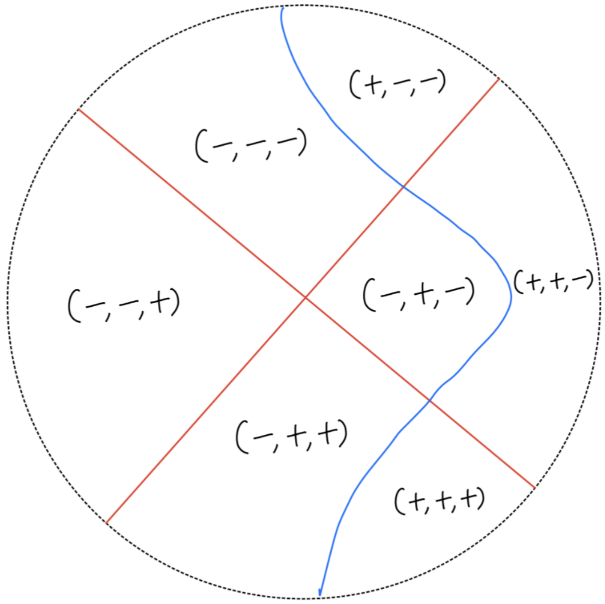

Figure 3.207

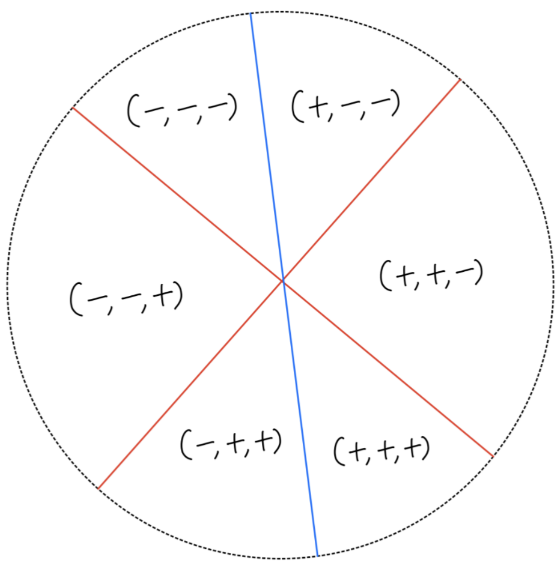

Figure 3.208

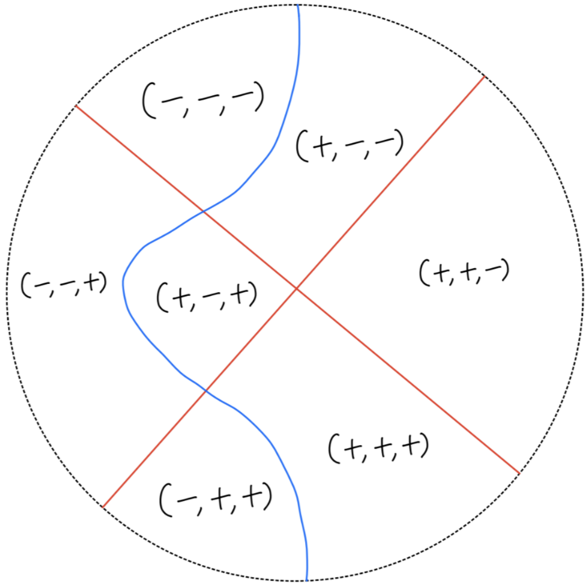

Figure 3.209

- 3 dimensional strata:

  $\{s_\bullet(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\} \text{ for i=1,2,3}\}$

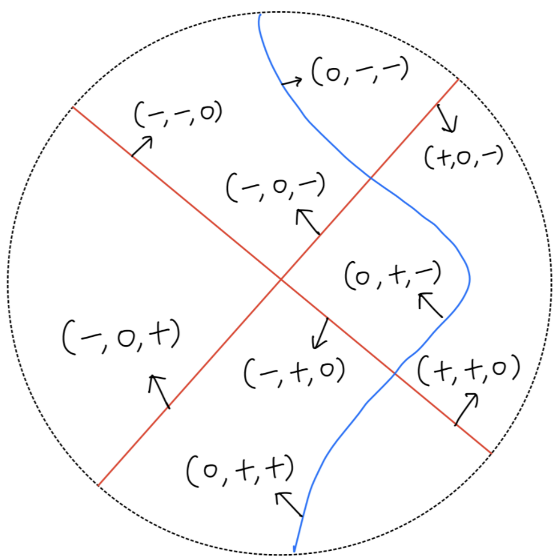

Figure 3.210

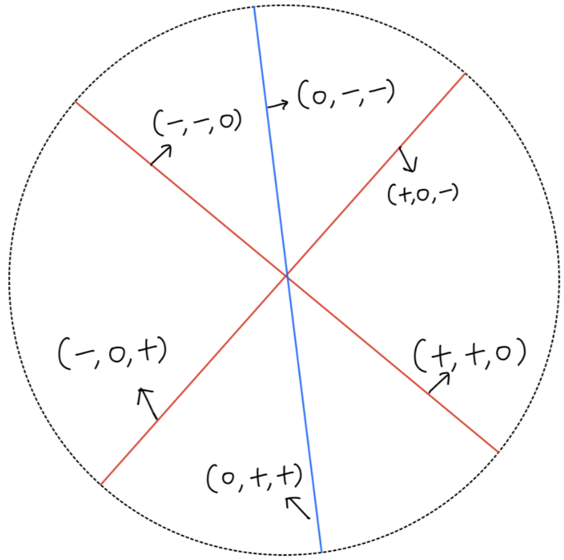

Figure 3.211

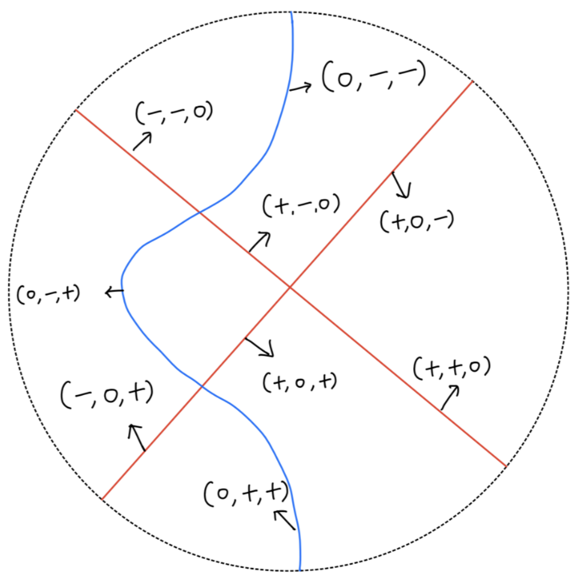

Figure 3.212

- 2 dimensional strata:

  $\{s_\bullet(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_\bullet(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3$\} \cup \{s_\bullet(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2$\}$

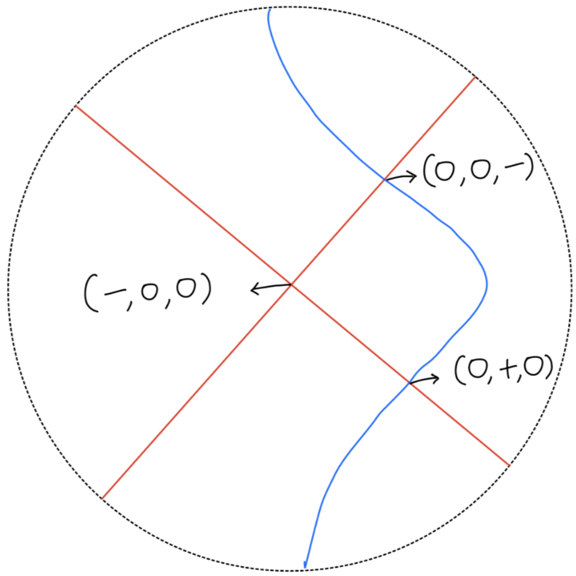

Figure 3.213

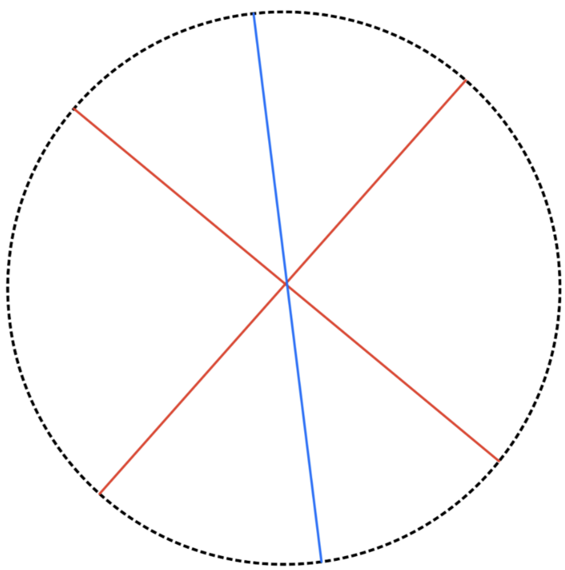

Figure 3.214

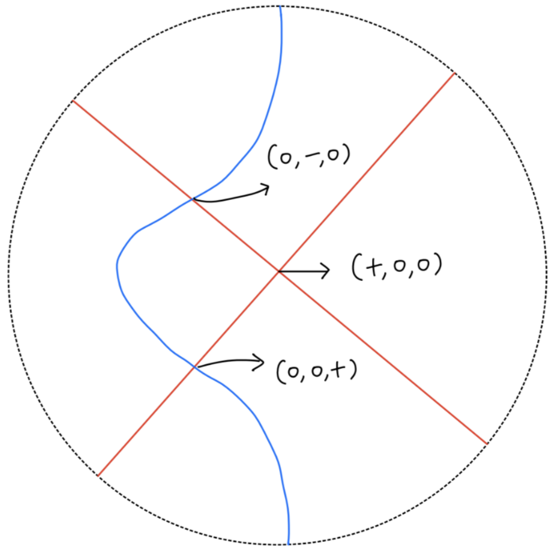

Figure 3.215

- 1 dimensional strata:

  $\{s_\bullet(sgn_1, 0, 0) \mid sgn_1 \in \{-,+\}\} \cup \{s_\bullet(0, sgn_2, 0) \mid sgn_2 \in \{-,+\}\} \cup \{s_\bullet(0, 0, sgn_3) \mid sgn_3 \in \{-,+\}\}$

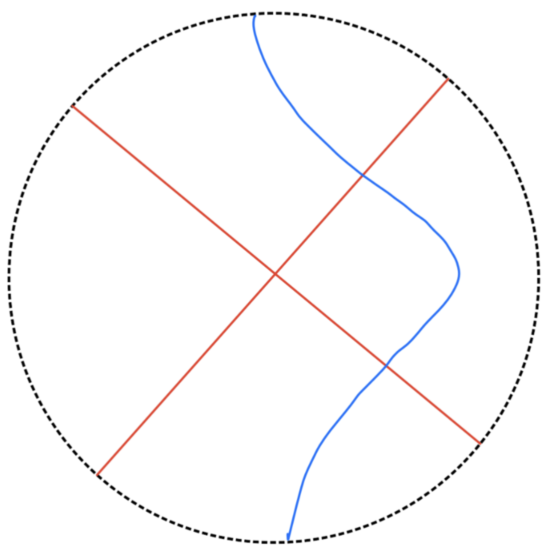

Figure 3.216

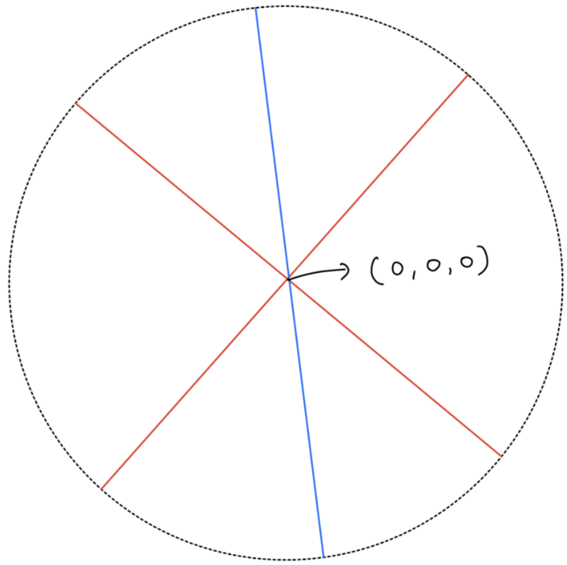

Figure 3.217

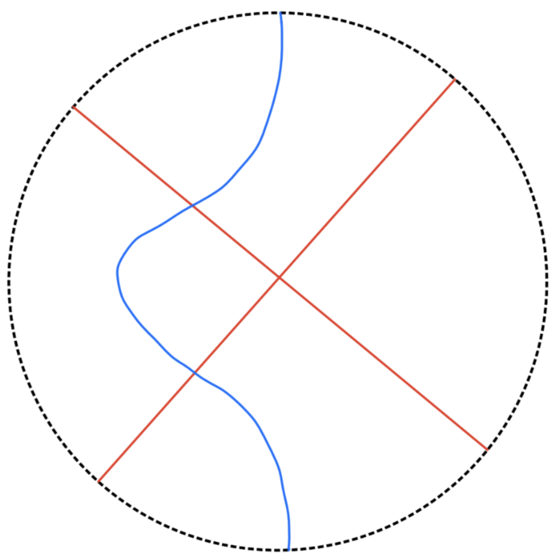

Figure 3.218

- 0 dimensional strata:

  $s_\bullet(0,0,0)$

Definition 167. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 168. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s.

Definition 169. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 170. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 171. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \text{ codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 172. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \text{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$,

then

$$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.
- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

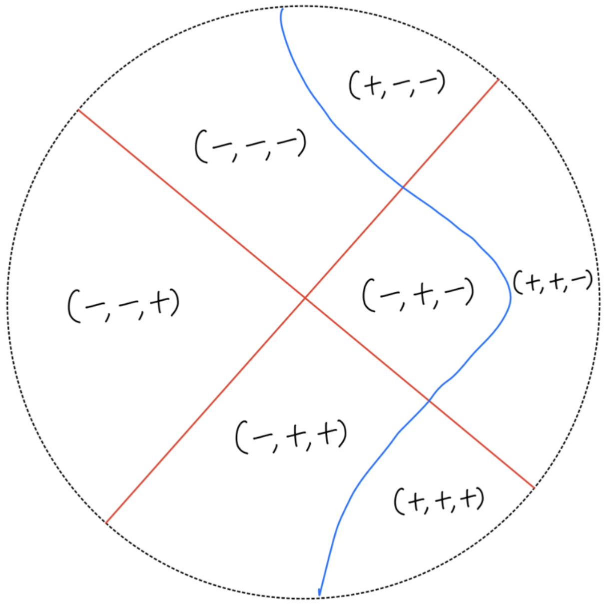

Figure 3.219

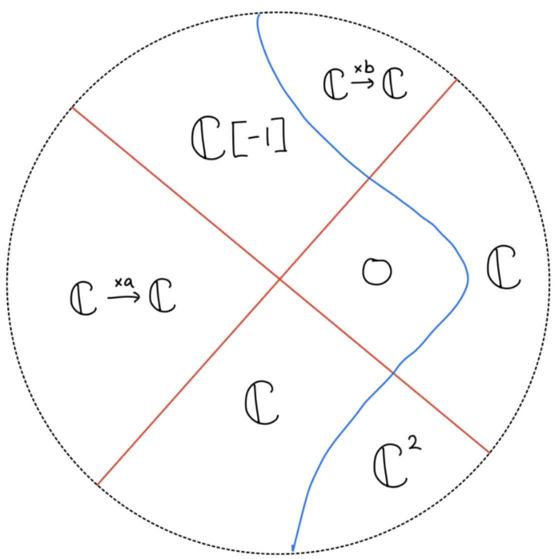

Figure 3.220

- $F_0(-,-,-) := \mathbb{C}[-1]$
- $F_0(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_0(-,-,+) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(-,+,-) := 0$

- $F_0(+,+,-) := \mathbb{C}$
- $F_0(-,+,+) := \mathbb{C}$
- $F_0(+,+,+) := \mathbb{C}^2$

Generization maps:

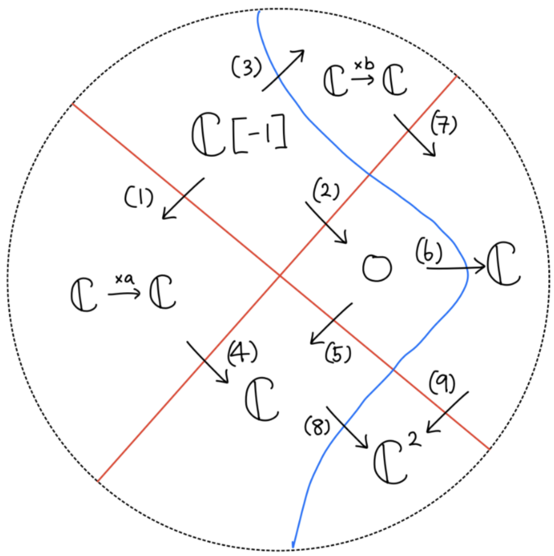

Figure 3.221

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(5)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(6)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(9)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord_4$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

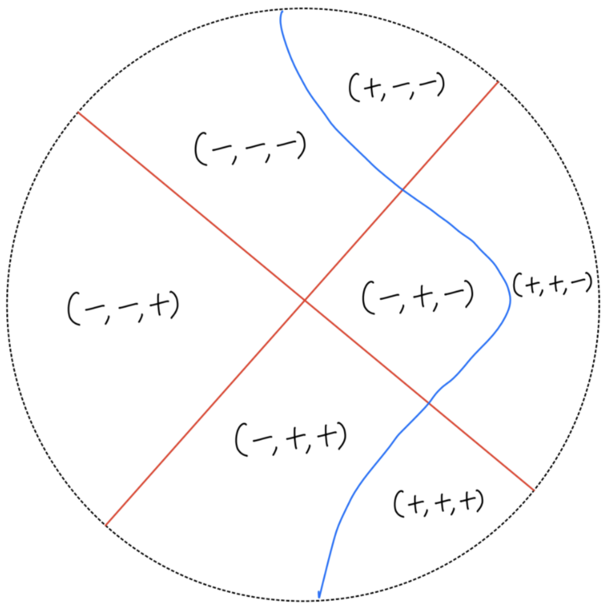

Figure 3.222

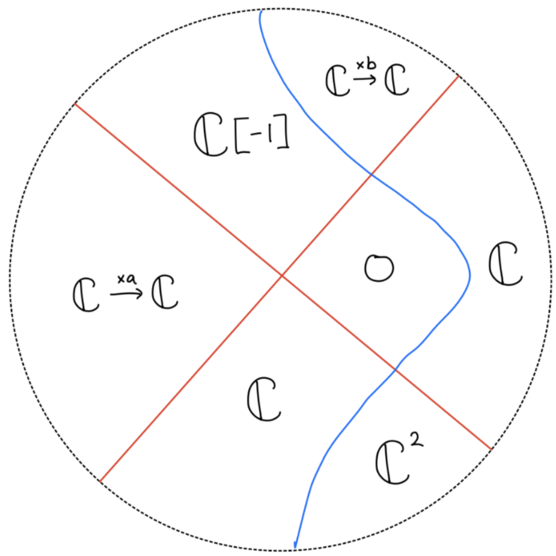

Figure 3.223

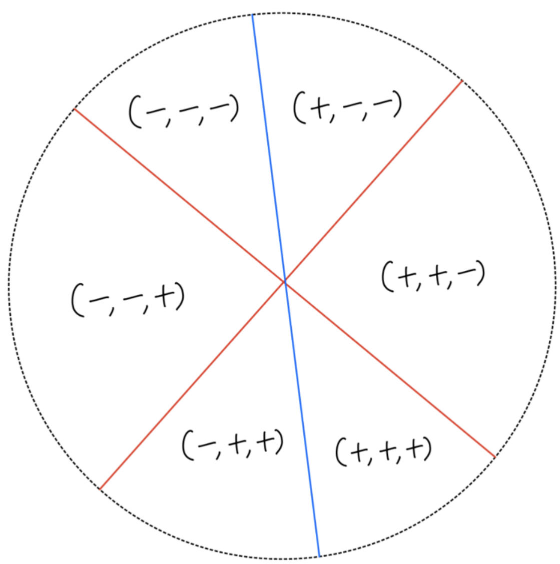

Figure 3.224

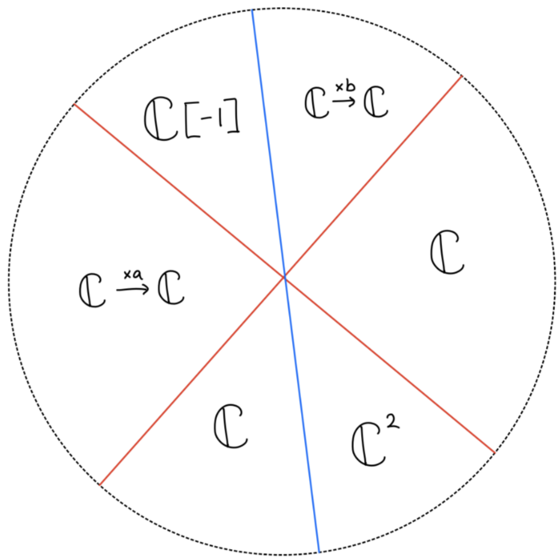

Figure 3.225

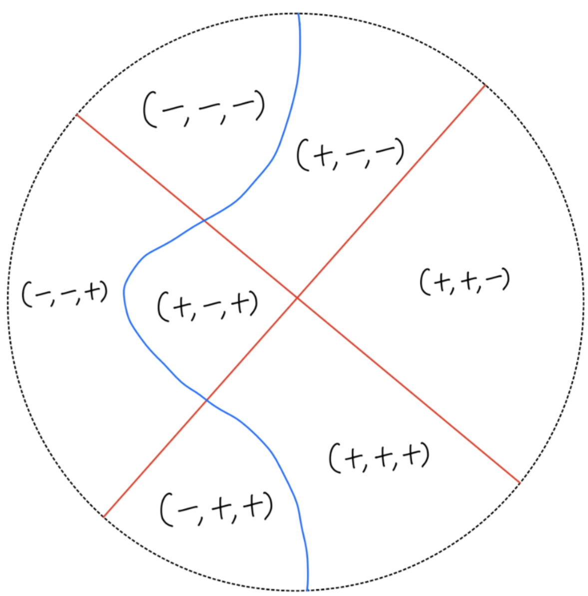

Figure 3.226

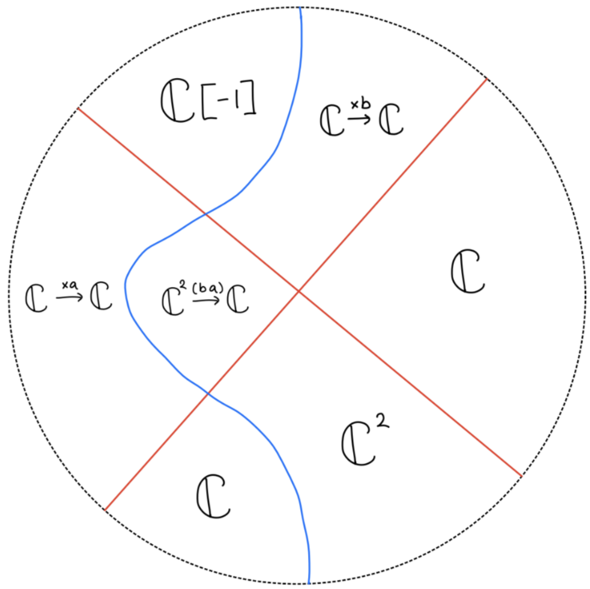

Figure 3.227

- $F_\bullet(-,-,-) := \mathbb{C}[-1]$
- $F_\bullet(-,-,+) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_\bullet(-,+,-) := 0$
- $F_\bullet(-,+,+) := \mathbb{C}$
- $F_\bullet(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_\bullet(+,-,+) := \mathbb{C}^2 \xrightarrow{\times(b\ \ a)} \mathbb{C}$
- $F_\bullet(+,+,-) := \mathbb{C}$
- $F_\bullet(+,+,+) := \mathbb{C}^2$

Generization maps:

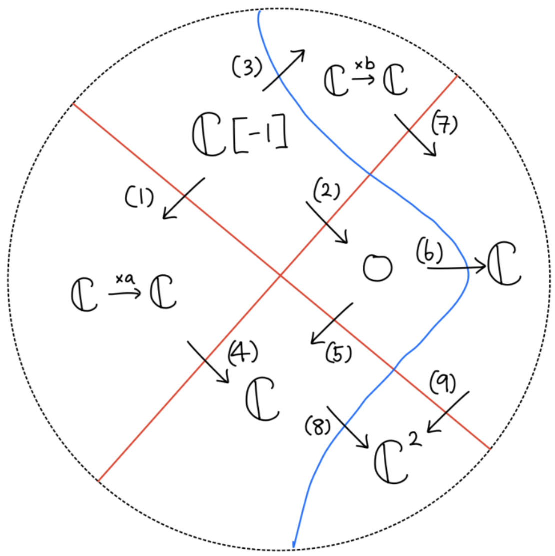

Figure 3.228

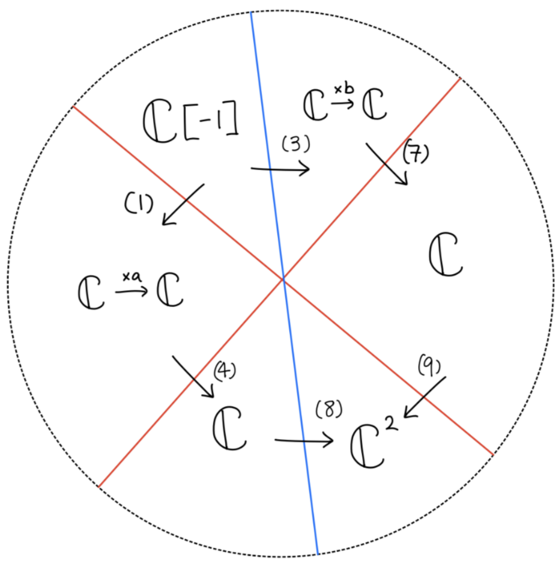

Figure 3.229

Here!!!!!

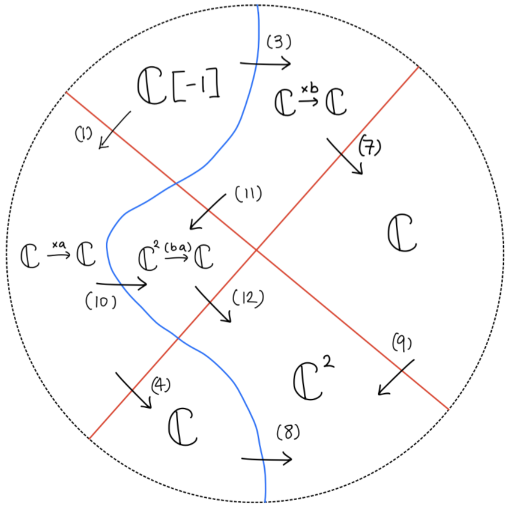

Figure 3.230

(1) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(2) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

(3) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(4) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(5) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(6) $$\begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(7)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(8)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(9)
$$\begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(10)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(11)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(12)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$ using the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$ where $pr_1 : (U\cap V)\times[0,1] \to (U\cap V)$ is the projection onto the first argument.

Definition 173. we define $\gamma_\bullet$ to be the composition

$$(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U\cap V)\times[0,1]} = \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times [0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V \times [0,1] \arrow[d, "pr_1"] \\
(U \cap V) \arrow[r, hook] & V
\end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 174. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of M.
Note that there is a diffeomorphism beteween $D_{r=2} \times (0,1)$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}[-1]$
- $F^3(-,-,+) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+,-,+) := \mathbb{C}^2 \xrightarrow{(b\ a)} \mathbb{C}$

- $F^3(-,+,-) := 0$
- $F^3(-,+,+) := \mathbb{C}$
- $F^3(+,+,-) := \mathbb{C}$
- $F^3(+,+,+) := \mathbb{C}^2$

Generization maps:

$$\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
 & s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}$$

(1) $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$

(2) $\begin{tikzcd} \mathbb{C} \arrow[r, "\times a"] & 0 \\ 0 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$

(3) $\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$

(4) $\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$

$$(5)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

$$(6)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(10)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

$$(11)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

$$(12)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

$$(\mathrm{i})\quad \begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \arrow[r, "\times a"] \arrow[d, "(\iota_1,\times 1)"] & \mathbb{C} \arrow[r, "(\times 1,\times 0)"] & \mathbb{C} \arrow[d, "\iota_1"] \\ \mathbb{C}^2 \arrow[r, "(b\ a)"] & \mathbb{C} \arrow[r, "(id,0)"] & \mathbb{C}^2 \end{tikzcd}$$

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[rr, "(\times 0,\times 1)"] \arrow[d, "(\times 0,\times 1)"] & & \mathbb{C} \arrow[r, "\times a"] & \mathbb{C} \arrow[d, "(\iota_1,\times 1)"] \\ \mathbb{C} \arrow[r, "\times b"] & \mathbb{C} \arrow[r, "(\iota_0,\times 1)"] & \mathbb{C}^2 \arrow[r, "(b\ a)"] & \mathbb{C} \end{tikzcd}$$

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} 0 \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C} \arrow[d, "\iota_1"] \\ \mathbb{C} \arrow[r, "\iota_0"] & \mathbb{C}^2 \end{tikzcd}$$

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}[-1] \arrow[r, "(\times 0,\times 1)"] \arrow[d] & \mathbb{C} \arrow[r, "\times a"] & \mathbb{C} \arrow[d, "(\times 1,\times 0)"] \\ 0 \arrow[rr] & & \mathbb{C} \end{tikzcd}$$

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C} \arrow[r, "\times b"] \arrow[d, "(\times 1,\times 0)"] & \mathbb{C} \arrow[r, "(\iota_0,\times 1)"] & \mathbb{C}^2 \arrow[r, "(b\ a)"] & \mathbb{C} \arrow[d, "(id,\times 0)"] \\ \mathbb{C} \arrow[rrr, "\iota_0"] & & & \mathbb{C}^2 \end{tikzcd}$$

(vii) the cubic diagram:

$$\begin{tikzcd} & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\ s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\ & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\ s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] & \end{tikzcd}$$

=

$$
\begin{tikzcd}
 & 0 \arrow[rr, "0"] \arrow[dd, "0"] & & \mathbb{C} \arrow[dd, "\iota_1"] \\
\mathbb{C}[-1] \arrow[ur, "0"] \arrow[rr, "(\times 0, \times 1)"] \arrow[dd, "(\times 0, \times 1)"'] & & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \arrow[ur, "(\times 1, \times 0)"'] \arrow[dd, "\iota_0"] & \\
 & \mathbb{C} \arrow[rr, "(\iota_1, \times 1)"] & & \mathbb{C}^2 \\
\mathbb{C} \xrightarrow{\times b} \mathbb{C} \arrow[ur, "(\times 1, \times 0)"] \arrow[rr, "(\iota_0, \times 1)"] & & \mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C} \arrow[ur, "(id, \times 0)"'] &
\end{tikzcd}
$$

whose total complex is

$$
\mathrm{Tot}\left(
\begin{tikzcd}
\mathbb{C} \arrow[r, "\times b"] & \mathbb{C} \arrow[r, "(M_1, \times 1)"] & \mathbb{C}^3 \arrow[r, "(0\ \ b\ \ a)"] & \mathbb{C} \arrow[r, "(M_2, \times 0)"] & \mathbb{C}^2 \\
0 \arrow[r] & \mathbb{C} \arrow[r, "(\times 0, \times 1)"] \arrow[u, "(\times 0, \times 1)"] & \mathbb{C} \arrow[r, "\times a"] \arrow[u, "(M_3, \times 1)"] & \mathbb{C} \arrow[r, "(\times(-1), \times 0)"] & \mathbb{C} \arrow[u, "\iota_1"]
\end{tikzcd}
\right)
$$

where

$$
M_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
$$

$$
M_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
M_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

This is equal to

$$\mathrm{Tot}\left(\begin{tikzcd} \mathbb{C} \arrow[r, "M_1"] & \mathbb{C}^2 \arrow[r, "M_2"] & \mathbb{C} \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & \mathbb{C}^2 \arrow[u, "M_3"] \arrow[r, "M_5"] & \mathbb{C} \arrow[u, "M_4"] \arrow[r, "M_6"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}\right)$$

$$M_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & b & a & 0 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which is the following cochain complex

$$0 \to \mathbb{C}^3 \xrightarrow{M_1} \mathbb{C}^6 \xrightarrow{M_2} \mathbb{C}^3$$

$$M_1 = \begin{pmatrix} 1 & -b & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & b & a & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The final cochain complex is acyclic because of the following reasons

- $M_2 \circ M_1 = 0$
- M_1 is a monomorphism
- M_2 is an epimorphism

□

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the $cobord_4$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_*\mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V \times \{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V\times\{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

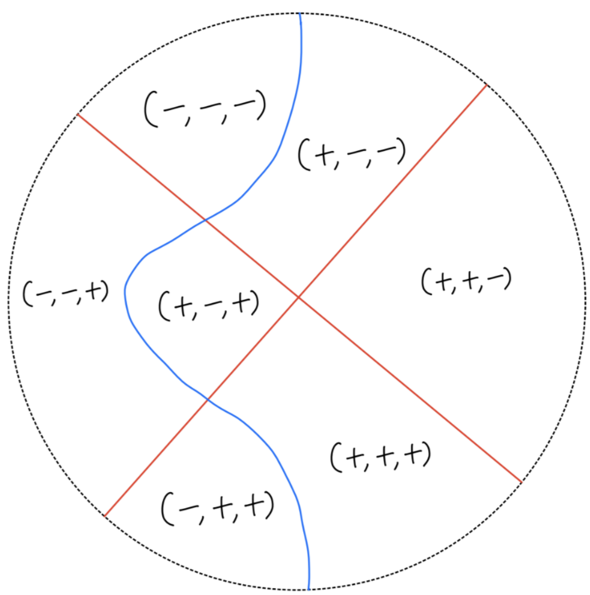

Figure 3.231

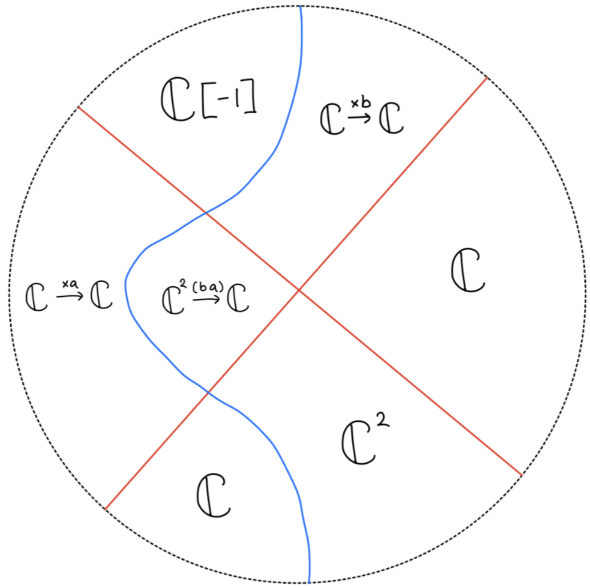

Figure 3.232

- $F_1(-,-,-) := \mathbb{C}[-1]$
- $F_1(+,-,-) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_1(-,-,+) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_1(+,-,+) := \mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C}$
- $F_1(+,+,-) := \mathbb{C}$
- $F_1(-,+,+) := \mathbb{C}$
- $F_1(+,+,+) := \mathbb{C}^2$

Generization maps:

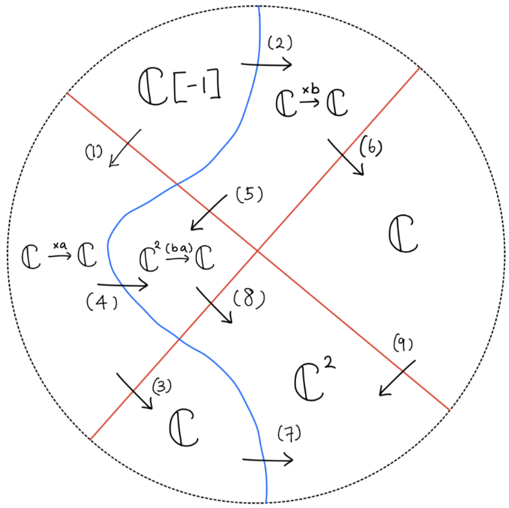

Figure 3.233

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times a"'] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\times b"'] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

(4)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b \ \ a)"'] \end{tikzcd}$$

(5)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b \ \ a)"'] \end{tikzcd}$$

(6)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times 1"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$(7)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

$$(8)\quad \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

$$(9)\quad \begin{tikzcd} 0 \arrow[r] & 0 \\ \mathbb{C} \arrow[u] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 175. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.9 4th sheaf cobordism'

In this section, we define $cobord'_4$, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism starting from

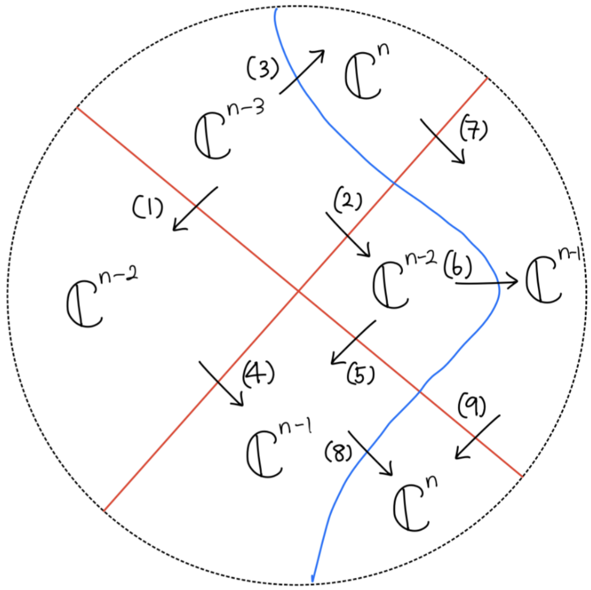

Figure 3.234

Generization maps:

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(5) ι_0

(6) ι_1

(7) ι_0

(8) ι_1

(9) ι_0

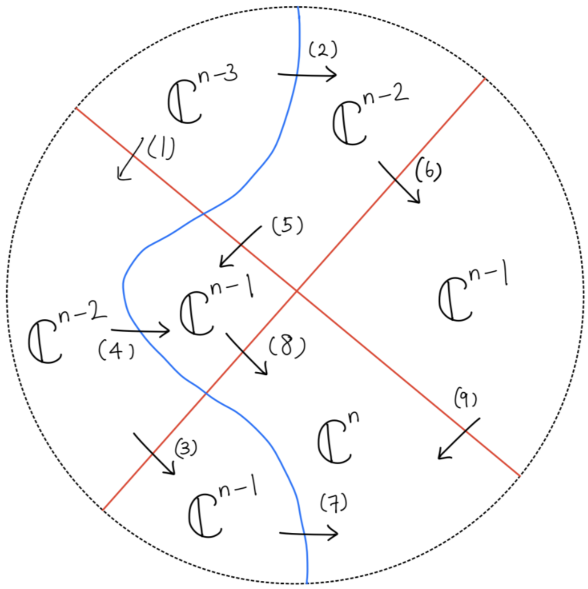

Figure 3.235

Generization maps:

(1) ι_0

(2) ι_1

(3) ι_0

(4) ι_1

(5) ι_0

(6) ι_0

(7) ι_1

(8) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(9) ι_0

Notations

Definition 176. M denotes a Riemann sphere with two punctures at 0 and ∞. M is diffeomorphic to a cylinder.

Definition 177. For $t_0 \in \{0, 1\}$ and $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{t_0}^{symbol}$ to be smooth maps

$$\Phi_{t_0}^{0} : (S^1)^n \to M$$

$$\Phi_{t_0}^{\infty} : (S^1)^m \to M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \to M$$

2. we denote $\Xi_{t_0}^{symbol}$ a co-orientation of $\Phi_{t_0}^{symbol}$.

3. we denote the pair $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$ as $\Lambda_{t_0}^{symbol}$. When $symbol \in \{0, \infty\}$, this could be thought as a front projection of a Legendrian living inside the cocircle bundle of M, so we will use $\Lambda_{t_0}^{symbol}$ to denote both

4. we denote the triple $(\Lambda_{t_0}^{0}, \Lambda_{t_0}^{\infty}, \Lambda_{t_0}^{squig})$ as Λ_{t_0} and call it the squiggly diagram at t_0. Later in the section, Λ_0 will be used to denote the squiggly diagram at the beginning of the isotopy underlying $cobord_4'$ and Λ_1 will be used to denote the squiggly diagram at the end of the isotopy underlying $cobord_4'$.

Definition 178. For $symbol \in \{0, \infty, squig\}$

1. we denote $\Phi_{\bullet}^{symbol}$ to be smooth maps

$$\Phi_{\bullet}^{0} : (S^1)^n \times [0, 1]_t \to M \times [0, 1]_t$$

$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0, 1]_t \to M \times [0, 1]_t$$

$$\Phi_{\bullet}^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \to M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript 't' to distinguish the time coordinates from the space coordinates.

2. we denote $\Xi_\bullet^{symbol}$ a co-orientation of $\Phi_\bullet^{symbol}$.

3. we denote the pair $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$ as $\Lambda_\bullet^{symbol}$. Later in the section, $\Lambda_\bullet^{symbol}$ will be used to denote the an isotopy from Λ_0^{symbol} to Λ_1^{symbol} underlying $cobord'_4$.

4. we denote the triple $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$ as $\Lambda_\bullet$ and call it a squiggly isotopy from Λ_0 to Λ_1. Later in the section, $\Lambda_\bullet$ will be used to denote the isotopy between squiggly diagrams starting from Λ_0 ending at Λ_1 underlying $cobord'_4$.

Definition 179. For $t \in [0,1]$, we define $\Psi_t : \mathbb{R} \to \mathbb{R}$ to be a bump function parametrized by t as follows

$$\Psi_t(x) = \begin{cases} e^{\left(\frac{x^2}{x^2-1}\right)}\left(\frac{1}{2}-t\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1,1]$ if $t \neq \frac{1}{2}$
- $\{(1,0),\ (-1,0), (0, \frac{1}{2}-t)\} \subset Graph(\Psi_t)$

Definition 180. We denote the standard open disk in $\mathbb{R}^2$ of radius r_0 centered at the origin as

$$D_{r=r_0} := \{(x,z) \to \mathbb{R}^2 \mid x^2+z^2 < r_0^2\}$$

For $t_0 \in [0,1]$, we canonically identify $D_{r=r_0} \times \{t_0\}$ with $D_{r=r_0}$ using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$
$$(x,z) \mapsto (x,z,t_0)$$

and with abuse of expression say that sheaves on $D_{r=r_0} \times \{t_0\}$ as sheaves on $D_{r=r_0}$.

Definition 181. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x,z) \in D_{r=2} \mid x = \Psi_0(z)\}$
- λ_0^∞ is the union of the following two components
  (i) $\{(x,z) \in D_{r=2} \mid z = x\}$
  (ii) $\{(x,z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_0^{symbol} of λ_0^{symbol} as follows

- ξ_0^0: hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_0^∞: hairs are pointing towards the left i.e. coefficients of dx are positive.

Definition 182. 1. We define the following subsets of $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x,z) \in D_{r=2} \mid x = \Psi_1(z)\}$
- λ_1^∞ is the union of the following two components
  (i) $\{(x,z) \in D_{r=2} \mid z = x\}$
  (ii) $\{(x,z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations ξ_1^{symbol} of λ_1^{symbol} as follows

- ξ_1^0: hairs are pointing towards the left i.e. coefficients of dx are negative.
- ξ_1^∞: hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 183. 1. We define the following subsets of $D_{r=2} \times [0,1]$

- $\lambda_\bullet^0 := \{(x,z,t) \in D_{r=2} \times [0,1] \mid x = \Psi_t(z)\}$
- $\lambda_\bullet^\infty$ is the union of the following two components
  (i) $\{(x,z,t) \in D_{r=2} \times [0,1] \mid z = x\}$
  (ii) $\{(x,z,t) \in D_{r=2} \times [0,1] \mid z = -x\}$

2. We define co-orientations $\xi_\bullet^{symbol}$ of $\lambda_\bullet^{symbol}$ as follows

   - $\xi_\bullet^0$: hairs are pointing towards the left i.e. coefficients of dx are negative.
   - $\xi_\bullet^\infty$: hairs are pointing upward direction i.e. coefficients of dz are positive.

Definition 184. 1. Consider a stratification $\mathcal{S}_0$ on $D_{r=2}$ induced by λ_0 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_0$. Note that 1 dimensional strata has co-orientations inherited from λ_0.

2. Consider a stratification $\mathcal{S}_1$ on $D_{r=2}$ induced by λ_1 i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of $M - \lambda_1$. Note that 1 dimensional strata has co-orientations inherited from λ_1.

Consider a stratification $\mathcal{S}_\bullet$ on $D_{r=2} \times [0,1]$ induced by $\lambda_\bullet$ i.e. strata are non-empty finite intersections of $\lambda_\bullet^0$, $\lambda_\bullet^\infty$, and $\lambda_\bullet^{squig}$. Note that 2 dimensional strata has co-orientations inherited from $\lambda_\bullet$.

Now let's list the strata of $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\bullet$ using the following notations:

Definition 185. $\mathrm{sgn} : \mathbb{R} \to \{-, 0, +\}$ is defined as

$$\mathrm{sgn}(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

Definition 186. For $i = 1, 2, 3$, $t_0 = 0, 1$, and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := \; & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & \mathrm{sgn}(x - \Psi_{t_0}(z)) = sgn_1, \;\; \mathrm{sgn}(x - z) = sgn_2, \\ & \mathrm{sgn}((-x - z) = sgn_3\} \end{aligned}$$

Definition 187. For $i = 1, 2, 3$ and $sgn_i \in \{-, 0, +\}$, we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := \; & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & \mathrm{sgn}(x - \Psi_t(z)) = sgn_1, \;\; \mathrm{sgn}(x - z) = sgn_2, \\ & \mathrm{sgn}((-x - z) = sgn_3\} \end{aligned}$$

Definition 188. Now I will describe $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_{\bullet}$ using the above notations:

1. $\mathcal{S}_0$:

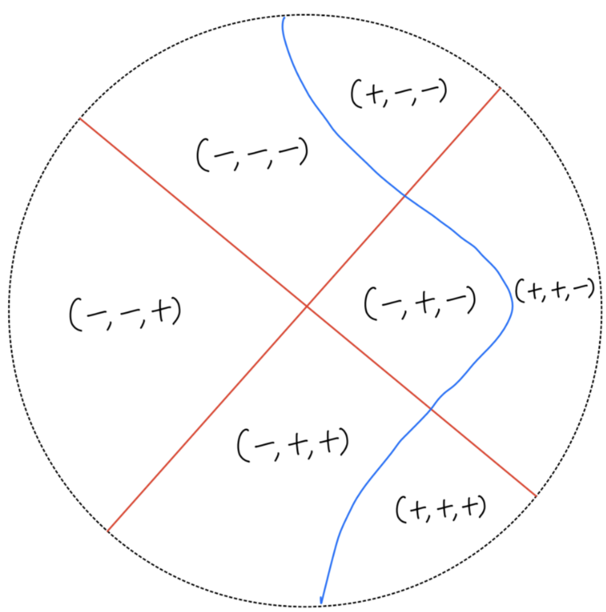

Figure 3.236

- 2 dimensional strata:

  $\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\}$ for i=1,2,3, except $s_0(+, -, +)\}$

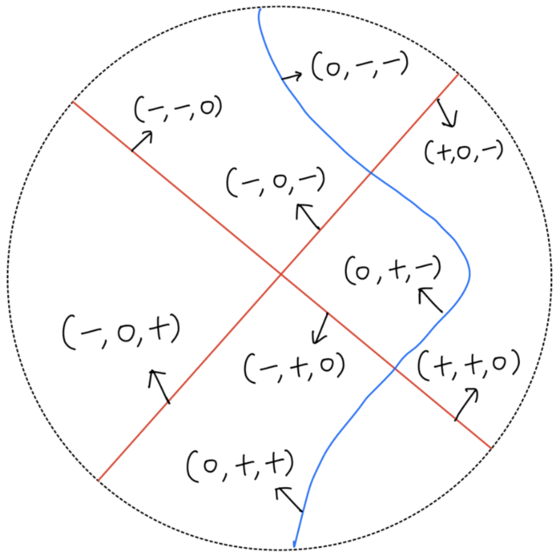

Figure 3.237

- 1 dimensional strata:

  $\{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3, except $s_0(0,-,+)\}$

  $\cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3, except $s_0(+,0,+)\}$

  $\cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_0(+,-,0)\}$

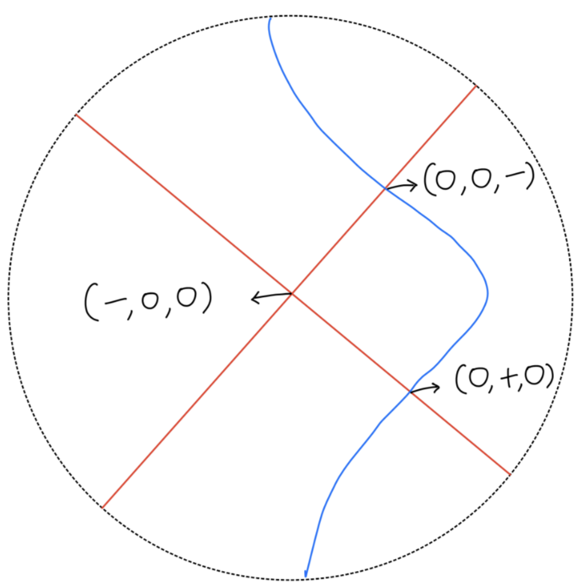

Figure 3.238

- 0 dimensional strata:

$s_0(-,0,0),\ s_0(0,0,-),\ s_0(0,+,0)$

2. $\mathcal{S}_1$:

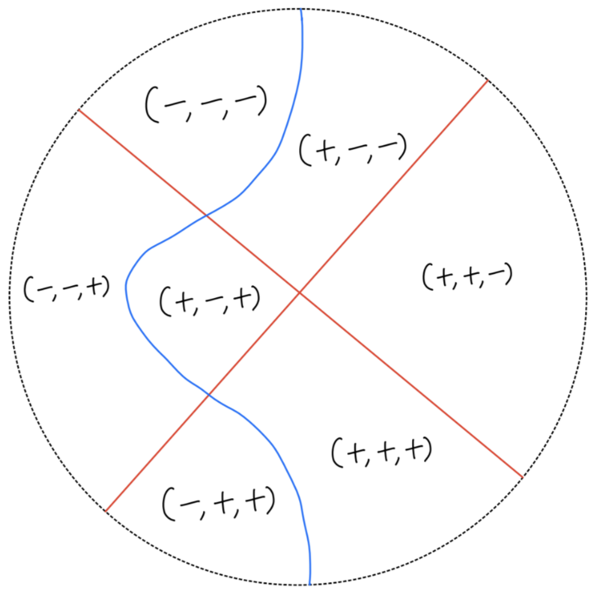

Figure 3.239

- 2 dimensional strata:

  $\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3, except $s_1(-,+,-)\}$

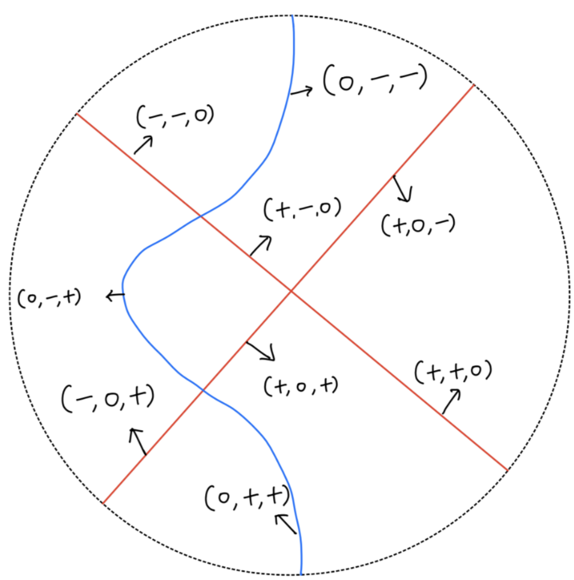

Figure 3.240

- 1 dimensional strata:

  $\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3, except $s_1(0,+,-)\}$

  $\cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3, except $s_1(-,0,-)\}$

  $\cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2, except $s_1(-,+,0)\}$

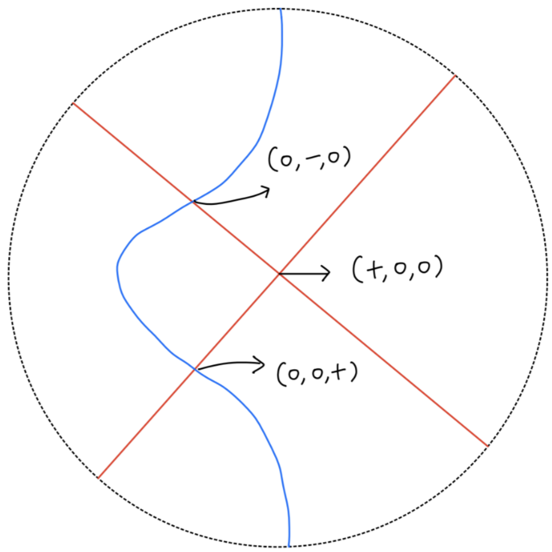

Figure 3.241

- 0 dimensional strata:

  $s_1(0,-,0)$, $s_1(+,0,0)$, $s_1(0,0,+)$

3. $\mathcal{S}_\bullet$:

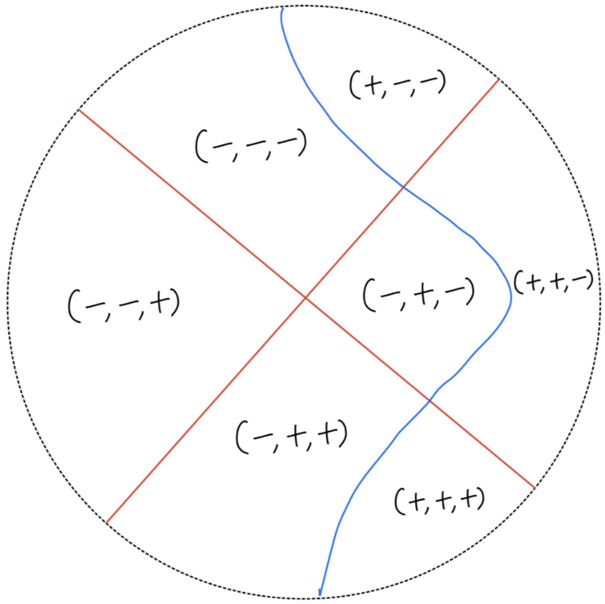

Figure 3.242

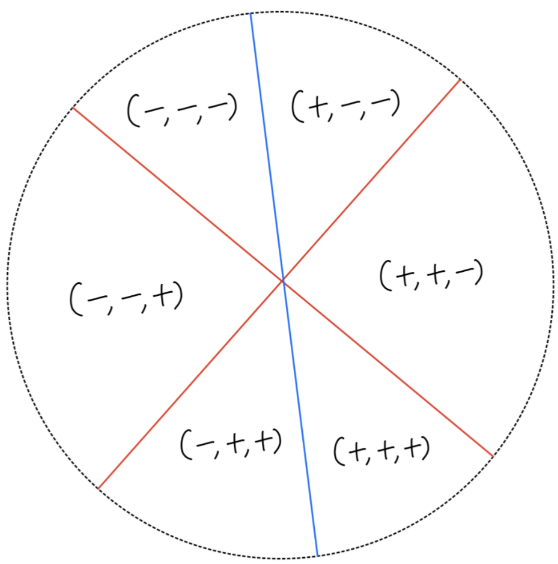

Figure 3.243

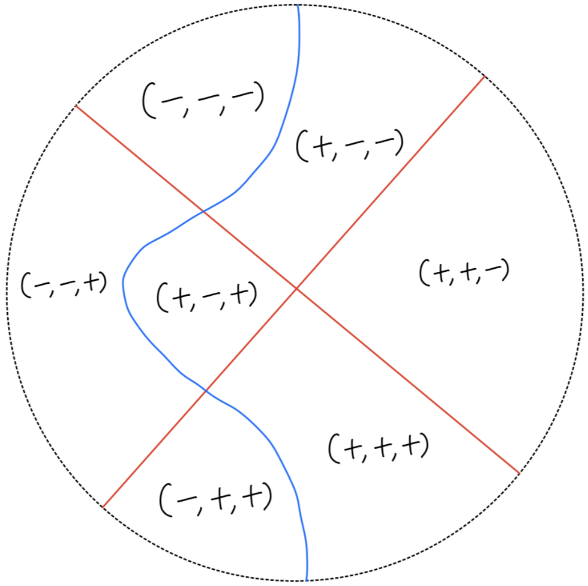

Figure 3.244

- 3 dimensional strata:

  $\{s_\bullet(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,2,3$\}$

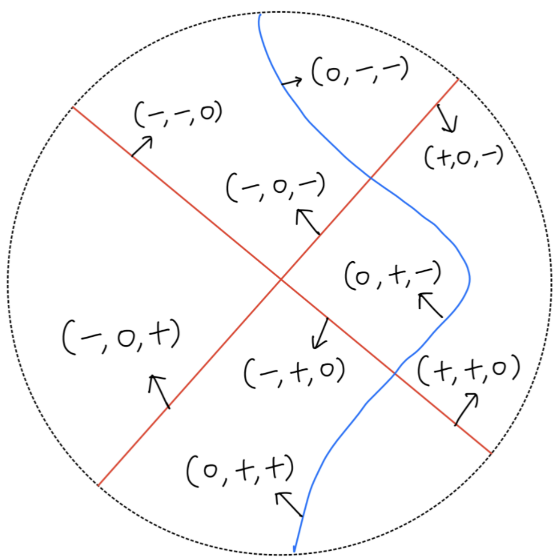

Figure 3.245

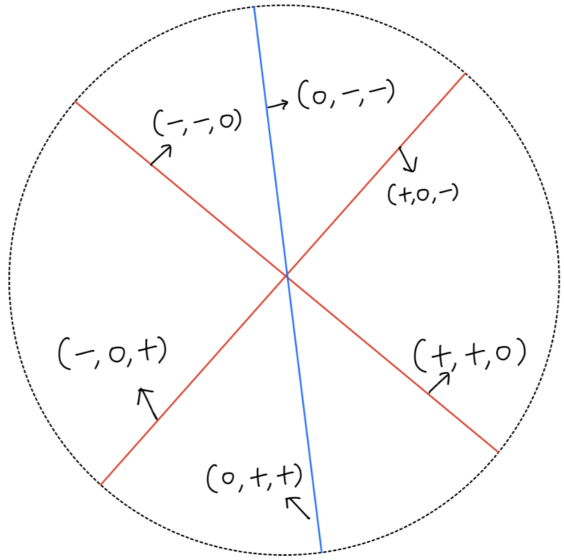

Figure 3.246

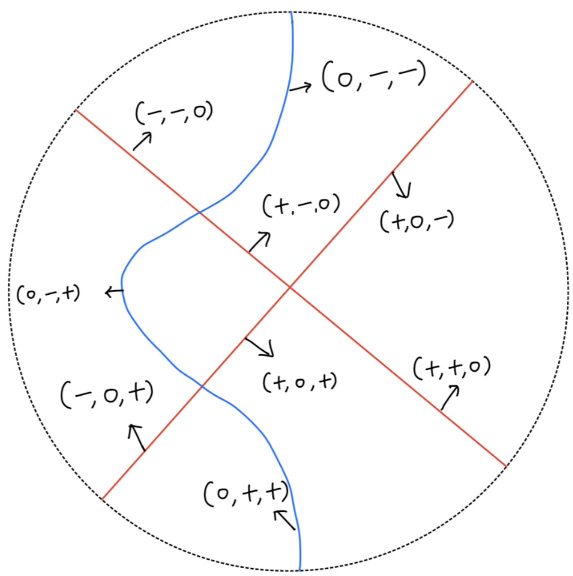

Figure 3.247

- 2 dimensional strata:

  $\{s_\bullet(0, sgn_2, sgn_3) \mid sgn_i \in \{-,+\}$ for i=2,3$\} \cup \{s_\bullet(sgn_1, 0, sgn_3) \mid sgn_i \in \{-,+\}$ for i=1,3$\} \cup \{s_\bullet(sgn_1, sgn_2, 0) \mid sgn_i \in \{-,+\}$ for i=1,2$\}$

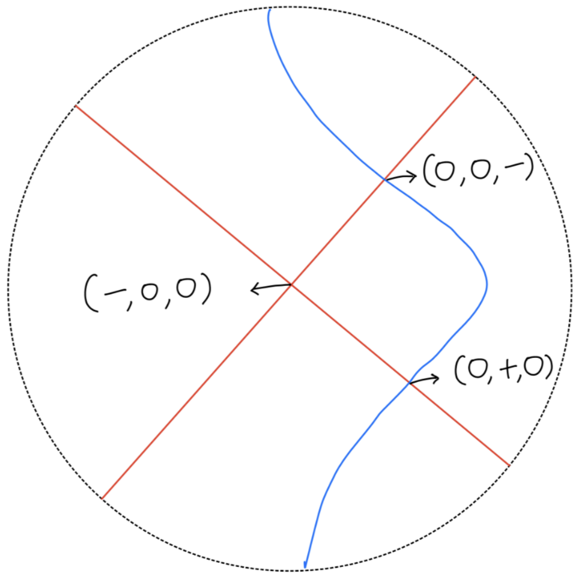

Figure 3.248

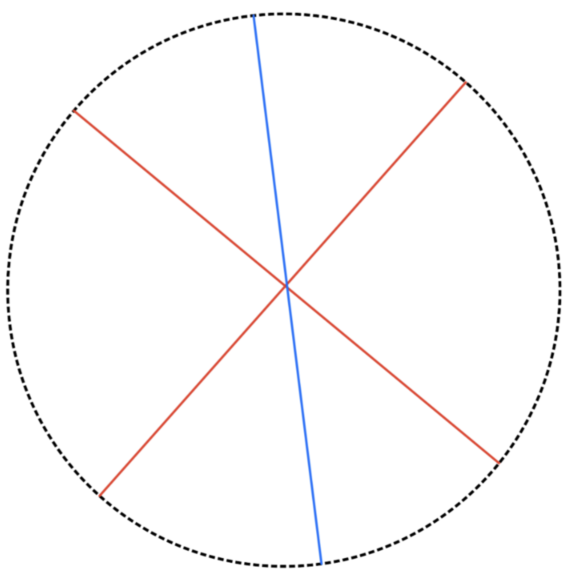

Figure 3.249

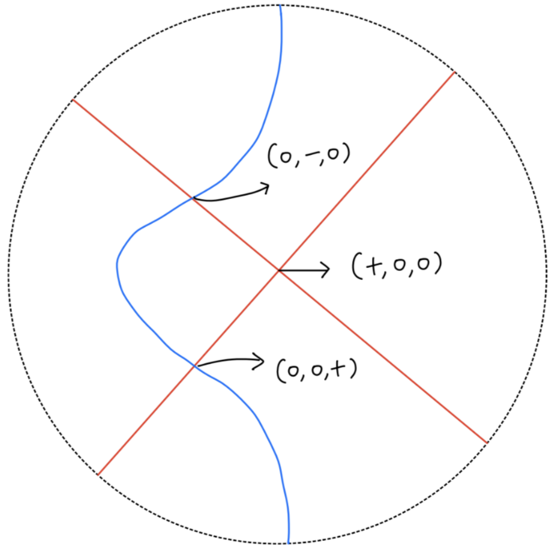

Figure 3.250

- 1 dimensional strata:

  $\{s_\bullet(sgn_1, 0, 0) \mid sgn_1 \in \{-,+\}\} \cup \{s_\bullet(0, sgn_2, 0) \mid sgn_2 \in \{-,+\}\} \cup \{s_\bullet(0, 0, sgn_3) \mid sgn_3 \in \{-,+\}\}$

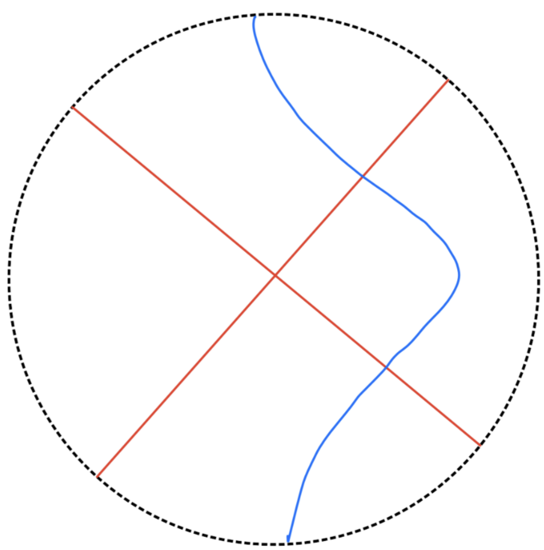

Figure 3.251

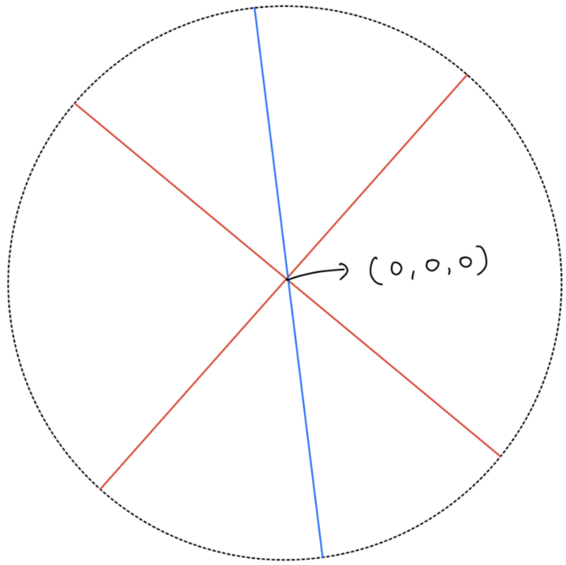

Figure 3.252

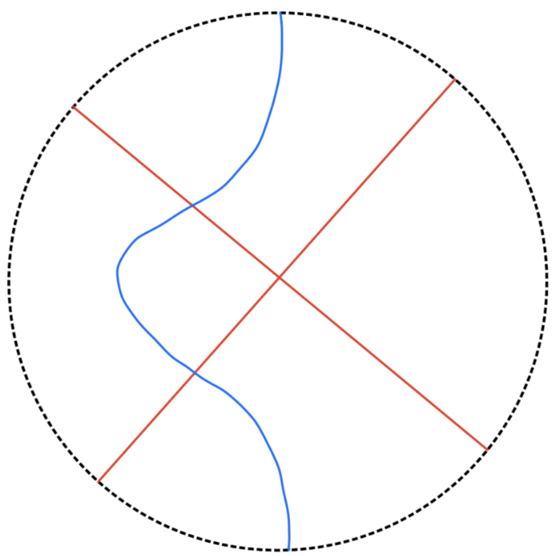

Figure 3.253

- 0 dimensional strata:

  $s_\bullet(0,0,0)$

Definition 189. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to $\mathcal{S}$, say $Q_{\mathcal{S}}$, to be a quiver

- whose vertices corresponds to codimension 0 strata of $\mathcal{S}$.
- whose arrows corresponds to codimension 1 strata of $\mathcal{S}$.
- the source of an arrow corresponding to $s \in \mathcal{S}$ is vertex corresponding to the region where the hairs of s are pointing at and the target is the other region contained in the star(s).

Definition 190. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then for each $s \in \mathcal{S}$, we define the subquiver of $Q_{\mathcal{S}}$ associated to s, say $Q_{\mathcal{S},s}$, to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of s.

Definition 191. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

then $\mathcal{S}$ is a legible stratification if for all $s \in \mathcal{S}$, $Q_{\mathcal{S},s}$ has the initial vertex. We say the quiver $Q_{\mathcal{S}}$ associated to $\mathcal{S}$ is legible if $\mathcal{S}$ is.

Definition 192. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is a legible representation if

- $\mathcal{S}$ is legible.
- for any $v, v' \in Vert(Q_{\mathcal{S}})$ and any paths $(a_1, a_2, \cdots, a_k)$,$(a'_1, a'_2, \cdots, a'_{k'})$ from v to v', $F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \cdots F_{\mathcal{S}}(a'_1)$ i.e. the composition is path independent.

Definition 193. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Supoose $\mathcal{S}$ is legible, then we define $\rho : \mathcal{S} \to \{s \in \mathcal{S} \mid \text{ codim}(s) = 0\}$ as

$$\rho(s) := \text{the codimension 0 stratum corresponding to the initial vertex of } Q_{\mathcal{S},s}$$

Definition 194. Suppose we have a manifold R with stratification $\mathcal{S}$ such that

- for each codimension 1 stratum $s \in \mathcal{S}$, there are exactly two regions(= codimension 0 strata) contained in star(s).
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation $F_{\mathcal{S}}$ of $Q_{\mathcal{S}}$ is legible, then we define the associated functor $\overline{F_{\mathcal{S}}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$ as follows:

- for $s \in \mathcal{S}$, $\overline{F_{\mathcal{S}}} := F_{\mathcal{S}}(\rho(s))$.
- for $s_1, s_2 \in \mathcal{S}$ where $s_2 \subset \text{star}(s_1)$, then $\overline{F_{\mathcal{S}}}(s_1 \to s_2)$ is defined as follows: choose a path from the vertex corresponding to $\rho(s_1)$ to $\rho(s_2)$ in $Q_{\mathcal{S}}$, say $(a_1, \cdots, a_k)$,

then

$$\overline{F_{\mathcal{S}}}(s_1 \to s_2) := F_{\mathcal{S}}(a_k) \circ \cdots F_{\mathcal{S}}(a_1)$$

This is well-defined because $F_{\mathcal{S}}$ is legible.

Setting

Suppose on M, we have

- a squiggly diagram Λ_0 on M
- nested regions $U' \subset U \subset M$. Note that if we define $V := M - \overline{U'}$, $\{U, V\}$ form an open cover of M.
- a smooth chart from $D_{r=2}$, say $f : D \to U \subset M$

such that

- $D_{r=1}$ is mapped to U'
- λ_0^0 is mapped to $\Lambda_0^0|_U$
- λ_0^∞ is mapped to $\Lambda_0^\infty|_U$
- λ_0^{squig} is mapped to $\Lambda_0^{squig}|_U$

Sheaf at the Beginning

Suppose we have a sheaf $\mathscr{F}_0$ singular supported on Λ_0 such that $f^*\mathscr{F}_0$ is isomorphic to the sheaf described by the following squiggly legible diagram F_0.

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

Stalks:

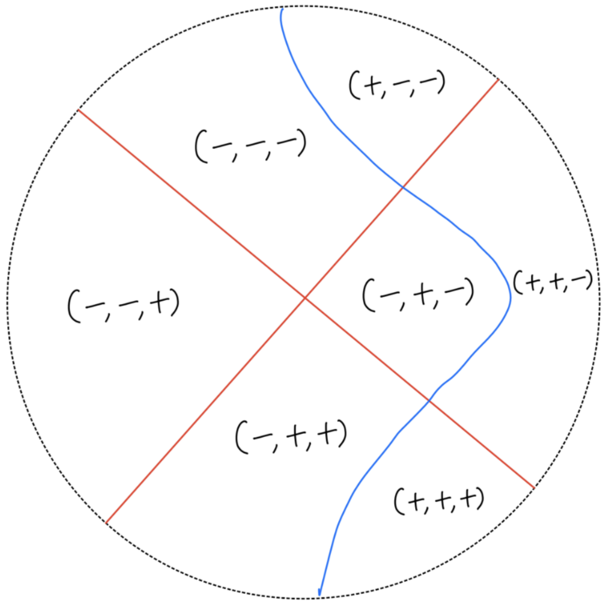

Figure 3.254

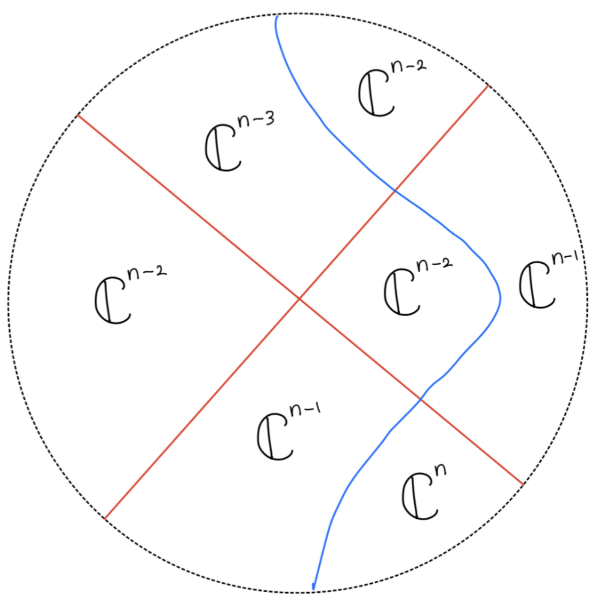

Figure 3.255

- $F_0(-,-,-) := \mathbb{C}^{n-3}$
- $F_0(+,-,-) := \mathbb{C}^{n-2}$
- $F_0(-,-,+) := \mathbb{C}^{n-2}$
- $F_0(-,+,-) := \mathbb{C}^{n-2}$

- $F_0(+,+,-) := \mathbb{C}^{n-1}$
- $F_0(-,+,+) := \mathbb{C}^{n-1}$
- $F_0(+,+,+) := \mathbb{C}^{n}$

Generization maps:

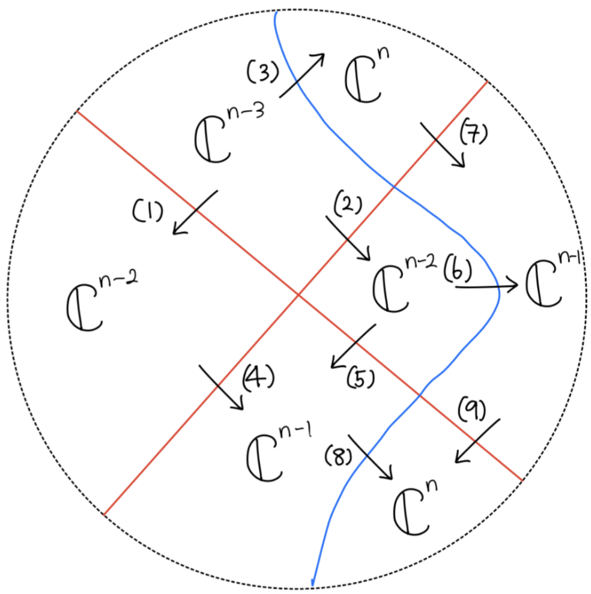

Figure 3.256

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(5) ι_0

(6) ι_1

(7) ι_0

(8) ι_1

(9) ι_0

Legendrian Cobordism

Then define a Legendrian cobordism $\mathscr{F}_\bullet$ starting from $\mathscr{F}_0$, say $cobord'_4$, that is supported on $\overline{U'}$ as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on $V \times [0,1]$, say $\mathscr{F}_{V\times[0,1]}$
- a sheaf on $D_{r=2} \times [0,1]$, say $\mathscr{F}_{D_{r=2}\times[0,1]}$
- a gluing isomorphsim, i.e. $\gamma_\bullet : (f_* \mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]} \xrightarrow{\sim} \mathscr{F}_{V\times[0,1]}|_{(U\cap V)\times[0,1]}$.

A. Sheaf on $V \times [0,1]$

First, I will define $\mathscr{F}_{V\times[0,1]}$ to be $pr_1^*(\mathscr{F}_0|_V)$ where $pr_1 : V\times[0,1] \to V$ is the projection onto the first argument.

B. Sheaf on $D_{r=2} \times [0,1]$

Next, I will describe $\mathscr{F}_{D_{r=2}\times[0,1]}$ as $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$ i.e. a functor from $\mathcal{S}_\bullet$ to the category of perfect $\mathbb{C}$-modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

Stalks:

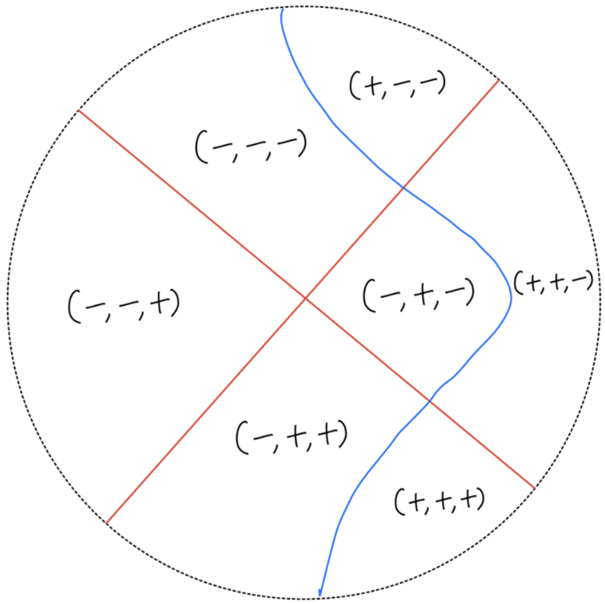

Figure 3.257

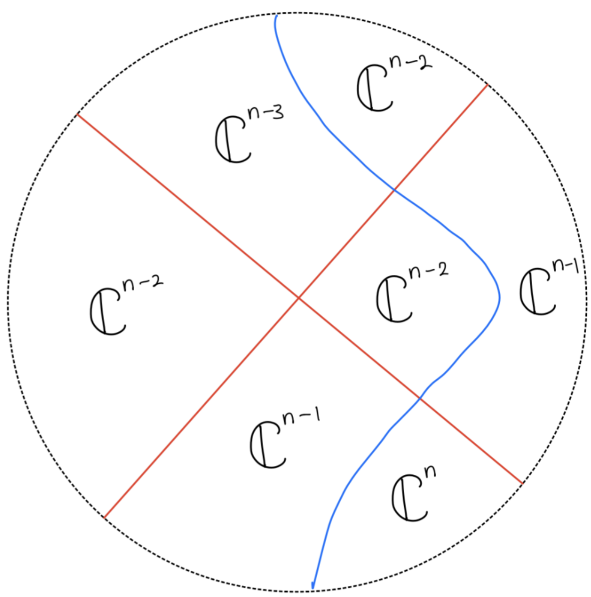

Figure 3.258

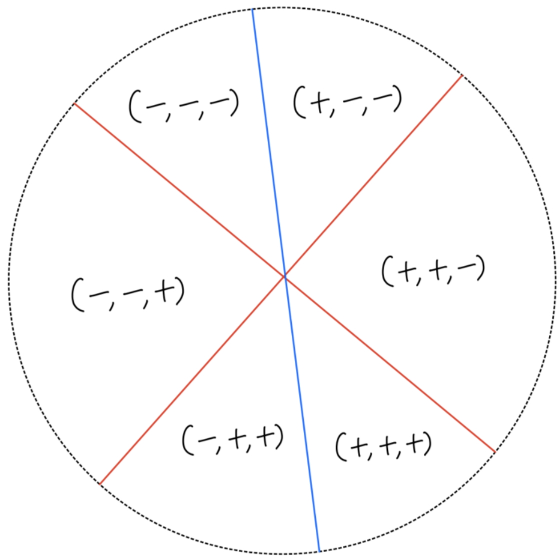

Figure 3.259

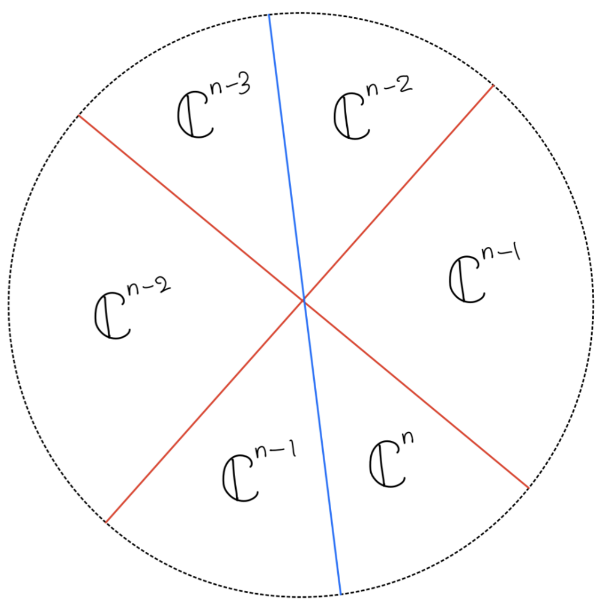

Figure 3.260

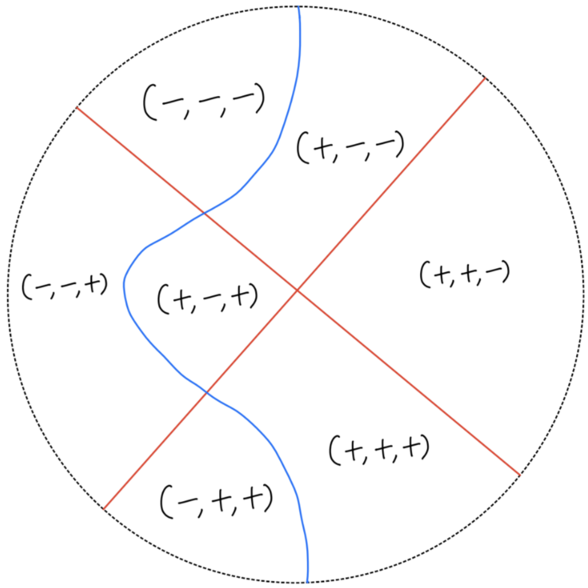

Figure 3.261

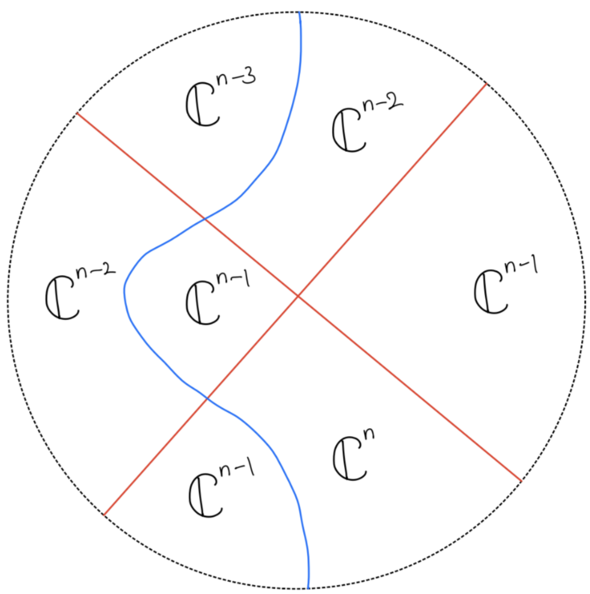

Figure 3.262

- $F_\bullet(-,-,-) := \mathbb{C}^{n-3}$
- $F_\bullet(-,-,+) := \mathbb{C}^{n-2}$
- $F_\bullet(-,+,-) := \mathbb{C}^{n-2}$
- $F_\bullet(-,+,+) := \mathbb{C}^{n-1}$

- $F_\bullet(+,-,-) := \mathbb{C}^{n-2}$
- $F_\bullet(+,-,+) := \mathbb{C}^{n-1}$
- $F_\bullet(+,+,-) := \mathbb{C}^{n-1}$
- $F_\bullet(+,+,+) := \mathbb{C}^{n}$

Generization maps:

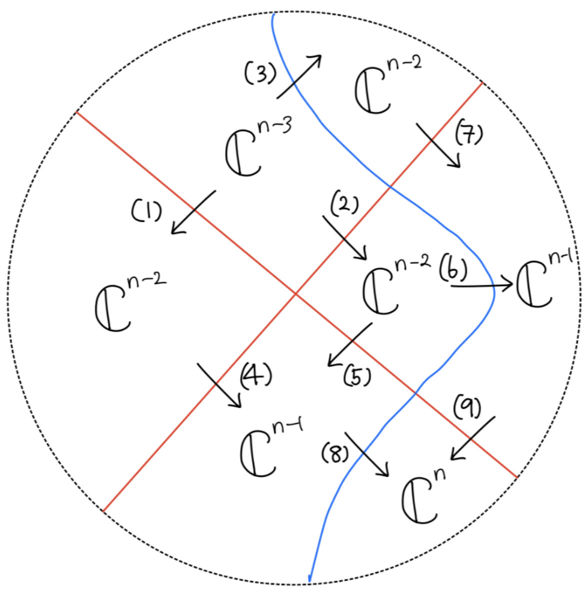

Figure 3.263

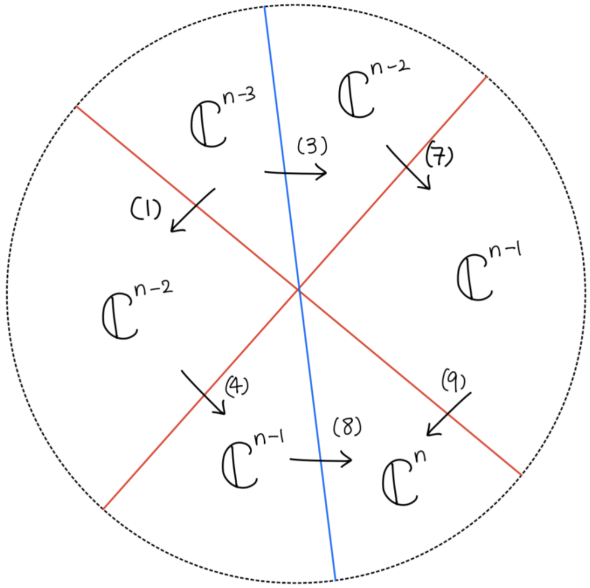

Figure 3.264

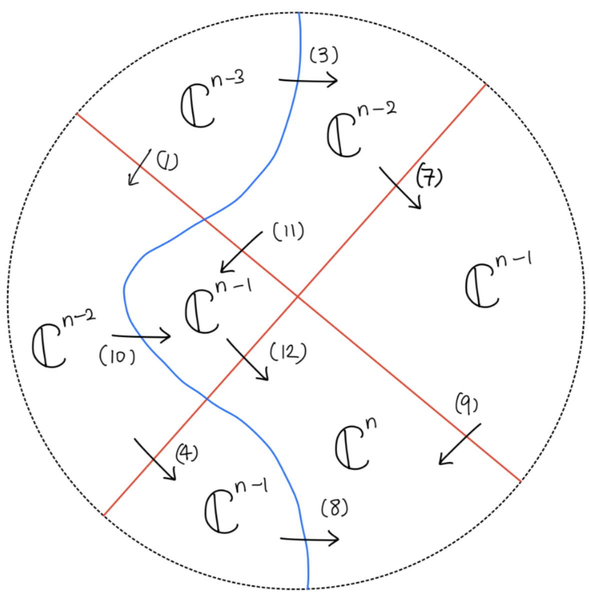

Figure 3.265

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ diag(1, \cdots, 1) + e'I + n, n-1$

(5) ι_0

(6) ι_1

(7) ι_0

(8) ι_1

(9) ι_0

(10) ι_1

(11) ι_0

(12) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism $\gamma_\bullet : (f_* \mathscr{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathscr{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$ using the fact that $(f_* \mathscr{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U \cap V})$ where $pr_1 : (U \cap V) \times [0,1] \to (U \cap V)$ is the projection onto the first argument.

Definition 195. we define $\gamma_\bullet$ to be the composition

$$(f_* \mathscr{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathscr{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times [0,1] \arrow[r, hook] \arrow[d, "pr_1"] & V \times [0,1] \arrow[d, "pr_1"] \\
(U \cap V) \arrow[r, hook] & V
\end{tikzcd}$$

Now we have defined a cobordism $\mathscr{F}_\bullet$, we show that this is a Legendrian cobordism.

Proposition 196. $\mathscr{F}_\bullet$ is a Legendrian cobordism i.e. $\mathscr{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$.

Proof. To prove the claim, I will show that the microlocal stalks of $\mathscr{F}_\bullet$ vanishes at every points on a contangent bundle of M.

Note that there is a diffeomorphism beteween $D_{r=2} \times (0,1)$ and $\mathbb{R}^3$ that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of $\mathscr{F}^\bullet$ along the above diffeomorphism vanishes at every points of $T^*\mathbb{R}^n$. The pullback of $\mathscr{F}^\bullet$ along the diffeomorphism could be described using the following legible diagram, say F^3. To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

Stalks:

- $F^3(-,-,-) := \mathbb{C}^{n-3}$
- $F^3(-,-,+) := \mathbb{C}^{n-2}$
- $F^3(+,-,-) := \mathbb{C}^{n-2}$
- $F^3(+,-,+) := \mathbb{C}^{n-2}$
- $F^3(-,+,-) := \mathbb{C}^{n-2}$
- $F^3(-,+,+) := \mathbb{C}^{n-1}$
- $F^3(+,+,-) := \mathbb{C}^{n-1}$

- $F^3(+,+,+) := \mathbb{C}^n$

Generization maps:

$$\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "(6)"] \arrow[dd, "(1)"] & & s(-,+,+) \arrow[dd, "(12)"] \\
s(-,-,-) \arrow[ur, "(2)"] \arrow[rr, "(7)"] \arrow[dd, "(3)"] & & s(-,-,+) \arrow[ur, "(4)"] \arrow[dd, "(10)"] & \\
 & s(+,+,-) \arrow[rr, "(5)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "(9)"] \arrow[rr, "(8)"] & & s(+,-,+) \arrow[ur, "(11)"] &
\end{tikzcd}$$

(1) ι_0

(2) ι_0

(3) ι_1

(4) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(5) ι_1

(6) ι_0

(7) ι_1

(8) ι_0

(9) ι_0

(10) ι_0

(11) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(12) ι_1

To prove that microlocal stalk vanishes everywhere, by Lemma 60, it is enough to show that the total complexes of F^3 restricted to the following squares and cubes are acyclic

(i)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,+,-) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,+,-) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-3} \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C}^{n-2} \arrow[d, "\iota_1"] \\ \mathbb{C}^{n-2} \arrow[r, "\iota_0"] & \mathbb{C}^{n-1} \end{tikzcd}$$

(ii)
$$\begin{tikzcd} F^3(-,-,+) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,-,+) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-2} \arrow[r, "(*)"] \arrow[d, "\iota_1"] & \mathbb{C}^{n-1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{n-1} \arrow[r, "(**)"] & \mathbb{C}^{n} \end{tikzcd}$$

(iii)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(+,-,-) \arrow[r] & F^3(+,-,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-3} \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C}^{n-2} \arrow[d, "\iota_1"] \\ \mathbb{C}^{n-2} \arrow[r, "\iota_0"] & \mathbb{C}^{n-1} \end{tikzcd}$$

(iv)
$$\begin{tikzcd} F^3(-,+,-) \arrow[r] \arrow[d] & F^3(-,+,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-2} \arrow[r, "\iota_0"] \arrow[d, "\iota_1"] & \mathbb{C}^{n-1} \arrow[d, "\iota_1"] \\ \mathbb{C}^{n-1} \arrow[r, "\iota_0"] & \mathbb{C}^{n} \end{tikzcd}$$

(v)
$$\begin{tikzcd} F^3(-,-,-) \arrow[r] \arrow[d] & F^3(-,-,+) \arrow[d] \\ F^3(-,+,-) \arrow[r] & F^3(-,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-3} \arrow[r, "\iota_0"] \arrow[d, "\iota_0"] & \mathbb{C}^{n-2} \arrow[d, "(*)"] \\ \mathbb{C}^{n-2} \arrow[r, "\iota_0"] & \mathbb{C}^{n-1} \end{tikzcd}$$

(vi)
$$\begin{tikzcd} F^3(+,-,-) \arrow[r] \arrow[d] & F^3(+,-,+) \arrow[d] \\ F^3(+,+,-) \arrow[r] & F^3(+,+,+) \end{tikzcd} = \begin{tikzcd} \mathbb{C}^{n-2} \arrow[r, "\iota_0"] \arrow[d, "\iota_0"] & \mathbb{C}^{n-1} \arrow[d, "(**)"] \\ \mathbb{C}^{n-1} \arrow[r, "\iota_0"] & \mathbb{C}^{n} \end{tikzcd}$$

where

- $(*) = \iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$
- $(**) = \iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

The total complexes of $(i) - (vi)$ are acyclic because they are cartesian.

(vii) the cubic diagram:

$$\begin{tikzcd}
 & s(-,+,-) \arrow[rr, "s(-,+,0)"] \arrow[dd, "s(0,+,-)"] & & s(-,+,+) \arrow[dd, "s(0,+,+)"] \\
s(-,-,-) \arrow[ur, "s(-,0,-)"] \arrow[rr, "s(-,-,0)"] \arrow[dd, "s(0,-,-)"] & & s(-,-,+) \arrow[ur, "s(-,0,+)"] \arrow[dd, "s(0,-,+)"] & \\
 & s(+,+,-) \arrow[rr, "s(+,+,0)"] & & s(+,+,+) \\
s(+,-,-) \arrow[ur, "s(+,0,-)"] \arrow[rr, "s(+,-,0)"] & & s(+,-,+) \arrow[ur, "s(+,0,+)"] &
\end{tikzcd}$$

$=$

$$\begin{tikzcd}
 & \mathbb{C}^{n-2} \arrow[rr, "\iota_0"] \arrow[dd, "\iota_1"] & & \mathbb{C}^{n-1} \arrow[dd, "\iota_1"] \\
\mathbb{C}^{n-3} \arrow[ur, "\iota_0"] \arrow[rr, "\iota_0"] \arrow[dd, "\iota_1"] & & \mathbb{C}^{n-2} \arrow[ur, "(*)"] \arrow[dd, "\iota_1"] & \\
 & \mathbb{C}^{n-1} \arrow[rr, "\iota_0"] & & \mathbb{C}^{n} \\
\mathbb{C}^{n-2} \arrow[ur, "\iota_0"] \arrow[rr, "\iota_0"] & & \mathbb{C}^{n-1} \arrow[ur, "(**)"] &
\end{tikzcd}$$

the total complex is acyclic because by $(i), \cdots, (vi)$, the total complexes of the square diagrams are acyclic and the configuration of three planes

- $\mathbb{C}^{n-1} \xrightarrow{\iota_0} \mathbb{C}$
- $\mathbb{C}^{n-1} \xrightarrow{\iota_1} \mathbb{C}$
- $\mathbb{C}^{n-1} \xrightarrow{(**)} \mathbb{C}$

corresponding to $s(+,+,-,)$, $s(-,+,+)$, $s(+,-,+)$ are in general position.

□

Sheaf at the End

In this subsection, I will describe the sheaf $\mathscr{F}_1$ at the end of the $cobord_4'$. By Mayer-Vietoris, $\mathscr{F}_1 := \mathscr{F}_\bullet|_{M\times\{1\}}$ on $M \cong M \times \{1\}$ is equivalent to the following data

- a sheaf on V, say $\mathscr{F}_V$
- a sheaf on $D_{r=2}$, say $\mathscr{F}_{D_{r=2}}$
- a gluing isomorphsim $\gamma_1 : f_*\mathscr{F}_{D_{r=2}}|_{U\cap V} \xrightarrow{\sim} \mathscr{F}_V|_{U\cap V}$.

A. Sheaf on V

First, a sheaf on $V \cong V\times\{1\}$ is the restriction of $\mathscr{F}_{V\times[0,1]}$ to $V\times\{1\}$, i.e. $\mathscr{F}_{V\times[0,1]}|_{V\times\{1\}} = pr_1^*(\mathscr{F}_0|_V)|_{V\times\{1\}} = \mathscr{F}_0|_V$.

B. Sheaf on $D_{r=2}$

Next, a sheaf on $D_{r=2} \cong D_{r=2} \times \{1\}$ is the restriction of $\mathscr{F}_{D_{r=2}\times[0,1]}$ to $D_{r=2} \times \{1\}$, i.e. $\mathscr{F}_{D_{r=2}\times[0,1]}|_{D_{r=2}\times\{1\}}$. I will describe it as a squiggly legible diagram F_1 which is the restriction of $F_\bullet$ defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

Stalks:

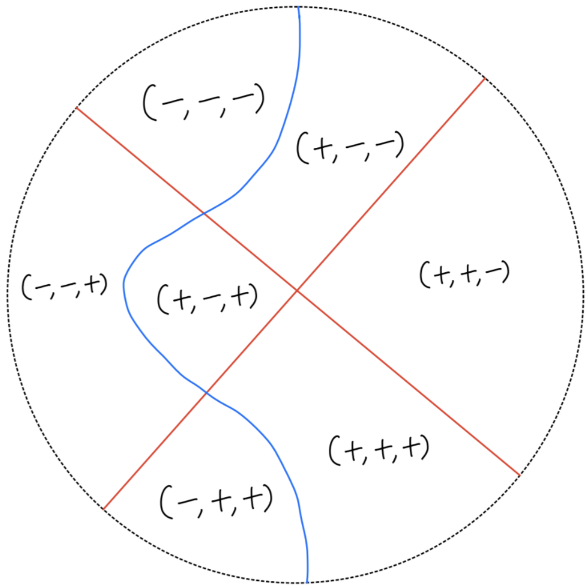

Figure 3.266

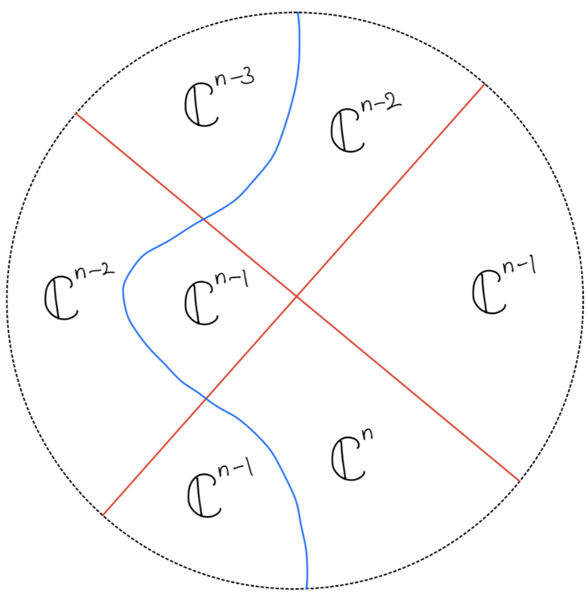

Figure 3.267

- $F_1(-,-,-) := \mathbb{C}^{n-3}$
- $F_1(+,-,-) := \mathbb{C}^{n-2}$
- $F_1(-,-,+) := \mathbb{C}^{n-2}$
- $F_1(+,-,+) := \mathbb{C}^{n-1}$

- $F_1(+,+,-) := \mathbb{C}^{n-1}$
- $F_1(-,+,+) := \mathbb{C}^{n-1}$
- $F_1(+,+,+) := \mathbb{C}^{n}$

Generization maps:

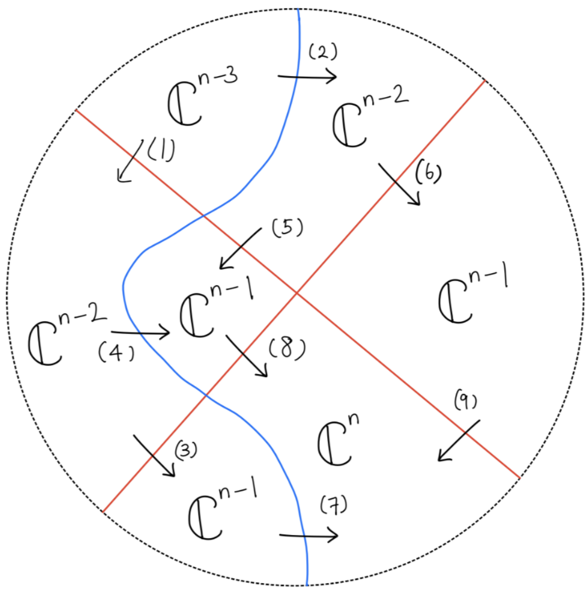

Figure 3.268

(1) ι_0

(2) ι_1

(3) ι_0

(4) ι_1

(5) ι_0

(6) ι_0

(7) ι_1

(8) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(9) ι_0

C. Gluing Isomorphism

Lastly, the gluing isomorphism $\gamma_1 := \gamma_\bullet|_{(U\cap V)\times\{1\}} : f_*\mathscr{F}_{D_{r=2}}|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$ is described as follows.

Definition 197. we define γ_1 to be the composition

$$(f_*\mathscr{F}_{D_{r=2}})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} pr_1^*(\mathscr{F}_0|_{U\cap V})|_{(U\cap V)\times\{1\}} \xrightarrow{\sim} \mathscr{F}_0|_{U\cap V}$$

where

- the first isomorphism follows from the fact that $(f_*\mathscr{F}_{D_{r=2}\times[0,1]})|_{(U\cap V)\times[0,1]}$ is isomorphic to $pr_1^*(\mathscr{F}_0|_{U\cap V})$.
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U\cap V) \xrightarrow{\sim} (U\cap V)\times\{1\} \hookrightarrow (U\cap V)\times[0,1] \twoheadrightarrow (U\cap V)$$

3.10 5th sheaf cobordism

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-1,1)_x \times (-n-1,2)_z$ under f

- Λ_0^0 gets mapped to $\{(x,z) \in R \mid z = 1\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup\limits_{k=1}^{n} \{(x,z) \in R \mid z = -k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x,z) \in R \mid x = 0\}$, co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

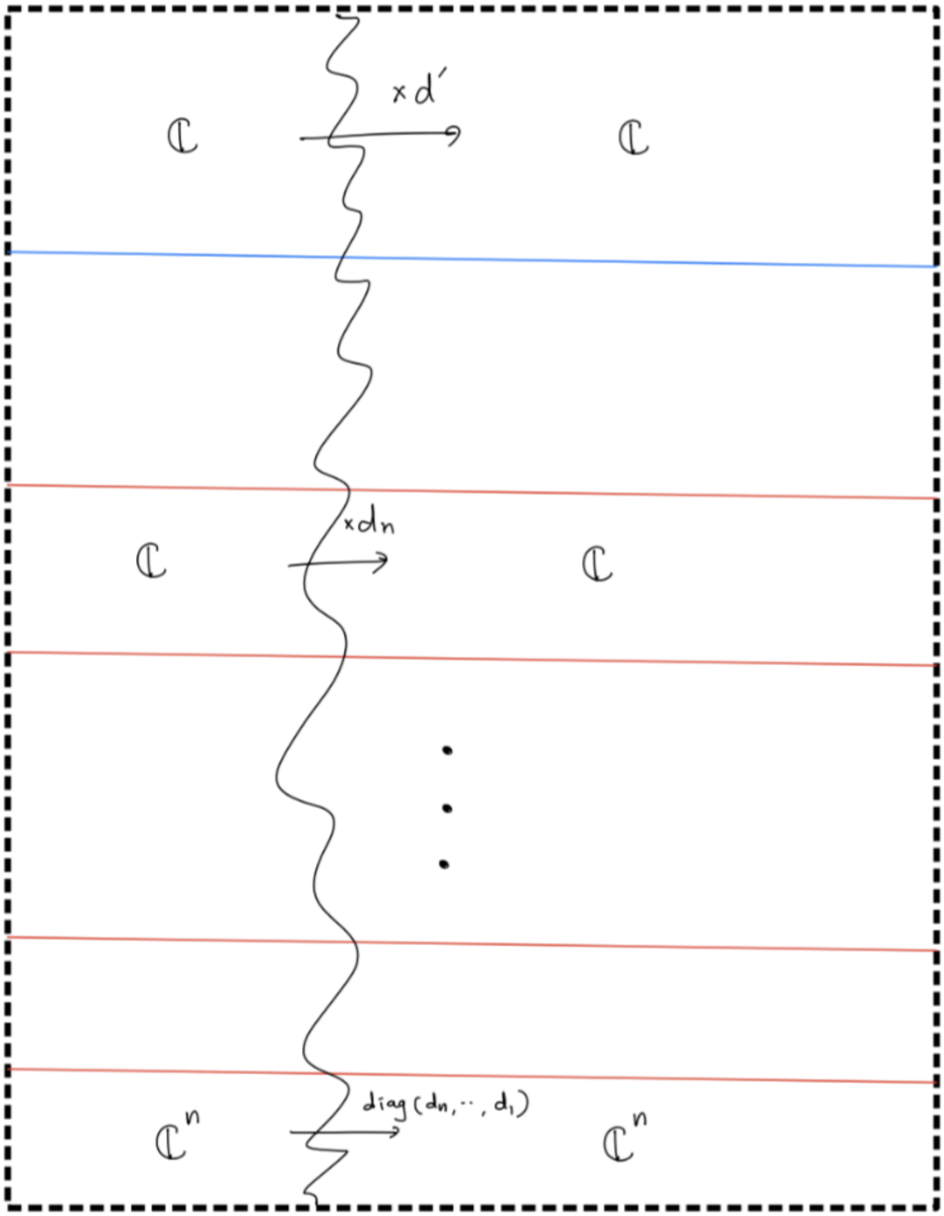

Figure 3.269

Then we define a cobordism starting from the above sheaf, say $cobord_5(n)$ supported on U. At the end of the cobordism , the sheaf, under the same chart f, is described as the following squiggly legible diagram.

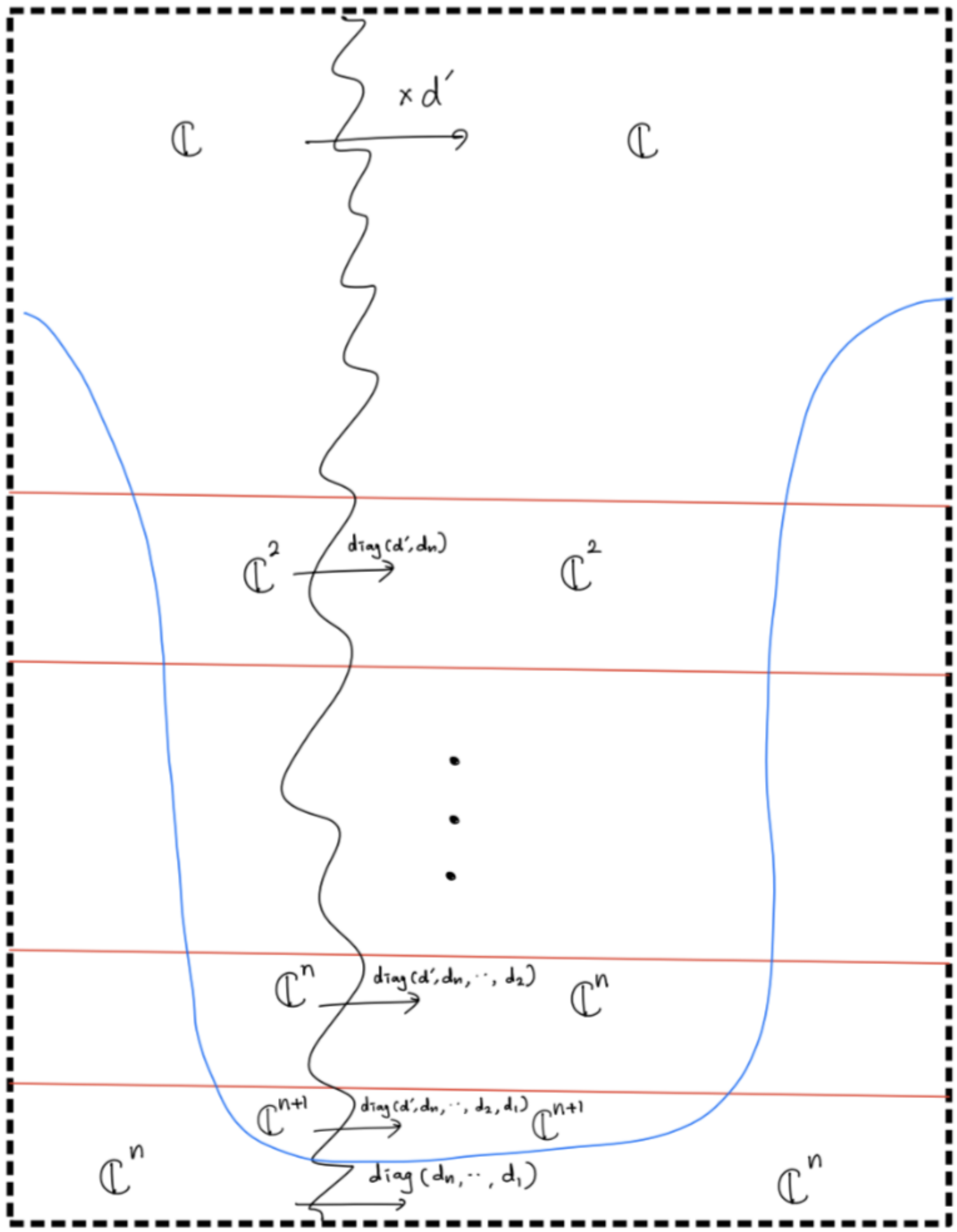
$\times d'$
$\mathbb{C}$
$\mathbb{C}$
diag(d', dn)
$\mathbb{C}^2$
$\mathbb{C}^2$
diag(d', dn, ··, d2)
$\mathbb{C}^n$
$\mathbb{C}^n$
diag(d', dn, ··, d2, d1)
$\mathbb{C}^{n+1}$
$\mathbb{C}^{n+1}$
$\mathbb{C}^n$
diag(dn, ··, d1)
$\mathbb{C}^n$

Figure 3.270

We define $cobord_5(n)$ inductively as follows.

(i) For $n = 1$, we define $cobord_5(1)$ to be $cobord_1$ from

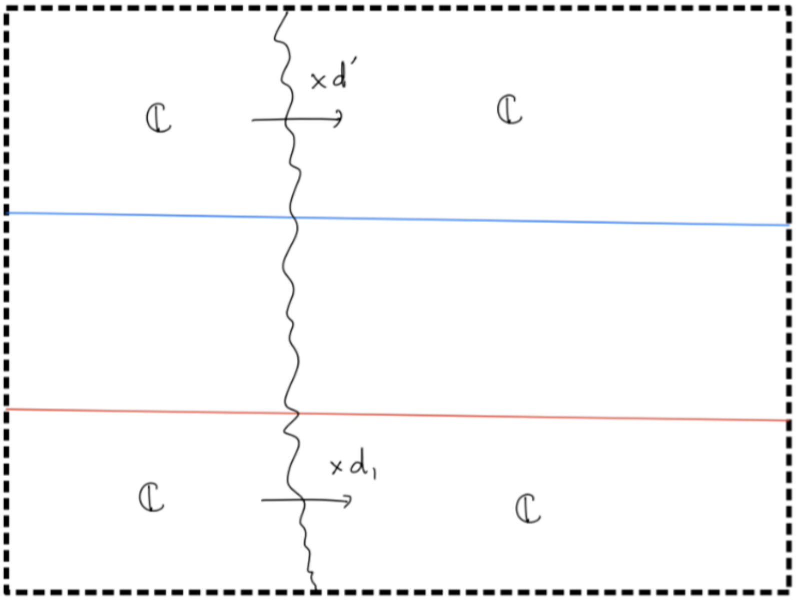

Figure 3.271

to

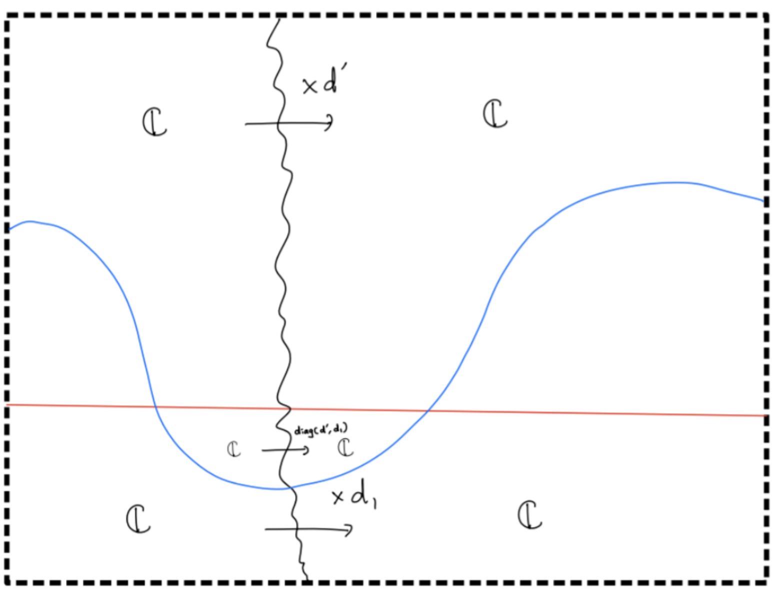

Figure 3.272

(ii) For $n > 0$,

(Step 1) we first apply $cobord_5(n-1)$ to the square region surrounded by a purple dotted line.

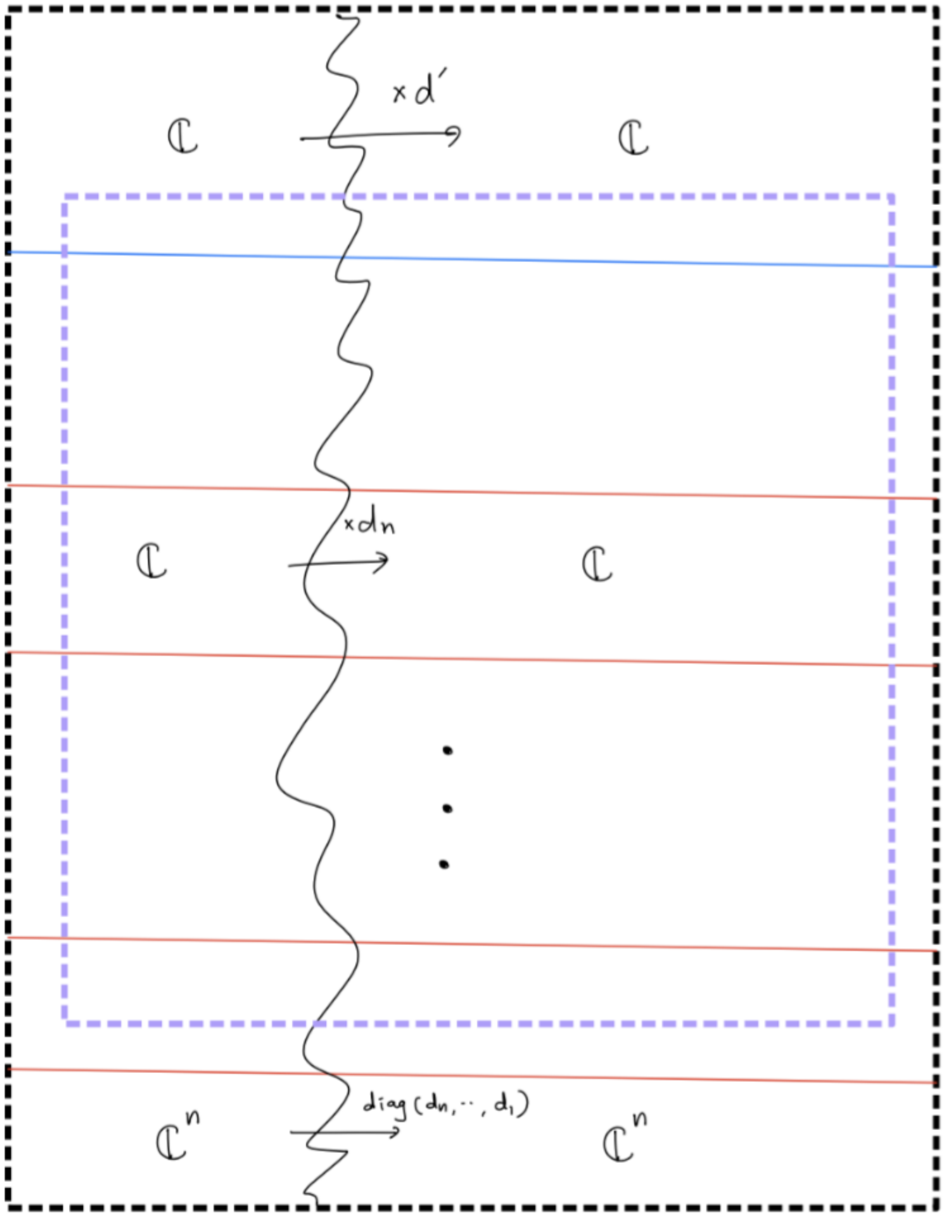

Figure 3.273

by induction hypothesis, we get

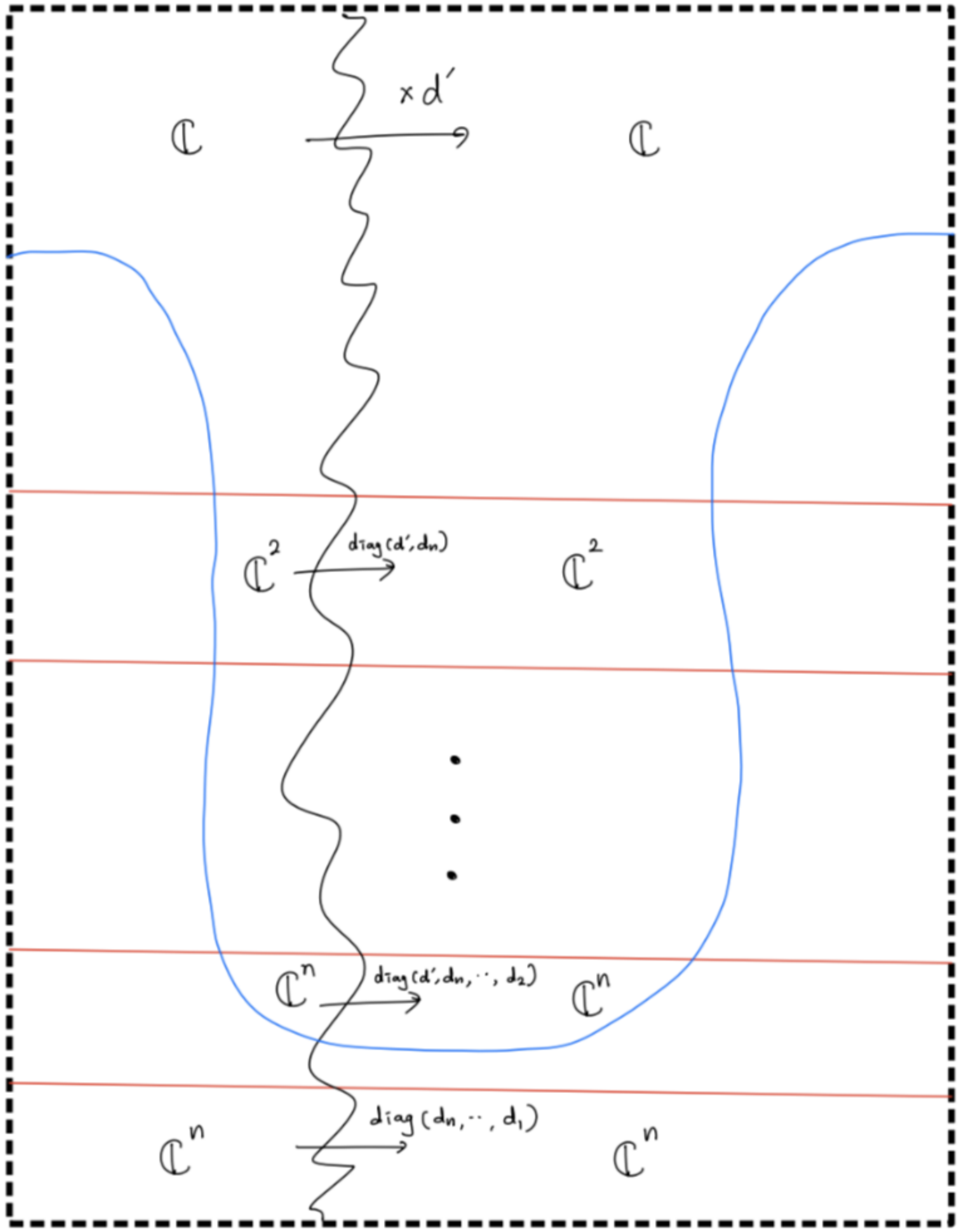

Figure 3.274

(Step 2) apply $cobord_1$ to the square region surrounded by purple dotted lines.

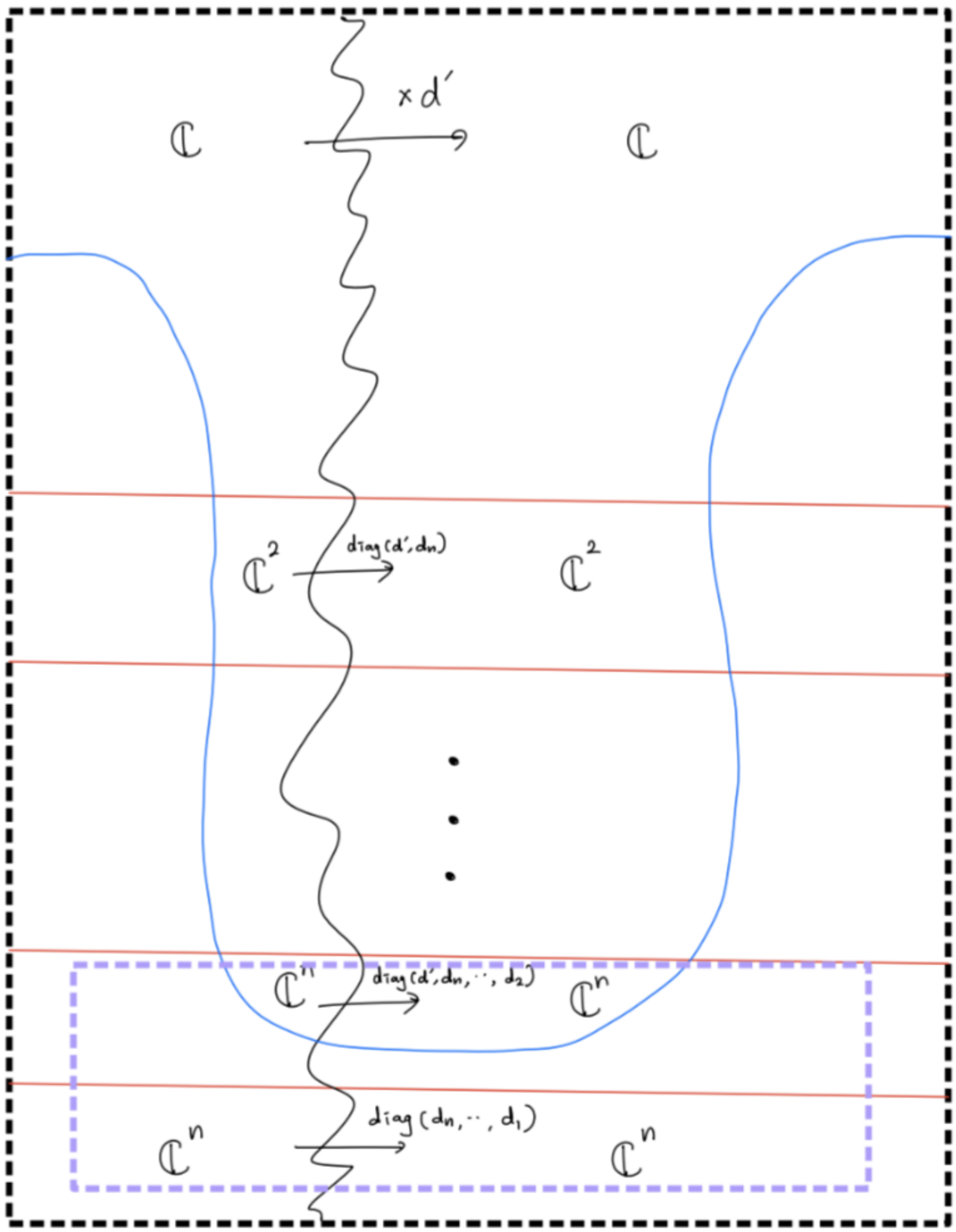

Figure 3.275

we get the final sheaf

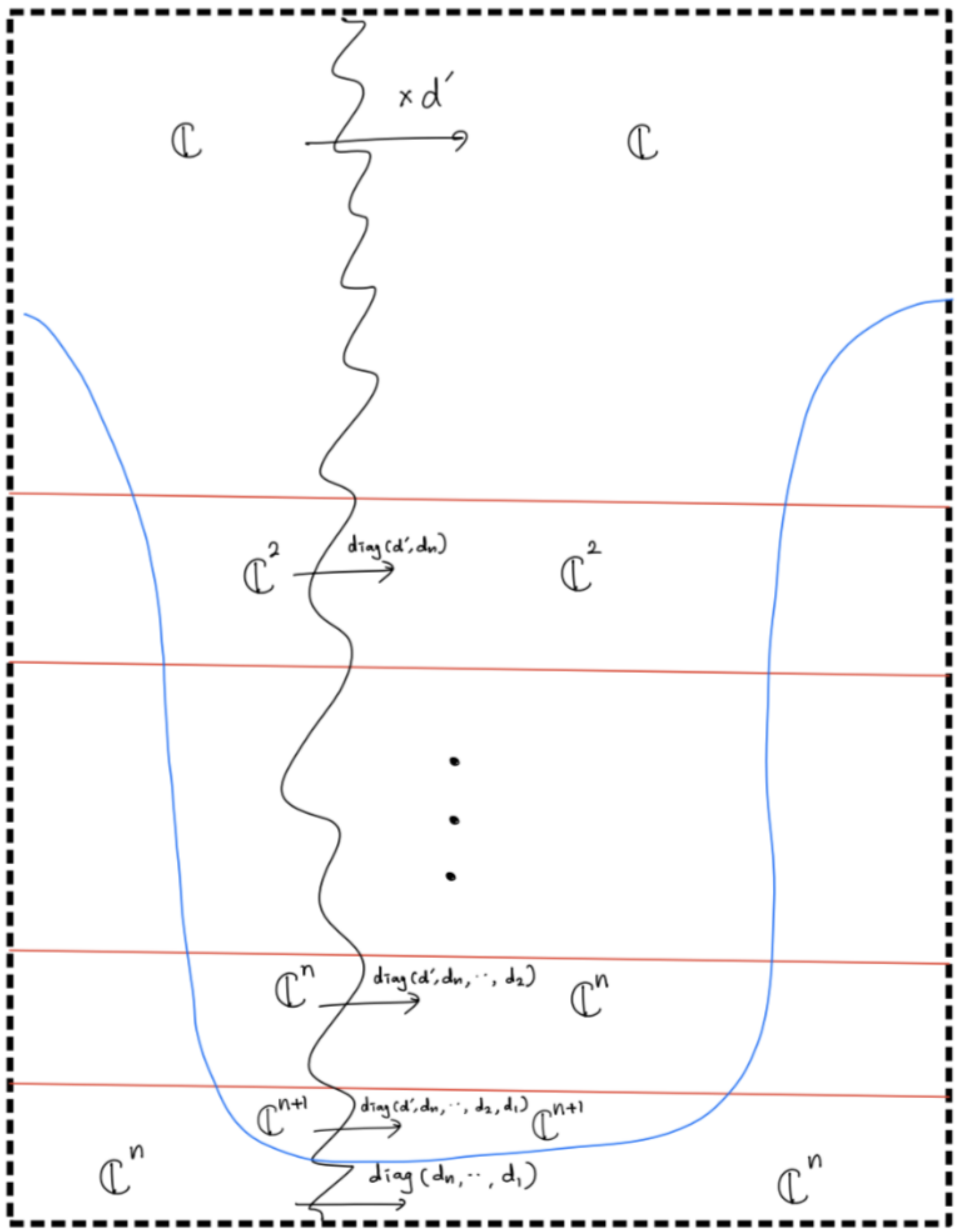

Figure 3.276

3.11 6th sheaf cobordism

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-1,1)_x \times (-2, n+1)_z$ under f

- Λ_0^0 gets mapped to $\bigcup\limits_{k=1}^{n} \{(x,z) \in R \mid z = k\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\{(x,z) \in R \mid z = -1\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x,z) \in R \mid x = 0\}$, co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

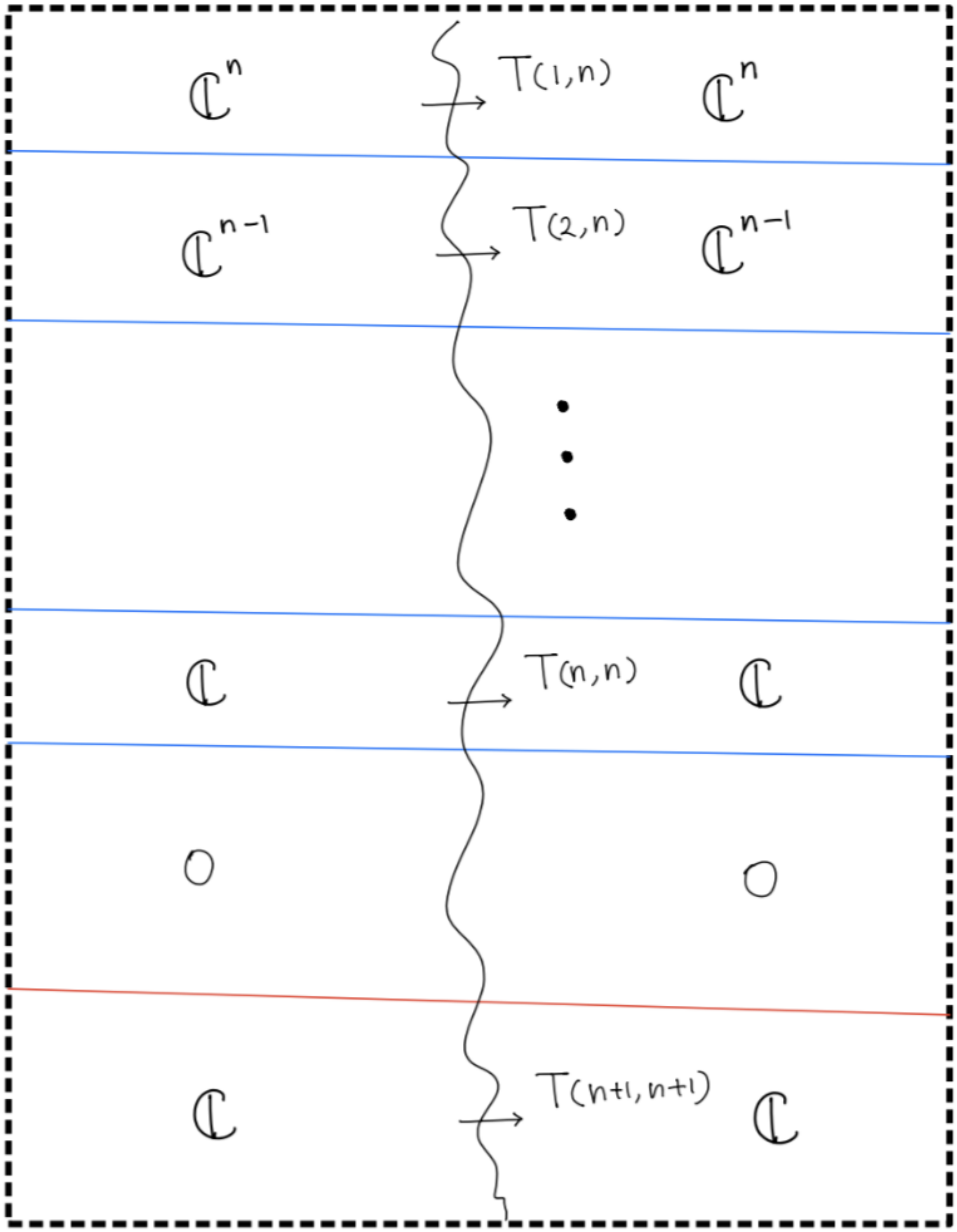

Figure 3.277

where $T \in M^{(n+1)\times(n+1)}$ such that preserves the following flags

- $\mathbb{C} \subset_{\iota_0} \cdots \subset_{\iota_0} \mathbb{C}^n$
- $\mathbb{C} \subset_{\iota_1} \mathbb{C}^n$

Then we define a cobordism starting from the above sheaf, say $cobord_6(n)$ supported on U, where n is the number of blue strands. At the end of the cobordism, the sheaf, under the same chart f, is described as the following squiggly legible diagram.

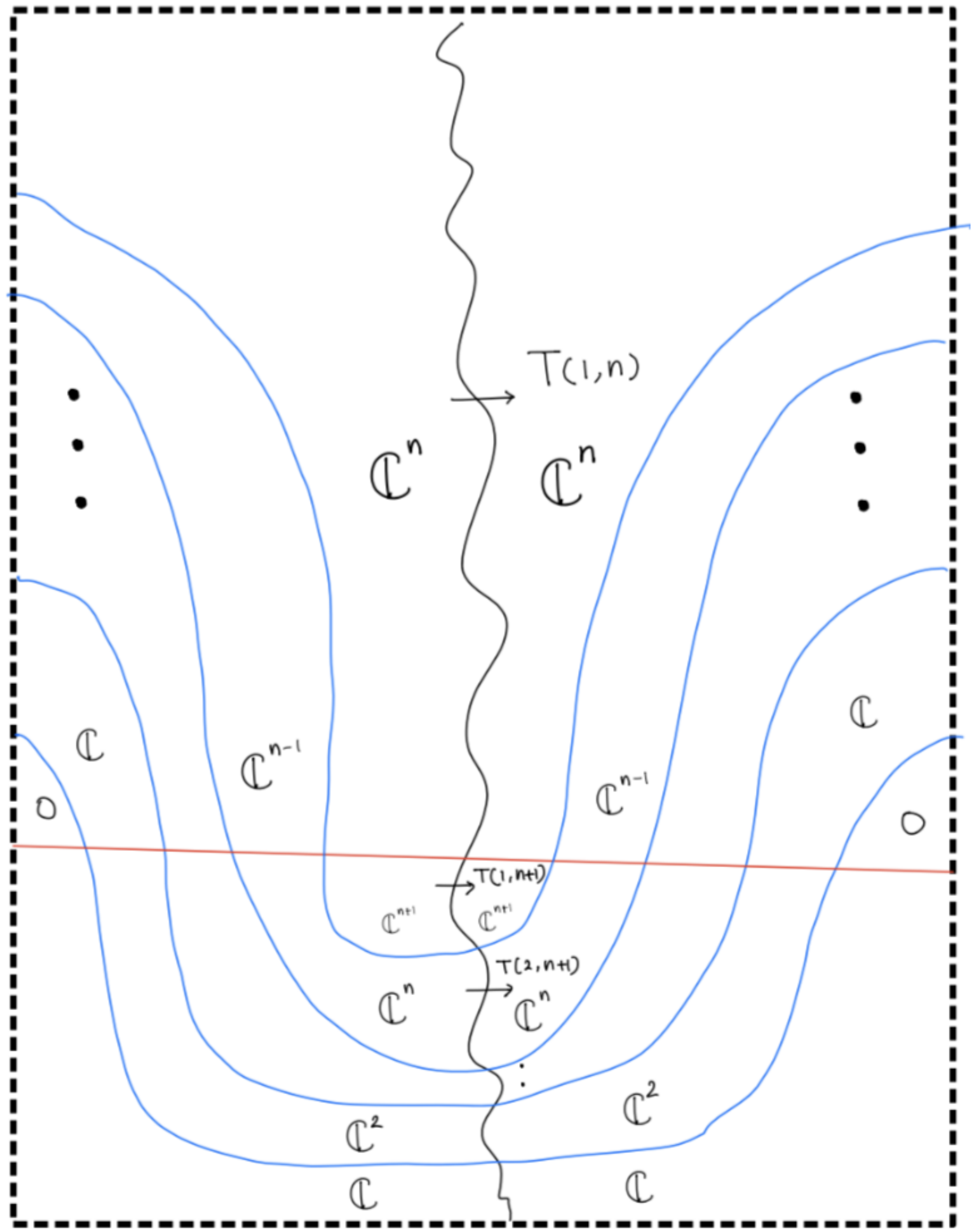

Figure 3.278

We define $cobord_6(n)$ inductively as follows.

(i) For $n = 1$, we define $cobord_6(1)$ to be $cobord_1$ from

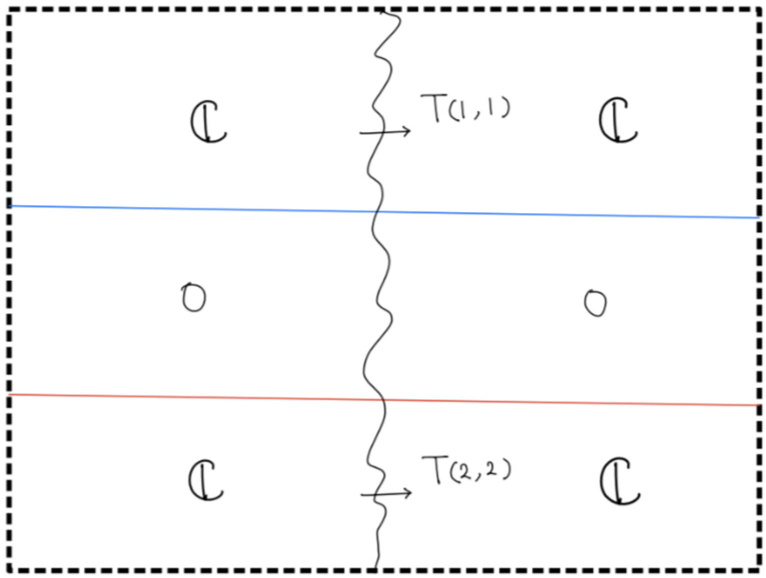

Figure 3.279

to

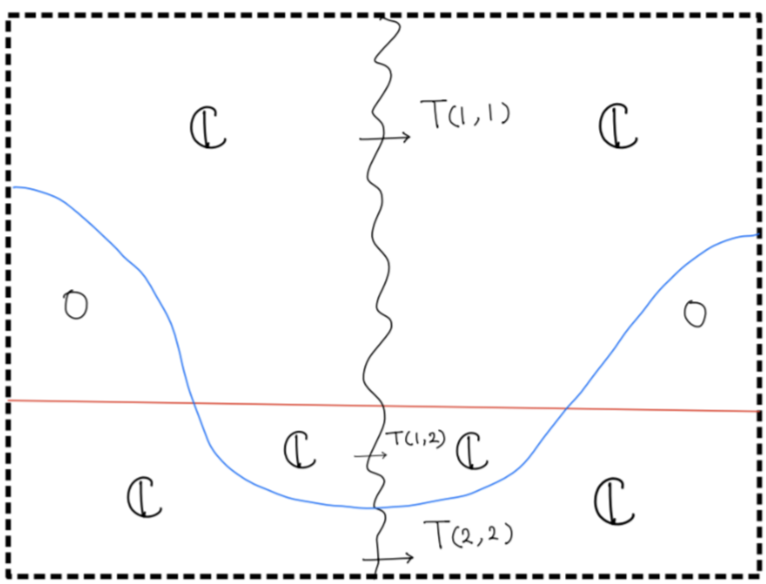

Figure 3.280

(ii) For $n > 0$,

(Step 1) we first apply $cobord_6(n-1)$ to the square region surrounded by purple dotted lines.

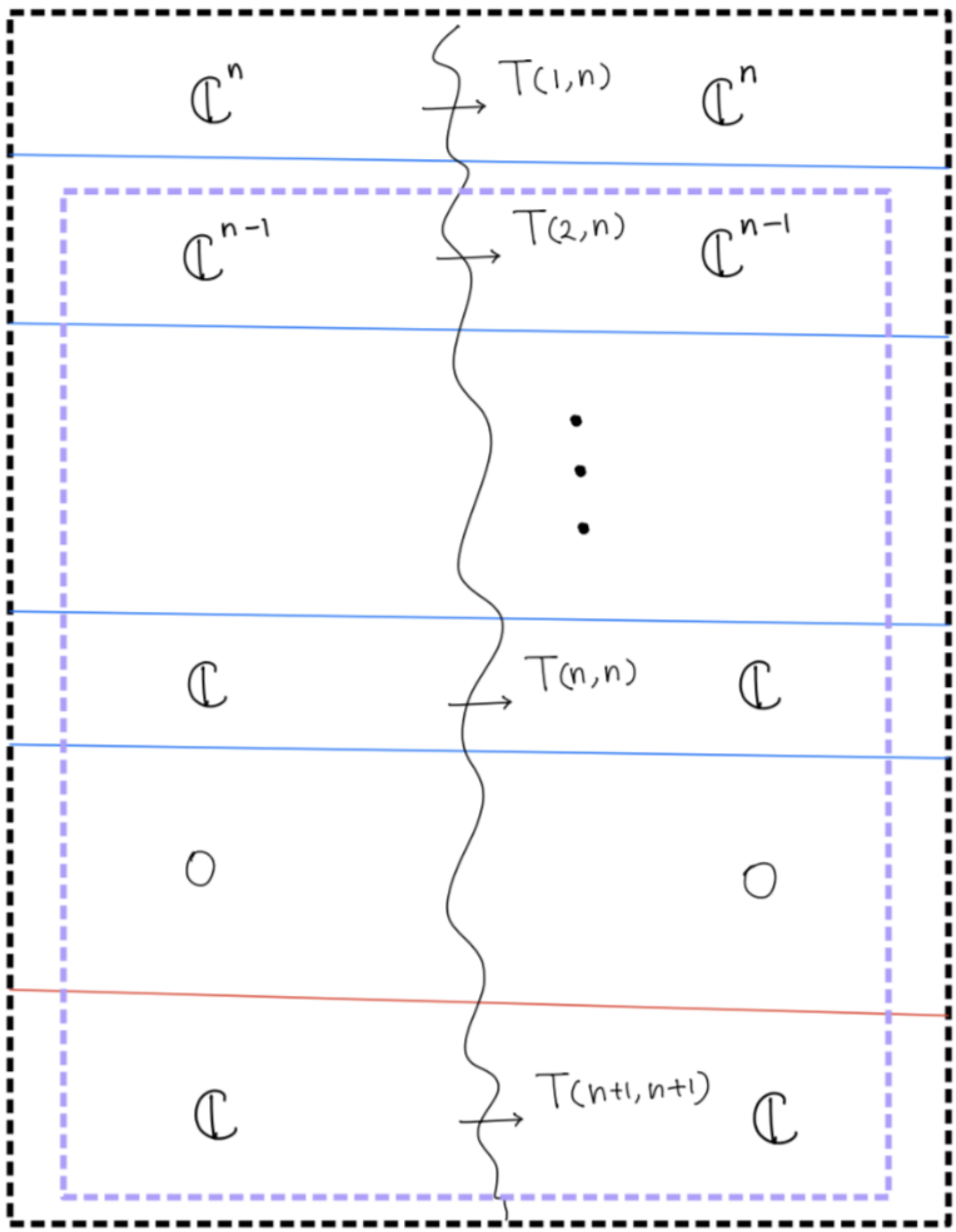

Figure 3.281

by induction hypothesis, we get

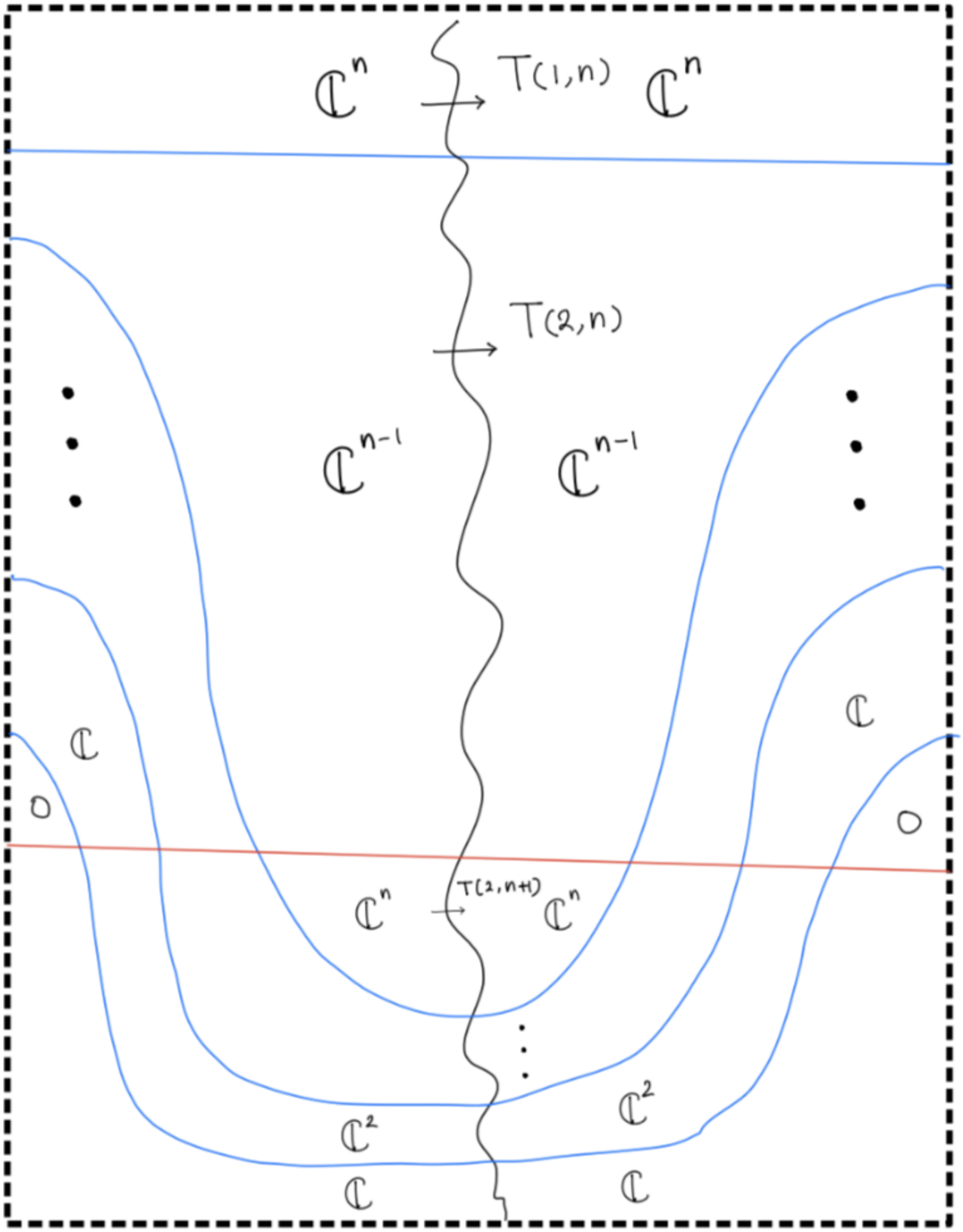

Figure 3.282

(Step 2) apply $cobord_1$ to the square region surrounded by purple dotted lines.

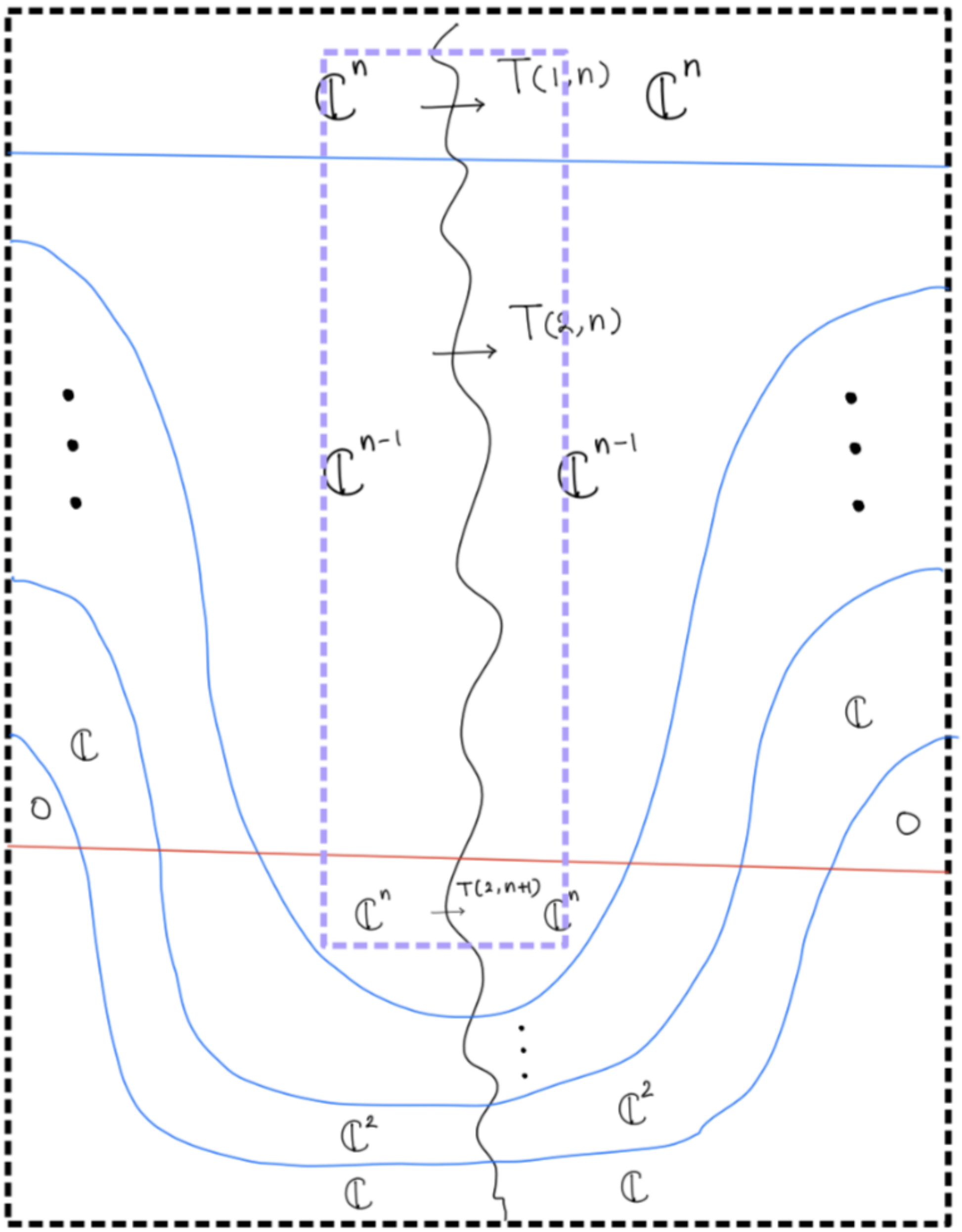

Figure 3.283

we get the final sheaf

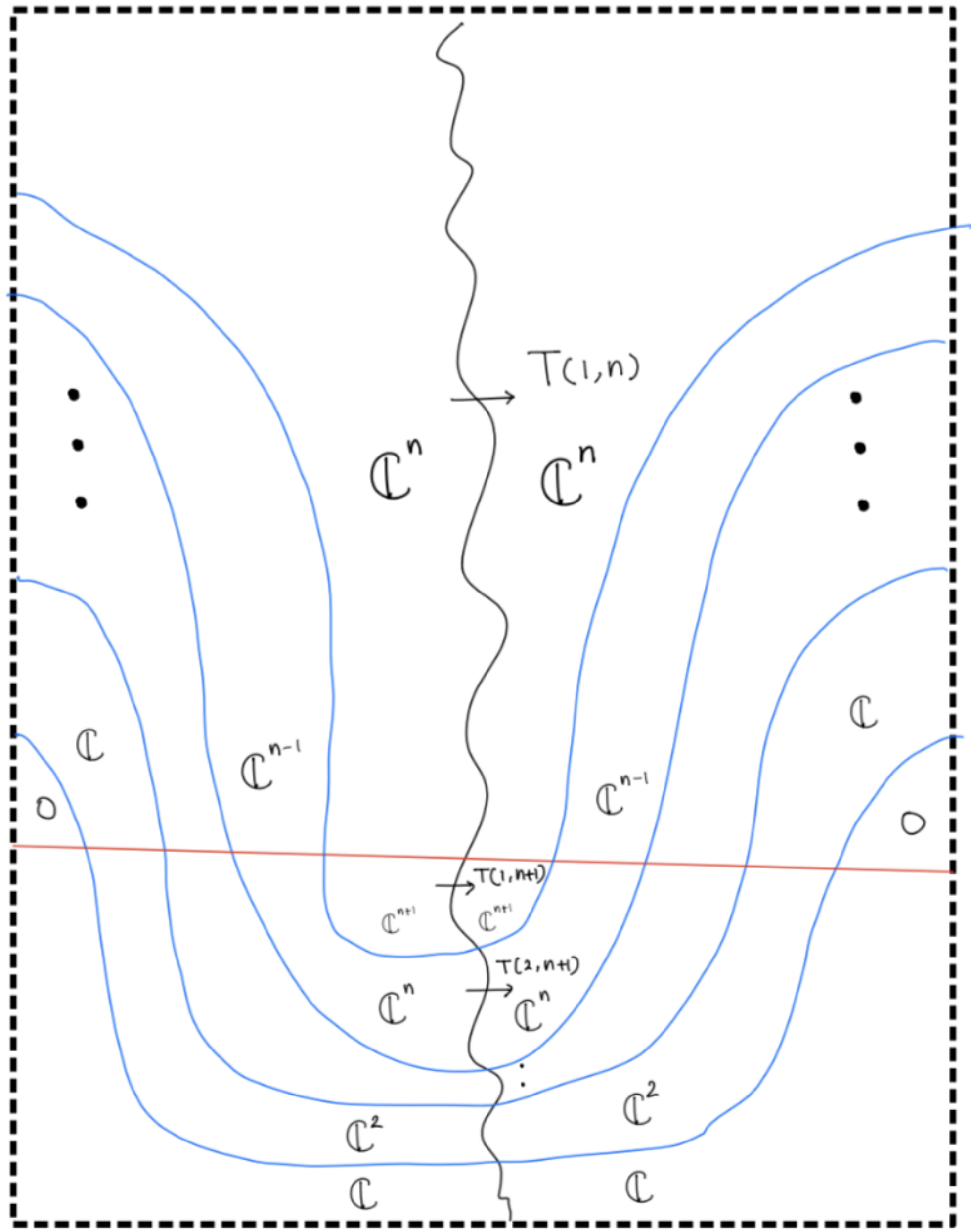

Figure 3.284

3.12 7th sheaf cobordism

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-1,1)_x \times (0, n+1)_z$ under f

- Λ_0^0 gets mapped to $\{(x,z) \in R \mid z = \Psi(\frac{1}{2}, n+\frac{1}{2})(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup\limits_{k=1}^{n} \{(x,z) \in R \mid z = -k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to ϕ.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

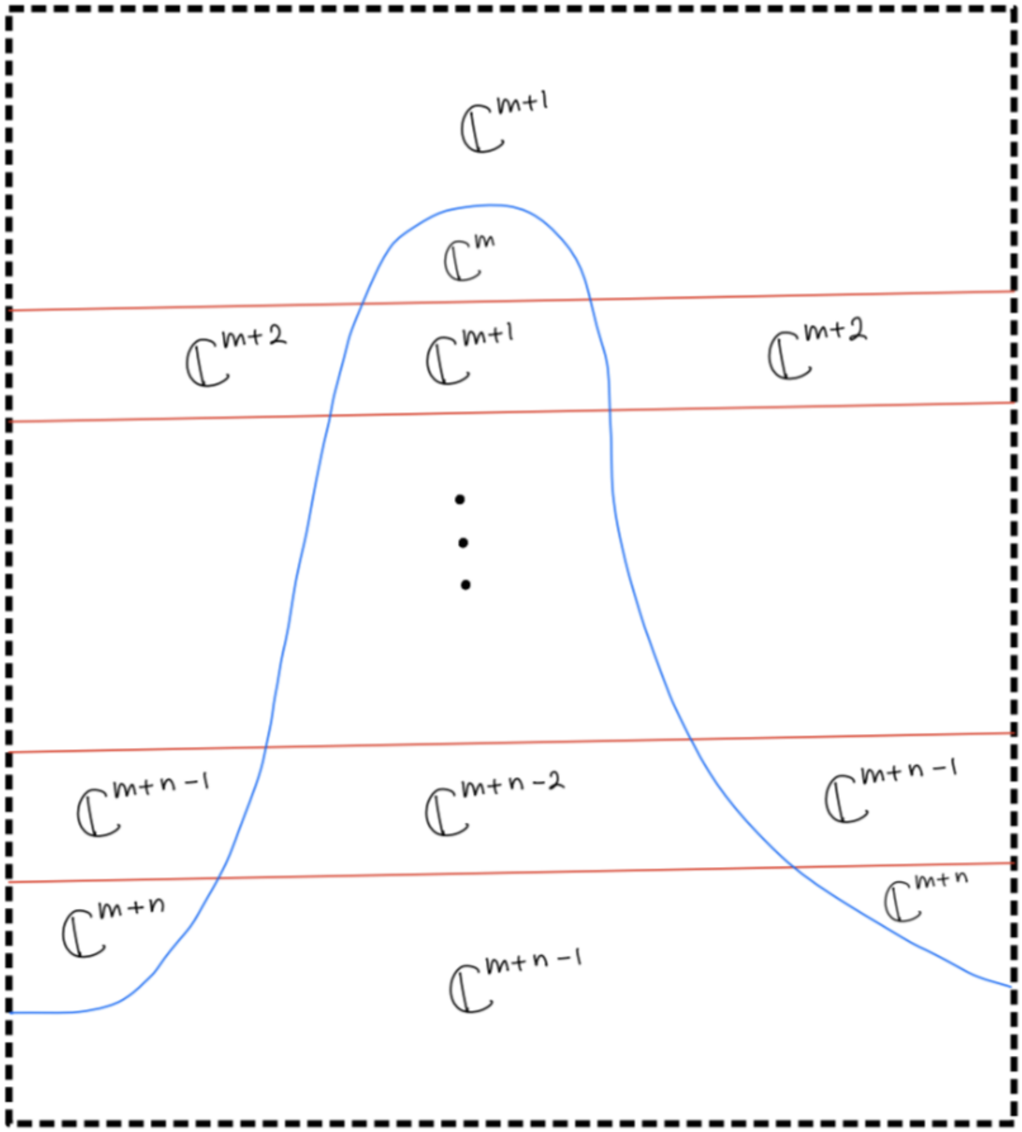

Figure 3.285

Then we define a cobordism starting from the above sheaf, say $cobord_5(n)$ supported on U, where n is the number of red strands. At the end of the cobordism , the sheaf, under the same chart f, is described as the following squiggly legible diagram.

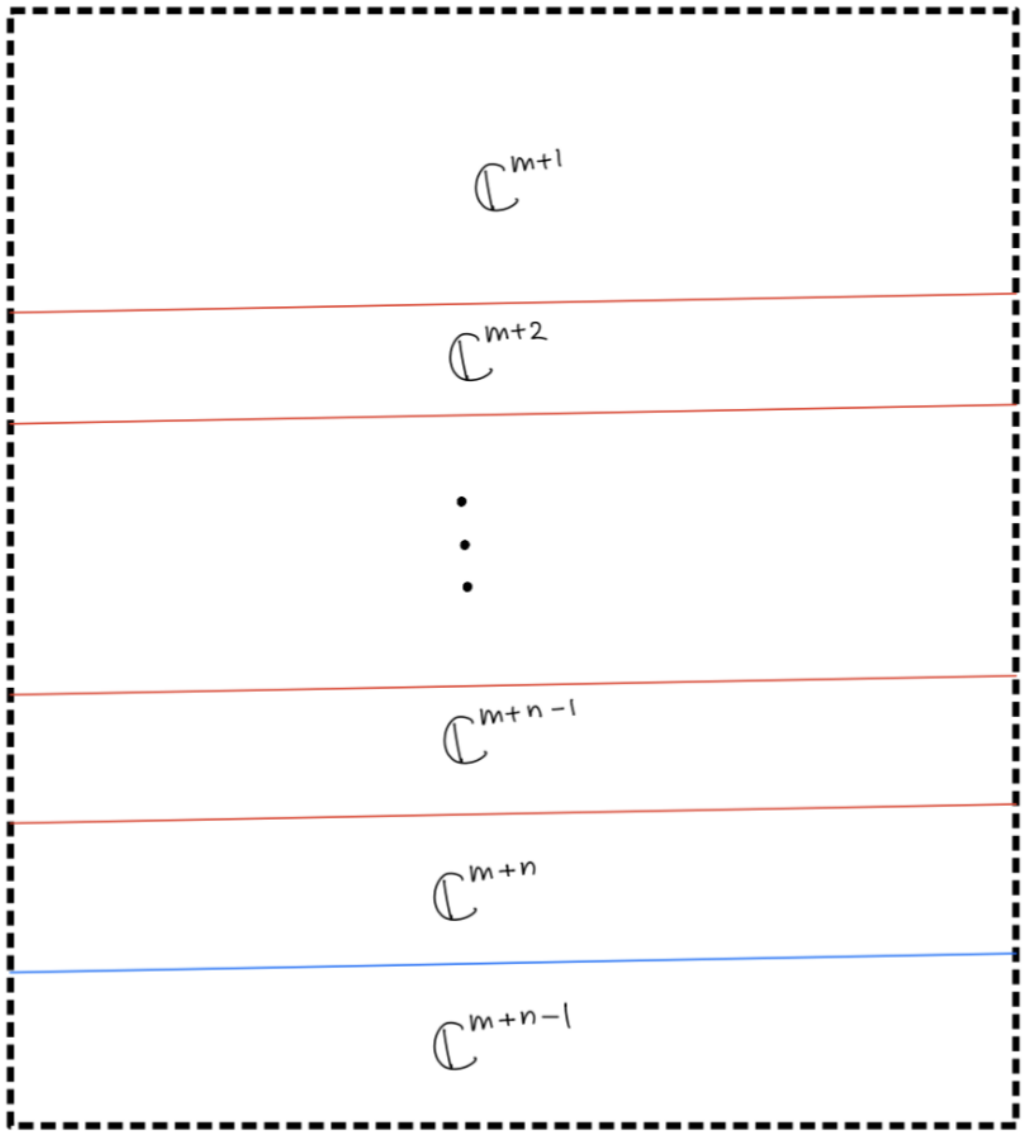

Figure 3.286

We define $cobord_7(n)$ inductively as follows.

(i) For $n = 1$, we define $cobord_7(1)$ to be $cobord_3$ from

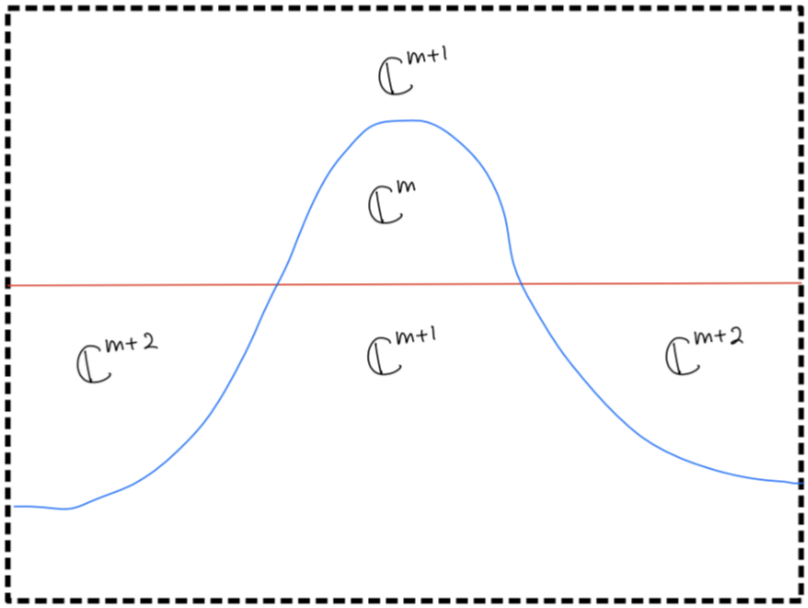

Figure 3.287

to

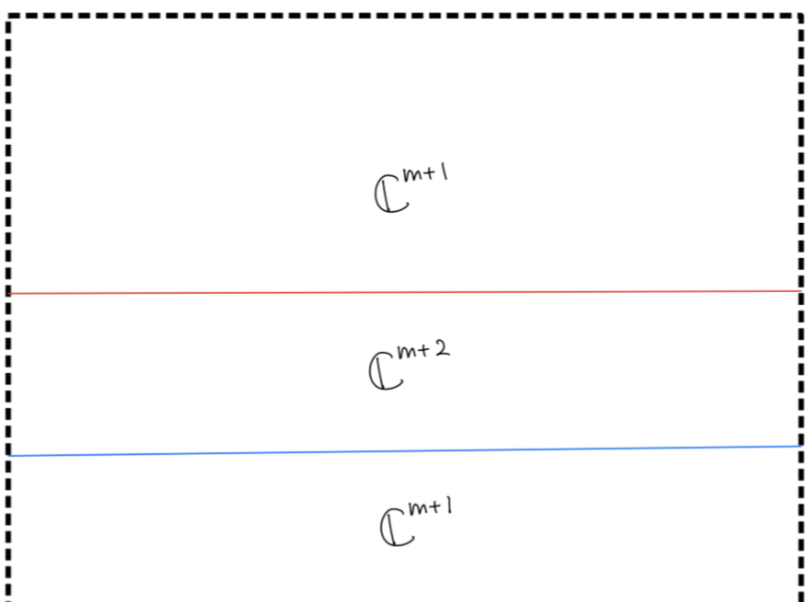

Figure 3.288

(ii) For $n > 0$,

(Step 1) we first apply $cobord_7(n-1)$ to the square region surrounded by purple dotted lines.

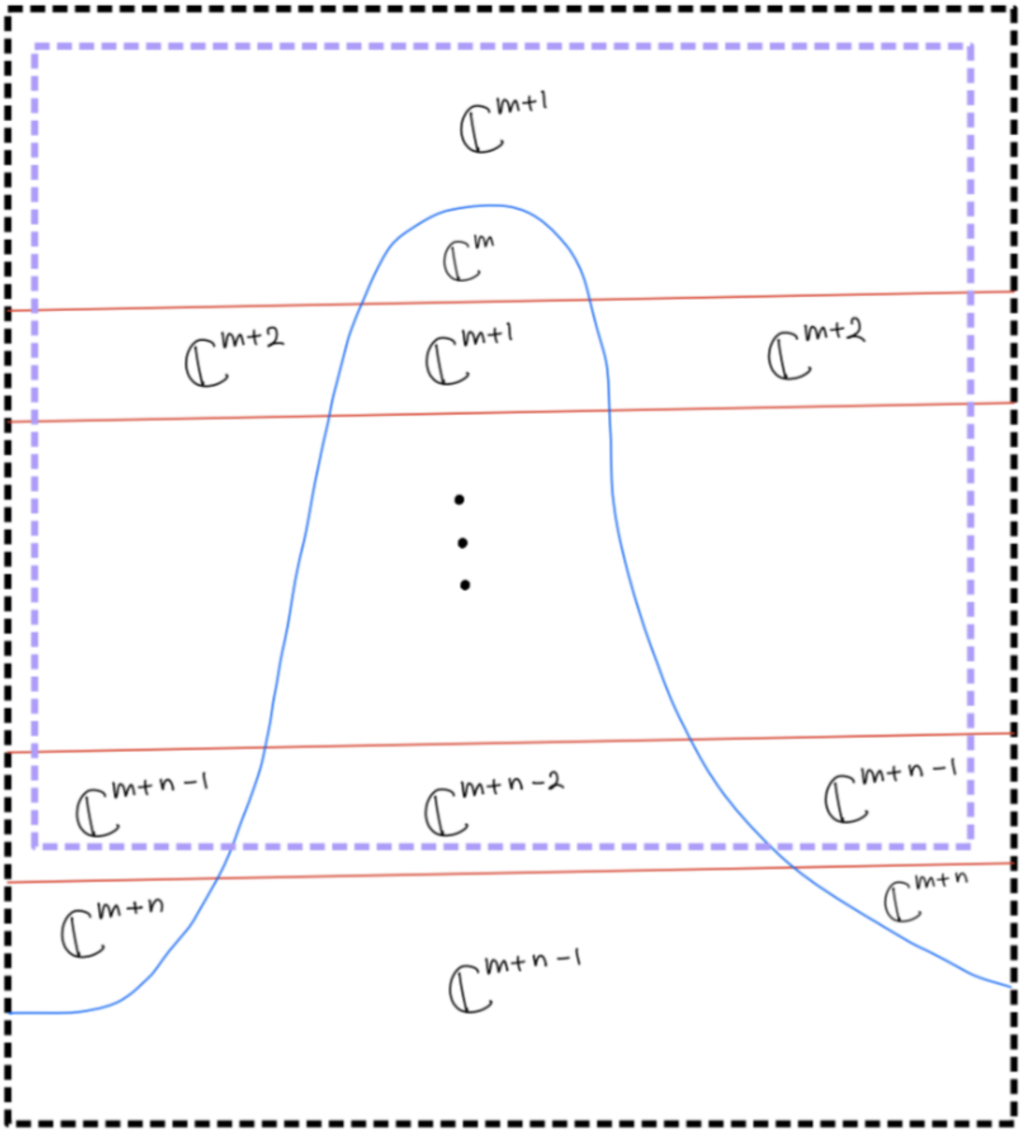

Figure 3.289

by induction hypothesis, we get

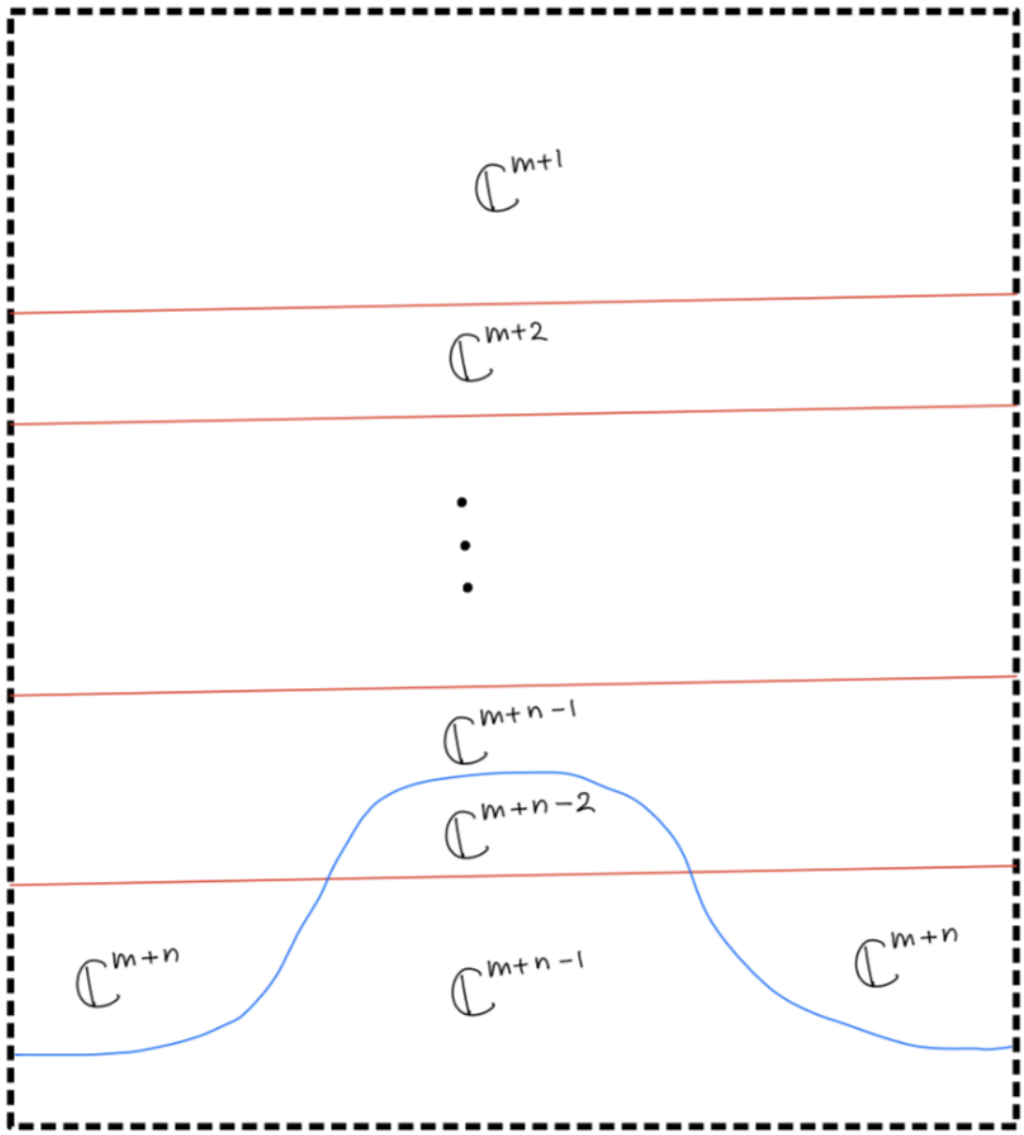

Figure 3.290

(Step 2) apply $cobord_3$ to the square region surrounded by purple dotted lines.

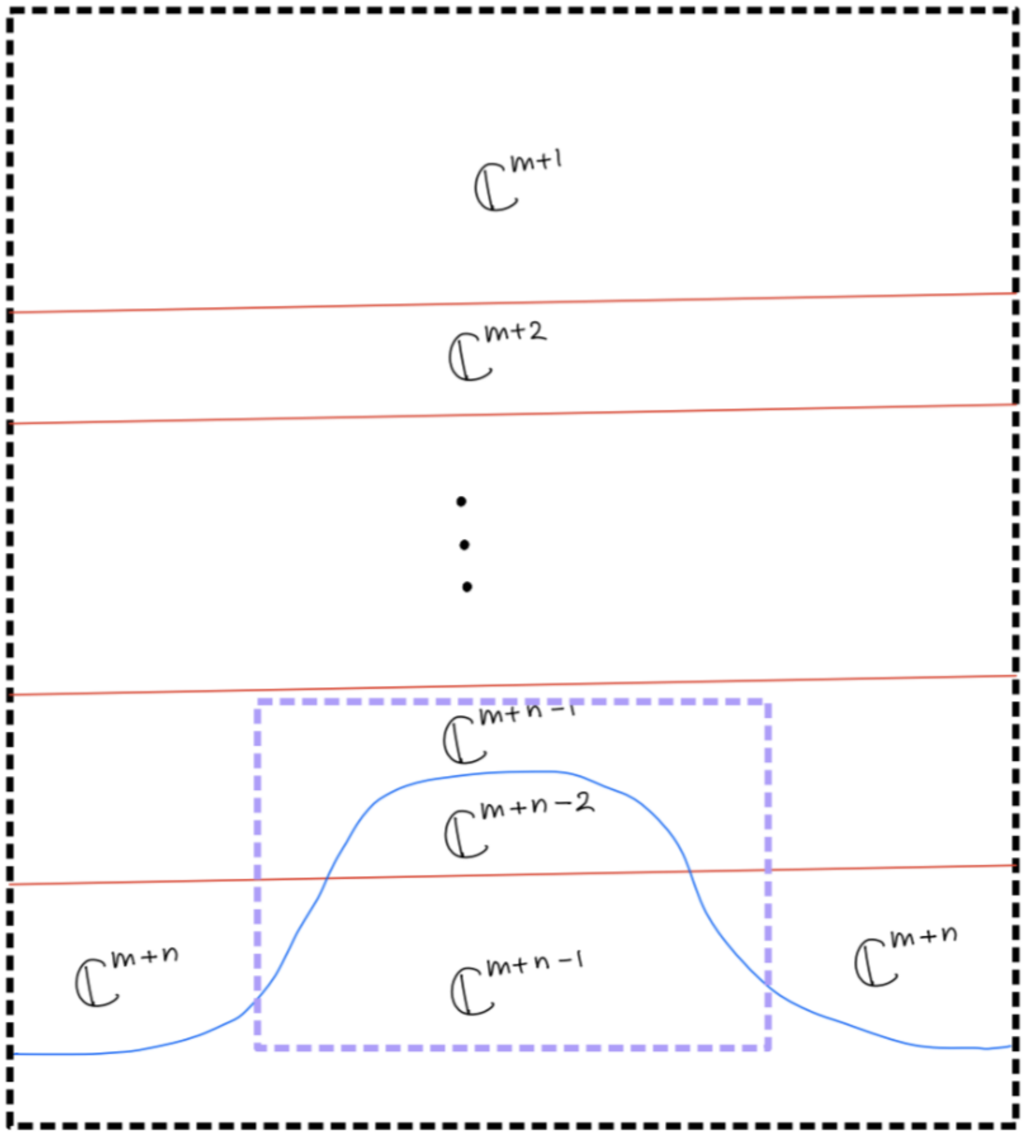

Figure 3.291

we get the final sheaf

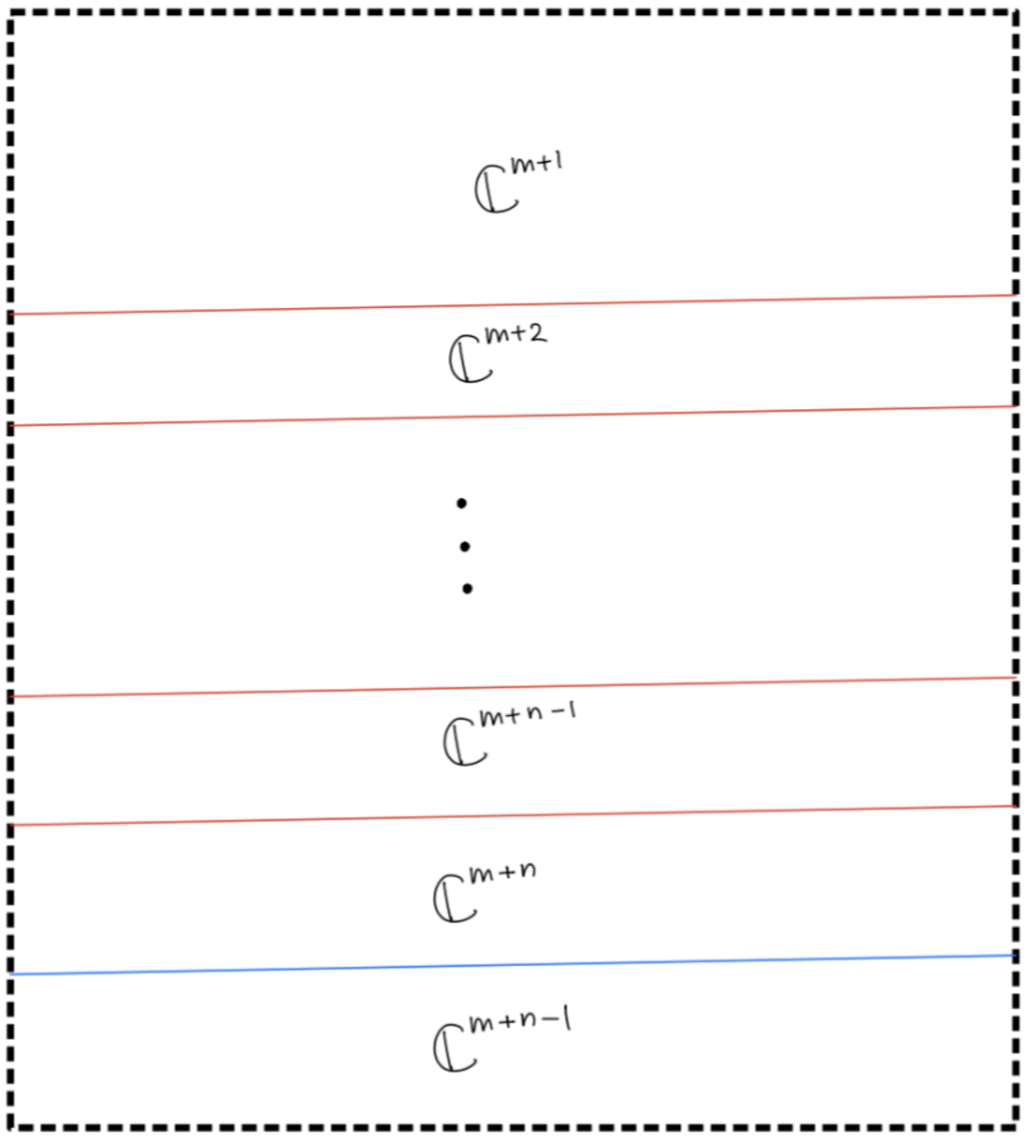

Figure 3.292

3.13 8th sheaf cobordism

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-2,2)_x \times (-1,1)_z$ under f

- Λ_0^0 gets mapped to blue strands in the below figure, co-oriented upward.
- Λ_0^∞ gets mapped to red strands in the below figure, co-oriented downward.
- Λ_0^{squig} gets mapped to squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

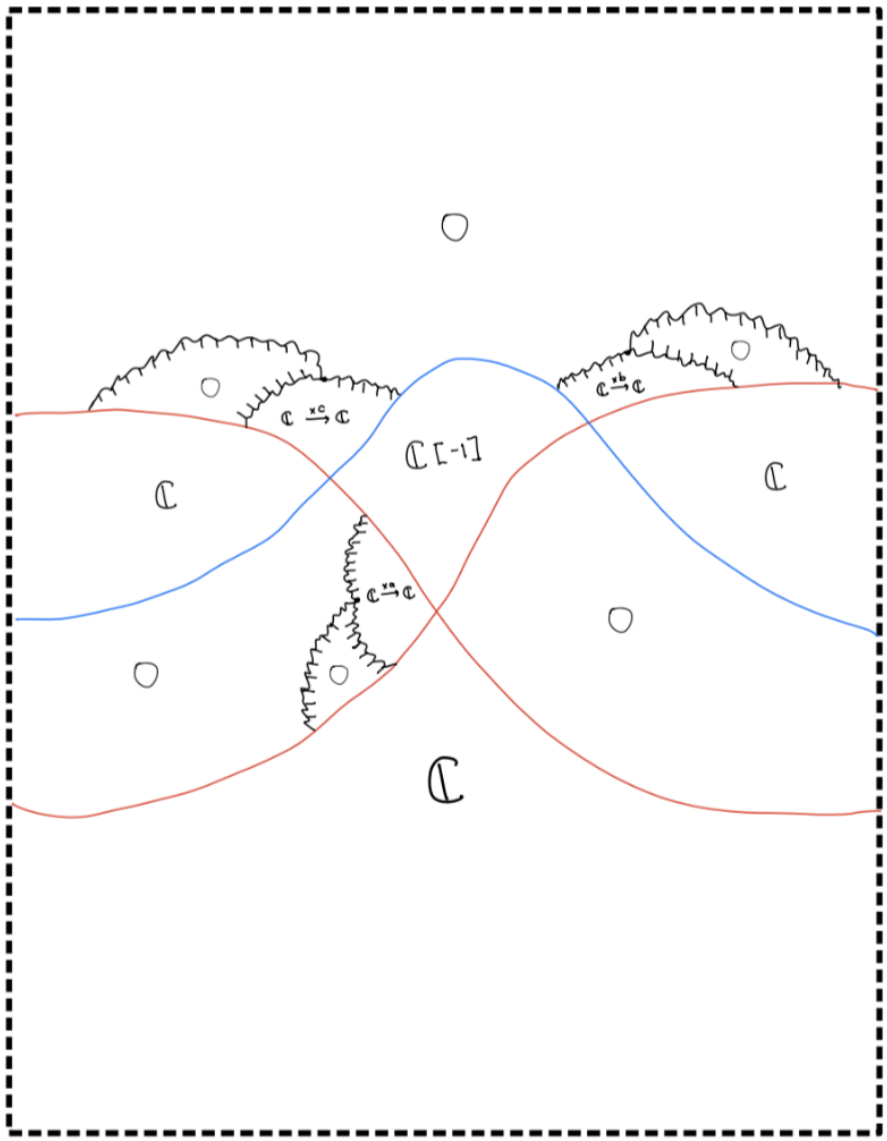

Figure 3.293

Then we define a cobordism starting from the above sheaf, say $cobord_8$ supported on U. At the end of the cobordism, the sheaf, under the same chart f, is described as the following squiggly legible diagram.

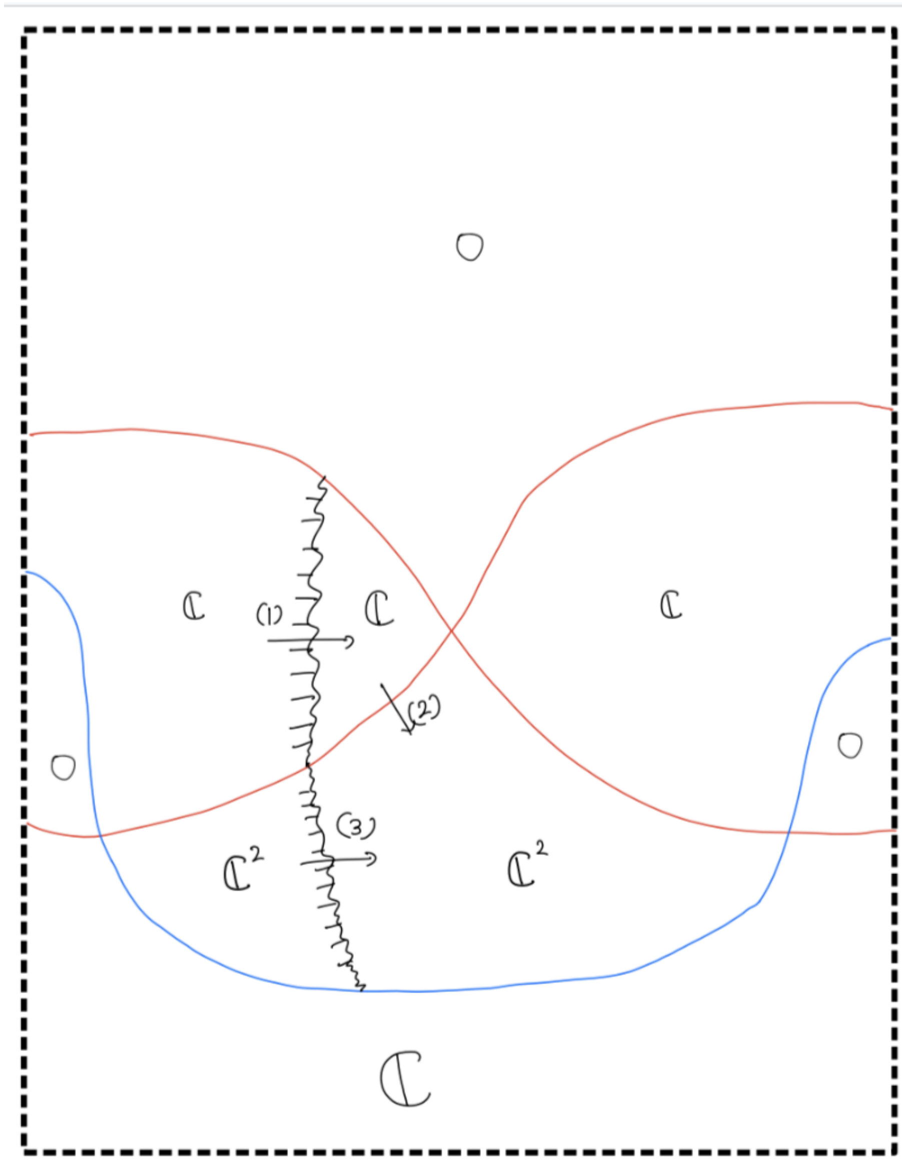

Figure 3.294

Generizations maps:

(1) $\times b^{-1}c$

(2) $\begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$

(3) $\begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$

We define $cobord_8$ as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

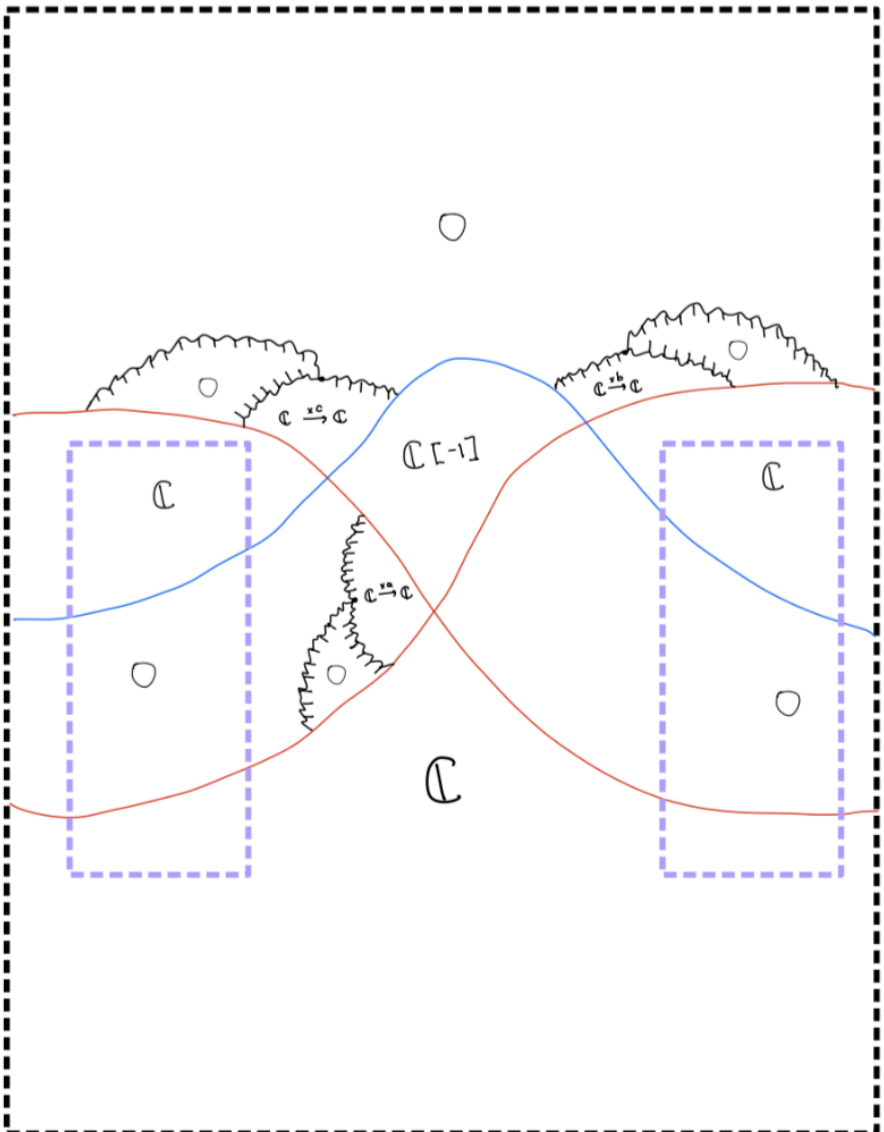

Figure 3.295

we get

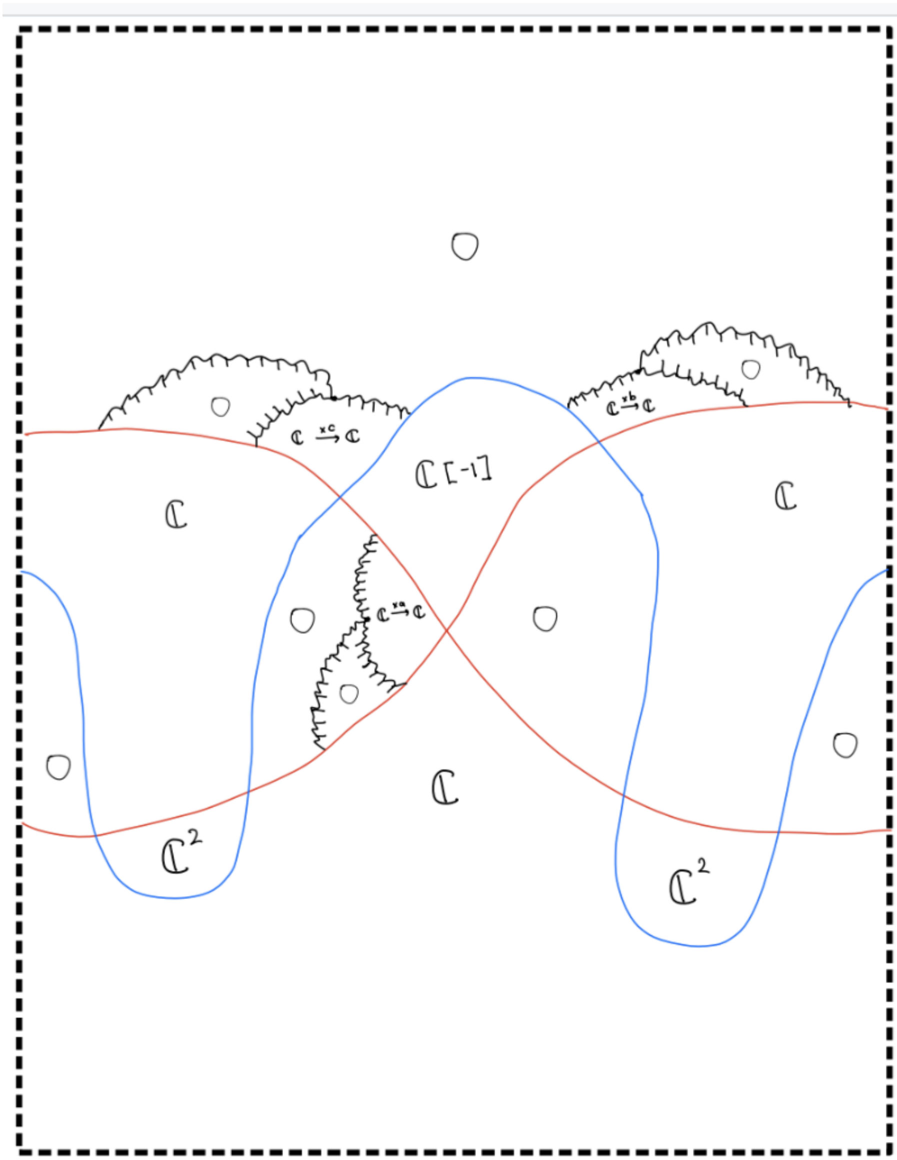

Figure 3.296

(Step 2) apply $cobord_4$ to the region surrounded by a purple dotted line.

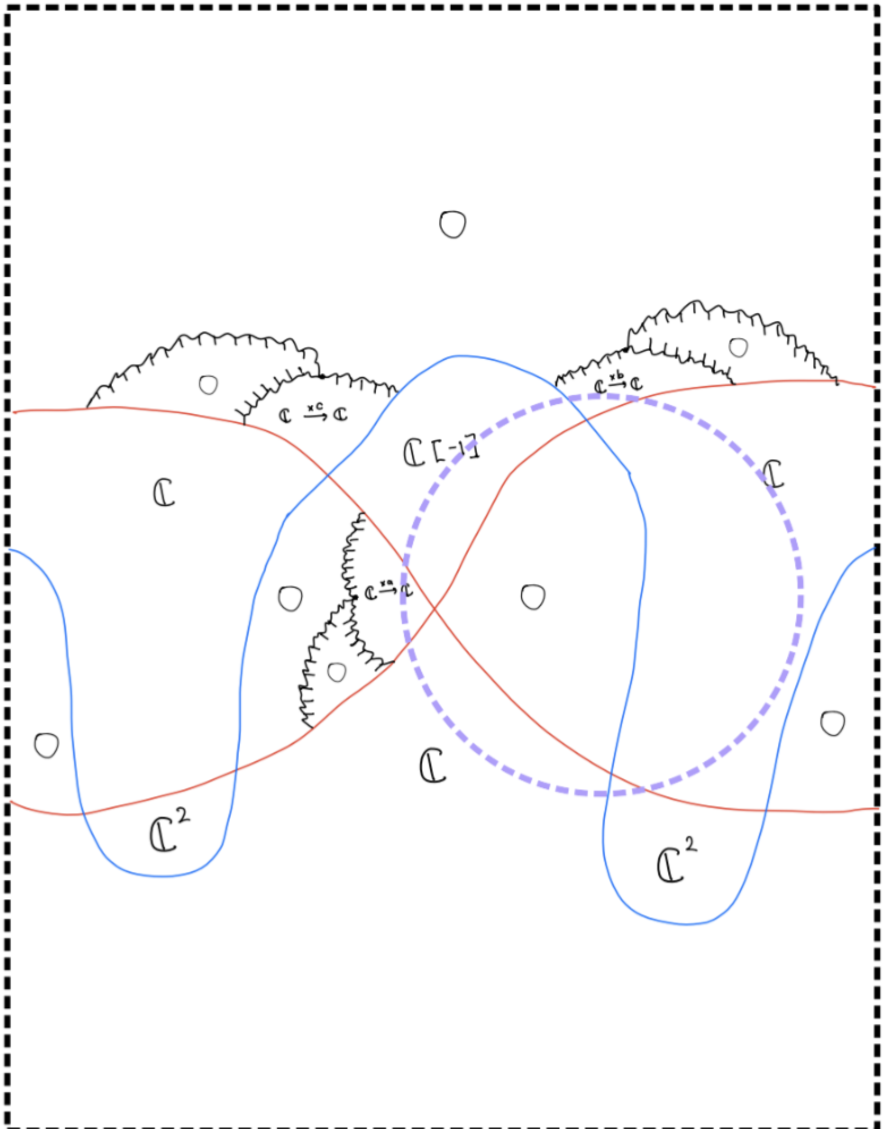

Figure 3.297

we get

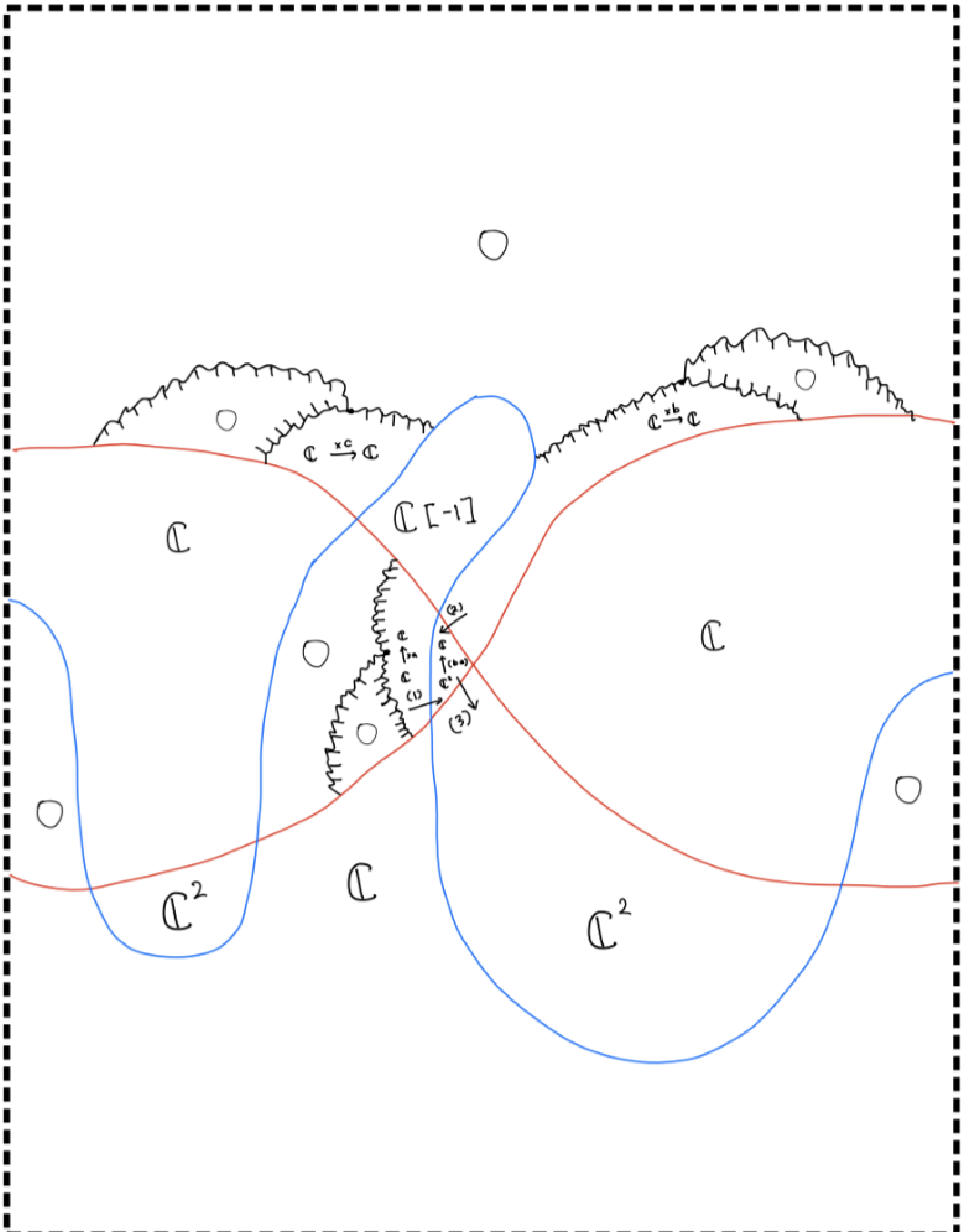

Figure 3.298

Generizations maps:

(1)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(3)
$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(Step 3) apply $cobord_2'$ to the square region surrounded by purple dotted lines.

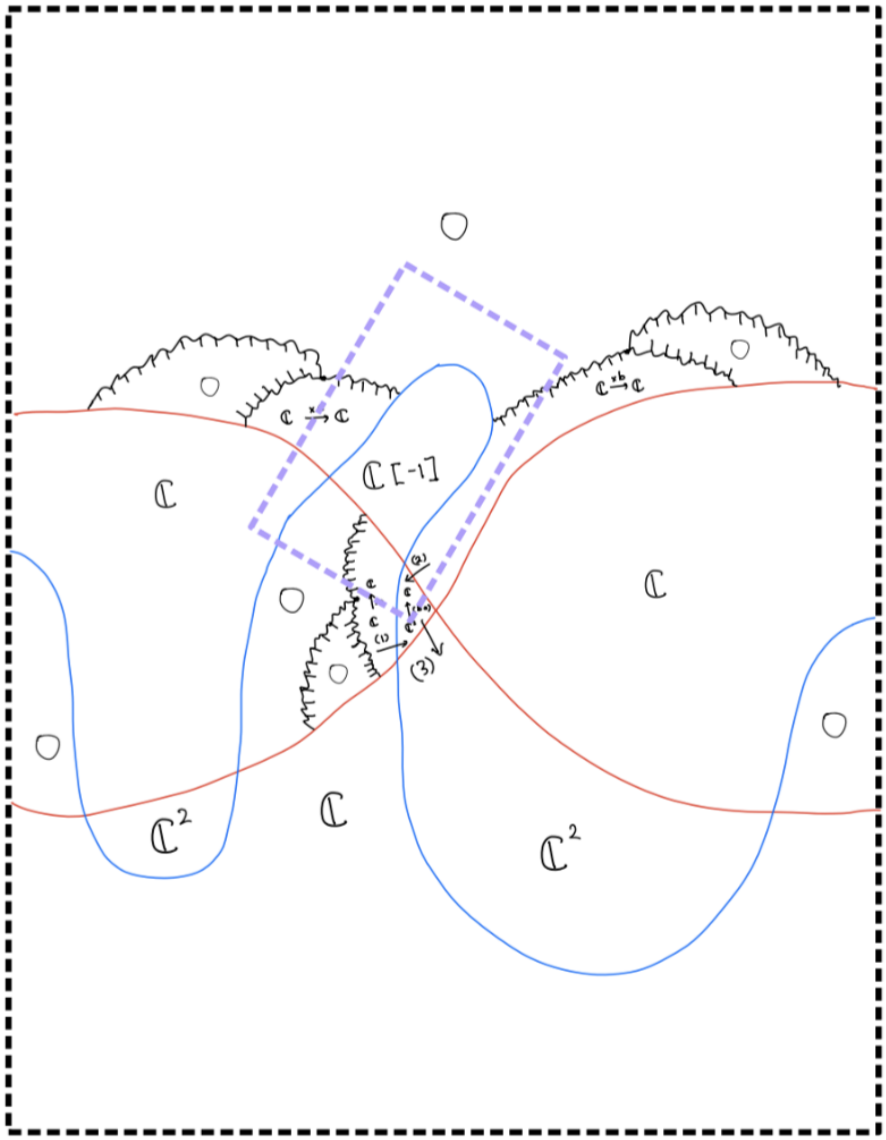

Figure 3.299

we get

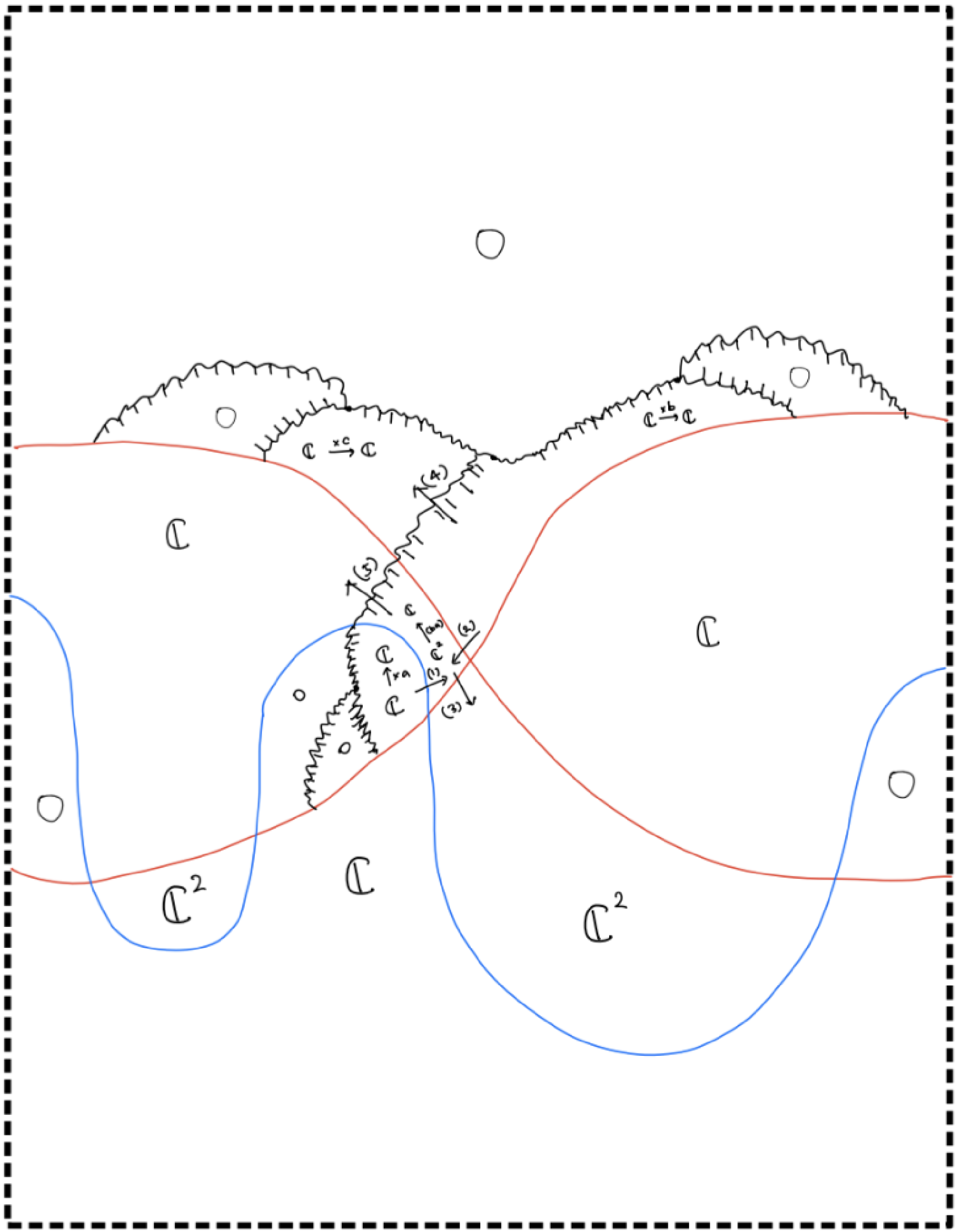

Figure 3.300

Generizations maps:

(1) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a"] \arrow[r, "\iota_1"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(2) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\iota_0"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$$

(3) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C}^2 \arrow[u, "(b\ \ a)"] \arrow[r, "id"] & \mathbb{C}^2 \arrow[u] \end{tikzcd}$$

(4) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b"] \arrow[r, "\times bc^{-1}"] & \mathbb{C} \arrow[u, "c"'] \end{tikzcd}$$

(5) $$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \\ \mathbb{C} \arrow[u, "(b\ \ a)"] \arrow[r, "(bc^{-1}\ \ 0)"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

After identifying $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$, $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$, $\mathbb{C} \xrightarrow{\times c} \mathbb{C}$ with 0 and then identifying $\mathbb{C}$ with $\mathbb{C}^2 \xrightarrow{(b\ \ a)} \mathbb{C}$ via $\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r, "(1\ -a^{-1}b)^T"] & \mathbb{C}^2 \arrow[u, "(b\ \ a)"'] \end{tikzcd}$

the above sheaf is quasi-isomorphic to

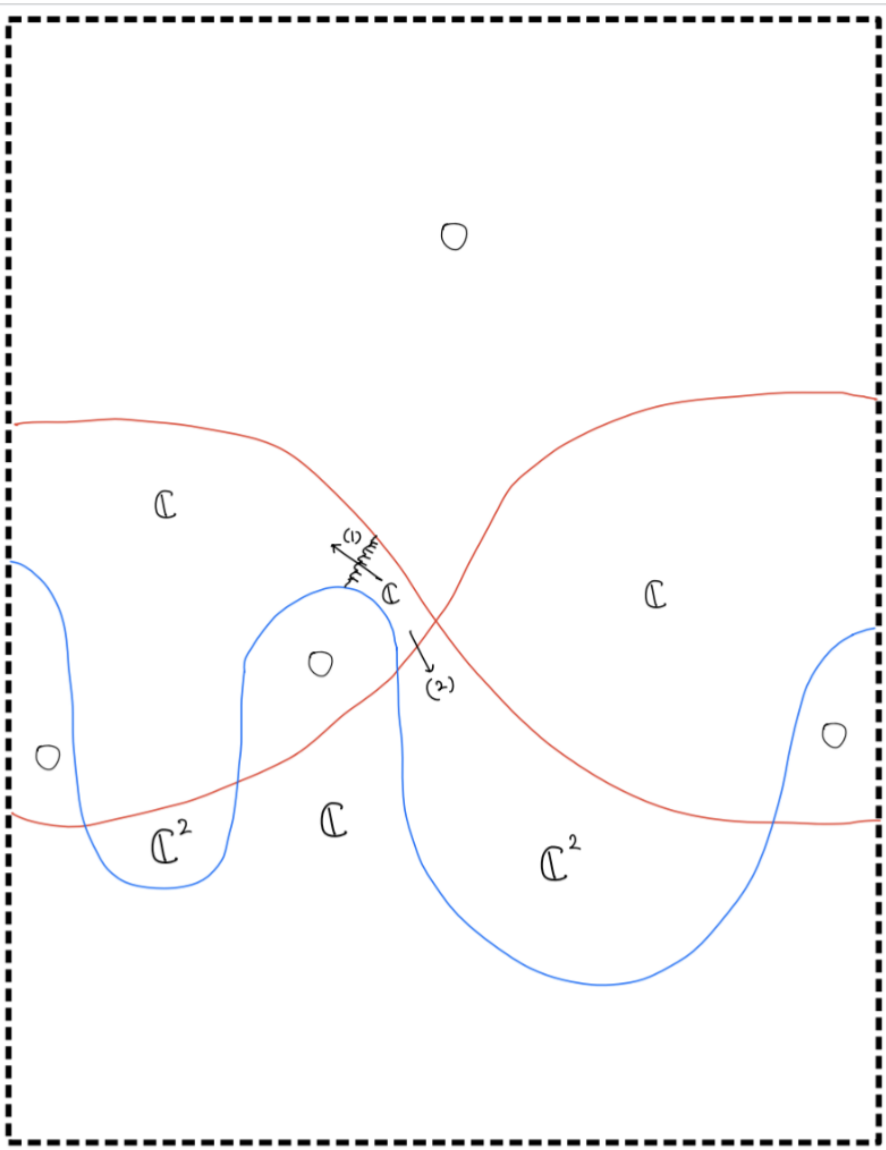

Figure 3.301

Generizations maps:

(1) $\times bc^{-1}$

(2) $\begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$

which is quasi-isomorphic to

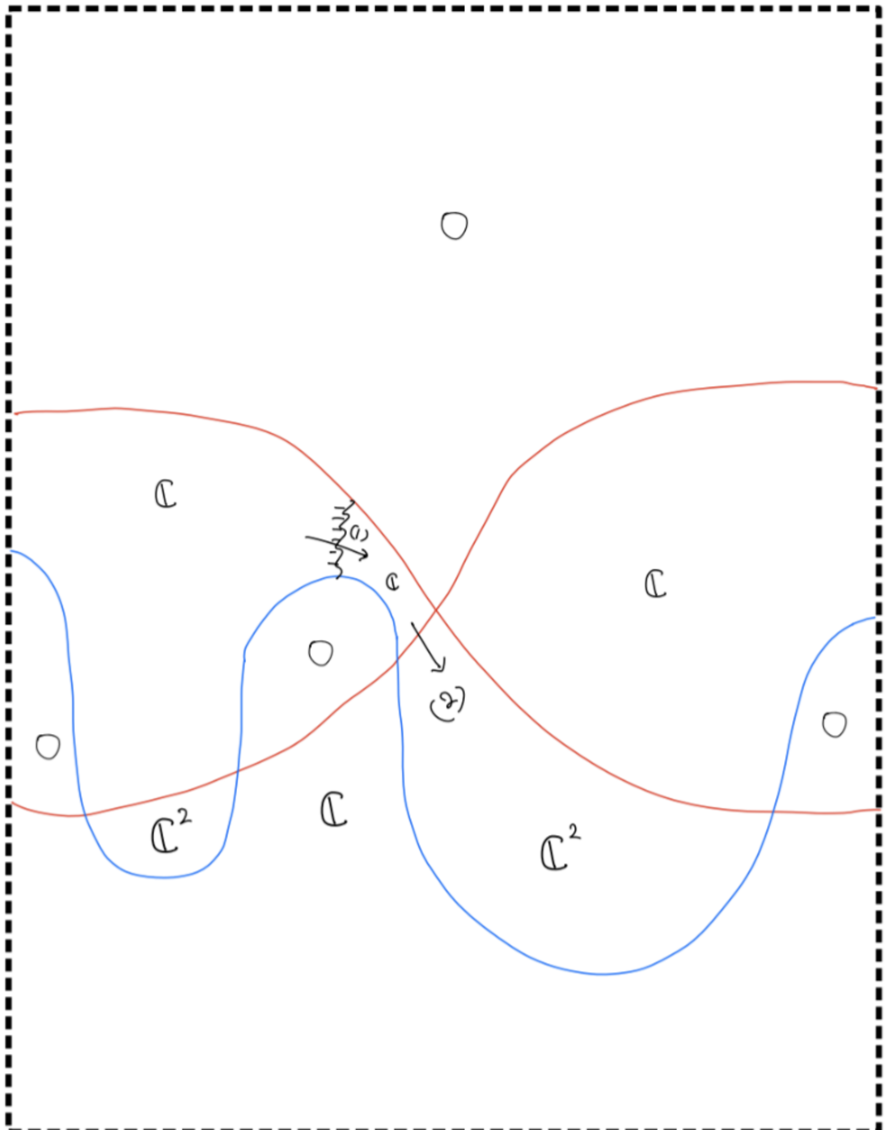

Figure 3.302

Generizations maps:

(1) $\times b^{-1}c$

(2) $\begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$

(Step 4) apply $cobord_3$ to the region surrounded by a purple dotted line

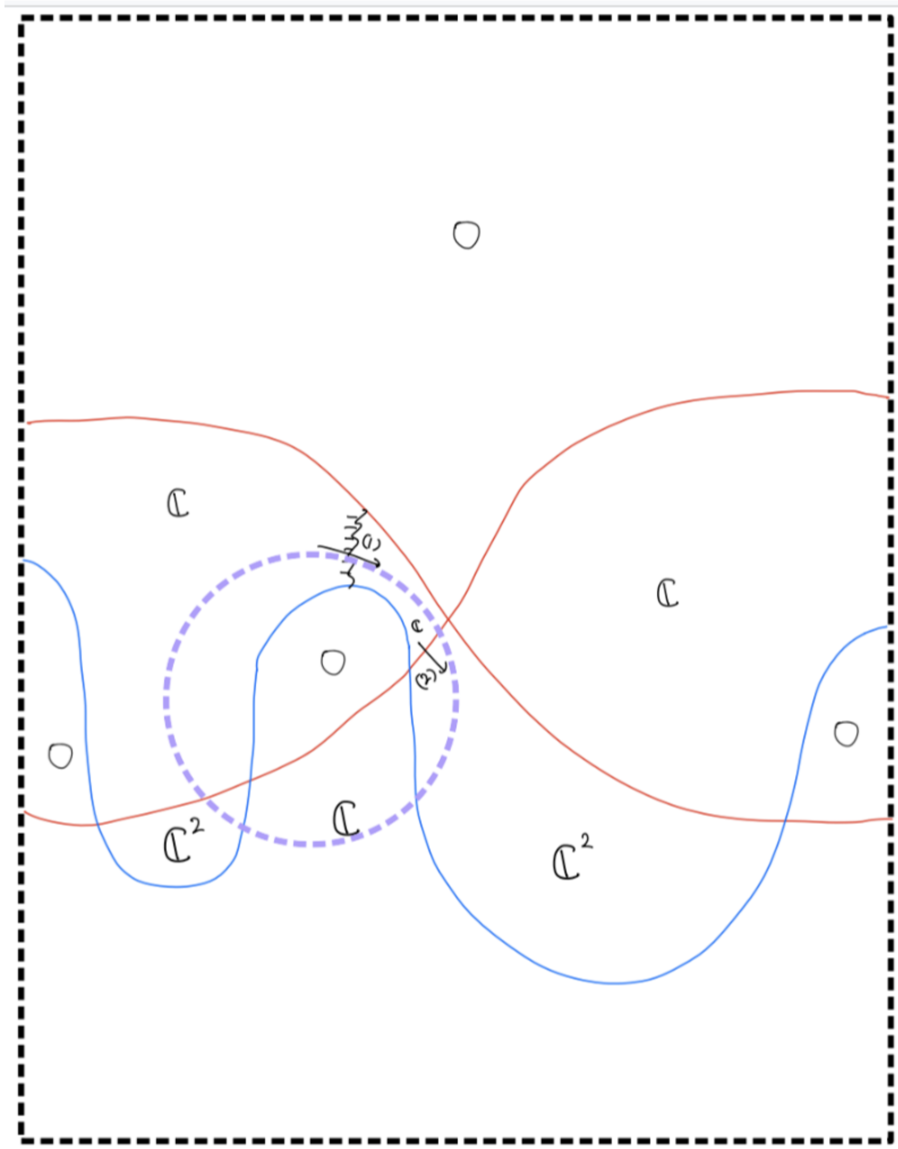

Figure 3.303

we get

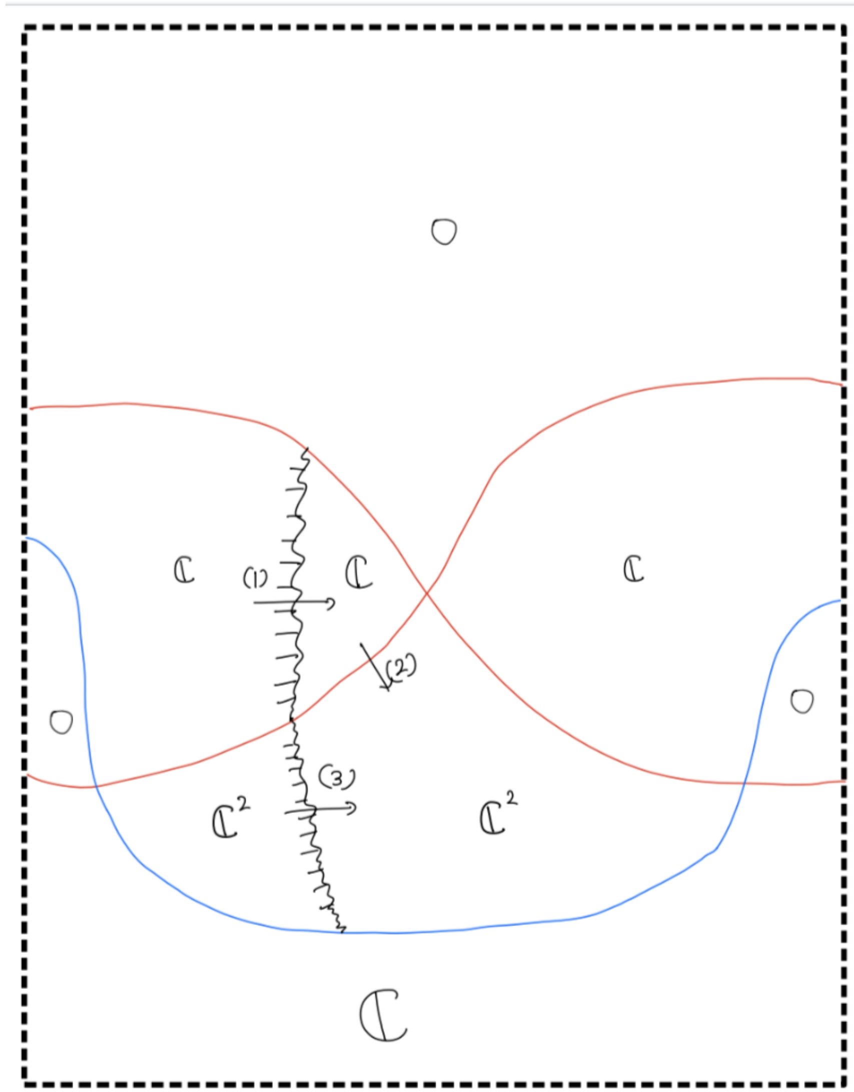

Figure 3.304

Generizations maps:

(1) $\times b^{-1}c$

(2) $\begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$

(3) $\begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$

3.14 8th sheaf cobordism'

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-2, 2)_x \times (-1, 1)_z$ under f

- Λ_0^0 gets mapped to $\{(x, z) \in R \mid z = \Psi(z_{lo} = 0, z_{hi} = 1)(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\{(x, z) \in R \mid z = \tau(\frac{x+1}{2}) - \frac{1}{2}\} \cup \{(x, z) \in R \mid z = -\tau(\frac{x+1}{2}) + \frac{1}{2}\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x, z) \in R \mid x = -c - \epsilon\}$ where $c > 0$, $\Psi(0, 1)(c) = \tau(\frac{c+1}{2}) - \frac{1}{2}$ and $\epsilon > 0$ is a small number such that $c > \epsilon$.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

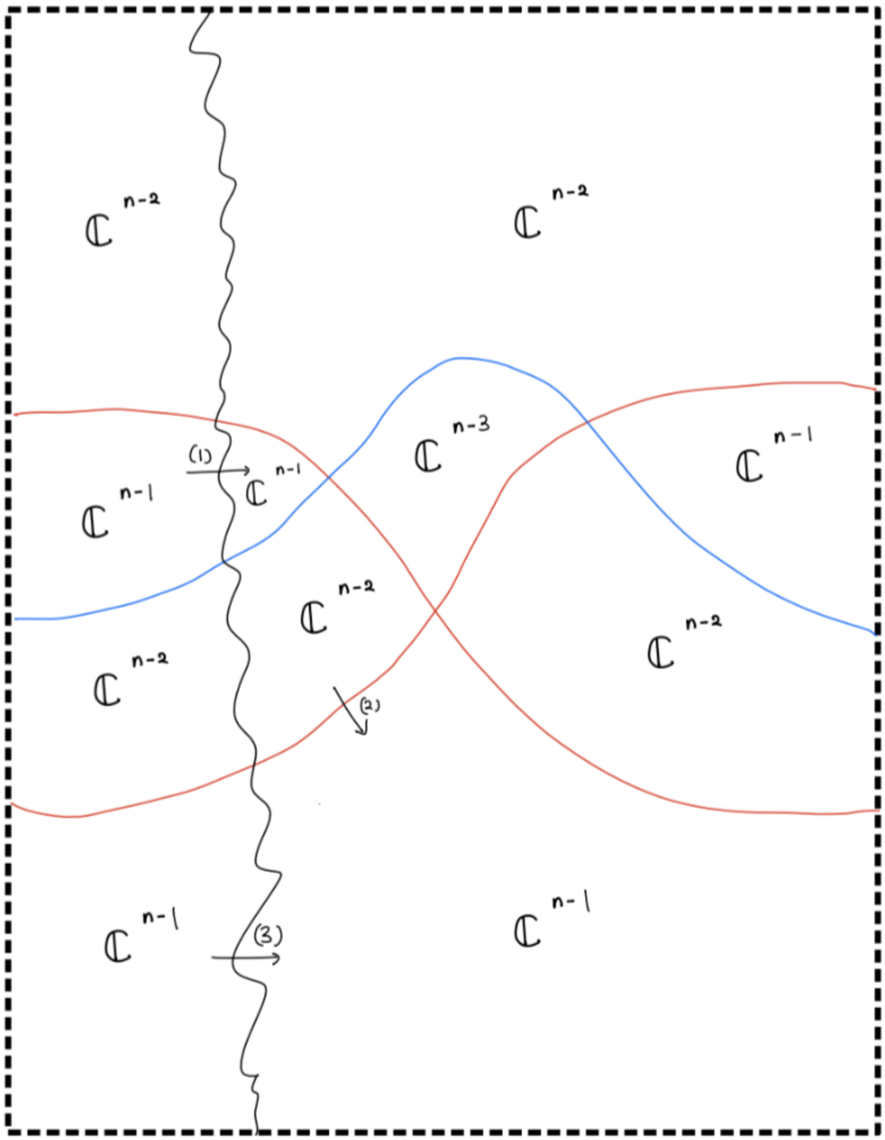

Figure 3.305

Generizations maps:

(1) $diag(d_n, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + e I_{n-1,n-2}$

where $e' = d_2^{-1} e$. Then we define a cobordism starting from the above sheaf, say $cobord_8'$ supported on U. At the end of the cobordism, the sheaf, under the same chart f, is described as the following squiggly legible diagram.

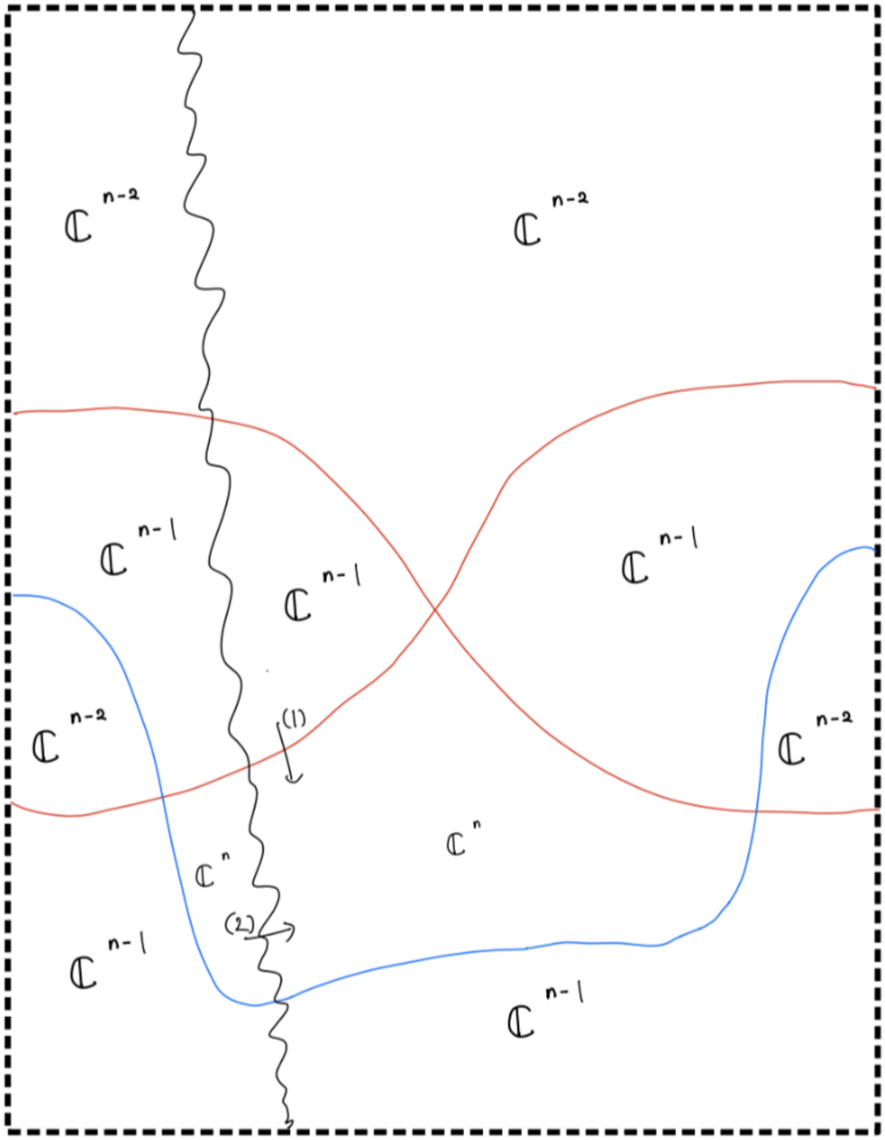

Figure 3.306

Generizations maps:

(1) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(2) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

We define $cobord'_8$ as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

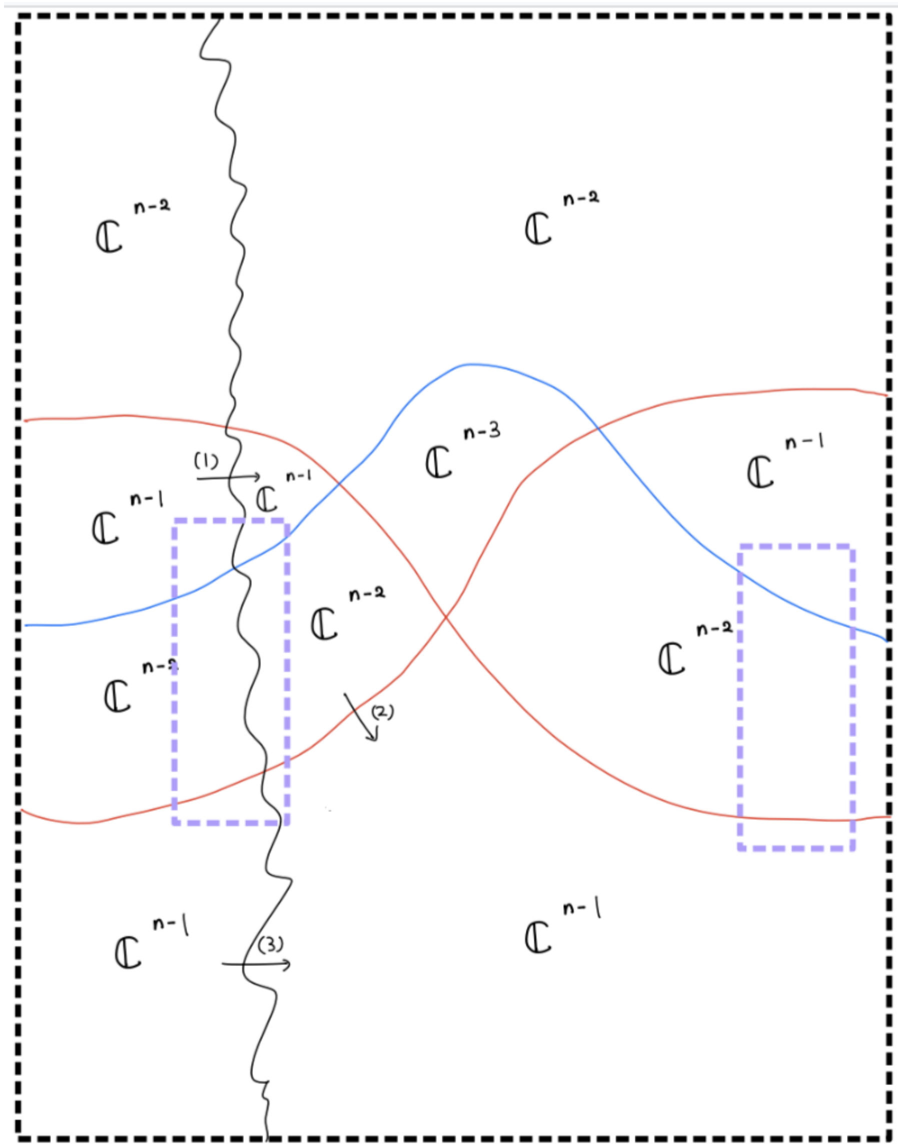

Figure 3.307

we get

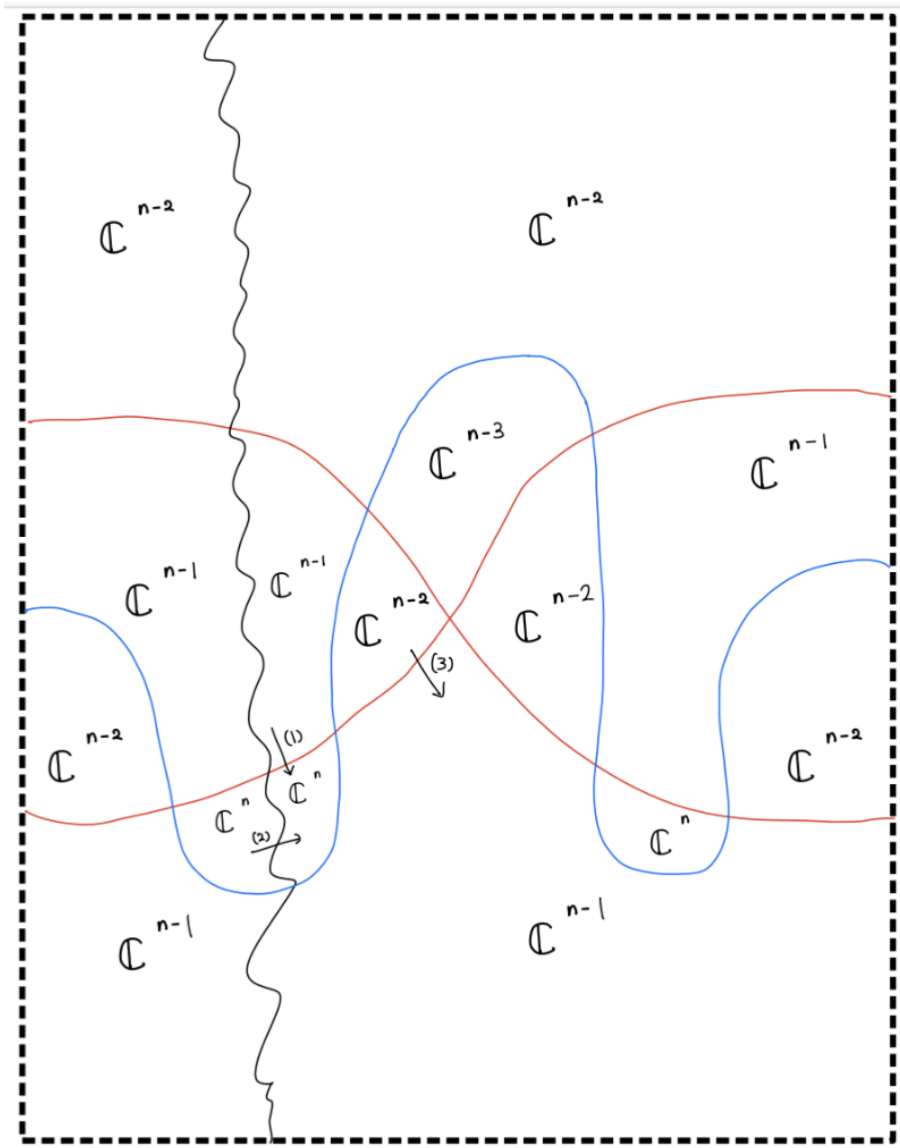

Figure 3.308

Generizations maps:

(1) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(2) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

(3) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(Step 2) apply $cobord'_4$ to the region surrounded by a purple dotted line.

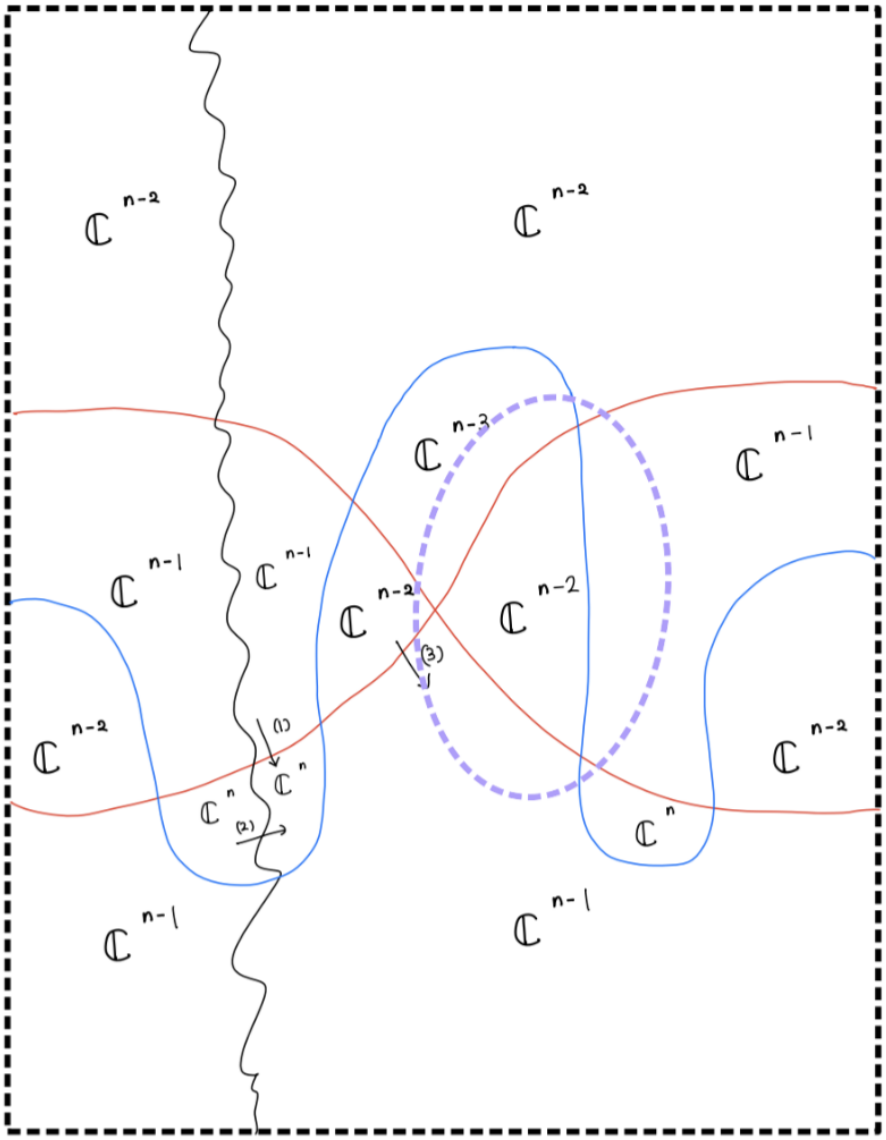

Figure 3.309

we get

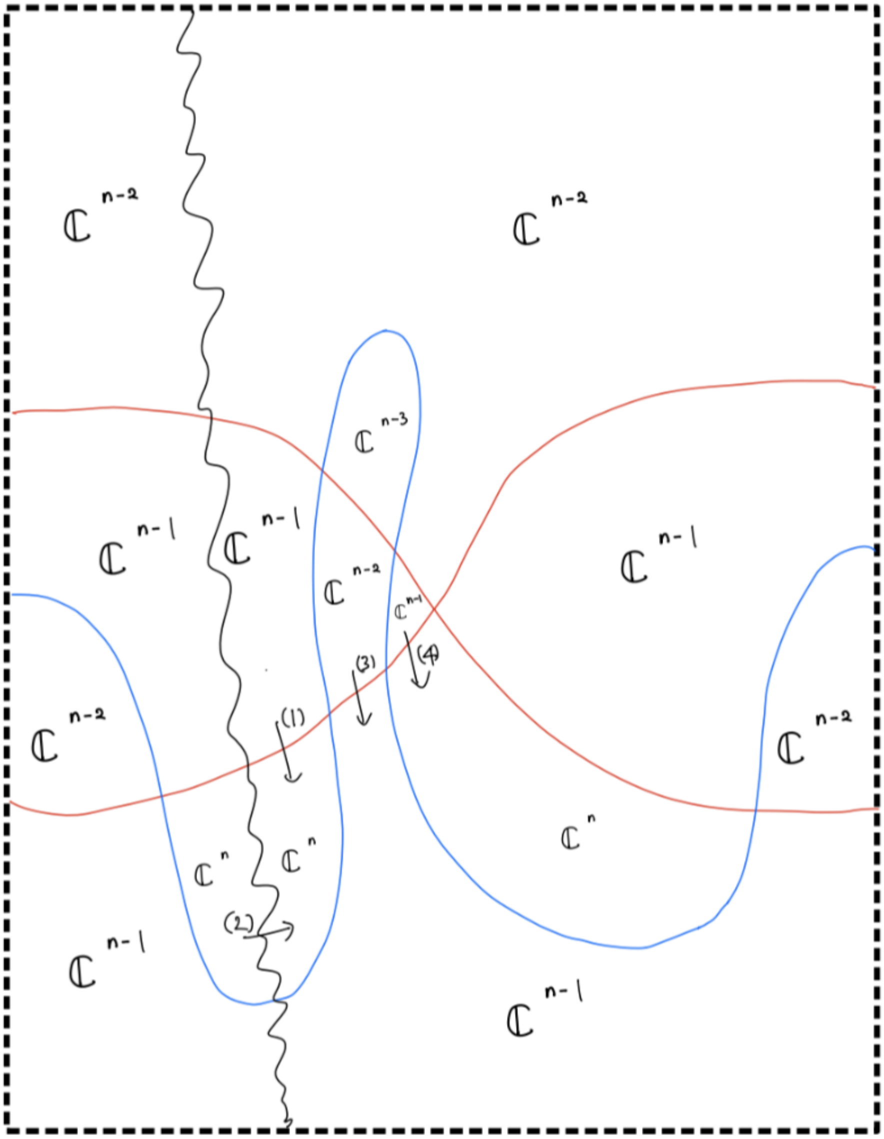

Figure 3.310

Generizations maps:

(1) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(2) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

(3) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(4) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(Step 3) apply $cobord_3$ to the region surrounded by a purple dotted line.

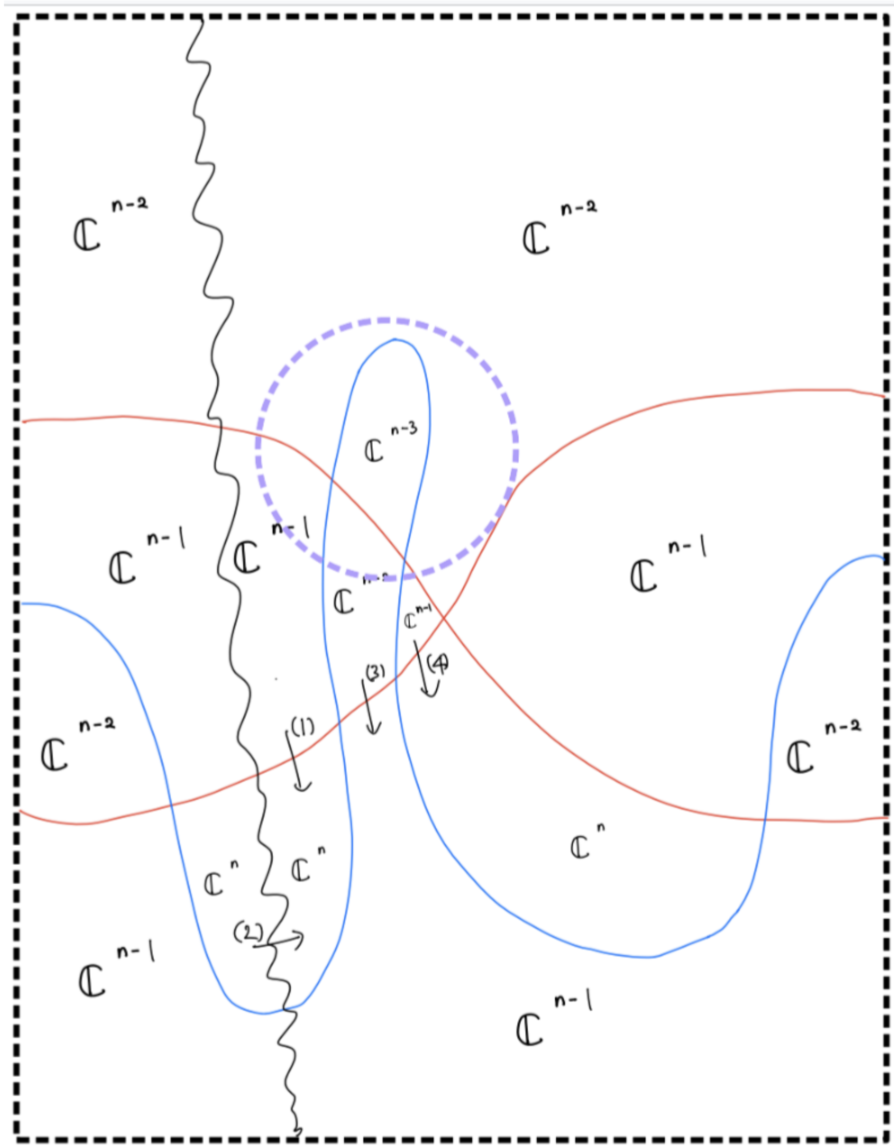

Figure 3.311

we get

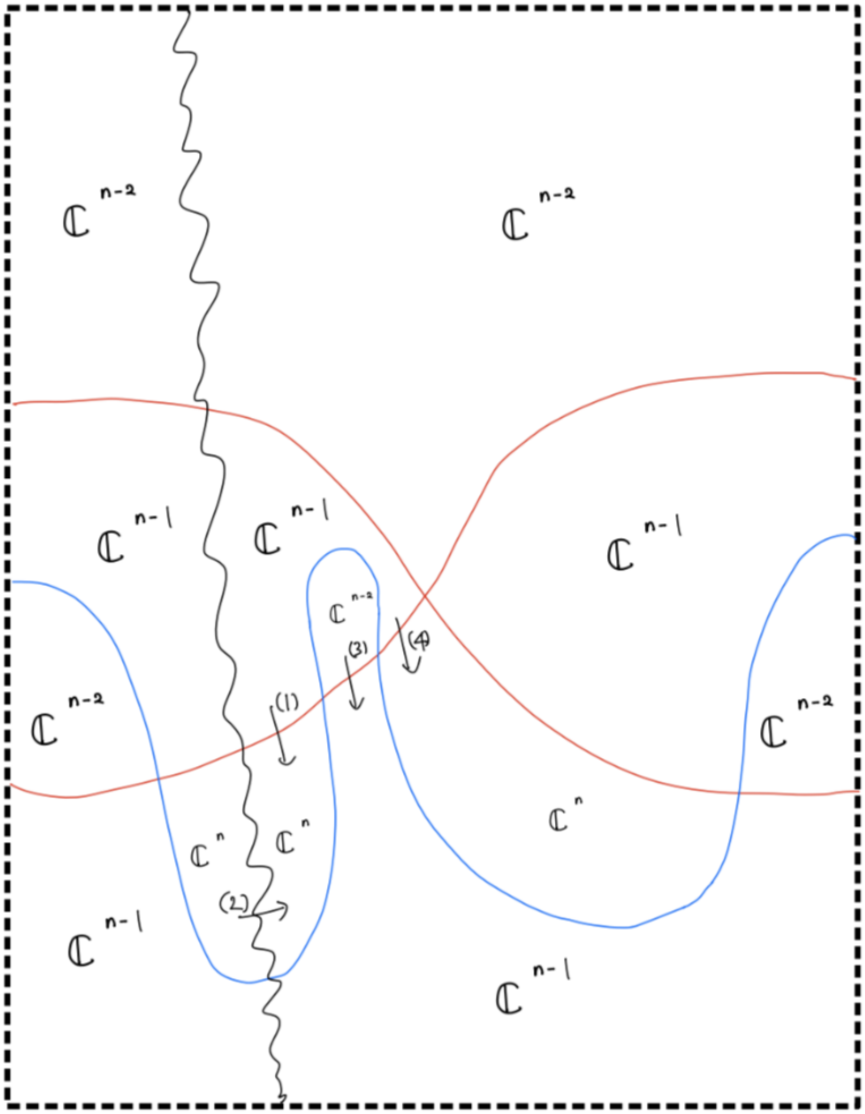

Figure 3.312

Generizations maps:

(1) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(2) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

(3) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(4) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(Step 4) apply $cobord_3$ to the region surrounded by a purple dotted line

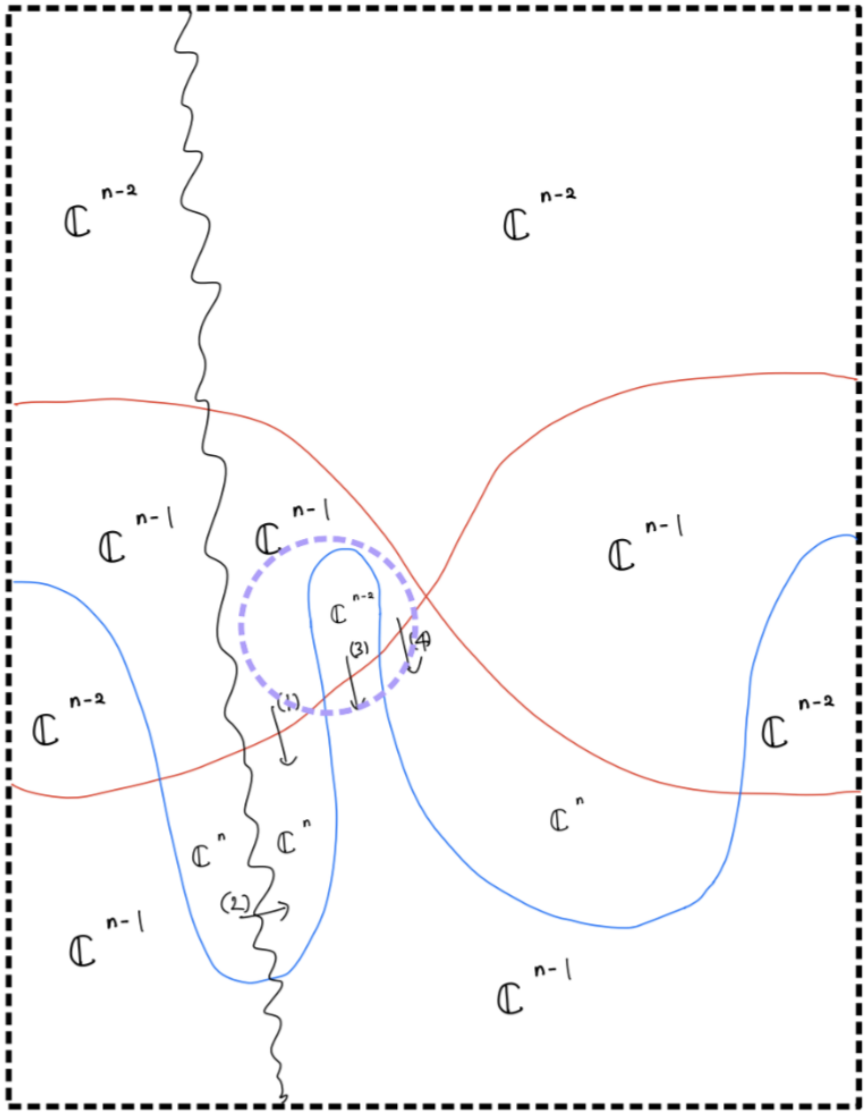

Figure 3.313

we get the final sheaf

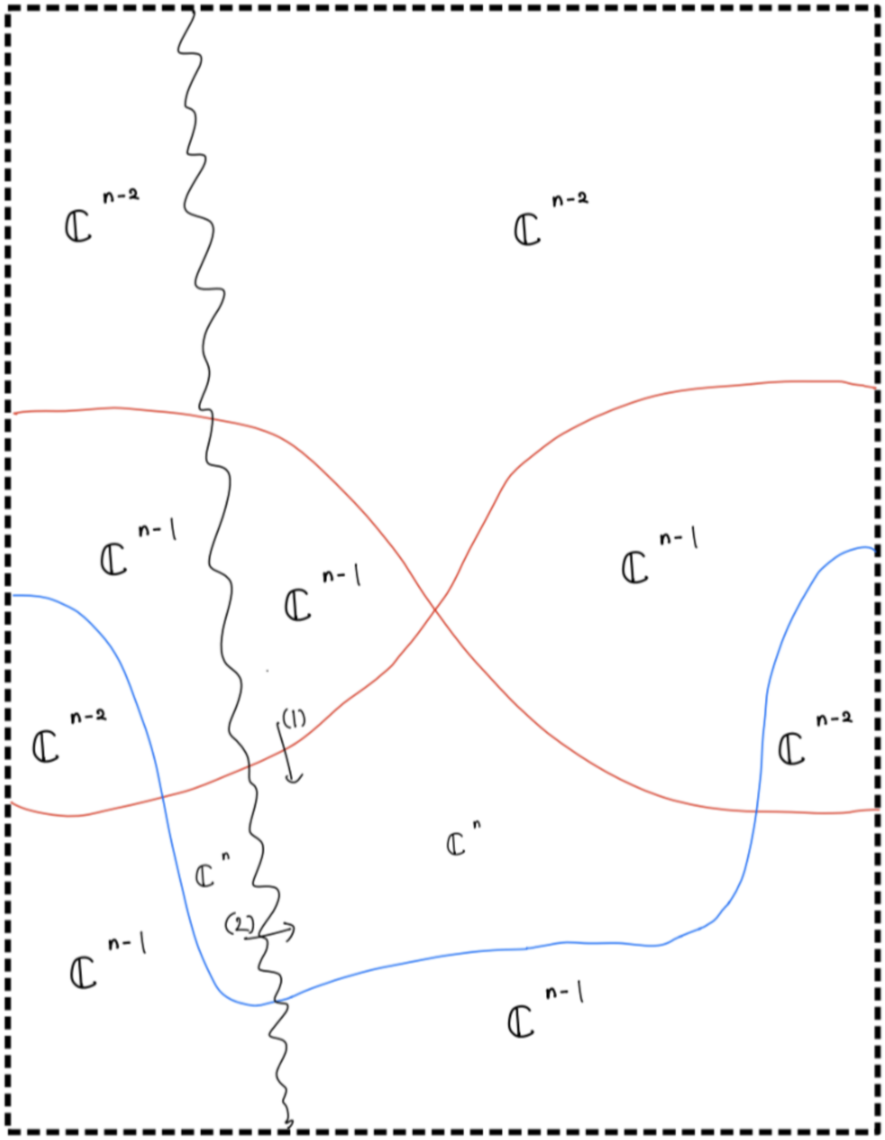

Figure 3.314

Generizations maps:

(1) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(2) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

3.15 Sheaf cobordism on generator regions

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-1,1)_x \times (-n-1, n+1)_z$ under f

- Λ_0^0 gets mapped to the red strands in the figure below, co-oriented upward.
- Λ_0^∞ gets mapped to the blue strands in the figure below, co-oriented downward.
- Λ_0^{squig} gets mapped to the squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

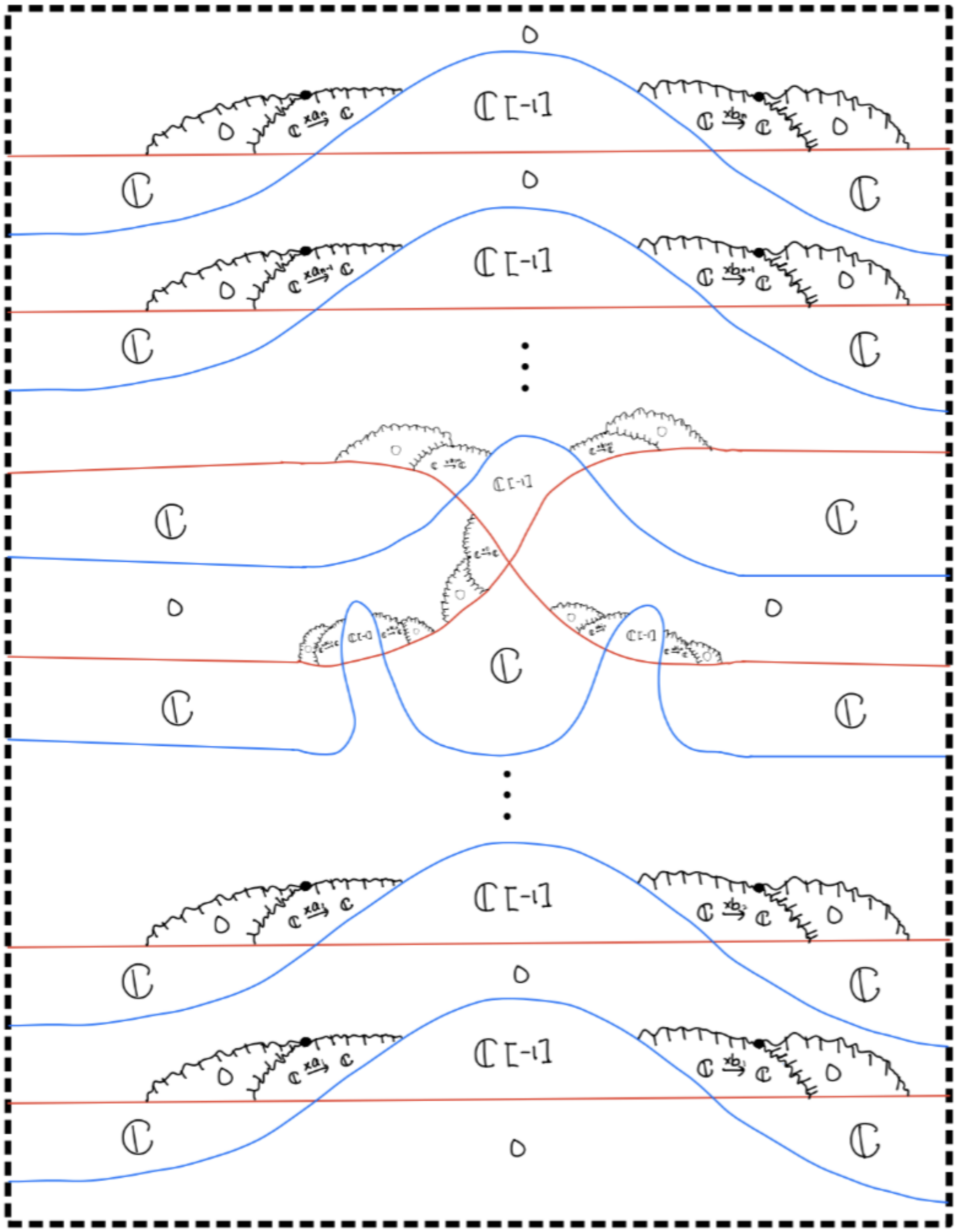

Figure 3.315

Then we define a cobordism starting from the above sheaf, say $cobord_{gen}(n)$ supported on U, where n is the number of blue strands(equivalently, red strands). At the end of the cobordism, the sheaf, under the same chart f, is described as the following squiggly legible diagram.

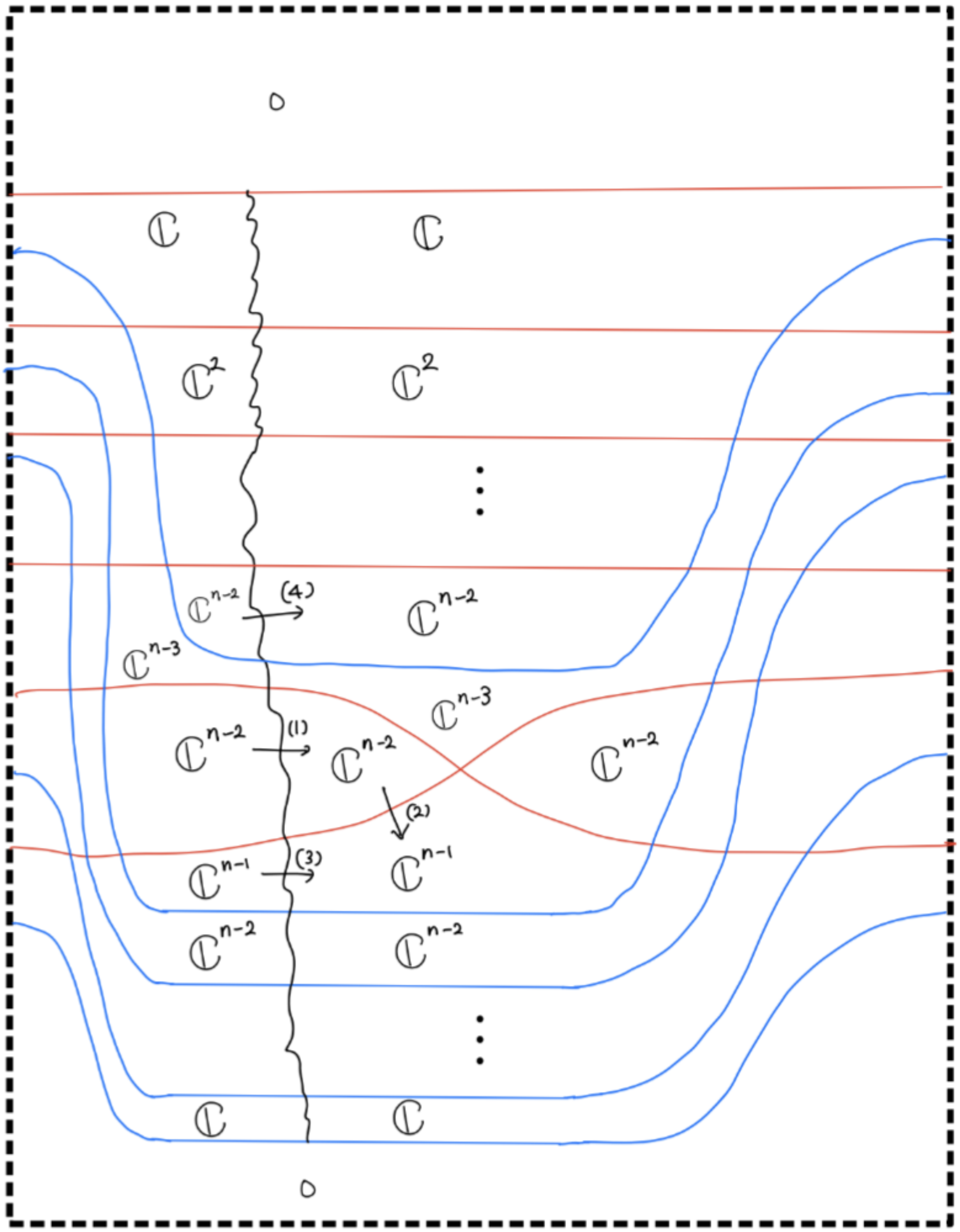
0
$\mathbb{C}$
$\mathbb{C}$
$\mathbb{C}^2$
$\mathbb{C}^2$
$\mathbb{C}^{n-2}$
(4)
$\mathbb{C}^{n-2}$
$\mathbb{C}^{n-3}$
$\mathbb{C}^{n-3}$
$\mathbb{C}^{n-2}$
(1)
$\mathbb{C}^{n-2}$
$\mathbb{C}^{n-2}$
(2)
$\mathbb{C}^{n-1}$
(3)
$\mathbb{C}^{n-1}$
$\mathbb{C}^{n-2}$
$\mathbb{C}^{n-2}$
$\mathbb{C}$
$\mathbb{C}$
0

Figure 3.316

Generization maps

(1) $diag(d_n, \cdots, d_{i+1})$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(3) $diag(d_n, \cdots, d_{i+1}) + e I_{n-i+1,n-i}$

where

- $a_i = a_{i,1} a_{i,2}^{-1}$ and $b_i = b_{i,1}^{-1} b_{i,2}$
- $d_r = a_r b_r^{-1}$ for $r = 1, \cdots, n$
- $e = -a_{i+1} b_i^{-1} c$
- $e' = d_{i+1}^{-1} e$

We define $cobord_{gen}(n)$ inductively as follows.

(i) For $n = 2$, we define $cobord_{gen}$ starting from the sheaf below

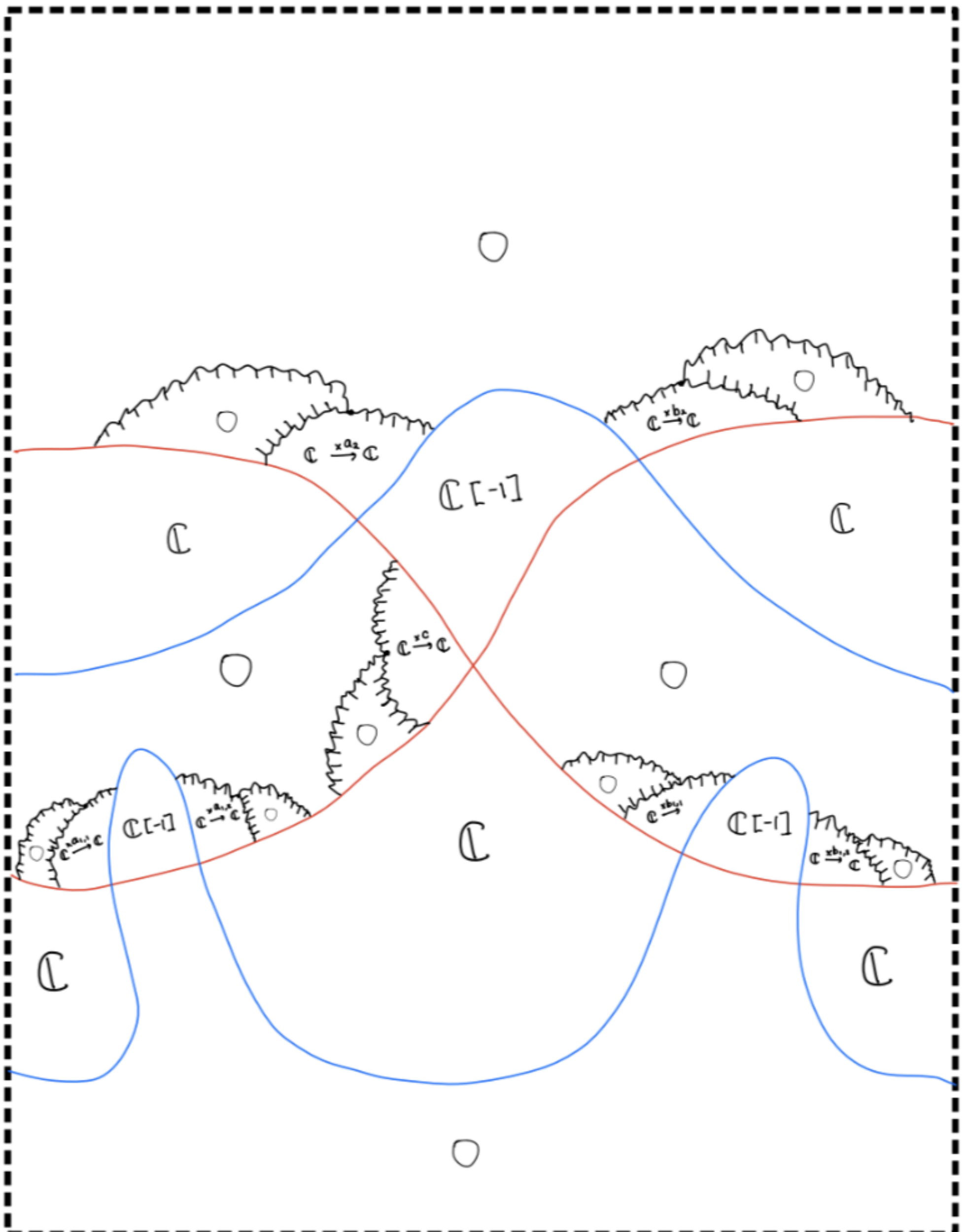

Figure 3.317

as follows.

(Step 1) we apply $cobord_1$ to the square regions surrounded by purple dotted lines.

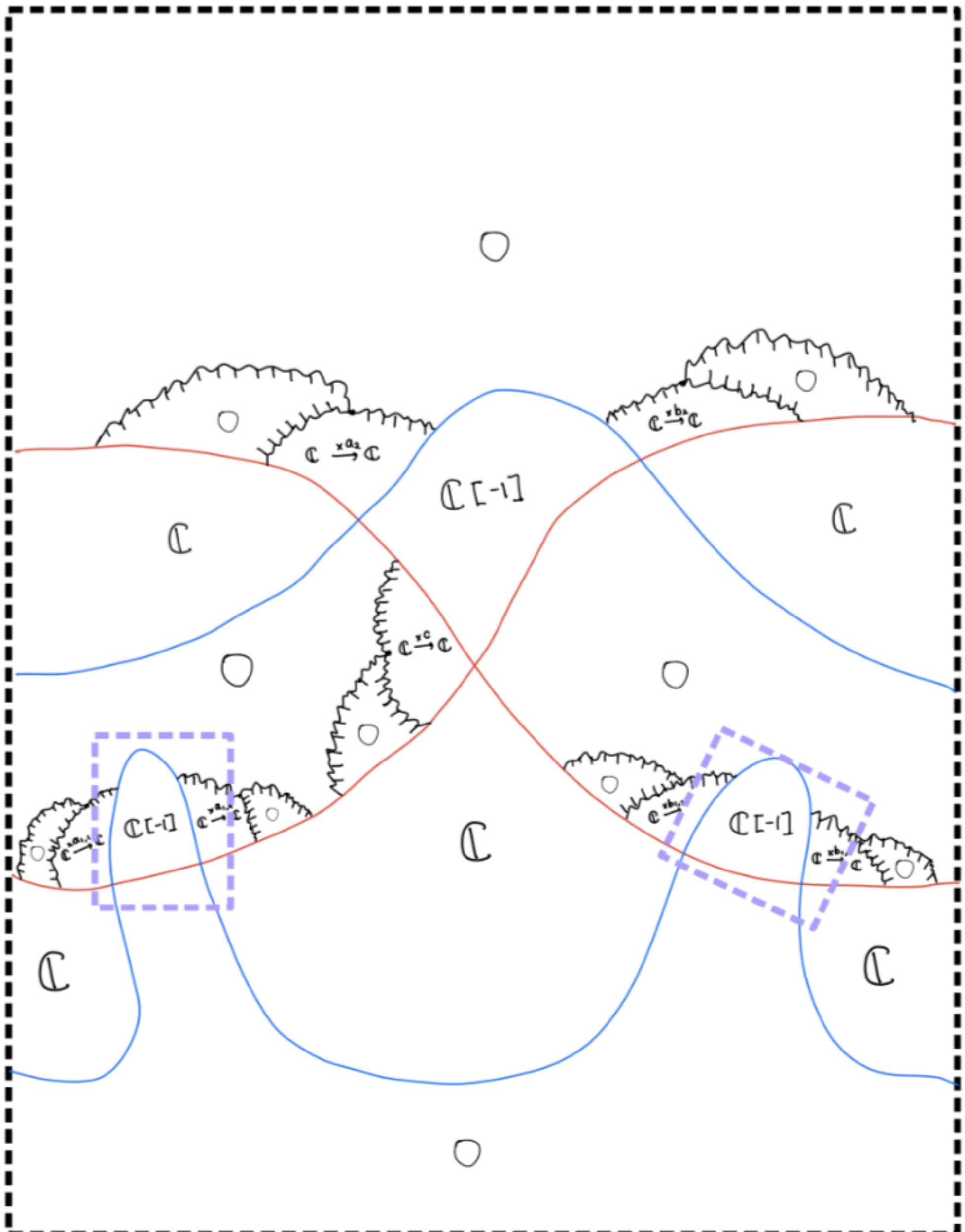

Figure 3.318

we get

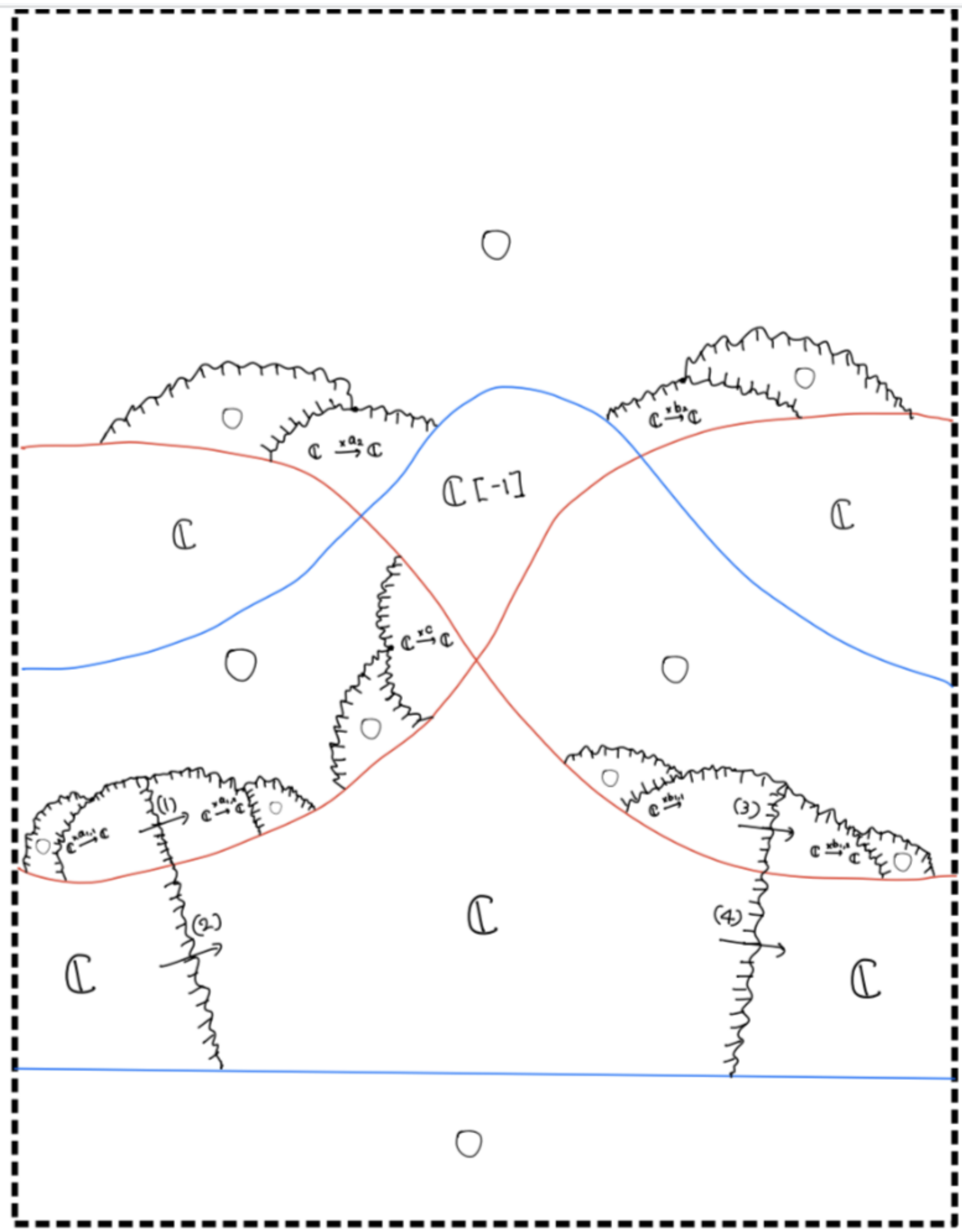

Figure 3.319

Generization maps

(1) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a_1"] \arrow[r, "\times a_{1,1}"] & \mathbb{C} \arrow[u, "\times a_{1,2}"'] \end{tikzcd}$$

(2) $\times a_1$

(3) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times b_1^{-1}"] \arrow[r, "\times b_{1,1}"] & \mathbb{C} \arrow[u, "\times b_{1,2}"'] \end{tikzcd}$$

(4) $\times b_1^{-1}$

where

- $a_1 = a_{1,1}a_{1,2}^{-1}$
- $b_1 = b_{1,1}^{-1}b_{1,2}$

which is quasi-isomorphic to

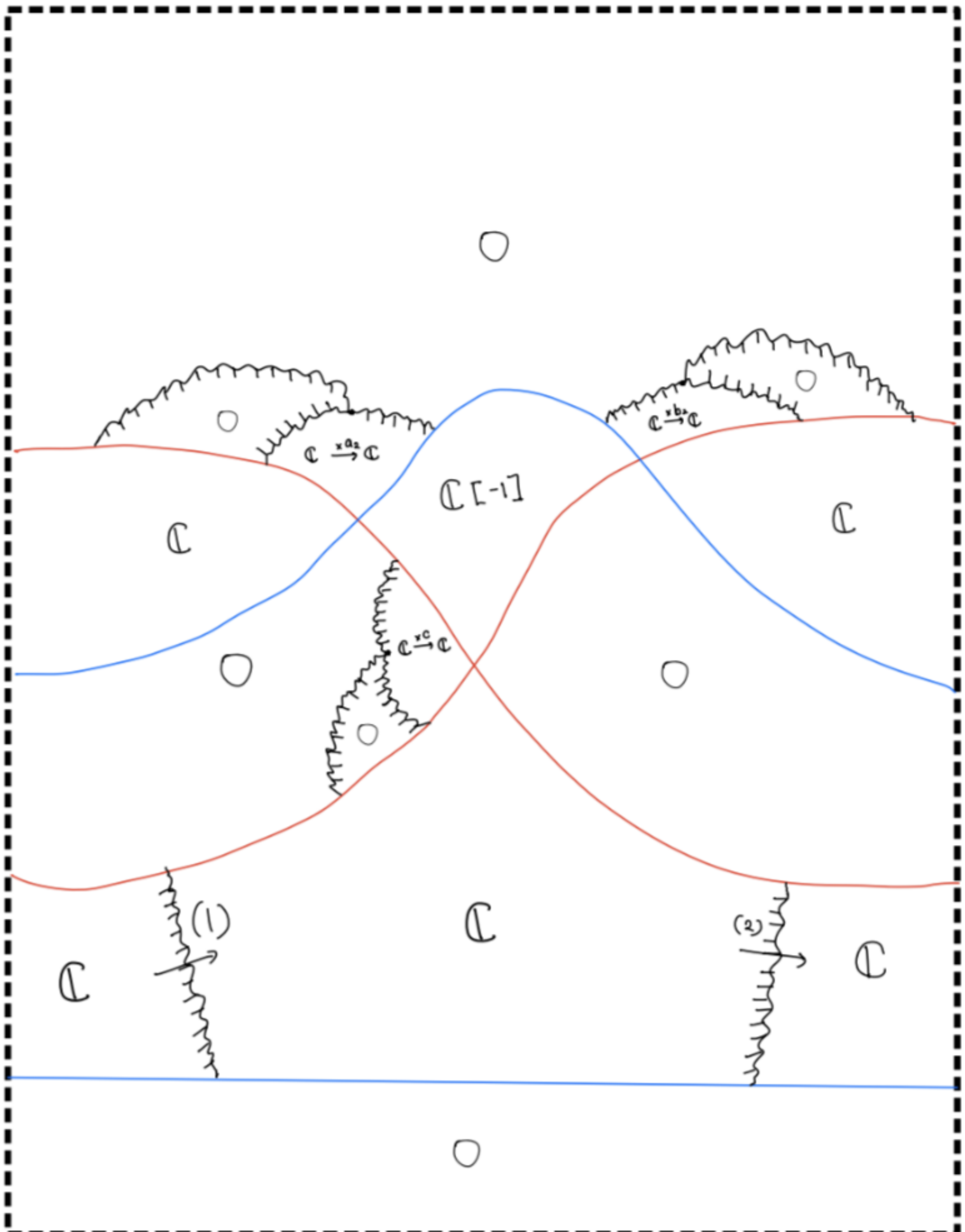

Figure 3.320

Generization maps

(1) $\times a_1$

(2) $\times b_1^{-1}$

(Step 2) apply $cobord_8$ to the region surrounded by a purple dotted line

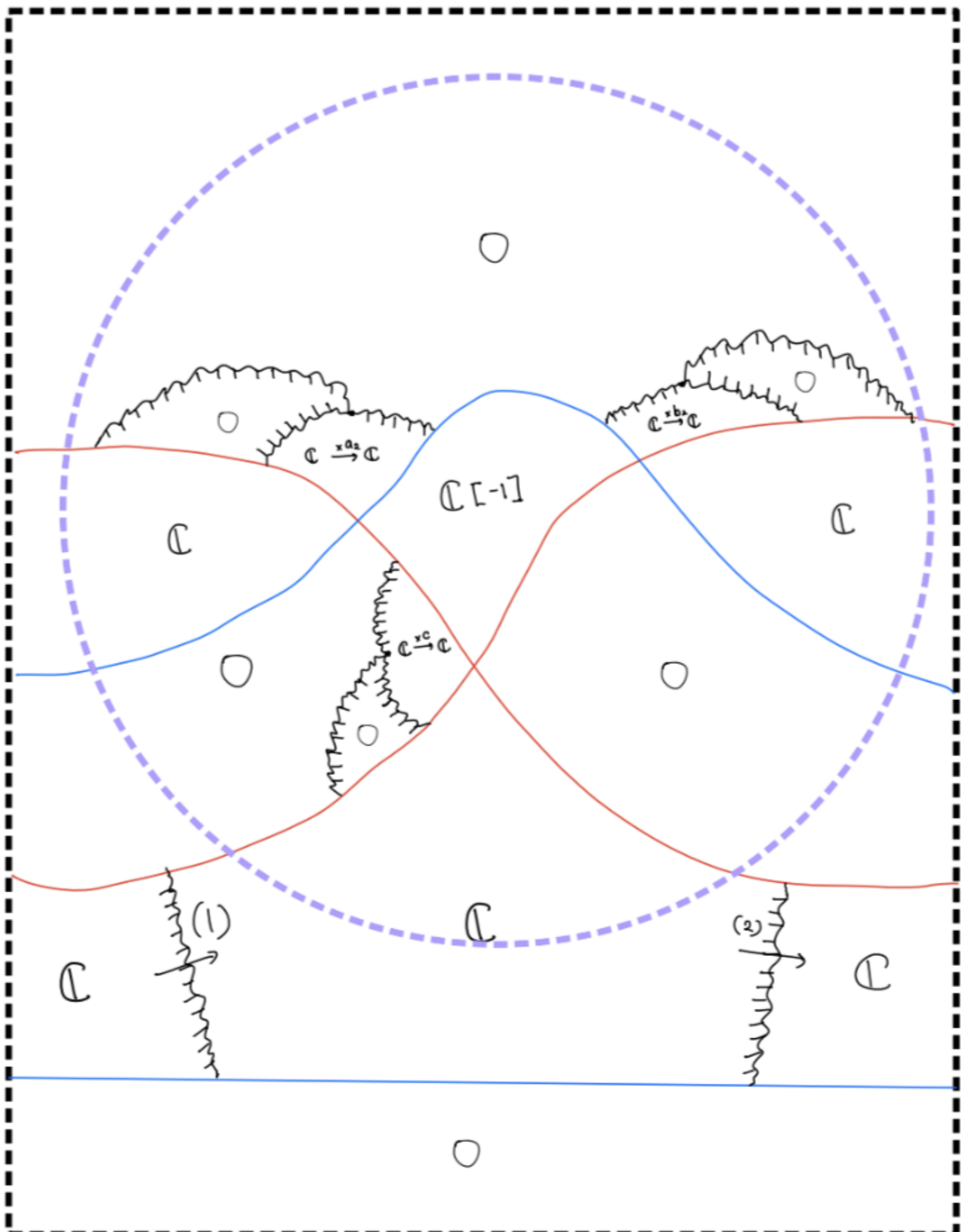

Figure 3.321

we get

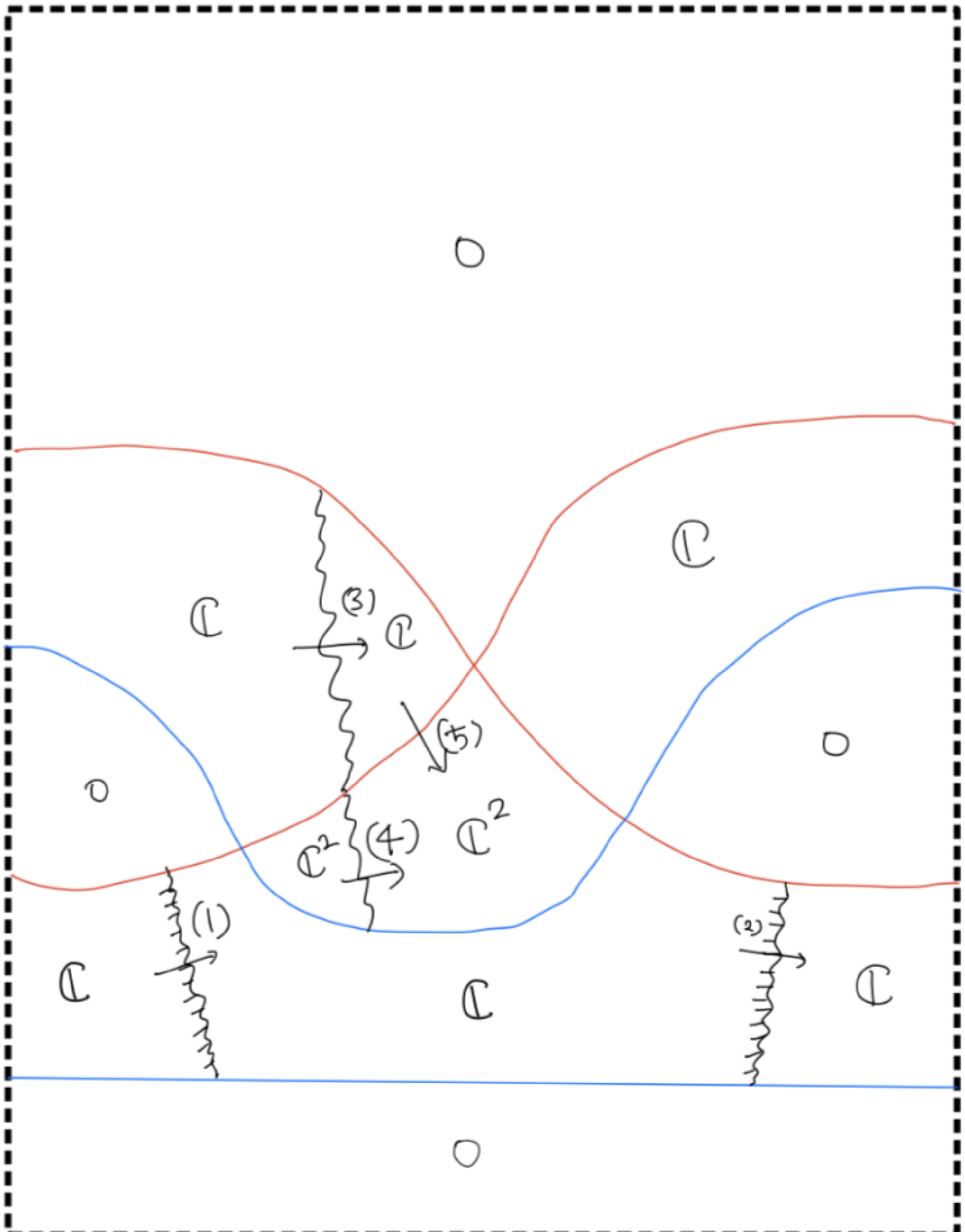

Figure 3.322

Generization maps

(1) $\times a_1$

(2) $\times b_1^{-1}$

(3) $\times a_2 b_2^{-1}$

(4) $\begin{pmatrix} a_2 b_2^{-1} & 0 \\ -a_2 c^{-1} & 1 \end{pmatrix}$

(5) $\begin{pmatrix} 1 \\ -b_2 c^{-1} \end{pmatrix}$

which is quasi-isomorphic to

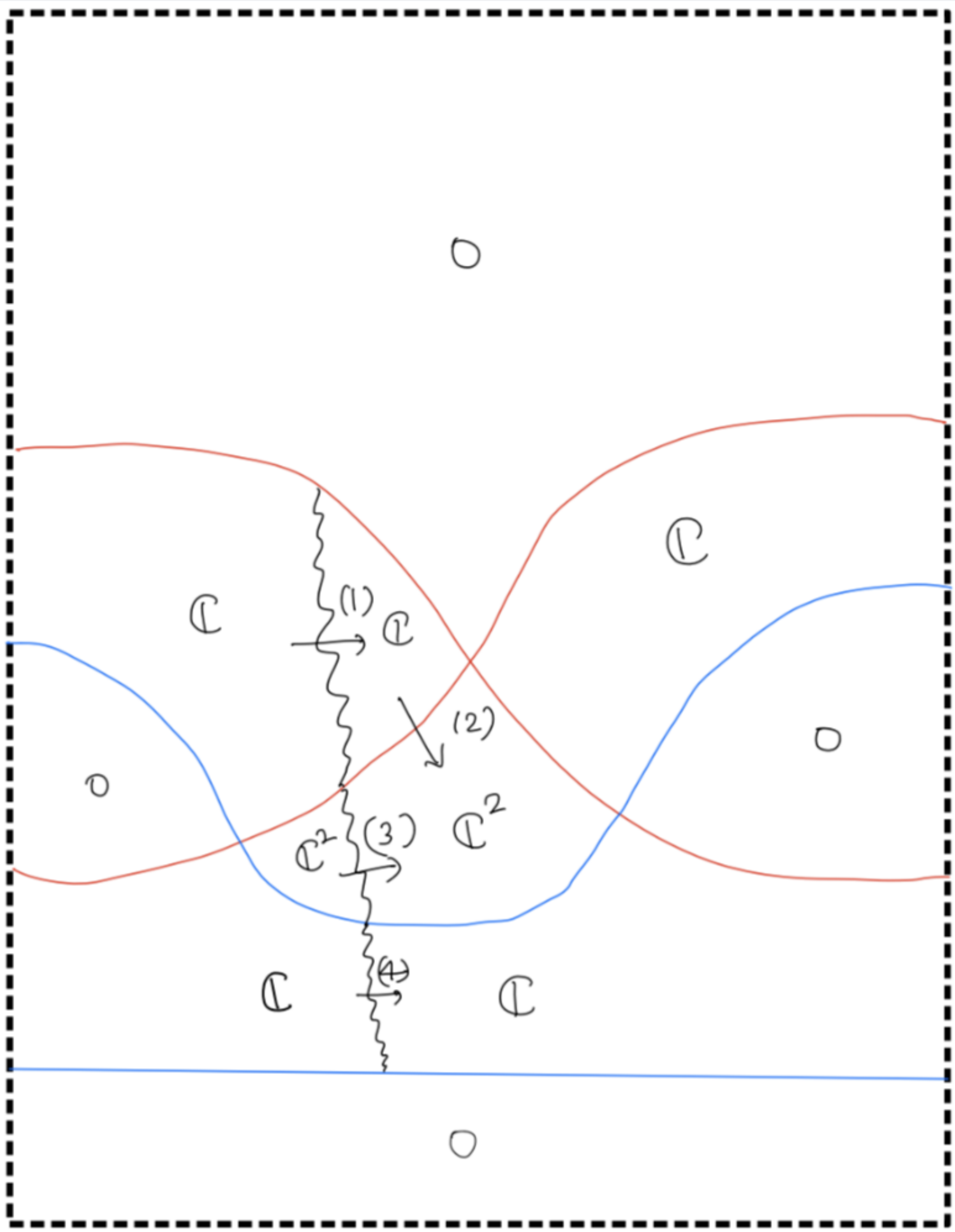

Figure 3.323

Generization maps

(1) $\times d_2$

(2) $\begin{pmatrix} 1 \\ e' \end{pmatrix}$

(3) $\begin{pmatrix} d_2 & 0 \\ e & d_1 \end{pmatrix}$

(4) $\times d_1$

where

- $d_r = a_r b_r^{-1}$
- $e = -a_2 b_1^{-1} c$
- $e' = d_2^{-1} e$

(ii) For $n > 2$,

(Case 1) if the generator s_i is $i \neq 1$,

(Step 1) we apply $cobord_{gen}(n-1)$ to the square region surrounded by purple dotted lines.

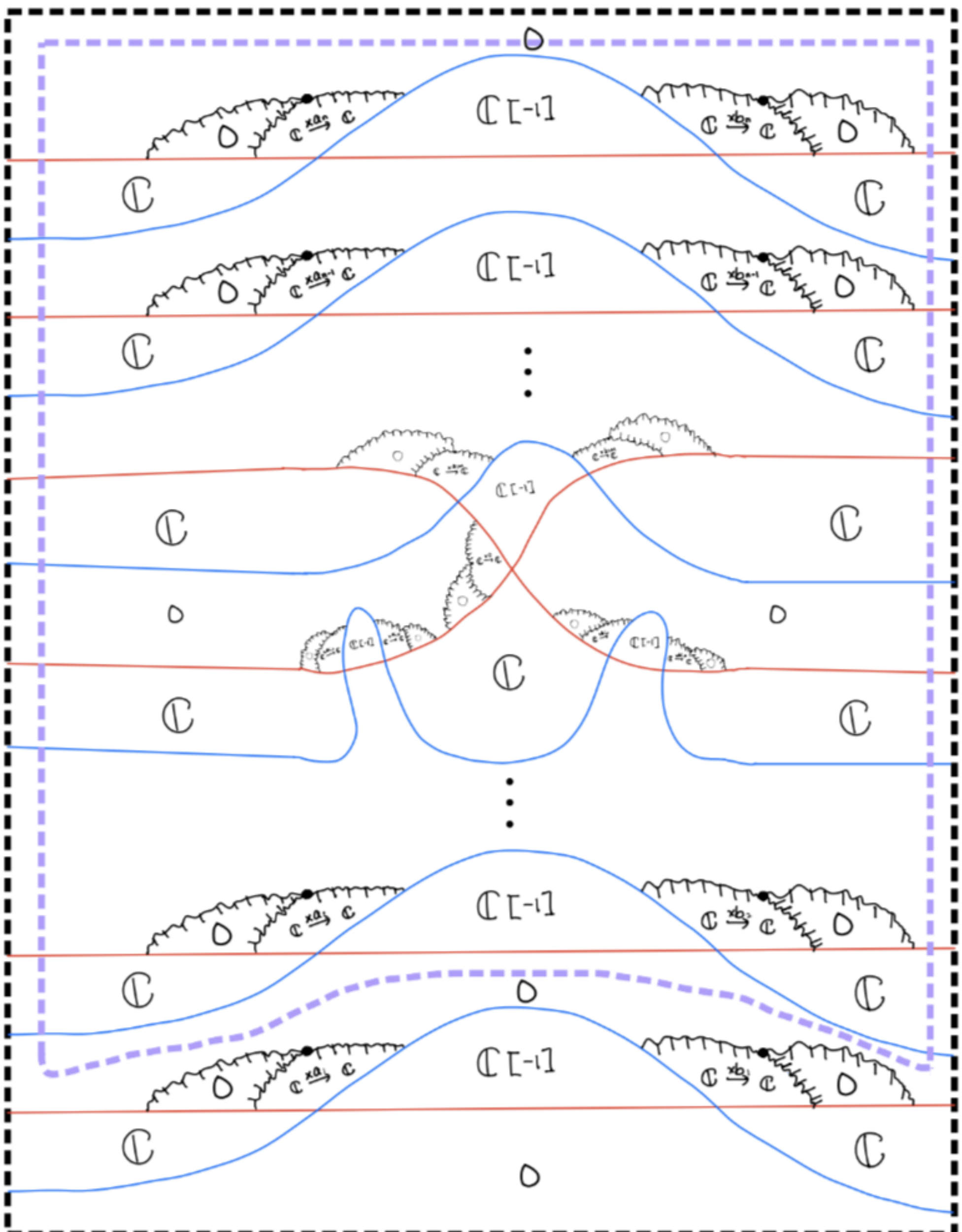

Figure 3.324

by induction hypothesis, we get

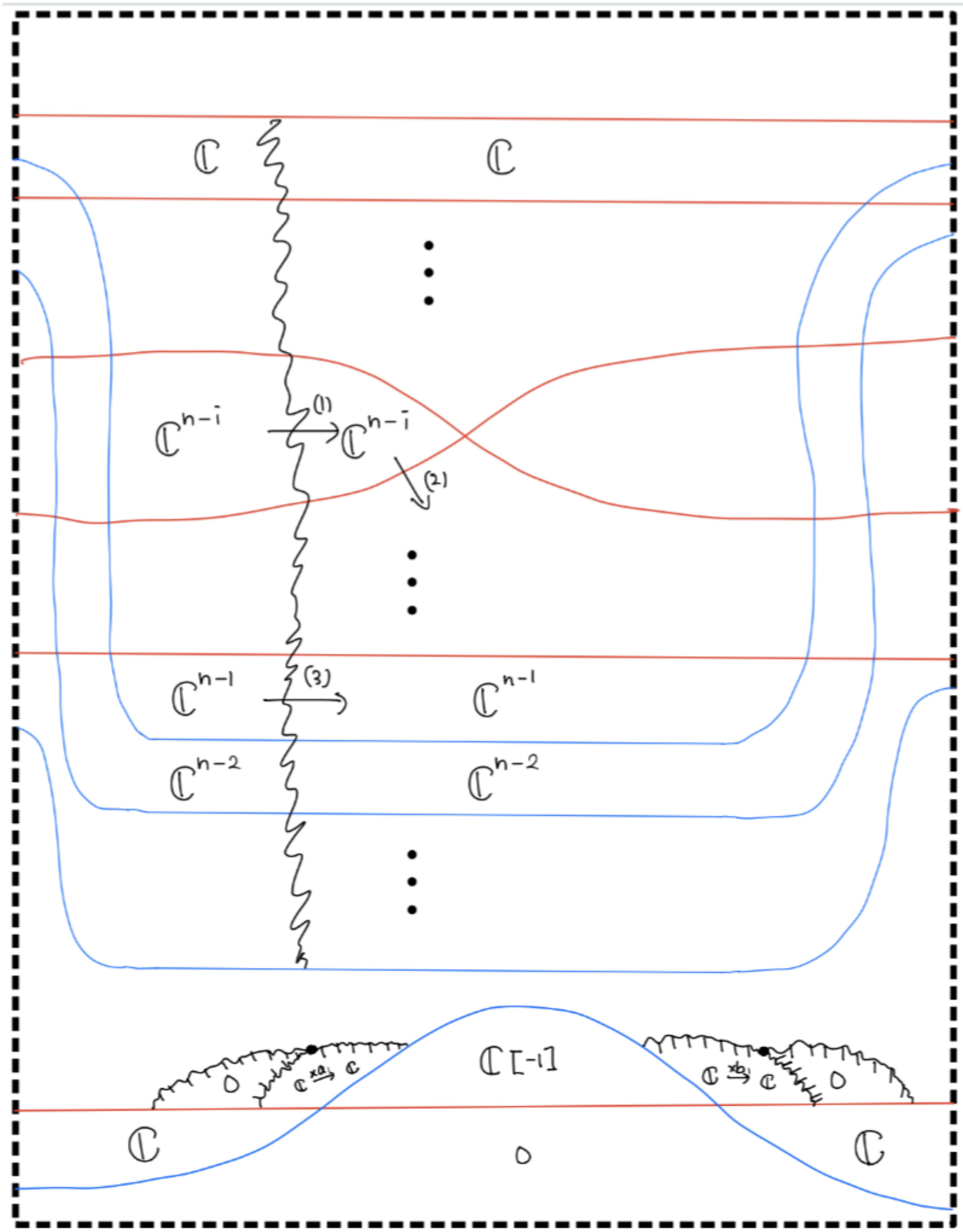

Figure 3.325

Generization maps

(1) $diag(d_n, \cdots, d_{i+1})$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(3) $diag(d_n, \cdots, d_2) + e I_{n-i+1,n-i}$

(Step 2) apply $cobord_2$ to the region surrounded by a purple dotted line.

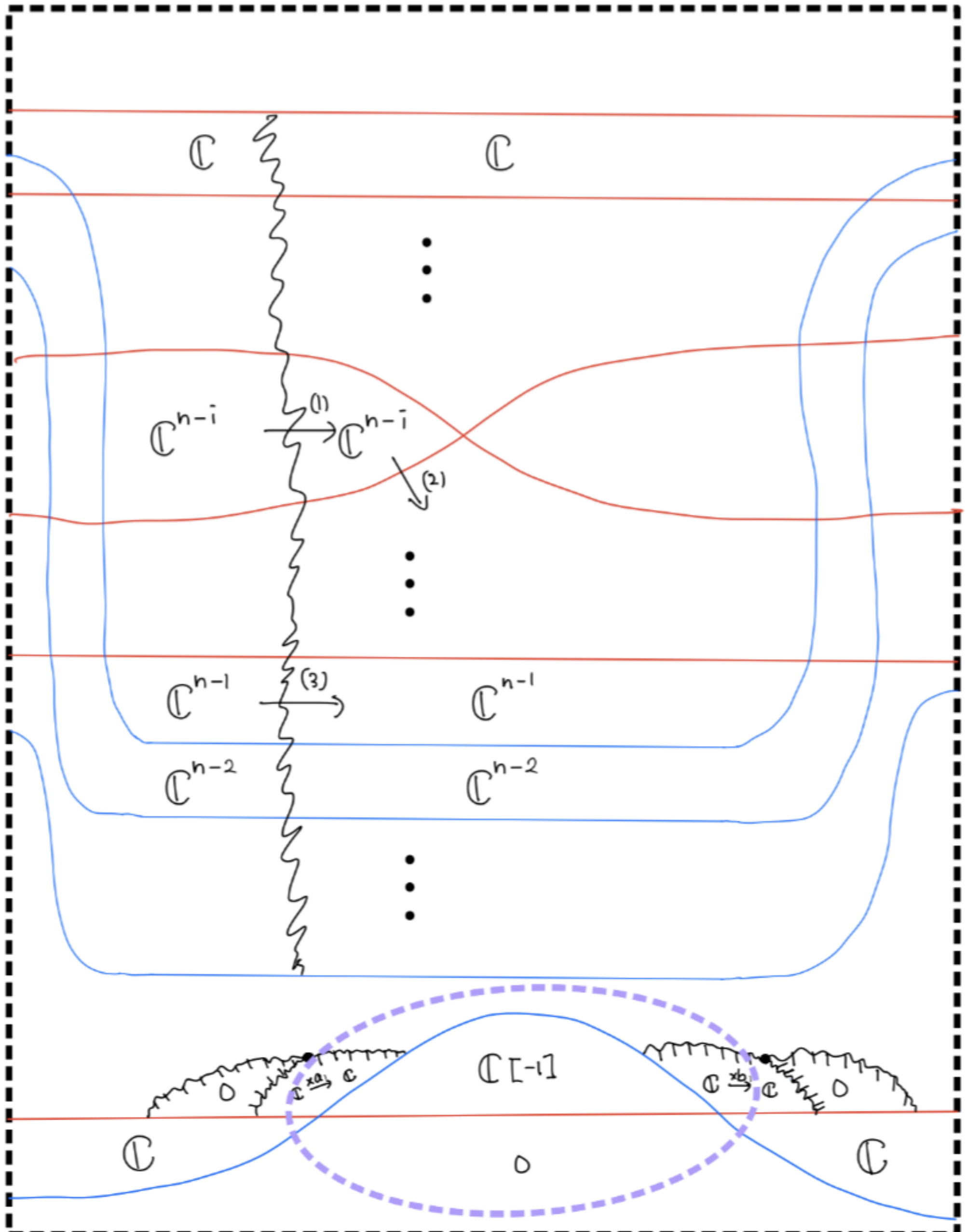

Figure 3.326

we get

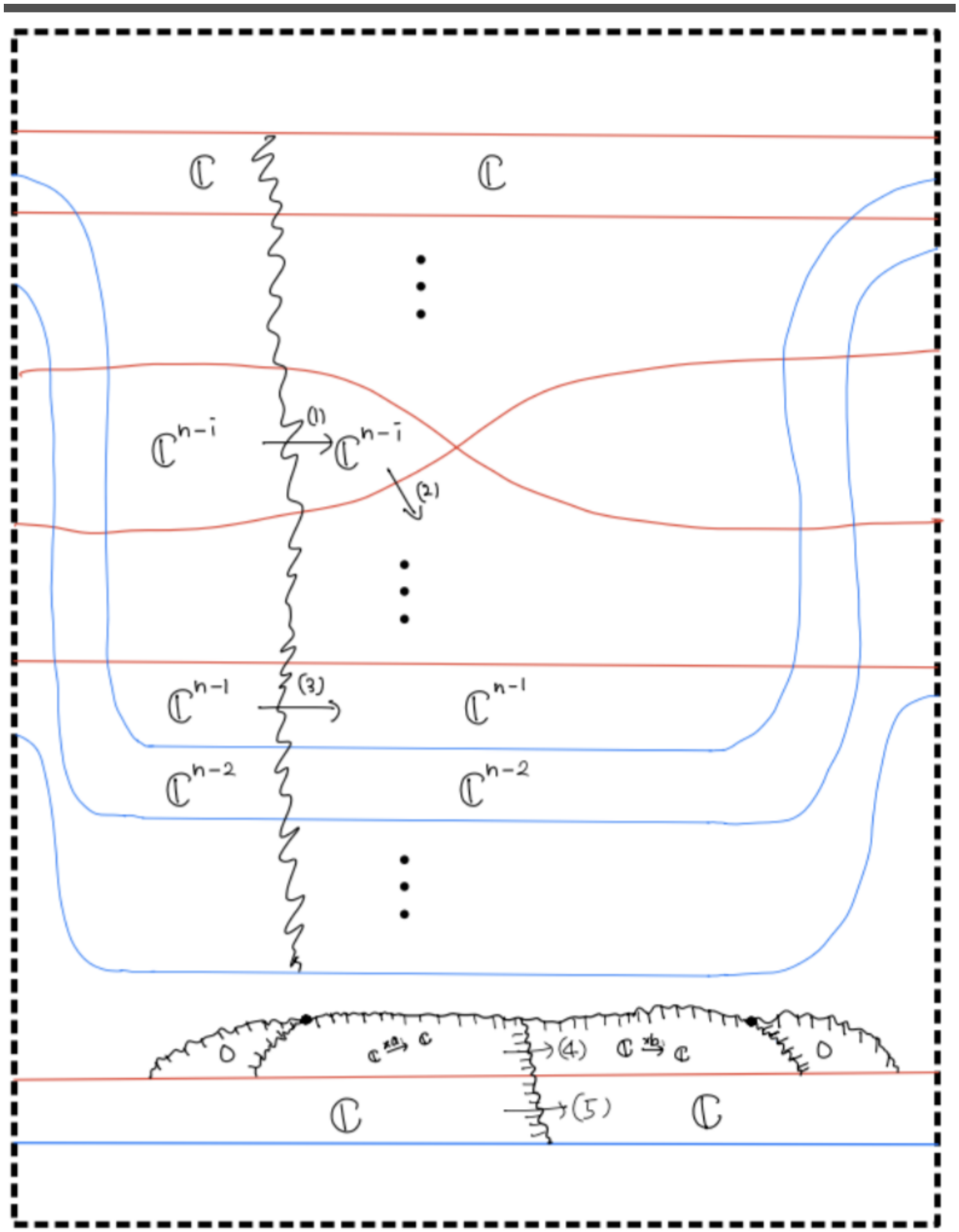

Figure 3.327

Generization maps

(1) $diag(d_n, \cdots, d_{i+1})$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(3) $diag(d_n, \cdots, d_2) + e I_{n-i+1,n-i}$

(4) $$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times a_1"] \arrow[r, "\times d_1"] & \mathbb{C} \arrow[u, "\times b_1"'] \end{tikzcd}$$

(5) $\times d_1$

which is quasi-isomorphic to

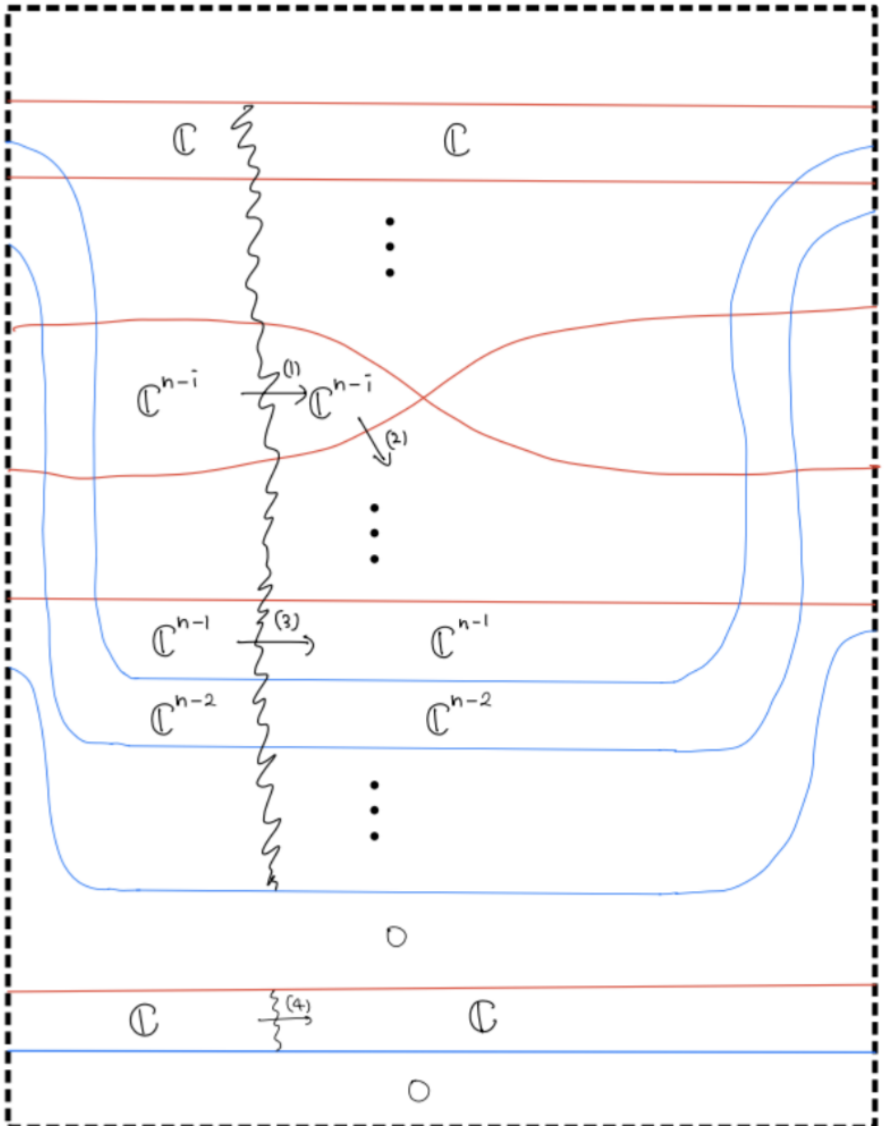

Figure 3.328

Generization maps

(1) $diag(d_n, \cdots, d_{i+1})$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(3) $diag(d_n, \cdots, d_2) + e I_{n-i+1,n-i}$

(4) $\times d_1$

(Step 3) apply $cobord_6$ to the square region surrounded by purple dotted lines.

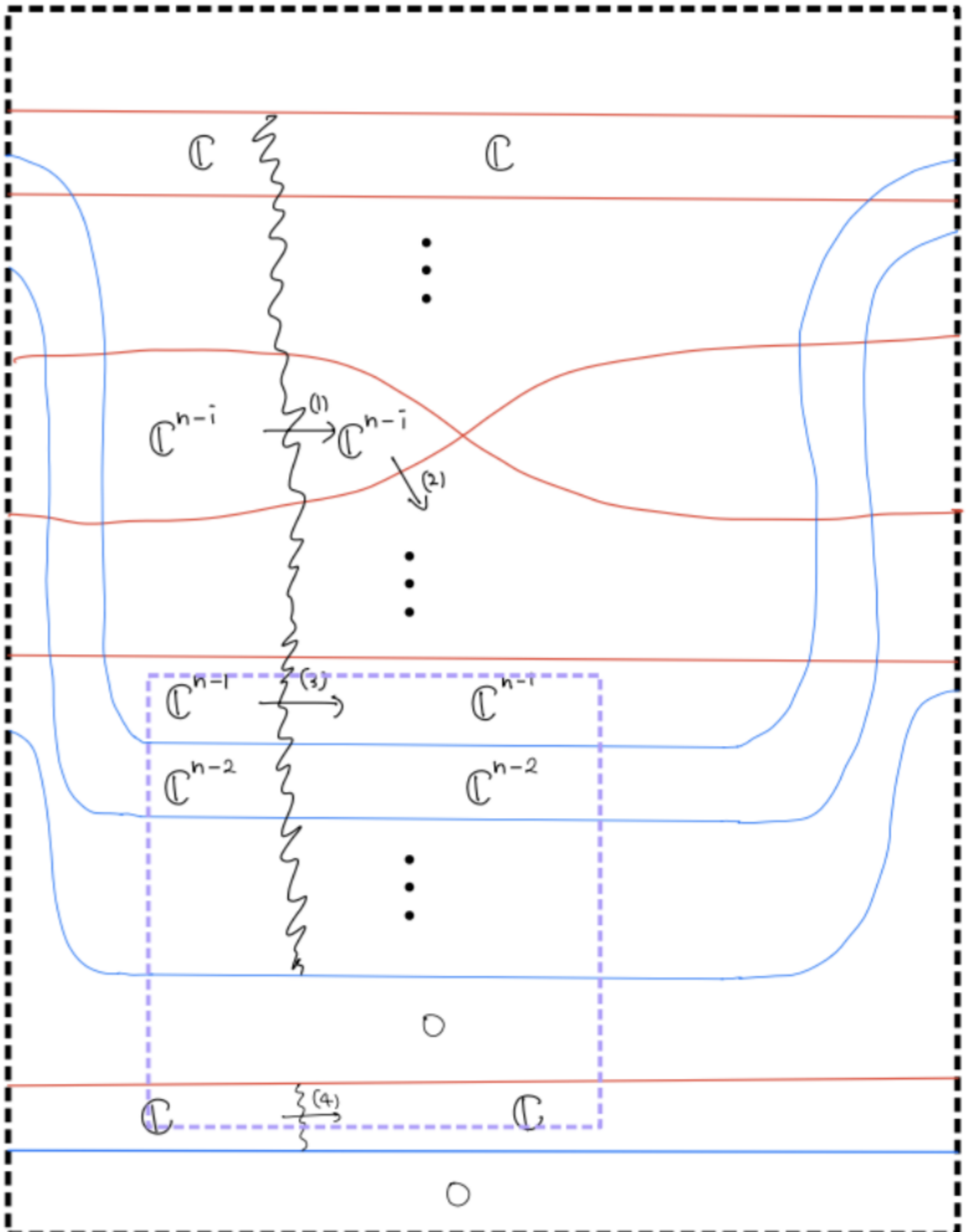

Figure 3.329

we get the final sheaf

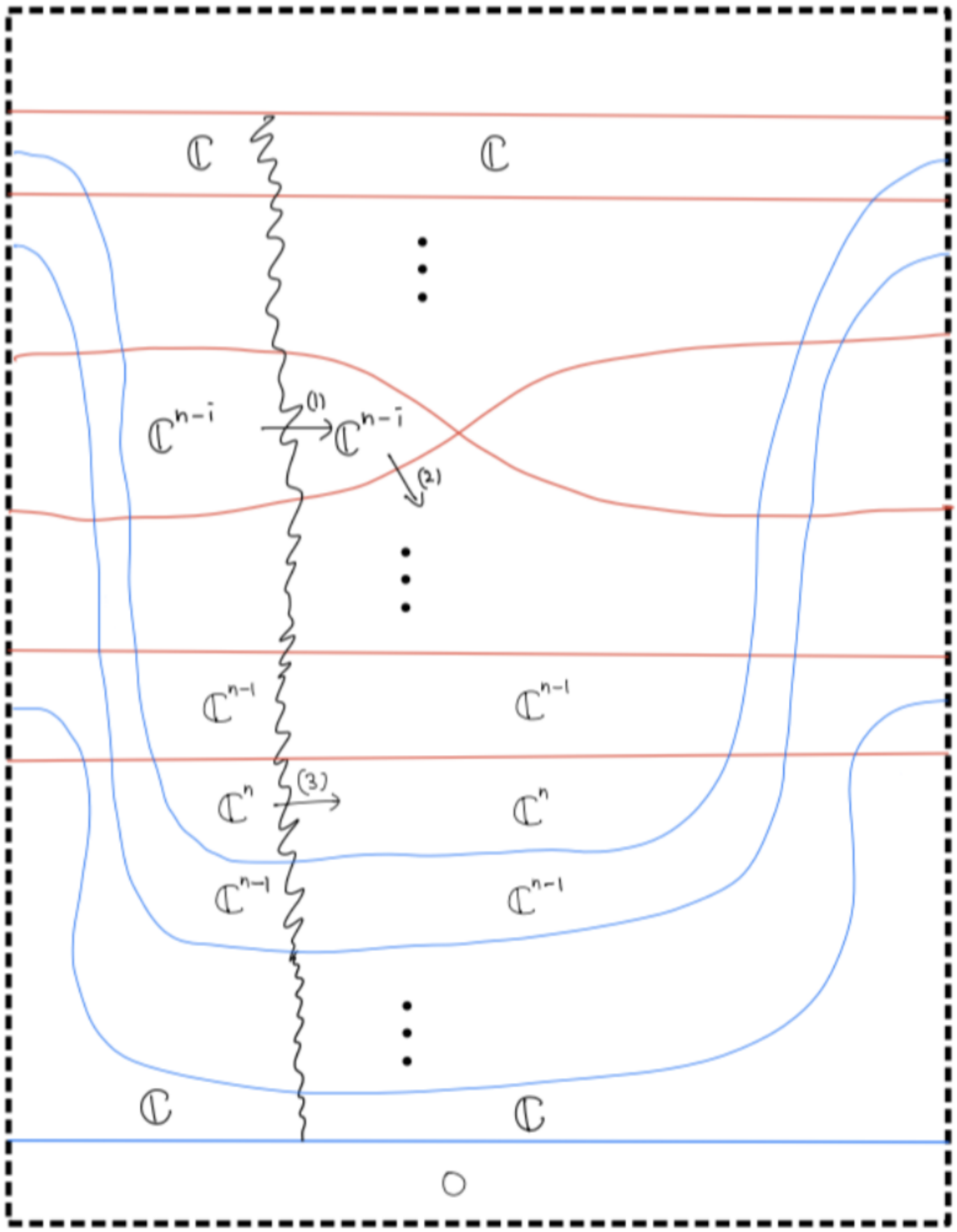

Figure 3.330

Generization maps

(1) $diag(d_n, \cdots, d_{i+1})$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(3) $diag(d_n, \cdots, d_1) + e I_{n-i+1,n-i}$

(Case 2) If the generator s_i is $i = 1$,

(Step 1) apply $cobord_{gen}(n-1)$ to the square region surrounded by purple dotted lines

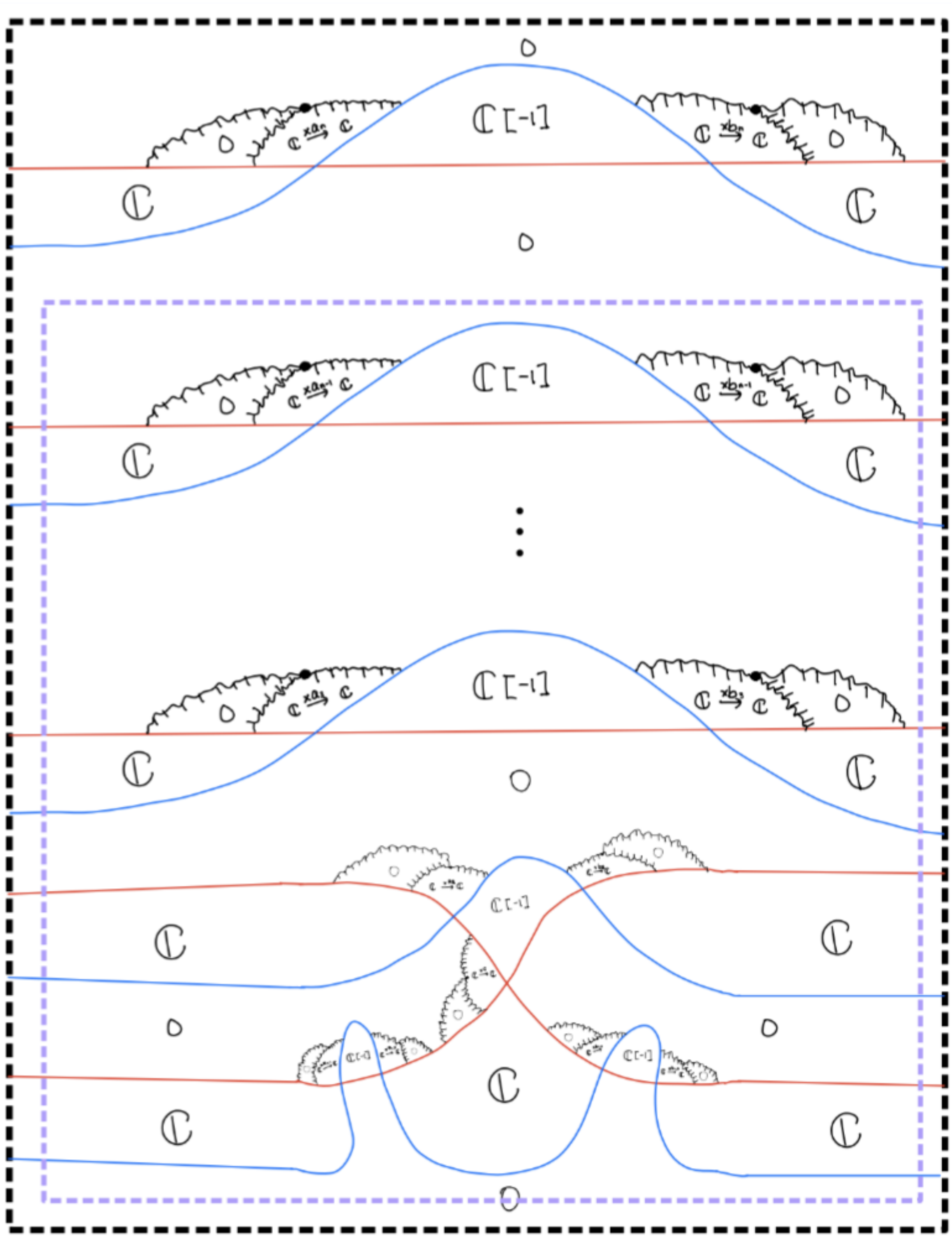

Figure 3.331

we get

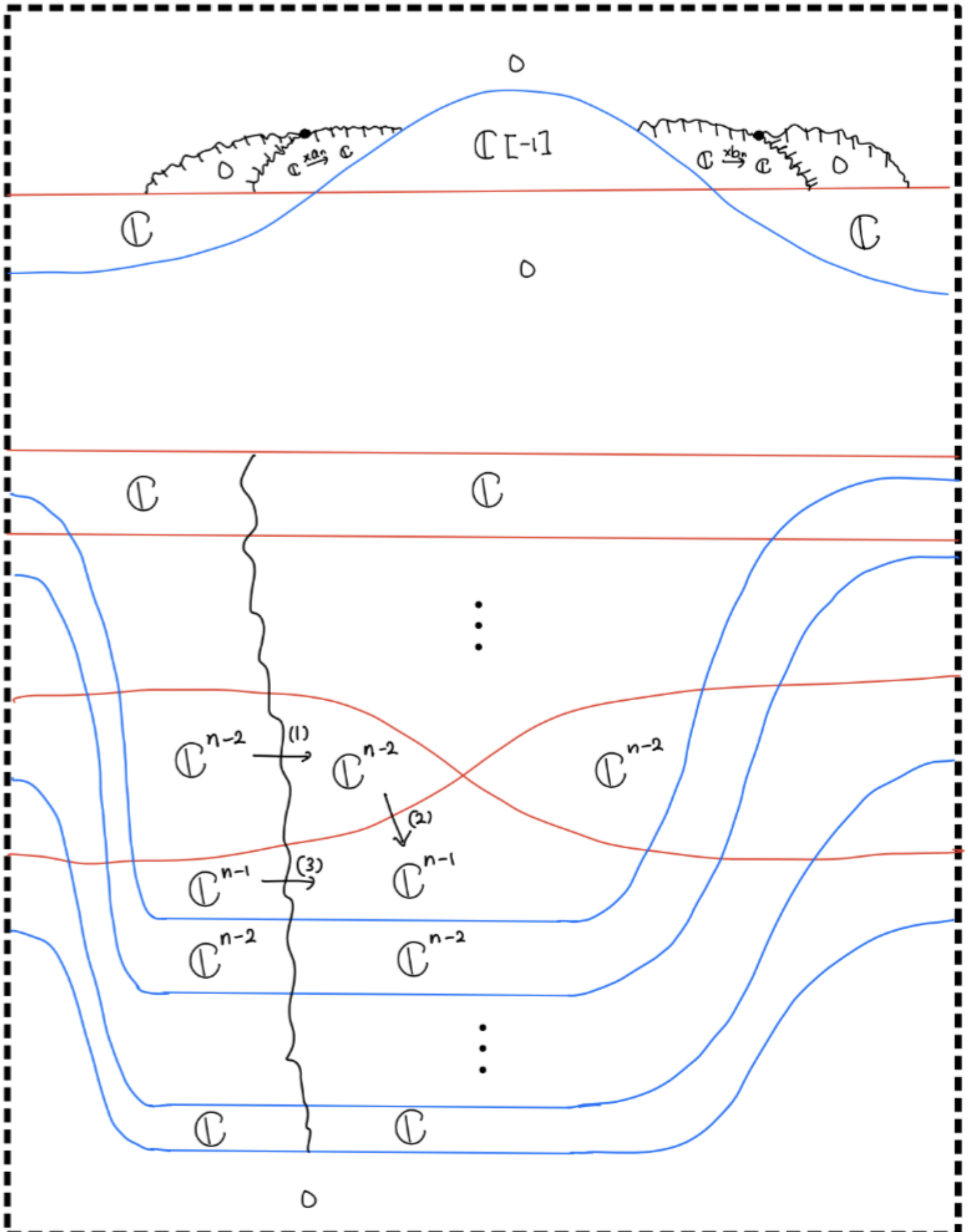

Figure 3.332

Generization maps

(1) $diag(d_{n-1}, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + e I_{n-1,n-2}$

(Step 2) apply $cobord_2$ to the region surrounded by a purple dotted line

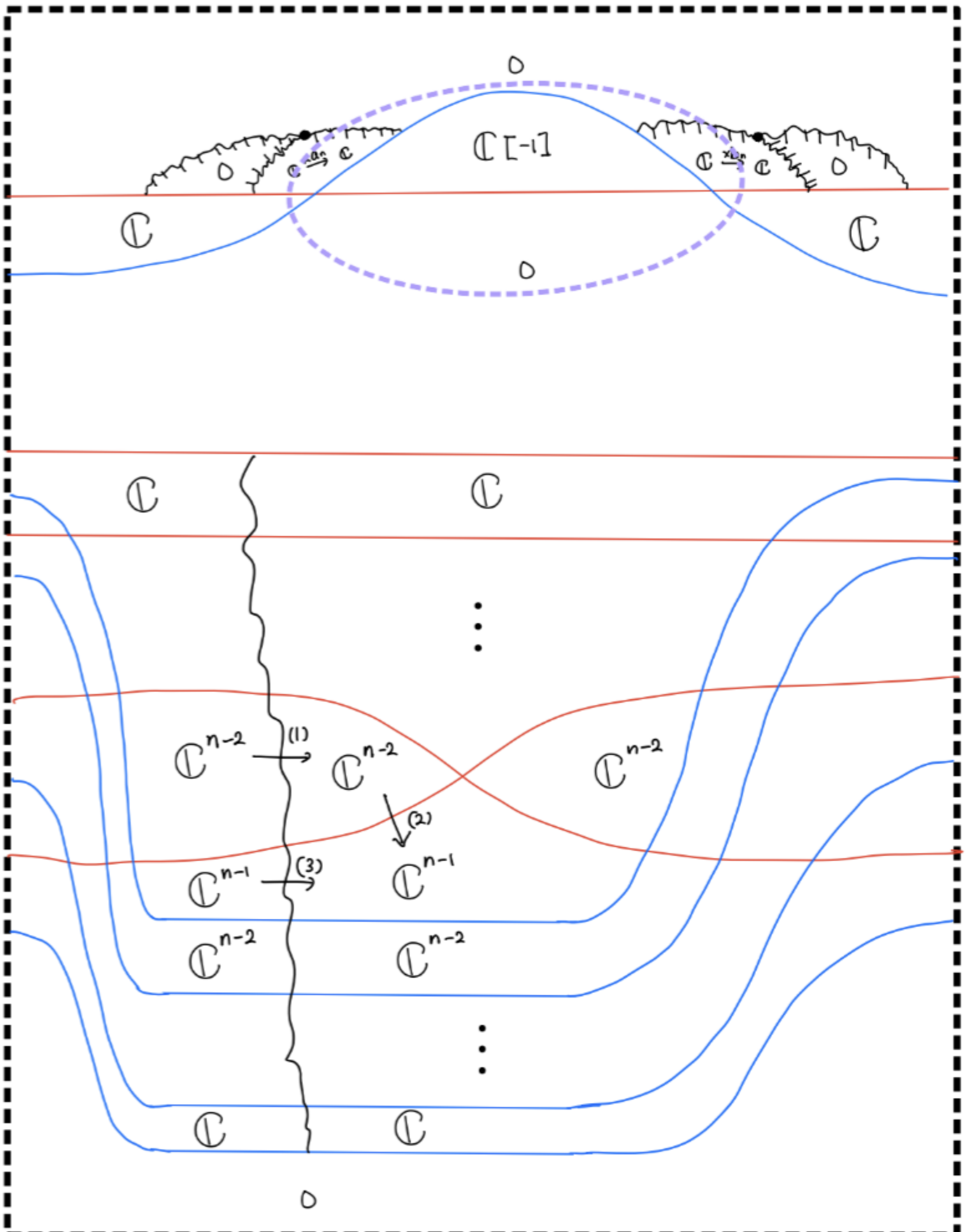

Figure 3.333

we get

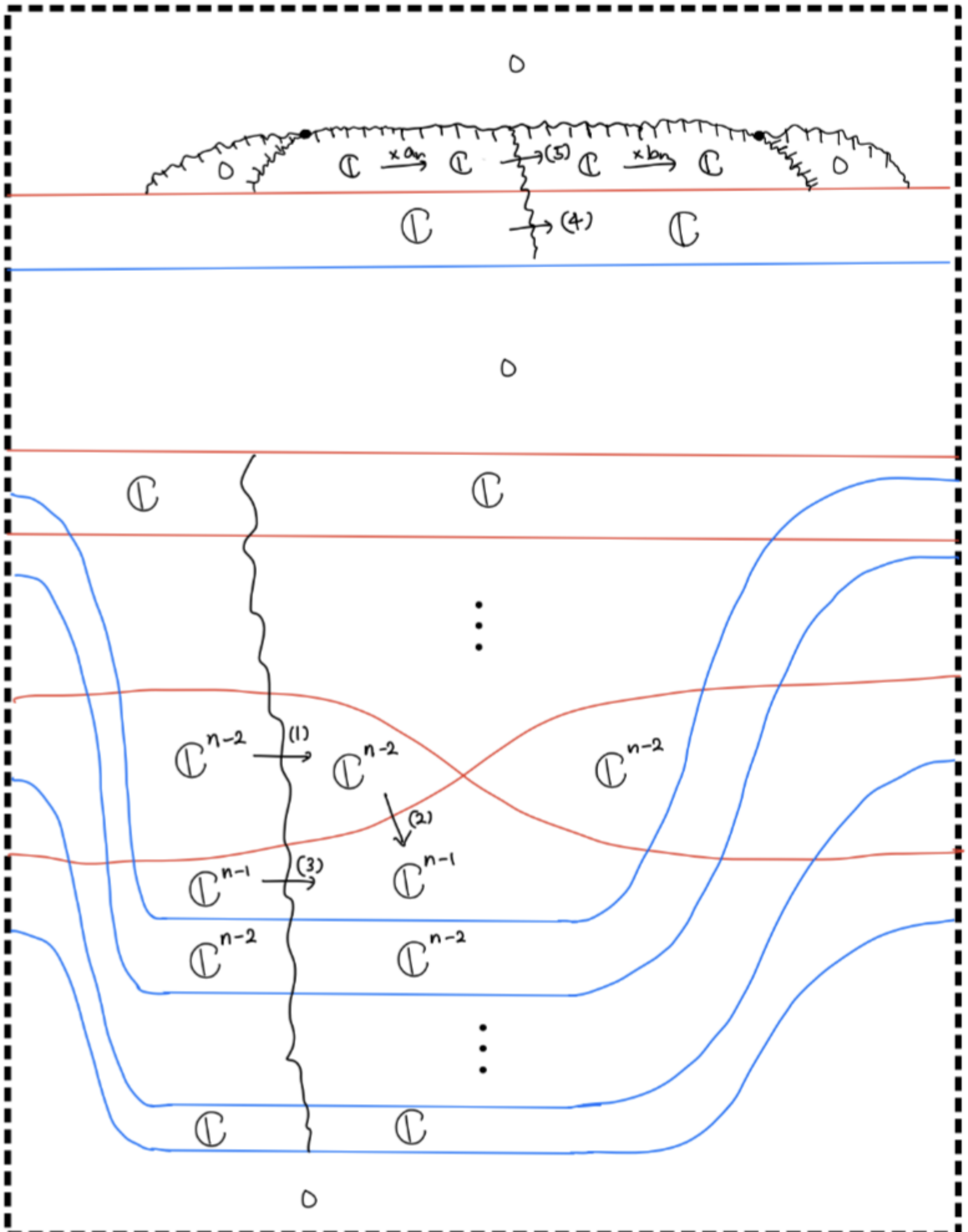

Figure 3.334

Generization maps

(1) $diag(d_{n-1}, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + e I_{n-1,n-2}$

(4) $\times d_n$

(5)
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\times 1"] & \mathbb{C} \\ \mathbb{C} \arrow[u, "\times d_n"] \arrow[r, "\times d_n"] & \mathbb{C} \arrow[u, "\times b_n"'] \end{tikzcd}$$

which is quasi-isomorphic to

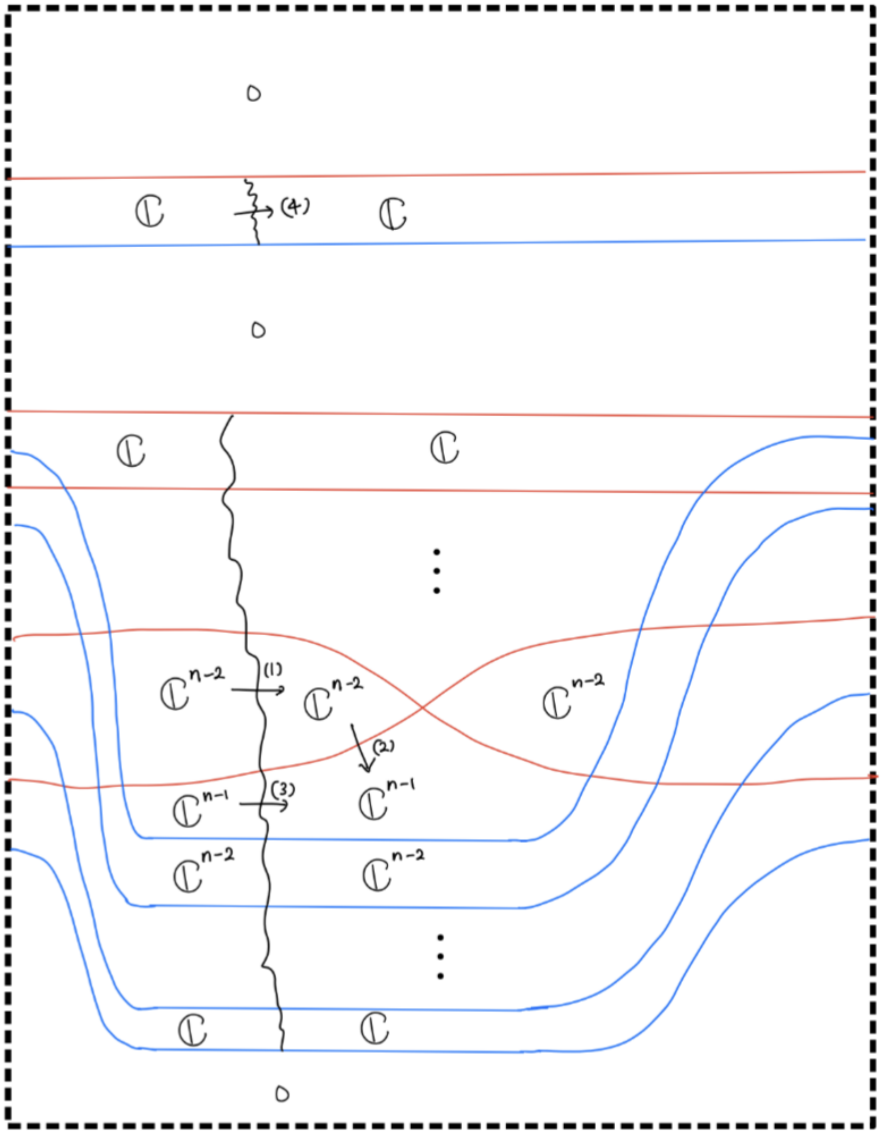

Figure 3.335

Generization maps

(1) $diag(d_{n-1}, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + e I_{n-1,n-2}$

(4) $\times d_n$

(Step 3) apply $cobord_5$ to the square region surrounded by purple dotted lines

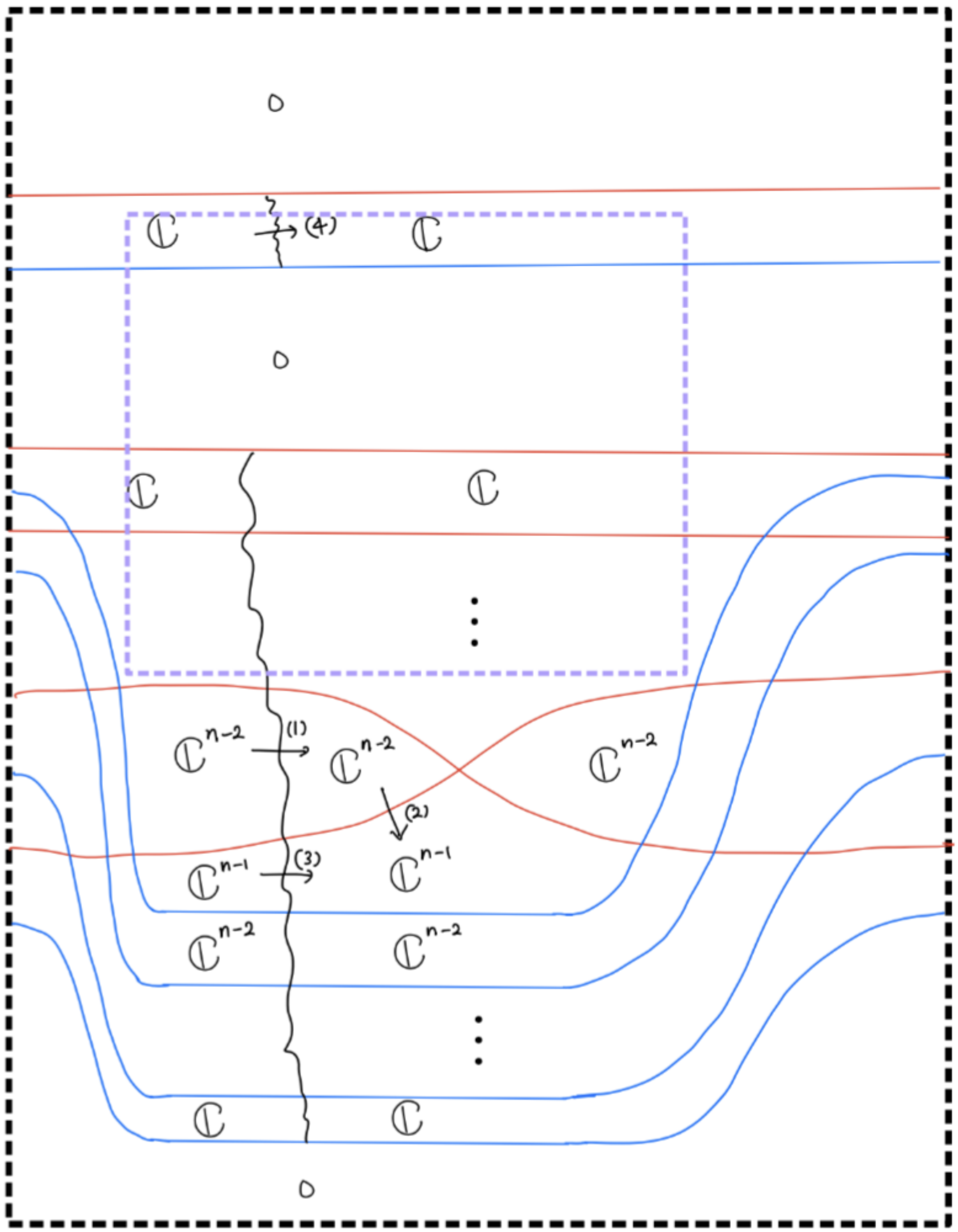

Figure 3.336

we get

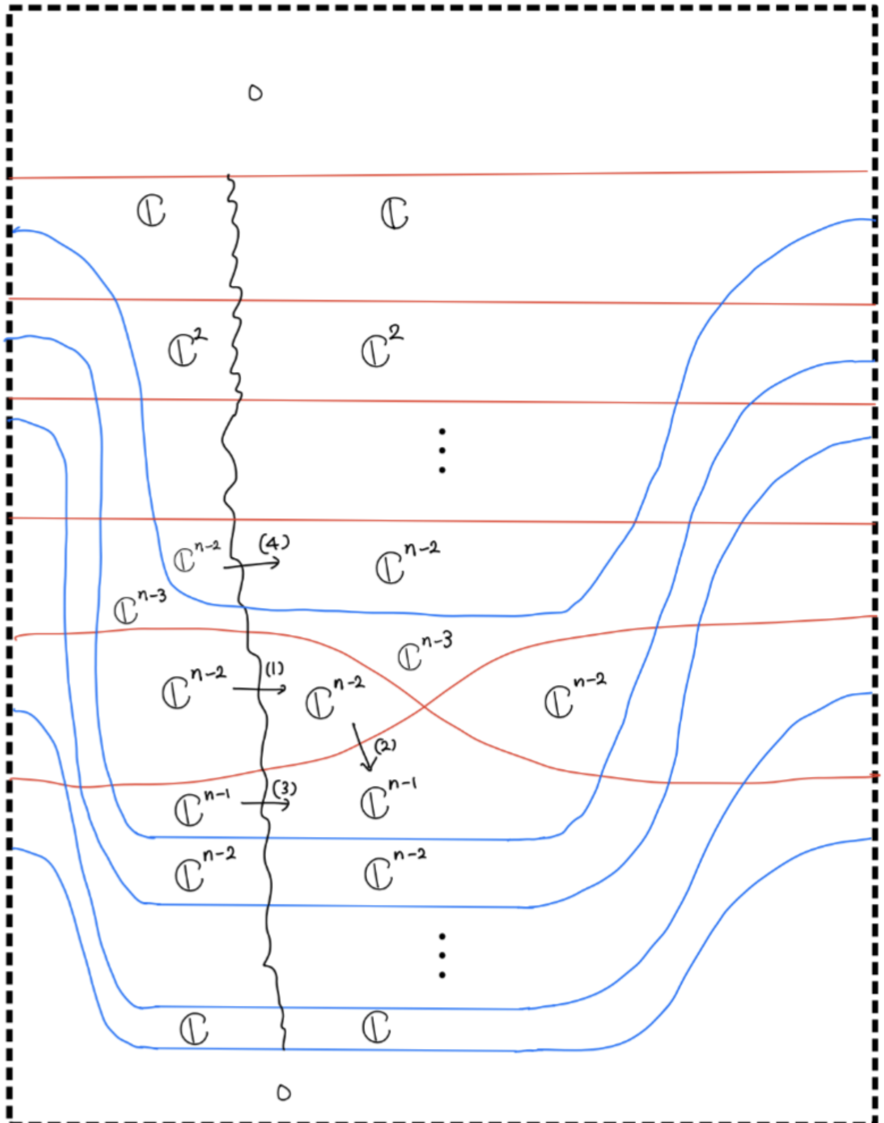

Figure 3.337

Generization maps

(1) $diag(d_{n-1}, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + e I_{n-1,n-2}$

(4) $diag(d_n, \cdots, d_3)$

(Step 4) apply $cobord_1$ to the regions surrounded by purple dotted lines

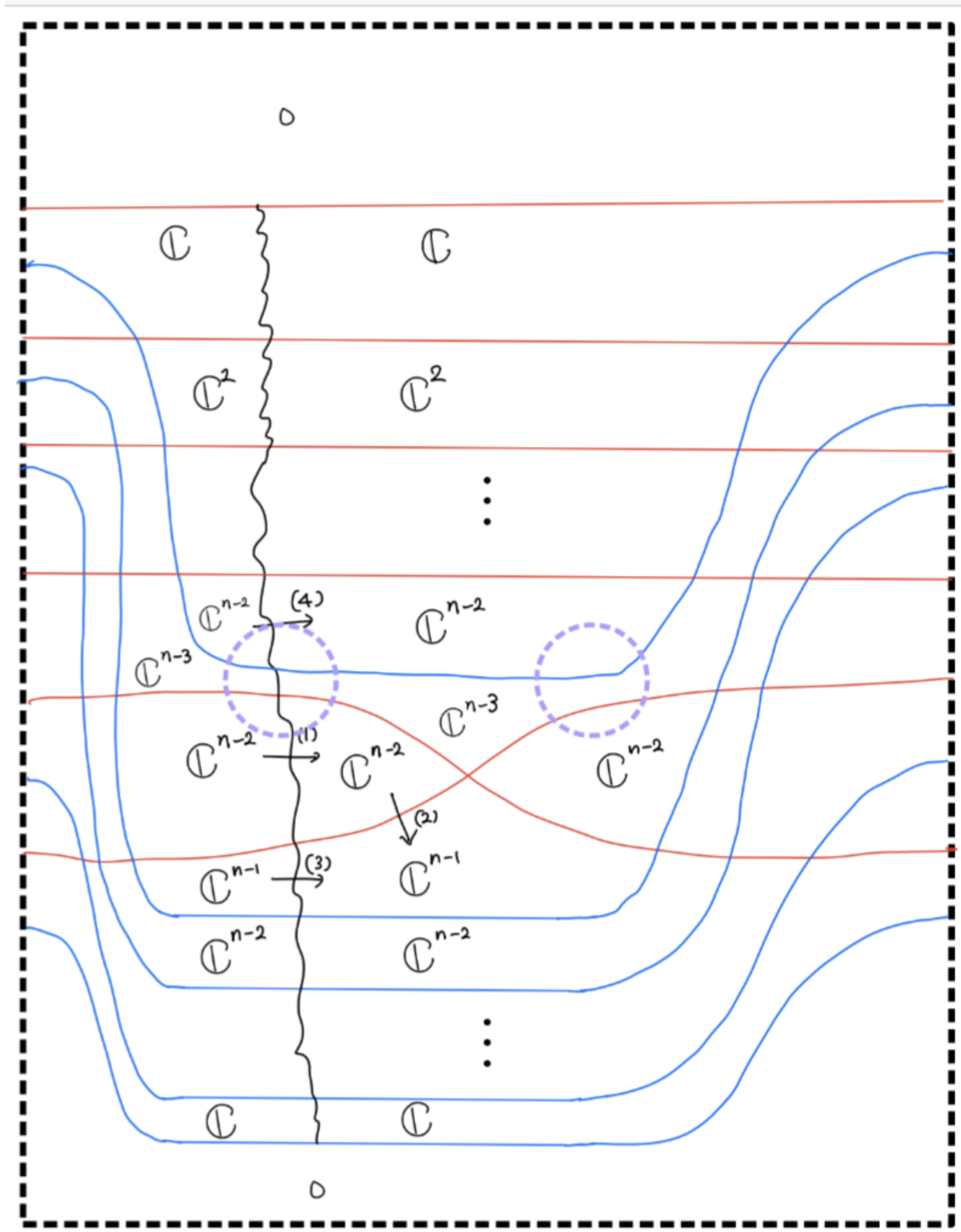

Figure 3.338

we get

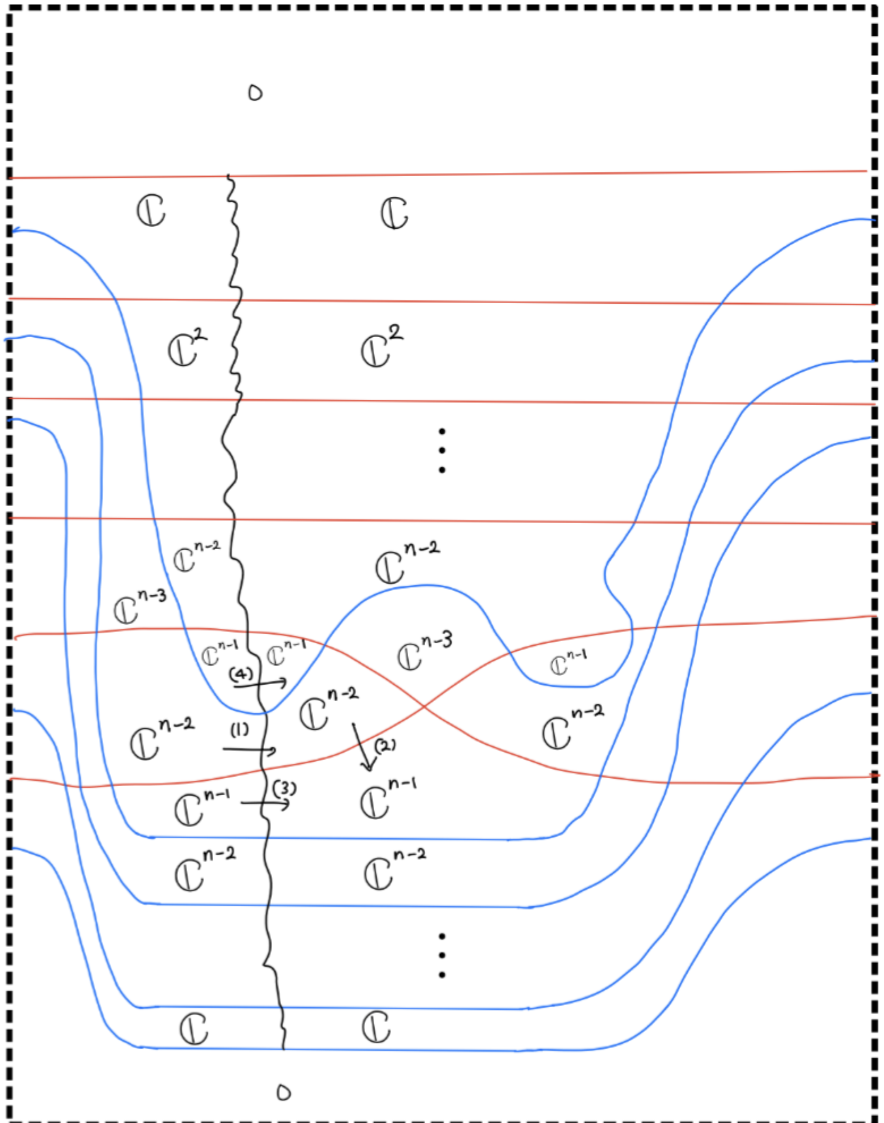

Figure 3.339

Generization maps

(1) $diag(d_{n-1}, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-1,n-2}$

(3) $diag(d_{n-1}, \cdots, d_1) + eI_{n-1,n-2}$

(4) $diag(d_n, \cdots, d_2)$

(Step 5) apply $cobord'_8$ to the region surrounded by purple dotted lines

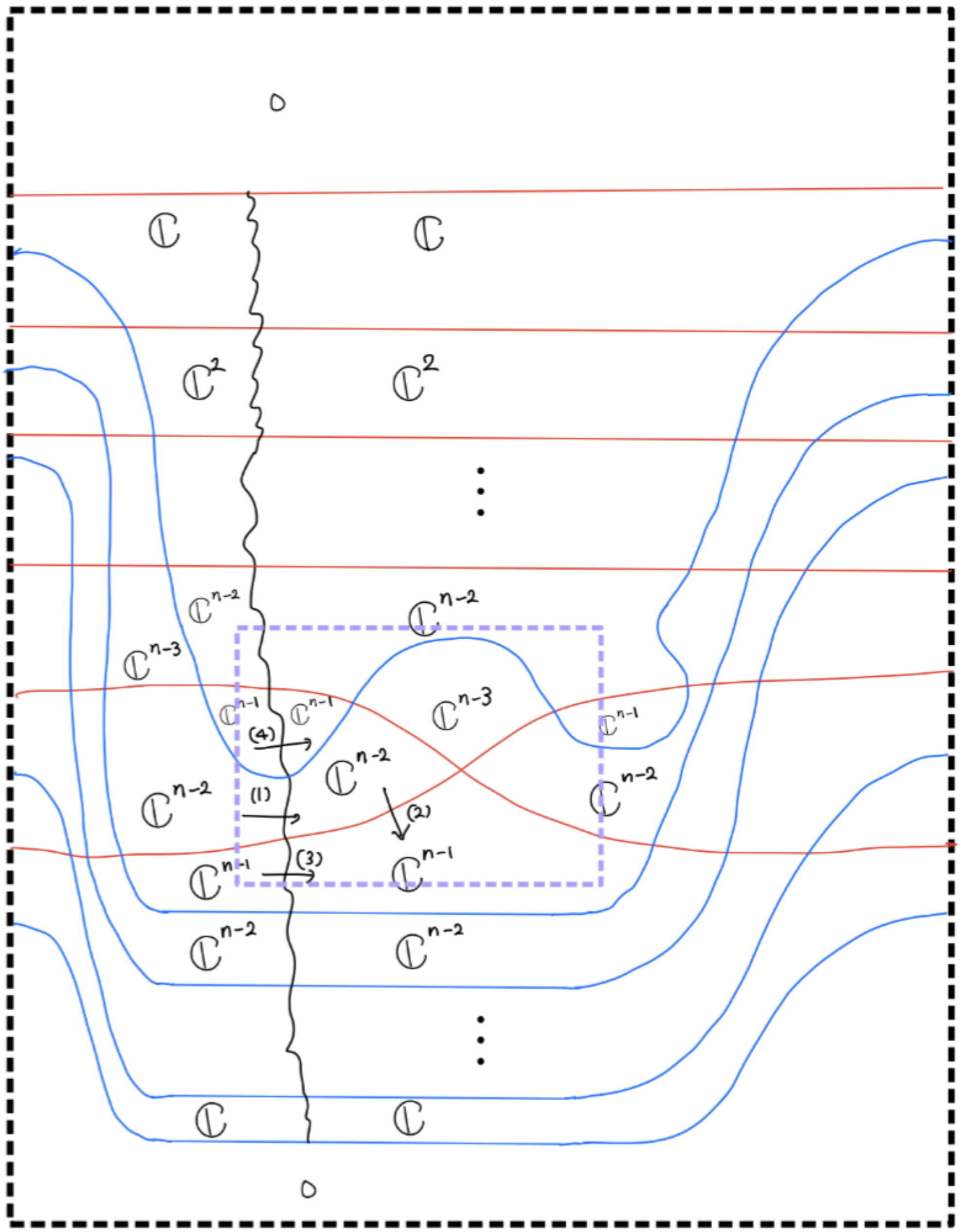

Figure 3.340

we get the final sheaf

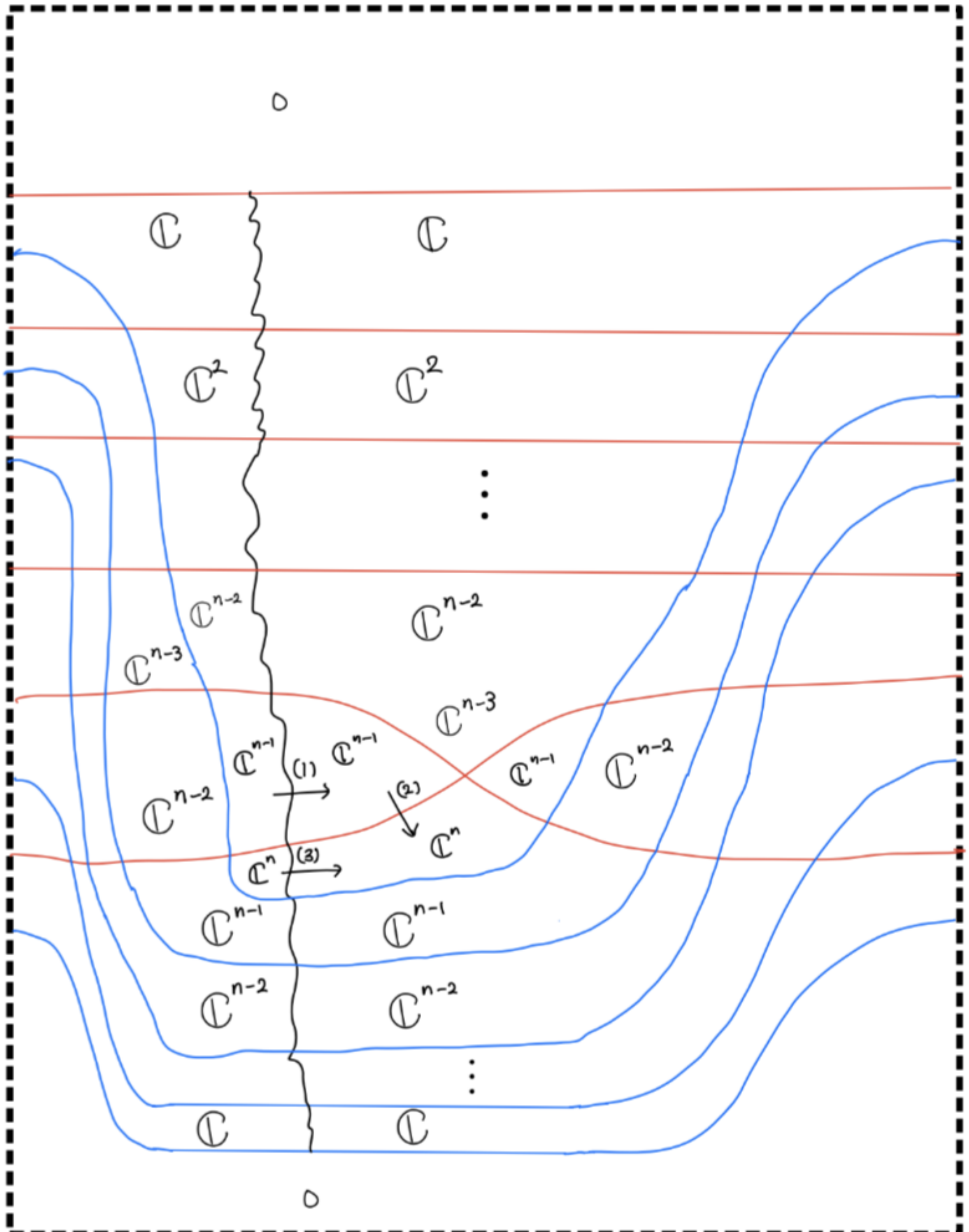

Figure 3.341

Generization maps

(1) $diag(d_n, \cdots, d_2)$

(2) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n,n-1}$

(3) $diag(d_n, \cdots, d_1) + e I_{n,n-1}$

3.16 Sheaf cobordism on inter-generator regions

Suppose we have a punctured Riemann sphere M and Λ_0^0, Λ_0^∞, Λ_0^{squig}, a nested regions $U \subset U' \subset M$, and a chart $f : U \to \mathbb{R}^2$ such that U' maps to $R := (-1,1)_x \times (-n-1, n+1)_z$ under f

- Λ_0^0 gets mapped to $\bigcup_{k=1}^{n}\{(x,z) \in R \mid z = \Psi(-k, n-k+\frac{1}{2})(x)\}$, co-oriented upward.
- Λ_0^∞ gets mapped to $\bigcup_{k=1}^{n}\{(x,z) \in R \mid z = k\}$, co-oriented downward.
- Λ_0^{squig} gets mapped to $\{(x,z) \in R \mid x = 0\}$, co-oriented toward the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are ι_1 and the red strands ι_0 otherwise stated. I have omitted these maps from the diagram.

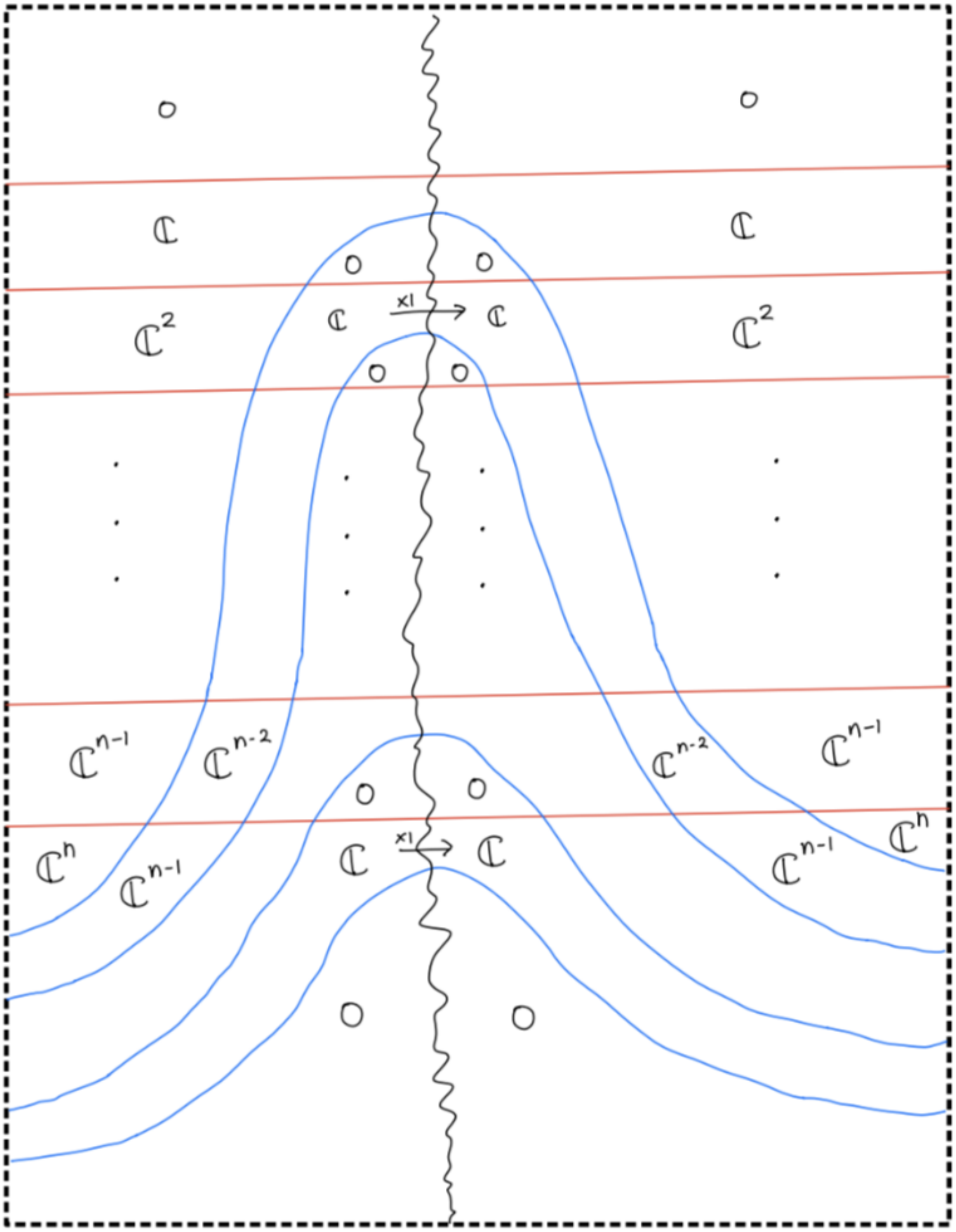
0
0
$\mathbb{C}$
$\mathbb{C}$
0
0
$\mathbb{C}^2$
$\mathbb{C}$
$\times 1$
$\mathbb{C}$
$\mathbb{C}^2$
0
0
$\mathbb{C}^{n-1}$
$\mathbb{C}^{n-2}$
$\mathbb{C}^{n-2}$
$\mathbb{C}^{n-1}$
0
0
$\mathbb{C}^n$
$\mathbb{C}^{n-1}$
$\mathbb{C}$
$\times 1$
$\mathbb{C}$
$\mathbb{C}^{n-1}$
$\mathbb{C}^n$
0
0

Figure 3.342

which is quasi-isomorphic to

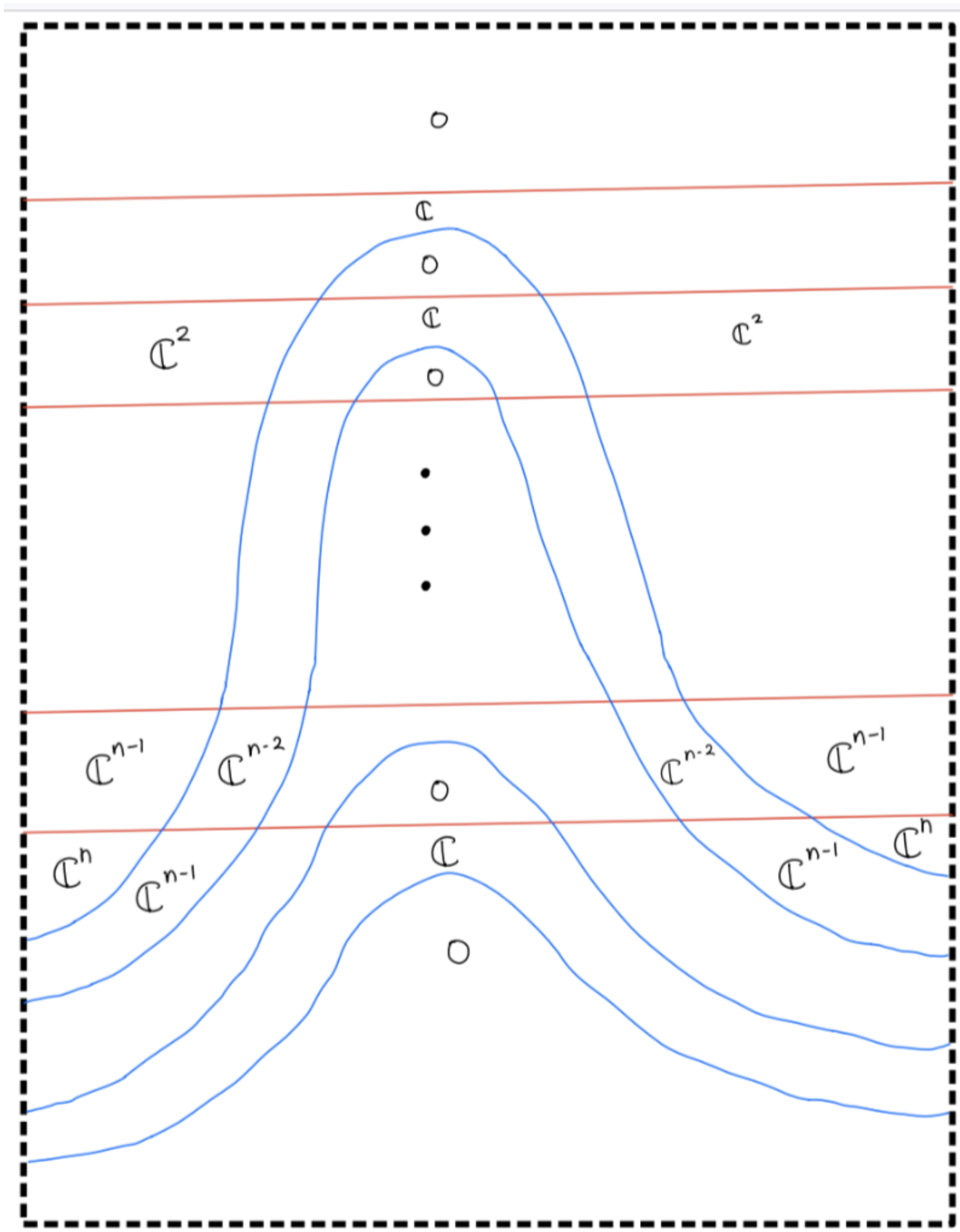

Figure 3.343

Then we define a cobordism starting from the above sheaf, say $cobord_{inter}(n)$ supported on U, where n is the number of blue strands(equivalently red strands). At the end of the cobordism, the sheaf, under the same chart f, is described as the following squiggly legible diagram.

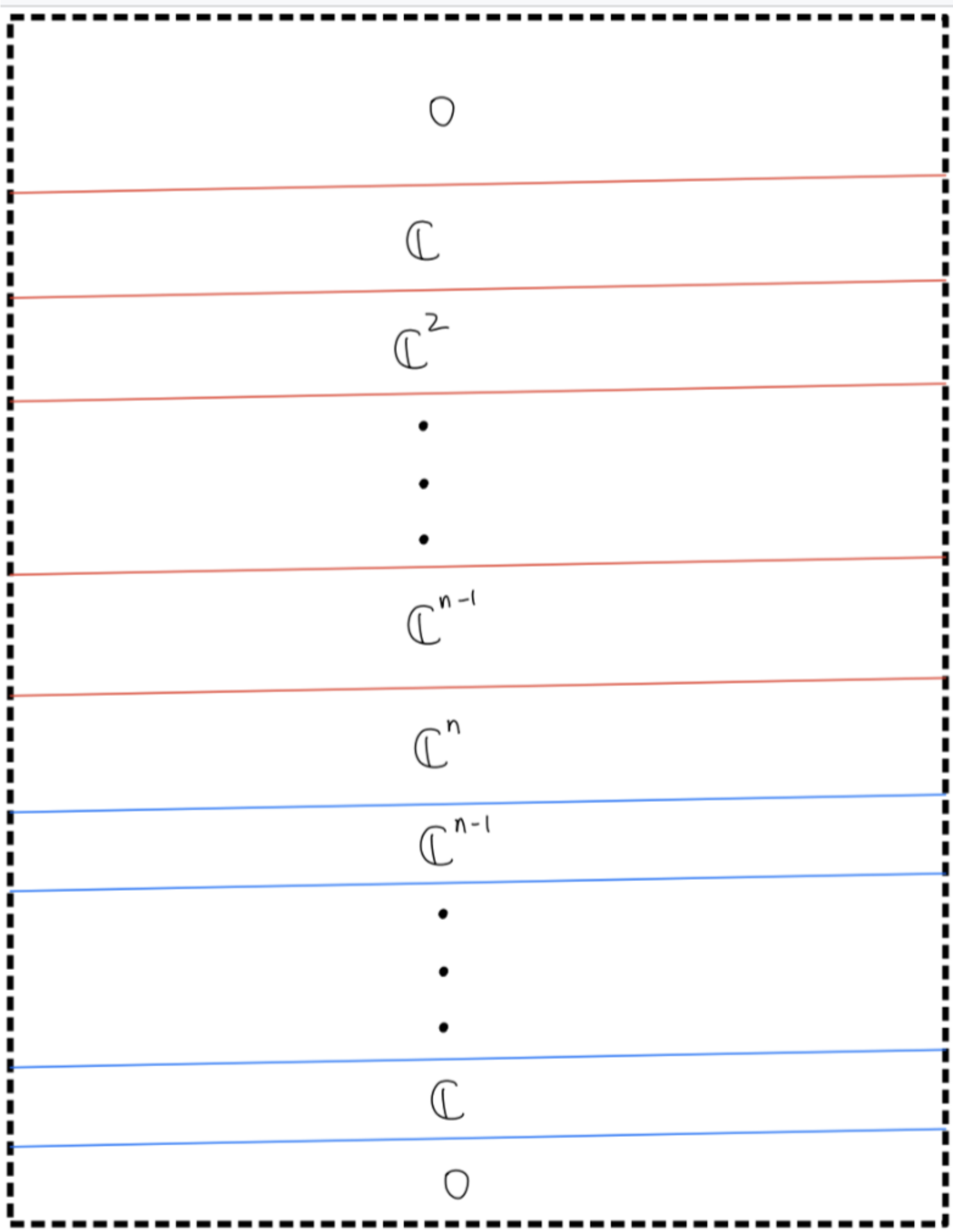
0
$\mathbb{C}$
$\mathbb{C}^2$
$\mathbb{C}^{n-1}$
$\mathbb{C}^n$
$\mathbb{C}^{n-1}$
$\mathbb{C}$
0

Figure 3.344

We define $cobord_{inter}(n)$ inductively as follows.

(i) For $n = 1$, we define $cobord_{inter}(1)$ to be the null cobordism from

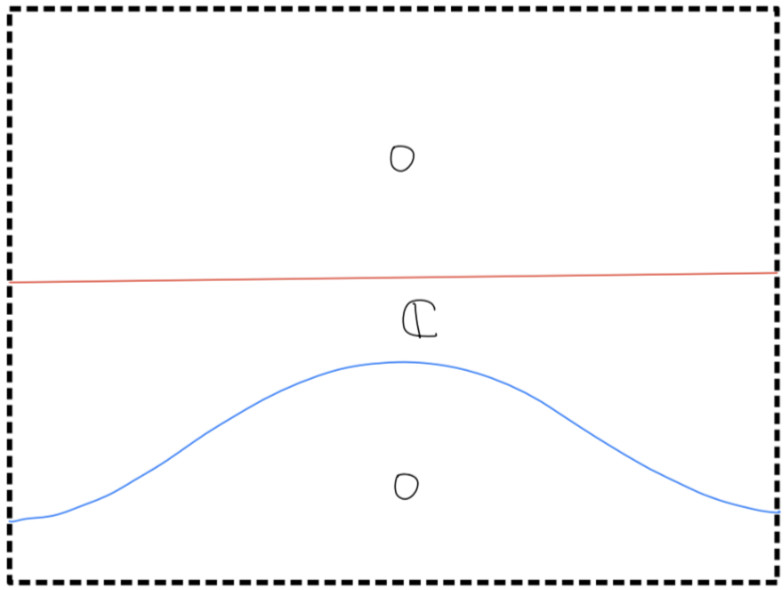

Figure 3.345

to

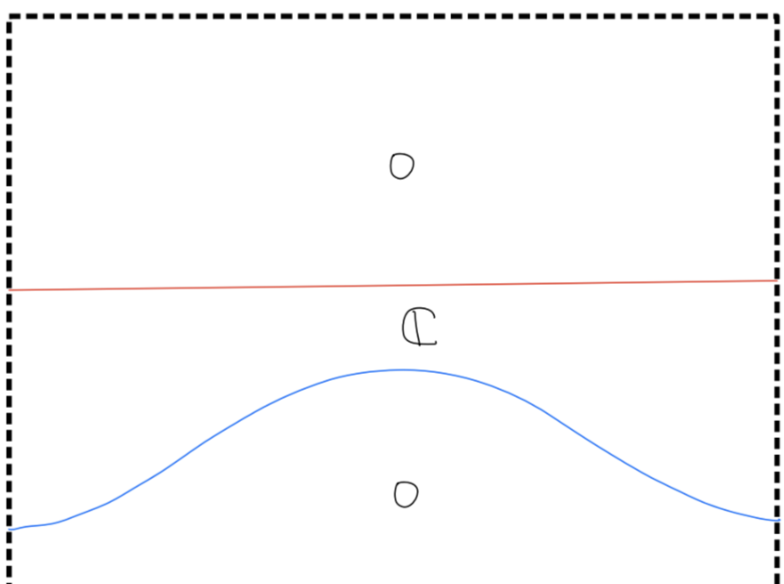

Figure 3.346

(ii) For $n > 1$,

(Step 1) we first apply $cobord_{inter}(n-1)$ to the square region surrounded by purple dotted lines.

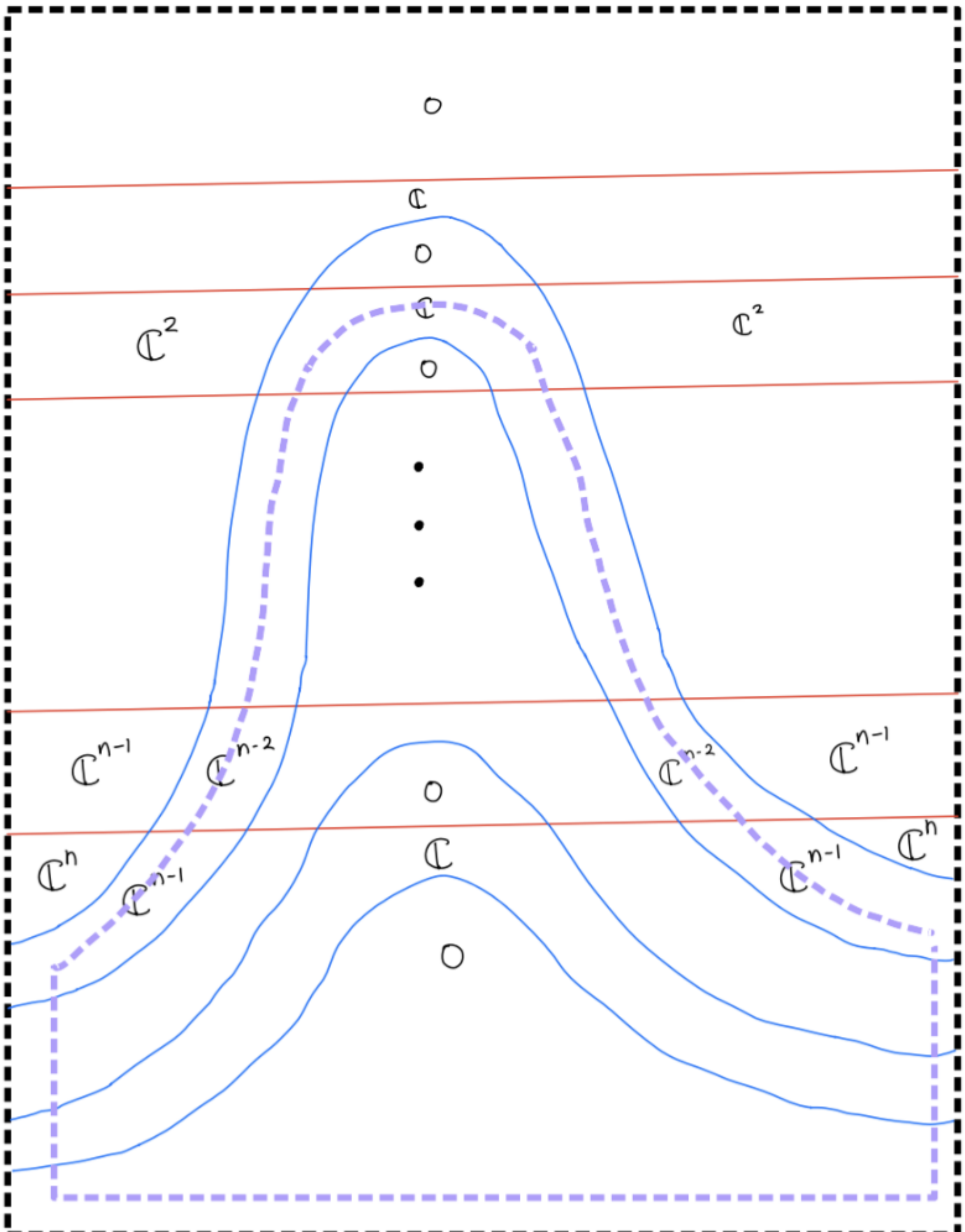

Figure 3.347

by induction hypothesis, we get

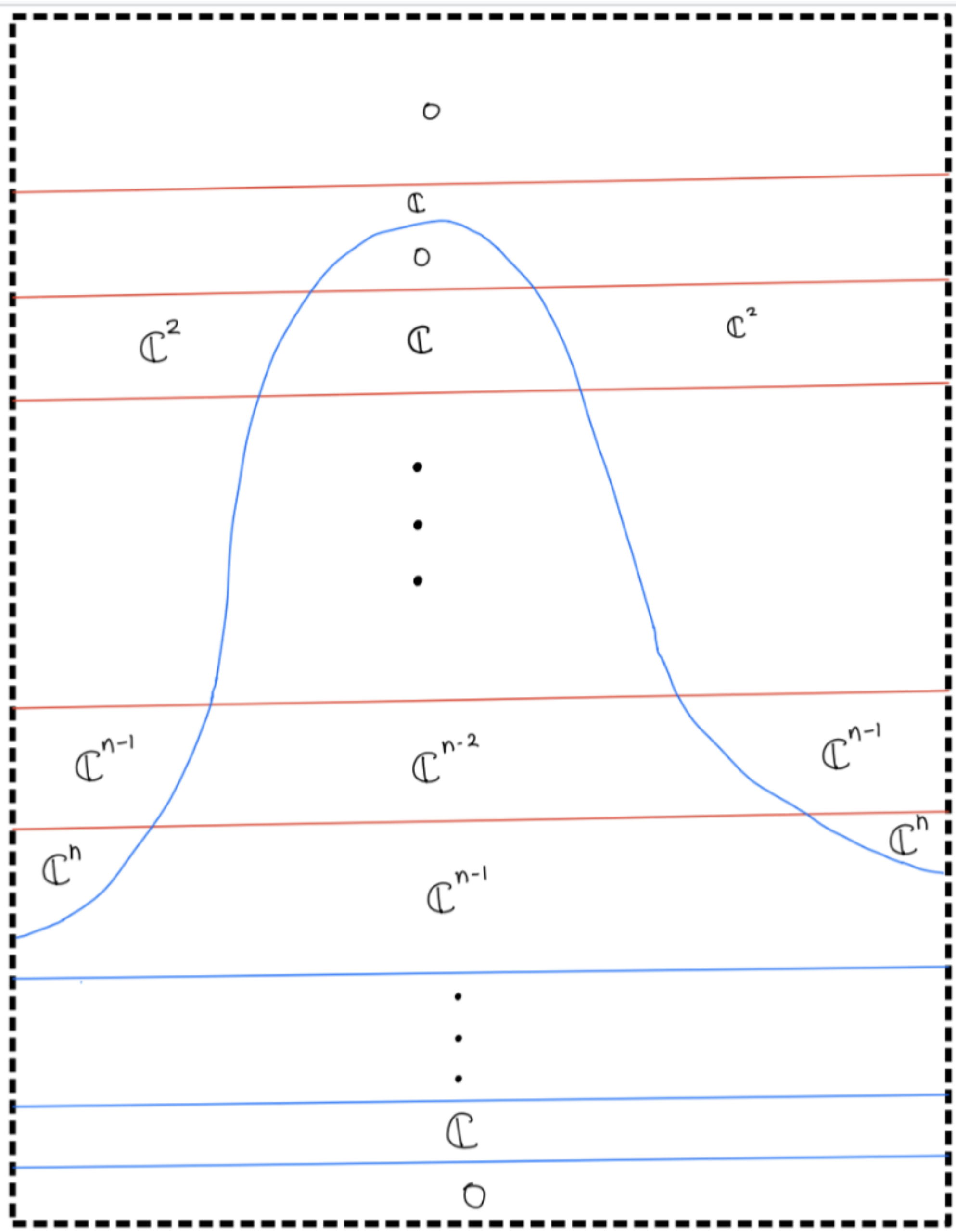

Figure 3.348

(Step 2) apply $cobord_7$ to the square region surrounded by purple dotted lines.

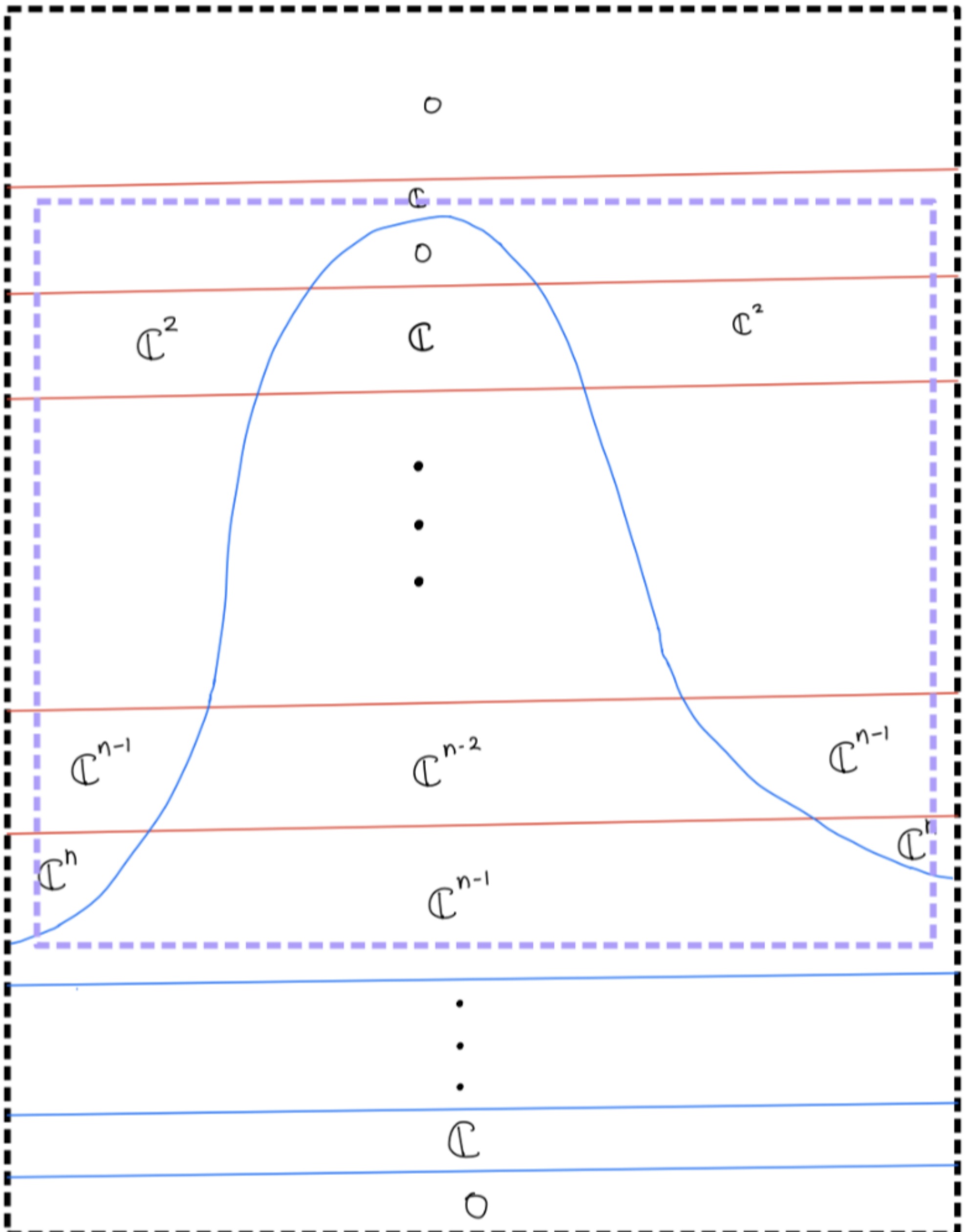

Figure 3.349

we get

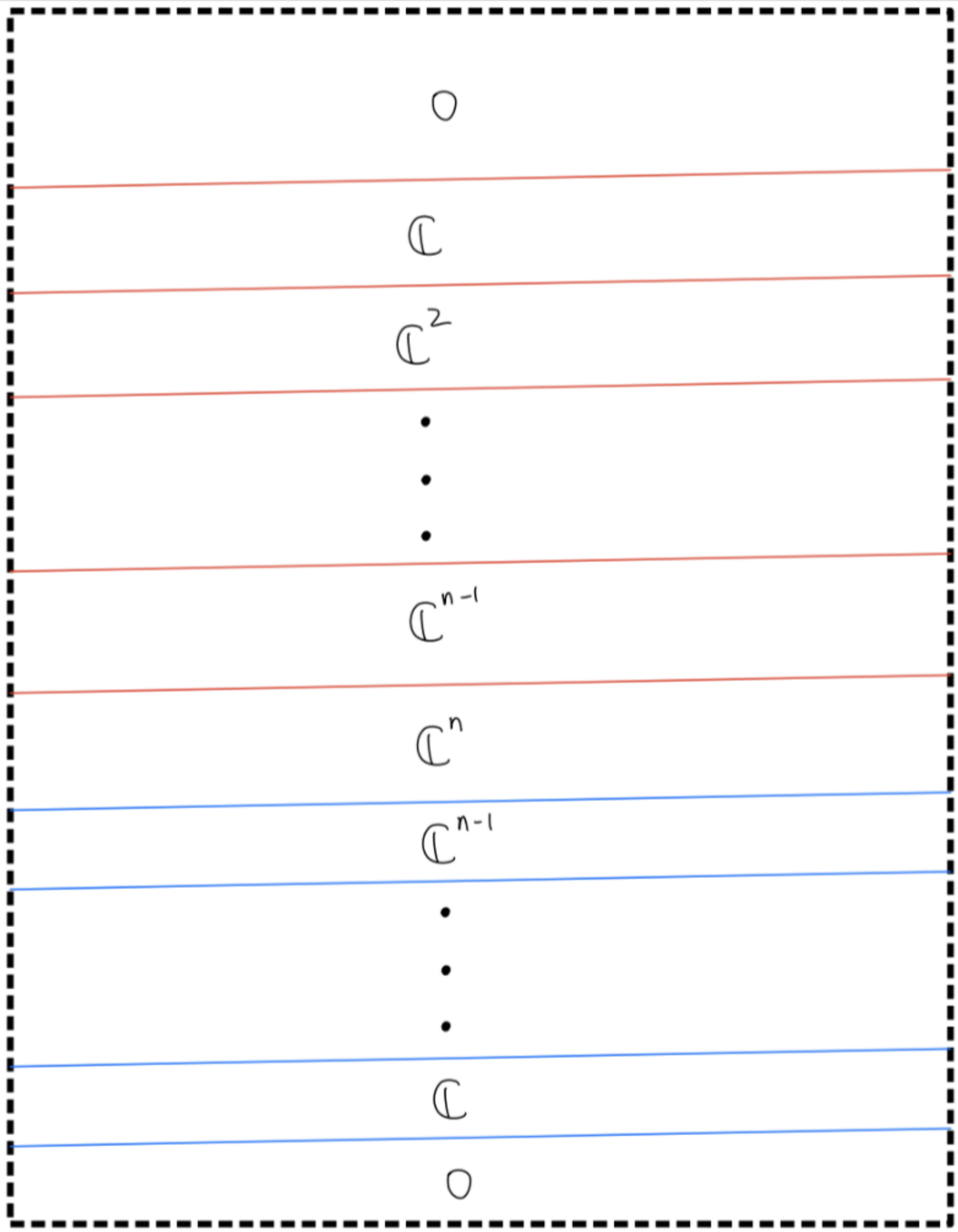

Figure 3.350

which is isomorphic to the final sheaf

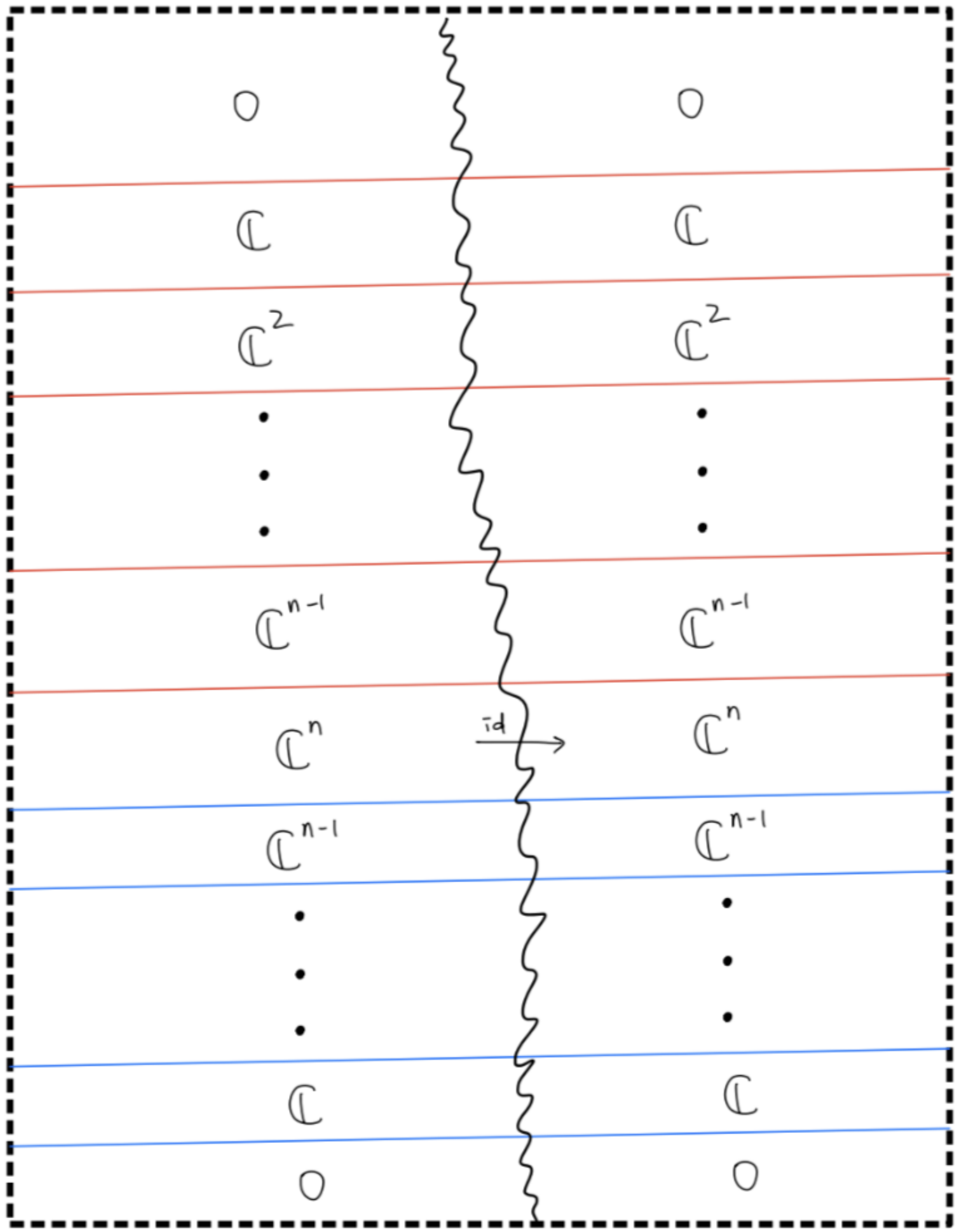

Figure 3.351

3.17 Full sheaf cobordism

Suppose we have a positive braid word $\omega = s_{i_1} \cdots s_{i_k}$ on n strands, then we have an embedding of the cylindrical closures of ω and the trivial braid $\omega_\emptyset$ in a Riemann sphere with punctures at $0, \infty$ described in section ??. Suppose we have an alternating sheaf on the natural alternating diagram associated to $\omega \coprod \omega_\emptyset$ when restricted to the j^{th} generator region is described by the following legible diagram

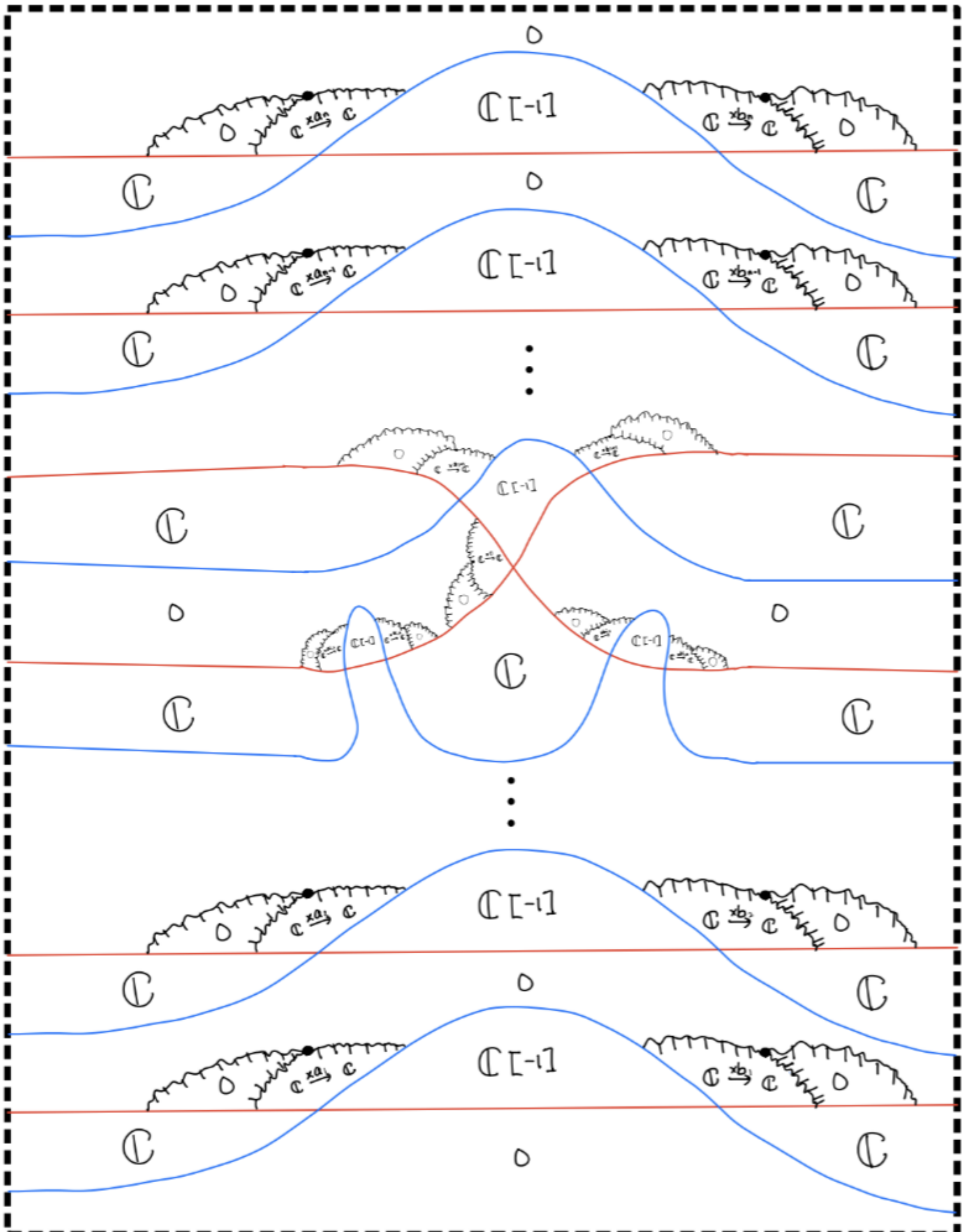

Figure 3.352

and when restricted to inter-generator regions is described by the following squiggly legible diagram

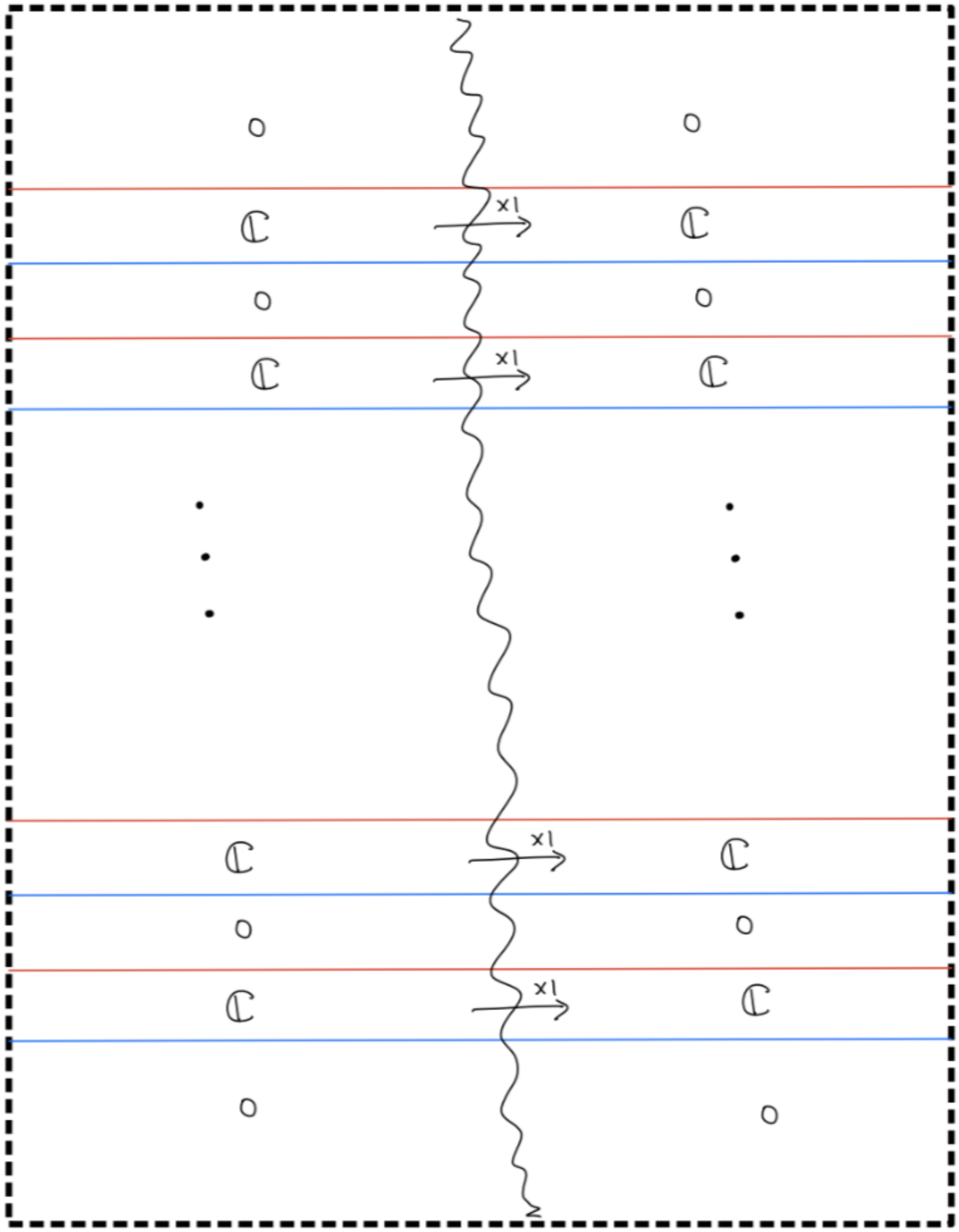

Figure 3.353

then we define a sheaf cobordism from the above alternating sheaf whose underlying Legendrian isotopy and at the separated diagram of $\omega \coprod \omega_{\emptyset}$, thereby describing a cluster coordinate.

(Step 1) to the j^{th} generator regions($j = 1, \cdots, k$), apply $cobord_{gen(n)}$, we get on the j^{th} generator region

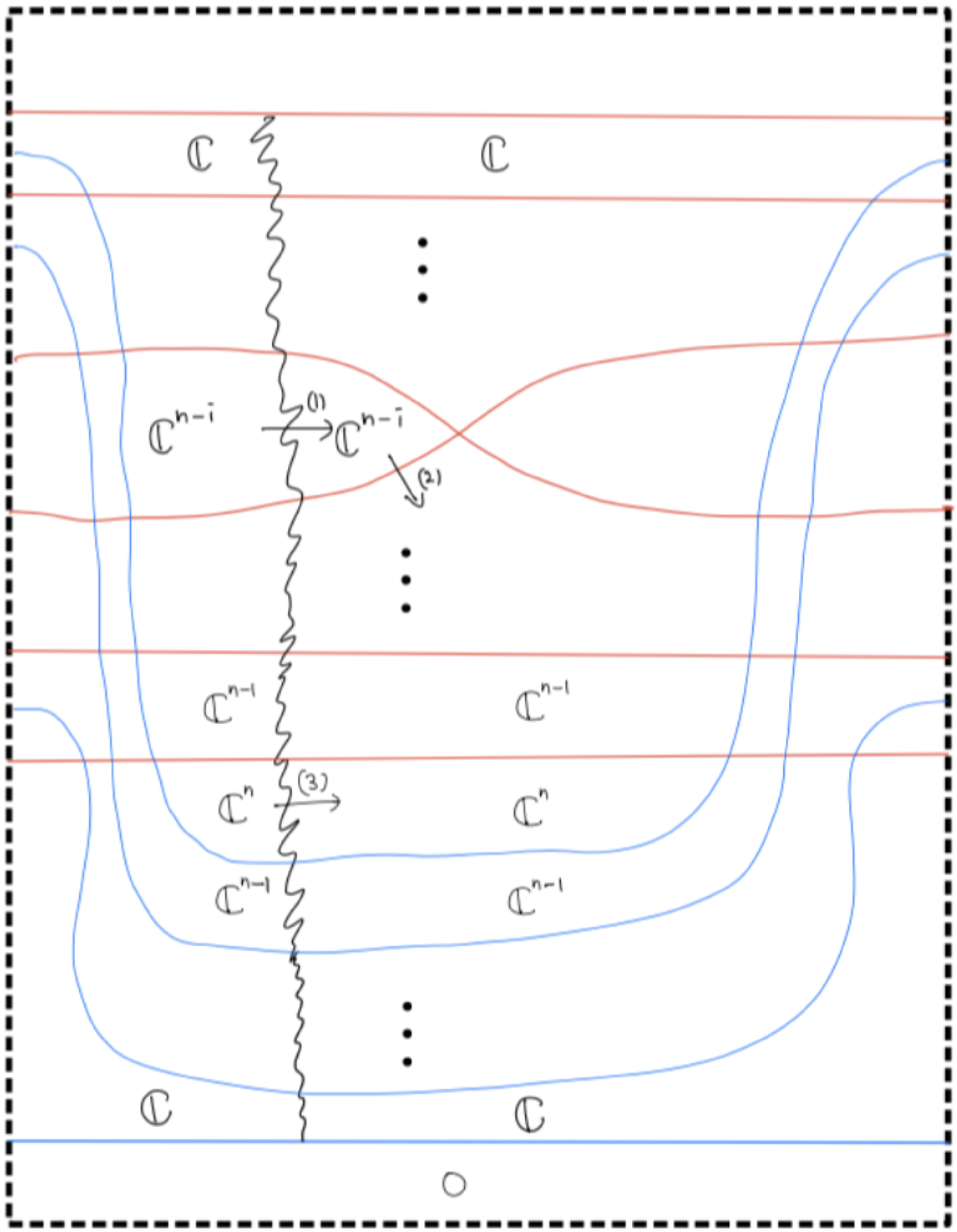

Figure 3.354

Generization maps

(a) $diag(d_n^j, \cdots, d_{i+1}^j)$

(b) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(c) $diag(d_n^j, \cdots, d_1^j) + e I_{n-i+1,n-i}$

where

- $a_i = a_{i,1} a_{i,2}^{-1}$ and $b_i = b_{i,1}^{-1} b_{i,2}$
- $d_r = a_r b_r^{-1}$
- $e = -a_{i+1} b_i^{-1} c$
- $e' = d_{i+1}^{-1} e$

and on the inter-generator regions we get

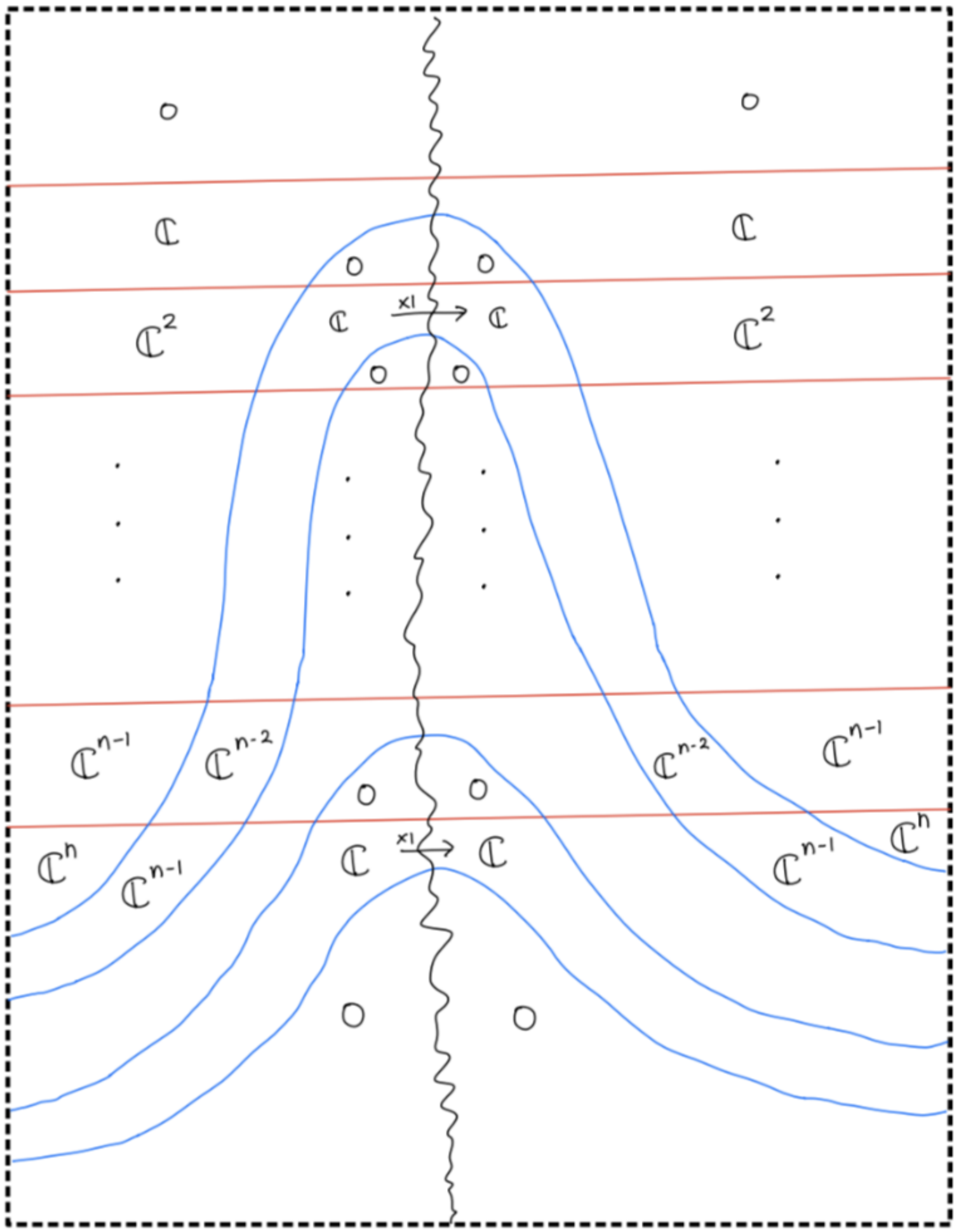

Figure 3.355

(Step 2) to the inter-generator regions, we apply $cobord_{inter}(n)$, on the j^{th} generator region, we get

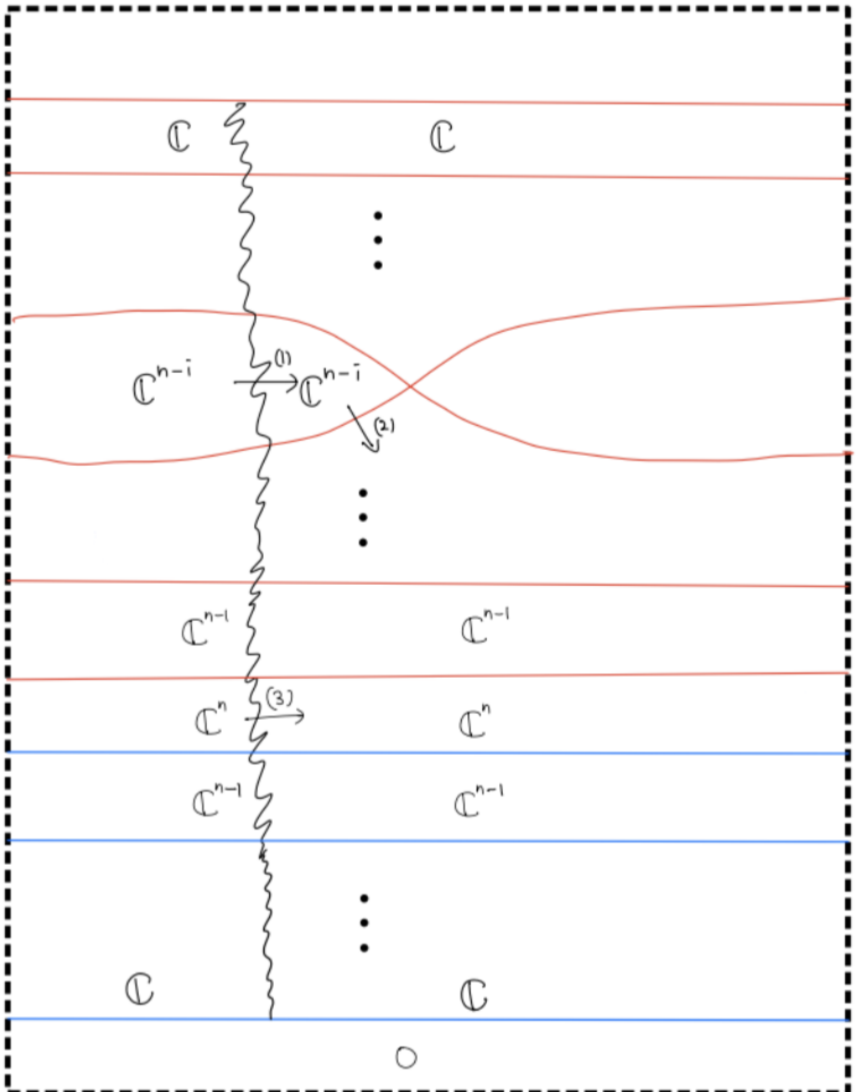

Figure 3.356

Generization maps

(a) $diag(d_n^j, \cdots, d_{i+1}^j)$

(b) $\iota_0 \circ diag(1, \cdots, 1) + e' I_{n-i+1,n-i}$

(c) $diag(d_n^j, \cdots, d_1^j) + e I_{n-i+1,n-i}$

and on the inter-generator regions, we get

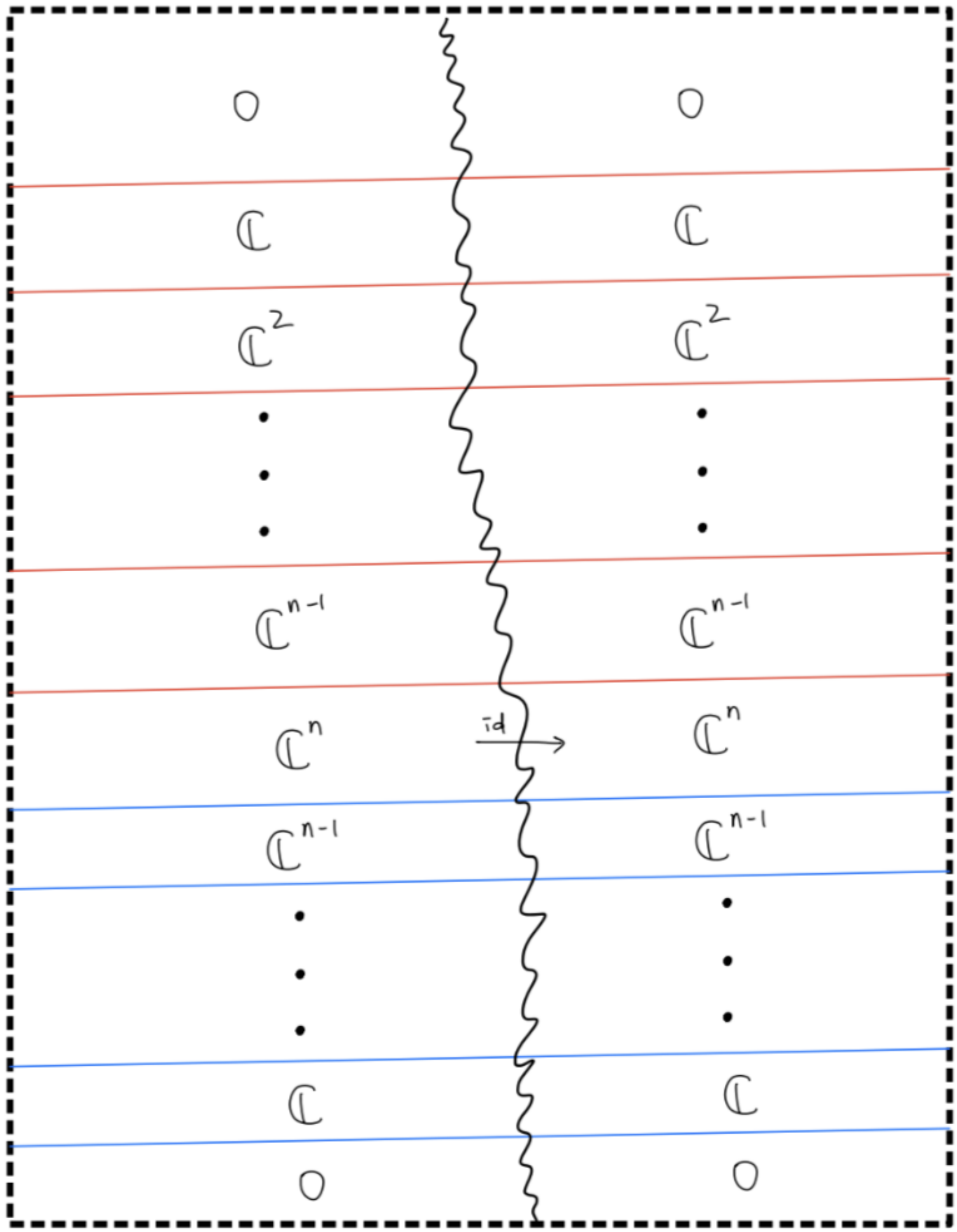

Figure 3.357

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