

# Moduli Spaces associated to Legendrian Links in the Cocircle Bundle of a Cylinder from Positive Braids

Jee Uhn Kim

A dissertation submitted to the Faculty of the  
Department of Mathematics in partial fulfillment of the  
requirements for the degree of Doctor of Philosophy

Boston College  
Morrissey College of Arts and Sciences  
Graduate School

May 2025

©2025 Jee Uhn Kim

**Moduli Spaces associated to Legendrian Links in the Cocircle Bundle of  
a Cylinder from Positive Braids**

Jee Uhn Kim

Advisor: David Treumann, Ph.D.

Abstract

# Contents

<b>Contents</b>	i
<b>List of Figures</b>	ii
<b>Acknowledgements</b>	iii
<b>1 Introduction</b>	1
1.1 Background and context . . . . .	1
1.2 Summary of results . . . . .	19
<b>2 Multivalent Braids and Generalized Sibuya Spaces</b>	28
2.1 Basic terminologies . . . . .	28
2.2 Multivalent braid words . . . . .	33
2.3 Bivalent braid words . . . . .	39
2.4 Examples : generalized Sibuya spaces . . . . .	45
<b>3 Natural Cluster Coordinates on Braid Moduli Spaces</b>	73

# List of Figures

1.1	7
1.2	10
1.3	12
1.4	13
1.5	14
1.6	15
1.7	20
1.8	21
1.9	23
1.10	24

## **Acknowledgments**

Acknowledgements

# Chapter 1

## Introduction

### 1.1 Background and context

#### Contact Geometry

First, we review basics of contact geometry, see [Etn05][Gei08] for details. A contact structure on a  $(2n - 1)$ -dimensional manifold  $X$  is a maximally nonintegrable distribution of  $(2n - 2)$ -planes. A contact structure defined globally as a kernel for a chosen 1-form  $\alpha$  is said to be co-oriented and  $\alpha$  is said to be its co-orientation. An  $(n - 1)$ -dimensional submanifold  $\Lambda \subset X$  is said to be Legendrian if its tangent bundle is contained in the contact hyperplanes. Now let's look at several examples of contact manifolds.

- (1) the cotangent bundle  $T^*M$  of a manifold  $M$  carries a canonical 1-form which is in local coordinates  $(q_i, p_i)$ ,  $\theta := \sum_i p_i dq_i$ . This form is invariant under dialation in cotangent directions, therefore,  $\theta$  descends to a well-defined 1-form (unique up to positive scalar function) on the cosphere bundle  $T^\infty M := (T^*M - 0_M)/\mathbb{R}_+$  where  $0_M$  is the zero section and  $\mathbb{R}_+$ -action is the dialation action along cotangent directions. Therefore,  $T^\infty M$  is equipped with a natural contact structure.

(2)  $\mathbb{R}_{x,y,z}^3$  with the global 1-form  $\alpha = dz - ydx$  defines a co-oriented contact structure. We can embed  $\mathbb{R}_{x,y,z}^3$  in  $T^{\infty,-}\mathbb{R}_{x,z}^2$  as an open contact submanifold as follows:

$$\begin{aligned}\mathbb{R}_{x,y,z}^3 &\rightarrow \mathbb{R}_{x,z}^2 \times S^1 \\ (x, y, z) &\mapsto (x, z; y, -1)\end{aligned}$$

The image is the contact submanifold  $T^{\infty,-}\mathbb{R}_{x,z}^2$  of "downward" covectors.

(3) Note that construction (2) is invariant under translation  $x \mapsto x + 1$ . Let  $S_x^1 := \mathbb{R}_x/\mathbb{Z}$ , then standard 1-form  $dz - ydx$  descends to  $S_x^1 \times \mathbb{R}_{y,z}$  and it embeds as the contact submanifold  $T^{\infty,-}(S_x^1 \times \mathbb{R}_z)$  in  $T^\infty(S_x^1 \times \mathbb{R}_z) \cong S_x^1 \times \mathbb{R}_z \times S_y^1$ . The "cocircle bundle" in the title of the thesis refers to  $T^\infty(S_x^1 \times \mathbb{R}_z)$ .

## Front Projection

Now let's restrict our attention to the case when the base manifold  $M$  is either  $\mathbb{R}_{x,z}^2$  or  $S_x^1 \times \mathbb{R}_z$ . We call the map  $\pi : T^\infty M \rightarrow M$  the "front projection" and  $\pi(\Lambda)$  the "front diagram".

- when  $M = \mathbb{R}_{x,z}^2$ ,  $\pi : \mathbb{R}_{x,y,z}^3 \subset T^\infty \mathbb{R}_{x,z}^2 \rightarrow \mathbb{R}_{x,z}^2$  is the projection onto  $x, z$  coordinates. We call  $\mathbb{R}_{x,z}^2$  a front plane.
- similarly, when  $M = S_x^1 \times \mathbb{R}_z$ ,  $\pi : S_x^1 \times \mathbb{R}_{y,z}^2 \subset T^\infty(S_x^1 \times \mathbb{R}_z) \rightarrow S_x^1 \times \mathbb{R}_z$  is the projection onto  $x, z$  coordinates. We call  $S_x^1 \times \mathbb{R}_z$  a front cylinder.

A Legendrian  $\Lambda \subset T^\infty M$  at immersed points of  $\pi(\Lambda)$  can be recovered from  $\pi(\Lambda)$  because  $\Lambda$  vanishes on the contact form  $dz - ydx$ , we have  $y = \frac{dz}{dx}$  as long as it is in general position. Under Hamiltonian isotopy,  $\Lambda$  can be put in general position, i.e.

- $\pi|_\Lambda$  is locally injective.

- there are only finitely many points on  $M$  at which  $\pi(\Lambda)$  is not an embedded submanifold. These are either
  - cusps where  $\pi|_{\Lambda}$  is injective and  $\frac{dz}{dx}$  has a well-defined limit of 0.
  - crossing where  $\pi(\Lambda)$  is locally a transverse intersection of two smooth curves.

In this paper, we only allow  $\pi(\Lambda)$  to have crossings. We won't consider  $\pi(\Lambda)$  with cusps. There are no interesting examples in  $\mathbb{R}_{x,y,z}^3$  but there are many interesting ones in the cocircle bundle of a cylinder. Suppose we have a smooth parametrized curve  $\Phi$  in  $M$

- which is an immersion
- which has finitely many self-intersections and they are transverse.

and a choice of preferred side, i.e. a co-orientation, diagrammatically described using hairs normal to the  $\Phi$  pointing the preferred direction. We say that the region on the preferred side lies beneath the arc and the region on the other side lies above the arc. We can lift it to a Legendrian in  $T^\infty M$ . Therefore, the Legendrian in  $T^\infty M$  amounts to the datum of the co-oriented front diagram.

## Constructible sheaves

Let  $M$  be a manifold and  $\mathcal{S}$  Whitney a stratification of  $M$ . A sheaf of  $\mathbb{C}$ -vector space  $\mathcal{F}$  on  $M$  is  $\mathcal{S}$ -constructible if the restriction of  $\mathcal{F}$  to each stratum of  $s \in \mathcal{S}$ ,  $\mathcal{F}|_s$ , is a locally constant sheaf.  $\mathcal{F}$  is called constructible if there exists a Whitney stratification  $\mathcal{S}$  such that  $\mathcal{F}$  is  $\mathcal{S}$ -constructible.  $Sh_{naive}^\bullet(M; \mathbb{C})$  denotes the triangulated dg category whose objects are cochain complexes of sheaves of  $\mathbb{C}$ -vector spaces of bounded cohomology whose cohomology sheaves are constructible sheaves. If we localize  $Sh_{naive}^\bullet(M; \mathbb{C})$  with respect to acyclic complexes, in the sense of [Dri04], we get

$Sh^\bullet(M; \mathbb{C})$ . We denote  $Sh_S^\bullet(M; \mathbb{C})$  the full subcategory of  $Sh^\bullet(M; \mathbb{C})$  of complexes whose cohomology sheaves are constructible with respect to  $\mathcal{S}$ . From now on we will call objects of  $Sh^\bullet(M; \mathbb{C})$  simply as sheaves.

## Singular Support

To each  $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$ , we assign a closed conical subset  $SS(\mathcal{F}) \subset T^*M$ , called the singular support of  $\mathcal{F}$  following [GM83][Sch12][STZ17]. For more general treatment, see [KS13]. Fix a Riemannian metric on  $M$  and choose an  $\epsilon$ -ball  $B_\epsilon(x)$  around a point  $x \in M$ . The following constructions are independent of the choice of metrics. Let  $\mathcal{F}$  be an  $\mathcal{S}$ -constructible sheaf on  $M$ . Fix a point  $x \in M$  and a smooth function  $f$  on  $B_\epsilon(x)$ . For  $\epsilon, \delta > 0$ , we define the local Morse group to be

$$Mo_{x,f,\epsilon,\delta}(\mathcal{F}) := H^*(B_\epsilon(x) \cap f^{-1}((-\infty, f(x) + \delta)), B_\epsilon(x) \cap f^{-1}((-\infty, f(x) - \delta)); \mathcal{F})$$

For  $\epsilon' < \epsilon$  and  $\delta' < \delta$ , there is a canonical restriction map

$$Mo_{x,f,\epsilon,\delta}(\mathcal{F}) \rightarrow Mo_{x,f,\epsilon',\delta'}(\mathcal{F})$$

The above restriction is an isomorphism if  $\epsilon$  and  $\delta$  are sufficiently small and  $f$  is stratified Morse at  $x$ . This allows us to define  $Mo_{x,f}(\mathcal{F})$  for  $f$  suitably generic with respect to  $\mathcal{S}$ [KS13, Prop. 7.5.3]. In fact,  $Mo_{x,f}(\mathcal{F})$  depends only on the Hessian of  $f$  at  $x$ . Let  $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$ , then  $(x, \xi) \in T^*M$  is characteristic with respect to  $\mathcal{F}$  if for some stratified Morse function  $f$  with  $df_x = \xi$ , the local Morse group  $Mo_{x,f}(\mathcal{F})$  is nonzero. Then we define the singular support of  $\mathcal{F}$  to be the closure of the set of characteristic covectors for  $\mathcal{F}$  in  $T^*M$ . Singular support is a conic Lagrangian i.e. stable under  $\mathbb{R}_+$ -dialation along contangent fibers.

Fix a conic Lagrangian  $L \subset T^*M$ , then we define  $Sh_L^\bullet(M; \mathbb{C})$  to be the full subcate-

gory of  $Sh_L^\bullet(M; \mathbb{C})$  whose objects are sheaves singular supported in  $L$ . If  $\Lambda \subset T^\infty M$  is a Legendrian, then we define  $Sh_\Lambda^\bullet(M; \mathbb{C}) := Sh_{\mathbb{R}_+ \Lambda \cup 0_M}^\bullet(M; \mathbb{C})$  where  $0_M \subset T^*M$  is the zero section.

## Combinatorial Model of $Sh_\Lambda^\bullet(M; \mathbb{C})$

The first necessary condition for  $\mathcal{F} \in Sh^\bullet(M; \mathbb{C})$  to belong to  $Sh_L^\bullet(M; \mathbb{C})$  is that  $\mathcal{F}$  is constructible with respect to the stratification on  $M$  induced by the front projection of  $\Lambda$ . That is the Whitney stratification whose zero dimensional strata are crossings, one dimensional strata are the arcs of  $\pi(\Lambda)$  between crossings and two dimensional strata are the connected components of the complement of  $\pi(\Lambda)$  in  $M$ .

**Definition 1.** Given a stratification  $\mathcal{S}$ , the star of a stratum  $s \in \mathcal{S}$  is the union of strata that contain  $s$  in their closure. We view  $\mathcal{S}$  as a poset category  $s \leq t$  if and only if  $s \subset \bar{t}$ . We call the map from  $s$  to  $t$  as a generization map. We say  $\mathcal{S}$  is a regular cell complex if every stratum is contractible and moreover the star of each stratum is contractible.

**Definition 2.** We define  $Fun_{naive}^\bullet(\mathcal{S}, \mathbb{C})$  to be the dg category of functors from  $\mathcal{S}$  to the category whose objects are cochain complexes of  $\mathbb{C}$ -vector spaces and whose maps are cochain maps. We define  $Fun^\bullet(\mathcal{S}, \mathbb{C})$  the dg quotient [Dri04] of  $Fun_{naive}^\bullet(\mathcal{S}, \mathbb{C})$  by the thick subcategory of functors taking values in acyclic complexes.

**Proposition 3.** [Kas84], [She85], [Nad09, Lemma 2.3.2]. Let  $\mathcal{S}$  be a Whitney stratification of  $M$ . Consider the functor

$$\Gamma_{\mathcal{S}} : Sh_{\mathcal{S}}^\bullet(M; \mathbb{C}) \rightarrow Fun^\bullet(\mathcal{S}, \mathbb{C})$$

$$\mathcal{F} \mapsto [s \mapsto \Gamma(star(s); \mathcal{F})]$$

If  $\mathcal{S}$  is a regular cell complex, then  $\Gamma_{\mathcal{S}}$  is a quasi-equivalence.

However, the stratification induced by the front diagram is not necessarily a regular cell complex. Therefore, we choose an  $\mathcal{S}$  that refines of the stratification induced by  $\pi(\Lambda)$ . In figures, we draw the new line segments in squiggly lines.

The restriction of  $\Gamma_{\mathcal{S}}$  to  $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$  is quasi-fully faithful. We will describe the essential image of  $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$  under  $\Gamma_{\mathcal{S}}$ .

**Definition 4.** Let  $\mathcal{S}$  be a regular cell complex refining the stratification induced by the front diagram. we define  $Fun_{\Lambda}^{\bullet}(\mathcal{S}, \mathbb{C})$  to be the full subcategory of  $Fun^{\bullet}(\mathcal{S}, \mathbb{C})$  whose objects satisfy the following properties.

- every map from a zero dimensional stratum in  $\mathcal{S}$  which is not a crossing is sent to quasi-isomorphism.
- every map from a one dimensional stratum which is not contained in an arc is sent to quasi-isomorphism.
- for each crossing  $c \in \mathcal{S}$ , Let  $N, E, S, W$  be the north, east, south, and west regions adjoining  $c$ , and let  $nw, ne, sw, se$  be the arcs separating them.

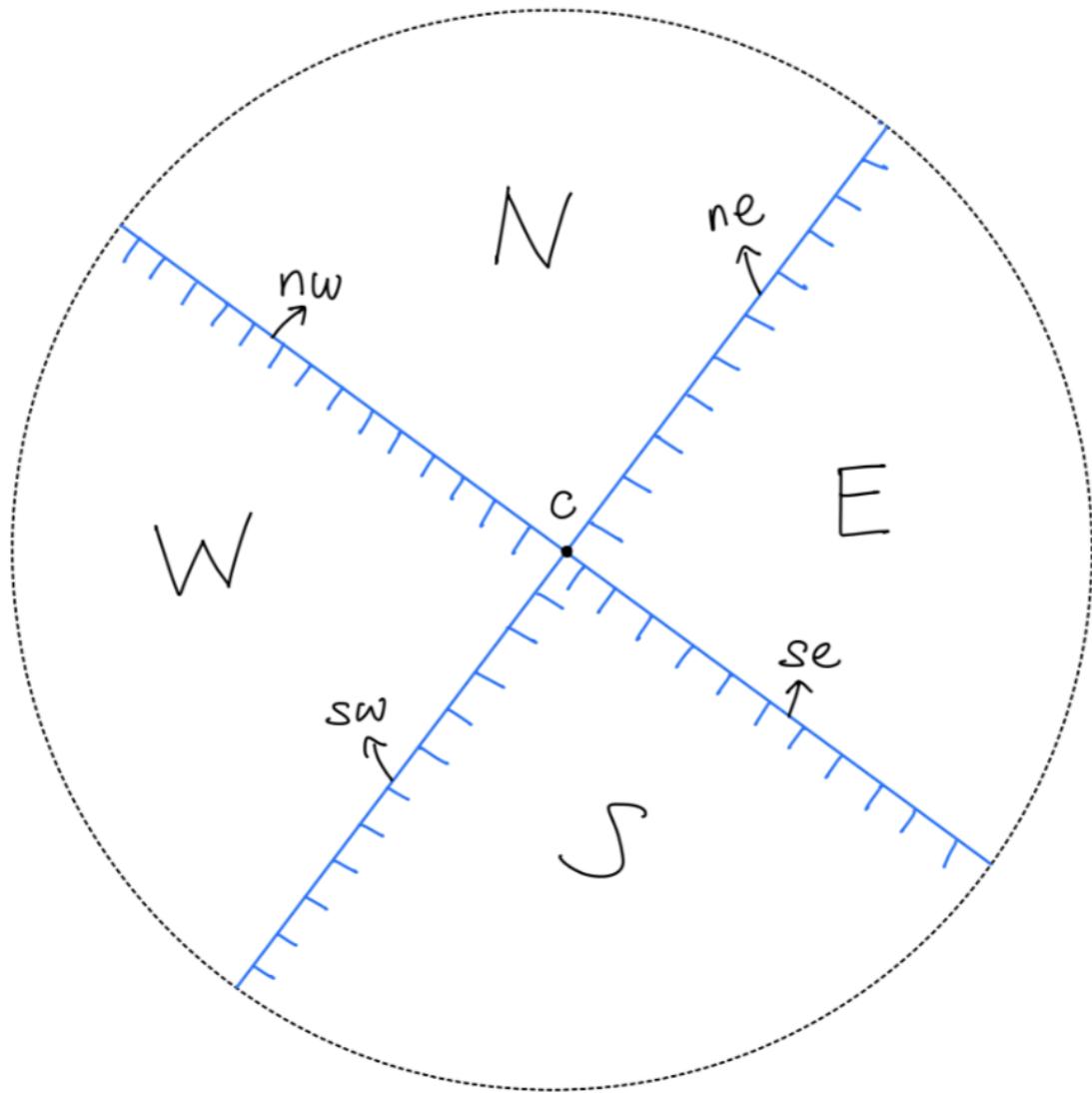
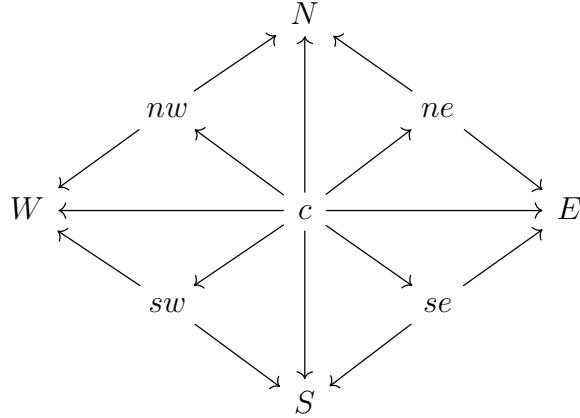


Figure 1.1

Then the following maps are sent to quasi-isomorphisms.

- maps between  $c$ ,  $sw$ ,  $se$ , and  $S$ .
- a map from  $nw$  to  $W$ .
- a map from  $ne$  to  $E$ .
- for each crossing  $c \in \mathcal{S}$ , we restrict the poset structure to the strata containing

$c$  in its closure, we get



Then all triangles in the diagram should commute and the total complex of the following bicomplex

$$F(c) \rightarrow F(nw) \oplus F(ne) \rightarrow F(N)$$

should be acyclic.

A standard argument shows that the essential image of  $Sh_{\Lambda}^{\bullet}(M, \mathbb{C})$  under  $\Gamma_S$  is  $Fun_{\Lambda}^{\bullet}(\mathcal{S}, \mathbb{C})$  [STZ17].

## Legible Object

Let  $M$  be a front surface,  $\Phi$  a front diagram,  $\Lambda \subset T^{\infty}M$  the associated Legendrian knot, and  $\mathcal{S}$  a refinement of the stratification induced by  $\Phi$  by adding squiggly lines. In this section, we define a “legible quiver” and a “squiggly legible diagram” to get a better handle on  $Fun_{\Lambda}^{\bullet}(M; \mathbb{C})$ .

Suppose we impose an auxiliary co-orientation on the squiggly lines as well, then this induces a quiver  $Q$

**Definition 5.** Suppose we have a regular cell complex  $\mathcal{S}$  that refines the stratification

induced by  $\Phi$  with auxiliary co-orientations on the added squiggly lines segments, then we define the associated quiver  $Q$  where

- the vertices of  $Q$  are 2 dimensional strata
- the arrows of  $Q$  are 1 dimensional strata where the sources and the targets are regions below and above the strata.

For a stratum  $s \in \mathcal{S}$ , we define  $Q_s$  to be the full sub-quiver of  $Q$  whose vertices are the regions incident with  $s$ .

**Definition 6.** We say that the quiver associated to  $\mathcal{S}$  is *legible* if for every point stratum  $p \in \mathcal{S}$ , the sub-quiver  $Q_p$  is a poset with the smallest element.

**Definition 7.** A squiggly legible diagram on  $\mathcal{S}$  is the representation  $F^\bullet$  of the quiver  $Q$  associated to  $\mathcal{S}$  valued in the cochain complex of  $\mathbb{C}$ -vector spaces subject to the condition that

- the maps corresponding to squiggly lines are quasi-isomorphisms.
- for fixed a source and a target vertices  $s$  and  $t$ , suppose we have two distinct paths in  $Q$  (a sequence of arrows) from  $s$  to  $t$ , say  $(a_1, a_2, \dots, a_k)$  and  $(a'_1, a'_2, \dots, a'_l)$ , then  $F^\bullet(a_k) \circ \dots \circ F^\bullet(a_1) = F^\bullet(a'_l) \circ \dots \circ F^\bullet(a'_1)$  i.e. the composition of cochain maps are path independent.
- at each crossing, we have surrounding region  $N, S, W, E$

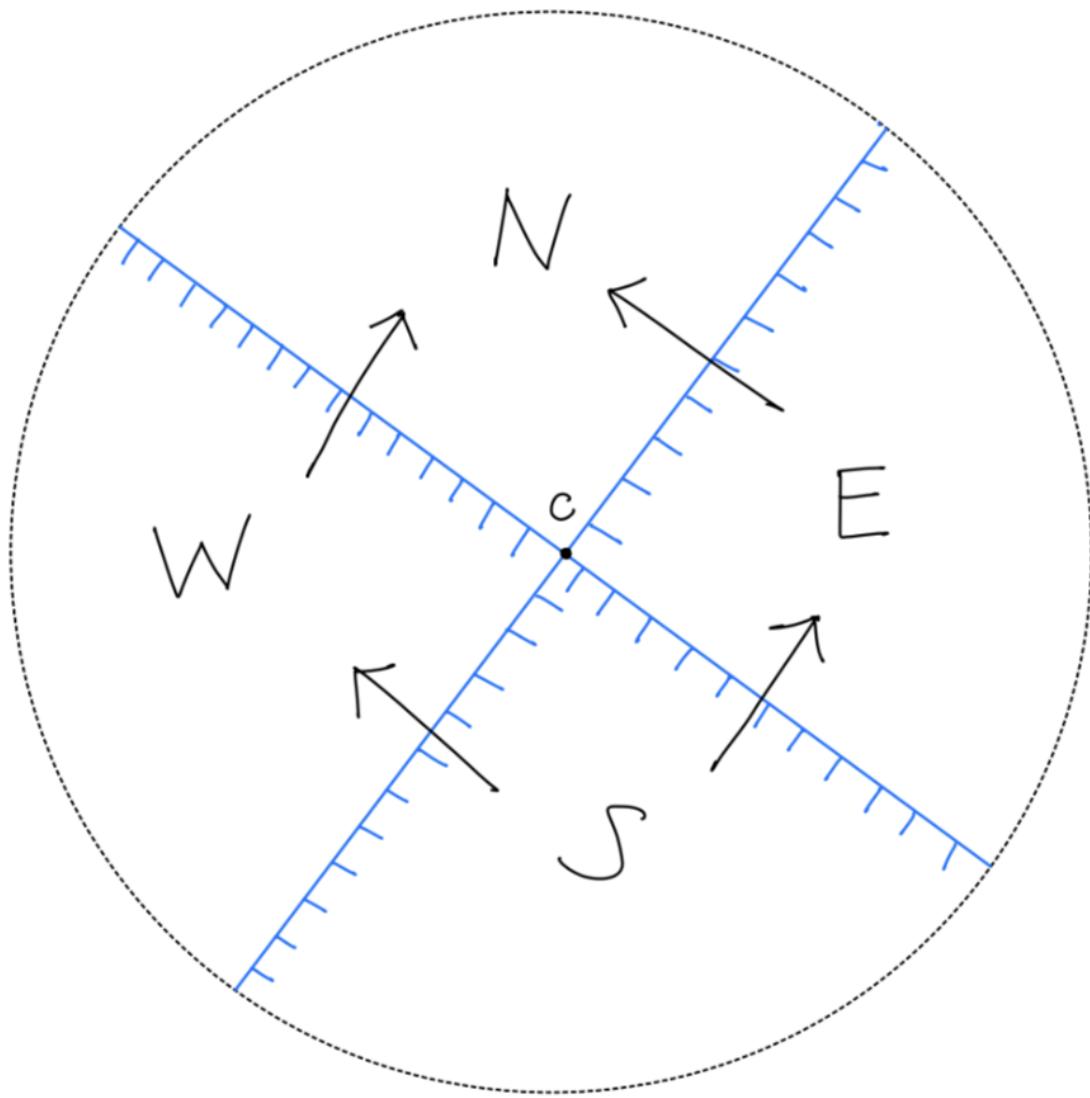


Figure 1.2

then the total complex of

$$F^\bullet(S) \rightarrow F^\bullet(W) \oplus F^\bullet(E) \rightarrow F^\bullet(N)$$

is acyclic.

Suppose we have  $\mathcal{S}$  a refinement of the stratification induced by  $\Phi$  and a squiggly legible diagram  $F^\bullet$ , then we can make an object  $\bar{F}^\bullet \in \text{Fun}_\Lambda^\bullet(\mathcal{S}, \mathbb{C})$  out of it in the following way.

**Definition 8.**  $\rho : \mathcal{S} \rightarrow \{R \in \mathcal{S} \mid \dim_{\mathbb{R}}(R) = 2\}$  is a function that assigns  $s \in \mathcal{S}$  to the smallest element in  $\{R \in \text{Vert}(Q) \mid s \subset \bar{R}\}$ .

**Definition 9.** Suppose  $F^\bullet$  is a squiggly legible diagram on  $\mathcal{S}$ , then we define  $\bar{F}^\bullet \in \text{Fun}_\Lambda^\bullet(\mathcal{S}, \mathbb{C})$  where

- for  $s \in \mathcal{S}$ ,  $\bar{F}^\bullet(s) := F^\bullet(\rho(s))$
- for  $s_1, s_2 \in \mathcal{S}$  such that  $s_2 \subset \text{star}(s_1)$ ,  $\bar{F}^\bullet(s_1 \rightarrow s_2) := F^\bullet(a_k) \circ \cdots F^\bullet(a_1)$  where  $(a_1, a_2, \dots, a_k)$  is a path from  $\rho(s_1)$  to  $\rho(s_2)$  in the quiver associated to  $\mathcal{S}$ . This is well-defined because the composition of cochain maps are path independent by the definition of squiggly legible diagram and the existence of a path from  $\rho(s_1)$  to  $\rho(s_2)$  is guaranteed by the fact that  $Q_{s_2}$  is a subquiver of  $Q_{s_1}$  and  $\rho(s_2), \rho(s_1)$  are the smallest elements from each of them.

In this paper, we only consider  $\Phi$ ,  $\Lambda$ , and  $\mathcal{S}$  that objects of  $\text{Fun}^\bullet(\mathcal{S}, \mathbb{C})$  arises from squiggly legible diagrams. A variant of [STZ17, Prop 3.22.] shows that every objects of  $\text{Sh}_\Lambda^\bullet(M; \mathbb{C})$  is quasi-isomorphic to a squiggly legible diagram.

## Microllocal Monodromy and Rank 1 Object

In this section, we review the notion of microllocal monodromy of sheaves following [STZ17, Section 5.1]. For more general treatment, see [KS13, Ch. IV]. Suppose on  $M$  we have a front diagram  $\Phi$  and a stratification  $\mathcal{S}$  that refines the one induced by  $\Phi$  by adding squiggly line segments. Suppose we are given a sheaf  $F^\bullet$  in terms of squiggly legible diagram, we define the associated sheaf on  $\Lambda$ , called microllocal monodromy denoted  $\mu\text{mon}(F^\bullet)$ , by assigning cochain complexes of  $\mathbb{C}$ -vector spaces for each one dimensional stratum contained in  $\Phi$  and cochain maps between them for each point stratum contained in  $\Phi$  as follows:

- if  $s$  is a one dimensional stratum contained in  $\Phi$ ,  $\mu\text{mon}(F^\bullet)(s) := \text{Cone}(F^\bullet(s))$

- if  $p$  is a point stratum in  $\Phi$ , then  $p$  is either a crossing or the endpoint of a squiggly line. We define the cochain map for each case as follows:
  - when  $p$  is a crossing, locally near  $p$  the diagram looks like

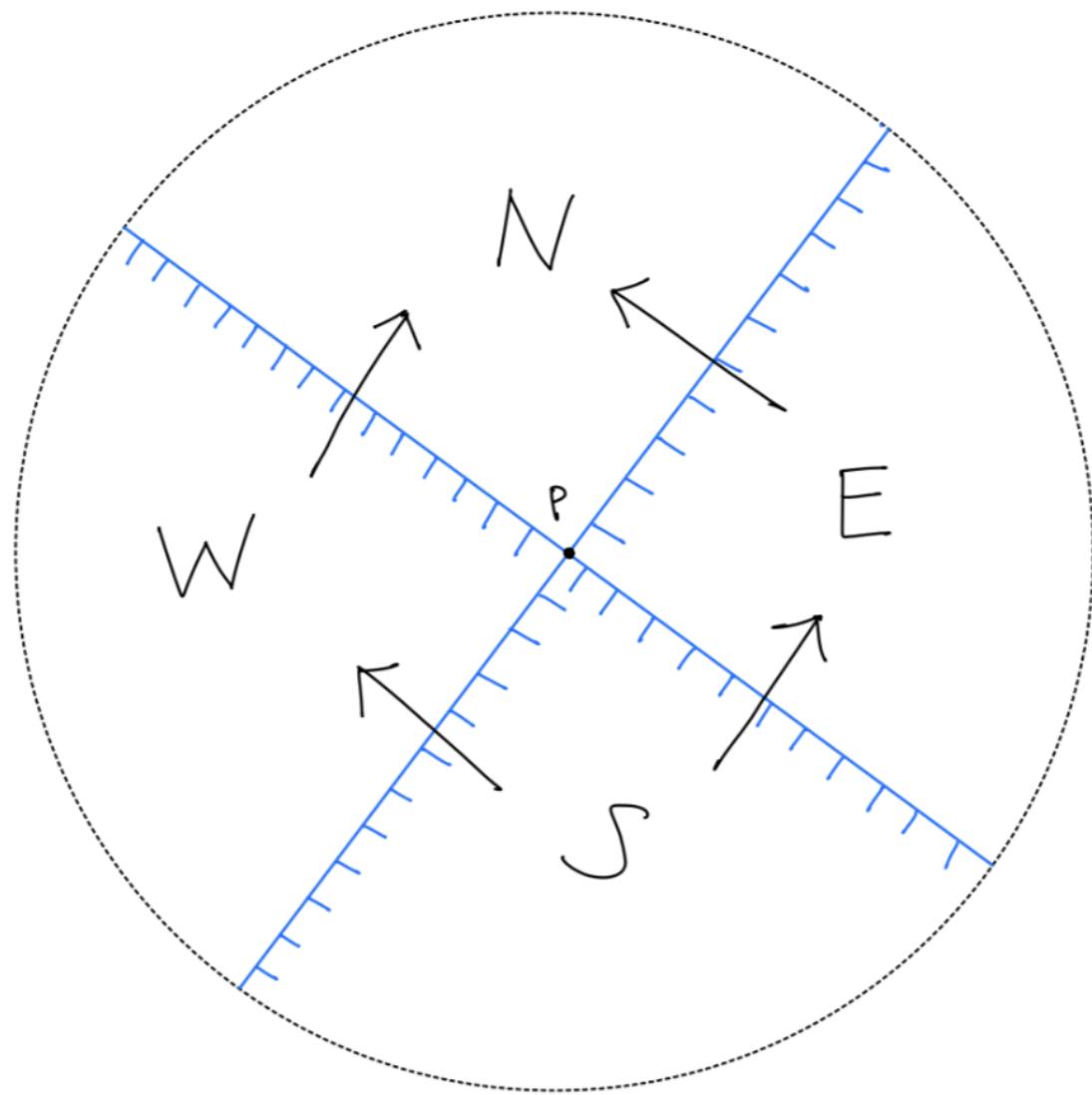


Figure 1.3

then  $p$  induces maps

$$\text{Cone}(S \rightarrow W) \rightarrow \text{Cone}(E \rightarrow W)$$

$$\text{Cone}(S \rightarrow E) \rightarrow \text{Cone}(W \rightarrow N)$$

- when  $p$  is the endpoint of a squiggly line segment, locally near  $p$  the diagram looks like one of the following diagrams

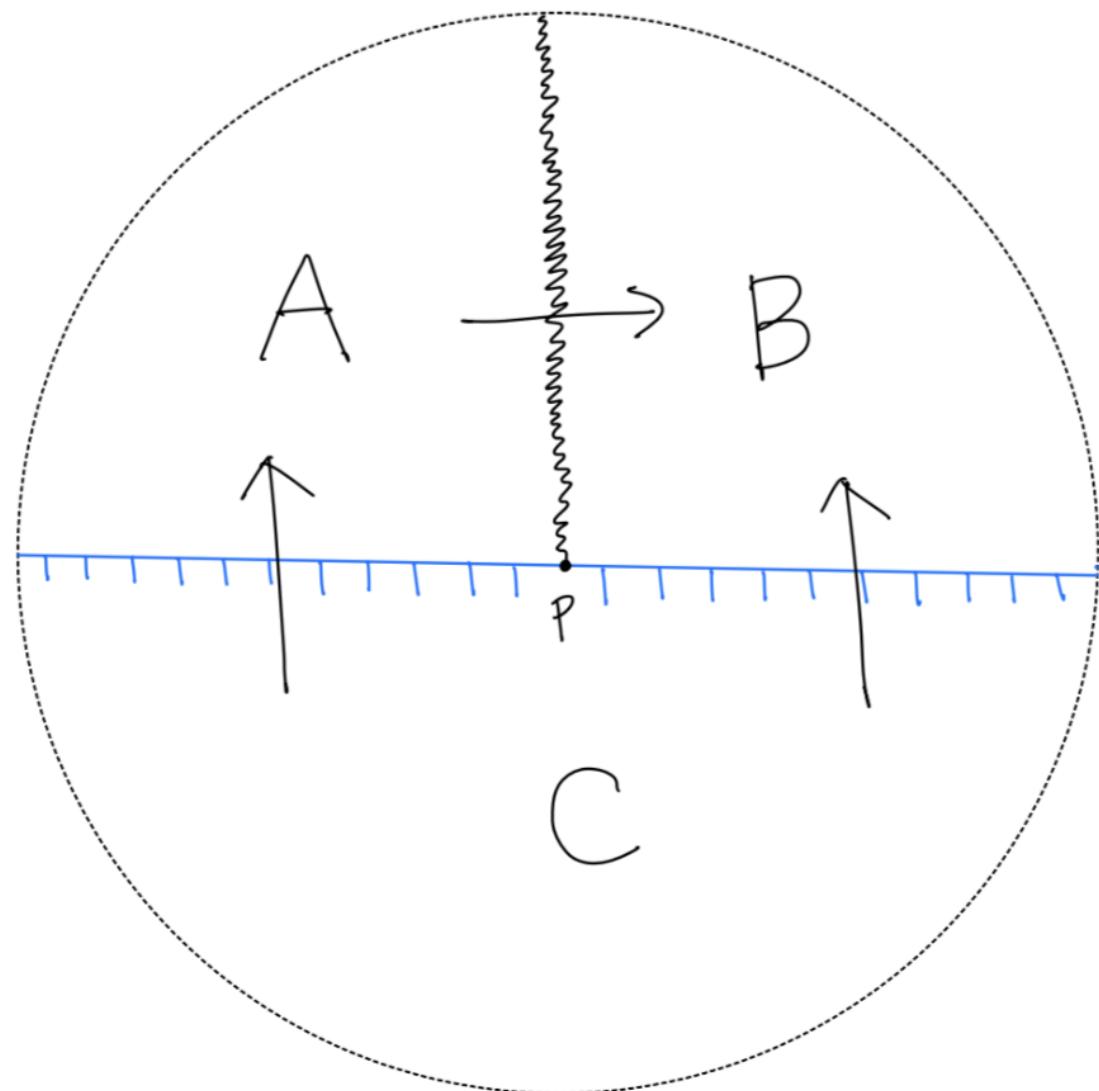


Figure 1.4

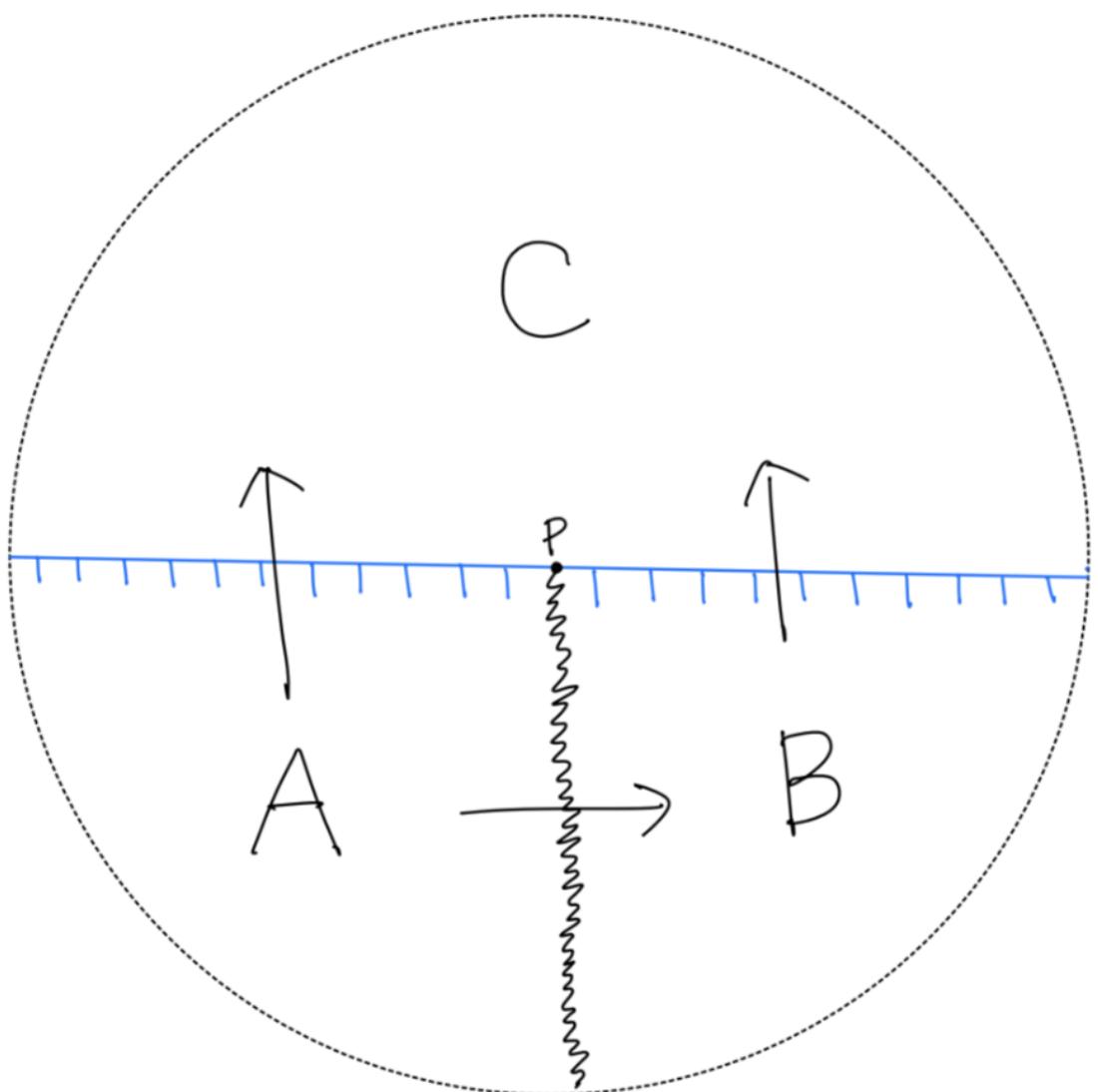


Figure 1.5

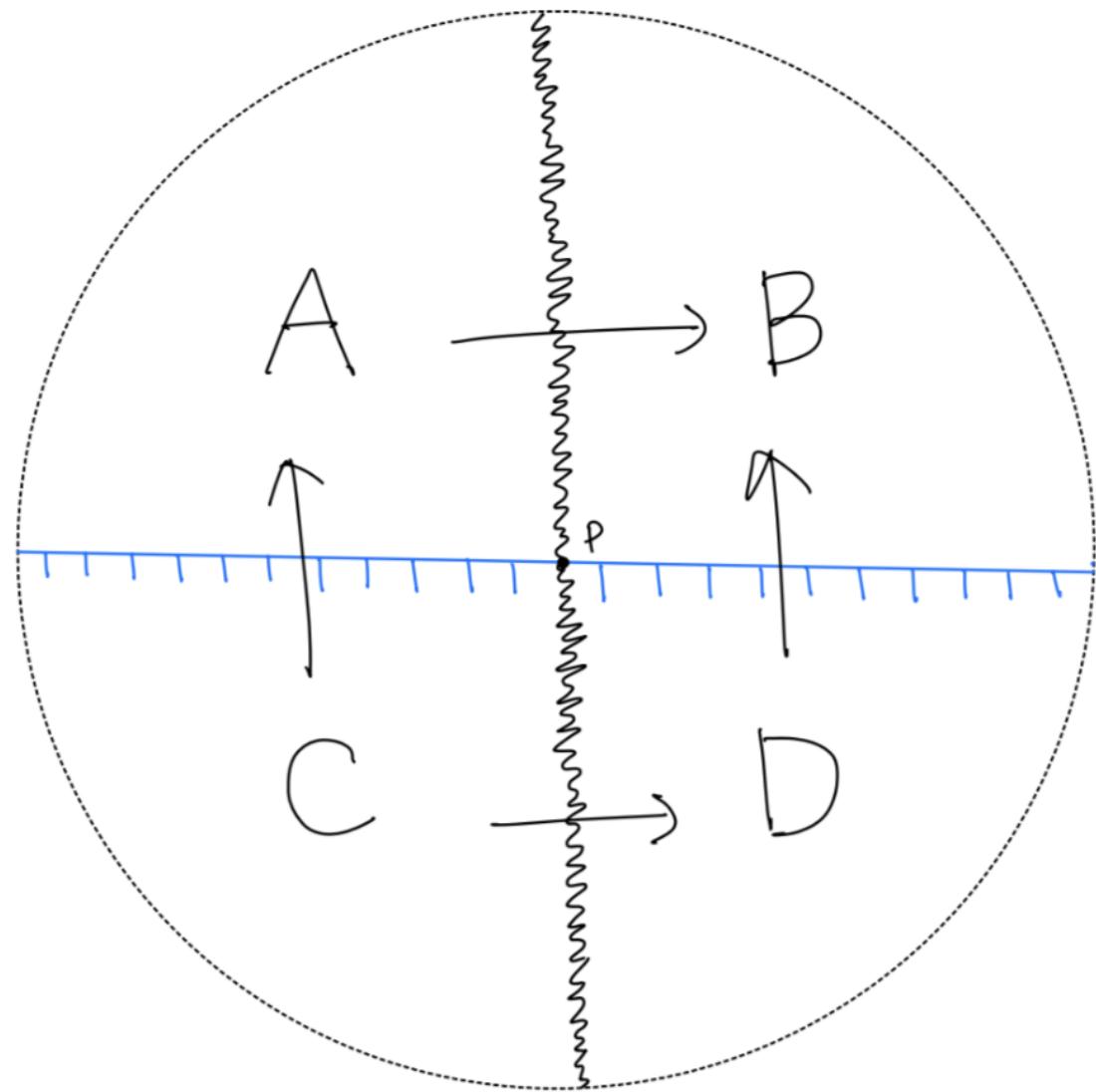


Figure 1.6

, then we have a map

$$\text{Cone}(C \rightarrow A) \rightarrow \text{Cone}(C \rightarrow B)$$

$$\text{Cone}(A \rightarrow C) \rightarrow \text{Cone}(B \rightarrow C)$$

$$\text{Cone}(C \rightarrow A) \rightarrow \text{Cone}(D \rightarrow B)$$

respectively.

The maps between cones are quasi-isomorphisms because of the crossing condition and the fact that the map corresponding to squiggly lines are quasi-isomorphism.

**Definition 10.**  $F^\bullet$  is called *rank n object* if  $\mu\text{mon}(F^\bullet)$  is a rank  $n$  local system concentrated in degree 0 on  $\Lambda$ .

**Definition 11.** The full subcategory of  $Sh_\Lambda^\bullet(M; \mathbb{C})$  containing rank  $n$  objects is denoted  $\mathcal{C}_n(M, \Lambda; \mathbb{C})$  and the moduli stack classifying such objects  $\mathcal{M}_n(M, \Lambda; \mathbb{C})$ . Additionally, let  $\{\sigma_1, \dots, \sigma_k\}$  be points in  $M$  that are contained in one of the regions separated by  $\Phi$ . Then we define  $\mathcal{C}_n(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$  to be the full subcategory of sheaves with vanishing stalks at  $\{\sigma_1, \dots, \sigma_k\}$ ,  $\mathcal{M}(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$  the moduli space classifying such objects.

## Moduli Spaces associated to Positive Braids

In this section, we define the moduli spaces associated to positive braids following [STWZ19, Section 3]. For background we refer to the survey [Toë14] and the foundational works [Lur04] [Toë09] [TV04] [TV05] [TV08]. Suppose we have a positive braid word  $\omega$  of  $n$  strands(i.e. a word freely generated by  $s_1, \dots, s_{n-1}$ ), Then we can think of its cylindrical closure in  $S_\theta^1 \times (0, 1)_r = (\mathbb{R}_\theta / \mathbb{Z}) \times (0, 1)_r$ . In this paper, we consider two kinds of moduli spaces associated to the braid which are the main objects of study in Chapter2 and Chapter3 respectively.

- (1) First, we embed the cylinder containing cylindrical closure of  $\omega$  in  $M = S_x^1 \times \mathbb{R}_z$  via  $(\theta, r) \mapsto (\theta, r - 1)$ . We can think of the cylindrical closure of  $\omega$  in  $S_x^1 \times \mathbb{R}_z$  co-oriented downward( $z < 0$ ) to be the front projection of a Legendrian knot  $\Lambda_\omega$  living in  $T^\infty M$ . Let  $\sigma_{z \ll 0}$  be a point in the non-compact region where  $z \ll 0$ . In Chapter2, we consider the moduli space  $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z \ll 0}\}; \mathbb{C})$ .
- (2) Next, we embed the cylinder containing the cylindrical closure of  $\omega$  in  $M = S_x^1 \times \mathbb{R}_z$  via  $(\theta, r) \mapsto (\theta, 1 - r)$  and embed the cylinder containing the cylindrical

closure of the trivial braid  $\omega_\emptyset$  via  $(\theta, r) \mapsto (\theta, r-1)$ . We get an embedding of the cylindrical closure of  $\omega \coprod \omega_\emptyset$  in  $S_x^1 \times \mathbb{R}_z$  where the embedding of the cylindrical closure of  $\omega$  is co-oriented upward and the embedding of the cylindrical closure of  $\omega_\emptyset$  is co-oriented downward. We will call the embedding the separated diagram of  $\omega$  and  $\omega_\emptyset$  which we consider as the front projection of a Legendrian link  $\Lambda_\infty \coprod \Lambda_0 \subset T^\infty M$ . Let  $\sigma_{z \ll 0}, \sigma_{z \gg 0}$  be points in the non-compact regions where  $z \ll 0$  and  $z \gg 0$ . In Chapter 3, we consider the moduli space  $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z \ll 0}, \sigma_{z \gg 0}\}; \mathbb{C})$ .

## Legendrian Isotopy and Sheaf Cobordism

Invariance of the category  $\mathcal{C}_1(M, \Lambda, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$  under Legendrian isotopy follows from the main theorem of [GKS12].

**Theorem 12.** [GKS12]. Suppose  $\Lambda_0 \subset T^\infty M$  is a Legendrian and  $\Lambda_\bullet \subset T^\infty \times [0, 1]$  a Legendrian isotopy where  $\Lambda_\bullet|_{T^\infty \times \{0\}}$  is equal to  $\Lambda_0$  under the natural identification  $T^\infty \times \{0\} \cong T^\infty$ , then the restriction functor

$$Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C}) \rightarrow Sh_{\Lambda_0}^\bullet(M; \mathbb{C})$$

is a quasi-equivalence. Furthermore, this quasi-equivalence preserves rank  $n$  objects.

**Definition 13.** Suppose we have a Legendrian isotopy  $\Lambda_\bullet \subset T^\infty \times [0, 1]$  from  $\Lambda_0 = \Lambda_\bullet|_{T^\infty \times \{0\}}$  to  $\Lambda_1 = \Lambda_\bullet|_{T^\infty \times \{1\}}$  and a sheaf  $\mathcal{F}_\bullet \in Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C})$  that restricts to  $\mathcal{F}_0$  and  $\mathcal{F}_1$  at  $M \times \{0\}$  and  $M \times \{1\}$ , then we say  $\mathcal{F}_\bullet$  is a *sheaf cobordism* from  $\mathcal{F}_0$  to  $\mathcal{F}_1$ .

**Definition 14.** Suppose  $\Lambda_0, \Lambda_1 \subset T^\infty M$  are Legendrian links and there is a Legendrian isotopy  $\Lambda_\bullet \subset T^{\infty,-} M \times [0, 1]$  where  $\Lambda_\bullet|_{T^{\infty,-} \times \{0\}}$  is equal to  $\Lambda_0$  under the natural identification  $T^{\infty,-} \times \{0\} \cong T^{\infty,-}$  and  $\Lambda_\bullet|_{T^{\infty,-} \times \{1\}}$  is equal to  $\Lambda_1$  under the natural

identification  $T^{\infty,-} \times \{1\} \cong T^{\infty,-}$ , then we have quasi-equivalence

$$Sh_{\Lambda_0}^\bullet(M; \mathbb{C}) \xrightarrow{\sim} Sh_{\Lambda_1}^\bullet(M; \mathbb{C})$$

from the following correspondence

$$Sh_{\Lambda_0}^\bullet(M; \mathbb{C}) \xleftarrow{\sim} Sh_{\Lambda_\bullet}^\bullet(M \times [0, 1]; \mathbb{C}) \xrightarrow{\sim} Sh_{\Lambda_1}^\bullet(M; \mathbb{C})$$

Furthermore, this quasi-equivalence restricts to

$$\mathcal{C}_n(M, \Lambda_0; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}(M, \Lambda_1; \mathbb{C})$$

When we have points  $\{\sigma_1, \dots, \sigma_k\} \subset M$  that are disjoint from the front projections of the Legendrian isotopy  $\Lambda_\bullet$ , we can further restrict the quasi-equivalence to

$$\mathcal{C}_n(M, \Lambda_0, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C}) \xrightarrow{\sim} \mathcal{C}_n(M, \Lambda_1, \{\sigma_1, \dots, \sigma_k\}; \mathbb{C})$$

## Alternating Legendrian

**Definition 15.** Let  $M$  be a surface and  $\Lambda \subset T^\infty M$  a Legendrian link. An *alternating coloring* for  $\Lambda$  is the datum of, for each region in the complement of the front diagram, a label of black, white, or null, subject to the following conditions

- the boundary of a black region is co-oriented inward.
- the boundary of a white region is co-oriented outward.
- the boundary of the null region have both inward and outward co-orientations.
- no black region shares a one dimensional border with white region and no null region shares a one dimensional border with another null region.

An *alternating Legendrian* is a Legendrian equipped with an alternating coloring and their front projection is called an *alternating strand diagram*. The *bipartite graph* of the alternating Legendrian is the graph whose vertices are black and white regions. Edges are connected if their closure intersect and are of distinct color.

Let  $\hat{M}$  denote the real blow up of  $M$  at the crossings of the front projection of  $\Lambda$ . The blow down map  $\hat{M} \rightarrow M$  is a diffeomorphism away from the crossing and the fiber above a crossing is the  $\mathbb{RP}^1$  of lines tangent to the crossing. We define  $W \subset M$  ( $B \subset M$  resp.) the union of the interiors of the white(black resp.) regions of the complement of the front projection.

**Definition 16.**  $\bar{L}$  denote the closure of the preimage of  $W \cup B$  in  $\hat{M}$ . It is a smooth surface with boundary and we refer to its interior  $L$  as the *conjuate surface* of  $\Lambda$ .

**Definition 17.** An *alternating sheaf* is an object of  $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$  whose support is contained in the closure of the union of the white and black regions.

**Theorem 18.** [STWZ19, Thm. 4.16][STWZ19, Cor. 4.17]. The full subcategory of  $Sh_{\Lambda}^{\bullet}(M; \mathbb{C})$  consisting of alternating sheaves is equivalent to the category of locally constant sheaves on  $L$ . Under this correspondence, rank 1 local systems of corresponds to rank 1 alternating sheaves.

## 1.2 Summary of results

### Chapter 2

We define a class of braids whose associated moduli spaces  $\mathcal{M}_{\omega} := \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_{\omega}, \{\sigma_{z << 0}\}; \mathbb{C})$  have simple descriptions, called Sibuya braids. In stead of providing definition, I will illustrate the concept with an example. Consider a braid word  $\omega = (s_1 s_2)^2$  on 3 strands, which is a power of the Coxeter braid. Then we have an embedding of  $\omega$  into the fundamental domain of  $S_x^1 \times \mathbb{R}_z$  given in the below figure

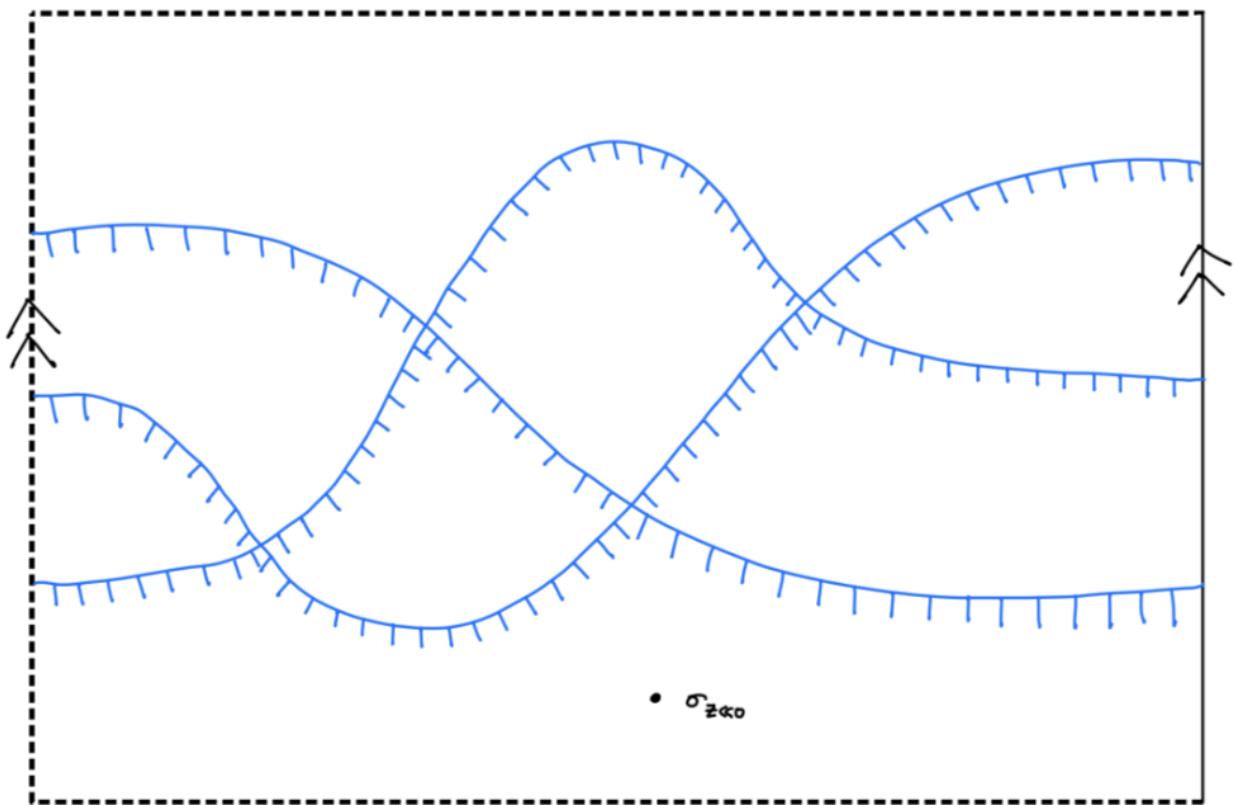


Figure 1.7

Since the stratification induced by  $\omega$  is not a regular cell complex, for example non-compact regions at the top and the bottom are not contractible, we refine it by adding squiggly lines co-oriented towards left

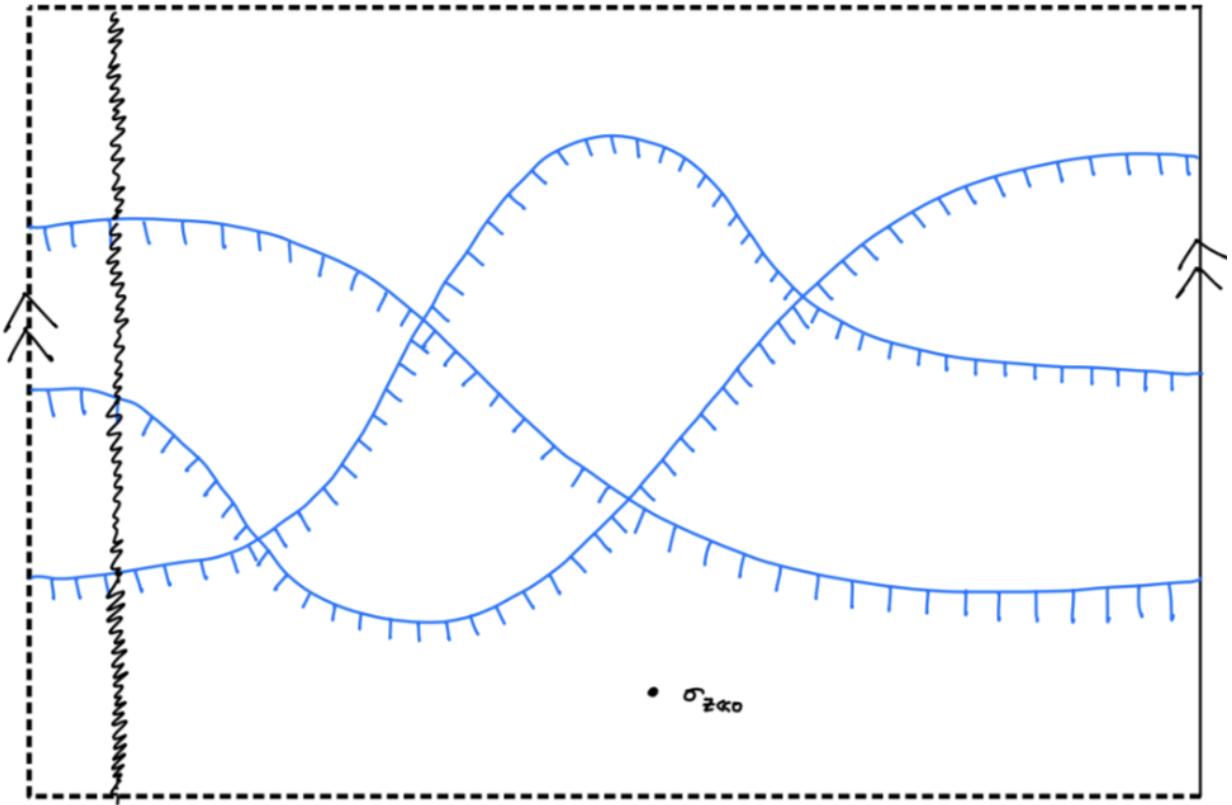


Figure 1.8

Consider an object of  $C_1(S^1_x \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z<<0}\}; \mathbb{C})$  which could be described as a squiggly legible diagram by assigning cochain complexes of  $\mathbb{C}$ -vector spaces to regions and cochain maps to arcs subject to the conditions mentioned in (legible object subsection). Let's check how those conditions translates to our situation:

- the cochain complex assigned to the bottom region, which we call height 0 region, is acyclic because of the vanishing condition at  $\sigma_{z<<0}$ . We are considering cochain complexes of sheaves upto quasi-isomorphism, so we assign 0 stalk to the region.
- the stalk at the regions adjacent to the height 0 region that are not a height 0 region, which we call hieght 1 regions, should be quasi-isomorphic to dimension 1 vector space  $\mathbb{C}$  because of the microlocal rank 1 condition on blue arcs separating height 0 region and height 1 regions i.e. the mapping cone of the cochain map

corresponding to blue arcs should be rank 1 vector spaces concentrated in degree 0. Therefore, we assign  $\mathbb{C}$  to these height 1 regions and incoming maps from the height 0 region zero maps.

- the stalk at the regions adjacent to the height 1 regions that are not height 0, 1 regions, which we call height 2 regions, should be quasi-isomorphic to dimension 2 vector space  $\mathbb{C}^2$  and incoming maps from the height 1 region to be monomorphisms because of the microlocal rank 1 condition again. Therefore, we assign  $\mathbb{C}^2$  to the height 2 regions and incoming maps from the height 1 regions to be inclusion maps.
- Now, we are left with one non-compact region at the top which we call height 3 region. Again by microlocal rank 1 condition, the stalk should be  $\mathbb{C}^3$  and the incoming maps from height 2 regions are inclusion maps.
- the maps corresponding to squiggly lines are isomorphisms of adjacent vector spaces.
- all diagrams should commute.
- for each crossing the crossing condition translates to

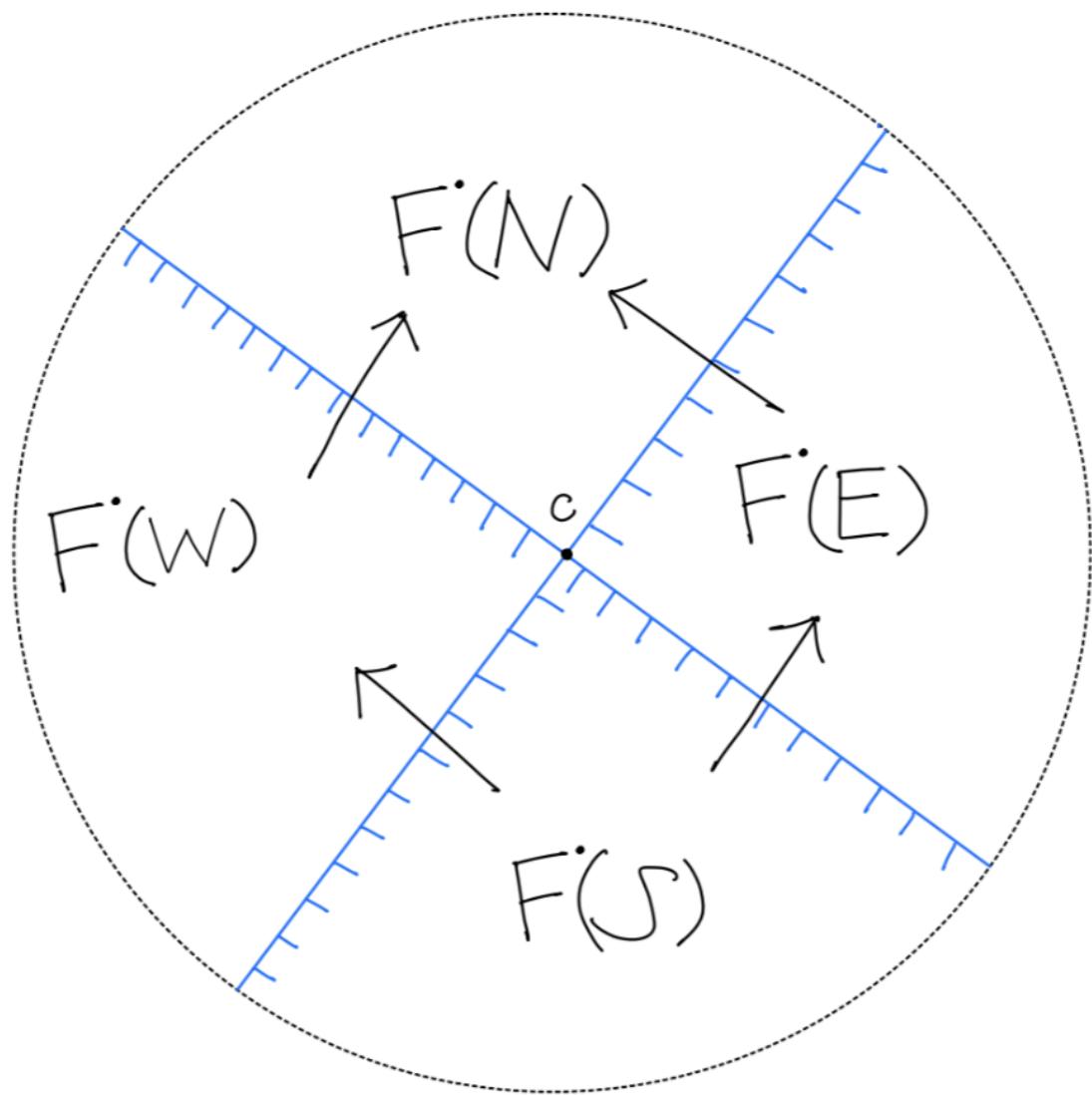


Figure 1.9

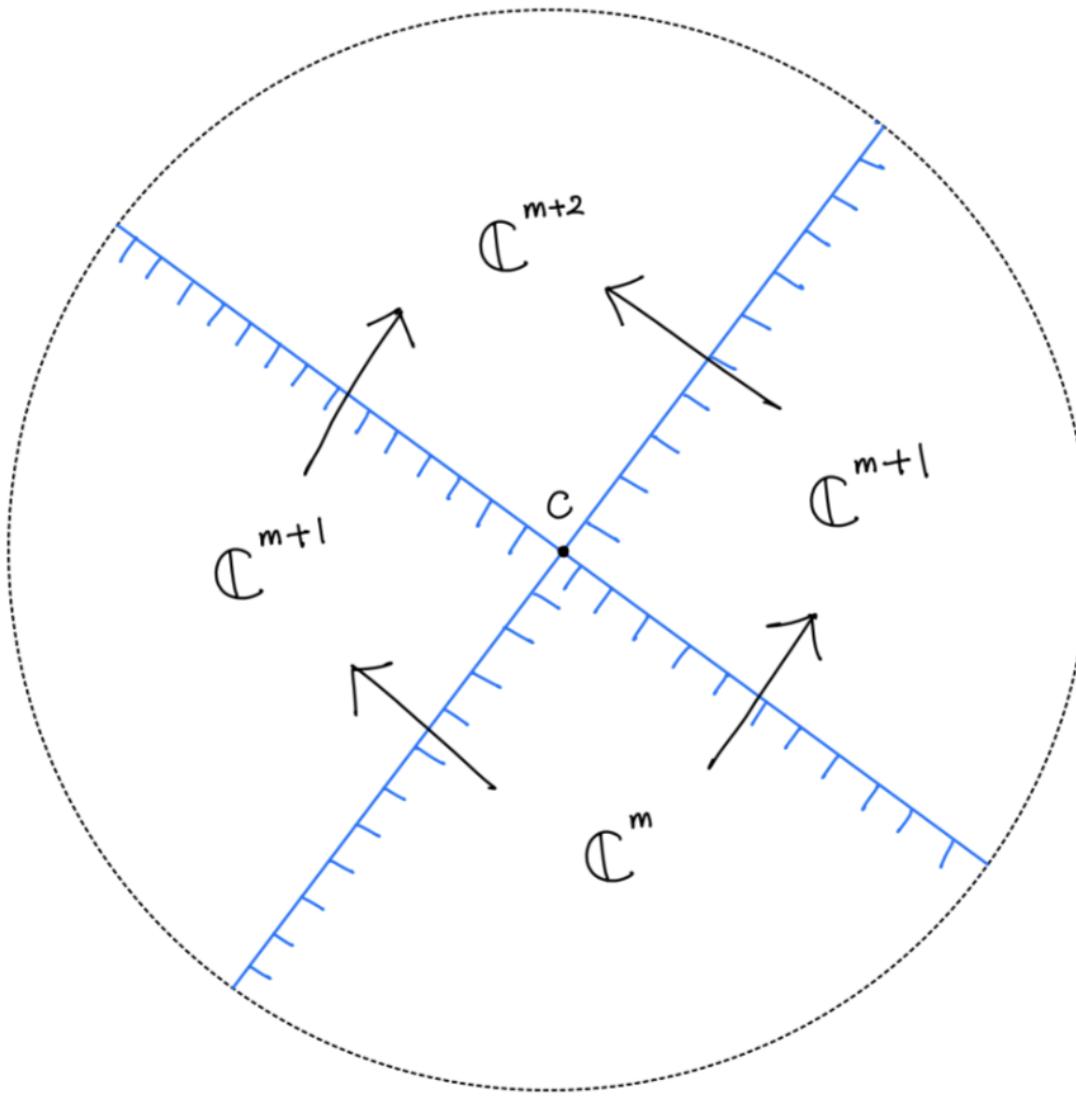


Figure 1.10

the inclusion of  $F^\bullet(W)$  and  $F^\bullet(E)$  in  $F^\bullet(N)$  intersects transversely in  $F^\bullet(S)$ .

In summary, an object of  $C_1(S_x^1 \times \mathbb{R}_z, \Lambda, \{\sigma_{z << 0}\}; \mathbb{C})$  amounts to the data of assigning

- dimension  $i$  subspaces of  $\mathbb{C}^3$  to height  $i$  regions subject to the condition that  $F^\bullet(W) + F^\bullet(E) = F^\bullet(N)$  and  $F^\bullet(W) \cap F^\bullet(E) = F^\bullet(S)$ .
- an element  $g \in GL_3(\mathbb{C})$  that maps the subspaces corresponding to the region to the left of squiggly lines to the subspaces corresponding to the region to the

right of the squiggly lines.

One thing to note in the above example is that once we assign the dimension 1 subspaces to the height 1 regions, all the other subspaces are determined by the crossing conditions. More precisely, we can construct subspaces that are assigned to the regions of heights greater than 1 as follows: Suppose  $R$  is a region, then the subspace we assign to  $R$  is the subspace of  $\mathbb{C}^3$  spanned by the 1 dimensional subspaces assigned to the height 1 regions that have paths to  $R$ . I will describe what this means in Chapter2. This fact(the sheaf is completely determined by its rank 1 stalks) relies on the fact that the diagram induced by  $\omega$  has the property that every region of height bigger than 1 has more than or equal to 2 different regions of height 1 less adjacent to it. We call a braid word with the above property as Sibuya braid word and a braid with Sibuya braid word representatio as Sibuya braid.

But also note that not all assignments of 1 dimensional subspaces of  $\mathbb{C}^3$  to height 1 regions gives rise to legitimate sheaves. In Chapter2, I will show how to restate the crossing conditions using only rank 1 stalks.

Under this interpretation, the moduli space could be thought of as parametrizing configurations of points on a Projective space subject to certain non-degeneracy conditions and could be thought of as a generalization of Sibuya space that Boalch studied in [Sib75][Boa15].

At the end of the Chapter2, we will explicitly compute examples of the generalized Sibuya space.

## Chapter 3

We study a special type of coordinate system of the moduli space  $\mathcal{M}_\omega \coprod_{\omega_\emptyset} := \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$  called cluster coordinates.

Let's review how cluster coordinates are constructed. Suppose we have a n alternating Legendrian  $\Lambda_{alt}$  that are Legendrian isotopic to  $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset}$  via a Legendrian isotopy

$\Lambda_\bullet$  whose front projection does not touch  $\{\sigma_{z<<0}, \sigma_{z>>0}\}$ , then , by GKS thoerem, induces an isomorphism between moduli spaces

$$\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C}) \rightarrow \mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

Furthermore, once we have alternating Legendrian, we can construct its conjugate surface  $L$  which deformation retracts to the bipartite graph  $\Gamma_{\Lambda_{alt}}$  of alternating coloring on  $\Lambda_{alt}$ . Therefore, there is a sequence of isomorphisms

$$H^1(L, \mathbb{C}^*) \cong H^1(\Gamma_{\Lambda_{alt}}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})}$$

where  $b_1(\Gamma_{\Lambda_{alt}})$  is the 1<sup>st</sup> Betti number of  $\Gamma_{\Lambda_{alt}}$ .

Also, local systems on  $L$  corresponds to alternating sheaves in  $M_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$  i.e. we have an inclusion

$$H^1(L, \mathbb{C}^*) \hookrightarrow M_1(S_x^1 \times \mathbb{R}_z, \Lambda_{alt}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

In conclusion, we have an inclusion

$$(\mathbb{C}^*)^{b_1(\Gamma_{\Lambda_{alt}})} \hookrightarrow M_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega \coprod \Lambda_{\omega_\emptyset}, \{\sigma_{z<<0}, \sigma_{z>>0}\}; \mathbb{C})$$

which is called a cluster coordinate of the moduli space.

In Chapter3,

- (i) we will systematically construct a natural alternating Legendrian(equivalently, alternating strand diagram) that are Legendrian isotopic to the separated Legendrian  $\Lambda_\omega \coprod \Lambda_{\omega_\emptyset}$ .
- (ii) we will construct a Legendrian isotopy between the alternating Legendrians and the separated Legendrians.

(iii) we will explicitly describe the sheaf cobordism induced by the above isotopy.

Conventions and notation

# Chapter 2

## Multivalent Braids and Generalized Sibuya Spaces

### 2.1 Basic terminologies

**Definition 19.** Let  $\omega$  be a braid word, then we have an embedding of the braid diagram in  $[0, 1]_x \times \mathbb{R}_z$ , co-oriented below, as an *open front diagram* of  $\omega$ .

**Definition 20.** Let  $\omega$  be a braid word representing a braid  $\beta \in Br_n^+$  i.e.  $[\omega] = \beta$ . Then we define  $Q_\omega^o$  to be the quiver

- whose vertices are labeled by the regions of the open front projection
- whose arrows are labeled by pairs of vertices whose corresponding regions are adjacent(bordered by the front projection of the braid) subject to the condition that the arrows always go against the co-orientation(hairs).

There are two distinguished vertices of  $Q_\omega^o$ . We denote the vertex corresponding to the region  $z \rightarrow \infty$ (resp.  $z \rightarrow -\infty$ ) as  $U$ (resp.  $D$ ).

**Definition 21.** Locally for each crossing  $c$ , there is a region all the hairs are pointing outward, we call this  $N_c$ (read north of  $c$ ). Starting from  $N_c$ , as we move counter-

clockwise about the crossing, we call the corresponding regions  $N_c, E_c, S_c, W_c$  respectively(read north of, east of, south of, west of  $c$ ).

**Theorem 22.** Suppose we have a fixed braid word  $\omega$  and  $v$  is a vertex corresponding to a region given by the front projection of  $\omega$ . Any path from  $d$  to a vertex  $v$  have same length.

*Proof.* we prove the statement by the induction on the length of the braid words. The statement is trivial for trivial braid word because there is only one path from  $d$  to any vertex. Suppose the statement is true for all quivers associated to braid words whose length is less than  $n$ . Suppose  $\omega$  is a braid word of length  $n$ . Suppose there are two distinct paths  $p_1$  and  $p_2$  from  $d$  to  $v$ . Let  $\omega = s_{i_1} \cdots s_{i_n}$ . There are two cases to consider :

(case1)  $v$  is the region east of the new crossing generated by  $s_{i_n}$ . Then the paths to  $v$  must have passed the region right below the region of  $v$  because that's the only possible way to get to  $v$  under the constraint that the arrow always go against the co-orientation. If we remove the last edge and vertex from the paths  $p_1, p_2$ , we have paths  $p'_1, p'_2$  ending at the same vertex(i.e. south of the crossing generated by  $s_{i_n}$ ) and are entirely contained in the subquiver  $Q_{\omega'}^o$  where  $\omega' = s_{i_1} \cdots s_{i_{n-1}}$ . Therefore, by the induction hypothesis the lengths of  $p'_1$  and  $p'_2$  are the same which immediately implies  $\text{length}(p_1) = \text{length}(p'_1) + 1 = \text{length}(p'_2) + 1 = \text{length}(p_2)$ .

(case2) Suppose  $v$  is not the region that is the east of the crossing generated by  $s_{i_n}$ . Without loss of generality, we can assume that two paths do not pass through the region east of the crossing, because we can always replace the part of the path  $s_c \rightarrow e_c \rightarrow n_c$  with  $S_c \rightarrow W_c \rightarrow N_c$  having the same length. Then once we know that two paths  $p_1$  and  $p_2$  do not pass through the region east of the crossing, we know that they are entirely contained in the subquiver  $\omega' = s_{i_1} \cdots s_{i_{n-1}}$ . Then the  $\text{length}(p_1) = \text{length}(p_2)$  by the induction hypothesis.  $\square$

**Definition 23.** The valency of the vertex of a quiver is the number of incoming

arrows. The height of the vertex of a quiver is the length of a path starting from  $d$  ending at that vertex which is well-defined by the previous theorem. Note that for every crossing  $c$ ,  $E_c$  and  $W_c$  have the same height.

**Definition 24.** We say two vertices are adjacent if there is a crossing  $c$  such that two vertices are  $E_c$  and  $W_c$  of this crossing. Let  $k$  be a positive integer, then we define a natural ordering on the set of all height  $k$  vertices generated by the relations, for each crossing  $W_c \leq E_c$ .

**Theorem 25.** The above ordering is well-defined and is a total ordering on the set of all height  $k$  vertices.

*Proof.* To prove the claim, we have to prove the following facts :

- (i) For any two distinct points of height  $k$ , there is a chain of crossings connecting two points.
- (ii) there is no chain of crossings starting at a point and ending at the same point.

We prove the claim by induction on the length of braid words. For the trivial braid, (i) holds because there is only one vertex for each height and (ii) holds because there is no arrow starting from that unique and ending at the point.

Now suppose the claim holds for all length  $< n$  braids and suppose  $i$  does not hold for the braid word  $\omega = s_{i_1} \cdots s_{i_n}$ . Then one of the two vertices should be the vertex corresponding to  $E_c$  of the crossing generated by  $s_{i_n}$ . Let's call this vertex  $v$  and the other as  $v'$ . Then we know that by the induction hypothesis there is a chain of crossing connecting  $v'$  and  $W_c$ . Since  $W_c$  is connected by the crossing  $c$  to  $E_c = v$  and  $v, v'$  are connected by a chain of crossings which is a contradiction.

Now suppose the claim holds for all length  $< n$  braids and suppose  $ii$ . Suppose there is a point where there is a chain of crossing starting and ending at the same point. By the induction hypothesis, the chain of crossing generated by  $s_{i_n}$  along the way.

Without loss of generality, using cyclic shift we can assume the chain of crossings starts from  $e_c$  which is a contradiction because  $E_c$  is not the  $W_{c'}$  of any crossing  $c'$ .  $\square$

**Definition 26.** There are two distinguished  $A_n$  sub-quivers of  $Q_\omega^o$ . We denote  $R_\omega$ (resp.  $L_\omega$ ) to be the full sub-quiver containing all the vertices corresponding to the rightmost(resp. leftmost) regions. Alternatively, the path following the largest(resp. smallest) arrows at each step. We denote the vertex of  $R_\omega$ (resp.  $L_\omega$ ) of height  $k$  by  $R_\omega^k$ (resp.  $L_\omega^k$ ).

**Definition 27.** Let  $\mathcal{M}_\omega^{fr}$  be the framed moduli space classifying pairs  $(F, g)$  where  $F$  is the representation of  $Q_\omega^o$  and  $g \in \mathrm{GL}_n(\mathbb{C})$  subject to the following conditions:

- For each vertex the vectorspace associated to it is a subspace of  $\mathbb{C}^n$  of dimension equal to its height.
- All maps are inclusion maps.
- $gF(R_\omega^k) = F(L_\omega^k)$  for all  $k = 0, 1, \dots, n$ .
- For each crossing  $c$ , then the sequence  $0 \rightarrow F(S_c) \rightarrow F(E_c) \oplus F(W_c) \rightarrow F(N_c) \rightarrow 0$  is a short exact sequence where maps are induced by the inclusion maps.

*Remark 28.* There is a natural left action of  $x \in \mathrm{GL}_n(\mathbb{C})$  on  $(F, g) \in \mathcal{M}_\omega^{fr}$ , that is,  $x \cdot (F, g) = (xF, xGx^{-1})$  where  $xF$  is left translation on quiver representation and  $xGx^{-1}$  is conjugation.

**Theorem 29.** [STZ17, Prop. 1.5]  $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z \ll 0}\}; \mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) \backslash \mathcal{M}_\omega^{fr}$ .

Now consider the following map

**Definition 30.** Suppose we have the braid word  $\omega$  of a braid  $\beta$ . Let  $\{v_1, v_2, \dots, v_m\}$  be the complete list of all height 1 vertices of  $Q_\omega^o$  such that  $v_i < v_j$  if and only if  $i < j$ . We define  $\iota_\omega$  to be the forgetful map

$$\begin{aligned}\iota_\omega : \mathcal{M}_\omega^{fr} &\rightarrow (\mathbb{P}^{n-1})^m \times \mathrm{GL}_n(\mathbb{C}) \\ (F, g) &\mapsto ([F(v_1)], \dots, [F(v_m)], g)\end{aligned}$$

**Definition 31.** Let  $\omega$  be a braid word. We denote the infinite cyclic copies of the original braid word  $\omega$  by  $\omega^\infty$ . More precisely, suppose  $\omega$  is a braid word given by a collection of sections  $\{\sigma_i : [0, 1]_x \rightarrow [0, 1]_x \times \mathbb{R}_{y,z}^2\}_{i=1,\dots,n}$ , then  $\omega^\infty$  is given by the collection  $\{\bar{\sigma}_i : \mathbb{R}_x \rightarrow \mathbb{R}_{x,y,z}^3 \mid \bar{\sigma}(x, y, z) = \sigma(x - [x], y, z)\}_{i=1,\dots,n}$ . Again, I will abuse the notation  $\omega^\infty$  to denote its front projection onto  $\mathbb{R}_{x,z}^2$ .

**Definition 32.** Let  $\omega$  be a braid word, then we have the quiver  $Q_\omega^o$  associated to it. We define the quiver  $Q_\omega^\infty$  to be the quotient of  $\prod_{i \in \mathbb{Z}} Q_\omega^o$  by the relations  $\{R_{\omega,i} = L_{\omega,i+1}\}_{i \in \mathbb{Z}}$  where  $R_{\omega,i}$  (resp.  $L_{\omega,i+1}$ ) is the subquiver  $R_\omega$  (resp.  $L_\omega$ ) in the  $i^{th}$  (resp.  $i + 1^{th}$ ) copy of  $Q_\omega^o$  in  $\prod_{i \in \mathbb{Z}} Q_\omega^o$ . Therefore, we have the quotient map of quivers  $q : \prod_{i \in \mathbb{Z}} Q_\omega^o \rightarrow Q_\omega^\infty$ . Let the signature function  $\sigma' : \mathrm{Vert}(\prod_{i \in \mathbb{Z}} Q_\omega^o) \rightarrow \mathbb{Z}$  be  $\sigma'(v) = i$  if  $v$  is in the  $i^{th}$  copy of  $Q_\omega^o$  in  $\prod_{i \in \mathbb{Z}} Q_\omega^o$ . Define  $\sigma : Q_\omega^\infty \rightarrow \mathbb{Z}$  to be  $\sigma(v) = \min_{w \in q^{-1}(v)} \sigma'(w)$  if  $|q^{-1}(v)| < \infty$  and 0 otherwise. Note that for  $v \in \mathrm{Vert}(Q_\omega^\infty)$ ,  $q^{-1}(v)$  is infinite if and only if  $R_\omega^{\mathrm{height}(v)} = L_\omega^{\mathrm{height}(v)}$  i.e. there is a unique vertex of height( $v$ ) in  $Q_\omega^o$ . We can think of  $Q_\omega^o$  as the full subquiver of  $Q_\omega^\infty$  spanned by the signature 0 vertices.

**Definition 33.** Suppose we have a quiver representation  $F_\omega$  of  $Q_\omega^o$ , then we define the induces quiver representation  $F_\omega^\infty$  of the quiver  $Q_\omega^\infty$  to be  $F_\omega^\infty(v) := g^{\sigma(v)} \cdot F_\omega(v - \sigma(v))$ .

Under the above correspondence, we get an isomorphism  $\mathcal{M}_\omega^{fr} \cong \mathcal{M}_\omega^{fr, \infty}$ .

**Definition 34.**  $\Upsilon_0^k$  is the set of all height  $k$ , signature zero vertices in  $Q_\omega^\infty$ . For each vertex  $v$  of  $Q_\omega^\infty$ , we define  $I_k(v)$  to be the set of all the height  $k$  vertices that have paths to  $v$ .

## 2.2 Multivalent braid words

**Definition 35.** A braid word  $\omega$  is multivalent if and only if all the vertices of height greater than 1 in  $Q_\omega^\infty$  has valencies greater than 1.

**Theorem 36.** If the braid word  $\omega$  is multivalent, then  $\iota_\omega$  is an embedding.

*Proof.* It is enough to prove that once we specify vectorspaces to height 1 vertices of  $Q_\omega^o$  and  $g \in \mathrm{GL}_n(\mathbb{C})$ , the quiver representation of  $Q_\omega^o$  extending the above datum is unique. Since  $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$ , it is enough to prove once we specify vectorspaces to signature 0 height 1 vertices of  $Q_\omega^\infty$  and  $g \in \mathrm{GL}_n(\mathbb{C})$ , the quiver representation  $F_\omega^\infty$  of  $Q_\omega^\infty$  extending the above datum is unique. Since  $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$ , for  $v \in \mathrm{Vert}(Q_\omega^\infty)$ ,  $F_\omega^\infty(v) = g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))$ . If  $\mathrm{height}(v) = 1$ , then  $v - \sigma(v)$  is height 1 signature 0 vertex. Therefore,  $F_\omega^\infty(v)$  is uniquely determined.

Now we prove the statement by induction on the heights of vertices. Assume  $\forall v \in \mathrm{Vert}(Q_\omega^\infty)$  with  $\mathrm{height}(v) < h$ ,  $F_\omega^\infty(v)$  are determined. Suppose  $v \in \mathrm{Vert}(Q_\omega^\infty)$  such that  $\mathrm{height}(v) = h > 1$ , then there are at least two vertices of height  $h - 1$  that have arrows to  $v$ , say  $v'$  and  $v''$ . Without loss of generality,  $v' \leq v''$ . Then by  $\omega^\infty$ ,  $Q_\omega^\infty$ -analogue of Theorem 25, there is a chain of crossings  $c_1, \dots, c_k$  and  $v' = v_0, \dots, v_k = v''$  where  $v_{i-1}, v_i$  are west and east of the crossing  $c_i$ . Therefore, if we choose any  $c_i$  to be  $c$ , then  $v = N_c$ . By the induction hypothesis,  $F_\omega^\infty(W_c)$ ,  $F_\omega^\infty(E_c)$ .and  $F_\omega^\infty(S_c)$  have already been specified because heights of  $W_c, E_c$ ,and  $S_c$  are  $h - 1$ ,  $h - 1$ ,and  $h - 2$  respectively. By the crossing condition,

$$0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

$F_\omega^\infty(N_c)$  is uniquely determined.  $\square$

**Theorem 37.** If the braid word  $\omega$  is multivalent, then the image of  $\iota_\omega$  is

$$X_\omega = \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}_v^{n-1} \times \mathrm{GL}_n(\mathbb{C}) \mid$$

$$\forall u \in \mathrm{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(u),$$

$$\forall c \in \mathrm{Cross}(\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(N_c) \}$$

where  $\mathbb{P}_v^{n-1}$  is a copy of  $\mathbb{P}^{n-1}$  labeled by  $v \in \Upsilon_0^1$ .

*Proof.* Instead of proving  $\mathrm{Im}(\iota_\omega) = X_\omega$ , I will prove that for  $\iota'$ , the map obtained by pre-composing the canonical isomorphism  $\mathcal{M}_\omega^{fr,\infty} \cong \mathcal{M}_\omega^{fr}$  to  $\iota_\omega$ ,  $\mathrm{Im}(\iota') = X_\omega$ .

First, let's prove  $\mathrm{Im}(\iota') \subset X_\omega$ . Recall that

$$\mathcal{M}_\omega^{fr,\infty} = \{(F_\omega^\infty, g) \in \mathrm{Rep}(Q_\omega^\infty) \times \mathrm{GL}_n(\mathbb{C}) \mid \text{All the maps of } F_\omega^\infty \text{ are inclusion maps,}$$

$$\forall v \in \mathrm{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}}(F_\omega^\infty(v)) = \mathrm{height}(v),$$

$$\forall v \in \mathrm{Vert}(Q_\omega^\infty), F_\omega^\infty(v+n) = g^n \cdot F_\omega^\infty(v),$$

$$\forall c \in \mathrm{Cross}(\omega^\infty), 0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

$$\text{are short exact sequences}\}$$

We need to show that for  $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$ ,  $\iota'_\omega((F_\omega^\infty, g)) = ((F_\omega^\infty(v))_{v \in \Upsilon_0^1}, g) \in X_\omega$  i.e.

- $\forall u \in \mathrm{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = \mathrm{height}(u)$
- $\forall c \in \mathrm{Cross}(\omega^\infty), \dim_{\mathbb{C}}(\sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v))) = \mathrm{height}(N_c)$

It is enough to prove that

- $\forall u \in \mathrm{Vert}(Q_\omega^\infty), \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(u)$
- $\forall c \in \mathrm{Cross}(\omega^\infty), \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(N_c)$

because of the condition  $\forall u \in \text{Vert}(Q_\omega^\infty)$ ,  $\dim_{\mathbb{C}}(F_\omega^\infty(u)) = \text{height}(u)$  defining  $\mathcal{M}_\omega^{fr,\infty}$ . Moreover,  $g^{\sigma(v)} \cdot F_\omega^\infty(v - \sigma(v)) = F_\omega^\infty(v)$  by the definition of  $\mathcal{M}_\omega^{fr,\infty}$ . Therefore, we need to prove that,

- (i)  $\sum_{v \in I_1(u)} F_\omega^\infty(v) = F_\omega^\infty(u)$
- (ii)  $\sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) = F_\omega^\infty(N_c)$

Proof of (i) : For each  $v \in I_1(u)$ , there is a path from  $v$  to  $u$ , say  $v = v_0 \rightarrow \dots \rightarrow v_k = u$ . Since  $F_\omega^\infty(v_i) \subset F_\omega^\infty(v_{i+1})$ ,  $F_\omega^\infty(v) \subset F_\omega^\infty(u)$ . Therefore,  $\sum_{v \in I_1(u)} F_\omega^\infty(v) \subset F_\omega^\infty(u)$ . Conversely, we prove  $F_\omega^\infty(u) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$  by induction on the height of  $u$ . If  $\text{height}(u) = 1$ , the statement is trivial. Now suppose the statement holds for all  $u \in \text{Vert}(Q_\omega^\infty)$  of  $\text{height}(u) = h$ . Since  $\omega$  is multivalent,  $u$  has at least two vertices of height  $h - 1$  that have arrows to  $u$ , say  $v'$  and  $v''$ . Without loss of generality  $v' \leq v''$ . Then by  $\omega^\infty$ ,  $Q_\omega^\infty$ -analogue of Theorem 25, there is a chain of crossings  $c_1, \dots, c_k$  and  $v' = v_0, \dots, v_k = v''$  where  $v_{i-1}, v_i$  are west and east of the crossing  $c_i$ . Therefore, if we choose any  $c_i$  to be  $c$ , then  $u = N_c$  i.e.  $v$  is the north of the crossing  $c$ . Then by the crossing condition of  $M_\omega^{fr,\infty}$

$$0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

we have  $F_\omega^\infty(u) = F_\omega^\infty(N_c) = F_\omega^\infty(E_c) + F_\omega^\infty(W_c)$  by the induction hypothesis,

$$\begin{aligned} F_\omega^\infty(E_c) &= \sum_{v \in I_1(E_c)} F_\omega^\infty(v), \quad F_\omega^\infty(W_c) = \sum_{v \in I_1(W_c)} F_\omega^\infty(v) \\ \Rightarrow F_\omega^\infty(u) &= \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \end{aligned}$$

because  $E_c$  and  $W_c$  have arrows to  $u = N_c \Rightarrow I_1(E_c) \cup I_1(W_c) \subset I_1(u)$ . Therefore,

$$F_\omega^\infty(u) = \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \subset \sum_{v \in I_1(u)} F_\omega^\infty(v)$$

Proof of (ii) : By (i), we have

$$\begin{aligned}
 & \sum_{v \in I_1(E_c) \cup I_1(W_c)} F_\omega^\infty(v) \\
 &= \sum_{v \in I_1(E_c)} F_\omega^\infty(v) + \sum_{v \in I_1(W_c)} F_\omega^\infty(v) \\
 &= F_\omega^\infty(E_c) + F_\omega^\infty(W_c)
 \end{aligned}$$

This is equal to  $F_\omega^\infty(N_c)$  by the crossing condition of  $\mathcal{M}_\omega^{fr,\infty}$ . Therefore, the proof of (ii) is complete.

Now let's prove  $X_\omega \subset \text{Im}(\iota'_\omega)$ . Let  $((z_v)_{v \in \Upsilon_0^1}, g)$  be an arbitrary point of  $X_\omega$ . I will define a point  $(F_\omega^\infty, g)$  such that  $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$ . We define a quiver representation  $F_\omega^\infty$  to be

$$F_\omega^\infty(u) := \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}$$

and all the arrows of  $F_\omega^\infty$  are inclusion maps. The inclusion maps are well-defined because if there is an arrow from  $u$  to  $u'$ , then  $I_1(u) \subset I_1(u')$ .

Note that if  $u \in \Upsilon_0^1$  i.e.  $\text{height}(u) = 1$  and  $\sigma(u) = 0$ , then

$$F_\omega^\infty(u) := \sum_{v \in \{u\}} g^{\sigma(v)} \cdot z_{v-\sigma(v)} = g^{\sigma(u)} \cdot z_{u-\sigma(u)}$$

If  $(F_\omega^\infty, g)$  is indeed a point of  $\mathcal{M}_\omega^{fr,\infty}$ , then  $\iota'_\omega((F_\omega^\infty, g)) = ((z_v)_{v \in \Upsilon_0^1}, g)$ . Thus, it is enough to prove that  $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr,\infty}$  i.e.

- (i) All maps of  $F_\omega^\infty$  are inclusion maps
- (ii)  $\forall v \in \text{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}}(F_\omega^\infty(v)) = \text{height}(v)$
- (iii)  $\forall v \in \text{Vert}(Q_\omega^\infty) \text{ and } n \in \mathbb{Z}, F_\omega^\infty(v+n) = F_\omega^\infty(v)$

- (iv)  $\forall c \in \text{Cross}(\omega^\infty)$ ,  $0 \rightarrow F_\omega^\infty(S_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$  are short exact sequences

We get (i) immediately from the definition of  $F_\omega^\infty$ .

To prove (ii), note that  $\dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \text{height}(u)$  because  $((z_v)_{v \in \Upsilon_0^1}, g) \in X_\omega$ . Therefore,  $\dim_{\mathbb{C}}(F_\omega^\infty(u)) = \dim_{\mathbb{C}}(\sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)}) = \text{height}(u)$ .

To prove (iii), note that  $\forall u \in \text{Vert}(Q_\omega^\infty)$  and  $n \in \mathbb{Z}$ ,  $I_1(u+n) = I_1(u) + n$ . Therefore,

$$\begin{aligned} F_\omega^\infty(u+n) &= \sum_{v \in I_1(u+n)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v \in I_1(u)+n} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v-n \in I_1(u)} (g^{\sigma(v)} \cdot z_{v-\sigma(v)}) \\ &= \sum_{v' \in I_1(u)} (g^{\sigma(v'+n)} \cdot z_{(v'+n)-\sigma(v'+n)}) \\ &= g^n \cdot (\sum_{v' \in I_1(u)} (g^{\sigma(v')} \cdot z_{v'-\sigma(v')})) \\ &= g^n \cdot F_\omega^\infty(u) \end{aligned}$$

Finally, let's prove (iv). Let  $c \in \text{Cross}(\omega^\infty)$ , then

$$\begin{aligned} \text{height}(S_c) + 2 &= \text{height}(E_c) + 1 = \text{height}(W_c) + 1 = \text{height}(N_c) \\ \Rightarrow \dim_{\mathbb{C}}(F_\omega^\infty(S_c)) + 2 &= \dim_{\mathbb{C}}(F_\omega^\infty(E_c)) + 1 = \dim_{\mathbb{C}}(F_\omega^\infty(W_c)) + 1 = \dim_{\mathbb{C}}(F_\omega^\infty(N_c)) \end{aligned}$$

Since there are arrows from  $S_c$  to  $E_c$ ,  $W_c$  and from  $E_c$ ,  $W_c$  to  $N_c$ ,  $I_1(S_c) \subset I_1(E_c)$ ,  $I_1(W_c) \subset I_1(N_c)$ . Therefore,  $F_\omega^\infty(S_c) \subset F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c)$  and  $F_\omega^\infty(E_c) \cup F_\omega^\infty(W_c) \subset F_\omega^\infty(N_c)$ .

By the condition that

$$\begin{aligned} \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) &= \text{height}(N_c) \\ \Rightarrow \dim_{\mathbb{C}} (F_\omega^\infty(E_c) + F_\omega^\infty(W_c)) &= \text{height}(N_c) = \dim_{\mathbb{C}} (F_\omega^\infty(N_c)) \\ \Rightarrow F_\omega^\infty(E_c) + F_\omega^\infty(W_c) &= F_\omega^\infty(N_c) \end{aligned}$$

we have surjection part of the short exact sequence i.e.  $F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$  is exact. Thus we have a short exact sequence

$$0 \rightarrow F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(E_c) \oplus F_\omega^\infty(W_c) \rightarrow F_\omega^\infty(N_c) \rightarrow 0$$

I claim that  $F_\omega^\infty(E_c) \cap F_\omega^\infty(W_c) = F_\omega^\infty(S_c)$ . Since  $F_\omega^\infty(E_c)$ ,  $F_\omega^\infty(W_c)$ , and  $F_\omega^\infty(S_c)$  are codimension 1, 1, and 2 inside of  $F_\omega^\infty(N_c)$  respectively, it is enough to show that  $F_\omega^\infty(E_c)$  and  $F_\omega^\infty(W_c)$  intersect transversely in  $F_\omega^\infty(N_c)$  i.e.  $F_\omega^\infty(E_c) \neq F_\omega^\infty(W_c)$ . This is true because otherwise

$$\begin{aligned} \text{height}(N_c) &= \dim_{\mathbb{C}} (F_\omega^\infty(N_c)) \\ &= \dim_{\mathbb{C}} (F_\omega^\infty(E_c) + F_\omega^\infty(W_c)) \\ &= \dim_{\mathbb{C}} (F_\omega^\infty(E_c)) \\ &= \text{height}(E_c) \end{aligned}$$

which is a contradiction.

Therefore,  $(F_\omega^\infty, g)$  satisfy all the condition (i)-(iv) i.e.  $(F_\omega^\infty, g) \in \mathcal{M}_\omega^{fr, \infty}$ .  $\square$

## 2.3 Bivalent braid words

**Definition 38.** A braid word  $\omega$  is bivalent if it is multivalent and the valencies of all the vertices of height  $1, \dots, n - 1$  in  $Q_\omega^\infty$  are equal to 2 where  $n$  is the number of strands of  $\omega$ .

**Lemma 39.** Let  $\omega$  be a braid word and  $v \in \text{Vert}(Q_\omega)$  ( $v \in \text{Vert}(Q_\omega^\infty)$  resp.), then there is a path from  $v$  to the unique height  $n$  vertex  $U$ . The path passes through a vertex of height  $n - 1$ .

*Proof.* If we prove the statement for  $Q_\omega^o$ , then the proof for the  $Q_\omega^\infty$  case follows immediately. Let's prove the claim by induction on the length of the braid word  $\omega$ . If  $\text{length}(\omega) = 0$  i.e. trivial braid word, there is a path from  $D$  to  $U$  and along the way it passes through all the points. Therefore, restricting this path to start from the point that we are interested in gives the desired path. Now let's assume that the statement holds for all braid words of length less than  $k$ . Let  $\omega = s_{i_1} \cdots s_{i_k}$ , then define  $\omega' = s_{i_1} \cdots s_{i_{k-1}}$ . We get that  $Q_{\omega'}^o$  is the subquiver of  $Q_\omega^o$  where  $\text{Vert}(Q_\omega^o) - \text{Vert}(Q_{\omega'}^o) = \{v\}$  is a singleton where  $v$  is the east of the crossing added by  $s_{i_k}$ , say  $c$ . Note that  $v = E_c$  has an arrow to  $N_c$  and  $N_c$  has a path to  $U$  by the induction hypothesis because  $N_c \in \text{Vert}(Q_{\omega'}^o)$ . Therefore, we can extend the path from  $N_c$  to  $U$  to start from  $E_c$  i.e.  $E_c \rightarrow (N_c \rightarrow \cdots \rightarrow U)$ . □

**Lemma 40.** Let  $\omega$  be a bivalent braid word and  $u \in \text{Vert}(Q_\omega^\infty)$  of height( $u$ )  $< n$ , then  $|I_1(u)| = \text{height}(u)$ . In particular, if  $c \in \text{Cross}(\omega^\infty)$ , then  $I_1(S_c) = \text{height}(S_c)$ ,  $I_1(E_c) = \text{height}(E_c)$ , and  $I_1(W_c) = \text{height}(W_c)$ .

*Proof.* We prove the statement by induction on the height of  $u$ . If  $\text{height}(u) = 1$ , then  $I_1(u) = \{u\} \Rightarrow |I_1(u)| = 1 = \text{height}(u)$  holds. Now suppose the statement holds for vertices of heights less than  $h$  and  $\text{height}(u) = h$  where  $h < n$ . Since  $\omega$  is bivalent, we have exactly two vertices of height  $h - 1$  and a crossing  $c$ , where those

two vertices are the east and west of the crossing  $c$ . By the induction hypothesis,  $|I_1(E_c)| = |I_1(W_c)| = h - 1$ ,  $|I_1(S_c)| = h - 2$  and  $I_1(S_c) \subset I_1(E_c), I_1(W_c)$ . Let

$$I_1(E_c) - I_1(S_c) = \{v_1\}$$

$$I_1(W_c) - I_1(S_c) = \{v_2\}$$

Since  $\omega$  is bivalent,

$$\begin{aligned} I_1(N_c) &= I_1(E_c) \cup I_1(W_c) \\ &= \{v_1, v_2\} \cup I_1(S_c) \end{aligned}$$

If  $v_1 \neq v_2 \Leftrightarrow I_1(E_c) \neq I_1(W_c)$ , then  $|I_1(N_c)| = 2 + (h - 2) = h$ . Therefore, it is enough to prove that  $\forall c \in \text{Cross}(\omega^\infty)$ ,  $I_1(E_c) \neq I_1(W_c)$ . This follows from Lemma 41 below.  $\square$

**Lemma 41.** Let  $\omega$  be a bivalent braid word and  $u, v$  be distinct vertices of  $Q_\omega^\infty$  of the same height, then  $I_1(u) \neq I_1(v)$ . Note that the height of  $u, v$  cannot be  $n$  because there is only one vertex of height  $n$ , say  $U$ .

*Proof.* We prove the claim by the induction on the height of  $u$  and  $v$ . If  $\text{height}(u) = \text{height}(v) = 1$ , then the claim holds because  $I_1(u) = \{u\}$  and  $I_1(v) = \{v\}$ .

Now suppose the claim holds for vertices of heights less than  $h$ . Then there are exactly two height  $h - 1$  vertices for each, say  $\{u_1, u_2\}, \{v_1, v_2\}$ , that have arrows to  $u$  and  $v$ . Note that there are crossings  $c, c'$  such that  $\{u_1, u_2\} = \{E_c, W_c\}, \{v_1, v_2\} = \{E_{c'}, W_{c'}\}$ .  $\{u_1, u_2\} \neq \{v_1, v_2\}$ , otherwise  $c = c' \Rightarrow u = N_c = N_{c'} = v$  which is a contradiction. Therefore, by the induction hypothesis,  $I_1(u) = I_1(u_1) \cup I_1(u_2) \neq I_1(v_1) \cup I_1(v_2) = I_1(v)$ .  $\square$

**Lemma 42.** Let  $\omega$  be a bivalent braid word and  $c \in \text{Cross}(\omega^\infty)$ , then  $I_1(E_c) \cap I_1(W_c) = I_1(S_c)$  and  $|I_1(E_c) \cup I_1(W_c)| = \text{height}(N_c)$ .

*Proof.*  $I_1(S_c) \subset I_1(E_c) \cap I_1(W_c)$  because there are arrows from  $S_c$  to  $E_c, W_c$ . Since  $E_c \neq W_c$ , by Lemma 35,  $I_1(E_c) \neq I_1(W_c)$ . Therefore,

$$\begin{aligned} |I_1(E_c) \cap I_1(W_c)| &\leq |I_1(E_c)| - 1 = |I_1(S_c)| \\ \Rightarrow I_1(E_c) \cap I_1(W_c) &= I_1(S_c) \end{aligned}$$

As a consequence,

$$\begin{aligned} |I_1(E_c) \cup I_1(W_c)| &= |I_1(E_c)| + |I_1(W_c)| - |I_1(E_c) \cap I_1(W_c)| \\ &= |I_1(E_c)| + |I_1(W_c)| - |I_1(S_c)| \\ &= (h-1) + (h-1) - (h-2) = h \\ &= |I_1(N_c)| \\ &= \text{height}(N_c) \end{aligned}$$

□

**Definition 43.** Suppose  $L_1, L_2, \dots, L_k \subset \mathbb{C}^n$  are lines(1 dimensional subspaces). We say  $\{L_1, L_2, \dots, L_k\}$  are linearly independent if and only if for nonzero  $v_i \in L_i$ ,  $\{v_1, v_2, \dots, v_k\}$  are linearly independent. Note that the definition does not depend on the choice of  $v_i$ 's.

**Theorem 44.** Let  $\omega$  be a bivalent braid word, then  $\iota_\omega$  is an open embedding whose image is

$$\begin{aligned} X'_\omega &= \{((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}^{n-1} \times \text{GL}_n(\mathbb{C}) \mid \\ &\quad \forall c \in \text{Cross}(\omega) \subset \text{Cross}(\omega^\infty) \text{ with } \text{height}(N_c) = n, \\ &\quad \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n \} \end{aligned}$$

*Proof.* Since bivalent braids are multivalent, by Theorem 37, the image of  $\iota_\omega$  is given by

$$\begin{aligned} X_\omega = \{ & ((z_v)_{v \in \Upsilon_0^1}, g) \in \prod_{v \in \Upsilon_0^1} \mathbb{P}_v^{n-1} \times \mathrm{GL}_n(\mathbb{C}) \mid \\ & \forall u \in \mathrm{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(u), \\ & \forall c \in \mathrm{Cross}(\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(N_c) \} \end{aligned}$$

I claim that  $X_\omega = X'_\omega$ . Since the condition

$$\forall c \in \mathrm{Cross}(\omega) \subset \mathrm{Cross}(\omega^\infty) \text{ with } \mathrm{height}(N_c) = n, \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n$$

defining  $X'_\omega$  is subsumed in the condition

$$\forall c \in \mathrm{Cross}(\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(N_c)$$

defining  $X_\omega$ ,  $X_\omega \subset X'_\omega$ .

Now let's prove  $X'_\omega \subset X_\omega$  i.e. suppose  $((z_v)_{v \in \Upsilon_0^1}, g)$  satisfy

$$(i) \quad \forall c \in \mathrm{Cross}(\omega) \subset \mathrm{Cross}(\omega^\infty) \text{ with } \mathrm{height}(N_c) = n, \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n$$

then it also satisfy

$$(ii) \quad \forall u \in \mathrm{Vert}(Q_\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(u)$$

$$(iii) \quad \forall c \in \mathrm{Cross}(\omega^\infty), \dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \mathrm{height}(N_c)$$

Let  $u \in \mathrm{Vert}(Q_\omega^\infty)$  and  $c \in \mathrm{Cross}(\omega^\infty)$ . First let's assume  $\mathrm{height}(u) < n$ . Since  $\omega$

is bivalent, by Lemma 40,  $|I_1(u)| = \text{height}(u)$  and by Lemma 40 and Lemma 42,  $|I_1(E_c) \cup I_1(W_c)| = |I_1(N_c)| = \text{height}(N_c)$ . Therefore, the condition (i) is equivalent to saying that

$\forall c \in \text{Cross}(\omega)$  with  $\text{height}(N_c) = n$ ,  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$  are linearly independent

Likewise, the conditions (ii),(iii) are equivalent to saying that

$\forall u \in \text{Vert}(Q_\omega^\infty)$ ,  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$  are linearly independent

$\forall c \in \text{Cross}(\omega^\infty)$ ,  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$  are linearly independent

Let's prove (i) implies (ii) and (iii) using these paraphrased statements. Assume (i) holds for  $((z_v)_{v \in \Upsilon_0^1}, g)$ . Suppose for some  $u \in \text{Vert}(Q_\omega^\infty)$ ,  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$  are linearly dependent. Let  $u' = u - \sigma(u)$ , then  $I_1(u') = I_1(u) - \sigma(u)$ . Therefore,

$$\begin{aligned} \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u')} &= \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v+\sigma(u) \in I_1(u)} \\ &= \{g^{\sigma(v')-\sigma(u)} \cdot z_{v'-\sigma(v')}\}_{v' \in I_1(u)} \\ &= g^{-\sigma(u)} \cdot \{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)} \end{aligned}$$

are also linearly dependent. Therefore, without loss of generality, we can assume  $\sigma(u) = 0$  i.e.  $u \in \text{Vert}(Q_\omega) \subset \text{Vert}(Q_\omega^\infty)$ . By Lemma 39, there is a path from  $u$  to a height  $n - 1$  vertex  $p$ . Thus,  $I_1(u) \subset I_1(p)$ . Since  $\omega$  is bivalent, for some  $c \in \text{Cross}(\omega)$ ,  $p = E_c$  or  $W_c$ . By the condition (i),  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(E_c) \cup I_1(W_c)}$  are linearly independent, thus its subset  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(u)}$  are linearly independent as well. Therefore,  $((z_v)_{v \in \Upsilon_0^1}, g)$  satisfy (ii).

Now let's show that  $((z_v)_{v \in \Upsilon_0^1}, g)$  satisfy (iii). Suppose  $c \in \text{Cross}(\omega^\infty)$ . If  $\text{height}(N_c) = n$ , then the condition (i) is equal to the condition (iii), there is nothing to prove. Suppose  $\text{height}(N_c) < n$ , then  $I_1(E_c) \cup I_1(W_c) = I_1(N_c)$ . Thus, the condition (iii)

translates to  $\{g^{\sigma(v)} \cdot z_{v-\sigma(v)}\}_{v \in I_1(N_c)}$  are linearly independent, which follows from (ii) that we already proved.

Now let's show that (iii) holds when  $\text{height}(u) = n$  i.e.  $u$  is the unique height  $n$  point  $U$ . Since  $\omega$  is bivalent, there is a crossing  $c$  such that  $u = N_c$ . Therefore,

$$\dim_{\mathbb{C}} \left( \sum_{v \in I_1(E_c) \cup I_1(W_c)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = \text{height}(N_c) = n$$

Since  $I_1(E_c) \cup I_1(W_c) \subset I_1(N_c) = I_1(u)$ ,

$$\dim_{\mathbb{C}} \left( \sum_{v \in I_1(u)} g^{\sigma(v)} \cdot z_{v-\sigma(v)} \right) = n = \text{height}(u)$$

□

**Theorem 45.** (Combinatorial characterization) A braid word is multivalent(bivalent resp.) if and only if in between  $s_i$ 's there is at least one(exactly one resp.)  $s_{i-1}$  upto cyclic shifts for  $2 \leq i < n$ .

*Proof.* The statement follows from the fact that

- there is a one to one correspondence between the set of region of height  $k$  and  $s_k$ 's in the braid word for  $1 \leq k < n$ .
- the valency of a region corresponding to a certain  $s_k$  is equal to the number of  $s_{k-1}$  in between that certain  $s_k$  and the next  $s_k$ .

□

**Definition 46.**  $\mathcal{M}_1(S_x^1 \times \mathbb{R}_z, \Lambda_\omega, \{\sigma_{z \ll 0}\}; \mathbb{C})$  is called *Generalized Sibuya space* if  $\omega$  is a bivalent braid.

## 2.4 Examples : generalized Sibuya spaces

By nequation, I mean a formula that expresses the non-equality of two expressions.

Whenever I denote capital  $X$  with subscript i.e.  $X_j$ , I mean an element of some projective space.

Lower case  $x$ 's with subscripts are used to denote the homogeneous coordinates of  $X_j$ 's.

For example  $X_j = [x_{1,j} : \dots : x_{n,j}]$ .

I will also denote

$$(X_1, \dots, X_{n-1}) = ([x_{1,1} : \dots : x_{n,1}], [x_{1,2} : \dots : x_{n,2}], \dots, [x_{1,n-1} : \dots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

We will use

$$\left( \begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_n \end{array} \right) = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

to denote the matrix whose entries are  $x_{i,j}$  which are sections of the line bundle  $O_{\mathbb{P}^k}(1)$ .

Thereby

$$\det \left( \begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_n \end{array} \right) = \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{k+1,1} & \cdots & x_{k+1,n} \end{pmatrix}$$

is a section of the line bundle  $\underbrace{O_{\mathbb{P}^k}(1) \boxtimes \cdots \boxtimes O_{\mathbb{P}^k}(1)}_{n-\text{copies}}$  on  $\underbrace{\mathbb{P}^k \times \cdots \times \mathbb{P}^k}_{n-\text{copies}}$ .

**Example 47. (Empty Set)** In this section we compute 2 kinds of moduli spaces

that are empty.

In the first case, we consider the moduli space of braids represented by a bivalent braid word that the number of  $s_1$  in the expression is less than the number of strands with unipotent monodromy  $u = I$

The framed moduli space is

$$\mathcal{M}^{fr} = \{(X_1, \dots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{k\text{-copies}} \mid \det \begin{pmatrix} X_1 & X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k} \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} X_2 & \cdots & X_k & X_1 & \cdots & X_{n-k+1} \end{pmatrix} \neq 0, \dots,$$

$$\det \begin{pmatrix} X_k & X_1 & \cdots & X_k & X_1 & \cdots & X_{n-k-1} \end{pmatrix} \neq 0\}$$

Take the first nequation and subtract 1<sup>st</sup>column from the  $(k + 1)^{th}$ column we get

$$\det \begin{pmatrix} X_1 & X_2 & \cdots & X_k & 0 & X_2 & \cdots & X_{n-k} \end{pmatrix} \neq 0$$

which is never true. Therefore, the framed moduli space  $\mathcal{M}^{fr} = \emptyset$  and so is the moduli space.

In the second case, we consider the moduli space of braids represented by a bivalent braid word that the number of  $s_1$  in the expression is less than (the number of

strands)–1 with unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The framed moduli space is

$$\mathcal{M}^{fr} = \left\{ (X_1, \dots, X_k) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{k-\text{copies}} \mid \det \begin{pmatrix} X_1 & X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k} \end{pmatrix} \neq 0, \right.$$

$$\det \begin{pmatrix} X_2 & \cdots & X_k & uX_1 & \cdots & uX_{n-k+1} \end{pmatrix} \neq 0, \dots,$$

$$\left. \det \begin{pmatrix} X_k & uX_1 & \cdots & uX_k & u^2X_1 & \cdots & u^2X_{n-k-1} \end{pmatrix} \neq 0 \right\}$$

Take the first nequation and subtract  $1^{st}$ (resp.  $2^{nd}$ ) column from the  $(k+1)^{th}$ (resp.  $(k+2)^{th}$ ) column we get

$$\det \begin{pmatrix} & & & & 0 & 0 & & & \\ X_1 & X_2 & \cdots & X_k & \vdots & \vdots & & & \\ & & & & 0 & 0 & uX_3 & \cdots & uX_{n-k} \\ & & & & x_{n,1} & x_{n,2} & & & \\ & & & & 0 & 0 & & & \end{pmatrix} \neq 0$$

which is never true. Therefore, the framed moduli space  $\mathcal{M}^{fr} = \emptyset$  and so is the moduli space.

**Example 48. (Points)** In this section, we compute the moduli space of braids represented by the braid word

$$(s_1 s_2 \cdots s_{n-1})^{n-1}$$

and the unipotent monodromy

$$u = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

i.e. a unipotent matrix given by the partition  $(\underbrace{1, 1, \cdots, 1}_{n-1}, 2)$ . The framed moduli space associated to the above data is given as follows

$$\begin{aligned} \mathcal{M}^{fr} &= \{(X_1, X_2, \cdots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \left( \begin{array}{c|c|c|c|c} X_1 & X_2 & \cdots & X_{n-1} & uX_1 \end{array} \right) \neq 0, \\ &\quad \det \left( \begin{array}{c|c|c|c|c} X_2 & \cdots & X_n & uX_1 & uX_2 \end{array} \right) \neq 0, \cdots, \det \left( \begin{array}{c|c|c|c|c} X_{n-1} & uX_1 & \cdots & uX_{n-2} & uX_{n-1} \end{array} \right) \neq 0\} \end{aligned}$$

Using the elementary column operation of subtracting the first column from the last

column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{(n-1)-copies} \mid \det \begin{pmatrix} & & & & 0 \\ X_1 & X_2 & \dots & X_{n-1} & 0 \\ & & & x_{n,1} & \\ & & & 0 & \end{pmatrix} \neq 0,$$

$$\det \begin{pmatrix} & & & 0 \\ X_2 & \dots & X_{n-1} & uX_1 \\ & & & \vdots \\ & & & x_{n,2} \\ & & & 0 \end{pmatrix} \neq 0, \dots, \det \begin{pmatrix} & & & 0 \\ X_{n-1} & uX_1 & \dots & uX_{n-2} \\ & & & \vdots \\ & & & x_{n,n-1} \\ & & & 0 \end{pmatrix} \neq 0\}$$

Applying the cofactor expansion formula with respect to the last column we get

$$\mathcal{M}^{fr} = \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{n-copies} \mid x_{n,1} \cdot \det \begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \dots & x_{n-2,n-1} \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix} \neq 0,$$

$$x_{n,2} \cdot \det \begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \dots & x_{n-2,n-1} \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix} \neq 0, \dots, x_{n,n-1} \cdot \det \begin{pmatrix} x_{1,1} & \dots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \dots & x_{n-2,n-1} \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix} \neq 0\}$$

$$\begin{aligned}
 &= \{(X_1, X_2, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{n\text{-copies}} \mid x_{n,i} \neq 0 \text{ for } i = 1, \dots, n-1 \\
 &\quad , \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \neq 0\}
 \end{aligned}$$

To get the moduli space out of the framed moduli space above, we have to quotient it out by the centralizer subgroup of  $u$ , that is,

$$\begin{aligned}
 C_{\mathrm{GL}_n(\mathbb{C})}(u) = \{ & \left( \begin{array}{ccc|cc} c_{1,1} & \cdots & c_{1,n-2} & 0 & c_{1,n} \\ \vdots & \ddots & \vdots & 0 & c_{1,n} \\ \hline c_{n-2,1} & \cdots & c_{n-2,n-2} & 0 & c_{n-2,n} \\ \hline 0 & \cdots & 0 & 0 & c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-2} & c_{n,n-1} & c_{n,n} \end{array} \right) \in \mathrm{GL}_n(\mathbb{C}) | \\
 & \det \begin{pmatrix} c_{1,1} & \cdots & c_{1,n-2} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n-2} \end{pmatrix}, c_{n,n-1}, c_{n-1,n} \neq 0 \}
 \end{aligned}$$

It acts diagonally on  $\underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_{(n-1)\text{-copies}}$  where the action on each coordinate is given by left multiplication.

To simplify the notation, I will denote

$$(X_1, \dots, X_{n-1}) = ([x_{1,1} : \dots : x_{n,1}], [x_{1,2} : \dots : x_{n,2}], \dots, [x_{1,n-1} : \dots : x_{n,n-1}])$$

as

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n-1} \end{bmatrix}$$

I claim that for any  $(X_1, \dots, X_{n-1}) \in \underbrace{\mathbb{P}^{n-1} \times \dots \mathbb{P}^{n-1}}_{(n-1)-copies}$ , there exists  $A \in C_{\text{GL}_n(\mathbb{C})}(u)$

such that

$$A \cdot X = \left[ \begin{array}{c|c} I & 0 \\ \hline & \vdots \\ & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array} \right]$$

Let

$$A_1 = \left( \begin{array}{ccc|cc} x_{1,1} & \cdots & x_{1,n-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-2} & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 \\ a_1 & \cdots & a_{n-2} & a_{n-1} & a_n \end{array} \right)$$

with  $a_{n-1} \neq 0$  such that

$$\sum_{i=1}^n a_i \cdot (x_{i,1}, \dots, x_{i,n}) = (1, \dots, 1)$$

We can always find such because we know that

$$\det \left( \begin{array}{ccc} x_{1,1} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-2,1} & \cdots & x_{n-2,n-1} \\ x_{n,1} & \cdots & x_{n,n} \end{array} \right) \neq 0$$

Then we get

$$A_1 \cdot X = \left[ \begin{array}{c|cc} I & b_1 \\ \hline & \vdots & b_{n-1} \\ x_{n,1} & \cdots & x_{n,n-2} & x_{n,n-1} \\ \hline 1 & \cdots & 1 & 1 \end{array} \right]$$

Since we know that  $x_{n,i} \neq 0$  for  $i = 1, \dots, n-1$ , without loss of generality we put  $x_{n,i} = 1$  for  $i = 1, \dots, n-1$ . Then we get

$$A_1 \cdot X = \left[ \begin{array}{c|cc} I & b_1 \\ \hline & \vdots & b_{n-1} \\ \hline 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & 1 \end{array} \right]$$

such that  $b_1 + \cdots + b_{n-1} + 1 \neq 0$ . Now let

$$A_2 = \left( \begin{array}{ccc|cc} a_{1,1} & \cdots & a_{1,n-2} & 0 & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline a_{n-2,1} & \cdots & a_{n-2,n-2} & 0 & a_{n-2,n} \\ \hline 0 & \cdots & 0 & & I \\ 0 & \cdots & 0 & & \end{array} \right)$$

such that

$$a_{i,1} \cdot (1, 0, \dots, 0, b_1) + a_{i,2} \cdot (0, 1, 0, \dots, 0, b_2) + \dots + a_{i,n-2} \cdot (0, \dots, 0, 1, b_{n-2}) + a_{i,n} \cdot (1, \dots, 1)$$

$$= (0, \cdot, 0, \overset{i^{th}}{\downarrow} 1, 0, \dots, 0)$$

Again, we can find such because

$$\det \left[ \begin{array}{ccc|c} & & & b_1 \\ & I & & \vdots \\ \hline & & & b_{n-1} \\ 1 & \cdots & 1 & 1 \end{array} \right] \neq 0$$

Then  $A_2 \cdot A_1$  is the desired  $A$ .

**Example 49. (Regular Unipotent Fibers)** Finite type integral schemes. Also, it has a stratification into rational varieties at most 1 stratum in each dimension.  $(s_1)^3$ ,  $(s_1 s_2)^2$

### Example1

In this section, we compute the moduli space associated to the braid word :  $(s_1 s_2)^2$  with regular unipotent monodromy. The framed moduli space is given by

$$\mathcal{M}^{fr} = \{([x:y:z], [a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0\}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$C := C_{\text{GL}_3(\mathbb{C})}(u) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in \text{GL}_3(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

We can cover  $\mathcal{M}^{fr}$  with open subsets  $U_1, U_2$  where

$$U_1 = \{([x:y:z], [a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, z \neq 0\}$$

$$U_2 = \{([x:y:z], [a:b:c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, c \neq 0\}$$

Therefore, we have a pushout square

$$\begin{array}{ccc} U_1 & \longrightarrow & \mathcal{M}^{fr} \\ \downarrow & & \downarrow \\ U_1 \cap U_2 & \longrightarrow & U_2 \end{array}$$

Quotienting out by the centralizer subgroup  $C$ , we get

$$\begin{array}{ccc} \overline{U}_1 := C \setminus U_1 & \longrightarrow & \mathcal{M} = C \setminus \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ \overline{U}_1 \cap \overline{U}_2 := C \setminus U_1 \cap U_2 & \longrightarrow & \overline{U}_2 := C \setminus U_2 \end{array}$$

Also note that, for the action of  $C$ , the centralizer subgroup of any element of  $\mathcal{M}^{fr}$  is the set of scalar multiplication matrices.

First let's simplify,  $\overline{U}_1$ . Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{y}{z}$$

$$\gamma = \frac{y^2}{z^2} - \frac{x}{z}$$

This expression makes sense because  $z \neq 0$  in  $U_1$ . If we take an element of  $U_1$  with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each  $C$ -orbit, we see that

$$\begin{aligned}\overline{U}_1 &\cong \{([0 : 0 : 1], [a : b : c]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} 0 & a & 0 \\ 0 & b & 1 \\ 1 & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} 0 & a & b \\ 0 & b & c \\ 1 & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\ &\cong \{[a : b : c] \in \mathbb{P}^2 \mid a \neq 0, \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0\} \\ &\cong \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}\end{aligned}$$

Under this identification,

$$\overline{U}_1 \cap \overline{U}_2 \cong \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

Now let's simplify,  $\overline{U}_2$ . Suppose

$$\begin{bmatrix} x & a \\ y & b \\ z & c \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is

$$\begin{aligned}\alpha &= 1 \\ \beta &= -\frac{b}{c} \\ \gamma &= \frac{b^2}{c^2} - \frac{a}{c}\end{aligned}$$

This expression makes sense because  $c \neq 0$  in  $U_1$ . If we take an element of  $U_2$  with

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each  $C$ -orbit, we see that

$$\begin{aligned}\overline{U}_2 &\cong \{([x:y:z], [0:0:1]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det \begin{pmatrix} x & a & y \\ y & b & z \\ z & c & 0 \end{pmatrix} \neq 0, \det \begin{pmatrix} x & a & b \\ y & b & c \\ z & c & 0 \end{pmatrix} \neq 0, 1 \neq 0\} \\ &\cong \{[x:y:z] \in \mathbb{P}^2 \mid x \neq 0, \det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \neq 0\} \\ &\cong \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}\end{aligned}$$

Under these identifications, the pushout square above becomes

$$\begin{array}{ccc}\{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} & \xrightarrow{\quad} & \mathcal{M} \\ \uparrow & & \uparrow \\ \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} & \xrightarrow{f} & \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}\end{array}$$

where

$$f : \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} \rightarrow \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}$$

$$(b, c) \mapsto \left(-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}\right)$$

Now define a variety  $V$  to be

$$V := \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0\} - \{(0, 0, 0)\}$$

I claim that  $V$  is isomorphic to  $\mathcal{M}$ . More precisely, I claim that

$$\begin{array}{ccc} \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} & \xrightarrow{i} & V \\ \uparrow g & & \uparrow \iota \\ \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\} & \xrightarrow{f} & \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \end{array}$$

is a pushout square where

$$i : \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\} \rightarrow V$$

$$(b, c) \mapsto \left(-\frac{bc}{b^2 - c}, \frac{c^2}{b^2 - c}, b\right)$$

$$\iota : \{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\} \rightarrow V$$

$$(y, z) \mapsto (y, z, \frac{yz}{z - y^2})$$

It is easy to check the square commutes.  $f, g$  are inclusion maps by construction.

$\iota$  is also an inclusion map because we can recover  $(y, z)$  from  $\iota(y, z)$  by projecting onto the 1<sup>st</sup>&2<sup>nd</sup> coordinates.

For  $i$ , we can recover  $b$  from  $i(b, c)$  by projecting onto the 3<sup>rd</sup> coordinate. We can recover  $c$  from  $i(b, c)$  by multiplying 2<sup>nd</sup>&3<sup>rd</sup> coordinates and dividing with -(the 1<sup>st</sup>

coordinate).

Let's check that the images of  $i$  and  $\iota$  form an open cover of  $V$ . The images of  $i$  and  $\iota$  are

$$i(\{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0\}) = \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, YW + Z + W^2 \neq 0\}$$

$$\begin{aligned} \iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) &= \{(Y, Z, W) \in \mathbb{A}^3 \mid Y^2W - ZW + YZ = 0, Y^2 - Z \neq 0\} \\ &= \{(Y, Z, W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z - Y^2}, Y^2 - Z \neq 0\} \end{aligned}$$

Clearly, they are open subsets of  $V$ .

Let's check that if  $(Y, Z, W) \in V$  and  $Y^2 = Z$ , then  $YW + Z + W^2 \neq 0$ . If  $Y^2 = Z$ , then the equation  $Y^2W - ZW + YZ = 0$  becomes  $YZ = 0$ . Therefore, we get  $Y = Z = 0$ . Since  $(0, 0, 0)$  is not contained in  $V$ ,  $W$  can only take non-zero values. Therefore,  $YW + Z + W^2 = W^2 \neq 0$ . We conclude that the images of  $i$  and  $\iota$  cover  $V$ .

Now let's check that

$$i^{-1}(\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

The image of  $\iota$  is

$$\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) = \{(Y, Z, W) \in \mathbb{A}^3 \mid W = \frac{YZ}{Z - Y^2}, Y^2 - Z \neq 0\}$$

we have

$$\begin{aligned}
 i(b, c) \in \iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\}) \\
 \iff (-\frac{bc}{b^2 - c})^2 \neq \frac{c^2}{b^2 - c} \\
 \iff b^2 c^2 \neq c^2(b^2 - c) \\
 \iff c^3 \neq 0 \\
 \iff c \neq 0
 \end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(y, z) \in \mathbb{A}^2 \mid z - y^2 \neq 0\})) = \{(b, c) \in \mathbb{A}^2 \mid c - b^2 \neq 0, c \neq 0\}$$

as desired.

Therefore,  $V$  is isomorphic to  $\mathcal{M}$ .

**Example 50.** In this section, we compute the moduli space associated to the braid word :  $s_1^3$  with regular unipotent monodromy. The framed moduli space is given by

$$\begin{aligned}
 \mathcal{M}^{fr} = \{([x : y], [z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid & \det \begin{pmatrix} x & z \\ y & w \end{pmatrix} \neq 0, \\
 & \det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0, \quad \det \begin{pmatrix} a & x + y \\ b & y \end{pmatrix} \neq 0 \}
 \end{aligned}$$

The centralizer subgroup of the unipotent matrix

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is

$$C := \mathrm{C}_{\mathrm{GL}_2(\mathbb{C})}(u) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

From this point on, we will use the following notation

$$\infty := [1 : 0]$$

$$X := [x : y]$$

$$Z := [z : w]$$

$$A := [a : b]$$

We can cover  $\mathcal{M}^{fr}$  with open subsets  $U_1, U_2$  where

$$U_1 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid X \neq \infty\}$$

$$U_2 = \{(X, Z, A) \in \mathcal{M}^{fr} \mid Z \neq \infty\}$$

Therefore, we have a pushout square

$$\begin{array}{ccc} U_1 & \longrightarrow & \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ U_1 \cap U_2 & \longrightarrow & U_2 \end{array}$$

Quotienting out by the centralizer subgroup  $C$ , we get

$$\begin{array}{ccc} \overline{U}_1 := C \setminus U_1 & \longrightarrow & \mathcal{M} = C \setminus \mathcal{M}^{fr} \\ \uparrow & & \uparrow \\ \overline{U_1 \cap U_2} := C \setminus U_1 \cap U_2 & \longrightarrow & \overline{U}_2 := C \setminus U_2 \end{array}$$

Also note that, for the action of  $C$ , the centralizer subgroup of any element of  $\mathcal{M}^{fr}$

is the set of scalar multiplication matrices.

First let's simplify,  $\bar{U}_1$ . Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_1$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{x}{y}$$

This expression makes sense because  $y \neq 0$  in  $U_1$ . If we take an element of  $U_1$  with

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each  $C$ -orbit, we see that

$$\begin{aligned} \bar{U}_1 &\cong \{([0 : 1], [z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} 0 & z \\ 1 & w \end{pmatrix} \neq 0, \det \begin{pmatrix} z & a \\ w & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \neq 0\} \\ &\cong \{([z : w], [a : b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z \neq 0, bz - aw \neq 0, a - b \neq 0\} \\ &\cong \{(w, [a : b]) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid b - aw \neq 0, a - b \neq 0\} \end{aligned}$$

Change variables  $a' := a + b, b' := a - b$  we get

$$\begin{aligned}\overline{U_1} &\cong \{(w, [a' : b']) \in \mathbb{A}^1 \times \mathbb{P}^1 \mid \frac{a' - b'}{2} - \frac{(a' + b')w}{2} \neq 0, b' \neq 0\} \\ &\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a' - 1) - (a' + 1)w \neq 0\} \\ &\cong \{(w, a'') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid (a'' + 1)(1 - w) - 2 \neq 0\} \\ &\cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1\}\end{aligned}$$

Under this identification,

$$\overline{U_1 \cap U_2} \cong \{(w', a''') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid a'''w' \neq 1, 2w' \neq 1\}$$

Now let's simplify,  $\overline{U}_2$ . Suppose

$$\begin{bmatrix} x & z & a \\ y & w & b \end{bmatrix} \in U_2$$

Then there exists a

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is

$$\alpha = 1$$

$$\beta = -\frac{z}{w}$$

This expression makes sense because  $w \neq 0$  in  $U_2$ . If we take an element of  $U_2$  with

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

as a representative from each  $C$ -orbit, we see that

$$\begin{aligned} \overline{U}_2 &\cong \{([x:y], [0:1], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid \det \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \neq 0, \det \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \neq 0, \det \begin{pmatrix} a & x+y \\ b & y \end{pmatrix} \neq 0\} \\ &\cong \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x \neq 0, a \neq 0, ay - b(x+y) \neq 0\} \\ &\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y - b(1+y) \neq 0\} \\ &\cong \{(y,b) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y(1-b) - b \neq 0\} \end{aligned}$$

Change variables  $b' := 1 - b, y' := y + 1$  we get

$$\overline{U}_2 \cong \{(y', b') \in \mathbb{A}^1 \times \mathbb{A}^1 \mid yb' \neq 1\}$$

Under these identifications, the pushout square above becomes

$$\begin{array}{ccc} \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1\} & \xrightarrow{\quad} & \mathcal{M} \\ \uparrow & & \uparrow \\ \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} & \xrightarrow{f} & \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \end{array}$$

where

$$\begin{aligned} f : \{(x,y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} &\rightarrow \{(a,b) \in \mathbb{A}^2 \mid ab \neq 1\} \\ (x,y) &\mapsto (2x, \frac{4x+y-4}{2xy-2}) \end{aligned}$$

Now define a variety  $V$  to be

$$V := \{(A, B, C) \in \mathbb{A}^3 \mid (AC - 2)B = 2A + C - 4\} - \{(1, 1, 2)\}$$

I claim that  $V$  is isomorphic to  $\mathcal{M}$ . More precisely, I claim that

$$\begin{array}{ccc} \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\} & \xrightarrow{i} & V \\ \uparrow g & & \uparrow \iota \\ \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\} & \xrightarrow{f} & \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} \end{array}$$

is a pushout square where

$$\begin{aligned} i : \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\} &\rightarrow V \\ (x, y) &\mapsto (2x, \frac{4x + y - 4}{2xy - 2}, y) \end{aligned}$$

$$\begin{aligned} \iota : \{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\} &\rightarrow V \\ (a, b) &\mapsto (a, b, \frac{2a + 2b - 4}{ab - 1}) \end{aligned}$$

It is easy to see that the square commutes and  $f, g, i, \iota$  are inclusion maps.

Let's check that the images of  $i$  and  $\iota$  form an open cover of  $V$ .

The image of  $i$  and  $\iota$  are

$$i(\{(x, y) \in \mathbb{A}^2 \mid xy \neq 1\}) = \{(A, B, C) \in V \mid AC \neq 2\}$$

$$\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V \mid AB \neq 1\}$$

Clearly, they are open subsets of  $V$ .

Let's check that if  $(A, B, C) \in V$  and  $AC = 2$ , then  $AB \neq 1$ . If  $AC = 2$ , then the left hand side of the equation  $(AC - 2)B = 2A + C - 4$  becomes zero. Therefore, we get  $AC = 2$  and  $2A + C = 4$  which implies  $2A + \frac{2}{A} = 4 \Leftrightarrow A^2 - 2A + 1 = 0$ . Solving

the quadratic equation, we get  $A = 1, C = 2$ . Since  $(1, 1, 2)$  is not contained in  $V$ ,  $B$  can take any value except 1. Therefore,  $AB \neq 1$ . We conclude that the images of  $i$  and  $\iota$  cover  $V$ .

Now let's check that

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \mathbb{A}^2 \mid xy \neq 1, 2x \neq 1\}$$

The image of  $\iota$  is

$$\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) = \{(A, B, C) \in V \mid AB \neq 1\}$$

we have

$$\begin{aligned} i(x, y) &\in \iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\}) \\ &\iff 2x \cdot \left( \frac{4x + y - 4}{2xy - 2} \right) \neq 1 \\ &\iff 2x \cdot (4x + y - 4) \neq 2xy - 2 \\ &\iff 8x^2 - 8x + 2 \neq 0 \\ &\iff 2(2x - 1)^2 \neq 0 \\ &\iff 2x \neq 1 \end{aligned}$$

Therefore,

$$i^{-1}(\iota(\{(a, b) \in \mathbb{A}^2 \mid ab \neq 1\})) = \{(x, y) \in \overline{U}_1 \mid 2x \neq 1\}$$

as desired.

Therefore,  $V$  is isomorphic to  $\mathcal{M}$ .

**Example 51.** In this section, we prove that the moduli space of a bivalent braid with regular unipotent monodromy is finite type integral scheme over  $\mathbb{C}$  not necessarily separated. Suppose we have a bivalent braid word with  $n$ -strands and a unipotent

monodromy

$$u = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

The framed moduli space is given as

$$\mathcal{M}^{fr} = \{X = (X^1, X^2, \dots, X^k) \in (\mathbb{P}^{n-1})^k \mid f_1(X) \neq 0, \dots, f_m(X) \neq 0\}$$

where  $f_i$ 's are determinants with column vectors of the form  $u^s \cdot X_j$ . Note that the entries of the last row(i.e. the  $n^{th}$  row) is one of  $x_{n,i}$  ( $i = 1, \dots, k$ ). Thus  $x_{n,i}$ 's ( $i = 1, \dots, k$ ) cannot be identically zero otherwise all of the  $f_r$ 's will vanish. Therefore, we have an open cover of  $\mathcal{M}^{fr}$ , i.e.  $\{U_i\}_{i=1, \dots, k}$  where  $U_i := \{X \in \mathcal{M}^{fr} \mid x_{n,i} \neq 0\}$ .

To get the moduli space, we quotient the framed moduli space with the centralizer subgroup of  $u$  in  $\mathrm{GL}_n(\mathbb{C})$  i.e.

$$C := C_{\mathrm{GL}_n(\mathbb{C})} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \mid \alpha_1 \in \mathbb{C}^*, \alpha_i \in \mathbb{C} \text{ for } i = 2, \dots, n \right\}$$

Suppose we have an element

$$X = (X^1, \dots, X^k) = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \in U_i$$

Then there exists a

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \in C$$

such that

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix} \cdot \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

That is, recursively,

$$\alpha_1 = 1$$

$$\alpha_s = -\frac{1}{x_{n,i}}(x_{n-1,i}\alpha_{s-1} + x_{n-2,i}\alpha_{s-2} + \cdots + x_{n-s+1,i}\alpha_1) = -\frac{1}{x_{n,i}}\left(\sum_{t=1}^{s-1} x_{n-t,i} \cdot \alpha_{s-t}\right)$$

This expression makes sense because  $x_{n,i} \neq 0$  in  $U_i$ . If we take an element of  $U_i$  with

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as a representative from each  $C$ -orbit, we see that

$$\overline{U}_i := C \setminus U_i \cong \left\{ \begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \in (\mathbb{P}^{n-1})^k \mid f_1 \left( \begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \right) \neq 0, \dots, \right. \\ \left. f_m \left( \begin{bmatrix} x_{1,1} & 0 & x_{1,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ x_{n-1,1} & 0 & x_{n-1,n} \\ x_{n,1} & 1 & x_{n,n} \end{bmatrix} \right) \neq 0 \right\}$$

$\uparrow$   
i<sup>th</sup> column

which are finite type scheme over  $\mathbb{C}$ . In summary, we have found an finite open cover of  $\mathcal{M}$  i.e.  $\{\overline{U}_i\}_{i=1,\dots,k}$  such that each open is a finite type scheme over  $\mathbb{C}$ . Thus,  $\mathcal{M}$  is also a finite type scheme over  $\mathcal{C}$ . Now we have a smooth surjective map  $\pi : \mathcal{M}^{fr} \rightarrow \mathcal{M}$ .  $\mathcal{M}$  is irreducible because  $\mathcal{M}^{fr}$  is irreducible and  $\pi$  is surjective.  $\mathcal{M}$  is reduced because  $\mathcal{M}^{fr}$  is reduced and  $\pi$  is smooth. Therefore, we conclude that the moduli space associated attached to bivalent braid with regular unipotent monodromy is finite type integral scheme over  $\mathbb{C}$ . But it may not be separated.

**Example 52.** In this section, I will provide an example of the moduli space associated to bivalent braid with regular unipotent monodromy that is non-separated. Consider a 2-strand braid given by the braid word  $s_1^2$  and a regular unipotent monodromy

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The framed moduli space is given as follows

$$\begin{aligned}
 \mathcal{M}^{fr} &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], u \cdot [a:b] \neq [x:y]\} \\
 &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid [x:y] \neq [a:b], [a+b:b] \neq [x:y]\} \\
 &= \{([x:y], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid ay \neq bx, (a+b)y \neq bx\}
 \end{aligned}$$

Then we have an open cover  $\{U_i\}_{i=1,2}$  where

$$\begin{aligned}
 U_1 &:= \{([x:y], [a:b]) \in \mathcal{M}^{fr} \mid b \neq 0\} \\
 U_2 &:= \{([x:y], [a:b]) \in \mathcal{M}^{fr} \mid y \neq 0\}
 \end{aligned}$$

We get a pushout square

$$\begin{array}{ccc}
 U_1 & \longrightarrow & \mathcal{M}^{fr} \\
 \downarrow & & \downarrow \\
 U_1 \cap U_2 & \longrightarrow & U_2
 \end{array}$$

we take quotients of these opens with respect to the centralizer subgroup of  $u$  i.e.

$$C := C_{\mathrm{GL}_2(\mathbb{C})} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid \alpha \neq 0 \right\}$$

we get

$$\begin{aligned}
 \overline{U}_1 &:= C \setminus U_1 \cong \{([x:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x\} \\
 &\cong \{([1:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} \\
 \overline{U}_2 &:= C \setminus U_2 \cong \{([0:1], [a:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid a \neq 0, (a+b) \neq 0\} \\
 &\cong \{([0:1], [1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\} \\
 \overline{U}_1 \cap \overline{U}_2 &:= C \setminus U_1 \cong \{([x:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
 &\cong \{([x:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid 0 \neq x, y \neq x, y \neq 0\} \\
 &\cong \{([1:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\}
 \end{aligned}$$

and a pushout square

$$\begin{array}{ccc}
 \{([1:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1\} & \longrightarrow & \mathcal{M} \\
 \downarrow & & \downarrow \\
 \{([1:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} & \xrightarrow{f} & \cong \{([0:1], [1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}
 \end{array}$$

where

$$f : \{([1:y], [0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid y \neq 1, y \neq 0\} \rightarrow \{([0:1], [1:b]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid b \neq -1\}$$

$$([1:y], [0:1]) \mapsto ([0:1], [1:-y])$$

Now consider the map

$$\begin{aligned}
 g : \mathbb{A}^1 - \{0, 1\} &\longrightarrow \overline{U}_1 \cap \overline{U}_2 \subseteq \mathcal{M} \\
 y &\mapsto ([1:y], [0:1])
 \end{aligned}$$

This map extends to  $\mathbb{A}^1$  in two different ways i.e. we have two distinct  $h_1, h_2$  that fit

into the following commutative square

$$\begin{array}{ccc} \mathbb{A}^1 - \{0, 1\} & \xhookrightarrow{\quad} & \mathbb{A}^1 - \{1\} \\ & \searrow^g & \downarrow h_i \\ & & \mathcal{M} \end{array}$$

which are

$$h_1 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_1 \subseteq \mathcal{M}$$

$$y \mapsto ([1:y], [0:1])$$

$$h_2 : \mathbb{A}^1 - \{1\} \longrightarrow \overline{U}_2 \subseteq \mathcal{M}$$

$$y \mapsto ([0:1], [1:-y])$$

$h_1$  and  $h_2$  are distinct because  $h_1(0) \in U_1 - U_2$ . Therefore, by the valuative criterion for separatedness,  $\mathcal{M}$  is non-separated.

# Chapter 3

## Natural Cluster Coordinates on Braid Moduli Spaces

### 3.1 Local Morse Group Computation

In this section, we compute the local Morse groups of a cochain complex of sheaves whose cohomologies are constructible with respect to the coordinate stratifications- See Definition ??.

#### Definitions and Notations

**Definition 53.** We define  $sgn : \mathbb{R} \rightarrow \{-1, 0, 1\}$  such that

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

**Definition 54.** We define  $[n]$  to be a set of integers from 1 to  $n$  i.e.

$$[n] := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$$

**Definition 55.** Let  $X$  be a set, then we denote the power set of  $X$  as  $\mathcal{P}(X)$ .

**Definition 56.** We denote the  $i^{th}$  standard basis of  $\mathbb{R}^n$  to be  $e_i^n$ .

**Definition 57.** Let

$$s^n(sgn_1, \dots, sgn_n) := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \text{for } k = 1, \dots, n, sgn(x_k) = sgn_k\}$$

To simplify the notation, we use + instead of +1 and – instead of -1. For example,

$$s^n(+, -, 0) := s^n(1, -1, 0).$$

**Definition 58.** 1. We define a *coordinate stratification*  $\mathcal{S}^n$  on  $\mathbb{R}^n$  to be

$$\{s^n(sgn_1, \dots, sgn_n) \mid sgn_k \in \{-, 0, +\} \text{ for } k = 1, \dots, n\}$$

2. There is a natural poset structure of  $\mathcal{S}^n$  such that  $s^n(sgn_1, \dots, sgn_n) \leq s^n(sgn'_1, \dots, sgn'_n)$  if and only if  $s^n(sgn'_1, \dots, sgn'_n) \subset \text{star}(s^n(sgn_1, \dots, sgn_n))$ . This is equivalent to saying that  $|sgn_i| \leq |sgn'_i|$  for all  $i \in [n]$ .
3. There is also a natural poset structure on  $n$ -dimensional strata of  $\mathcal{S}^n$ , where  $s^n(sgn_1, \dots, sgn_n) \leq s^n(sgn'_1, \dots, sgn'_n)$  if and only if  $sgn_i \leq sgn'_i$  for all  $i \in [n]$ . Note that this is not the poset structure that is inherited from the poset structure mentioned above.
4. We give co-orientations to  $(n - 1)$ -dimensional stratum as follows: Note that codimension 1 strata can be expressed as  $s^n(sgn_1, \dots, sgn_n)$  where only one of

arguments is zero, say  $sgn_i$ . Then we define the co-orientation of  $s^n(sgn_1, \dots, sgn_n)$

to be  $-dx_i$  i.e. hairs are pointing the region  $s^n(sgn_1, \dots, \underset{i^{th}}{\uparrow}, \dots, sgn_n)$

5. Let  $x \in \mathbb{R}^n$ , then  $s_x^n$  is defined as the stratum in  $\mathcal{S}^n$  containing  $x$ .

Suppose we have  $\mathcal{F}^\bullet$  a complex of constructible sheaves on  $\mathbb{R}^n$  constructible with respect to the coordinate stratification. In this section, we will compute local Morse groups of  $\mathcal{F}^\bullet$  when given  $(x, \xi) \in T^*\mathbb{R}^n$  i.e. we will compute the stalk of the microlocalization of  $\mathcal{F}^\bullet (= \mu\mathcal{F}^\bullet)$  on  $T^*\mathbb{R}^n$ . Note that the singular support of  $\mathcal{F}^\bullet$ ,  $SS(\mathcal{F}^\bullet)$ , is contained in the union of  $2^n$  Lagrangian subspaces that are conormals of strata in  $\mathcal{S}^n$  i.e.  $\Lambda_{\mathcal{S}^n} = \bigcup_{s \in \mathcal{S}^n} N^*s \subset T^*\mathbb{R}^n$ . These Lagrangians intersect with each other so union of them form a singular Lagrangian and it has smooth part in it, say  $\Lambda_{\mathcal{S}^n}^{smooth} \subset \Lambda_{\mathcal{S}^n}$ . Now we know that the singular support  $SS(\mathcal{F}^\bullet) := supp(\mu\mathcal{F}^\bullet) \subset \Lambda_{\mathcal{S}^n}^{smooth}$  and  $\mu\mathcal{F}^\bullet$  is constant on each component of  $\Lambda_{\mathcal{S}^n}^{smooth}$ , we define the local Morse group of a component of  $\Lambda_{\mathcal{S}^n}^{smooth}$  with coefficient  $\mathcal{F}^\bullet$  to be the local Morse group of a point in the component with coefficient  $\mathcal{F}^\bullet$ . We will see later that the components are contractible so this is well-defined upto unique isomorphism. Note that  $\Lambda_{\mathcal{S}^n}^{smooth}$  has  $2^{2n}$  components which are labelled by  $\mathcal{P}([n]) \times \mathcal{P}([n])$ . More precisely, for each  $(I, J) \in \mathcal{P}([n]) \times \mathcal{P}([n])$ , we have a component  $comp_{(I, J)}^n \subset \Lambda_{\mathcal{S}^n}^{smooth}$  that is the image of

$$\iota_{(I, J)}^n : \mathbb{R}_{>0}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

where  $(x_1, \dots, x_n)^T$  is mapped to

$$((\delta_I(1)sgn_J(1)x_1, \dots, \delta_I(n)sgn_J(n)x_n)^T, ((1-\delta_I(1))sgn_J(1)x_1, \dots, (1-\delta_I(n))sgn_J(n)x_n)^T)$$

where

- $\delta_I : [n] \rightarrow \{0, 1\}$

$$\delta_I(k) = \begin{cases} 1 & \text{if } k \in I \\ 0 & \text{if } k \notin I \end{cases}$$

- $sgn_J : [n] \rightarrow \{-1, 1\}$

$$sgn_J(k) = \begin{cases} 1 & \text{if } k \in J \\ -1 & \text{if } k \notin J \end{cases}$$

I will choose the representative of the  $comp_{(I,J)}^n$  to be

$$(x_{(I,J)}, \xi_{(I,J)}) := \iota_{(I,J)}^n((1, \dots, 1)^T)$$

Also, we define  $G_{(I,J)}(\mathcal{F}^\bullet)$  to be the local Morse group of  $comp_{(I,J)}^n$  with coefficient  $\mathcal{F}^\bullet$  i.e.

$$H^*(N_x, N_x \cap \xi^{-1}((-\infty, c - \epsilon)); \mathcal{F}^\bullet)$$

where  $(x, \xi) \in comp_{(I,J)}^n$ ,  $N_x$  a regular neighborhood of  $x$ , and  $\epsilon$  a small positive number.

**Definition 59.** We define

- $Q^n$  to be the quiver associated to  $\mathcal{S}^n$  i.e. we have one vertex for each  $n$ -dimensional stratum and one arrow for each  $(n-1)$ -dimensional stratum where the direction goes against the co-orientation.
- $F^\bullet$  to be the legible diagram of  $\mathcal{F}^\bullet$  i.e. the quiver representation of  $Q^n$  induced by  $\mathcal{F}^\bullet$ .
- $Q_s^n$  to be the full suquiver of  $Q^n$  spanned by vertices whose corresponding strata are contained in the star of  $s \in \mathcal{S}^n$ .

Also, we simplify the notation as follows:

- $s_{(I,J)}^n := s_{x_{(I,J)}}^n$

- For  $k \notin I$ ,  $s_{(I,J)}^n(k) := s_{(x_{(I,J)} - sgn_J(k) \cdot e_k^n)}^n$

**Lemma 60.** Suppose we have a cochain complex of  $\mathcal{S}^n$ -constructible sheaf  $\mathcal{F}^\bullet$  on  $\mathbb{R}^n$  that could be described by a legible diagram  $F^\bullet$  on  $Q^n$ , then

$$G_{(I,J)}(\mathcal{F}^\bullet) = \begin{cases} 0 & \text{if } I^c \notin J \\ H^*(\text{Tot}(F^\bullet|_{Q_{s_{(I,J)}}^n})) \text{ up to an overall shift in degree} & \text{if } k \in J \end{cases}$$

*Proof.* We prove the Lemma by induction on  $n$ .

(i) suppose  $I$  is not empty i.e.  $x_{(I,J)} \neq 0$ . We want to compute  $G_{(I,J)}(\mathcal{F}^\bullet) = H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c-\epsilon]); \mathcal{F}^\bullet)$  where  $N_{x_{(I,J)}}$  is a regular neighborhood of  $x_{(I,J)}$ ,  $c = \xi_{(I,J)}(x_{(I,J)})$ , and  $\epsilon$  a small positive real number. Once and for all fix  $i \in I$ , consider the following linear map

$$\text{Lin}_i : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

where

$$\text{Lin}_i(e_j) = \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } j \geq i \end{cases}$$

and let  $\text{Aff}_i := \text{Lin}_i + e_i^n$  be the affine inclusion map. Define

- $I' = \{k \in I \mid k < i\} \cup \{k-1 \mid k > i, k \in I\}$
- $J' = \{k \in J \mid k < i\} \cup \{k-1 \mid k > i, k \in J\}$

and  $(x_{(I',J')}, \xi_{(I',J')}) = \iota_{(I',J')}^{n-1}((1, \dots, 1)^T)$ , then  $x_{(I,J)} \in \text{Im}(\text{Aff}_i)$  and  $\alpha_i^{-1}(x_{(I,J)}) = x_{(I',J')}$  and  $\xi_{(I,J)} \circ \text{Aff}_i$ . We let the regular neighborhood of  $x_{(I,J)}$ ,  $N_{x_{(I,J)}}$ , to be  $\text{star}(s_{(I,J)}^{n-1})$  and the regular neighborhood of  $x_{(I',J')}$ ,  $N_{x_{(I',J')}}$ , to be  $\text{star}(s_{(I',J')}^{n-1})$ ,  $c = \xi_{(I,J)}(x_{(I,J)})$ , and  $c' = \xi_{(I',J')}(x_{(I',J')})$

(Claim)  $H^*(N_{x_{(I,J)}}, N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c-\epsilon)); \mathcal{F}^\bullet) \cong H^*(N_{x_{(I',J')}}, N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c'-\epsilon)); \text{Aff}_i^* \mathcal{F}^\bullet)$

(proof) Note that when we have a stratified spaces  $(M, \mathcal{S}_M)$  and  $N$ , a subspace of  $M$ , we have an induced stratification  $\mathcal{S}_N$  on  $N$ . Then we can identify poset  $\mathcal{S}_N$  as the subposet of  $\mathcal{S}_M$  via  $(N \hookrightarrow M)_*$ .

Now consider the following cartesian diagram of inclusion maps

$$\begin{array}{ccc} N_{x_{(I,J)}} & \longleftrightarrow & N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c - \epsilon)) \\ \text{Aff}_i \uparrow & & \uparrow \\ N_{x_{(I',J')}} & \longleftrightarrow & N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon)) \end{array}$$

Note that  $\text{Aff}_i$  induces a poset isomorphism from the poset  $\mathcal{S}_{N_{x_{(I',J')}}}$  to the poset  $\mathcal{S}_{N_{x_{(I,J)}}}$  where via this isomorphism the subposet  $\mathcal{S}_{N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon))}$  gets identified with  $\mathcal{S}_{N_{x_{(I,J)}} \cap \xi_{(I,J)}^{-1}((-\infty, c - \epsilon))}$ . Furthermore, the representation of the poset  $\mathcal{S}_{N_{x_{(I',J')}}}$  that gives rise to the sheaf  $\text{Aff}_i^* \mathcal{F}^\bullet$  is exactly the representation of the poset  $\mathcal{S}_{N_{x_{(I,J)}}}$  that gives rise to the sheaf  $\mathcal{F}^\bullet$  under the poset isomorphism  $(\text{Aff}_i)_*$  and the relative cohomology is completely determined by the poset, subpost structures and their representations. Therefore, the claim is proved.

Therefore,  $G_{(I,J)}(\mathcal{F}^\bullet) \cong H^*(N_{x_{(I',J')}}, N_{x_{(I',J')}} \cap \xi_{(I',J')}^{-1}((-\infty, c' - \epsilon)); \text{Aff}_i^* \mathcal{F}^\bullet)$ . By the induction hypothesis if  $I'^c \not\subset J'^c$ , this is 0 and if  $I'^c \subset J'^c$ , this is equal to  $\text{Tot}(\text{Aff}_i^*(F^\bullet|_{\mathcal{S}_{N_{x_{(I',J')}}}})) = \text{Tot}(F^\bullet|_{\mathcal{S}_{N_{x_{(I,J)}}}}) = \text{Tot}(F^\bullet|_{Q_{s_{(I,J)}}^n})$ . Because  $I^c \subset J^c$  if and only if  $I'^c \subset J'^c$ , we proved the case when  $I \neq \phi$ .

(ii) now consider the case where  $I = \phi$ .

(Case1)  $I^c \not\subset J^c$  i.e.  $J = \phi$ , then we want to prove that  $G_{(I,J)}(\mathcal{F}^\bullet) = 0$ . Equivalently, we want to prove that the mapping cone of the following restriction map is acyclic.

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c - \epsilon)); \mathcal{F}^\bullet)$$

First, let's compute  $R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet)$ . We can use a singleton Čech cover  $\{N_{x_{(I,J)}} = \text{star}(s_{(I,J)}^n)\}$ . Therefore,  $R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \cong \text{Tot}(F^\bullet(N_{x_{(I,J)}}))$ .

Next, let's compute  $R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon]); \mathcal{F}^\bullet)$ . We can use the following Čech cover

$$\mathcal{U} = \{U_k\}_{k \in [n]}$$

where

$$U_k := \text{star}(s_{x_{(I,J)}}^n(k)) = \text{star}(s^n(0, \dots, 0, \underset{\substack{\uparrow \\ k^{th}}}{(-1)^{\delta_J(k)}}, 0, \dots, 0))$$

We define

$$U_{i_0 \dots i_p} := N_{x_{(I,J)}} \cap U_{i_0} \cap \dots \cap U_{i_p}$$

Note that

- if the index set in the subscript is empty, then  $U = N_{x_{I,J}} = \mathbb{R}^n$
- $U_{i_0 \dots i_p} = \text{star}(s^n(sgn_1, \dots, sgn_n))$  where

$$sgn_k = \begin{cases} (-1)^{\delta_J(k)} & \text{if } k \in \{i_0, \dots, i_p\} \\ 0 & \text{otherwise} \end{cases}$$

Since  $J \neq \emptyset$ , without loss of generality assume  $1 \in J$  (we can always relabel coordinates). Note that for all  $1 \leq i_0 < \dots < i_p \leq n$

$$F^\bullet(U_{i_0 \dots i_p}) = F^\bullet(U_{1 i_0 \dots i_p})$$

because  $F^\bullet$  a poset representation obtained from a legible diagram.

From the above Čech cover we get a Čech double complex

$$C^{p,q} = \bigoplus_{1 \leq i_0 < \dots < i_p \leq n} F^q(U_{i_0 \dots i_p})$$

where horizontal maps are Čech differentials( $\delta$ ) and vertical maps are cochain differentials( $d$ ).

The mapping cone of

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon)); \mathcal{F}^\bullet)$$

is the total complex of the following double complex

$$C_{ext}^{p,q} = \begin{cases} C^{p,q} & \text{if } p \geq 0 \\ F^q(N_{x_{(I,J)}}) = F^q(\mathbb{R}^n) & \text{if } p = -1 \\ 0 & \text{otherwise} \end{cases}$$

where  $\delta^{-1,q}$  is just a restriction map.

Now I will define a homotopy  $h^{\bullet,\bullet} : C_{ext}^{\bullet,\bullet} \rightarrow C_{ext}^{\bullet-1,\bullet}$  on the Čech double complex that induces homotopy  $h$  on the total complex of  $C_{ext}^{\bullet,\bullet}$ . Let  $c \in C^{p,q} := \bigoplus_{1 \leq i_0 < \dots < i_p \leq n} F^q(U_{i_0 \dots i_p})$ , then we define  $hc \in C^{p-1,q}$  so that

$$(hc)_{i_0 \dots i_p} = c_{1i_0 \dots i_p}$$

Here we are using the fact that if  $1 \notin \{i_0, \dots, i_p\}$

$$F^q(U_{i_0 \dots i_p}) = F^q(U_{1i_0 \dots i_p})$$

Note that if one of  $i_0, \dots, i_p$  is 1, then  $(hc)_{i_0 \dots i_p} = 0$ . Also note that  $h^{p,q-1}d^{p,q} = d^{p-1,q}h^{p,q}$ . Now let's compute  $hD_{Tot} + D_{Tot}h$  where  $D_{Tot}$  is the total differential. Suppose  $c \in C_{ext}^{-1,q}$ , then

- $(hD_{Tot}c) = (h(-d+\delta)c) = -(hdc) + (h\delta c)$ . Because  $hdc$  lands on degree  $p = -2$ ,  $(hdc) = 0$  and  $(hD_{Tot}c) = (h\delta c) = c|_{U_1} = c$ . Here, we use the identity map between  $F^q(N_{x_{(I,J)}}) = F^q(U_1)$  and think of  $c|_{U_1}$  as an element of  $F^q(N_{x_{(I,J)}})$  that uniquely extends  $c|_{U_1}$ , which is  $c$  itself.

- $(D_{Tot}hc) = D_{Tot}0 = 0$  because  $hc$  lands on degree  $p = -2$ .

Suppose  $c \in C_{ext}^{p,q}$  ( $p \geq 0$ ),

- $hD_{Tot} = h((-1)^p d + \delta) = (-1)^p hd + h\delta$
- $D_{Tot}h = ((-1)^{p-1} d + \delta)h = (-1)^{p-1} dh + \delta h$

Therefore,  $hD_{Tot} + D_{Tot}h = h\delta + \delta h$  and

$$\begin{aligned}
 (h\delta c + \delta hc)_{i_0 \dots i_p} &= (h\delta c)_{i_0 \dots i_p} + (\delta hc)_{i_0 \dots i_p} \\
 &= (\delta c)_{1i_0 \dots i_p} + \sum_{j=0}^p (-1)^j (hc)_{i_0 \dots \hat{i}_j \dots i_p} \\
 &= [c_{i_0 \dots i_p} + \sum_{j=0}^p (-1)^{j+1} c_{1i_0 \dots \hat{i}_j \dots i_p}] + \sum_{j=0}^p (-1)^j (hc)_{i_0 \dots \hat{i}_j \dots i_p} \\
 &= c_{i_0 \dots i_p}
 \end{aligned}$$

Therefore,  $h$  is a homotopy between identity cochain map on  $\text{Tot}(C_{ext}^{\bullet, \bullet})$  and a zero map. Therefore, we conclude that  $\text{Tot}(C_{ext}^{\bullet, \bullet})$  is acyclic.

(Case2)  $I^c \subset J^c$  i.e.  $J = \phi$ , then use the same Čech cover  $\mathcal{U}$  as in (case1). Then again the mapping cone of

$$R\Gamma(N_{x_{(I,J)}}; \mathcal{F}^\bullet) \rightarrow R\Gamma(N_{x_{(I,J)}} \cap \xi^{-1}((\infty, c-\epsilon)); \mathcal{F}^\bullet)$$

is the total complex of the following double complex up to a shift

$$F^\bullet(N_{x_{(I,J)}}) \rightarrow \bigoplus F^\bullet(U_{i_0}) \rightarrow \bigoplus F^\bullet(U_{i_0 i_1}) \rightarrow \dots$$

which is equal to

$$\begin{aligned} F^\bullet(\text{star}(s^n(0, \dots, 0))) &\rightarrow \oplus F^\bullet(\text{star}(s^n(\text{sgn}_{i_0} = +, \text{else } 0))) \rightarrow \\ &\oplus F^\bullet(\text{star}(s^n(\text{sgn}_{i_0} = \text{sgn}_{i_1} = +, \text{else } 0))) \rightarrow \dots \end{aligned}$$

which is equal to

$$\begin{aligned} F^\bullet(\text{star}(s^n(-, \dots, -))) &\rightarrow \oplus F^\bullet(\text{star}(s^n(\text{sgn}_{i_0} = +, \text{else } -))) \rightarrow \\ &\oplus F^\bullet(\text{star}(s^n(\text{sgn}_{i_0} = \text{sgn}_{i_1} = +, \text{else } -))) \rightarrow \dots \end{aligned}$$

whose total complex is the total complex of  $F^\bullet$  restricted to  $Q^n$ . Because  $Q^n = Q_{s^n(-, \dots, -)}^n$  the proof is complete.  $\square$

## 3.2 natural alternating diagram

Suppose we have a positive braid word  $\omega$ , then we can draw the associated braid diagram  $(i_1, \dots, i_{n-1} : [0, 1]_x \rightarrow [0, 1]_x \times (0, 1)_z)$  where  $i_k$  are smooth sections of the projection  $[0, 1]_x \times (0, 1)_z \rightarrow [0, 1]_x$  on  $[0, 1]_x \times (0, 1)_z$  and its cylindrical closure  $S_x^1 \times (0, 1)_z$ . For example, if  $\omega = s_1 s_1$ , a braid word on 3 strand, then the cylindrical closure of the associated braid diagram is shown in the figure below.

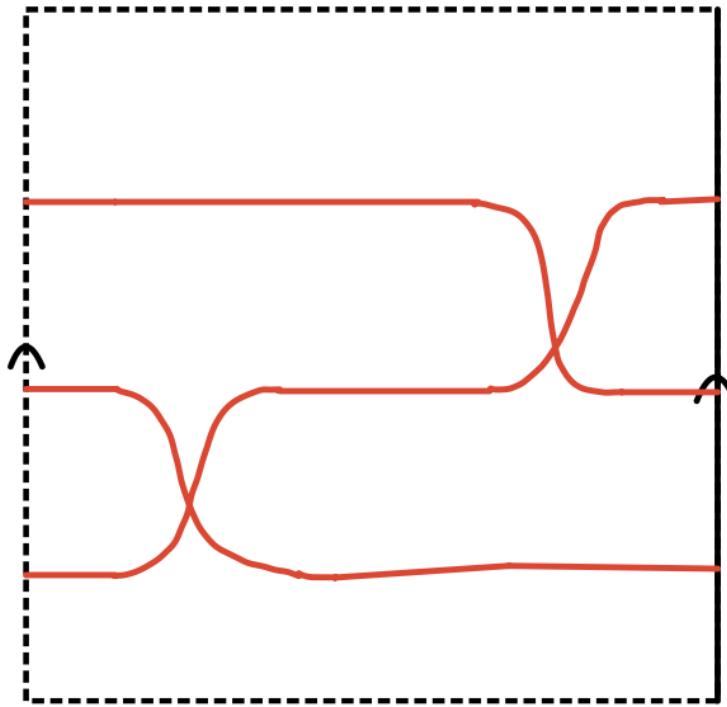


Figure 3.1

We will specify co-orientations of  $i_1, \dots, i_{n-1}$  so that we can think of the cylindrical closure of the braid word as the front projection of a Legendrian knot living inside the co-circle bundle of the cylindrical closure.

Let  $x_0 \in [0, 1]$ , we define the co-orientation at  $i_k(x_0)$  to be  $\xi = adx + cdz$  so that  $\xi$  vanishes at  $\frac{di_k}{dt}|_{x=x_0}$ ,  $\|(a, c)\| = 1$ , and  $c > 0$ . This can be visually represented as hairs pointing upward(i.e.  $c > 0$ ).

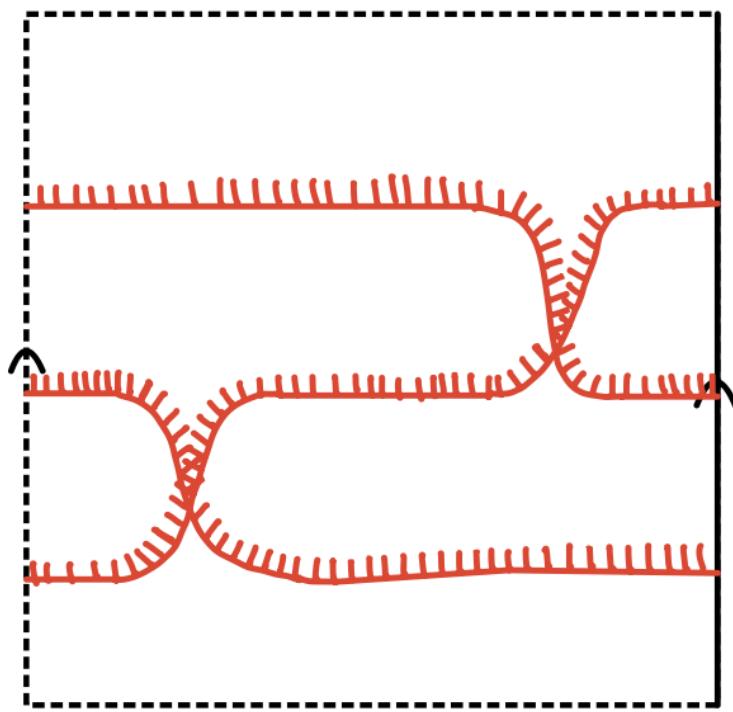


Figure 3.2

Suppose we have a Riemann sphere  $M$  with two punctures at  $0$  and  $\infty$ .  $M$  is diffeomorphic to the boundaryless cylinder as shown in the figure below.

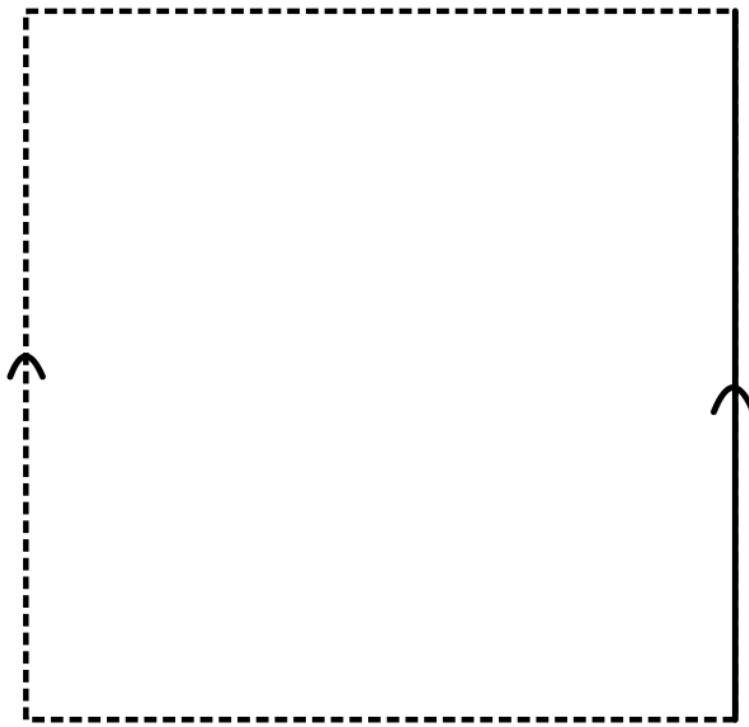


Figure 3.3

There are two distinguished ways of embedding the cylindrical closure of  $\omega$  into  $M$ . We can embed the cylindrical closure onto the hemisphere containing 0( $\infty$  resp.), i.e. the lower hemisphere(upper hemisphere resp.), in such a way that the embedding extends

- (i) to  $S^1 \times \{0\}$ ) as an isomorphism onto the equator of  $M$
- (ii) to  $S^1 \times \{1\}$  as a constant map to 0( $\infty$  resp.)

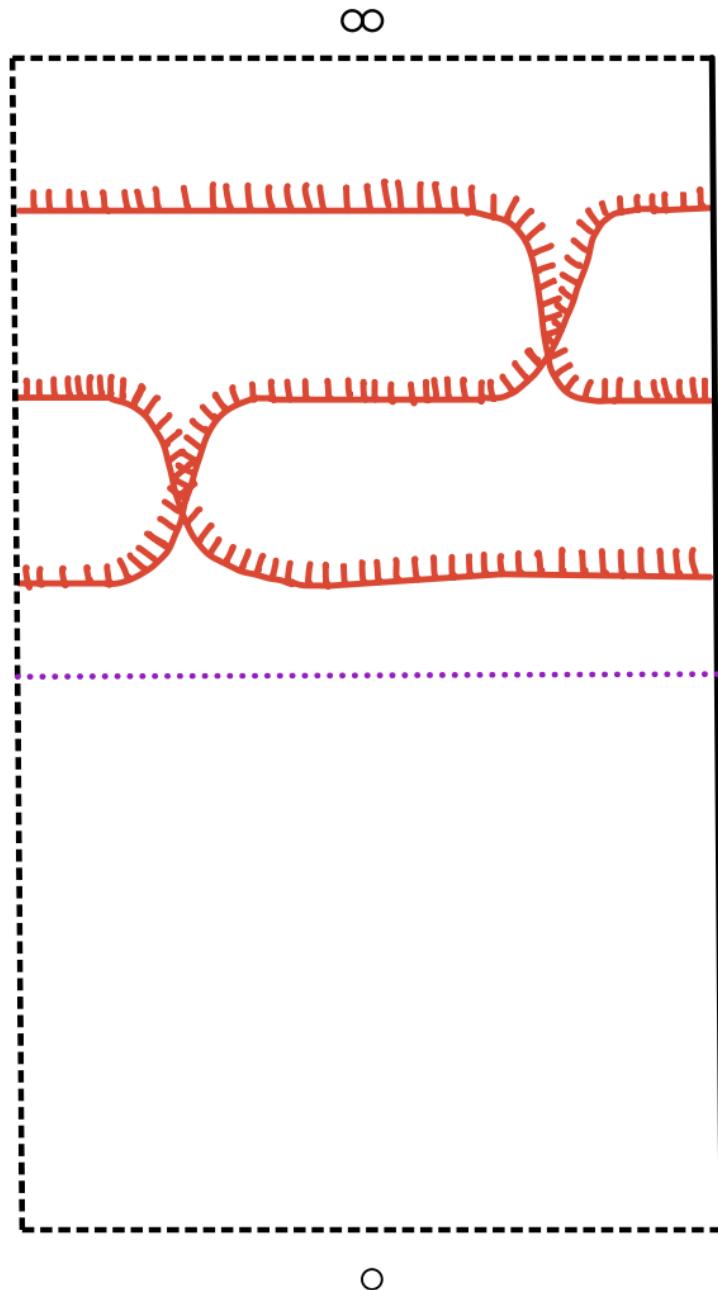


Figure 3.4

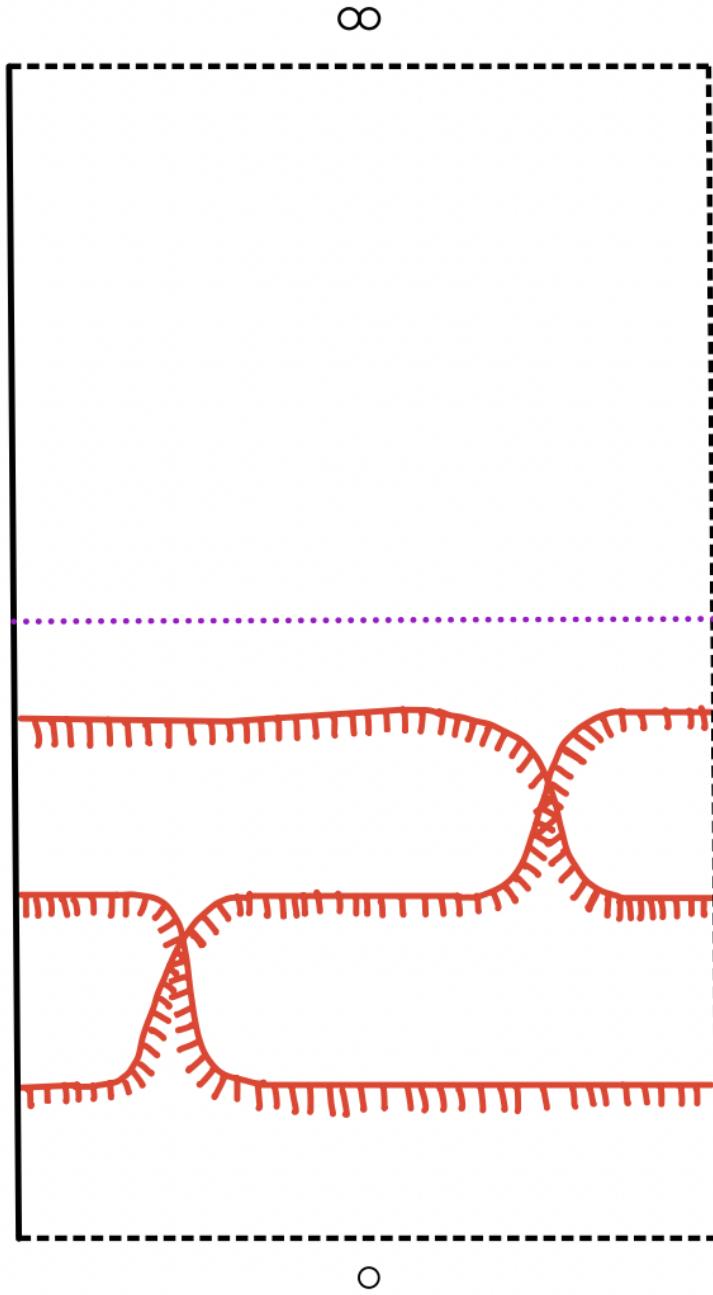


Figure 3.5

Suppose we have a positive braid word  $\omega$  on  $n$  strands, we have the following associated objects:

- $M$ : A Riemann sphere with two punctures at 0 and  $\infty$
- $\Phi_0 : \coprod_{i=1}^n S^1 \rightarrow M$ : the front projection induced by the embedding of the

cylindrical closure of the trivial braid word onto the lower hemisphere

- $\xi_0$ : the co-orientation of  $\Phi_0$
- $\Phi_\infty : \coprod_{i=1}^m S^1 \rightarrow M$  (where  $m \leq n$ ): the front projection given by the embedding of the cylindrical closure of the braid word  $\omega$  onto the upper hemisphere
- $\xi_\infty$  : the co-orientation of  $\iota_\infty$

To simplify the notation, we will denote the pair  $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$  resp.) as  $\Lambda_0(\Lambda_\infty$  resp.). Also, we will abuse  $\Lambda_0(\Lambda_\infty$  resp.) to denote the Legendrian associated to the pair  $(\Phi_0, \xi_0)((\Phi_\infty, \xi_\infty)$  resp.).

Now fix a positive braid word  $\omega$  and the object  $(M, \Lambda_0, \Lambda_\infty)$  associated with it which we call the separated diagram of  $\omega$ . I will define a natural alternating braid diagram  $(M, \Lambda'_0, \Lambda'_\infty)$  whose associated Legendrian is Legendrian isotopic to the Legendrian associated with  $(M, \Lambda_0, \Lambda_\infty)$ . I will construct an explicit Legendrian isotopy between them. Furthermore, I will construct cobordisms between constructible sheaves singular supported on  $(M, \Lambda_0, \Lambda_\infty)$  and  $(M, \Lambda'_0, \Lambda'_\infty)$  which will be the main result of this chapter. The isotopy will be only applied to  $\Lambda_0$ , so the  $\Lambda_\infty$  will remain fixed i.e.  $\Lambda_\infty = \Lambda'_\infty$ .

First, let's draw  $\Lambda'_\infty$  as in red on  $M$  as follows :

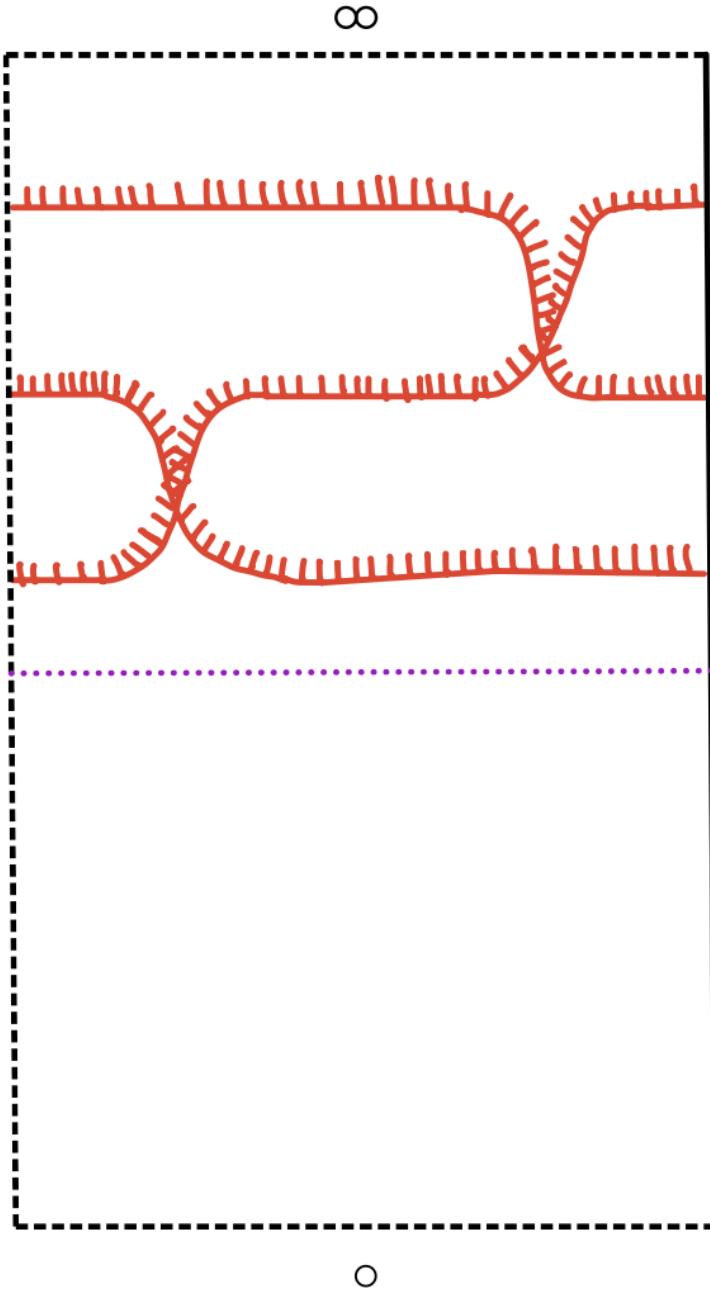


Figure 3.6

Now on the above diagram, let's draw  $\Lambda'_0$  the part that is Legendrian isotopic to  $\Lambda_0$  in blue. But before that, we need some definitions.

**Definition 61.** Suppose  $\omega = s_{1_1}, \dots, s_{i_k}$ , then the cylindrical closure can be parsed into a concatenation of  $k$  mutually disjoint regions where  $i^{th}$  region containing a part of the braid diagram corresponding to the generator  $s_{i_j}$  shown in the figure below. We

call the region corresponding to  $s_{ij}$  as the  $j^{th}$  generator region(also its image under the embedding into  $M$ ).

Below is the picture of the  $1^{st}$  generator region of the cylindrical closure of  $\omega = s_1 s_2$ .

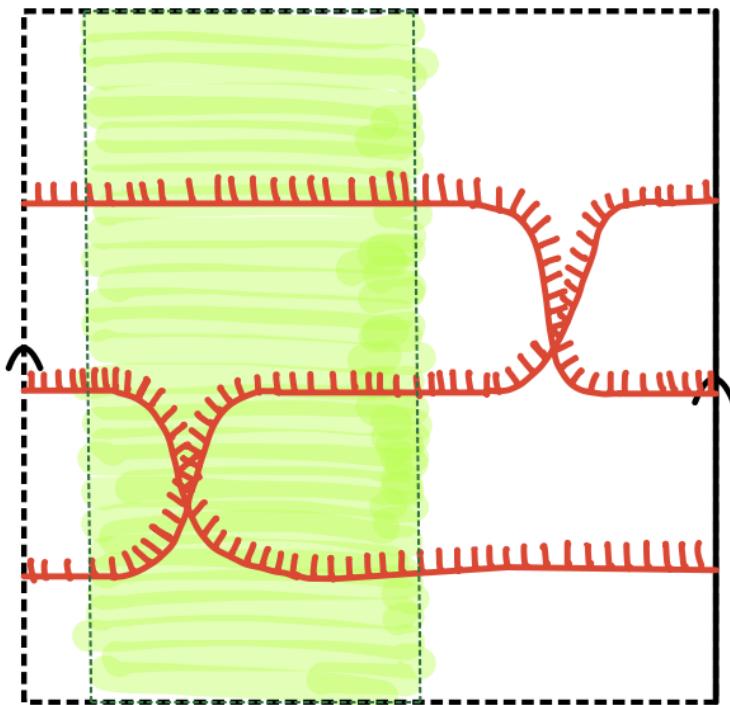


Figure 3.7

Below is the picture of the  $2^{nd}$  generator region of the cylindrical closure of  $\omega = s_1 s_2$ .

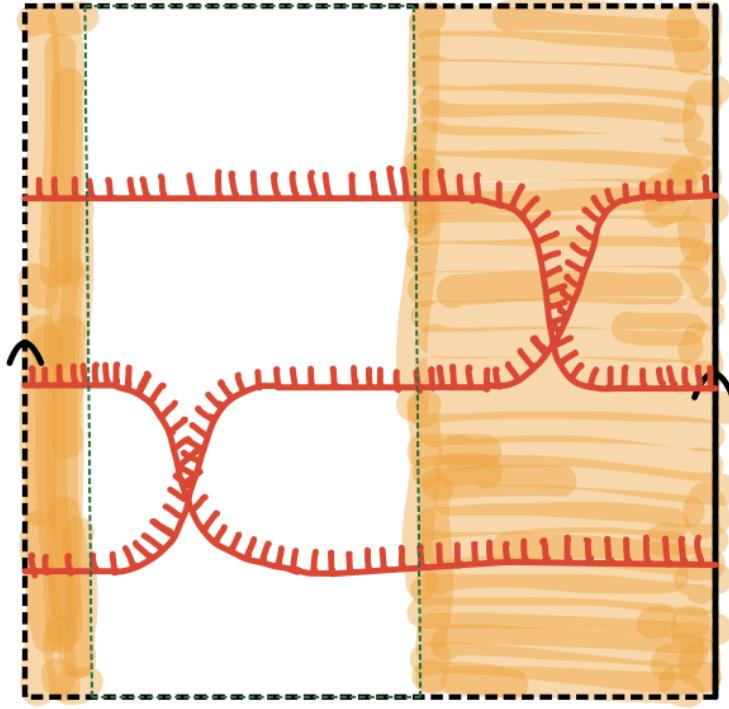


Figure 3.8

**Definition 62.** Suppose we set-theoretically subtract the union of all generator regions from the cylinder, we get  $k$  connected components. That is, for each  $j = 1, \dots, k$ , we have one component in between  $j^{th}$  and  $(j + 1 \pmod k)^{th}$  regions. We call the neighborhood of this component inside the cylinder as  $j^{th}$  inter-generator region(also its image inside the cylinder under the embedding into  $M$ ).

- inter-generator regions do not contain any crossing
- inter-generator regions are mutually disjoint
- $j^{th}$  intergenerator region intersects with  $j^{th}$  and  $j + 1^{th}$ (modulo  $k$ ) generator region

Below is the picture of the  $1^{st}$  inter-generator region of the cylindrical closure of  $\omega = s_1 s_2$ .

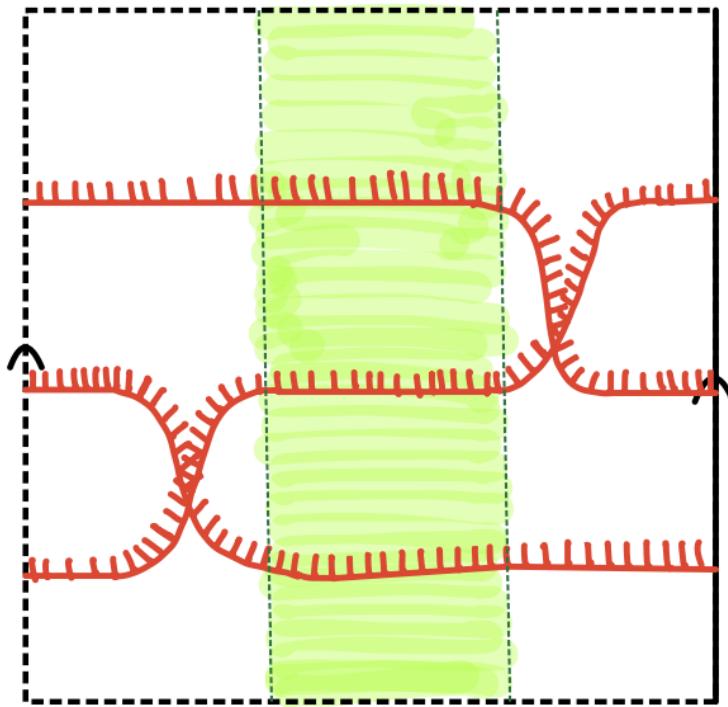


Figure 3.9

Below is the picture of the  $2^{nd}$  generator region of the cylindrical closure of  $\omega = s_1 s_2$ .

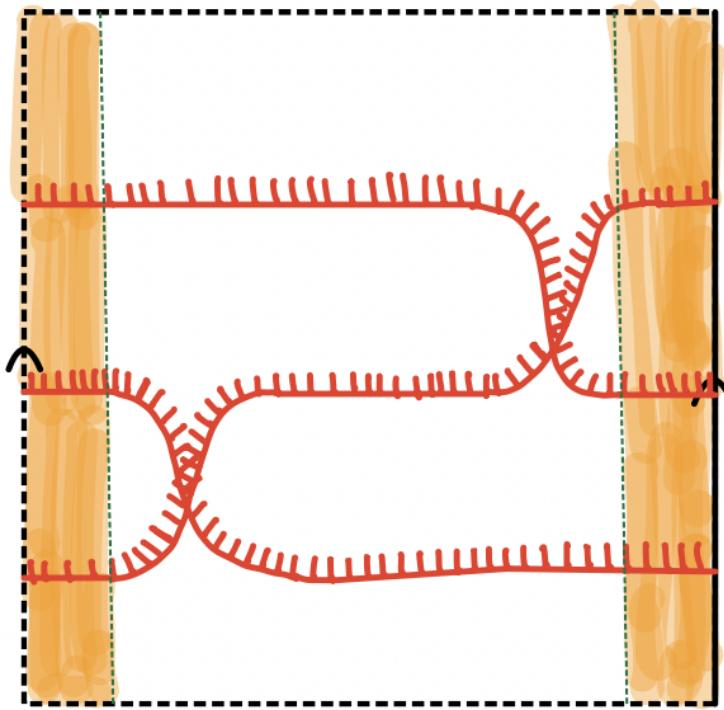


Figure 3.10

Now I will draw  $\Lambda'_0$  for each generator region so that they glue up to the whole  $\Lambda'_0$ .

First, we restrict the diagram to  $j^{th}$  generator region, we have the following diagram: Note that  $i_j^{th}$  and  $i_j + 1^{th}$  strands cross each other and all the other strands are horizontal.

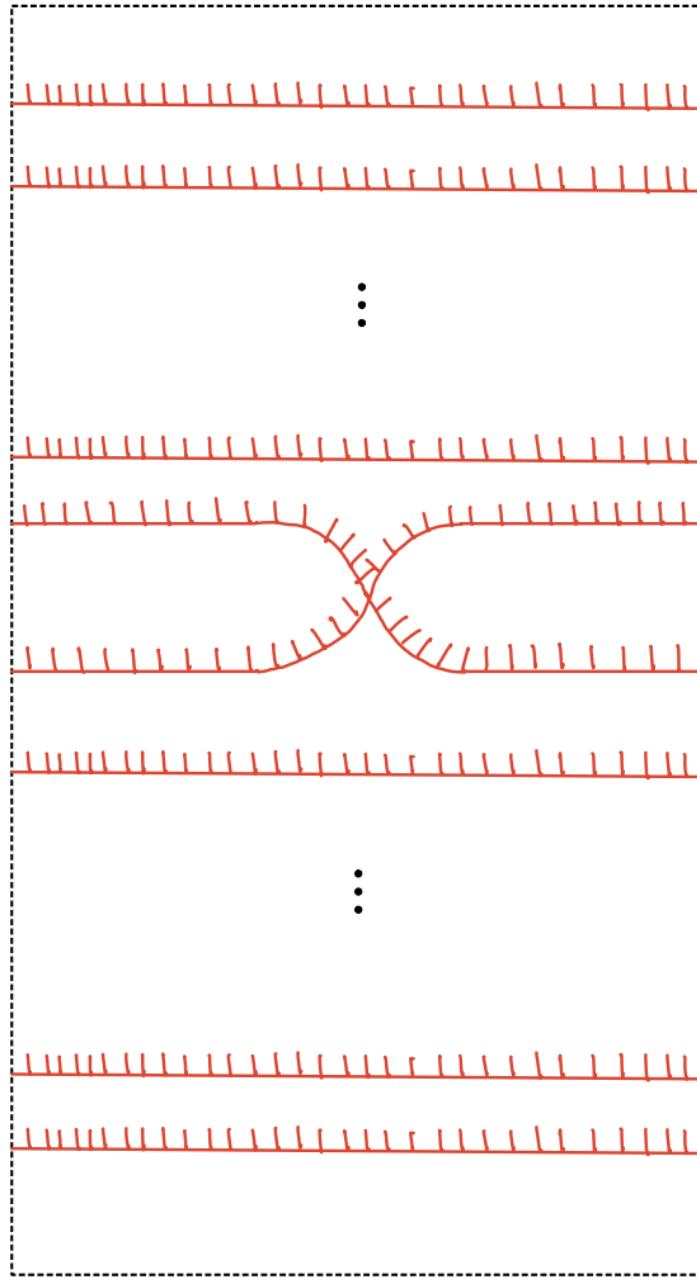


Figure 3.11

We label the strands from bottom to top using integers from 1 to  $n$  with reference to the left end points. This is the strand labelling scheme that I will use throughout this chapter.

I will draw  $\Lambda'_0$  as blue strand on it as follows :

- $l^{th}$  blue strand starts from the midpoint of the starting points of  $l - 1^{th}$  and  $l^{th}$  red strands and ends at the midpoint of the end points of  $l - 1^{th}$  and  $l^{th}$  red strands
- if  $l \neq i_j$  and  $i \neq i_j + 1$ , then along the way the  $l^{th}$  blue strand crosses up and down once
- if  $l = i_j + 1$ ,  $l^{th}$  blue strand crosses  $l^{th}$  red strand up in the part before the crossing and then crosses  $l - 1^{th}$  red strand down in the part after the crossing.
- if  $l = i_j$ ,  $l^{th}$  blue strand crosses  $l^{th}$  red strand up and down in the part before the crossing and then crosses  $l + 1^{th}$  red strand up and down in the part after the crossing.

The picture below overlays  $\Lambda'_0$ , drawn in blue strands with hairs pointing downward, on the previous diagram.

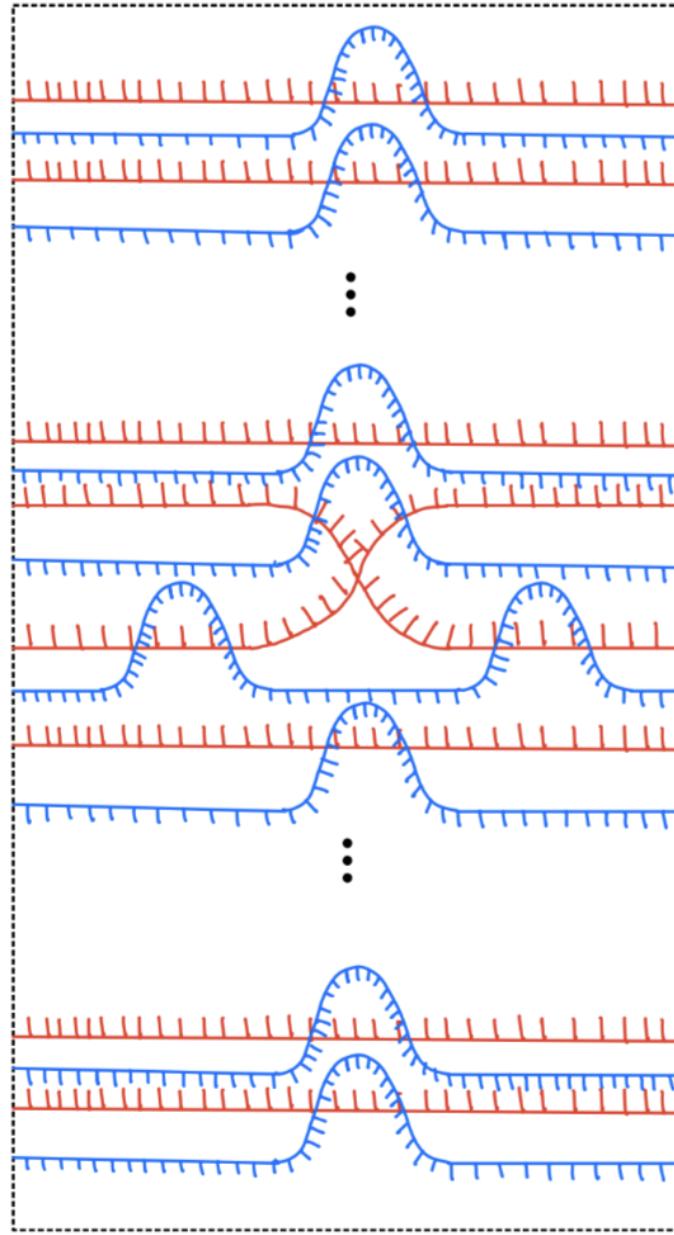


Figure 3.12

For the full alternating strand diagram, we take the closure of blue strands from the generator regions so that the end points from the bordering regions coincide.

The picture below shows how the global natural alternating strand diagram associated with  $\omega = s_1 s_2$  looks like after gluing together local alternating strand diagrams of  $s_1$  and  $s_2$ .

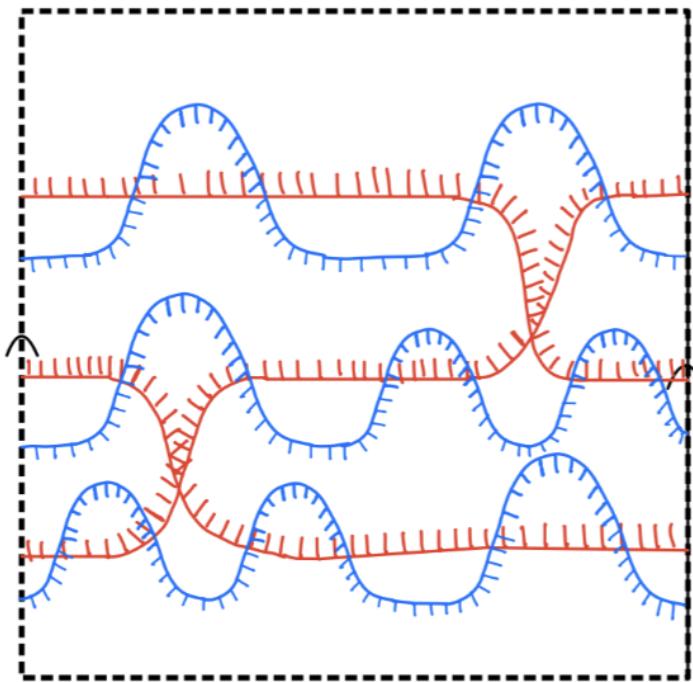


Figure 3.13

**Theorem 63.** The above defined strand diagram is alternating

*Proof.* we will denote

- the region with all the hairs pointing outward as  $\circ$
- the region with all the hairs pointing inward as  $\triangle$
- else with  $\times$

for the generator region we have the following figure :

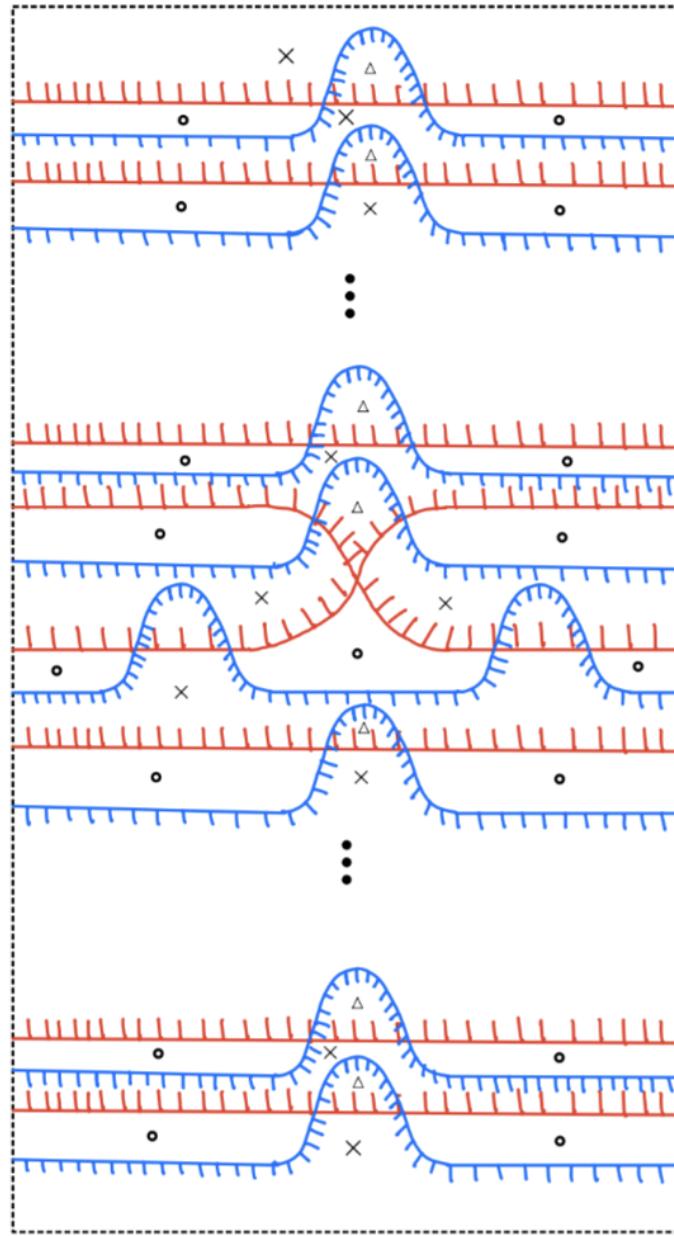


Figure 3.14

The above marking extends to the inter-generator region, we have the following figure:

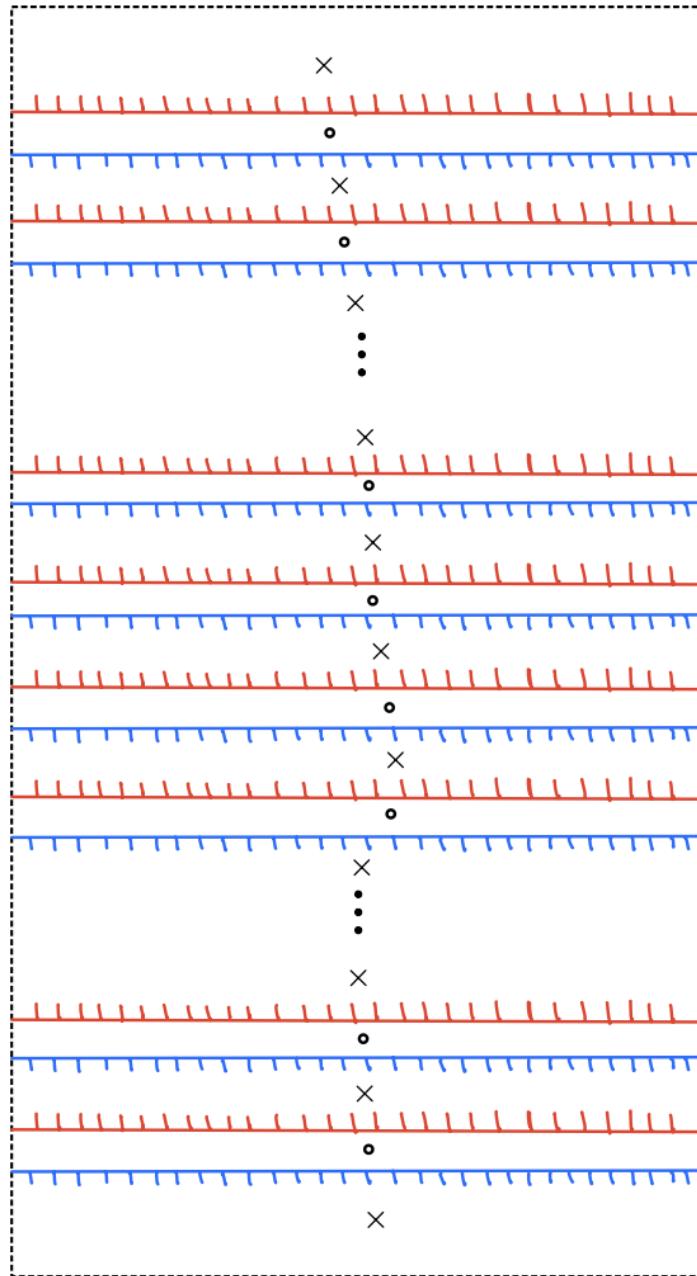


Figure 3.15

for each crossing, it satisfy the alternating condition. This diagram is indeed alternating.  $\square$

### 3.3 local systems on natural alternating diagrams

Suppose we have a positive braid word  $\omega$  then we have the associated natural alternating diagram  $(M, \Lambda'_0, \Lambda'_{\infty})$  defined in the previous section.

We can associate a quiver  $Q$  to the alternating diagram in such a way that

- we have one vertex for regions where all hairs are pointing outward/inward
- for each crossing, we have an arrow from the vertex corresponding to the region where all hairs pointing outward to inward

For example, for each generator region of a natural alternating strand diagram:

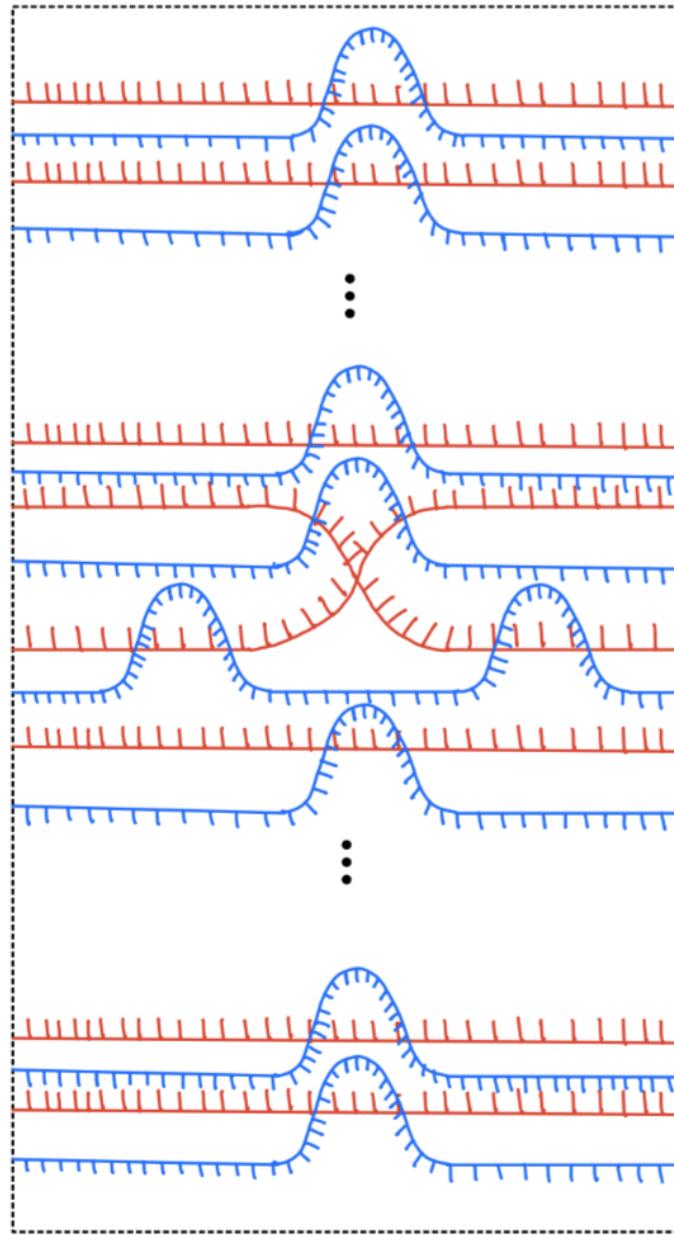


Figure 3.16

we have the following associated quiver:

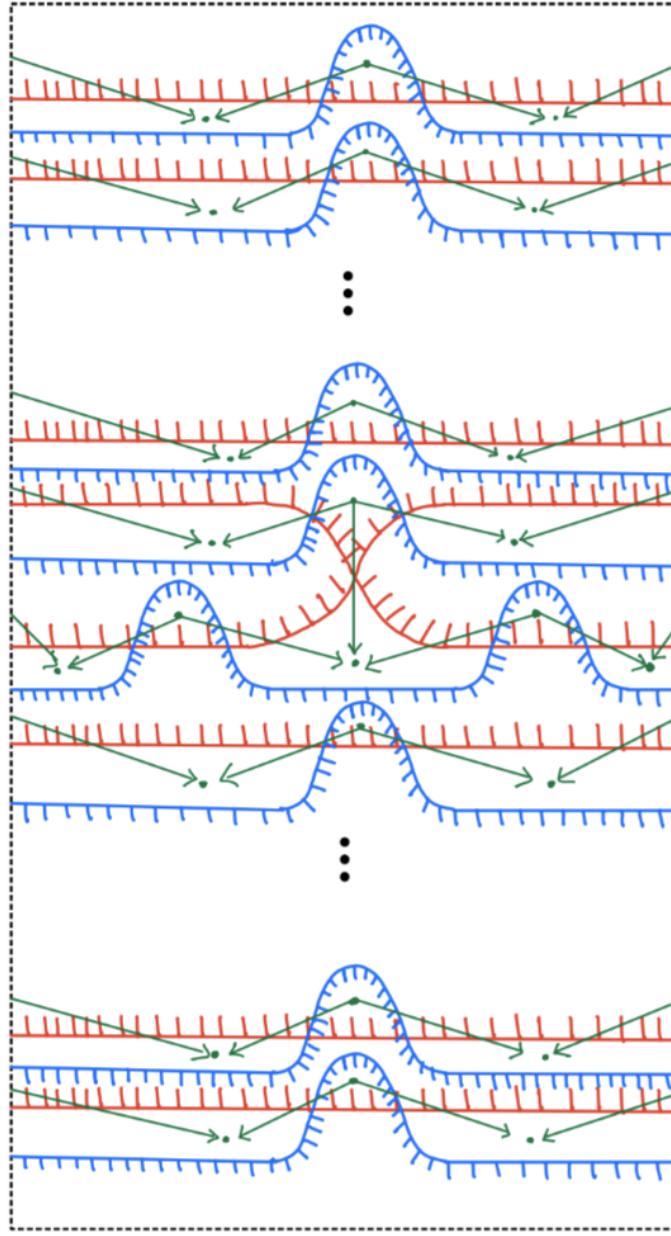


Figure 3.17

Once we have an alternating strand diagram, we have the associated conjugate surface  $L_{conj}$ . Furthermore, we can embed the underlying undirected graph  $\Gamma_{bi}$  of the quiver  $Q$  (i.e. the bipartite graph of the alternating coloring) in  $L_{conj}$  in such a way that  $L_{conj}$  deformation retracts to  $\Gamma_{bi}$ . Suppose we have a rank 1 local system on the conjugate surface associated with  $(M, \Lambda'_0, \Lambda'_{\infty})$ , then restricting to  $Q$ , we get a local

system on  $Q$ . Note that the pullback, induced by the restriction map, between the space of local systems  $H^1(L_{conj}, \mathbb{C}^*) \rightarrow H^1(\Gamma_{bi}, \mathbb{C}^*)$  is an isomorphism.

$H^1(\Gamma_{bi}, \mathbb{C}^*)$  is isomorphic to  $(\mathbb{C}^*)^{|Arr(Q)|} // (\mathbb{C}^*)^{|Vert(Q)|}$  here the group action is defined as the following : let  $g_v \in (\mathbb{C}^*)^{|Vert(Q)|}$  (more precisely,  $g_v := (g^{\delta_{w,v}})_{w \in Vert(Q)}$  where  $\delta$  is the Kronecker delta), then  $g_v \cdot (x_a)_{a \in Arr(Q)}$  is

- for entries with index  $a$  such that the source of  $a$  is  $v$  i.e.  $s(a) = v$ , we have

$$g_v \cdot x_a$$

- for entries with index  $a$  such that the target of  $a$  is  $v$  i.e.  $t(a) = v$ , we have

$$g_v^{-1} \cdot x_a$$

Now we define the associated alternating sheaf on some regular cell complex refinement of the natural alternating strand diagram associated with a rank 1 local systems on  $Q$ .

First, I will describe the special kind of regular cell complex associated with the alternating strand diagram called the regular cell complex refinement of the natural alternating strand diagram. I will define the refinement for each generator region and glue them to get the global regular cell complex.

**Definition 64.** Suppose we fix a generator region for the alternating strand diagram. Then we denote the  $j^{th}$  crossing(numbering starts from left to right) the  $i^{th}$  blue strand(numbering starts from top to bottom) crosses red strands as  $c_{i,j}$ . We will call the crossing between  $i^{th}$  and  $i + 1^{th}$  red strand as  $c$ .

(i) For each crossing  $c_{i,j}$  we add

- when  $j$  is odd, locally near  $c_{i,j}$  we have the following local diagram

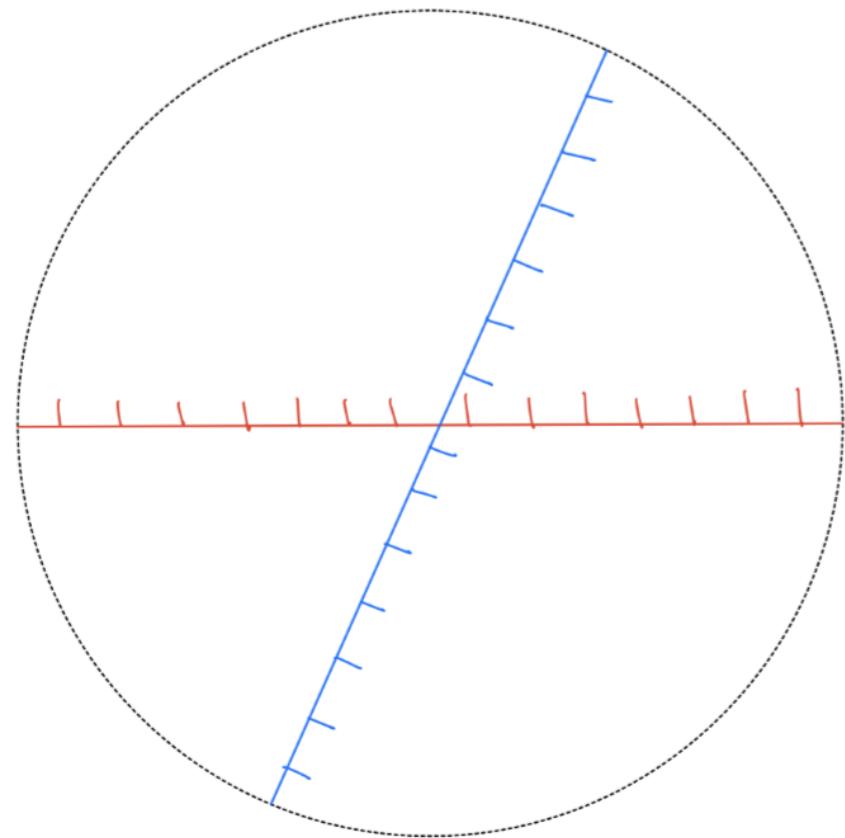


Figure 3.18

then we add squiggly lines with co-orientations and end points to get

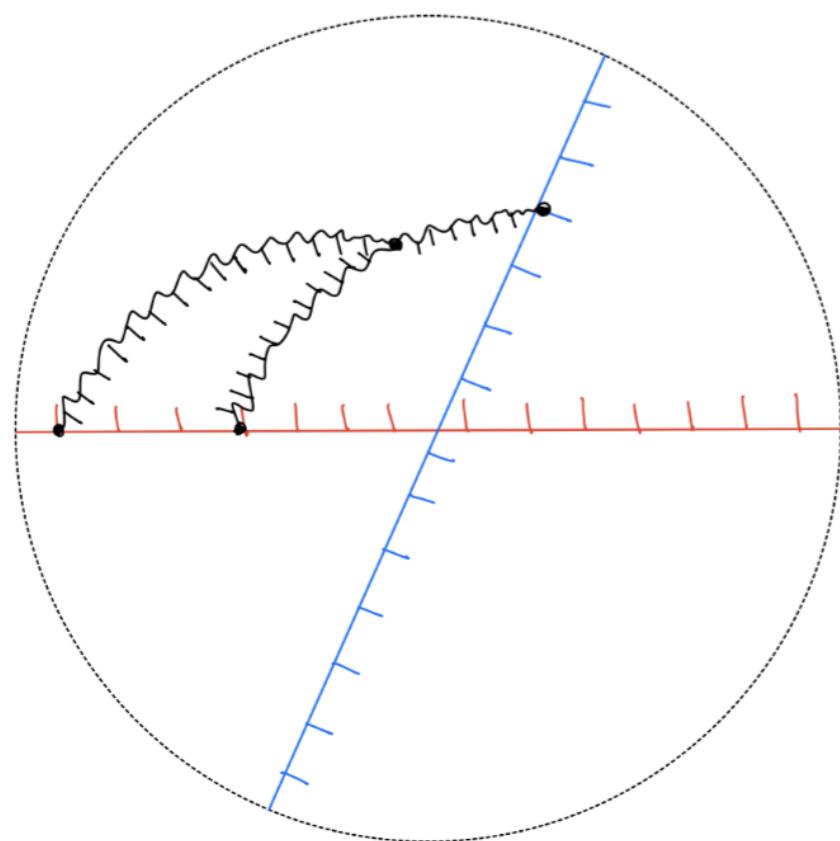


Figure 3.19

We call the region marked with \* a crossing region.

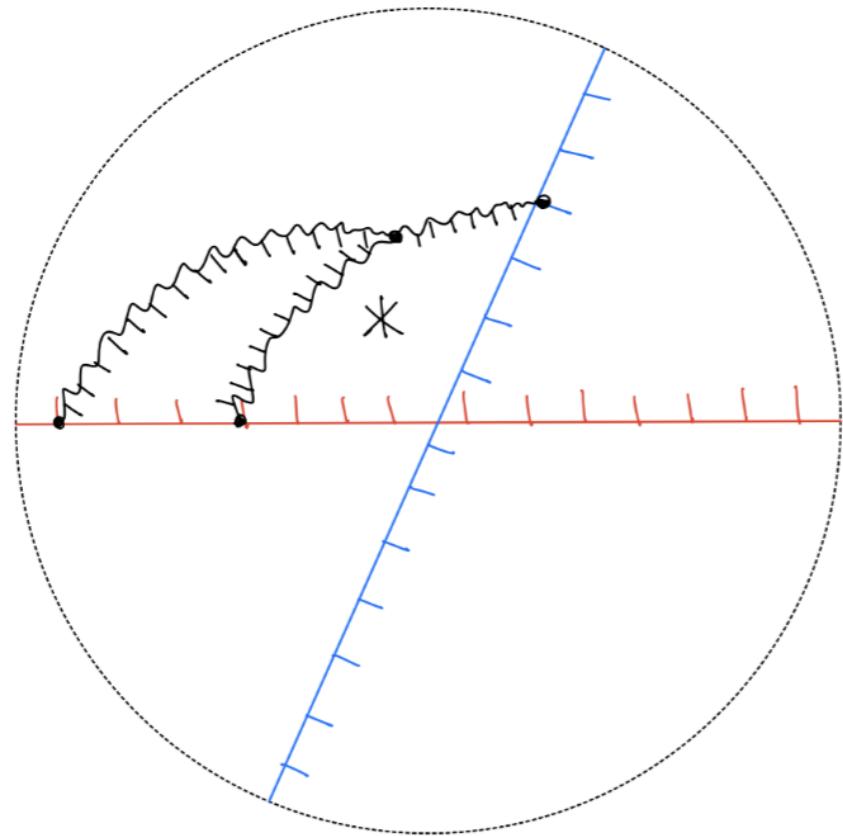


Figure 3.20

- when  $j$  is even, locally near  $c_{i,j}$  we have the following local diagram

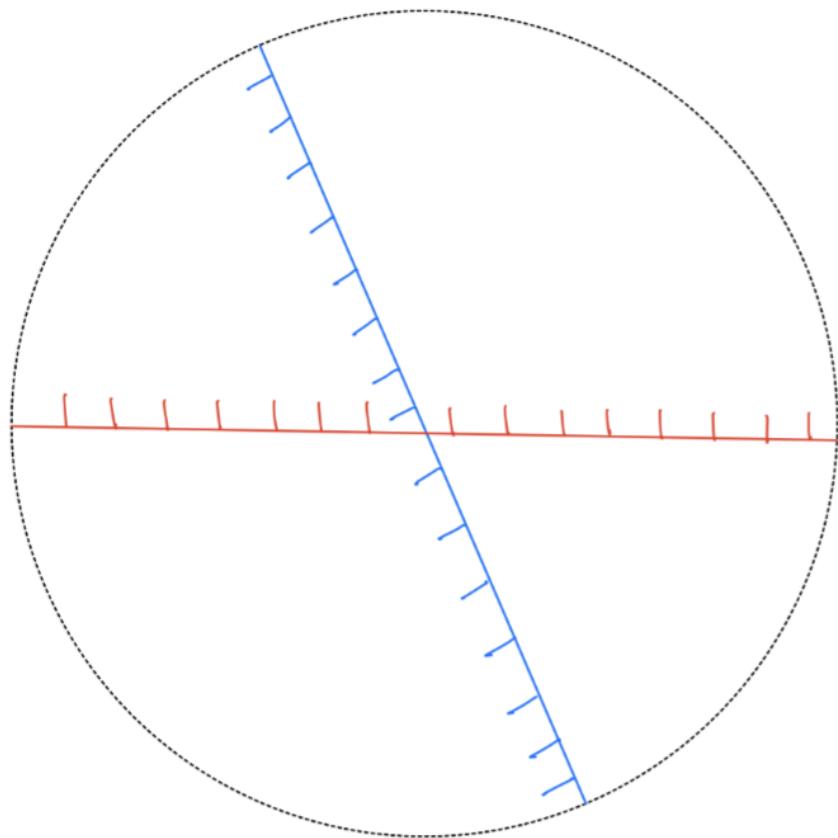


Figure 3.21

then we add squiggly lines with co-orientations and end points to get

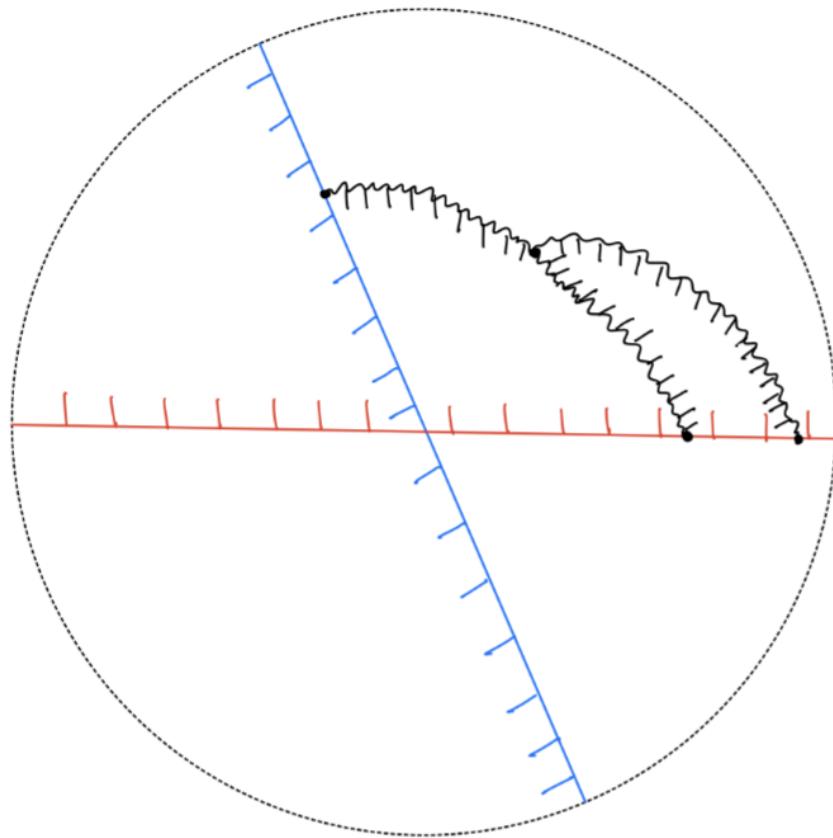


Figure 3.22

We call the region marked with \* a crossing region.

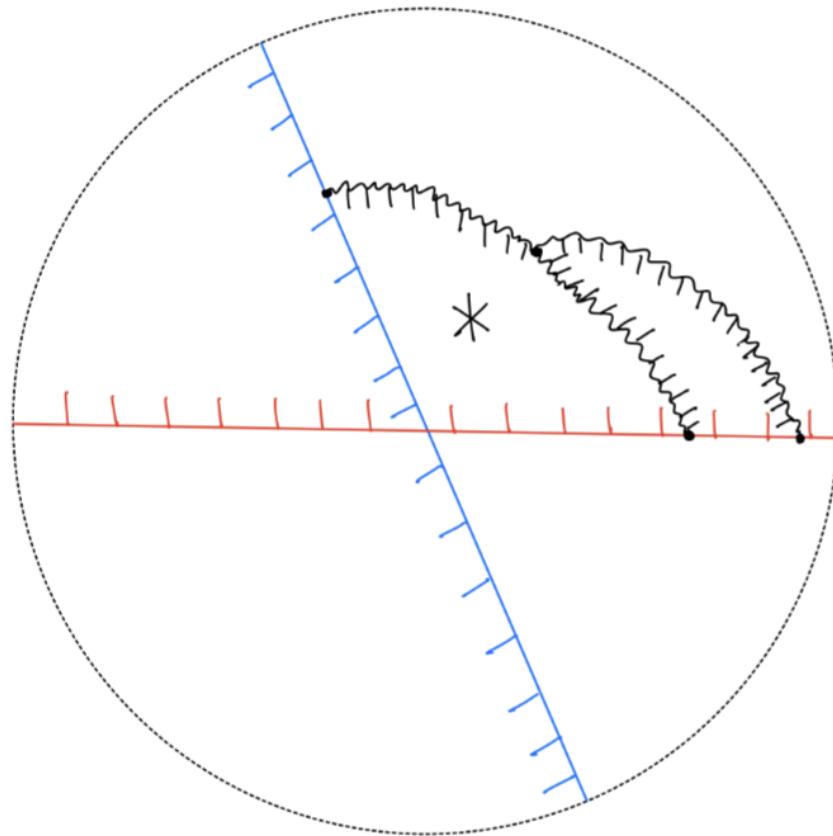


Figure 3.23

(ii) For the crossing  $c$ , locally near the crossing, we have the following local diagram

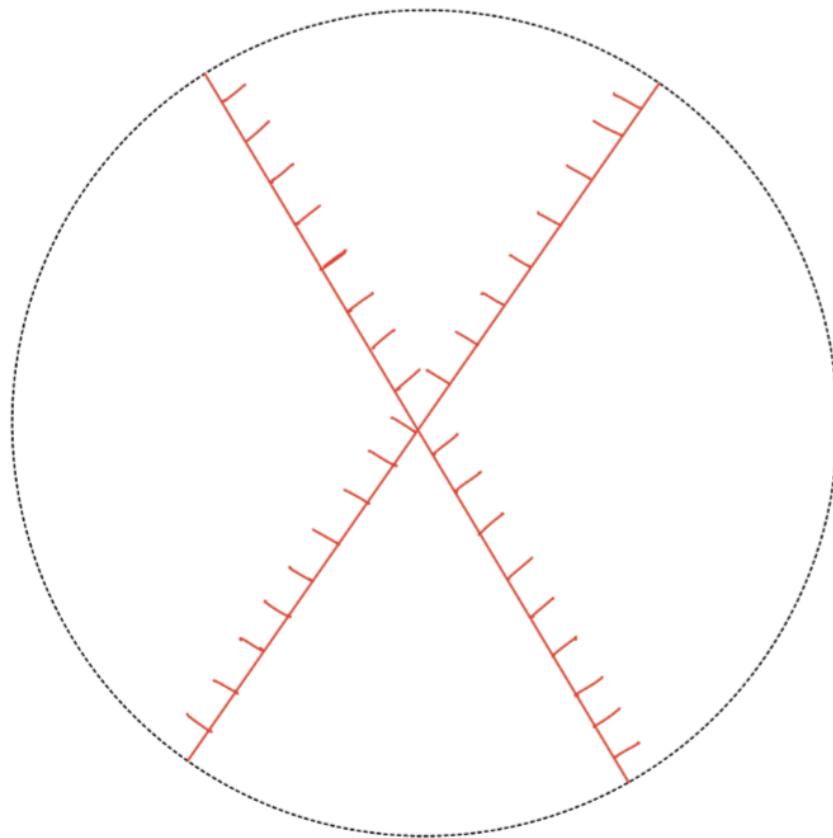


Figure 3.24

then we add squiggly lines with co-orientations and end points to get

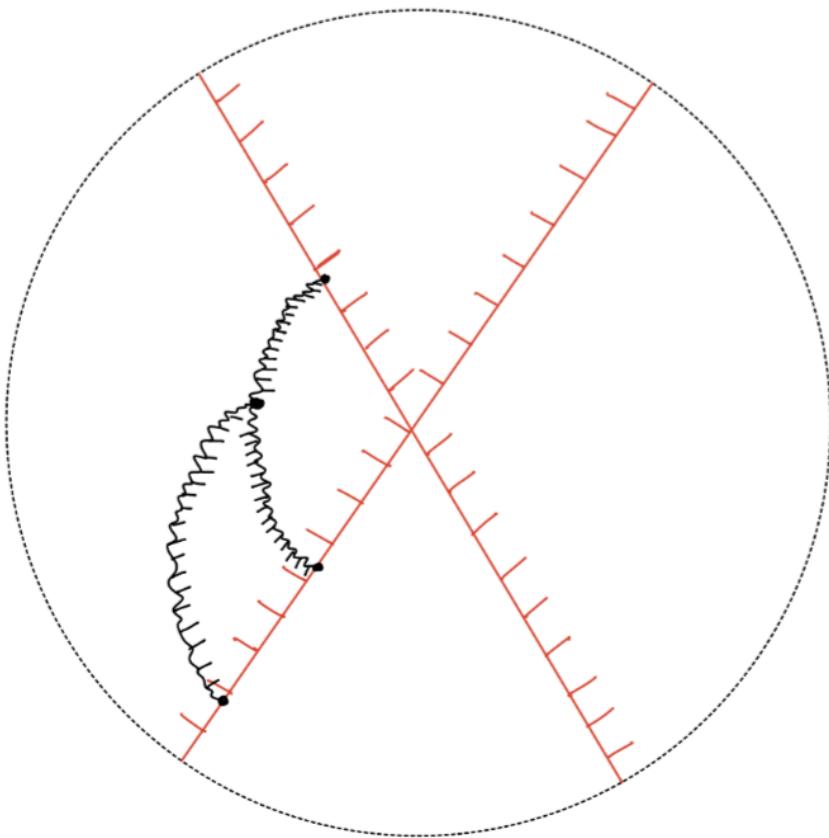


Figure 3.25

We call the region marked with \* a crossing region.

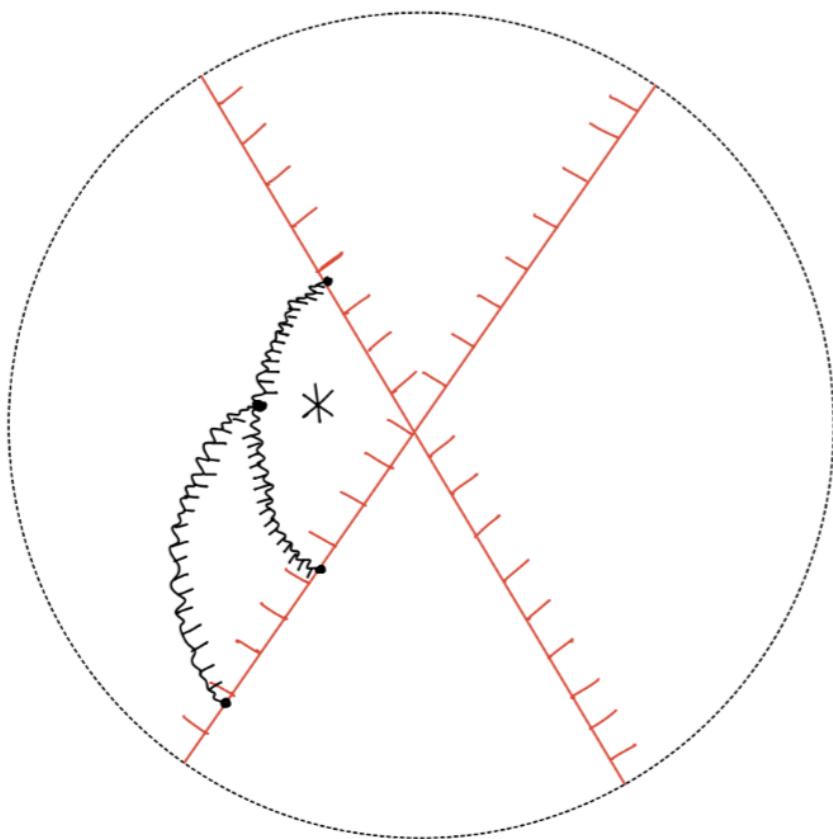


Figure 3.26

Below is the picture of a generator region of a natural alternating diagram:

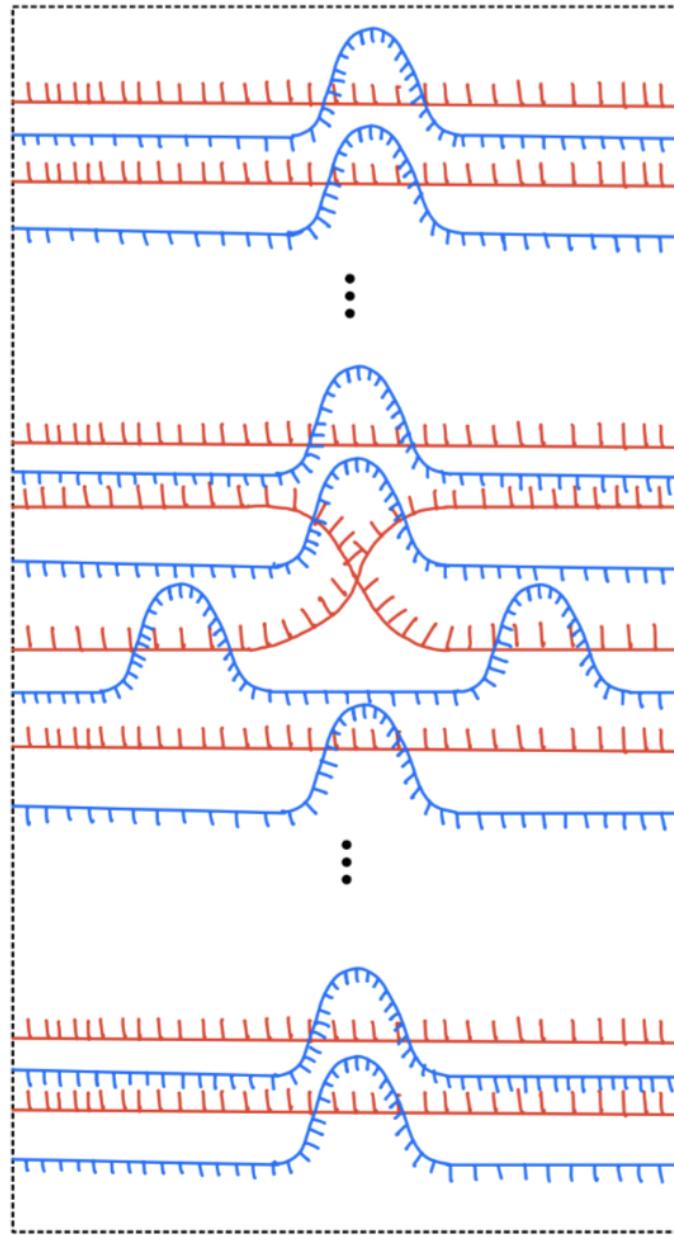


Figure 3.27

and below is the picture of the regular cell complex refinement in a generator region:

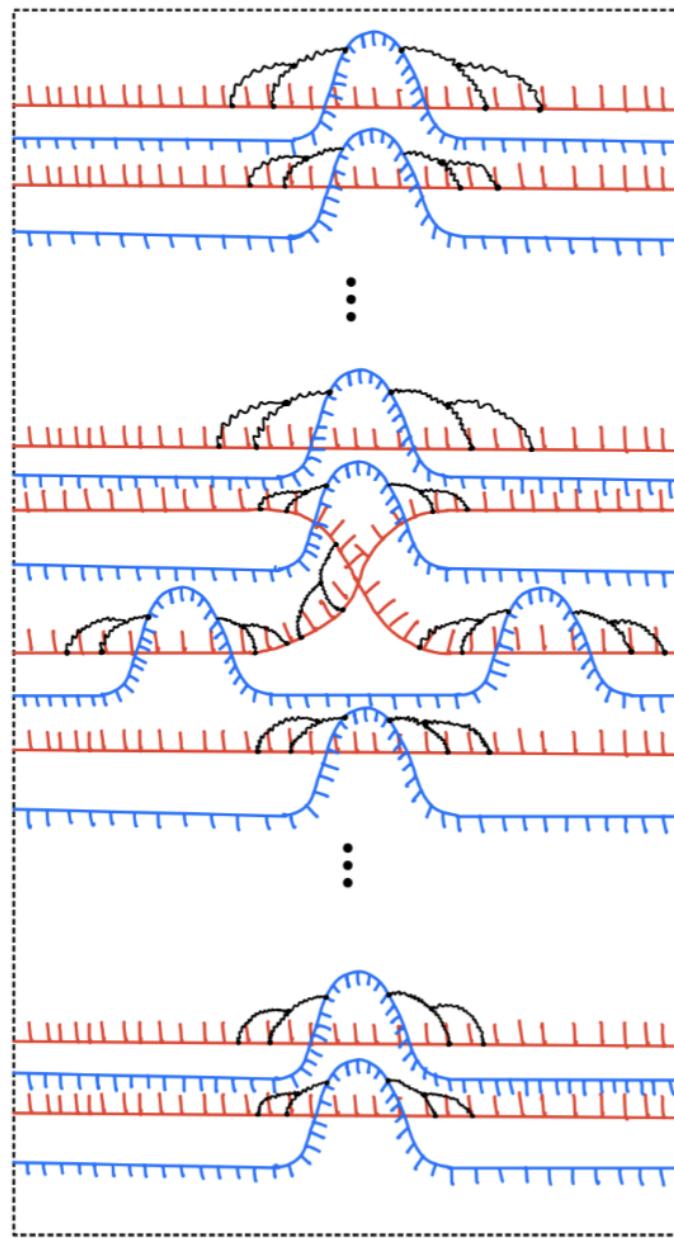


Figure 3.28

Next, we fix an inter-generator region

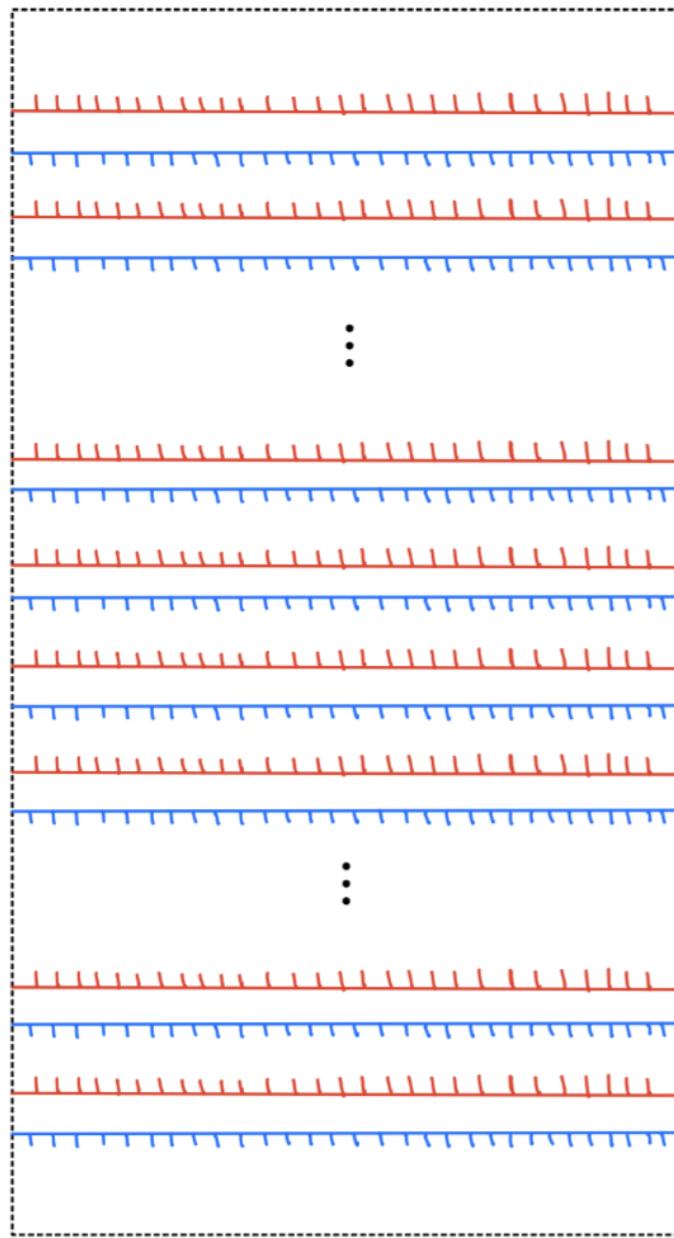


Figure 3.29

add a vertical squiggly line co-oriented towards the left

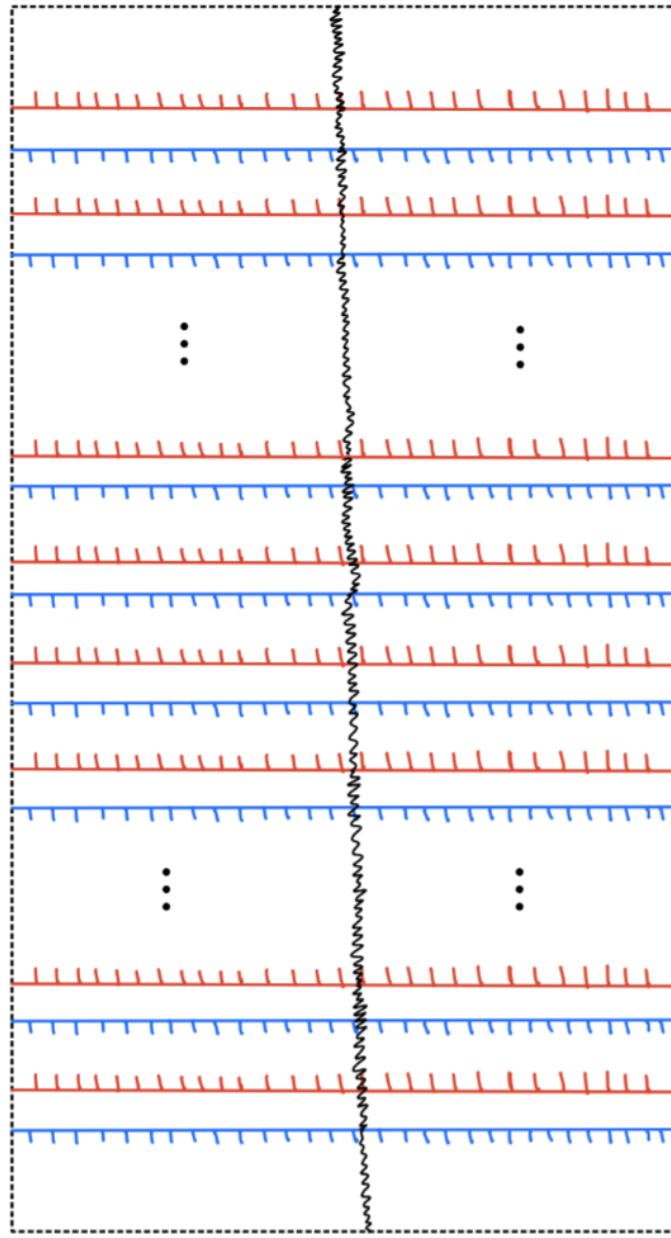


Figure 3.30

Now I will describe a way to specify a constructible sheaf on the above regular cell complex refinement associated with the local system on  $\Gamma_{bi}$ .

**Definition 65.** Suppose we have a local system on  $Q$  which can be represented as an element  $(x_A)_{a \in Arr(Q)} (\mathbb{C}^*)^{|Arr(Q)|}$ :

- (i) stalk of the region where all the hairs are pointing outward is  $\mathbb{C}[-1]$

- (ii) stalk of the region where all the hairs(except the hairs on the squiggly lines) are pointing inward is  $\mathbb{C}$
- (iii) stalk of the crossing regions is  $\mathbb{C} \xrightarrow{\times x_a} \mathbb{C}$  where  $a$  is the arrow corresponding to the associated crossing.
- (iv) rest of the stalks are 0
- (v) the only nonzero genrization maps are from regions of type (iii) to (i), from (i) to (ii), or from (ii) to (ii)(the ones corresponding to vertical squiggly lines in the inter-generator regions)

The maps from (iii) to (i) are

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

The maps from (i) to (ii) are

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

The maps from (ii) to (ii) are identity maps.

Note that the group action maps a constructible sheaf to the isomorphic constructible sheaf. Therefore, we have a well-defined map  $H^1(L_{conj}, \mathbb{C}^*) \rightarrow \mathcal{M}_1(M, \Lambda', \{\sigma_{z \ll 0}, \sigma_{z \gg 0}\}; \mathbb{C})$  where  $\Lambda' = \Lambda'_0 \coprod \Lambda'_\infty$ ,  $\sigma_{z \ll 0}$  is a point in the region containing 0, and  $z \gg 0$  is a point in the region containing  $\infty$ .

### 3.4 1st Sheaf Cobordism

In this section, we define *cobord*<sub>1</sub>, a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism: from

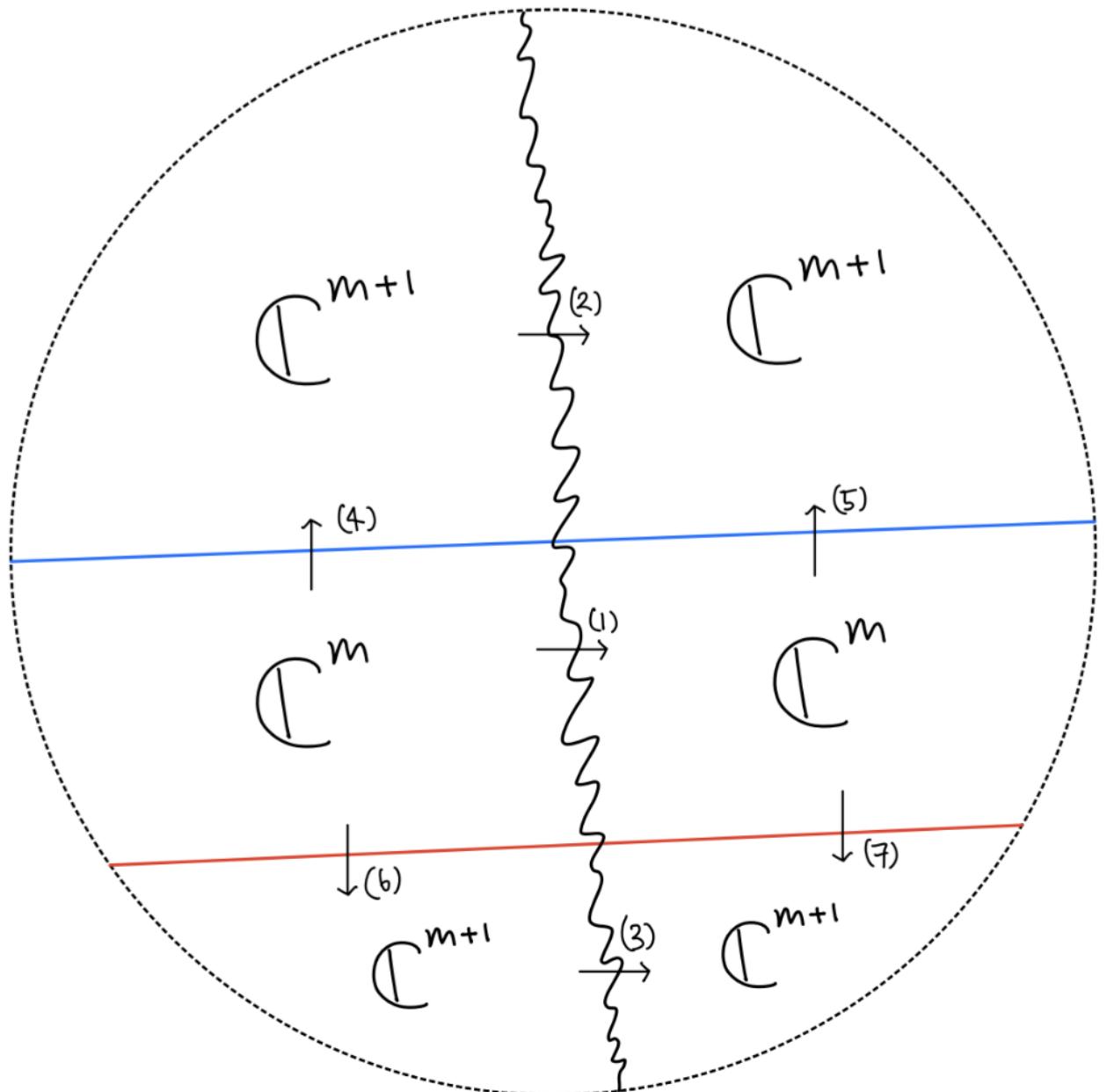


Figure 3.31

Generalization maps:

$$(1) \ \mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$$

$$(2) \ \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(3) \ \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(4) \ \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \ \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(6) \ \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(7) \ \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

to

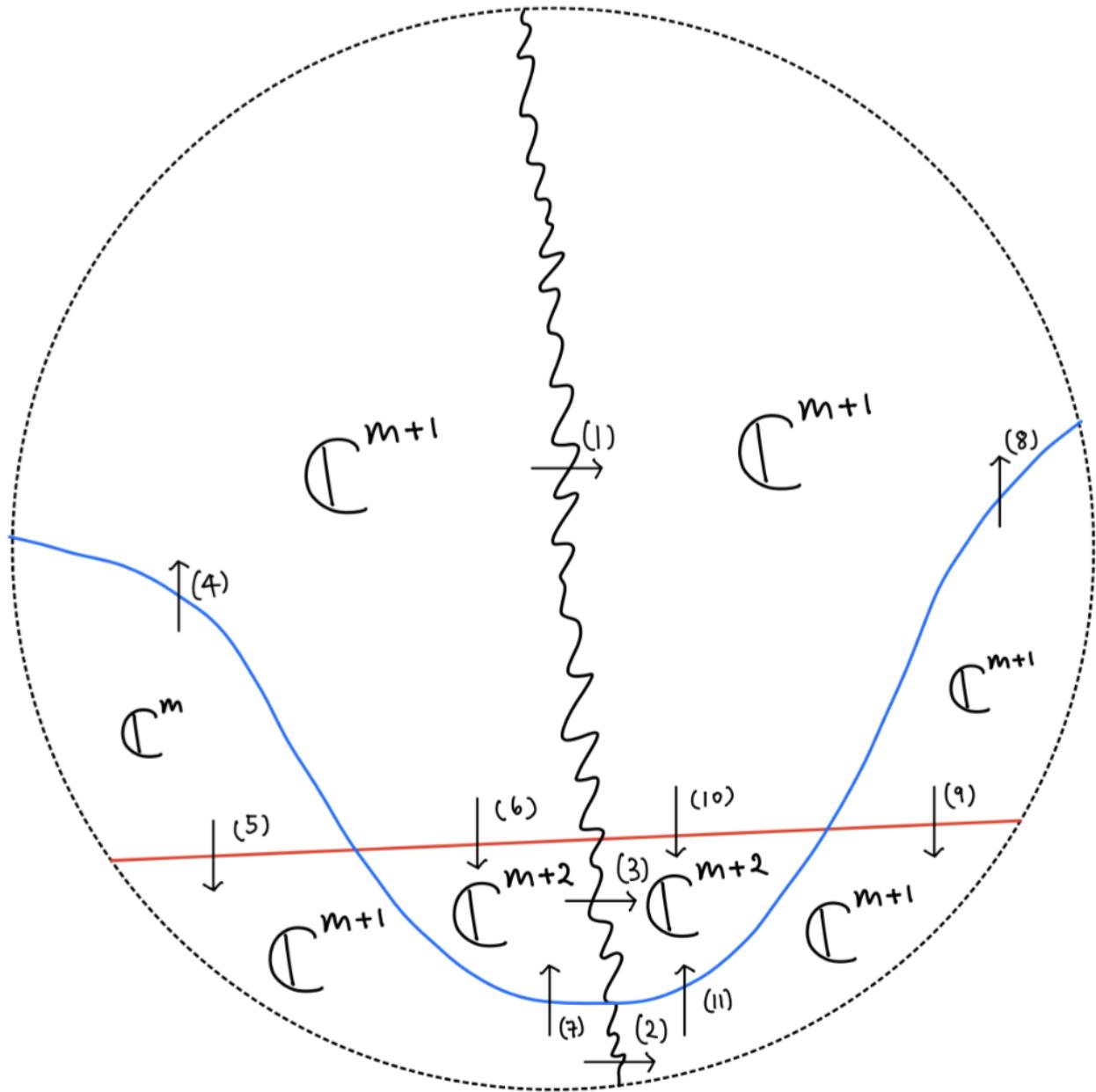


Figure 3.32

**Generalization maps:**

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(8) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(9) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(10) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(11) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

## Notations

**Definition 66.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 67.** For  $t_0 \in \{0, 1\}$  and  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{t_0}^{symbol}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{symbol}$  a co-orientation of  $\Phi_{t_0}^{symbol}$ .

3. we denote the pair  $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$  as  $\Lambda_{t_0}^{symbol}$ . When  $symbol \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{symbol}$  to denote both

4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the beginning of the isotopy underlying *cobord*<sub>1</sub> and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying *cobord*<sub>1</sub>.

**Definition 68.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_\bullet^{symbol}$  to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript '*t*' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_\bullet^{symbol}$  a co-orientation of  $\Phi_\bullet^{symbol}$ .

3. we denote the pair  $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$  as  $\Lambda_\bullet^{symbol}$ . Later in the section,  $\Lambda_\bullet^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>1</sub>.

4. we denote the triple  $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$  as  $\Lambda_\bullet$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_\bullet$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>1</sub>.

**Definition 69.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} -\frac{3}{4}e^{(\frac{x^2}{x^2-1})}t & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $\text{supp}(\Psi_t) = [-1, 1]$  if  $t \neq 0$
- $\{(1, 0), (-1, 0), (0, -\frac{3}{4}t)\} \subset \text{Graph}(\Psi_t)$

**Definition 70.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 71.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\} = \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_0^{\text{squig}} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations  $\xi_0^{\text{symbol}}$  of  $\lambda_0^{\text{symbol}}$  as follows

- $\xi_0^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_0^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_0^{\text{squig}}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 72.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\}$

- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
  - $\lambda_1^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$
2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows
- $\xi_1^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
  - $\xi_1^{squig}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 73.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
  - $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -\frac{1}{2}\}$
  - $\lambda_\bullet^{squig} := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$
2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows
- $\xi_\bullet^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
  - $\xi_\bullet^{squig}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are positive.

**Definition 74.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 75.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 76.** For  $i = 1, 2, 3$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(-\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

**Definition 77.** For  $i = 1, 2, 3$  and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3\} \end{aligned}$$

**Definition 78.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the above notations:

1.  $\mathcal{S}_0$ :

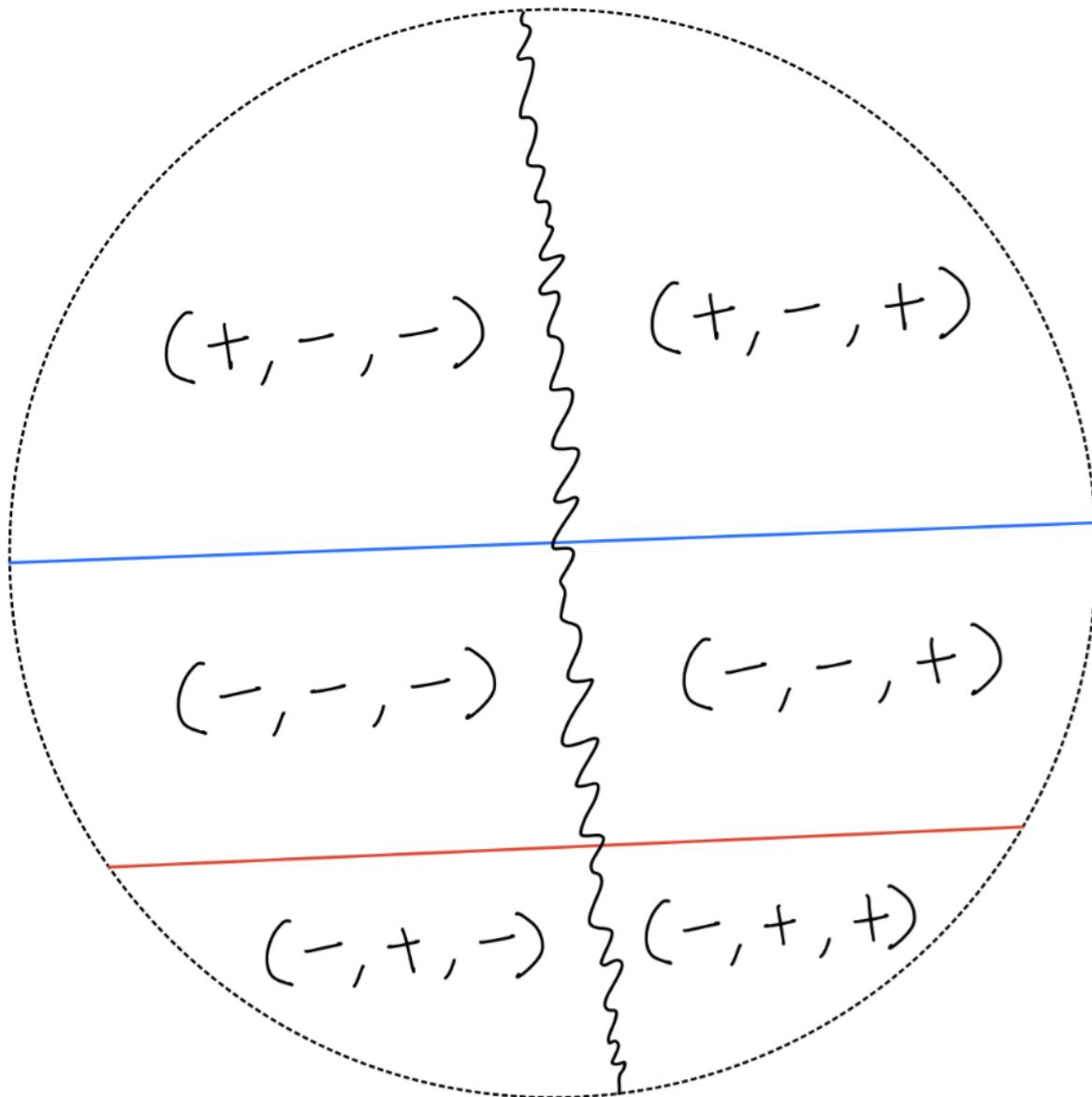


Figure 3.33

- 2 dimensional strata:

$$s_0(+, -, -), s_0(+, -, +), s_0(-, -, -), s_0(-, -, +), s_0(-, +, -), s_0(-, +, +)$$

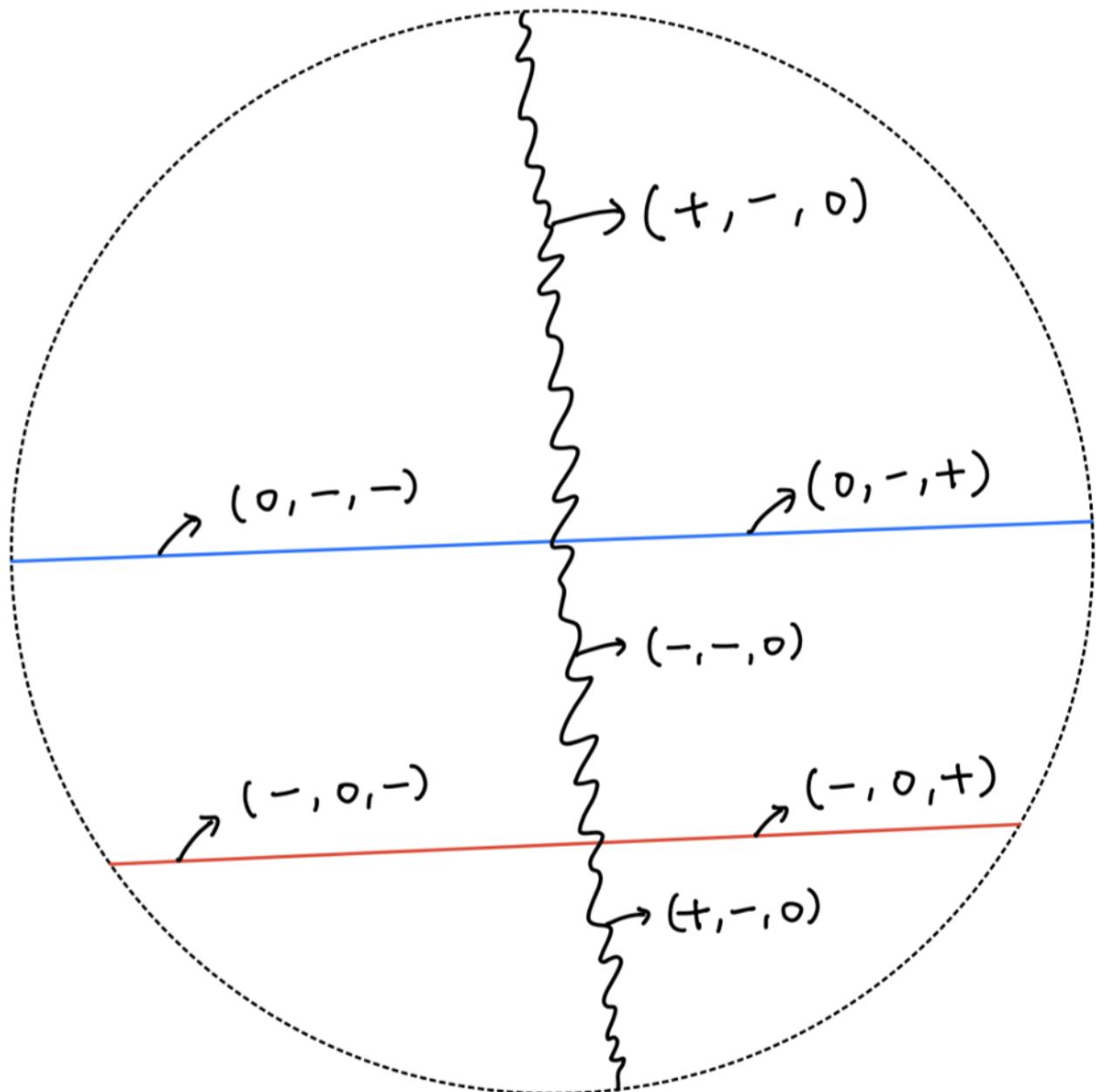


Figure 3.34

- 1 dimensional strata:

$$s_0(0, -, -), s_0(0, -, +), s_0(-, 0, -), s_0(-, 0, +), s_0(-, -, 0), s_0(-, +, 0), s_0(0, -, +)$$

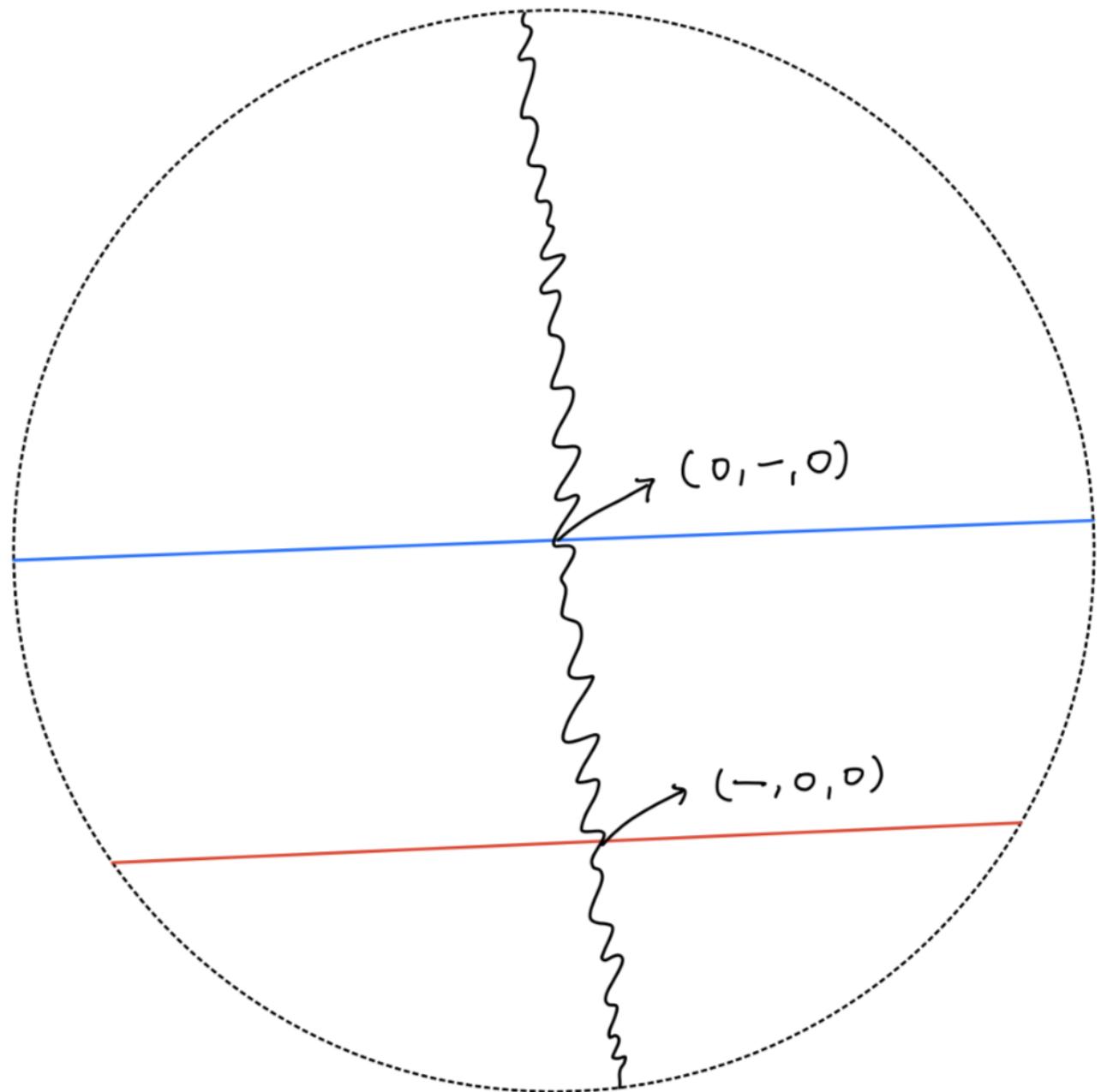


Figure 3.35

- 0 dimensional strata:

$$s_0(0, -, 0), s_0(-, 0, 0)$$

2.  $\mathcal{S}_1$ :

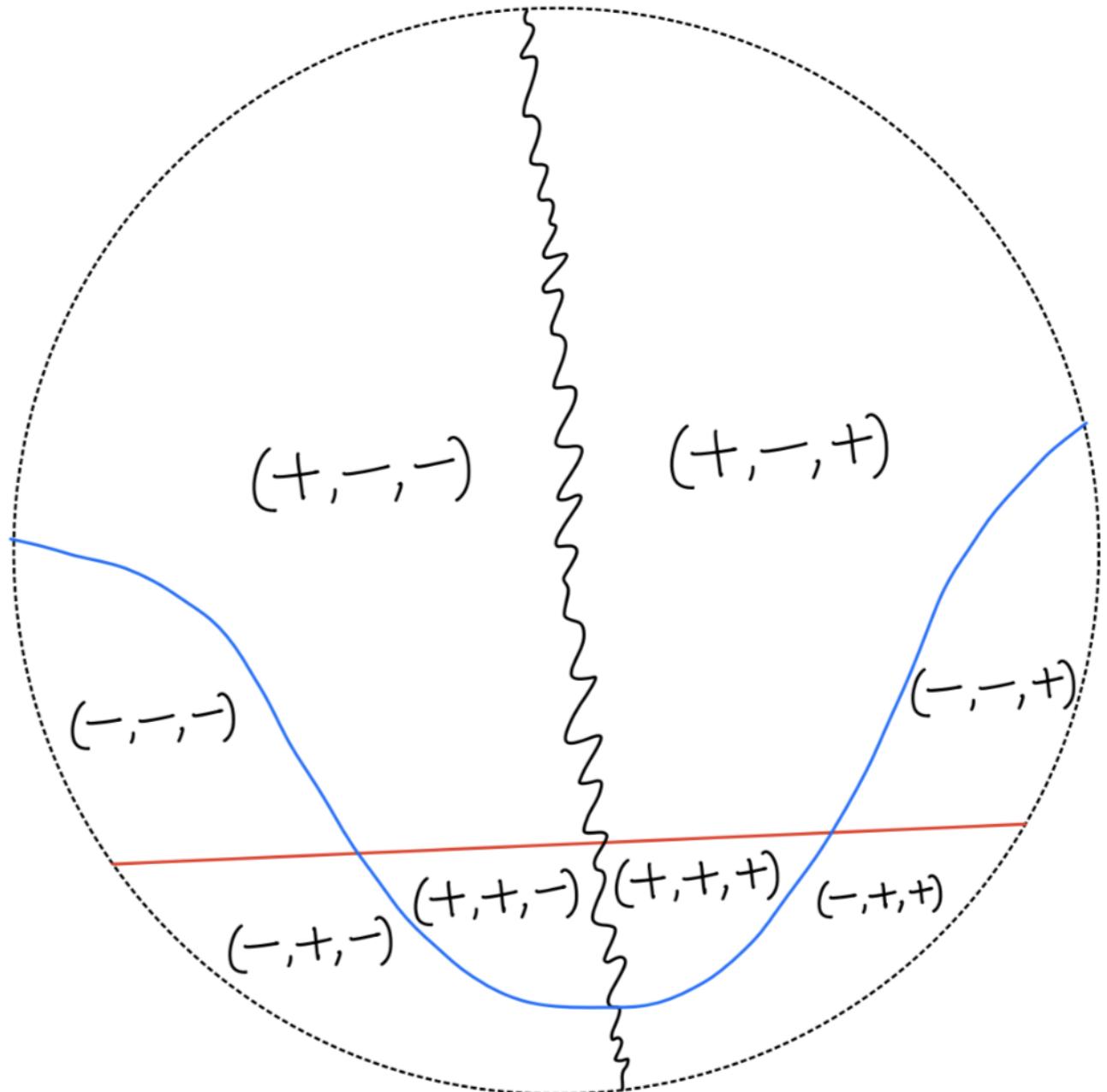


Figure 3.36

- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

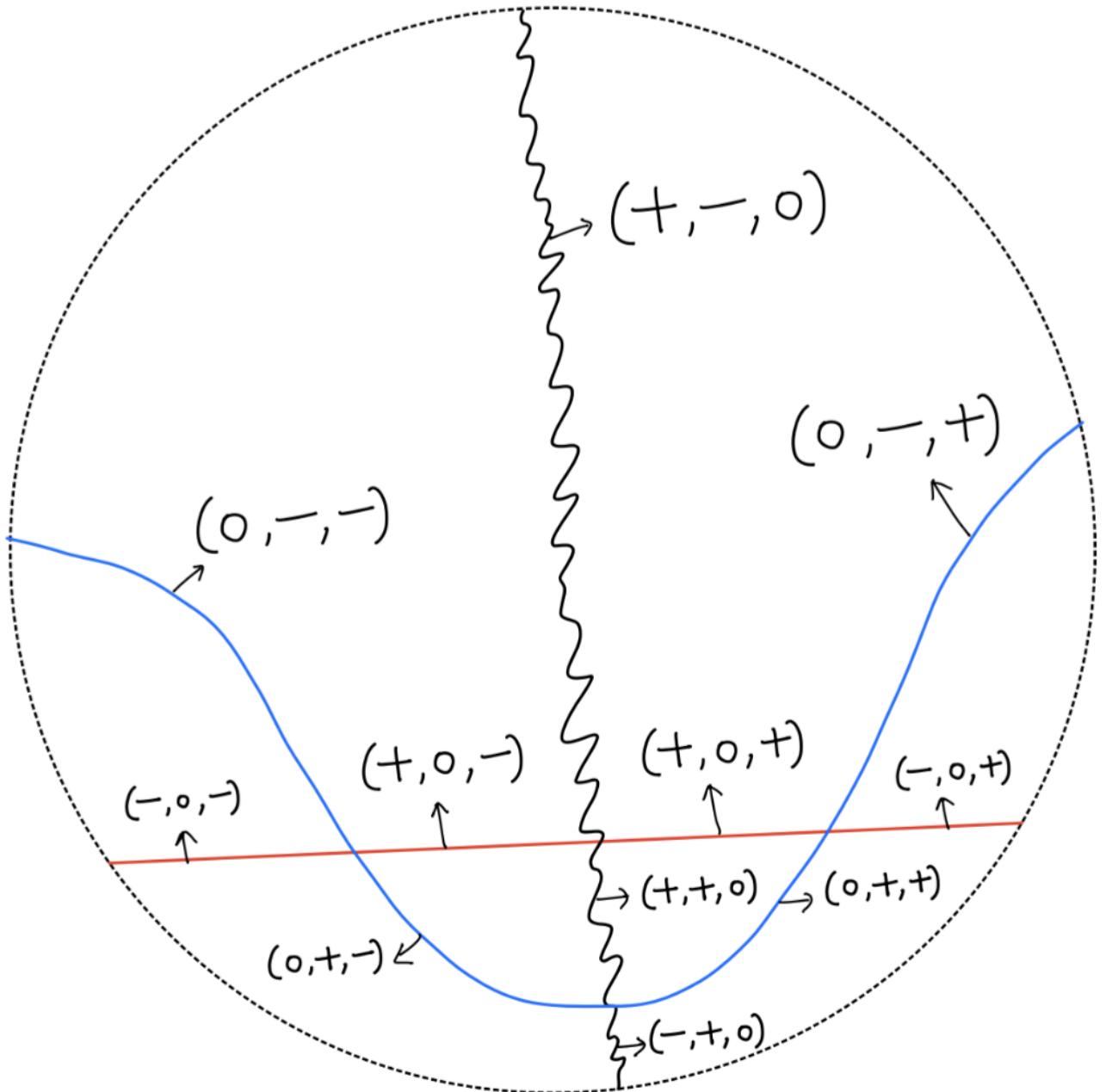


Figure 3.37

- 1 dimensional strata:

$\{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3\} \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(-, -, 0)\}$

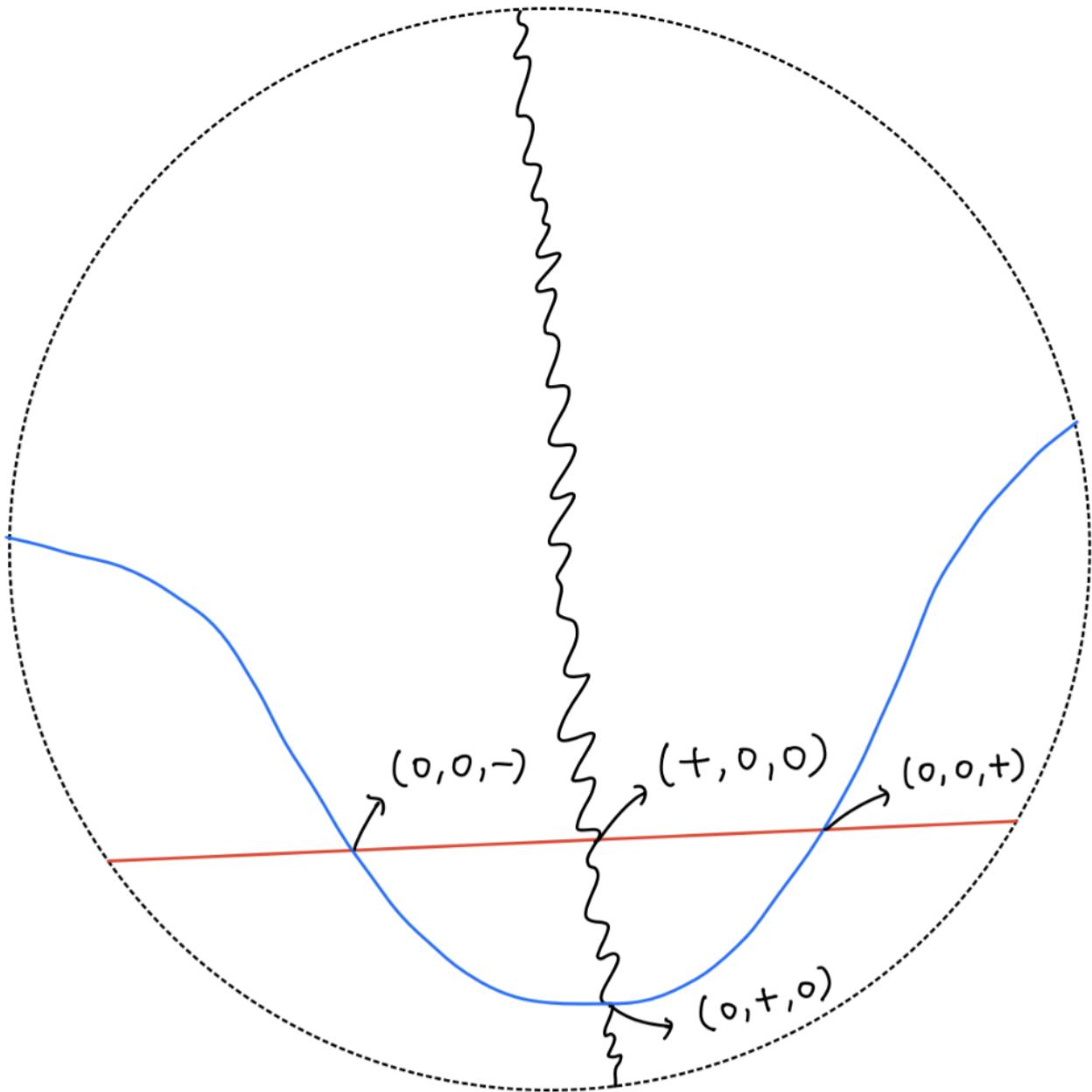


Figure 3.38

- 0 dimensional strata:

$$s_1(0,0,-), s_1(0,+0), s_1(0,0,+), s_1(+,0,0)$$

3.  $\mathcal{S}_\bullet$ :

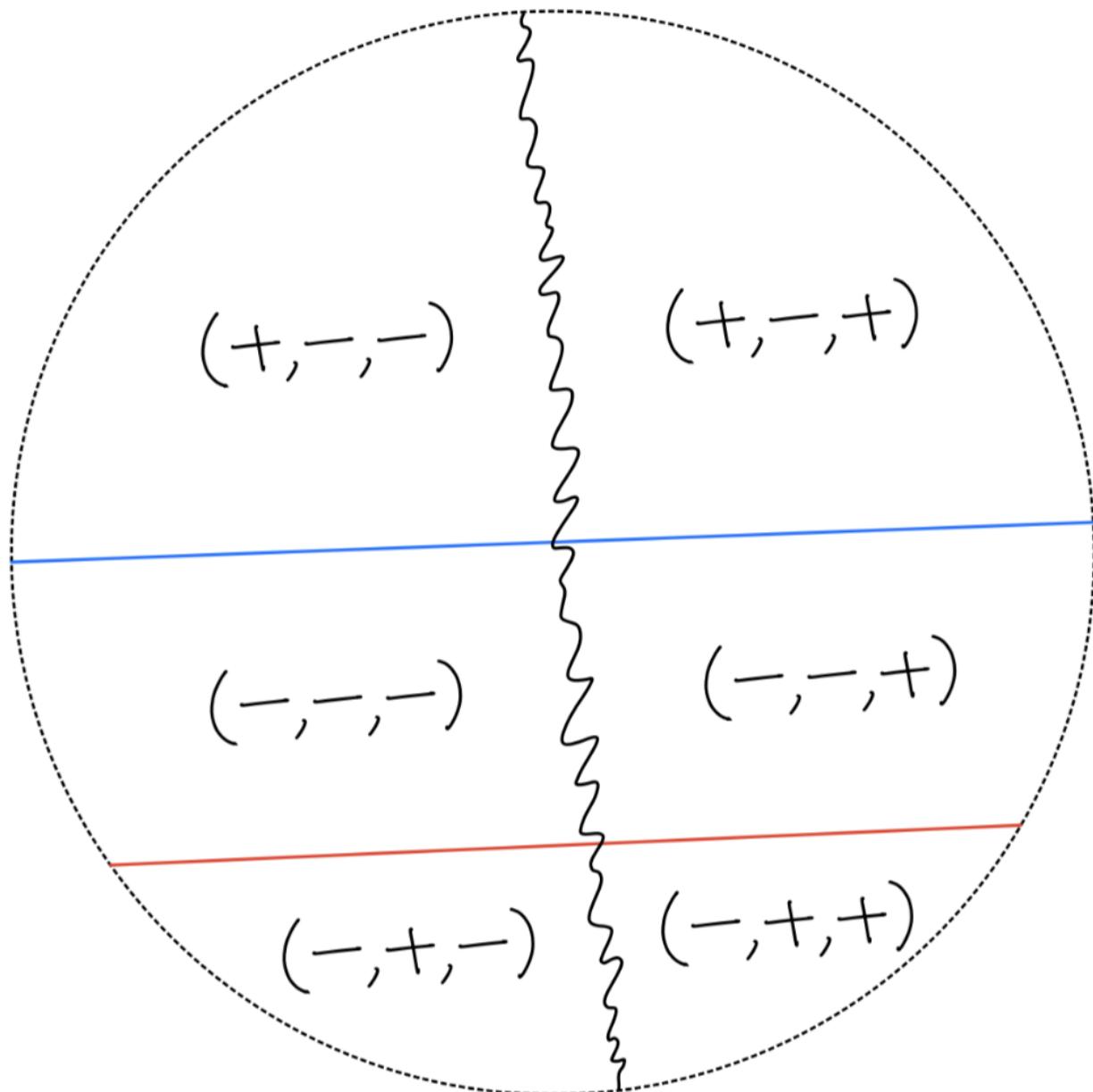


Figure 3.39

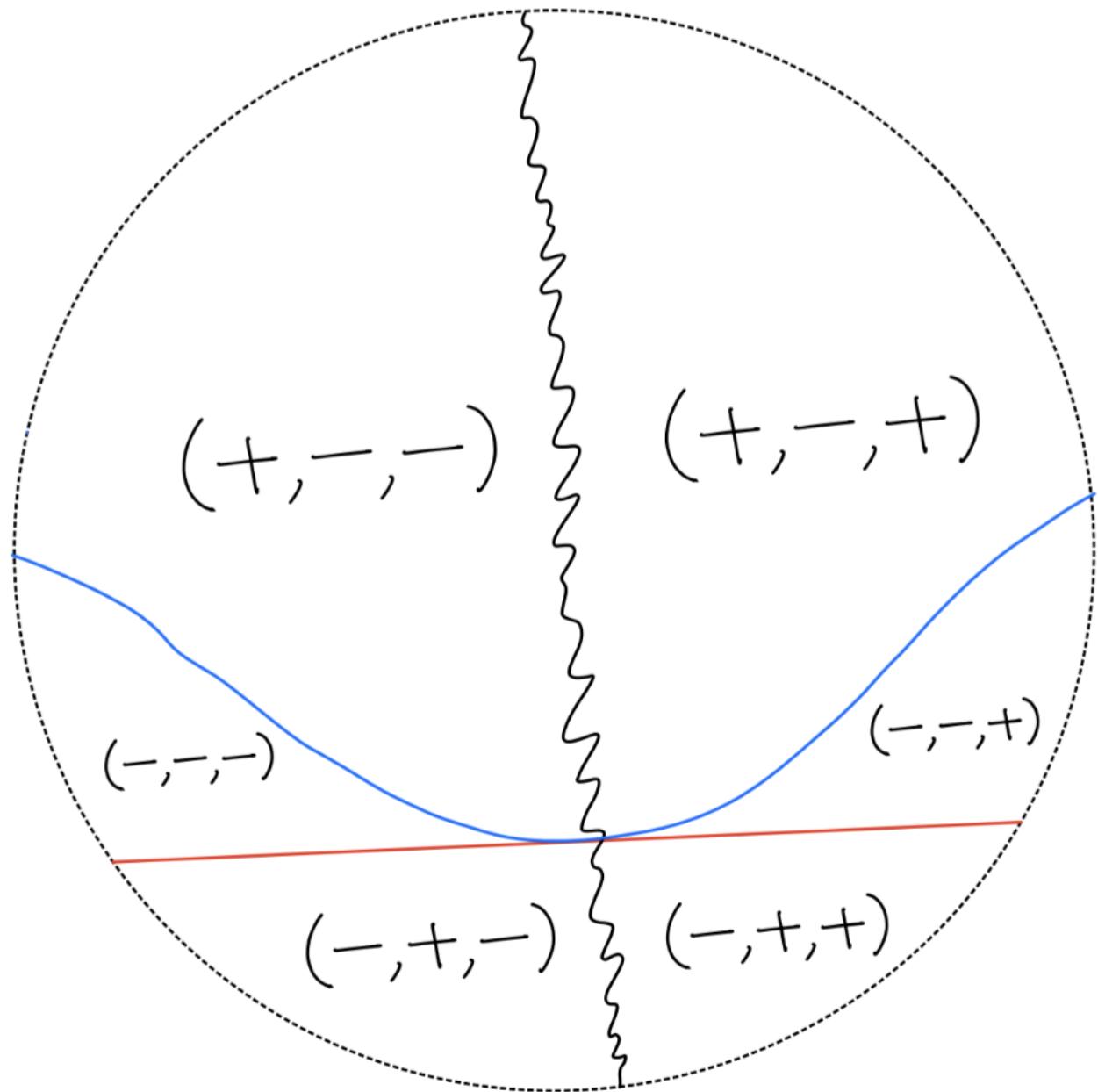


Figure 3.40

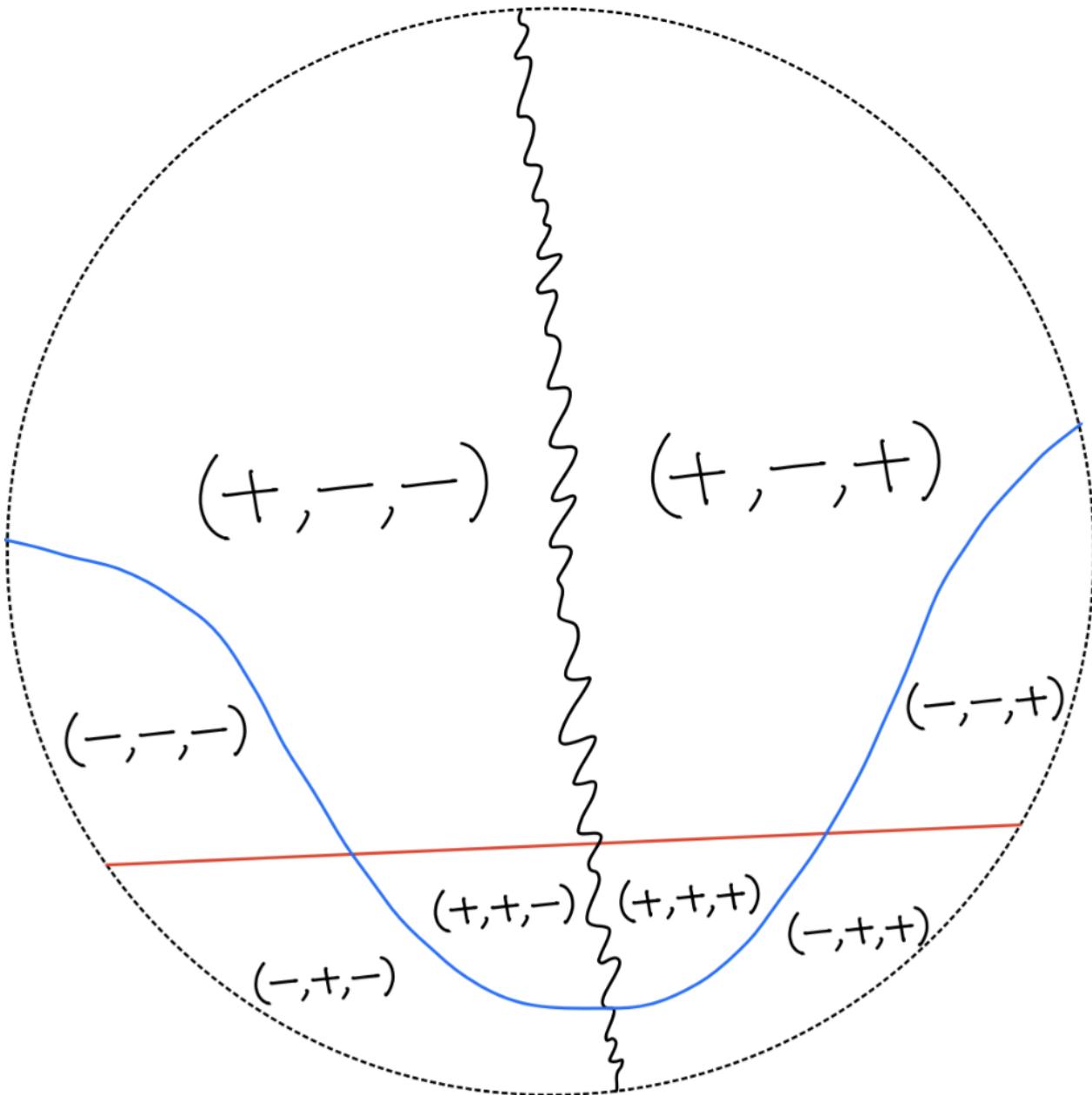


Figure 3.41

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

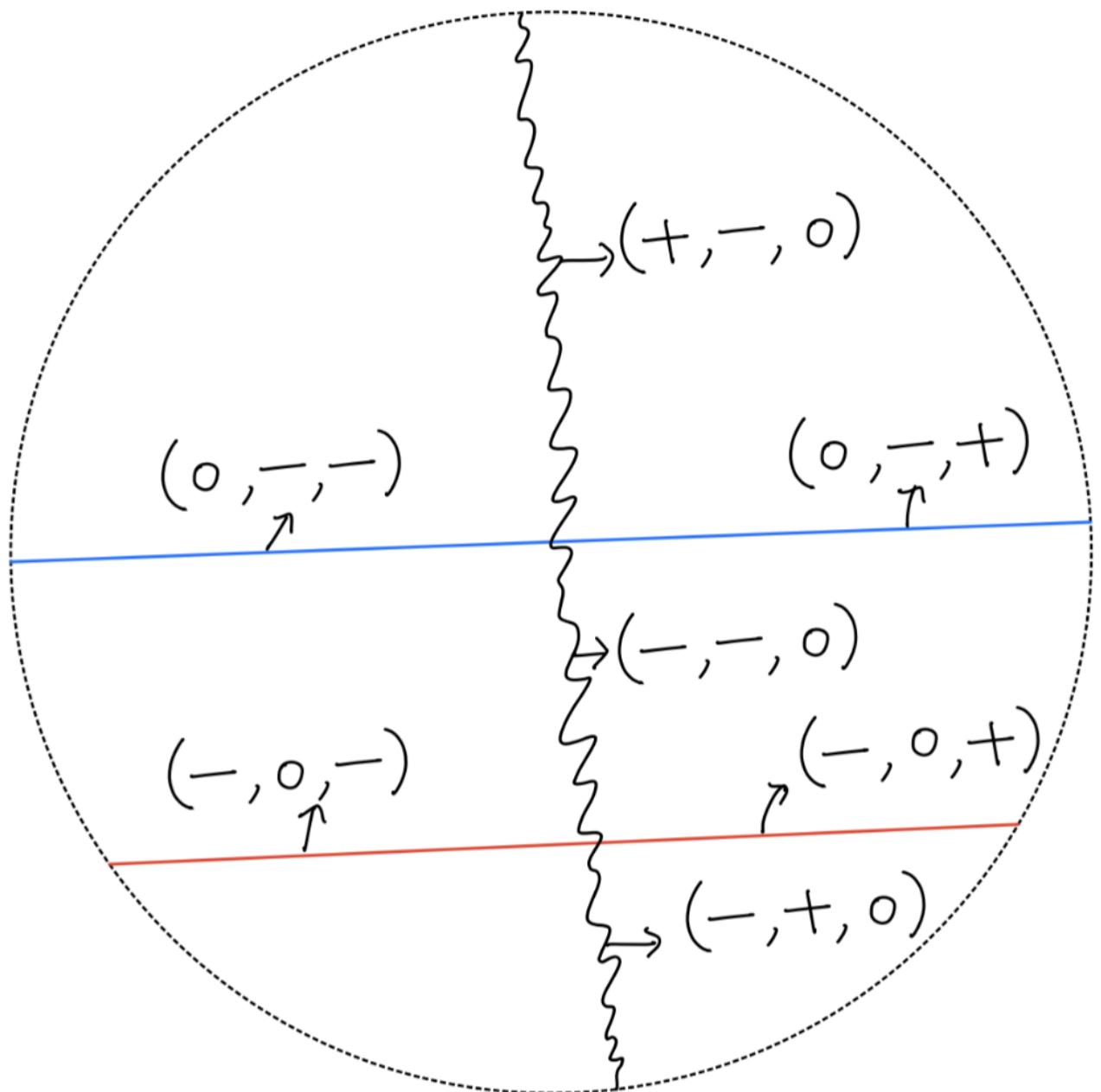


Figure 3.42

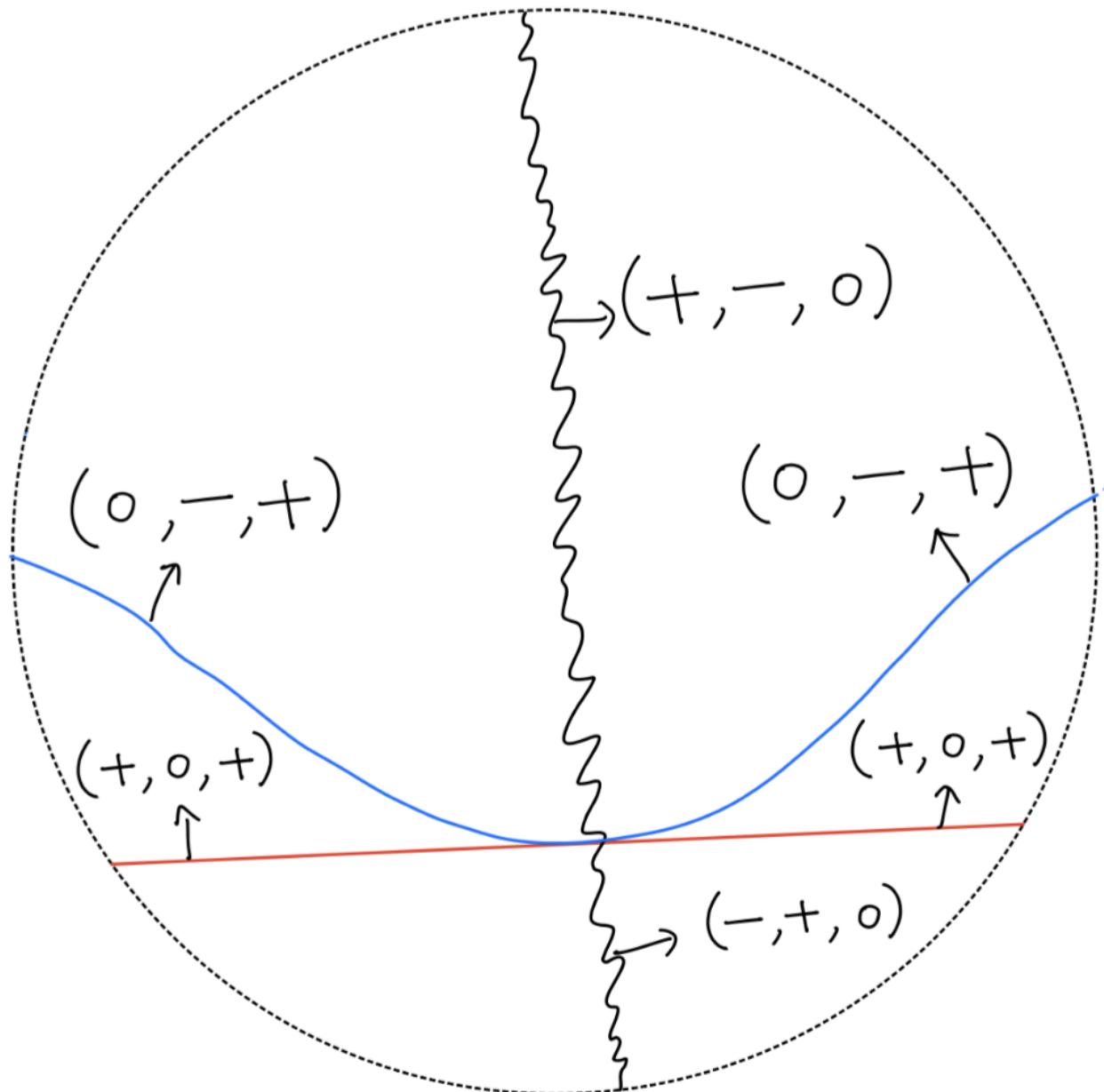


Figure 3.43

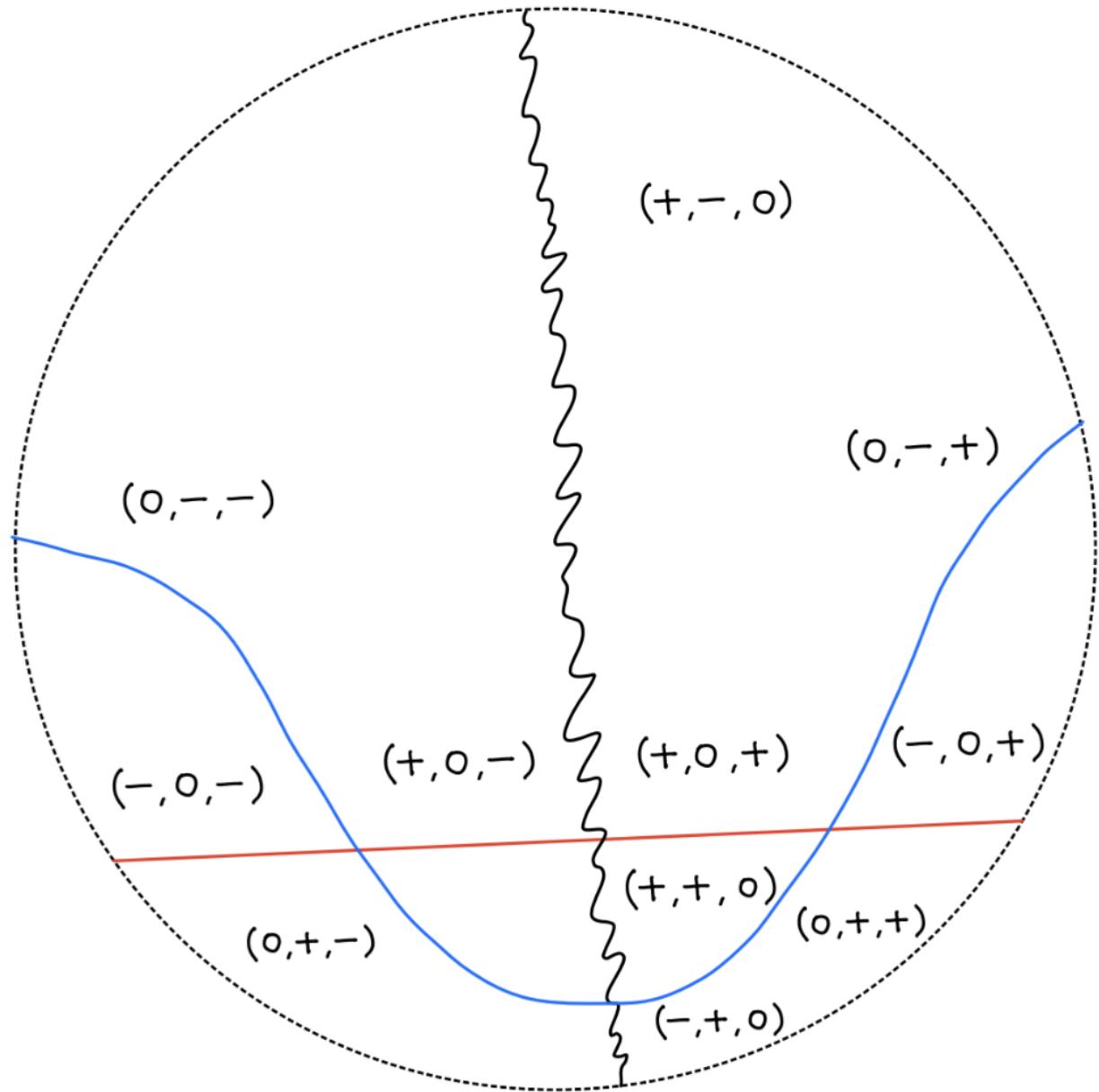


Figure 3.44

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

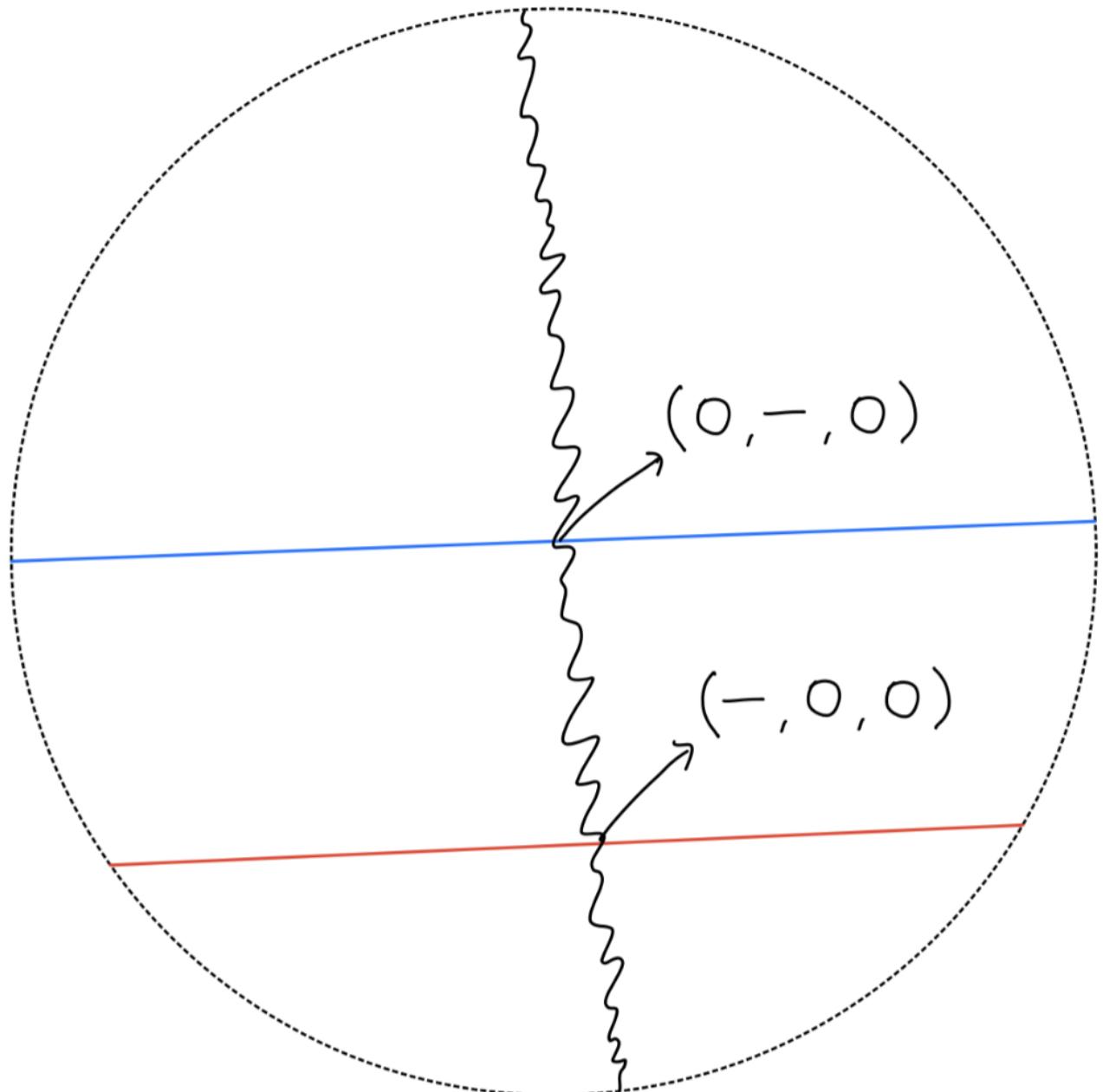


Figure 3.45

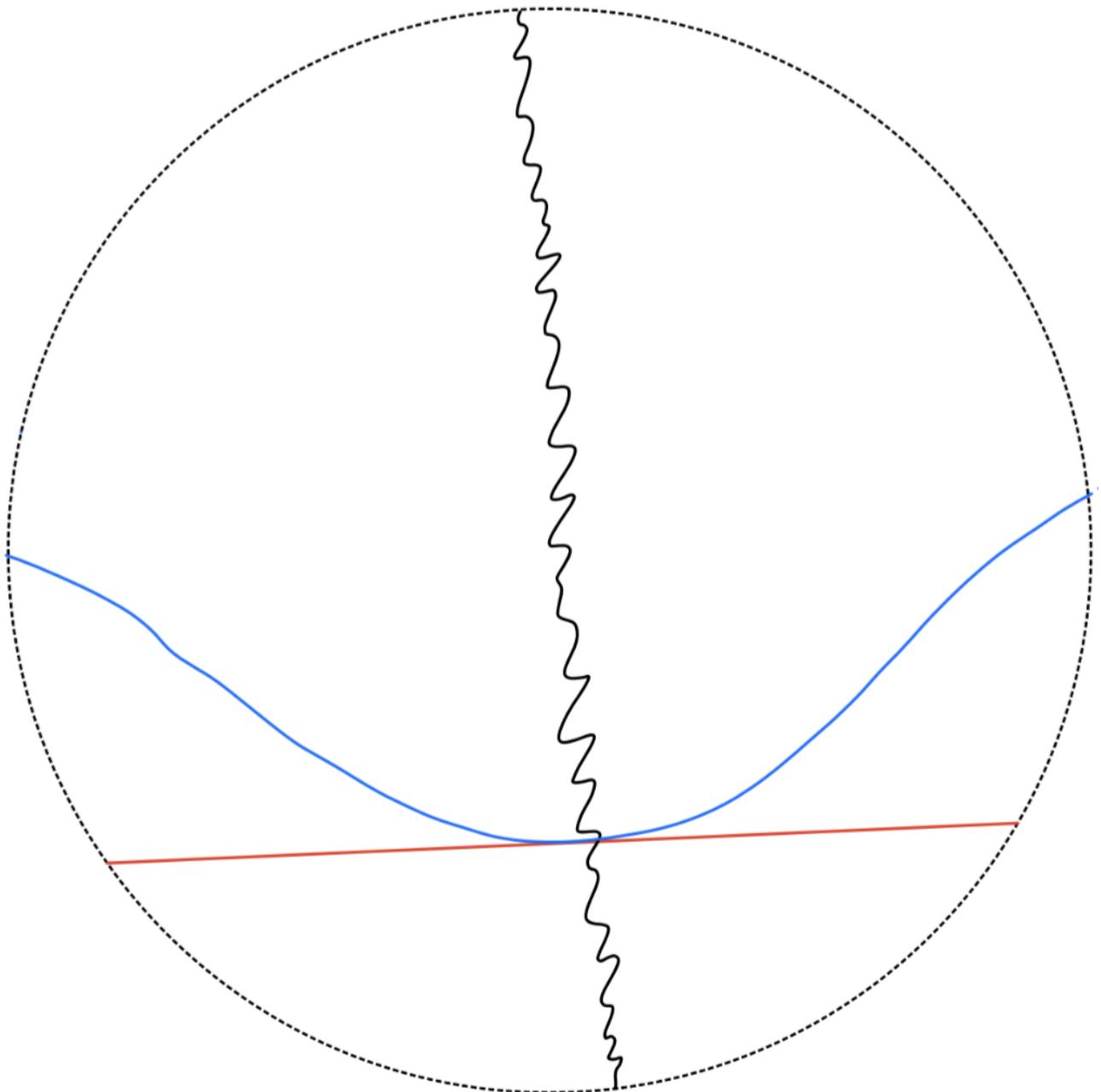


Figure 3.46

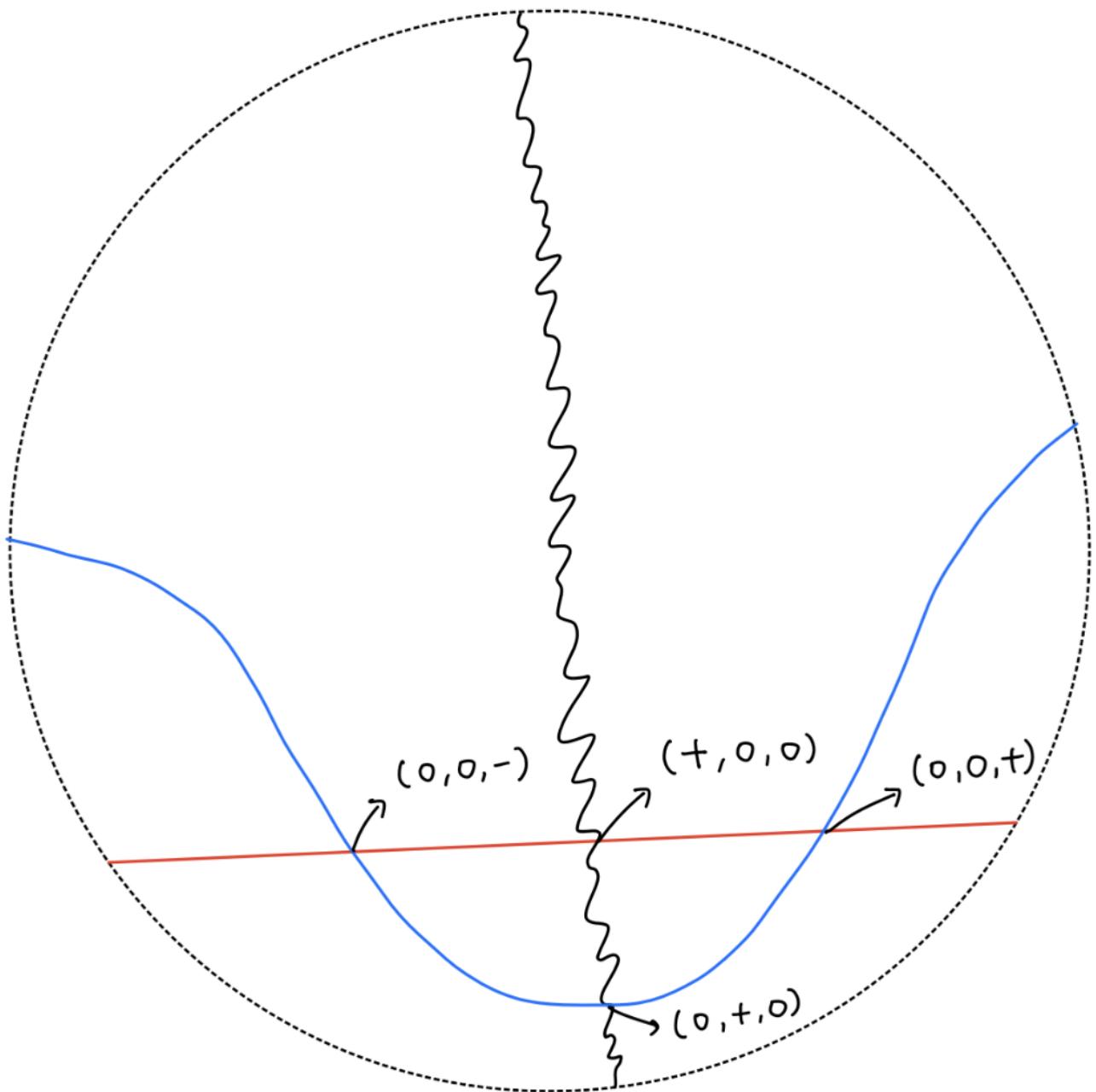


Figure 3.47

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

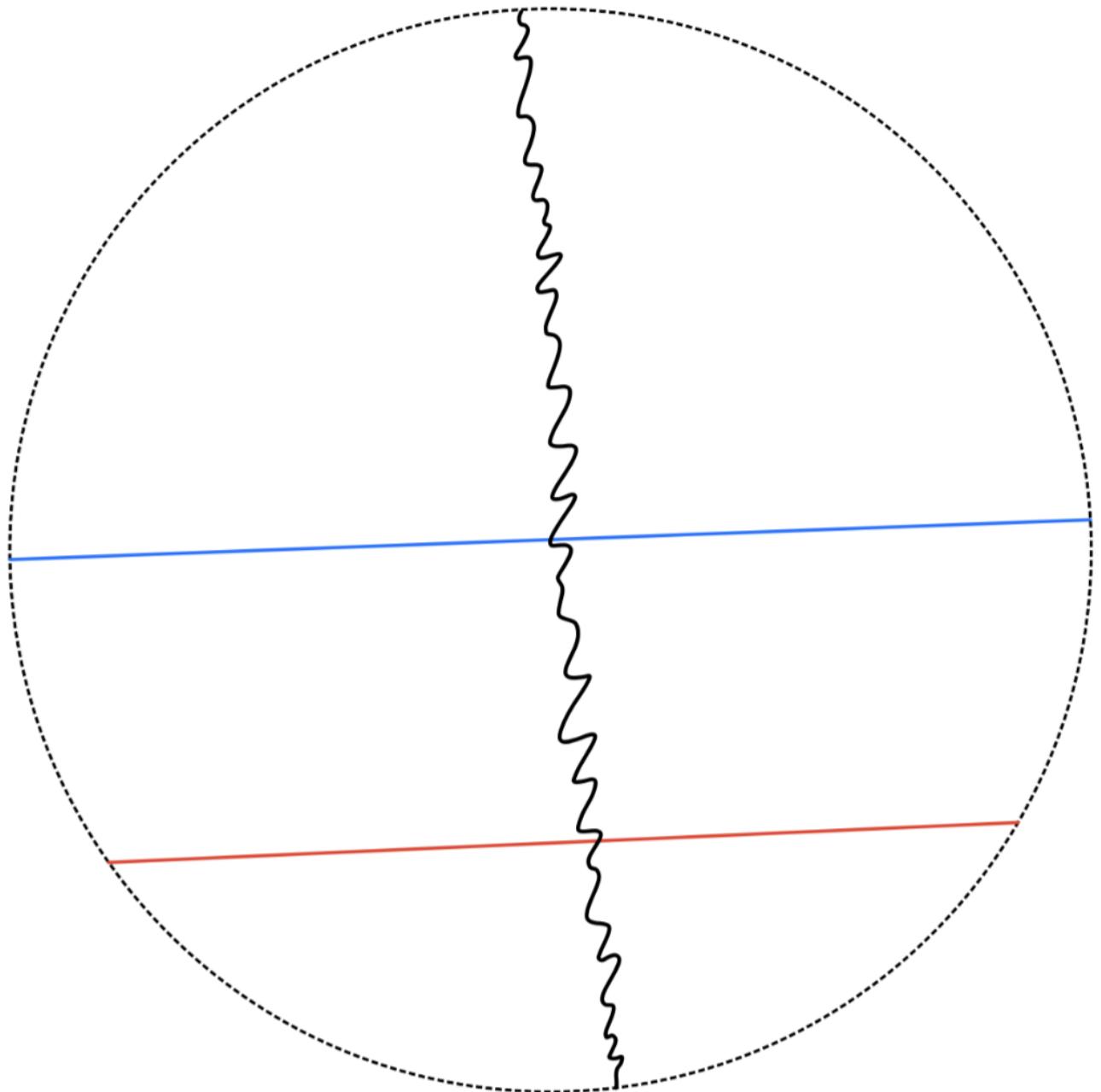


Figure 3.48

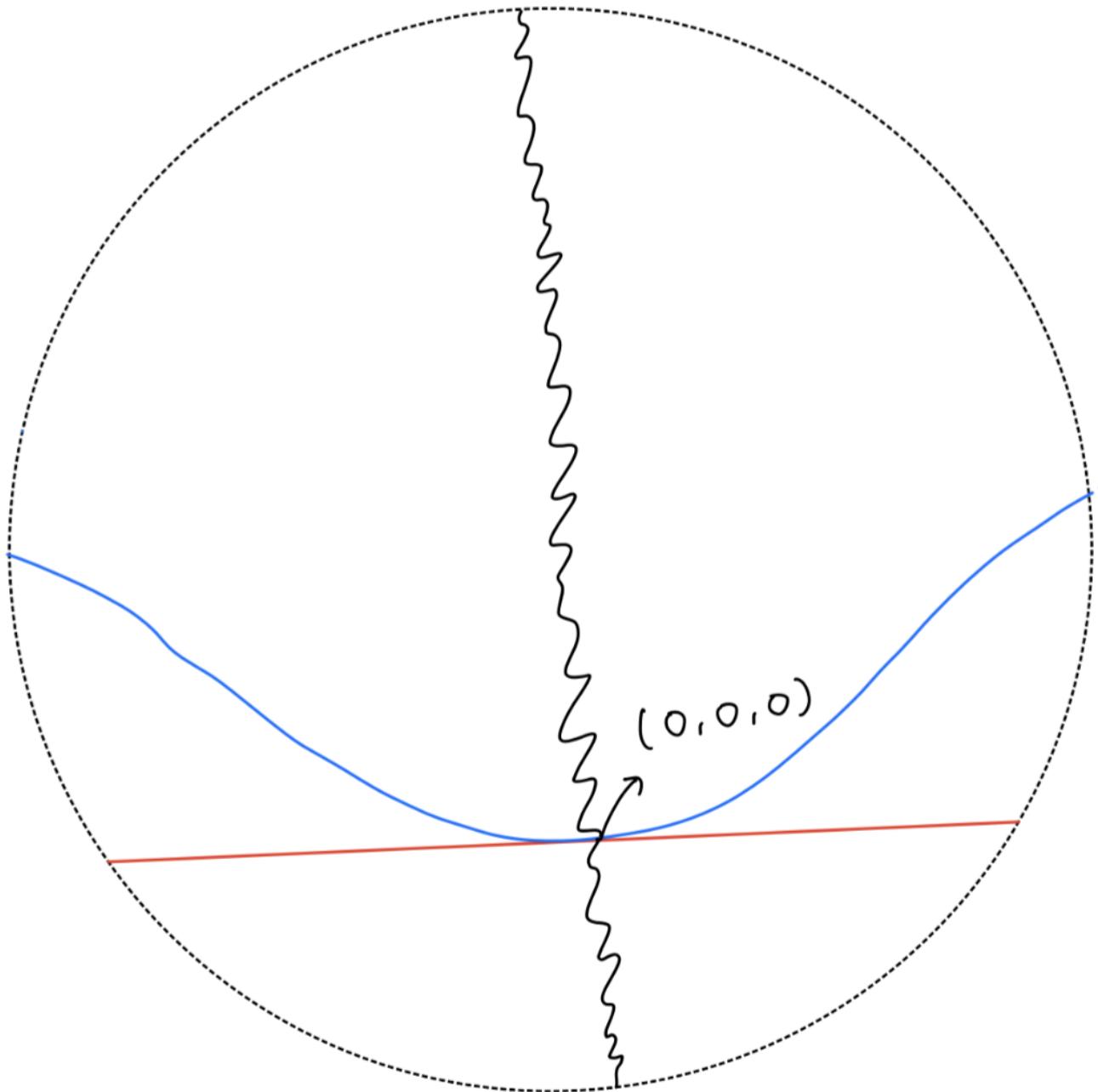


Figure 3.49

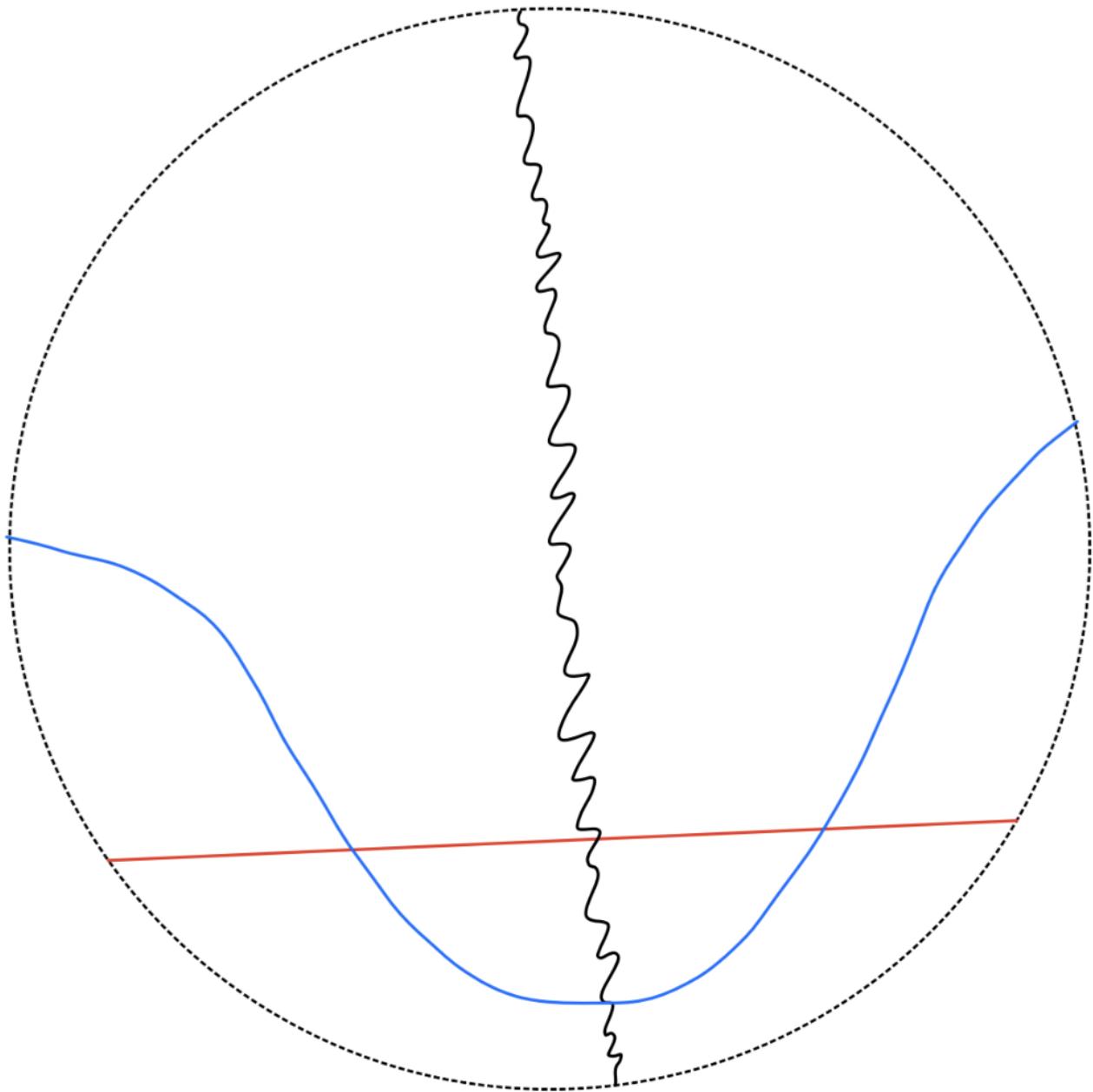


Figure 3.50

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

**Definition 79.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $\text{star}(s)$ .

**Definition 80.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of  $s$ .

**Definition 81.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 82.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 83.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 84.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 85.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

## Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

## Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

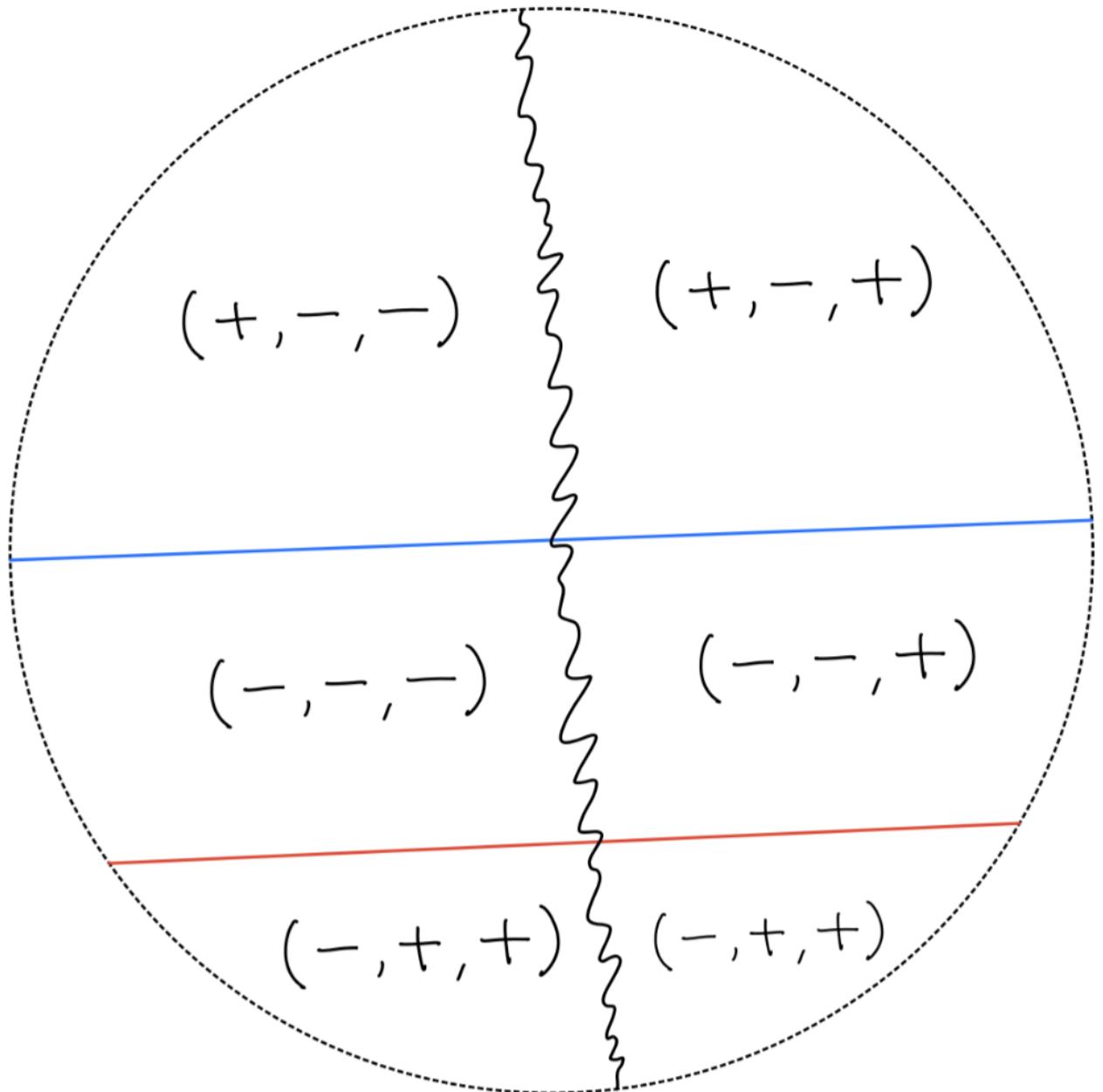


Figure 3.51

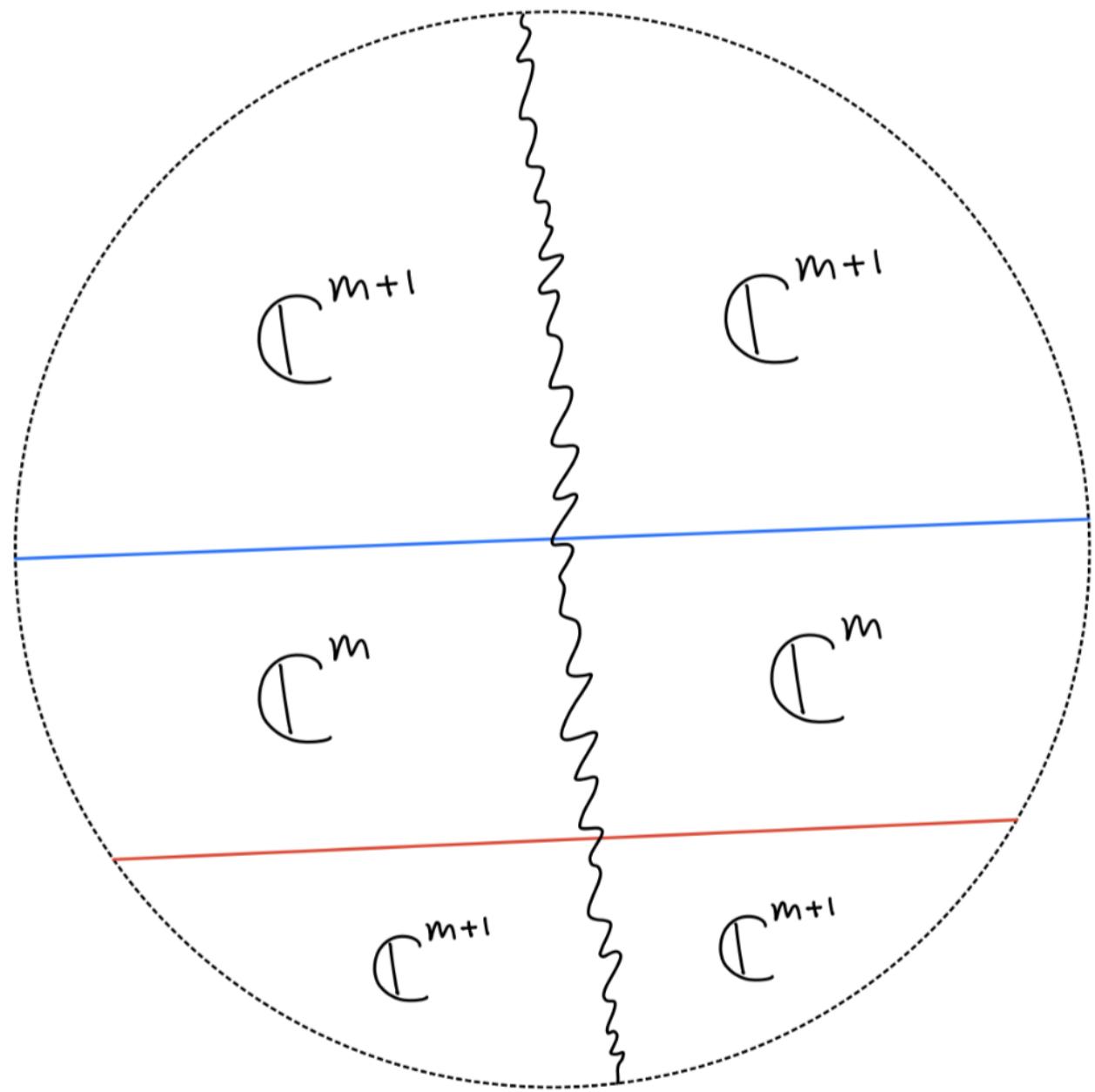


Figure 3.52

- $F_0(-, -, -) := \mathbb{C}^m$
- $F_0(-, -, +) := \mathbb{C}^m$
- $F_0(+, -, -) := \mathbb{C}^{m+1}$
- $F_0(+, -, +) := \mathbb{C}^{m+1}$

- $F_0(-, +, -) := \mathbb{C}^{m+1}$

- $F_0(-, +, +) := \mathbb{C}^{m+1}$

**Generalization maps:**

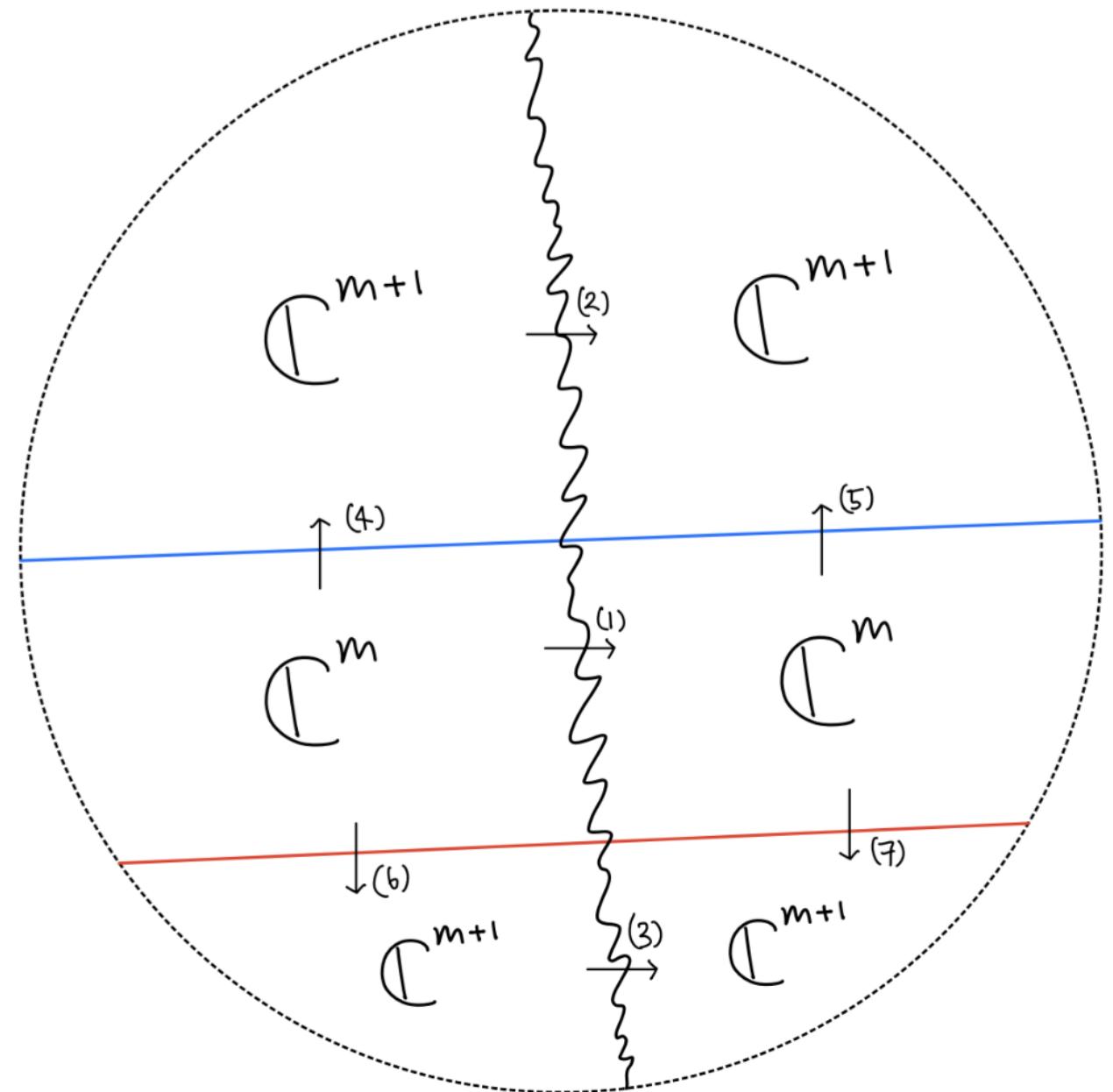


Figure 3.53

$$(1) \quad \mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

## Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say *cobord*<sub>1</sub>, that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

### A. Sheaf on $V \times [0, 1]$

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

### B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_{\bullet}(sgn_1, sgn_2, sgn_3) := F_{\bullet}(s_{\bullet}(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

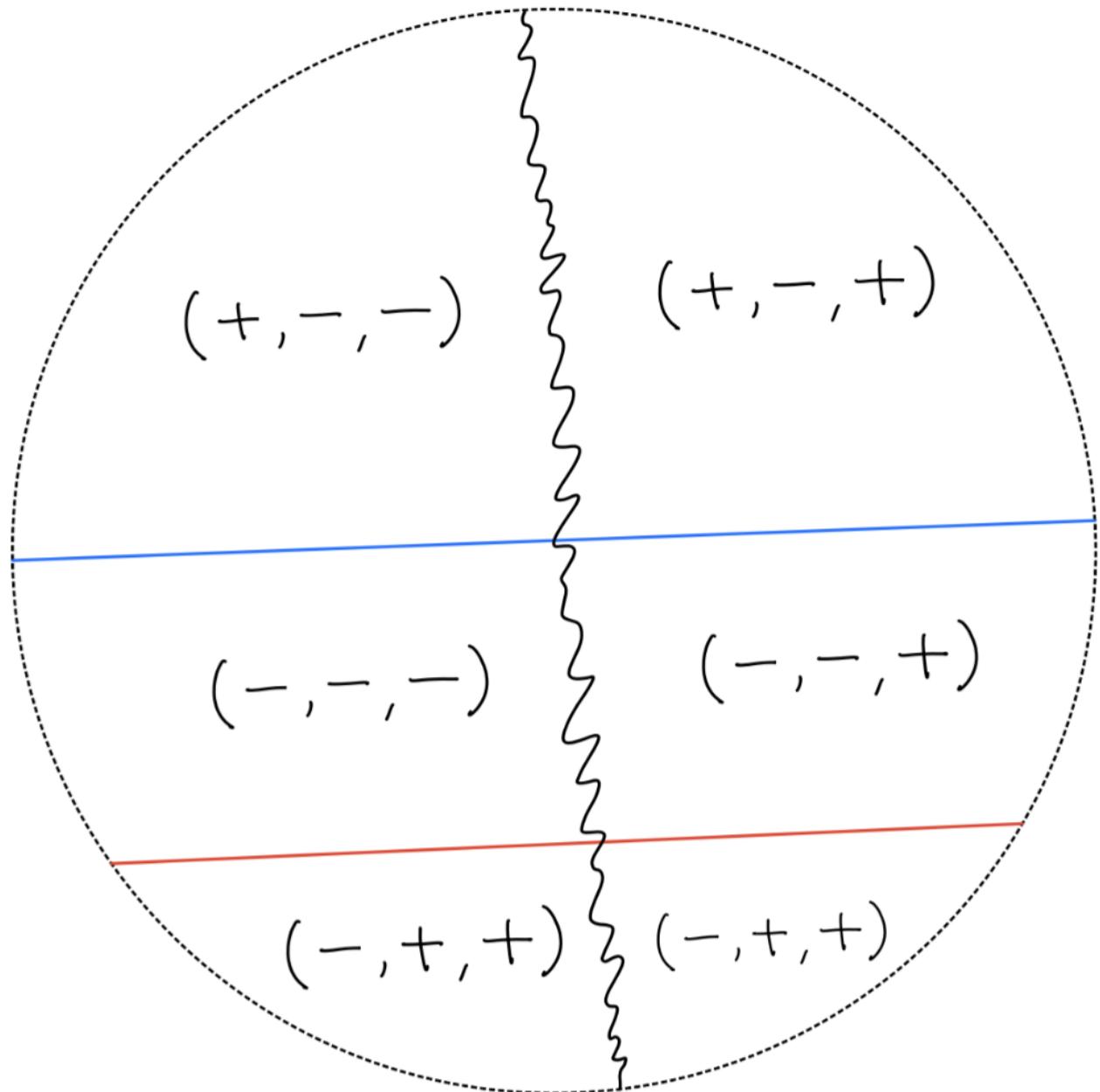


Figure 3.54

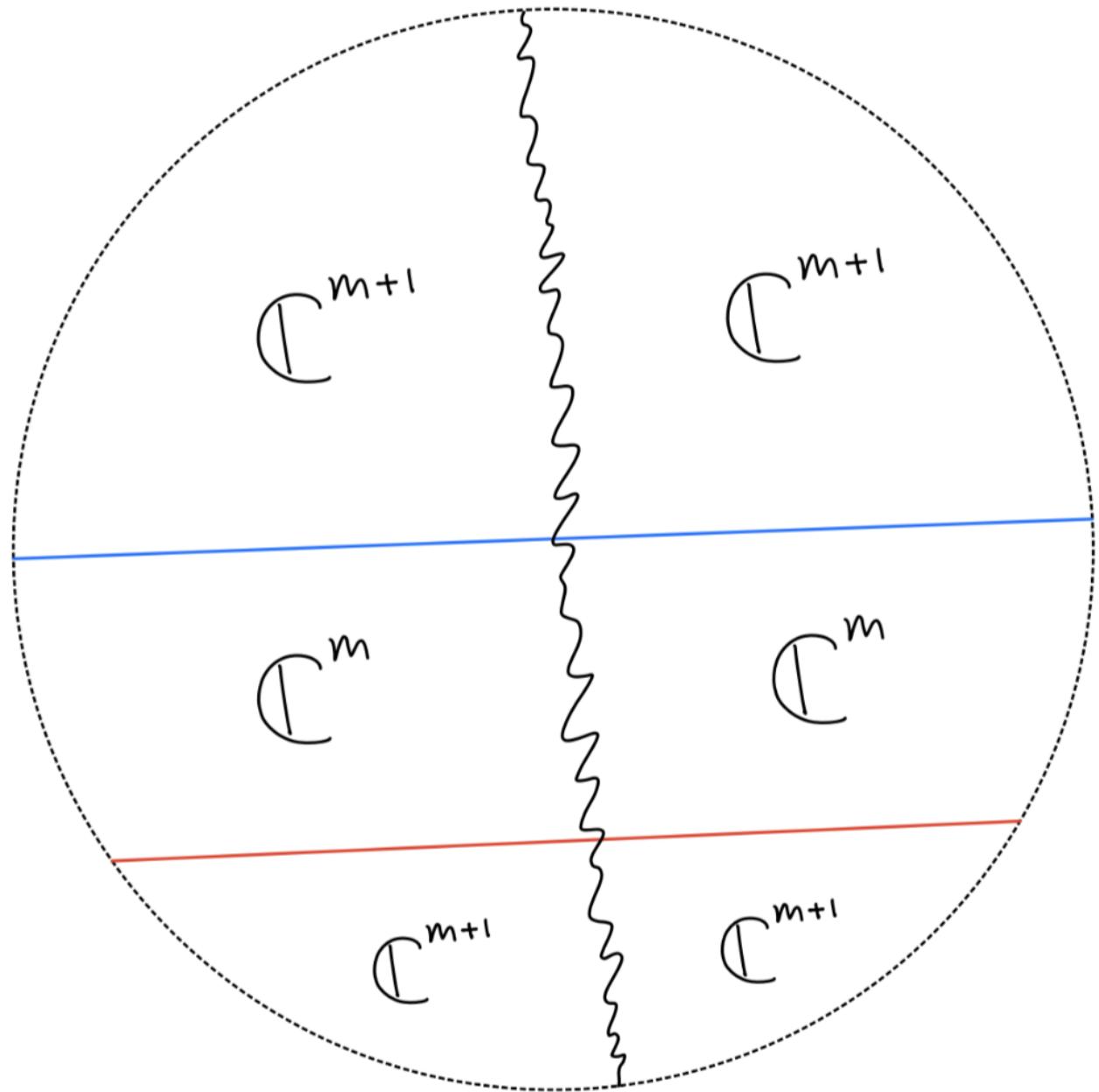


Figure 3.55

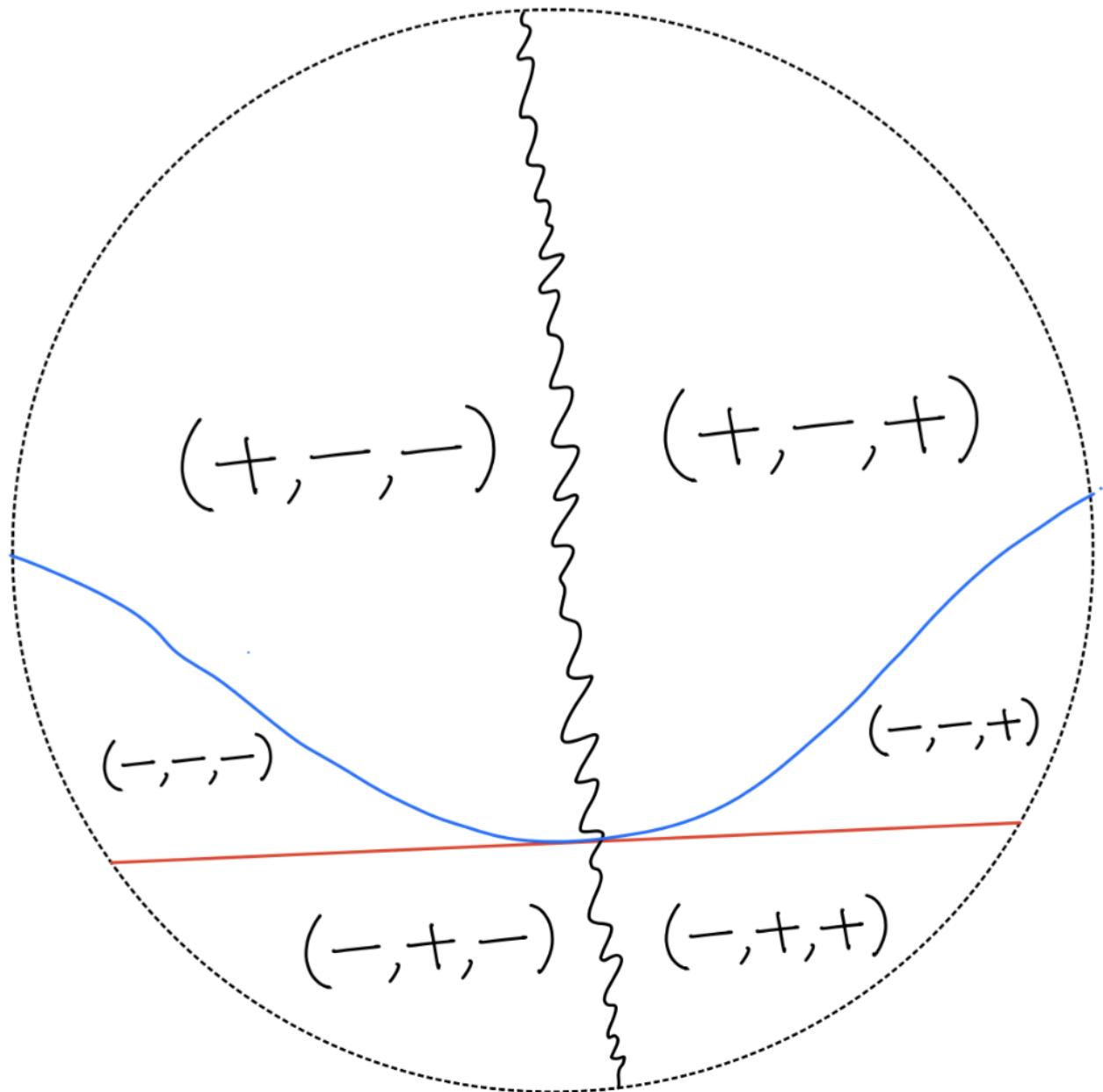


Figure 3.56

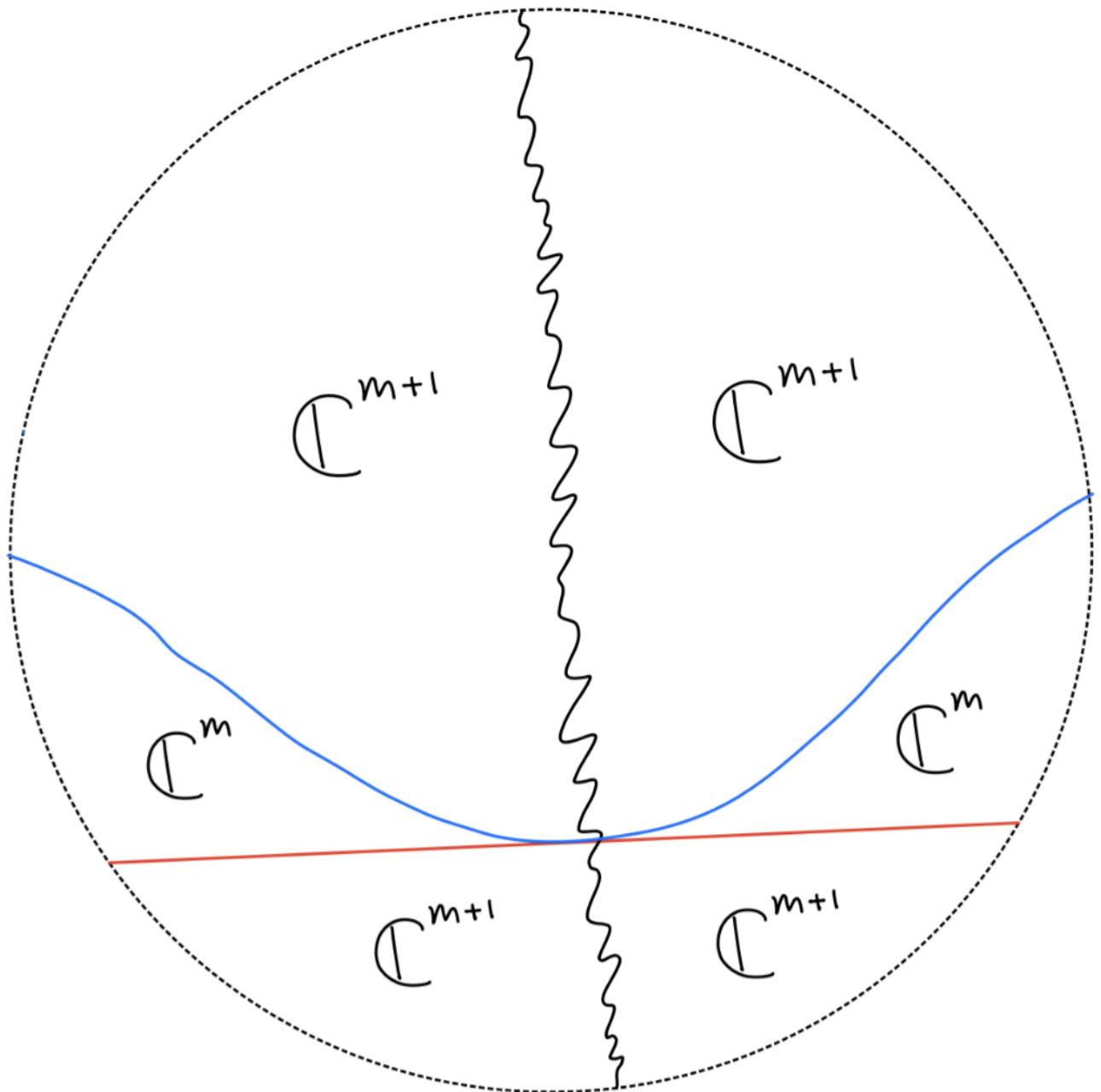


Figure 3.57

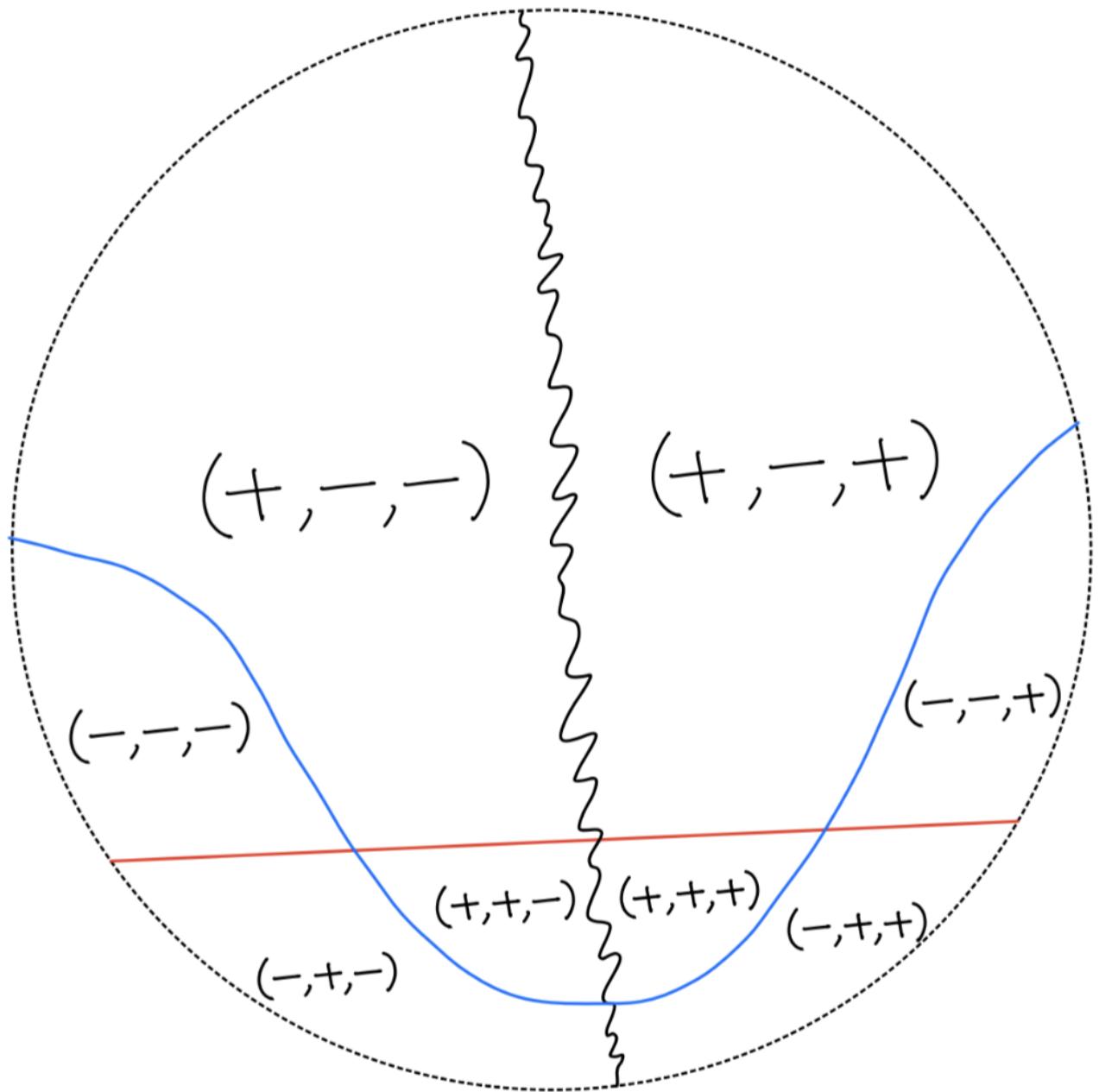


Figure 3.58

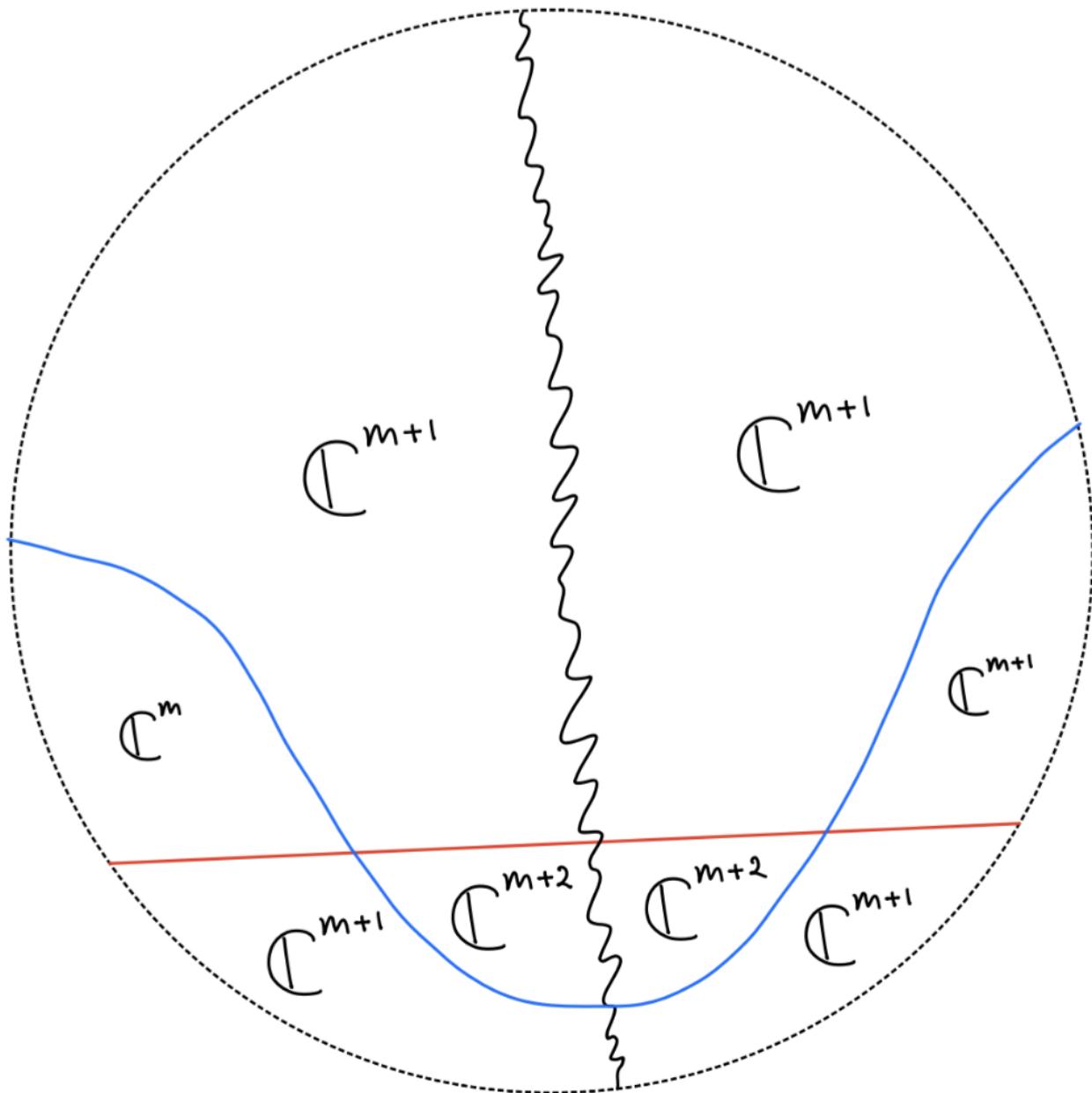


Figure 3.59

- $F_\bullet(-, -, -) := \mathbb{C}^m$
- $F_\bullet(-, -, +) := \mathbb{C}^m$
- $F_\bullet(+, -, -) := \mathbb{C}^{m+1}$
- $F_\bullet(+, -, +) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, -) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, +) := \mathbb{C}^{m+1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{m+2}$

- $F_\bullet(+, +, +) := \mathbb{C}^{m+2}$

**Generalization maps:**

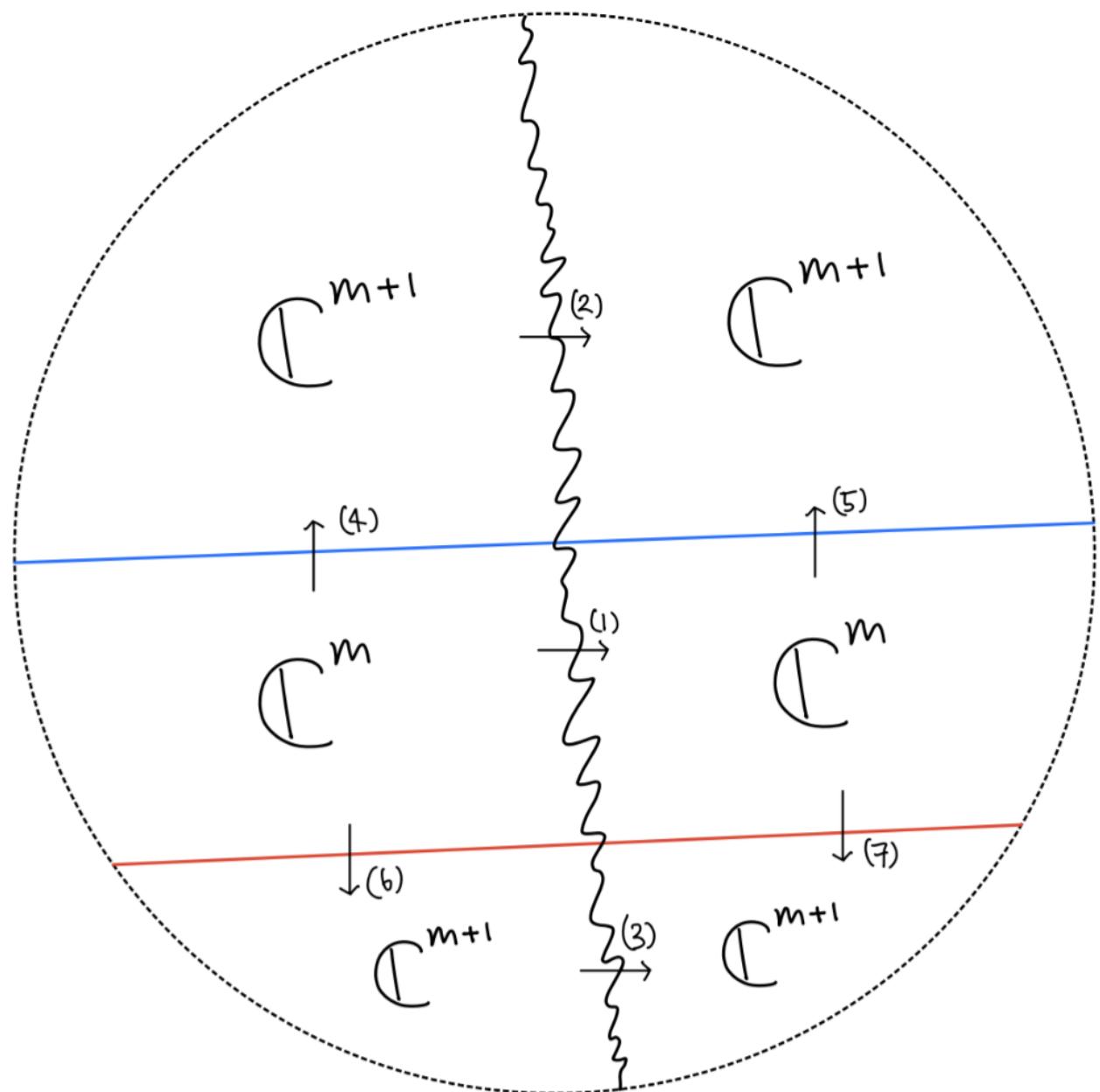


Figure 3.60

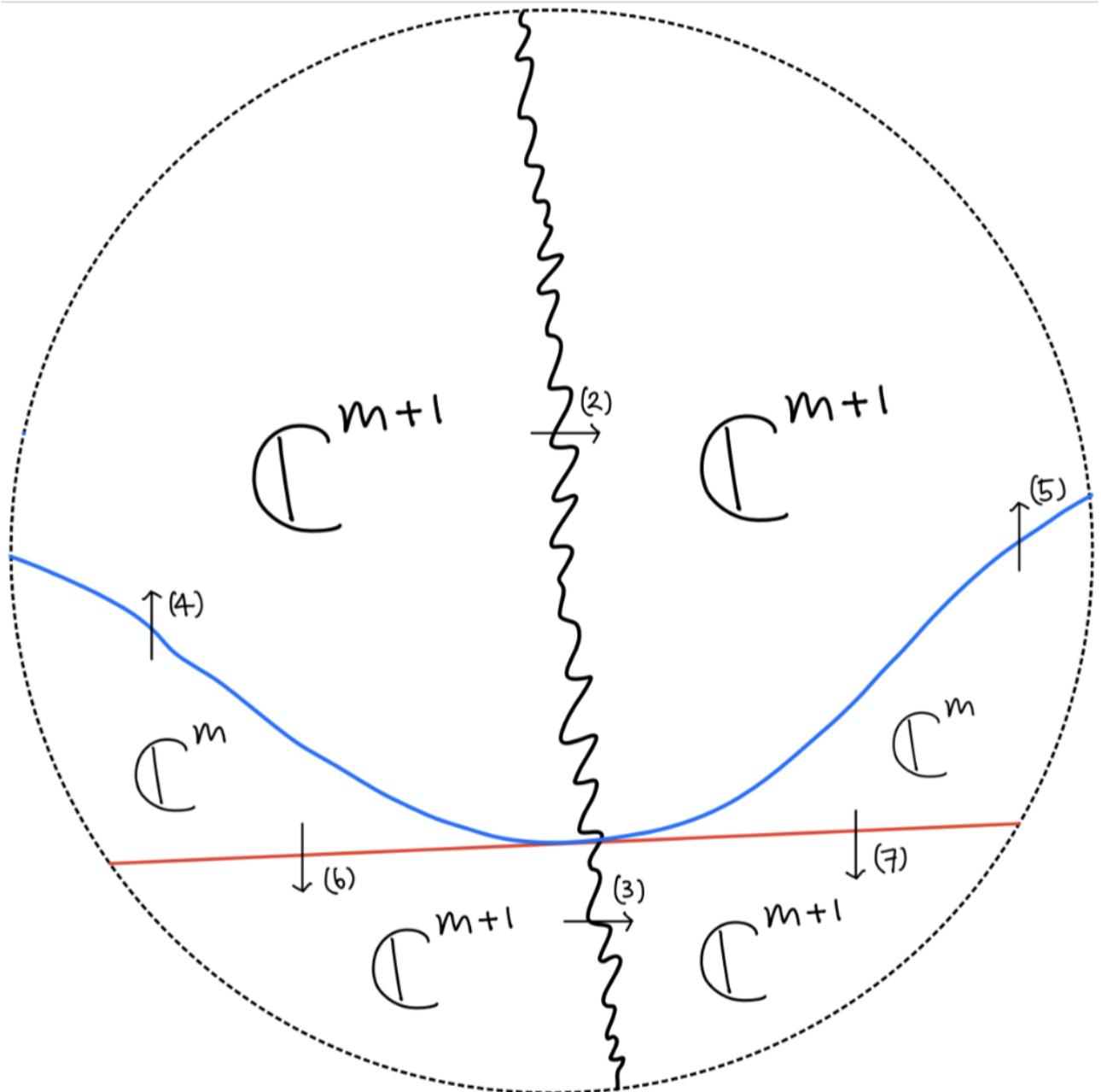


Figure 3.61

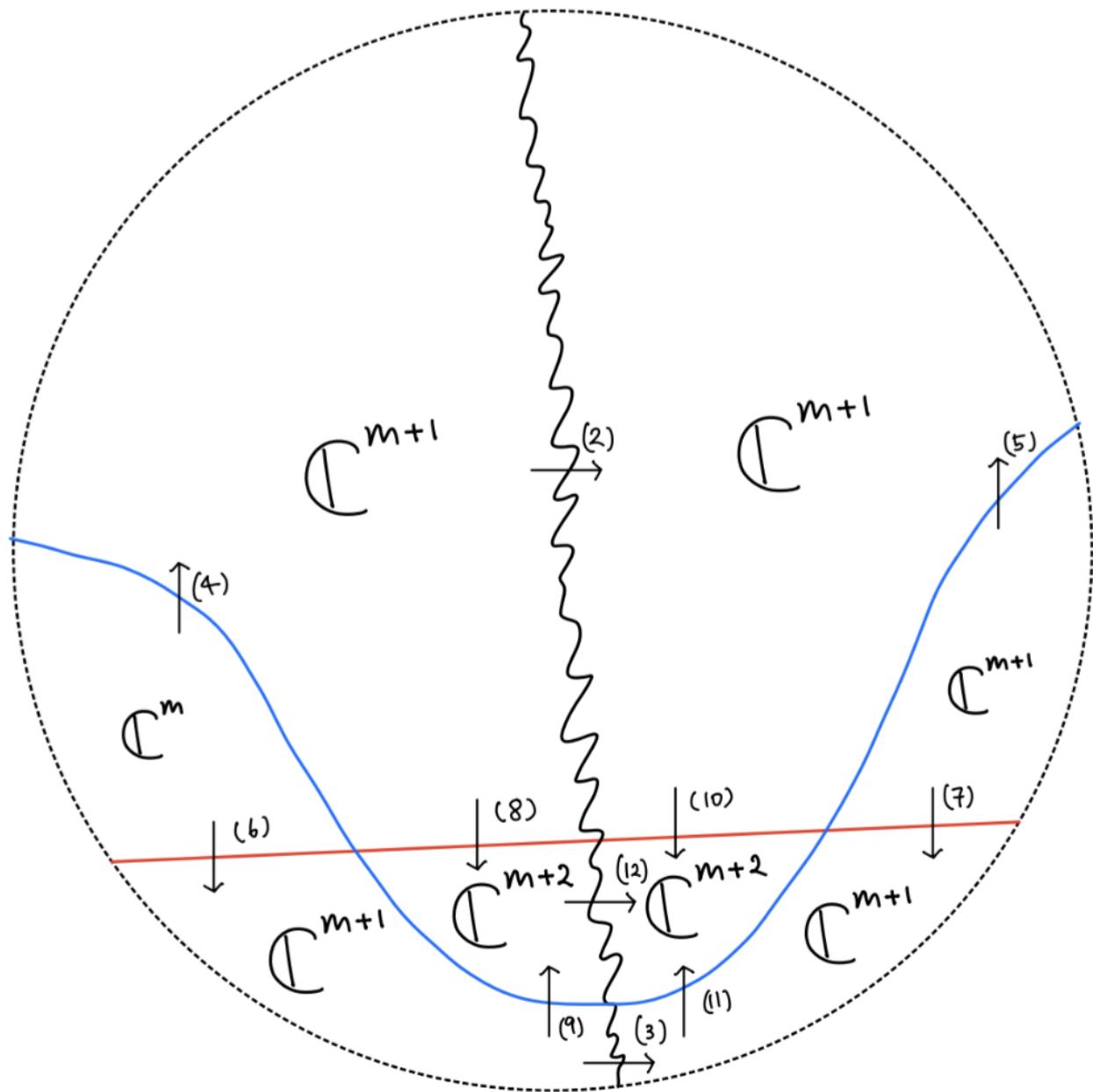


Figure 3.62

$$(1) \mathbb{C}^m \xrightarrow{T(2,2,m+1,m+1)} \mathbb{C}^m$$

$$(2) \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(3) \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(4) \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(11) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(12) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 86.** we define  $\gamma_\bullet$  to be the composition

$$(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \xhookrightarrow{\quad} & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 87.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M$ .

Note that there is a diffeomorphism between  $D_{r=2} \times (0, 1)$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

- $F^3(-, -, -) := \mathbb{C}^m$
- $F^3(-, -, +) := \mathbb{C}^m$
- $F^3(+, -, -) := \mathbb{C}^{m+1}$
- $F^3(+, -, +) := \mathbb{C}^{m+1}$
- $F^3(-, +, -) := \mathbb{C}^{m+1}$
- $F^3(-, +, +) := \mathbb{C}^{m+1}$
- $F^3(+, +, -) := \mathbb{C}^{m+2}$

- $F^3(+, +, +) := \mathbb{C}^{m+2}$

**Generalization maps:**

- $F^3(-, sgn_2, sgn_3) \rightarrow F^3(+, sgn_2, sgn_3) := \iota_1$

- $F^3(sgn_1, -, sgn_3) \rightarrow F^3(sgn_1, +, sgn_3) := \iota_0$

- $F^3(-, -, -) \rightarrow F^3(-, -, +) := T(2, m+1)$

- $F^3(+, -, -) \rightarrow F^3(+, -, +) := T(1, m+1)$

- $F^3(-, +, -) \rightarrow F^3(-, +, +) := T(2, m+2)$

- $F^3(+, +, -) \rightarrow F^3(+, +, +) := T$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(iv) \quad \begin{array}{ccc} F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^{m+1} & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(v) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^m \\ \downarrow \iota_0 & & \downarrow \iota_0 \\ \mathbb{C}^{m+1} & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$(vi) \quad \begin{array}{ccc} F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

(vii) the cubic diagram:

$$\begin{array}{ccccccc}
 & & s(-, +, -) & & & & s(-, +, +) \\
 & \nearrow s(-, 0, -) & \downarrow & & \searrow s(-, 0, +) & & \downarrow s(0, +, +) \\
 s(-, -, -) & \xrightarrow{s(-, -, -)} & s(-, 0, 0) & \xrightarrow{s(-, -, +)} & s(-, -, +) & & \\
 \downarrow s(0, -, -) & & \downarrow & & \downarrow s(0, -, +) & & \downarrow \\
 & & s(+, +, -) & & & & s(+, +, +) \\
 & \nearrow s(+, 0, -) & \downarrow & & \searrow s(+, 0, +) & & \downarrow \\
 s(+, -, -) & \xrightarrow{s(+, -, -)} & s(+, -, 0) & \xrightarrow{s(+, -, +)} & s(+, -, +) & & \\
 = & & & & & &
 \end{array}$$

$$\begin{array}{ccccc}
 & & \mathbb{C}^{m+1} & & \\
 & \swarrow \iota_0 & \downarrow & \searrow \iota_0 & \\
 \mathbb{C}^m & \xrightarrow{T(2,m+1)} & \mathbb{C}^m & \xrightarrow{T(2,m+2)} & \mathbb{C}^{m+1} \\
 & \downarrow \iota_1 & & \downarrow \iota_1 & \\
 & & \mathbb{C}^{m+2} & & \\
 & \swarrow \iota_0 & \downarrow T_{\iota_1} & \searrow \iota_0 & \\
 \mathbb{C}^{m+1} & \xrightarrow{T(1,m+1)} & \mathbb{C}^{m+1} & \xrightarrow{T_{\iota_1}} & \mathbb{C}^{m+2}
 \end{array}$$

Note that  $T(2, m+1), T(2, m+2), T(1, m+1), T$  are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

□

## Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the *cobord*<sub>1</sub>. By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

### A. Sheaf on $V$

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

**B. Sheaf on  $D_{r=2}$** 

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

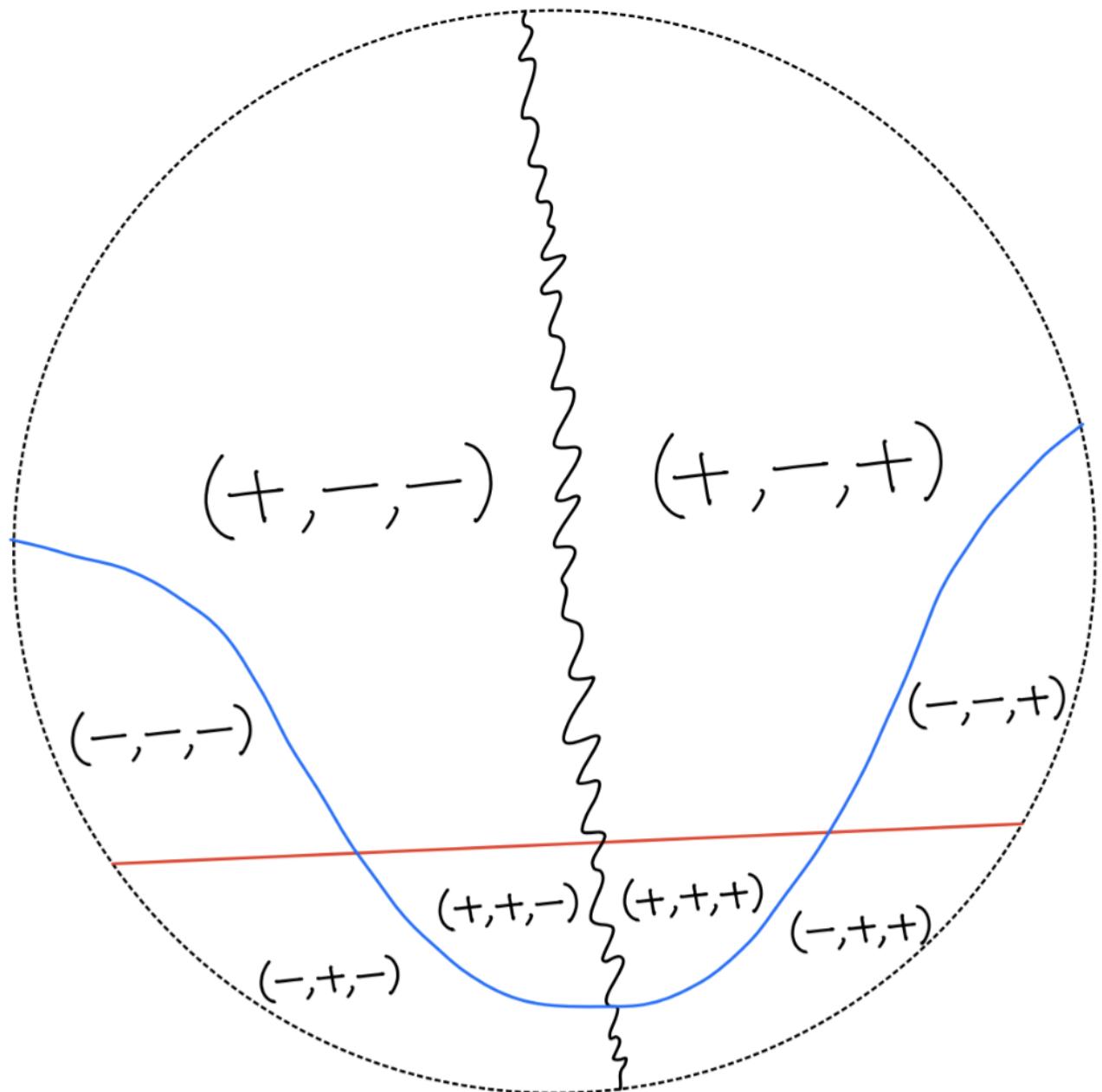


Figure 3.63

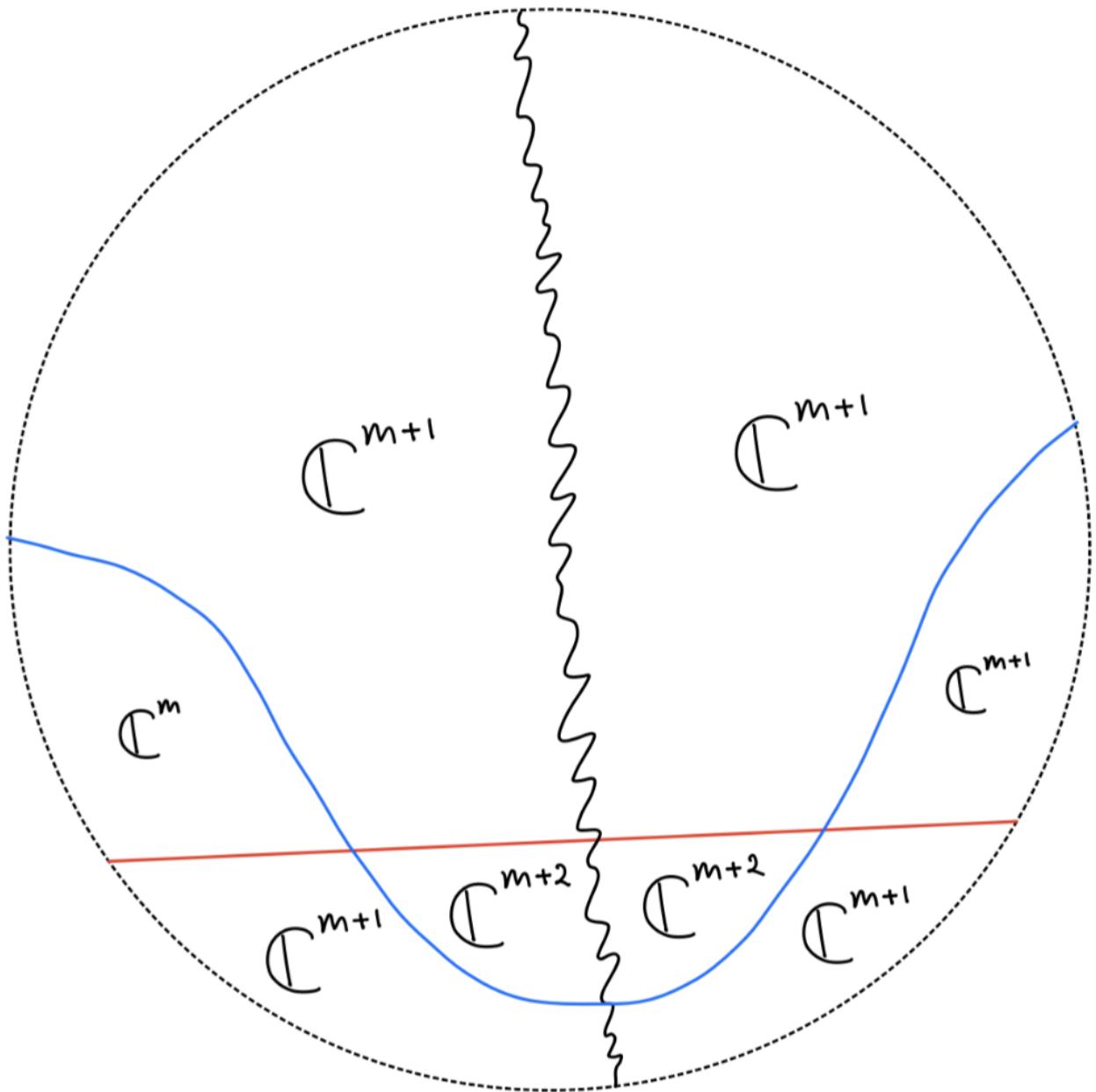


Figure 3.64

- $F_1(-, -, -) := \mathbb{C}^m$
- $F_1(-, -, +) := \mathbb{C}^m$
- $F_1(+, -, -) := \mathbb{C}^{m+1}$
- $F_1(+, -, +) := \mathbb{C}^{m+1}$

$$\bullet \ F_1(-, +, -) := \mathbb{C}^{m+1}$$

$$\bullet \ F_1(-, +, +) := \mathbb{C}^{m+1}$$

$$\bullet \ F_1(+, +, -) := \mathbb{C}^{m+2}$$

$$\bullet \ F_1(+, +, +) := \mathbb{C}^{m+2}$$

**Generalization maps:**

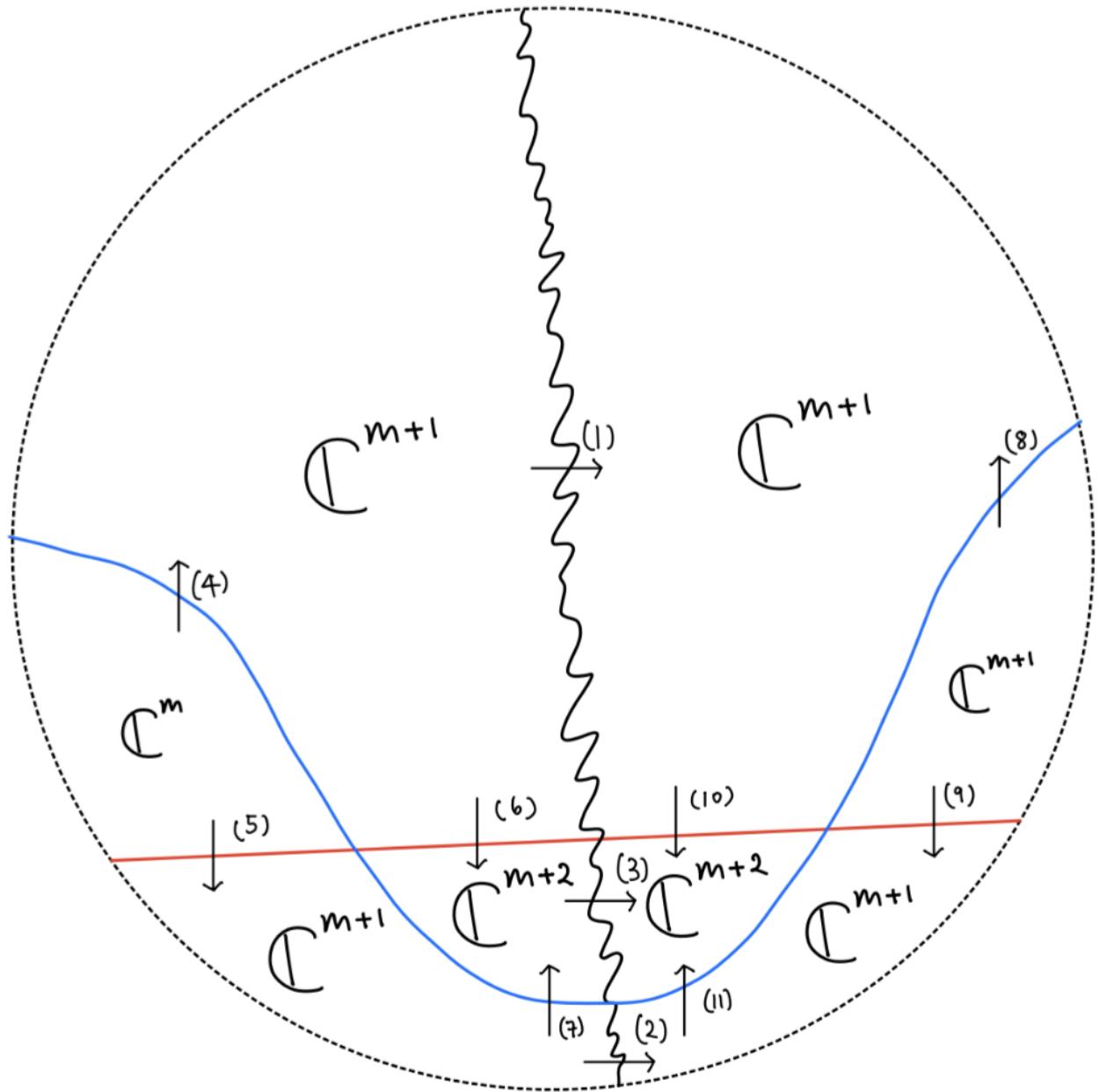


Figure 3.65

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{T(1,1,m+1,m+1)} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{T(2,2,m+2,m+2)} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(5) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(8) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(9) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(10) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(11) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 88.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0,1] \twoheadrightarrow (U \cap V)$$

### 3.5 2nd Sheaf Cobordism

In this section, we define  $cobord_2$ , a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism: from

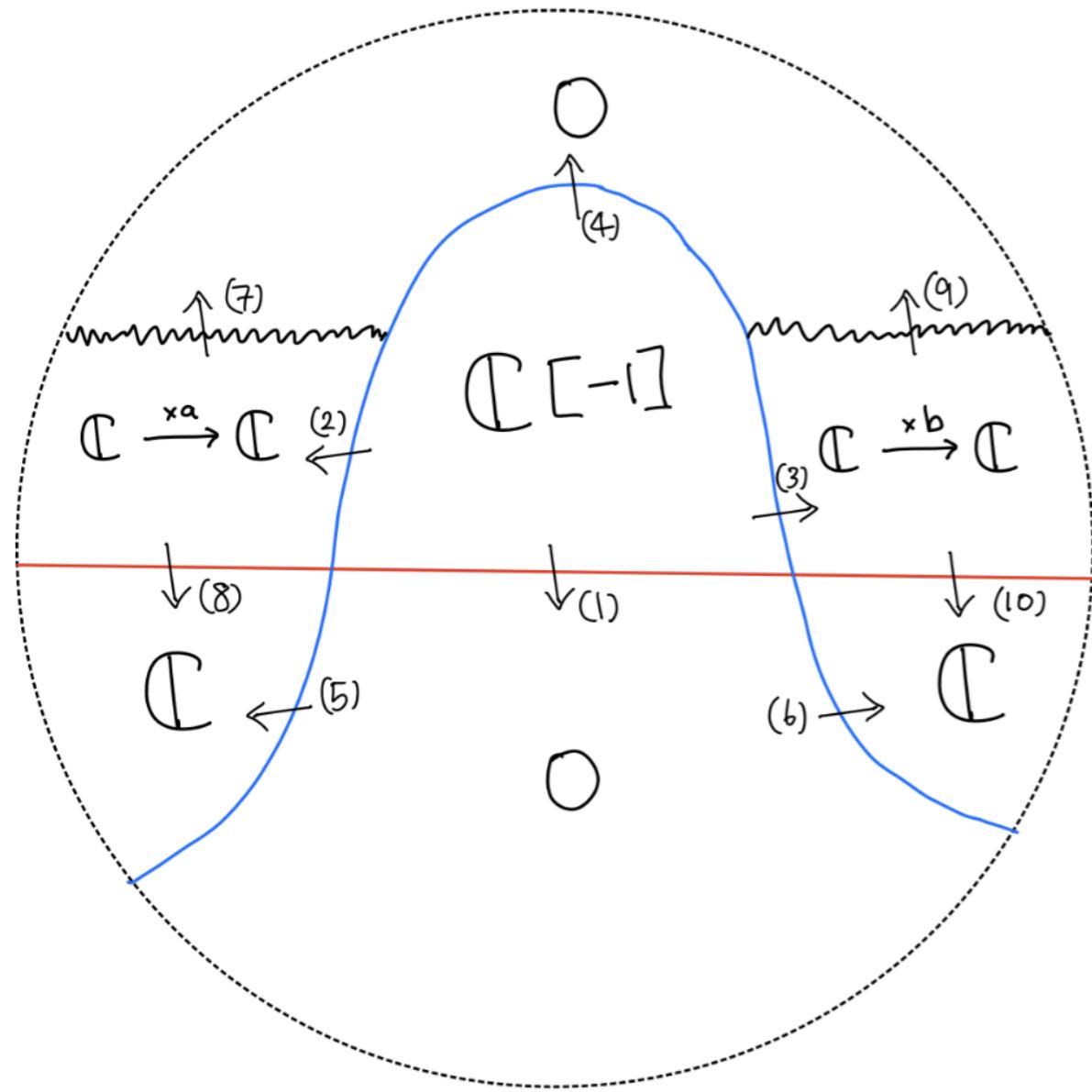


Figure 3.66

**Generalization maps:**

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

to

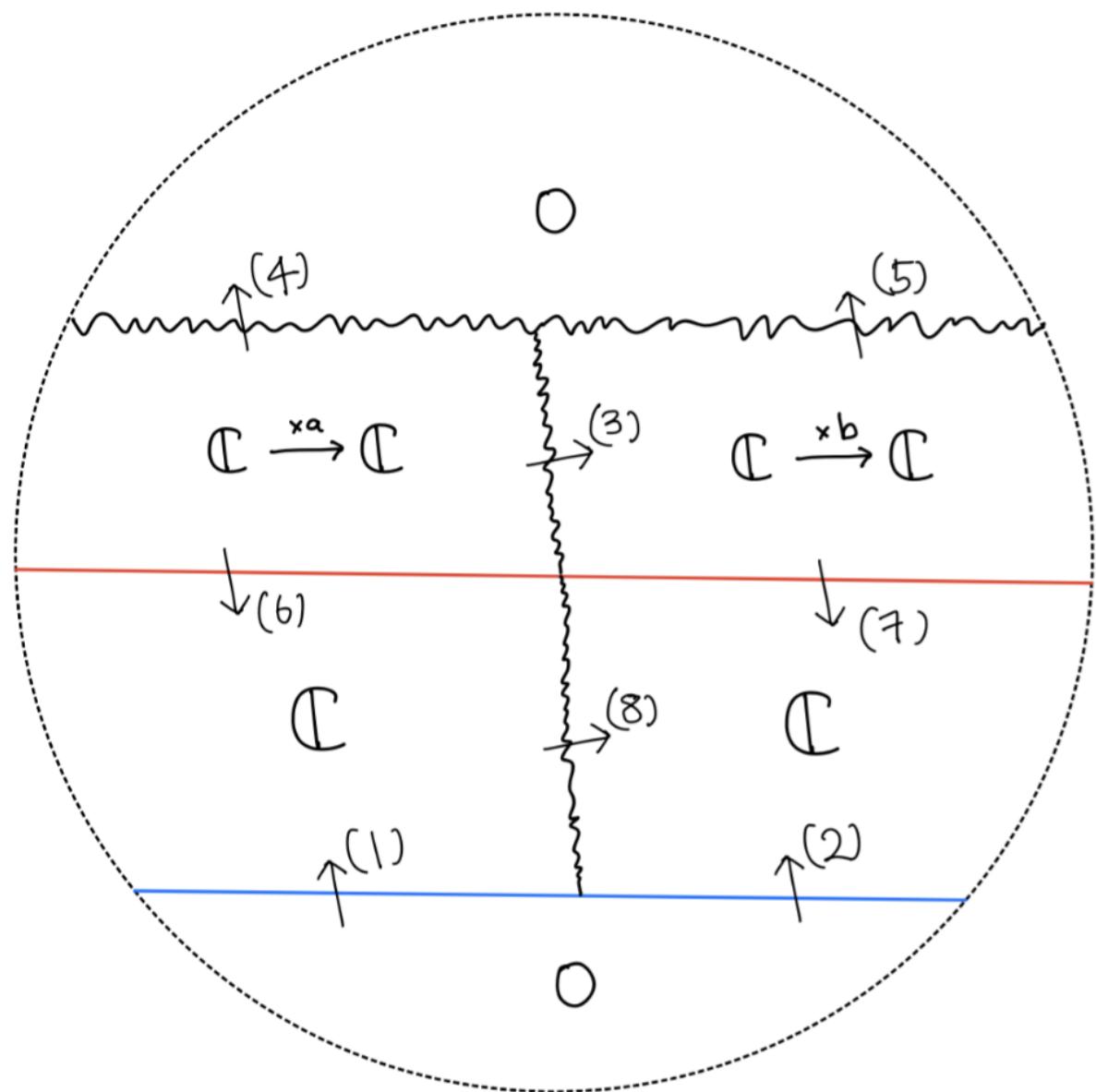


Figure 3.67

**Generalization maps:**

$$(1) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

## Notations

**Definition 89.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 90.** For  $t_0 \in \{0, 1\}$  and  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{t_0}^{symbol}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{symbol}$  a co-orientation of  $\Phi_{t_0}^{symbol}$ .

3. we denote the pair  $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$  as  $\Lambda_{t_0}^{symbol}$ . When  $symbol \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{symbol}$  to denote both
4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the beginning of the isotopy underlying  $cobord_2$  and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying  $cobord_2$ .

**Definition 91.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_\bullet^{symbol}$  to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript ' $t$ ' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_\bullet^{symbol}$  a co-orientation of  $\Phi_\bullet^{symbol}$ .

3. we denote the pair  $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$  as  $\Lambda_{\bullet}^{symbol}$ . Later in the section,  $\Lambda_{\bullet}^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>2</sub>.
4. we denote the triple  $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$  as  $\Lambda_{\bullet}$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_{\bullet}$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>2</sub>.

**Definition 92.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

**Definition 93.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 94.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_0^{squig}$  is the union of the following two components
  - (i)  $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  - (ii)  $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$

2. We define co-orientations  $\xi_0^{symbol}$  of  $\lambda_0^{symbol}$  as follows

- $\xi_0^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_0^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_0^{squig}$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.

**Definition 95.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_1^{squig}$  is the union of the following three components
  - (i)  $\{(x, \frac{1}{2}) \in D_{r=2}\}$
  - (ii)  $\{(0, z) \in D_{r=2} \mid -\frac{1}{2} < z < \frac{1}{2}\}$

2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows

- $\xi_1^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_1^{squig}$ :

- for (i), hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- for (ii), hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 96.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = 0\}$
- $\lambda_\bullet^{squig}$  is the union of the following two components
  - (i)  $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0, 1] \mid \frac{1}{2} > \Psi_t(x)\}$
  - (ii)  $\{(0, z, t) \in D_{r=2} \times [0, 1] \mid \Psi_t(0) < z < \frac{1}{2}\}$

2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows

- $\xi_\bullet^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_\bullet^{squig}$ :
  - for (i), hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - for (ii), hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 97.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 98.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 99.** For  $i = 1, 2, 3, 4$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ ,

1. we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, sgn(-z) = sgn_2, \\ & sgn(z - \frac{1}{2}) = sgn_3, \\ & sgn(x) = sgn_4\} \end{aligned}$$

2. we use \* as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{i^{th}}{\ast}, \cdots) := s_{t_0}(\cdots, \underset{i^{th}}{-}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{0}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{+}, \cdots)$$

3. we omit trailing \*'s e.g.  $s_0(+, -) = s_0(+, -, *, *)$
4. note that we do not omit \*'s located in between  $-$ ,  $0$ ,  $+$  e.g.  $s_0(+, -, *, -, -) \neq s_0(+, -, -, -)$

**Definition 100.** For  $i = 0, 1, 2, 3, 4$  and  $sgn_i \in \{-, 0, +\}$ ,

1. we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\ & sgn(z - \frac{1}{2}) = sgn_3, \\ & sgn(x) = sgn_4\} \end{aligned}$$

2. we use \* as a wild card sign i.e.

$$s_{\bullet}(\cdots, \underset{i^{th}}{*}, \cdots) := s_{\bullet}(\cdots, \underset{i^{th}}{-}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{0}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{+}, \cdots)$$

3. we omit trailing \*'s e.g.  $s_{\bullet}(+,-) = s_{\bullet}(+,-,*,*)$
4. note that we do not omit \*'s located in between  $-$ ,  $0$ ,  $+$  e.g.  $s_{\bullet}(+,-,*,-, -) \neq s_{\bullet}(+,-, -, -)$

**Definition 101.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_{\bullet}$  using the above notations:

1.  $\mathcal{S}_0$ :

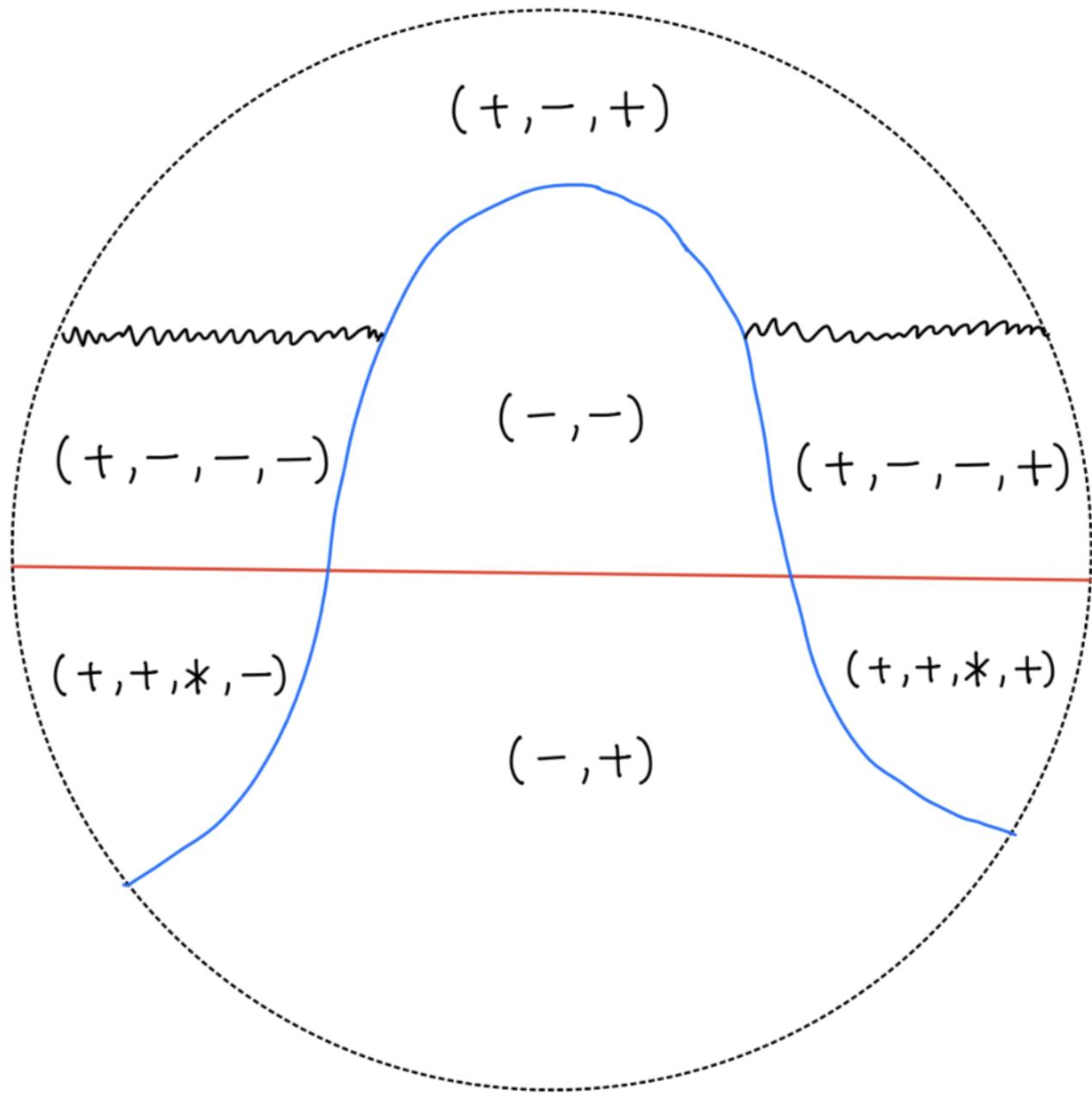


Figure 3.68

- 2 dimensional strata:

$$\begin{aligned}
 & s_0(-, +), s_0(-, -), s_0(+, -, -, -), s_0(+, -, -, +), s_0(+, -, +), s_0(+, +, *, -), \\
 & s_0(+, +, *, -)
 \end{aligned}$$

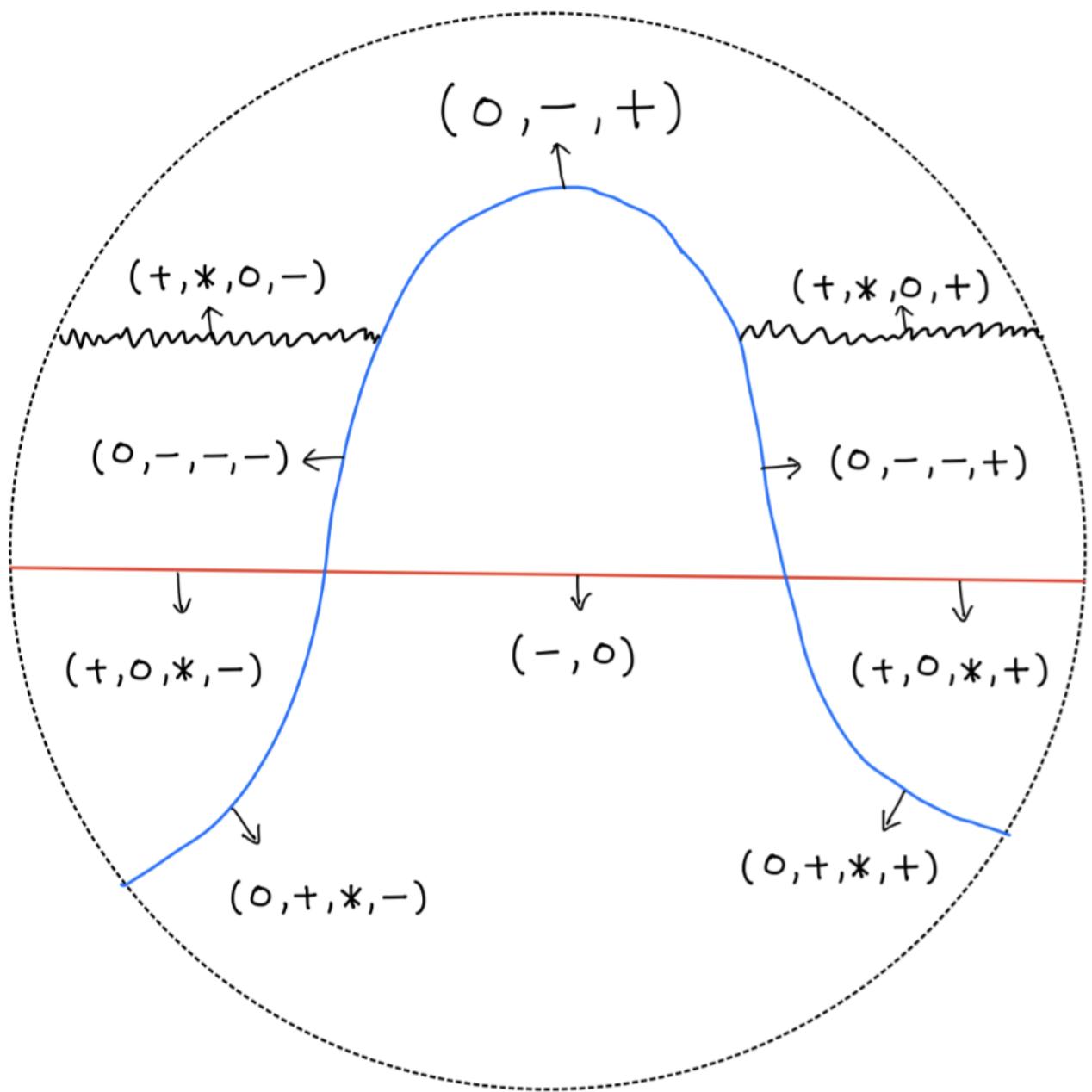


Figure 3.69

- 1 dimensional strata:

$$\begin{aligned}
 & s_0(0, +, *, -), s_0(0, -, -, -), s_0(0, -, +), s_0(0, -, -, +), s_0(0, +, *, +), s_0(+, 0, *, -), \\
 & s_0(-, 0), s_0(+, 0, *, +), s_0(+, *, 0, -), s_0(+, *, 0, +)
 \end{aligned}$$

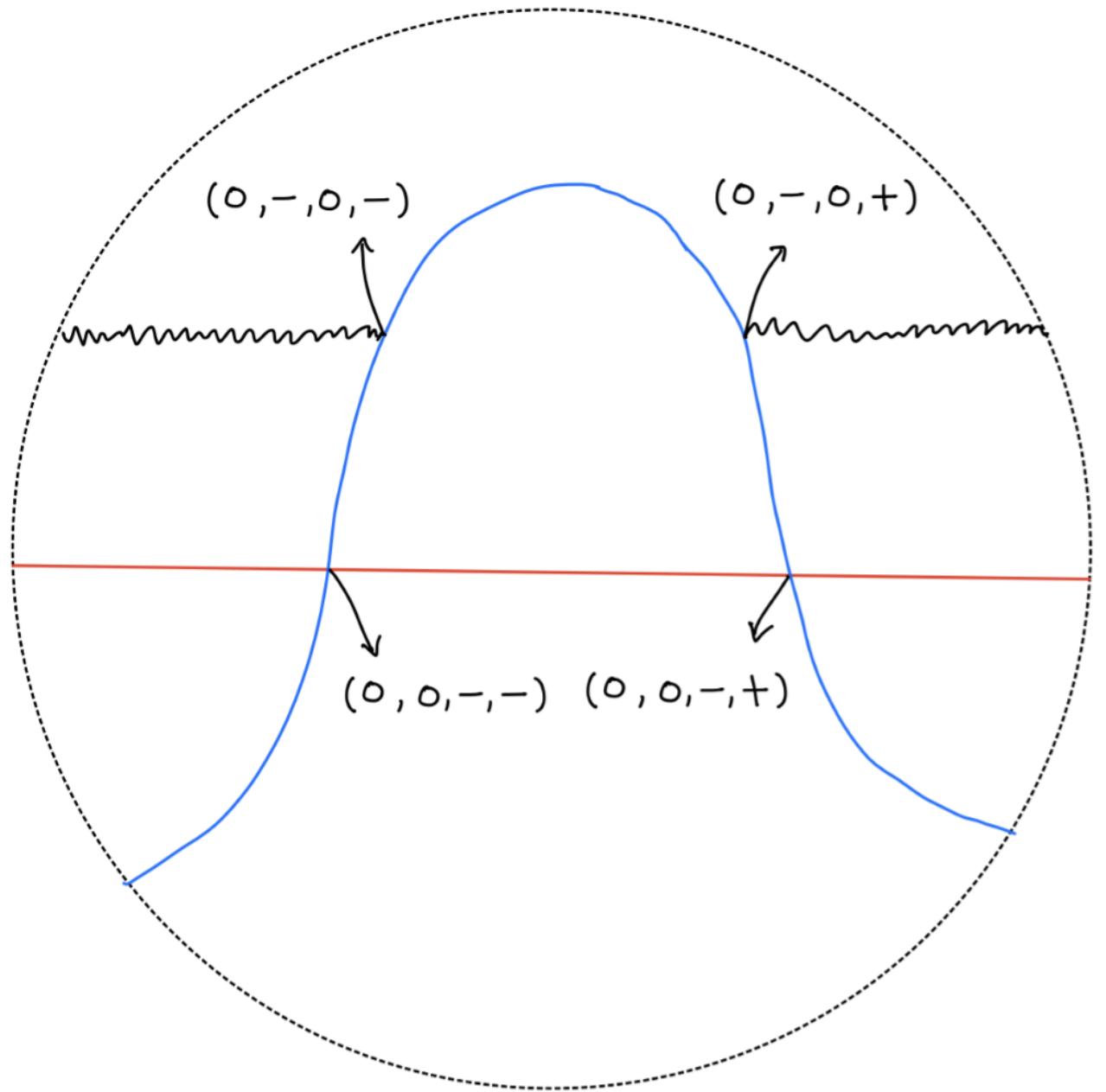


Figure 3.70

- 0 dimensional strata:

$$s_0(0, 0, -, -), s_0(0, -, 0, -), s_0(0, -, 0, +), s_0(0, 0, -, +)$$

2.  $\mathcal{S}_1$ :

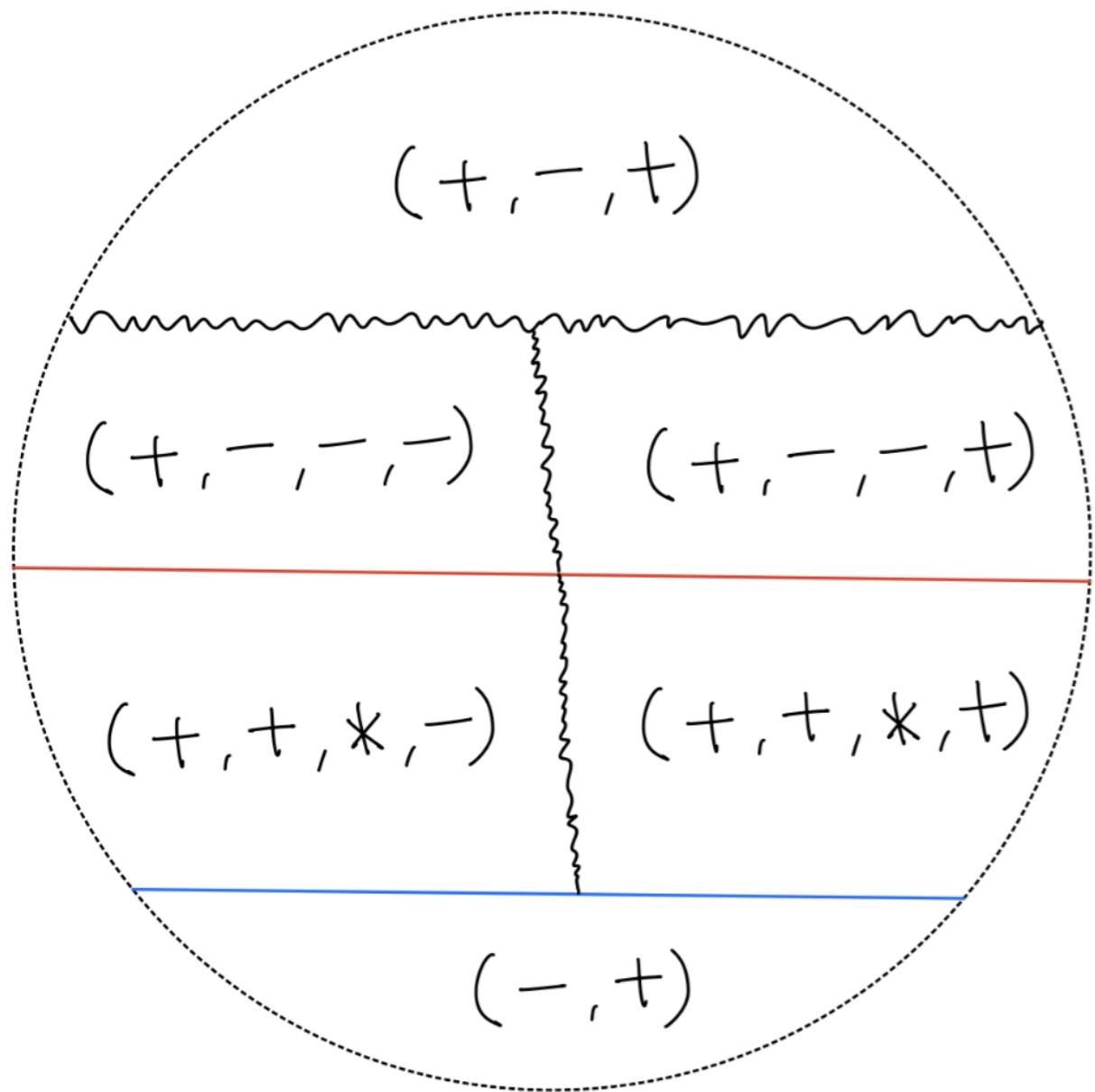


Figure 3.71

- 2 dimensional strata:

$$s_1(-, +), s_1(+, -, -, -), s_1(+, -, -, +), s_1(+, -, +), s_1(+, +, *, -), s_1(+, +, *, +)$$

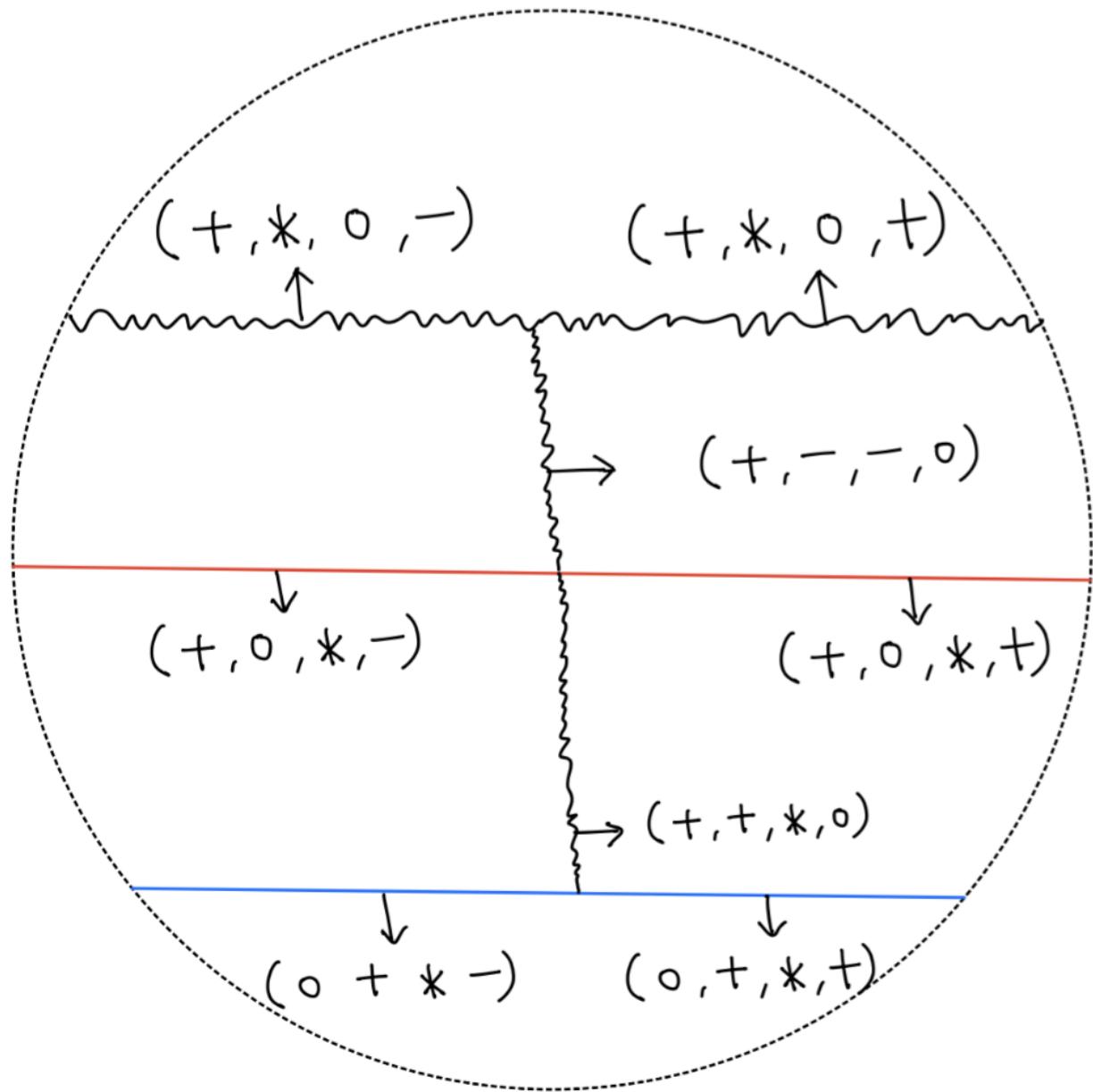


Figure 3.72

- 1 dimensional strata:

$$\begin{aligned}
 & s_1(0, +, *, -), s_1(0, +, *, +), s_1(+, 0, *, -), s_1(+, 0, *, +), s_1(+, *, 0, -), s_1(+, *, 0, +), \\
 & s_1(+, -, -, 0), s_1(+, +, *, 0)
 \end{aligned}$$

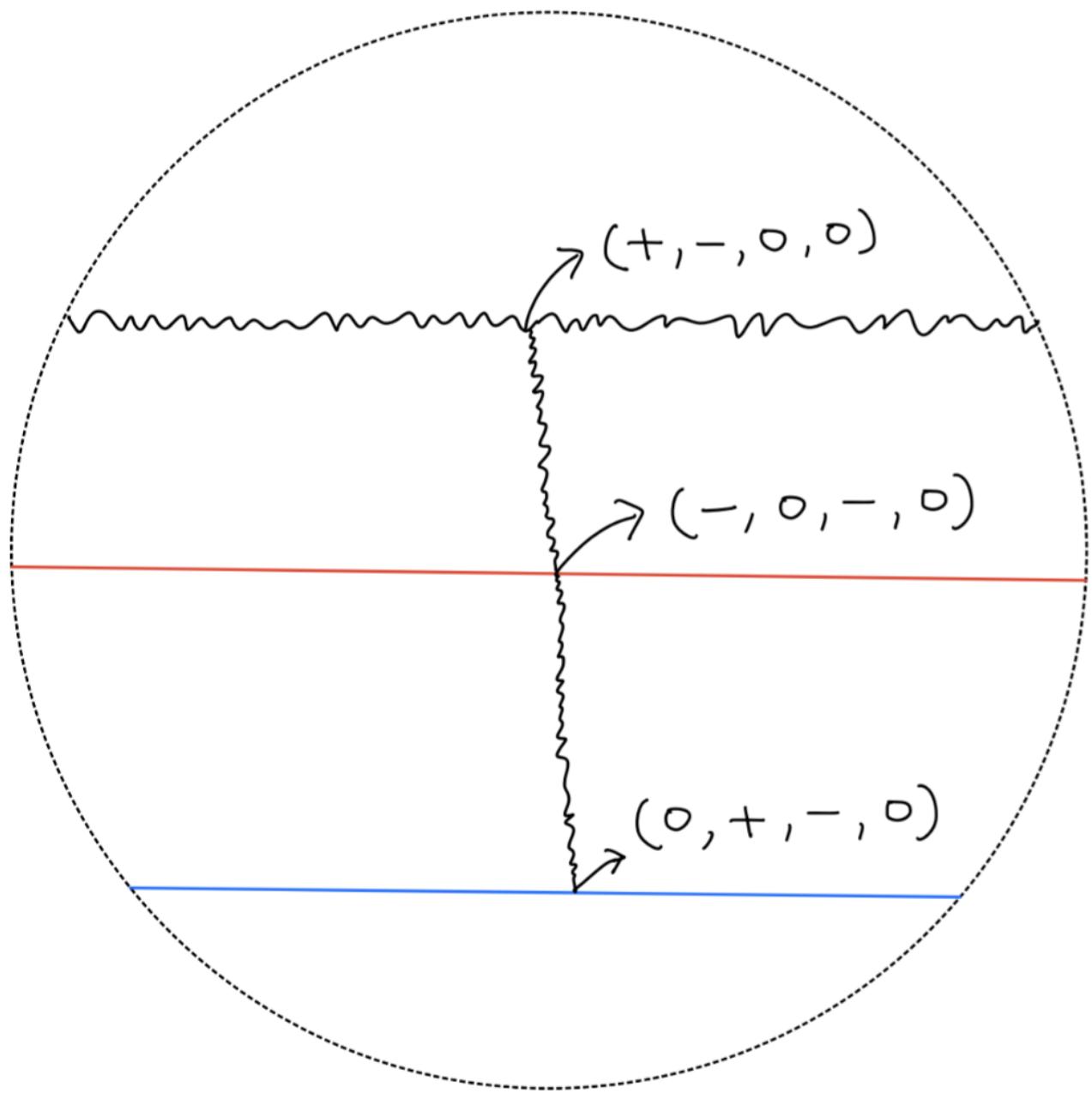


Figure 3.73

- 0 dimensional strata:

$$s_1(+, -, 0, 0), s_1(-, 0, -, 0), s_1(0, +, -, 0)$$

3.  $\mathcal{S}_\bullet$ :

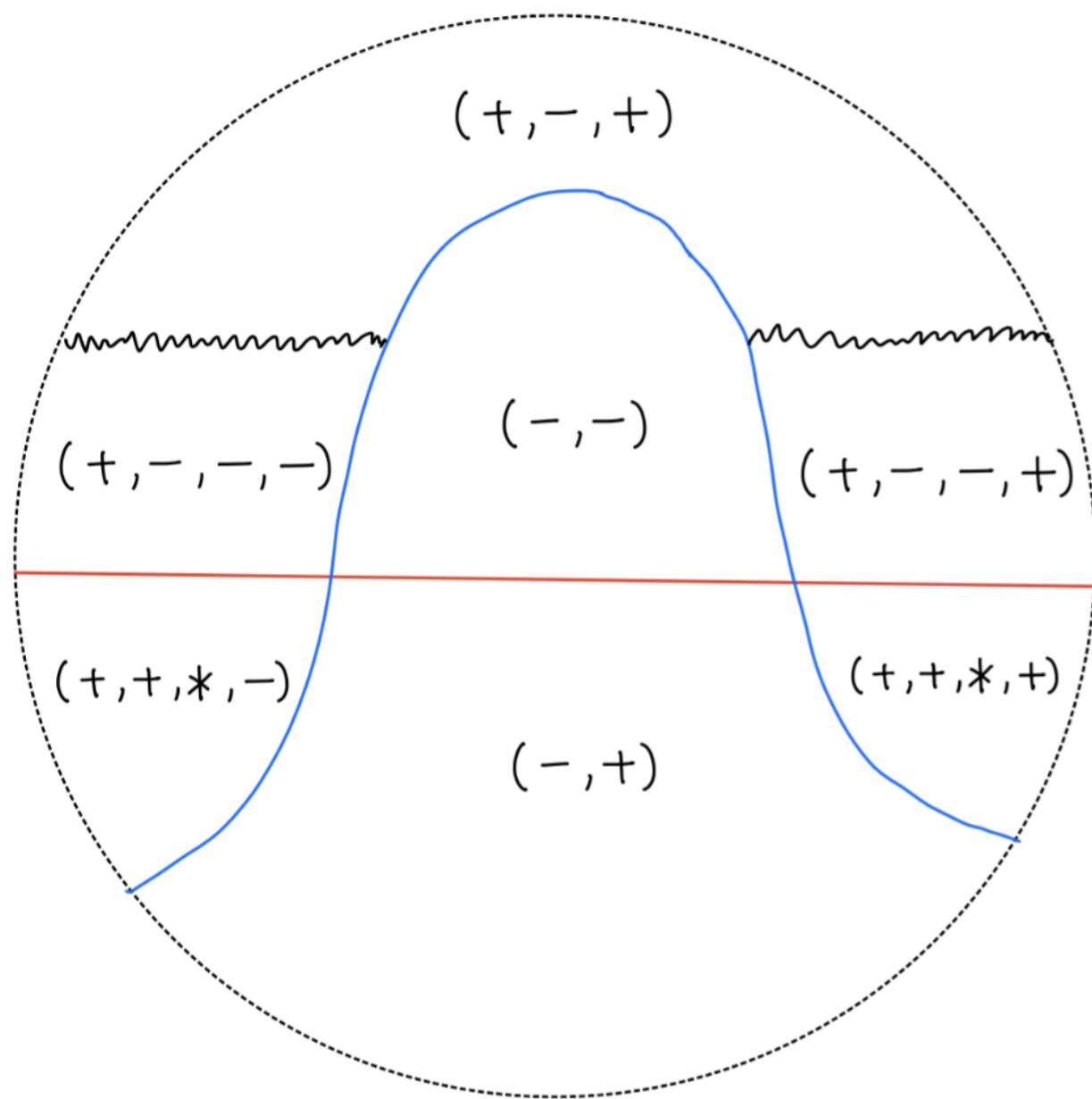


Figure 3.74

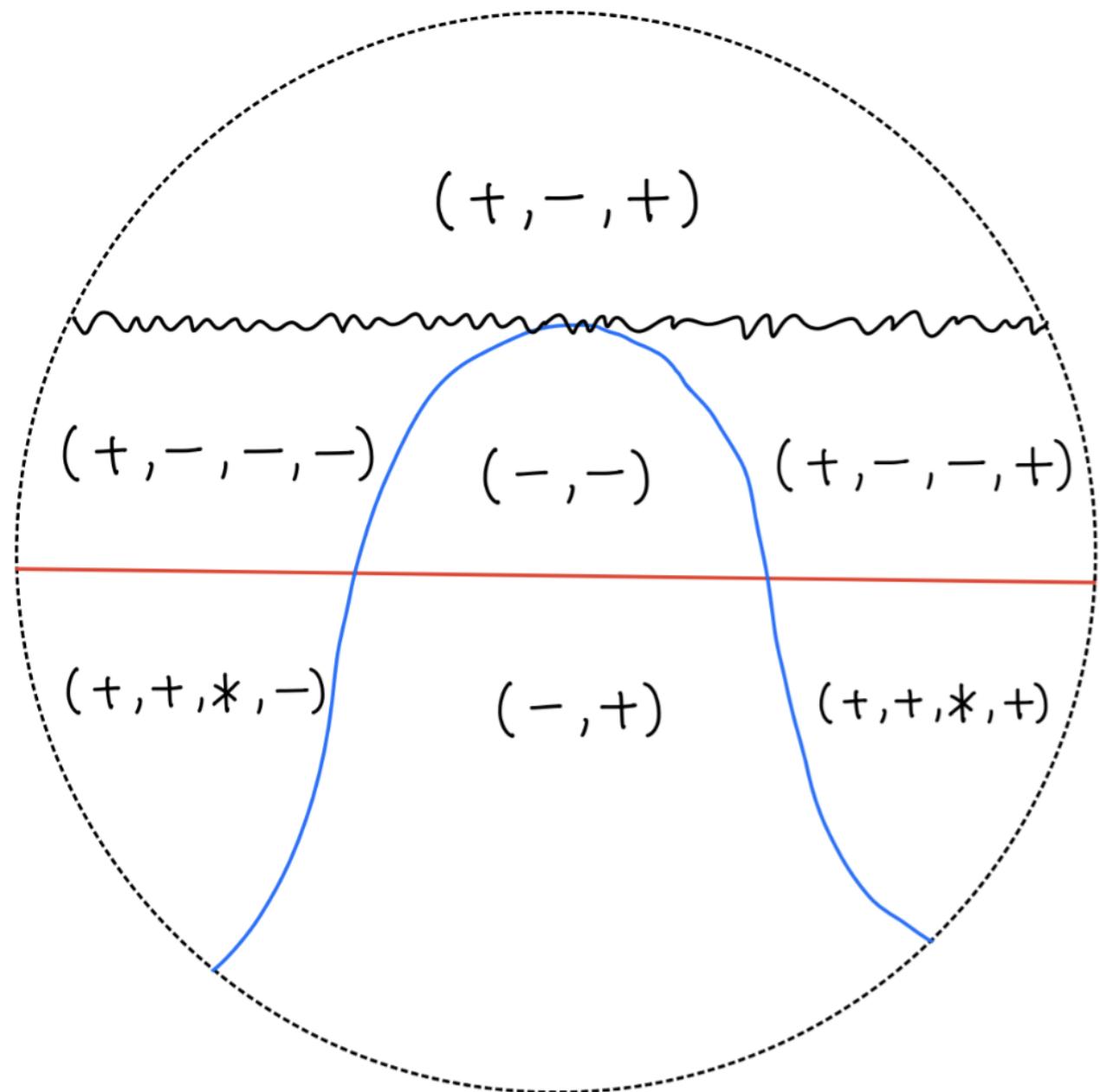


Figure 3.75

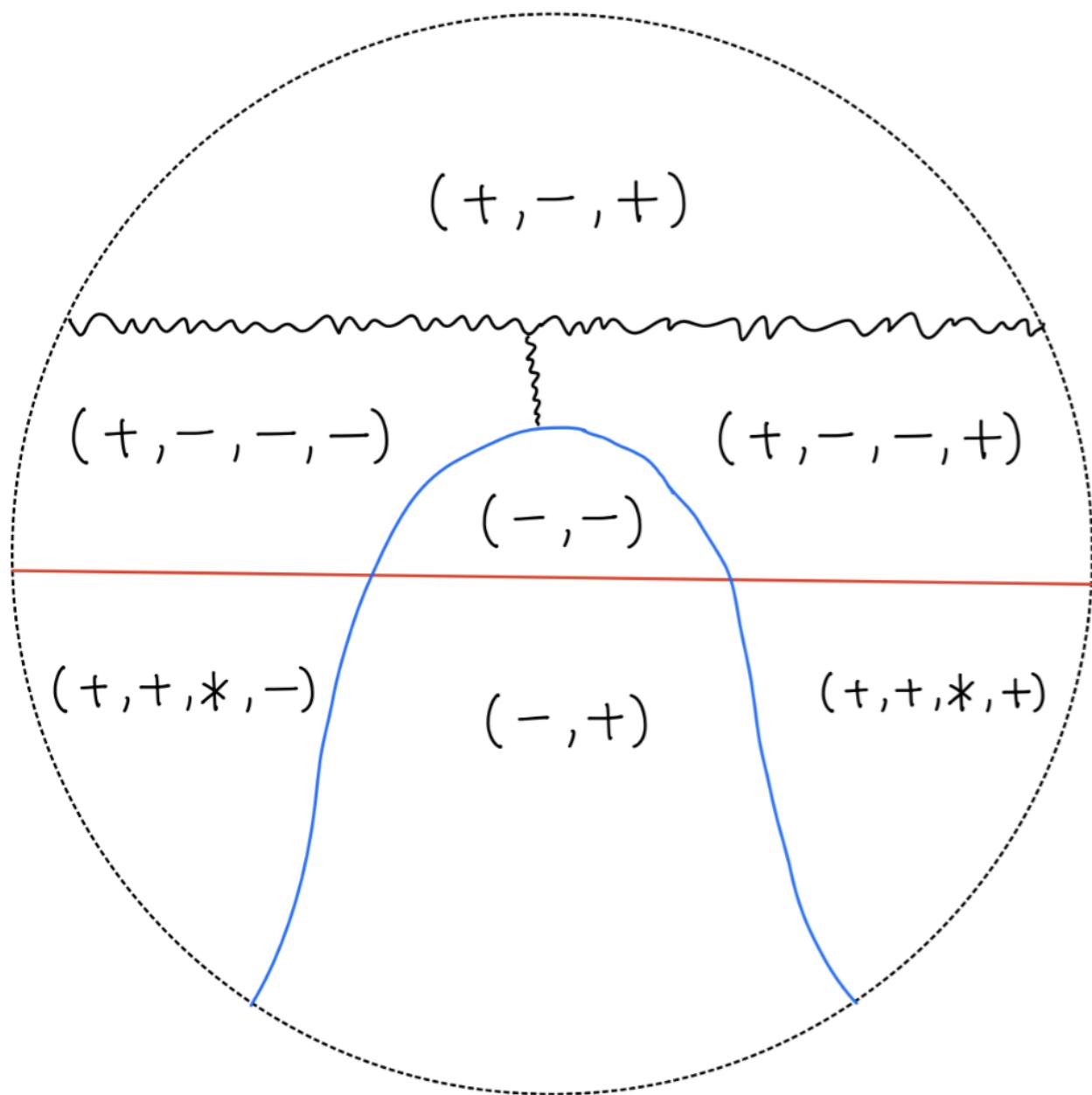


Figure 3.76

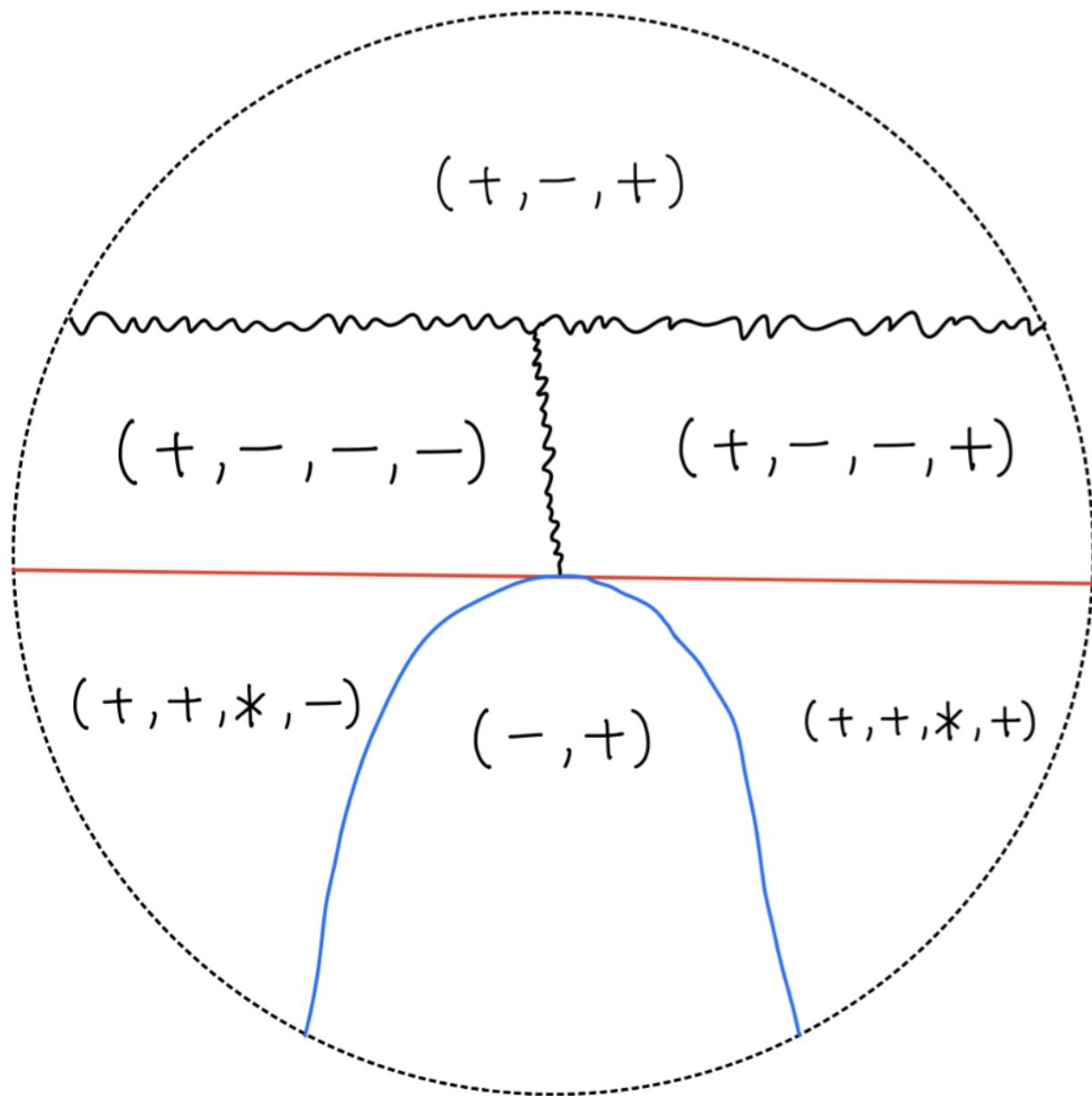


Figure 3.77

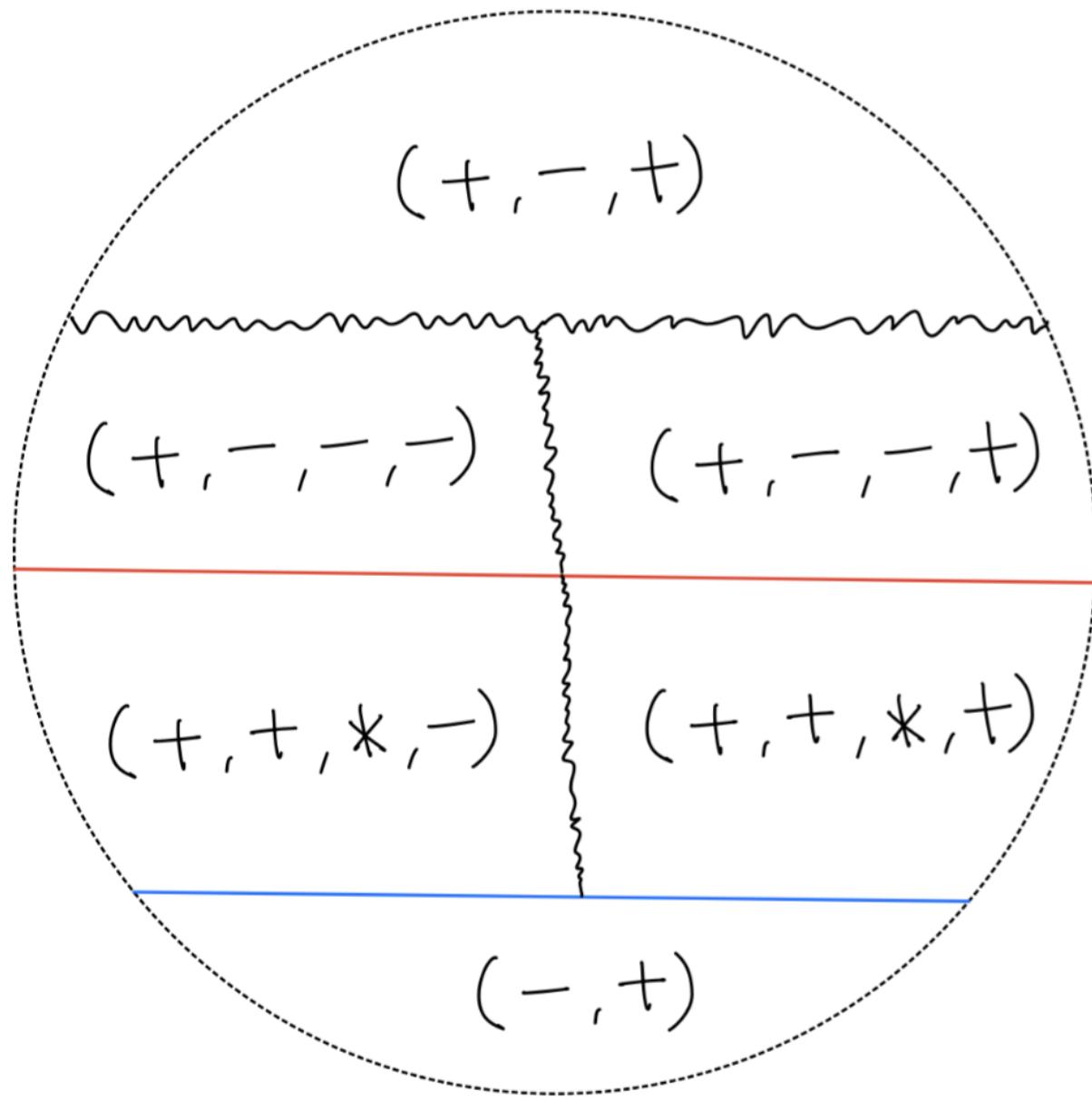


Figure 3.78

- 3 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(-, +), s_{\bullet}(-, -), s_{\bullet}(-, +), s_{\bullet}(+, -, -, -), s_{\bullet}(+, -, -, +), s_{\bullet}(+, -, +), \\
 & s_{\bullet}(+, +, *, -), s_{\bullet}(+, +, *, +)
 \end{aligned}$$

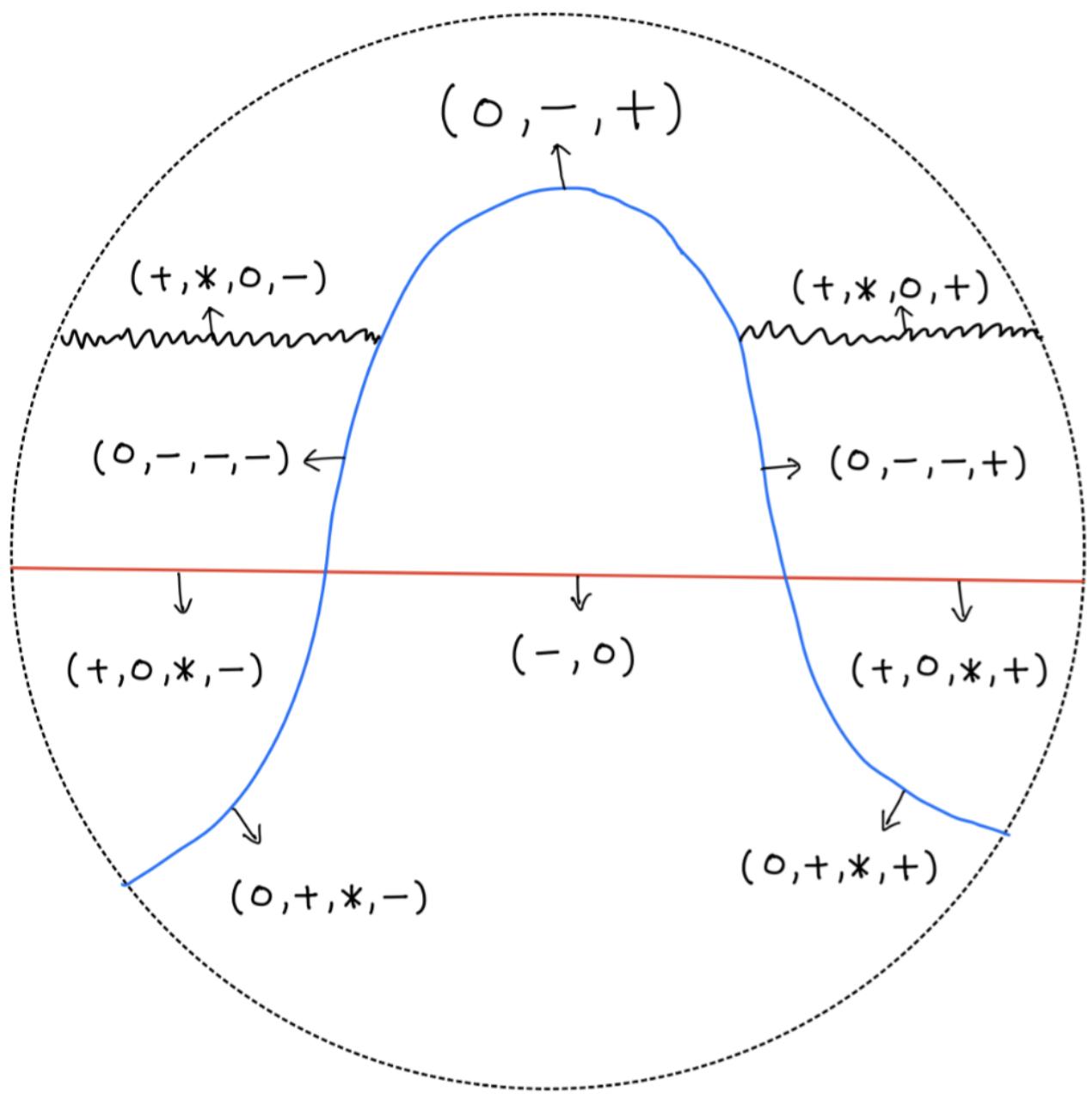


Figure 3.79

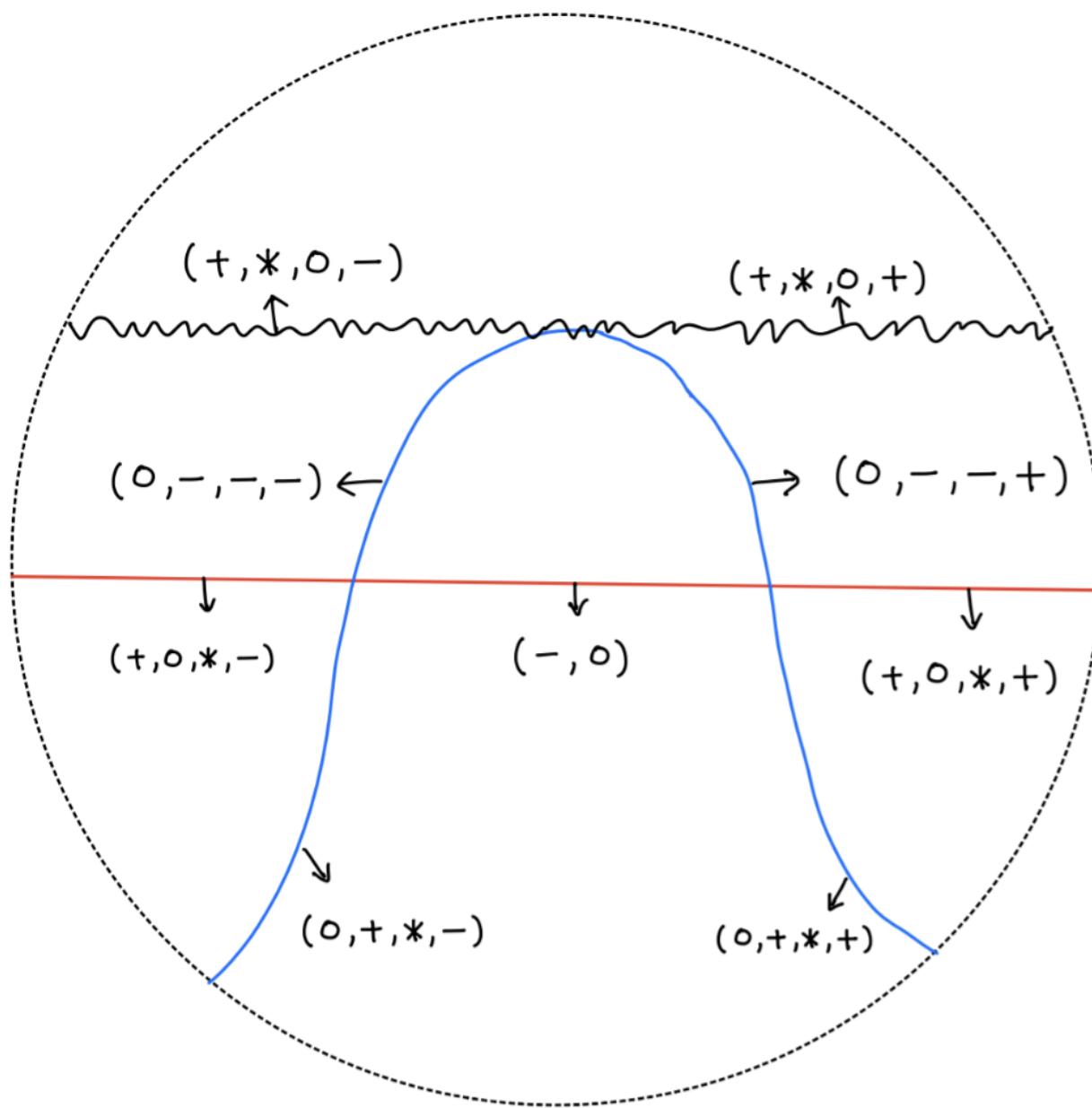


Figure 3.80

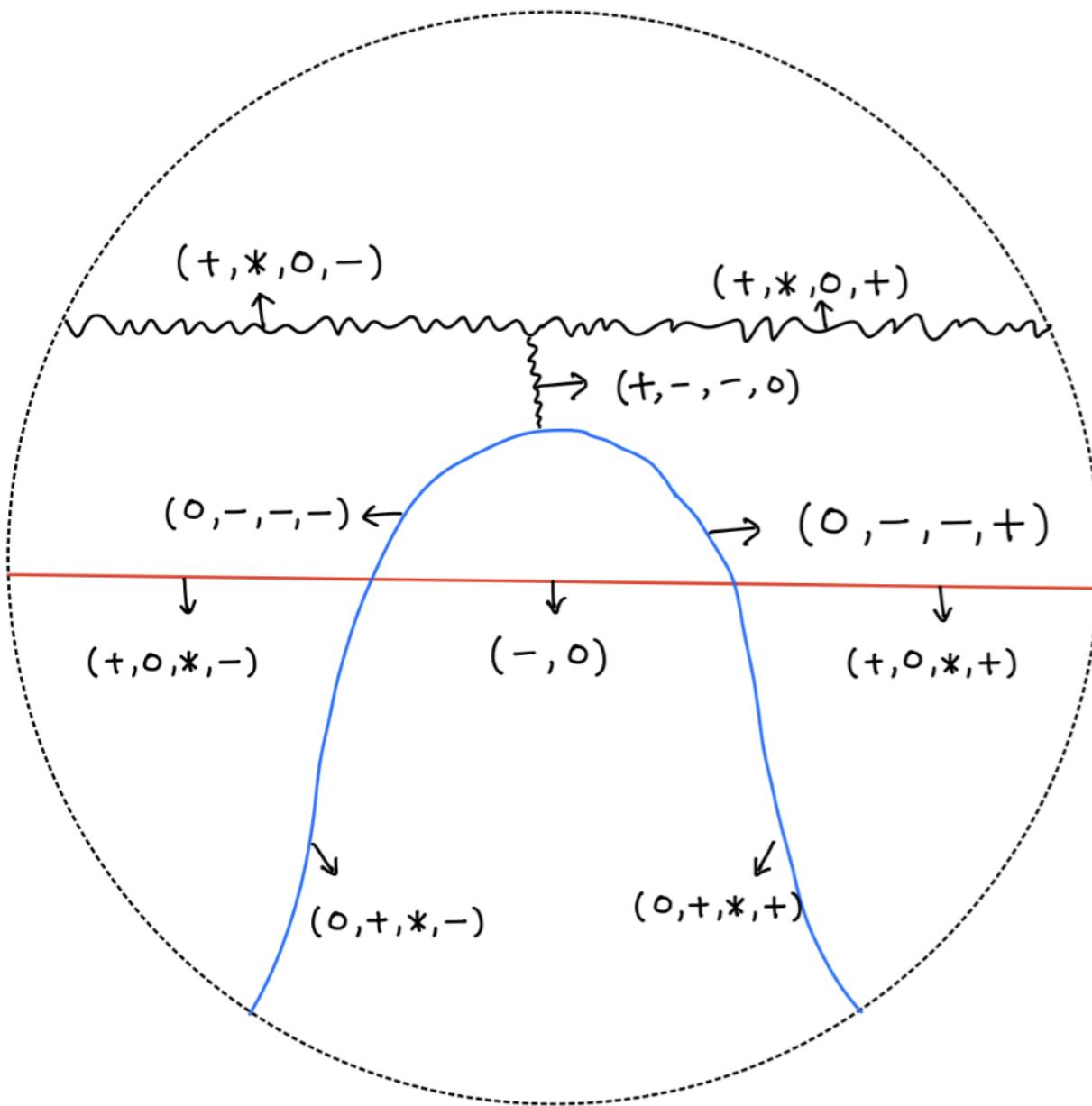


Figure 3.81

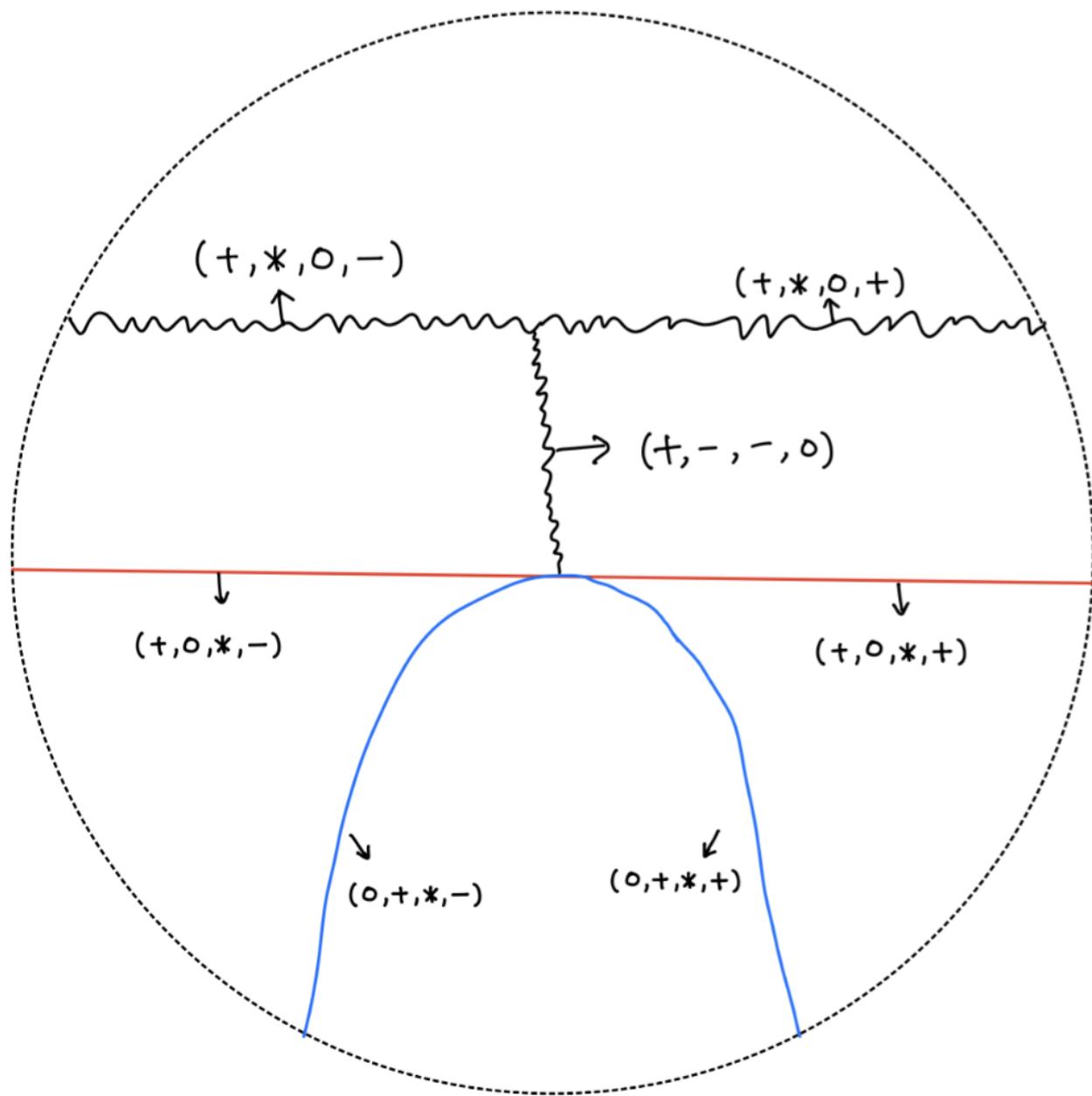


Figure 3.82

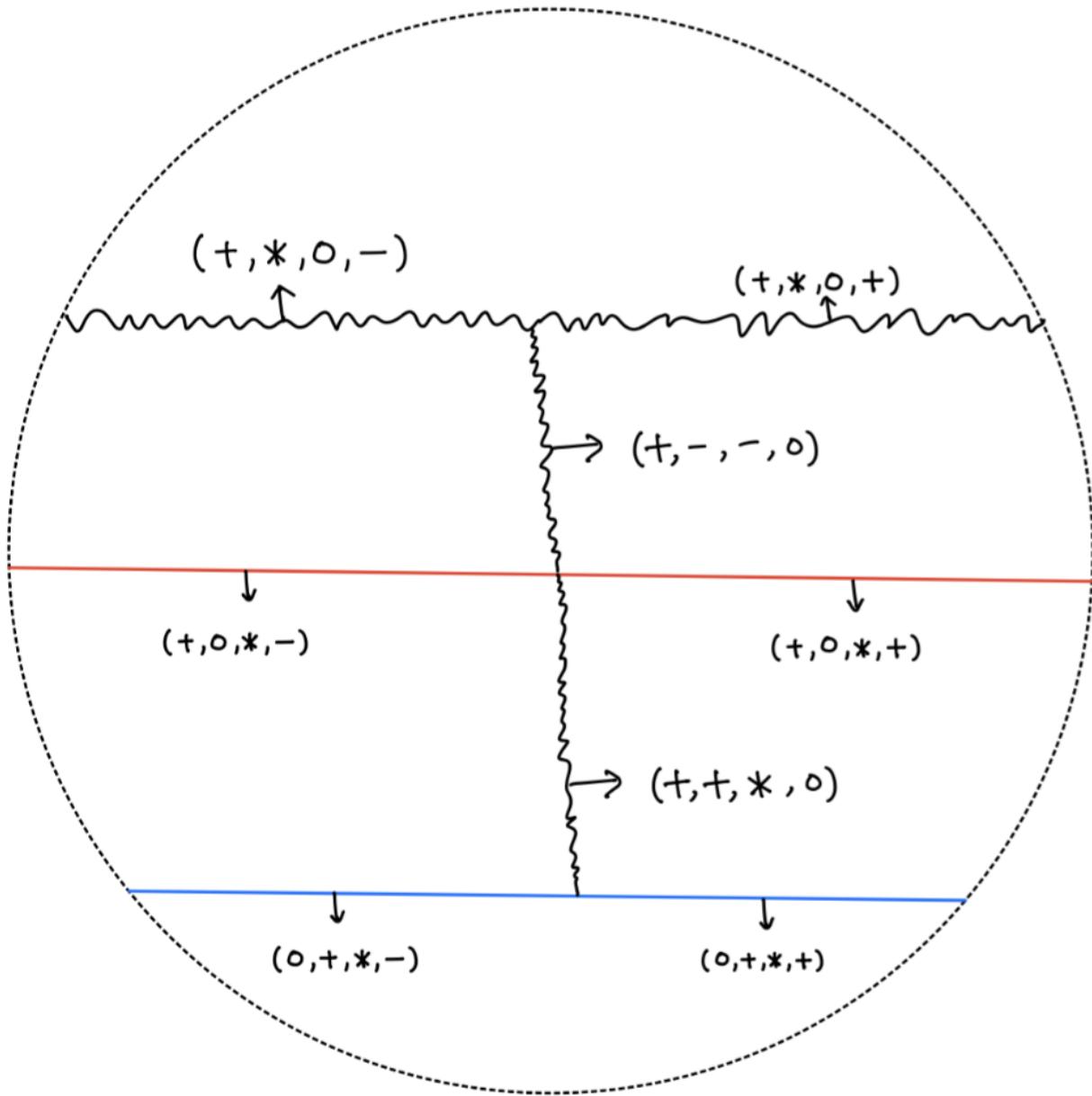


Figure 3.83

- 2 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, +, *, -), s_{\bullet}(0, -, -, -), s_{\bullet}(0, -, +), s_{\bullet}(0, -, -, +), s_{\bullet}(0, +, *, +), s_{\bullet}(+, 0, *, -), \\
 & s_{\bullet}(-, 0), s_{\bullet}(+, 0, *, +), s_{\bullet}(+, *, 0, -), s_{\bullet}(+, *, 0, +), s_{\bullet}(+, -, -, 0), s_{\bullet}(+, +, *, 0)
 \end{aligned}$$

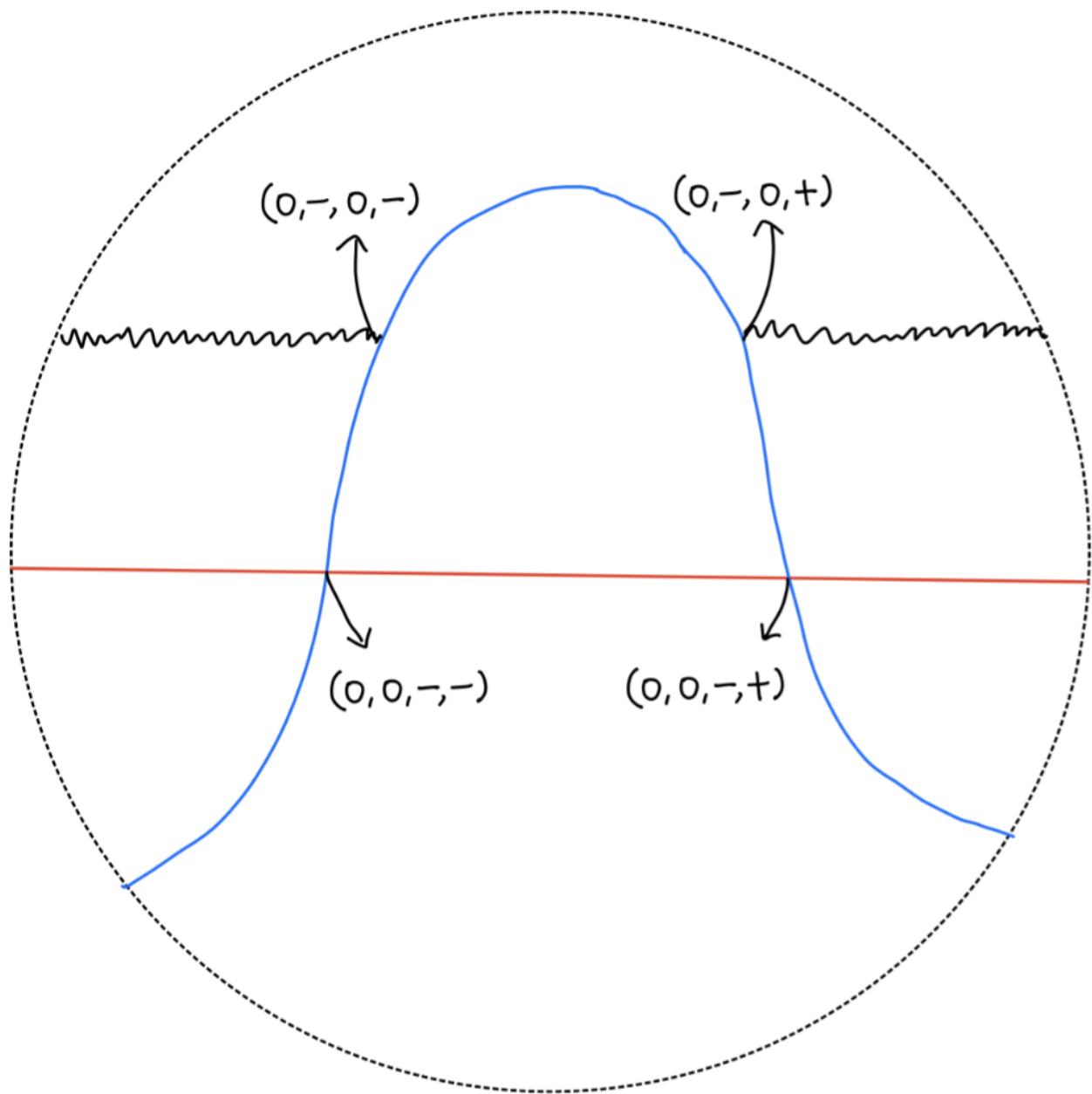


Figure 3.84

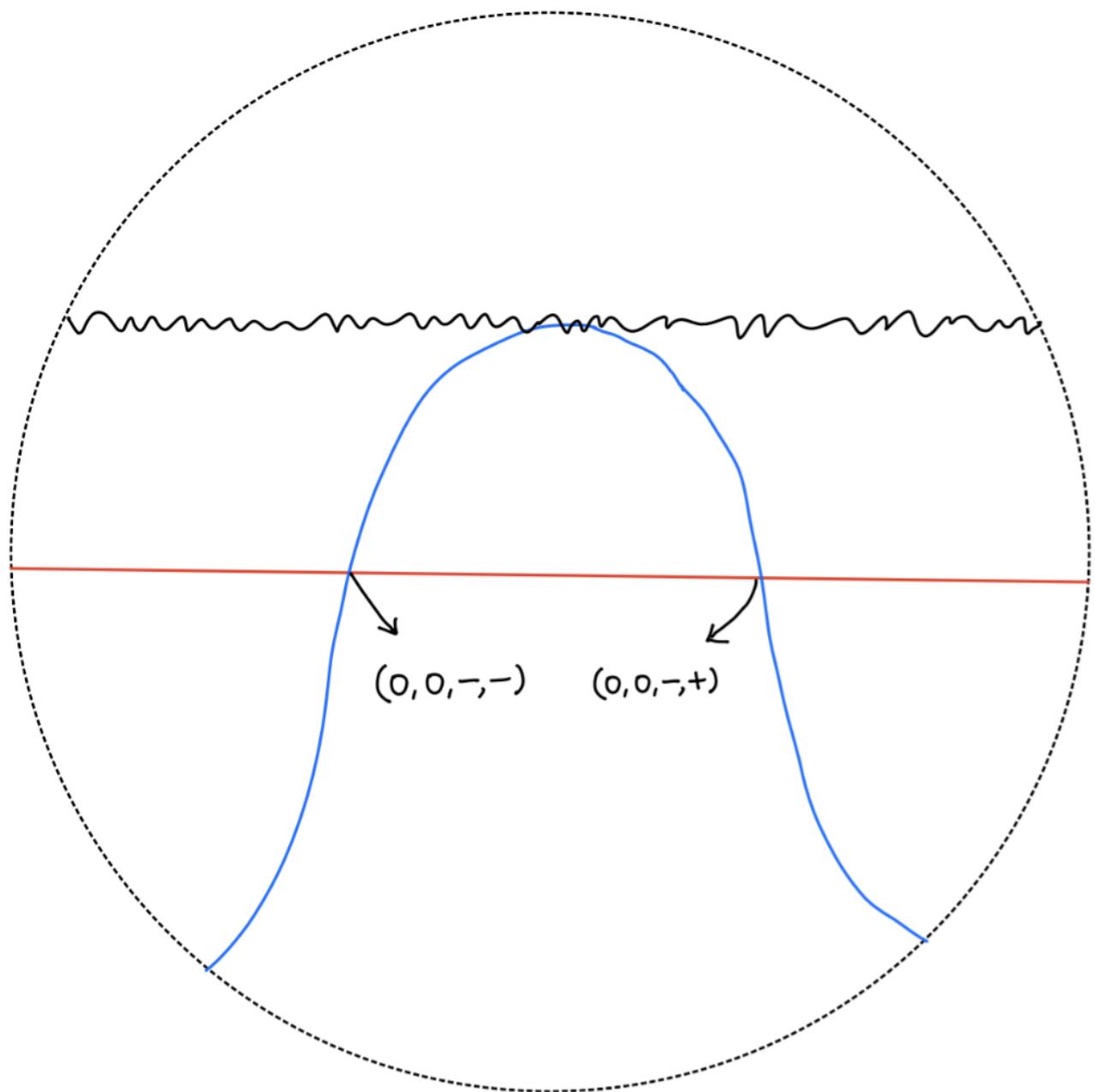


Figure 3.85

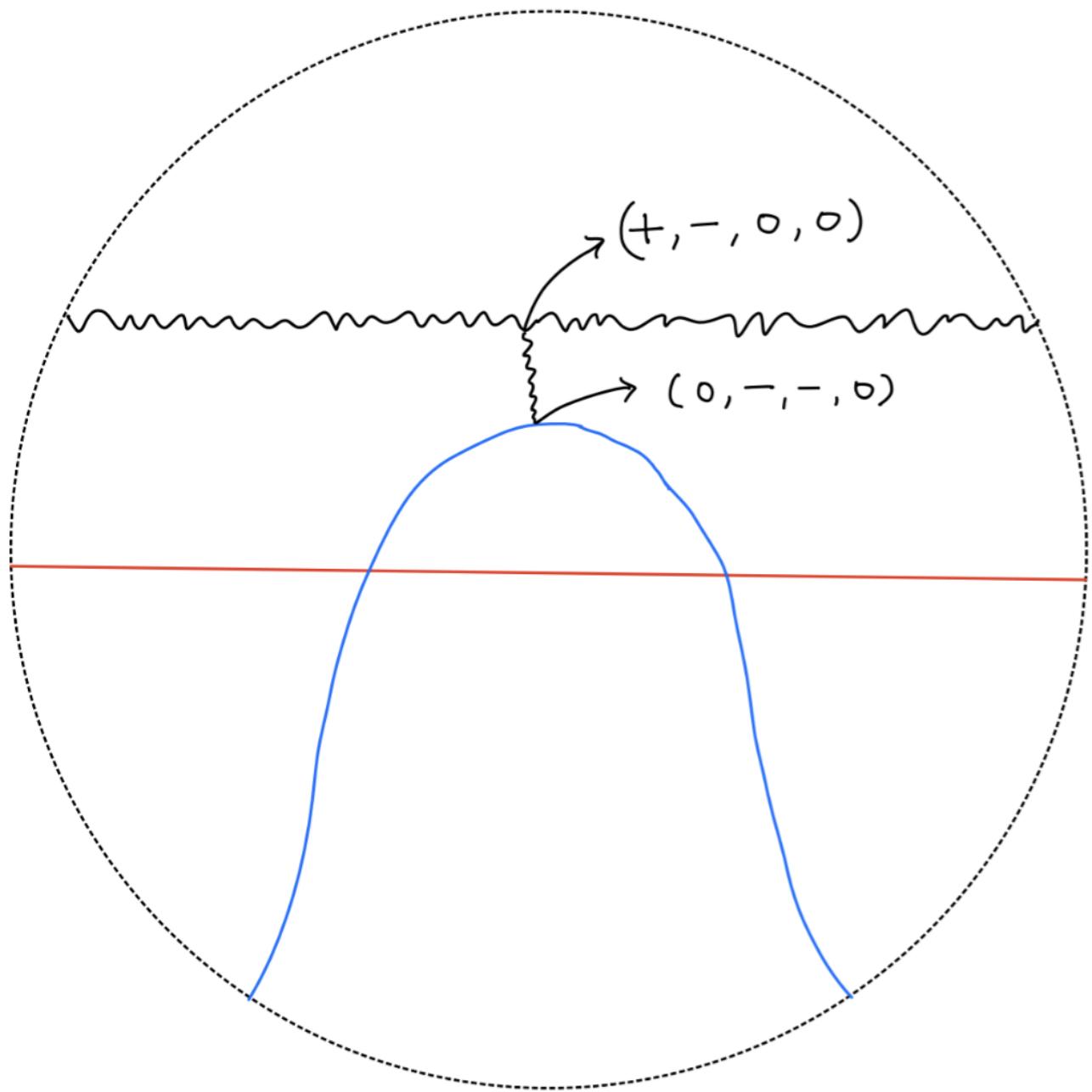


Figure 3.86

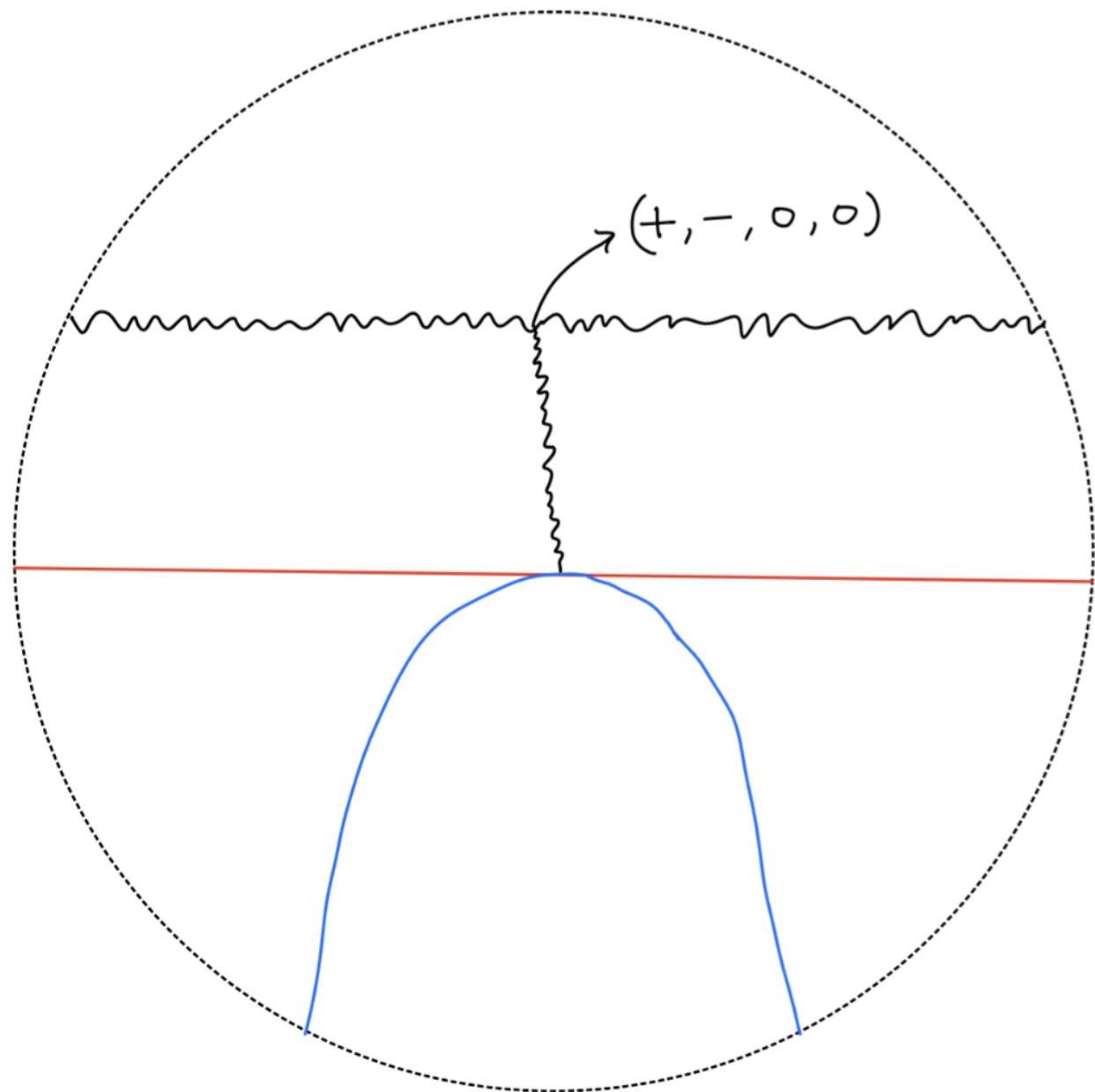


Figure 3.87

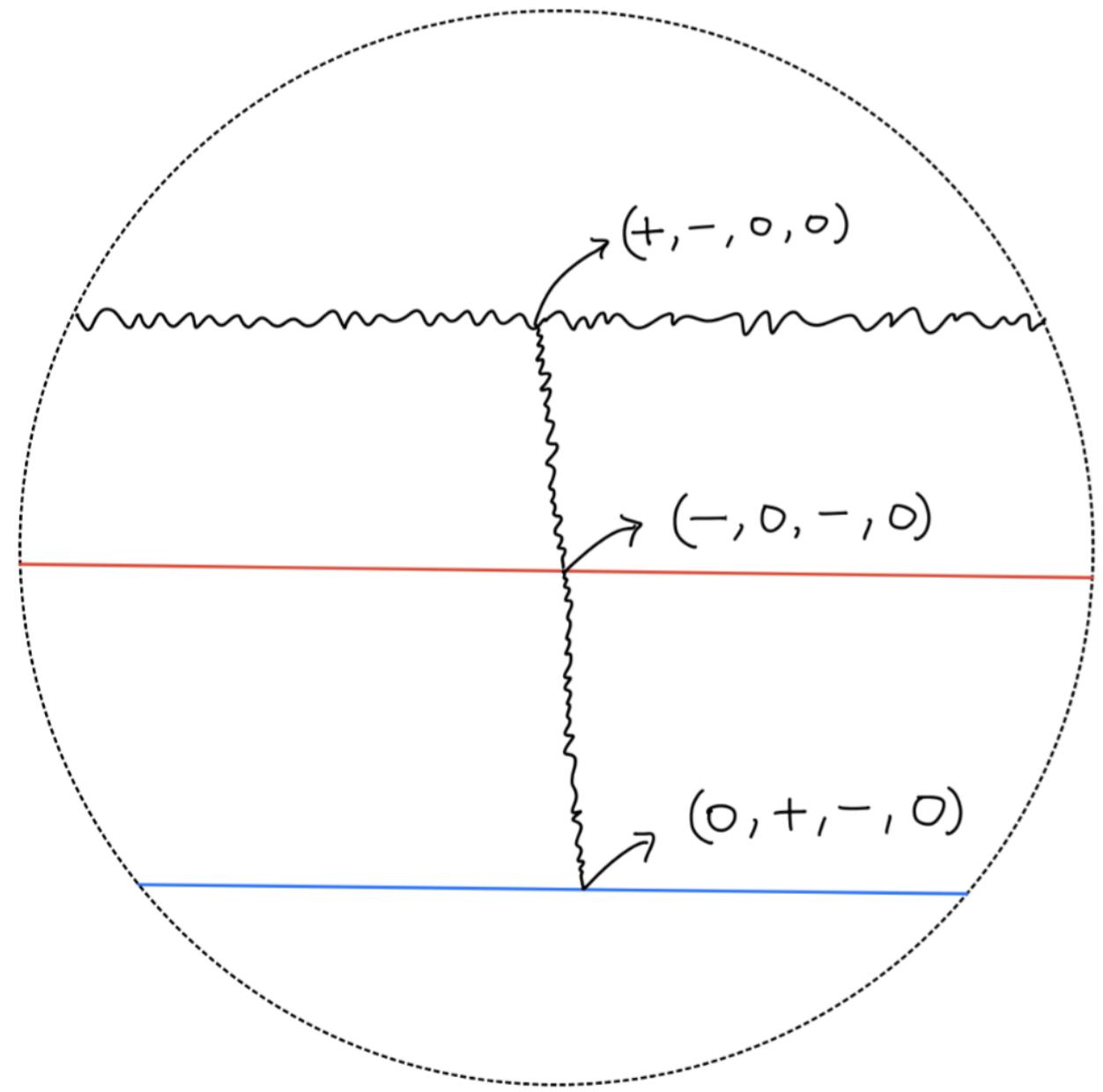


Figure 3.88

- 1 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, 0, -, -), s_{\bullet}(0, -, 0, -), s_{\bullet}(0, -, 0, +), s_{\bullet}(0, 0, -, +), s_{\bullet}(+, -, 0, 0), s_{\bullet}(0, -, -, 0), \\
 & s_{\bullet}(-, 0, -, 0), s_{\bullet}(0, +, -, 0)
 \end{aligned}$$

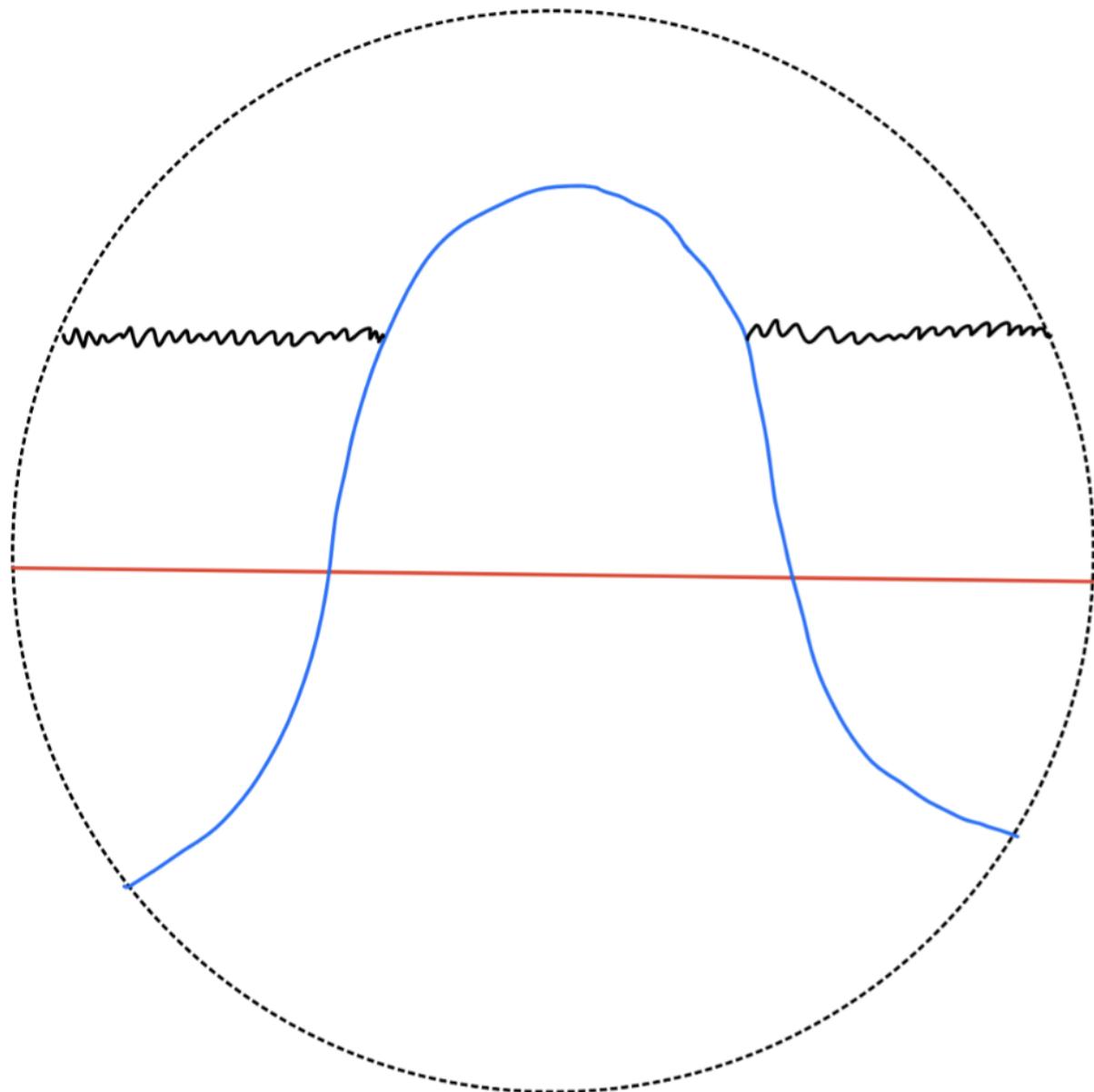


Figure 3.89

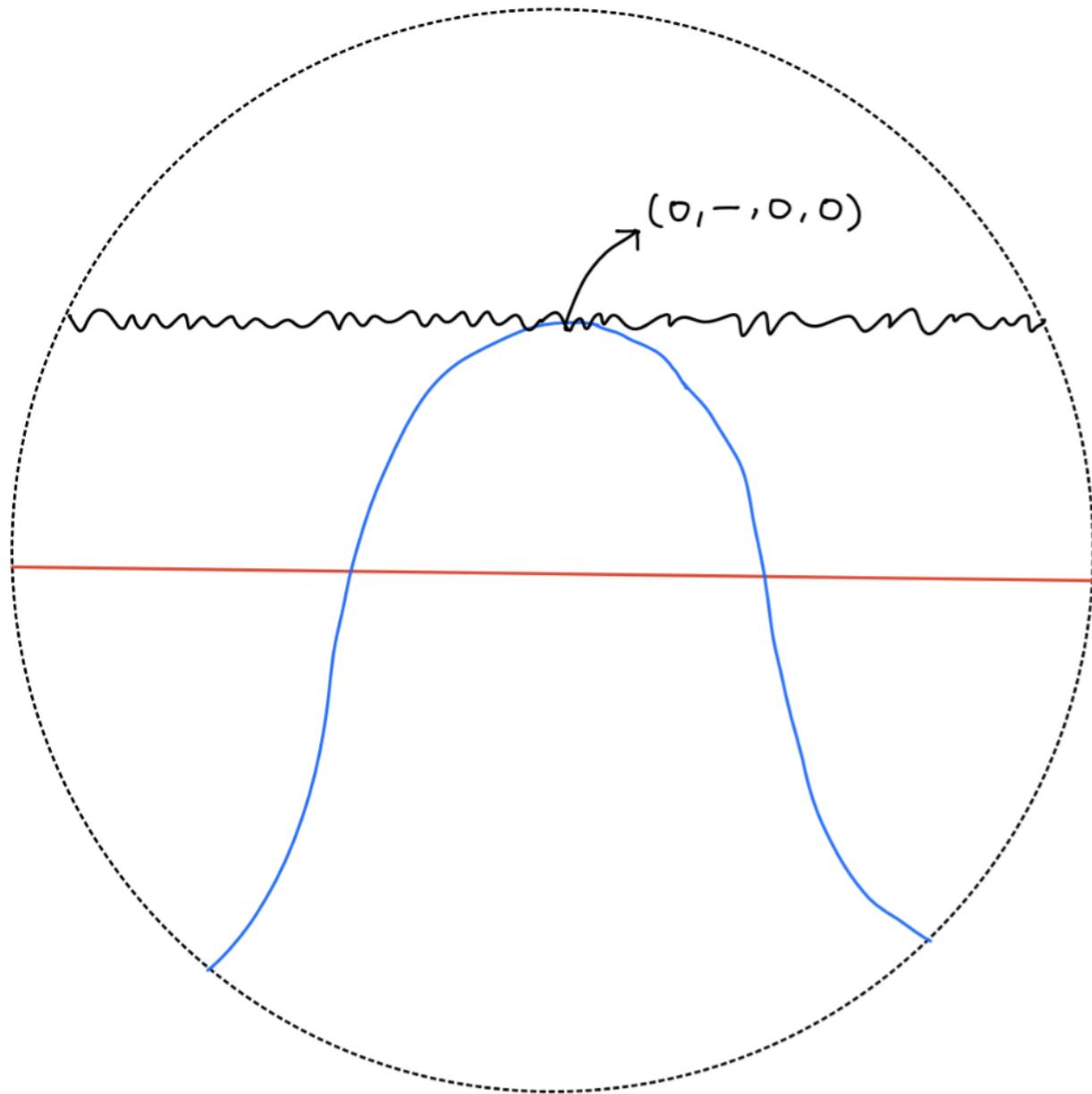


Figure 3.90

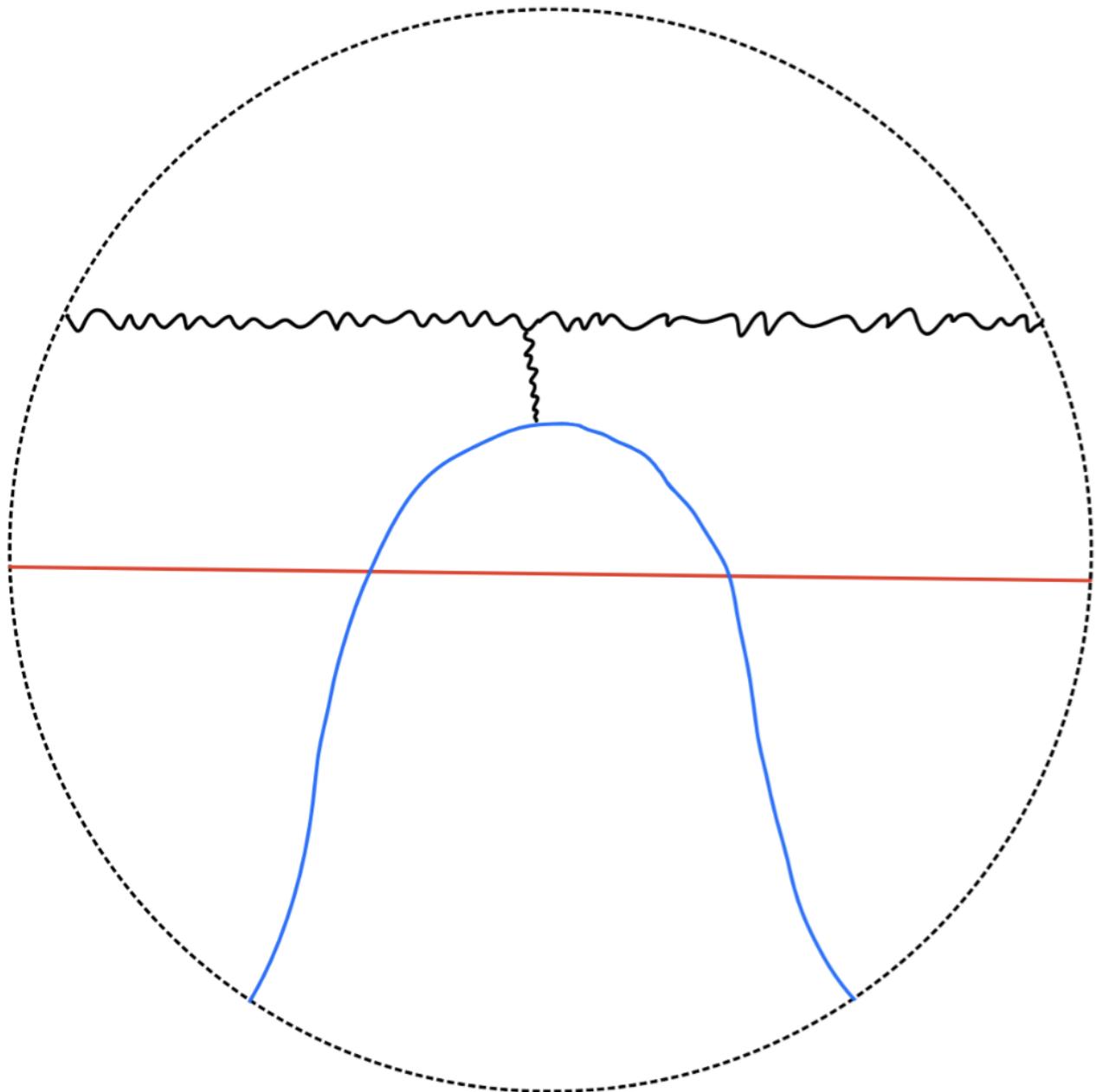


Figure 3.91

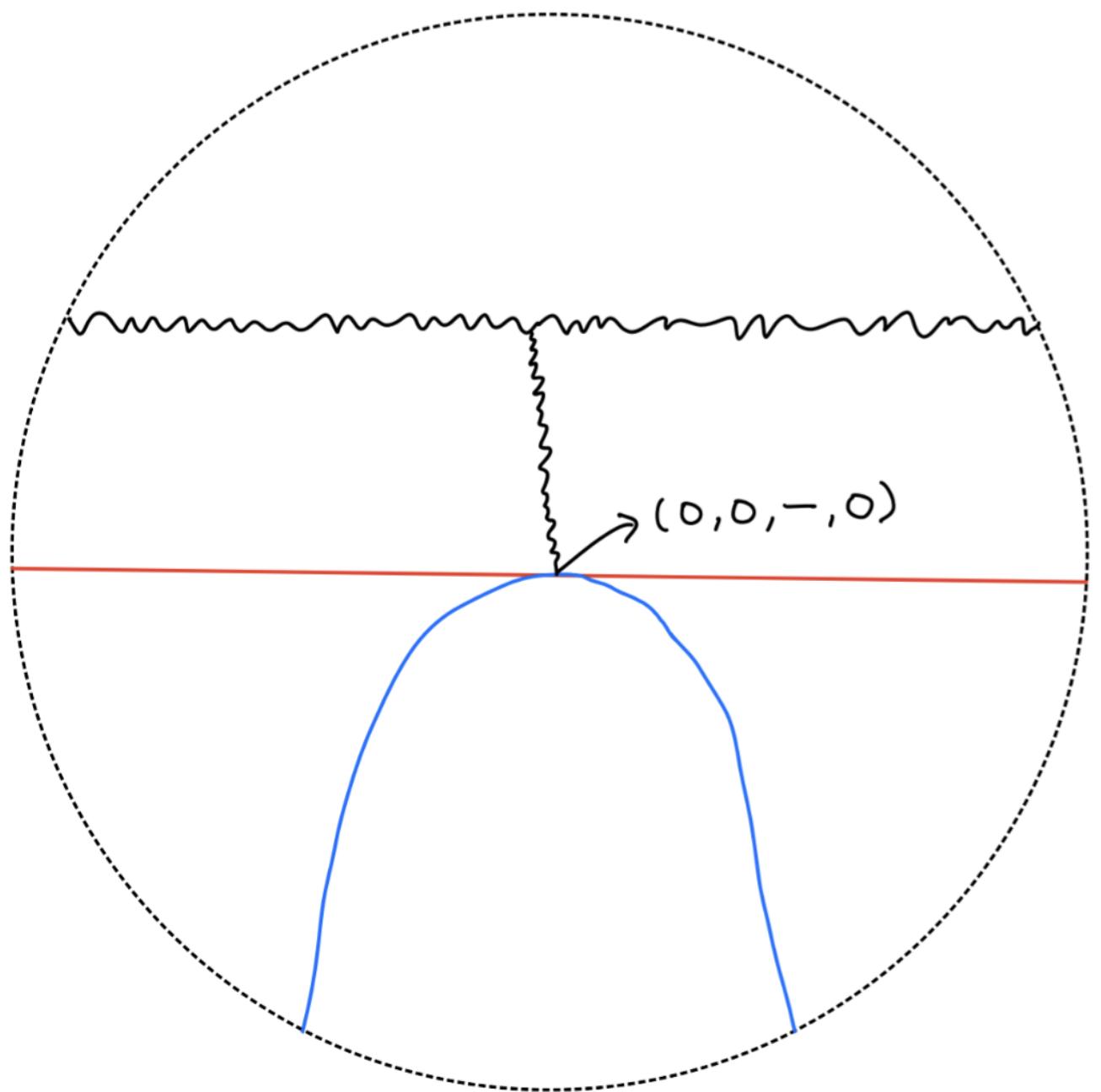


Figure 3.92

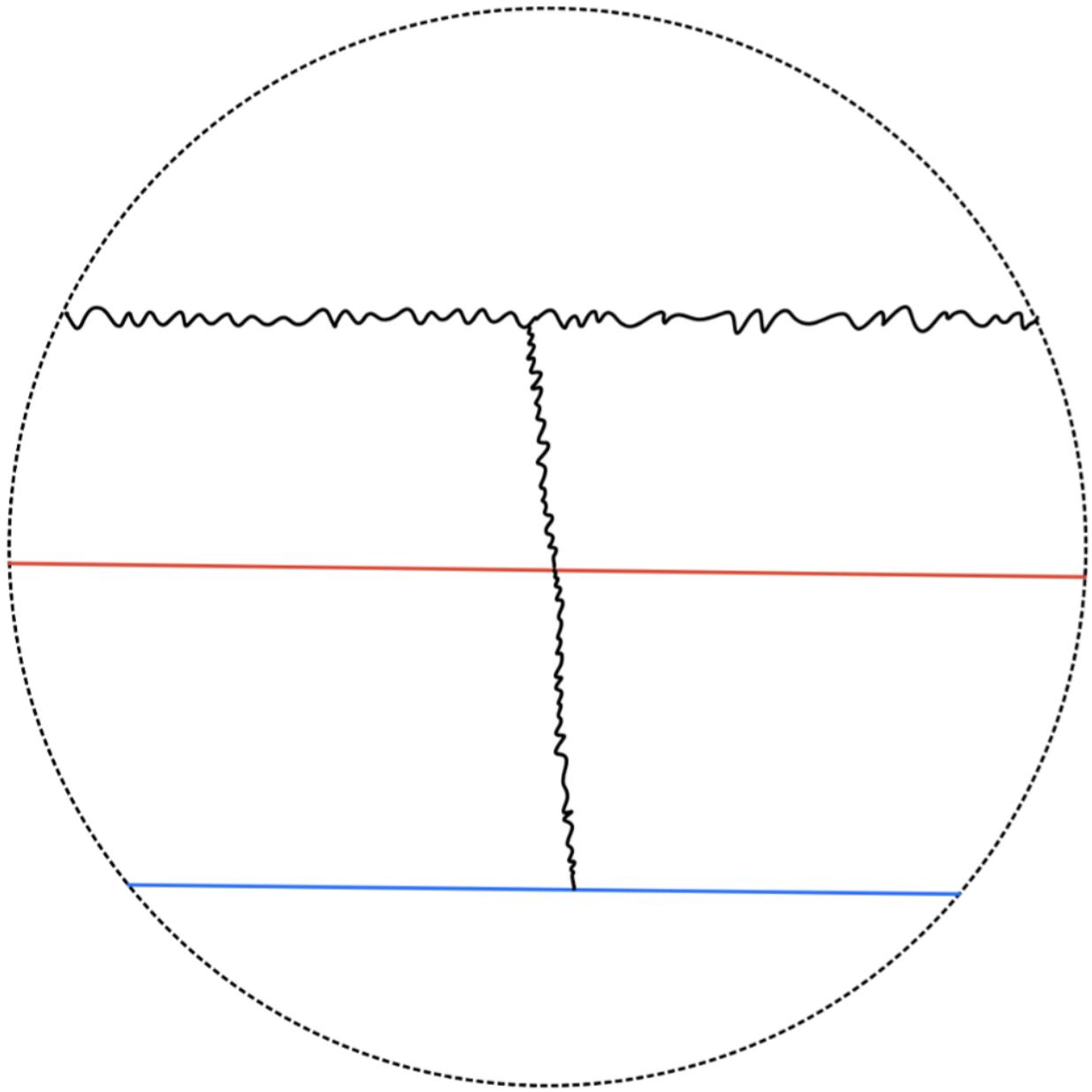


Figure 3.93

- 0 dimensional strata:

$$s_\bullet(0, -, 0, 0), s_\bullet(0, 0, -, 0)$$

**Definition 102.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $\text{star}(s)$ .

**Definition 103.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of  $s$ .

**Definition 104.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 105.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 106.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 107.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \dots \circ F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 108.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

### 3.5.1 Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

### 3.5.2 Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

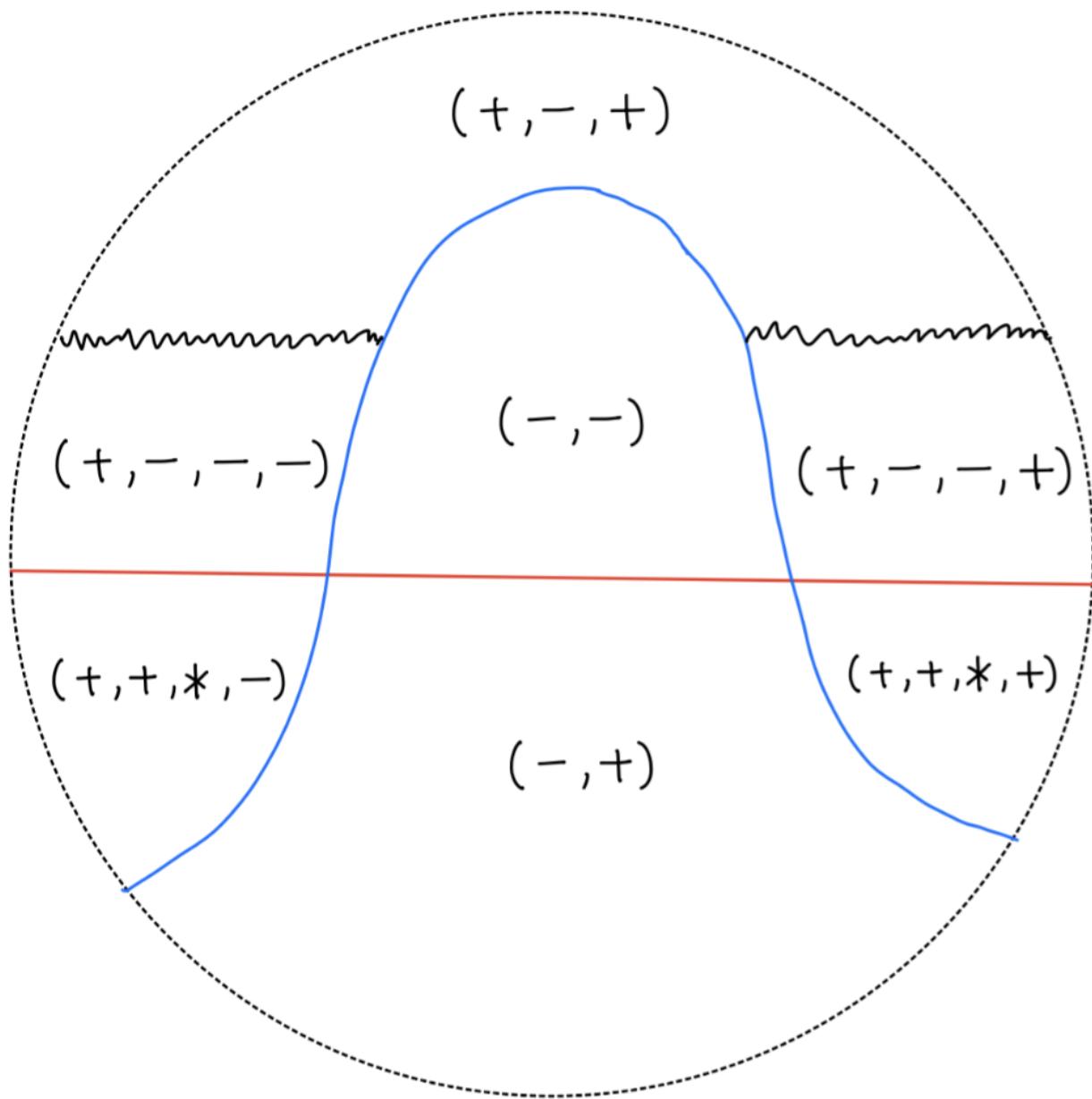


Figure 3.94

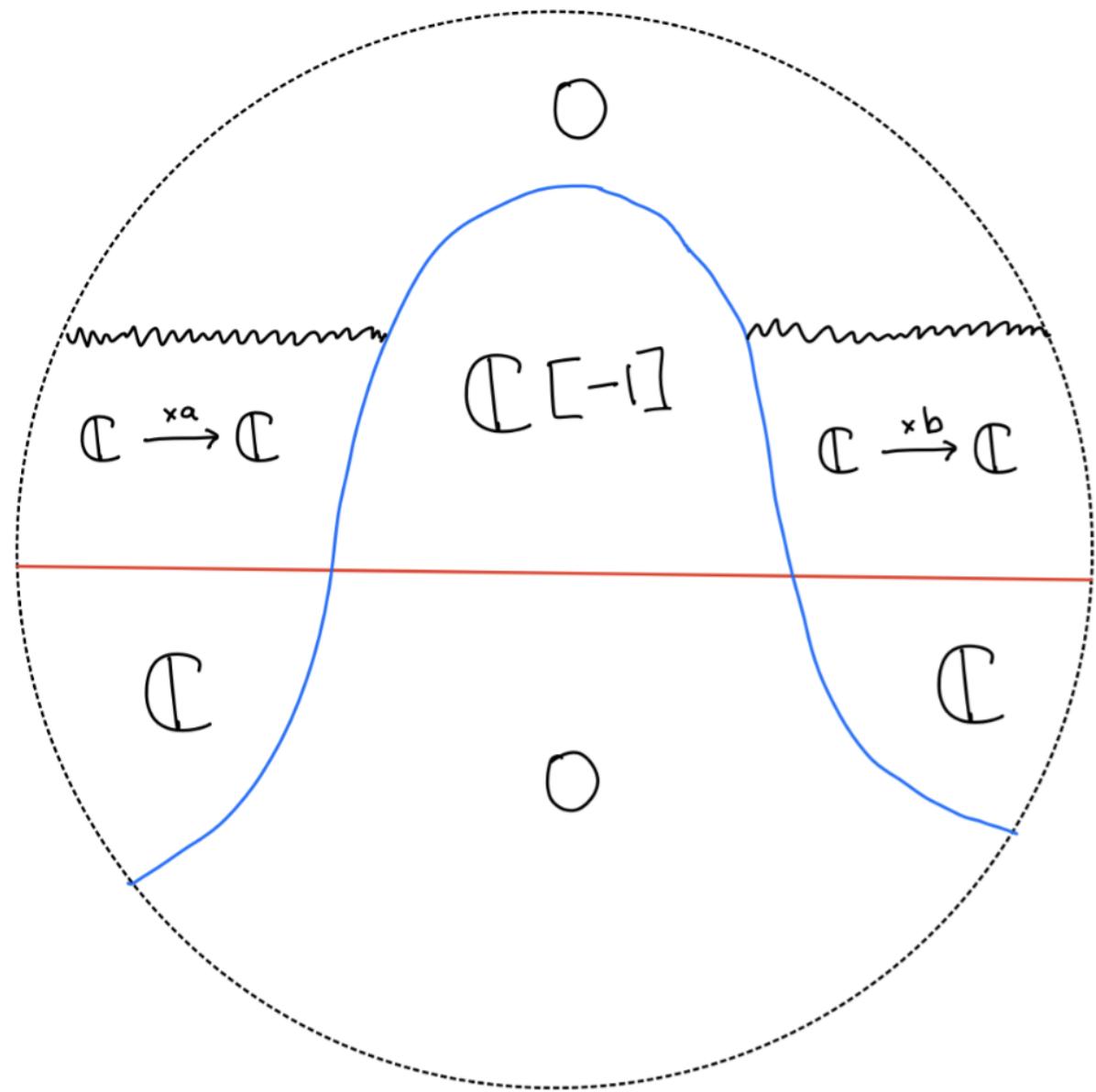


Figure 3.95

- $F_0(-, -) := \mathbb{C}[-1]$
- $F_0(-, +) := 0$
- $F_0(+, -, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(+, -, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_0(+, -, +) := 0$

- $F_0(+, +, *, -) := \mathbb{C}$

- $F_0(+, +, *, +) := \mathbb{C}$

Generalization maps:

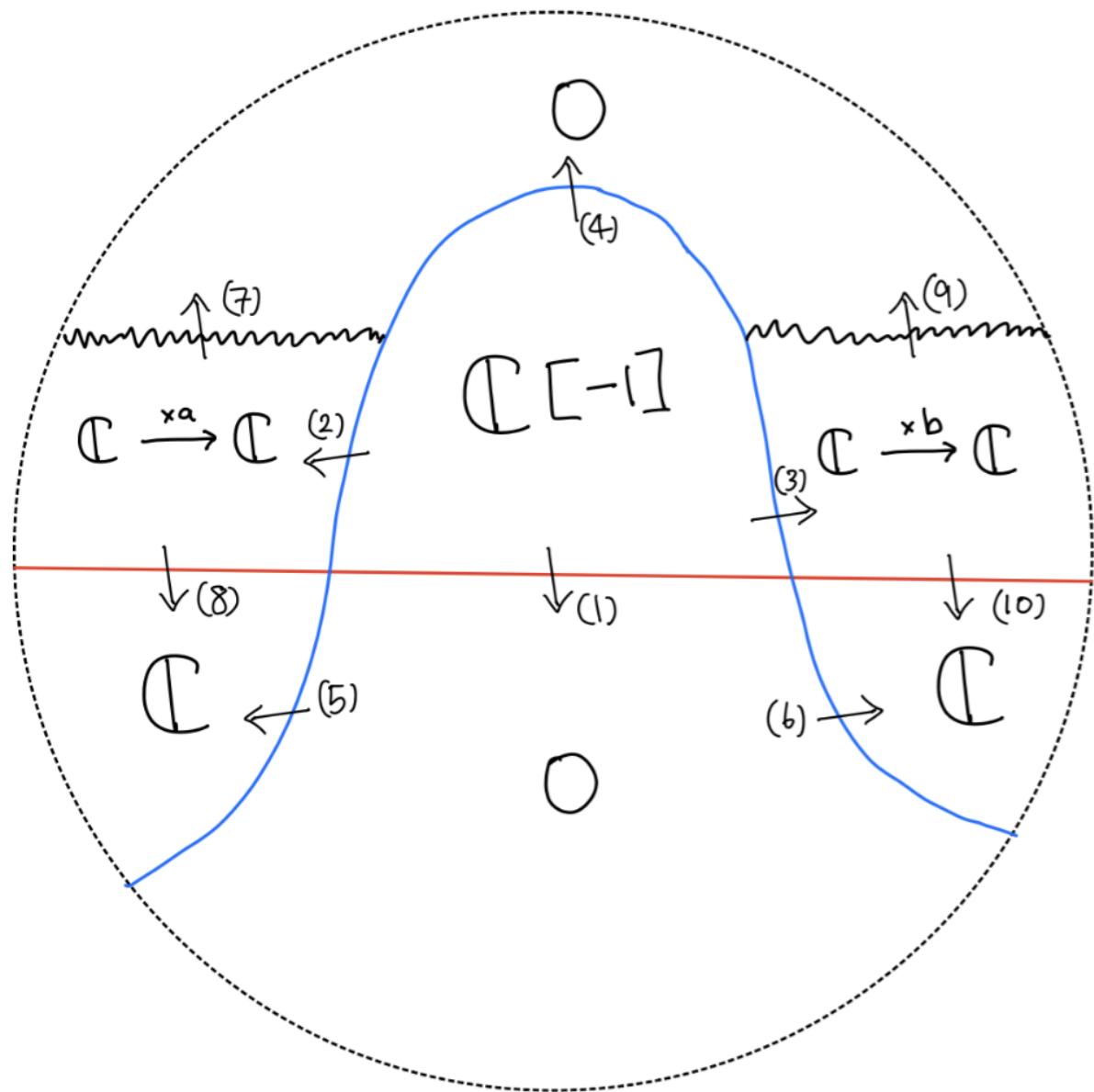


Figure 3.96

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

### 3.5.3 Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say *cobord*<sub>2</sub>, that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

#### A. Sheaf on $V \times [0, 1]$

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

#### B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3, sgn_4) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

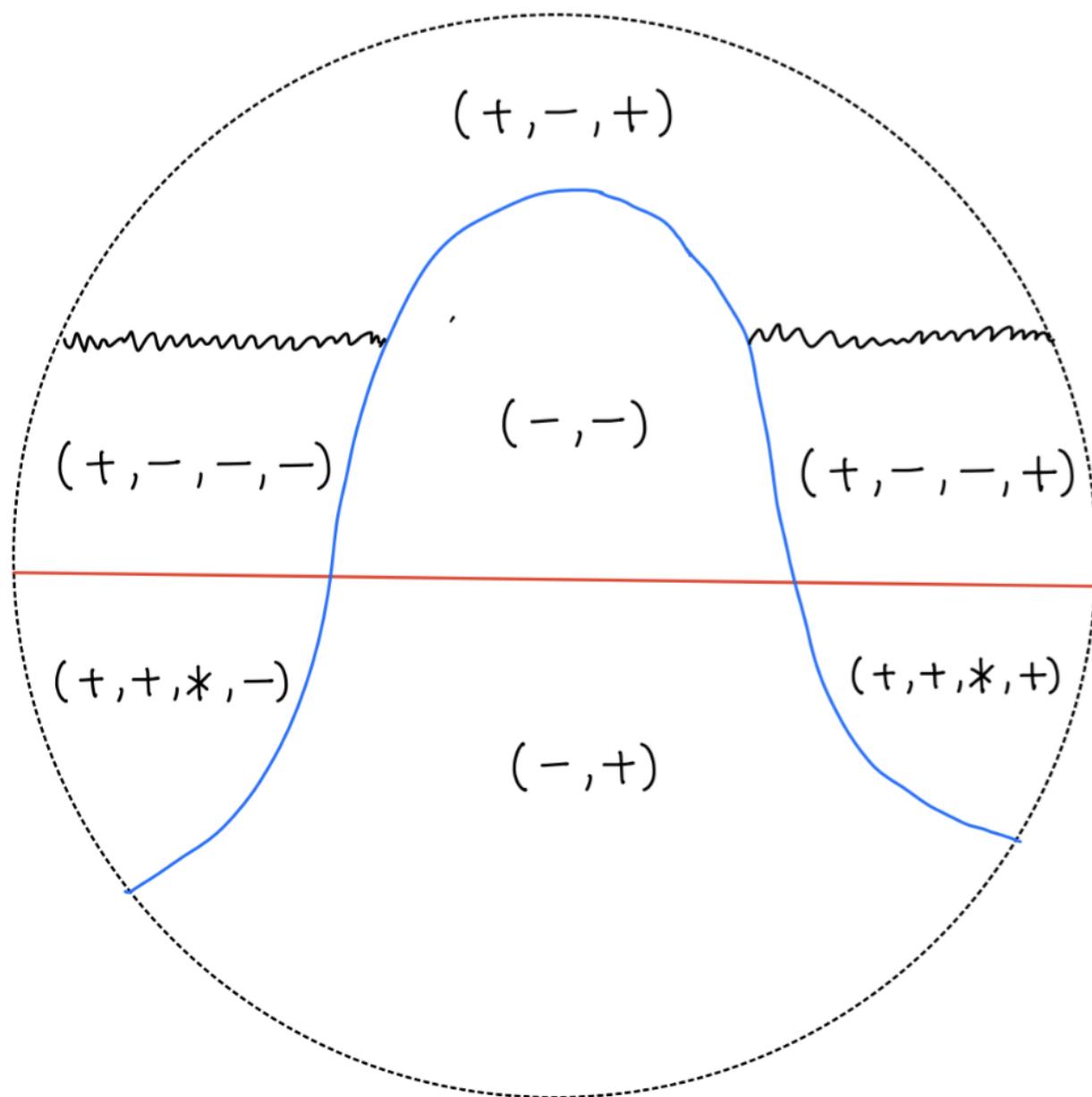


Figure 3.97

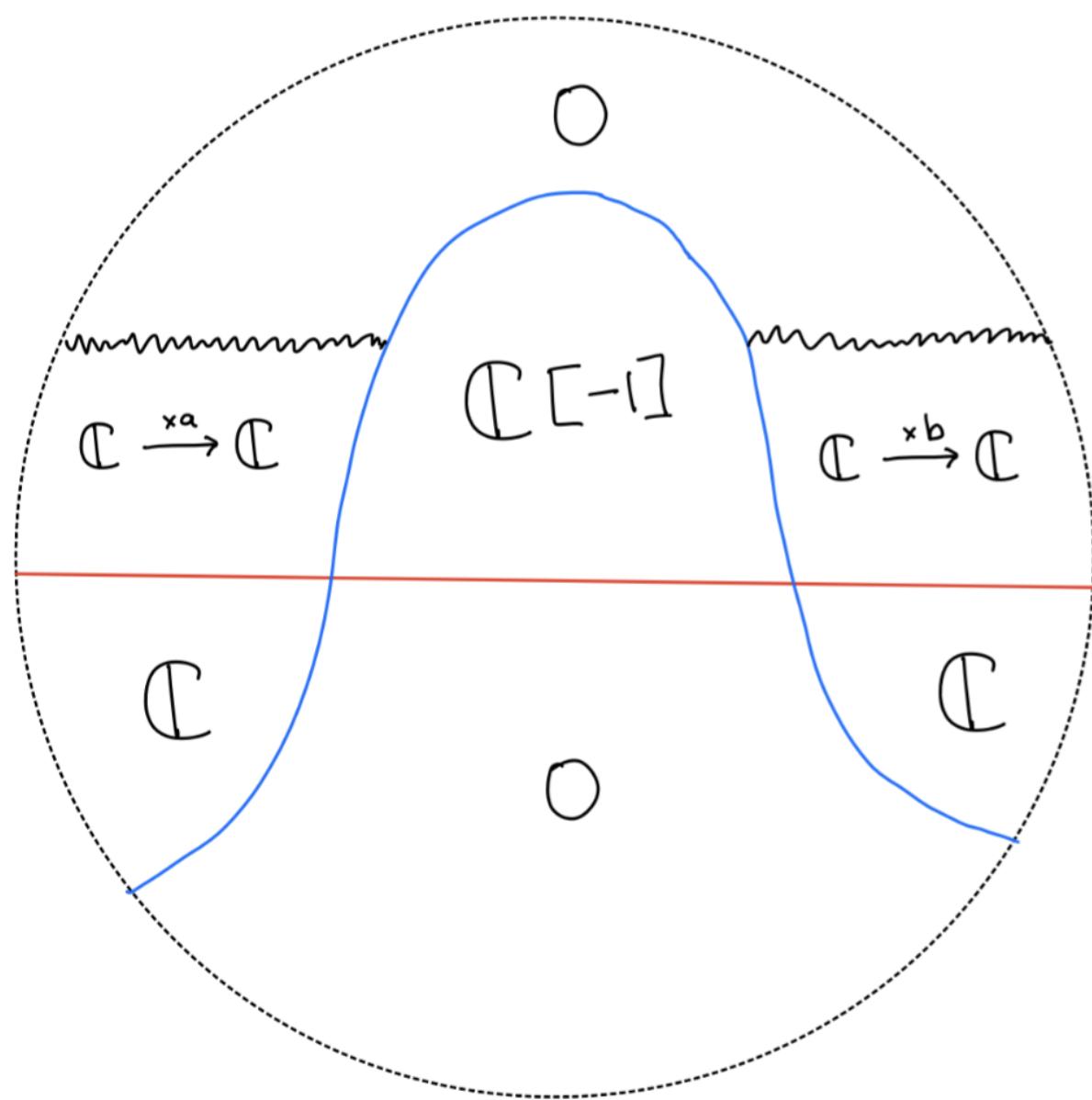


Figure 3.98

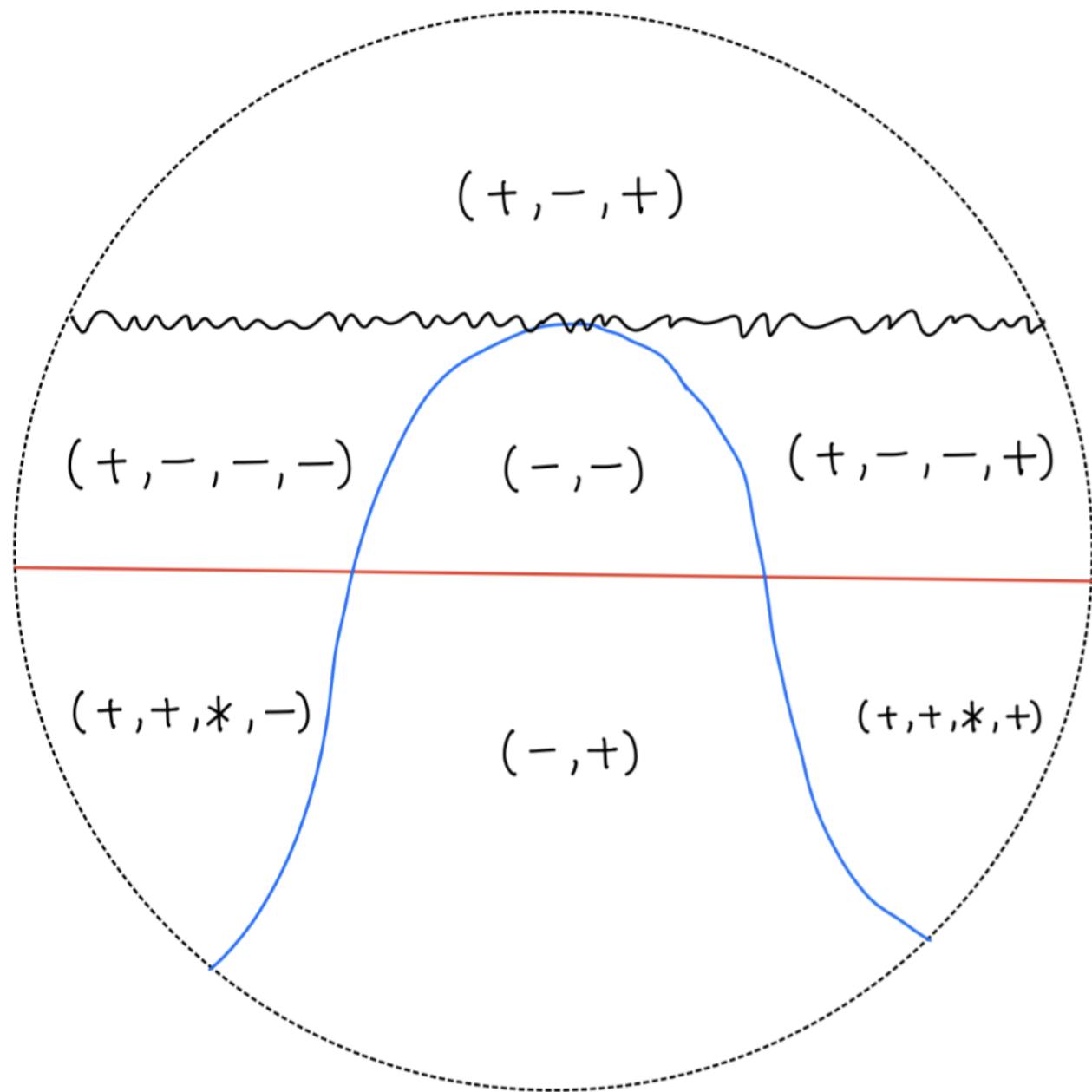


Figure 3.99

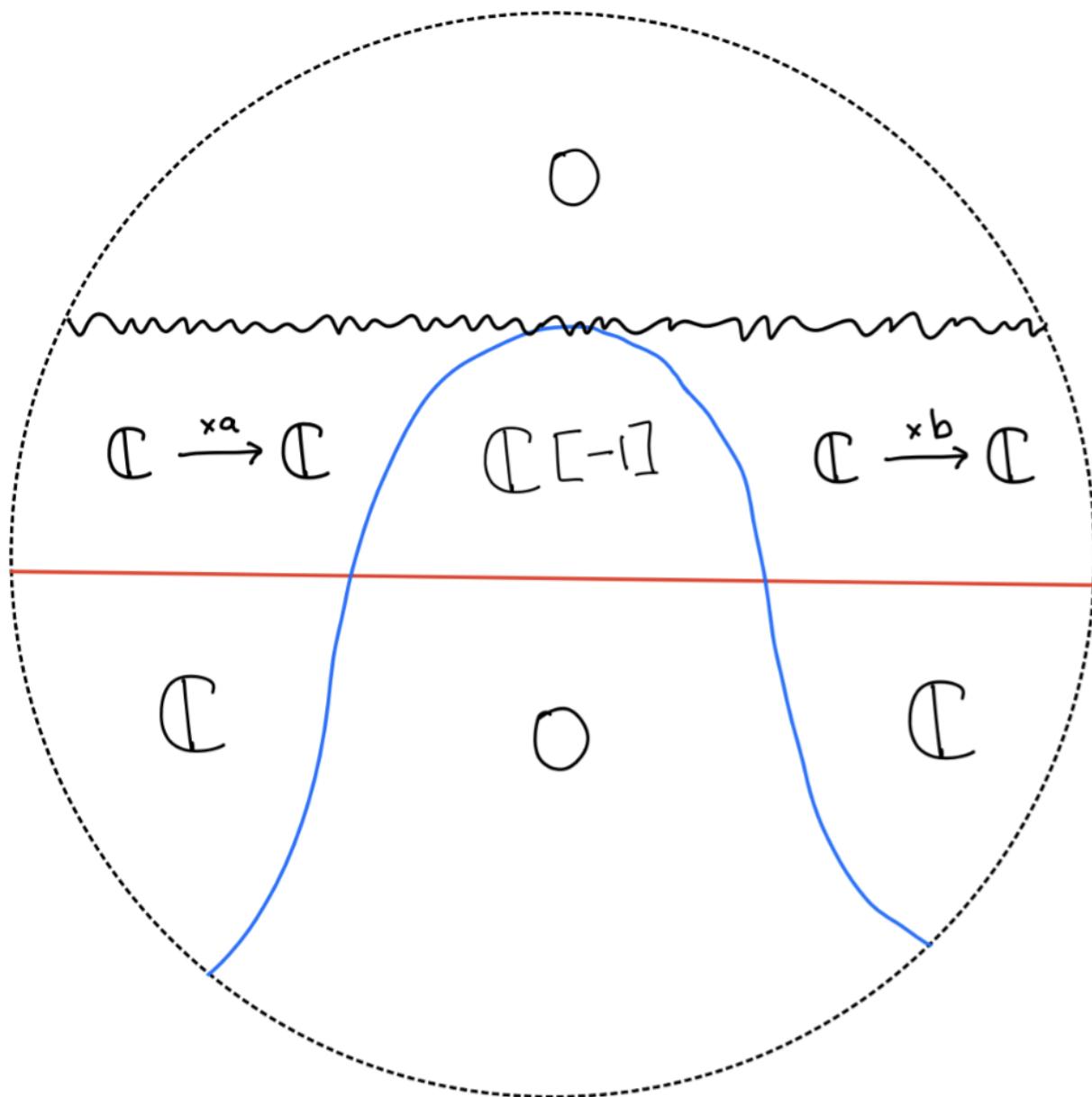


Figure 3.100

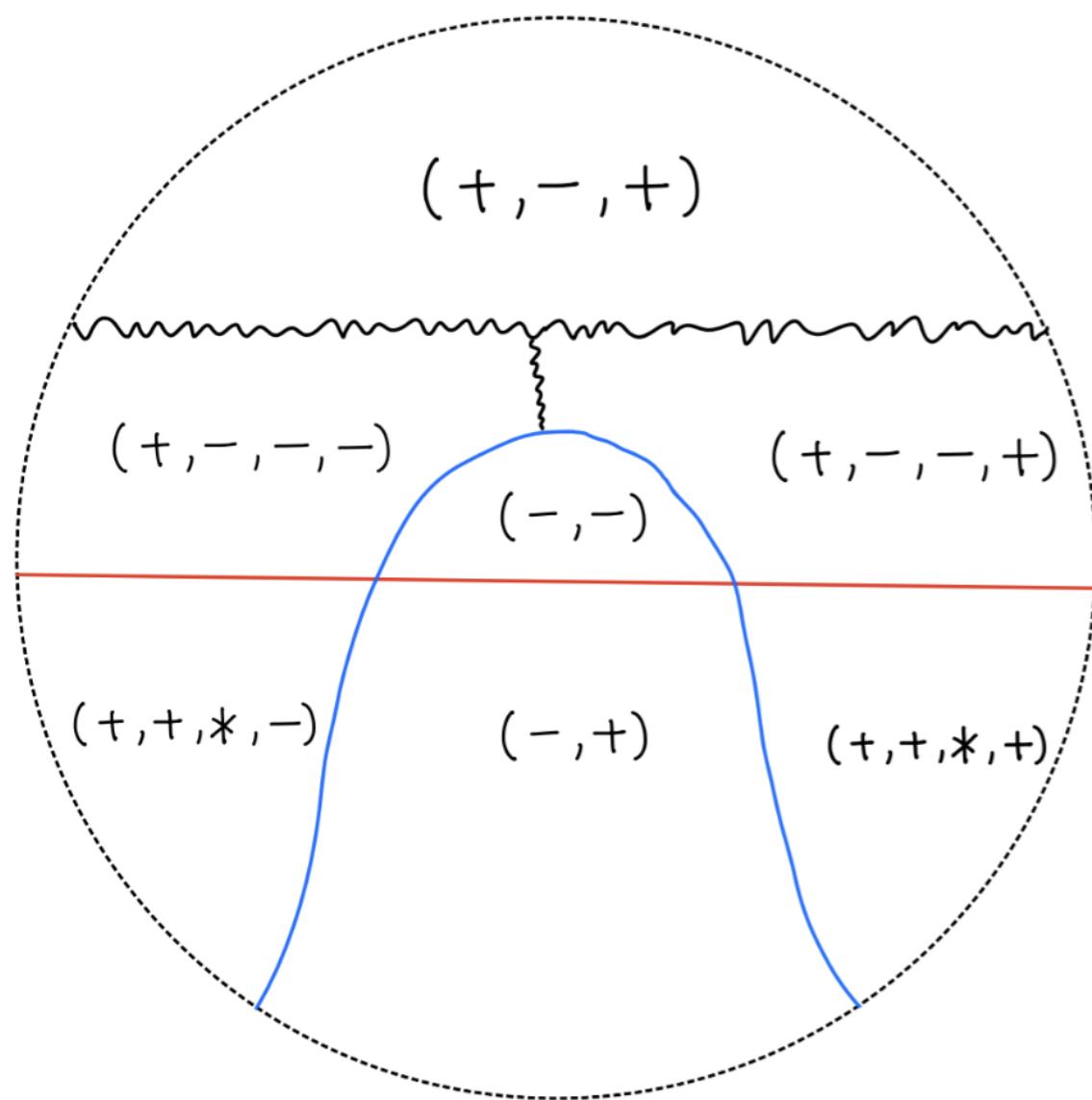


Figure 3.101

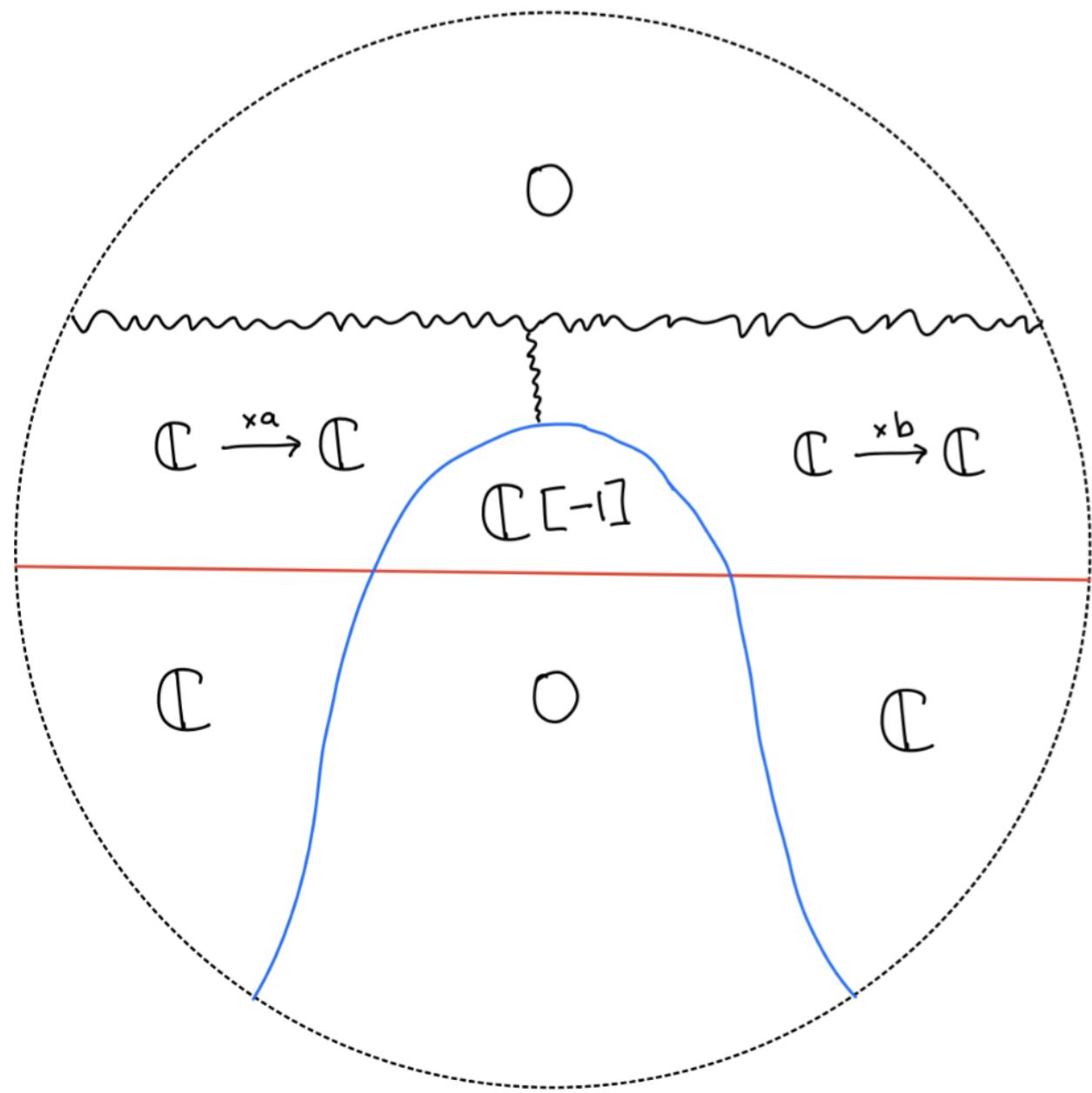


Figure 3.102

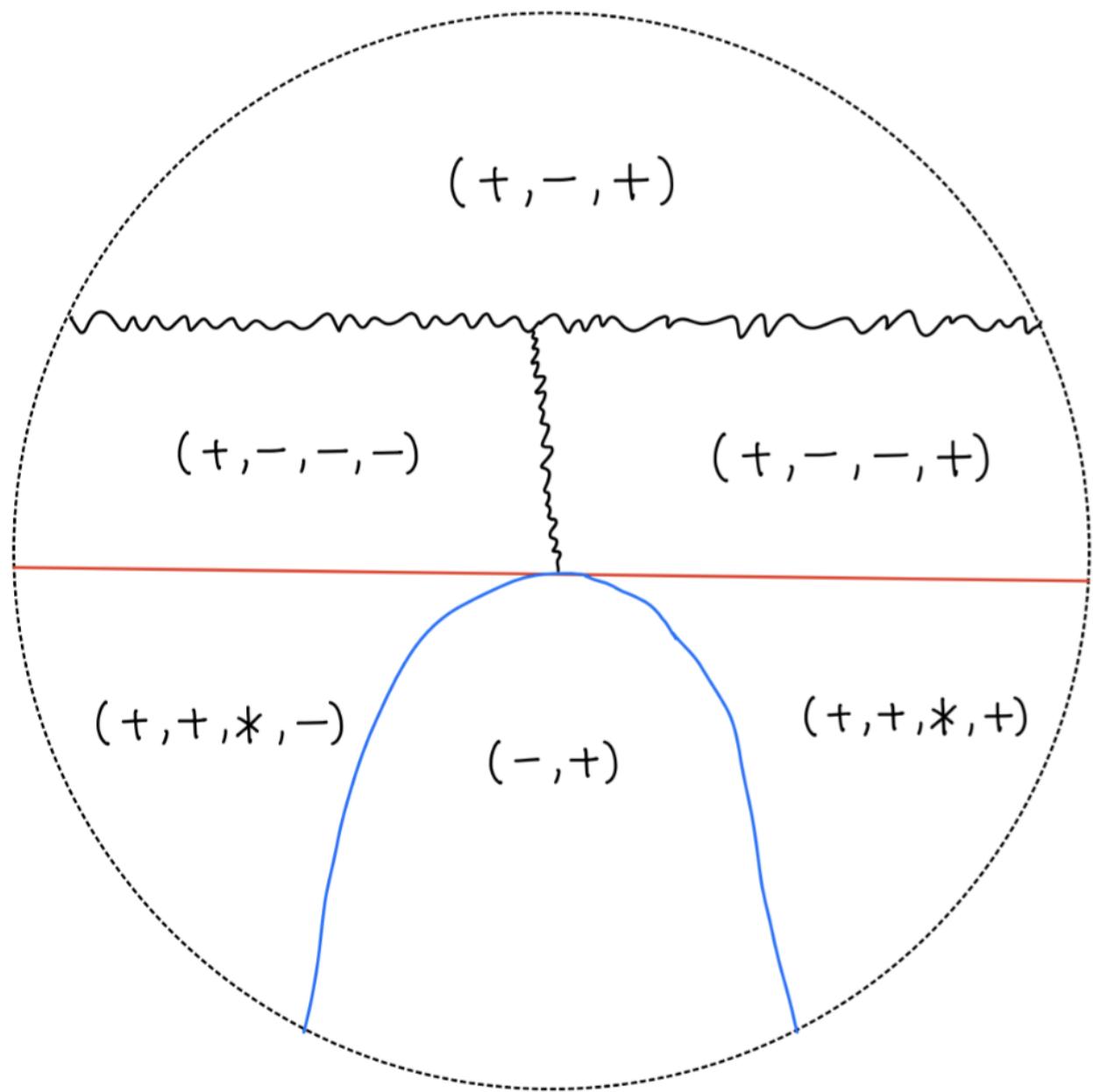


Figure 3.103

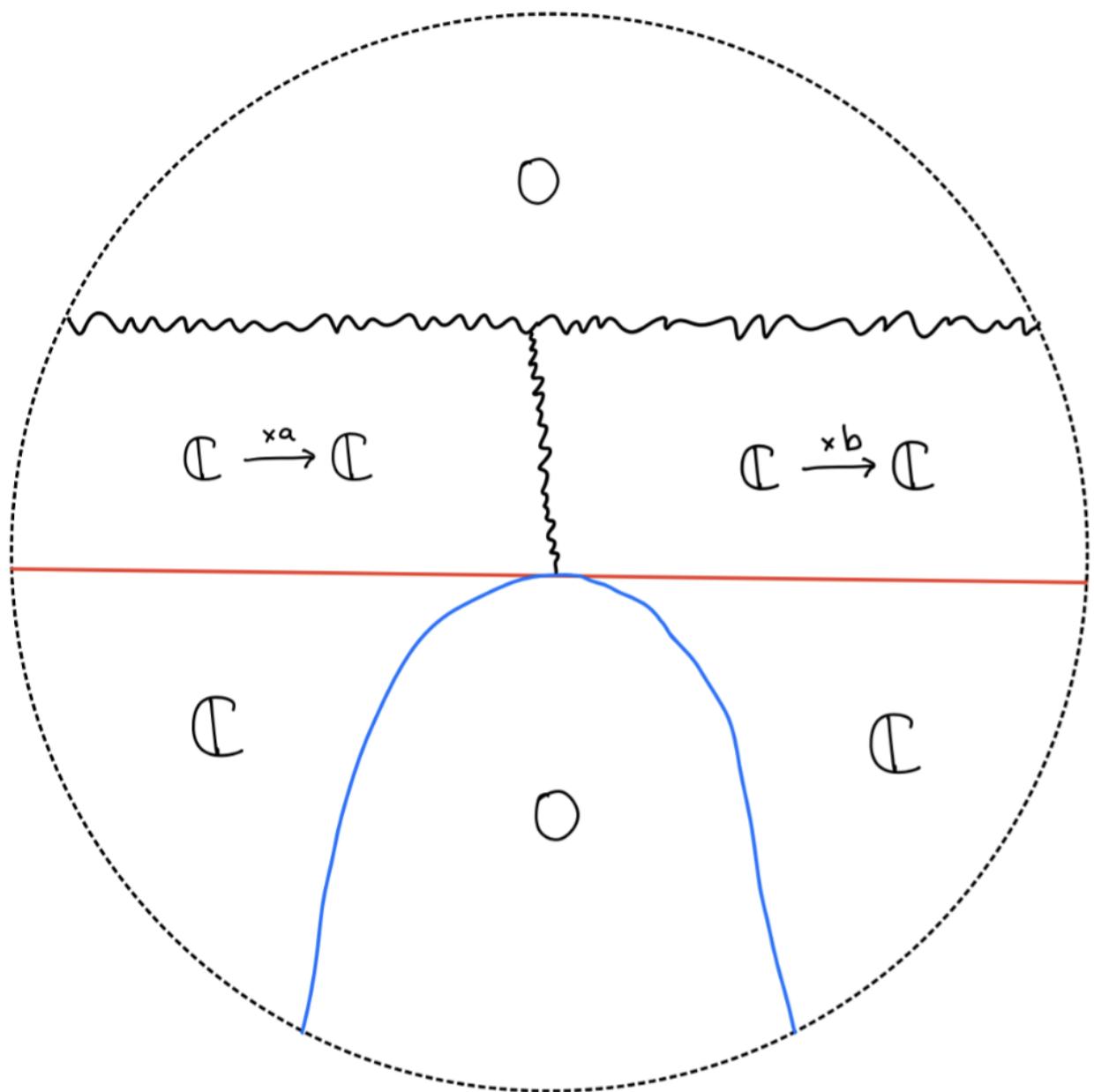


Figure 3.104

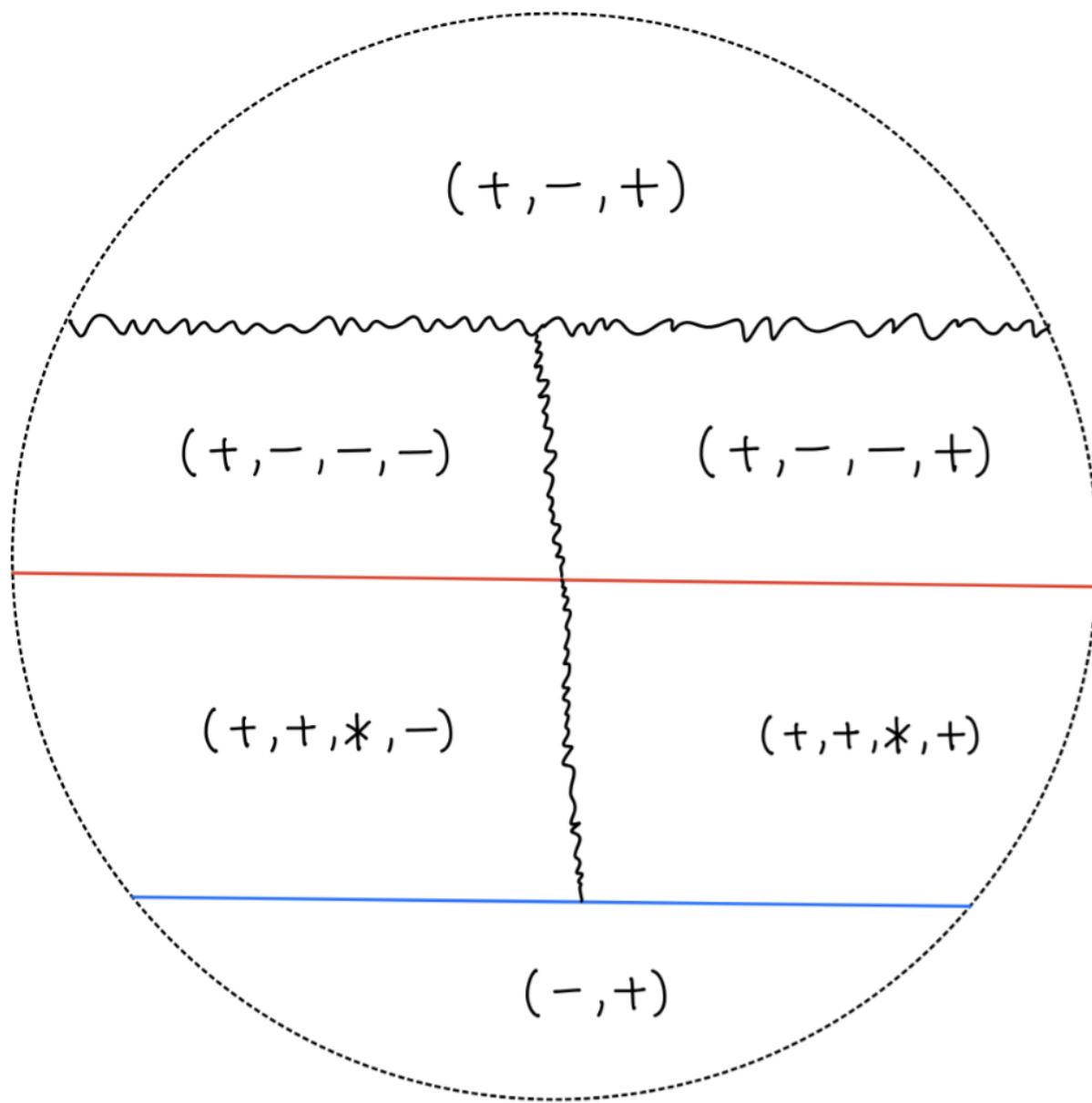


Figure 3.105

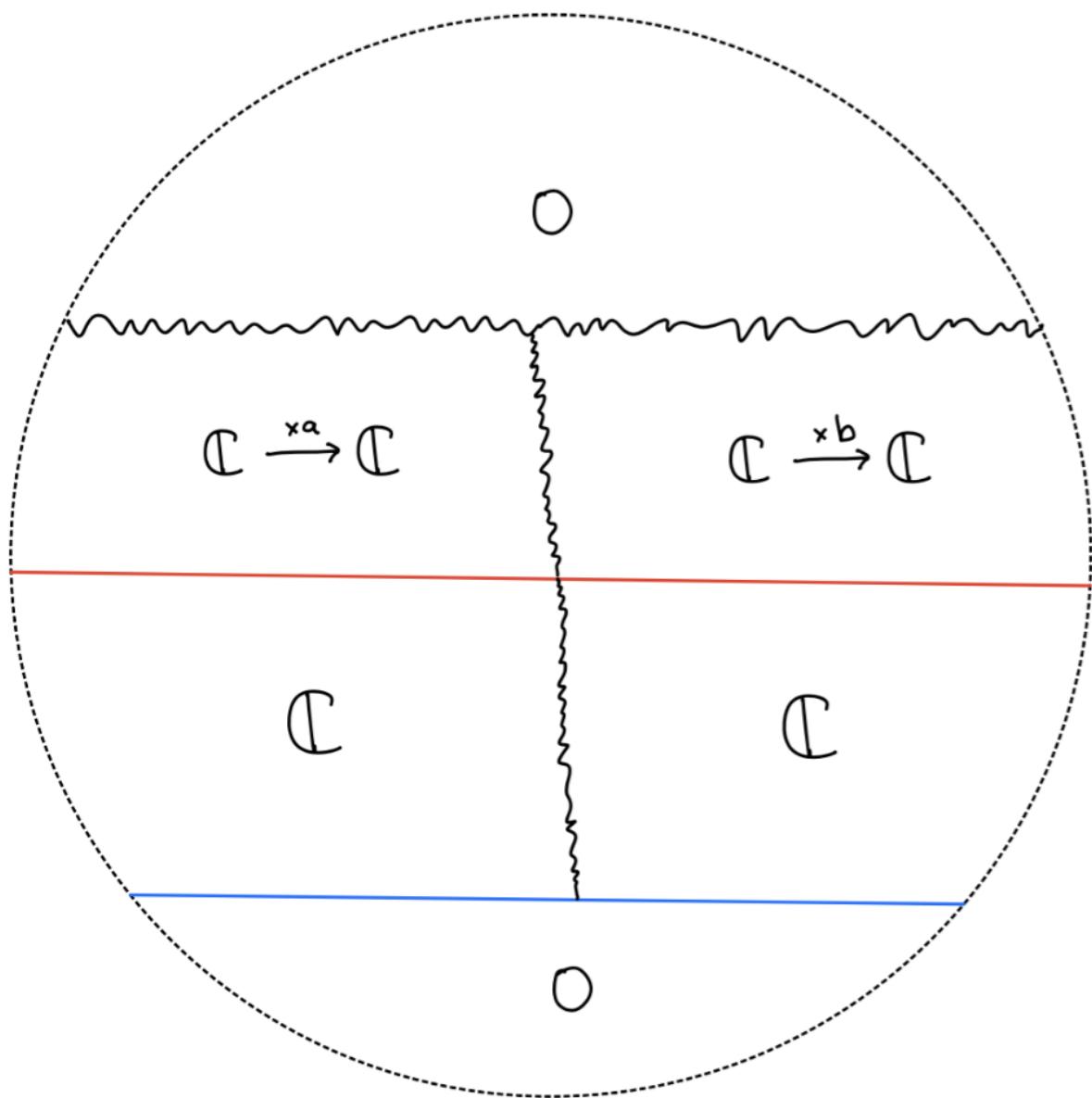


Figure 3.106

- $F_{\bullet}(-, -) := \mathbb{C}[-1]$
- $F_{\bullet}(-, +) := 0$
- $F_{\bullet}(+, -, -, -) := \mathbb{C} \xrightarrow{x^a} \mathbb{C}$
- $F_{\bullet}(+, -, -, +) := \mathbb{C} \xrightarrow{x^b} \mathbb{C}$
- $F_{\bullet}(+, -, +) := 0$

- $F_\bullet(+, +, *, -) := \mathbb{C}$

- $F_\bullet(+, +, *, +) := \mathbb{C}$

Generalization maps:

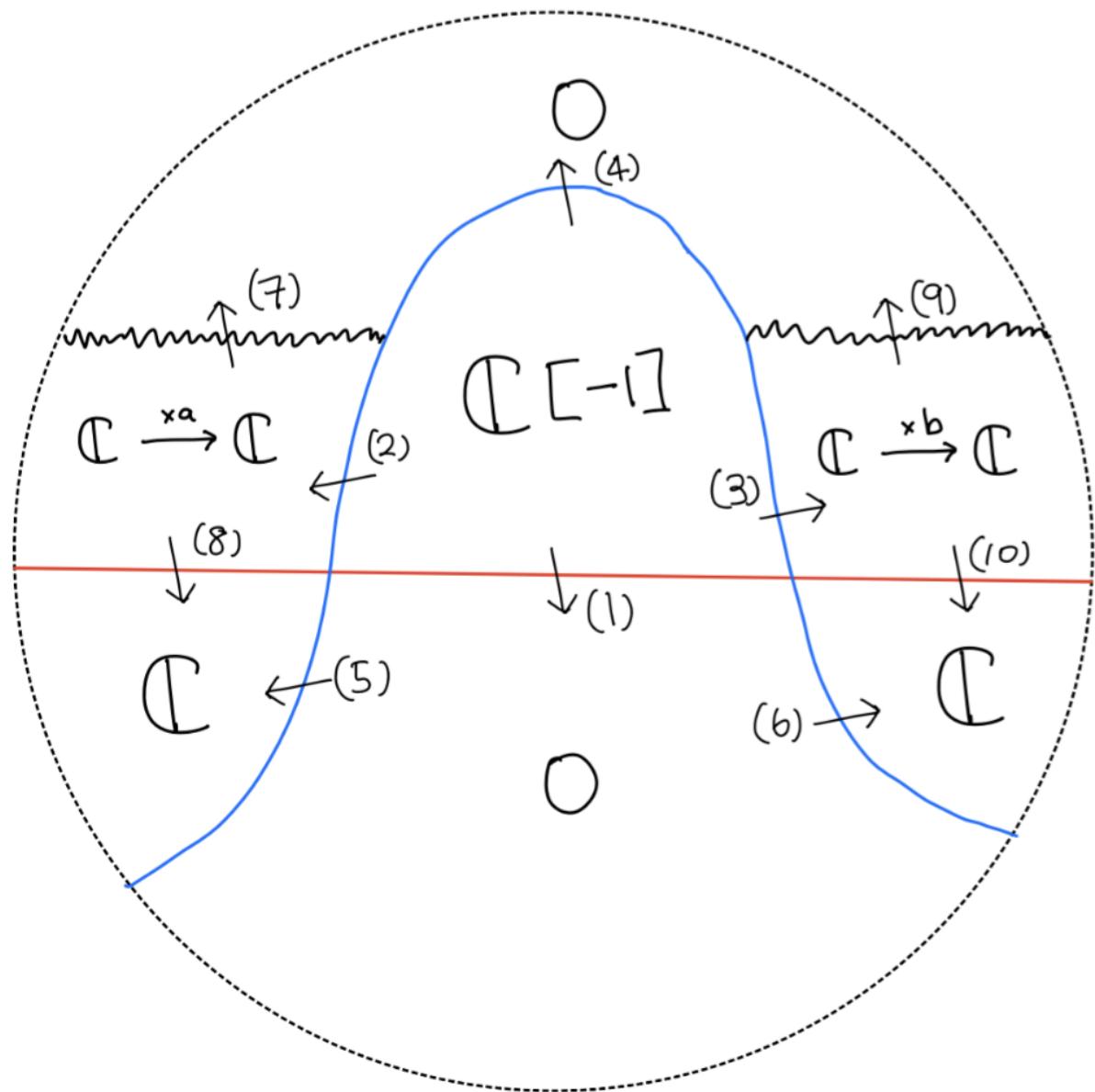


Figure 3.107

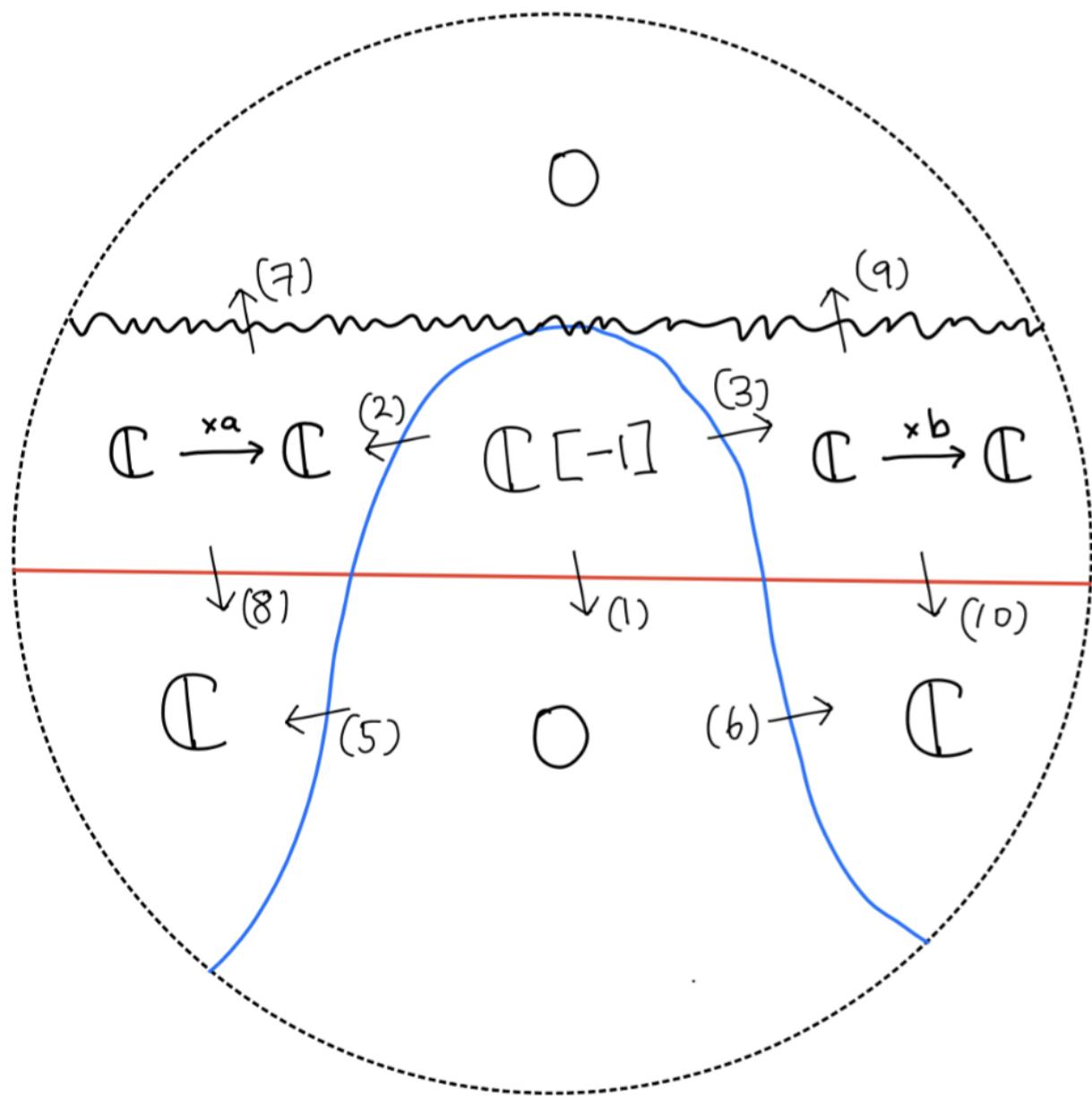


Figure 3.108

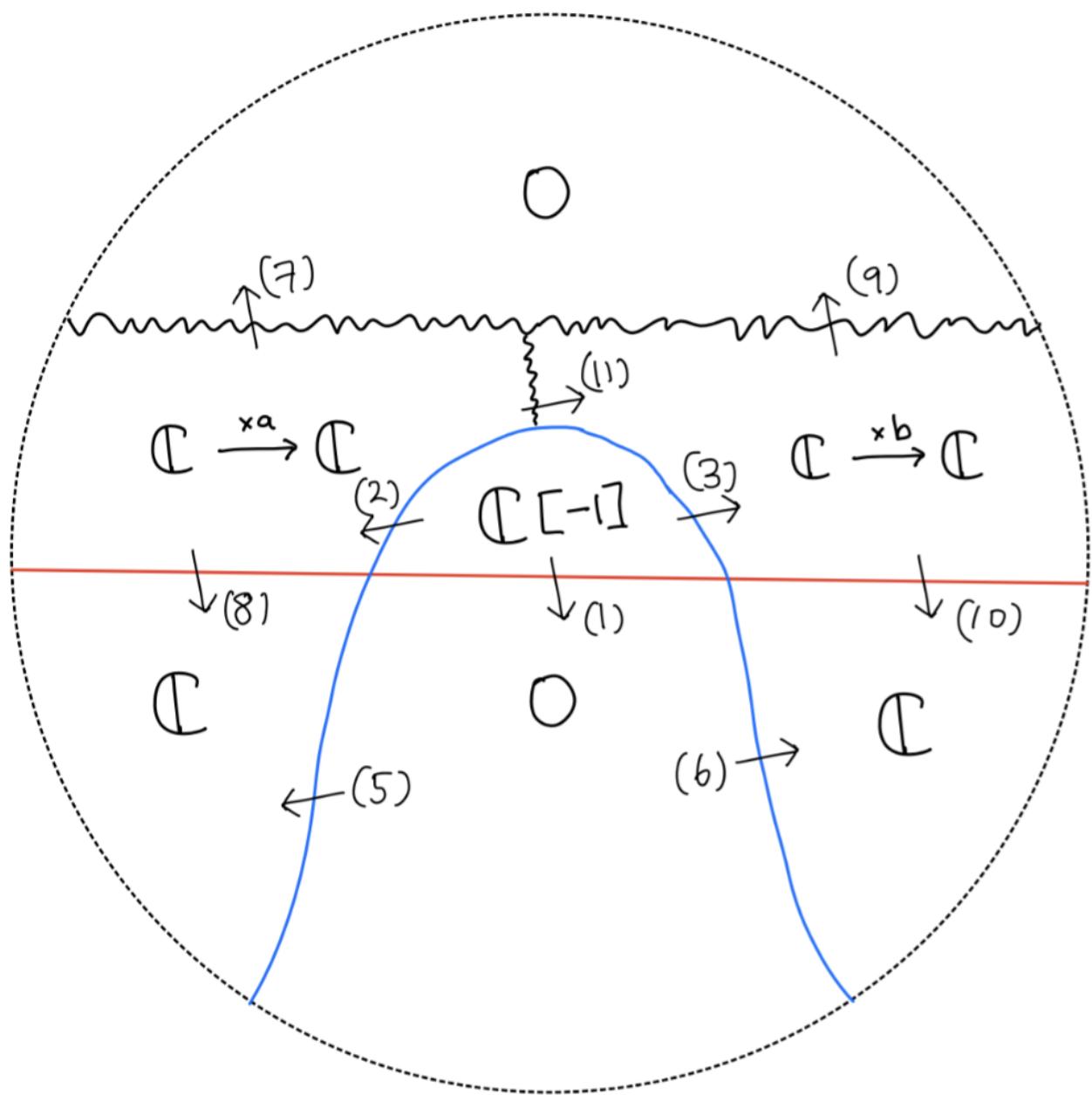


Figure 3.109

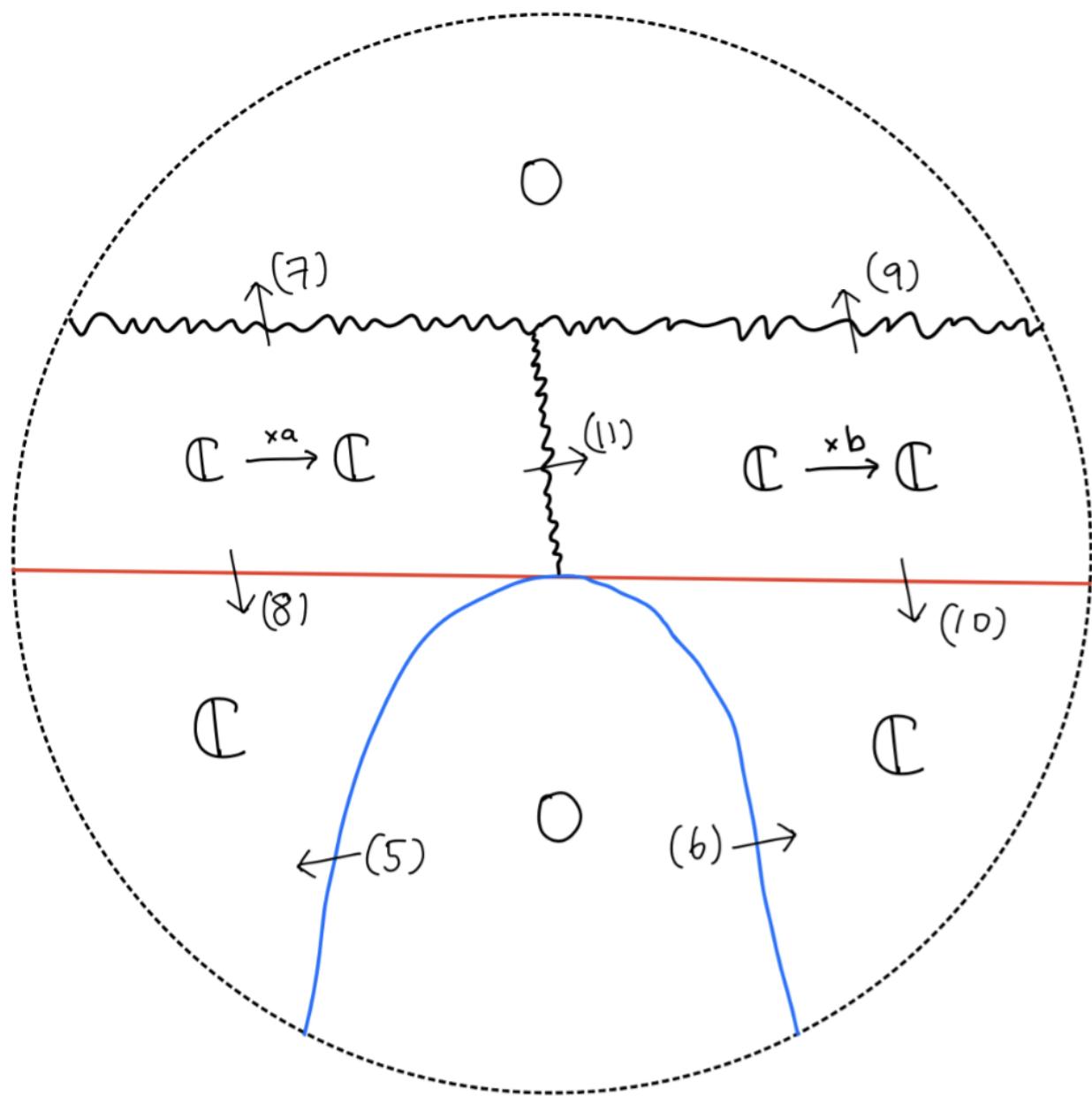


Figure 3.110

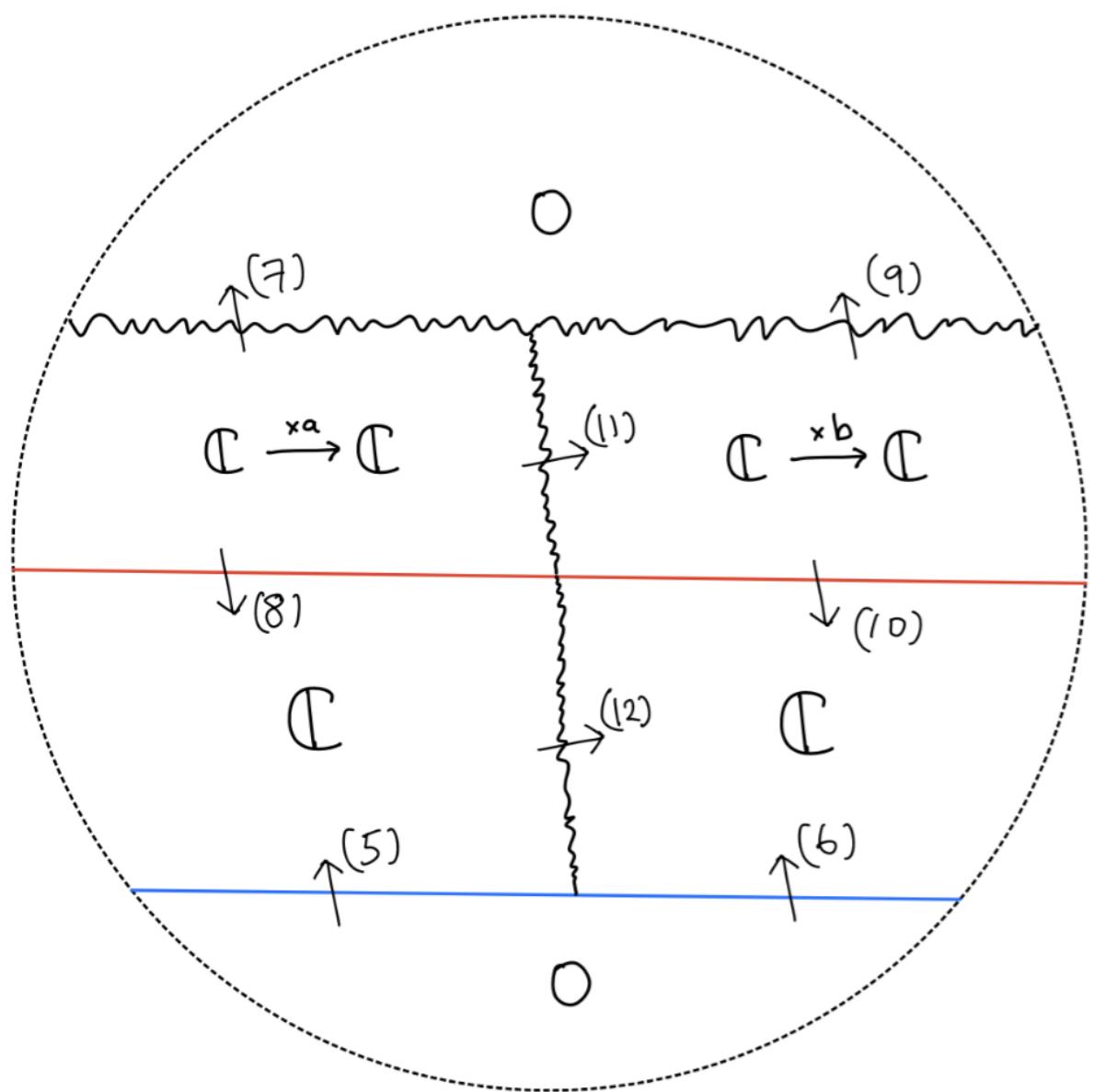


Figure 3.111

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} C & \xrightarrow{x^1} & C \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & C \end{array}$$

$$\qquad \qquad \qquad \begin{array}{c} \times a \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 109.** we define  $\gamma_\bullet$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \hookrightarrow & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \longrightarrow & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 110.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M \times [0,1], \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M \times [0,1]$ .

Since  $\mathcal{F}^\bullet$  is constant along the time coordinate on  $U'^c \times [0,1]$ , it is enough to check for the points of  $U \times [0,1] \cong D_{r=2} \times [0,1]$ . Now consider the following open cover of  $D_{r=2} \times [0,1]$

$$\{star(s_\bullet(0, -, 0, 0)), star(s_\bullet(0, 0, -, 0))\}$$

(1) First, let's show that the microlocal stalks of  $\mathcal{F}^\bullet|_{star(s_\bullet(0, -, 0, 0))}$  vanishes. Note that there is a diffeomorphism between  $star(s_\bullet(0, -, 0, 0))$  and  $\mathbb{R}^3$  that preserves there

stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

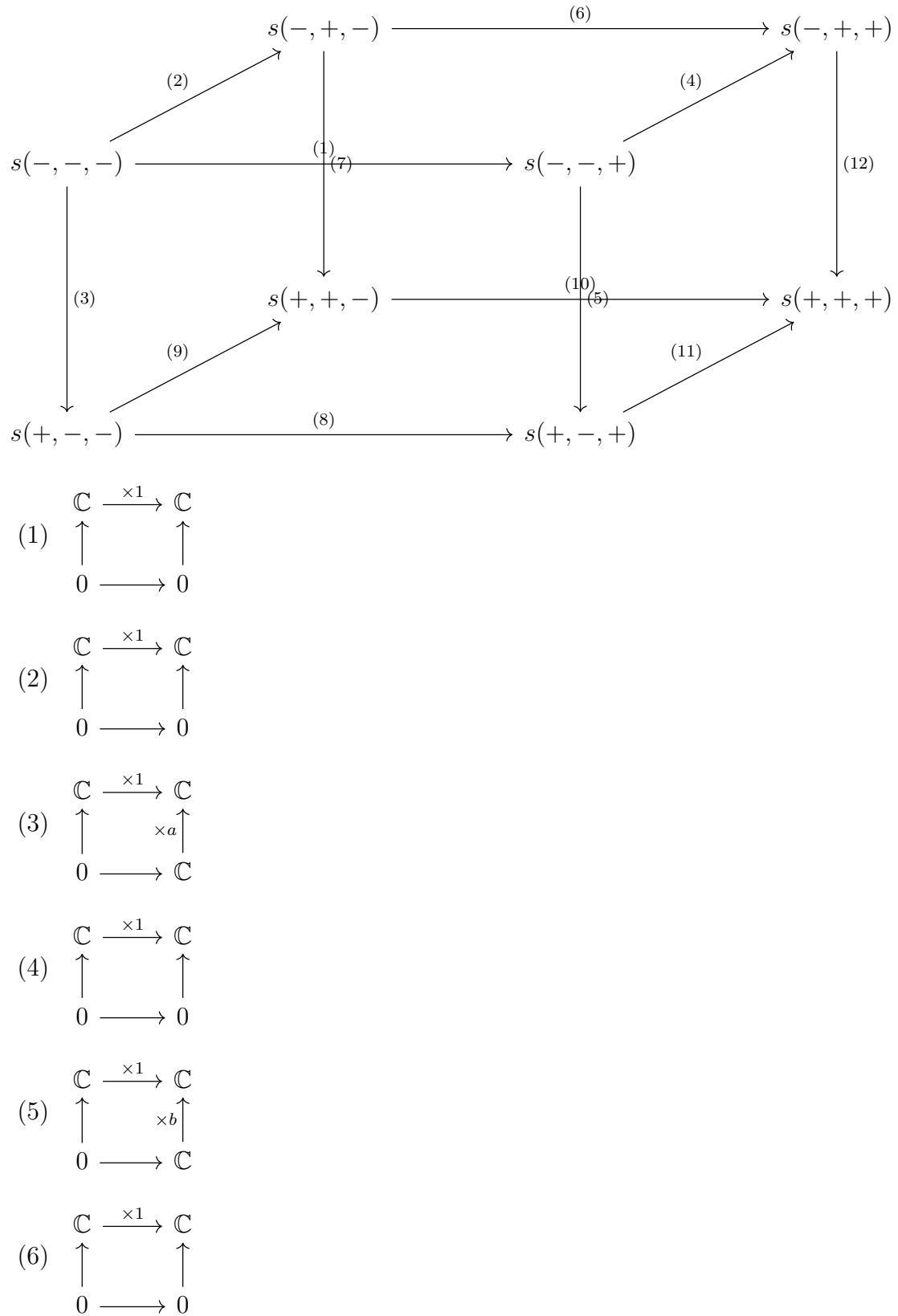
To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

### Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-, +, -) := \mathbb{C}[-1]$
- $F^3(-, +, +) := \mathbb{C}[-1]$
- $F^3(+, +, -) := 0$
- $F^3(+, +, +) := 0$

### Generalization maps:



$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times_a \uparrow & & \times_b \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \end{array} \longrightarrow 0$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \end{array} \longrightarrow 0$$

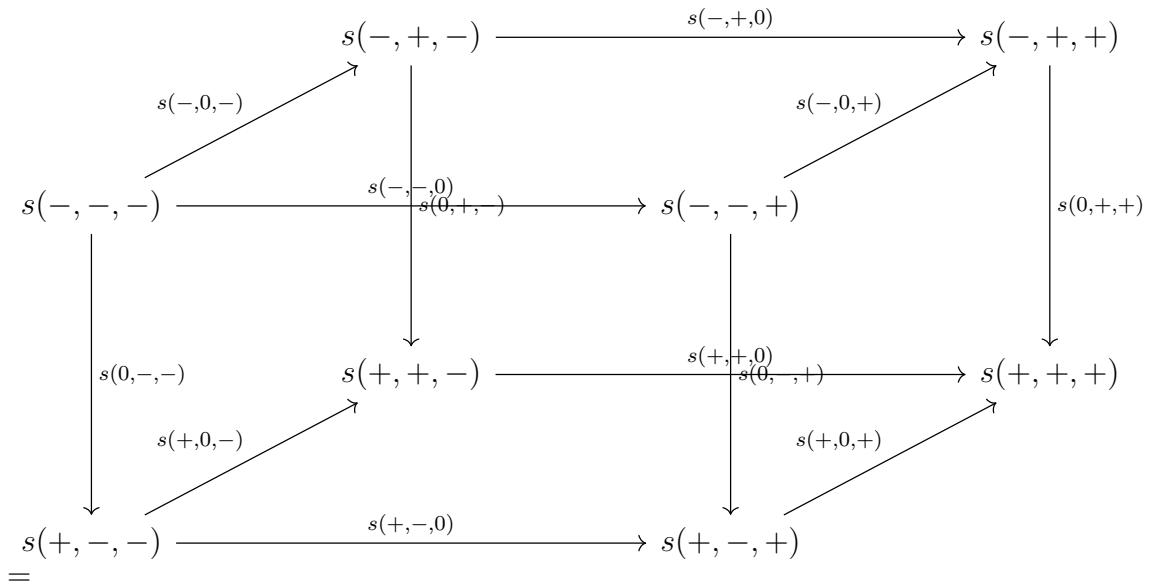
$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \end{array} \longrightarrow \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \end{array}$$

$$(iv) \quad \begin{array}{ccc} F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(v) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \end{array}$$

$$(vi) \quad \begin{array}{ccc} F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

(vii) the cubic diagram:



$$\begin{array}{ccccc}
 & \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \\
 \swarrow & & \downarrow & & \searrow \\
 \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C}[-1] \\
 \downarrow & & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & 0 & \\
 \searrow & & \downarrow & & \swarrow \\
 & \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \\
 \end{array}$$

For  
i-vi, horizontal cochain map in each degree are quasi-isomorphism. Therefore,

the total complex is acyclic.

For vii,  $\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} & , & \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow & , & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & , & 0 & \longrightarrow & 0 \end{array}$ ,  $\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times a & & \uparrow \times b \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$ ,  $\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$  are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of  $\mathcal{F}^\bullet|_{\text{star}(s_\bullet(0,0,-,0))}$  vanishes. Note that there is a diffeomorphism between  $\text{star}(s_\bullet(0,0,-,0))$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

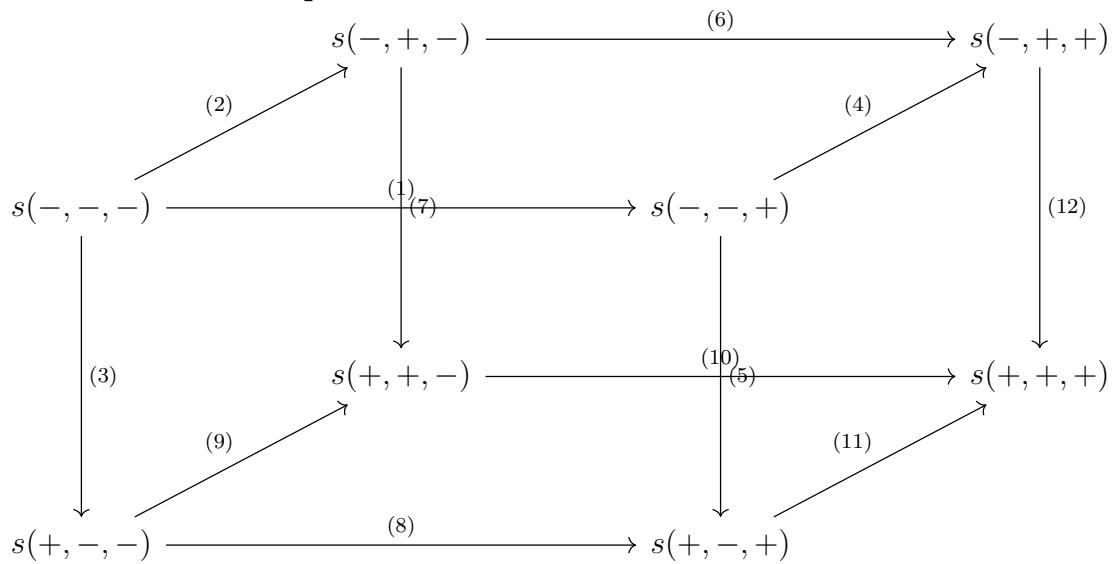
To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

### Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(-, +, -) := 0$
- $F^3(-, +, +) := 0$
- $F^3(+, +, -) := \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}$

### Generalization maps:



$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \times b \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{ab^{-1}} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

$$(iii) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times b} \mathbb{C} \end{array}$$

$$(iv) \quad \begin{array}{ccc} F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{ab^{-1}} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

$$(v) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\ \downarrow & & \downarrow \\ F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(vi) \quad \begin{array}{ccc} F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\ \downarrow & & \downarrow \\ F^3(+, +, -) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \xrightarrow{\times ab^{-1}, \times 1} \mathbb{C} \xrightarrow{\times b} \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-,+,-) & \xrightarrow{s(-,+0)} & s(-,+,+) \\
 s(-,0,-) \nearrow & \downarrow & & \searrow s(-,0,+)
 \\ 
 s(-,-,-) & \xrightarrow[s(-,0,+)]{s(0,+,-)} & s(-,-,+)
 & & \downarrow s(0,+,+)
 \\ 
 \downarrow s(0,-,-) & s(+,+,-) & \xrightarrow[s(+,0,+)]{s(0,-,+)} & s(+,+,+)
 & \downarrow
 \\ 
 s(+,0,-) \nearrow & \downarrow & & \searrow s(+,0,+)
 \\ 
 s(+,-,-) & \xrightarrow{s(+,-0)} & s(+,-,+)
 & & 
 \end{array}
 = 
 \begin{array}{ccc}
 0 & \xrightarrow{\quad} & 0 \\
 \downarrow & & \downarrow \\
 \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C}[-1]
 \\ 
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\quad} & \mathbb{C}
 \\ 
 \downarrow & & \downarrow \\
 \mathbb{C} \xrightarrow{a} \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \xrightarrow{b} \mathbb{C}
 \end{array}$$

For iii-vi,

horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For i and ii, straightforward calculation shows that the total complexes are acyclic.

For vii,  $\begin{array}{cccc} \mathbb{C} \xrightarrow{x^1} \mathbb{C} & 0 \longrightarrow 0 & \mathbb{C} \xrightarrow{x^1} \mathbb{C} & 0 \xrightarrow{x^1} 0 \\ \uparrow & \uparrow & \uparrow \xrightarrow{x_a} \uparrow \xrightarrow{x_b} \uparrow & \uparrow \xrightarrow{x_{ab^{-1}}} \uparrow \\ 0 \longrightarrow 0 & 0 \longrightarrow 0 & \mathbb{C} \xrightarrow{x_{ab^{-1}}} \mathbb{C} & \mathbb{C} \xrightarrow{x_{ab^{-1}}} \mathbb{C} \end{array}$  are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two

double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete.  $\square$

### 3.5.4 Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the *cobord*<sub>2</sub>. By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

#### A. Sheaf on $V$

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

#### B. Sheaf on $D_{r=2}$

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

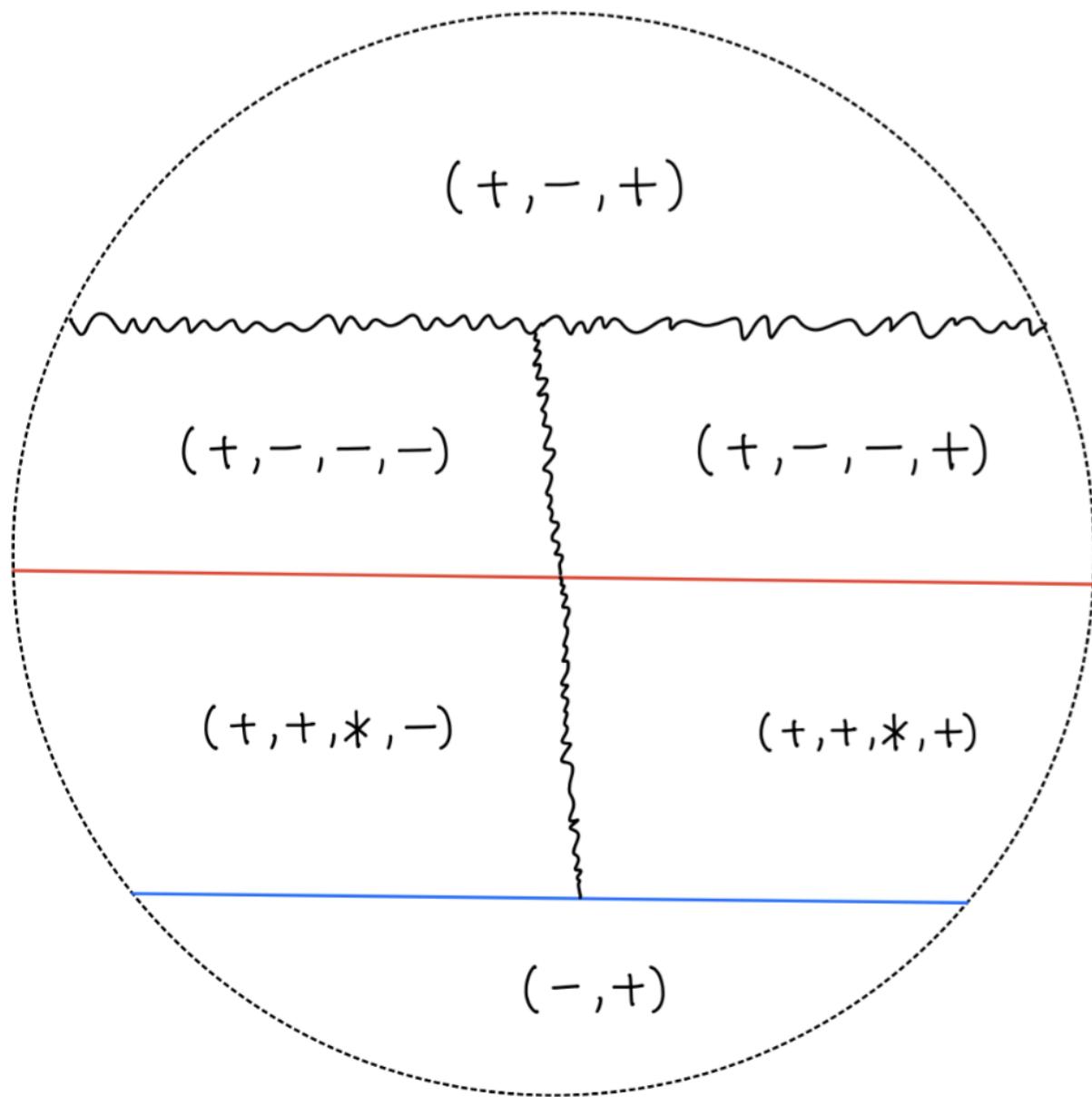


Figure 3.112

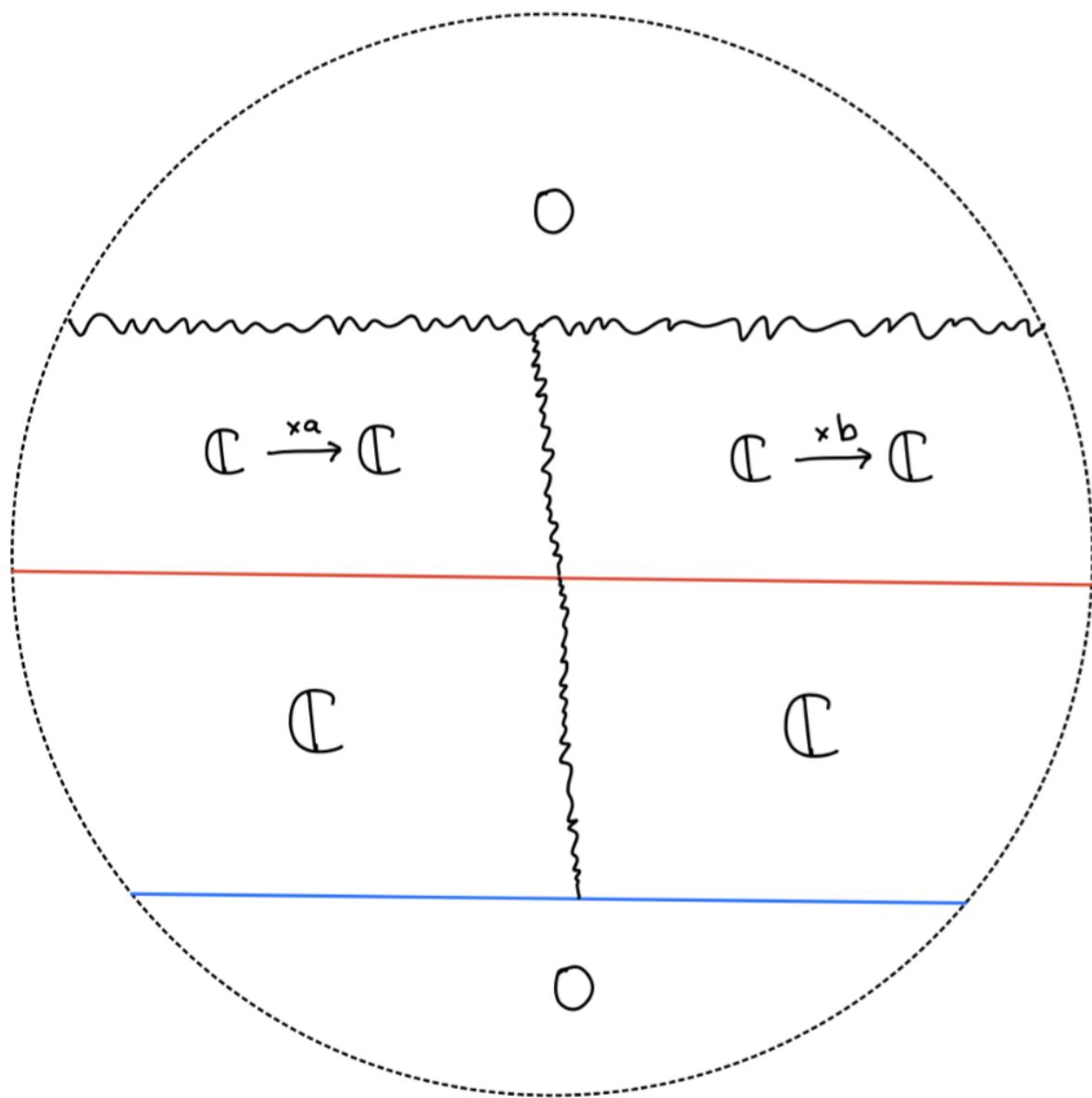


Figure 3.113

- $F_1(-, +) := 0$
- $F_1(+, -, -, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_1(+, -, -, +) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_1(+, -, +) := 0$
- $F_1(+, +, *, -) := \mathbb{C}$

- $F_1(+, +, *, +) := \mathbb{C}$

Generalization maps:

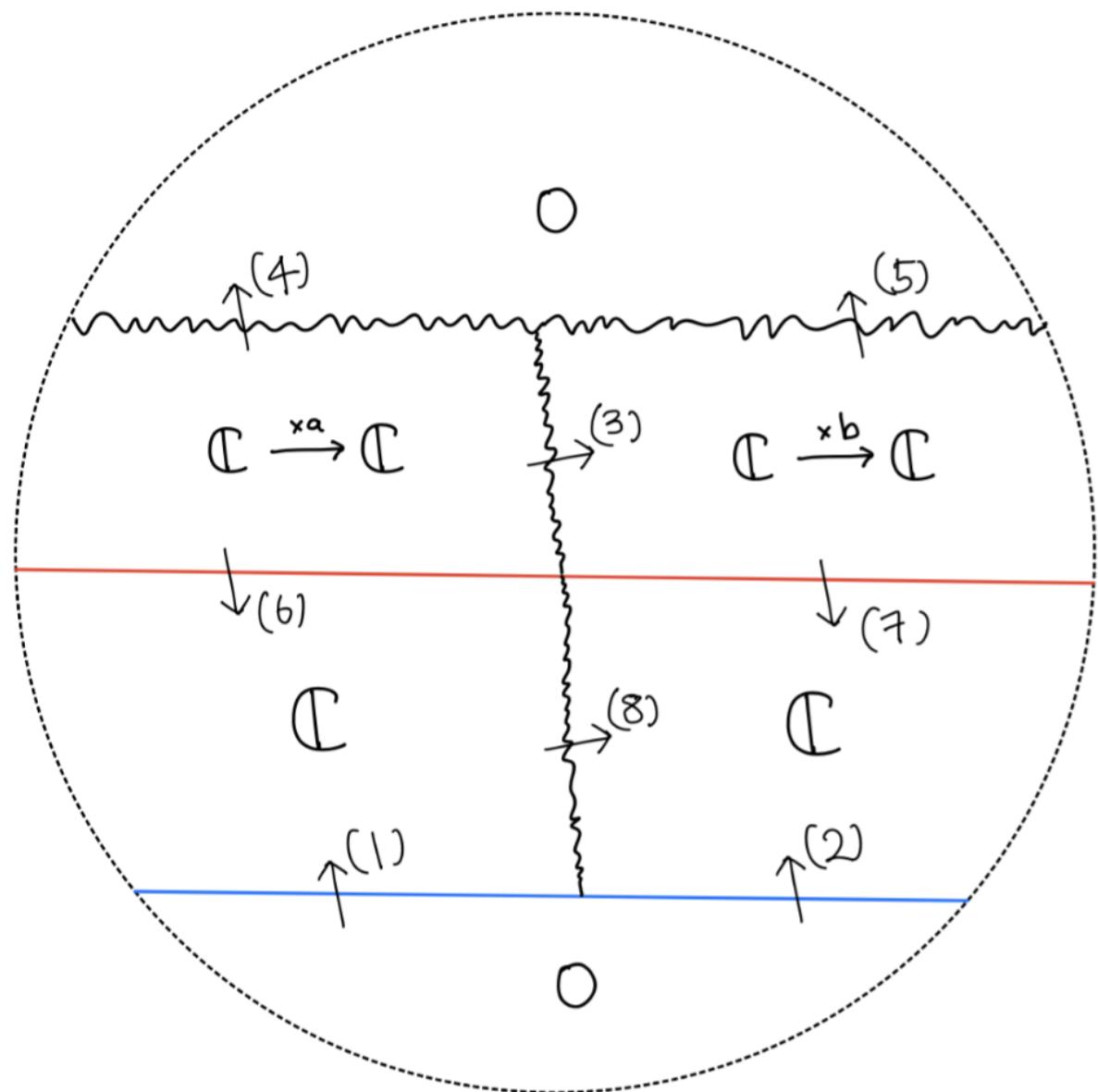


Figure 3.114

$$(1) \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \times b \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times ab^{-1}} & \mathbb{C} \end{array}$$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_{\bullet}|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 111.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

## 3.6 2nd Sheaf Cobordism'

In this section, we define  $cobord'_2$ , a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

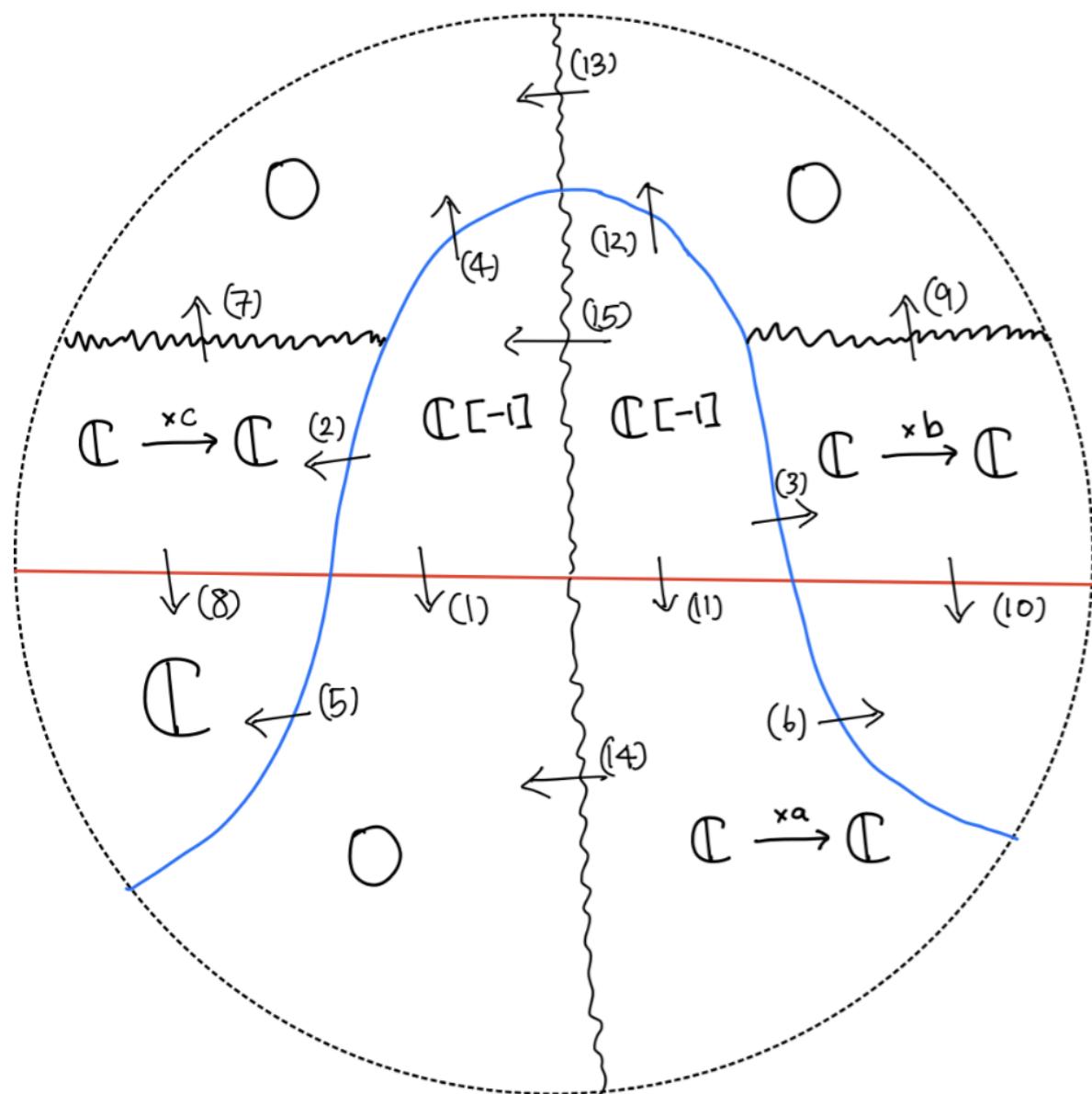


Figure 3.115

**Generalization maps:**

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} C & \xrightarrow{\times 1} & C \\ \uparrow & \uparrow \times_c & \\ 0 & \longrightarrow & C \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \times a \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(13) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(15) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

to

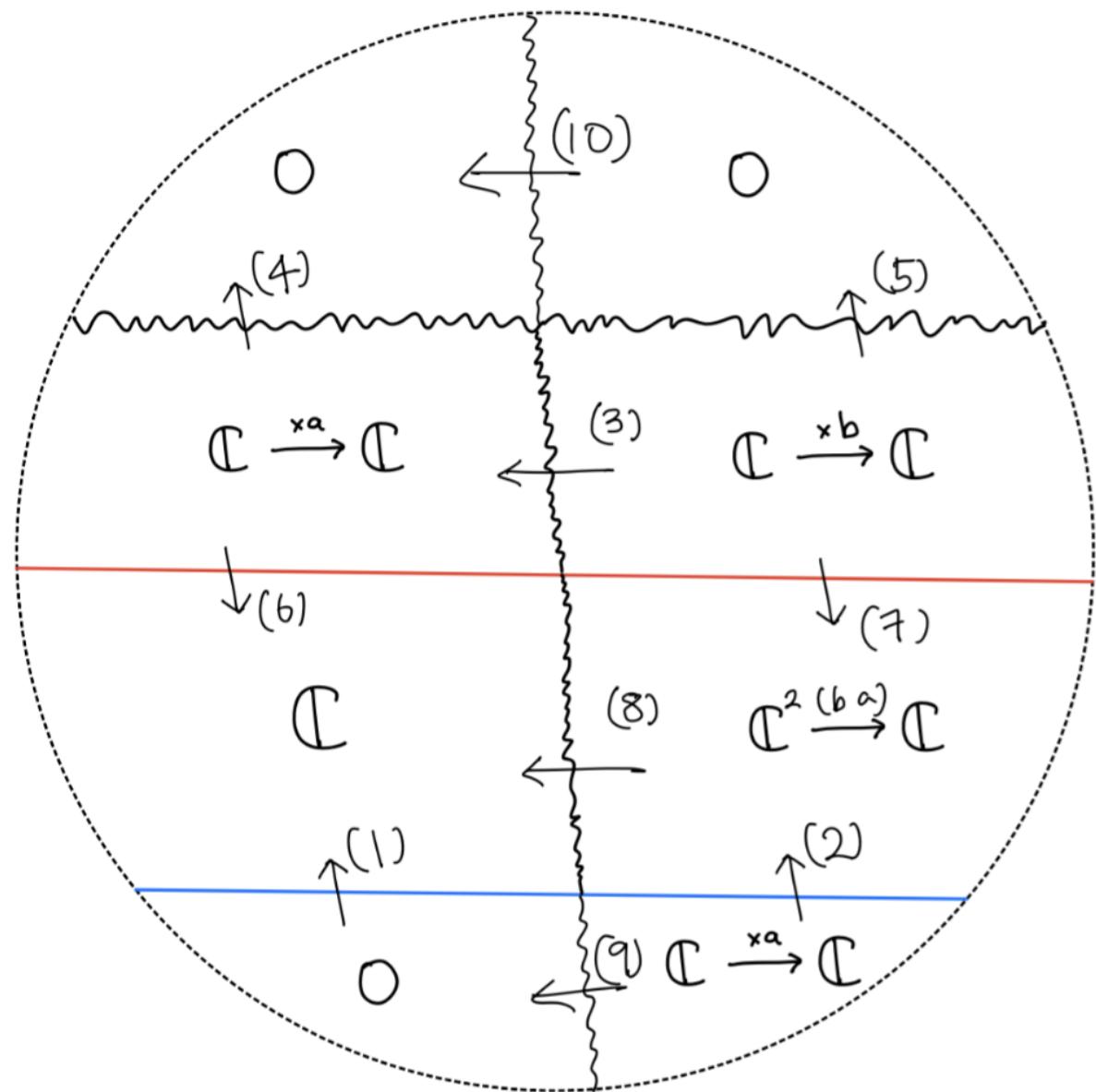


Figure 3.116

**Generalization maps:**

$$(1) \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \begin{array}{ccc} \mathbb{C} & \xrightarrow{x^1} & \mathbb{C} \\ \times_a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \times c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & (b \ a) \uparrow & \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

## Notations

**Definition 112.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 113.** For  $t_0 \in \{0, 1\}$  and  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{t_0}^{symbol}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{symbol}$  a co-orientation of  $\Phi_{t_0}^{symbol}$ .

3. we denote the pair  $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$  as  $\Lambda_{t_0}^{symbol}$ . When  $symbol \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{symbol}$  to denote both
4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the beginning of the isotopy underlying  $cobord_2$  and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying  $cobord_2$ .

**Definition 114.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_\bullet^{symbol}$  to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript ' $t$ ' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_\bullet^{symbol}$  a co-orientation of  $\Phi_\bullet^{symbol}$ .

3. we denote the pair  $(\Phi_\bullet^{symbol}, \Xi_\bullet^{symbol})$  as  $\Lambda_\bullet^{symbol}$ . Later in the section,  $\Lambda_\bullet^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>2</sub>.
4. we denote the triple  $(\Lambda_\bullet^0, \Lambda_\bullet^\infty, \Lambda_\bullet^{squig})$  as  $\Lambda_\bullet$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_\bullet$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>2</sub>.

**Definition 115.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} \frac{5}{4}e^{(\frac{4x^2}{4x^2-3})}(1-t) - \frac{1}{2} & \text{if } |x| < \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \text{if } |x| \geq \frac{\sqrt{3}}{2} \end{cases}$$

Note that

- $supp(\Psi_t) = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$
- $\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0, -\frac{5}{4}t + \frac{3}{4})\} \subset Graph(\Psi_t)$

**Definition 116.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 117.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_0^{squig}$  is the union of the following three components
  - (i)  $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \leq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  - (ii)  $\{(x, \frac{1}{2}) \in D_{r=2} \mid x \geq 0, \frac{1}{2} \geq \Psi_0(x)\}$
  - (iii)  $\{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations  $\xi_0^{symbol}$  of  $\lambda_0^{symbol}$  as follows

- $\xi_0^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_0^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_0^{squig}$ : the hairs of the components *i*, *ii* are pointing downward direction i.e. coefficients of  $dz$  are negative and the hairs of the component *iii* are pointing towards the right.

**Definition 118.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = -\frac{1}{2}\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_1^{squig}$  is the union of the following three components
  - (i)  $\{(x, \frac{1}{2}) \in D_{r=2}\}$
  - (ii)  $\{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows

- $\xi_1^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.

- $\xi_1^{squig}$ :

- for (i), hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- for (ii), hairs are pointing towards the right i.e. coefficients of  $dx$  are positive.

**Definition 119.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
- $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = 0\}$
- $\lambda_\bullet^{squig}$  is the union of the following two components
  - (i)  $\{(x, \frac{1}{2}, t) \in D_{r=2} \times [0, 1] \mid \frac{1}{2} > \Psi_t(x)\}$
  - (ii)  $\{(0, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$

2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows

- $\xi_\bullet^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_\bullet^{squig}$ :
  - for (i), hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - for (ii), hairs are pointing towards the right i.e. coefficients of  $dx$  are positive.

**Definition 120.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines

that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 121.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 122.** For  $i = 1, 2, 3, 4$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ ,

1. we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3, sgn_4) := \{ (x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\ sgn(z - \frac{1}{2}) = sgn_3, \\ sgn(-x) = sgn_4 \} \end{aligned}$$

2. we use \* as a wild card sign i.e.

$$s_{t_0}(\cdots, \underset{i^{th}}{\overset{*}{\uparrow}}, \cdots) := s_{t_0}(\cdots, \underset{i^{th}}{\overset{-}{\uparrow}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\overset{0}{\uparrow}}, \cdots) \cup s_{t_0}(\cdots, \underset{i^{th}}{\overset{+}{\uparrow}}, \cdots)$$

3. we omit trailing \*'s e.g.  $s_0(+, -) = s_0(+, -, *, *)$
4. note that we do not omit \*'s located in between  $-$ ,  $0$ ,  $+$  e.g.  $s_0(+, -, *, -) \neq s_0(+, -, -)$

**Definition 123.** For  $i = 0, 1, 2, 3, 4$  and  $sgn_i \in \{-, 0, +\}$ ,

1. we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(-z) = sgn_2, \\ & sgn(z - \frac{1}{2}) = sgn_3, \\ & sgn(-x) = sgn_4\} \end{aligned}$$

2. we use \* as a wild card sign i.e.

$$s_{\bullet}(\cdots, \underset{i^{th}}{\overset{*}{\uparrow}}, \cdots) := s_{\bullet}(\cdots, \underset{i^{th}}{\overset{-}{\uparrow}}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{\overset{0}{\uparrow}}, \cdots) \cup s_{\bullet}(\cdots, \underset{i^{th}}{\overset{+}{\uparrow}}, \cdots)$$

3. we omit trailing \*'s e.g.  $s_{\bullet}(+,-) = s_{\bullet}(+,-,*,*)$
4. note that we do not omit \*'s located in between  $-$ ,  $0$ ,  $+$  e.g.  $s_{\bullet}(+,-,*,-) \neq s_{\bullet}(+,-,-)$

**Definition 124.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_{\bullet}$  using the above notations:

1.  $\mathcal{S}_0$ :

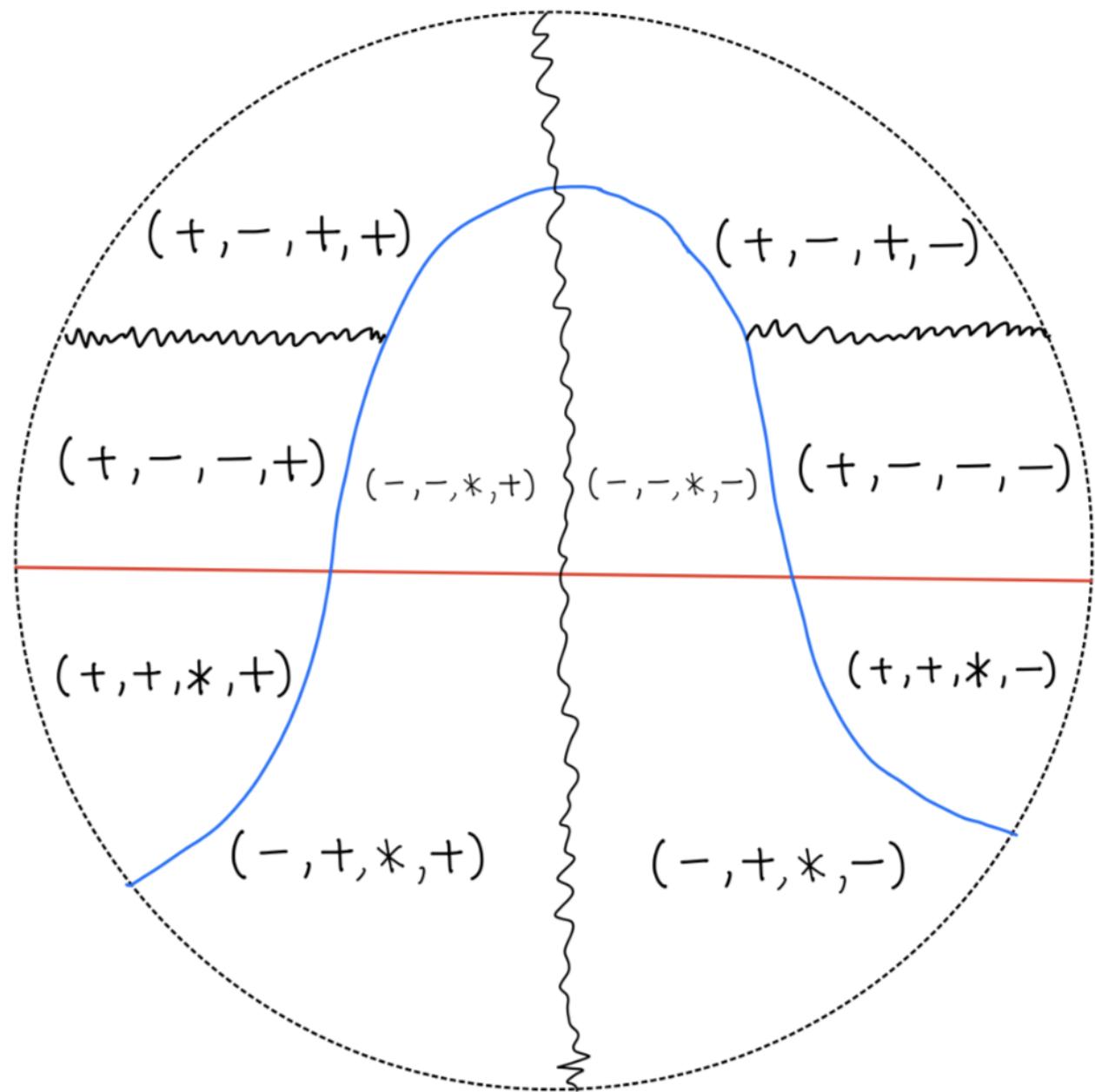


Figure 3.117

- 2 dimensional strata:

$$\begin{aligned}
 & s_0(-, -, *, -), s_0(-, -, *, +), s_0(+, -, -, -), s_0(+, -, -, +), s_0(+, -, +, -), s_0(+, -, +, +), \\
 & s_0(+, +, *, -), s_0(+, +, *, -), s_0(-, +, *, -), s_0(-, +, *, +)
 \end{aligned}$$

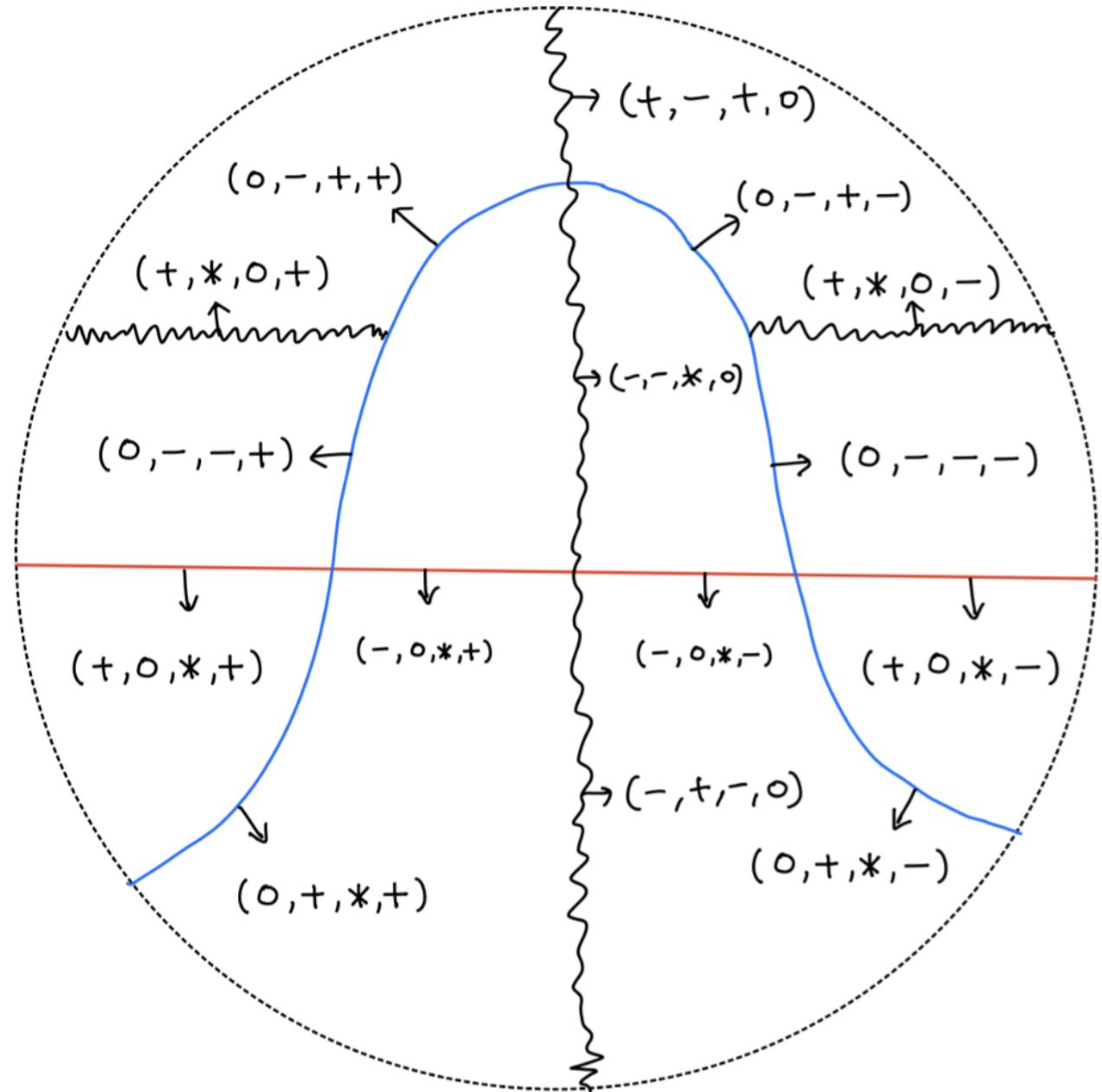


Figure 3.118

- 1 dimensional strata:

$s_0(0, +, *, -), s_0(0, -, -, -), s_0(0, -, +, -), s_0(0, -, +, +), s_0(0, -, -, +),$   
 $s_0(0, +, *, +), s_0(+, 0, *, -), s_0(-, 0, *, -), s_0(-, 0, *, +), s_0(+, 0, *, +), s_0(+, *, 0, -),$   
 $s_0(+, *, 0, +), s_0(+, -, +, 0), s_0(-, -, *, 0), s_0(-, +, -, 0)$

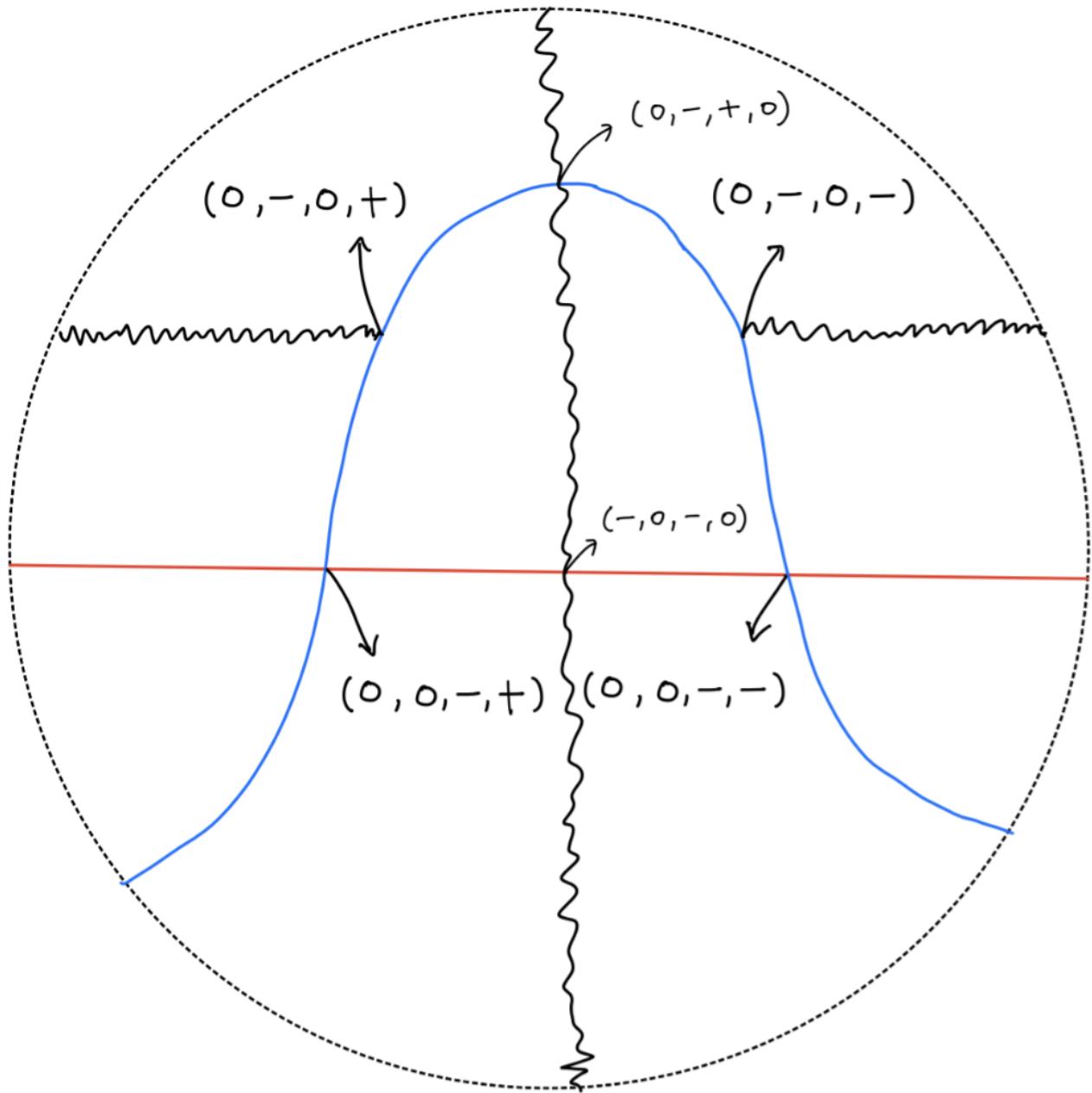


Figure 3.119

- 0 dimensional strata:

$$s_0(0, 0, -, -), s_0(0, -, 0, -), s_0(0, -, 0, +), s_0(0, 0, -, +), s_0(0, -, +, -), s_0(-, 0, -, 0)$$

2.  $\mathcal{S}_1$ :

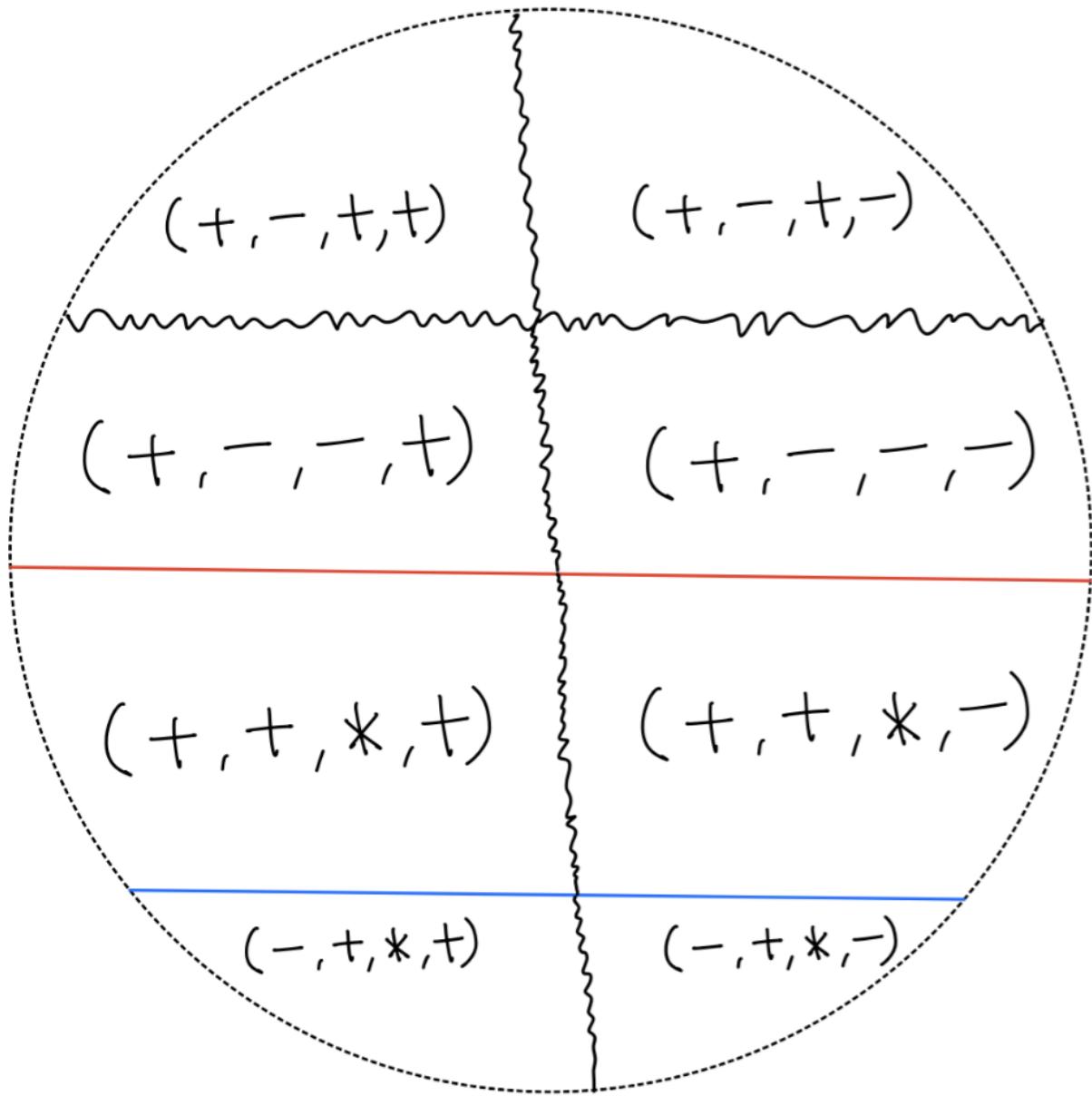


Figure 3.120

- 2 dimensional strata:

$$\begin{aligned}
 & s_1(-, +, *, -), s_1(-, +, *, +), s_1(+, -, -, -), s_1(+, -, -, +), s_1(+, -, +, -), \\
 & s_1(+, -, +, +), s_1(+, +, *, -), s_1(+, +, *, +)
 \end{aligned}$$

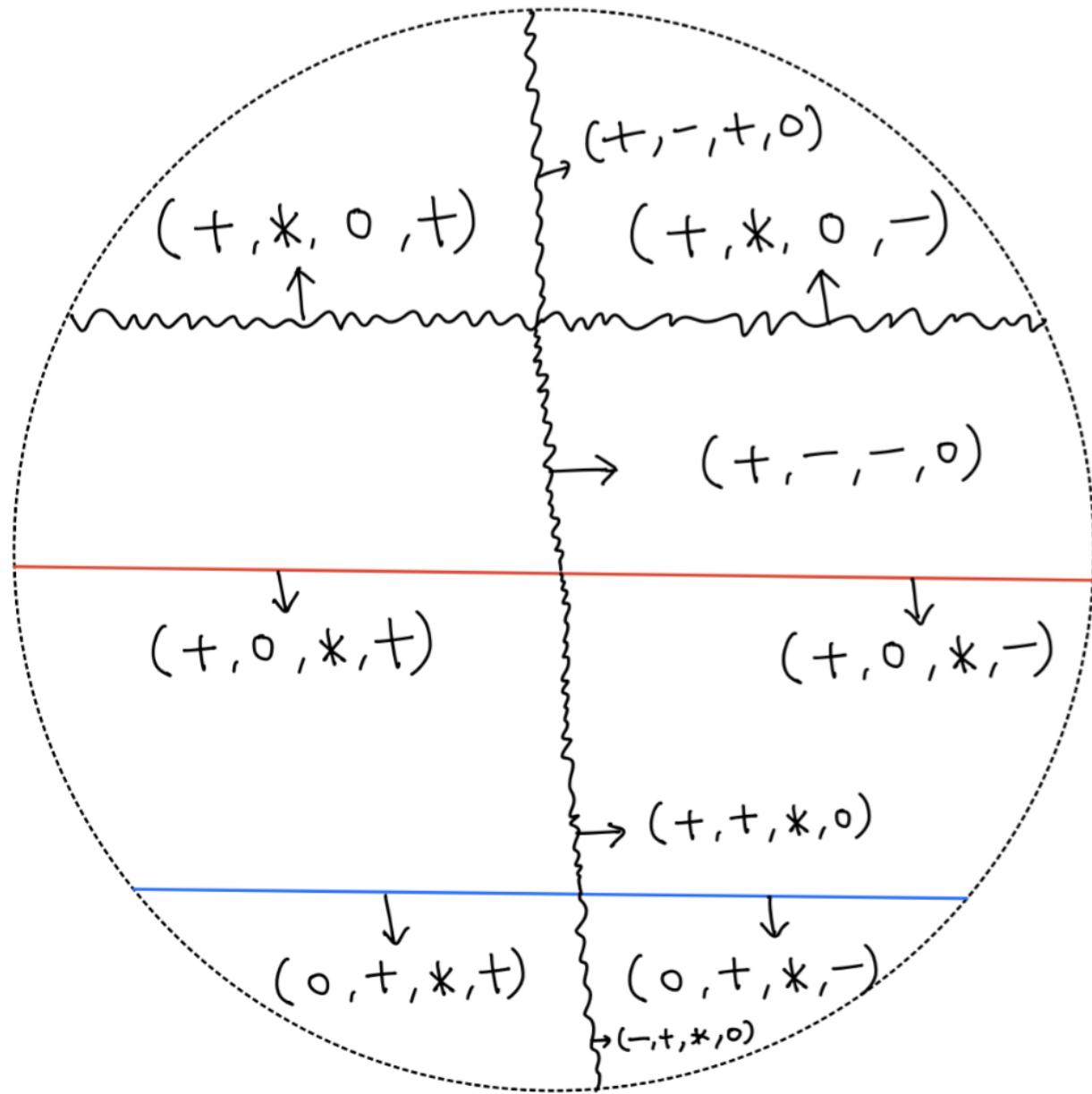


Figure 3.121

- 1 dimensional strata:

$$\begin{aligned}
 & s_1(0, +, *, -), s_1(0, +, *, +), s_1(+, 0, *, -), s_1(+, 0, *, +), s_1(+, *, 0, -), s_1(+, *, 0, +), \\
 & s_1(+, -, -, 0), s_1(+, +, *, 0), s_1(+, -, +, 0), s_1(-, +, *, 0)
 \end{aligned}$$

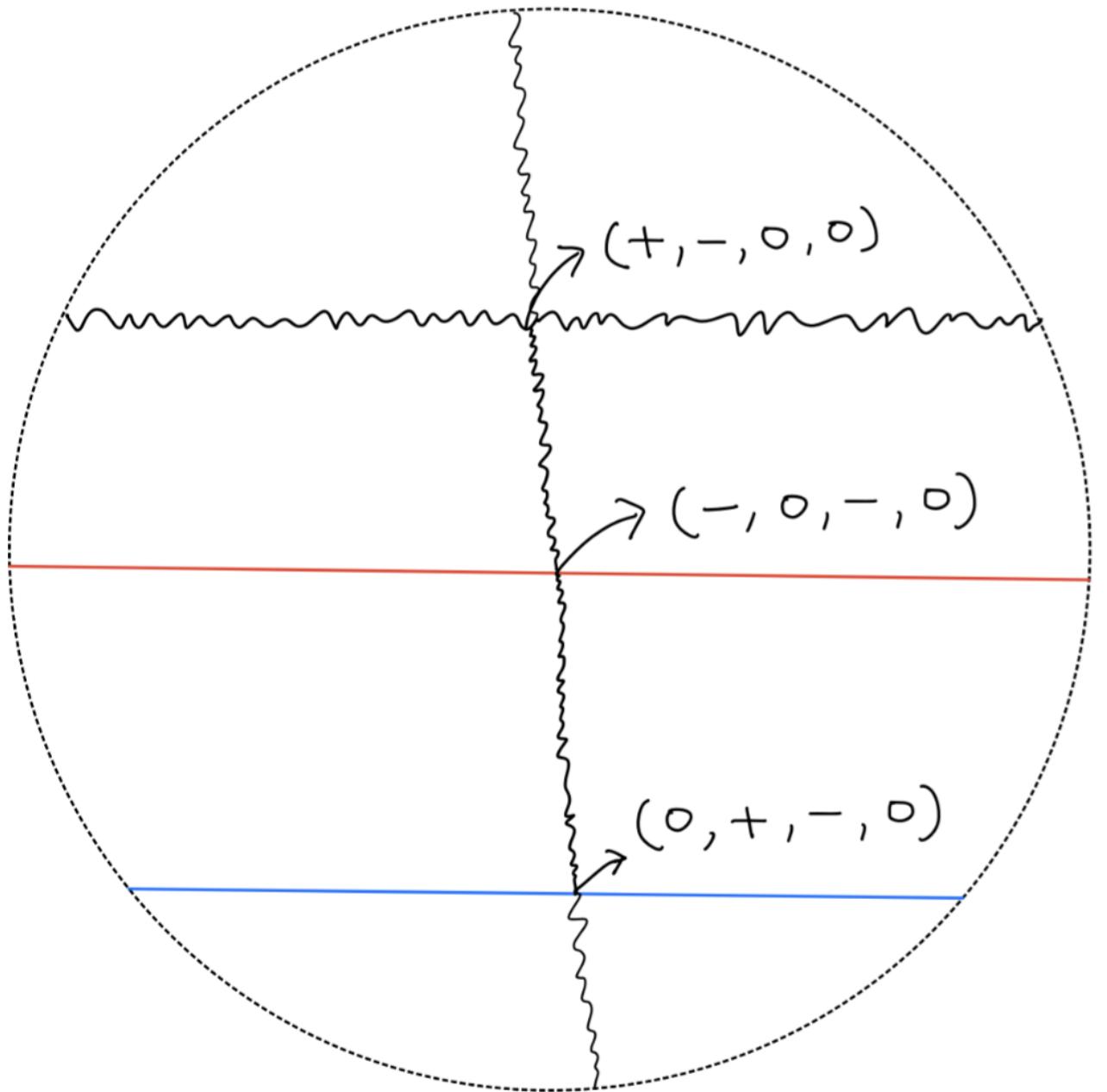


Figure 3.122

- 0 dimensional strata:

$$s_1(+, -, 0, 0), s_1(-, 0, -, 0), s_1(0, +, -, 0)$$

3.  $\mathcal{S}_\bullet$ :

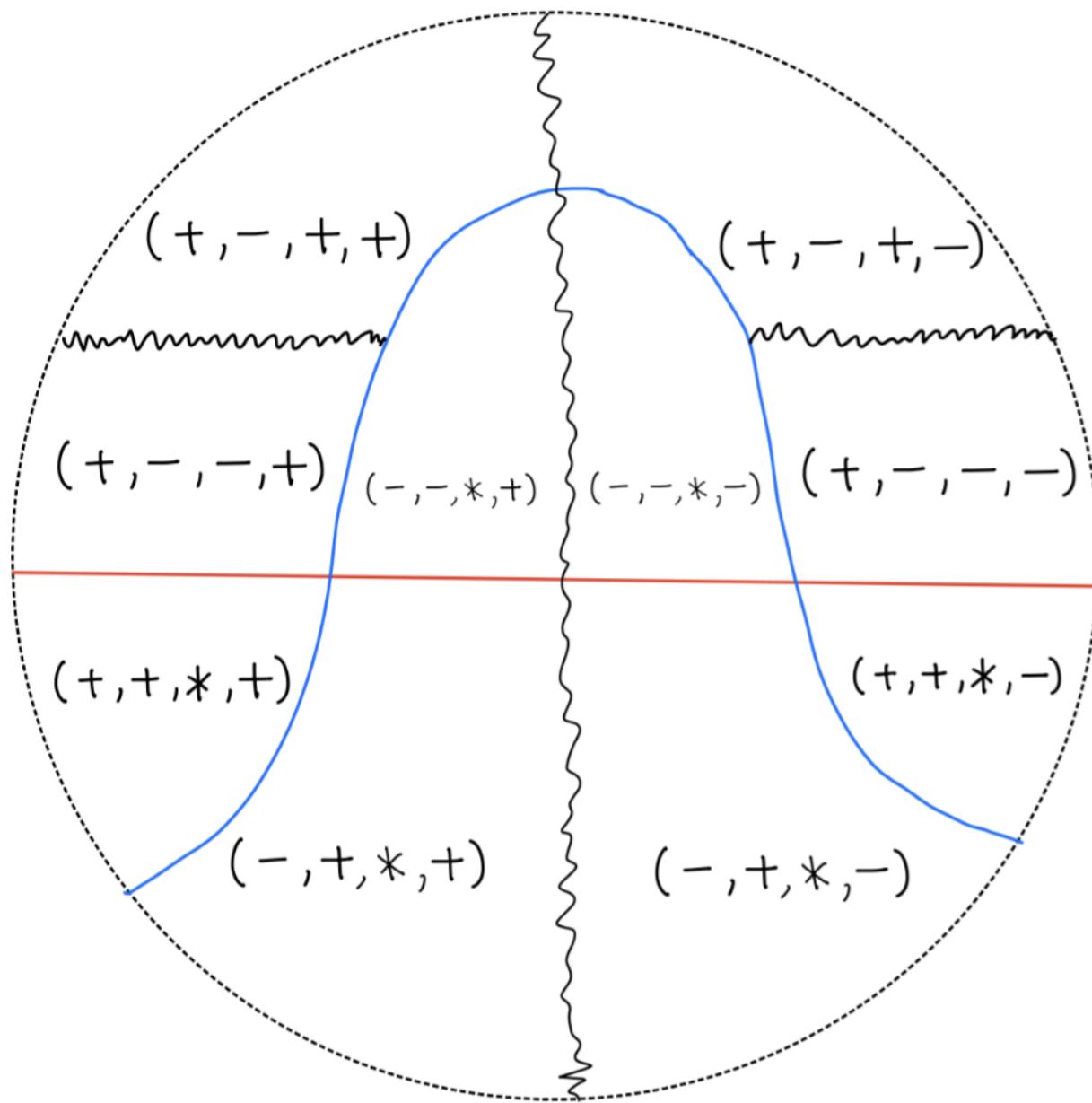


Figure 3.123

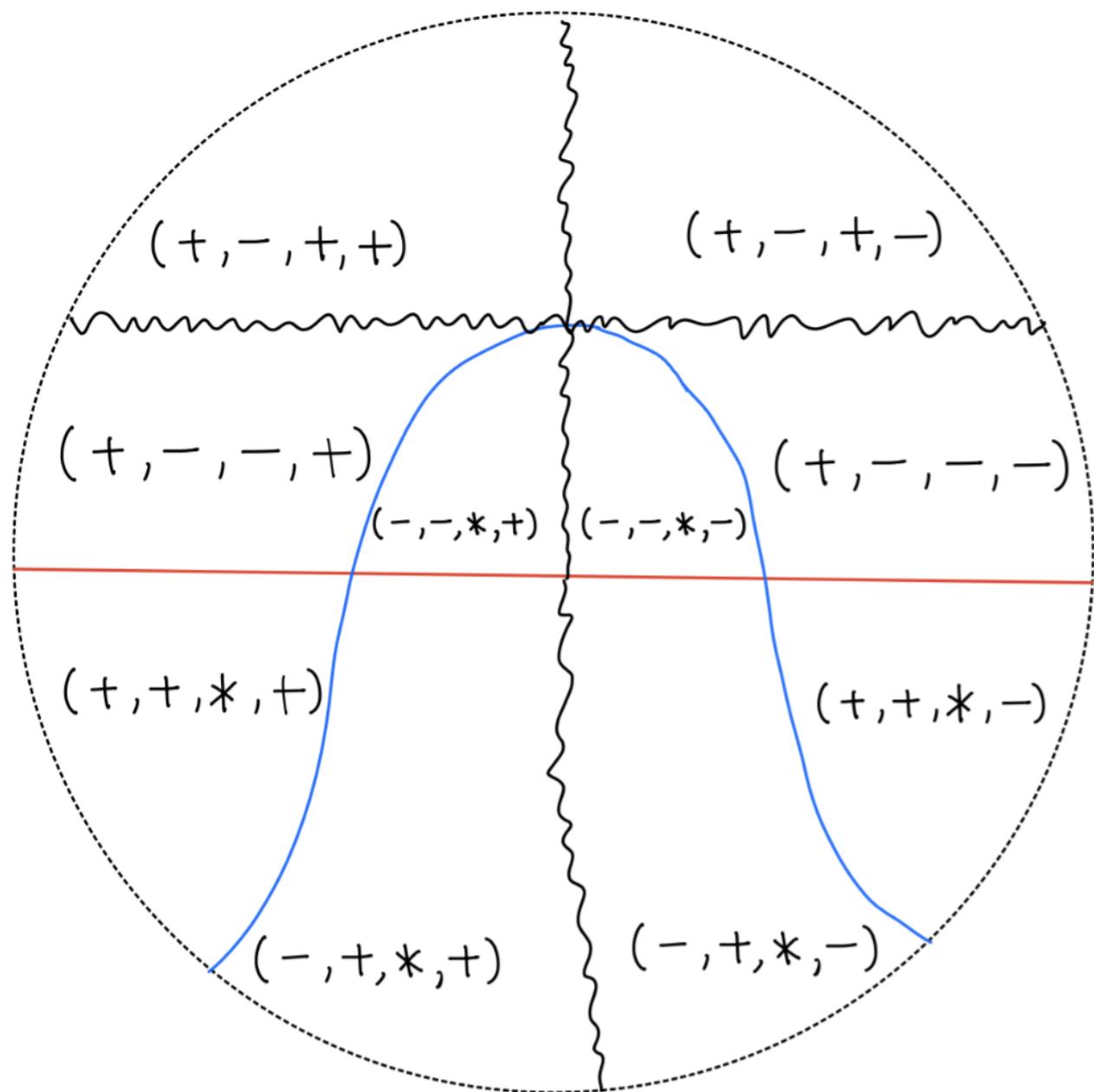


Figure 3.124

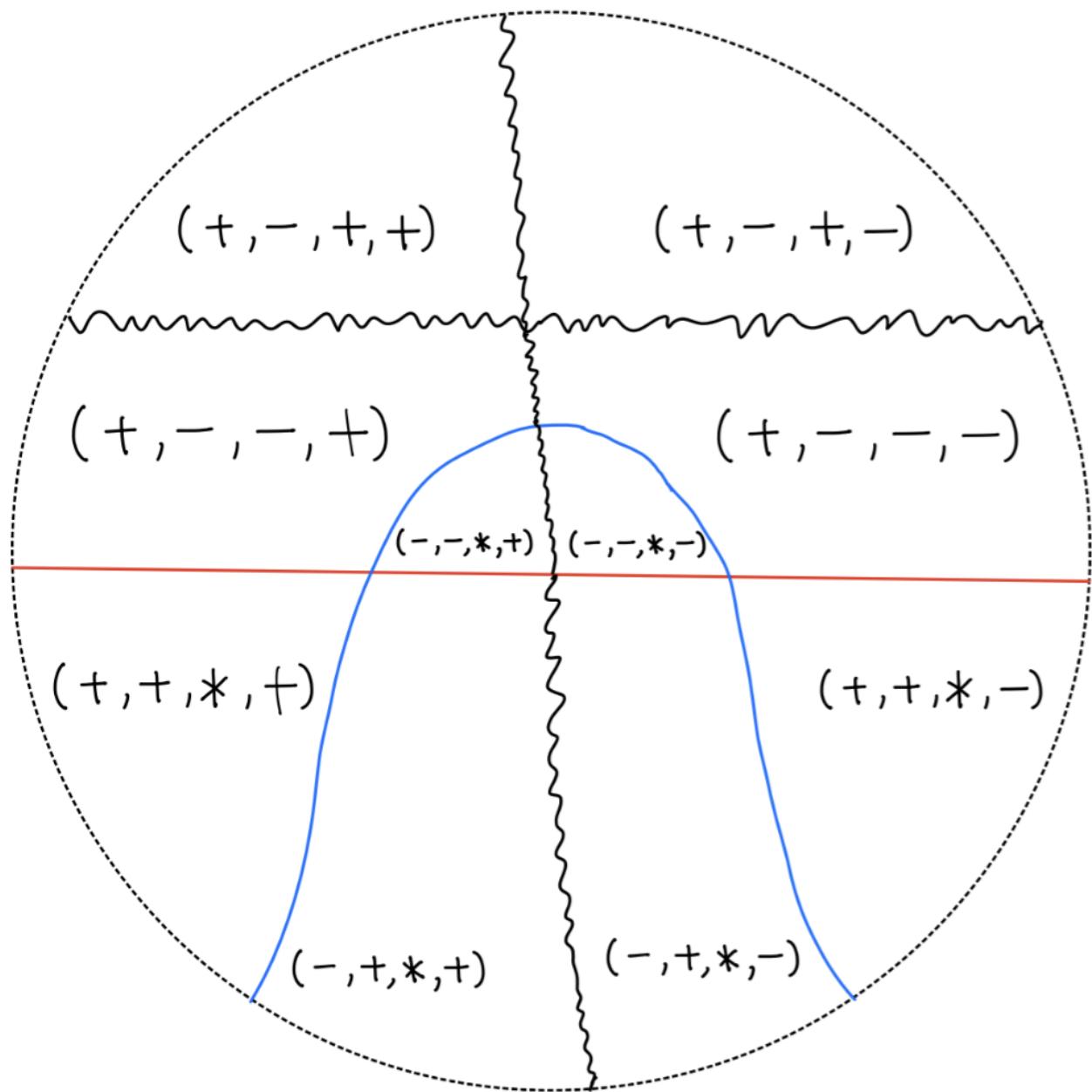


Figure 3.125

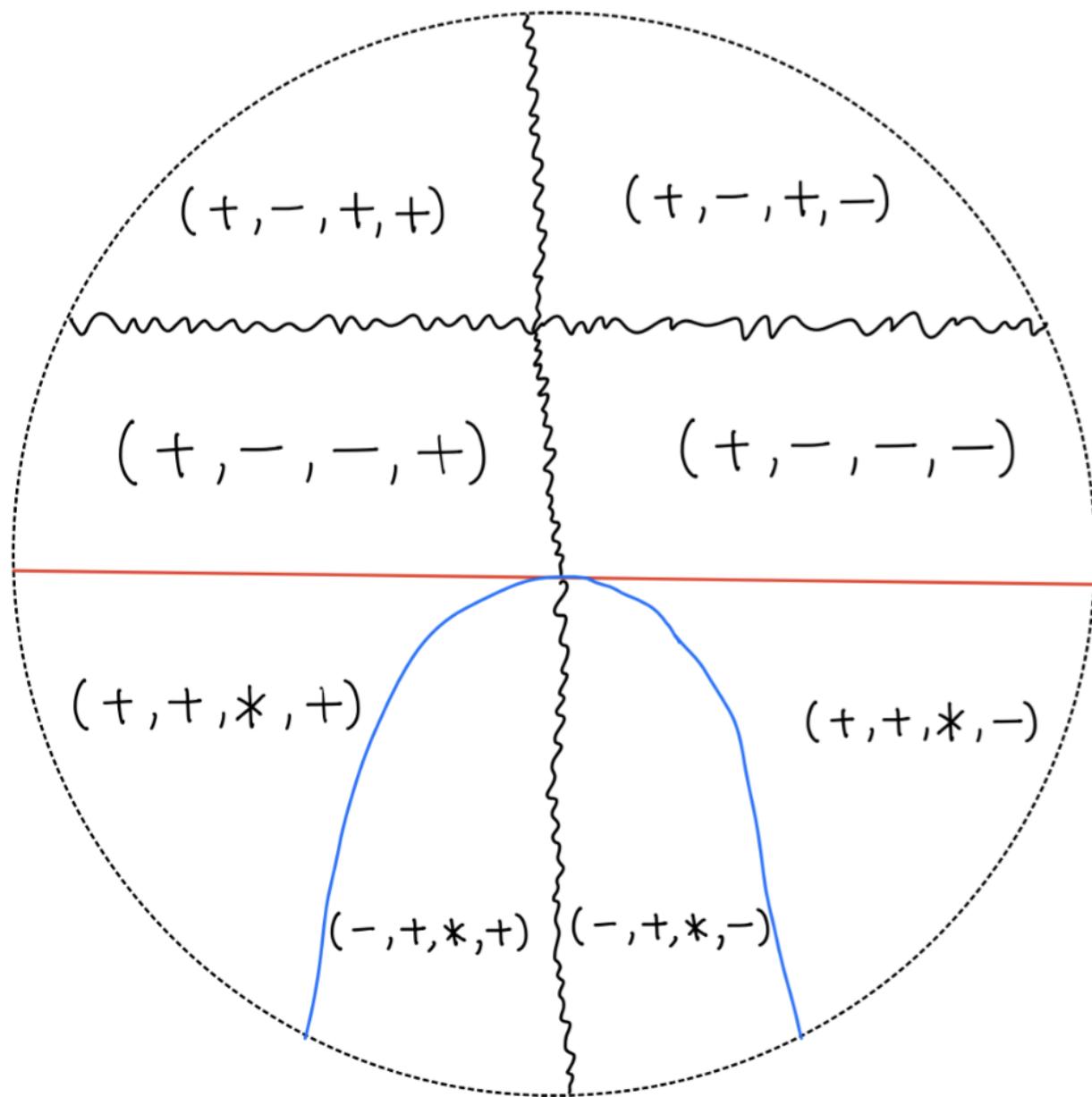


Figure 3.126

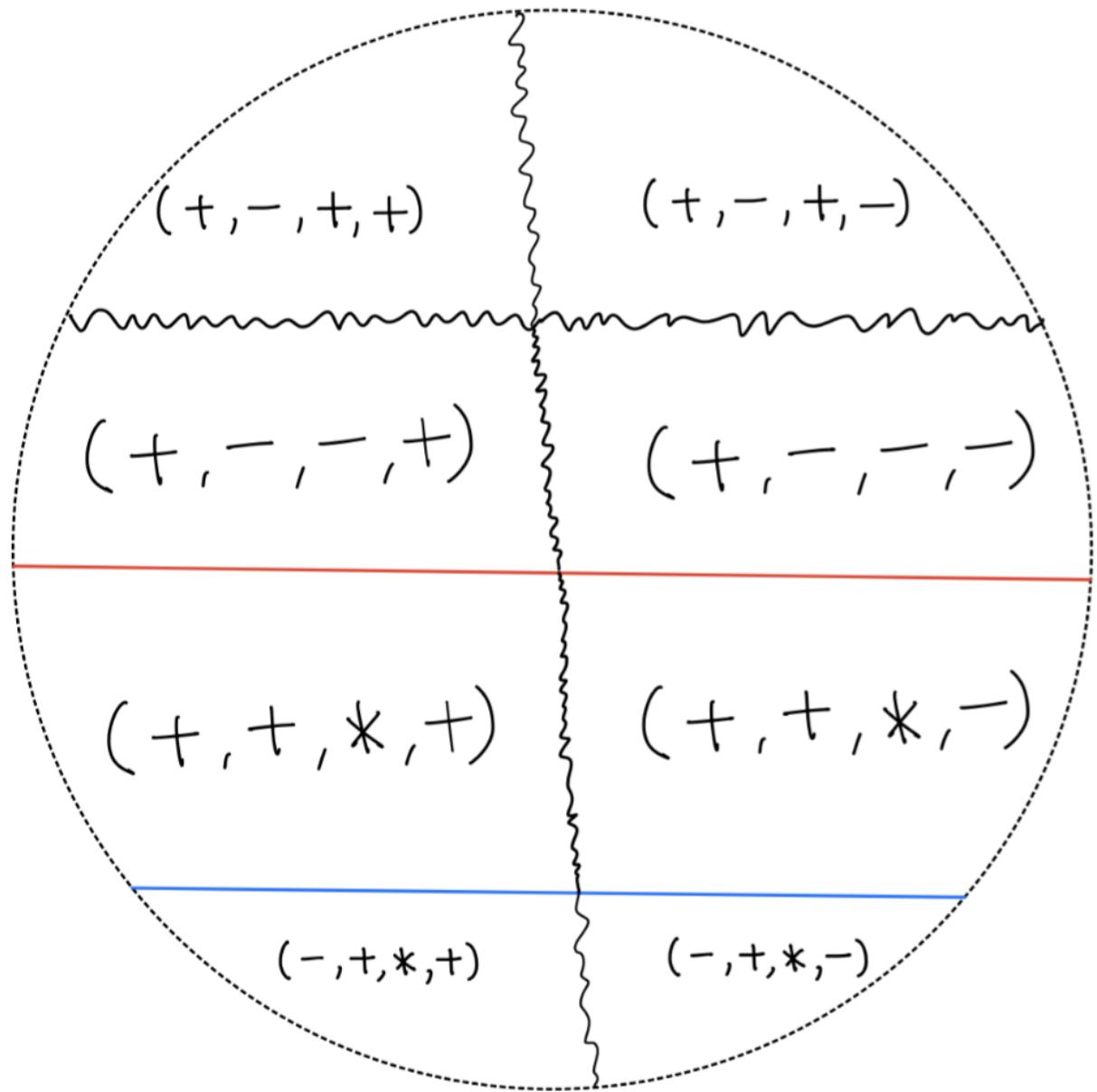


Figure 3.127

- 3 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(-, +, *, -), s_{\bullet}(-, +, *, +), s_{\bullet}(-, -, *, -), s_{\bullet}(-, -, *, +), s_{\bullet}(-, +), s_{\bullet}(+,-,-,-), \\
 & s_{\bullet}(+,-,-,+), s_{\bullet}(+,-,+,-), s_{\bullet}(+,-,+,+), s_{\bullet}(+,-,*,-), s_{\bullet}(+,-,*,+)
 \end{aligned}$$

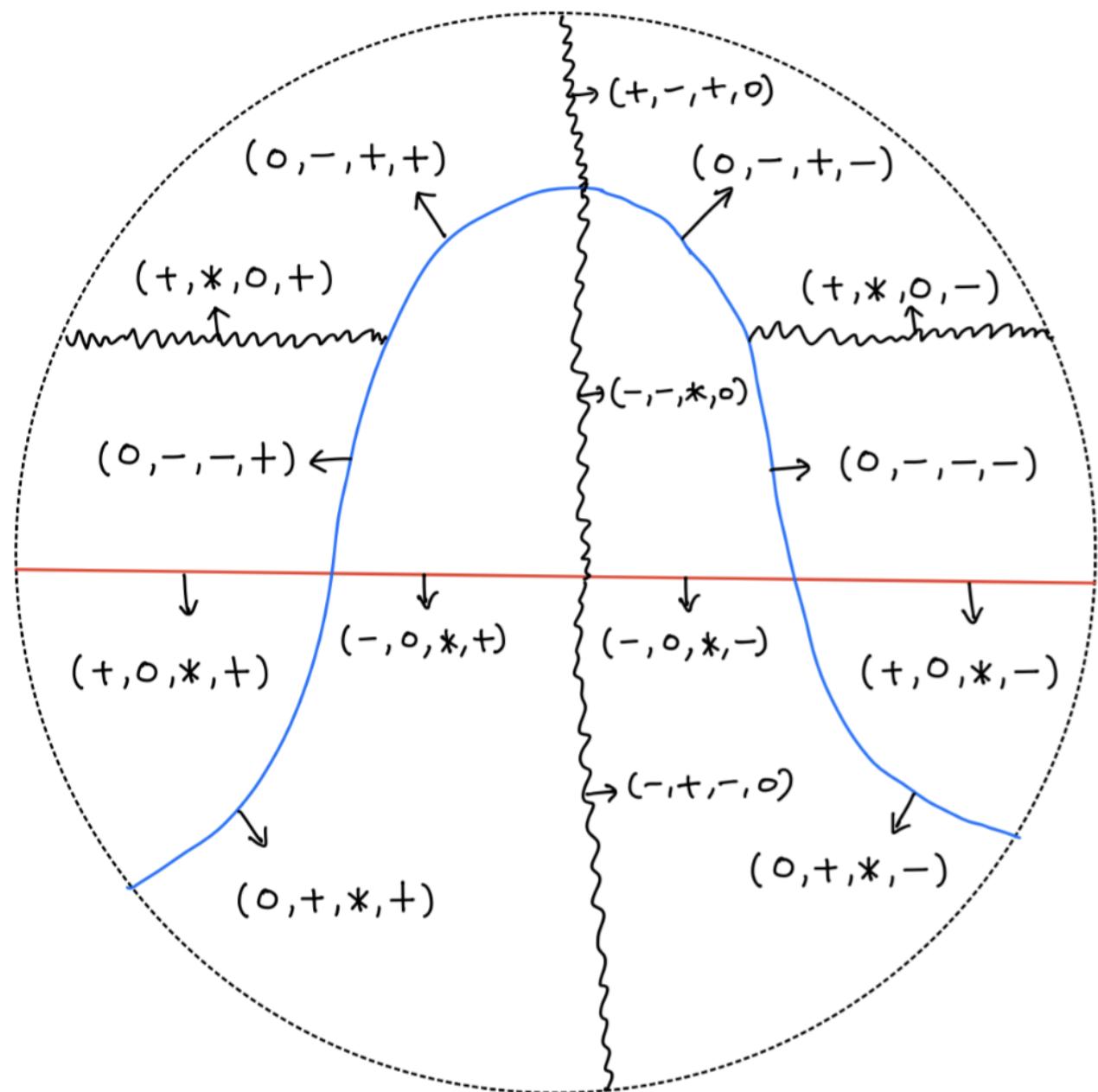


Figure 3.128

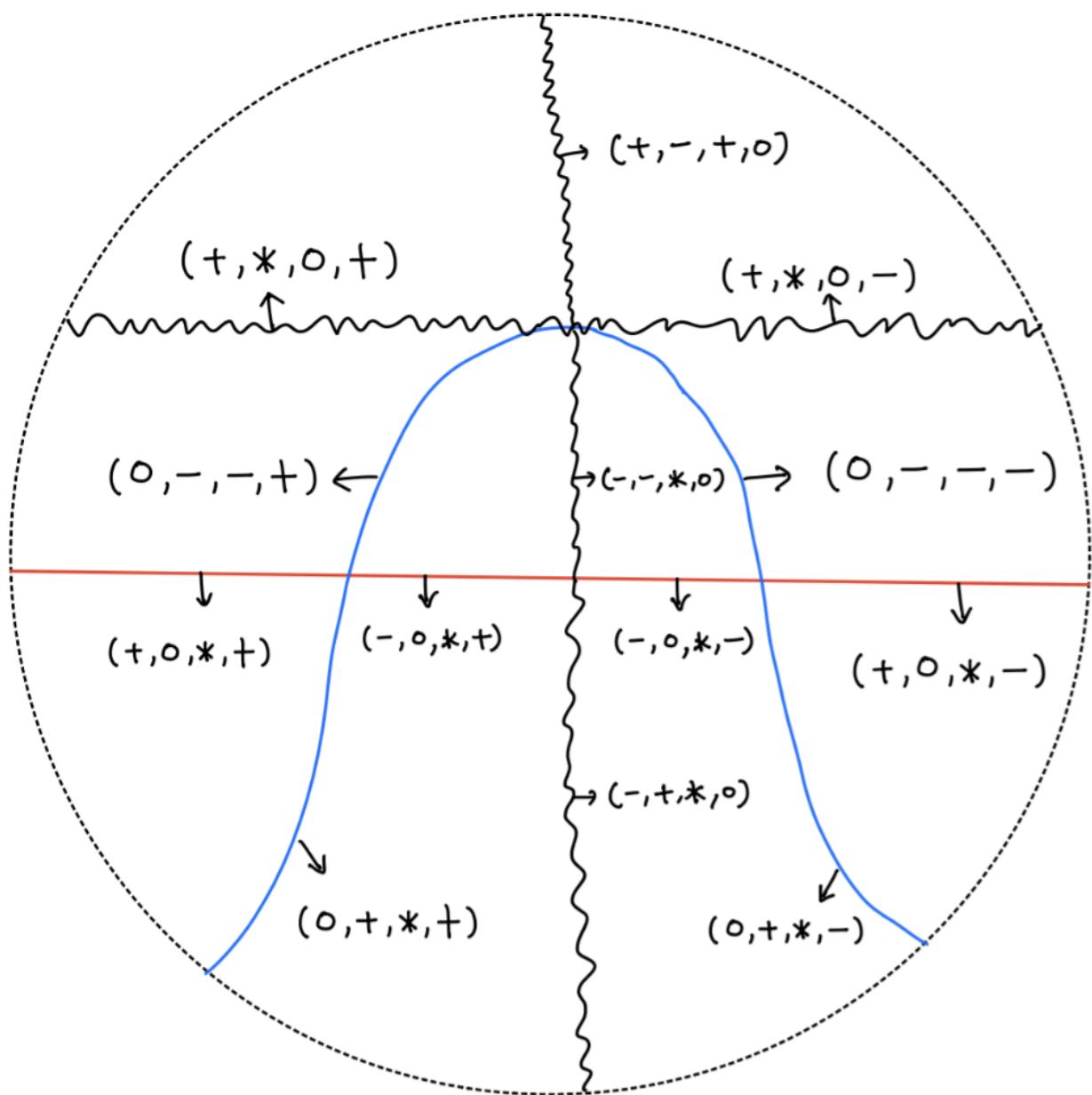


Figure 3.129

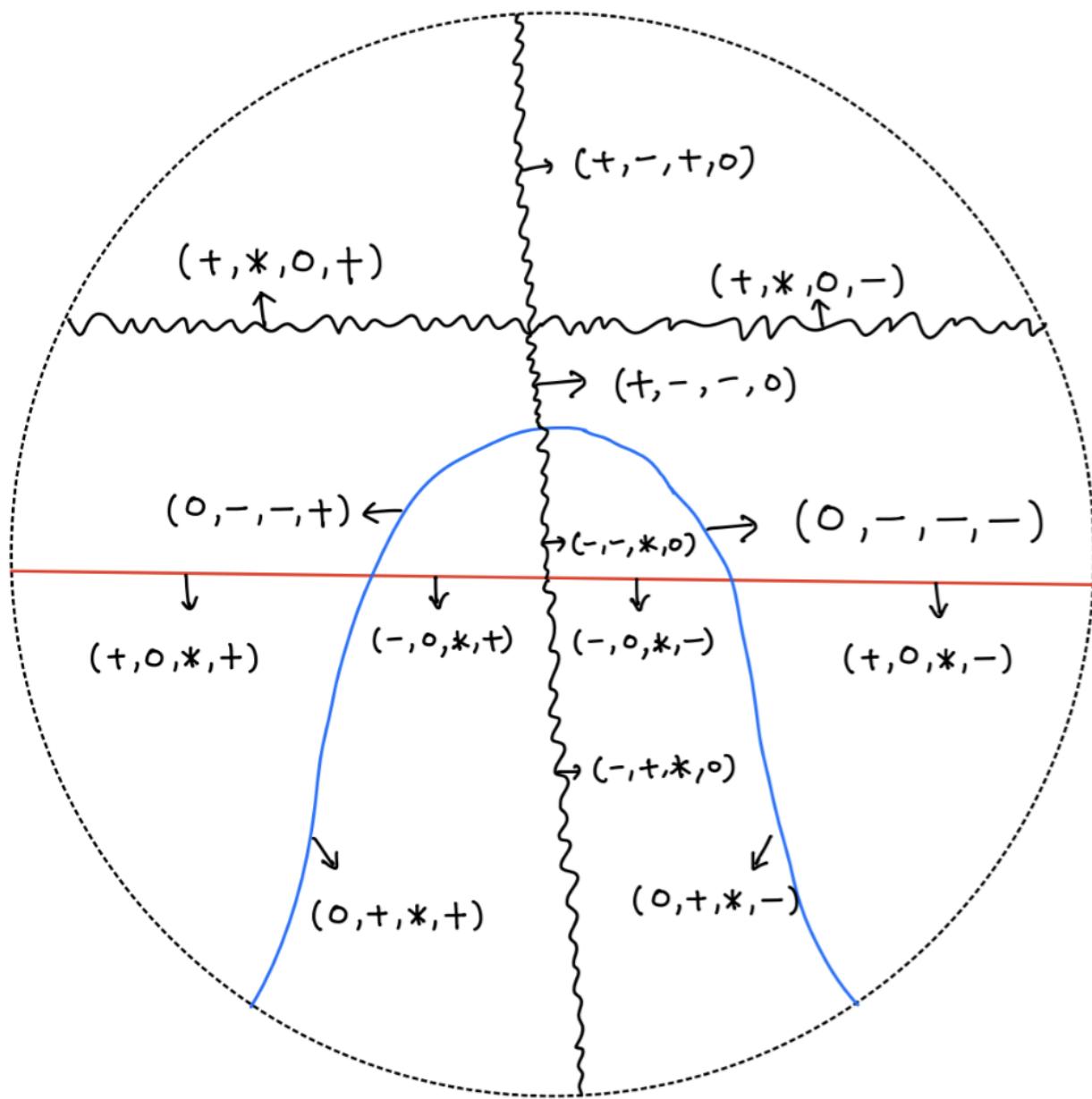


Figure 3.130

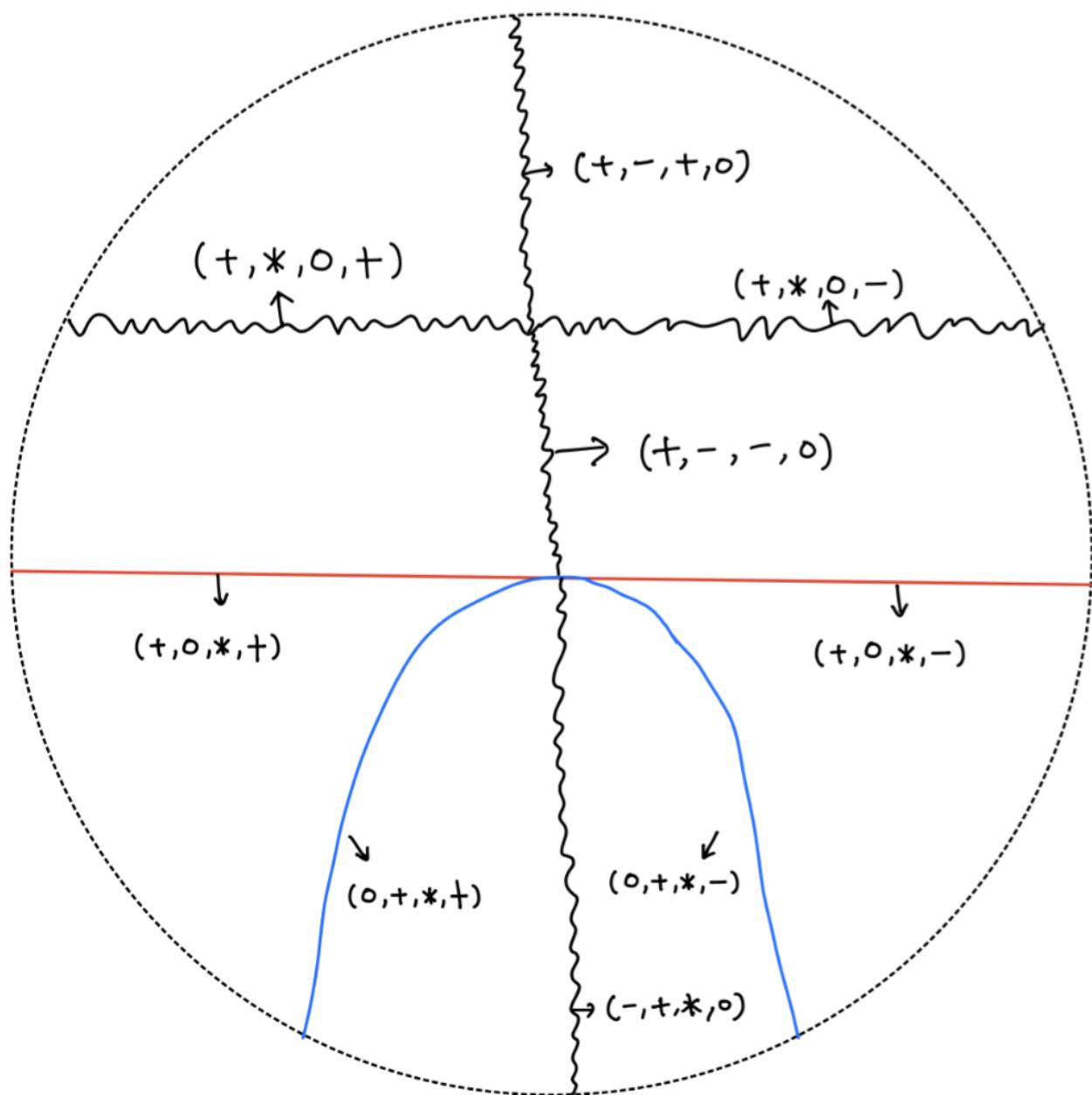


Figure 3.131

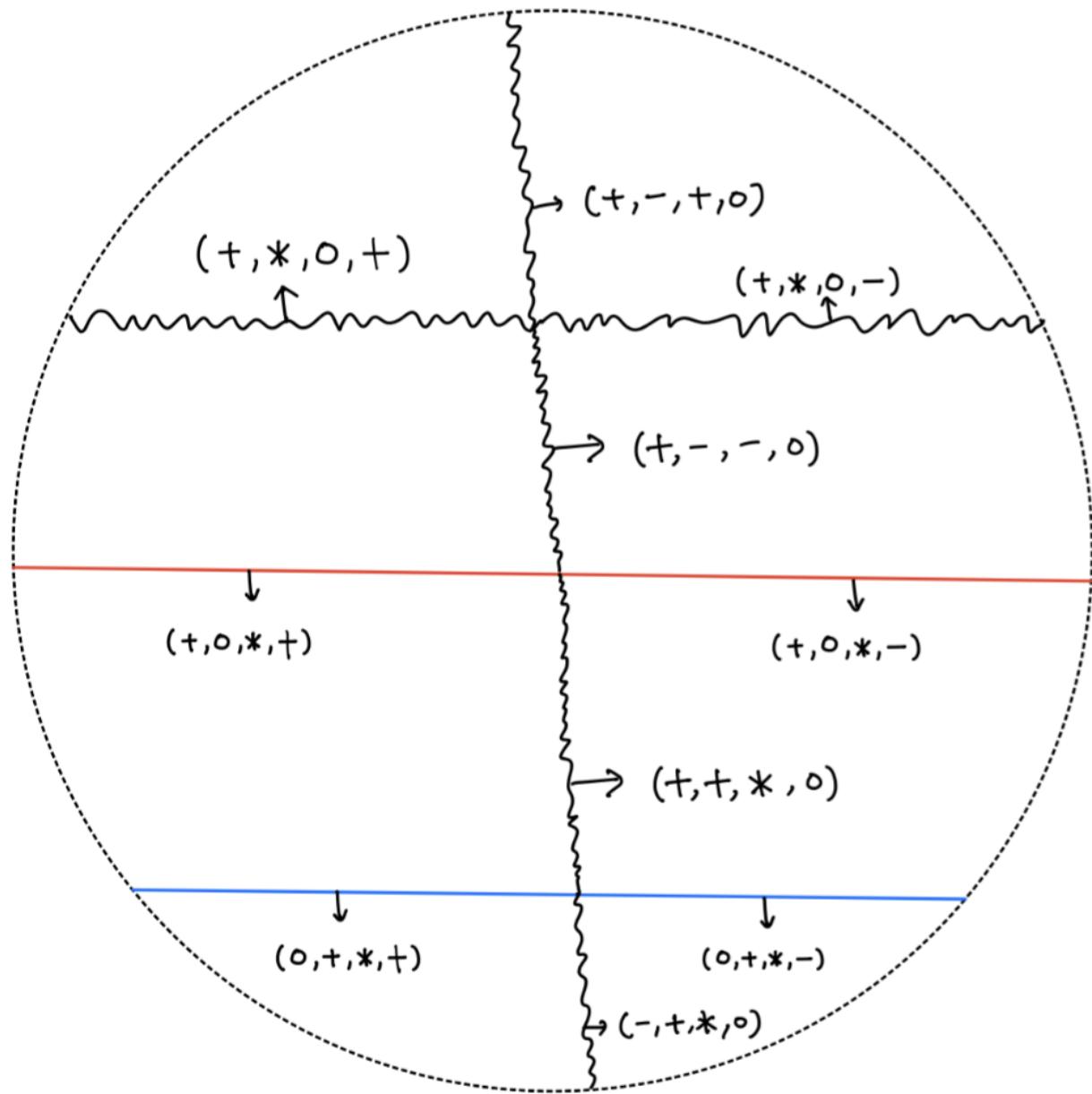


Figure 3.132

- 2 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, +, *, -), s_{\bullet}(0, -, -, -), s_{\bullet}(0, -, +), s_{\bullet}(0, -, +, -), s_{\bullet}(0, -, +, +), s_{\bullet}(0, -, -, +), \\
 & s_{\bullet}(0, +, *, +), s_{\bullet}(+, 0, *, -), s_{\bullet}(-, 0, *, -), s_{\bullet}(-, 0, *, +), s_{\bullet}(+, 0, *, +), s_{\bullet}(+, *, 0, -), \\
 & s_{\bullet}(+, *, 0, +), s_{\bullet}(+, -, -, 0), s_{\bullet}(+, +, *, 0), s_{\bullet}(+, -, +, 0), s_{\bullet}(-, +, *, 0), s_{\bullet}(-, -, *, 0)
 \end{aligned}$$

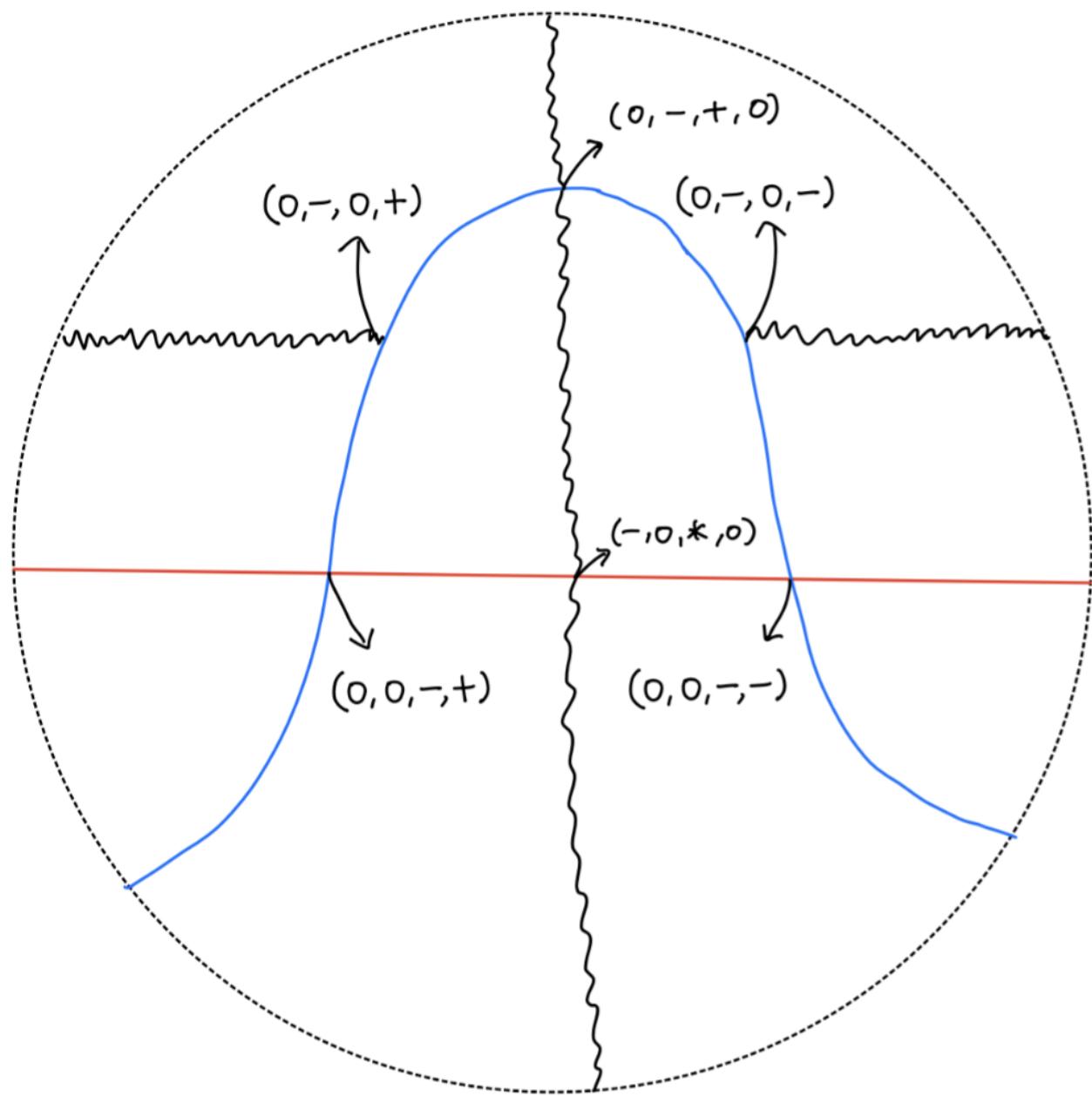


Figure 3.133

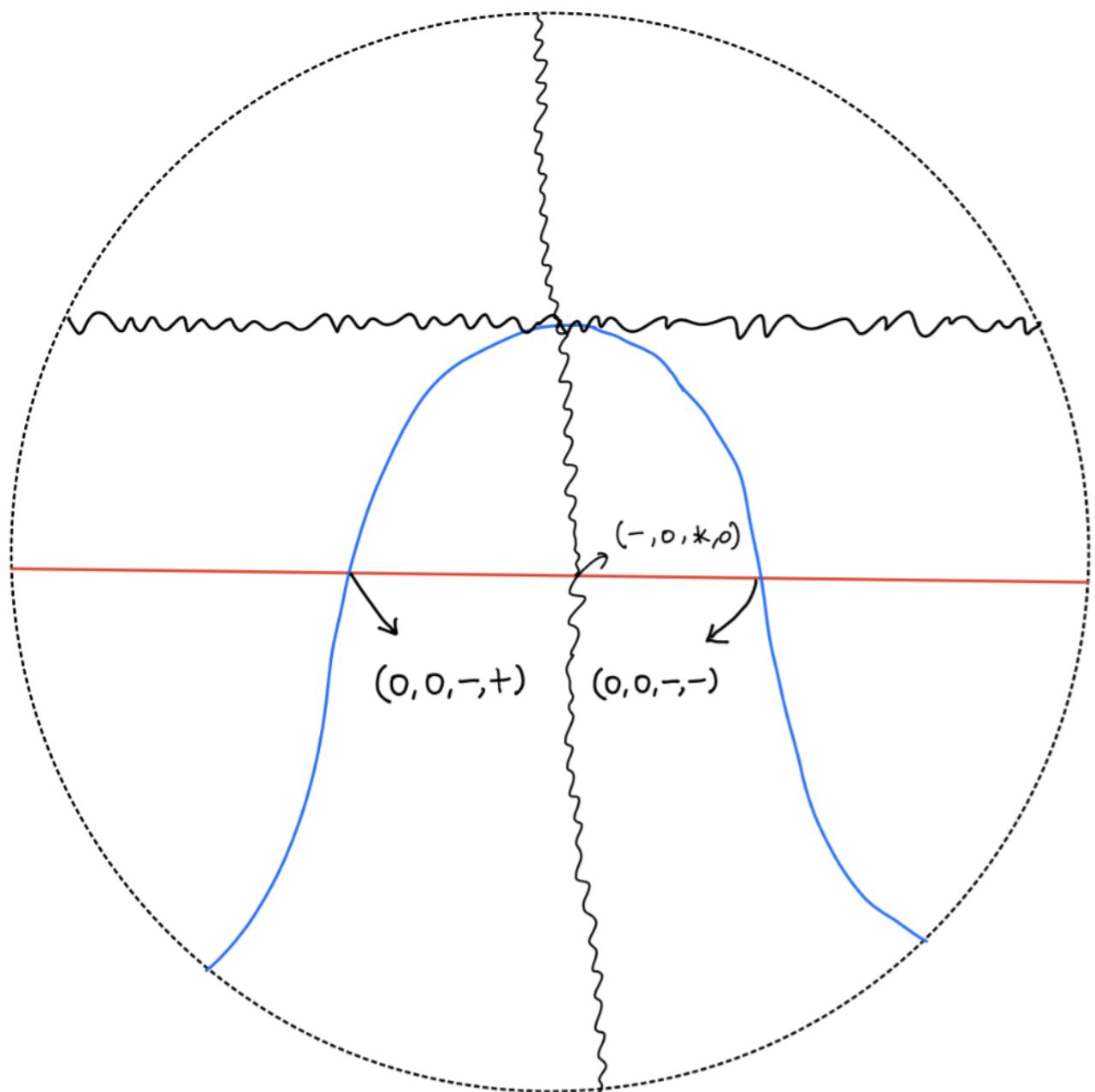


Figure 3.134

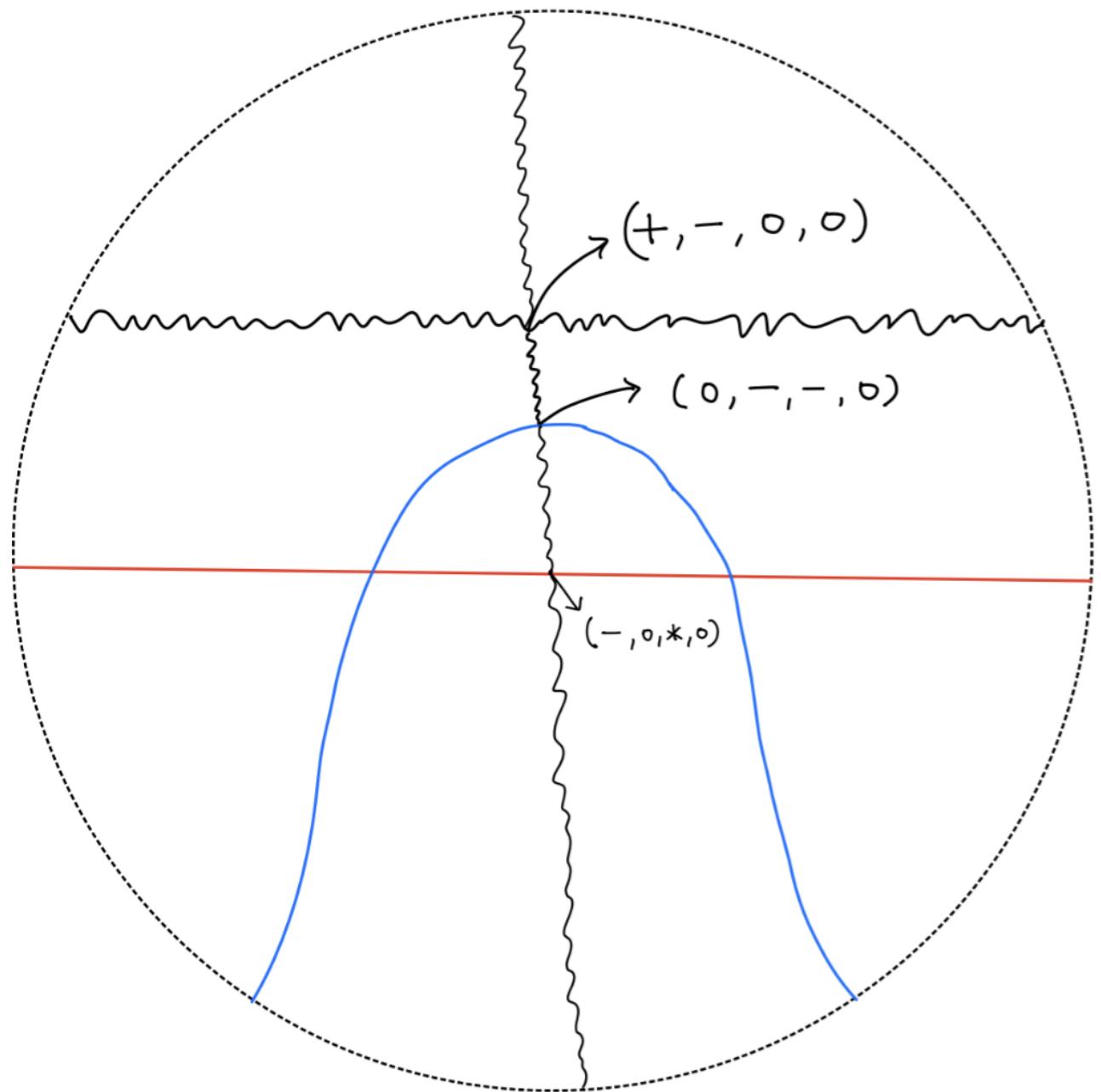


Figure 3.135

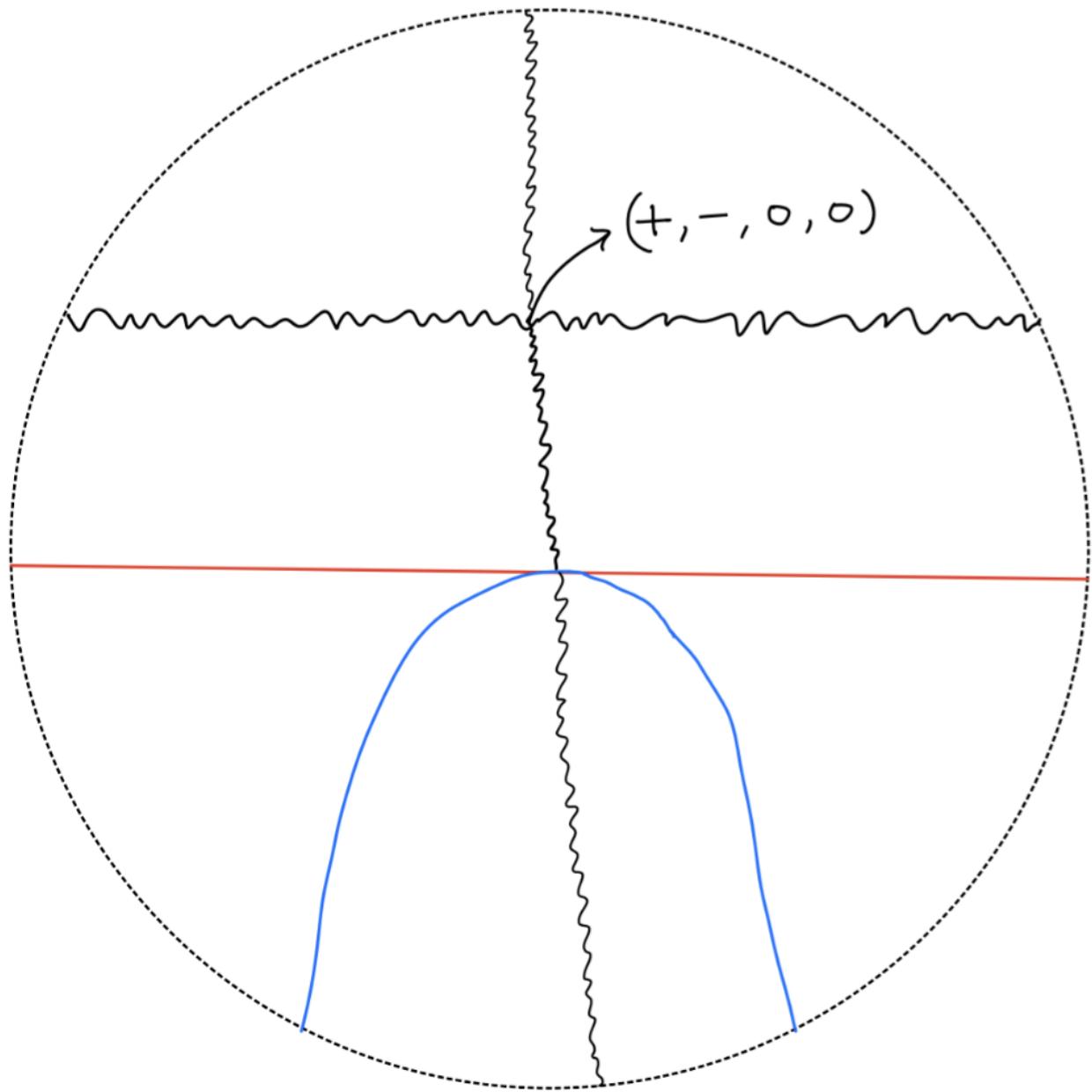


Figure 3.136

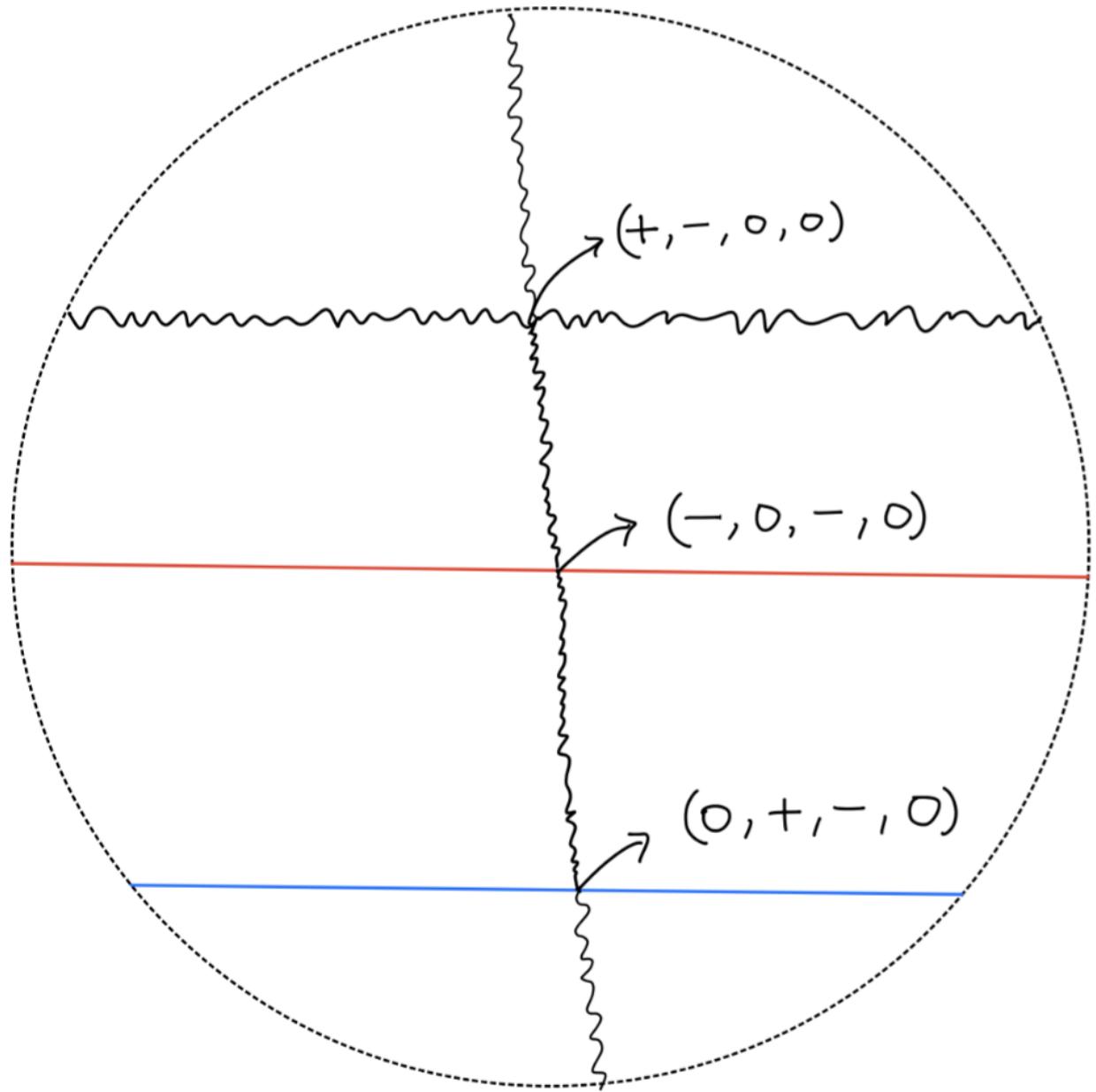


Figure 3.137

- 1 dimensional strata:

$$\begin{aligned}
 & s_{\bullet}(0, 0, -, -), s_{\bullet}(0, -, 0, -), s_{\bullet}(0, -, 0, +), s_{\bullet}(0, 0, -, +), s_{\bullet}(+, -, 0, 0), s_{\bullet}(0, -, -, 0), \\
 & s_{\bullet}(-, 0, -, 0), s_{\bullet}(0, +, -, 0), s_{\bullet}(0, -, +, 0), s_{\bullet}(-, 0, *, 0)
 \end{aligned}$$

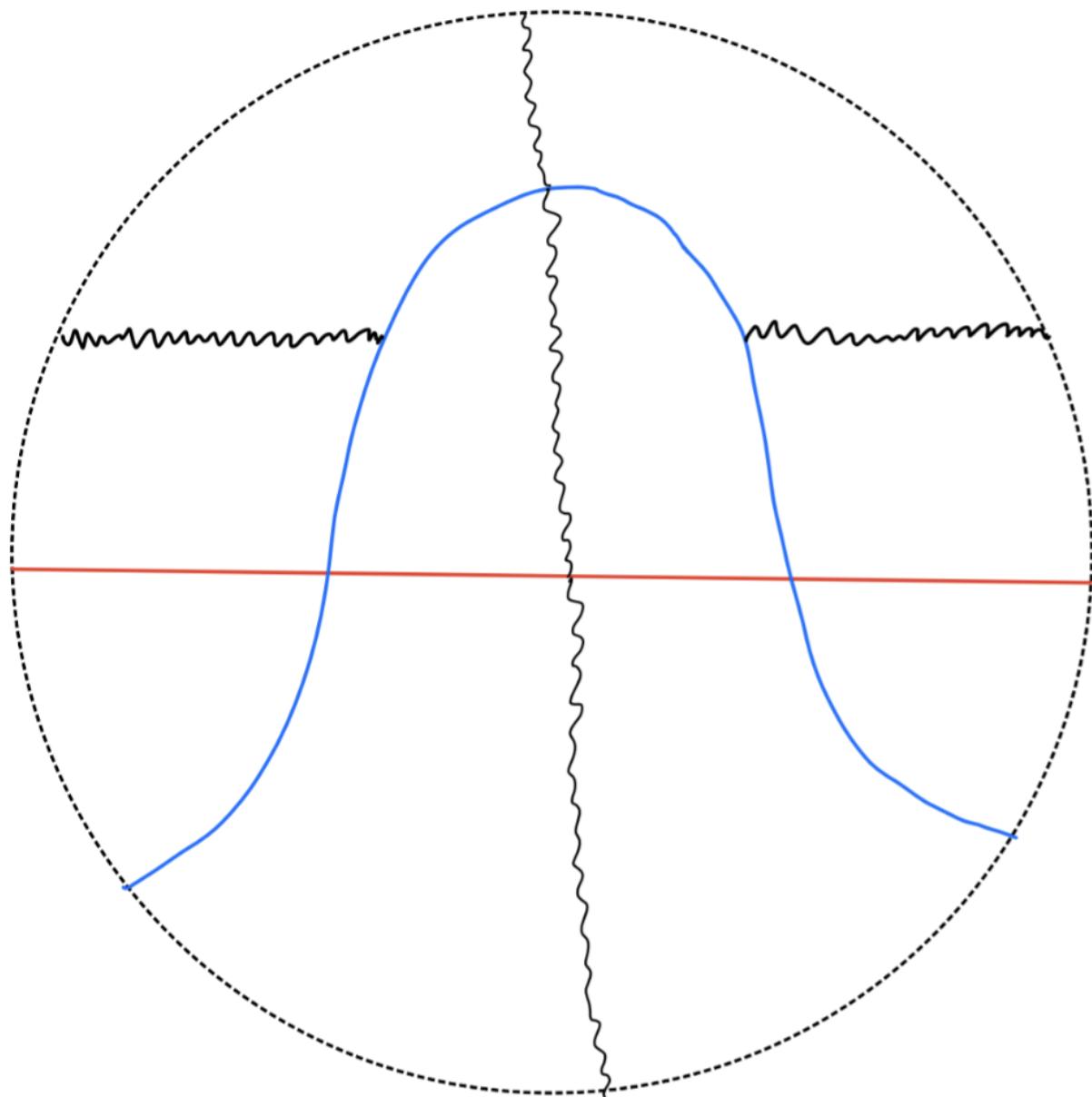


Figure 3.138

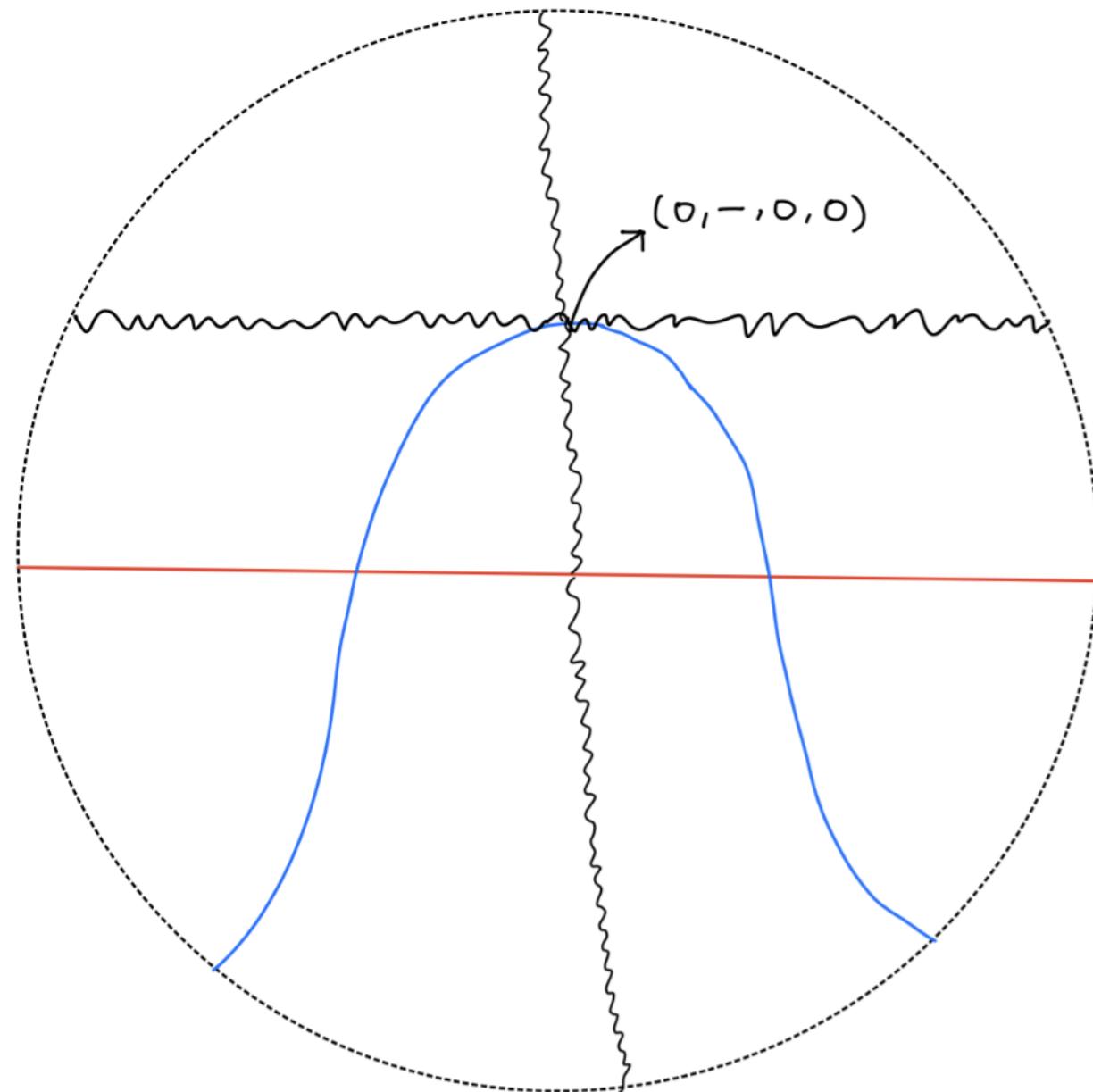


Figure 3.139

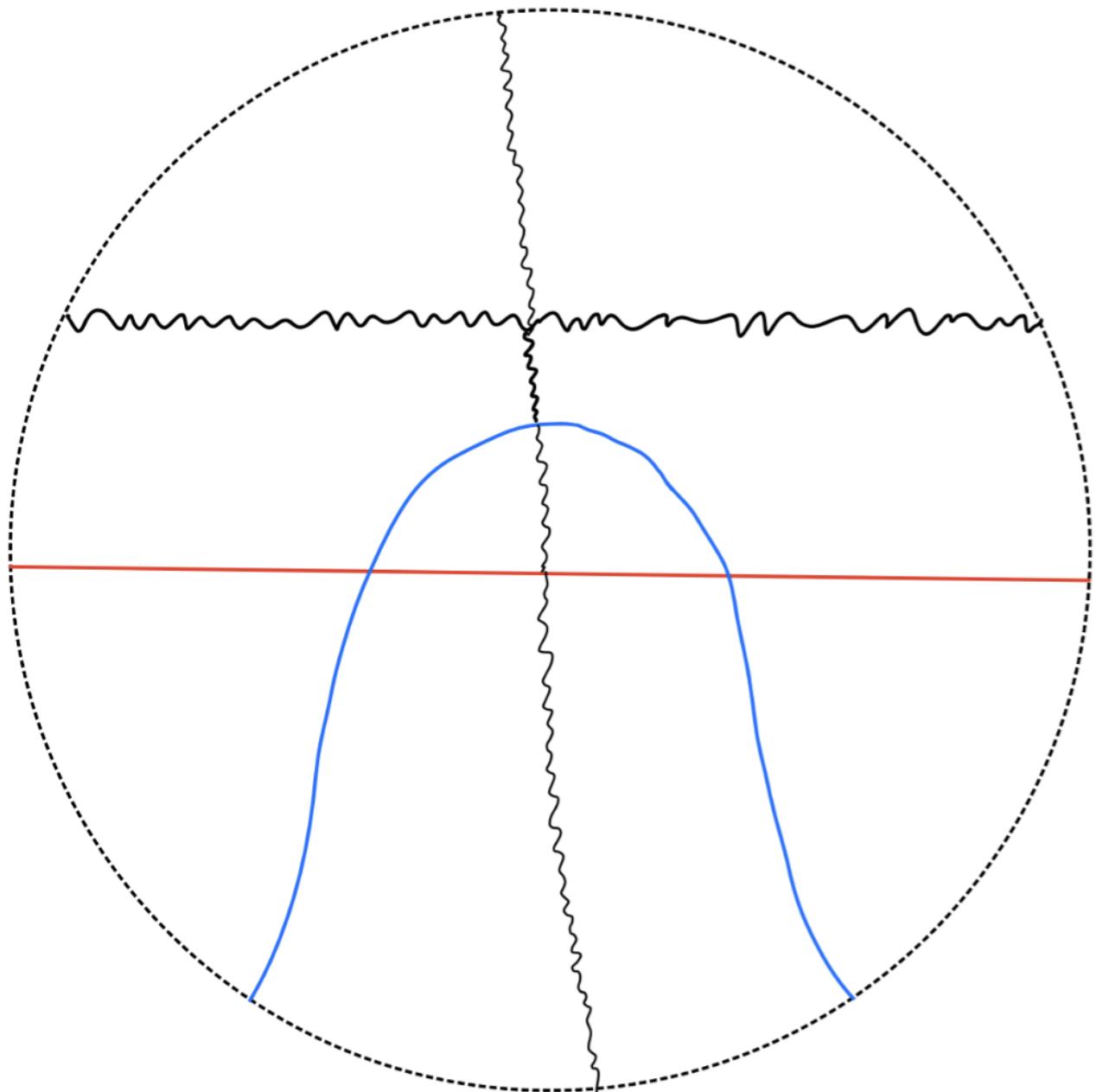


Figure 3.140

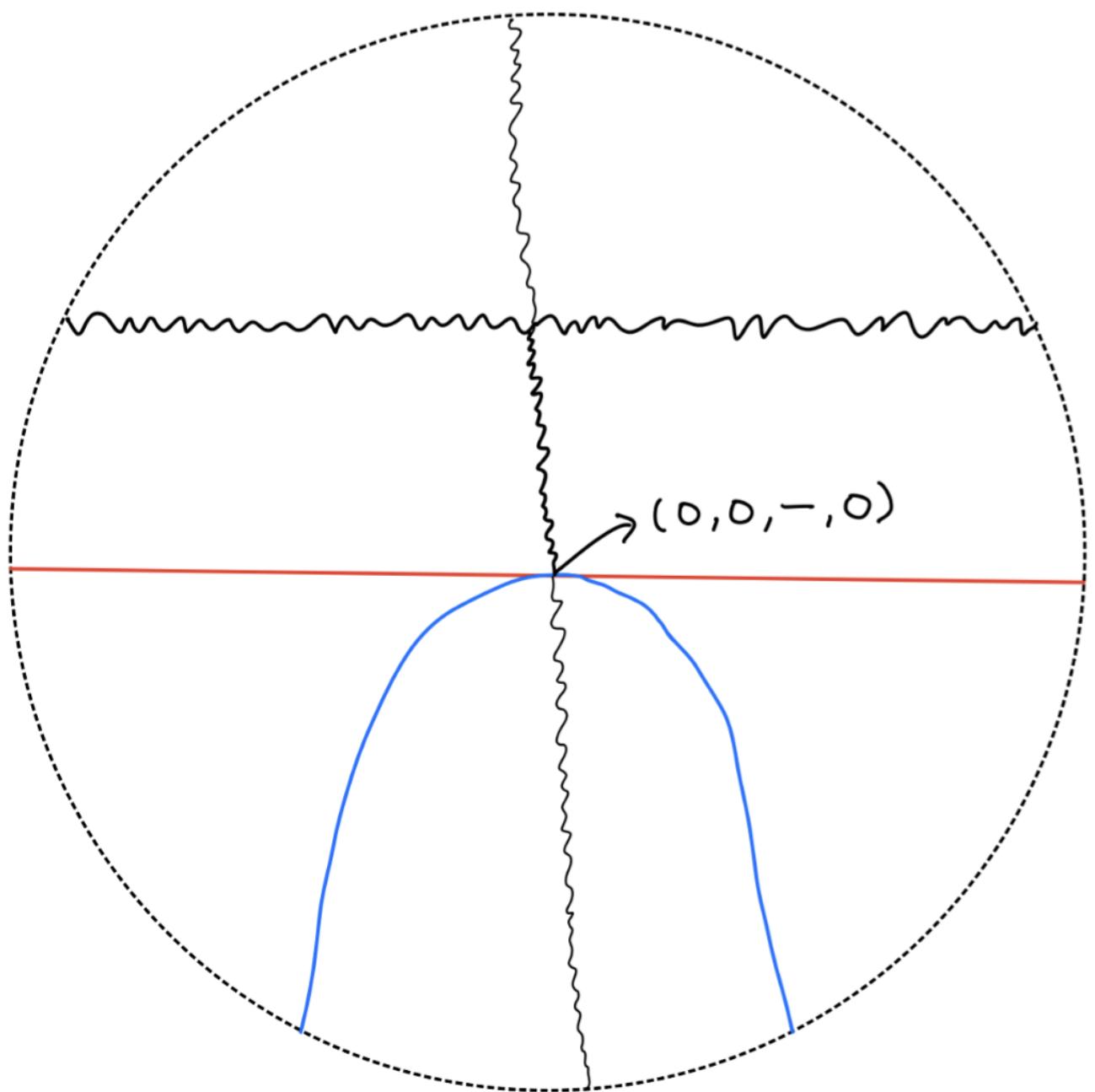


Figure 3.141

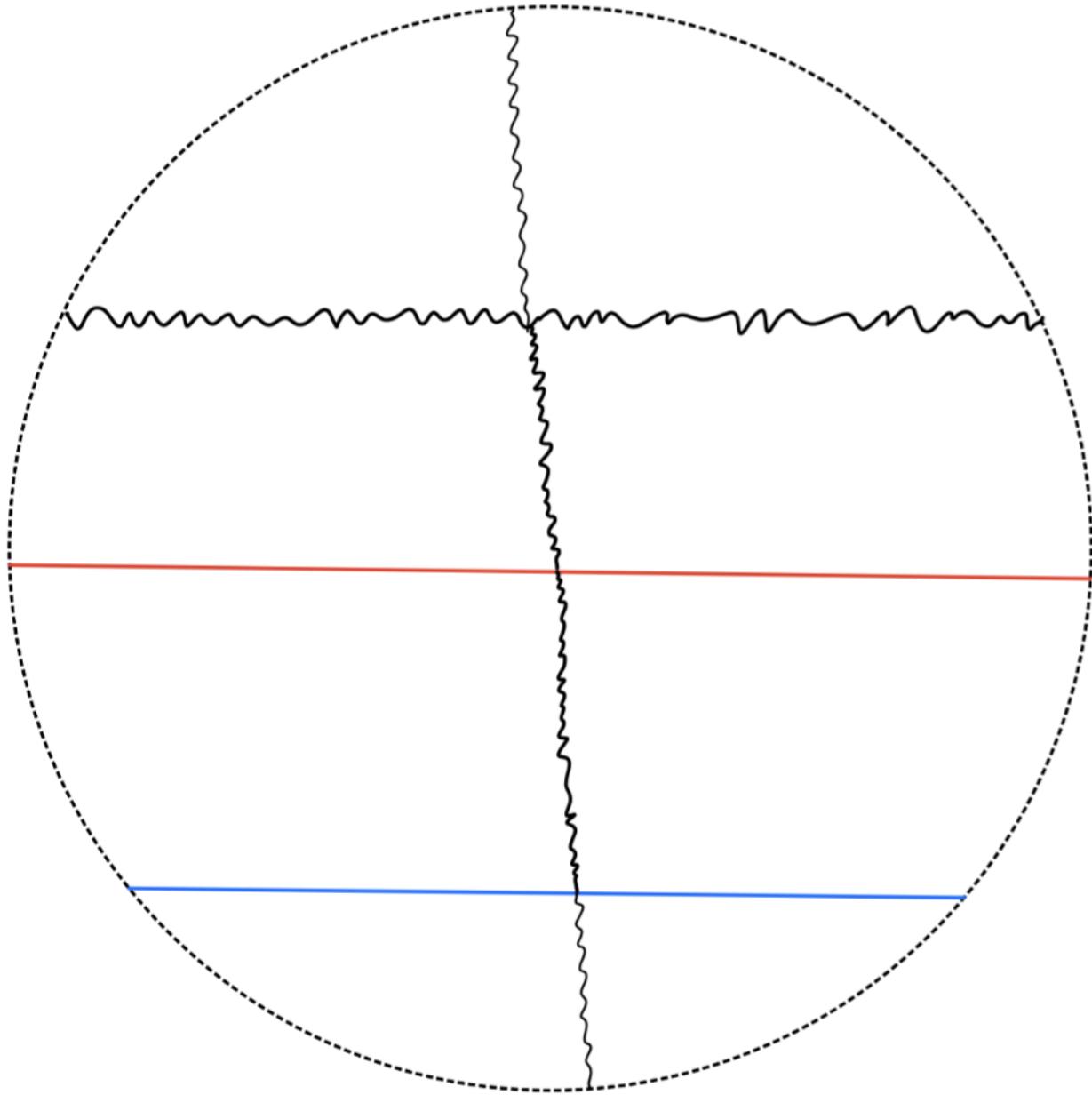


Figure 3.142

- 0 dimensional strata:

$$s_{\bullet}(0, -, 0, 0), s_{\bullet}(0, 0, -, 0)$$

**Definition 125.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $\text{star}(s)$ .

**Definition 126.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the star of  $s$ .

**Definition 127.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 128.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 129.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 130.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 131.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

## Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

## Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

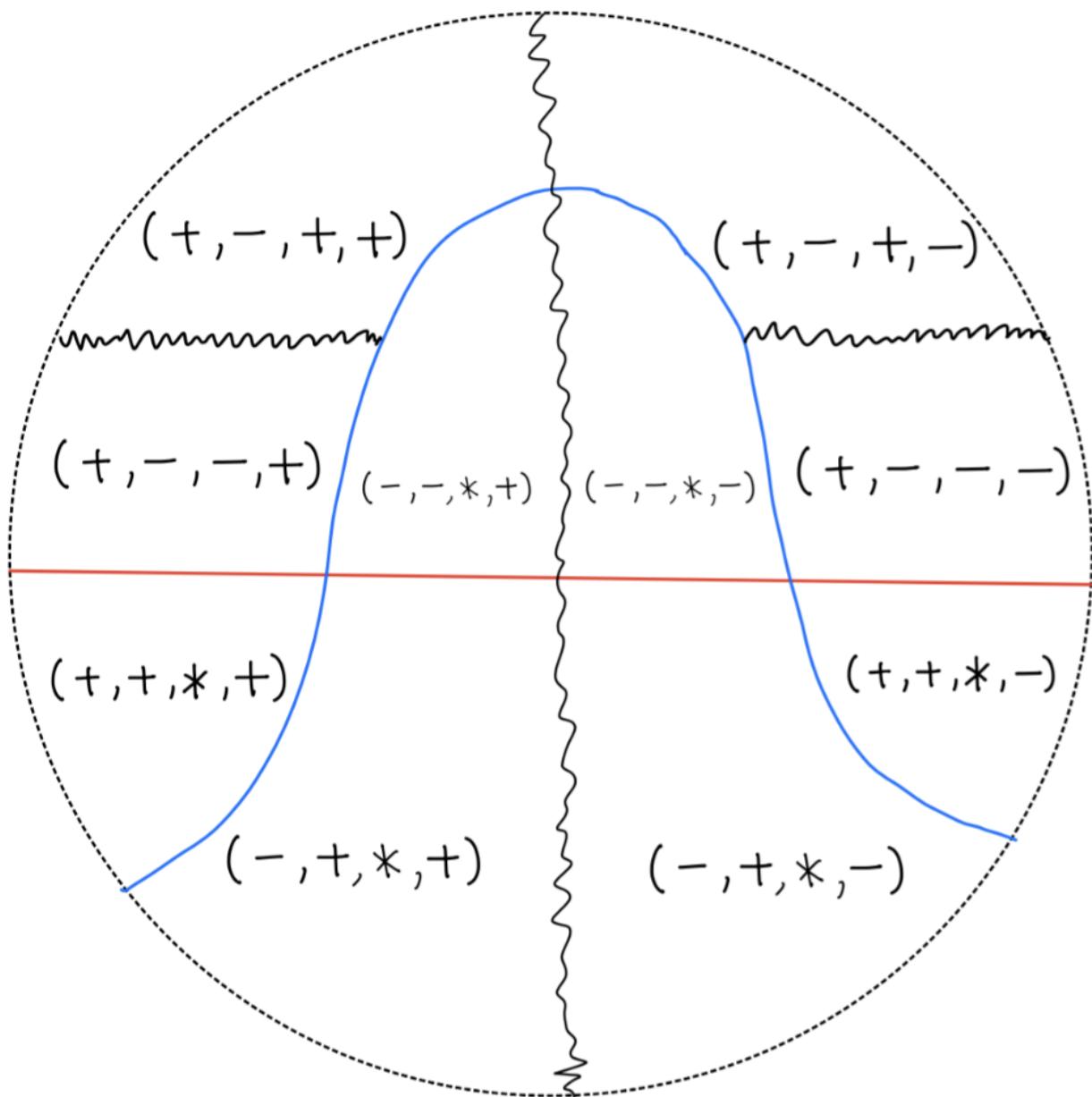


Figure 3.143

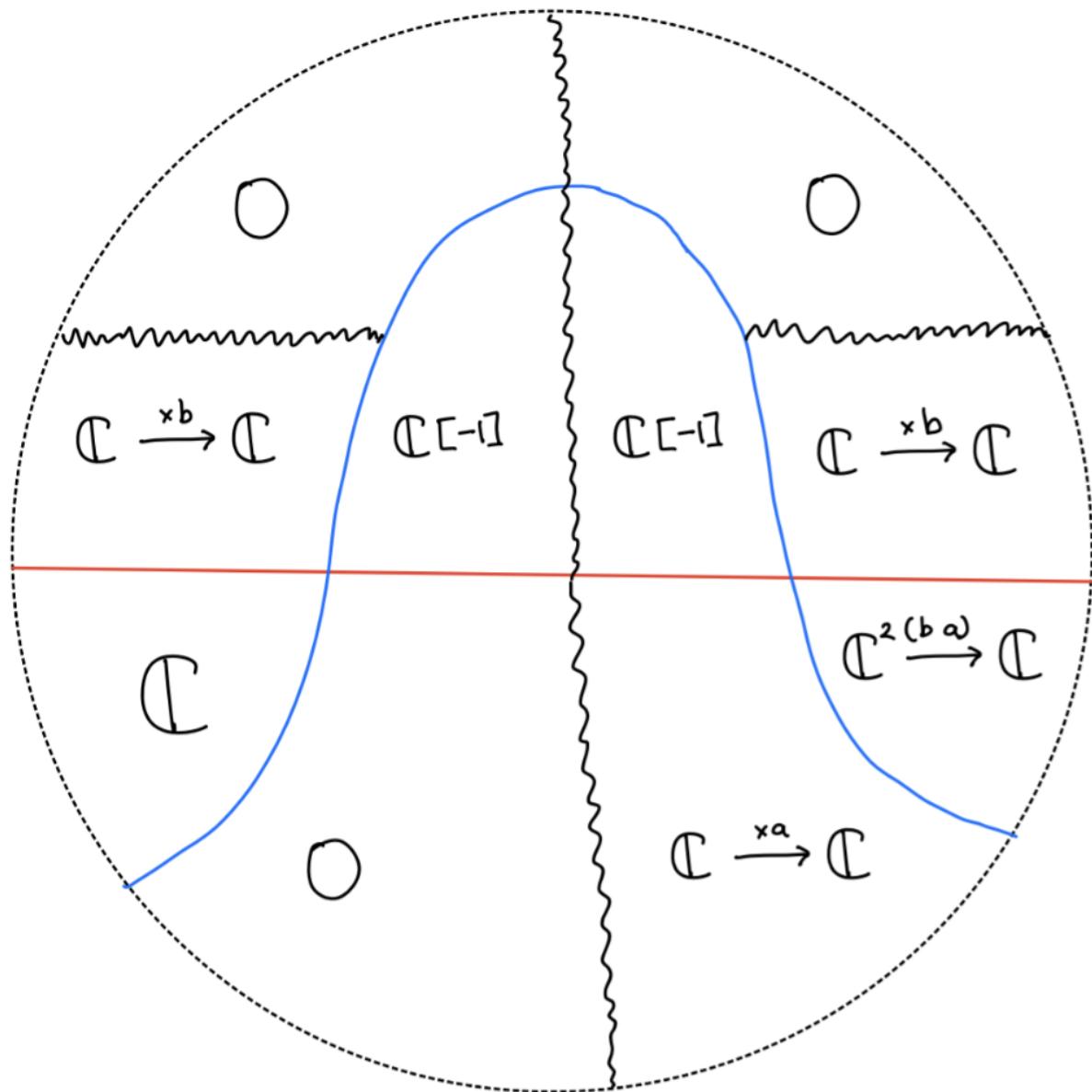


Figure 3.144

- $F_0(-, -, *, -) := \mathbb{C}[-1]$
- $F_0(-, -, *, +) := \mathbb{C}[-1]$
- $F_0(-, +, *, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_0(-, +, *, +) := 0$
- $F_0(+, -, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- 
- $F_0(+, -, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
  - $F_0(+, -, +, -) := 0$
  - $F_0(+, -, +, +) := 0$
  - $F_0(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
  - $F_0(+, +, *, +) := \mathbb{C}$

**Generalization maps:**

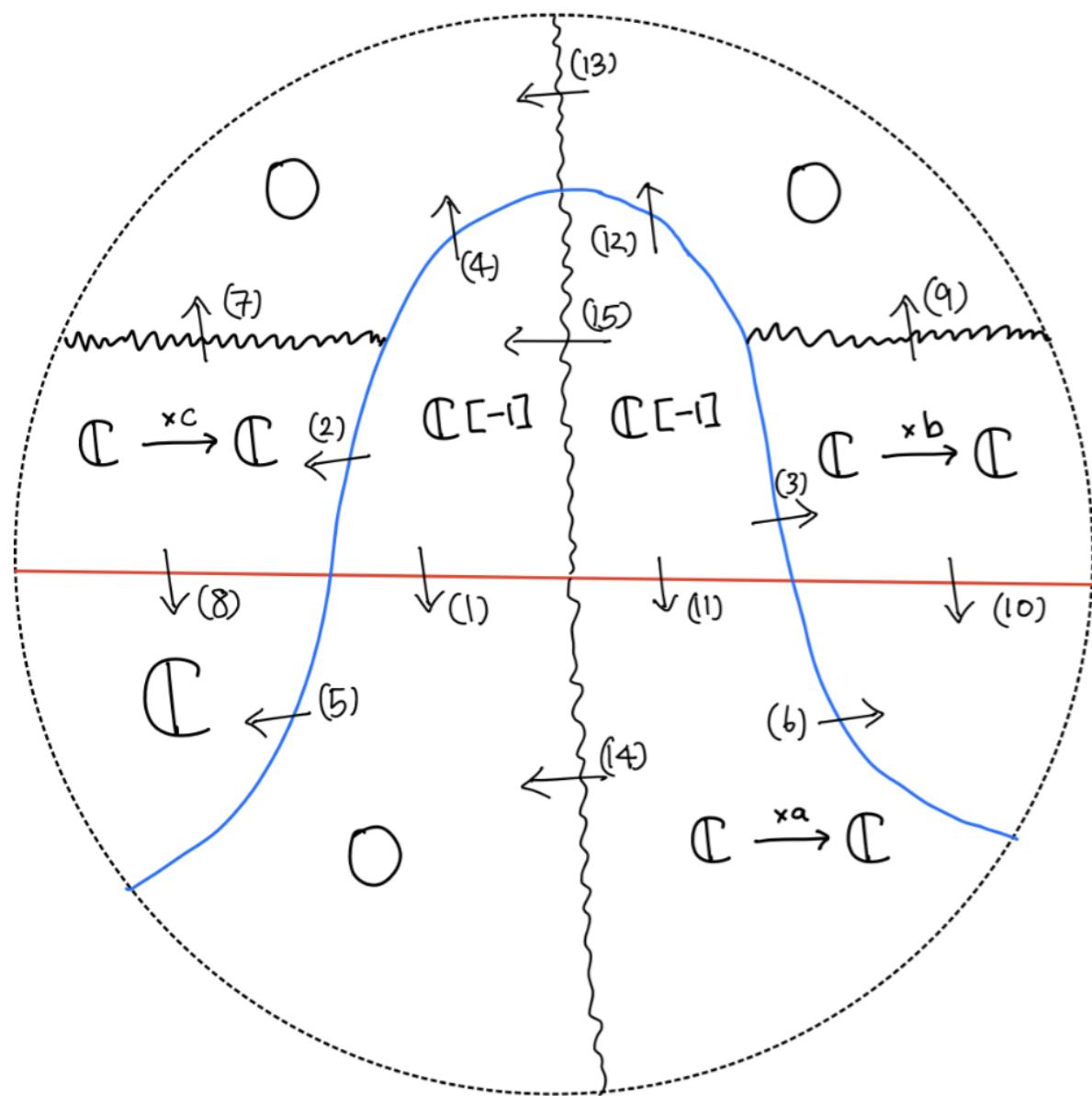


Figure 3.145

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} C & \xrightarrow{x_1} & C \\ \uparrow & \uparrow x_c & \uparrow \\ 0 & \longrightarrow & C \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \times a \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(13) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(15) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

## Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say *cobord*'<sub>2</sub>, that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

### A. Sheaf on $V \times [0, 1]$

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

### B. Sheaf on $D_{r=2} \times [0, 1]$

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4) := F_{\bullet}(s_{\bullet}(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

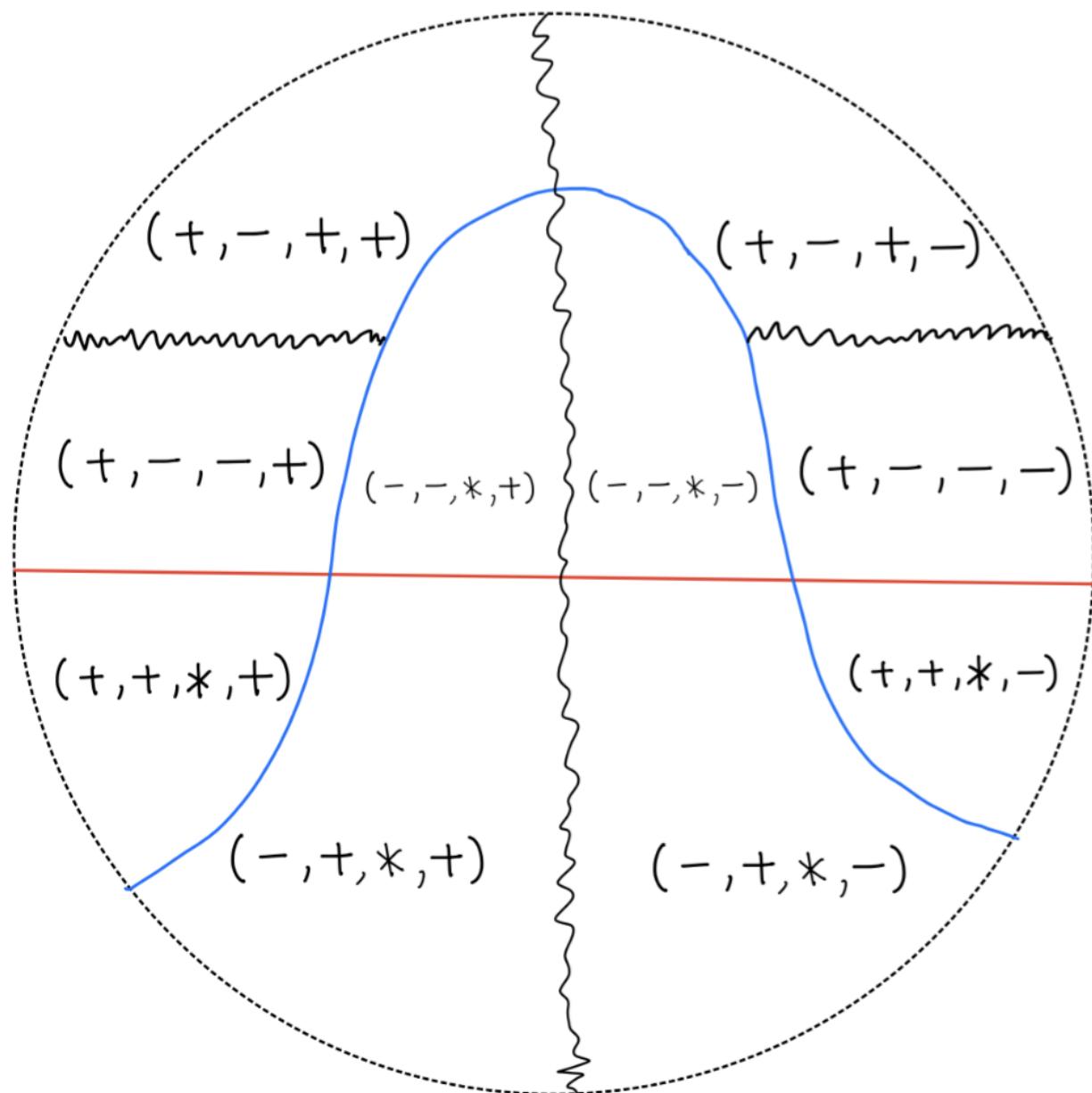


Figure 3.146

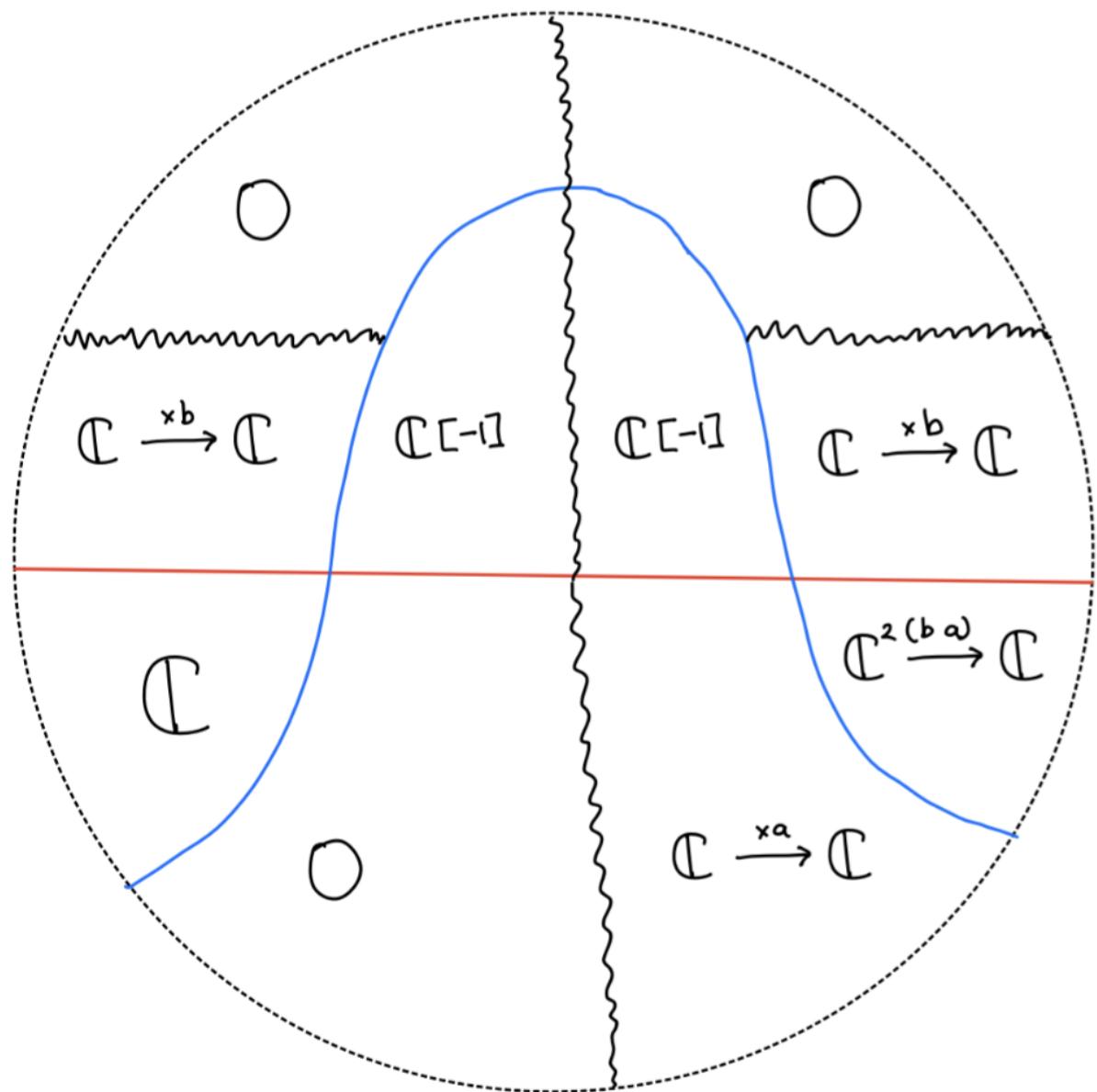


Figure 3.147

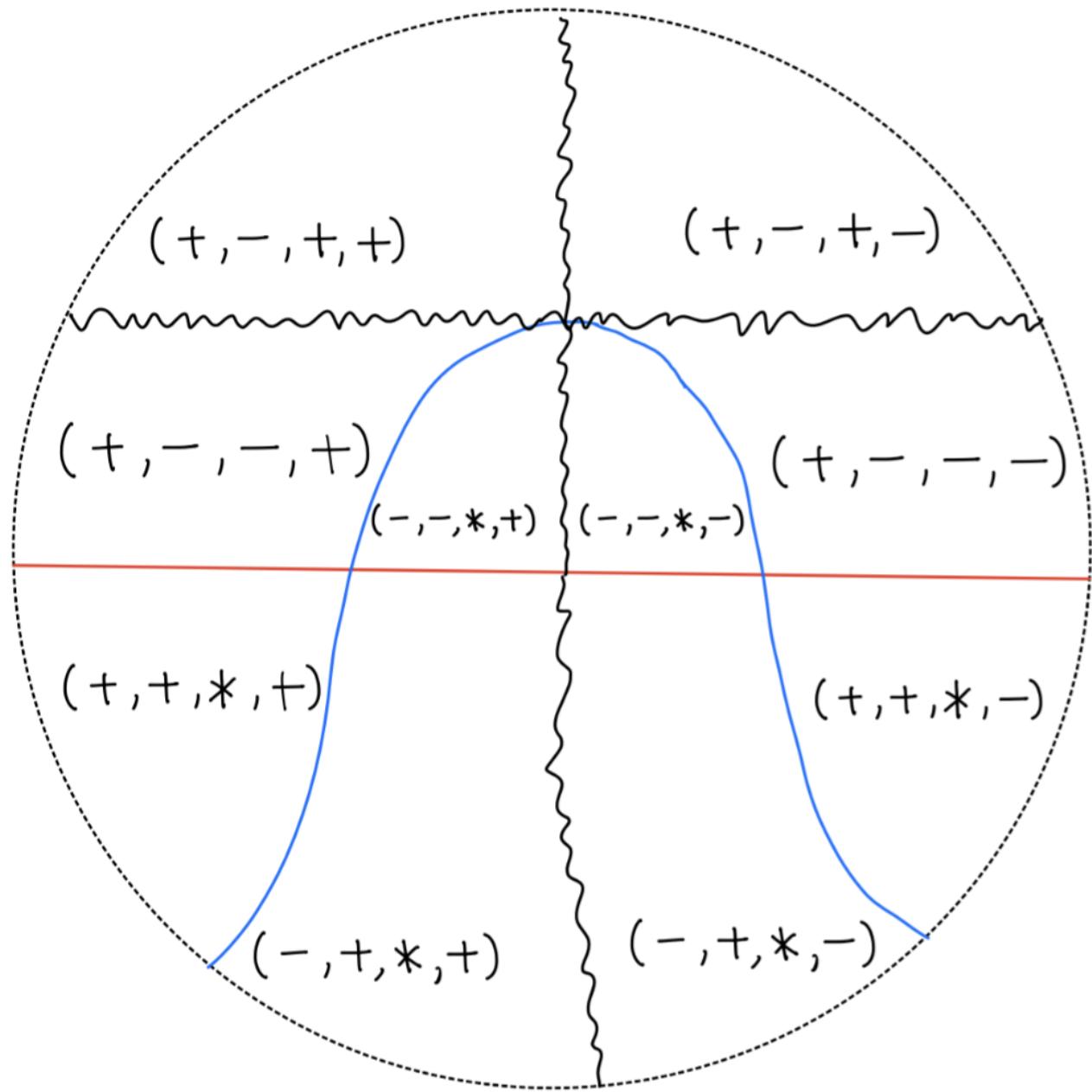


Figure 3.148

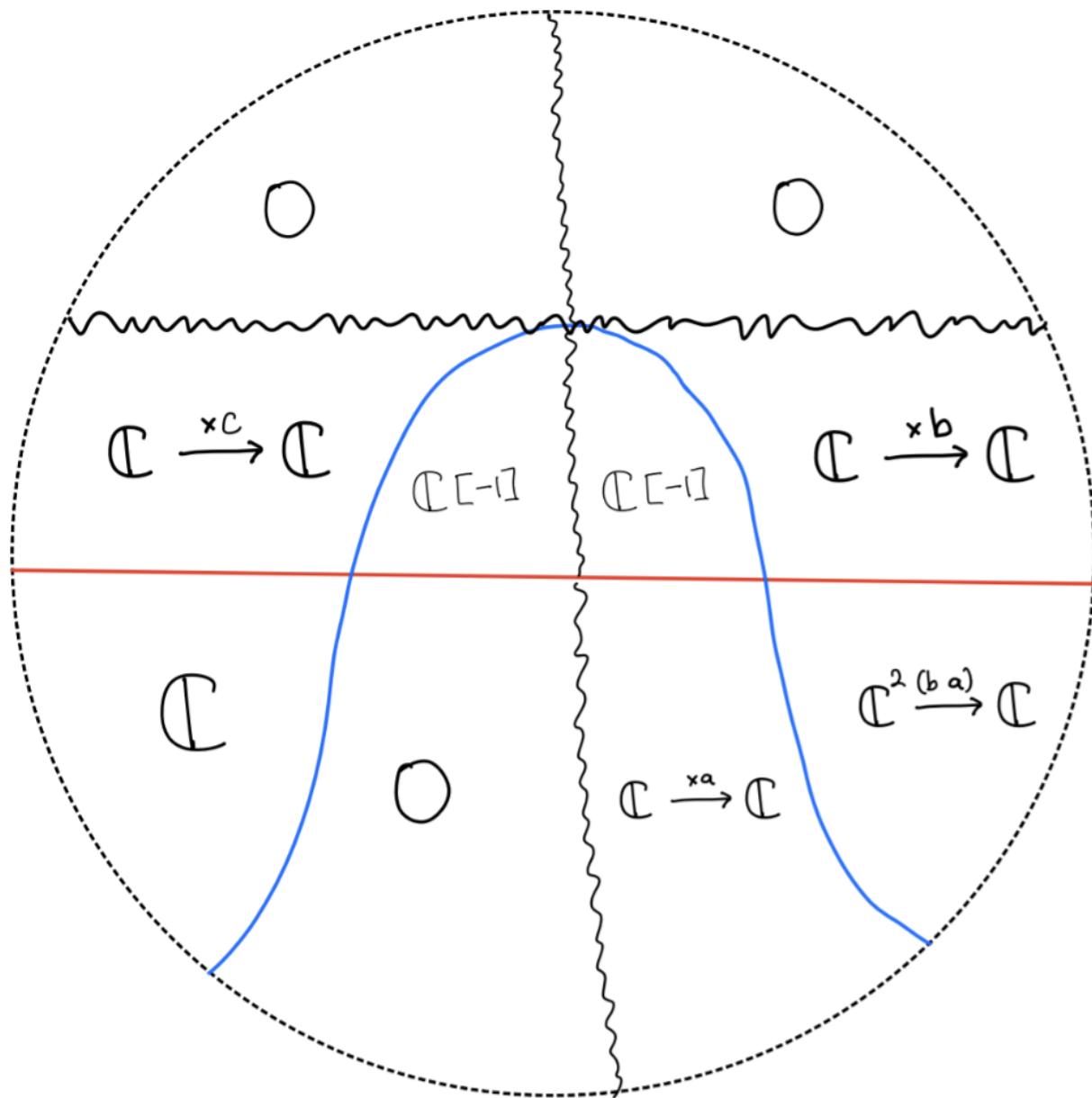


Figure 3.149

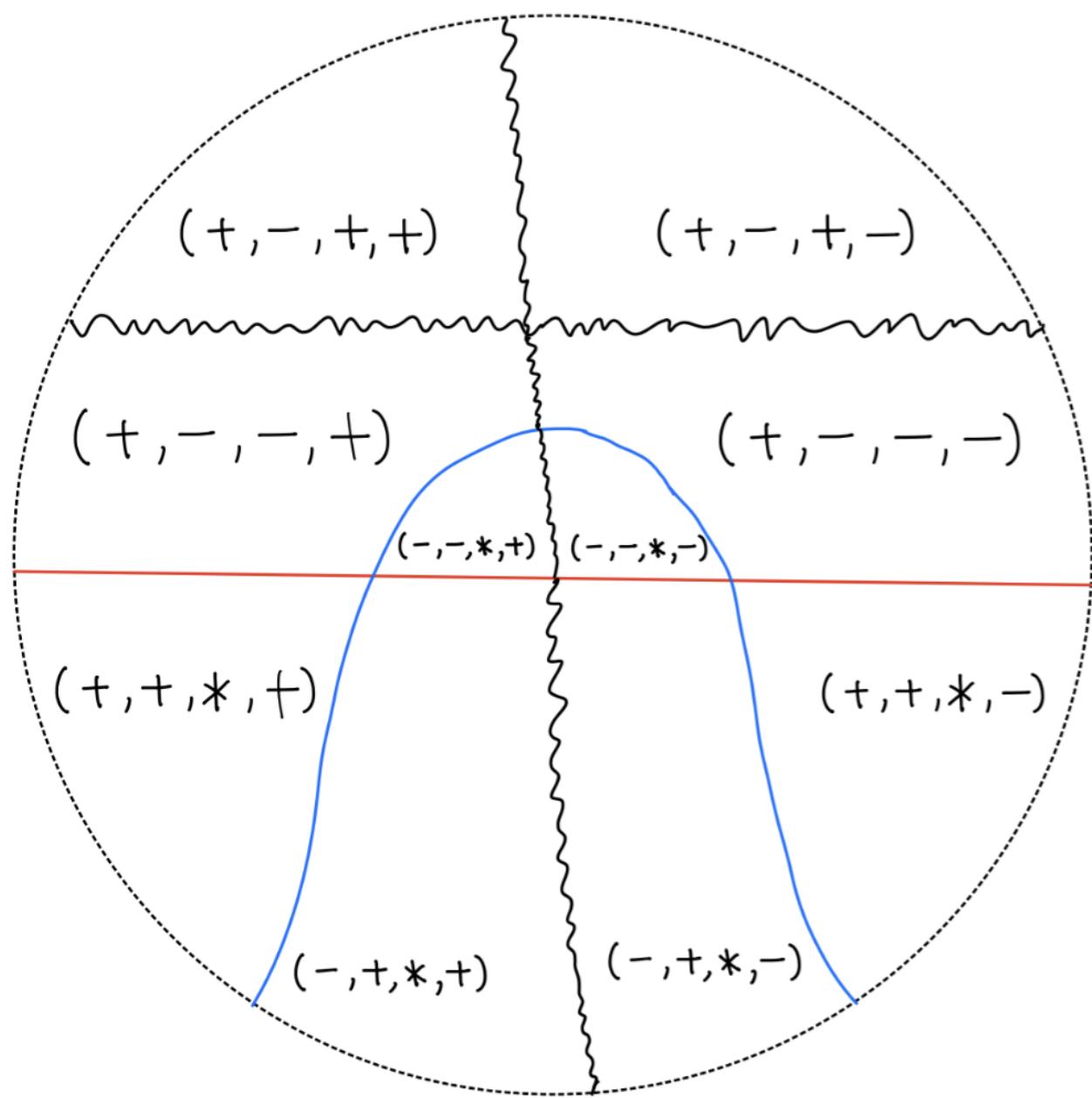


Figure 3.150

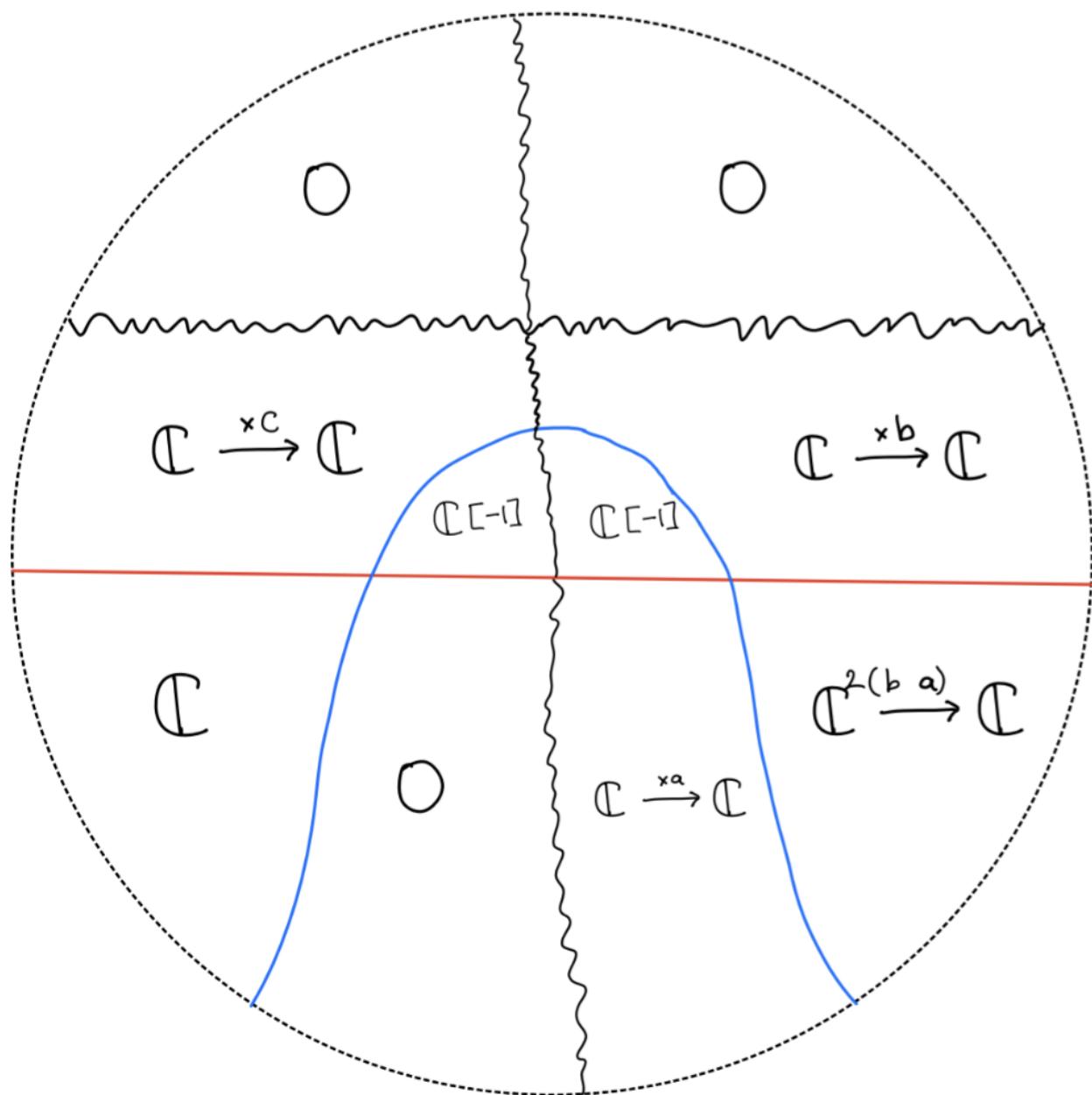


Figure 3.151

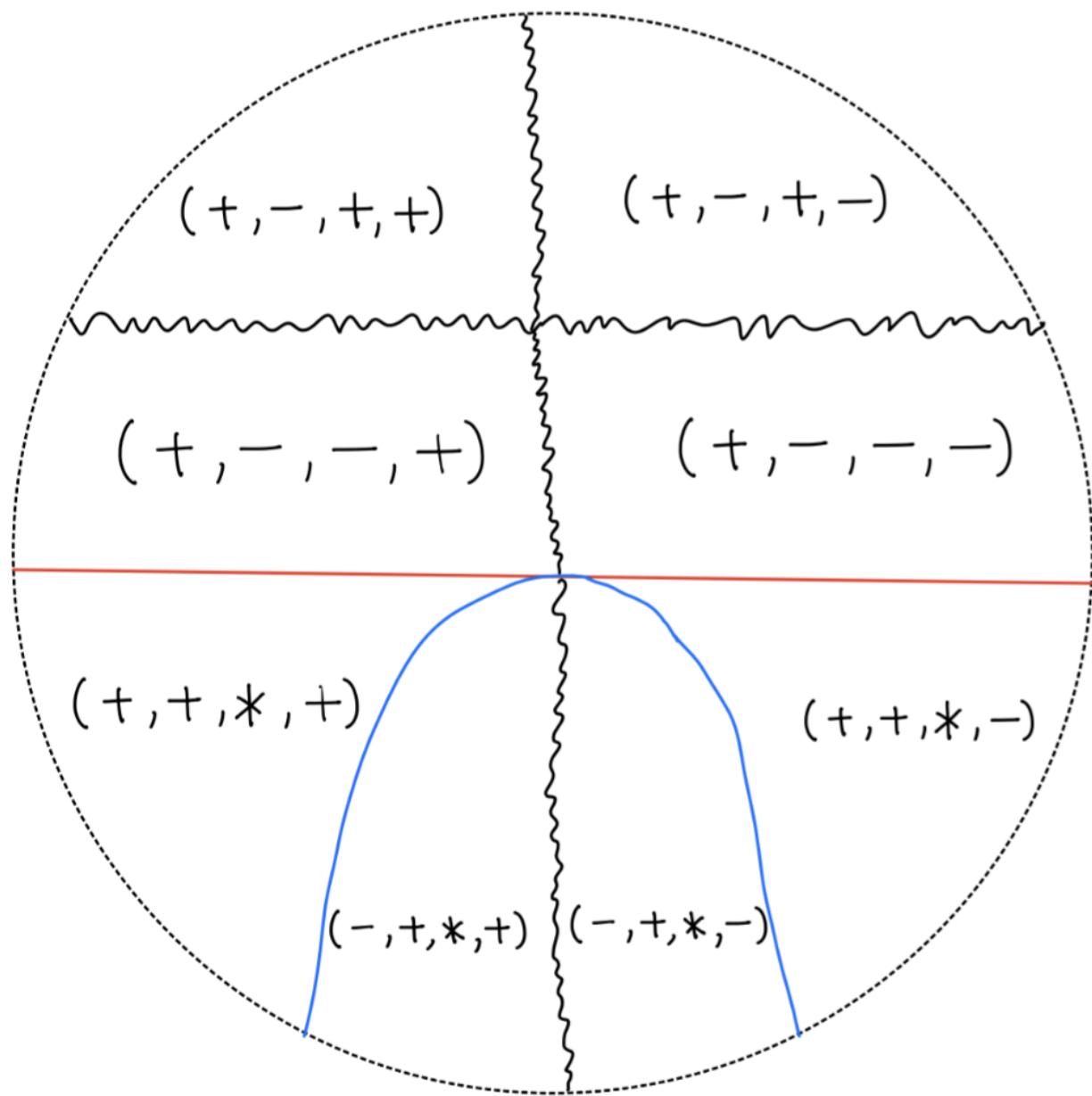


Figure 3.152

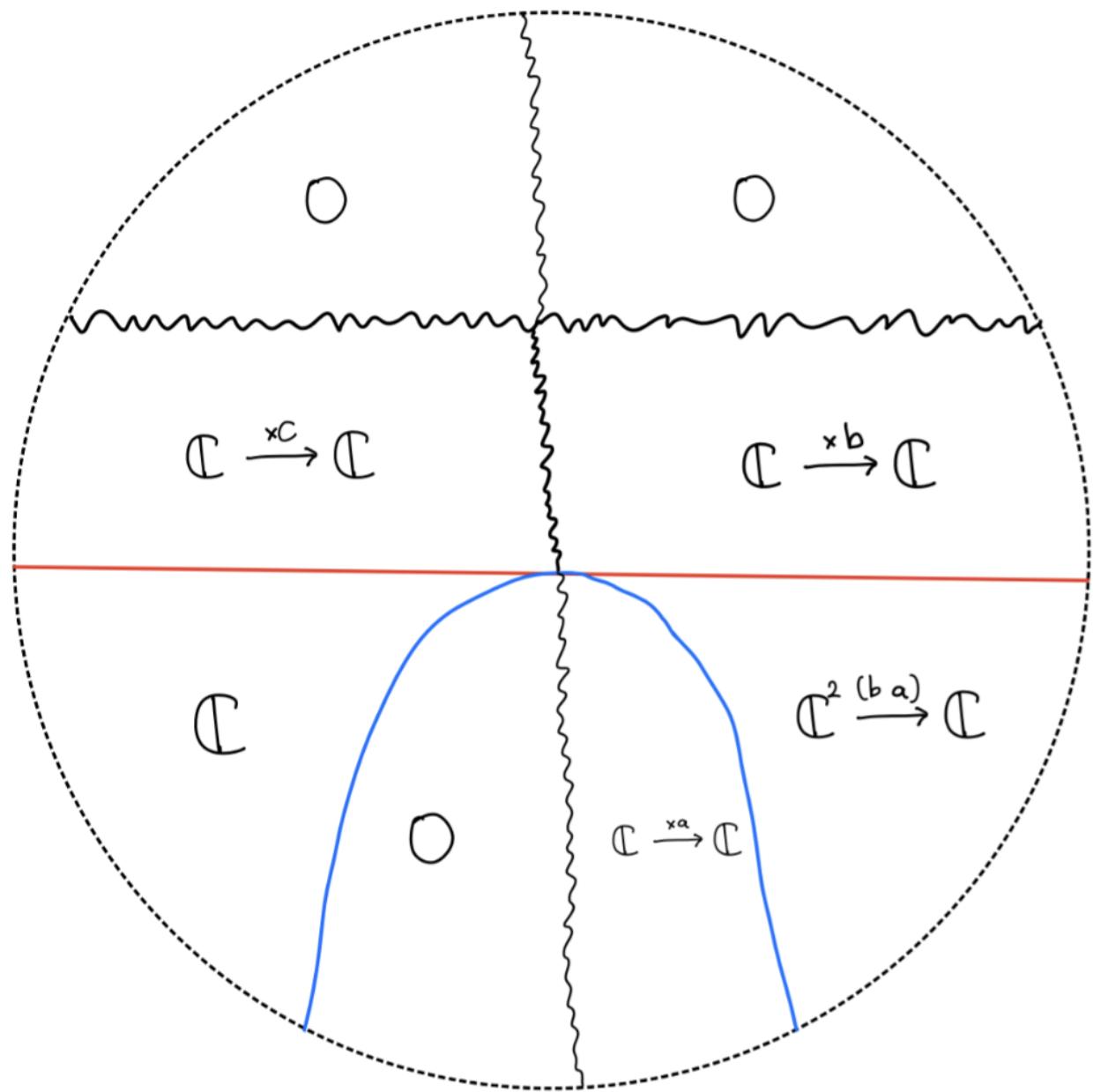


Figure 3.153

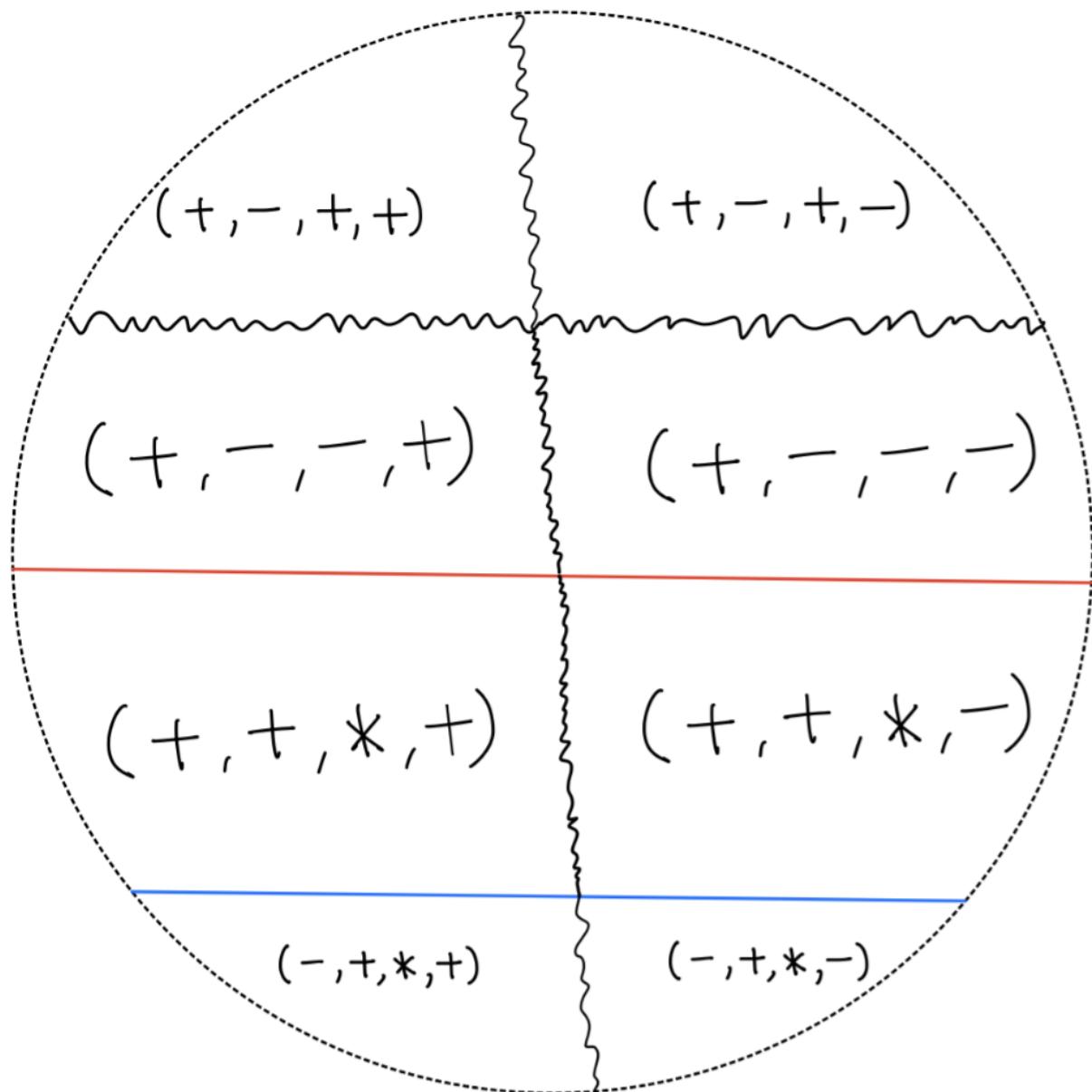


Figure 3.154

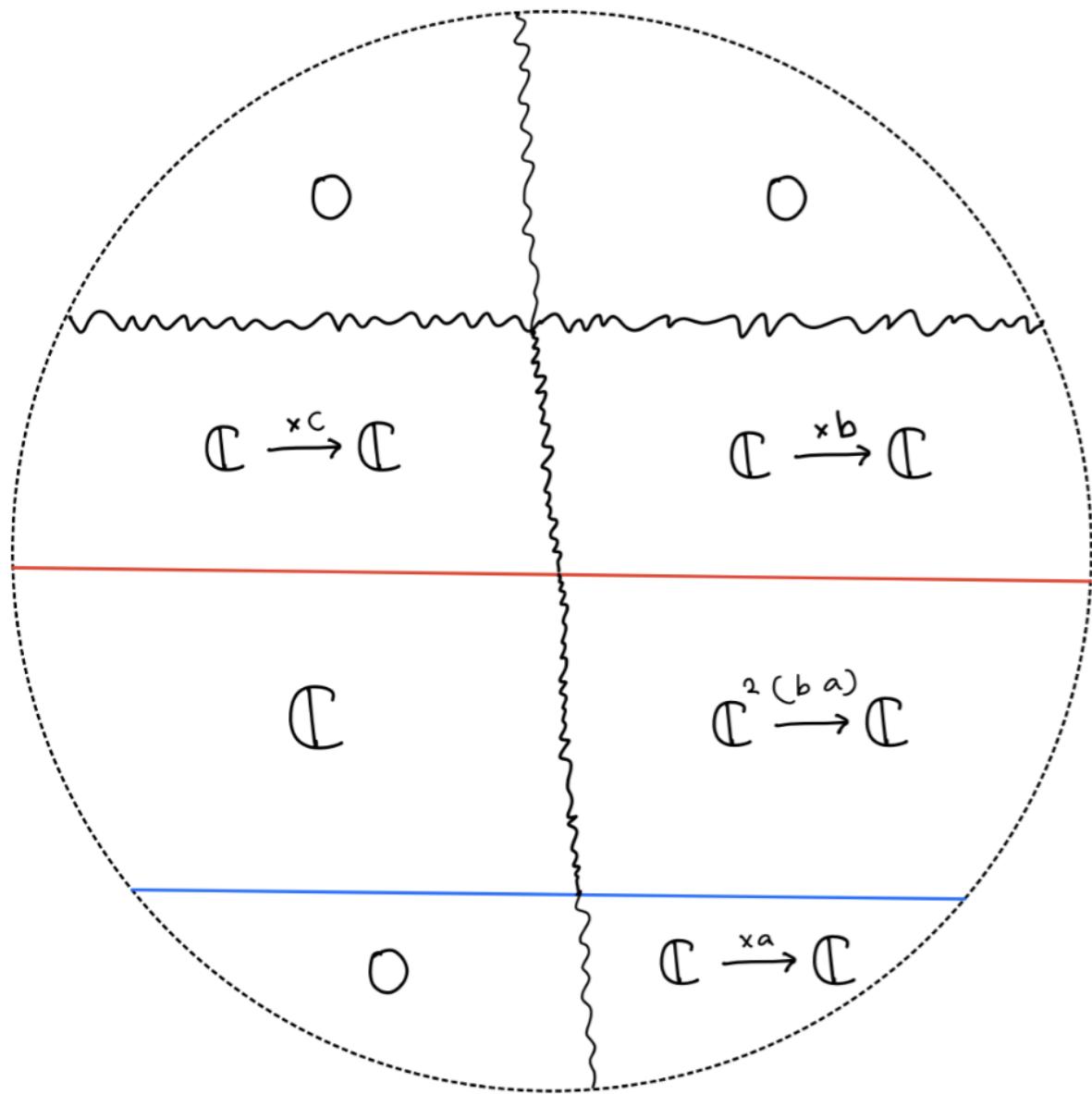


Figure 3.155

- $F_{\bullet}(-, -, *, -) := \mathbb{C}[-1]$
- $F_{\bullet}(-, -, *, +) := \mathbb{C}[-1]$
- $F_{\bullet}(-, +, *, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F_{\bullet}(-, +, *, +) := 0$
- $F_{\bullet}(+, -, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- $F_\bullet(+, -, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F_\bullet(+, -, +, -) := 0$
- $F_\bullet(+, -, +, +) := 0$
- $F_\bullet(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F_\bullet(+, +, *, +) := \mathbb{C}$

**Generalization maps:**

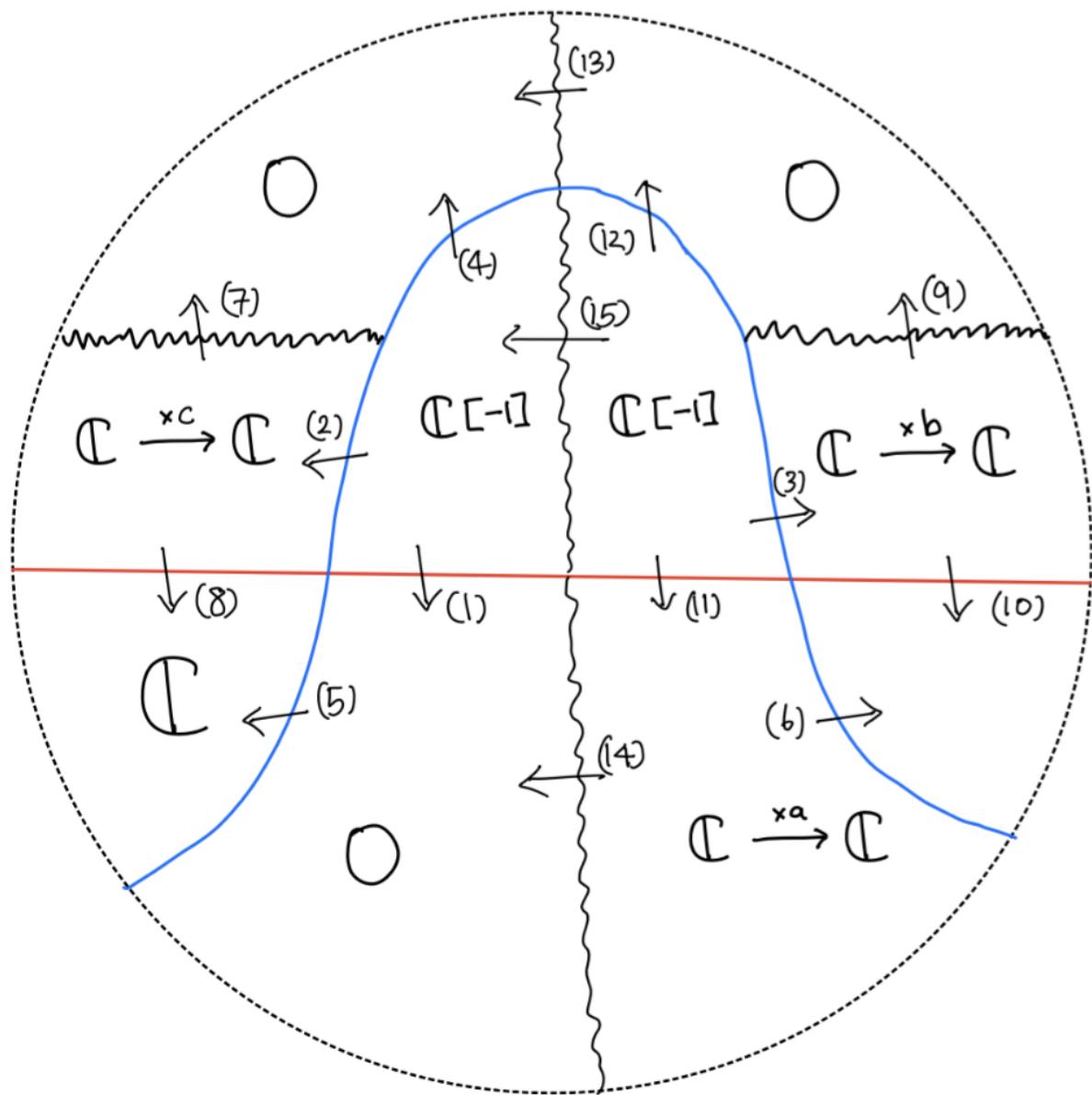


Figure 3.156

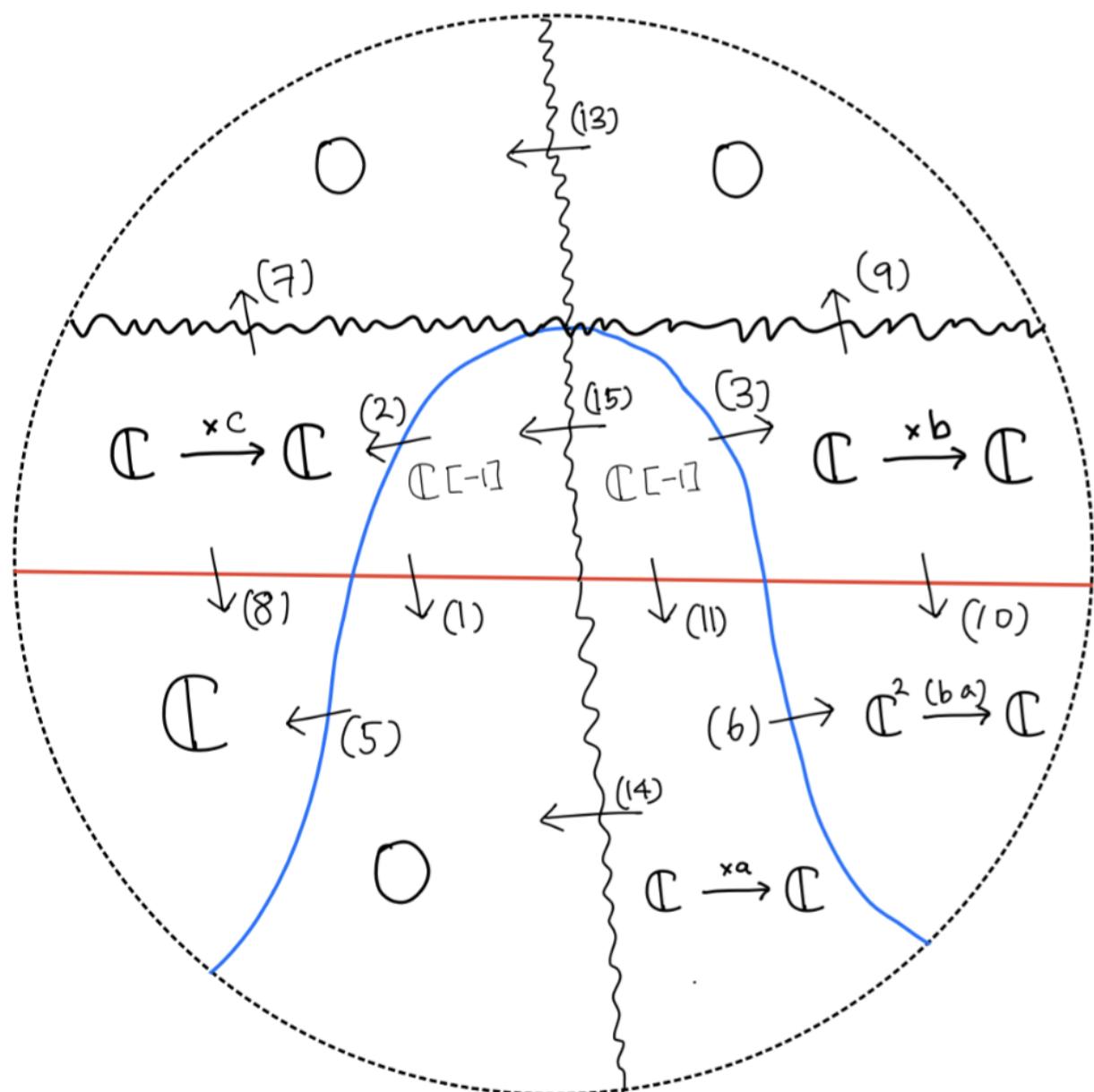


Figure 3.157

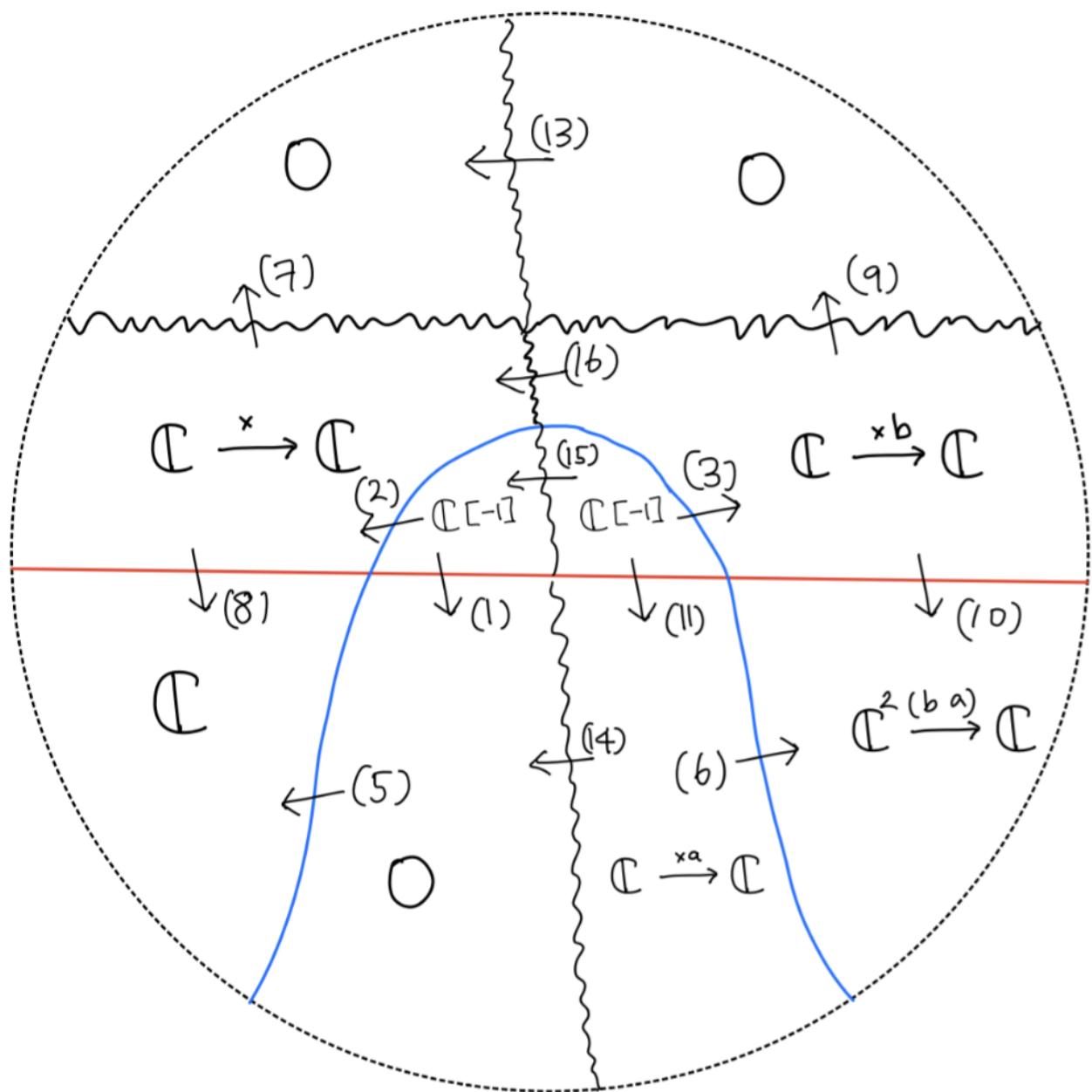


Figure 3.158

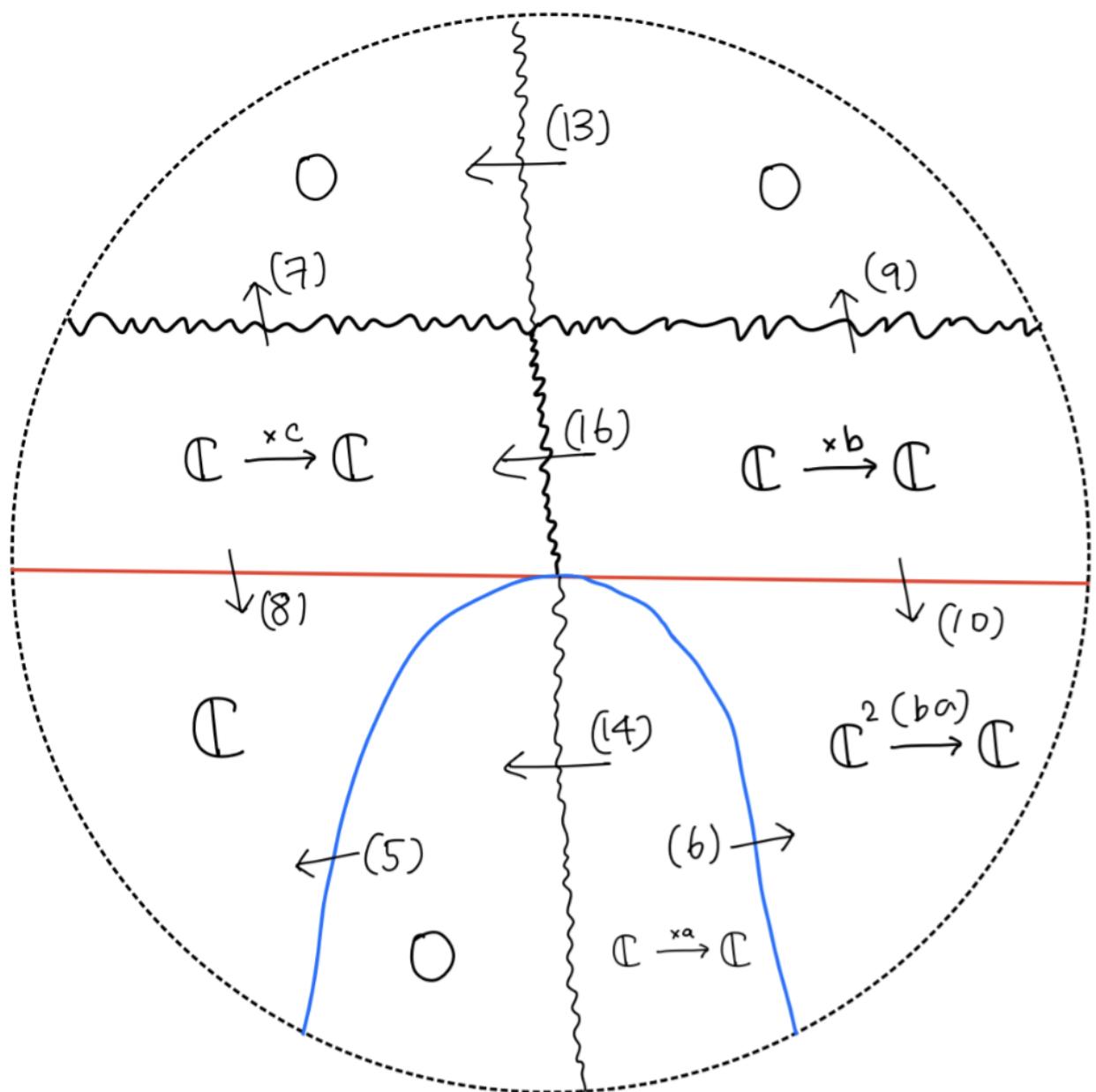


Figure 3.159

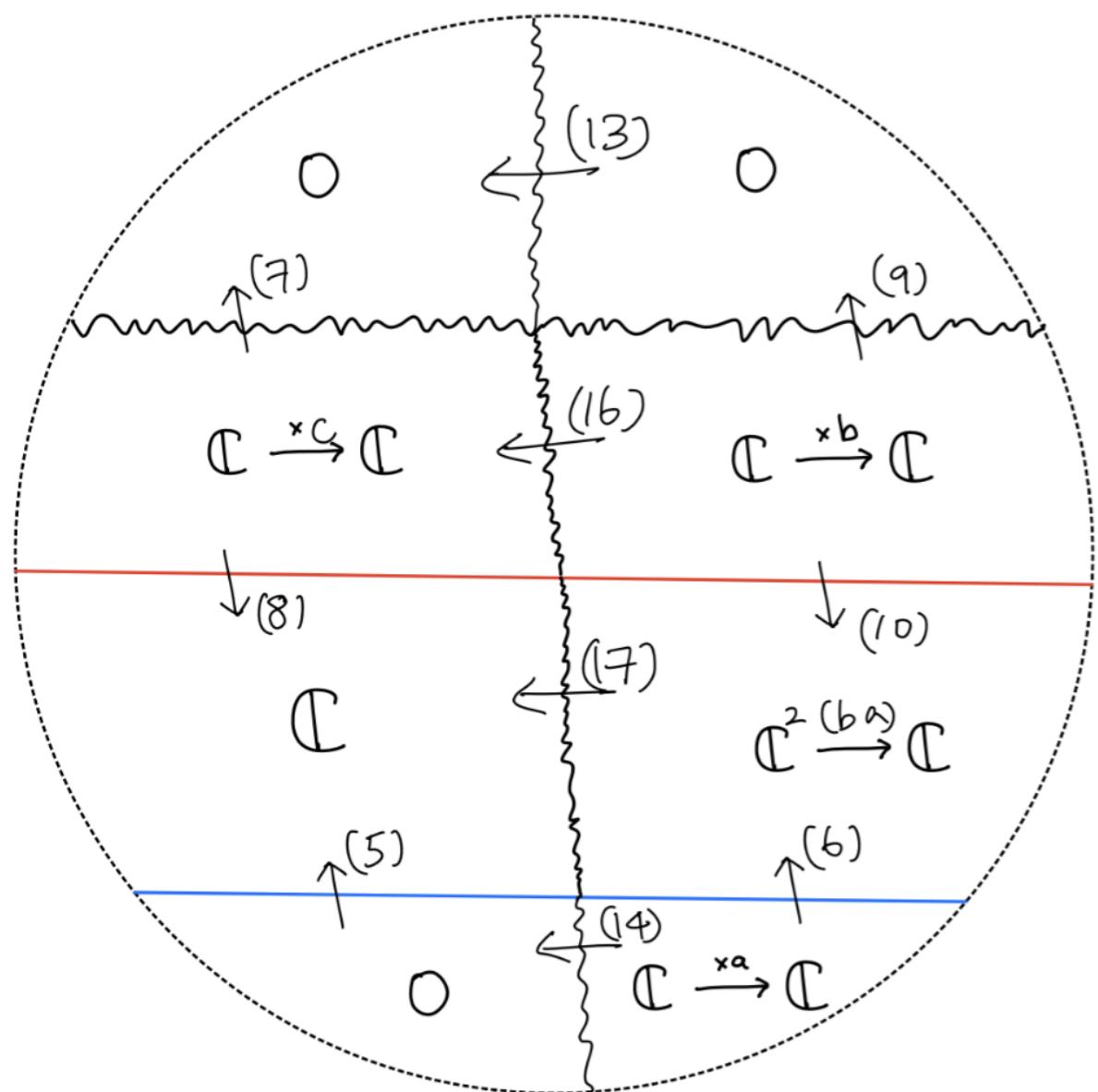


Figure 3.160

$$(1) \quad \begin{array}{ccc} C & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(2) \quad \begin{array}{ccc} C & \xrightarrow{x_1} & C \\ \uparrow & \uparrow x_c & \\ 0 & \longrightarrow & C \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \times a \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(13) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(14) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times a} & 0 \end{array}$$

$$(15) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(16) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times b & & \uparrow \times c \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & 0 \end{array}$$

$$(17) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow (b \ a) & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 132.** we define  $\gamma_\bullet$  to be the composition

$$(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 133.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M \times [0, 1], \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M \times [0, 1]$ .

Since  $\mathcal{F}^\bullet$  is constant along the time coordinate on  $U'^c \times [0, 1]$ , it is enough to check for the points of  $U \times [0, 1] \cong D_{r=2} \times [0, 1]$ . Now consider the following open cover of  $D_{r=2} \times [0, 1]$

$$\{star(s_\bullet(0, -, 0, 0)), star(s_\bullet(0, 0, -, 0))\}$$

(1) First, let's show that the microlocal stalks of  $\mathcal{F}^\bullet|_{star(s_\bullet(0, -, 0, 0))}$  vanishes. Note that there is a diffeomorphism between  $star(s_\bullet(0, -, 0, 0))$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, -, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

- $F^3(-, -, -) := \mathbb{C}[-1]$

- $F^3(-, -, +) := \mathbb{C}[-1]$

- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$

- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$

- $F^3(-, +, -) := \mathbb{C}[-1]$

- $F^3(-, +, +) := \mathbb{C}[-1]$

- $F^3(+, +, -) := 0$

- $F^3(+, +, +) := 0$

**Generalization maps:**

$$\begin{array}{ccccc}
 s(-, +, -) & \xrightarrow{(6)} & s(-, +, +) \\
 \downarrow & & \downarrow \\
 s(-, -, -) & \xrightarrow{(2)} & s(-, -, +) & \xrightarrow{(4)} & s(+, +, +) \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 s(+, +, -) & \xrightarrow{(1)} & s(+, -, +) & \xrightarrow{(10)} & s(+, +, +) \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 s(+, -, -) & \xrightarrow{(9)} & s(+, -, +) & \xrightarrow{(11)} & 
 \end{array}$$

$$\begin{array}{ccc}
 (1) & \mathbb{C} & \xrightarrow{\times 1} \mathbb{C} \\
 & \uparrow & \uparrow \\
 & 0 & \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 (2) & \mathbb{C} & \xrightarrow{\times 1} \mathbb{C} \\
 & \uparrow & \uparrow \\
 & 0 & \longrightarrow 0
 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times c \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow \times c \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(12) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-,-,-) & \longrightarrow & F^3(-,+,-) \\ \downarrow & & \downarrow \\ F^3(+,-,-) & \longrightarrow & F^3(+,+,+) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \end{array} \longrightarrow 0$$

$$(ii) \quad \begin{array}{ccc} F^3(-,-,+)&\longrightarrow&F^3(-,+,+) \\ \downarrow&&\downarrow \\ F^3(+,-,+)&\longrightarrow&F^3(+,+,+) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times c} & \mathbb{C} \end{array} \longrightarrow 0$$

$$(iii) \quad \begin{array}{ccc} F^3(-,-,-) & \longrightarrow & F^3(-,-,+) \\ \downarrow & & \downarrow \\ F^3(+,-,-) & \longrightarrow & F^3(+,-,+) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \end{array} \xrightarrow{\times c} \mathbb{C}$$

$$(iv) \quad \begin{array}{ccc} F^3(-,+,+) & \longrightarrow & F^3(-,+,+) \\ \downarrow & & \downarrow \\ F^3(+,+,+) & \longrightarrow & F^3(+,+,+) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(v) \quad \begin{array}{ccc} F^3(-,-,-) & \longrightarrow & F^3(-,-,+) \\ \downarrow & & \downarrow \\ F^3(-,+,+) & \longrightarrow & F^3(-,+,+) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\ \downarrow & & \downarrow \\ \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \end{array}$$

$$(vi) \quad \begin{array}{ccc} F^3(+,-,-) & \longrightarrow & F^3(+,-,+) \\ \downarrow & & \downarrow \\ F^3(+,+,+) & \longrightarrow & F^3(+,+,+) \end{array} = \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array} \xrightarrow{\times c} \mathbb{C}$$

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-, +, -) & \xrightarrow{s(-,+0)} & s(-, +, +) & \\
 s(-, 0, -) \nearrow & \downarrow & & \nearrow s(-, 0, +) & \downarrow s(0, +, +) \\
 s(-, -, -) & \xrightarrow[s(-,-,0)]{s(0,+,+)} & s(-, -, +) & & \\
 \downarrow s(0, -, -) & & \downarrow & & \\
 & s(+, +, -) & \xrightarrow[s(+,+0)]{s(0,+,+)} & s(+, +, +) & \\
 s(+, 0, -) \nearrow & \downarrow & \nearrow s(+, 0, +) & & \\
 s(+, -, -) & \xrightarrow{s(+,-,0)} & s(+, -, +) & & \\
 = & & & & \\
 & \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \\
 & \downarrow & & \downarrow & \\
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & \\
 \mathbb{C} \xrightarrow{x_b} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{x_c} \mathbb{C} & & 0
 \end{array}$$

For

i-vi, horizontal cochain map in each degree are quasi-isomorphism. Therefore,

the total complex is acyclic.

For vii,  $\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{x_1} & \mathbb{C} & , & \mathbb{C} & \xrightarrow{x_1} & \mathbb{C} \\ \uparrow & & \uparrow & , & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & , & 0 & \longrightarrow & 0 \\ & & & , & \uparrow x_b & & \uparrow x_c \\ & & & , & \mathbb{C} & \xrightarrow{x_{bc}^{-1}} & \mathbb{C} \\ & & & , & \uparrow & & \uparrow \\ & & & , & 0 & \longrightarrow & 0 \end{array}$  are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

(2) Next, let's show that the microlocal stalks of  $\mathcal{F}^\bullet|_{star(s_\bullet(0,0,-,0))}$  vanishes. Note

that there is a diffeomorphism between  $\text{star}(s_\bullet(0, 0, -, 0))$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, -, sgn_3)$$

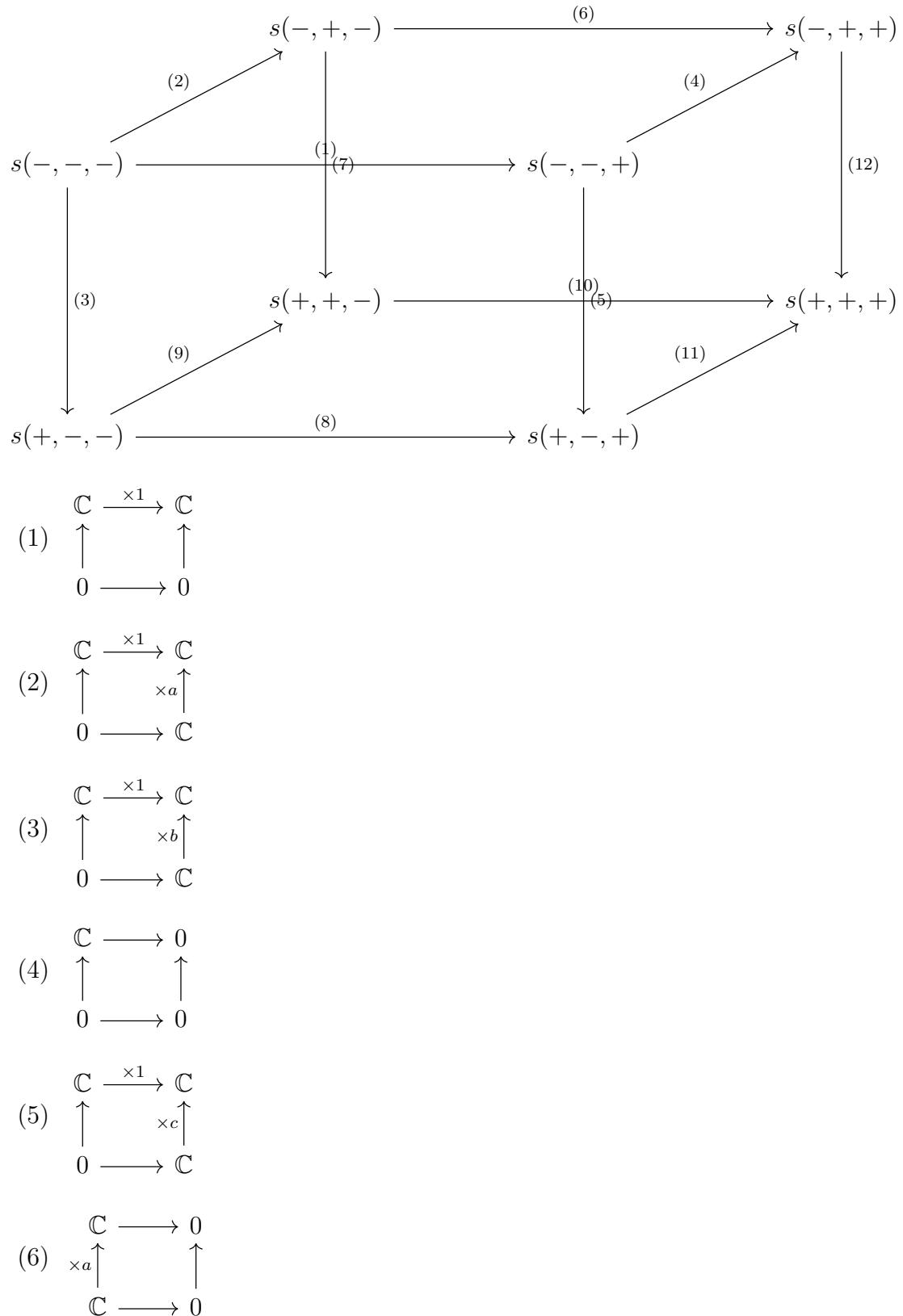
Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

### Stalks:

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C}[-1]$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C} \xrightarrow{\times c} \mathbb{C}$
- $F^3(-, +, -) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(-, +, +) := 0$
- $F^3(+, +, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}$

### Generalization maps:



$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \times_a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times_b \uparrow & & \times_c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times_b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times_c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$(i) \quad \begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \end{array}$$

$$(ii) \quad \begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}[-1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbb{C} \xrightarrow{\times c} \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

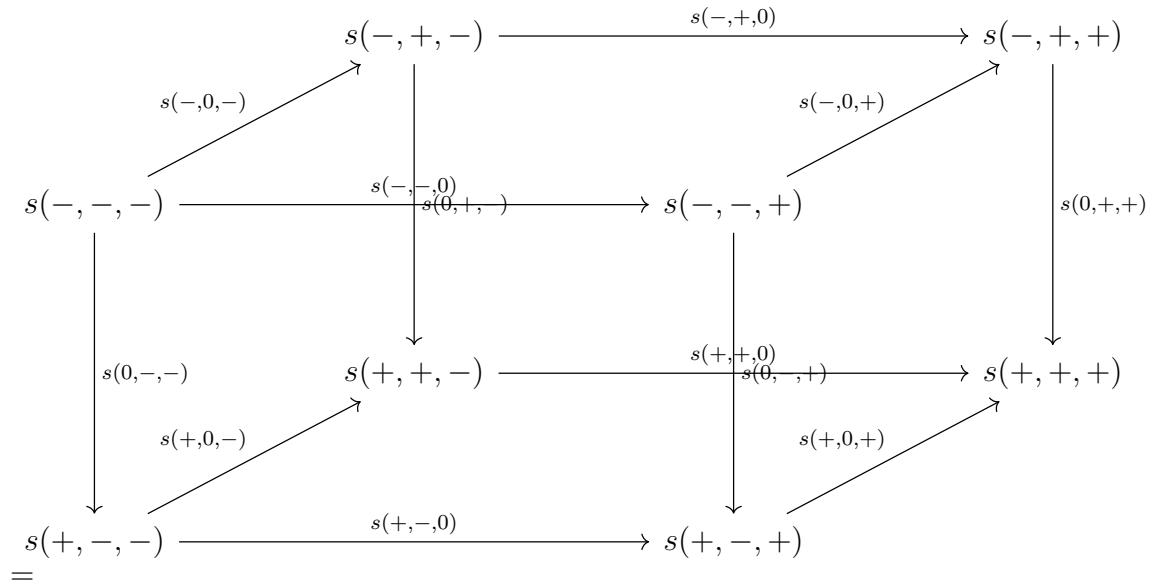
$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow \mathbb{C} \xrightarrow{\times c} \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \longrightarrow 0 \\
 \downarrow & & \downarrow \\
 \mathbb{C}^2 & \xrightarrow{(b \ a)} & \mathbb{C} \xrightarrow{(bc^{-1} \ 0)} \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}[-1] & \longrightarrow & \mathbb{C}[-1] \\
 \downarrow & & \downarrow \\
 \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} \longrightarrow 0
 \end{array}$$

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} \longrightarrow \mathbb{C} \xrightarrow{\times c} \mathbb{C} \\
 \downarrow & & \downarrow \\
 \mathbb{C}^2 & \xrightarrow{(b \ a)} & \mathbb{C} \longrightarrow \mathbb{C}
 \end{array}$$

(vii) the cubic diagram:



$$\begin{array}{ccccc}
 & \mathbb{C} \xrightarrow{\times a} \mathbb{C} & \longrightarrow & 0 \\
 & \downarrow & & & \downarrow \\
 \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C}[-1] & \xrightarrow{\quad} & \mathbb{C} \\
 & \downarrow & & & \downarrow \\
 & \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} & \longrightarrow & \mathbb{C} \\
 & \downarrow & & & \downarrow \\
 \mathbb{C} \xrightarrow{\times b} \mathbb{C} & \longrightarrow & \mathbb{C} \xrightarrow{\times c} \mathbb{C} & \longrightarrow & \mathbb{C}
 \end{array}$$

For

iii-vi, the horizontal cochain map in each degree are quasi-isomorphism. Therefore, the total complex is acyclic.

For i and ii, straightforward calculation shows that the total complexes are acyclic.

For vii,

$\mathbb{C} \xrightarrow{\times 1} \mathbb{C}$	$\mathbb{C} \longrightarrow 0$	$\mathbb{C} \xrightarrow{\times 1} \mathbb{C}$	$\mathbb{C} \longrightarrow 0$
$\uparrow$	$\uparrow$ , $\times a \uparrow$	$\uparrow$ , $\times b \uparrow$	$\uparrow$ , $\times c \uparrow$
$0 \longrightarrow 0$	$\mathbb{C} \longrightarrow 0$	$\mathbb{C} \xrightarrow{\times bc^{-1}} \mathbb{C}$	$\mathbb{C}^2 \xrightarrow{(bc^{-1} \ 0)} \mathbb{C}$

are quasi-isomorphisms. Therefore, we can think of the cube diagram as quasi-isomorphism of two double complexes. Therefore, the total complex is acyclic.

Therefore, the proof is complete.  $\square$

## Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the cobordism. By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$

- a gluing isomorphism  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

### A. Sheaf on $V$

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

### B. Sheaf on $D_{r=2}$

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3, sgn_4) := F_1(s_1(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

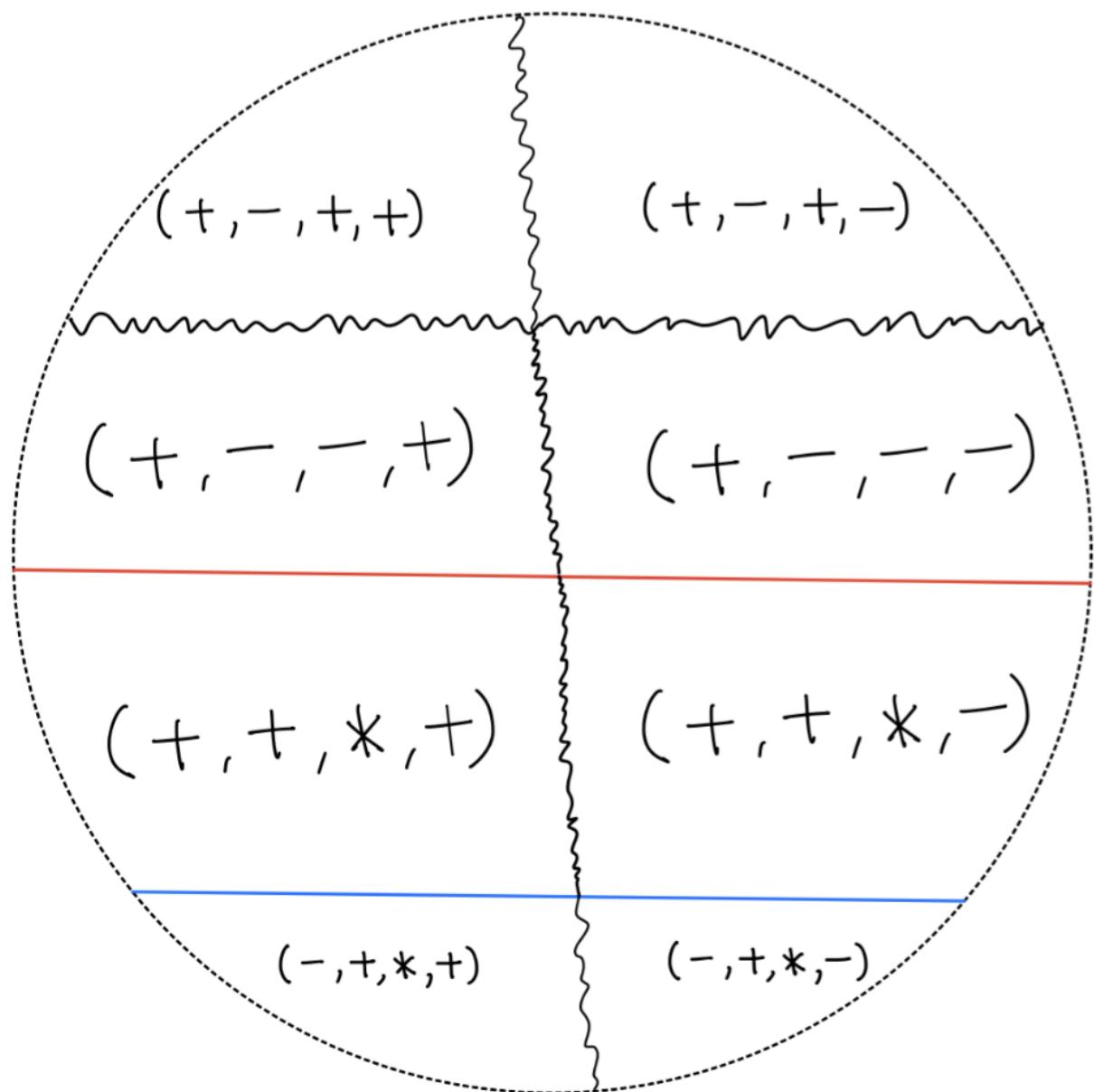


Figure 3.161

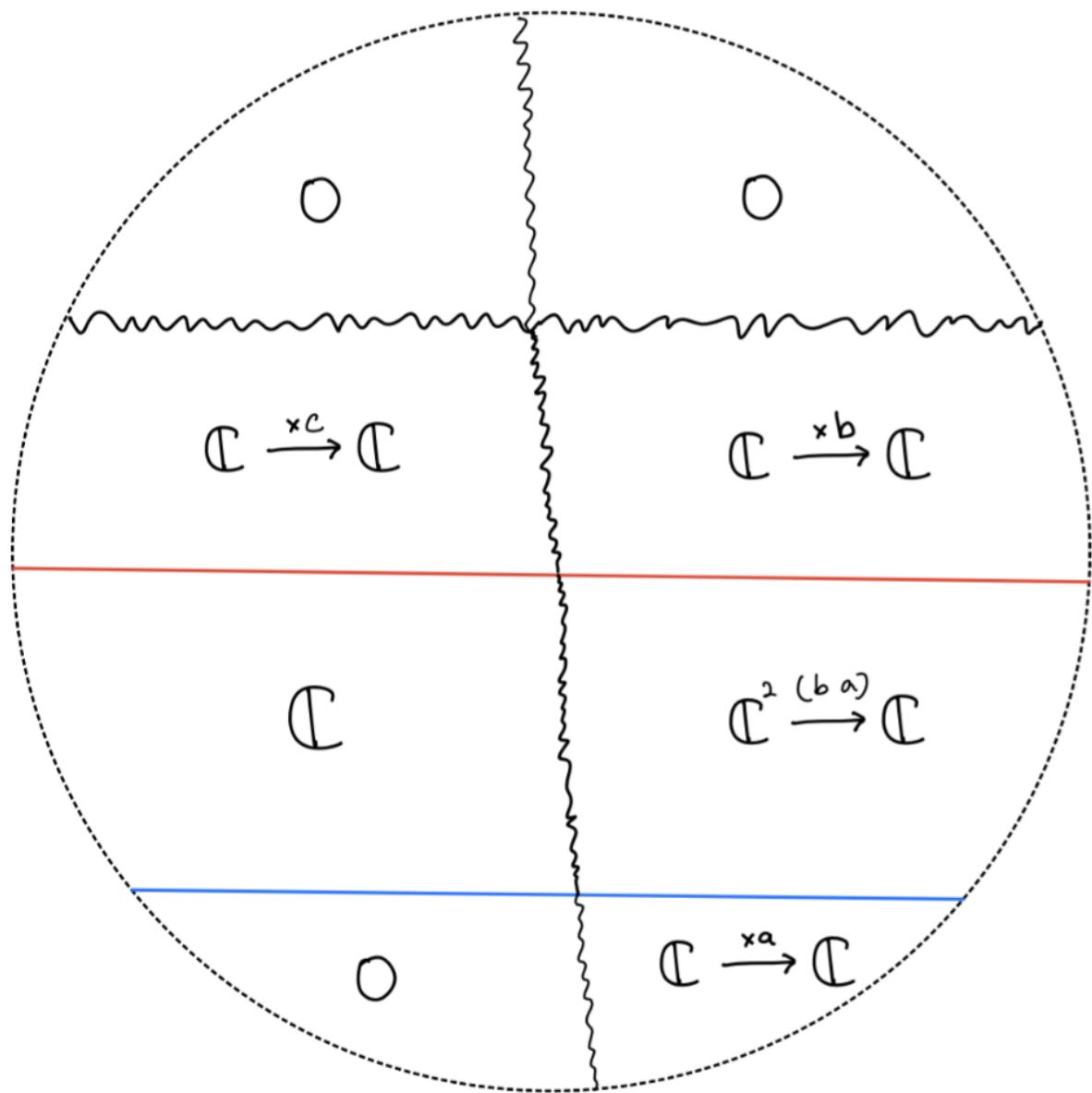


Figure 3.162

- $F_1(-, +, *, -) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$

- $F_1(-, +, *, +) := 0$

- $F_1(+, -, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$

- $F_1(+, -, -, +) := \mathbb{C} \xrightarrow{x_c} \mathbb{C}$

- $F_1(+, -, +, -) := 0$

- $F_1(+, -, +, +) := 0$
- $F_1(+, +, *, -) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F_1(+, +, *, +) := \mathbb{C}$

Generalization maps:

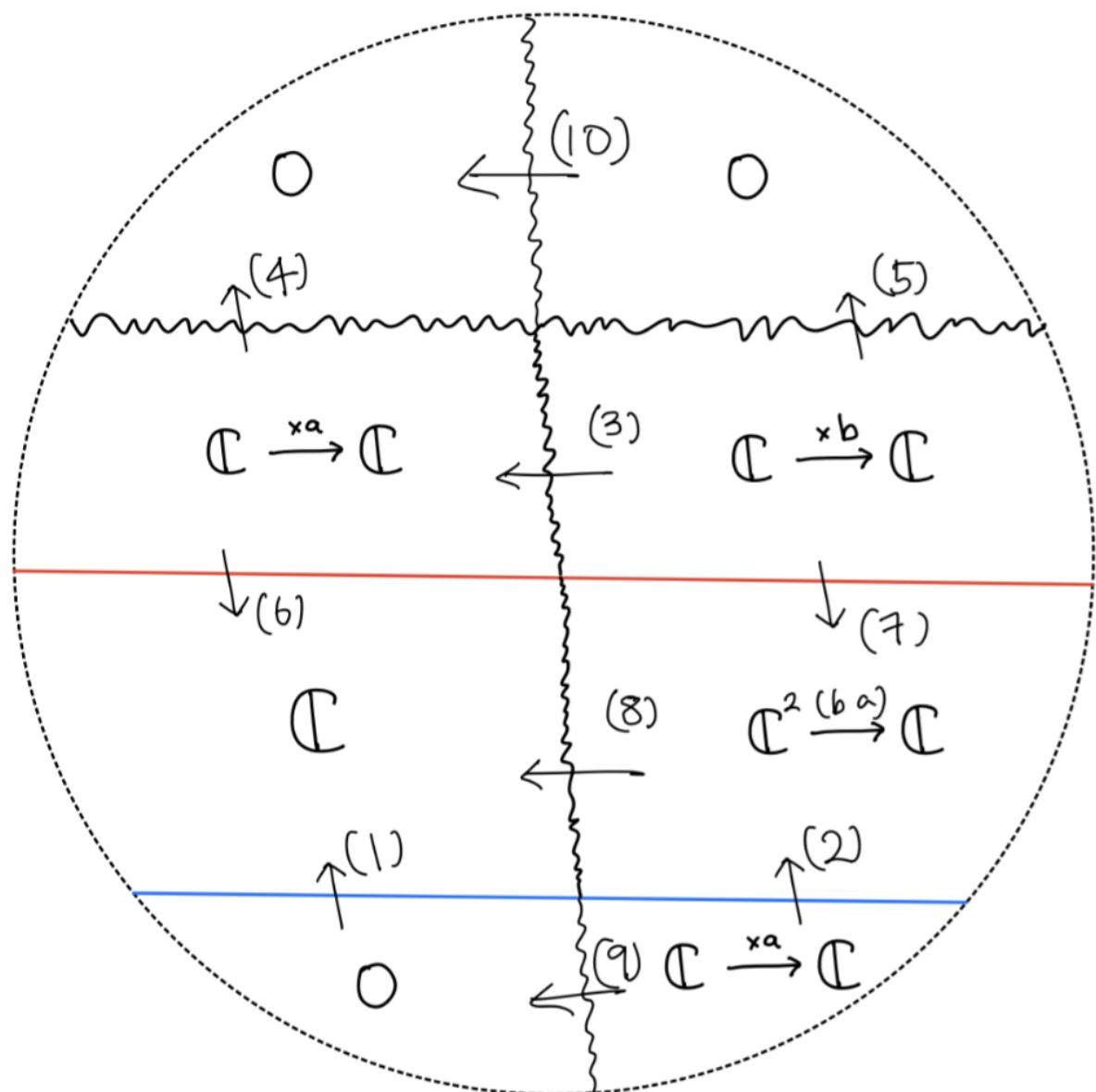


Figure 3.163

$$(1) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \times c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times c \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 134.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

## 3.7 3rd Sheaf Cobordism

In this section, we define  $cobord_3$ , a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

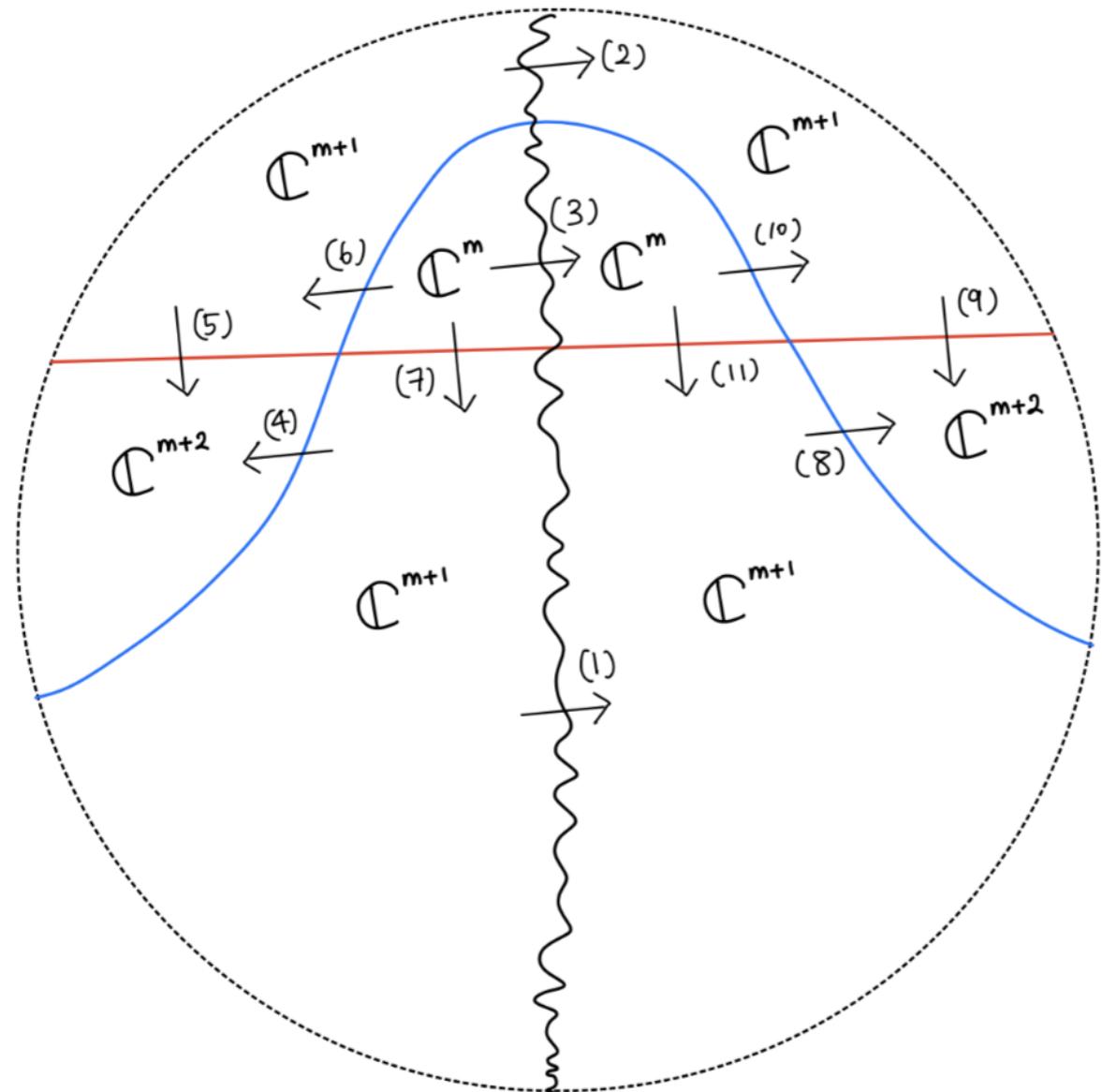


Figure 3.164

**Generalization maps:**

$$(1) \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \mathbb{C}^m \xrightarrow{\theta} \mathbb{C}^m$$

$$(4) \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$$

$$(11) \quad \mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$$

to

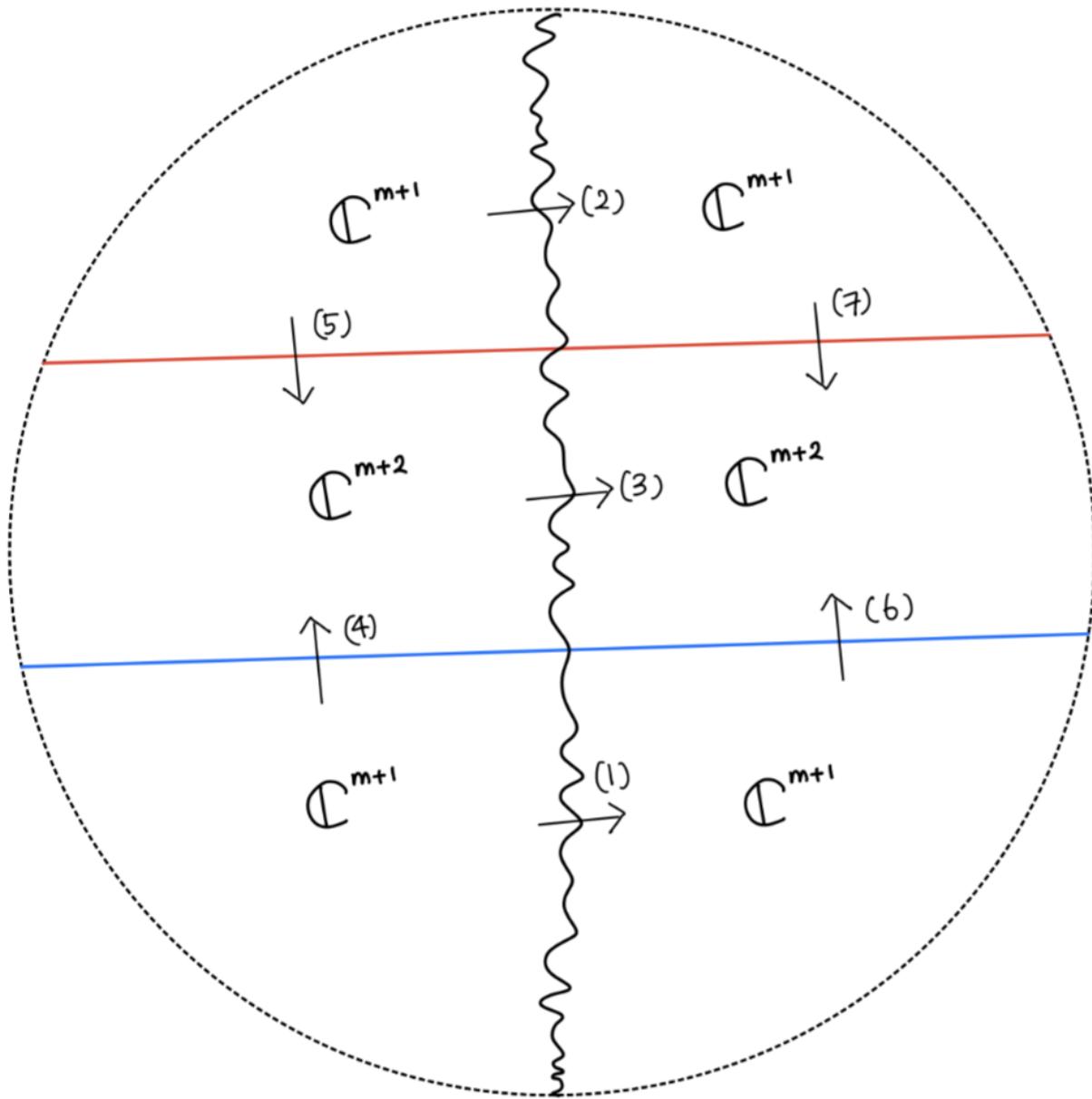


Figure 3.165

**Generalization maps:**

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$
- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

## Notations

**Definition 135.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 136.** For  $t_0 \in \{0, 1\}$  and  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{t_0}^{symbol}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{symbol}$  a co-orientation of  $\Phi_{t_0}^{symbol}$ .

3. we denote the pair  $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$  as  $\Lambda_{t_0}^{symbol}$ . When  $symbol \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{symbol}$  to denote both

4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the

beginning of the isotopy underlying *cobord*<sub>3</sub> and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying *cobord*<sub>3</sub>.

**Definition 137.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{\bullet}^{symbol}$  to be smooth maps

$$\Phi_{\bullet}^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript '*t*' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_{\bullet}^{symbol}$  a co-orientation of  $\Phi_{\bullet}^{symbol}$ .

3. we denote the pair  $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$  as  $\Lambda_{\bullet}^{symbol}$ . Later in the section,  $\Lambda_{\bullet}^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>3</sub>.

4. we denote the triple  $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$  as  $\Lambda_{\bullet}$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_{\bullet}$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>3</sub>.

**Definition 138.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} \frac{3}{4}e^{(\frac{x^2}{x^2-1})}(1-t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$  if  $t \neq 0$

- $\{(1, 0), (-1, 0), (0, \frac{3}{4}(1-t))\} \subset Graph(\Psi_t)$

**Definition 139.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$

$$(x, z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 140.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_0(x)\}$
- $\lambda_0^\infty := \{(x, z) \in D_{r=2} \mid z = \frac{1}{2}\}$
- $\lambda_0^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$

2. We define co-orientations  $\xi_0^{symbol}$  of  $\lambda_0^{symbol}$  as follows

- $\xi_0^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
- $\xi_0^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- $\xi_0^{squig}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 141.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid z = \Psi_1(x)\} = \{(x, z) \in D_{r=2} \mid z = 0\}$
- $\lambda_1^\infty := \{(x, z) \in D_{r=2} \mid z = \frac{1}{2}\}$

- $\lambda_1^{squig} := \{(x, z) \in D_{r=2} \mid x = 0\}$
2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows
- $\xi_1^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
  - $\xi_1^{squig}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

**Definition 142.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \Psi_t(x)\}$
  - $\lambda_\bullet^\infty := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = \frac{1}{2}\}$
  - $\lambda_\bullet^{squig} := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = 0\}$
2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows
- $\xi_\bullet^0$ : hairs are pointing downward direction i.e. coefficients of  $dz$  are negative.
  - $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
  - $\xi_\bullet^{squig}$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are positive.

**Definition 143.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are

separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 144.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 145.** For  $i = 1, 2, 3$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := \{ & (x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(z - \Psi_{t_0}(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3 \} \end{aligned}$$

**Definition 146.** For  $i = 1, 2, 3$  and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3) := \{ & (x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(z - \Psi_t(x)) = sgn_1, \quad sgn(\frac{1}{2} - z) = sgn_2, \\ & sgn((x) = sgn_3 \} \end{aligned}$$

**Definition 147.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the above notations:

1.  $\mathcal{S}_0$ :

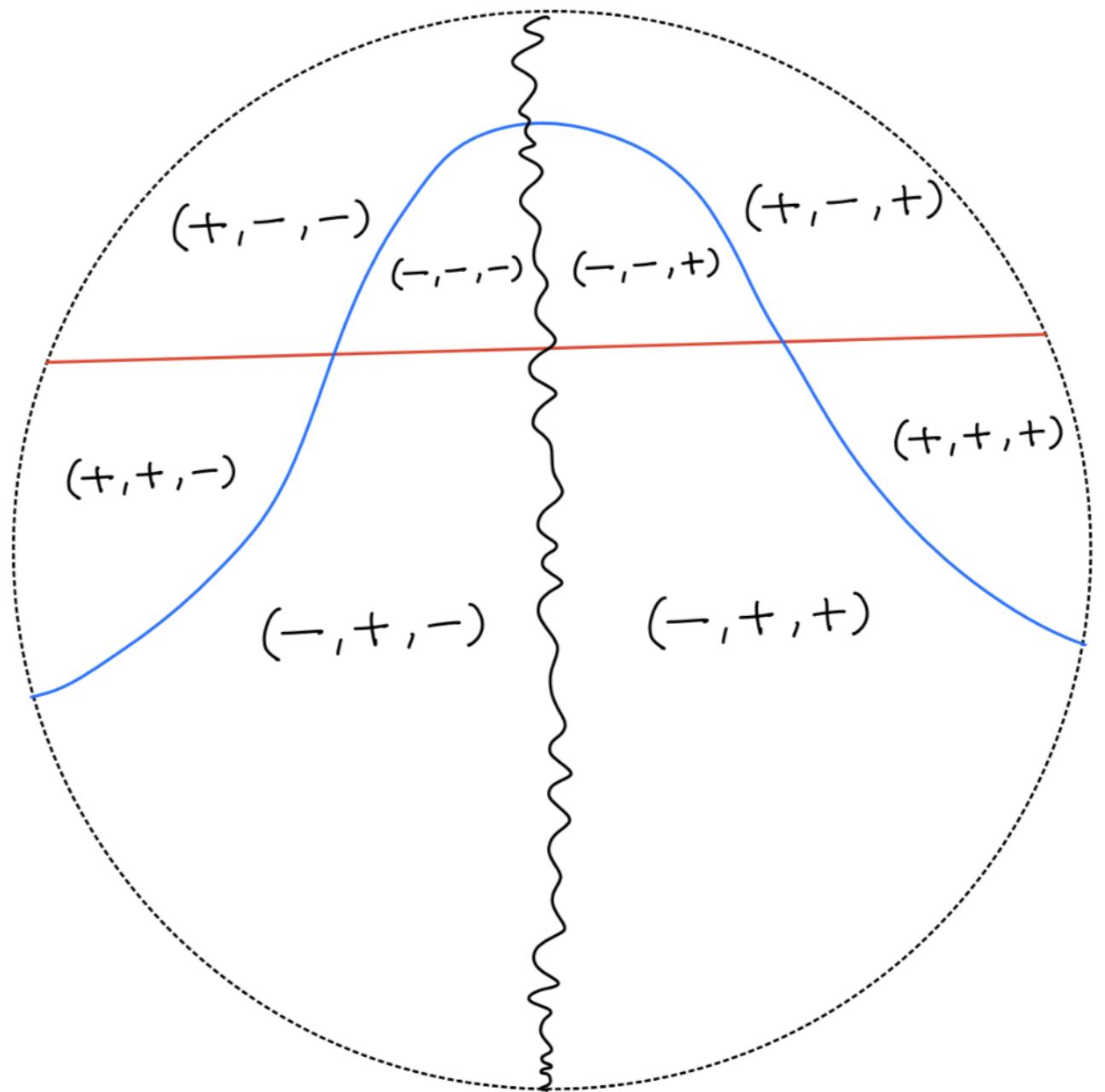


Figure 3.166

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

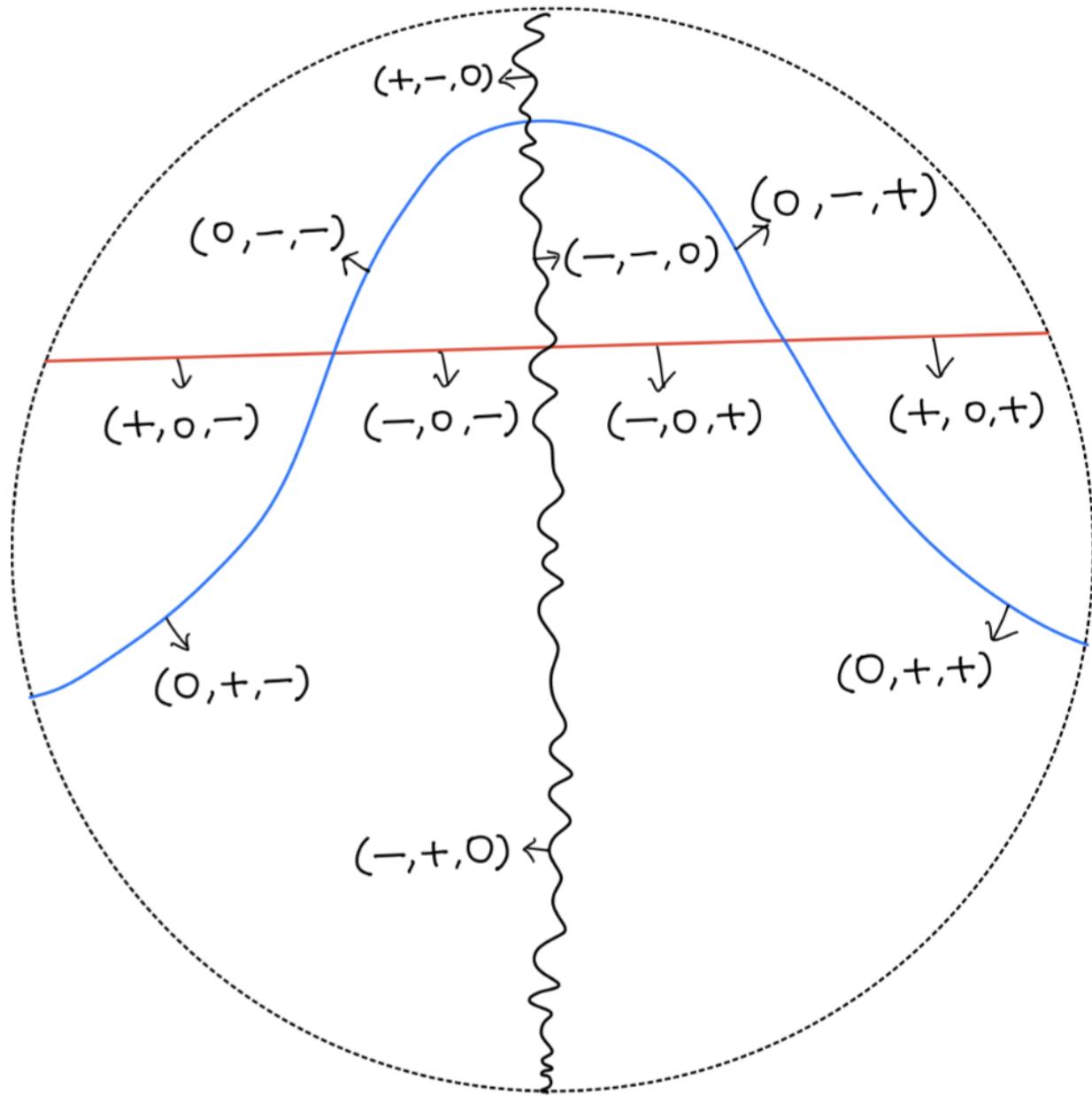


Figure 3.167

- 1 dimensional strata:

$$\begin{aligned} & \{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, +, 0)\} \end{aligned}$$

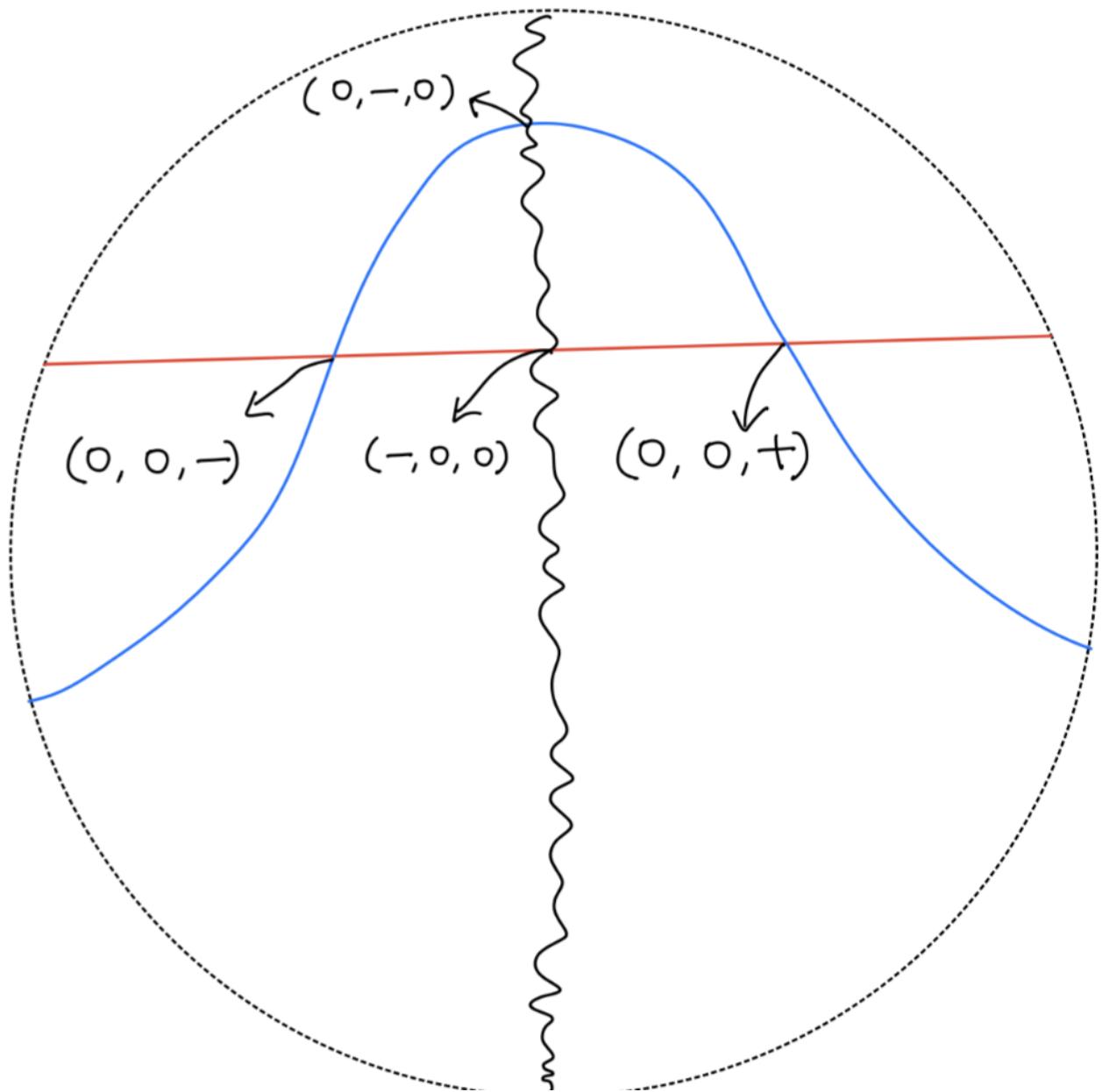


Figure 3.168

- 0 dimensional strata:

$$s_0(0, 0, +), s_0(-, 0, 0)$$

2.  $\mathcal{S}_1$ :

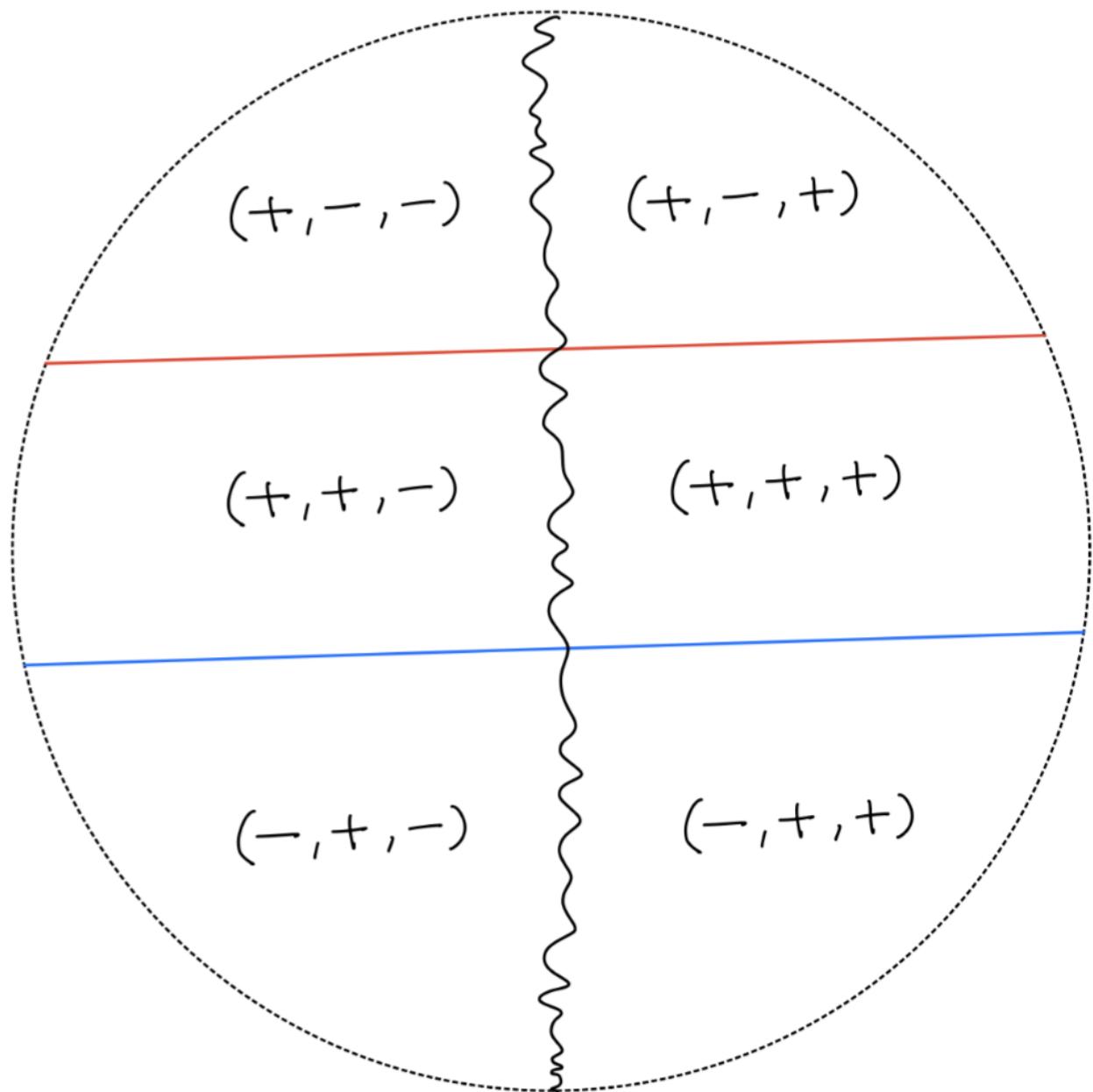


Figure 3.169

- 2 dimensional strata:

$$s_1(+, -, -), s_1(+, -, +), s_1(+, +, -), s_1(+, +, +), s_1(-, +, -), s_1(-, +, +)$$

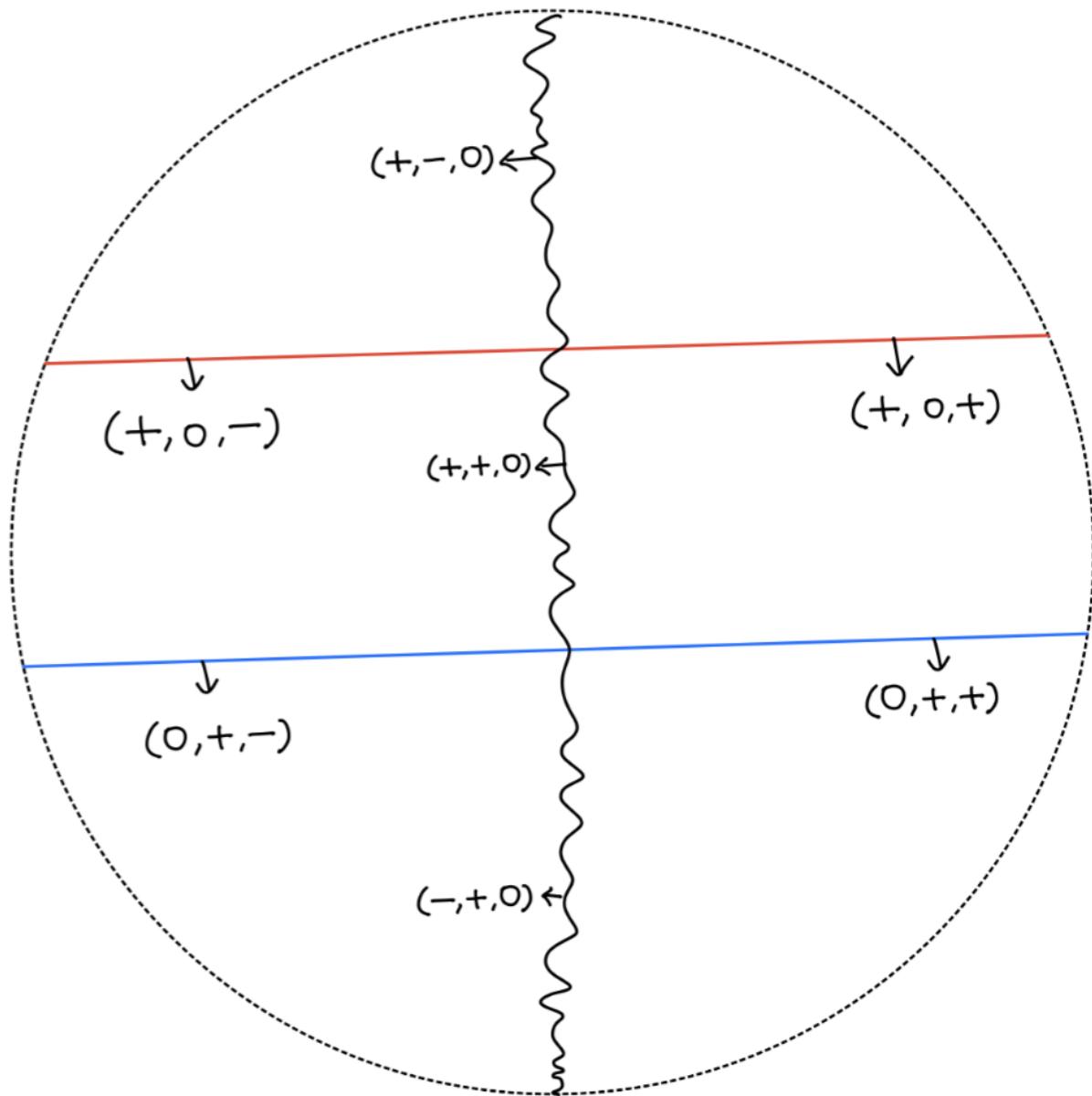


Figure 3.170

- 1 dimensional strata:

$$s_1(0, +, -), s_1(0, +, +), s_1(+, 0, -), s_1(+, 0, +), s_1(+, -, 0), s_1(+, +, 0), s_1(-, +, 0)$$

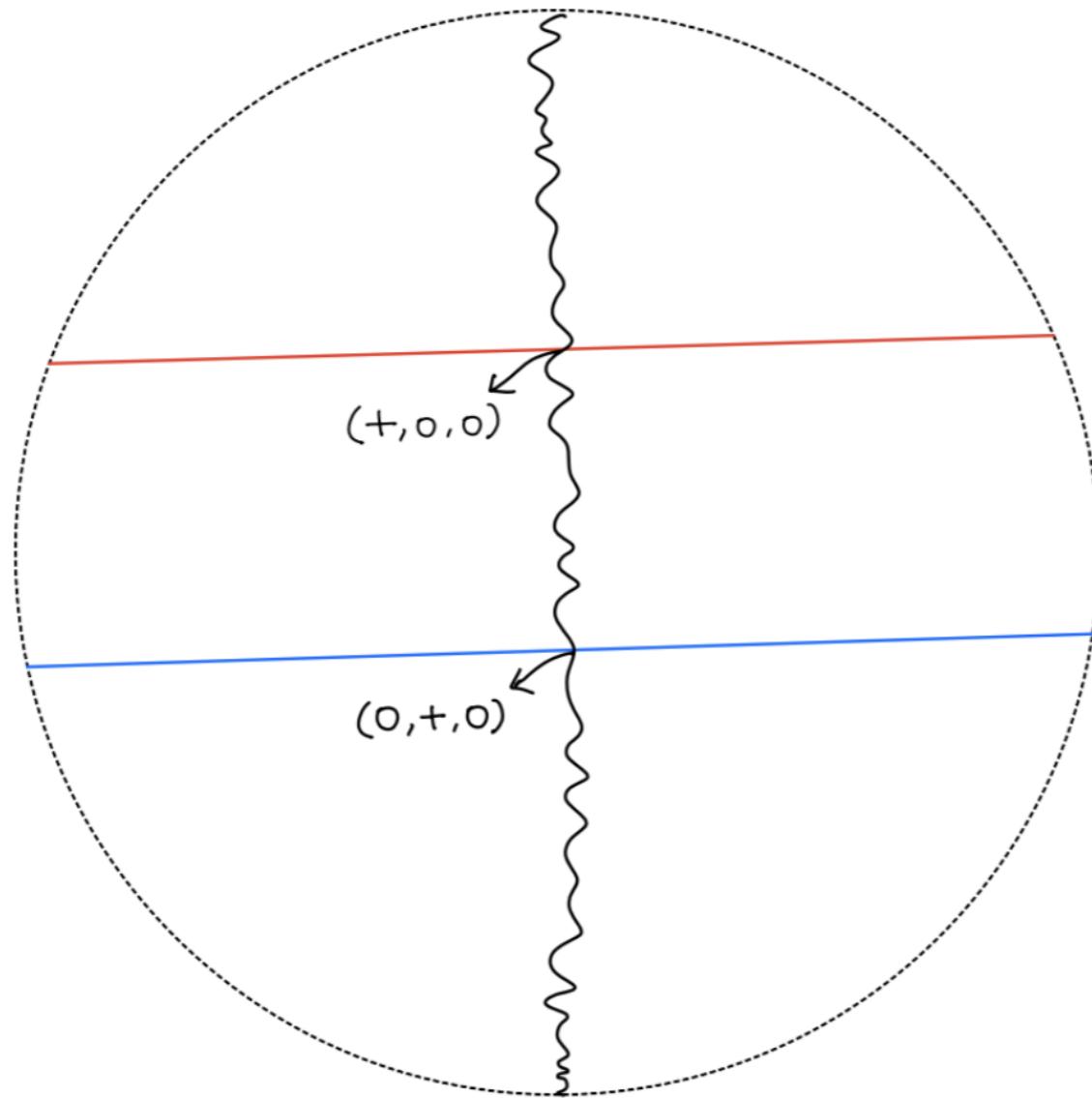


Figure 3.171

- 0 dimensional strata:

$$s_1(0, +, 0), s_1(+, 0, 0)$$

3.  $\mathcal{S}_\bullet$ :

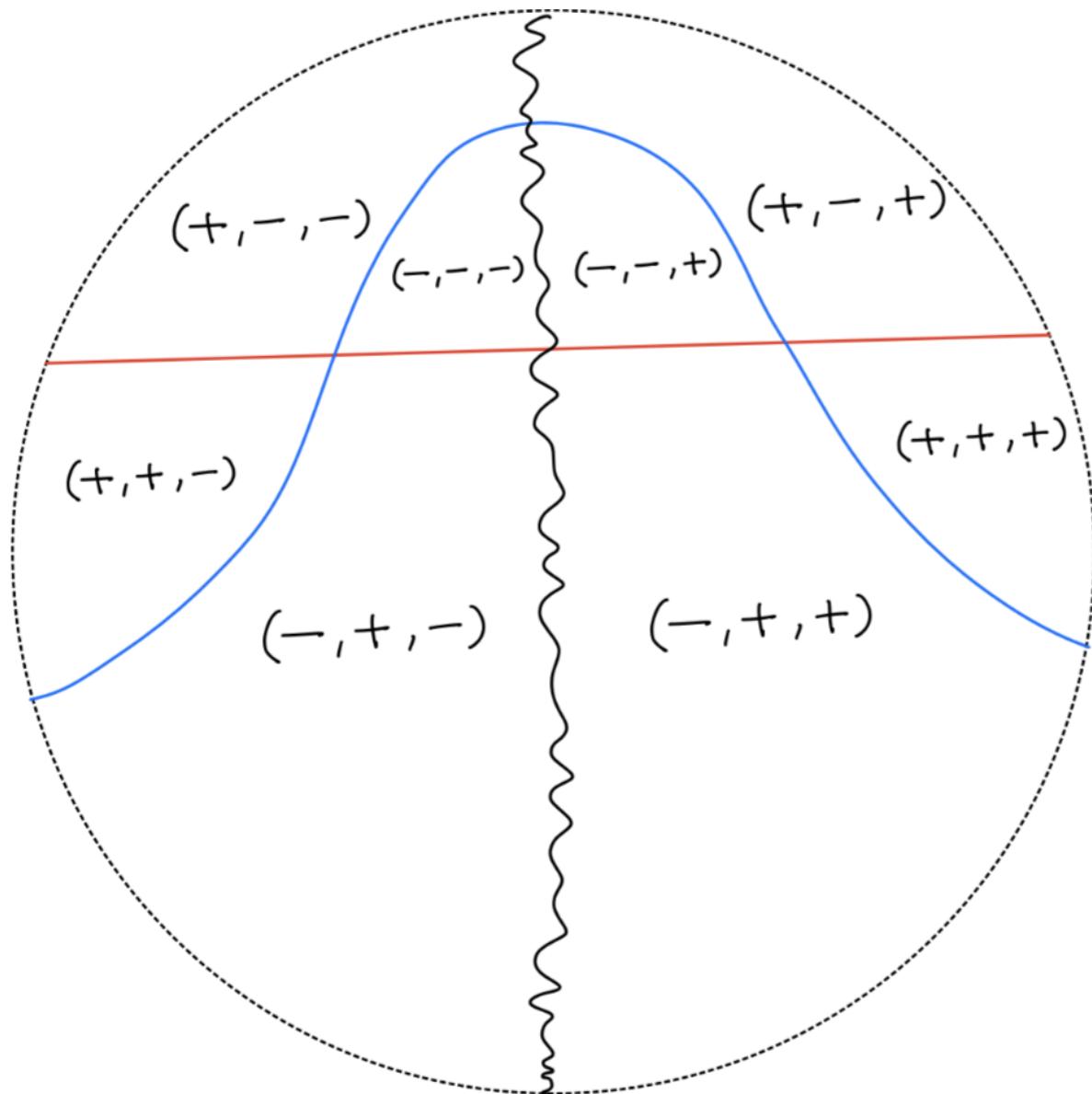


Figure 3.172

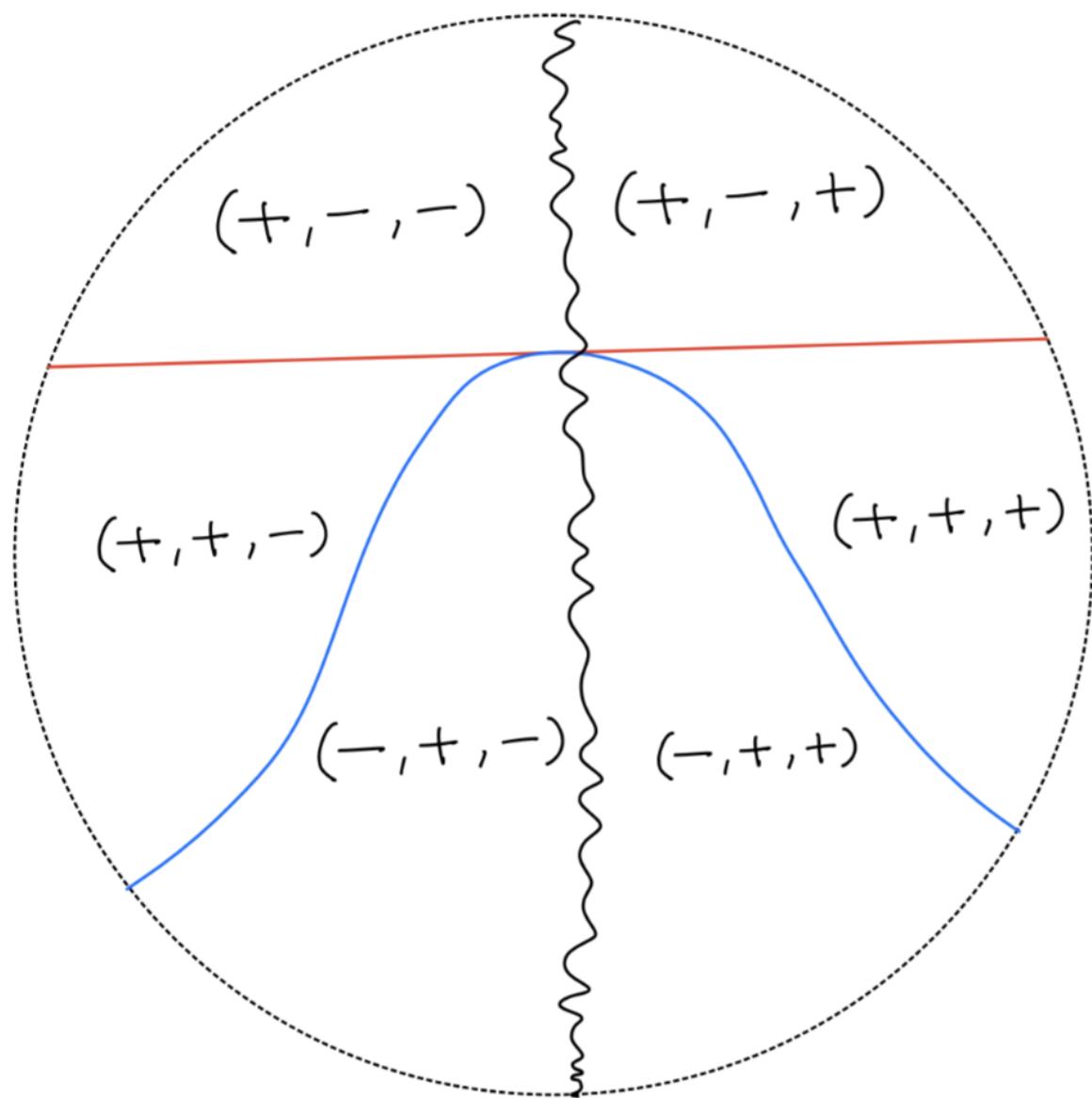


Figure 3.173

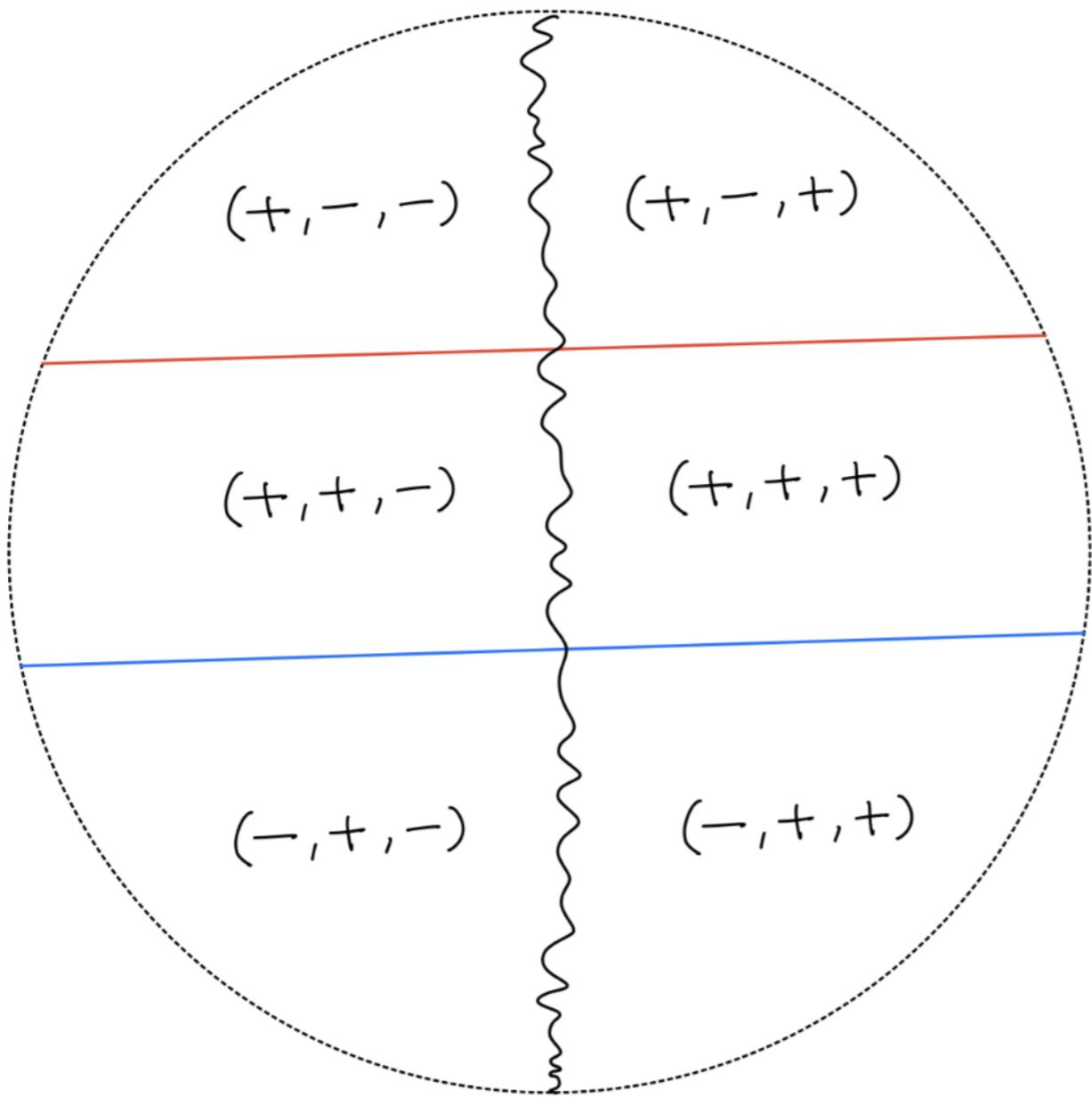


Figure 3.174

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

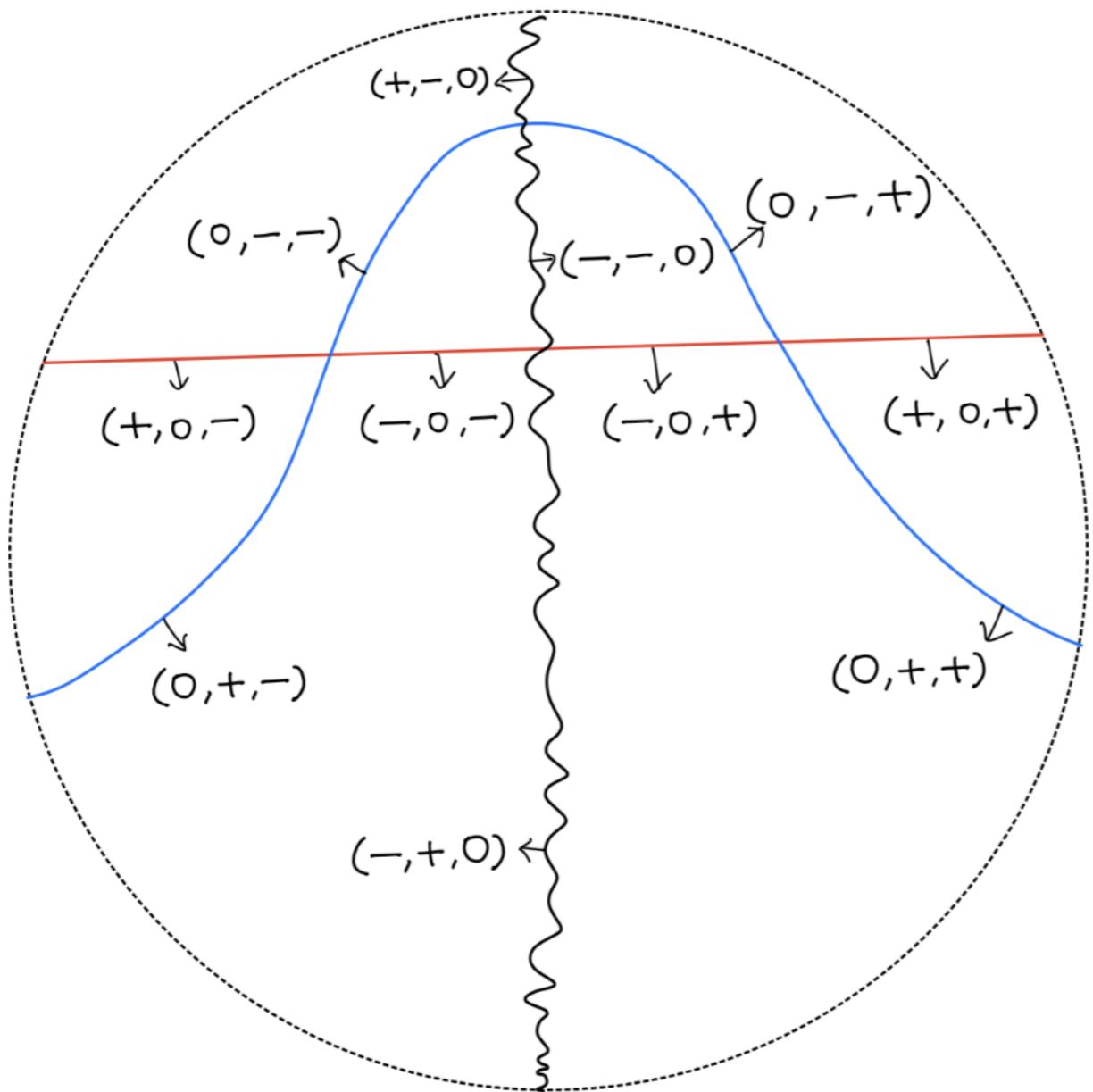


Figure 3.175

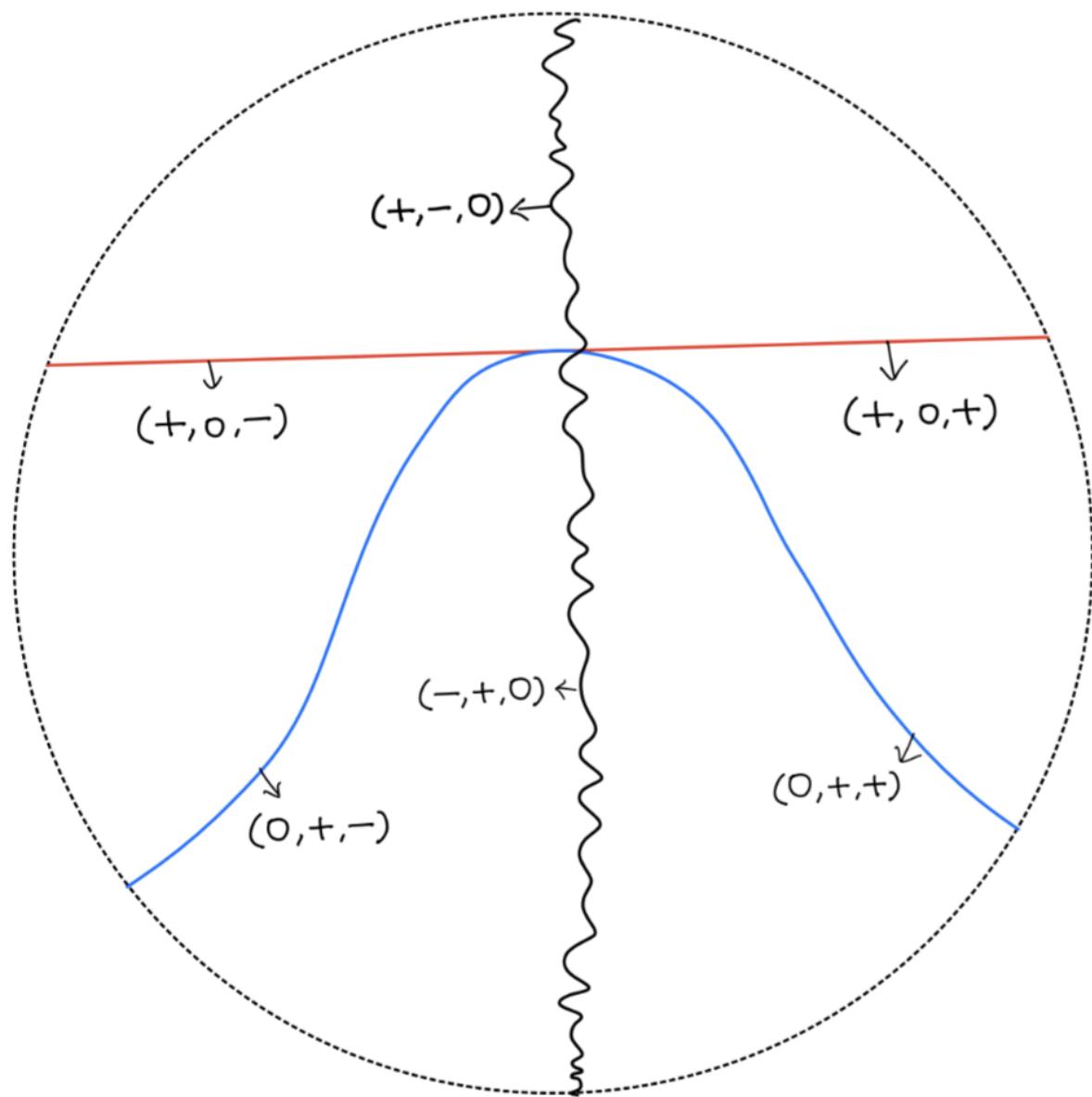


Figure 3.176

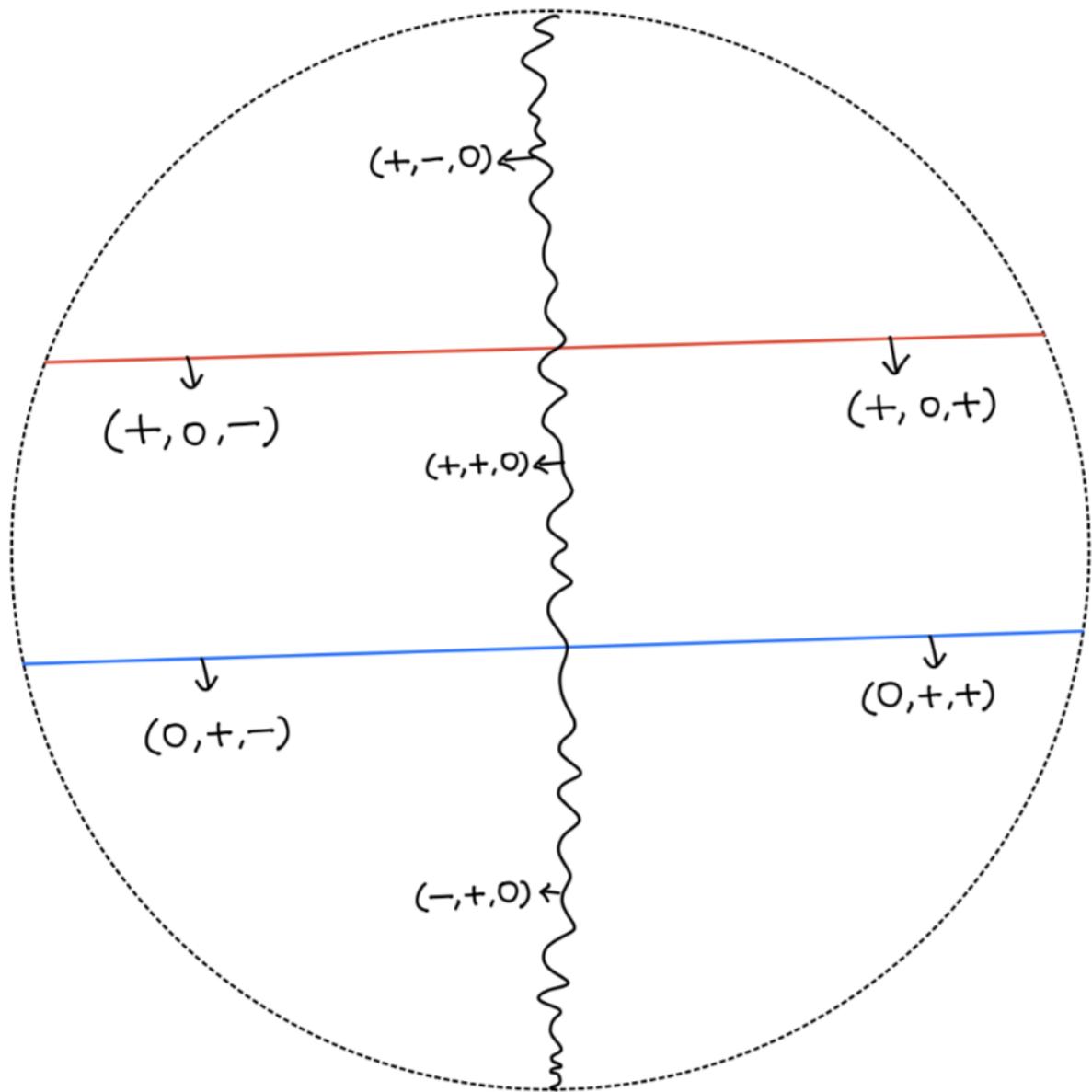


Figure 3.177

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

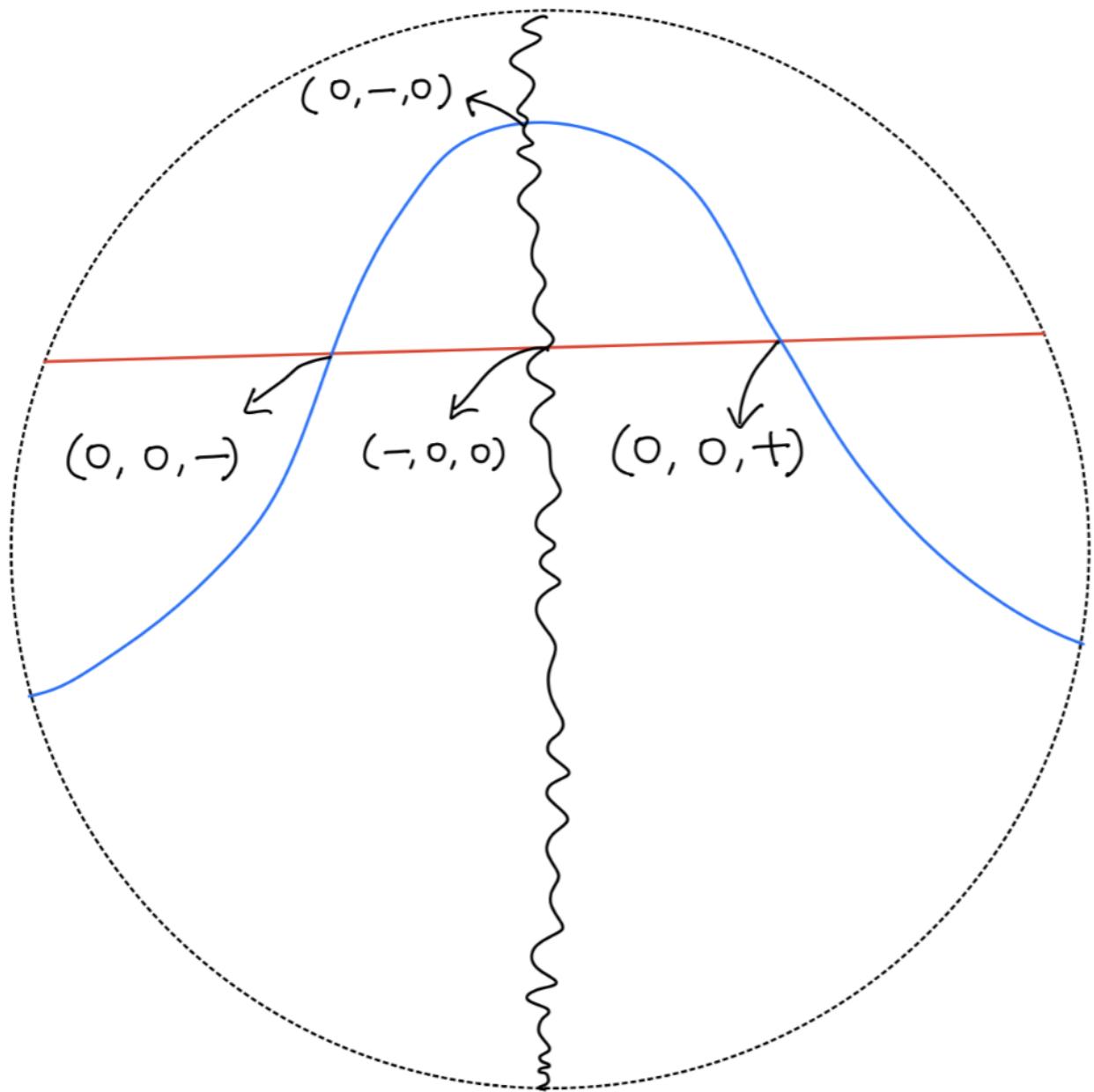


Figure 3.178

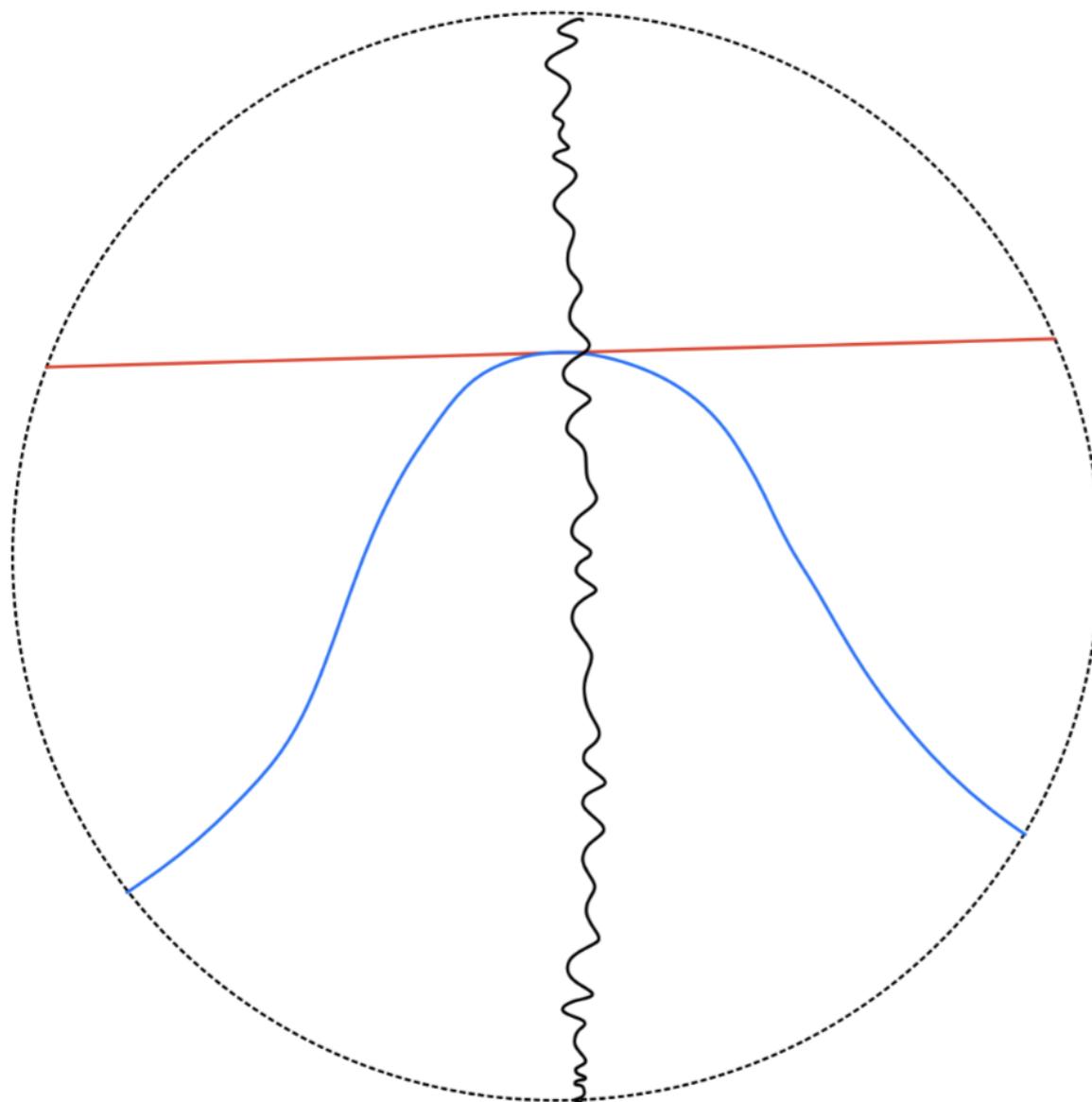


Figure 3.179

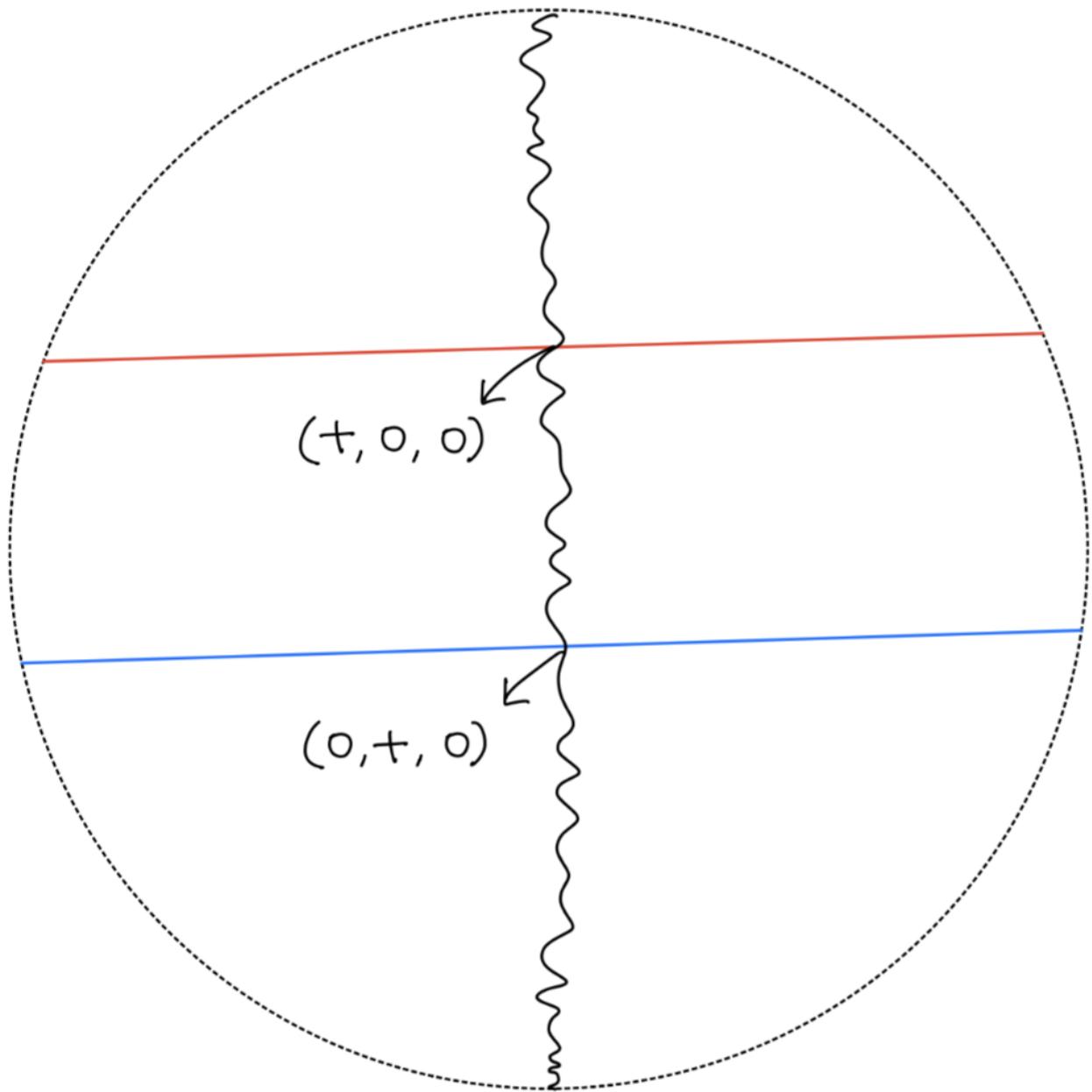


Figure 3.180

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

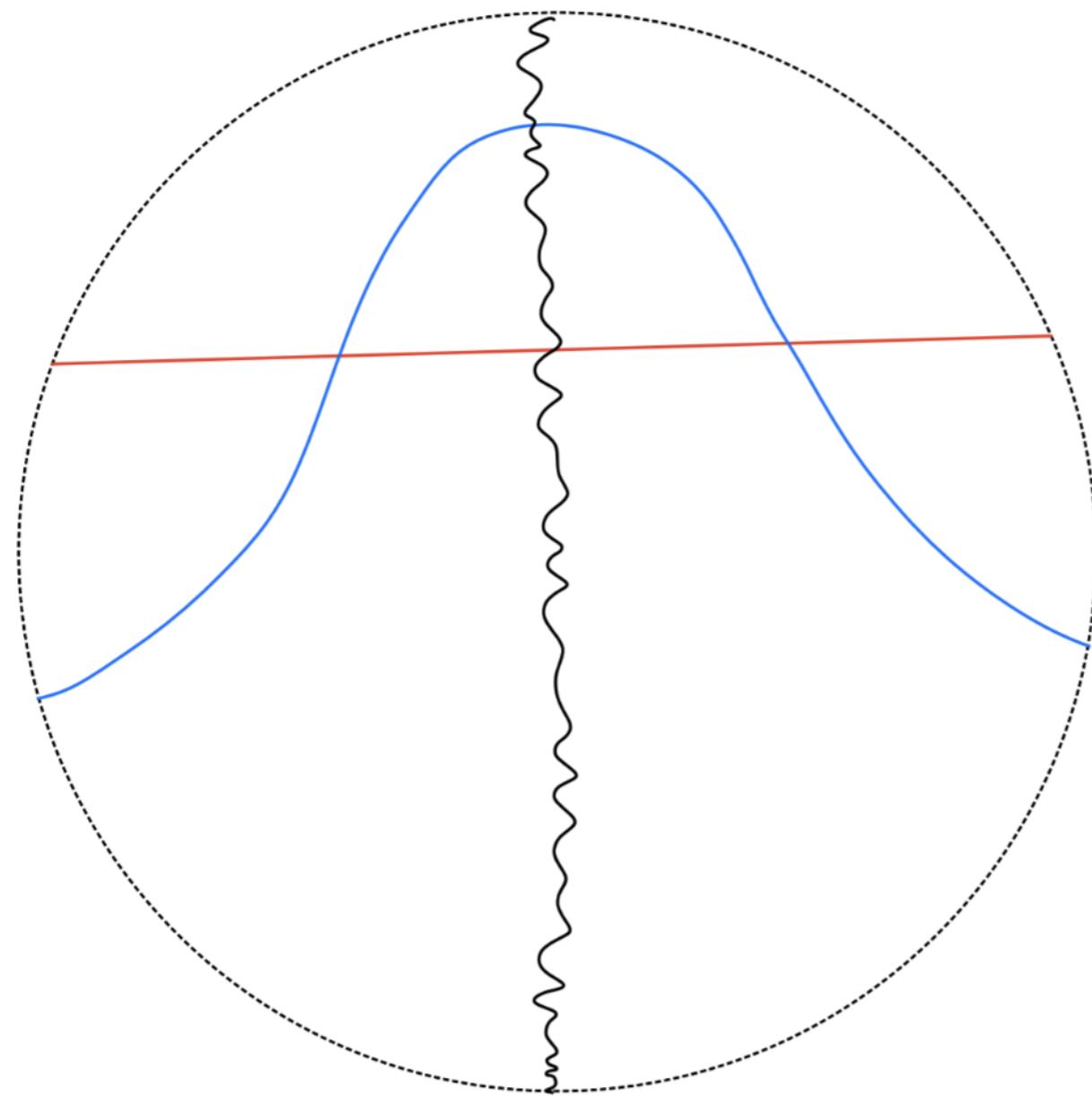


Figure 3.181

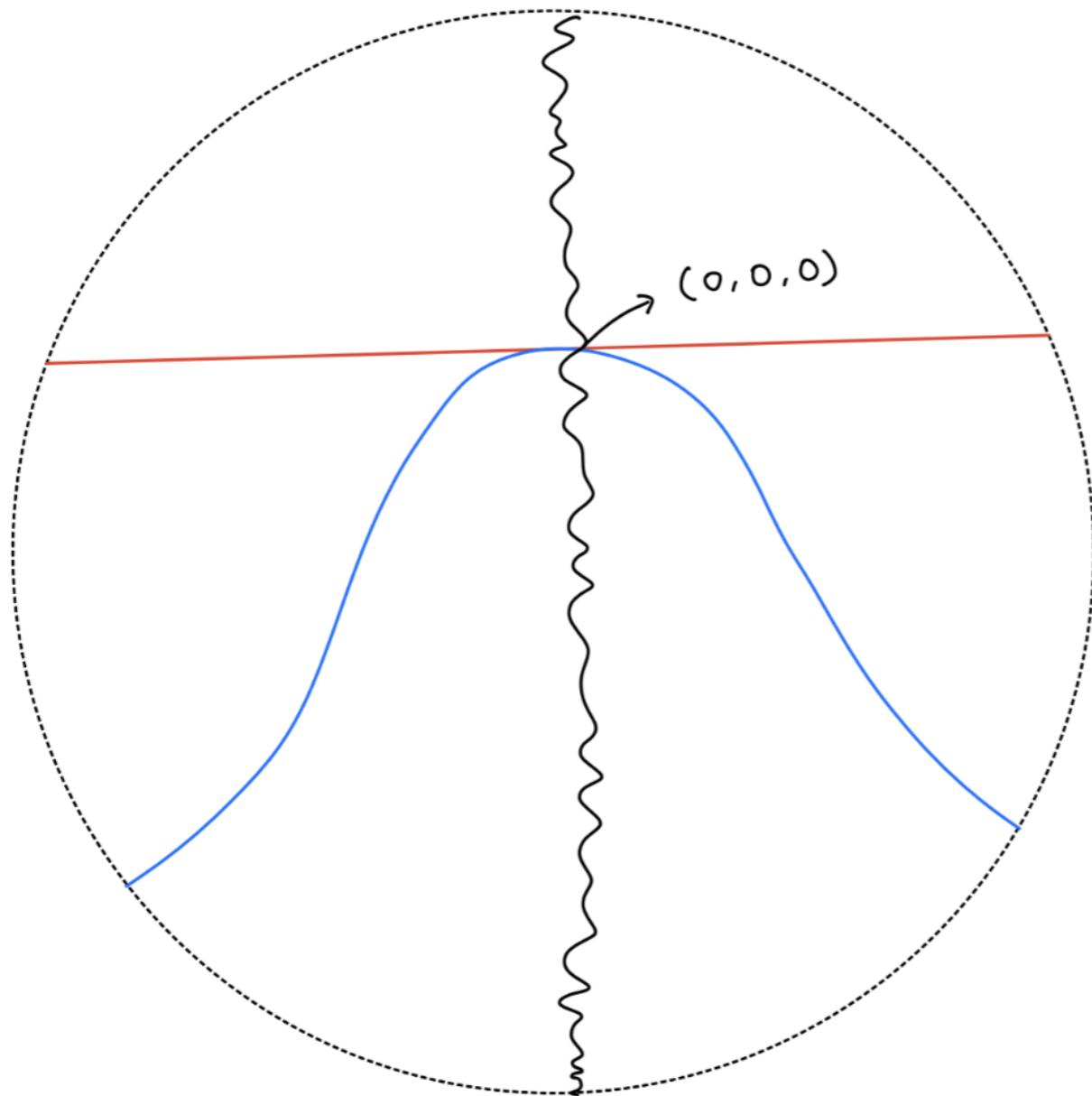


Figure 3.182

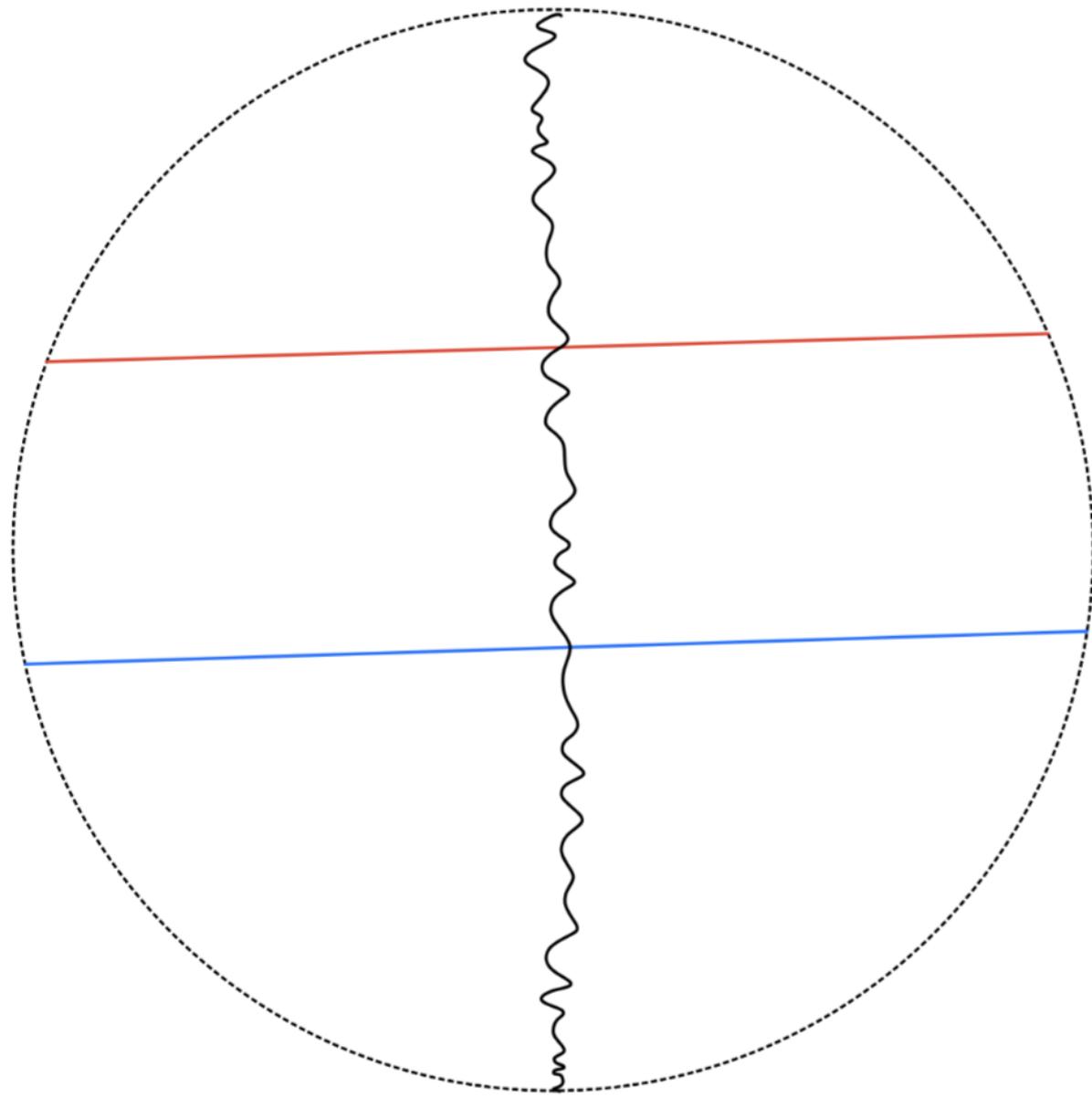


Figure 3.183

- 0 dimensional strata:

$$s_{\bullet}(0, 0, 0)$$

**Definition 148.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $\text{star}(s)$ .

**Definition 149.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of  $s$ .

**Definition 150.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 151.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $\text{star}(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 152.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 153.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \cdots F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 154.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

### 3.7.1 Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

### 3.7.2 Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3) := F_0(s_0(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

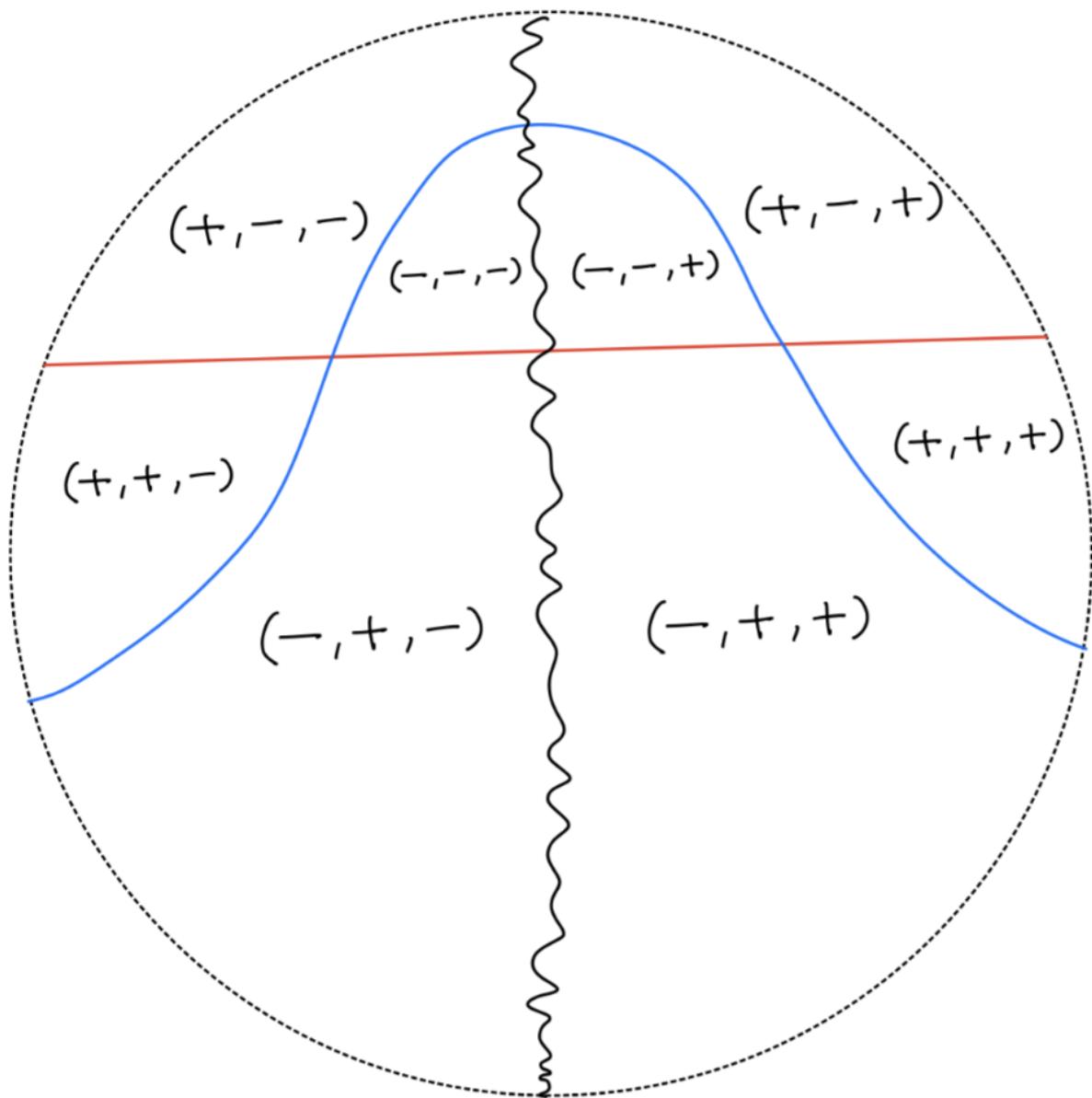


Figure 3.184

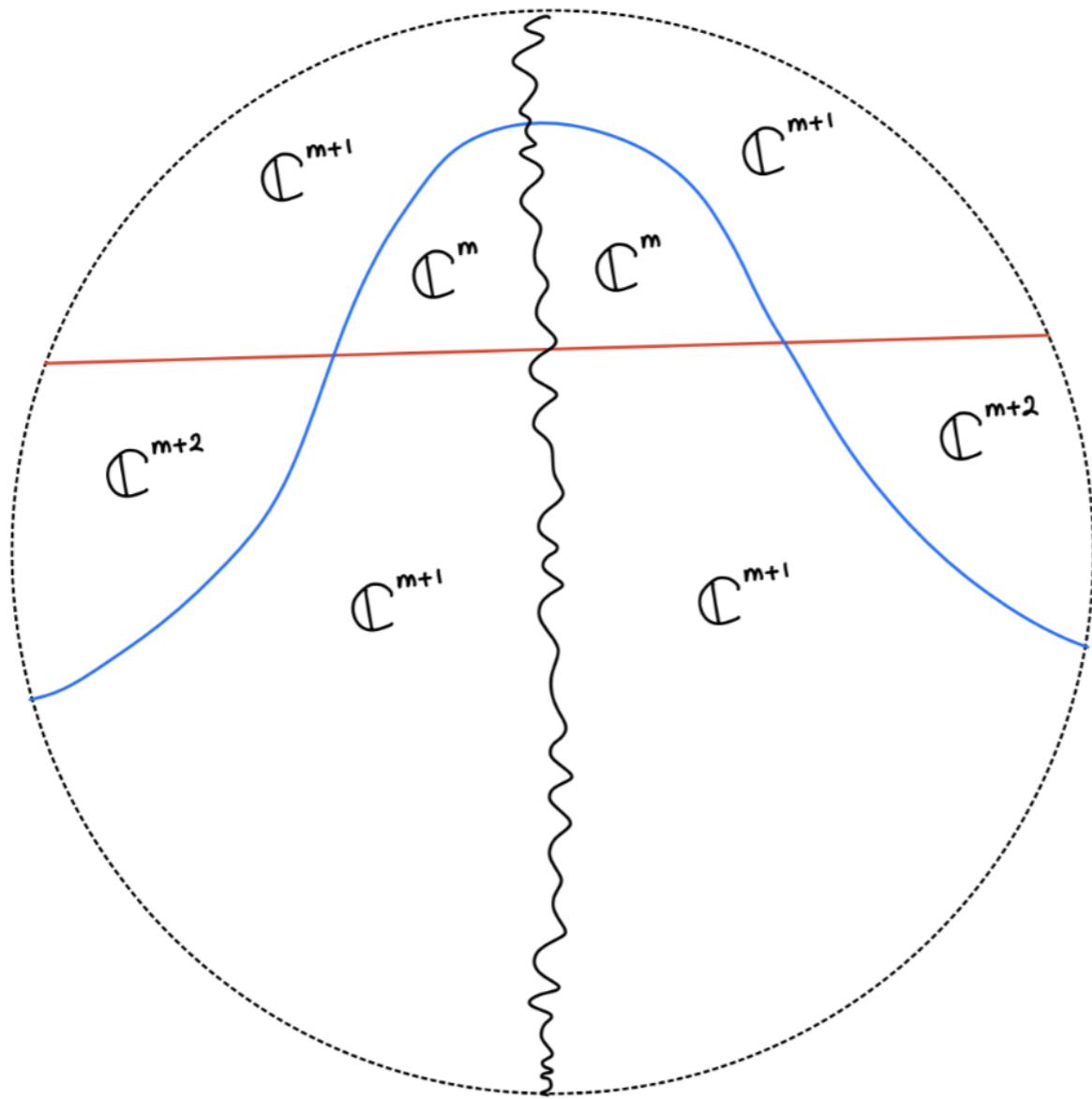


Figure 3.185

- $F_0(-, -, -) := \mathbb{C}^m$
- $F_0(-, -, +) := \mathbb{C}^m$
- $F_0(+, -, -) := \mathbb{C}^{m+1}$
- $F_0(+, -, +) := \mathbb{C}^{m+1}$
- $F_0(-, +, -) := \mathbb{C}^{m+1}$

- $F_0(-, +, +) := \mathbb{C}^{m+1}$

- $F_0(+, +, -) := \mathbb{C}^{m+2}$

- $F_0(+, +, +) := \mathbb{C}^{m+2}$

**Generalization maps:**

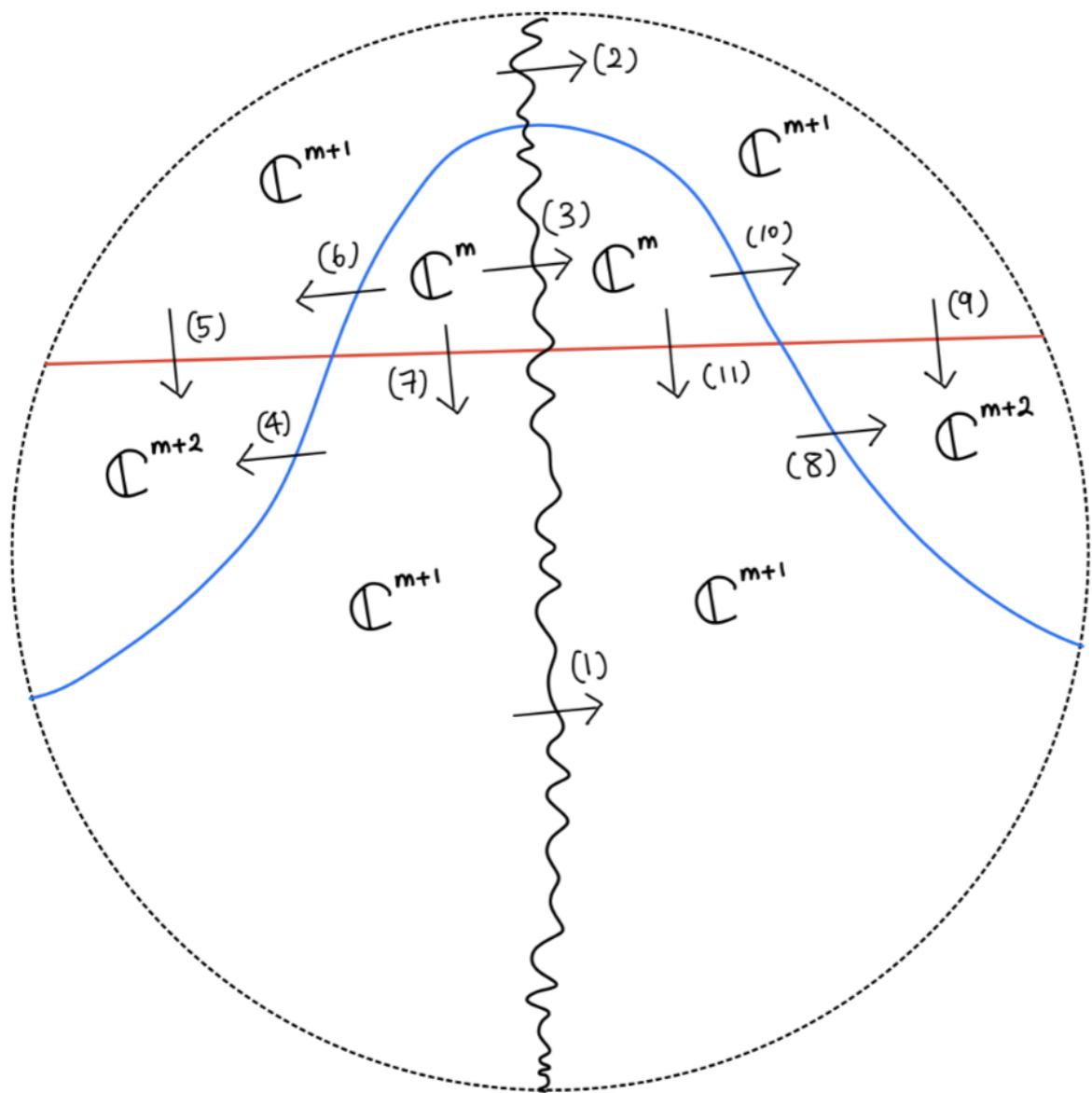


Figure 3.186

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^m \xrightarrow{\theta} \mathbb{C}^m$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$$

$$(11) \quad \mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$$

### 3.7.3 Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say  $cobord_3$ , that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

#### A. Sheaf on $V \times [0, 1]$

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

**B. Sheaf on  $D_{r=2} \times [0, 1]$** 

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

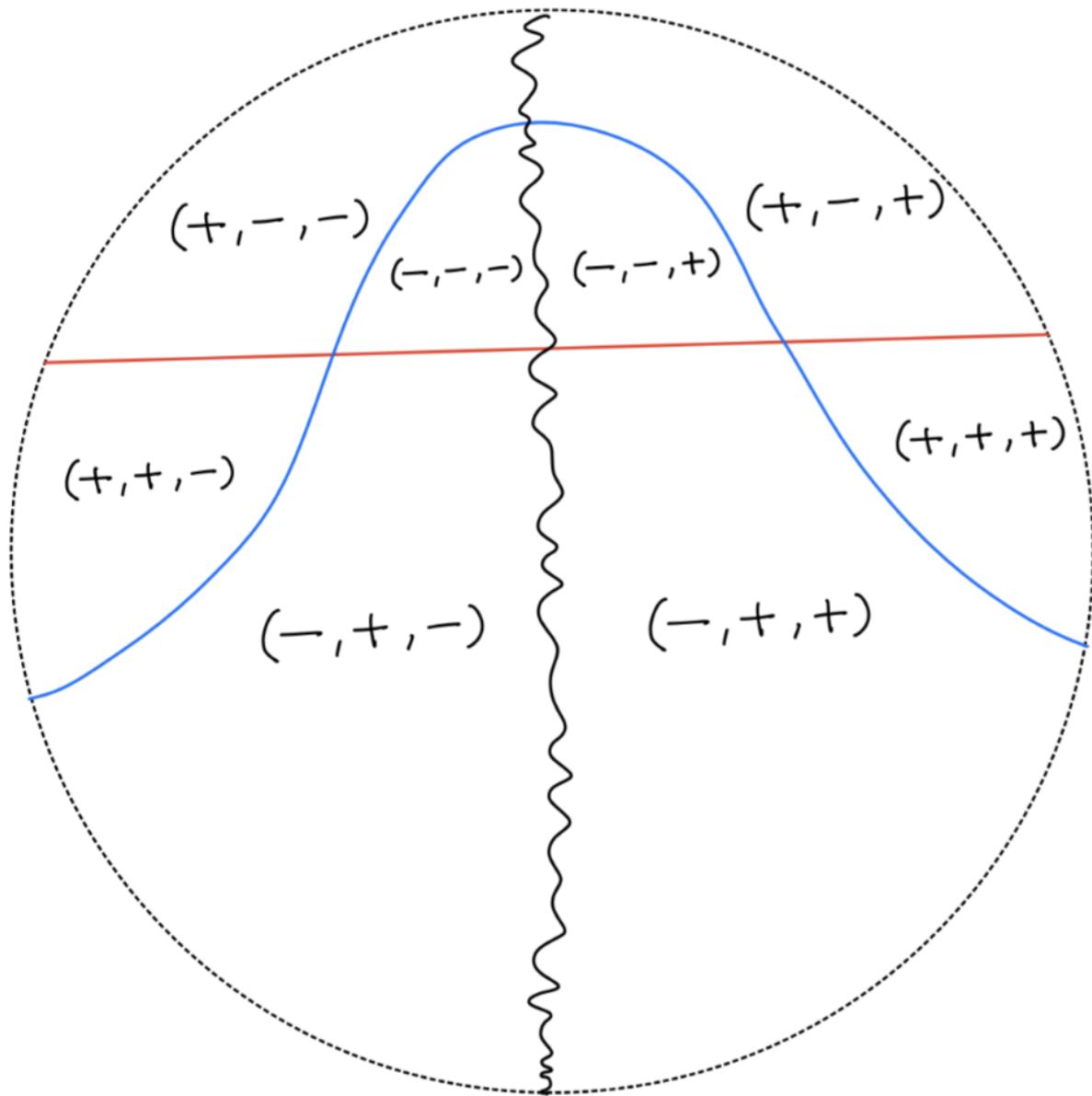


Figure 3.187

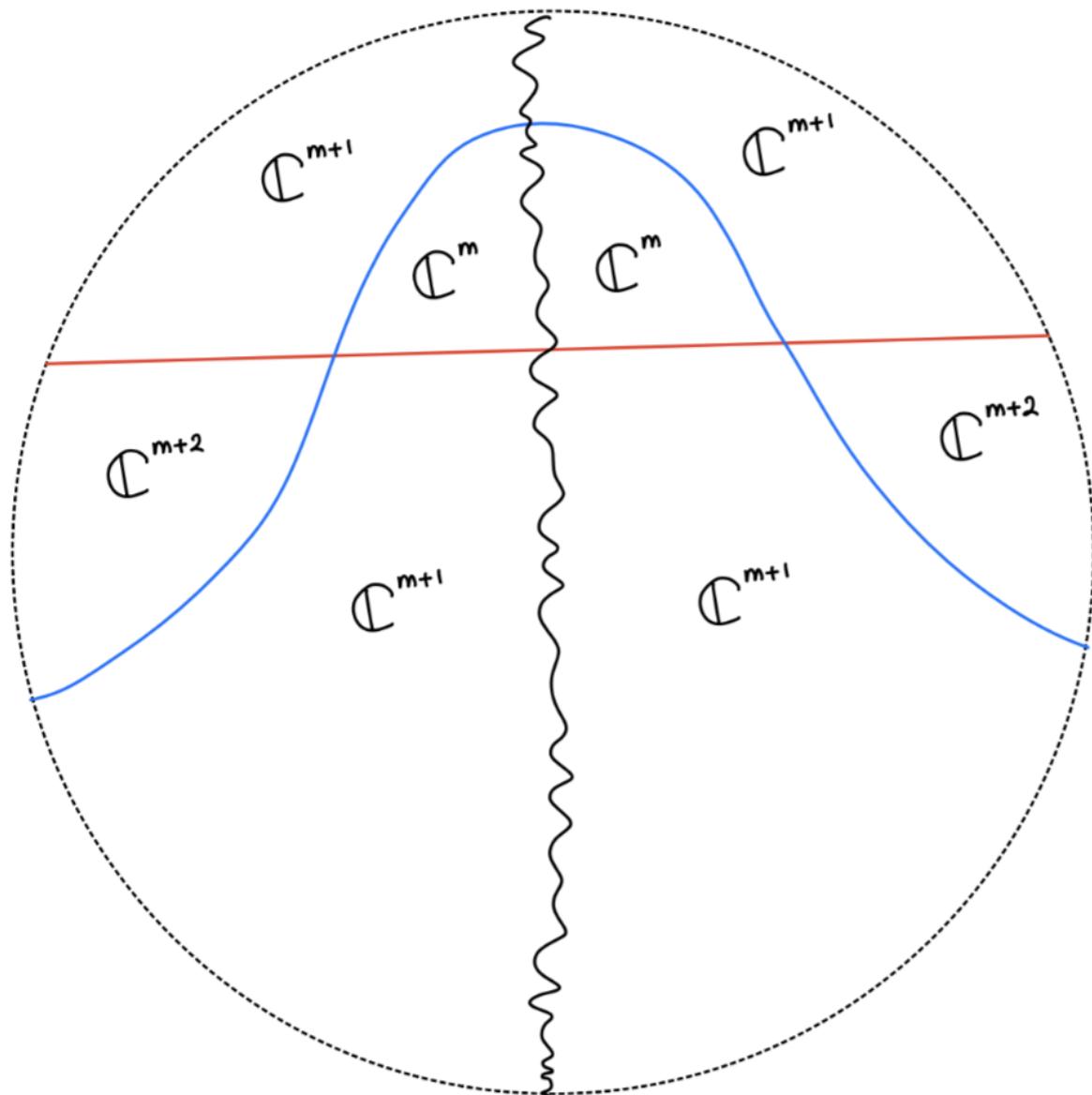


Figure 3.188

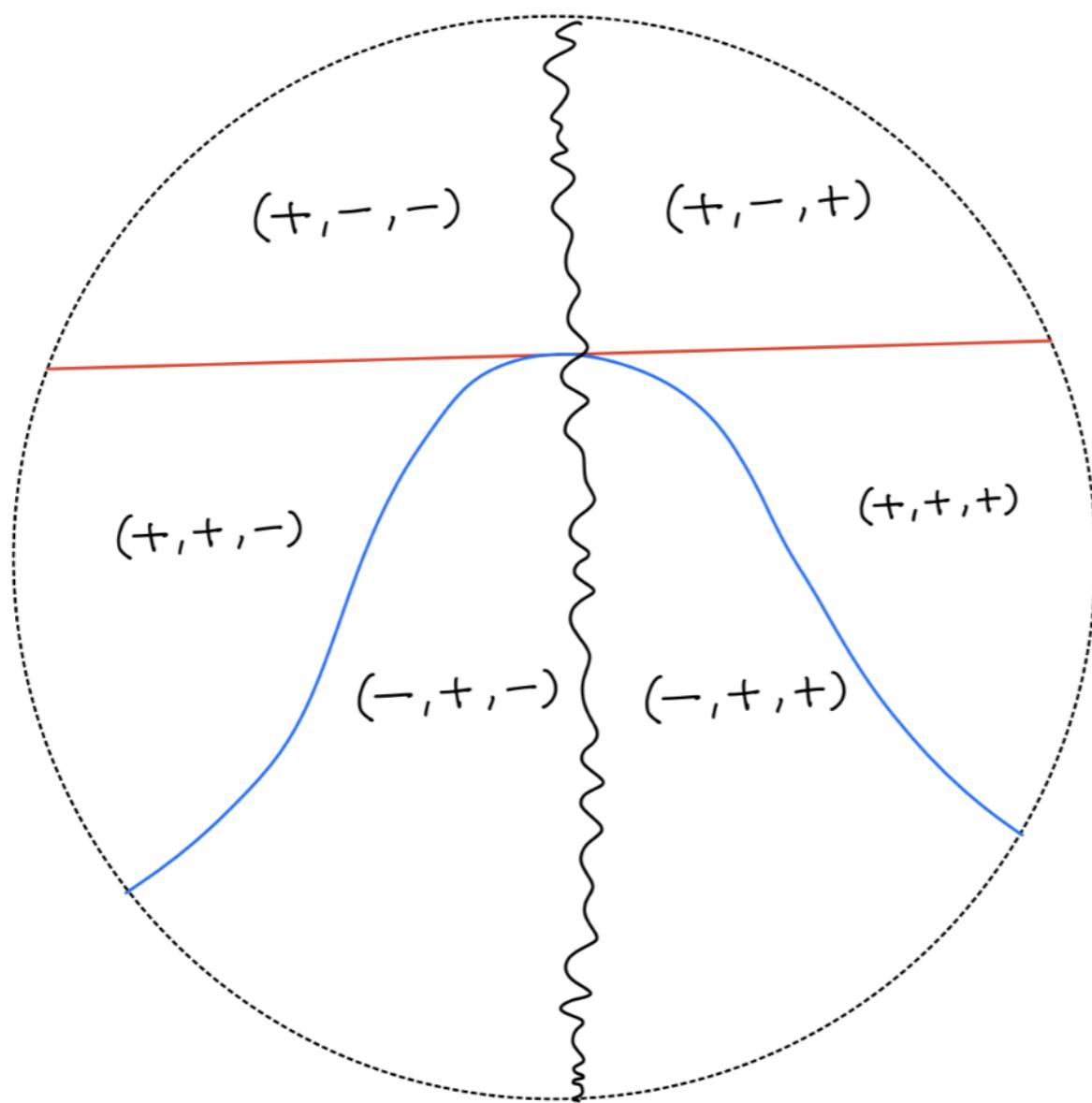


Figure 3.189

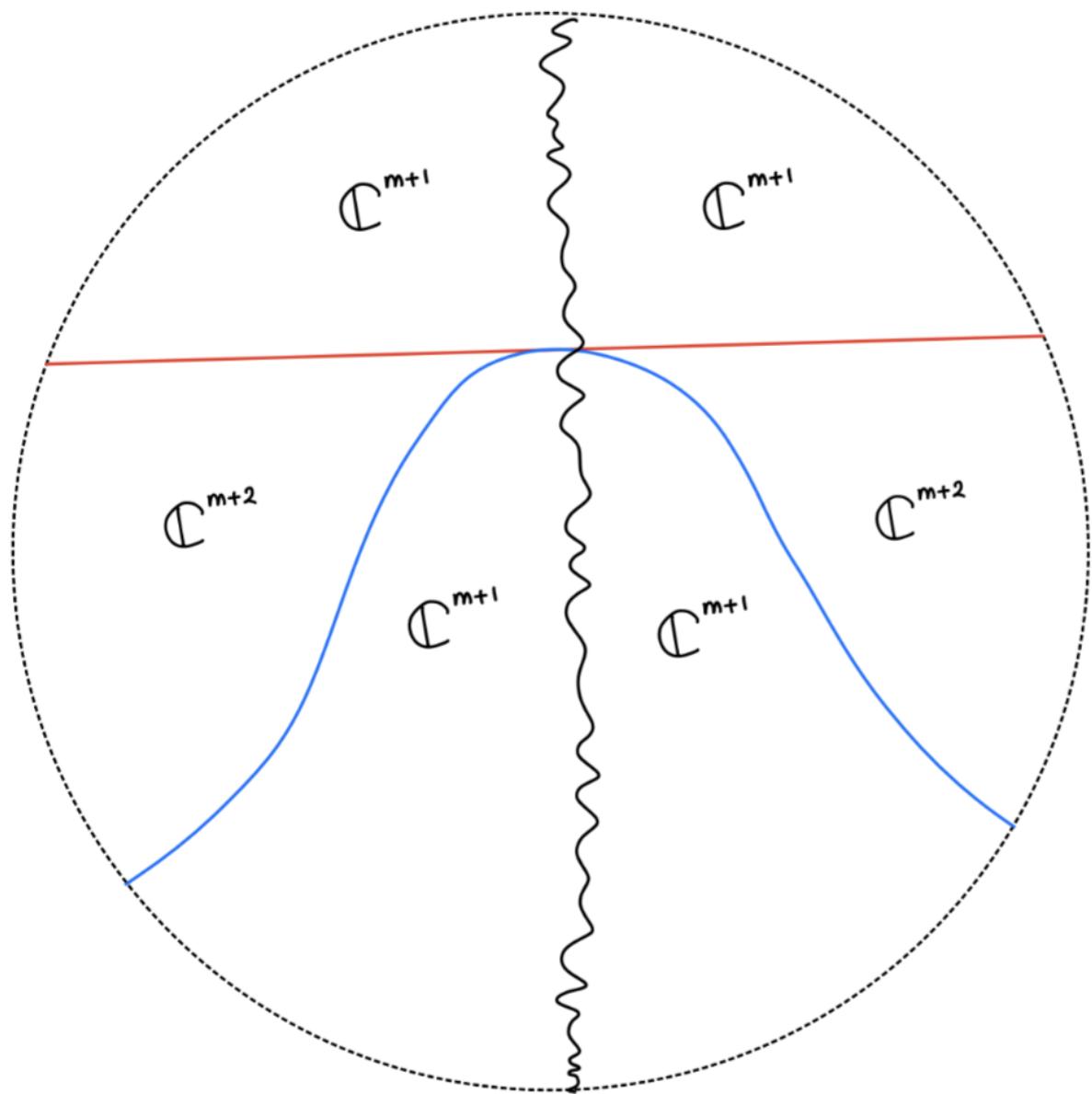


Figure 3.190

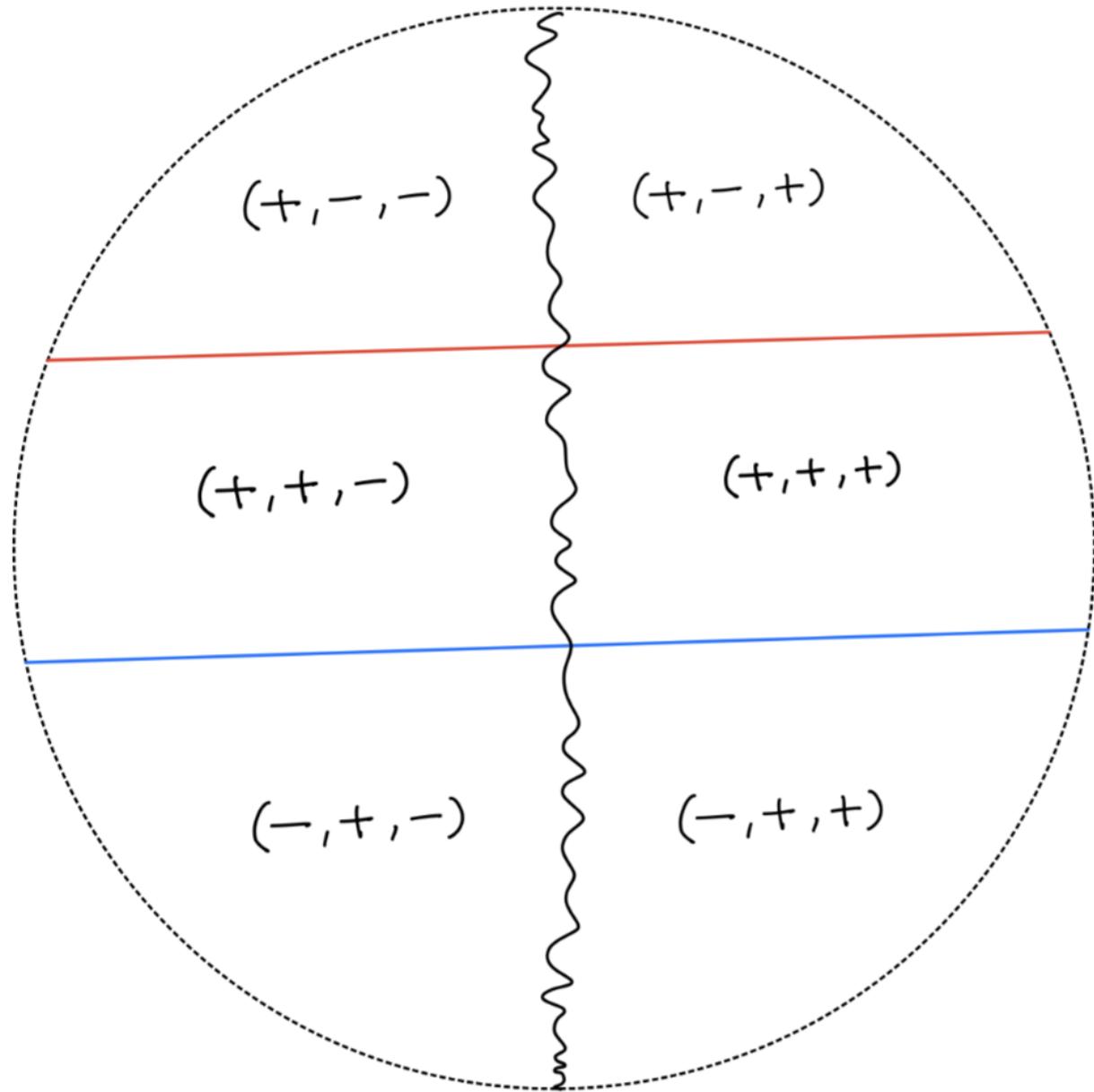


Figure 3.191

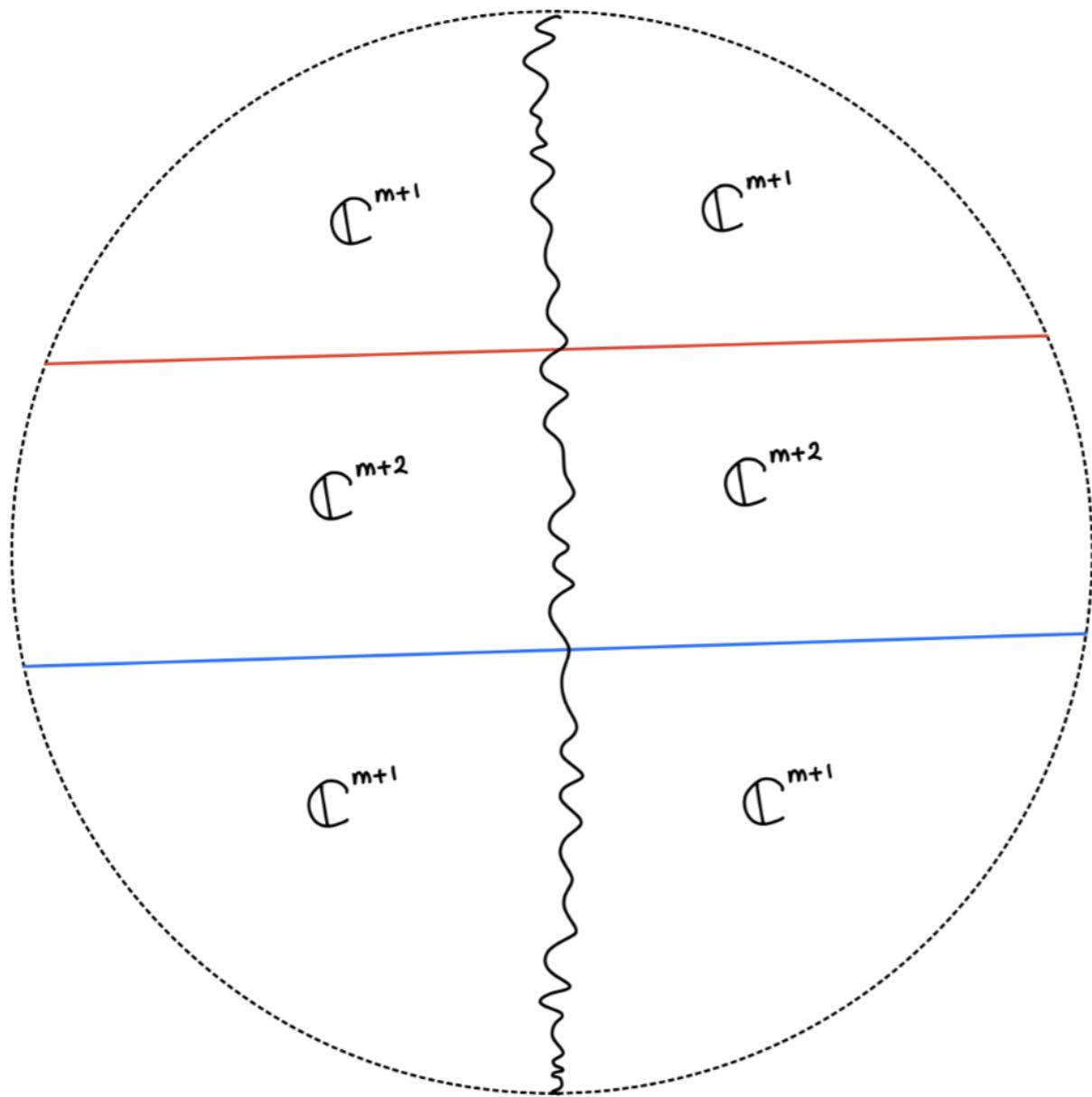


Figure 3.192

- $F_{\bullet}(-, -, -) := \mathbb{C}^m$
- $F_{\bullet}(-, -, +) := \mathbb{C}^m$
- $F_{\bullet}(+, -, -) := \mathbb{C}^{m+1}$
- $F_{\bullet}(+, -, +) := \mathbb{C}^{m+1}$
- $F_{\bullet}(-, +, -) := \mathbb{C}^{m+1}$

- $F_\bullet(-, +, +) := \mathbb{C}^{m+1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{m+2}$

- $F_\bullet(+, +, +) := \mathbb{C}^{m+2}$

**Generalization maps:**

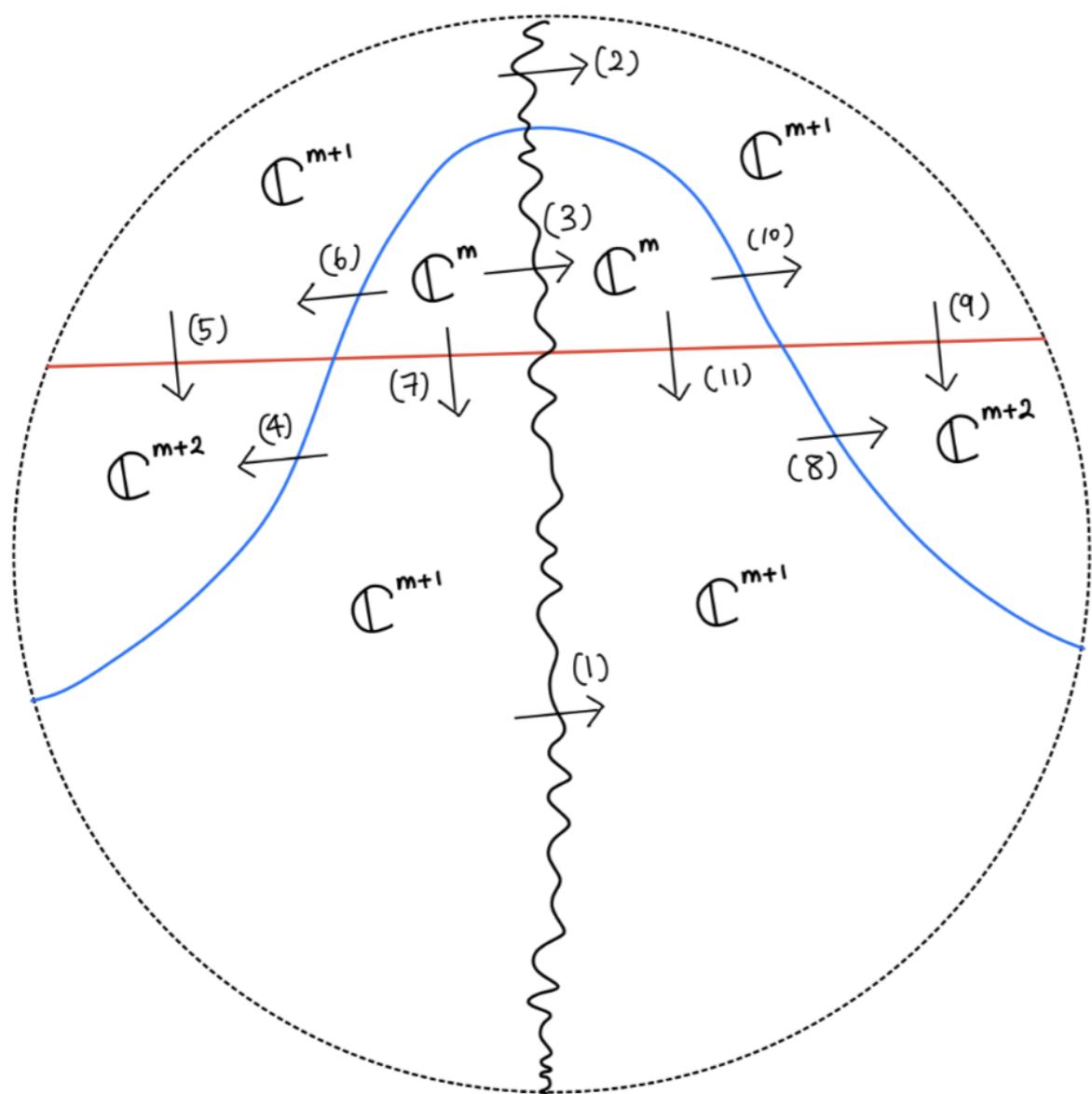


Figure 3.193

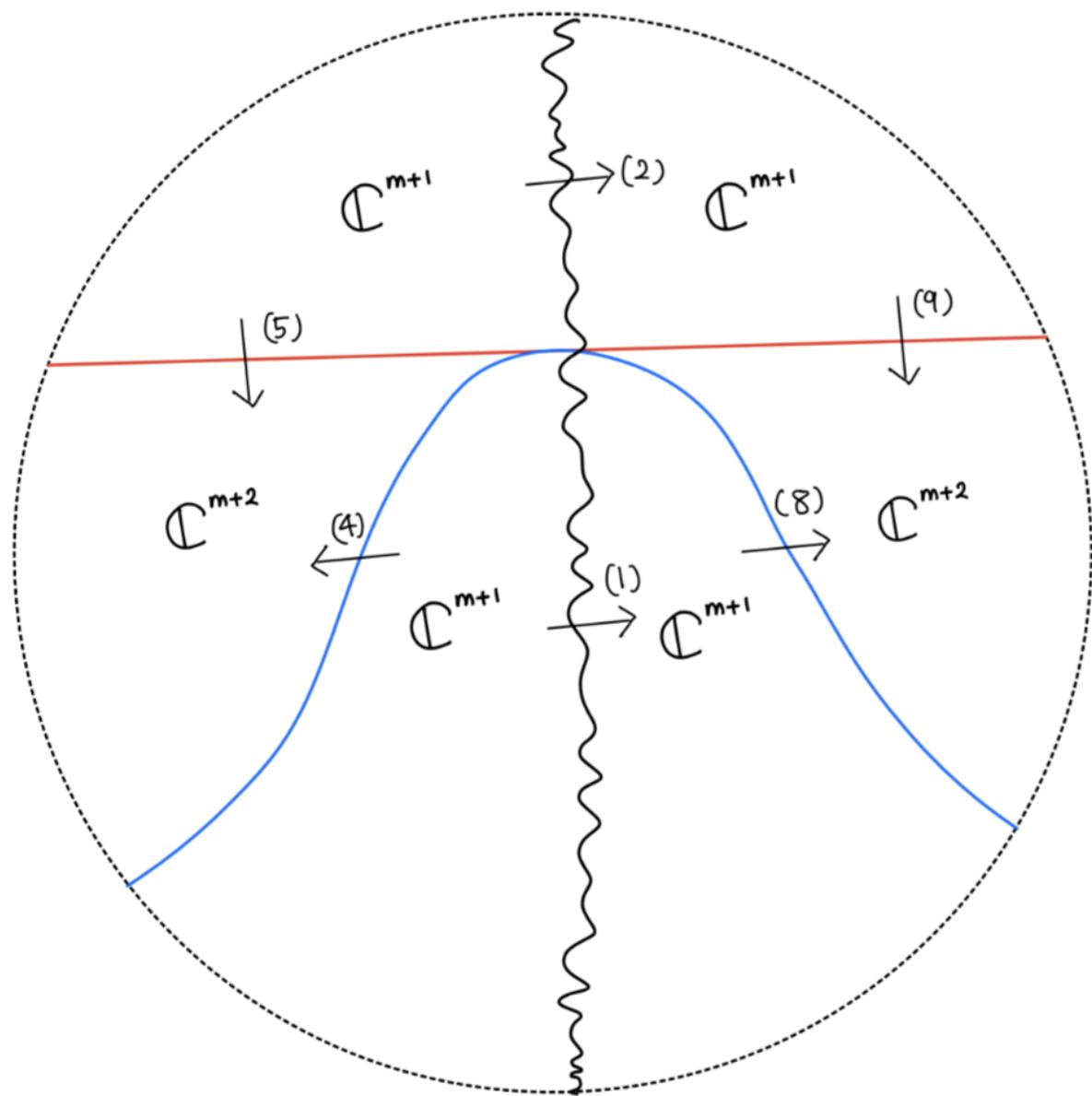


Figure 3.194

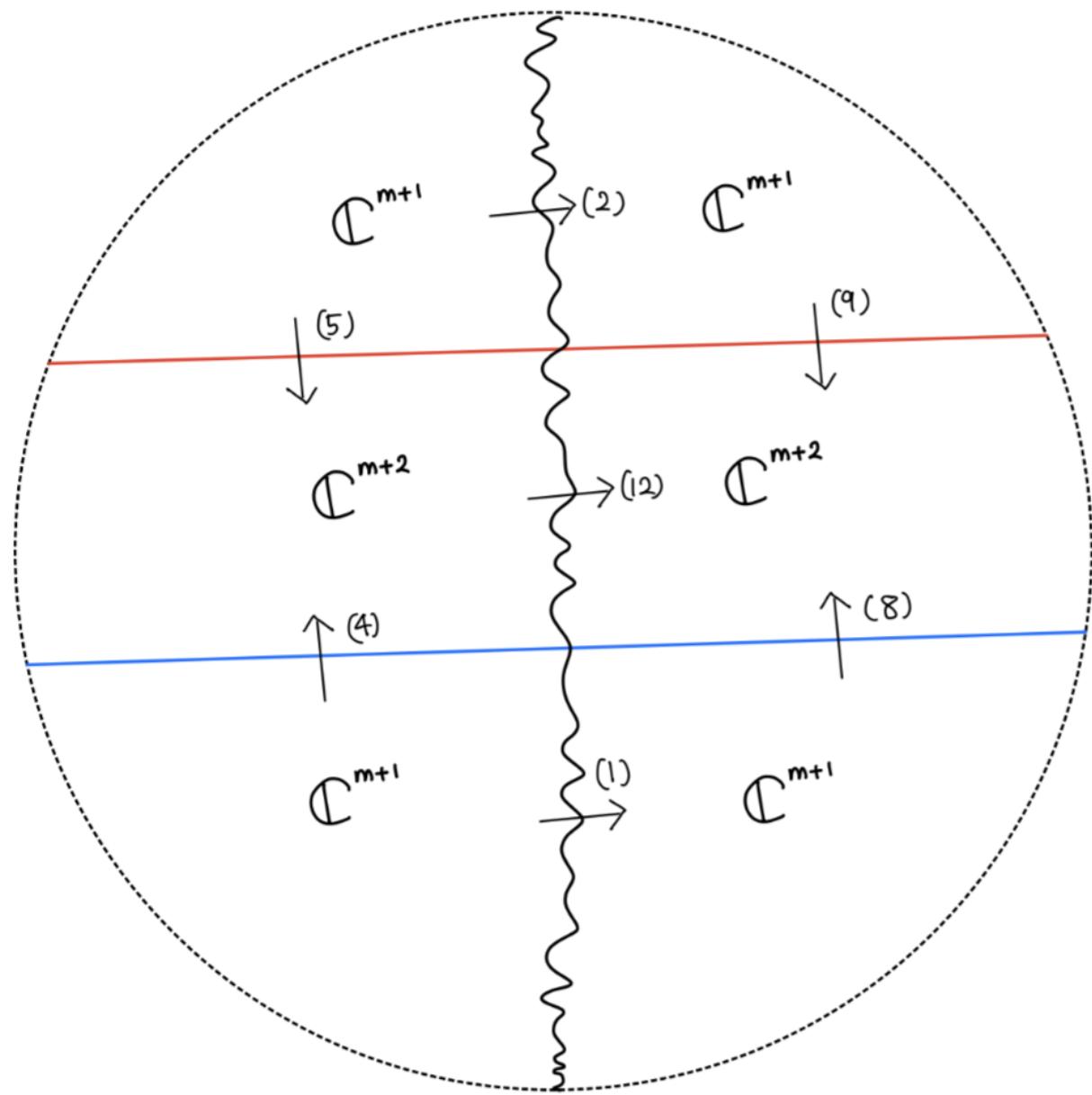


Figure 3.195

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^m \xrightarrow{\theta} \mathbb{C}^m$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^m \xrightarrow{\iota_1} \mathbb{C}^{m+1}$$

$$(7) \quad \mathbb{C}^m \xrightarrow{\iota_0} \mathbb{C}^{m+1}$$

$$(8) \quad \mathbb{C}^{m+1} \xrightarrow{\mu'} \mathbb{C}^{m+2}$$

$$(9) \quad \mathbb{C}^{m+1} \xrightarrow{\nu'} \mathbb{C}^{m+2}$$

$$(10) \quad \mathbb{C}^m \xrightarrow{\theta'} \mathbb{C}^{m+1}$$

$$(11) \quad \mathbb{C}^m \xrightarrow{\theta''} \mathbb{C}^{m+1}$$

$$(12) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

where

$$\bullet \quad T(1, 1, m+2, m+1) = \nu' \circ \nu$$

$$\bullet \quad T(1, 2, m+2, m+2) = \mu' \circ \mu$$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 155.** we define  $\gamma_\bullet$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.

- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0, 1] & \xhookrightarrow{\quad} & V \times [0, 1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \xhookrightarrow{\quad} & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 156.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M$ .

Note that there is a diffeomorphism between  $D_{r=2} \times (0, 1)$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ . To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

### Stalks:

- $F^3(-, -, -) := \mathbb{C}^m$
- $F^3(-, -, +) := \mathbb{C}^m$
- $F^3(+, -, -) := \mathbb{C}^{m+1}$
- $F^3(+, -, +) := \mathbb{C}^{m+1}$

- $F^3(-, +, -) := \mathbb{C}^{m+1}$

- $F^3(-, +, +) := \mathbb{C}^{m+1}$

- $F^3(+, +, -) := \mathbb{C}^{m+2}$

- $F^3(+, +, +) := \mathbb{C}^{m+2}$

**Generalization maps:**

- $F^3(-, sgn_2, sgn_3) \rightarrow F^3(+, sgn_2, sgn_3) := \iota_1$  except  $F^{(-, -, +)} \rightarrow F^3(+, -, +) := \theta'$   
and  $F^{(-, +, +)} \rightarrow F^3(+, +, +) := \mu'$

- $F^3(sgn_1, -, sgn_3) \rightarrow F^3(sgn_1, +, sgn_3) := \iota_0$  except  $F^{(-, -, +)} \rightarrow F^3(-, +, +) := \theta''$   
and  $F^{(+, -, +)} \rightarrow F^3(+, +, +) := \nu'$

- $F^3(-, -, -) \rightarrow F^3(-, -, +) := \theta$

- $F^3(+, -, -) \rightarrow F^3(+, -, +) := \nu$

- $F^3(-, +, -) \rightarrow F^3(-, +, +) := \mu$

- $F^3(+, +, -) \rightarrow F^3(+, +, +) := T$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$\begin{array}{ccc} F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\ \downarrow & & \downarrow \\ F^3(+, -, -) & \longrightarrow & F^3(+, +, -) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\iota_0} & \mathbb{C}^{m+1} \\ \downarrow \iota_1 & & \downarrow \iota_1 \\ \mathbb{C}^{m+1} & \xrightarrow{\iota_0} & \mathbb{C}^{m+2} \end{array}$$

: This is a cartesian diagram, therefore, the total complex is acyclic

$$\begin{array}{ccc} F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\ \downarrow & & \downarrow \\ F^3(+, -, +) & \longrightarrow & F^3(+, +, +) \end{array} = \begin{array}{ccc} \mathbb{C}^m & \xrightarrow{\theta''} & \mathbb{C}^{m+1} \\ \downarrow \theta' & & \downarrow \mu' \\ \mathbb{C}^{m+1} & \xrightarrow{\nu'} & \mathbb{C}^{m+2} \end{array}$$

: acyclicity follows from the crossing condition of  $F_0$

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^m & \xrightarrow{\theta} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \theta' \\
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+2}
 \end{array}$$

: acyclicity follows from the crossing condition of  $F_0$

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \mu' \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

:  $\mu, T$  are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

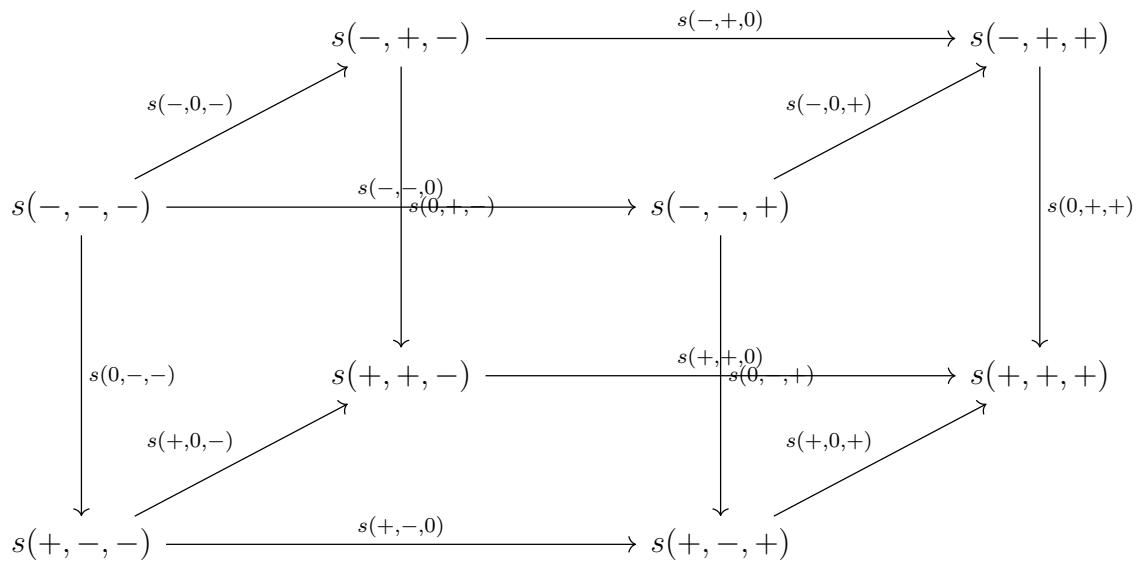
$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^m & \xrightarrow{\theta} & \mathbb{C}^m \\
 \downarrow \iota_0 & & \downarrow \theta'' \\
 \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1}
 \end{array}$$

:  $\theta, \mu$  are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{m+1} & \xrightarrow{\nu} & \mathbb{C}^{m+1} \\
 \downarrow \iota_1 & & \downarrow \nu' \\
 \mathbb{C}^{m+2} & \xrightarrow{T} & \mathbb{C}^{m+2}
 \end{array}$$

:  $\nu, T$  are isomorphisms. Therefore, we can think of them as an isomorphism of cochain complexes. Therefore, the total complex is acyclic.

(vii) the cubic diagram:



$$\begin{array}{ccccc}
 & = & & & \\
 & & \mathbb{C}^{m+1} & \xrightarrow{\mu} & \mathbb{C}^{m+1} \\
 & \swarrow \iota_0 & \downarrow & & \searrow \theta'' \\
 \mathbb{C}^m & \xrightarrow{\theta_{\mu_1}} & \mathbb{C}^m & & \downarrow \mu' \\
 & \downarrow \iota_1 & & & \downarrow \\
 & & \mathbb{C}^{m+2} & \xrightarrow{T_{\theta'}} & \mathbb{C}^{m+2} \\
 & \swarrow \iota_0 & & & \searrow \nu' \\
 \mathbb{C}^{m+1} & \xrightarrow{\nu} & \mathbb{C}^{m+1} & & 
 \end{array}$$

Note that  $\theta, \mu, \nu, T$  are isomorphisms. Therefore, we can think of the cube diagram as isomorphism of two double complexes. Therefore, the total complex is acyclic.

□

## Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the *cobord*<sub>3</sub>. By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphism  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

### A. Sheaf on $V$

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

**B. Sheaf on  $D_{r=2}$** 

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

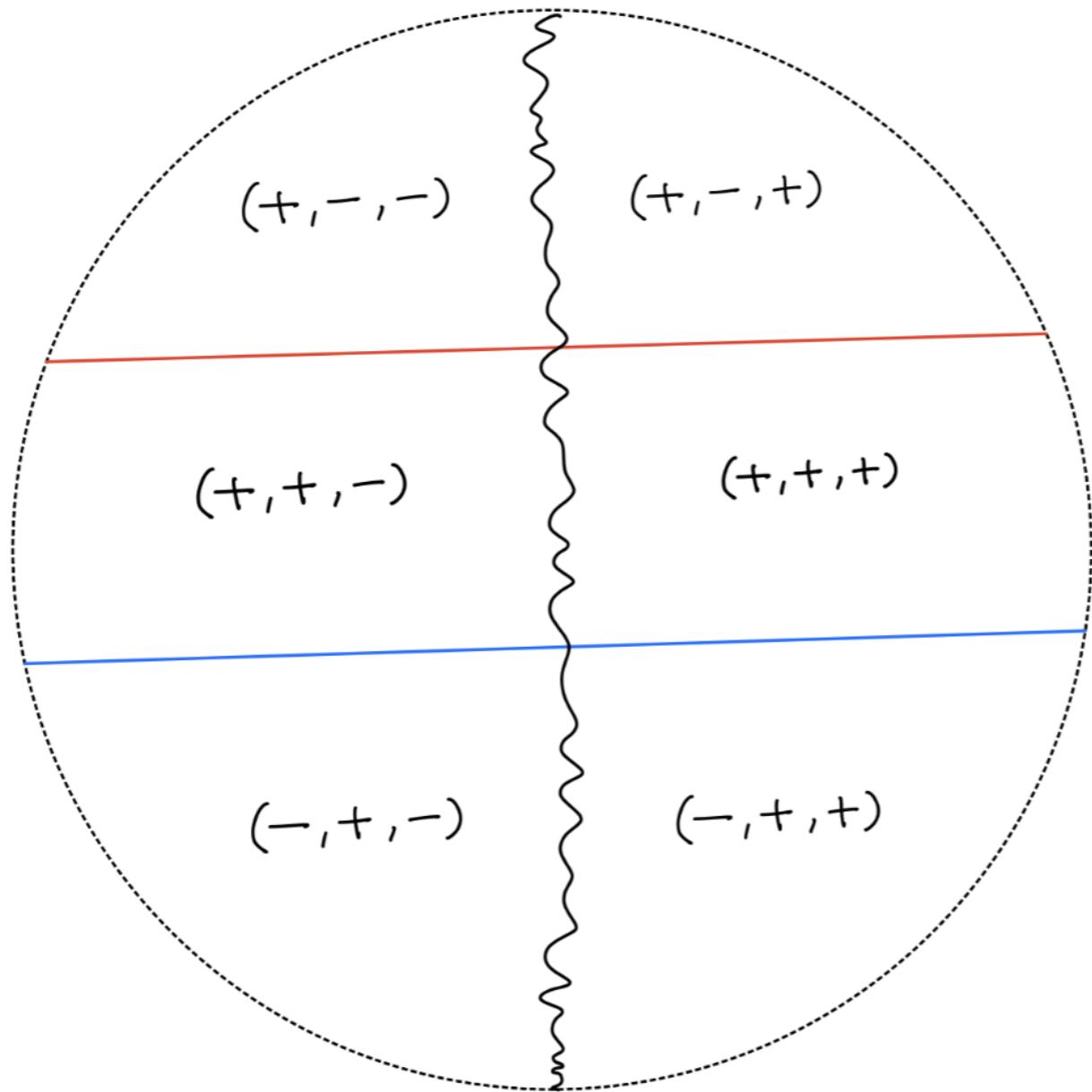


Figure 3.196

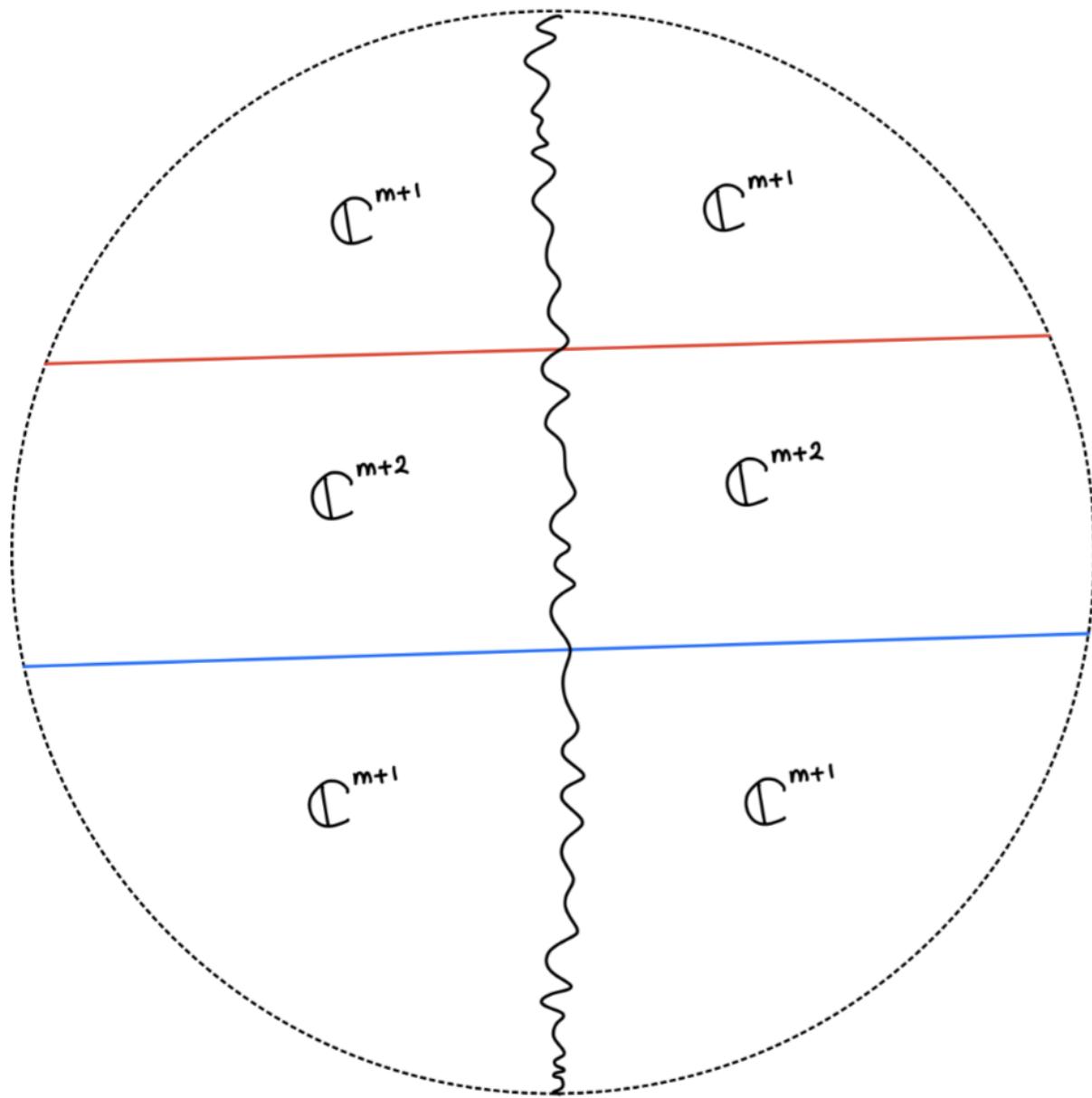


Figure 3.197

- $F_1(+, -, -) := \mathbb{C}^{m+1}$

- $F_1(+, -, +) := \mathbb{C}^{m+1}$

- $F_1(-, +, -) := \mathbb{C}^{m+1}$

- $F_1(-, +, +) := \mathbb{C}^{m+1}$

- $F_1(+, +, -) := \mathbb{C}^{m+2}$

- $F_1(+, +, +) := \mathbb{C}^{m+2}$

Generalization maps:

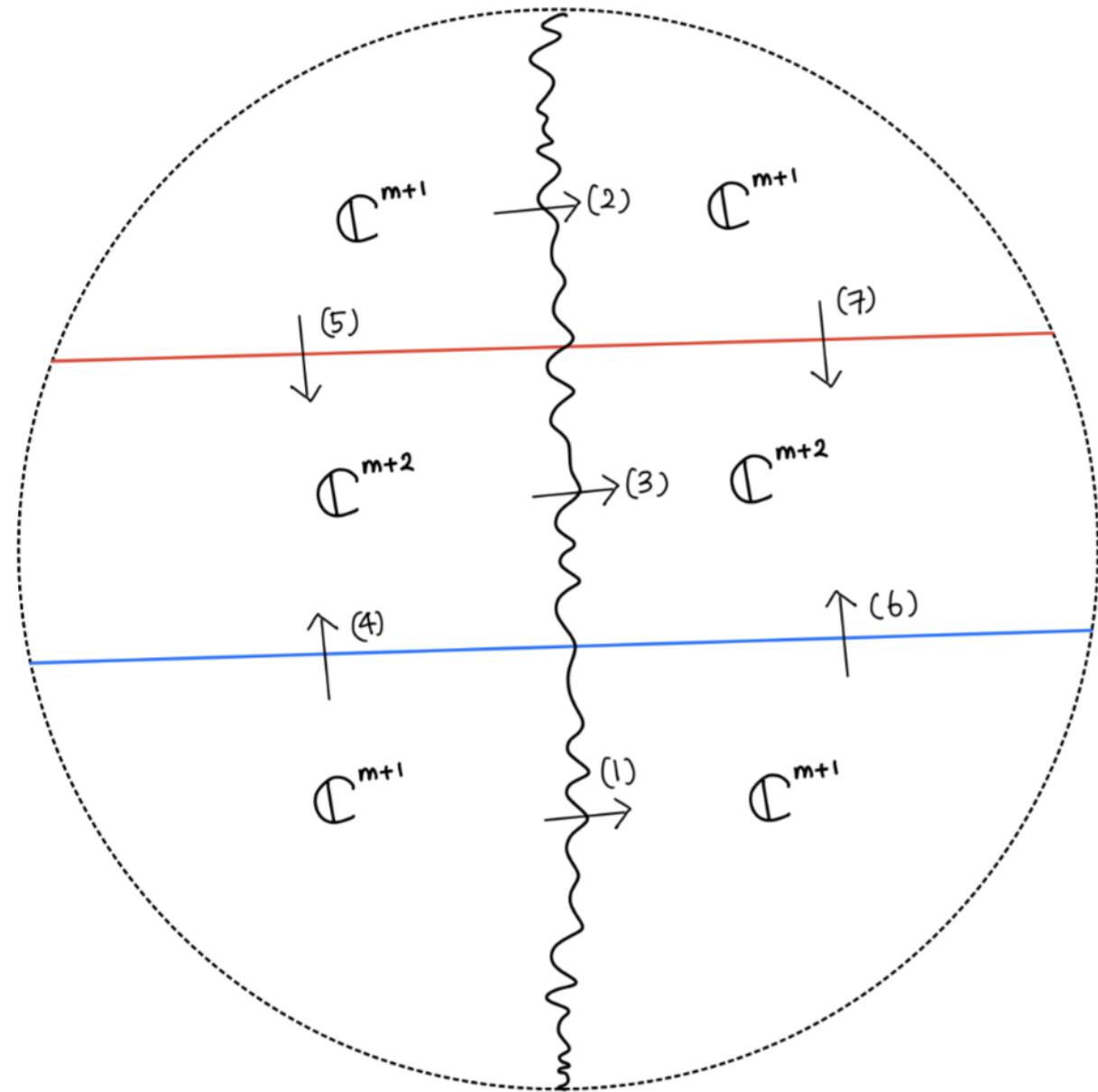


Figure 3.198

$$(1) \quad \mathbb{C}^{m+1} \xrightarrow{\mu} \mathbb{C}^{m+1}$$

$$(2) \quad \mathbb{C}^{m+1} \xrightarrow{\nu} \mathbb{C}^{m+1}$$

$$(3) \quad \mathbb{C}^{m+2} \xrightarrow{T} \mathbb{C}^{m+2}$$

$$(4) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(5) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

$$(6) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_1} \mathbb{C}^{m+2}$$

$$(7) \quad \mathbb{C}^{m+1} \xrightarrow{\iota_0} \mathbb{C}^{m+2}$$

where

- $T(1, 1, m+2, m+1) = \nu' \circ \nu$
- $T(1, 2, m+2, m+2) = \mu' \circ \mu$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 157.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

### 3.8 4th Sheaf Cobordism

In this section, we define  $cobord_4$ , a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism from

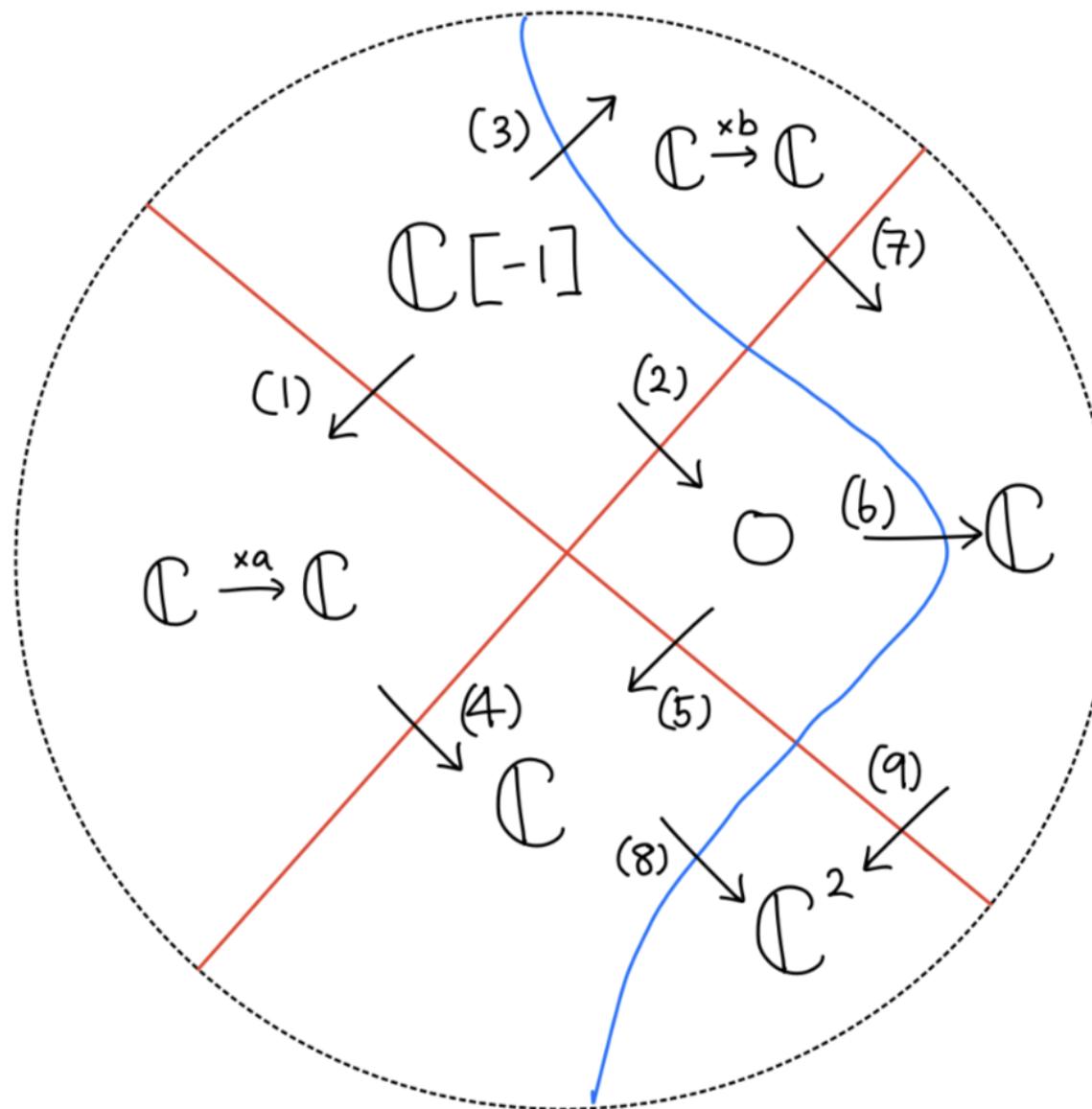


Figure 3.199

**Generalization maps:**

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{x^1} & \mathbb{C} \\ \uparrow & & \uparrow x_a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow_{\times b} \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow_{\times a} & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow_{\times b} & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

to

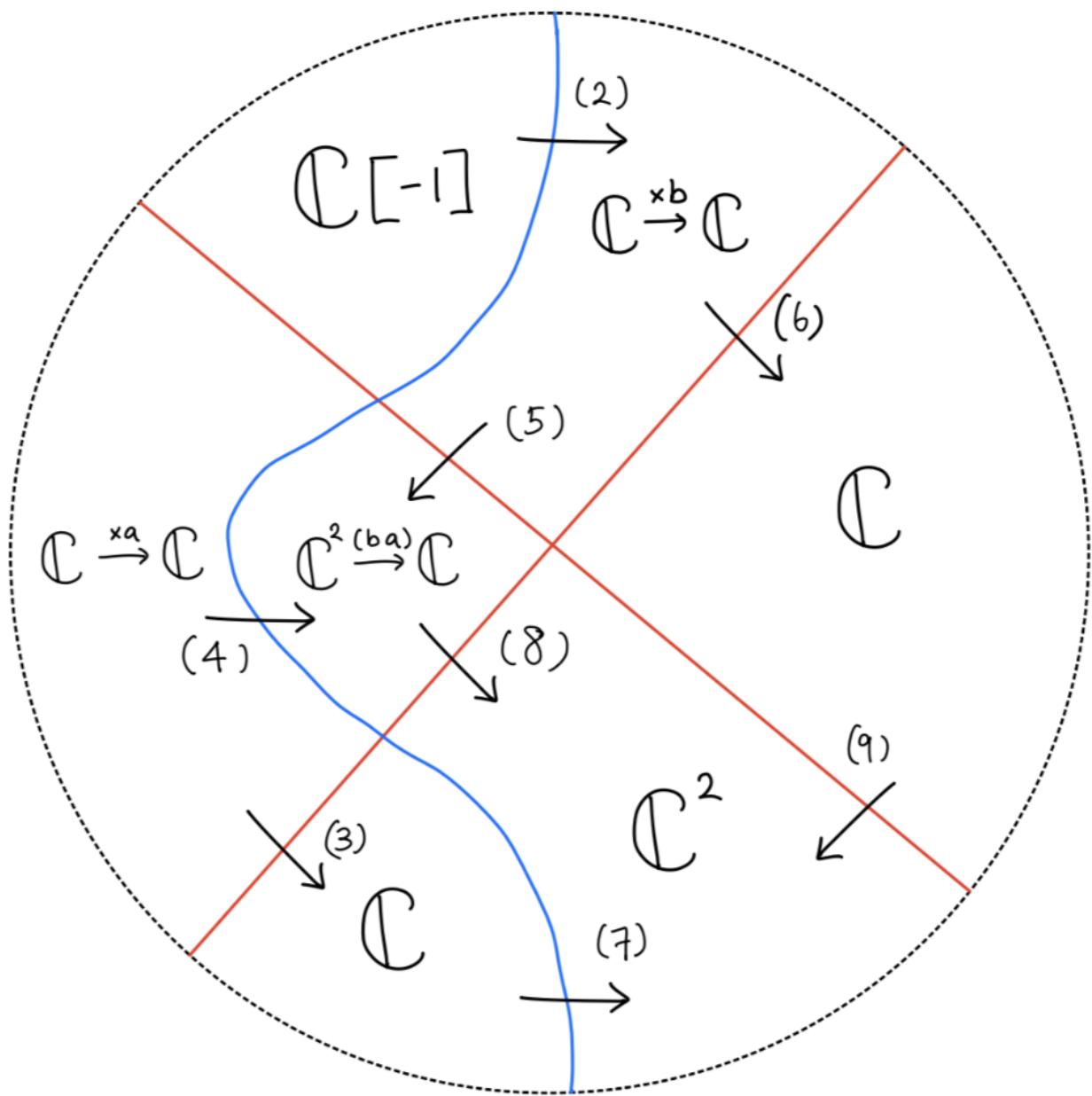


Figure 3.200

Generalization maps:

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{x_1} & \mathbb{C} \\ \uparrow & \uparrow x_a & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & \times b \uparrow & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

## Notations

**Definition 158.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 159.** For  $t_0 \in \{0, 1\}$  and  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{t_0}^{symbol}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{squig} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{symbol}$  a co-orientation of  $\Phi_{t_0}^{symbol}$ .

3. we denote the pair  $(\Phi_{t_0}^{symbol}, \Xi_{t_0}^{symbol})$  as  $\Lambda_{t_0}^{symbol}$ . When  $symbol \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{symbol}$  to denote both
4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{squig})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the beginning of the isotopy underlying  $cobord_4$  and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying  $cobord_4$ .

**Definition 160.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_\bullet^{symbol}$  to be smooth maps

$$\Phi_\bullet^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^\infty : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_\bullet^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript ' $t$ ' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_\bullet^{symbol}$  a co-orientation of  $\Phi_\bullet^{symbol}$ .

3. we denote the pair  $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$  as  $\Lambda_{\bullet}^{symbol}$ . Later in the section,  $\Lambda_{\bullet}^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>4</sub>.
4. we denote the triple  $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$  as  $\Lambda_{\bullet}$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_{\bullet}$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>4</sub>.

**Definition 161.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} e^{(\frac{x^2}{x^2-1})}(\frac{1}{2} - t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$  if  $t \neq \frac{1}{2}$
- $\{(1, 0), (-1, 0), (0, \frac{1}{2} - t)\} \subset Graph(\Psi_t)$

**Definition 162.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \rightarrow \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$\begin{aligned} D_{r=r_0} &\xrightarrow{\sim} D_{r=r_0} \times \{t_0\} \\ (x, z) &\mapsto (x, z, t_0) \end{aligned}$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 163.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_0(z)\}$
  - $\lambda_0^\infty$  is the union of the following two components
    - (i)  $\{(x, z) \in D_{r=2} \mid z = x\}$
    - (ii)  $\{(x, z) \in D_{r=2} \mid z = -x\}$
2. We define co-orientations  $\xi_0^{symbol}$  of  $\lambda_0^{symbol}$  as follows
- $\xi_0^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.
  - $\xi_0^\infty$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are positive.
- Definition 164.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$
- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_1(z)\}$
  - $\lambda_1^\infty$  is the union of the following two components
    - (i)  $\{(x, z) \in D_{r=2} \mid z = x\}$
    - (ii)  $\{(x, z) \in D_{r=2} \mid z = -x\}$
2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows
- $\xi_1^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.
  - $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.
- Definition 165.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$
- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = \Psi_t(z)\}$
  - $\lambda_\bullet^\infty$  is the union of the following two components
    - (i)  $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = x\}$
    - (ii)  $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -x\}$

2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows
- $\xi_\bullet^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.

- $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.

**Definition 166.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 167.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 168.** For  $i = 1, 2, 3$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(x - \Psi_{t_0}(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z) = sgn_3\} \end{aligned}$$

**Definition 169.** For  $i = 1, 2, 3$  and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_{\bullet}(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(x - \Psi_t(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z) = sgn_3\} \end{aligned}$$

**Definition 170.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_{\bullet}$  using the above notations:

1.  $\mathcal{S}_0$ :

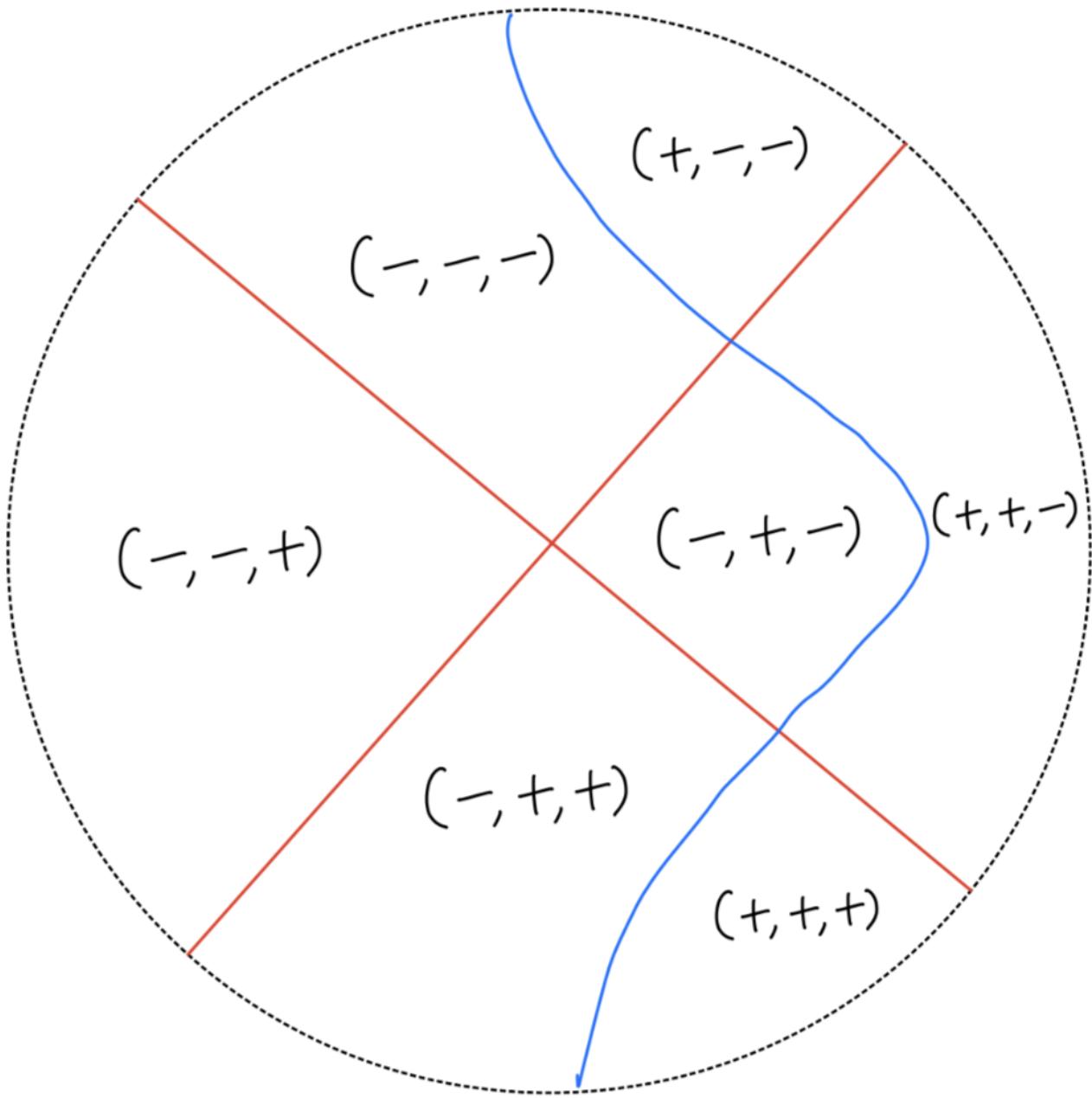


Figure 3.201

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_0(+, -, +)\}$$

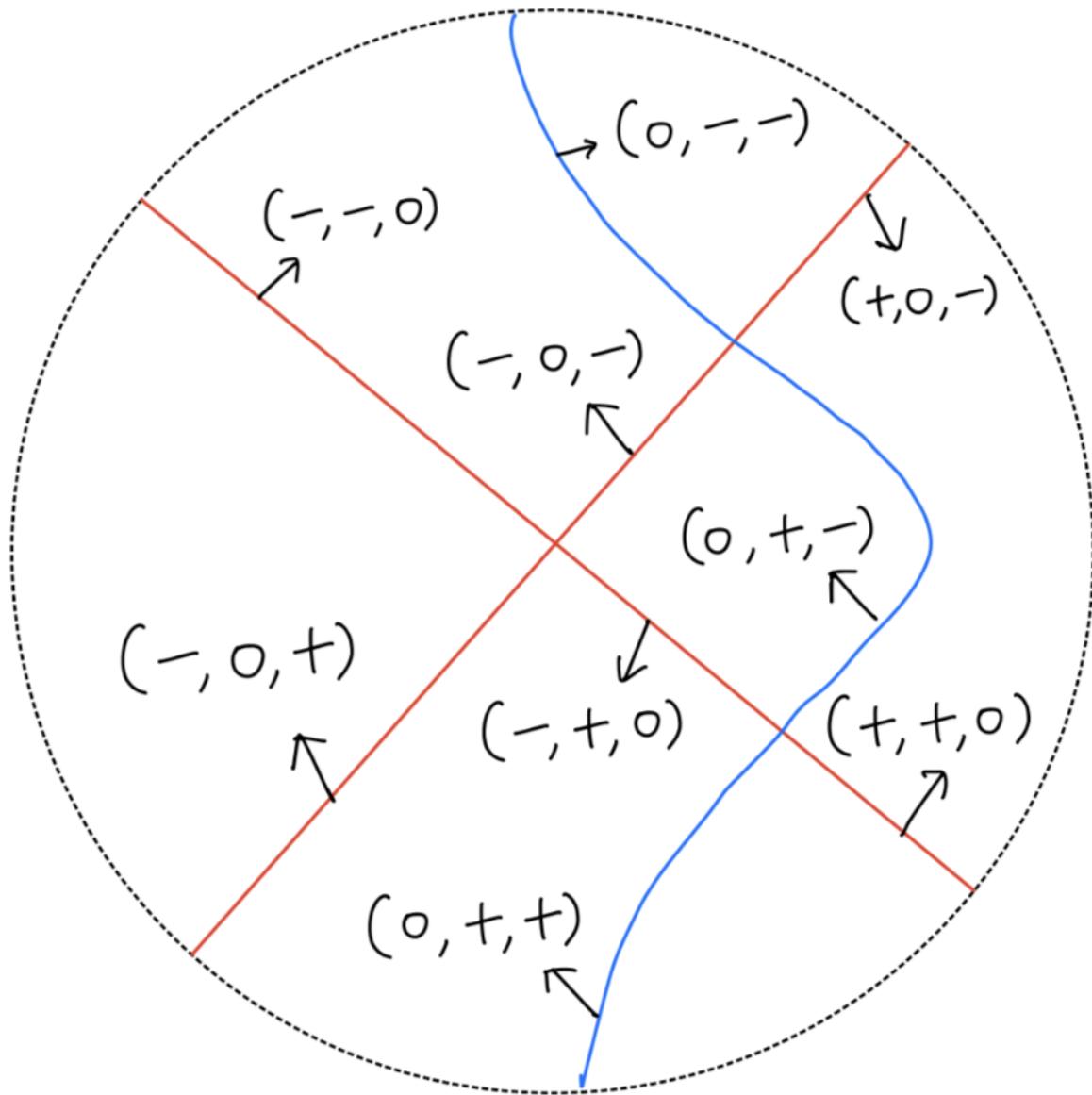


Figure 3.202

- 1 dimensional strata:

$$\begin{aligned} & \{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_0(0, -, +)\} \\ & \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_0(+, 0, +)\} \\ & \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, -, 0)\} \end{aligned}$$

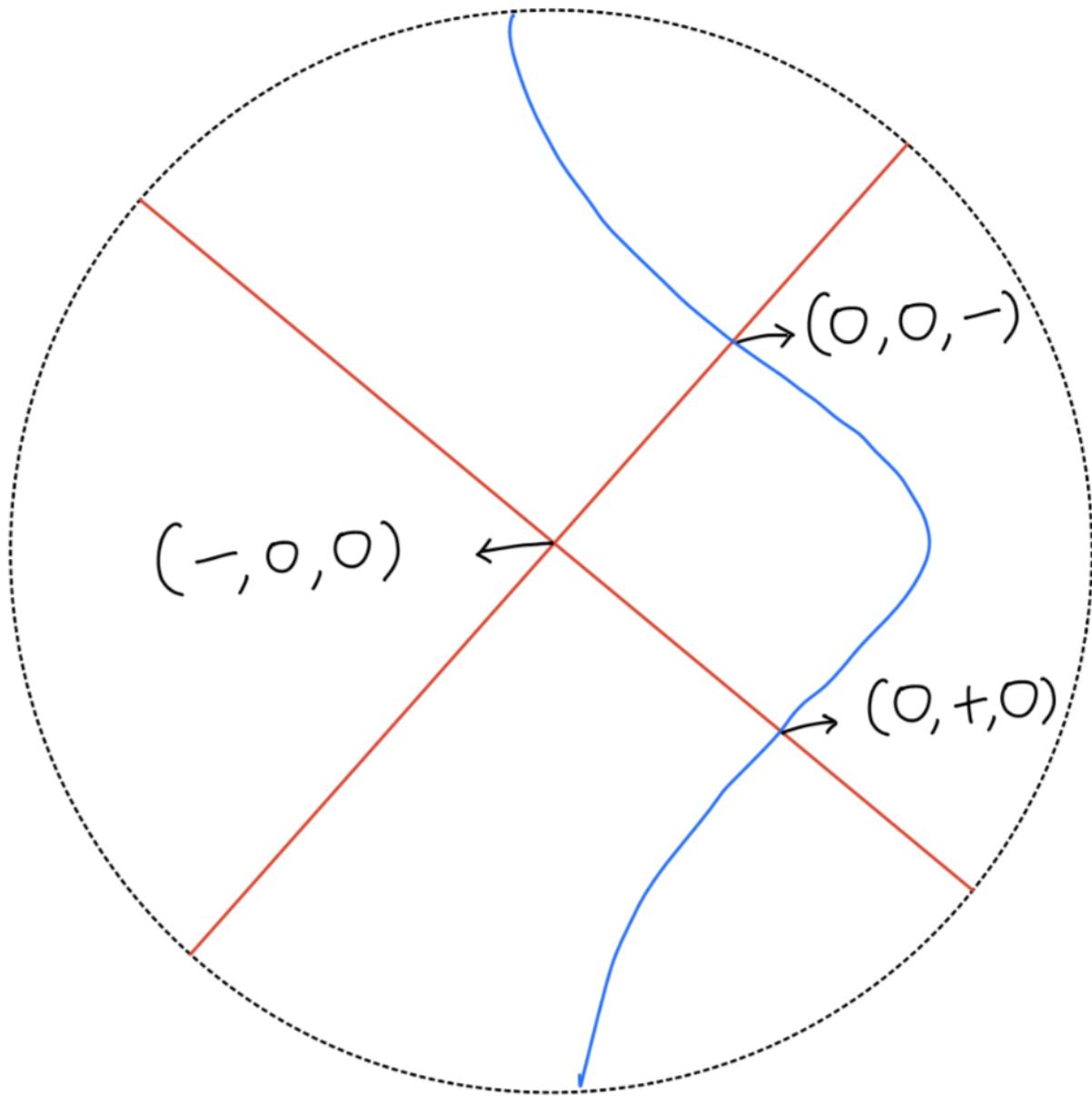


Figure 3.203

- 0 dimensional strata:

$$s_0(-, 0, 0), s_0(0, 0, -), s_0(0, +, 0)$$

2.  $\mathcal{S}_1$ :

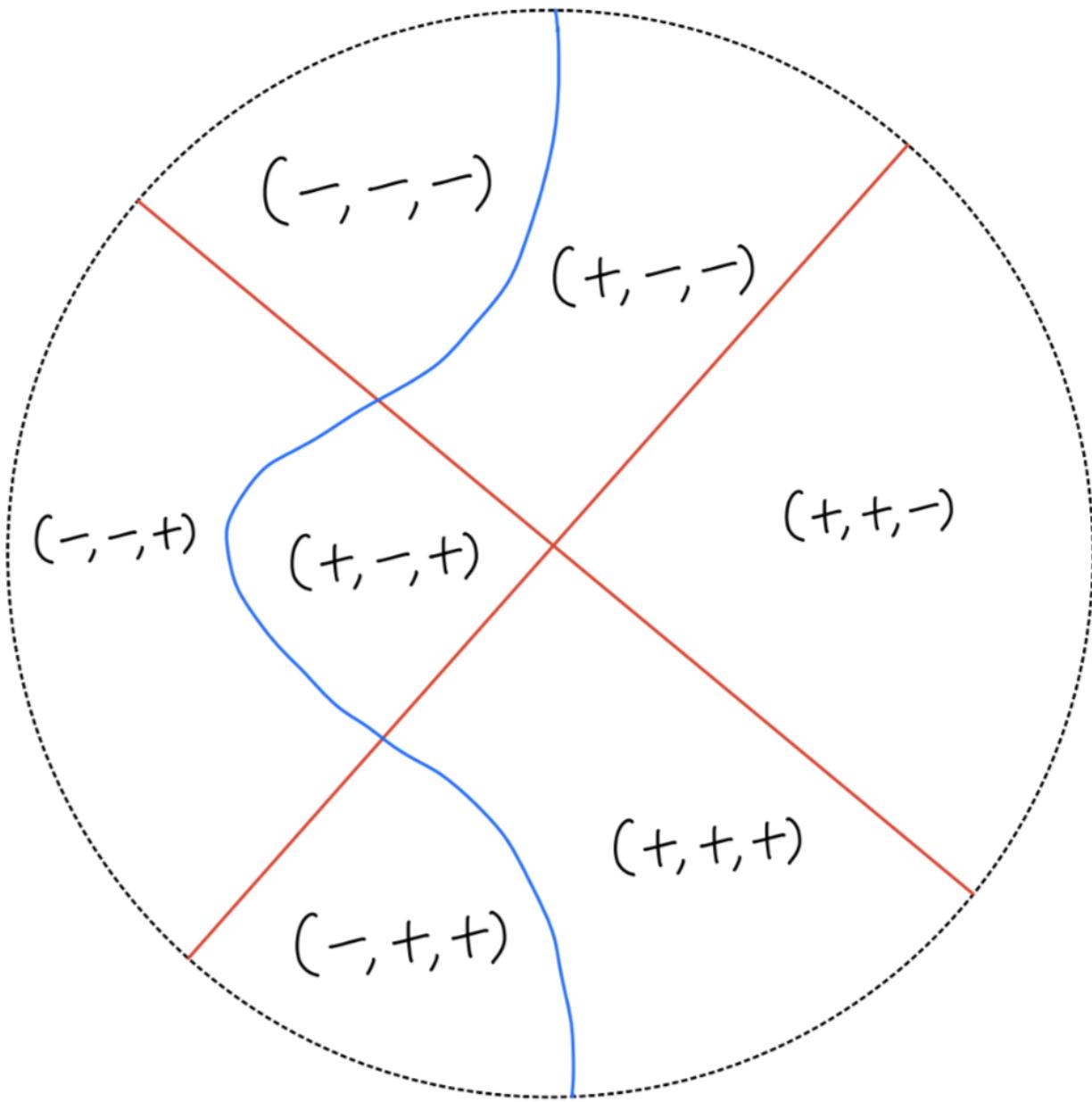


Figure 3.204

- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_1(-, +, -)\}$$

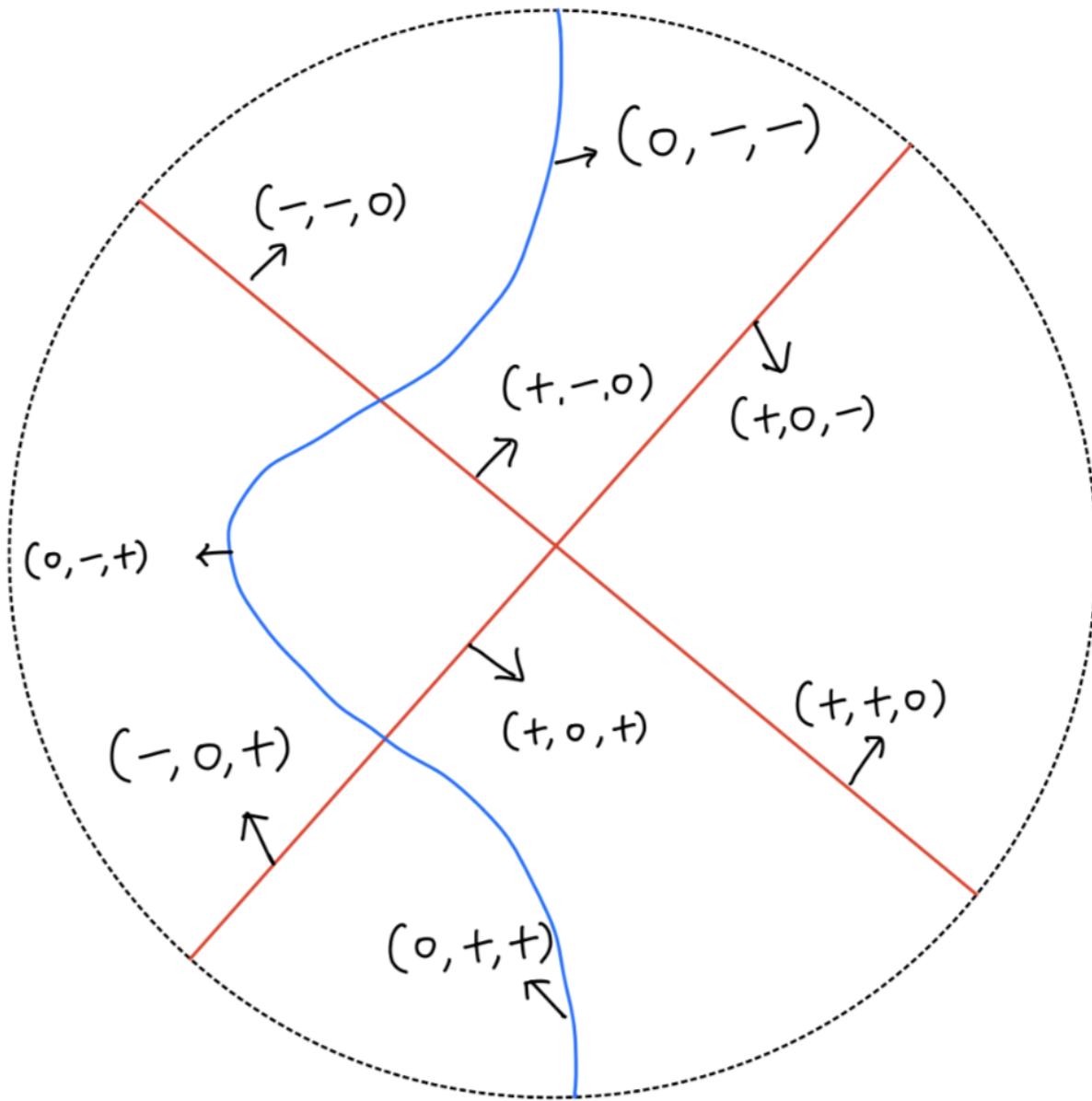


Figure 3.205

- 1 dimensional strata:

$$\begin{aligned}
 & \{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_1(0, +, -)\} \\
 & \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_1(-, 0, -)\} \\
 & \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(-, +, 0)\}
 \end{aligned}$$

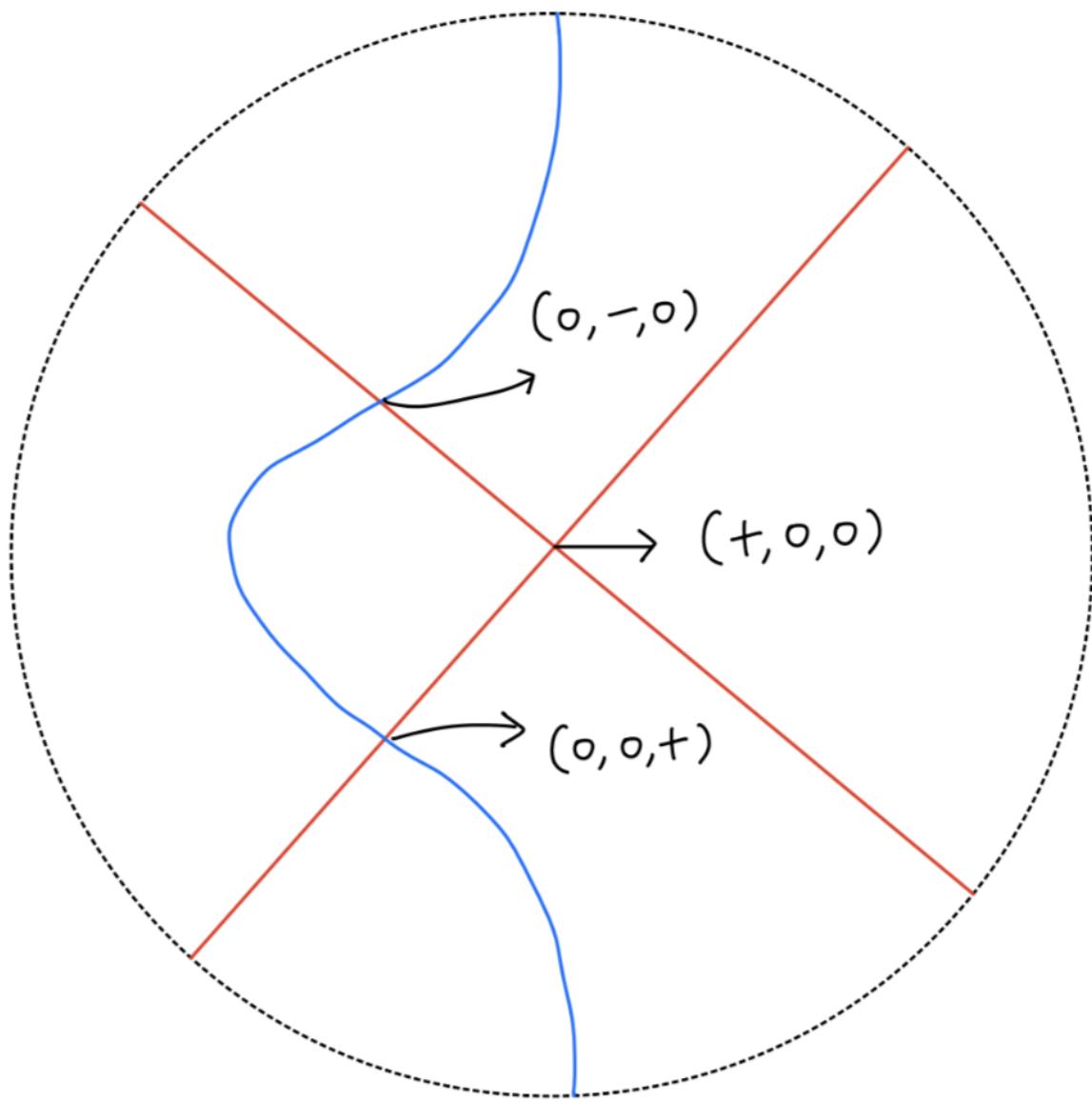


Figure 3.206

- 0 dimensional strata:

$$s_1(0, -, 0), s_1(+, 0, 0), s_1(0, 0, +)$$

3.  $\mathcal{S}_\bullet$ :

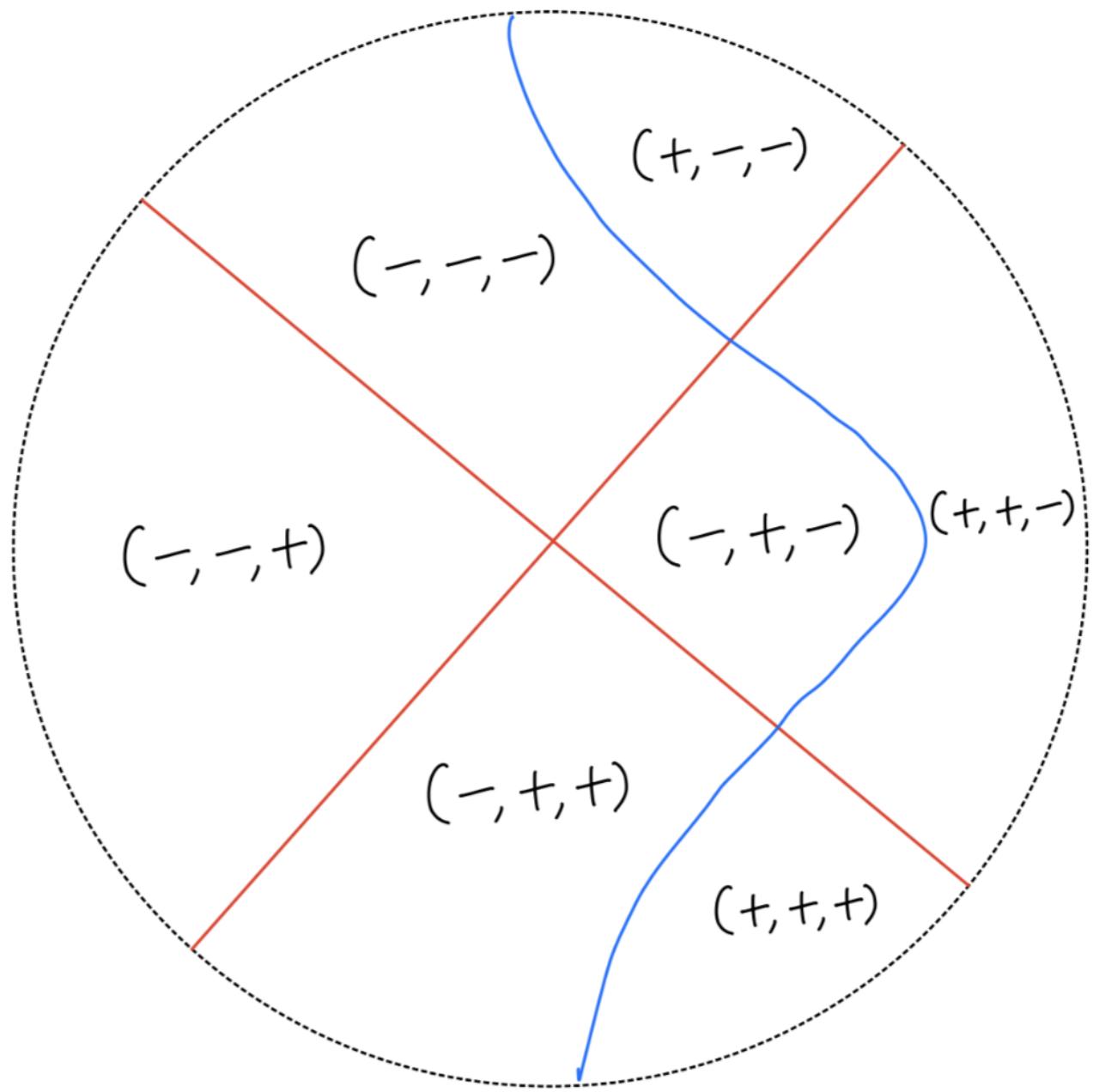


Figure 3.207

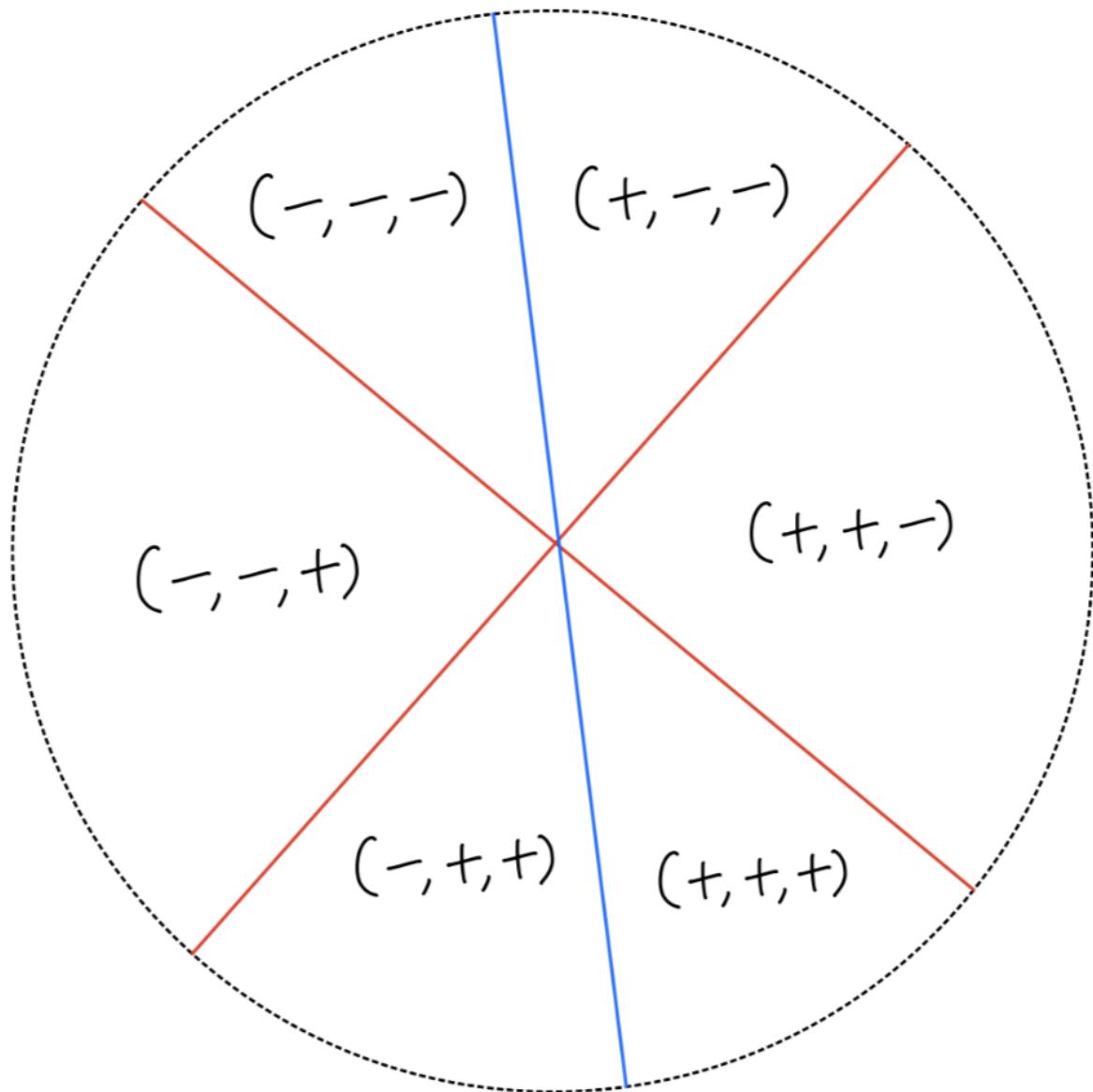


Figure 3.208

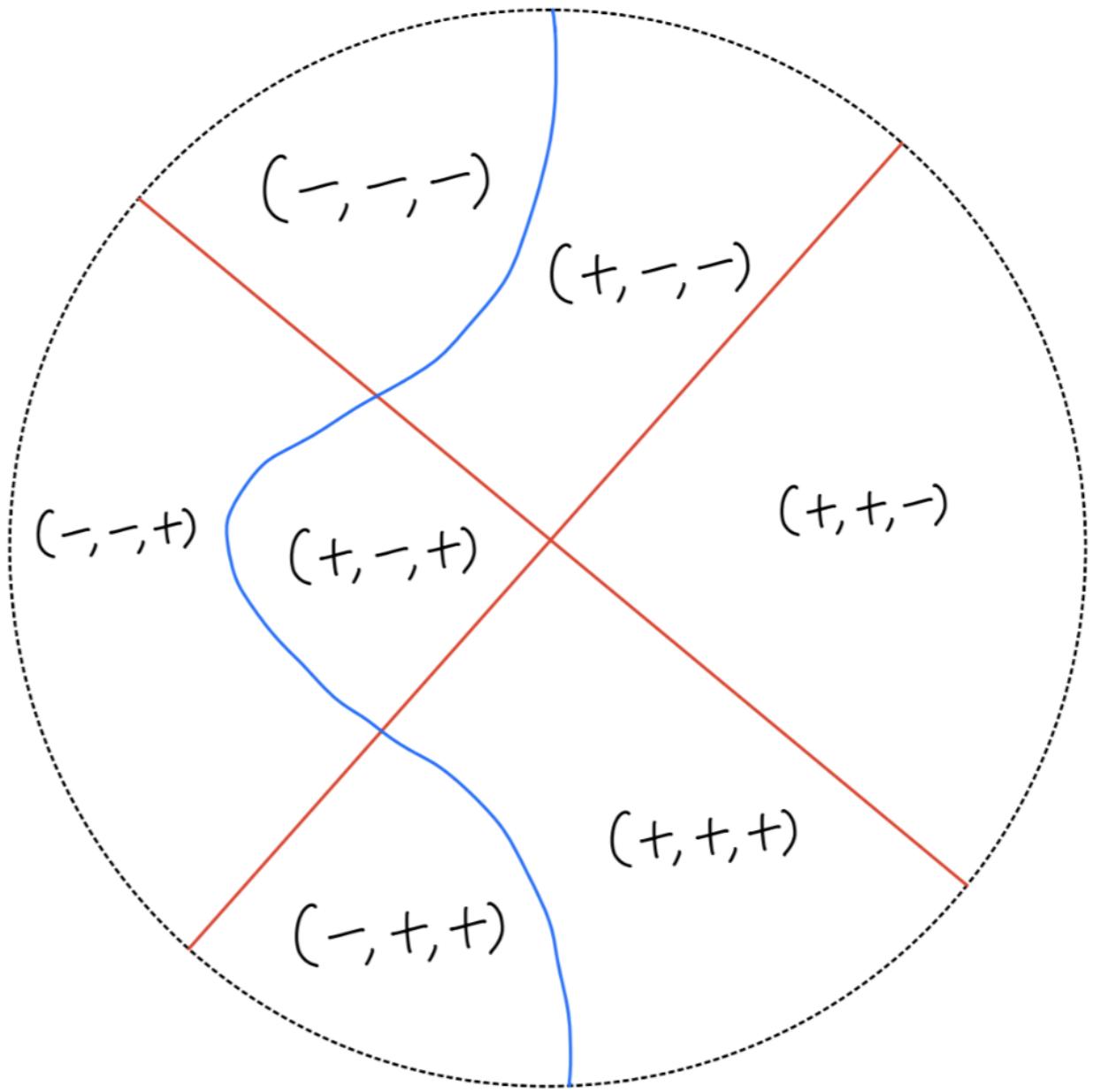


Figure 3.209

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

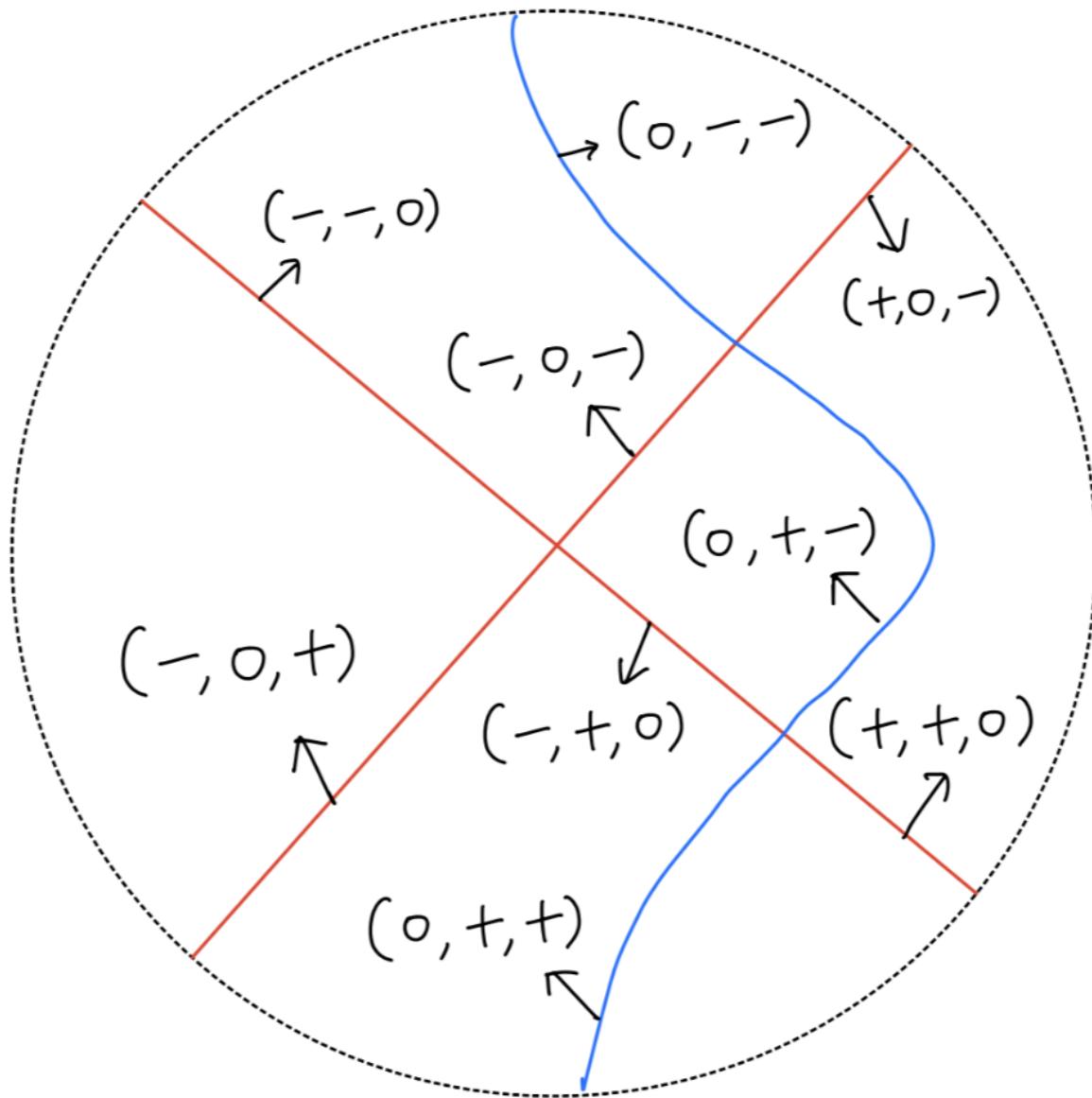


Figure 3.210

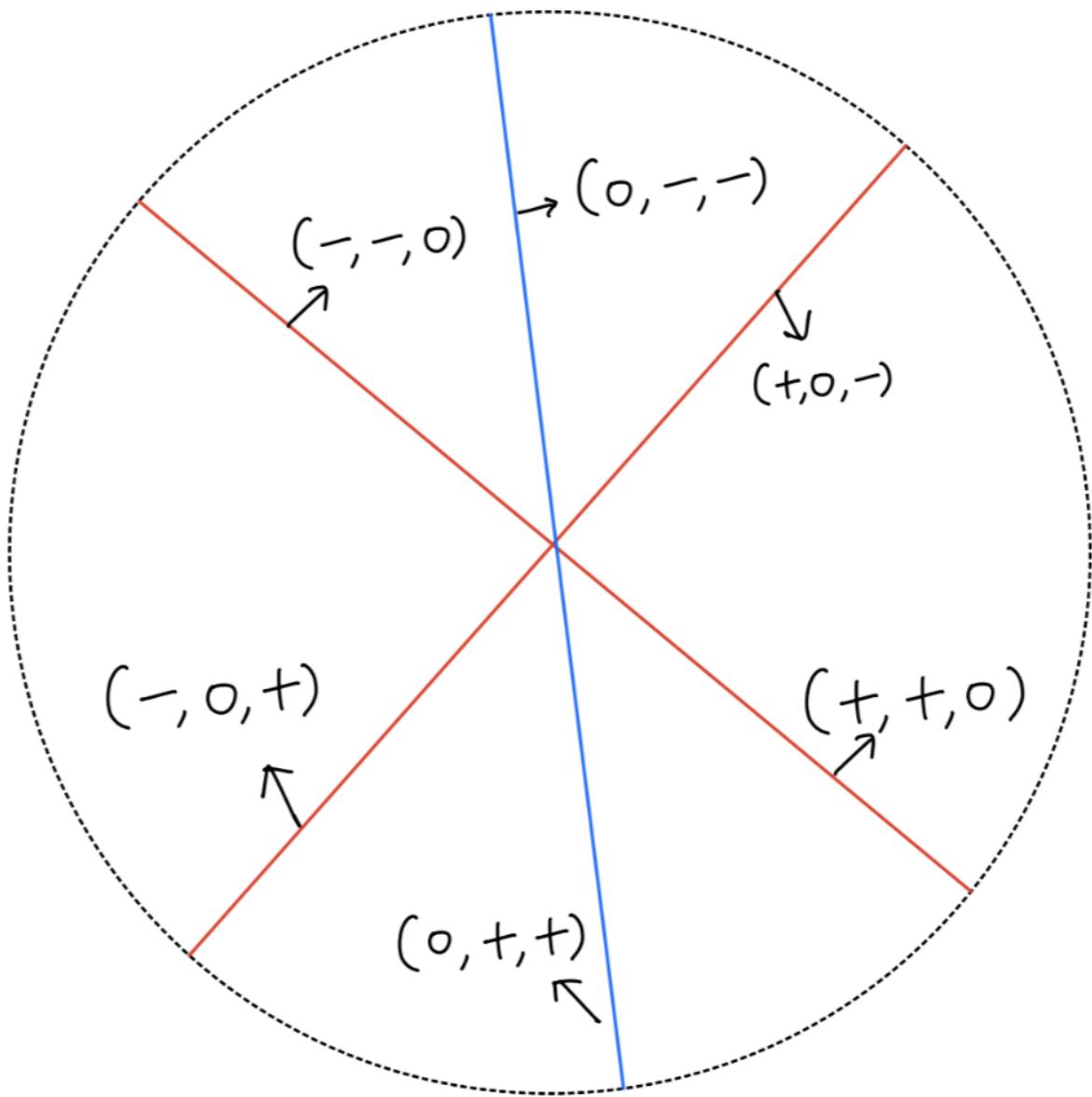


Figure 3.211

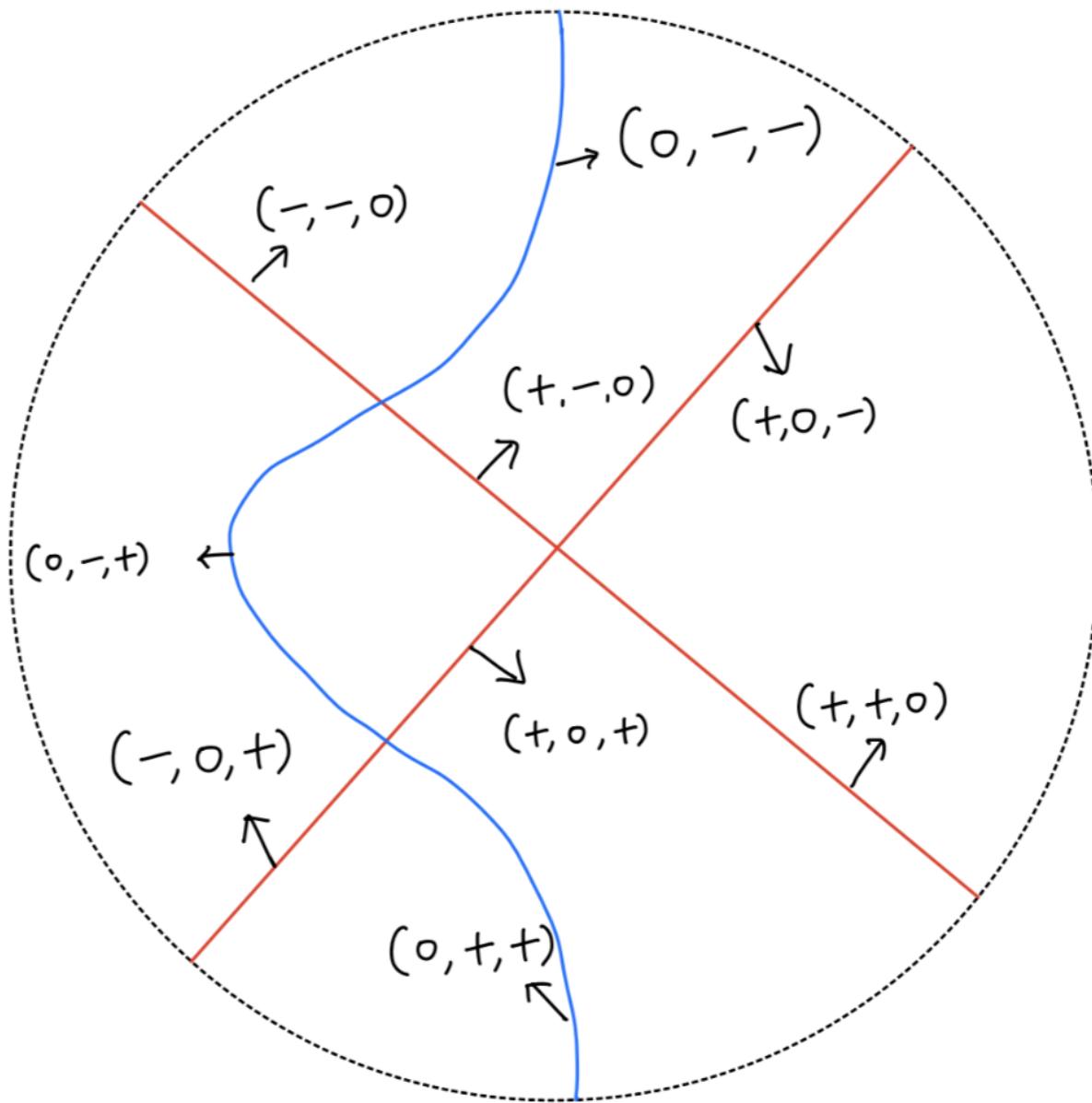


Figure 3.212

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

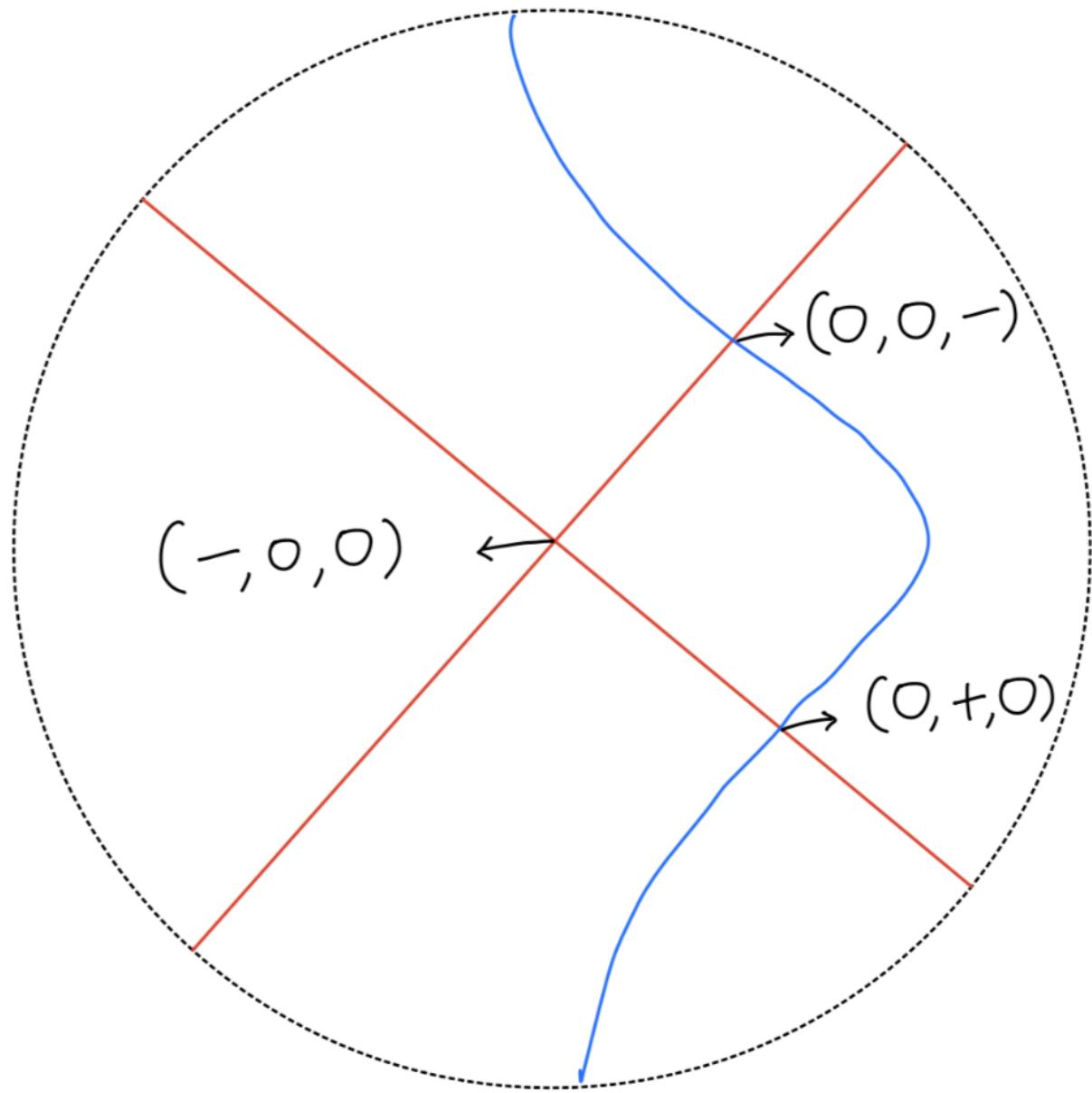


Figure 3.213

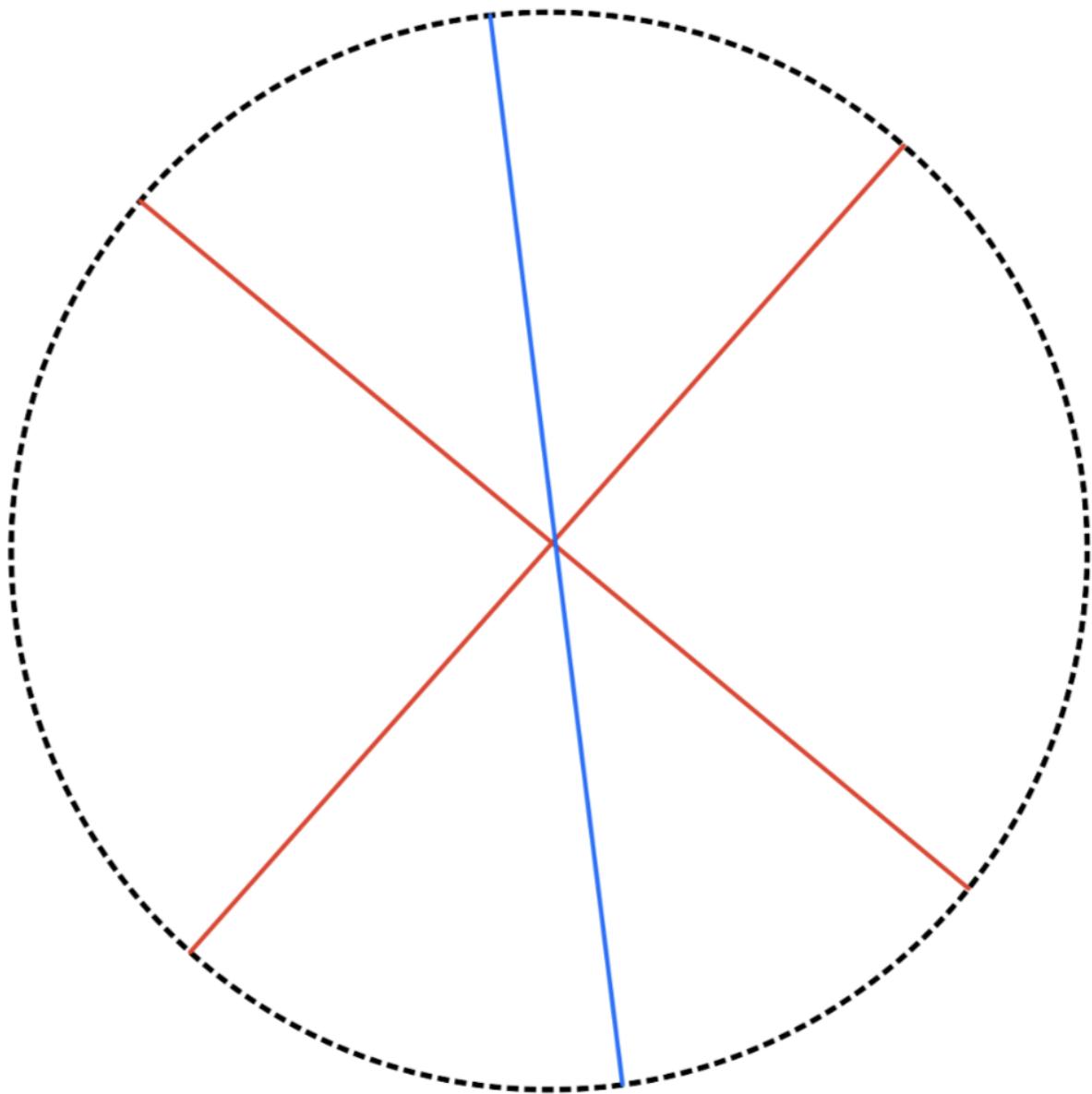


Figure 3.214

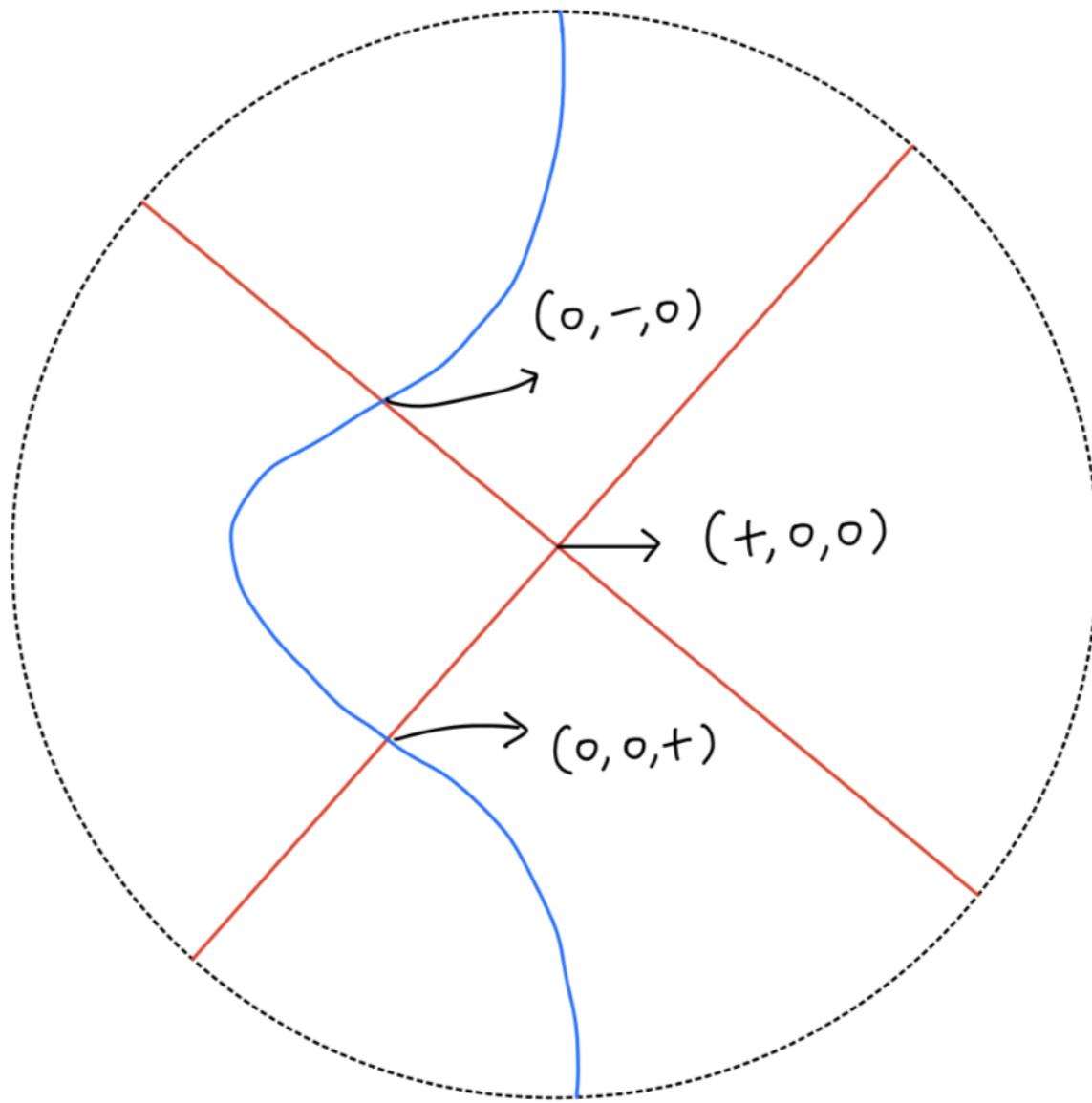


Figure 3.215

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

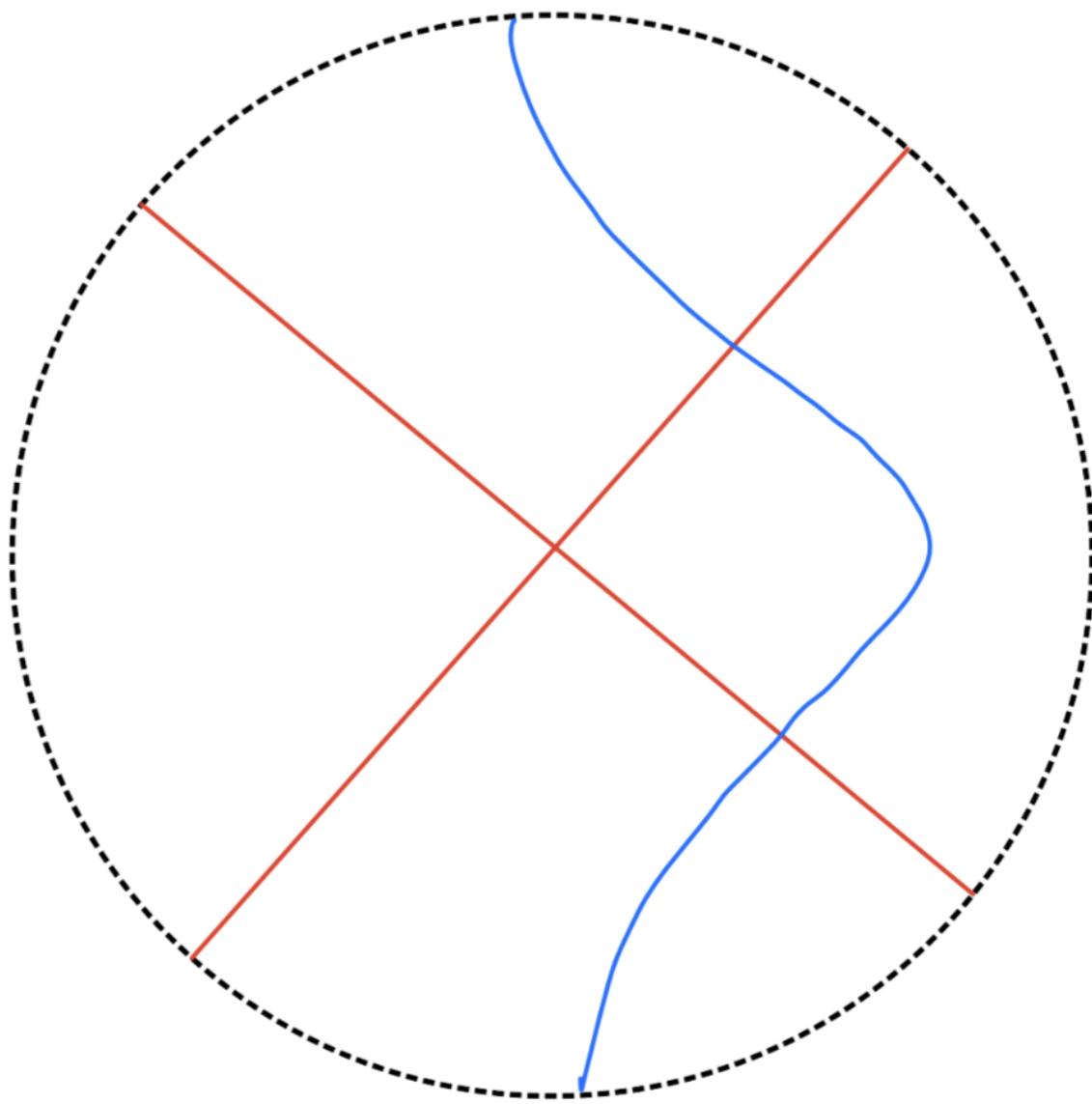


Figure 3.216

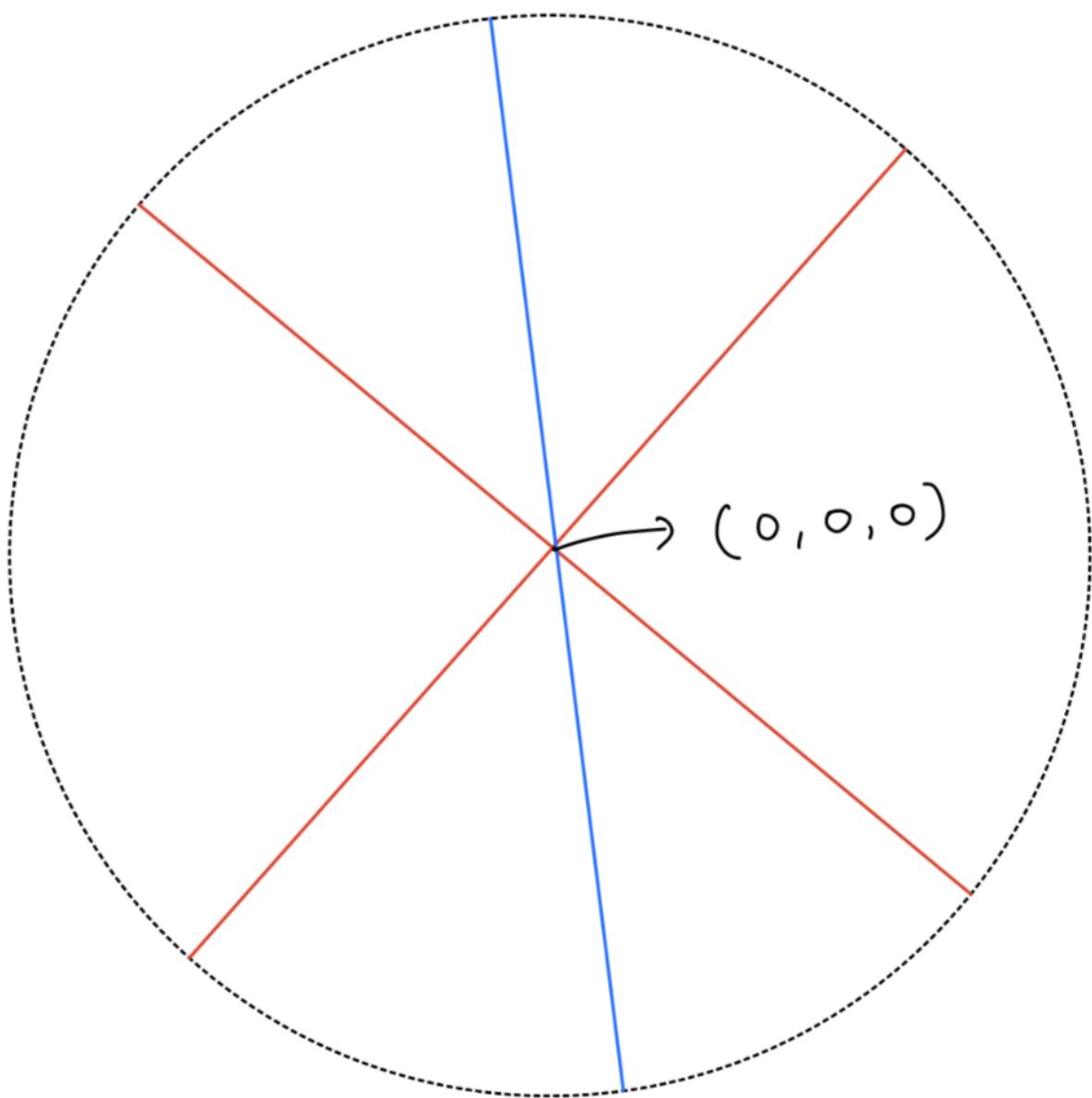


Figure 3.217

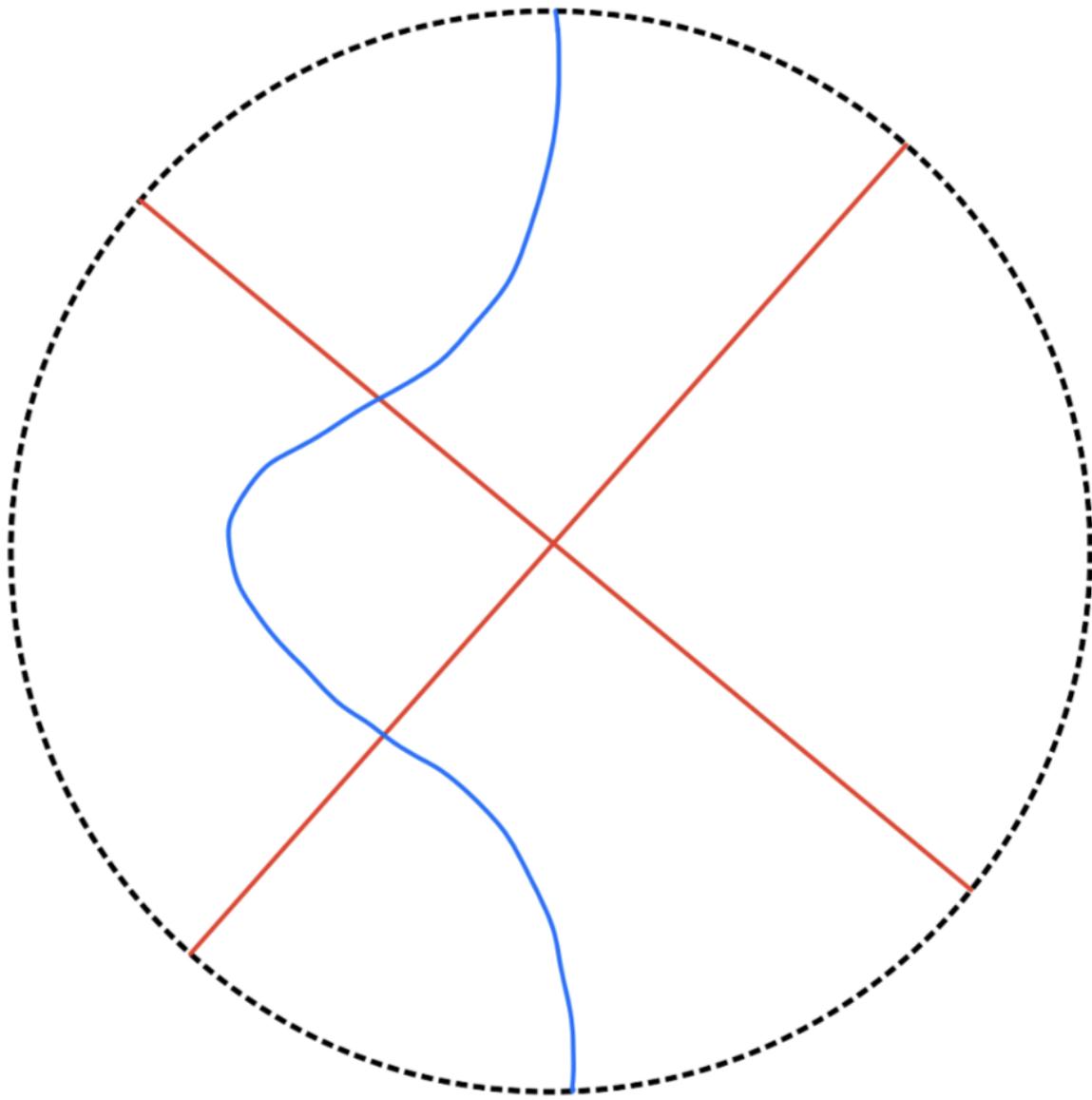


Figure 3.218

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

**Definition 171.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $star(s)$ .

**Definition 172.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of  $s$ .

**Definition 173.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 174.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 175.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 176.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \dots \circ F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 177.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

## Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

## Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

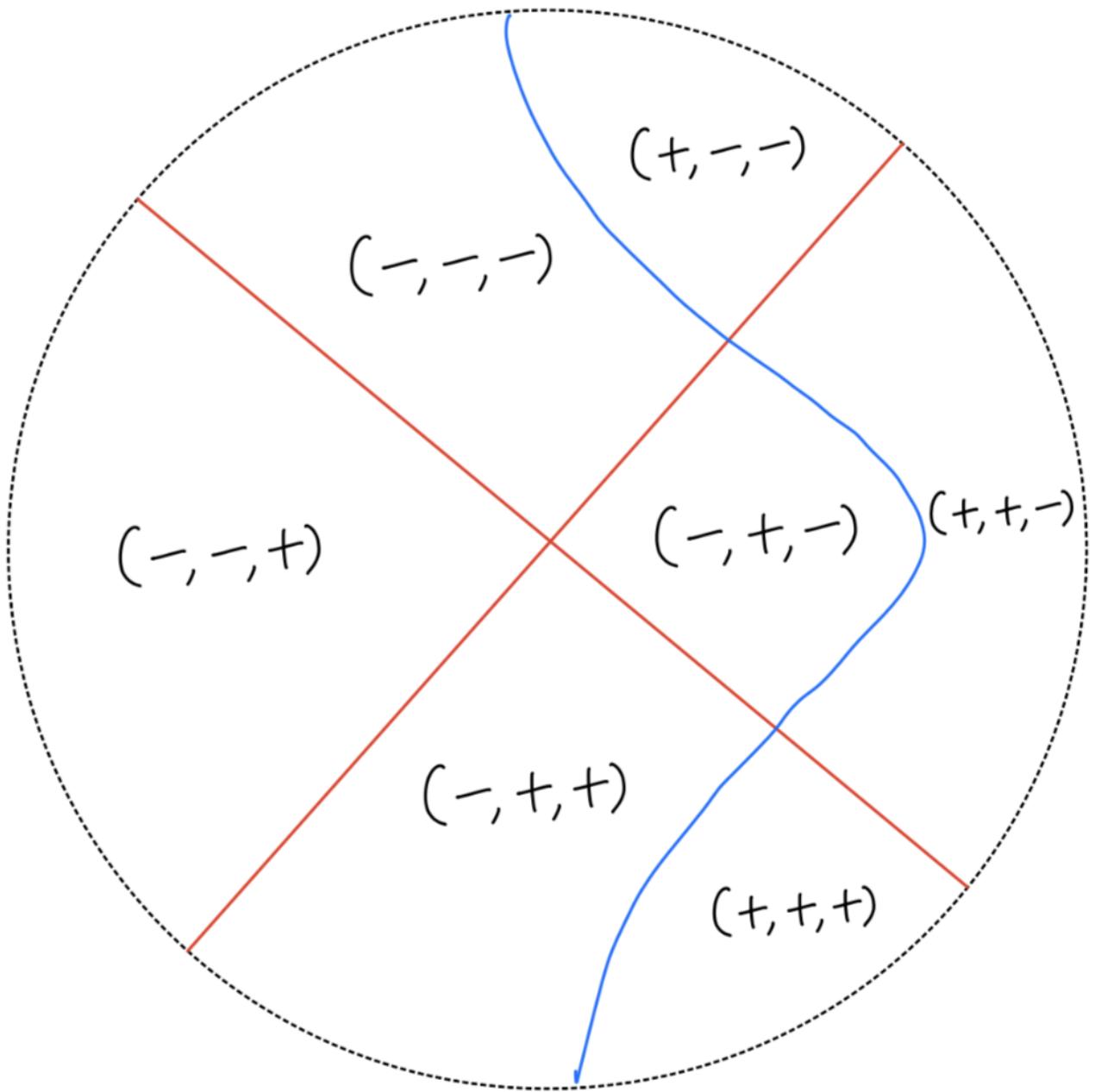


Figure 3.219

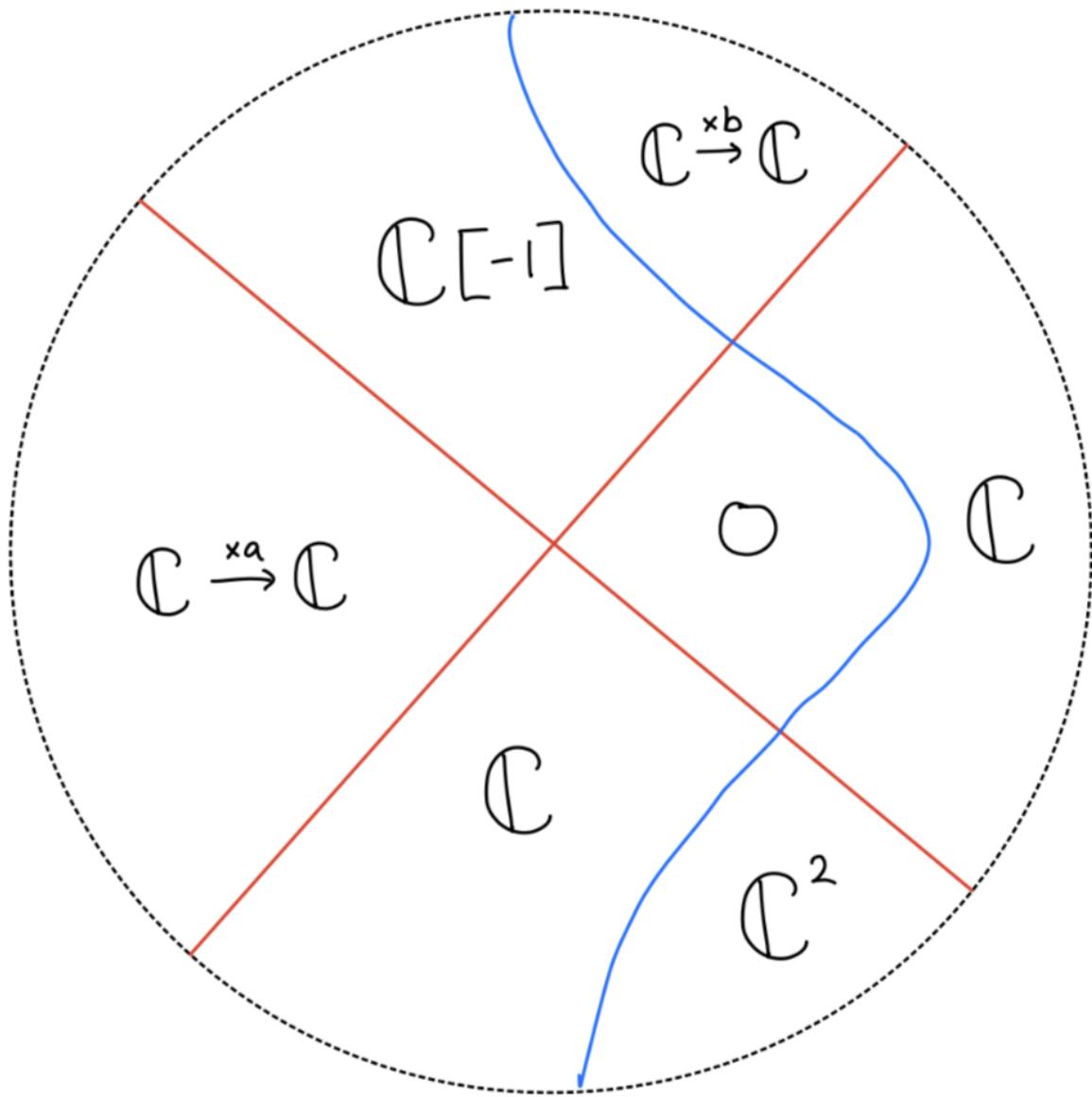


Figure 3.220

- $F_0(-, -, -) := \mathbb{C}[-1]$
- $F_0(+, -, -) := \mathbb{C} \xrightarrow{x_b} \mathbb{C}$
- $F_0(-, -, +) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_0(-, +, -) := 0$
- $F_0(+, +, -) := \mathbb{C}$

- $F_0(-, +, +) := \mathbb{C}$

- $F_0(+, +, +) := \mathbb{C}^2$

Generalization maps:

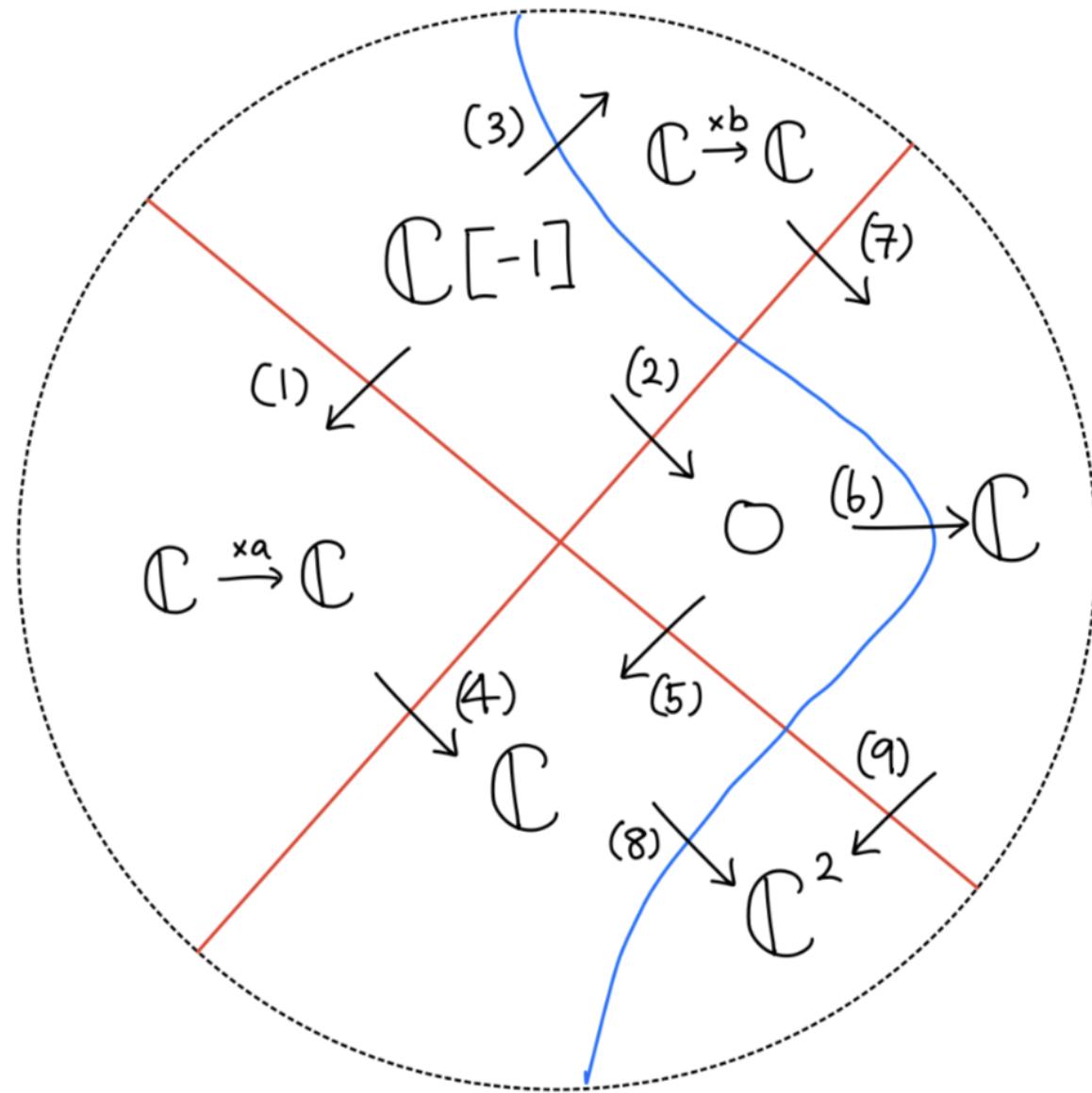


Figure 3.221

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & \uparrow \times a & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

## Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say *cobord*<sub>4</sub>, that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphism, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

**A. Sheaf on  $V \times [0, 1]$** 

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

**B. Sheaf on  $D_{r=2} \times [0, 1]$** 

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in Fun(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

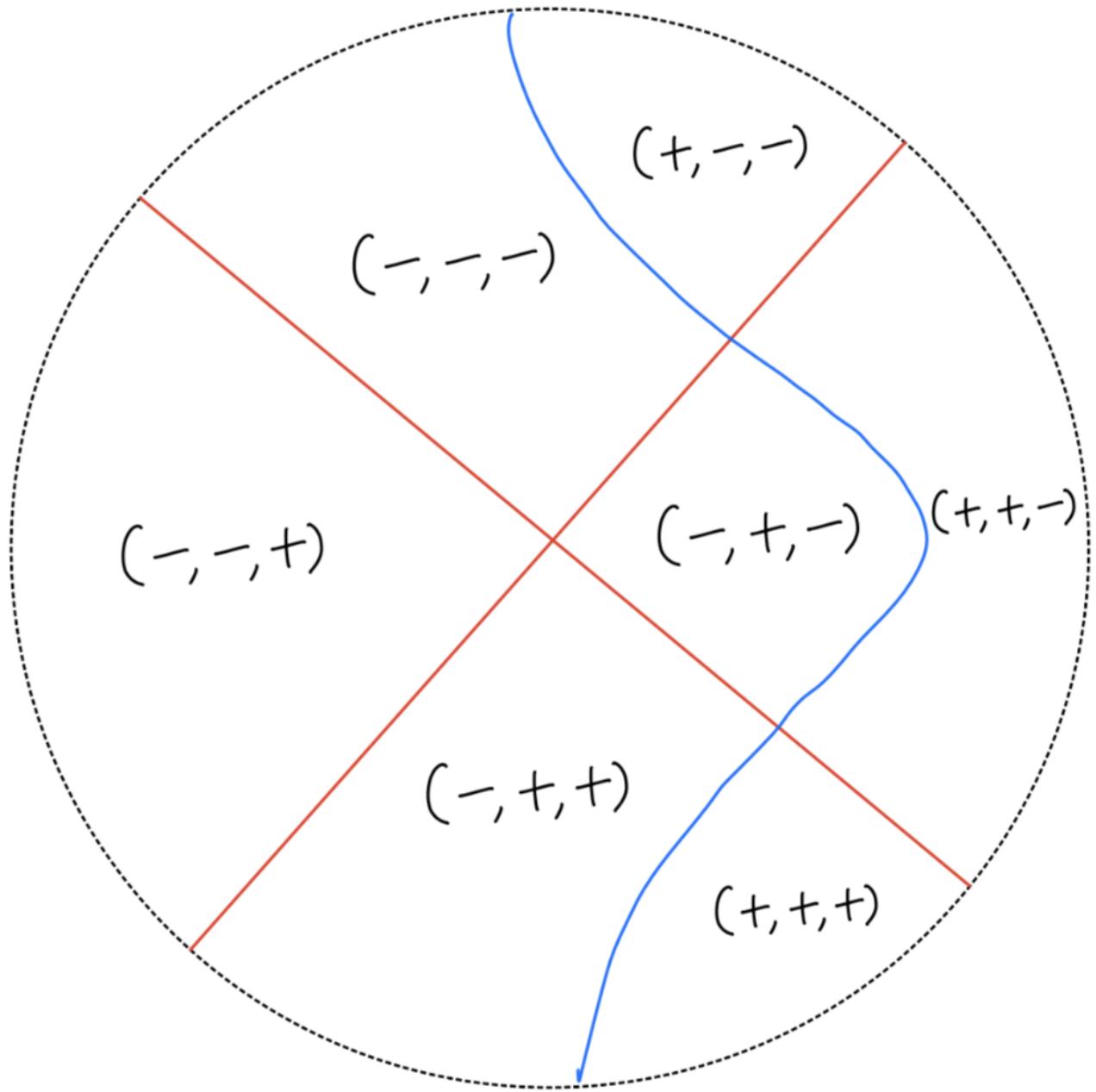


Figure 3.222

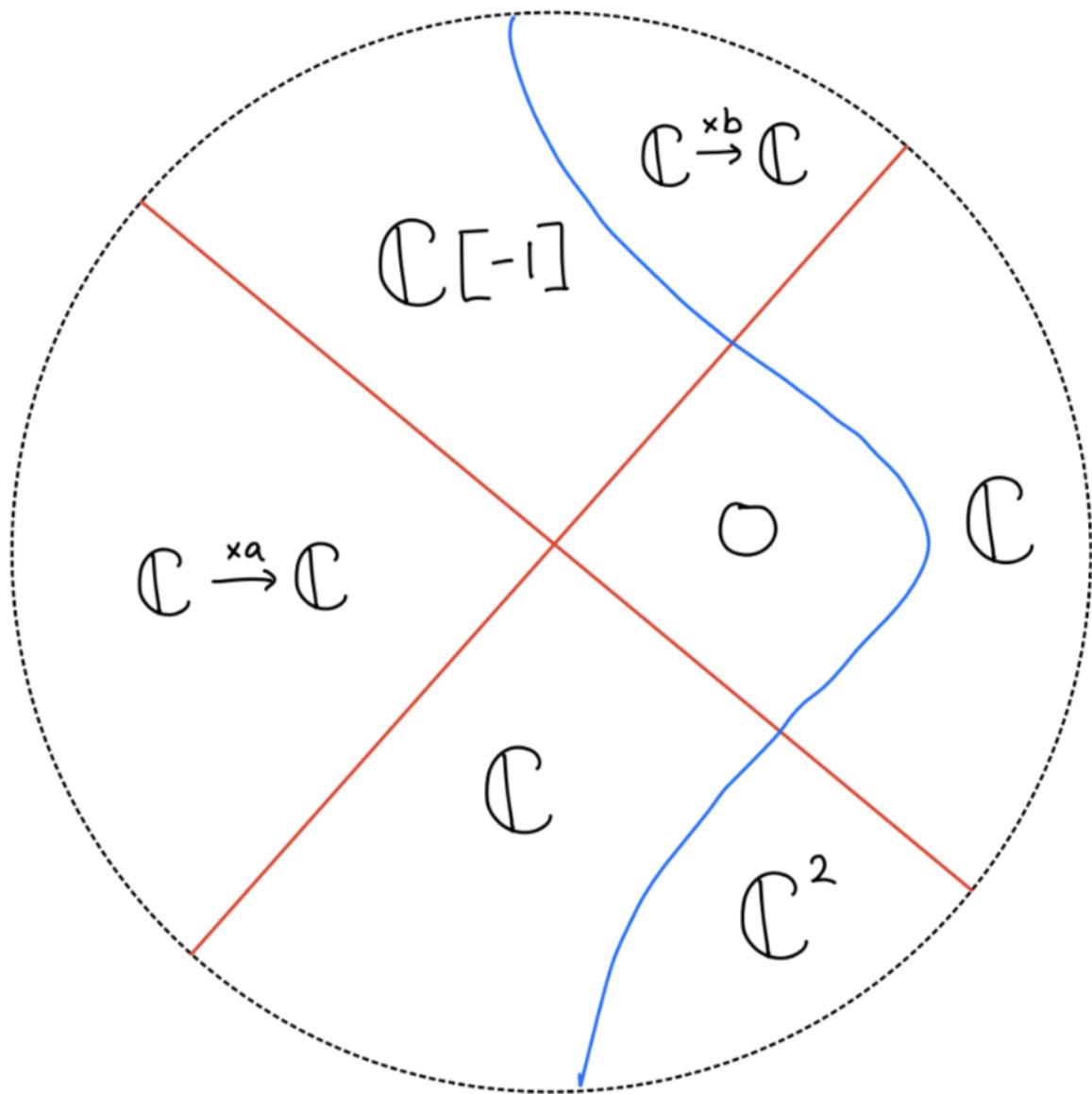


Figure 3.223

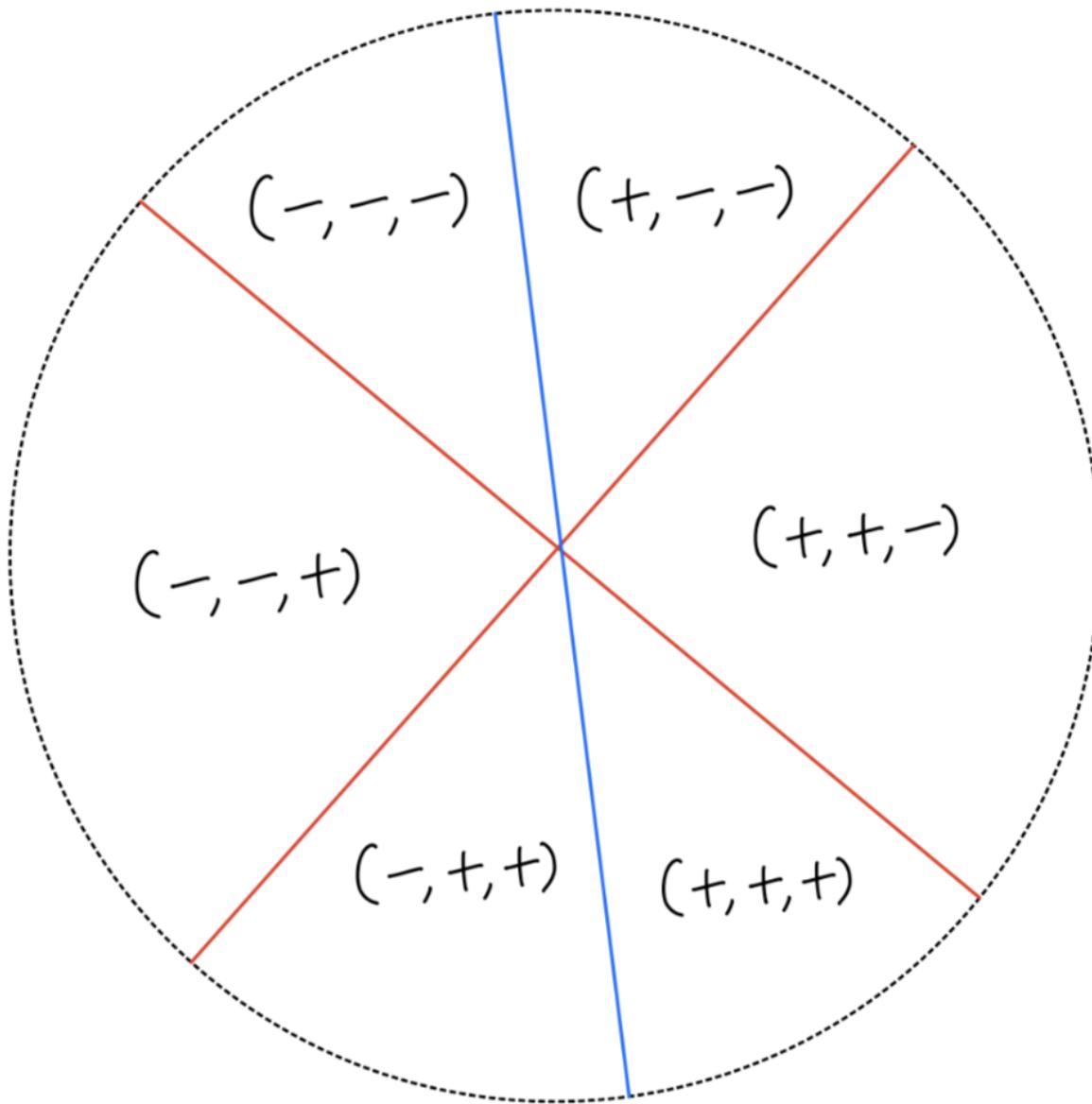


Figure 3.224

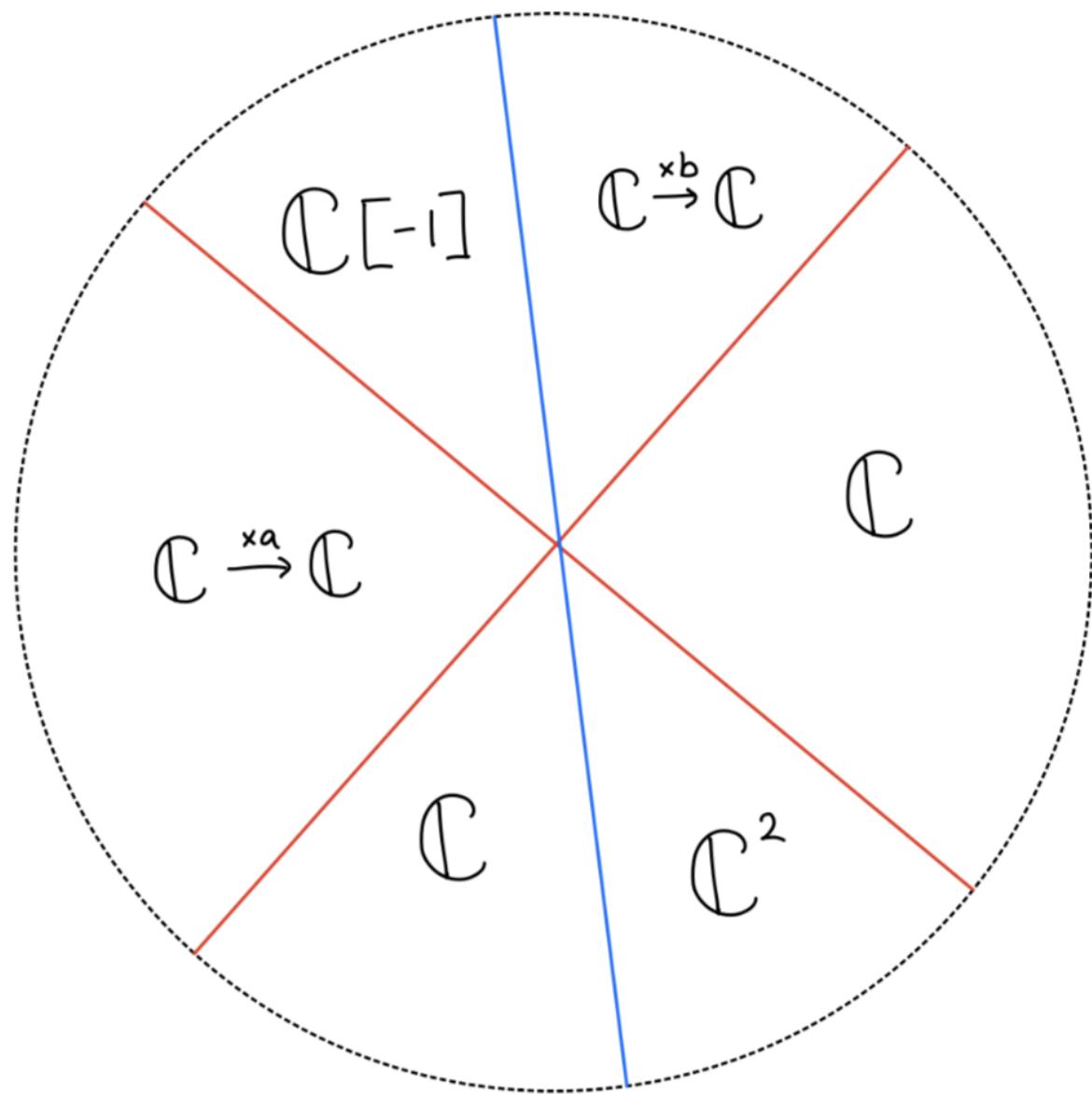


Figure 3.225

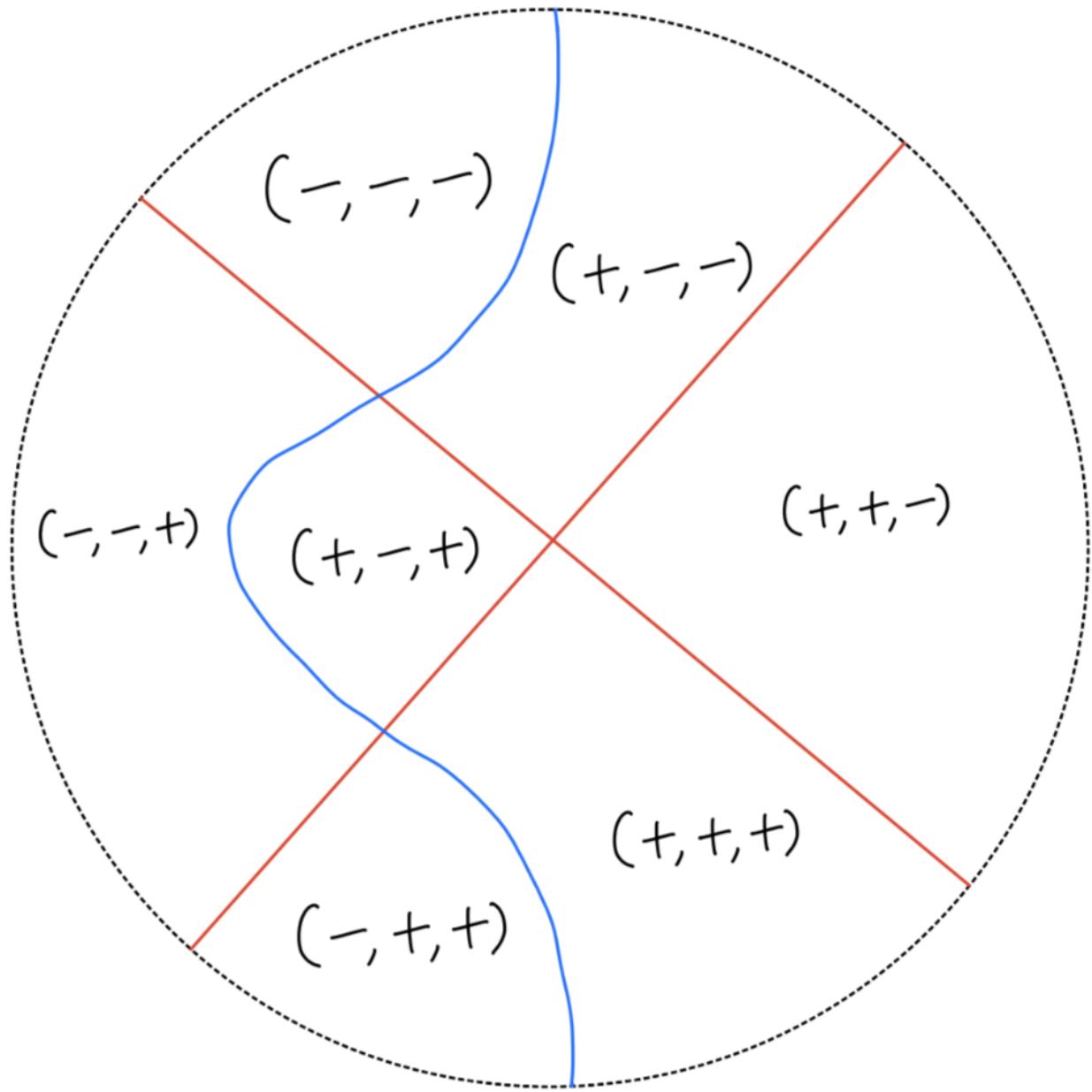


Figure 3.226

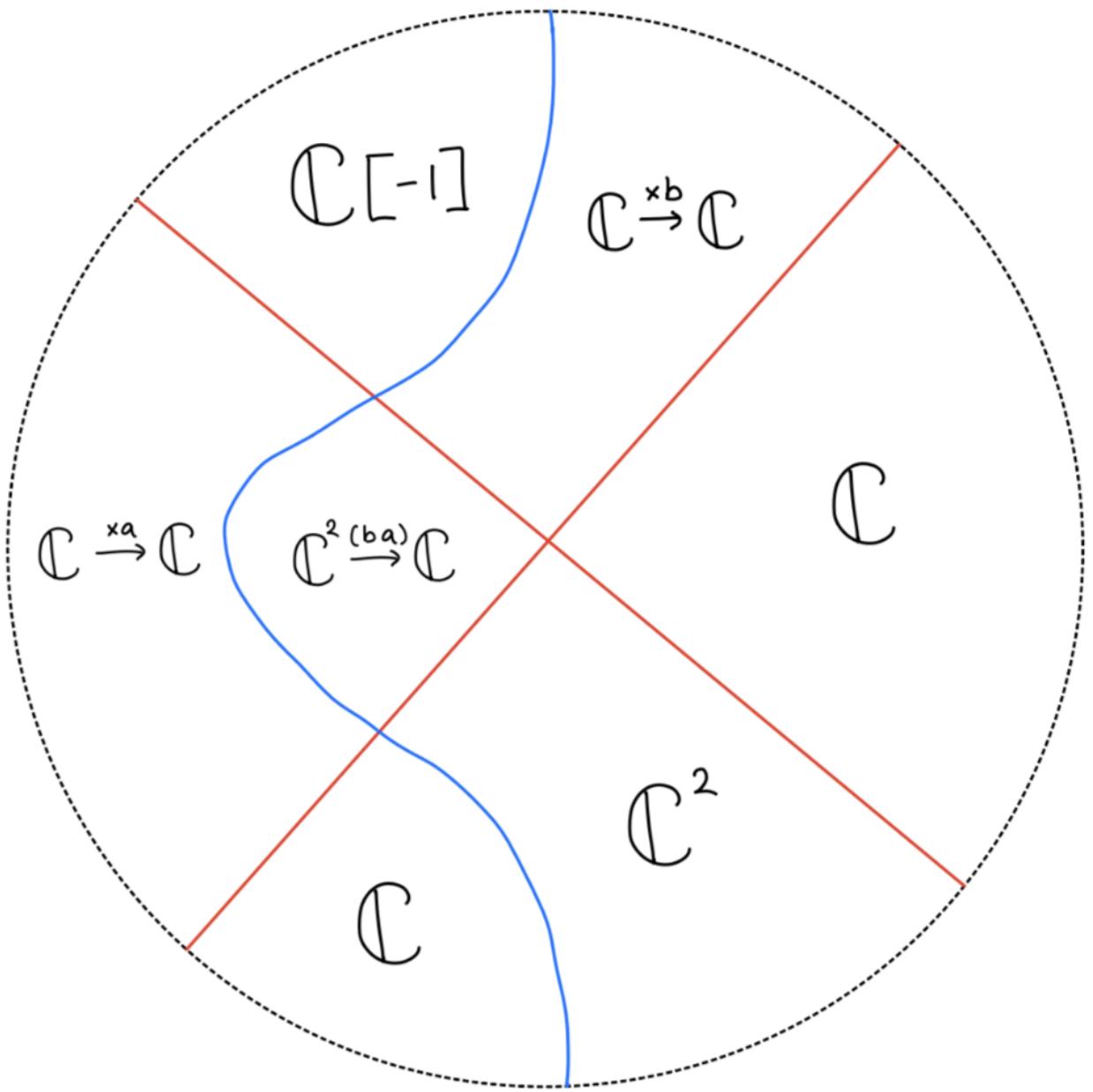


Figure 3.227

- $F_\bullet(-, -, -) := \mathbb{C}[-1]$
- $F_\bullet(-, -, +) := \mathbb{C} \xrightarrow{x_a} \mathbb{C}$
- $F_\bullet(-, +, -) := 0$
- $F_\bullet(-, +, +) := \mathbb{C}$

- $F_\bullet(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F_\bullet(+, -, +) := \mathbb{C}^2 \xrightarrow{\times(b\ a)} \mathbb{C}$
- $F_\bullet(+, +, -) := \mathbb{C}$
- $F_\bullet(+, +, +) := \mathbb{C}^2$

Generalization maps:

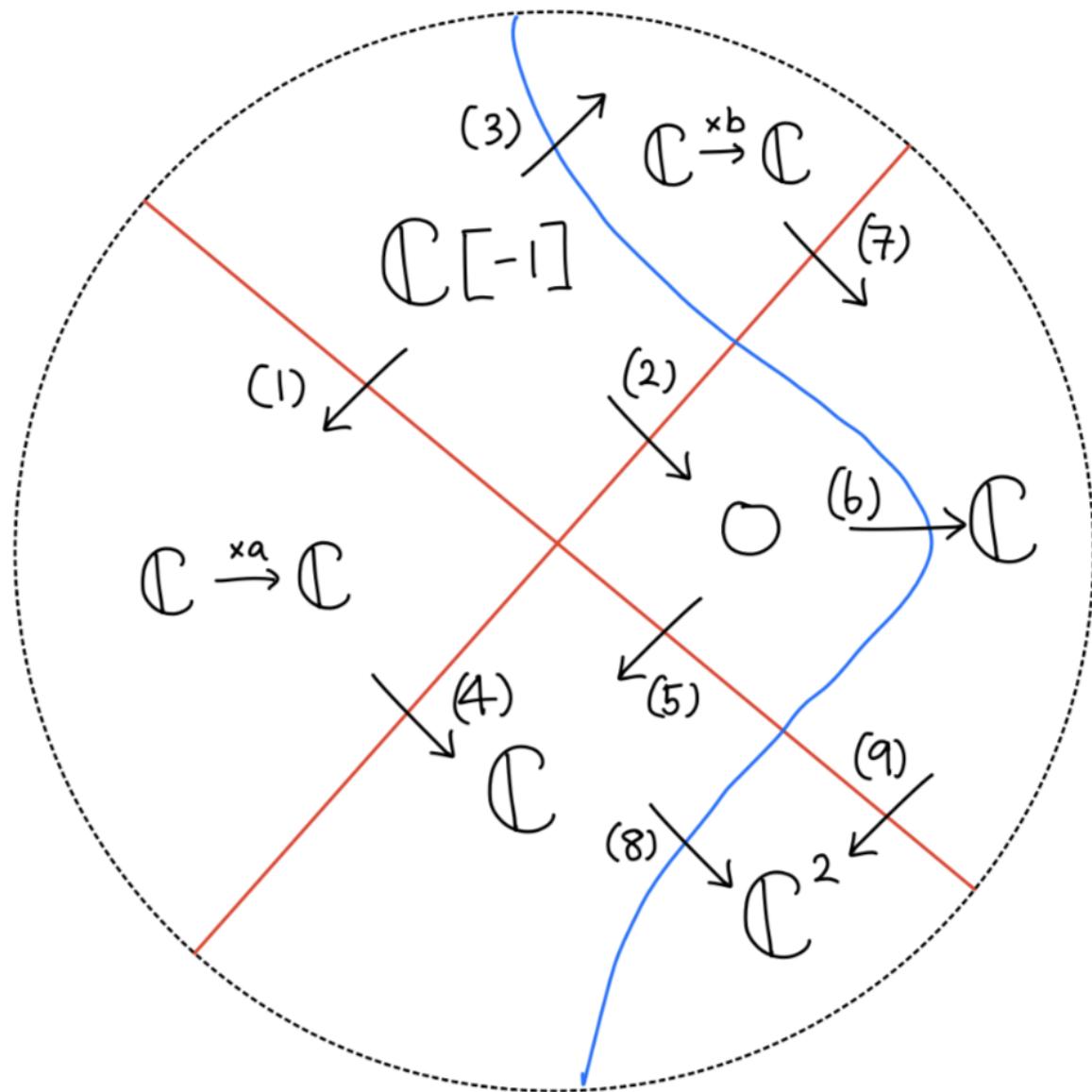


Figure 3.228

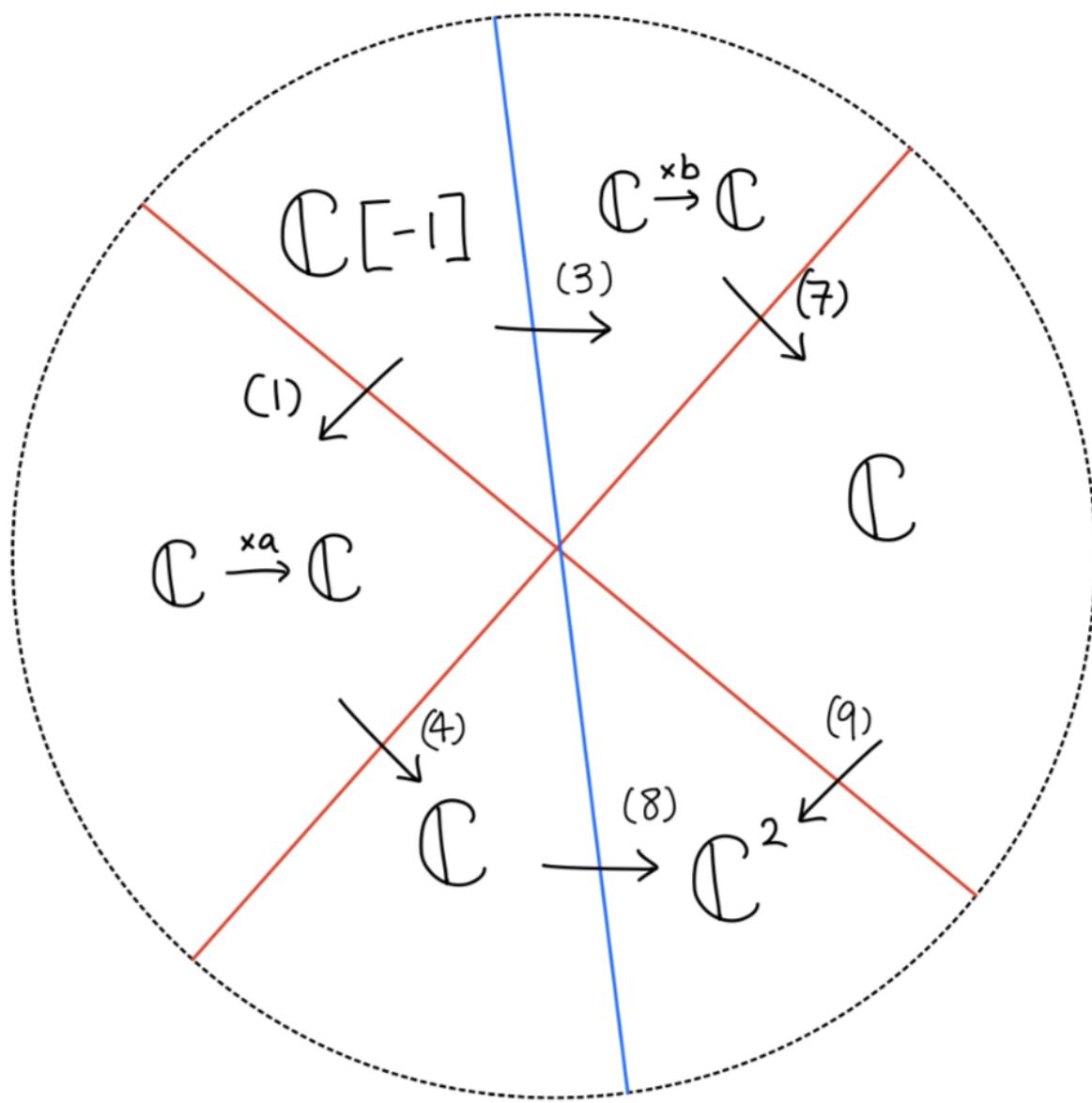


Figure 3.229

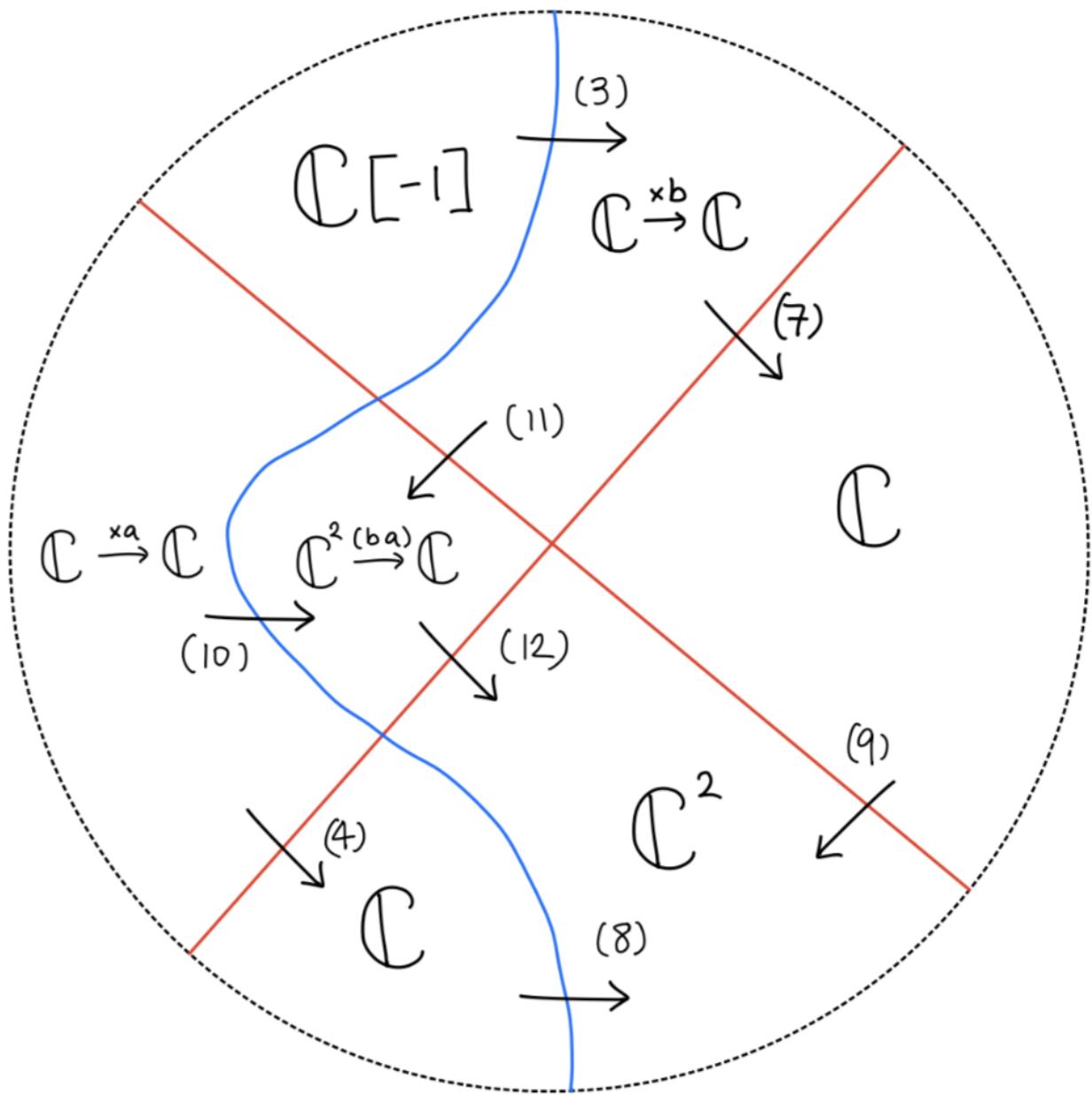


Figure 3.230

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & \uparrow \times a & \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times a \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \times b \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(10) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 178.** we define  $\gamma_\bullet$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \hookrightarrow & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \longrightarrow & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 179.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M$ .

Note that there is a diffeomorphism between  $D_{r=2} \times (0,1)$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along

the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

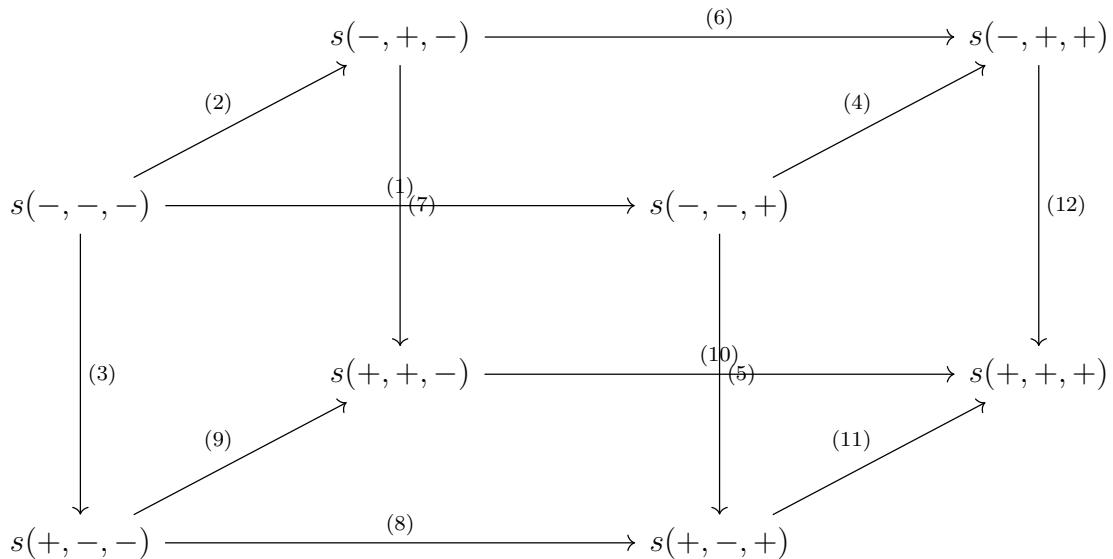
To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

- $F^3(-, -, -) := \mathbb{C}[-1]$
- $F^3(-, -, +) := \mathbb{C} \xrightarrow{\times a} \mathbb{C}$
- $F^3(+, -, -) := \mathbb{C} \xrightarrow{\times b} \mathbb{C}$
- $F^3(+, -, +) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$
- $F^3(-, +, -) := 0$
- $F^3(-, +, +) := \mathbb{C}$
- $F^3(+, +, -) := \mathbb{C}$
- $F^3(+, +, +) := \mathbb{C}^2$

**Generalization maps:**



$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times a} & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times a & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times a & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(6) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(8) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times b & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(9) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times b & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(10) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(11) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

$$(12) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$\begin{aligned} \text{(i)} \quad & F^3(-, -, -) \longrightarrow F^3(-, +, -) \quad \mathbb{C}[-1] \longrightarrow 0 \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(+, -, -) \longrightarrow F^3(+, +, -) \quad \mathbb{C} \xrightarrow{\times a} \mathbb{C} \longrightarrow \mathbb{C} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & F^3(-, -, +) \longrightarrow F^3(-, +, +) \quad \mathbb{C} \xrightarrow{\times a} \mathbb{C} \xrightarrow{(\times 1, \times 0)} \mathbb{C} \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(+, -, +) \longrightarrow F^3(+, +, +) \quad \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \xrightarrow{(id, 0)} \mathbb{C}^2 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & F^3(-, -, -) \longrightarrow F^3(-, -, +) \quad \mathbb{C}[-1] \xrightarrow{(\times 0, \times 1)} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(+, -, -) \longrightarrow F^3(+, -, +) \quad \mathbb{C} \xrightarrow{\times b} \mathbb{C} \xrightarrow{(\iota_0, \times 1)} \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad & F^3(-, +, -) \longrightarrow F^3(-, +, +) \quad 0 \xrightarrow{\iota_0} \mathbb{C} \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(+, +, -) \longrightarrow F^3(+, +, +) \quad \mathbb{C} \xrightarrow{\iota_0} \mathbb{C}^2 \end{aligned}$$

$$\begin{aligned} \text{(v)} \quad & F^3(-, -, -) \longrightarrow F^3(-, -, +) \quad \mathbb{C}[-1] \xrightarrow{(\times 0, \times 1)} \mathbb{C} \xrightarrow{\times a} \mathbb{C} \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(-, +, -) \longrightarrow F^3(-, +, +) \quad 0 \longrightarrow \mathbb{C} \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad & F^3(+, -, -) \longrightarrow F^3(+, -, +) \quad \mathbb{C} \xrightarrow{\times b} \mathbb{C} \xrightarrow{(\iota_0, \times 1)} \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C} \\ & \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ & F^3(+, +, -) \longrightarrow F^3(+, +, +) \quad \mathbb{C} \xrightarrow{\iota_0} \mathbb{C}^2 \end{aligned}$$

(vii) the cubic diagram:

$$\begin{array}{ccccc}
 & s(-,+,-) & \xrightarrow{s(-,+0)} & s(-,+,+) & \\
 s(-,0,-) \nearrow & \downarrow & & \nearrow s(-,0,+)& \\
 s(-,-,-) & \xrightarrow[s(-,-,0)]{s(0,+,+)} & s(-,-,+)& & \downarrow s(0,+,+) \\
 \downarrow s(0,-,-) & s(+,+,-) & \xrightarrow[s(+,+0)]{s(0,-,+)} & s(+,+,+)& \\
 s(+,0,-) \nearrow & \downarrow & \nearrow s(+,0,+)& & \\
 s(+,-,-) & \xrightarrow{s(+,-0)} & s(+,-,+)& & \\
 = & & & & \\
 & 0 & \xrightarrow{0} & \mathbb{C} & \\
 & \uparrow 0 & & \nearrow (\times 1, \times 0) & \\
 \mathbb{C}[-1] & \xrightarrow{(\times 0, \times 1)} & \mathbb{C} & \xrightarrow{\times a} \mathbb{C} & \downarrow \iota_1 \\
 & \downarrow (\times 0, \times 1) & & \downarrow \iota_0(\iota_1, \times 1) & \\
 & \mathbb{C} & \xrightarrow{(\times 1, \times 0)} & \mathbb{C}^2 & \\
 & \downarrow & & \nearrow (id, \times 0) & \\
 \mathbb{C} & \xrightarrow{\times b} \mathbb{C} & \xrightarrow{(\iota_0, \times 1)} & \mathbb{C}^2 & \xrightarrow{(b \ a)} \mathbb{C}
 \end{array}$$

whose total complex is

$$\begin{array}{ccccccc}
 \mathbb{C} & \xrightarrow{\times b} & \mathbb{C} & \xrightarrow{(M_1, \times 1)} & \mathbb{C}^3 & \xrightarrow{(0 \ b \ a)} & \mathbb{C} & \xrightarrow{(M_2, \times 0)} & \mathbb{C}^2 \\
 \uparrow (\times 0, \times 1) & & & & \uparrow (M_3, \times 1) & & & & \uparrow \iota_1 \\
 \text{Tot}( & & & & & & ) \\
 0 \rightarrow \mathbb{C} & \xrightarrow{(\times 0, \times 1)} & \mathbb{C} & \xrightarrow{\times a} & \mathbb{C} & \xrightarrow{(\times (-1), \times 0)} & \mathbb{C}
 \end{array}$$

where

$$M_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This is equal to

$$\text{Tot}( \begin{array}{ccccccc} \mathbb{C} & \xrightarrow{M_1} & \mathbb{C}^2 & \xrightarrow{M_2} & \mathbb{C} & \longrightarrow & 0 \\ \uparrow & & M_3 \uparrow & & M_4 \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C}^2 & \xrightarrow{M_5} & \mathbb{C} & \xrightarrow{M_6} & \mathbb{C}^2 \end{array} )$$

$$\begin{aligned}M_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\M_2 &= \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\M_3 &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \\M_4 &= \begin{pmatrix} & b & a & 0 \end{pmatrix} \\M_5 &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \\M_6 &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\end{aligned}$$

which is the following cochain complex

$$0 \rightarrow \mathbb{C}^3 \xrightarrow{M_1} \mathbb{C}^6 \xrightarrow{M_2} \mathbb{C}^3$$

$$M_1 = \begin{pmatrix} 1 & -b & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & b & a & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The final cochain complex is acyclic because of the following reasons

- $M_2 \circ M_1 = 0$
- $M_1$  is a monomorphism
- $M_2$  is an epimorphism

□

## Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the *cobord*<sub>4</sub>. By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphsim  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

**A. Sheaf on  $V$** 

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

**B. Sheaf on  $D_{r=2}$** 

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

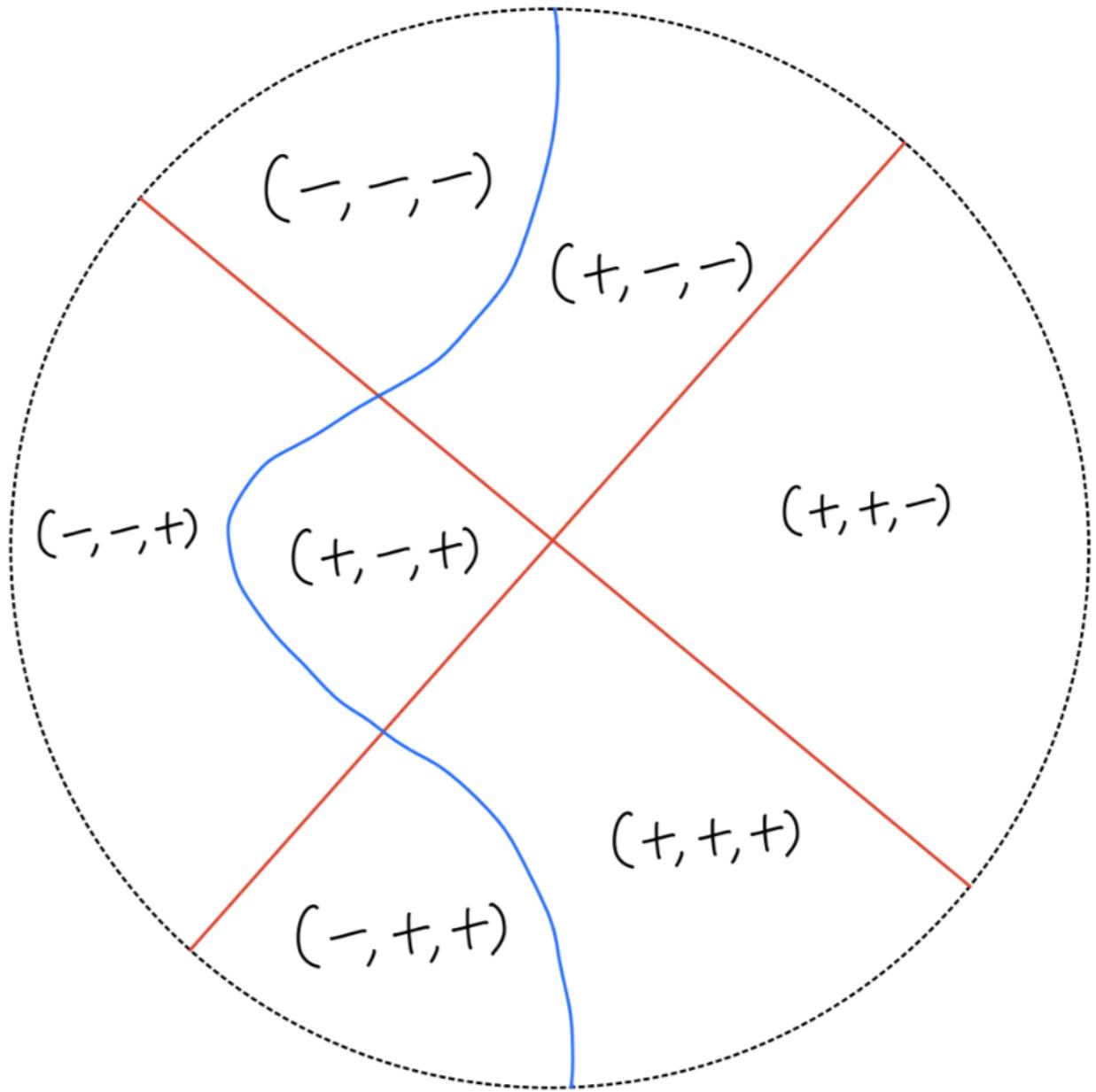


Figure 3.231

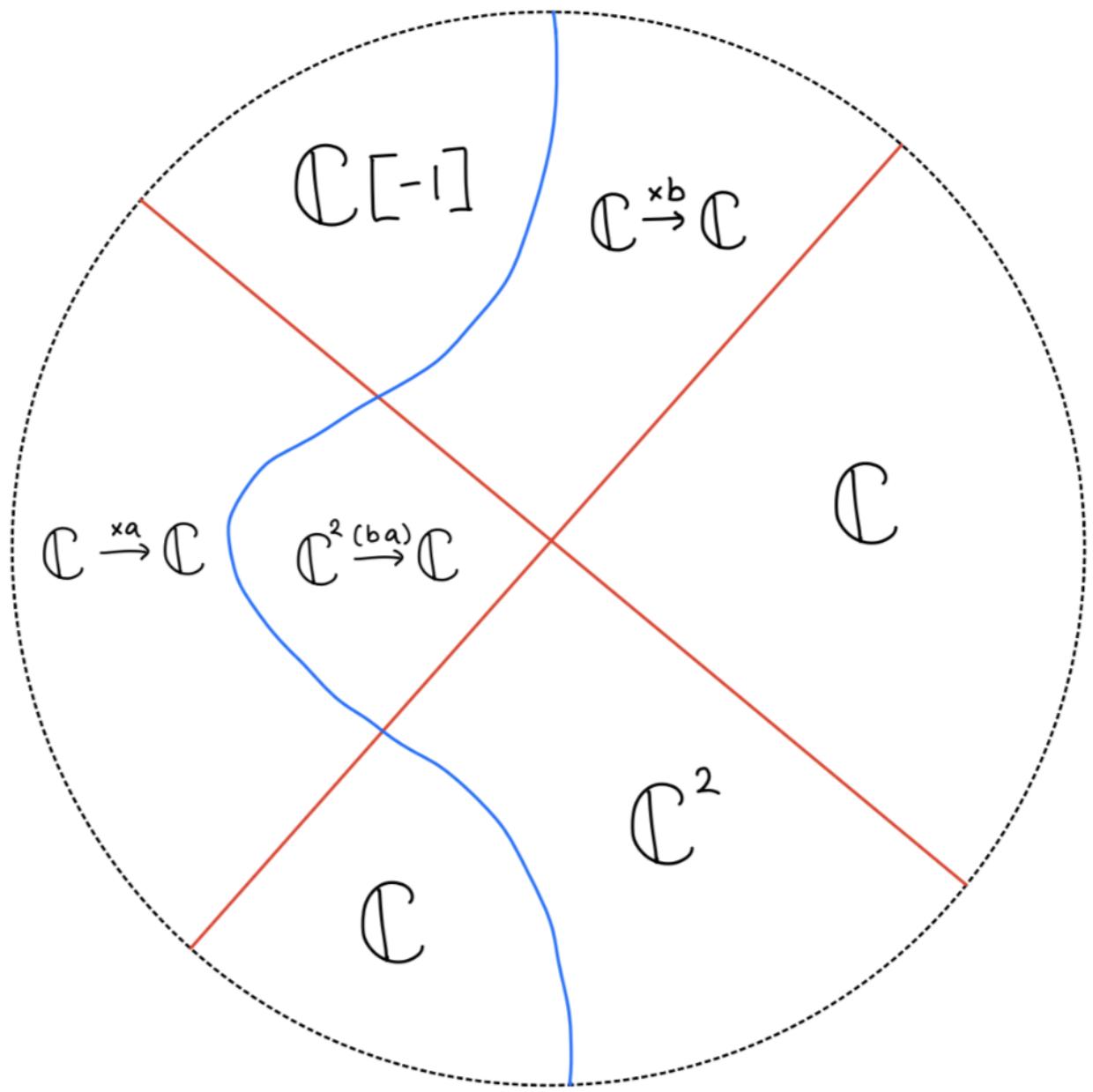


Figure 3.232

- $F_1(-, -, -) := \mathbb{C}[-1]$
- $F_1(+, -, -) := \mathbb{C} \xrightarrow{x^b} \mathbb{C}$
- $F_1(-, -, +) := \mathbb{C} \xrightarrow{x^a} \mathbb{C}$
- $F_1(+, -, +) := \mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$

- $F_1(+, +, -) := \mathbb{C}$

- $F_1(-, +, +) := \mathbb{C}$

- $F_1(+, +, +) := \mathbb{C}^2$

Generalization maps:

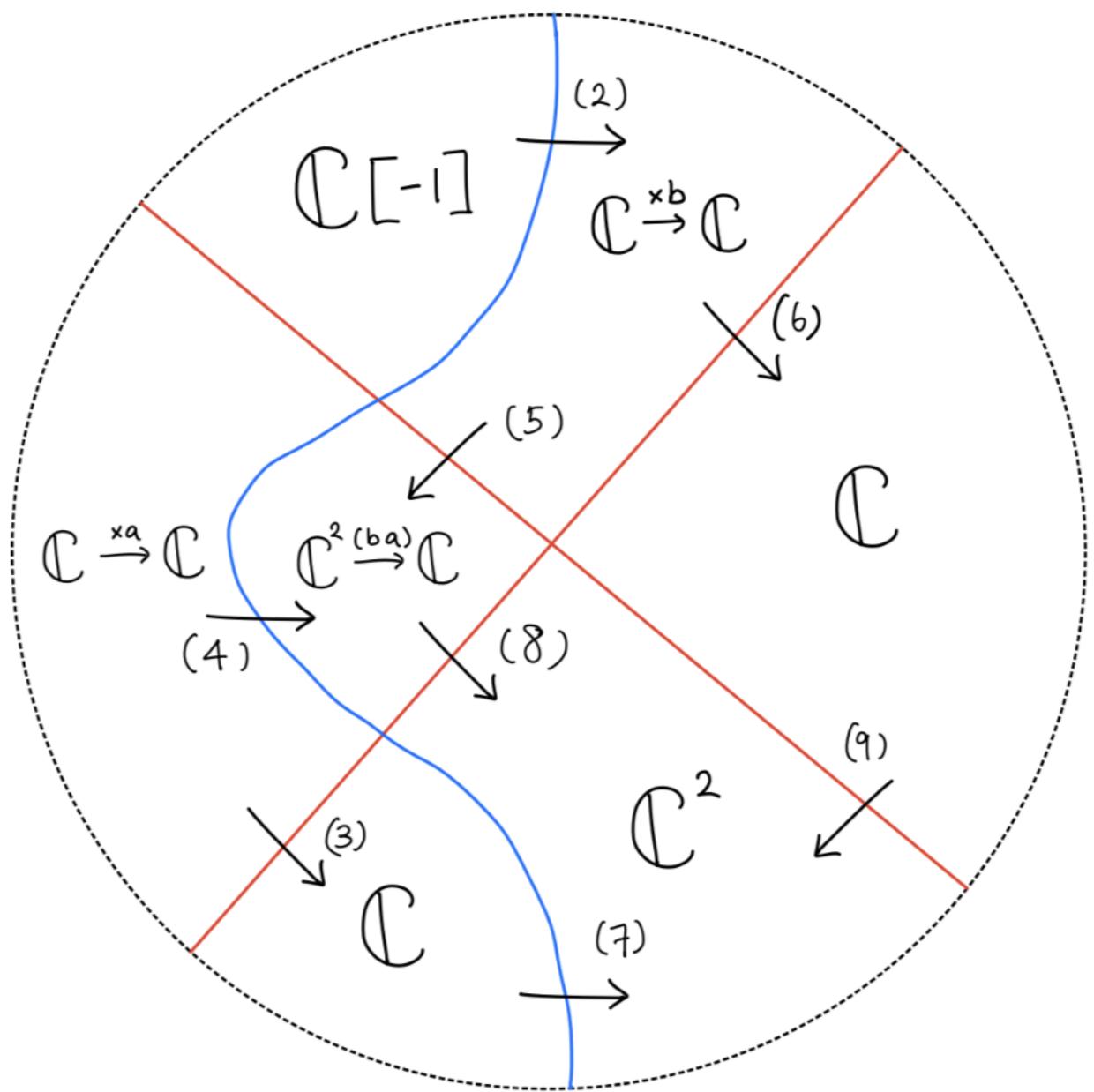


Figure 3.233

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times a \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow & & \uparrow \times b \\ 0 & \longrightarrow & \mathbb{C} \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times a & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times a & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \uparrow \times b & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(6) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ \uparrow \times b & & \uparrow \\ \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \end{array}$$

$$(7) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(8) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

$$(9) \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 180.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0, 1] \twoheadrightarrow (U \cap V)$$

## 3.9 4th Sheaf Cobordism'

In this section, we define  $cobord'_4$ , a compactly supported sheaf cobordism between the following squiggly legible diagrams on the support of the cobordism starting from

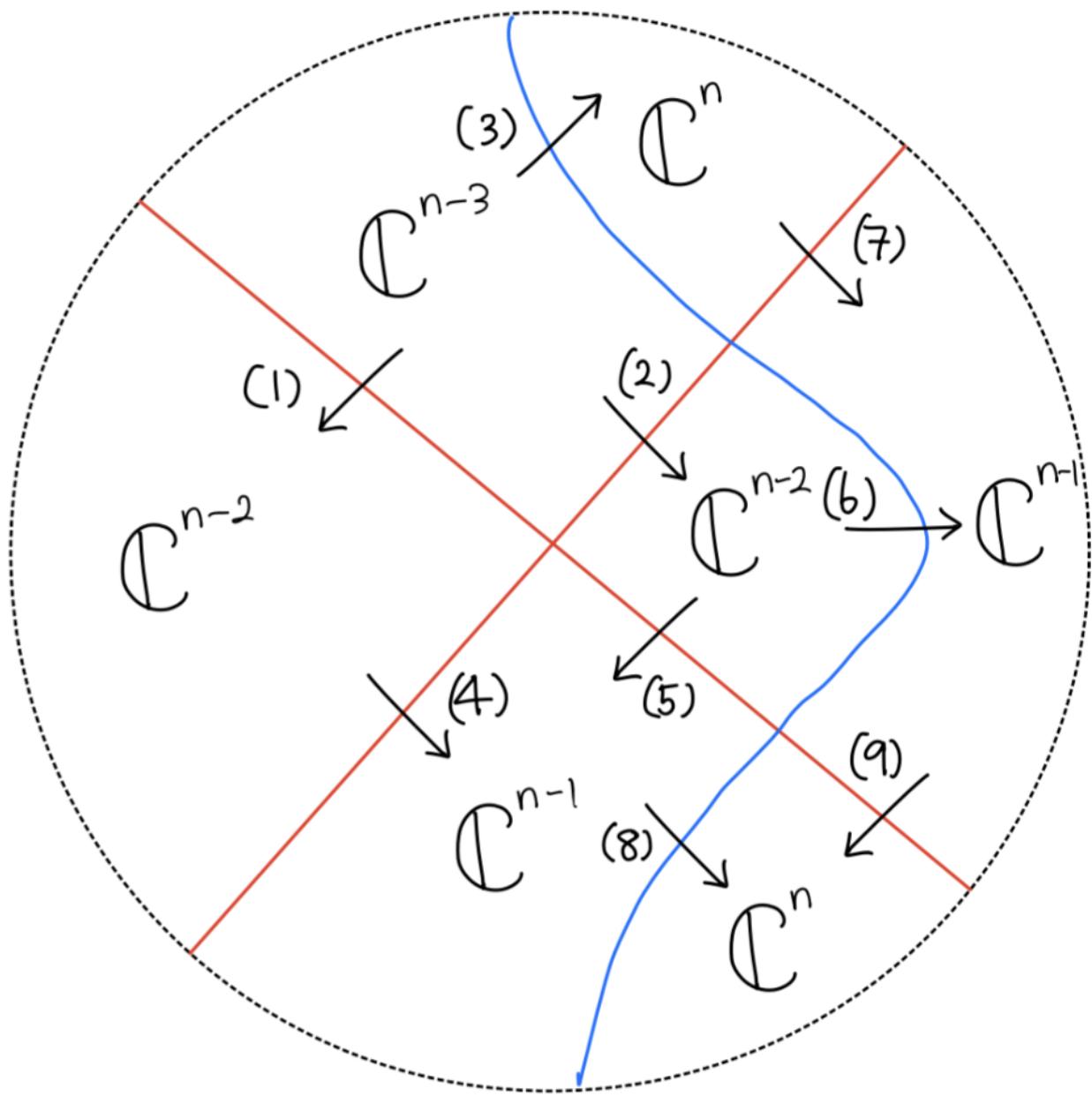


Figure 3.234

**Generalization maps:**

(1)  $\iota_0$

(2)  $\iota_0$

(3)  $\iota_1$

(4)  $\iota_0 \circ diag(1, \dots, 1) + e'I_{n,n-1}$

(5)  $\iota_0$

(6)  $\iota_1$

(7)  $\iota_0$

(8)  $\iota_1$

(9)  $\iota_0$

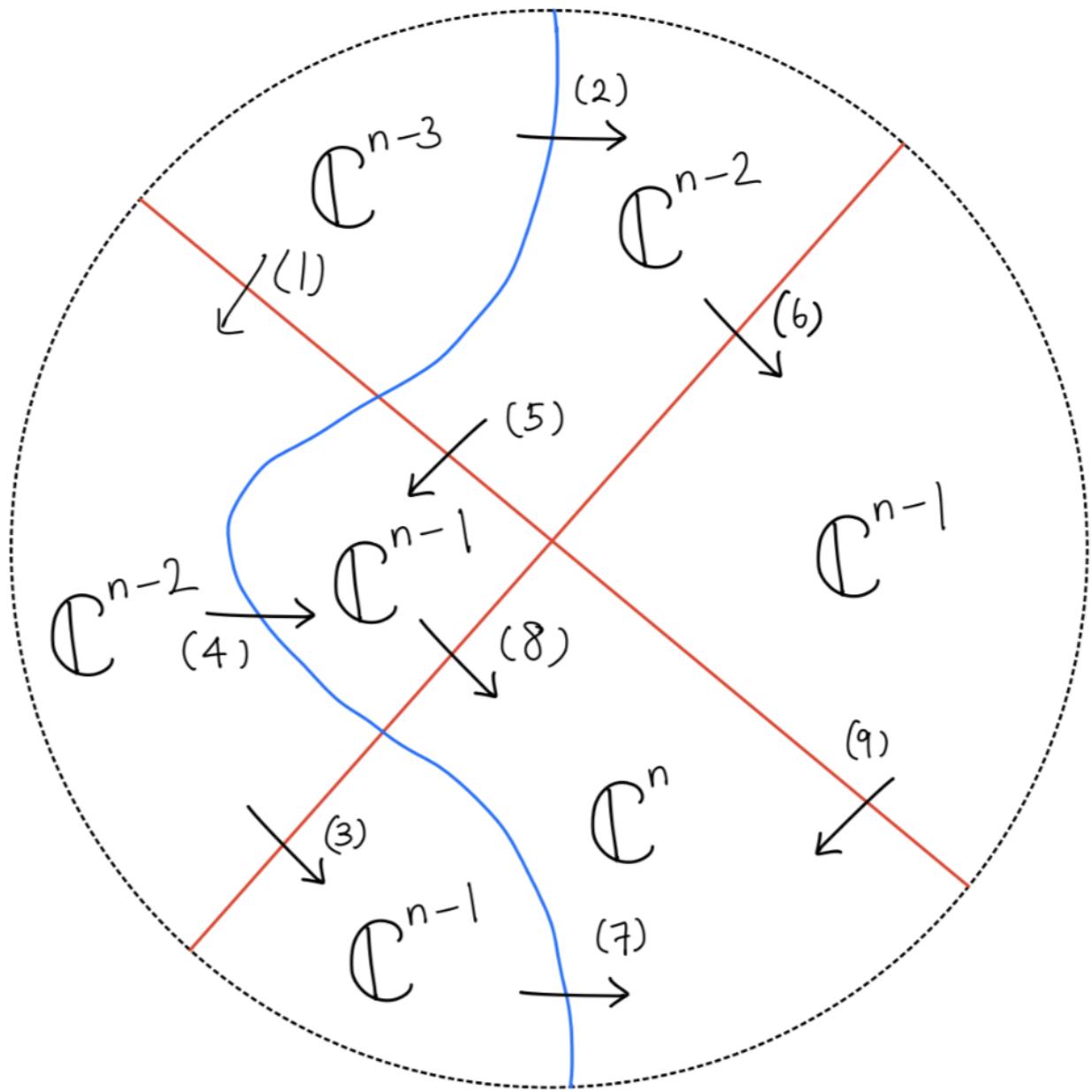


Figure 3.235

**Generalization maps:**

(1)  $\iota_0$

(2)  $\iota_1$

(3)  $\iota_0$

(4)  $\iota_1$

(5)  $\iota_0$

(6)  $\iota_0$

(7)  $\iota_1$

(8)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(9)  $\iota_0$

### 3.9.1 Notations

**Definition 181.**  $M$  denotes a Riemann sphere with two punctures at 0 and  $\infty$ .  $M$  is diffeomorphic to a cylinder.

**Definition 182.** For  $t_0 \in \{0, 1\}$  and  $\text{symbol} \in \{0, \infty, \text{squig}\}$

1. we denote  $\Phi_{t_0}^{\text{symbol}}$  to be smooth maps

$$\Phi_{t_0}^0 : (S^1)^n \rightarrow M$$

$$\Phi_{t_0}^\infty : (S^1)^m \rightarrow M$$

$$\Phi_{t_0}^{\text{squig}} : [0, 1]^{k_{t_0}} \rightarrow M$$

2. we denote  $\Xi_{t_0}^{\text{symbol}}$  a co-orientation of  $\Phi_{t_0}^{\text{symbol}}$ .

3. we denote the pair  $(\Phi_{t_0}^{\text{symbol}}, \Xi_{t_0}^{\text{symbol}})$  as  $\Lambda_{t_0}^{\text{symbol}}$ . When  $\text{symbol} \in \{0, \infty\}$ , this could be thought as a front projection of a Legendrian living inside the cocircle bundle of  $M$ , so we will use  $\Lambda_{t_0}^{\text{symbol}}$  to denote both

4. we denote the triple  $(\Lambda_{t_0}^0, \Lambda_{t_0}^\infty, \Lambda_{t_0}^{\text{squig}})$  as  $\Lambda_{t_0}$  and call it the squiggly diagram at  $t_0$ . Later in the section,  $\Lambda_0$  will be used to denote the squiggly diagram at the

beginning of the isotopy underlying *cobord*<sub>4</sub> and  $\Lambda_1$  will be used to denote the squiggly diagram at the end of the isotopy underlying *cobord*<sub>4</sub>.

**Definition 183.** For  $symbol \in \{0, \infty, squig\}$

1. we denote  $\Phi_{\bullet}^{symbol}$  to be smooth maps

$$\Phi_{\bullet}^0 : (S^1)^n \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{\infty} : (S^1)^m \times [0, 1]_t \rightarrow M \times [0, 1]_t$$

$$\Phi_{\bullet}^{squig} : \coprod_{1 \leq i \leq k} ([0, 1] \times [a_i, b_i]_t) \rightarrow M \times [0, 1]_t$$

where the maps are identity maps on the time coordinates. I added auxiliary subscript '*t*' to distinguish the time coordinates from the space coordinates.

2. we denote  $\Xi_{\bullet}^{symbol}$  a co-orientation of  $\Phi_{\bullet}^{symbol}$ .

3. we denote the pair  $(\Phi_{\bullet}^{symbol}, \Xi_{\bullet}^{symbol})$  as  $\Lambda_{\bullet}^{symbol}$ . Later in the section,  $\Lambda_{\bullet}^{symbol}$  will be used to denote the an isotopy from  $\Lambda_0^{symbol}$  to  $\Lambda_1^{symbol}$  underlying *cobord*<sub>4</sub>.

4. we denote the triple  $(\Lambda_{\bullet}^0, \Lambda_{\bullet}^{\infty}, \Lambda_{\bullet}^{squig})$  as  $\Lambda_{\bullet}$  and call it a squiggly isotopy from  $\Lambda_0$  to  $\Lambda_1$ . Later in the section,  $\Lambda_{\bullet}$  will be used to denote the isotopy between squiggly diagrams starting from  $\Lambda_0$  ending at  $\Lambda_1$  underlying *cobord*<sub>4</sub>.

**Definition 184.** For  $t \in [0, 1]$ , we define  $\Psi_t : \mathbb{R} \rightarrow \mathbb{R}$  to be a bump function parametrized by  $t$  as follows

$$\Psi_t(x) = \begin{cases} e^{(\frac{x^2}{x^2-1})} (\frac{1}{2} - t) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Note that

- $supp(\Psi_t) = [-1, 1]$  if  $t \neq \frac{1}{2}$

- $\{(1, 0), (-1, 0), (0, \frac{1}{2} - t)\} \subset Graph(\Psi_t)$

**Definition 185.** We denote the standard open disk in  $\mathbb{R}^2$  of radius  $r_0$  centered at the origin as

$$D_{r=r_0} := \{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 < r_0^2\}$$

For  $t_0 \in [0, 1]$ , we canonically identify  $D_{r=r_0} \times \{t_0\}$  with  $D_{r=r_0}$  using the following diffeomorphism

$$D_{r=r_0} \xrightarrow{\sim} D_{r=r_0} \times \{t_0\}$$

$$(x, z) \mapsto (x, z, t_0)$$

and with abuse of expression say that sheaves on  $D_{r=r_0} \times \{t_0\}$  as sheaves on  $D_{r=r_0}$ .

**Definition 186.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{0\}$

- $\lambda_0^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_0(z)\}$
- $\lambda_0^\infty$  is the union of the following two components
  - $\{(x, z) \in D_{r=2} \mid z = x\}$
  - $\{(x, z) \in D_{r=2} \mid z = -x\}$

2. We define co-orientations  $\xi_0^{symbol}$  of  $\lambda_0^{symbol}$  as follows

- $\xi_0^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.
- $\xi_0^\infty$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are positive.

**Definition 187.** 1. We define the following subsets of  $D_{r=2} \cong D_{r=2} \times \{1\}$

- $\lambda_1^0 := \{(x, z) \in D_{r=2} \mid x = \Psi_1(z)\}$
- $\lambda_1^\infty$  is the union of the following two components
  - $\{(x, z) \in D_{r=2} \mid z = x\}$

$$(ii) \quad \{(x, z) \in D_{r=2} \mid z = -x\}$$

2. We define co-orientations  $\xi_1^{symbol}$  of  $\lambda_1^{symbol}$  as follows

- $\xi_1^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.
- $\xi_1^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.

**Definition 188.** 1. We define the following subsets of  $D_{r=2} \times [0, 1]$

- $\lambda_\bullet^0 := \{(x, z, t) \in D_{r=2} \times [0, 1] \mid x = \Psi_t(z)\}$
- $\lambda_\bullet^\infty$  is the union of the following two components
  - (i)  $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = x\}$
  - (ii)  $\{(x, z, t) \in D_{r=2} \times [0, 1] \mid z = -x\}$

2. We define co-orientations  $\xi_\bullet^{symbol}$  of  $\lambda_\bullet^{symbol}$  as follows

- $\xi_\bullet^0$ : hairs are pointing towards the left i.e. coefficients of  $dx$  are negative.
- $\xi_\bullet^\infty$ : hairs are pointing upward direction i.e. coefficients of  $dz$  are positive.

**Definition 189.** 1. Consider a stratification  $\mathcal{S}_0$  on  $D_{r=2}$  induced by  $\lambda_0$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_0$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_0$ .

2. Consider a stratification  $\mathcal{S}_1$  on  $D_{r=2}$  induced by  $\lambda_1$  i.e. stratification where 0 dimensional strata are either crossings or end points of squiggly lines, 1 dimensional strata are sub-arcs of co-oriented links and squiggly lines that are separated by 0 dimensional strata, and 2 dimensional strata are exactly the connected components of  $M - \lambda_1$ . Note that 1 dimensional strata has co-orientations inherited from  $\lambda_1$ .

Consider a stratification  $\mathcal{S}_\bullet$  on  $D_{r=2} \times [0, 1]$  induced by  $\lambda_\bullet$  i.e. strata are non-empty finite intersections of  $\lambda_\bullet^0$ ,  $\lambda_\bullet^\infty$ , and  $\lambda_\bullet^{squig}$ . Note that 2 dimensional strata has co-orientations inherited from  $\lambda_\bullet$ .

Now let's list the strata of  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the following notations:

**Definition 190.**  $sgn : \mathbb{R} \rightarrow \{-, 0, +\}$  is defined as

$$sgn(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases}$$

**Definition 191.** For  $i = 1, 2, 3$ ,  $t_0 = 0, 1$ , and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_{t_0}(sgn_1, sgn_2, sgn_3) := & \{(x, z) \in D_{r=2} \cong D_{r=2} \times \{t_0\} \mid \\ & sgn(x - \Psi_{t_0}(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z)) = sgn_3\} \end{aligned}$$

**Definition 192.** For  $i = 1, 2, 3$  and  $sgn_i \in \{-, 0, +\}$ , we define

$$\begin{aligned} s_\bullet(sgn_1, sgn_2, sgn_3) := & \{(x, z, t) \in D_{r=2} \times [0, 1] \mid \\ & sgn(x - \Psi_t(z)) = sgn_1, \quad sgn(x - z) = sgn_2, \\ & sgn((-x - z)) = sgn_3\} \end{aligned}$$

**Definition 193.** Now I will describe  $\mathcal{S}_0$ ,  $\mathcal{S}_1$ , and  $\mathcal{S}_\bullet$  using the above notations:

1.  $\mathcal{S}_0$ :

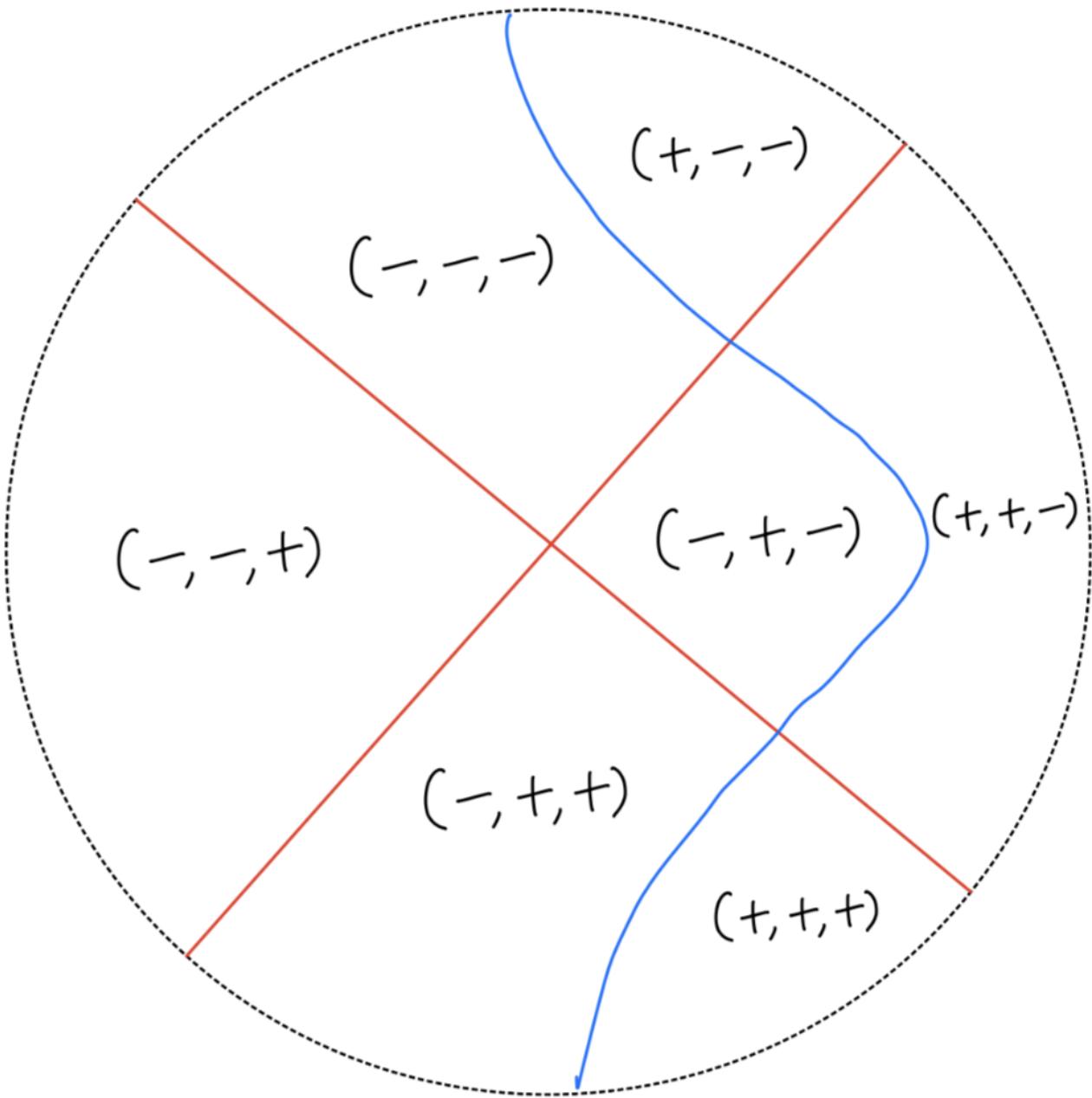


Figure 3.236

- 2 dimensional strata:

$$\{s_0(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_0(+, -, +)\}$$

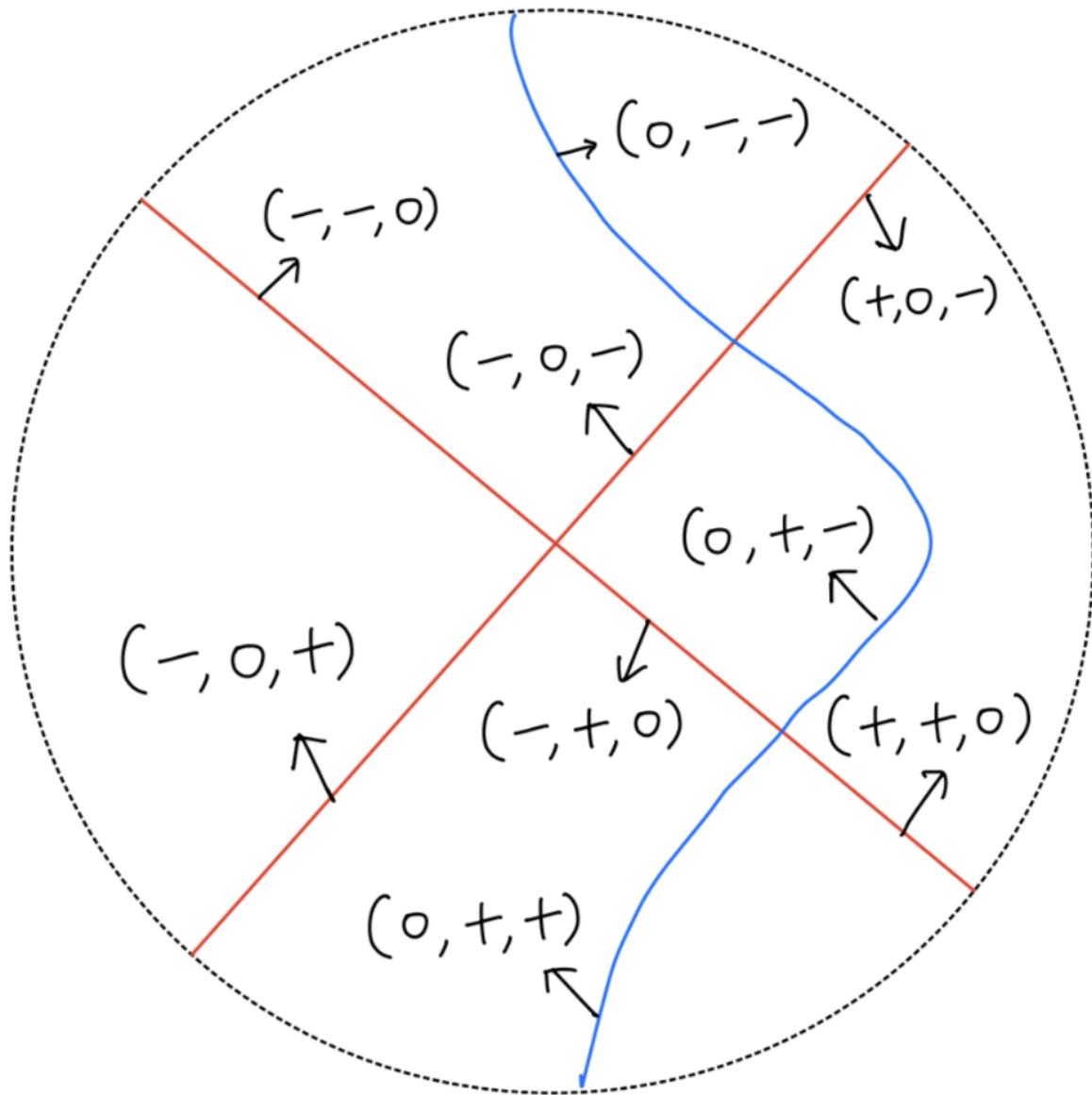


Figure 3.237

- 1 dimensional strata:

$$\begin{aligned}
 & \{s_0(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_0(0, -, +)\} \\
 & \cup \{s_0(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_0(+, 0, +)\} \\
 & \cup \{s_0(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_0(+, -, 0)\}
 \end{aligned}$$

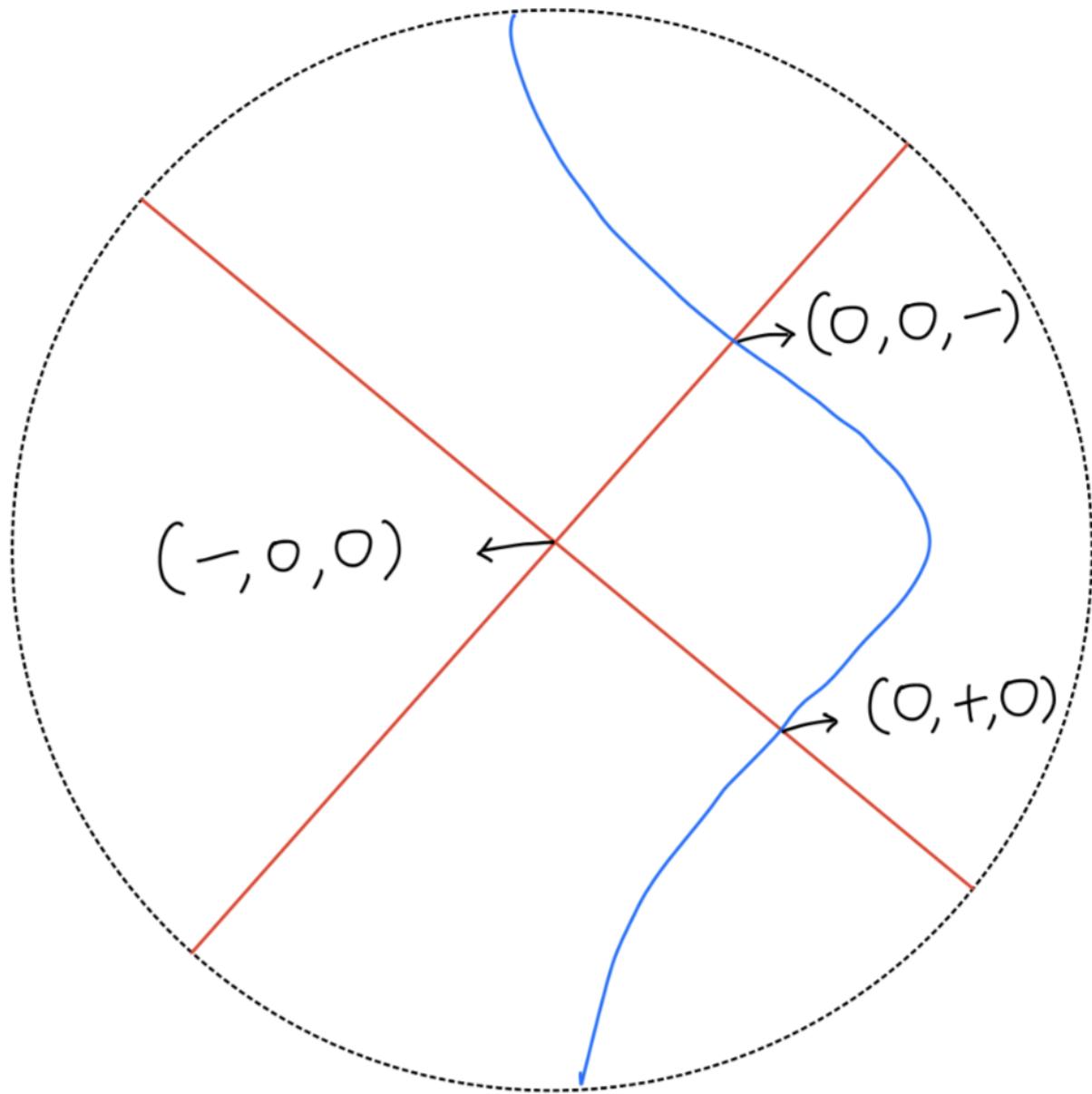


Figure 3.238

- 0 dimensional strata:

$$s_0(-, 0, 0), s_0(0, 0, -), s_0(0, +, 0)$$

2.  $\mathcal{S}_1$ :

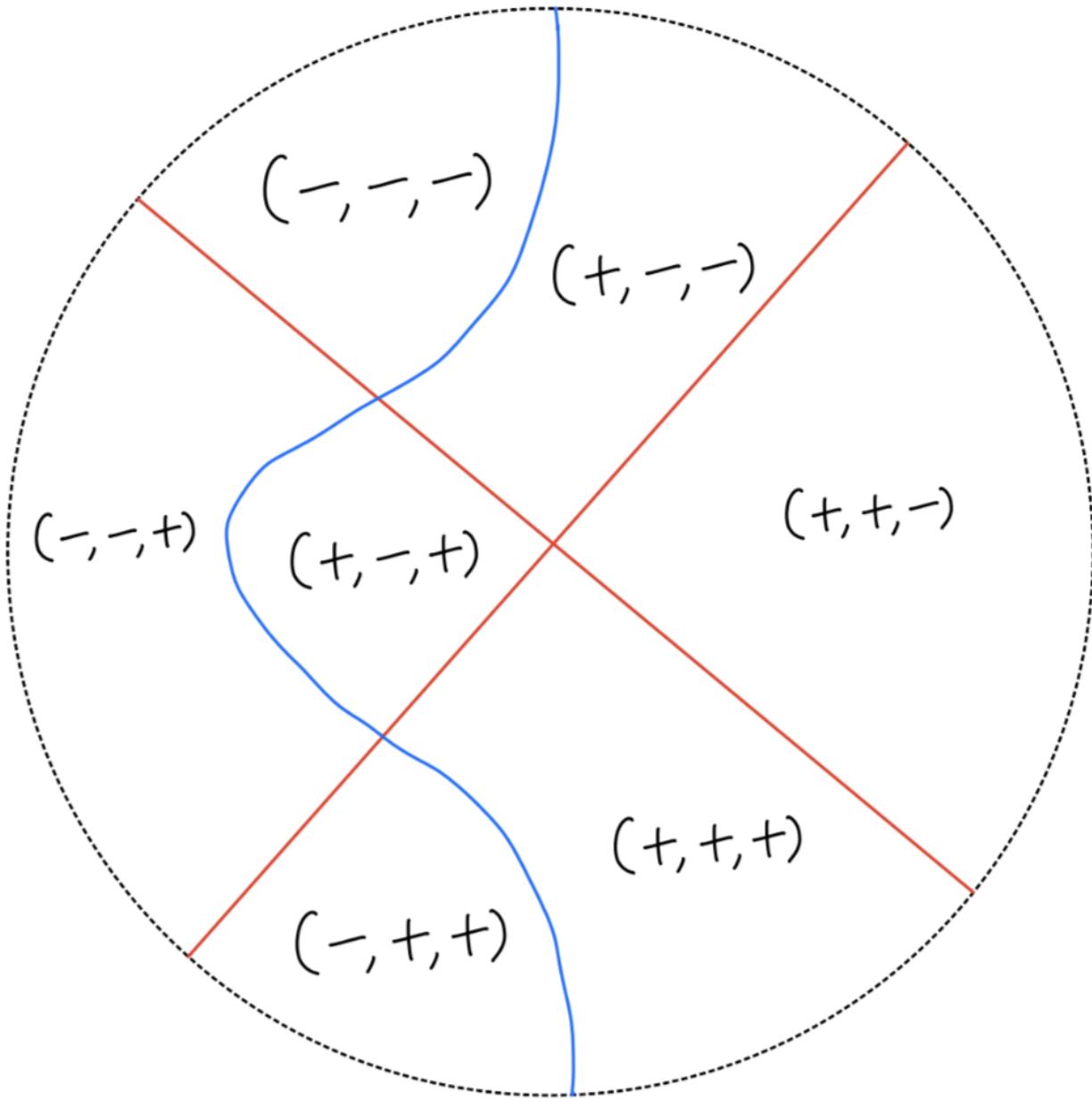


Figure 3.239

- 2 dimensional strata:

$$\{s_1(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3, \text{ except } s_1(-, +, -)\}$$

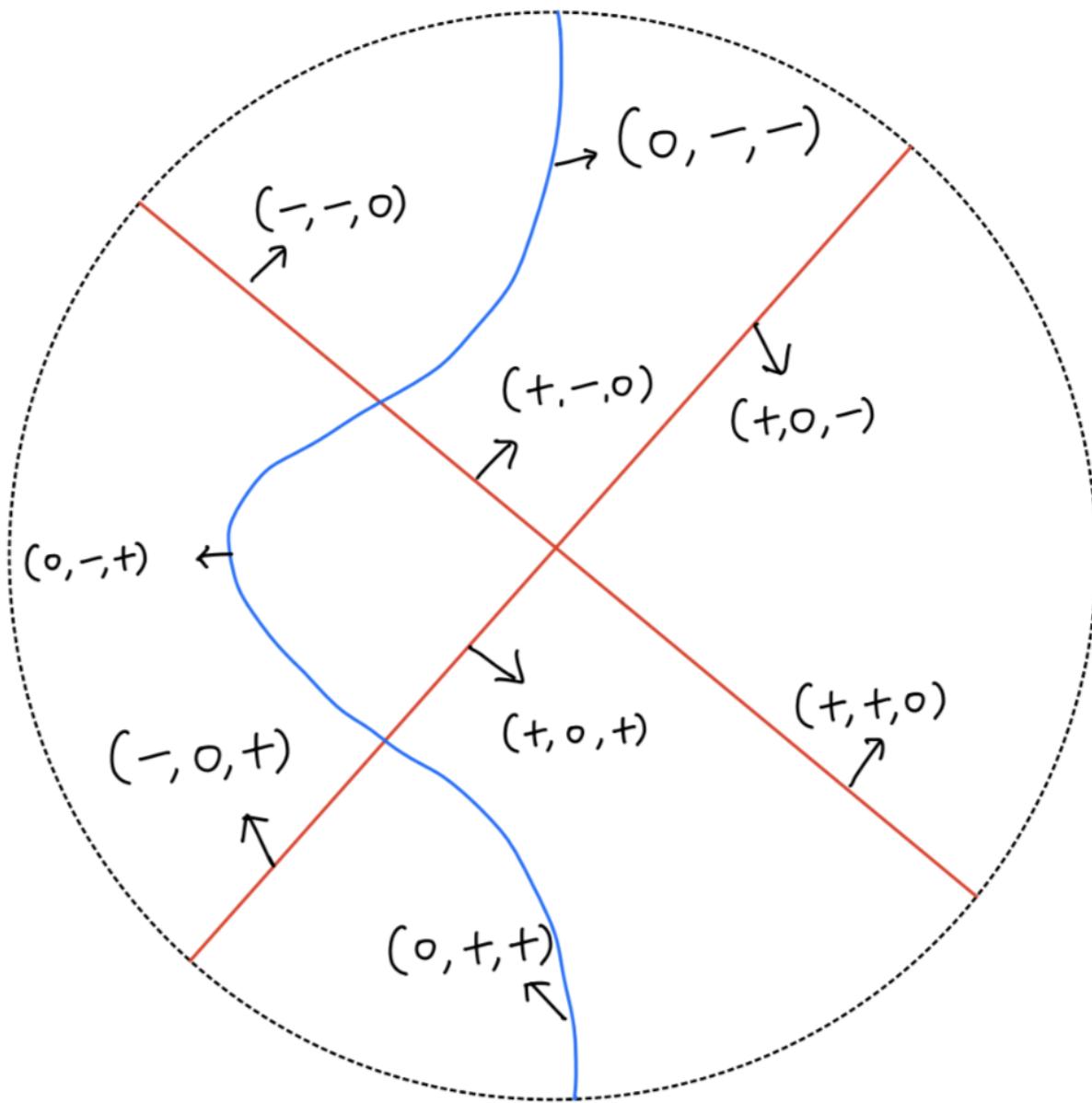


Figure 3.240

- 1 dimensional strata:

$$\begin{aligned}
 & \{s_1(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3, \text{ except } s_1(0, +, -)\} \\
 & \cup \{s_1(sgn_1, 0, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,3, \text{ except } s_1(-, 0, -)\} \\
 & \cup \{s_1(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2, \text{ except } s_1(-, +, 0)\}
 \end{aligned}$$

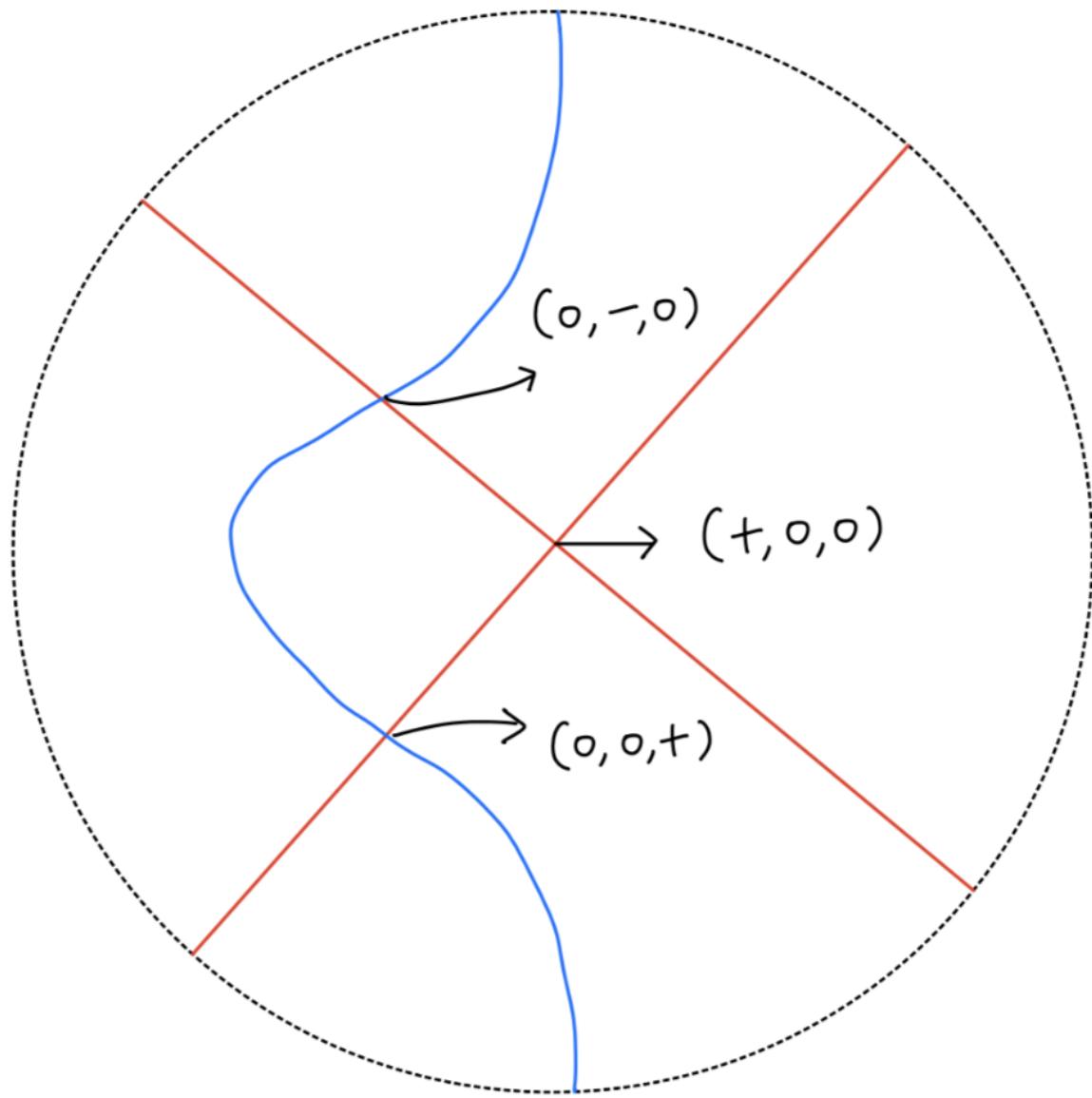


Figure 3.241

- 0 dimensional strata:

$$s_1(0, -, 0), s_1(+, 0, 0), s_1(0, 0, +)$$

3.  $\mathcal{S}_\bullet$ :

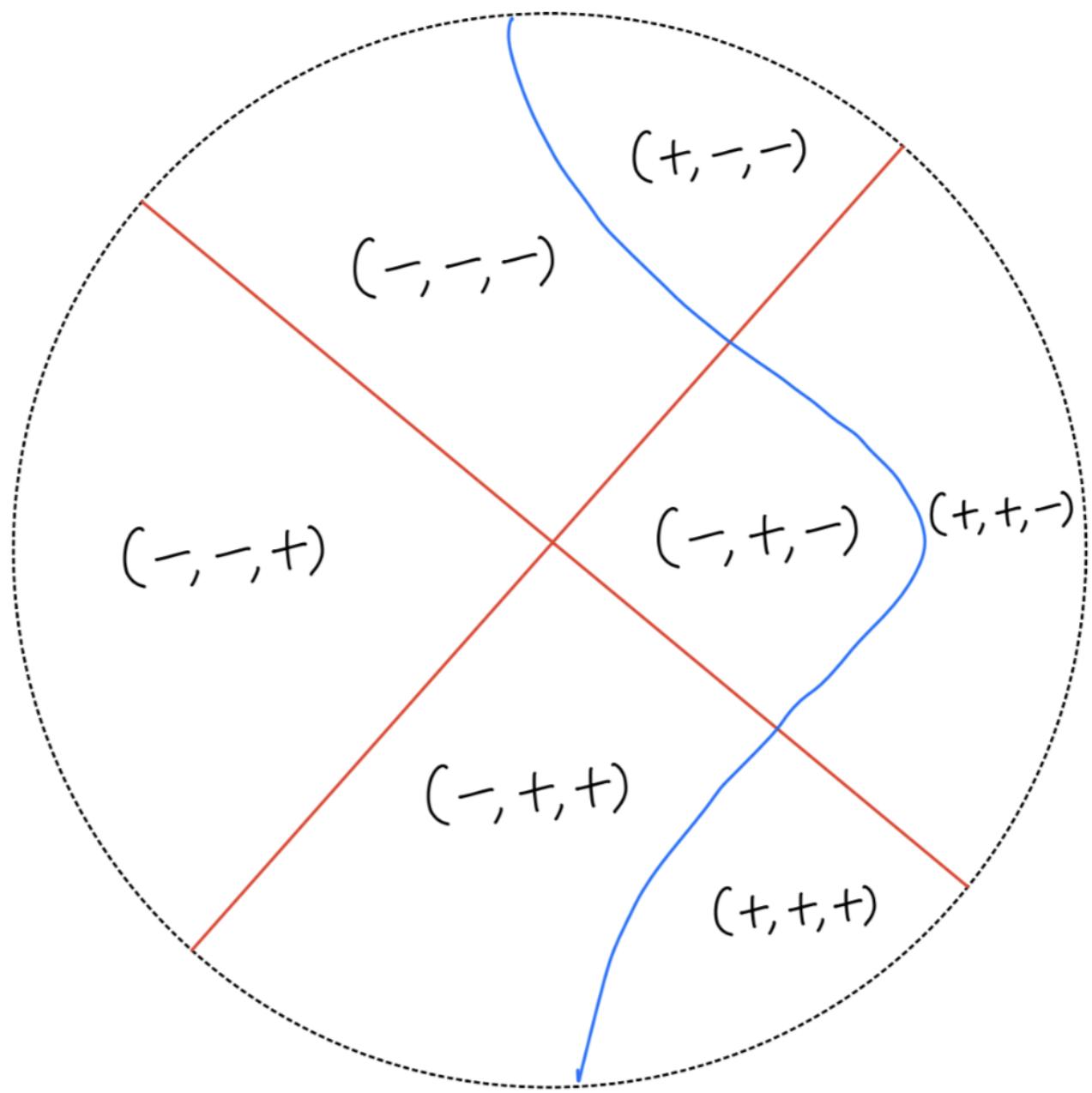


Figure 3.242

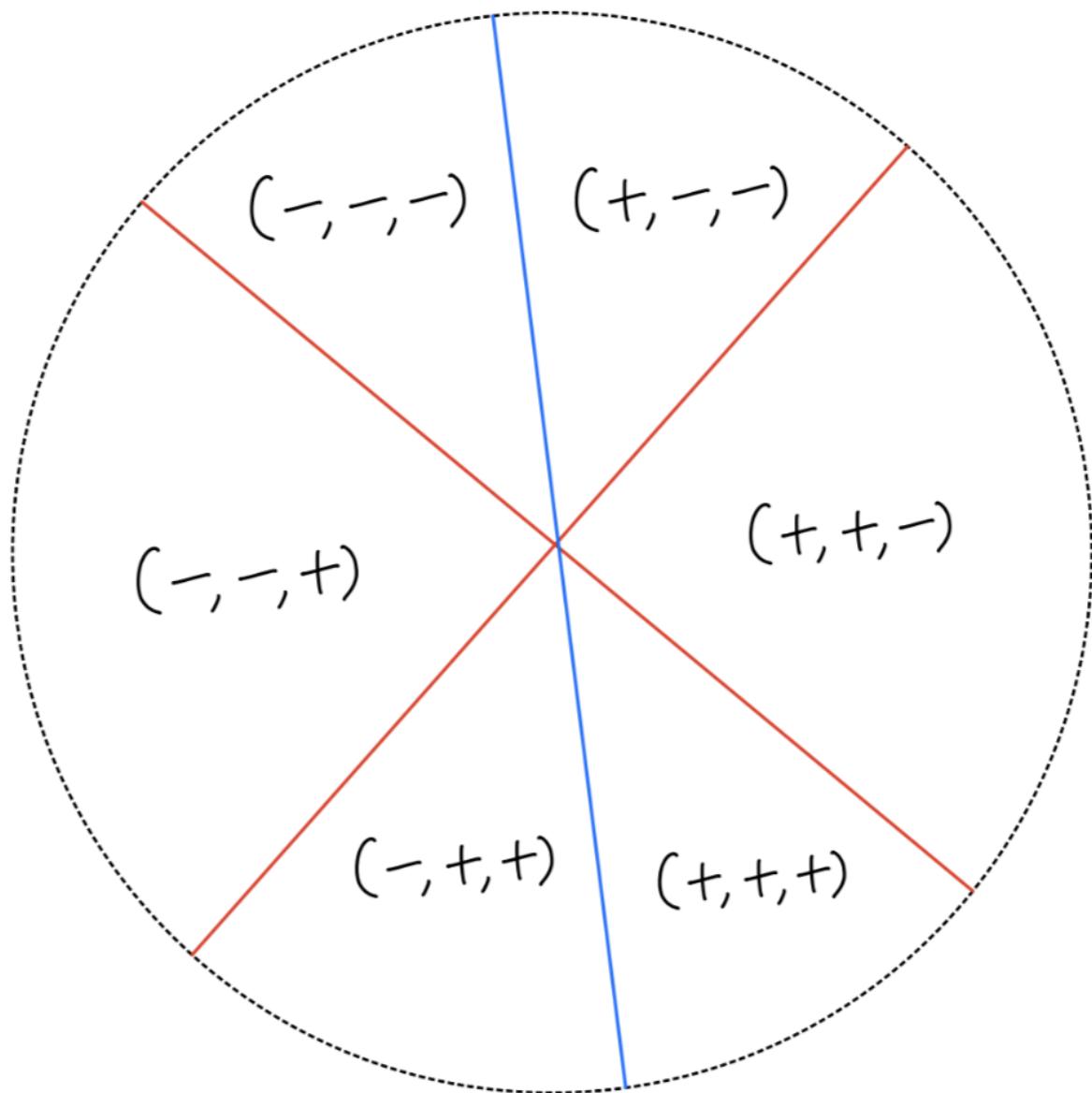


Figure 3.243

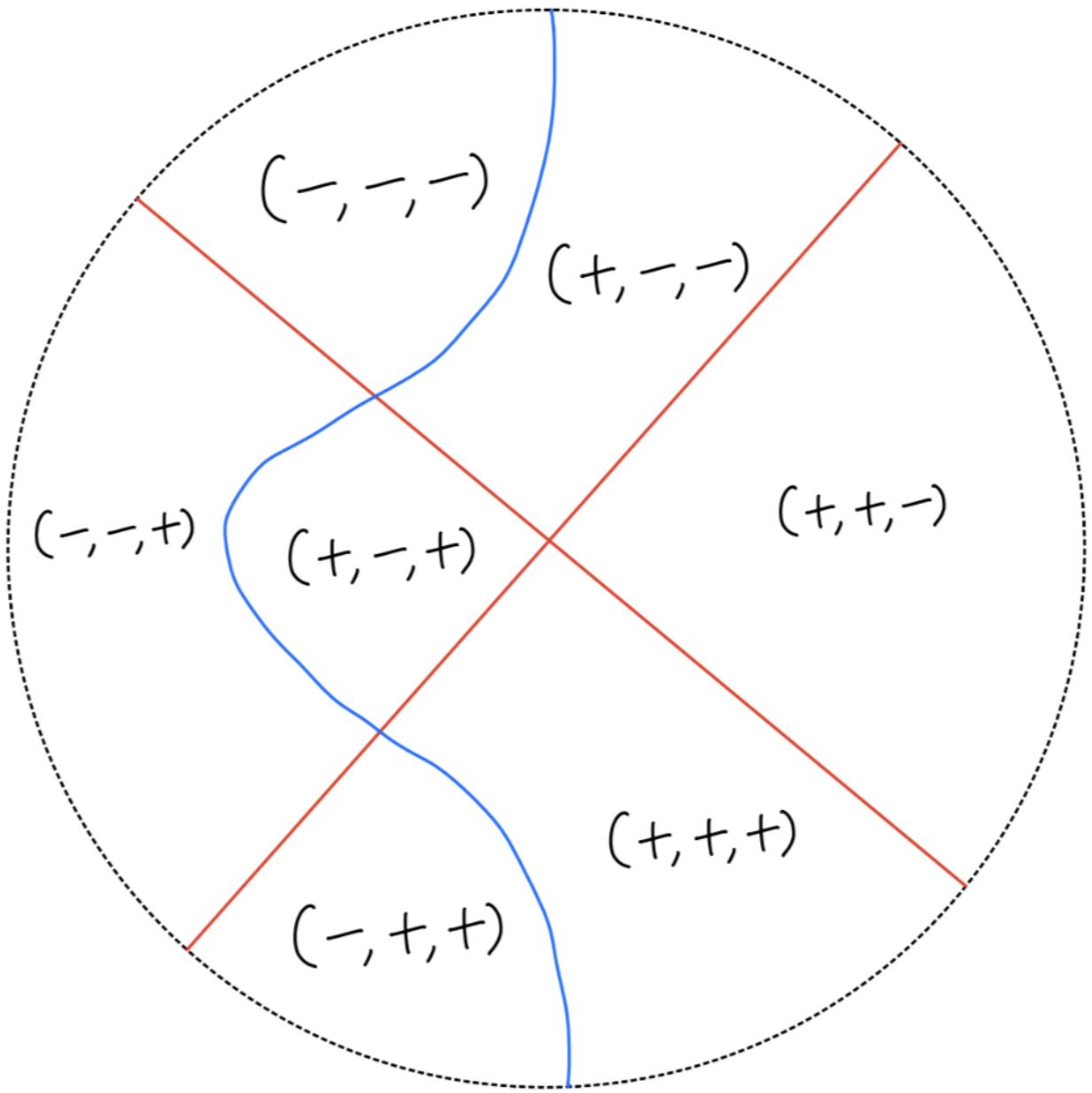


Figure 3.244

- 3 dimensional strata:

$$\{s_{\bullet}(sgn_2, sgn_1, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=1,2,3\}$$

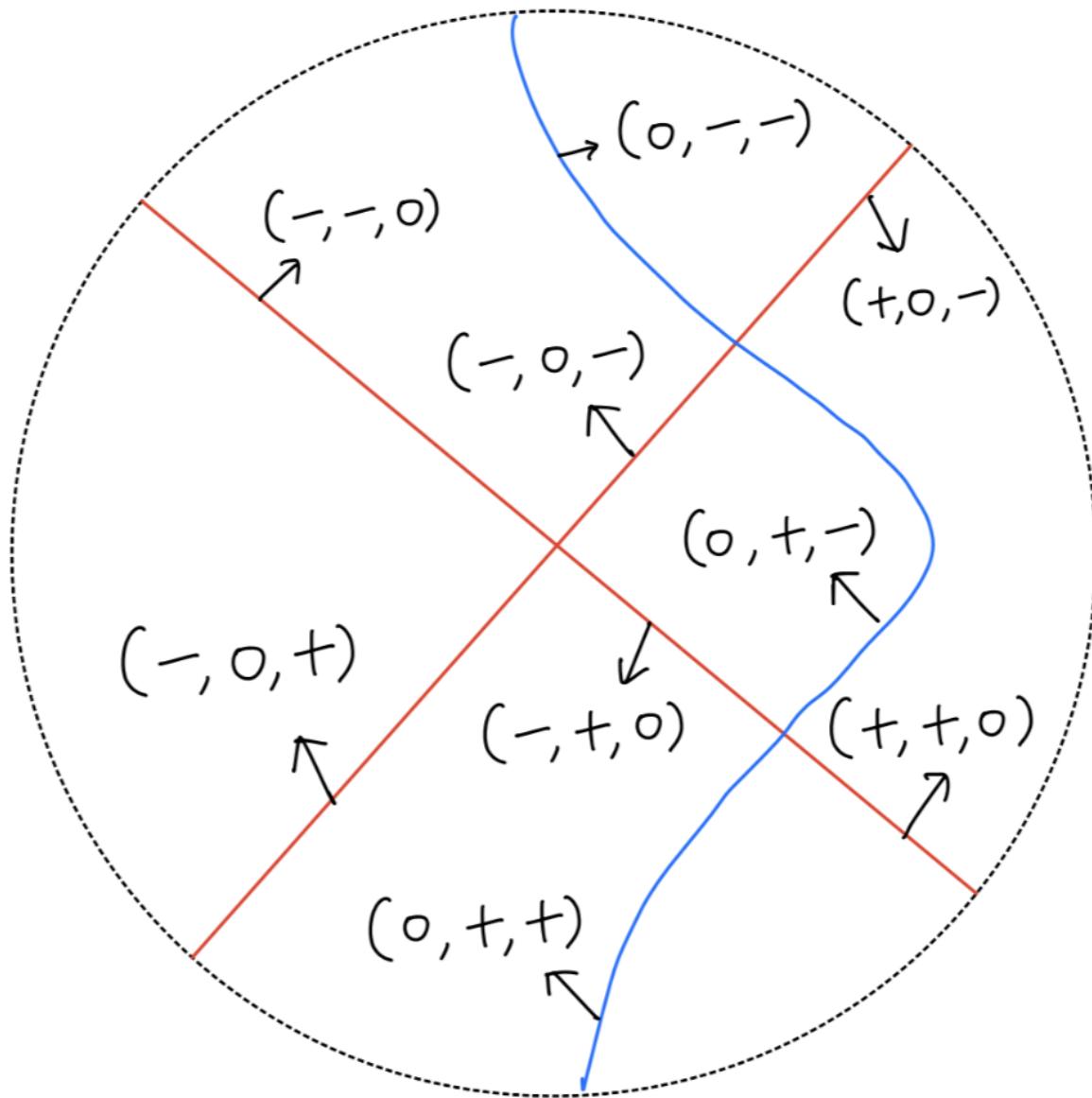


Figure 3.245

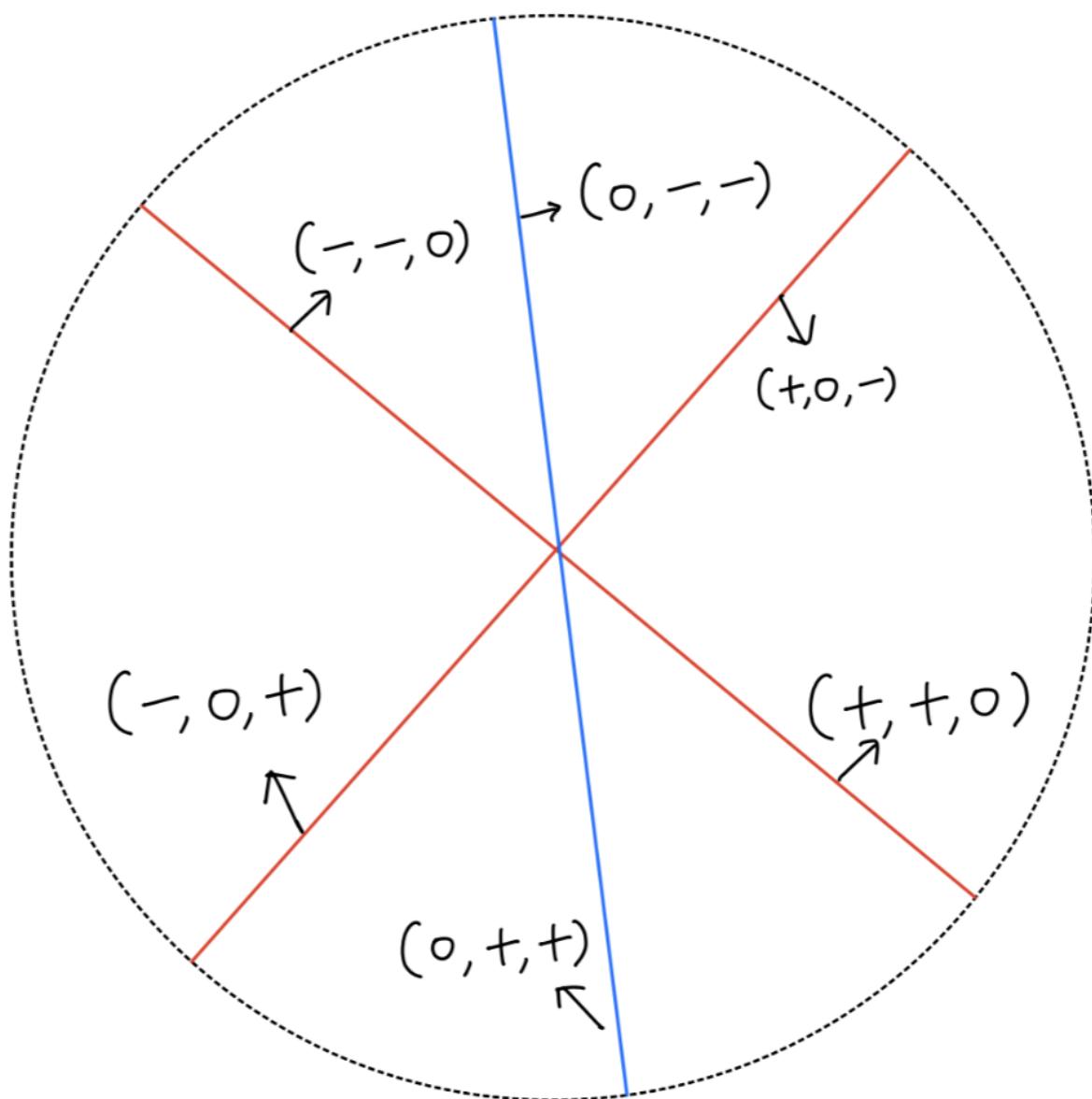


Figure 3.246

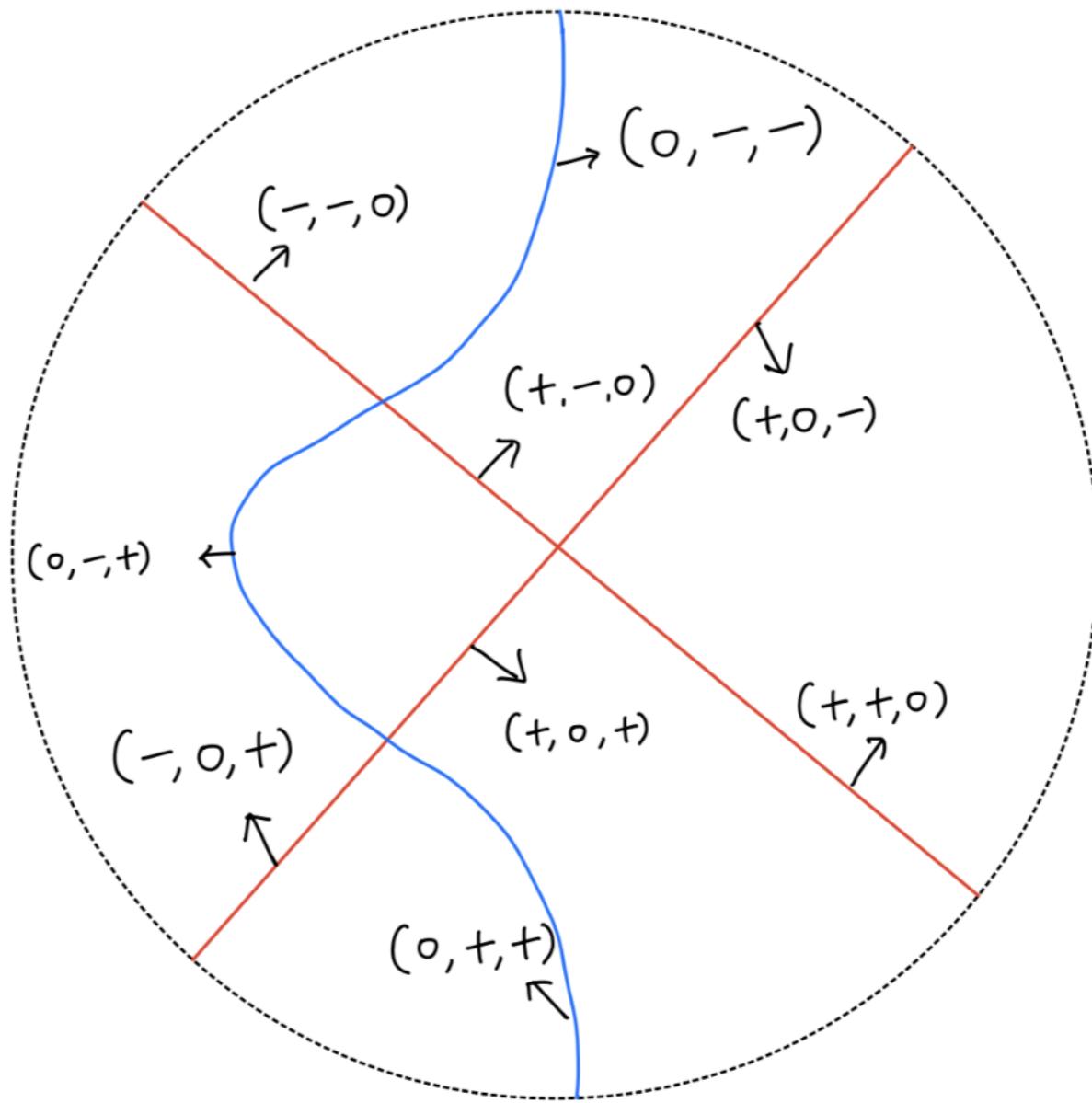


Figure 3.247

- 2 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(0, sgn_2, sgn_3) \mid sgn_i \in \{-, +\} \text{ for } i=2,3\} \cup \{s_{\bullet}(sgn_1, 0, sgn_3) \mid sgn_i \in \\ & \{-, +\} \text{ for } i=1,3\} \cup \{s_{\bullet}(sgn_1, sgn_2, 0) \mid sgn_i \in \{-, +\} \text{ for } i=1,2\} \end{aligned}$$

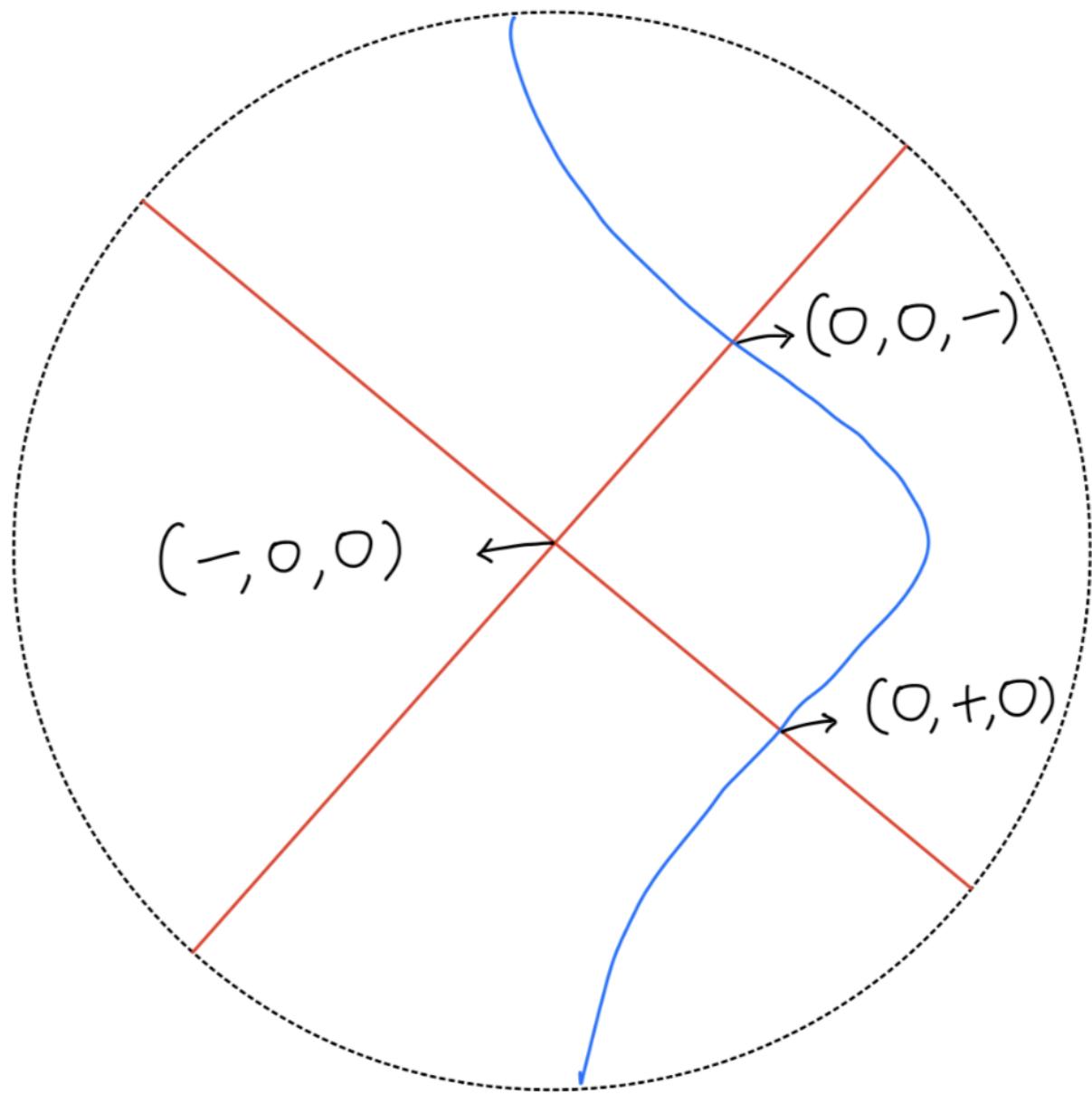


Figure 3.248

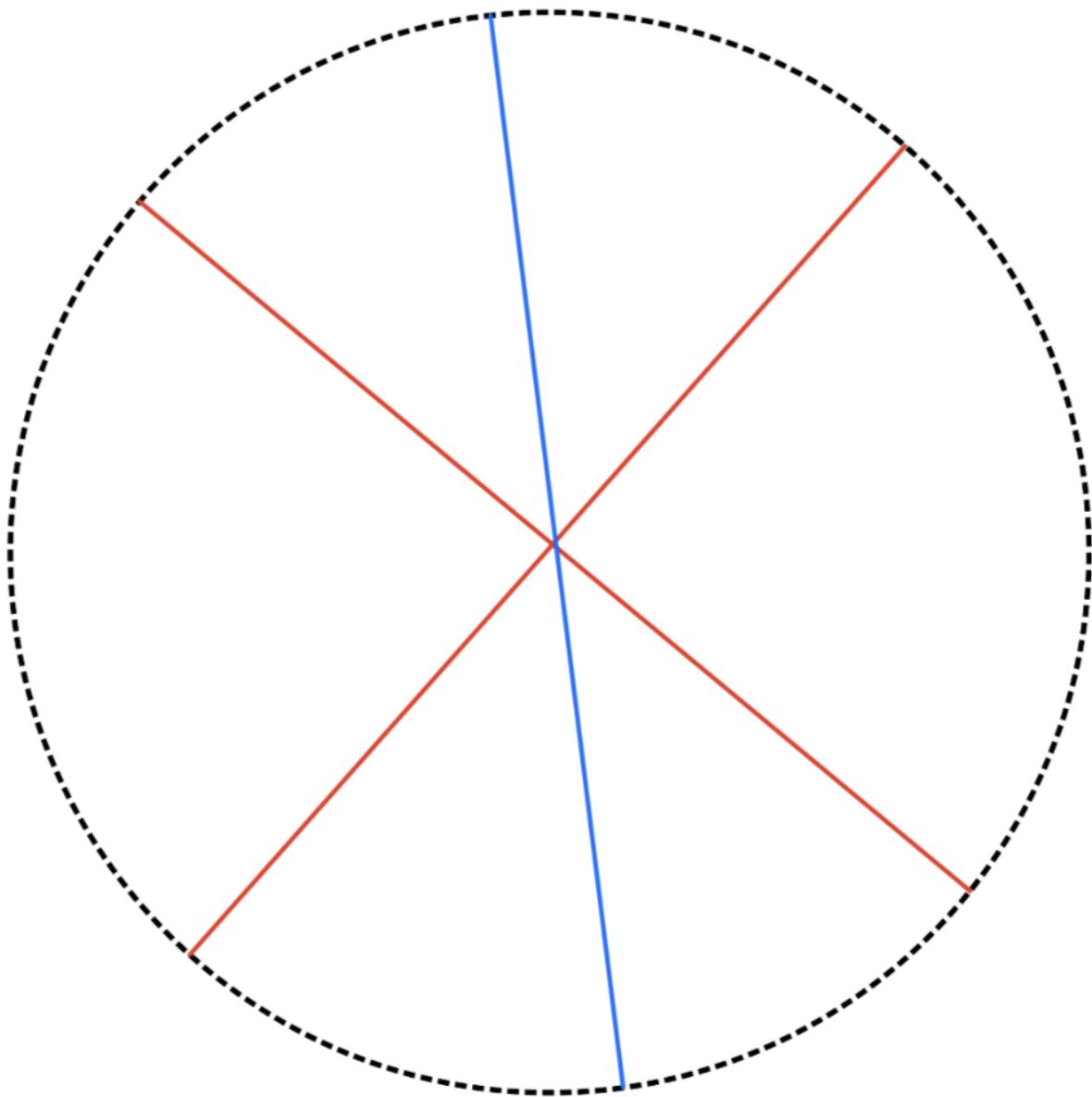


Figure 3.249

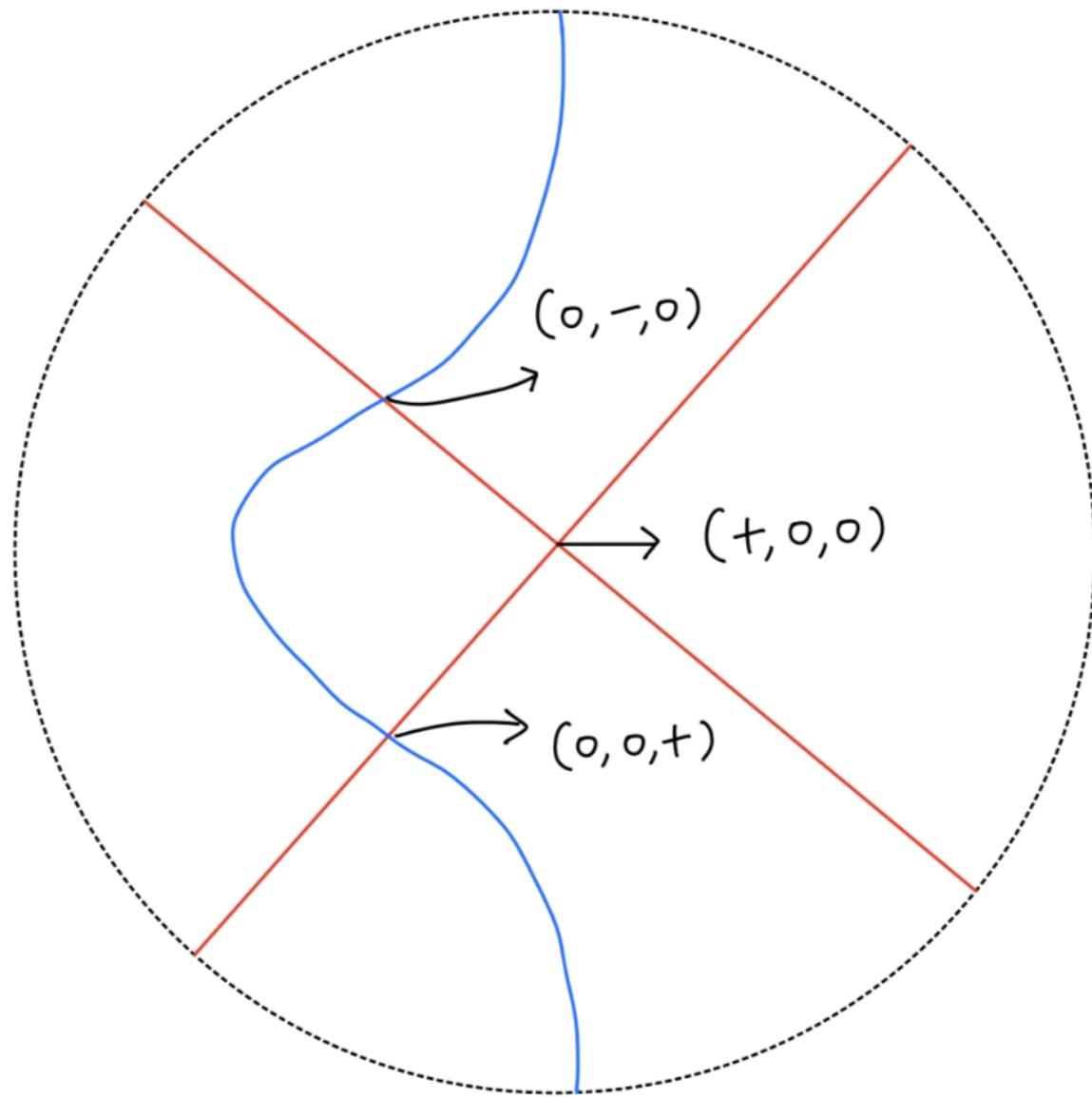


Figure 3.250

- 1 dimensional strata:

$$\begin{aligned} & \{s_{\bullet}(sgn_1, 0, 0) \mid sgn_1 \in \{-, +\}\} \cup \{s_{\bullet}(0, sgn_2, 0) \mid sgn_2 \in \{-, +\}\} \cup \\ & \{s_{\bullet}(0, 0, sgn_3) \mid sgn_3 \in \{-, +\}\} \end{aligned}$$

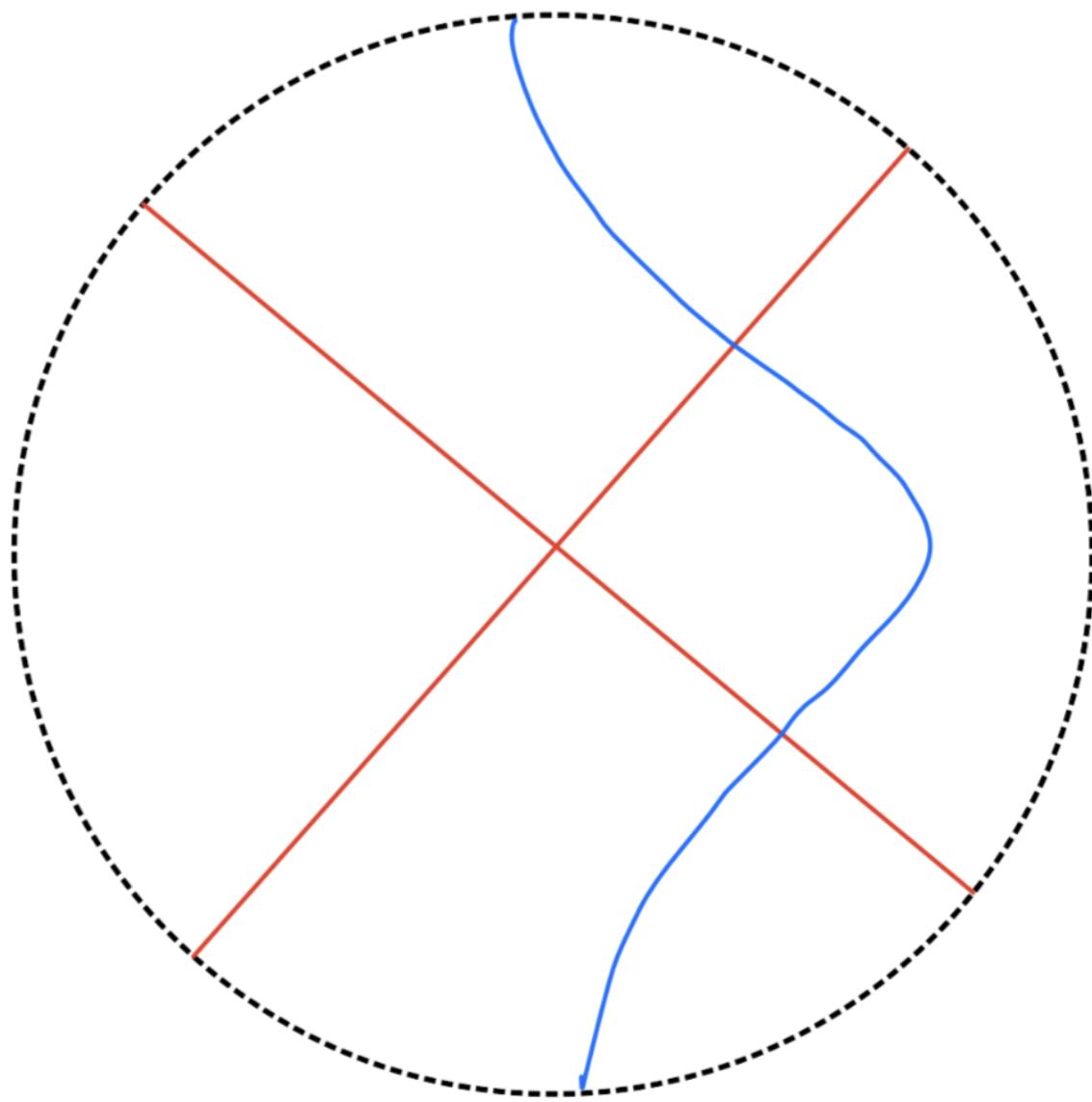


Figure 3.251

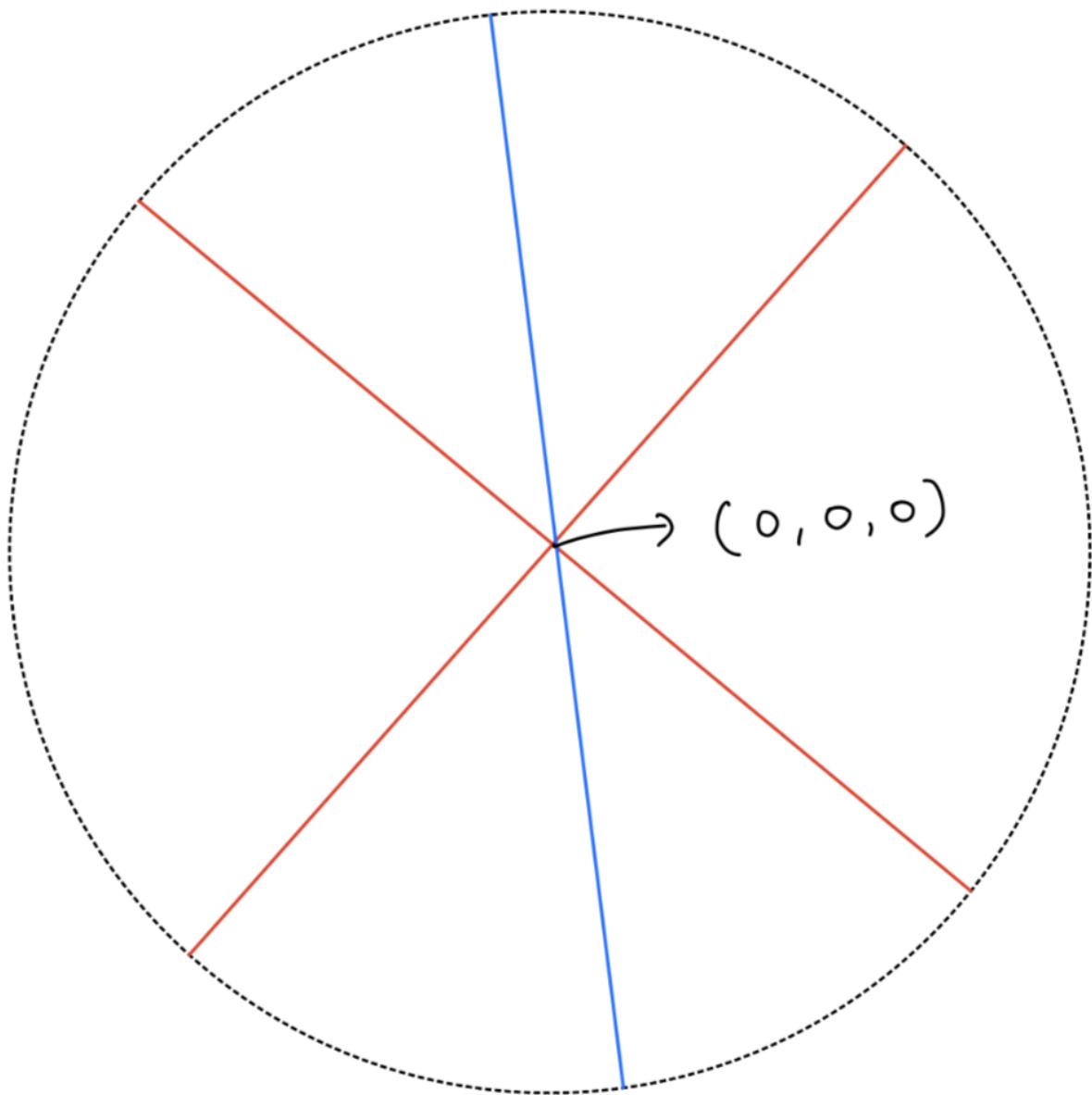


Figure 3.252

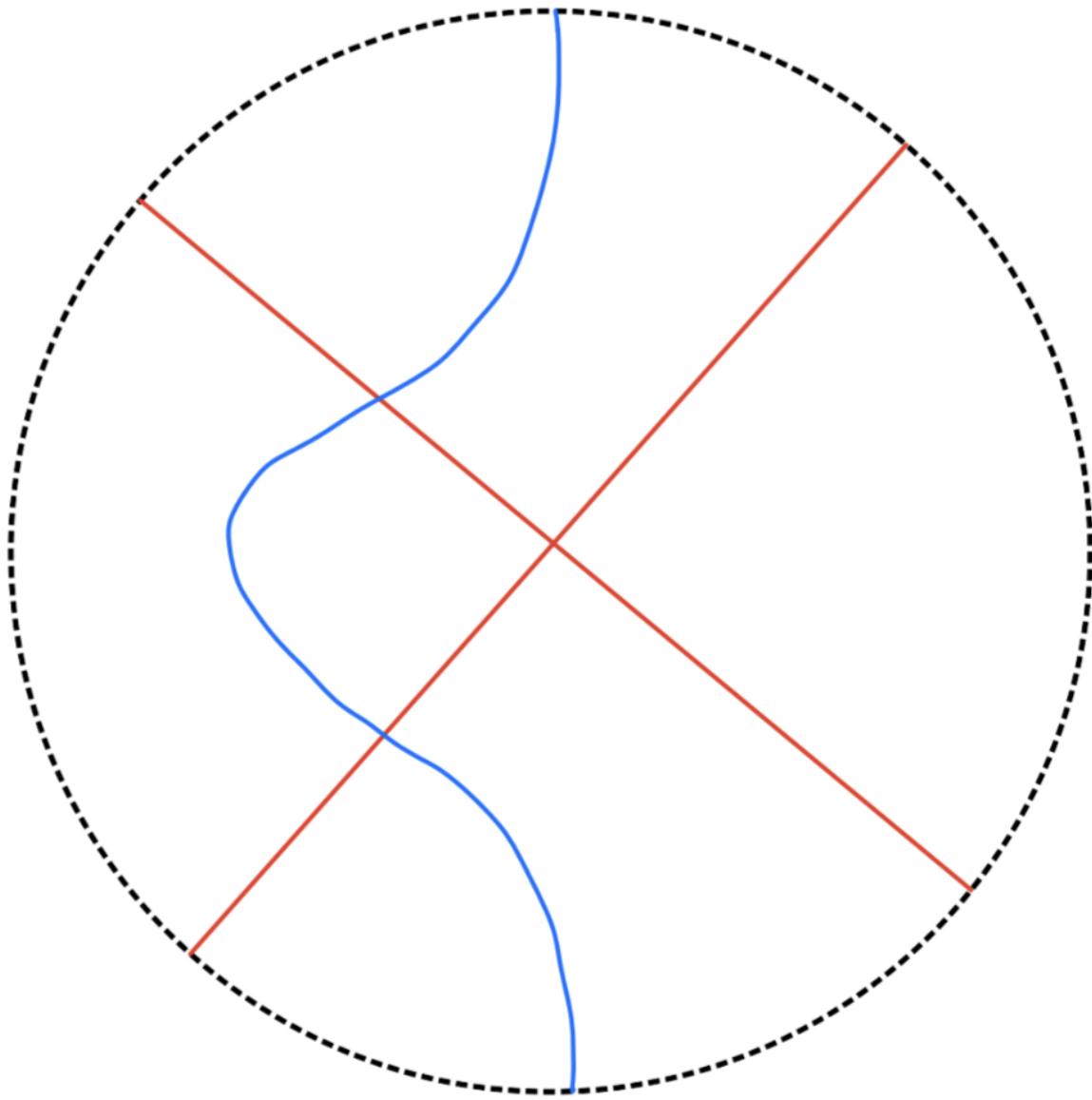


Figure 3.253

- 0 dimensional strata:

$$s_\bullet(0, 0, 0)$$

**Definition 194.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $\text{star}(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then we define the quiver associated to  $\mathcal{S}$ , say  $Q_{\mathcal{S}}$ , to be a quiver

- whose vertices corresponds to codimension 0 strata of  $\mathcal{S}$ .
- whose arrows corresponds to codimension 1 strata of  $\mathcal{S}$ .
- the source of an arrow corresponding to  $s \in \mathcal{S}$  is vertex corresponding to the region where the hairs of  $s$  are pointing at and the target is the other region contained in the  $star(s)$ .

**Definition 195.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then for each  $s \in \mathcal{S}$ , we define the subquiver of  $Q_{\mathcal{S}}$  associated to  $s$ , say  $Q_{\mathcal{S},s}$ , to be the full subquiver whose vertices are the ones that corresponds to the regions contained in the start of  $s$ .

**Definition 196.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

then  $\mathcal{S}$  is a legible stratification if for all  $s \in \mathcal{S}$ ,  $Q_{\mathcal{S},s}$  has the initial vertex. We say the quiver  $Q_{\mathcal{S}}$  associated to  $\mathcal{S}$  is legible if  $\mathcal{S}$  is.

**Definition 197.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions(= codimension 0 strata) contained in  $star(s)$ .

- each codimension 1 stratum is equipped with a co-orientation.

then we say the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is a legible representation if

- $\mathcal{S}$  is legible.
- for any  $v, v' \in Vert(Q_{\mathcal{S}})$  and any paths  $(a_1, a_2, \dots, a_k), (a'_1, a'_2, \dots, a'_{k'})$  from  $v$  to  $v'$ ,  $F_{\mathcal{S}}(a_k) \circ \dots \circ F_{\mathcal{S}}(a_1) = F_{\mathcal{S}}(a'_{k'}) \circ \dots \circ F_{\mathcal{S}}(a'_1)$  i.e. the composition is path independent.

**Definition 198.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Supoose  $\mathcal{S}$  is legible, then we define  $\rho : \mathcal{S} \rightarrow \{s \in \mathcal{S} \mid codim(s) = 0\}$  as

$\rho(s) :=$  the codimension 0 stratum corresponding to the initial vertex of  $Q_{\mathcal{S},s}$

**Definition 199.** Suppose we have a manifold  $R$  with stratification  $\mathcal{S}$  such that

- for each codimension 1 stratum  $s \in \mathcal{S}$ , there are exactly two regions (= codimension 0 strata) contained in  $star(s)$ .
- each codimension 1 stratum is equipped with a co-orientation.

Suppose the quiver representation  $F_{\mathcal{S}}$  of  $Q_{\mathcal{S}}$  is legible, then we define the associated functor  $\overline{F}_{\mathcal{S}} \in Obj(Fun(\mathcal{S}, \mathbb{C}))$  as follows:

- for  $s \in \mathcal{S}$ ,  $\overline{F}_{\mathcal{S}} := F_{\mathcal{S}}(\rho(s))$ .
- for  $s_1, s_2 \in \mathcal{S}$  where  $s_2 \subset start(s_1)$ , then  $\overline{F}_{\mathcal{S}}(s_1 \rightarrow s_2)$  is defined as follows: choose a path from the vertex corresponding to  $\rho(s_1)$  to  $\rho(s_2)$  in  $Q_{\mathcal{S}}$ , say

$(a_1, \dots, a_k)$ , then

$$\overline{F_S}(s_1 \rightarrow s_2) := F_S(a_k) \circ \dots \circ F_S(a_1)$$

This is well-defined because  $F_S$  is legible.

**Definition 200.** Let  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$  be the cochain complexes  $(C^\bullet, \delta^\bullet)$  supported on degree 0 and 1 and  $\phi^\bullet$  a morphism between  $(C^\bullet, \delta^\bullet)$  and  $(C'^\bullet, \delta'^\bullet)$ , then

1. we denote  $(C^\bullet, \delta^\bullet)$  as either

- $C^0 \xrightarrow{\delta^1} C^1$

or

- $$\begin{array}{ccc} & C^1 & \\ \bullet & \delta^1 \uparrow & \\ & C^0 & \end{array}$$

2. we denote  $\phi^\bullet$  as

- $$\begin{array}{ccc} C^1 & \xrightarrow{\phi^1} & C'^1 \\ \delta^1 \uparrow & & \delta'^1 \uparrow \\ C^0 & \xrightarrow{\phi^0} & C'^0 \end{array}$$

We omit coboundary maps or cochain maps if they are either zero map or identity map and could be inferred from the context.

### 3.9.2 Setting

Suppose on  $M$ , we have

- a squiggly diagram  $\Lambda_0$  on  $M$
- nested regions  $U' \subset U \subset M$ . Note that if we define  $V := M - \overline{U'}$ ,  $\{U, V\}$  form an open cover of  $M$ .
- a smooth chart from  $D_{r=2}$ , say  $f : D \rightarrow U \subset M$

such that

- $D_{r=1}$  is mapped to  $U'$
- $\lambda_0^0$  is mapped to  $\Lambda_0^0|_U$
- $\lambda_0^\infty$  is mapped to  $\Lambda_0^\infty|_U$
- $\lambda_0^{squig}$  is mapped to  $\Lambda_0^{squig}|_U$

### 3.9.3 Sheaf at the Beginning

Suppose we have a sheaf  $\mathcal{F}_0$  singular supported on  $\Lambda_0$  such that  $f^*\mathcal{F}_0$  is isomorphic to the sheaf described by the following squiggly legible diagram  $F_0$ .

For simplicity, we use the following notations

$$F_0(sgn_1, sgn_2, sgn_3, sgn_4) := F_0(s_0(sgn_1, sgn_2, sgn_3, sgn_4))$$

**Stalks:**

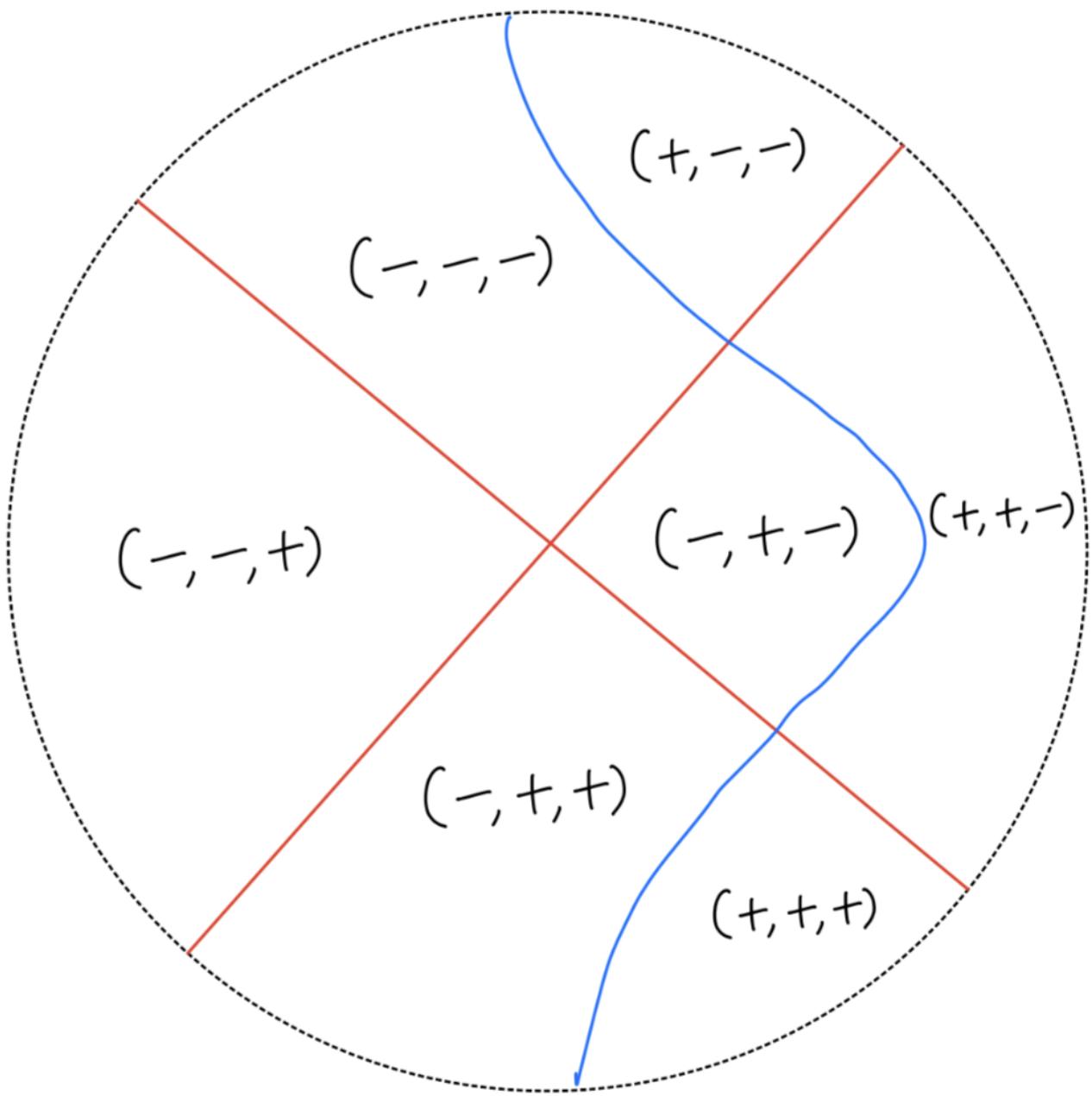


Figure 3.254

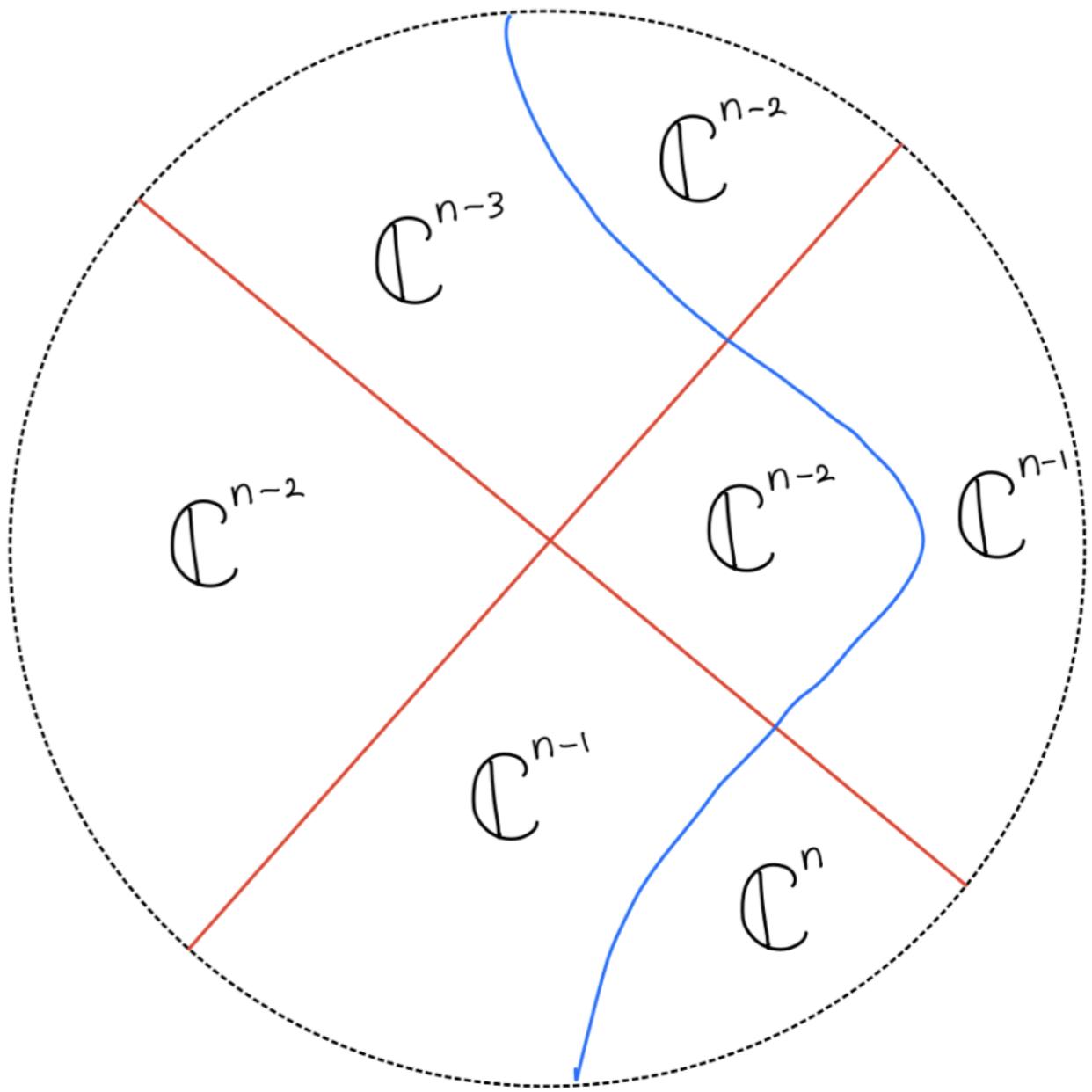


Figure 3.255

- $F_0(-, -, -) := \mathbb{C}^{n-3}$

- $F_0(+, -, -) := \mathbb{C}^{n-2}$

- $F_0(-, -, +) := \mathbb{C}^{n-2}$

- $F_0(-, +, -) := \mathbb{C}^{n-2}$

- $F_0(+, +, -) := \mathbb{C}^{n-1}$

- $F_0(-, +, +) := \mathbb{C}^{n-1}$

- $F_0(+, +, +) := \mathbb{C}^n$

Generalization maps:

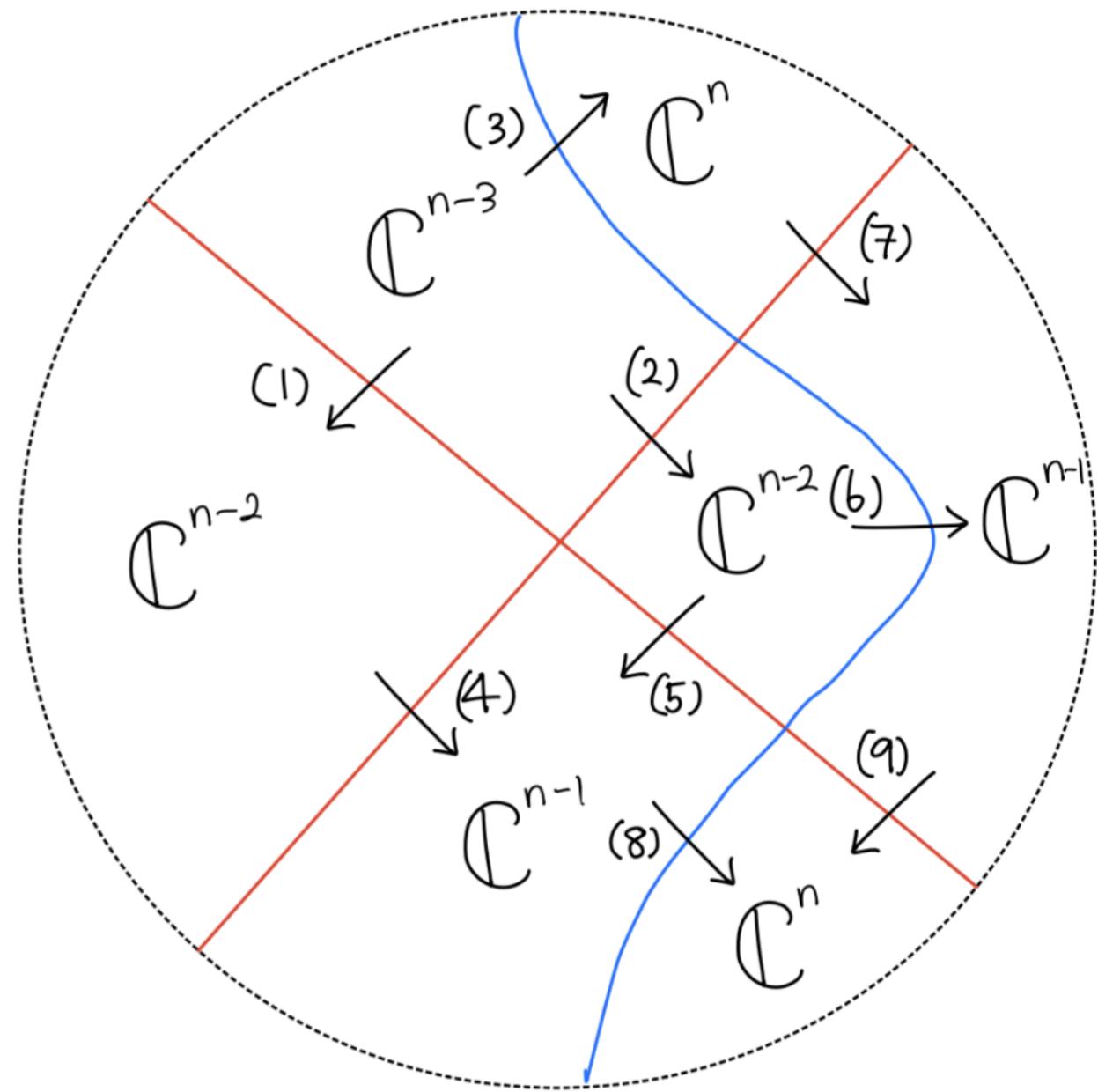


Figure 3.256

(1)  $\iota_0$

(2)  $\iota_0$

(3)  $\iota_1$

(4)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(5)  $\iota_0$

(6)  $\iota_1$

(7)  $\iota_0$

(8)  $\iota_1$

(9)  $\iota_0$

### 3.9.4 Legendrian Cobordism

Then define a Legendrian cobordism  $\mathcal{F}_\bullet$  starting from  $\mathcal{F}_0$ , say  $\text{cobord}'_4$ , that is supported on  $\overline{U'}$  as follows:

By Mayer-Vietoris, this equivalent to the following data

- a sheaf on  $V \times [0, 1]$ , say  $\mathcal{F}_{V \times [0, 1]}$
- a sheaf on  $D_{r=2} \times [0, 1]$ , say  $\mathcal{F}_{D_{r=2} \times [0, 1]}$
- a gluing isomorphsim, i.e.  $\gamma_\bullet : (f_* \mathcal{F}_{D_{r=2} \times [0, 1]})|_{(U \cap V) \times [0, 1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0, 1]}|_{(U \cap V) \times [0, 1]}$ .

#### A. Sheaf on $V \times [0, 1]$

First, I will define  $\mathcal{F}_{V \times [0, 1]}$  to be  $pr_1^*(\mathcal{F}_0|_V)$  where  $pr_1 : V \times [0, 1] \rightarrow V$  is the projection onto the first argument.

**B. Sheaf on  $D_{r=2} \times [0, 1]$** 

Next, I will describe  $\mathcal{F}_{D_{r=2} \times [0, 1]}$  as  $F_\bullet \in \text{Fun}(\mathcal{S}_\bullet, \mathbb{C})$  i.e. a functor from  $\mathcal{S}_\bullet$  to the category of perfect  $\mathbb{C}$ -modules as follows:

For simplicity, we use the following notations

$$F_\bullet(sgn_1, sgn_2, sgn_3) := F_\bullet(s_\bullet(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

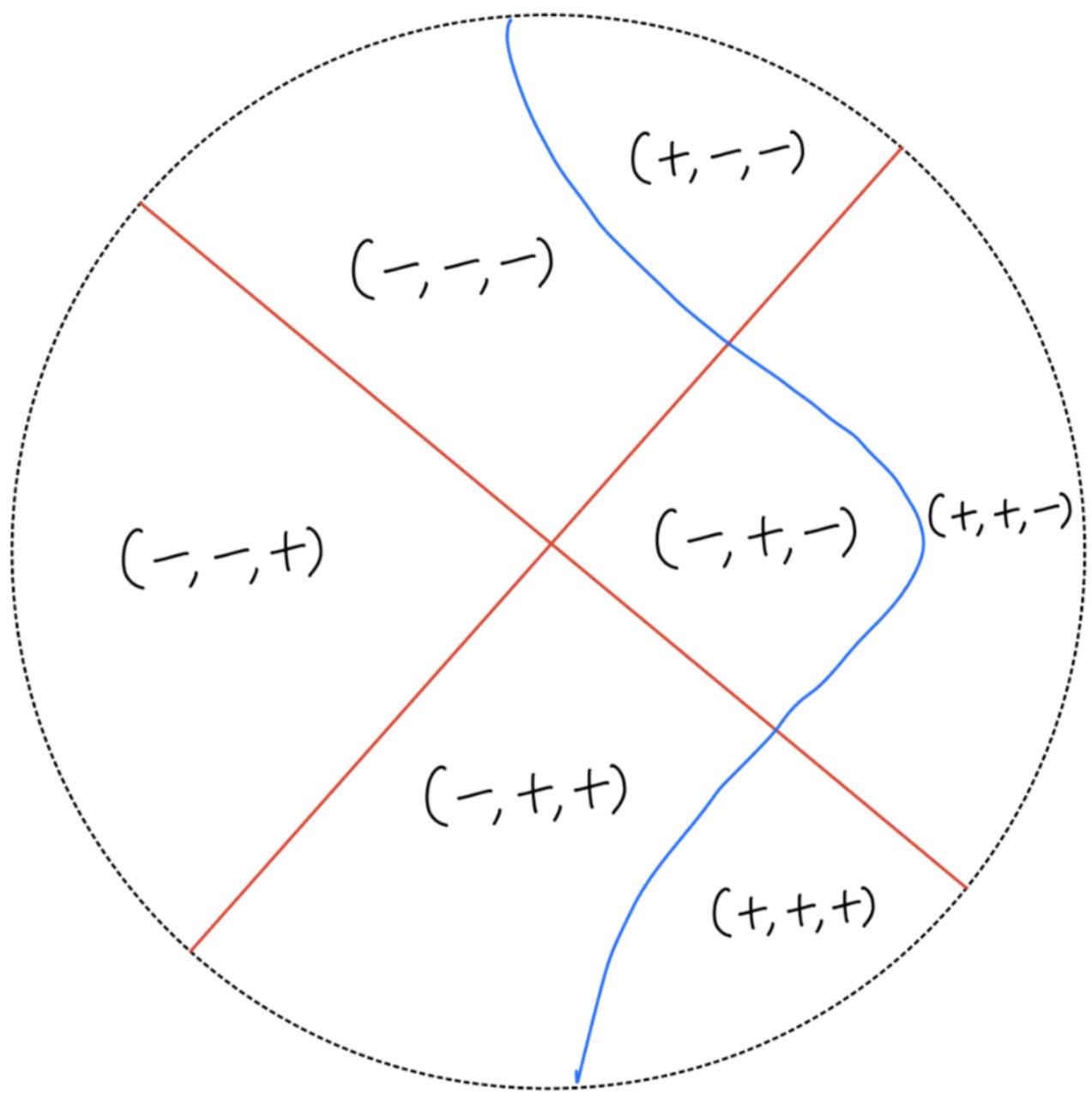


Figure 3.257

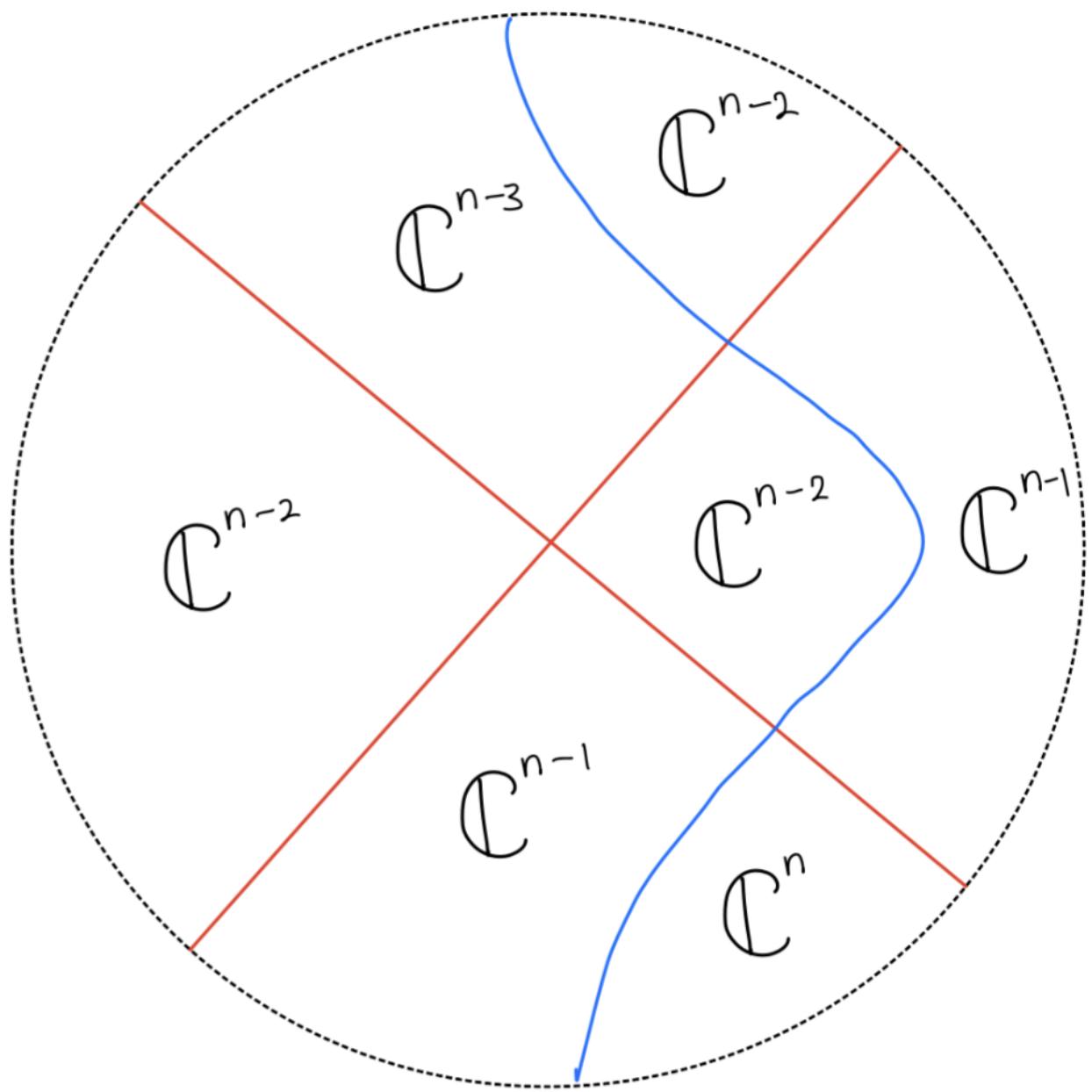


Figure 3.258

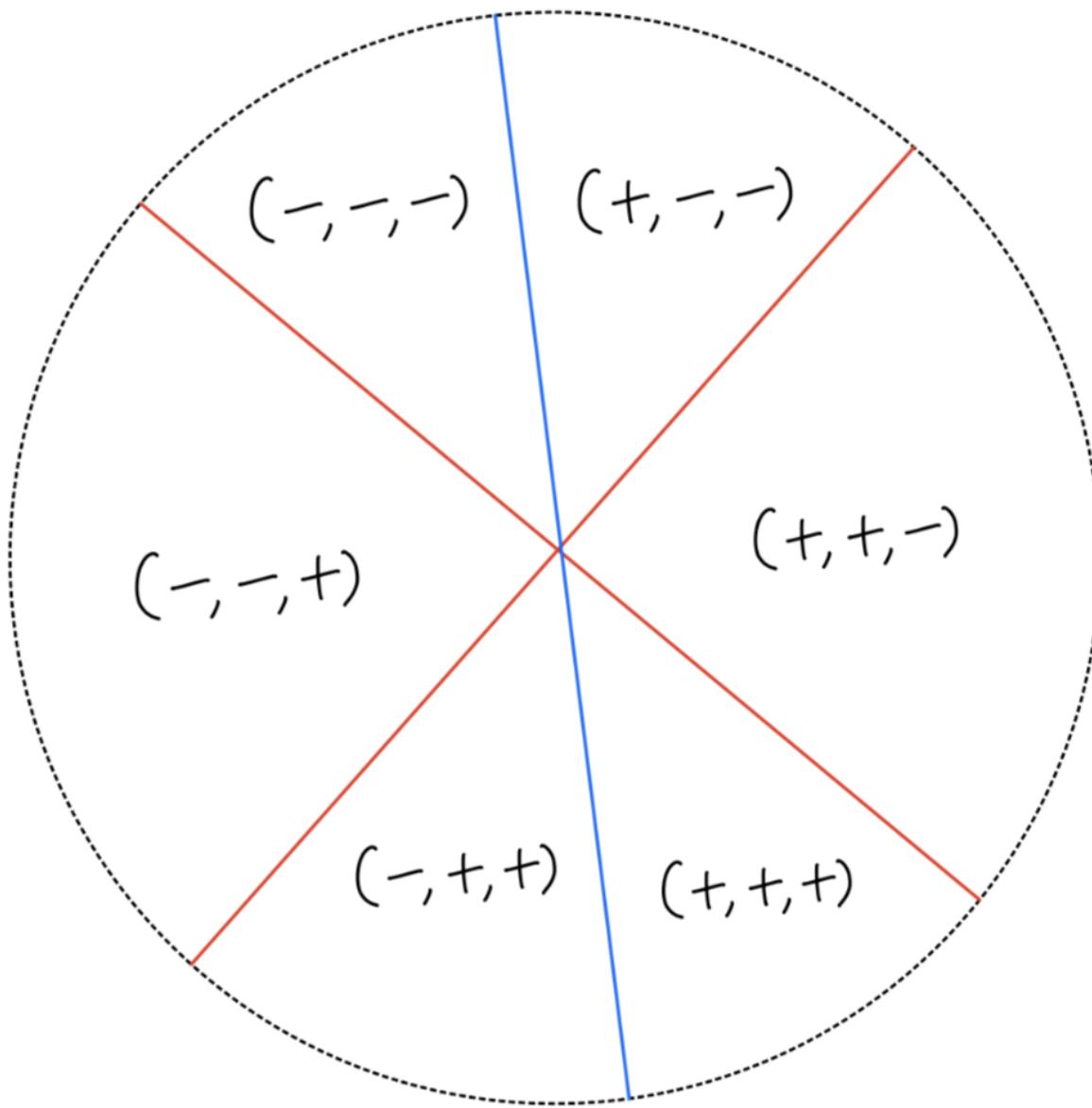


Figure 3.259

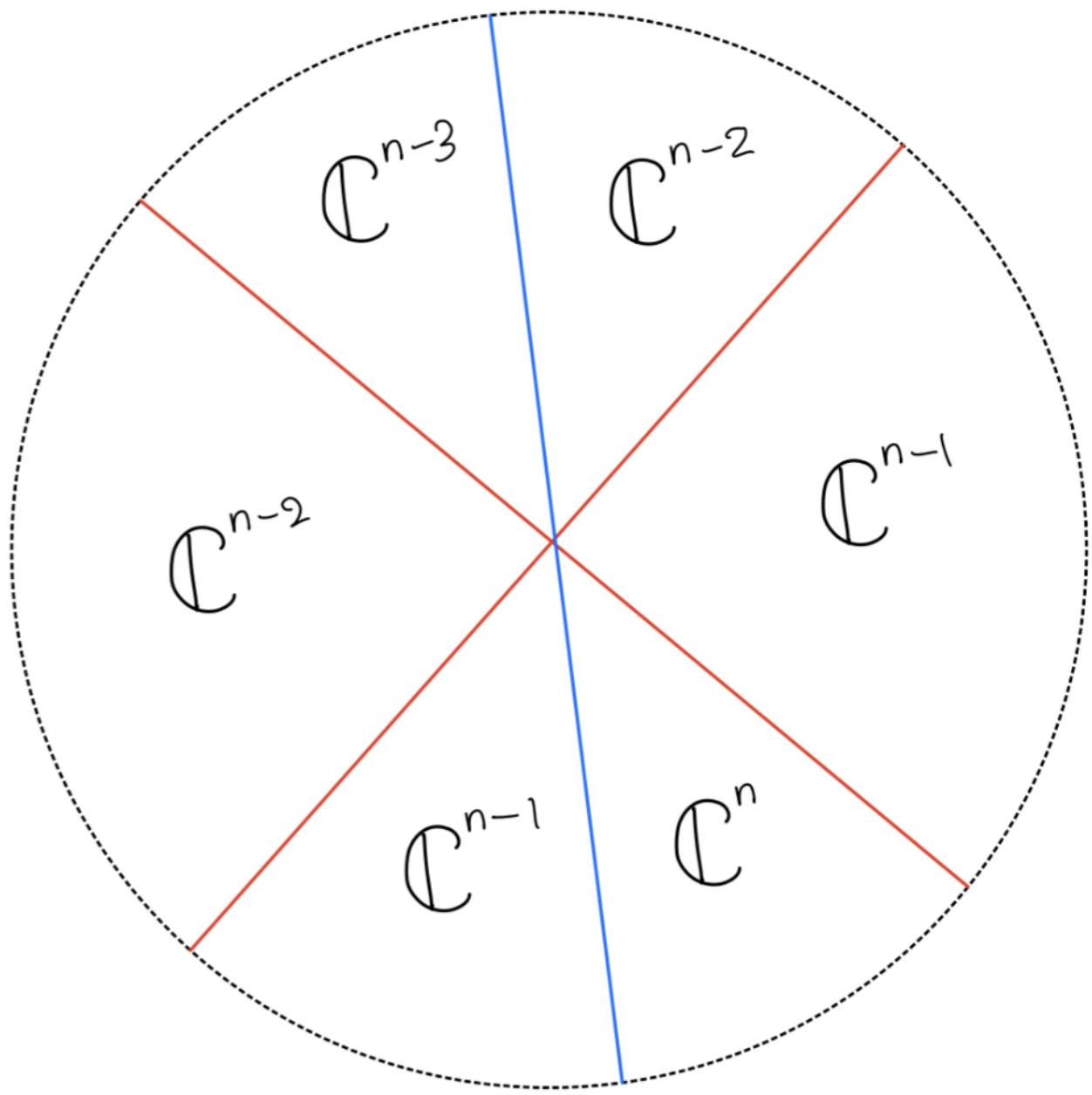


Figure 3.260

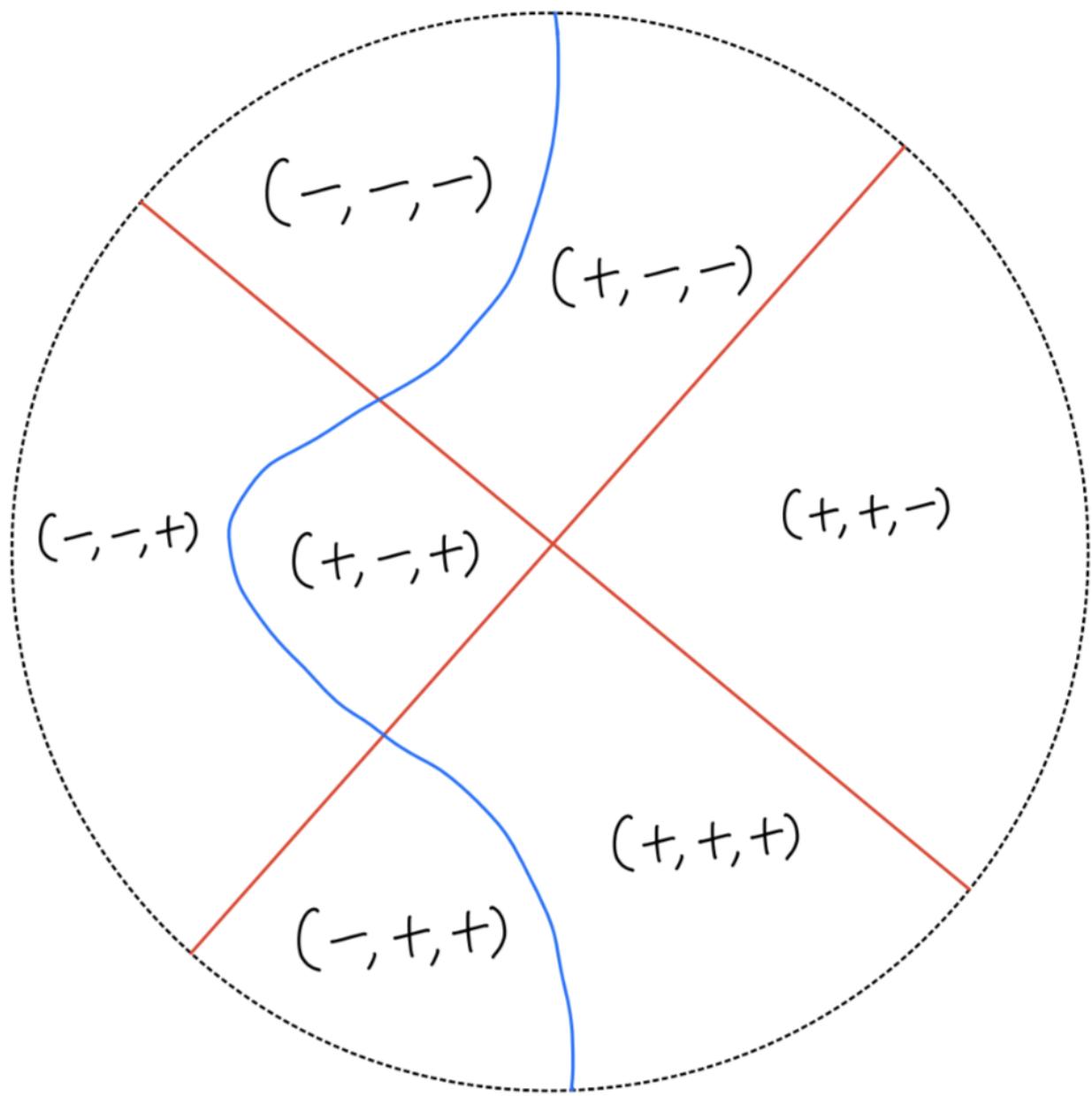


Figure 3.261

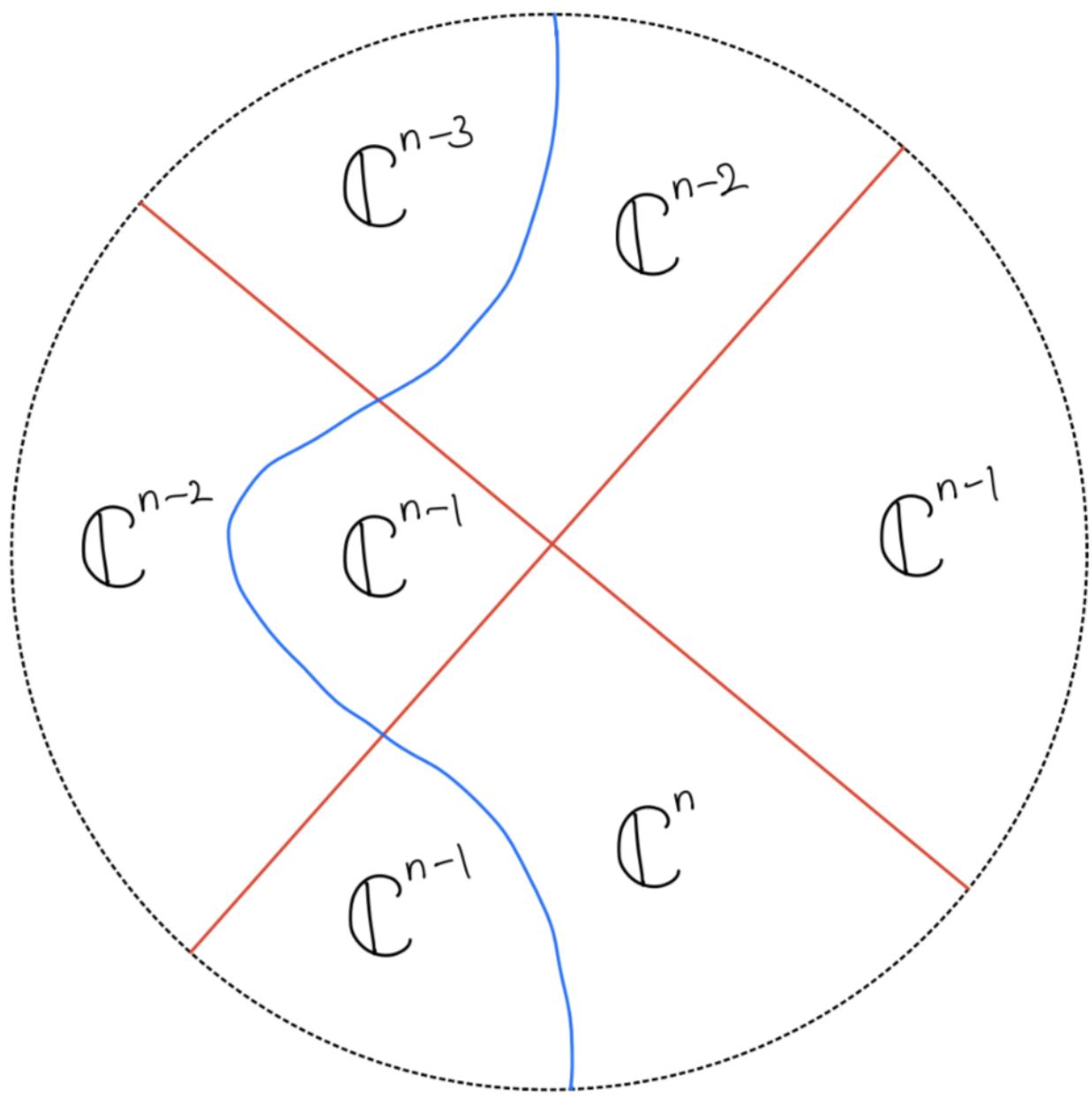


Figure 3.262

- $F_{\bullet}(-, -, -) := \mathbb{C}^{n-3}$

- $F_{\bullet}(-, -, +) := \mathbb{C}^{n-2}$

- $F_{\bullet}(-, +, -) := \mathbb{C}^{n-2}$

- $F_{\bullet}(-, +, +) := \mathbb{C}^{n-1}$

- $F_\bullet(+, -, -) := \mathbb{C}^{n-2}$

- $F_\bullet(+, -, +) := \mathbb{C}^{n-1}$

- $F_\bullet(+, +, -) := \mathbb{C}^{n-1}$

- $F_\bullet(+, +, +) := \mathbb{C}^n$

Generalization maps:

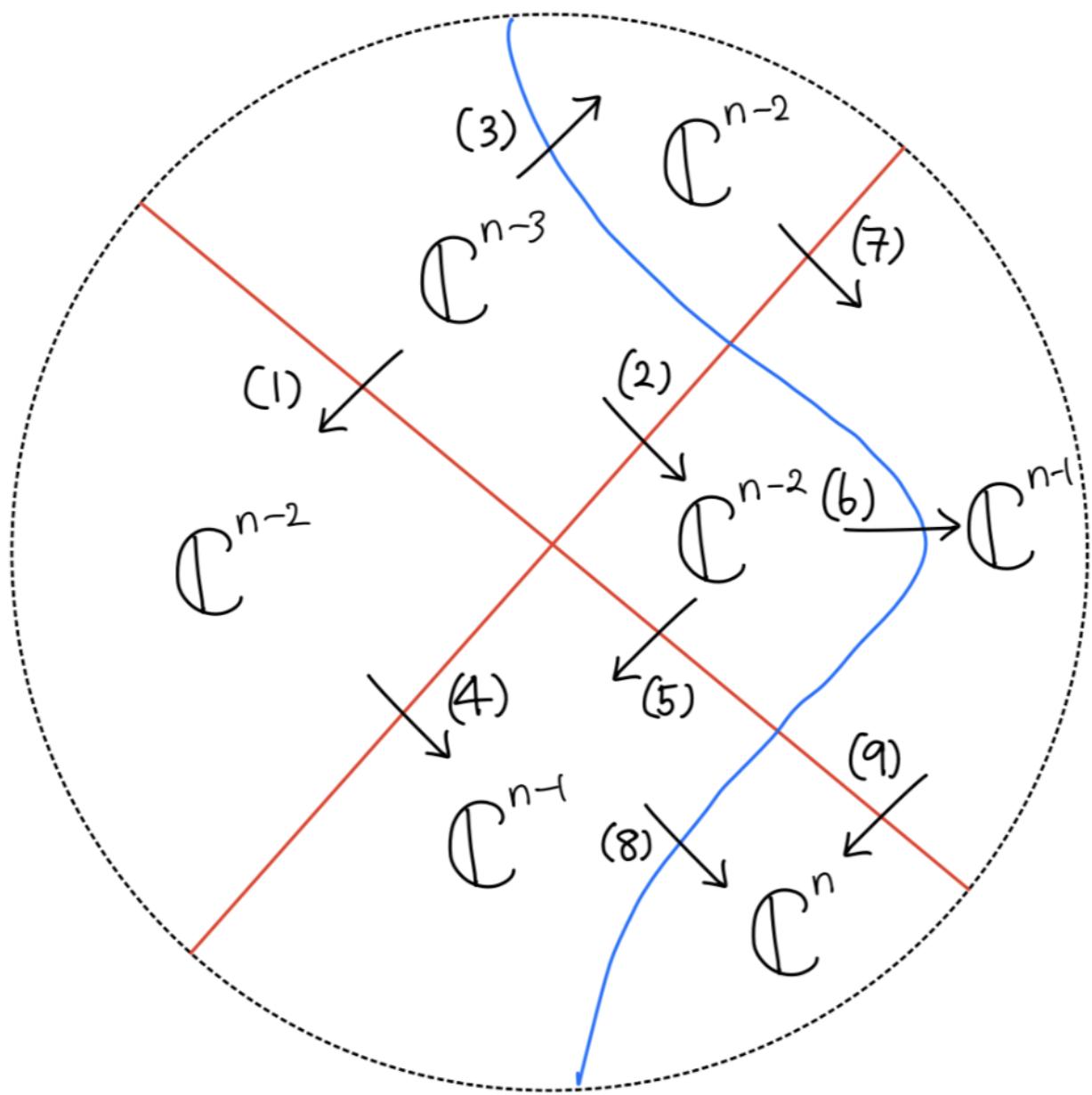


Figure 3.263

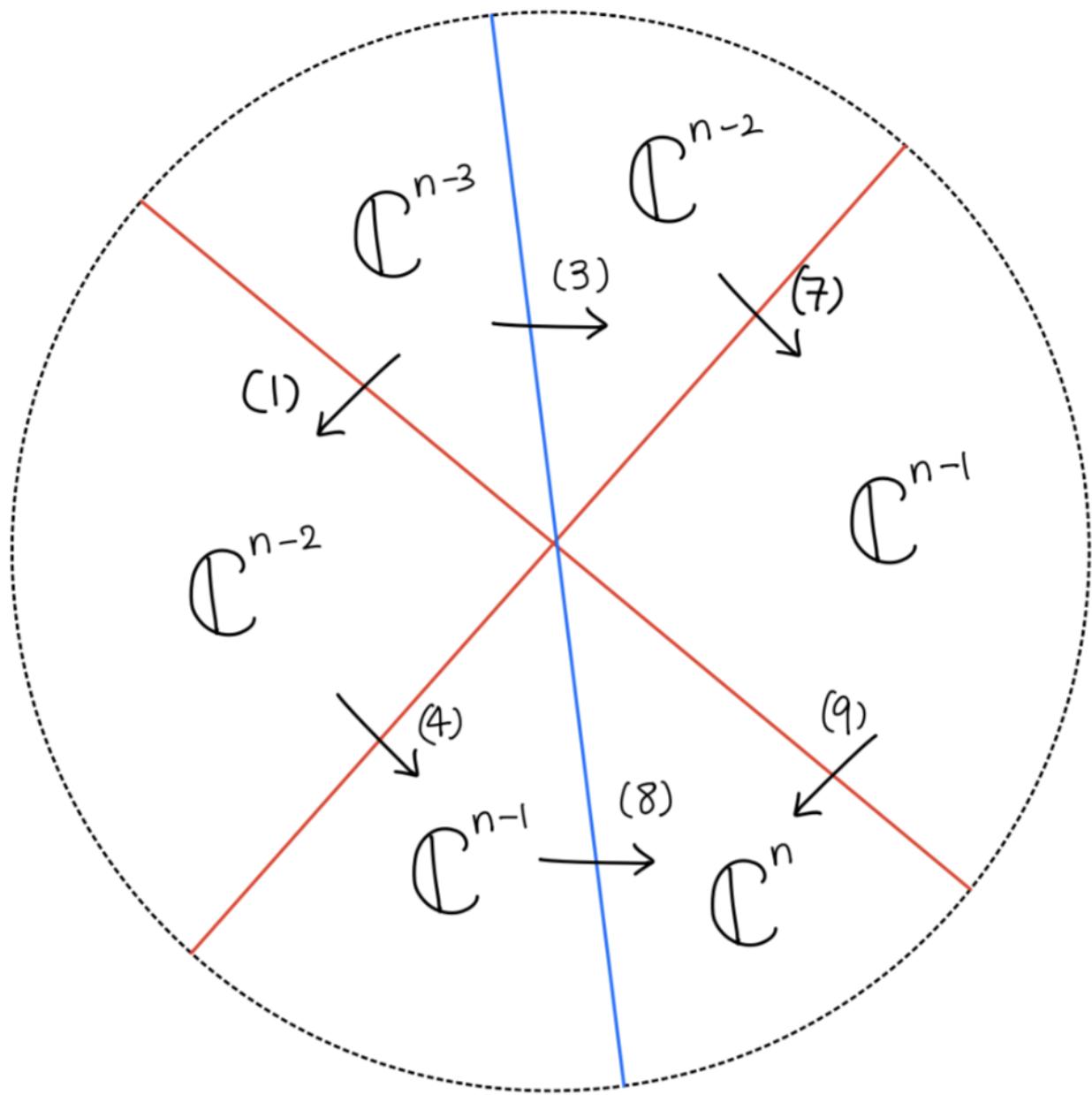


Figure 3.264

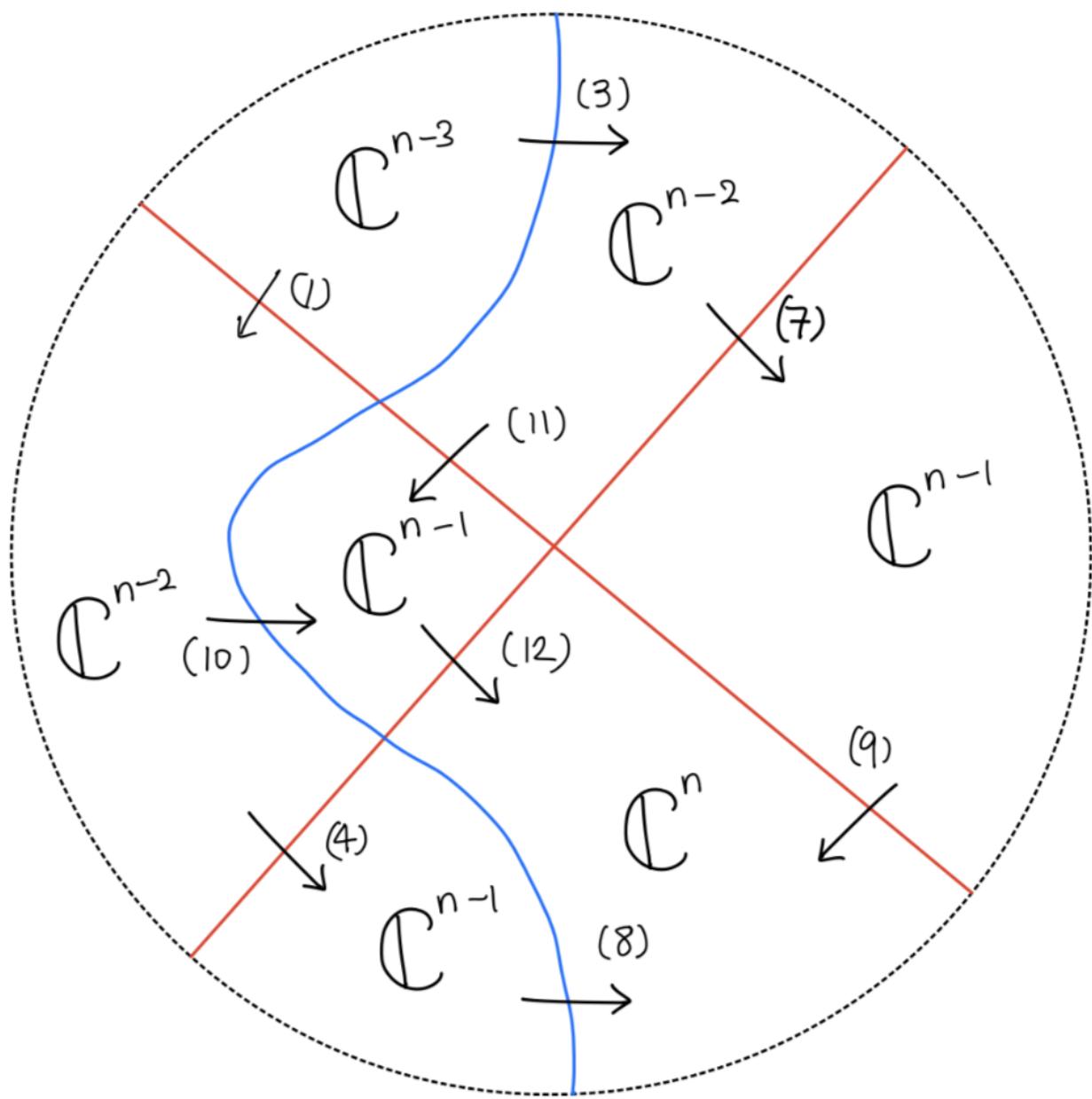


Figure 3.265

(1)  $\iota_0$ (2)  $\iota_0$ (3)  $\iota_1$ (4)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I + n, n - 1$

(5)  $\iota_0$

(6)  $\iota_1$

(7)  $\iota_0$

(8)  $\iota_1$

(9)  $\iota_0$

(10)  $\iota_1$

(11)  $\iota_0$

(12)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

### C. Gluing Isomorphism

Lastly, I will define a gluing isomorphism  $\gamma_\bullet : (f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$  using the fact that  $(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$  where  $pr_1 : (U \cap V) \times [0,1] \rightarrow (U \cap V)$  is the projection onto the first argument.

**Definition 201.** we define  $\gamma_\bullet$  to be the composition

$$(f_*\mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V}) \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_V)|_{(U \cap V) \times [0,1]} = \mathcal{F}_{V \times [0,1]}|_{(U \cap V) \times [0,1]}$$

where

- the first isomorphism is the one mentioned in the above proposition.
- the second isomorphism from the fact that the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times [0,1] & \hookrightarrow & V \times [0,1] \\ \downarrow pr_1 & & \downarrow pr_1 \\ (U \cap V) & \longrightarrow & V \end{array}$$

Now we have defined a cobordism  $\mathcal{F}_\bullet$ , we show that this is a Legendrian cobordism.

**Proposition 202.**  $\mathcal{F}_\bullet$  is a Legendrian cobordism i.e.  $\mathcal{F}_\bullet \in Sh_\Lambda(M, \mathbb{C})$ .

*Proof.* To prove the claim, I will show that the microlocal stalks of  $\mathcal{F}_\bullet$  vanishes at every points on a contangent bundle of  $M$ .

Note that there is a diffeomorphism between  $D_{r=2} \times (0, 1)$  and  $\mathbb{R}^3$  that preserves there stratification i.e.

$$s^3(sgn_1, sgn_2, sgn_3) \mapsto s_\bullet(sgn_1, sgn_2, sgn_3)$$

Then it is enough to prove that the microlocal stalk of the pullback of  $\mathcal{F}^\bullet$  along the above diffeomorphism vanishes at every points of  $T^*\mathbb{R}^n$ . The pullback of  $\mathcal{F}^\bullet$  along the diffeomorphism could be described using the following legible diagram, say  $F^3$ .

To simplify the notation, we denote

$$F^3(sgn_1, sgn_2, sgn_3) := F^3(s^3(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

- $F^3(-, -, -) := \mathbb{C}^{n-3}$

- $F^3(-, -, +) := \mathbb{C}^{n-2}$

- $F^3(+, -, -) := \mathbb{C}^{n-2}$

- $F^3(+, -, +) := \mathbb{C}^{n-2}$

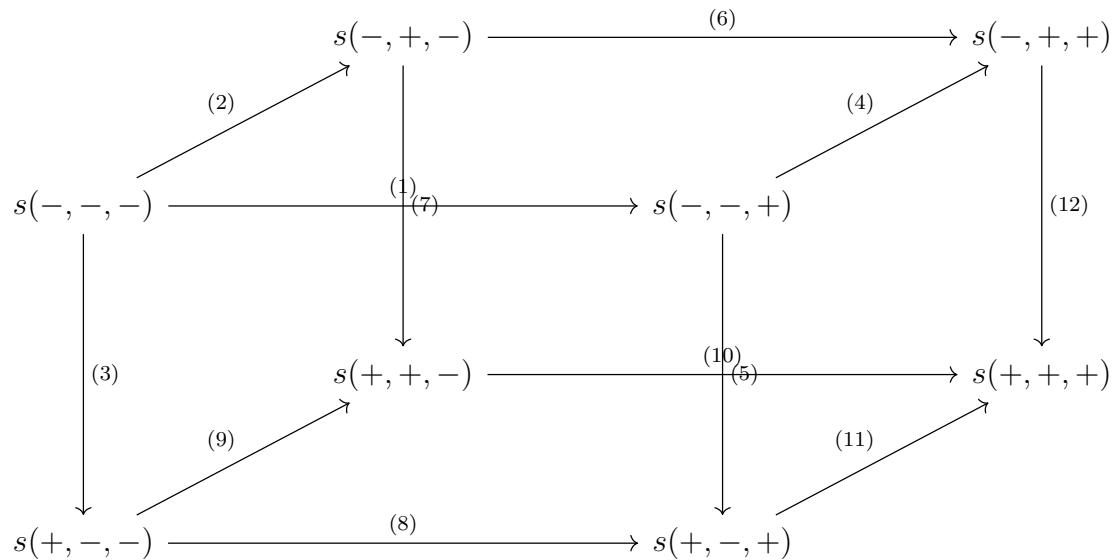
- $F^3(-, +, -) := \mathbb{C}^{n-2}$

- $F^3(-, +, +) := \mathbb{C}^{n-1}$

- $F^3(+, +, -) := \mathbb{C}^{n-1}$

- $F^3(+, +, +) := \mathbb{C}^n$

**Generalization maps:**



(1)  $\iota_0$

(2)  $\iota_0$

(3)  $\iota_1$

(4)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$

(5)  $\iota_1$

(6)  $\iota_0$

(7)  $\iota_1$

(8)  $\iota_0$

(9)  $\iota_0$

(10)  $\iota_0$

(11)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n, n-1}$

(12)  $\iota_1$

To prove that microlocal stalk vanishes everywhere, by Lemma ??, it is enough to show that the total complexes of  $F^3$  restricted to the following squares and cubes are acyclic

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, +, -) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, +, -)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$
  

$$\begin{array}{ccc}
 F^3(-, -, +) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, +) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{(*)} & \mathbb{C}^{n-1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-1} & \xrightarrow{(**)} & \mathbb{C}^n
 \end{array}$$
  

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$
  

$$\begin{array}{ccc}
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1} \\
 \downarrow \iota_1 & & \downarrow \iota_1 \\
 \mathbb{C}^{n-1} & \xrightarrow{\iota_0} & \mathbb{C}^n
 \end{array}$$
  

$$\begin{array}{ccc}
 F^3(-, -, -) & \longrightarrow & F^3(-, -, +) \\
 \downarrow & & \downarrow \\
 F^3(-, +, -) & \longrightarrow & F^3(-, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-3} & \xrightarrow{\iota_0} & \mathbb{C}^{n-2} \\
 \downarrow \iota_0 & & \downarrow (*) \\
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1}
 \end{array}$$
  

$$\begin{array}{ccc}
 F^3(+, -, -) & \longrightarrow & F^3(+, -, +) \\
 \downarrow & & \downarrow \\
 F^3(+, +, -) & \longrightarrow & F^3(+, +, +)
 \end{array} = \begin{array}{ccc}
 \mathbb{C}^{n-2} & \xrightarrow{\iota_0} & \mathbb{C}^{n-1} \\
 \downarrow \iota_0 & & \downarrow (** \\
 \mathbb{C}^{n-1} & \xrightarrow{\iota_0} & \mathbb{C}^n
 \end{array}$$

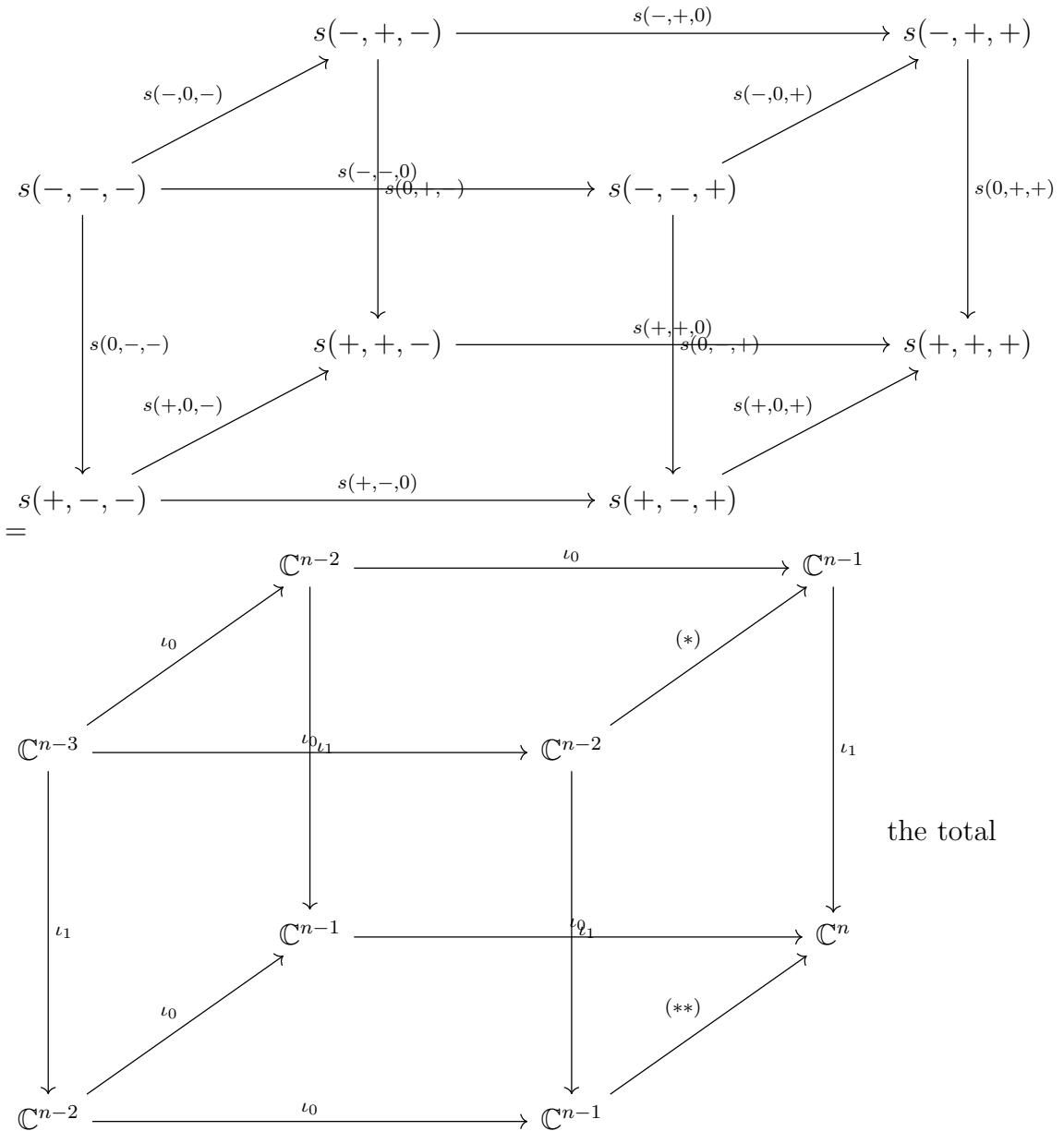
where

$$\bullet \quad (*) = \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$$

$$\bullet \quad (**) = \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n, n-1}$$

The total complexes of (i) – (vi) are acyclic because they are cartesian.

(vii) the cubic diagram:



complex is acyclic because by (i),  $\dots$ , (vi), the total complexes of the square diagrams are acyclic and the configuration of three planes

- $\mathbb{C}^{n-1} \xrightarrow{\iota_0} \mathbb{C}$

- $\mathbb{C}^{n-1} \xrightarrow{\iota_1} \mathbb{C}$

- $\mathbb{C}^{n-1} \xrightarrow{(**)} \mathbb{C}$

corresponding to  $s(+, +, -)$ ,  $s(-, +, +)$ ,  $s(+, -, +)$  are in general position.

□

### 3.9.5 Sheaf at the End

In this subsection, I will describe the sheaf  $\mathcal{F}_1$  at the end of the  $cobord'_4$ . By Mayer-Vietoris,  $\mathcal{F}_1 := \mathcal{F}_\bullet|_{M \times \{1\}}$  on  $M \cong M \times \{1\}$  is equivalent to the following data

- a sheaf on  $V$ , say  $\mathcal{F}_V$
- a sheaf on  $D_{r=2}$ , say  $\mathcal{F}_{D_{r=2}}$
- a gluing isomorphism  $\gamma_1 : f_* \mathcal{F}_{D_{r=2}}|_{U \cap V} \xrightarrow{\sim} \mathcal{F}_V|_{U \cap V}$ .

#### A. Sheaf on $V$

First, a sheaf on  $V \cong V \times \{1\}$  is the restriction of  $\mathcal{F}_{V \times [0,1]}$  to  $V \times \{1\}$ , i.e.  $\mathcal{F}_{V \times [0,1]}|_{V \times \{1\}} = pr_1^*(\mathcal{F}_0|_V)|_{V \times \{1\}} = \mathcal{F}_0|_V$ .

#### B. Sheaf on $D_{r=2}$

Next, a sheaf on  $D_{r=2} \cong D_{r=2} \times \{1\}$  is the restriction of  $\mathcal{F}_{D_{r=2} \times [0,1]}$  to  $D_{r=2} \times \{1\}$ , i.e.  $\mathcal{F}_{D_{r=2} \times [0,1]}|_{D_{r=2} \times \{1\}}$ . I will describe it as a squiggly legible diagram  $F_1$  which is the restriction of  $F_\bullet$  defined in the previous section.

For simplicity, we use the following notations

$$F_1(sgn_1, sgn_2, sgn_3) := F_1(s_1(sgn_1, sgn_2, sgn_3))$$

**Stalks:**

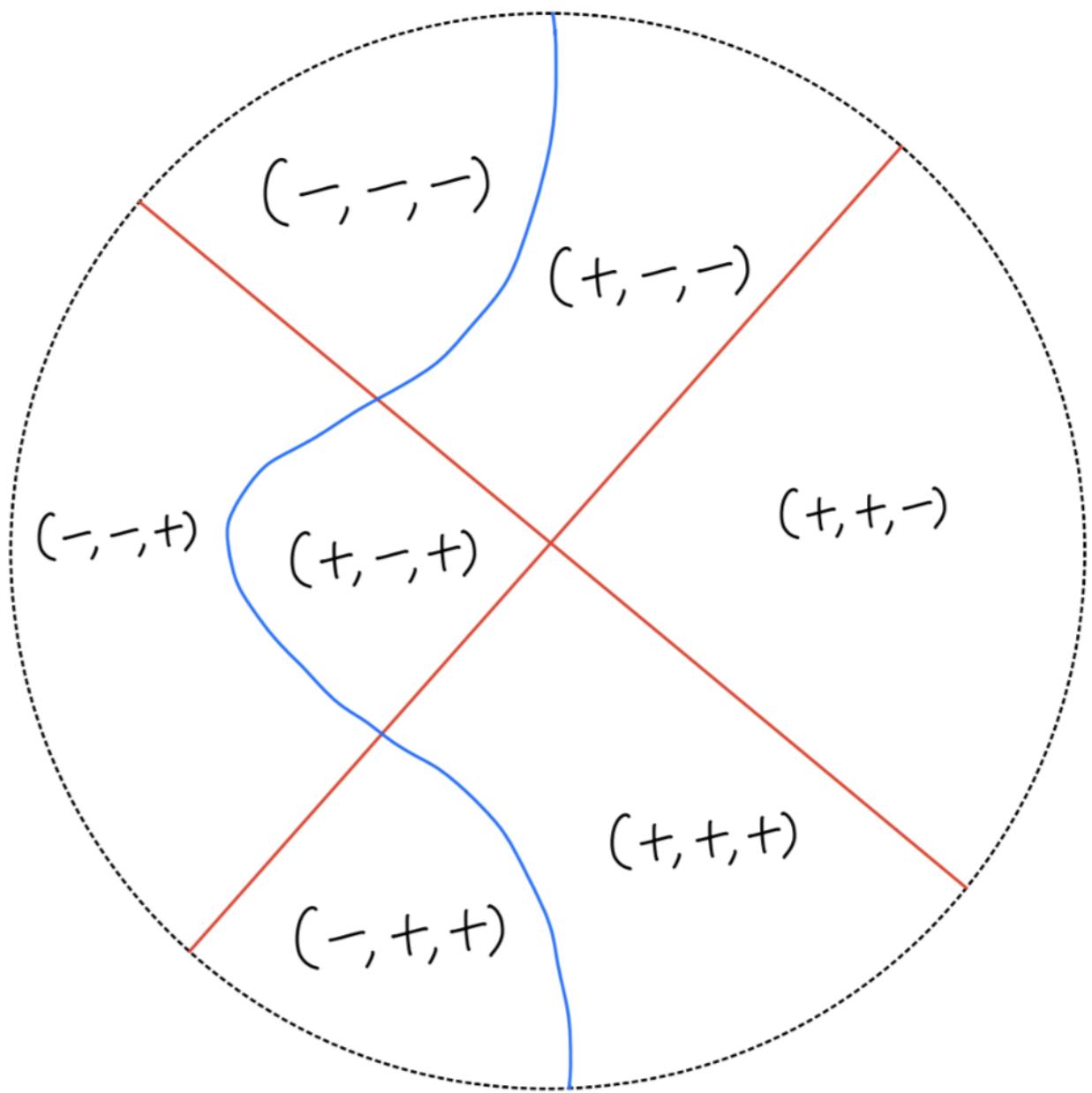


Figure 3.266

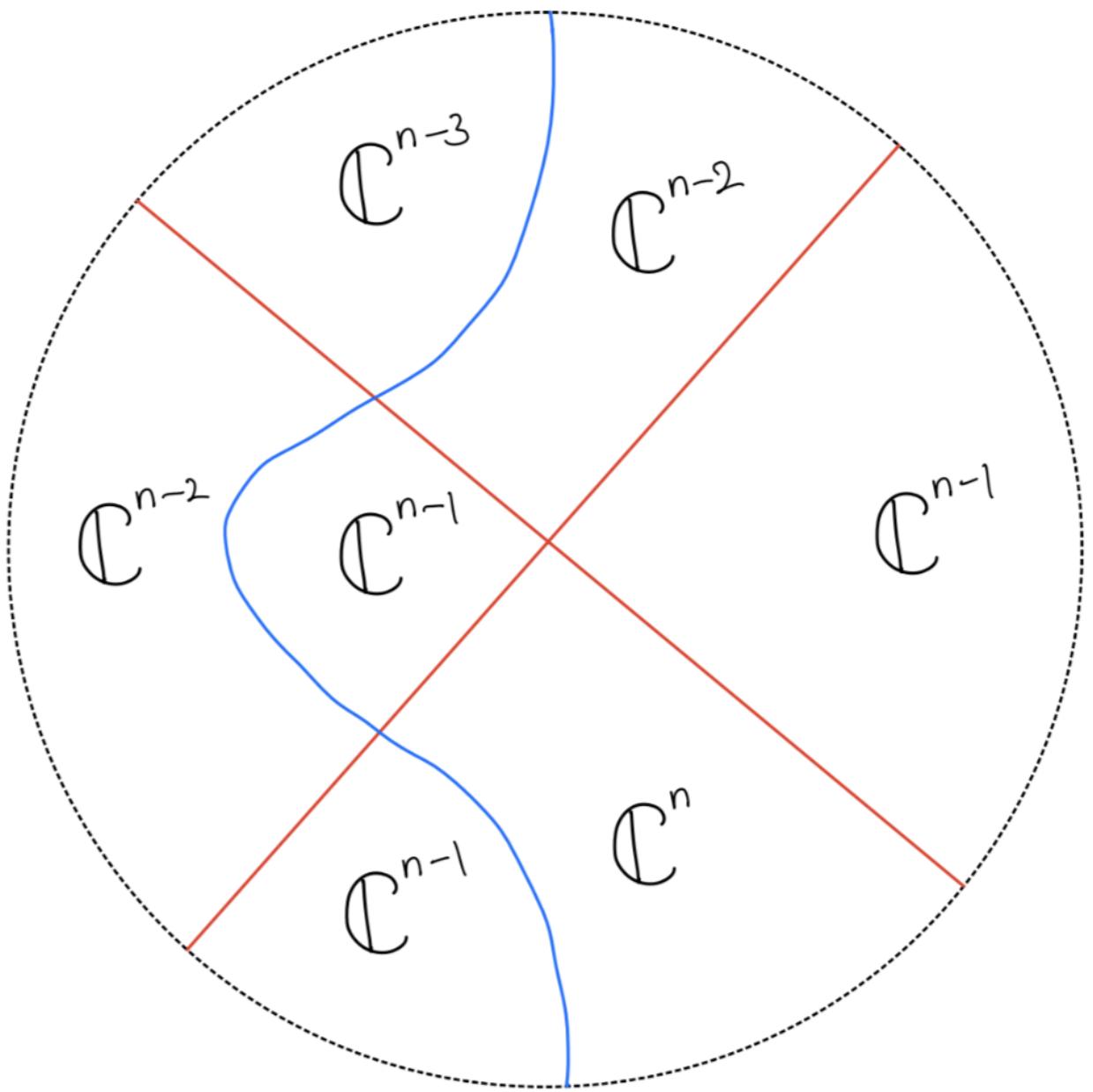


Figure 3.267

- $F_1(-, -, -) := \mathbb{C}^{n-3}$

- $F_1(+, -, -) := \mathbb{C}^{n-2}$

- $F_1(-, -, +) := \mathbb{C}^{n-2}$

- $F_1(+, -, +) := \mathbb{C}^{n-1}$

- $F_1(+, +, -) := \mathbb{C}^{n-1}$

- $F_1(-, +, +) := \mathbb{C}^{n-1}$

- $F_1(+, +, +) := \mathbb{C}^n$

Generalization maps:

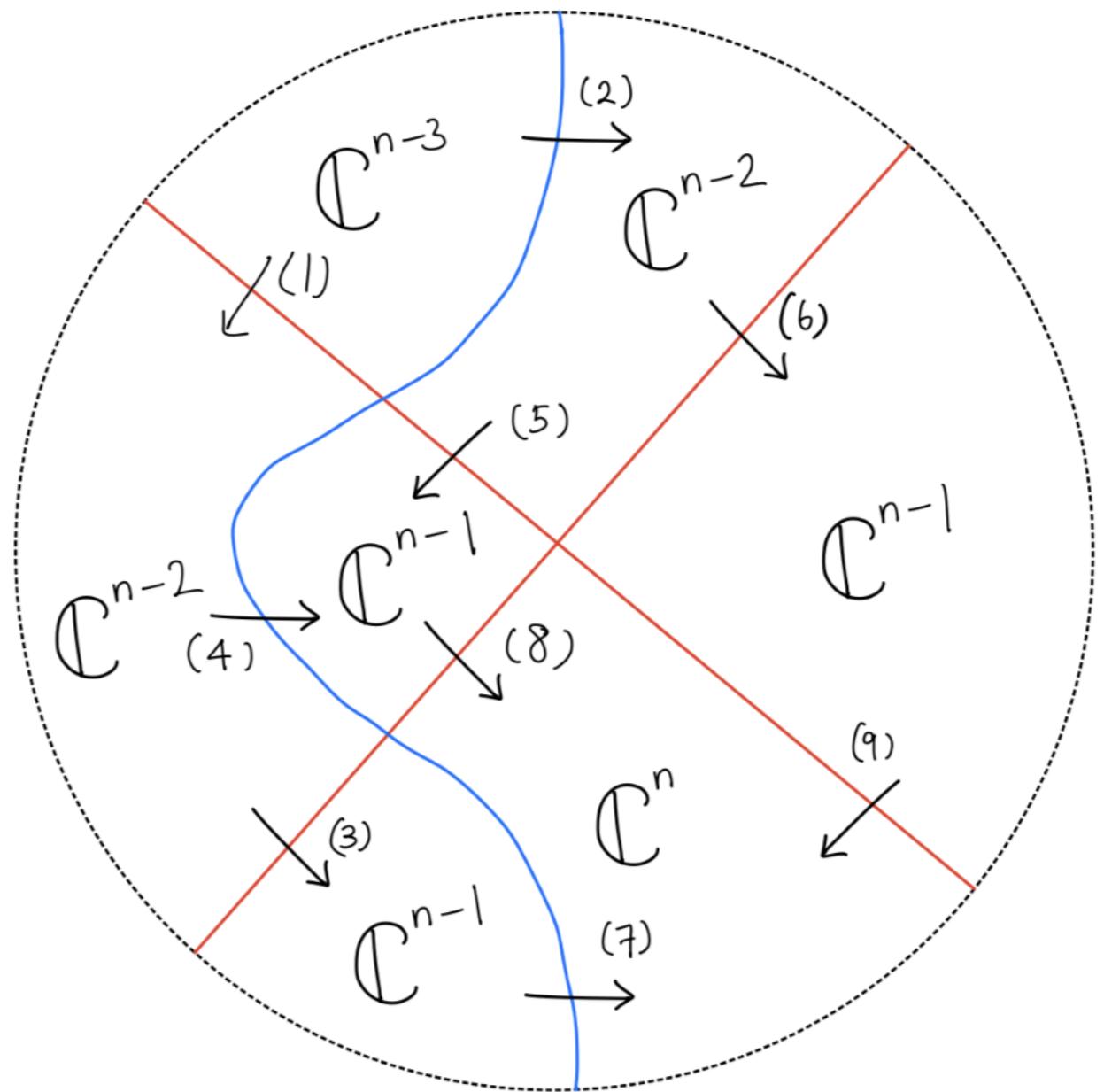


Figure 3.268

(1)  $\iota_0$

(2)  $\iota_1$

(3)  $\iota_0$

(4)  $\iota_1$

(5)  $\iota_0$

(6)  $\iota_0$

(7)  $\iota_1$

(8)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(9)  $\iota_0$

### C. Gluing Isomorphism

Lastly, the gluing isomorphism  $\gamma_1 := \gamma_\bullet|_{(U \cap V) \times \{1\}} : f_* \mathcal{F}_{D_{r=2}}|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$  is described as follows.

**Definition 203.** we define  $\gamma_1$  to be the composition

$$(f_* \mathcal{F}_{D_{r=2}})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} pr_1^*(\mathcal{F}_0|_{U \cap V})|_{(U \cap V) \times \{1\}} \xrightarrow{\sim} \mathcal{F}_0|_{U \cap V}$$

where

- the first isomorphism follows from the fact that  $(f_* \mathcal{F}_{D_{r=2} \times [0,1]})|_{(U \cap V) \times [0,1]}$  is isomorphic to  $pr_1^*(\mathcal{F}_0|_{U \cap V})$ .
- the second isomorphism follows from the fact that the following composition is an identity map:

$$(U \cap V) \xrightarrow{\sim} (U \cap V) \times \{1\} \hookrightarrow (U \cap V) \times [0,1] \twoheadrightarrow (U \cap V)$$

## 3.10 5th Sheaf Cobordism

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-1, 1)_x \times (-n-1, 2)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to  $\{(x, z) \in R \mid z = 1\}$ , co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to  $\bigcup_{k=1}^n \{(x, z) \in R \mid z = -k\}$ , co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to  $\{(x, z) \in R \mid x = 0\}$ , co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

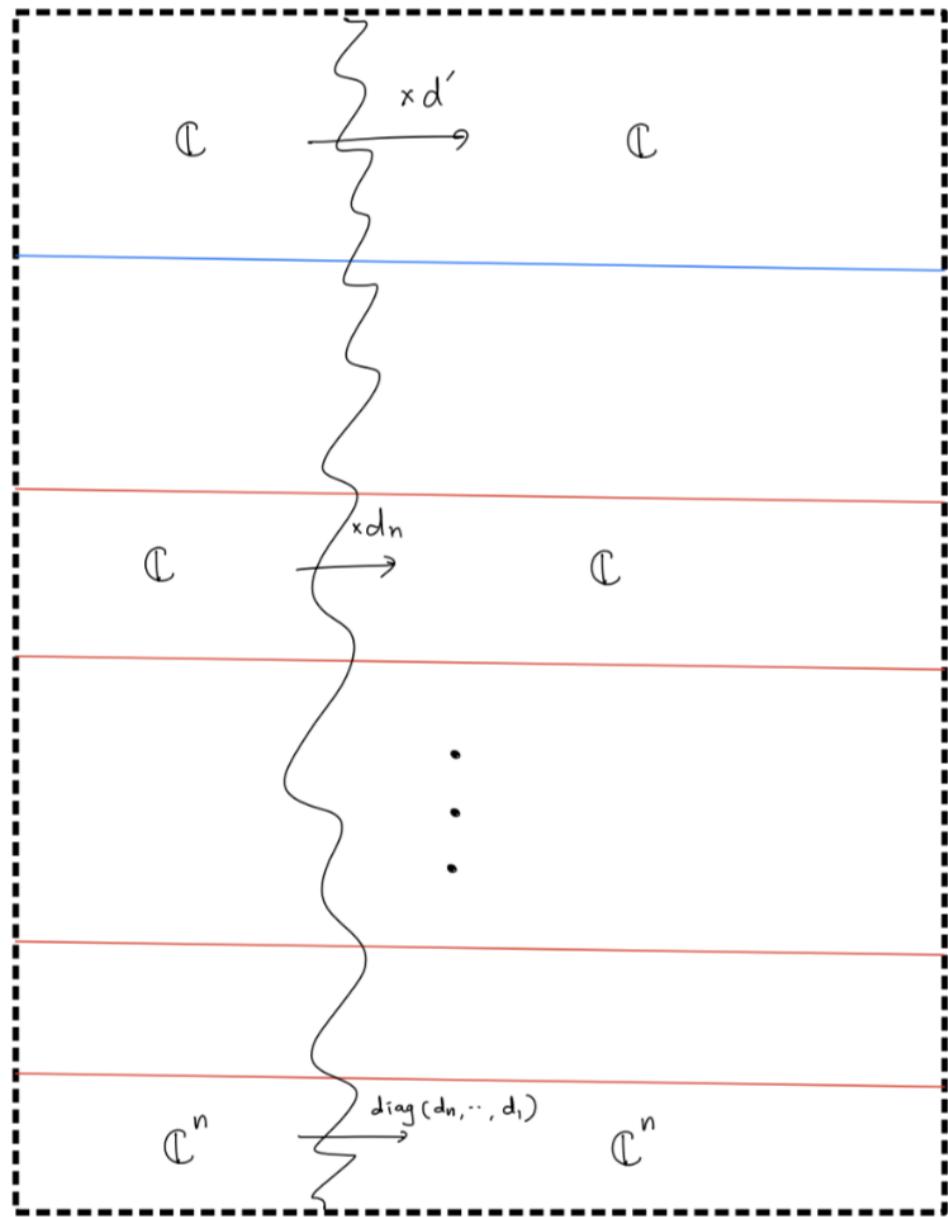


Figure 3.269

Then we define a cobordism starting from the above sheaf, say  $cobord_5(n)$  supported on  $U$ . At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

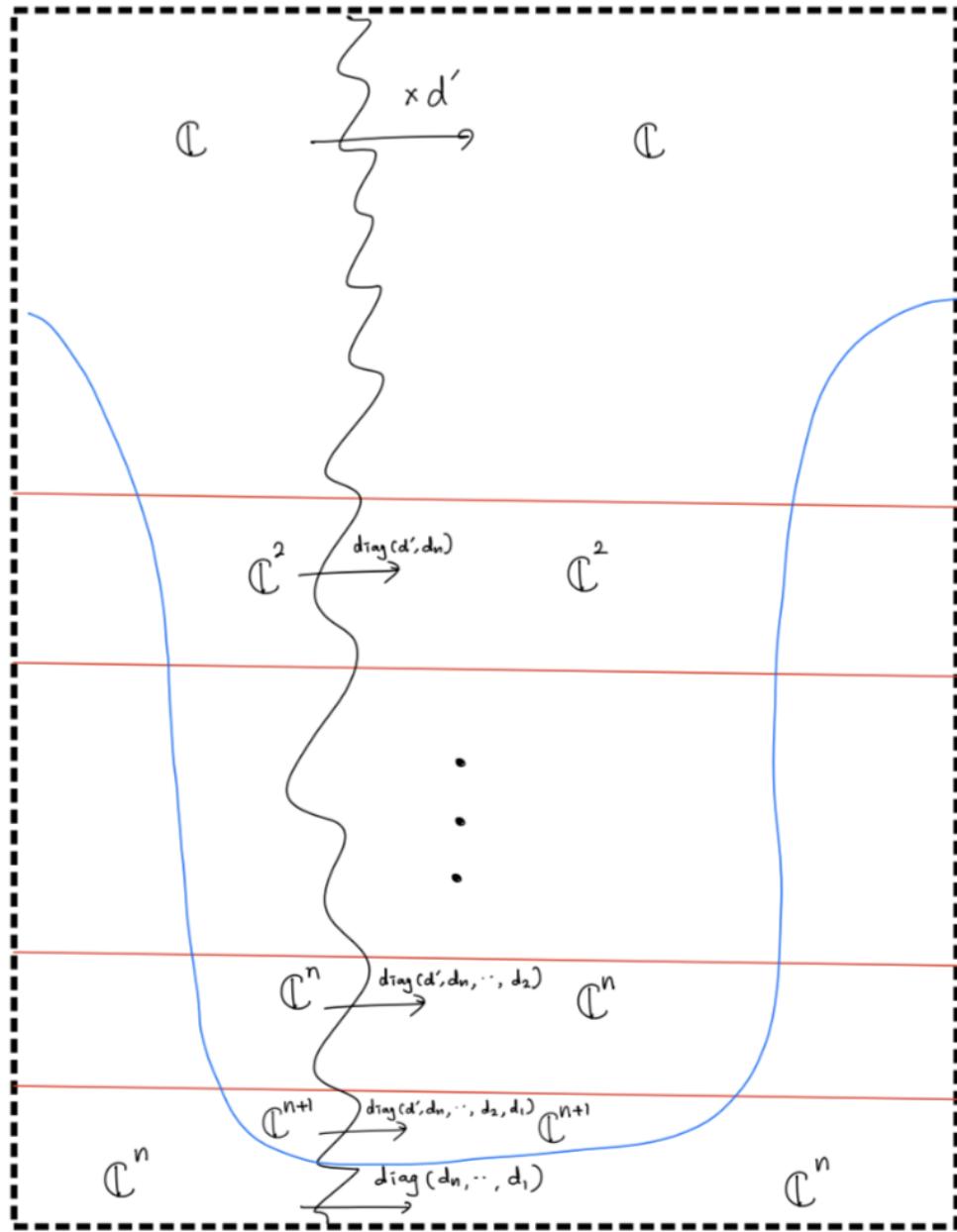


Figure 3.270

We define  $cobord_5(n)$  inductively as follows.

- (i) For  $n = 1$ , we define  $cobord_5(1)$  to be  $cobord_1$  from

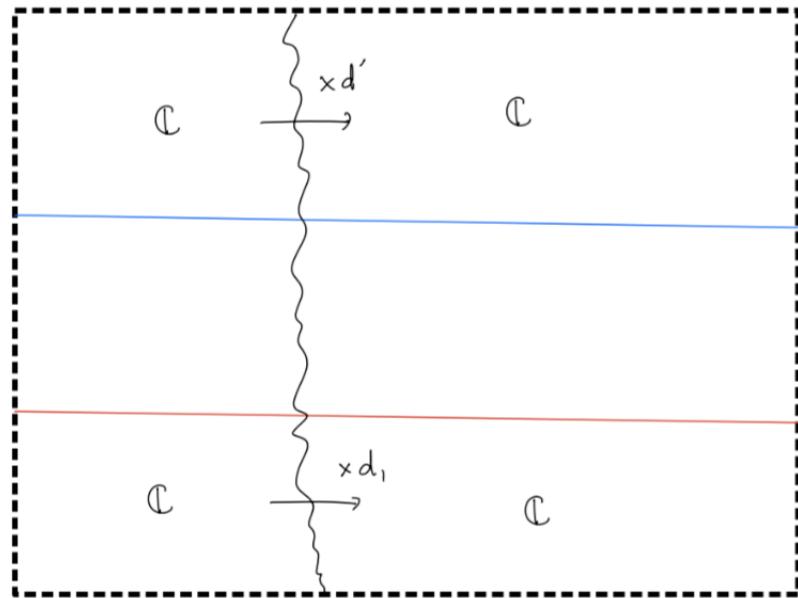


Figure 3.271

to

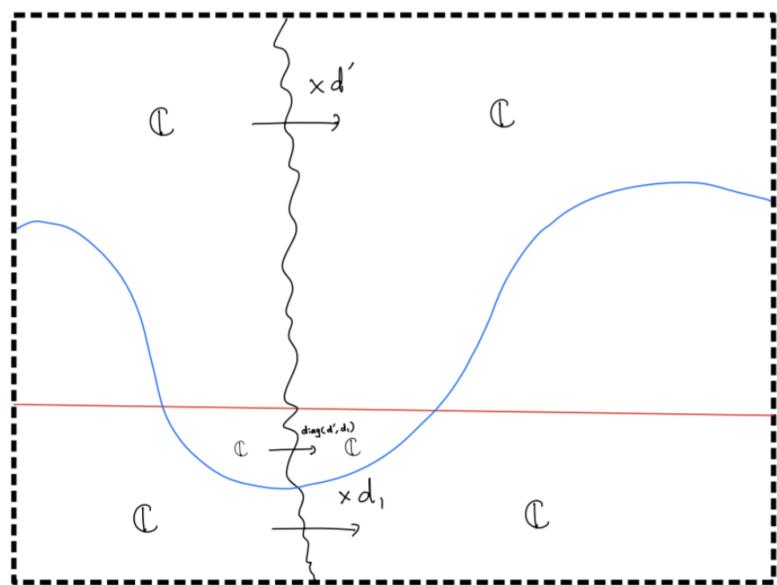


Figure 3.272

(ii) For  $n > 0$ ,

(Step 1) we first apply  $cobord_5(n - 1)$  to the square region surrounded by a purple

dotted line.

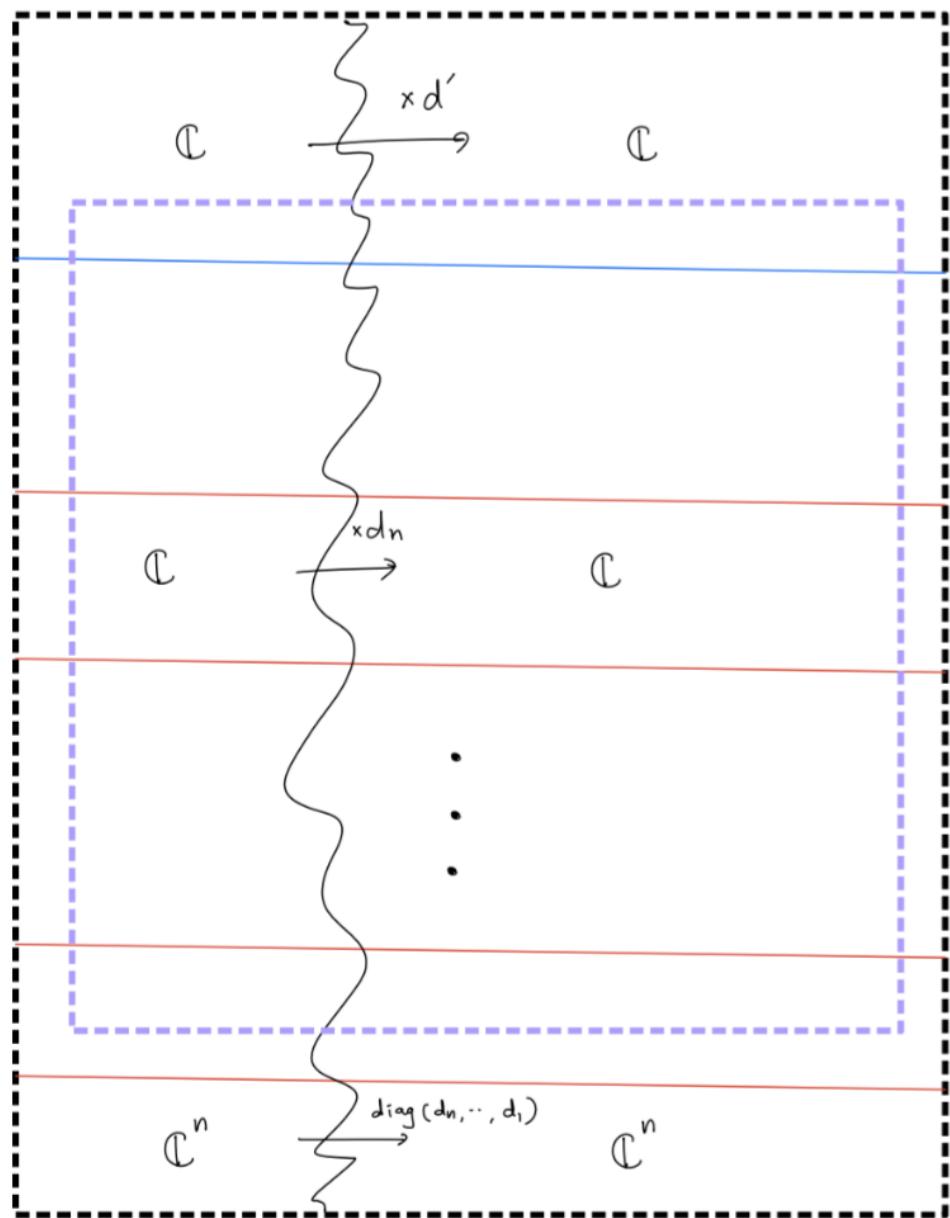


Figure 3.273

by induction hypothesis, we get

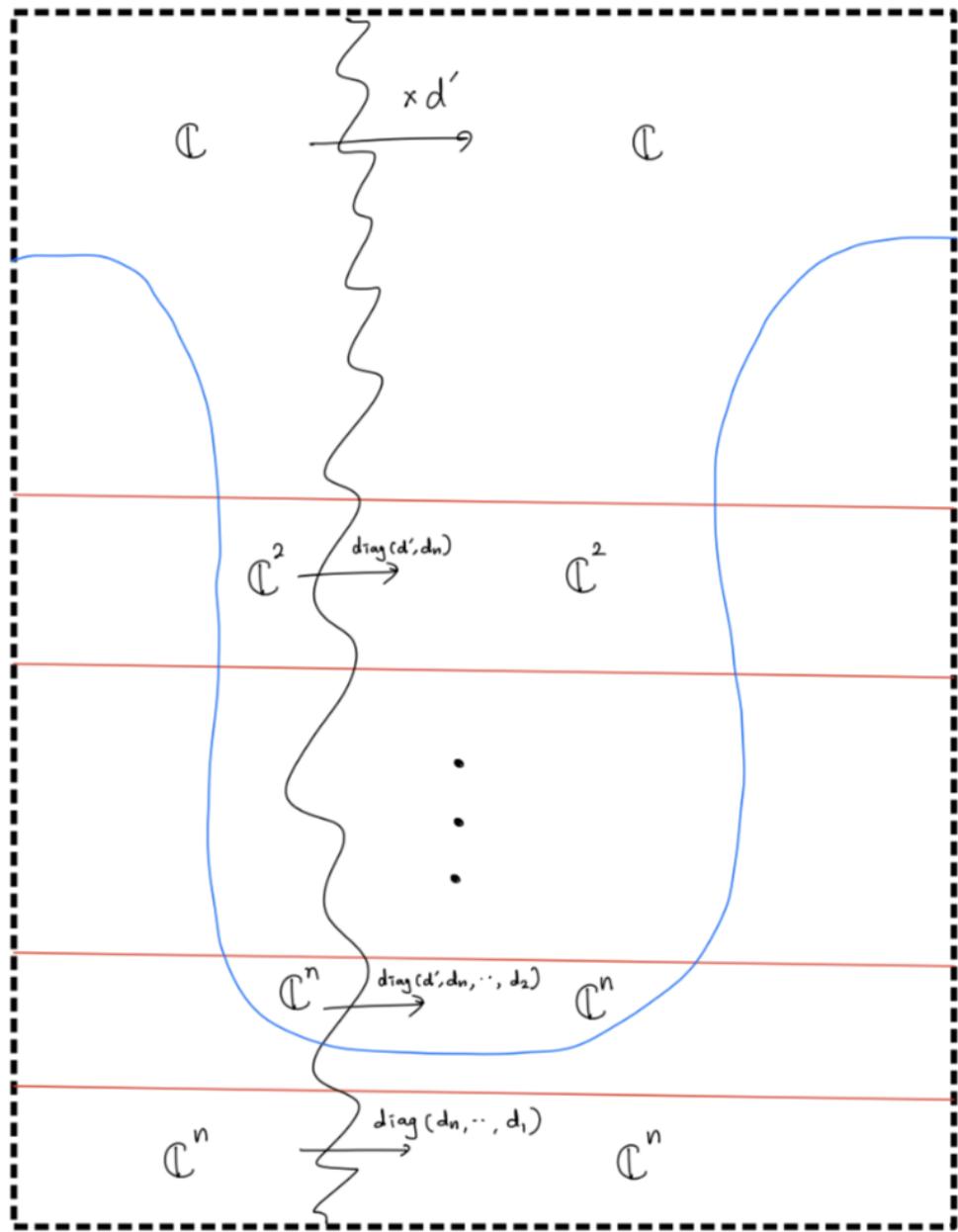


Figure 3.274

(Step 2) apply  $cobord_1$  to the square region surrounded by purple dotted lines.

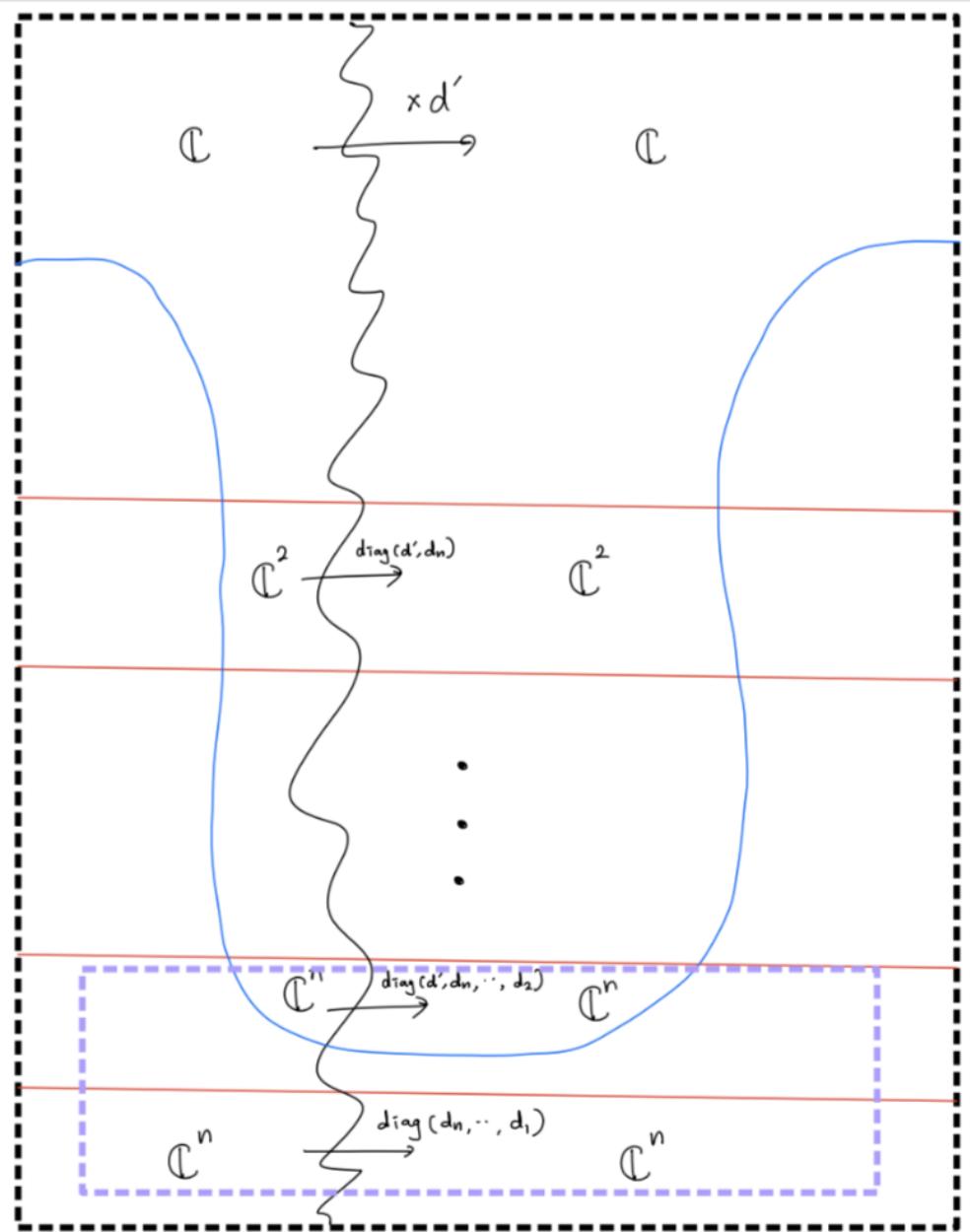


Figure 3.275

we get the final sheaf

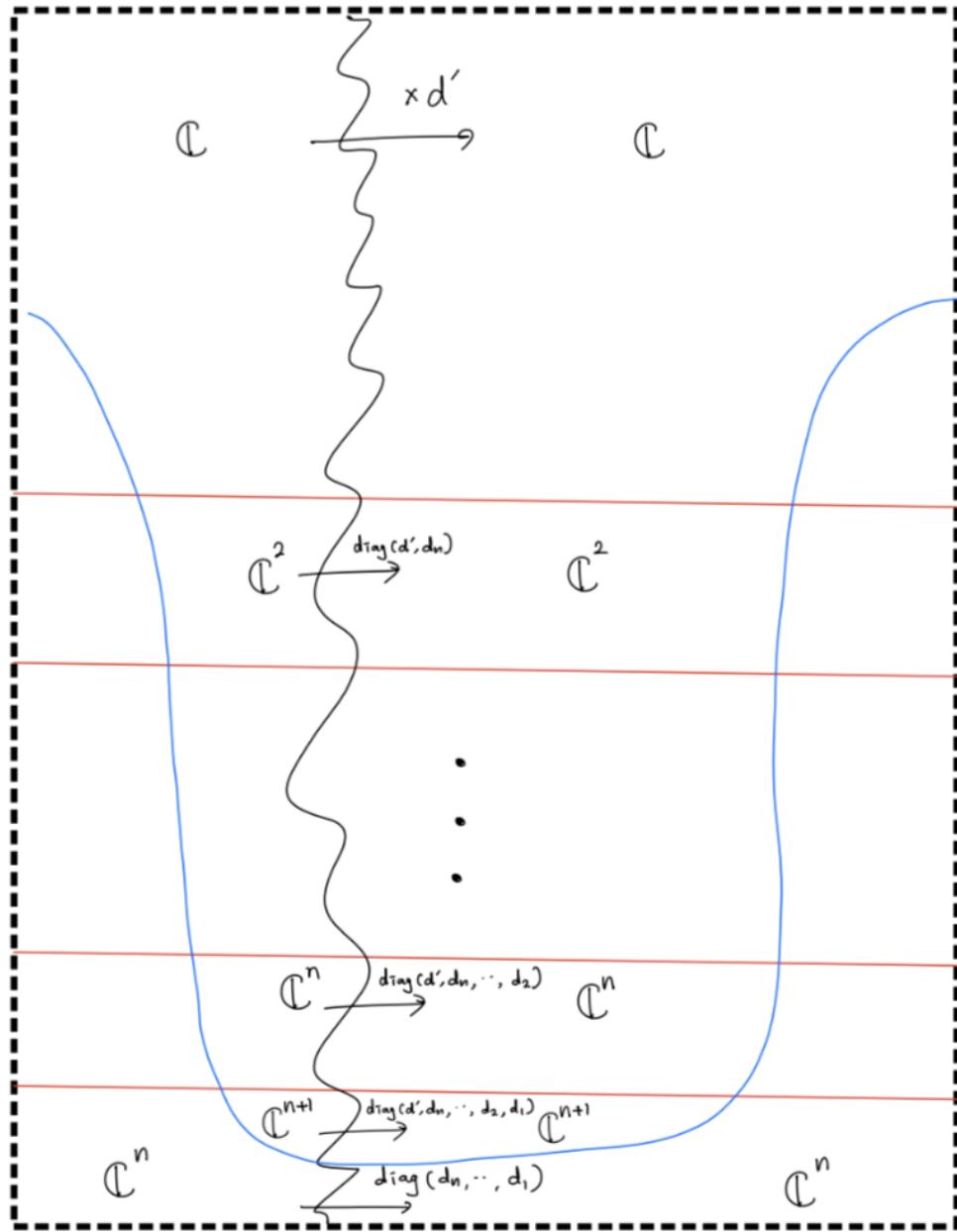


Figure 3.276

### 3.11 6th Sheaf Cobordism

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-1, 1)_x \times (-2, n+1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to  $\bigcup_{k=1}^n \{(x, z) \in R \mid z = k\}$ , co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to  $\{(x, z) \in R \mid z = -1\}$ , co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to  $\{(x, z) \in R \mid x = 0\}$ , co-oriented towards the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

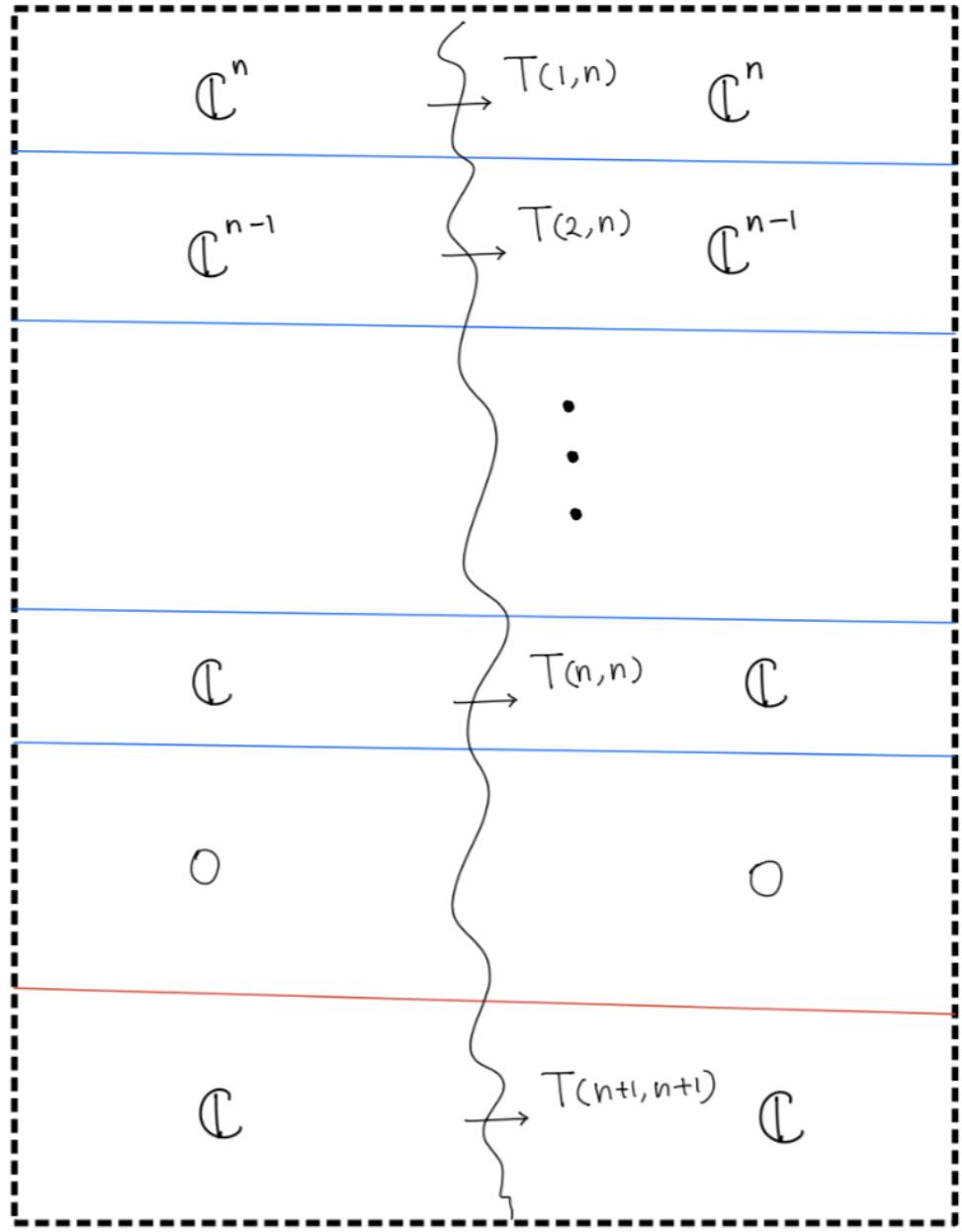


Figure 3.277

where  $T \in M^{(n+1) \times (n+1)}$  such that preserves the following flags

- $\mathbb{C} \subset_{\iota_0} \cdots \subset_{\iota_0} \mathbb{C}^n$
- $\mathbb{C} \subset_{\iota_1} \mathbb{C}^n$

Then we define a cobordism starting from the above sheaf, say  $cobord_6(n)$  supported on  $U$ , where  $n$  is the number of blue strands. At the end of the cobordism, the

sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

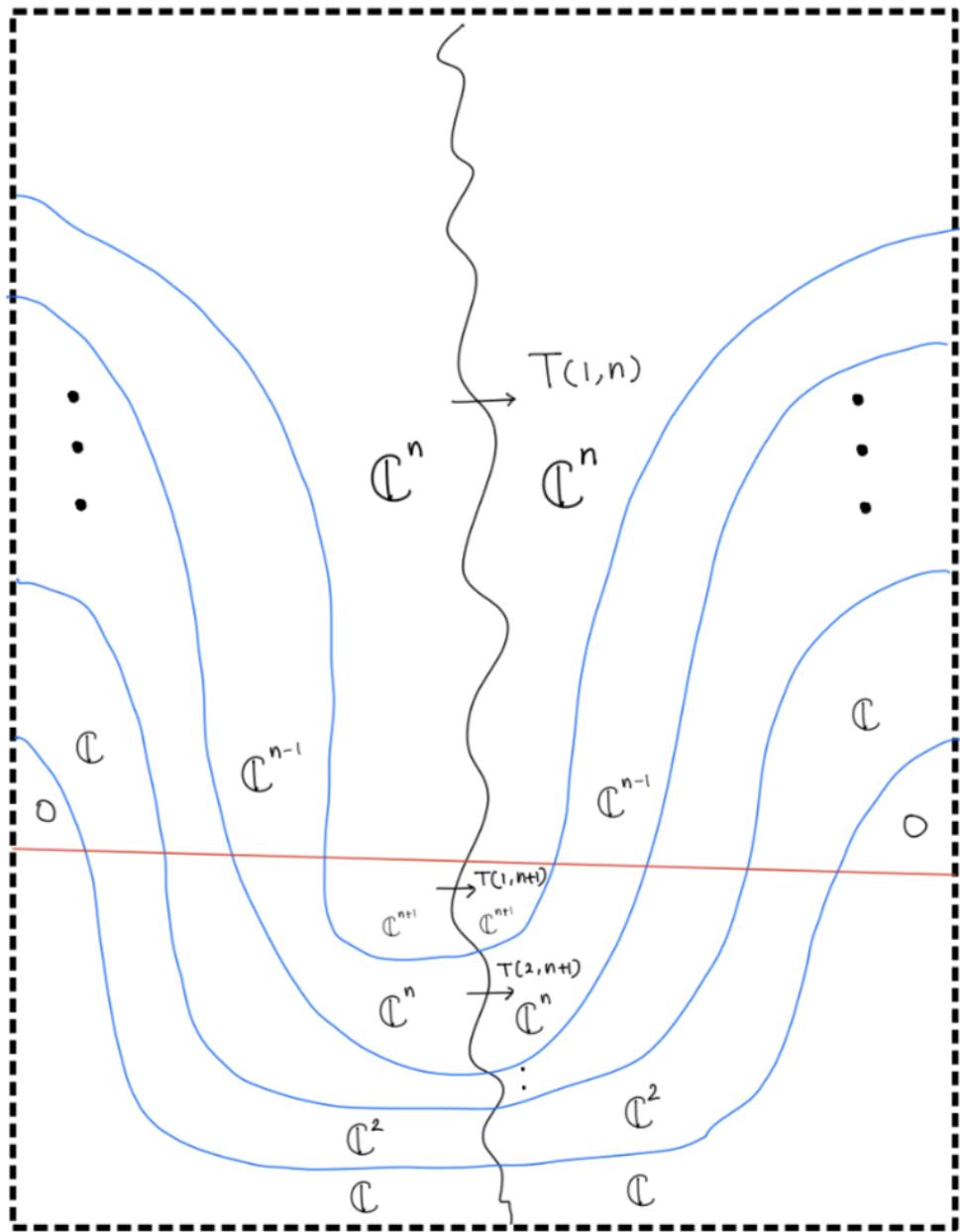


Figure 3.278

We define  $cobord_6(n)$  inductively as follows.

- (i) For  $n = 1$ , we define  $cobord_6(1)$  to be  $cobord_1$  from

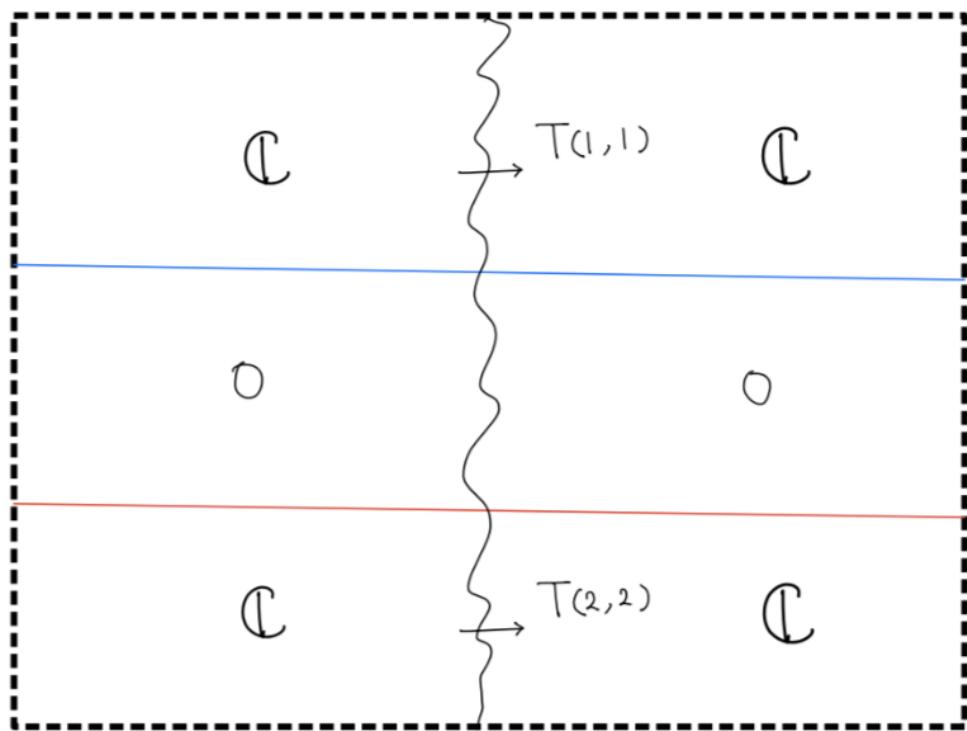


Figure 3.279

to

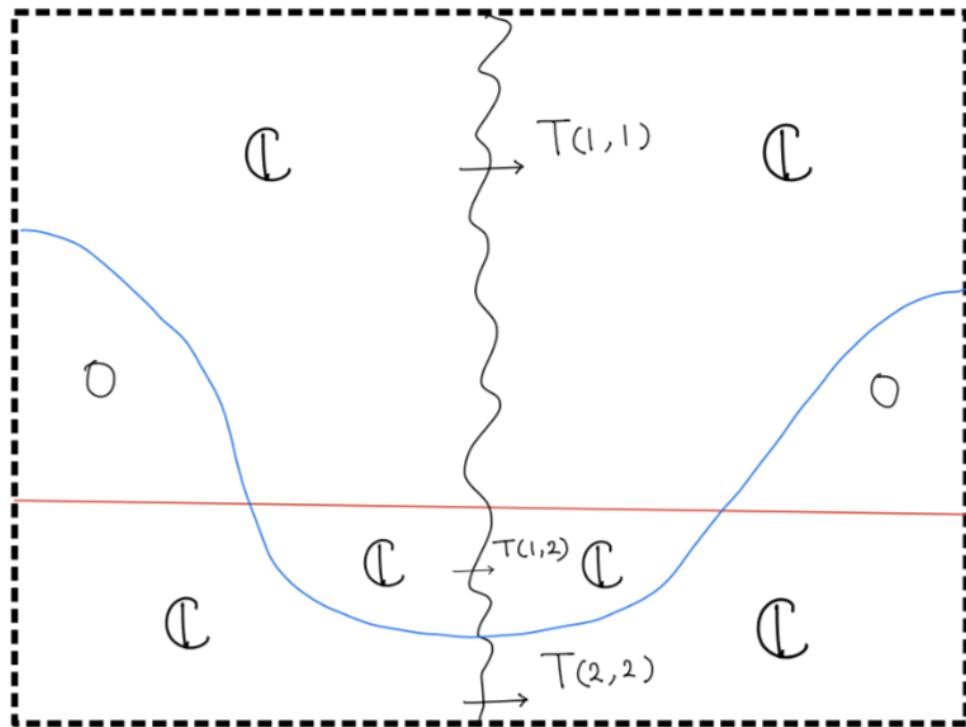


Figure 3.280

(ii) For  $n > 0$ ,

(Step 1) we first apply  $cobord_6(n - 1)$  to the square region surrounded by purple dotted lines.

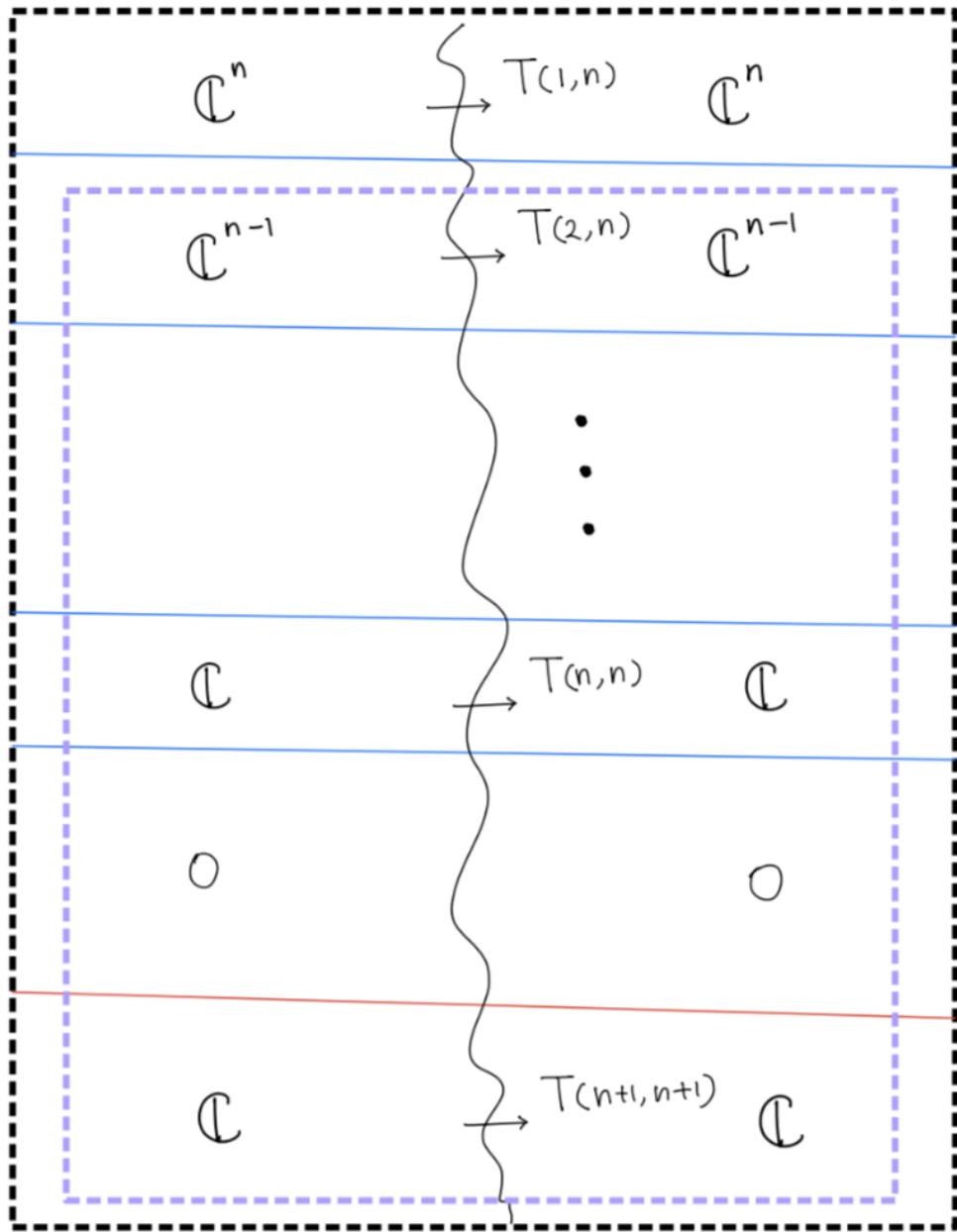


Figure 3.281

by induction hypothesis, we get

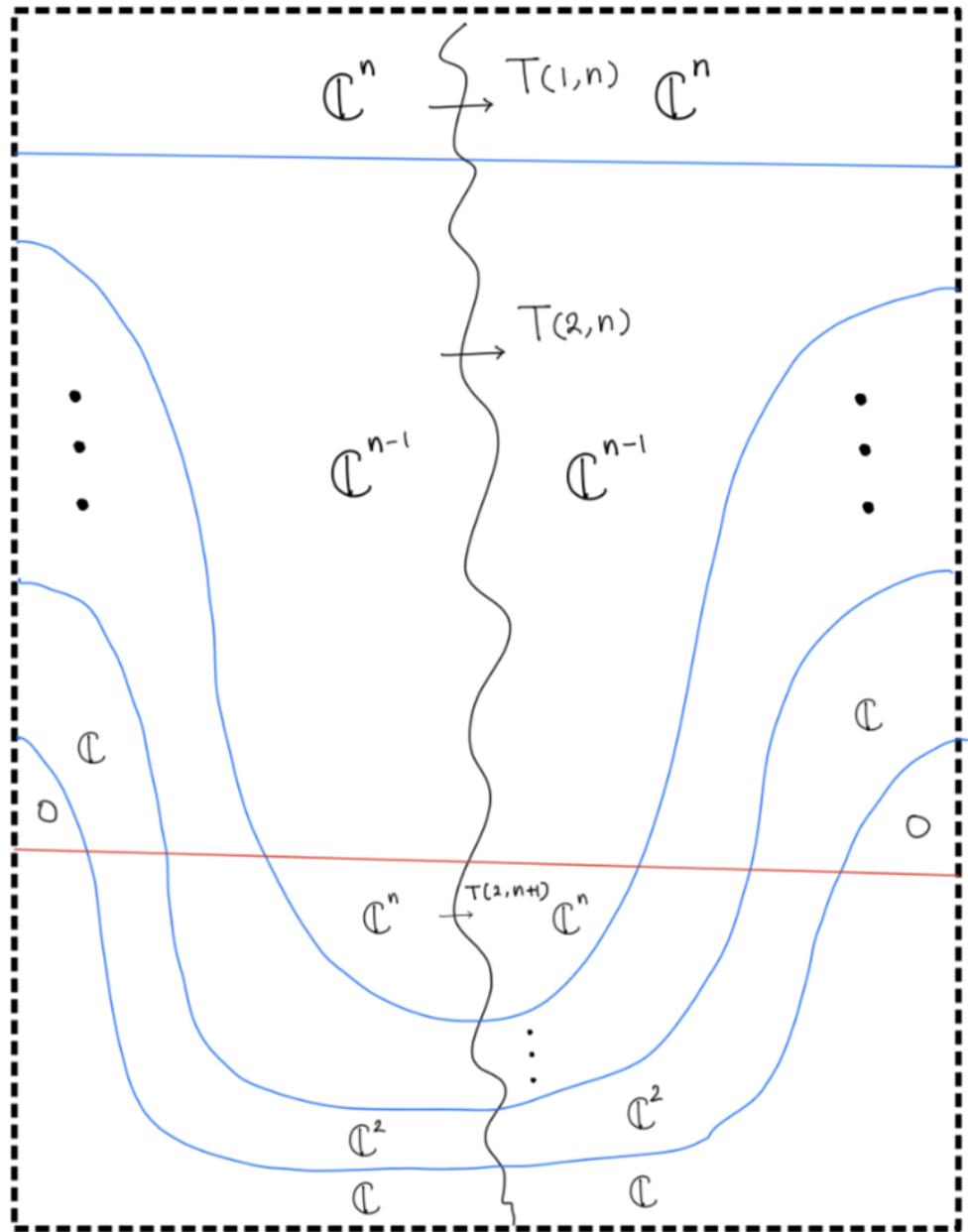


Figure 3.282

(Step 2) apply  $cobord_1$  to the square region surrounded by purple dotted lines.

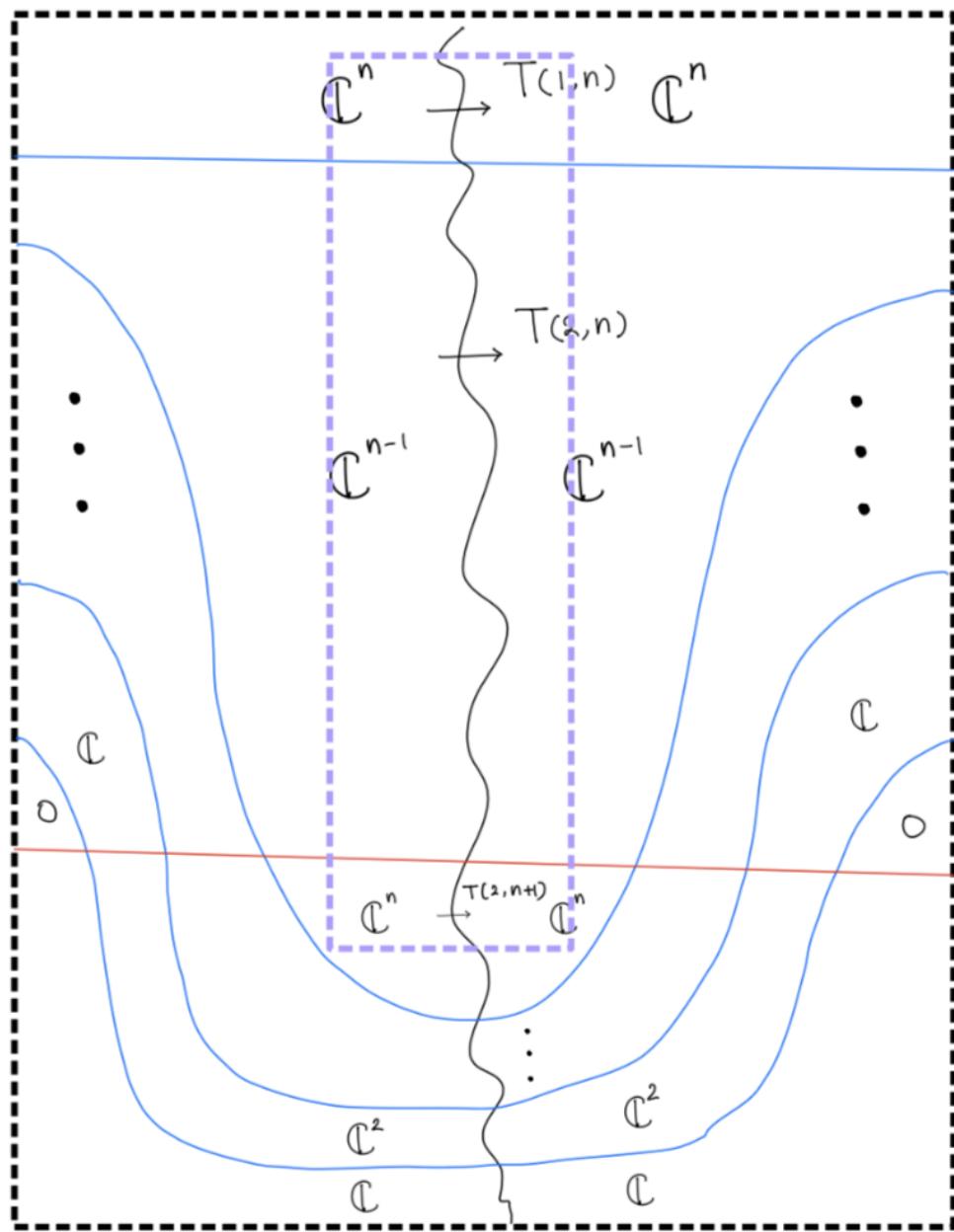


Figure 3.283

we get the final sheaf

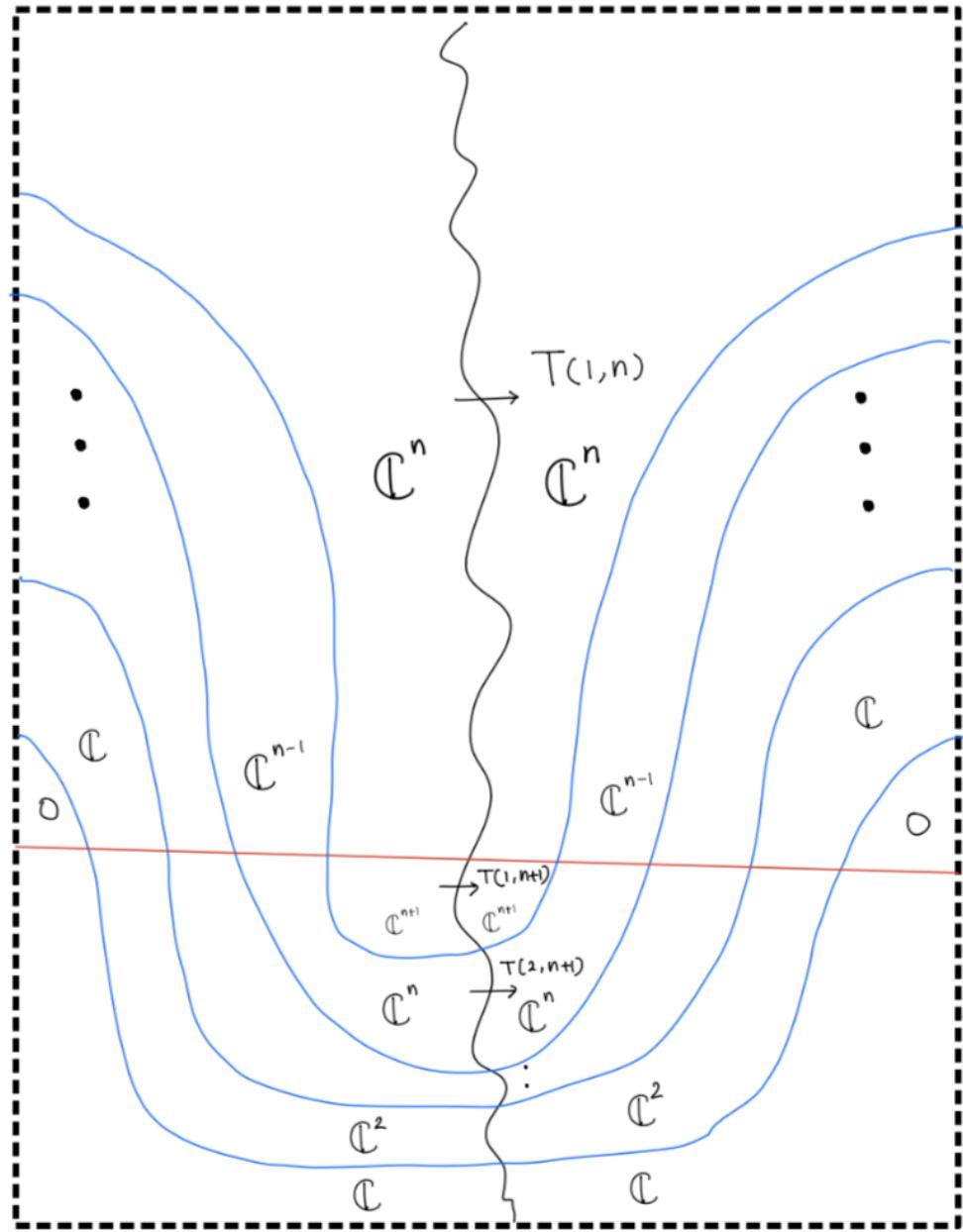


Figure 3.284

### 3.12 7th Sheaf Cobordism

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0, \Lambda_0^\infty, \Lambda_0^{sqwig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-1, 1)_x \times (0, n+1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to  $\{(x, z) \in R \mid z = \Psi(\frac{1}{2}, n + \frac{1}{2})(x)\}$ , co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to  $\bigcup_{k=1}^n \{(x, z) \in R \mid z = -k\}$ , co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to  $\phi$ .

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

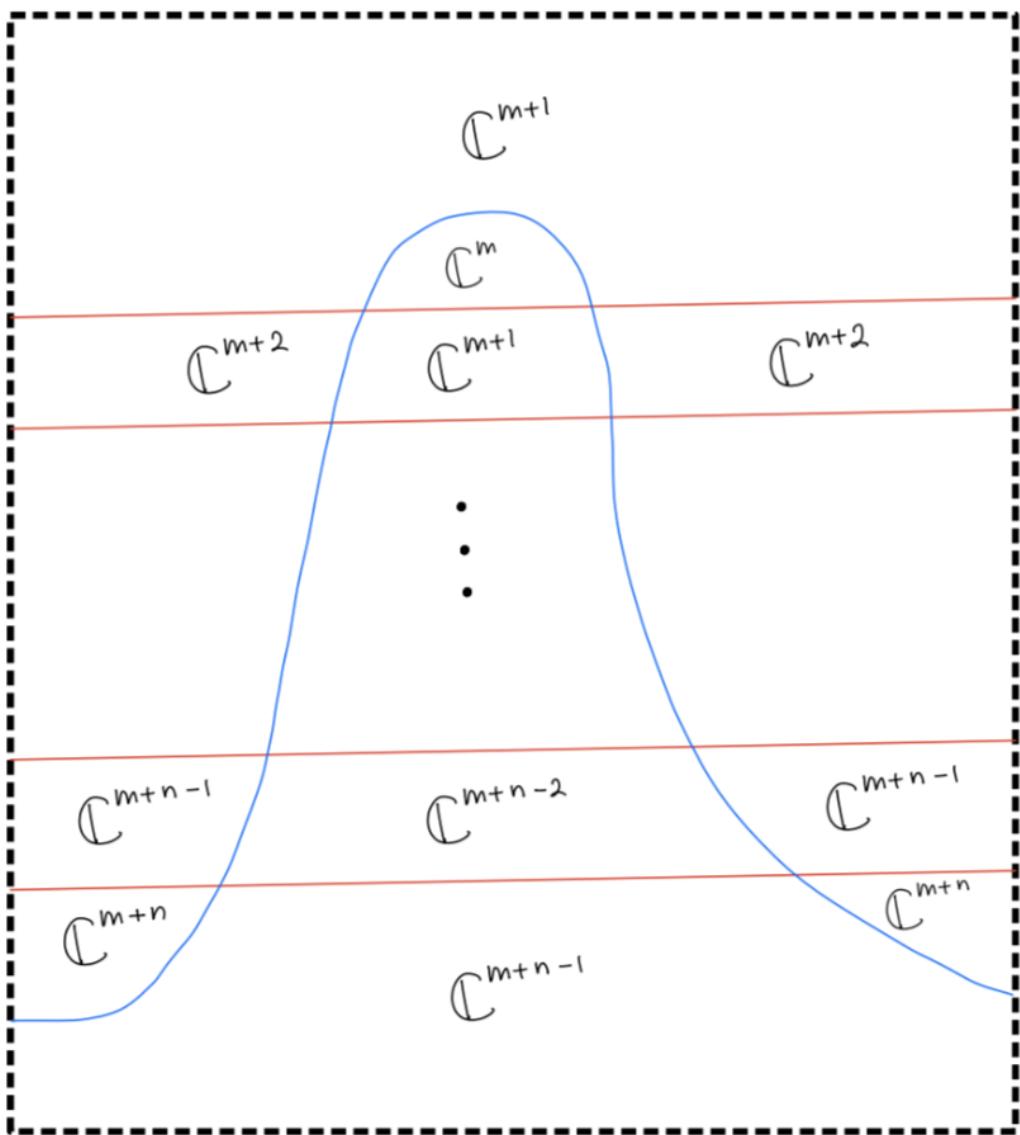


Figure 3.285

Then we define a cobordism starting from the above sheaf, say  $cobord_5(n)$  supported on  $U$ , where  $n$  is the number of red strands. At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

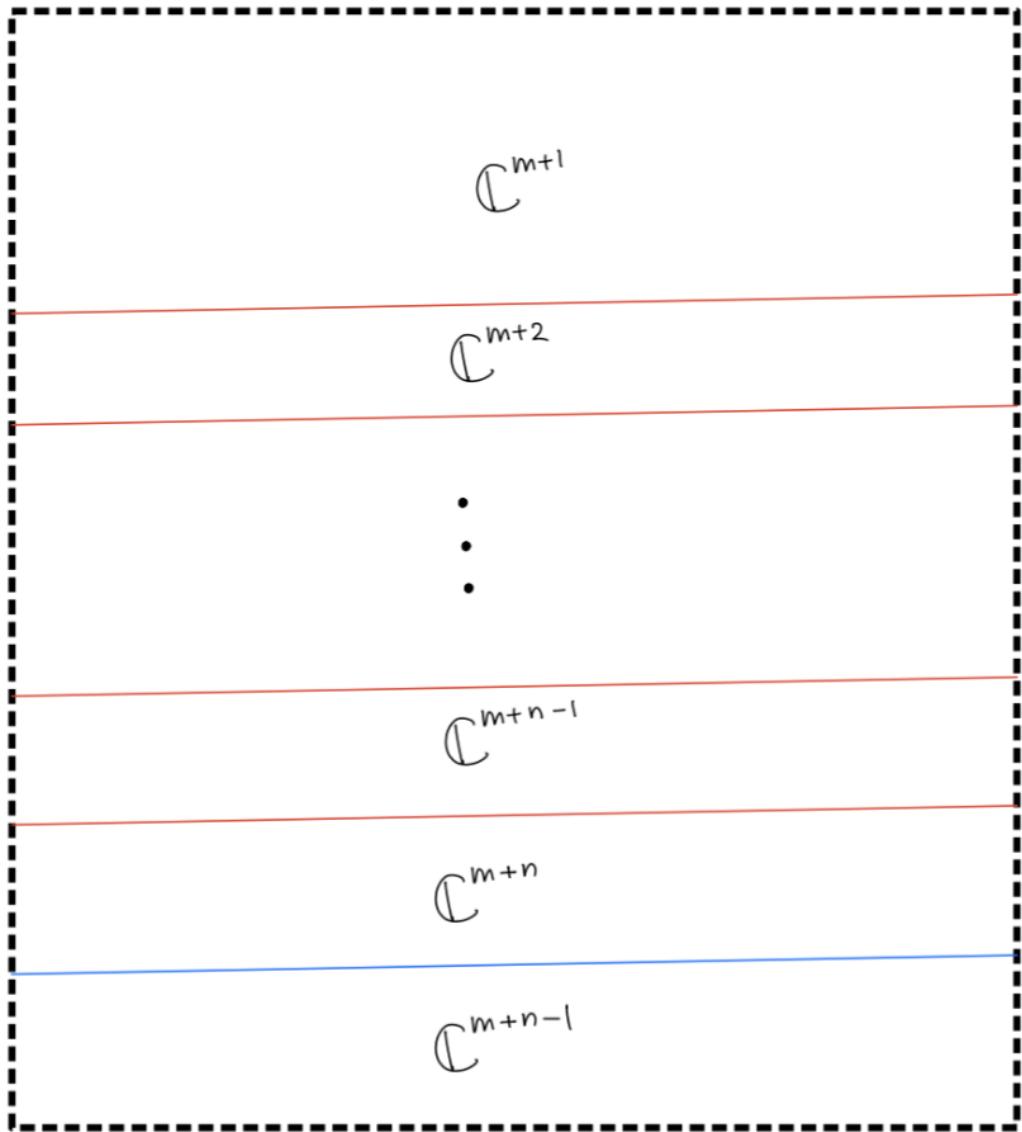


Figure 3.286

We define  $cobord_7(n)$  inductively as follows.

- (i) For  $n = 1$ , we define  $cobord_7(1)$  to be  $cobord_3$  from

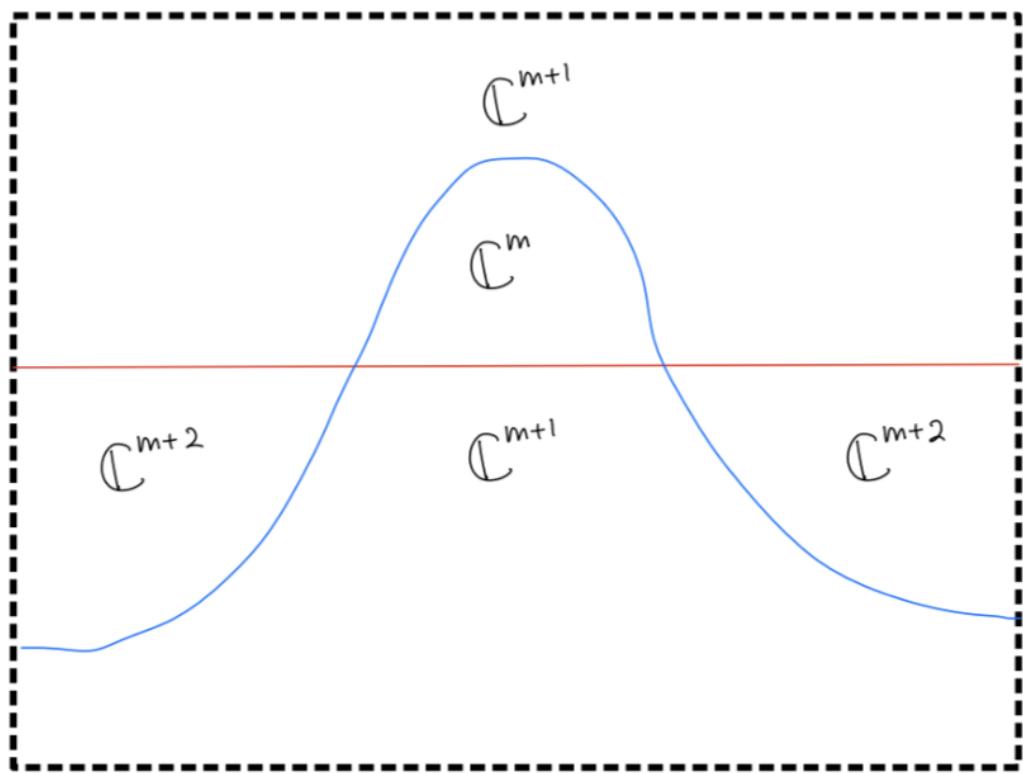


Figure 3.287

to

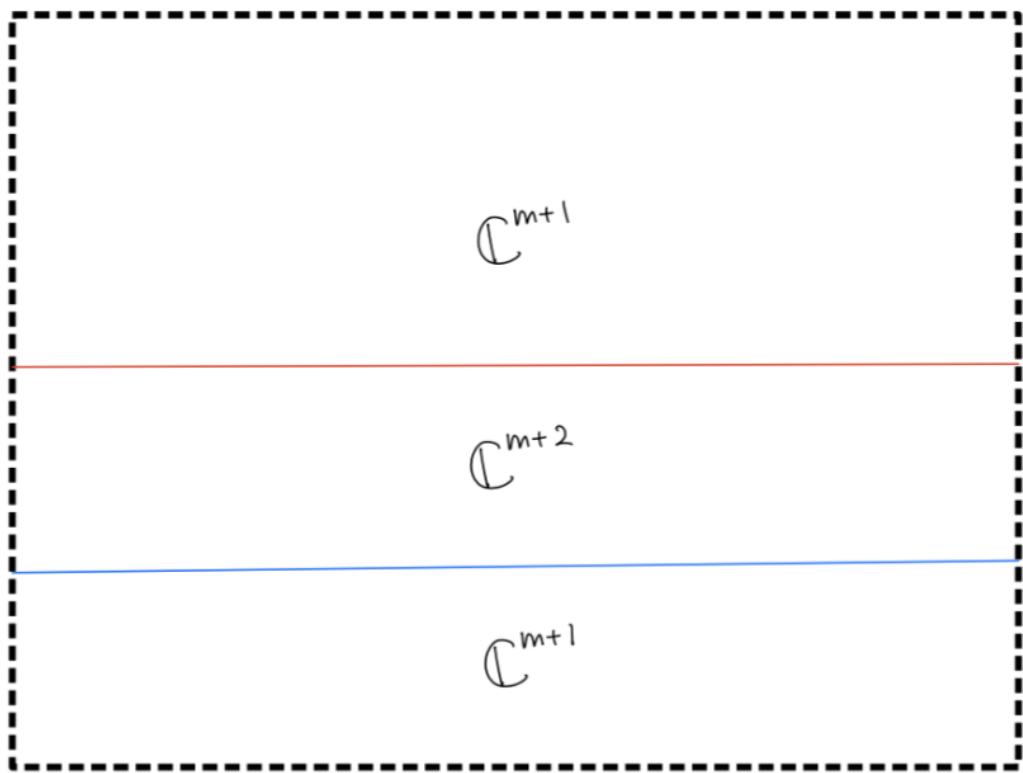


Figure 3.288

(ii) For  $n > 0$ ,

(Step 1) we first apply  $cobord_7(n - 1)$  to the square region surrounded by purple dotted lines.

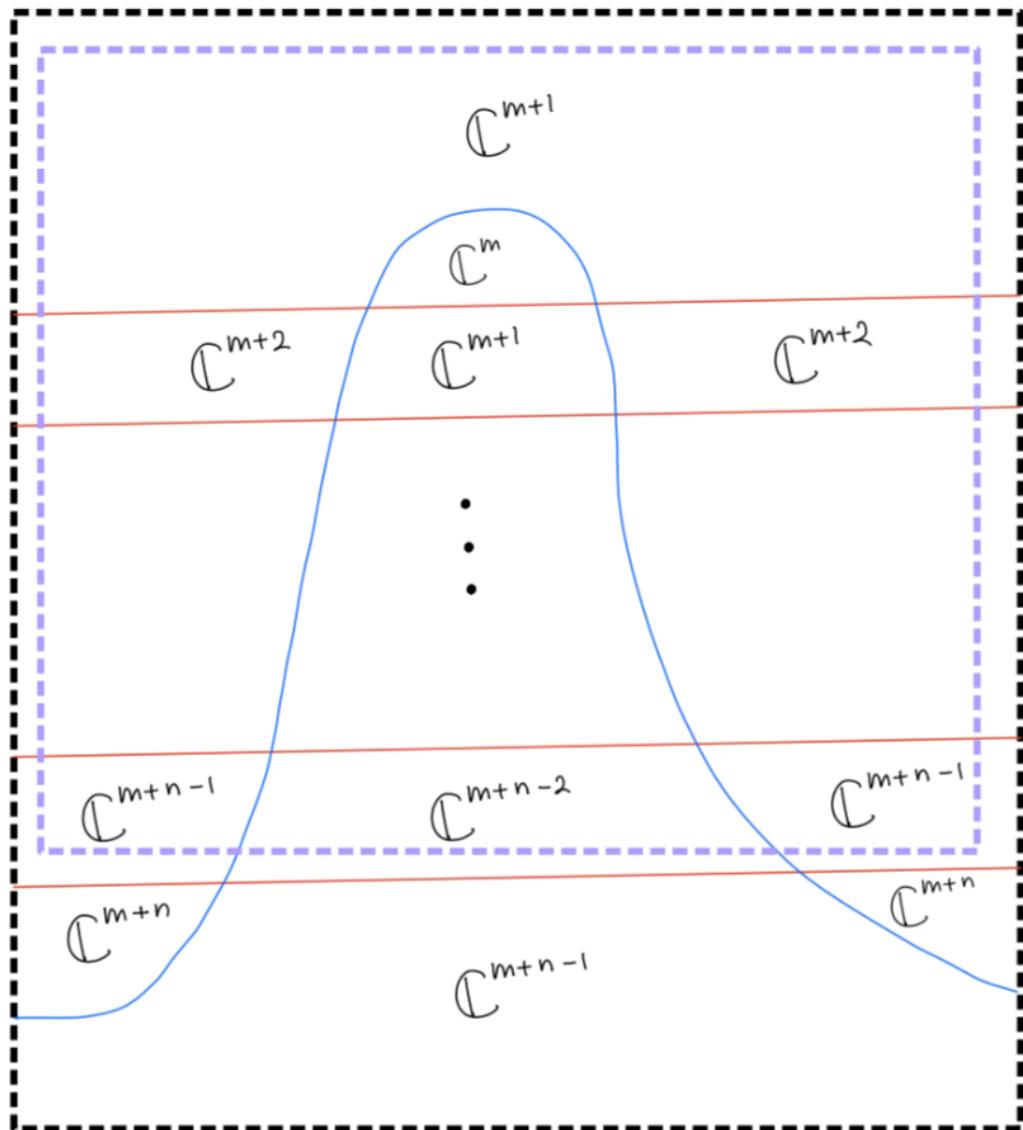


Figure 3.289

by induction hypothesis, we get

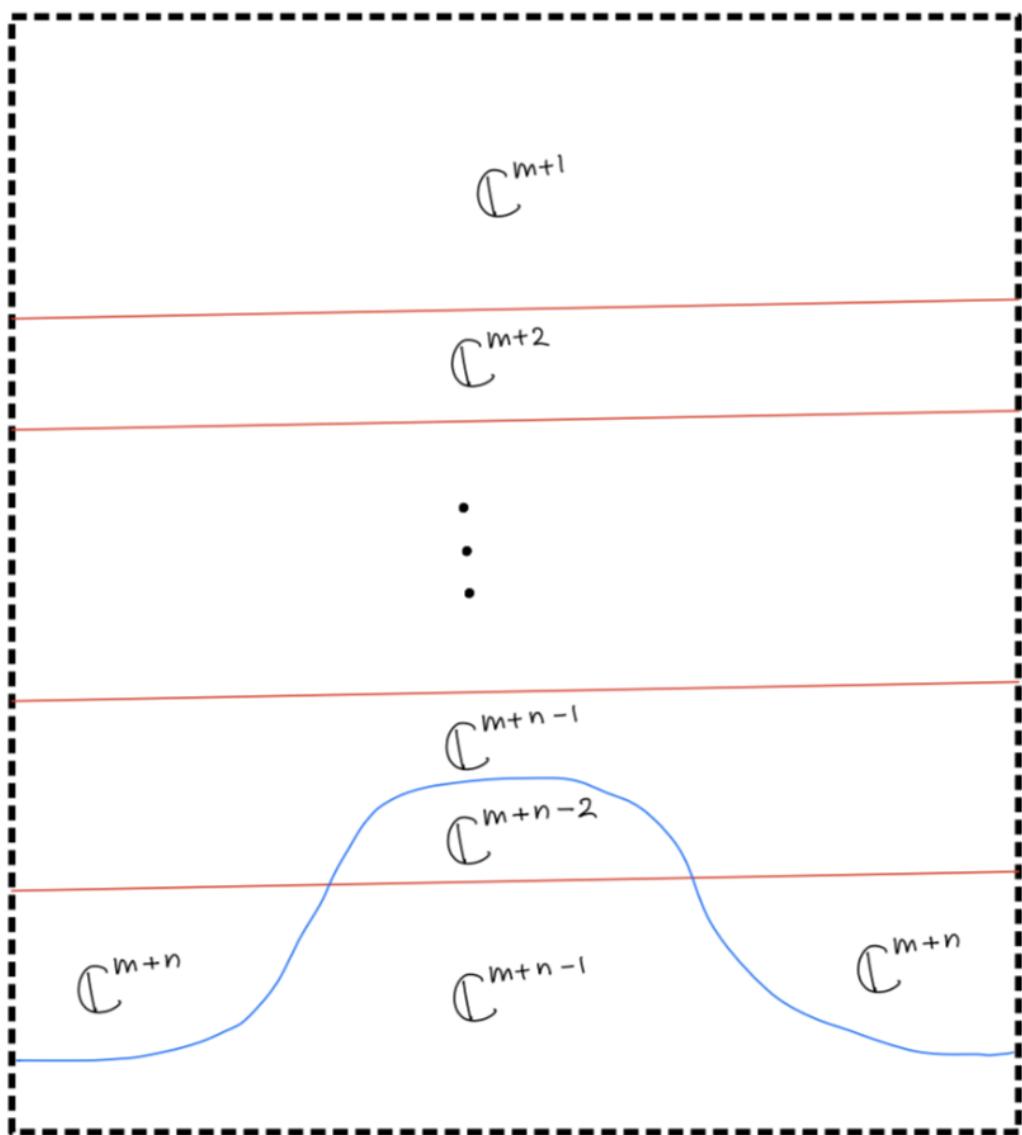


Figure 3.290

(Step 2) apply *cobord*<sub>3</sub> to the square region surrounded by purple dotted lines.

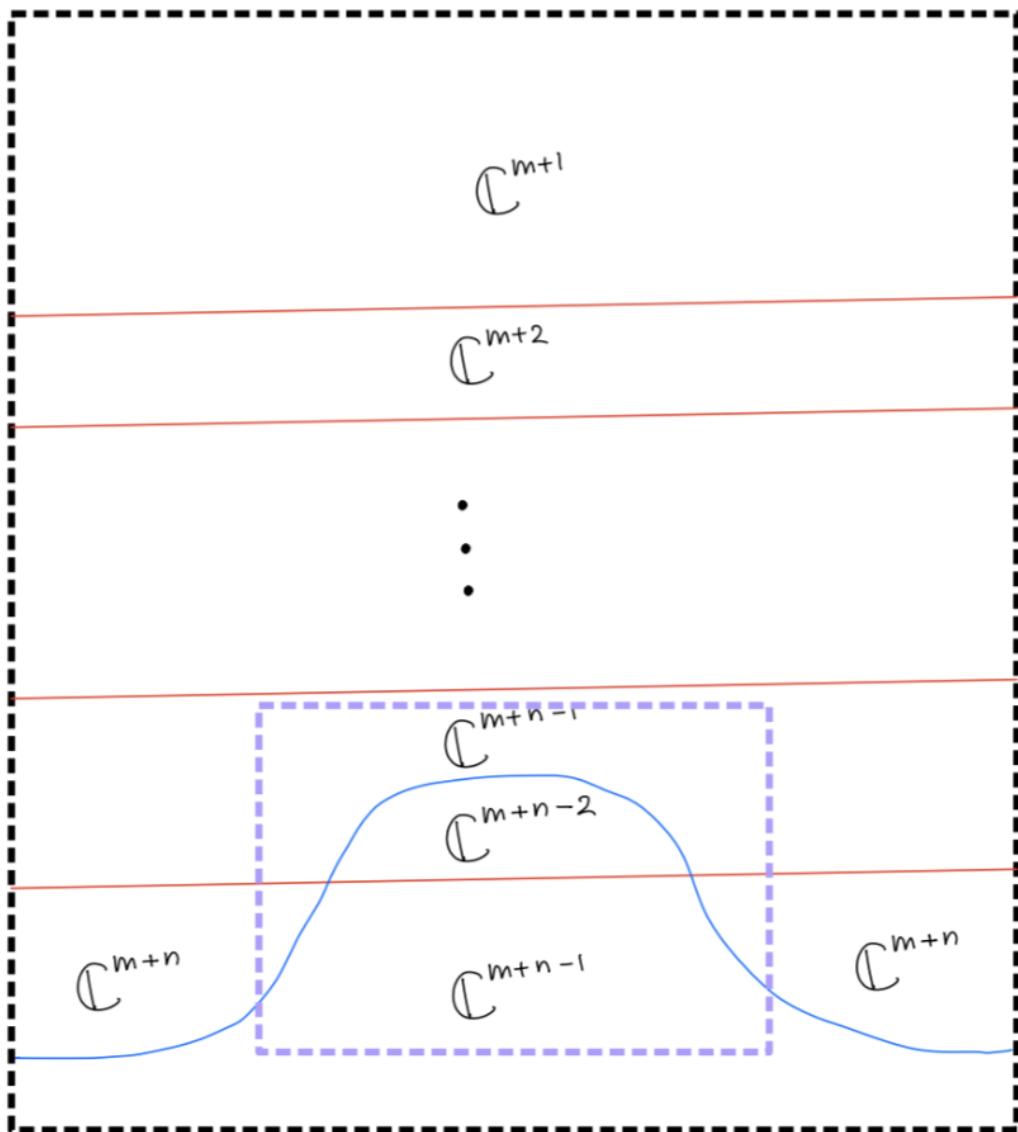


Figure 3.291

we get the final sheaf

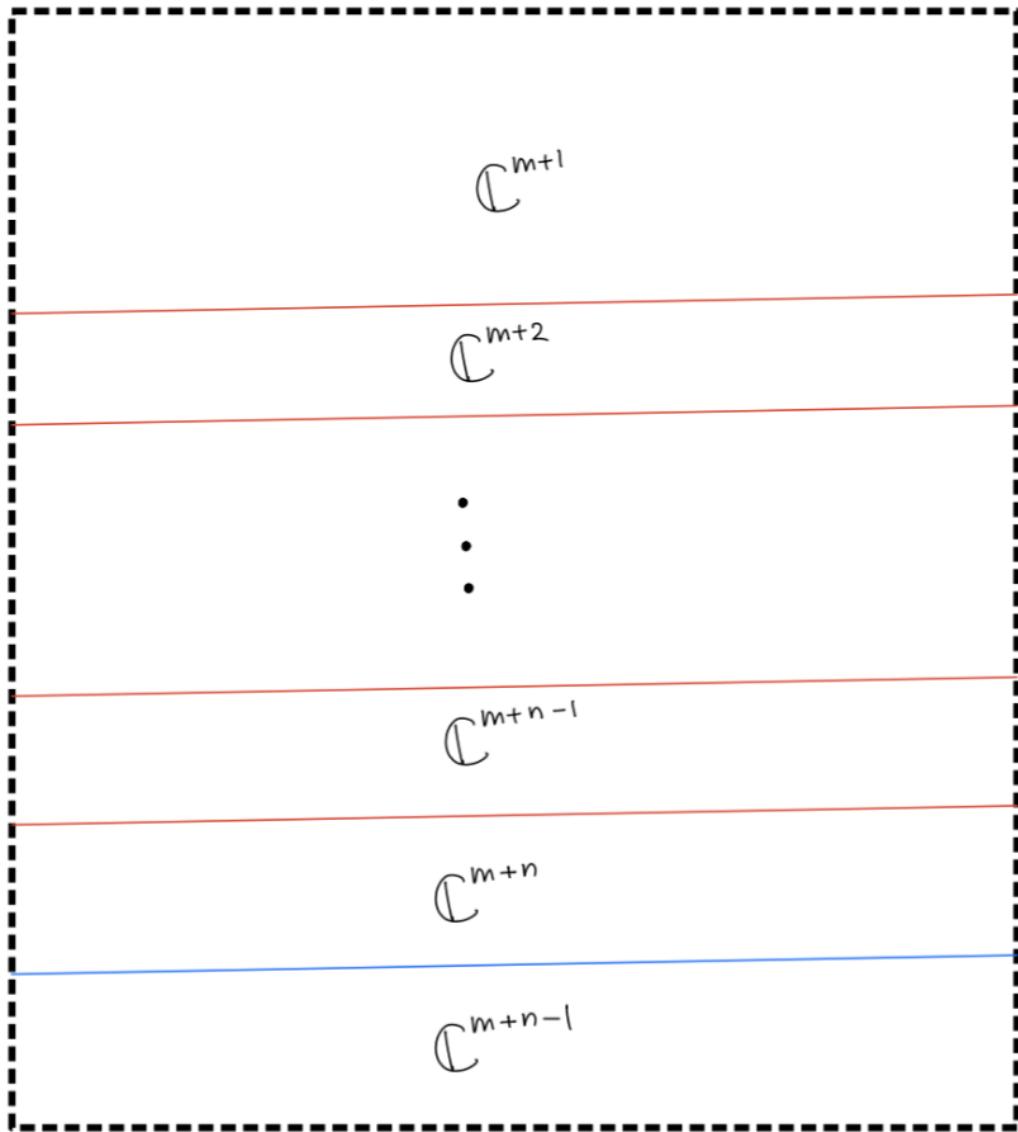


Figure 3.292

### 3.13 8th Sheaf Cobordism

**Definition 204.** (transition function) Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define a smooth transition function  $\tau$  to be

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0$ ,  $\Lambda_0^\infty$ ,  $\Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-2, 2)_x \times (-1, 1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to blue strands in the below figure, co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to red strands in the below figure, co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

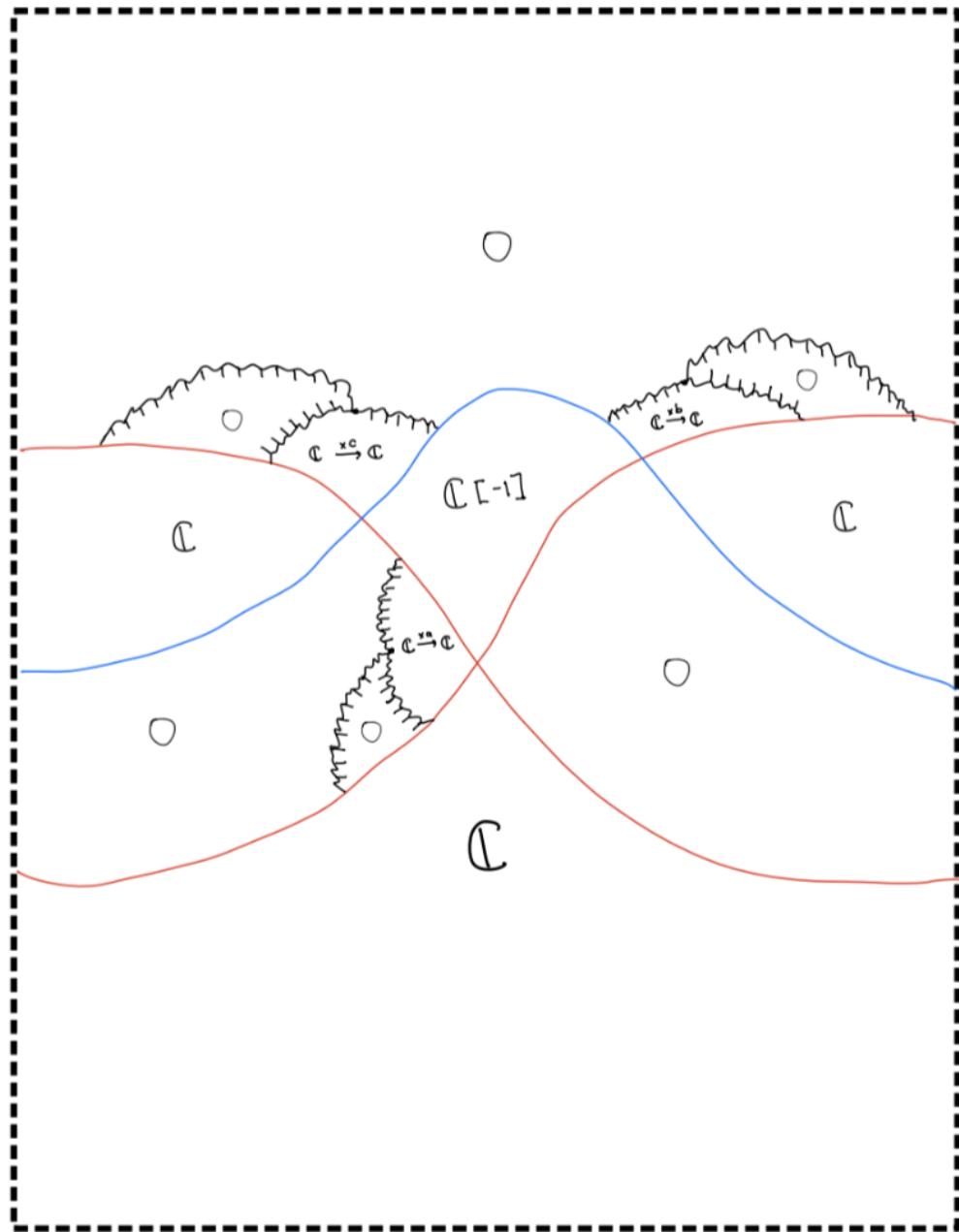


Figure 3.293

Then we define a cobordism starting from the above sheaf, say  $cobord_8$  supported on  $U$ . At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

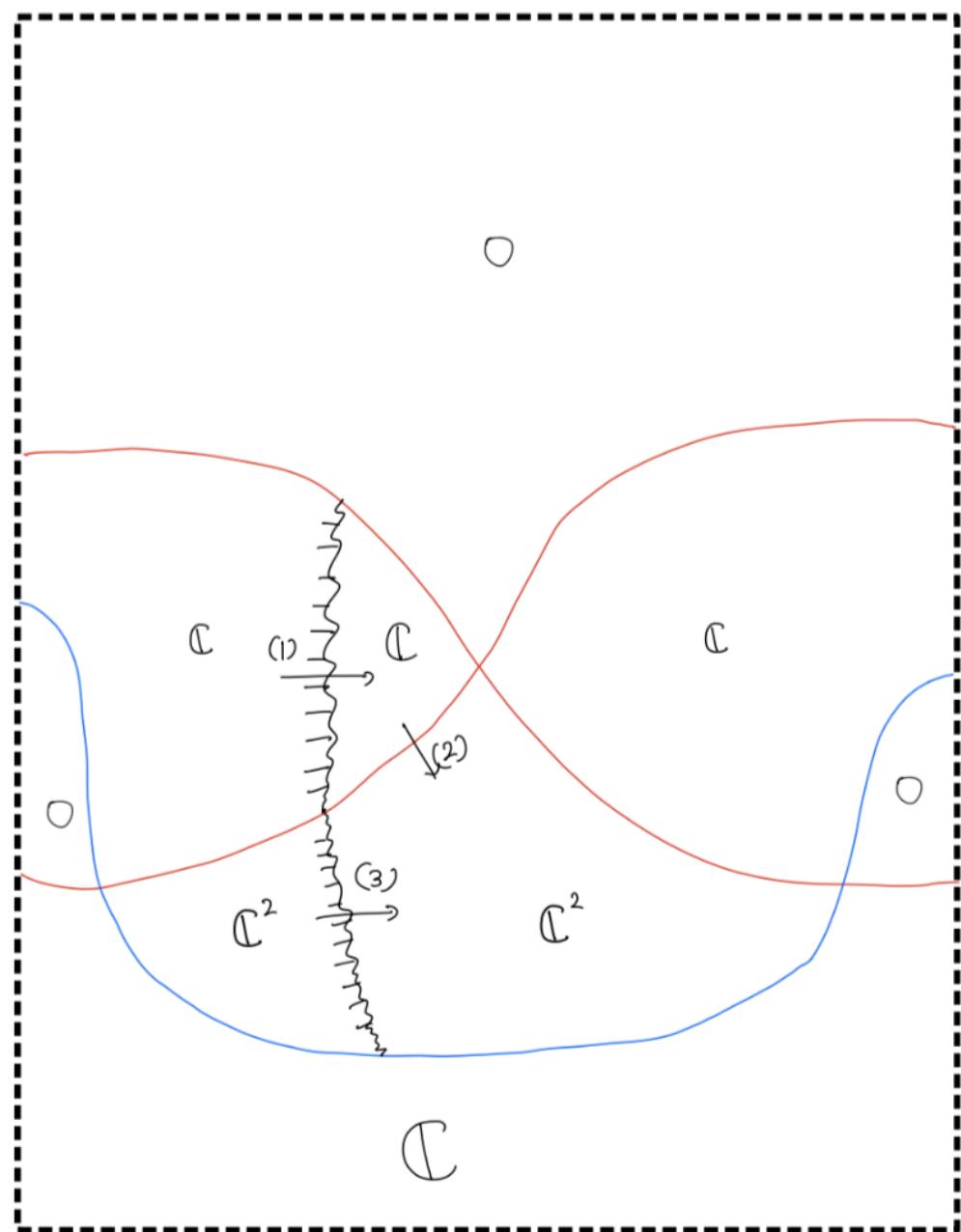


Figure 3.294

**Generalizations maps:**

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

$$(3) \begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$$

We define  $cobord_8$  as follows.

(Step 1) we apply  $cobord_1$  to the square regions surrounded by purple dotted lines.

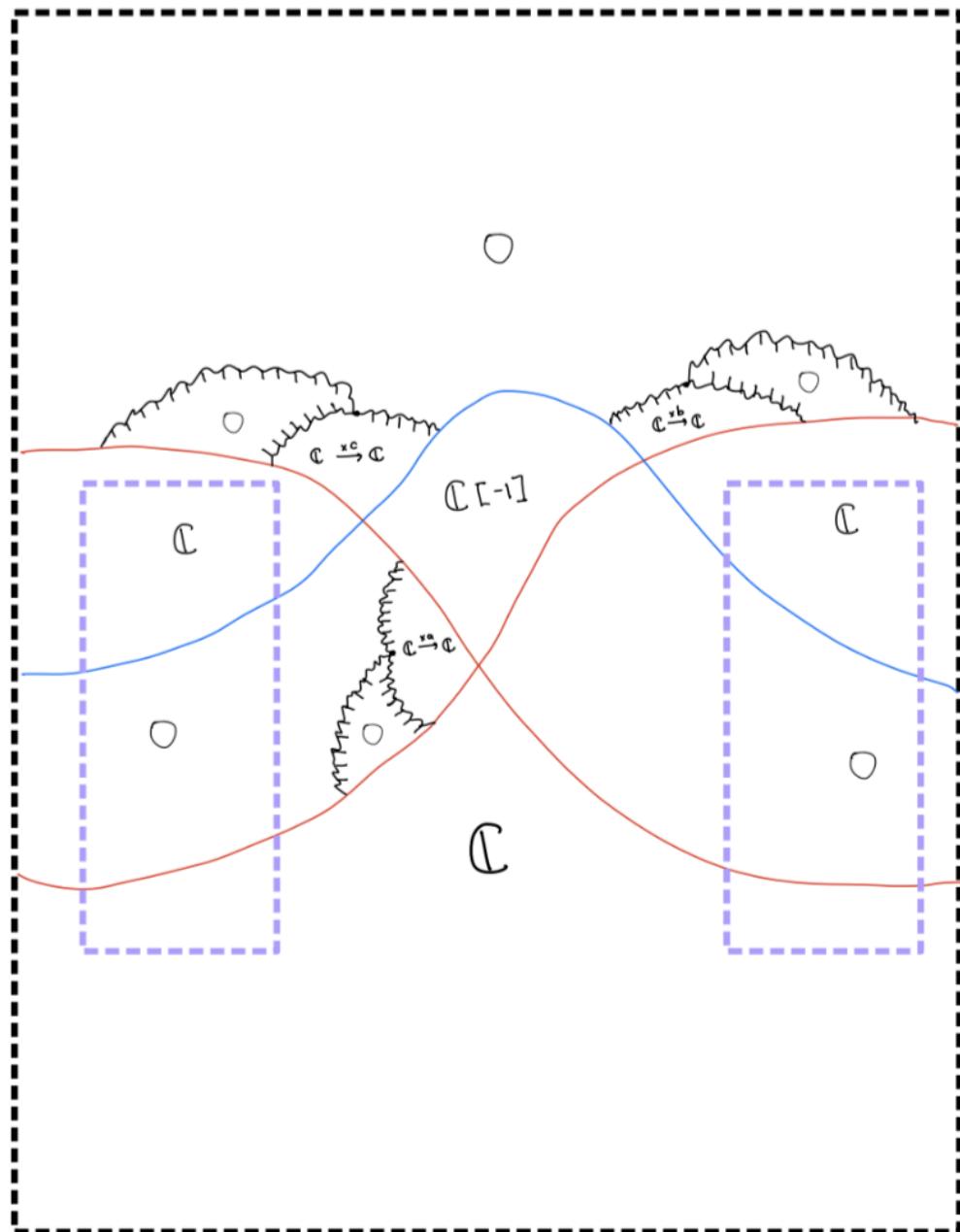


Figure 3.295

we get

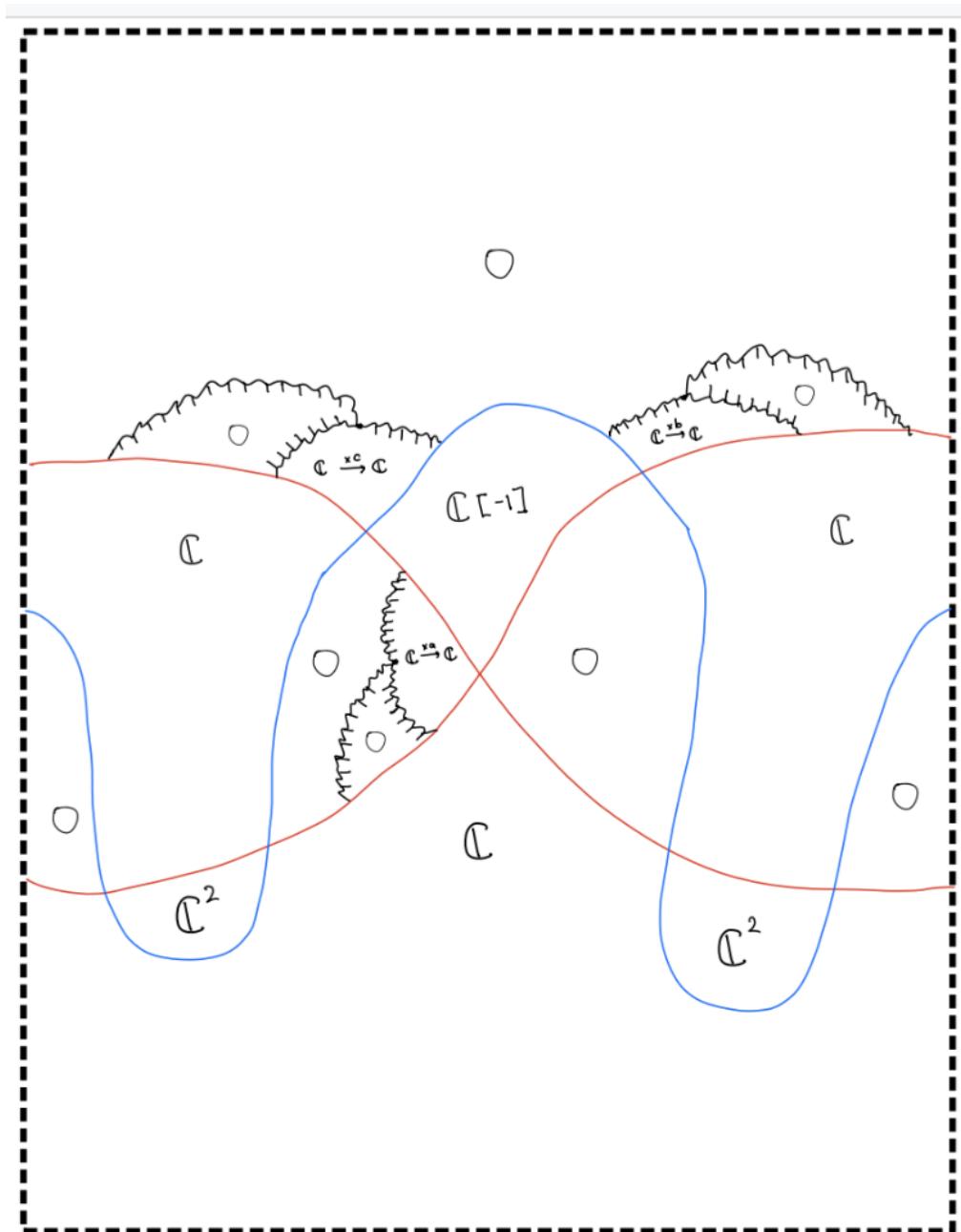


Figure 3.296

(Step 2) apply  $cobord_4$  to the region surrounded by a purple dotted line.

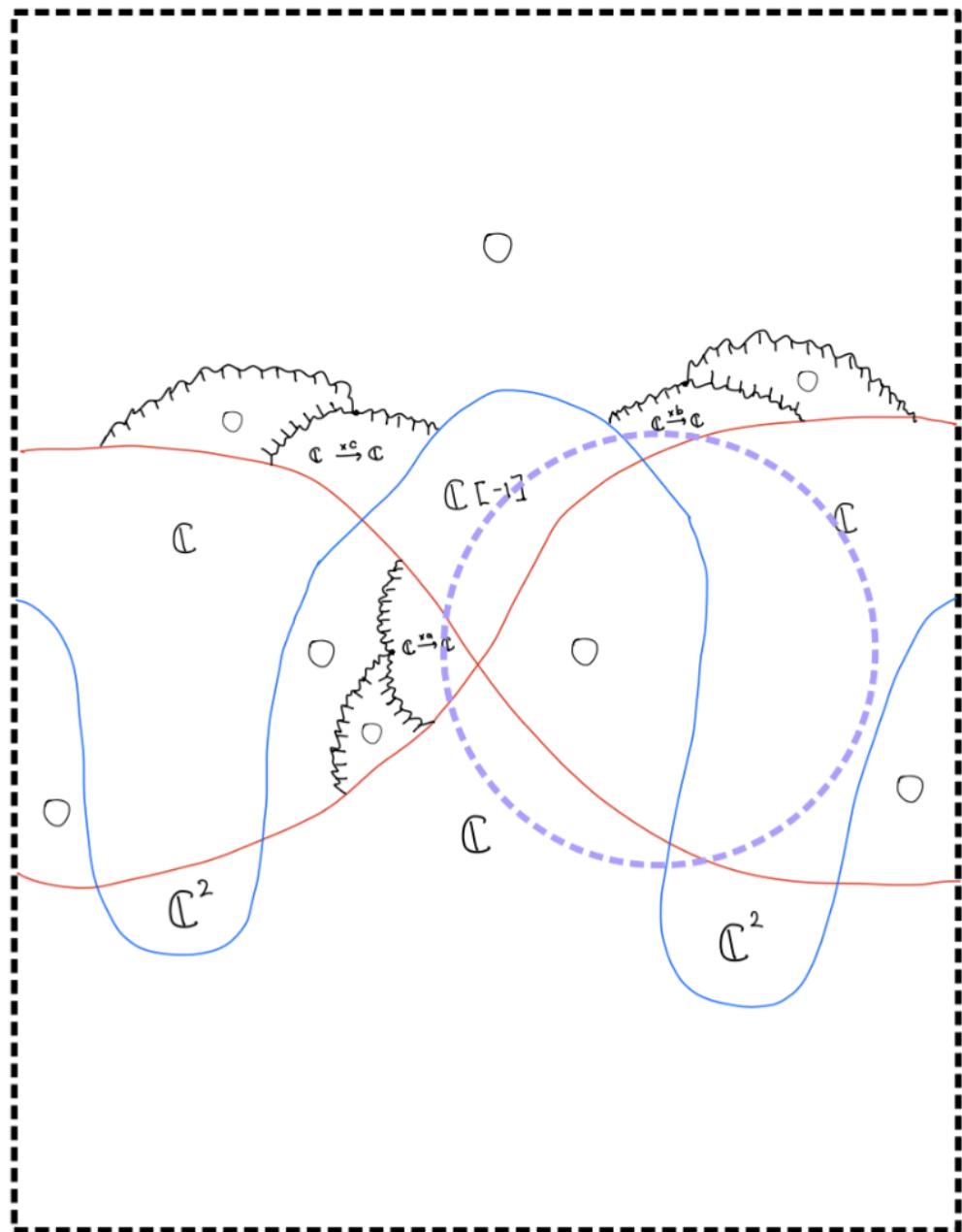


Figure 3.297

we get

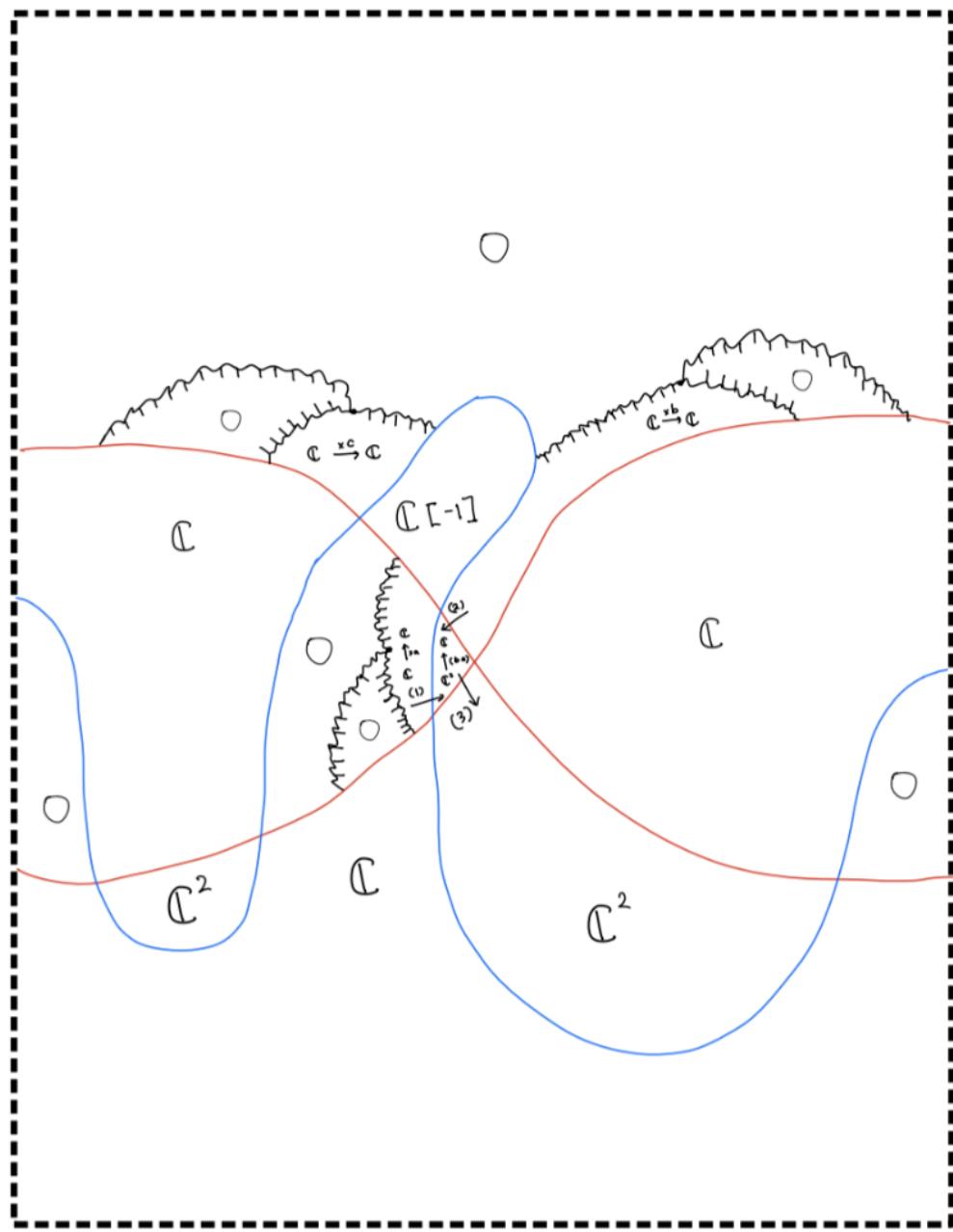


Figure 3.298

**Generalizations maps:**

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

(Step 3) apply  $cobord'_2$  to the square region surrounded by purple dotted lines.

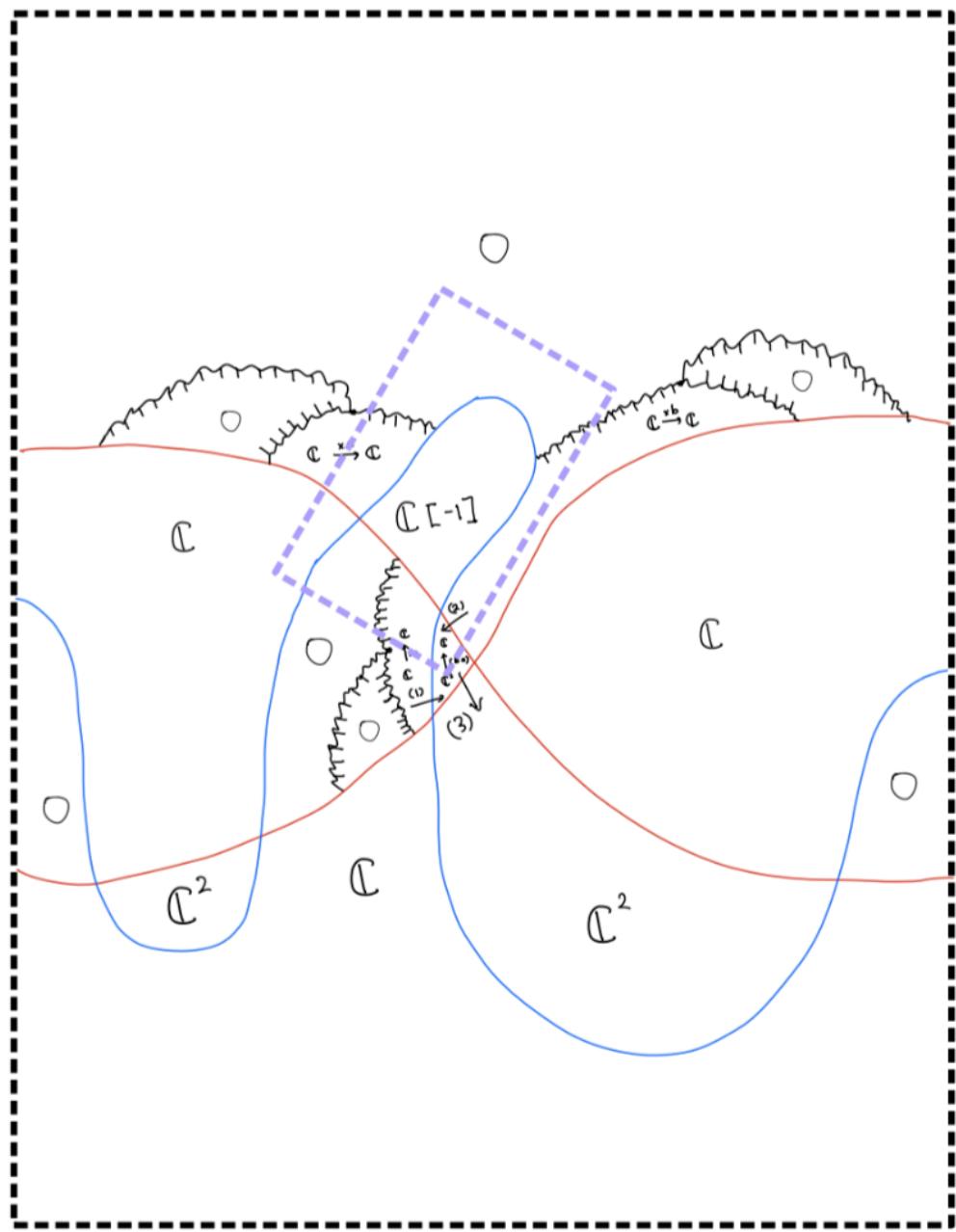


Figure 3.299

we get

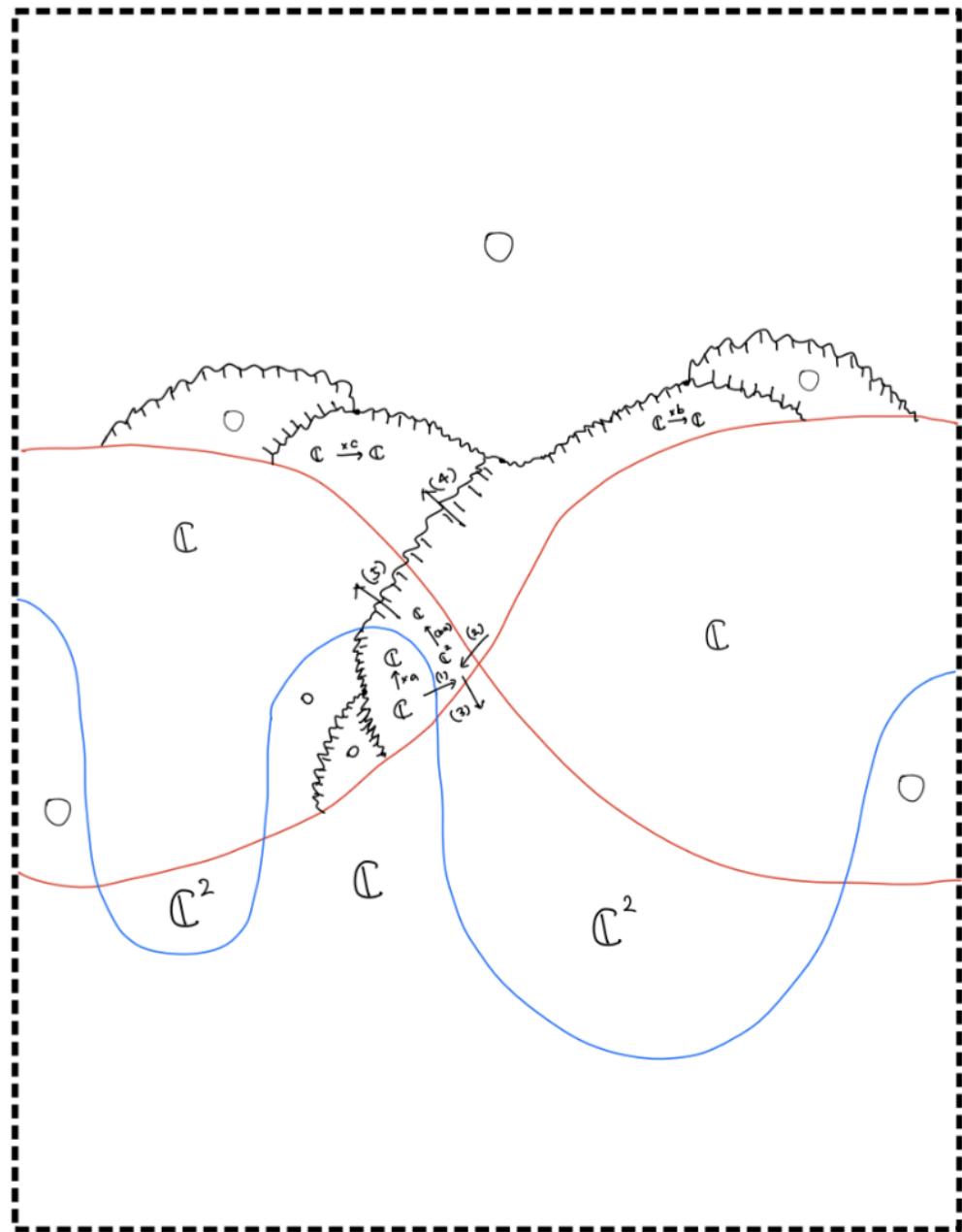


Figure 3.300

**Generalizations maps:**

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a \uparrow & & \uparrow (b \ a) \\ \mathbb{C} & \xrightarrow{\iota_1} & \mathbb{C}^2 \end{array}$$

$$(2) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & (b \ a) \uparrow \\ \mathbb{C} & \xrightarrow{\iota_0} & \mathbb{C}^2 \end{array}$$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C}^2 & \xrightarrow{id} & \mathbb{C}^2 \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b \uparrow & & c \uparrow \\ \mathbb{C} & \xrightarrow{\times bc^{-1}} & \mathbb{C} \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$$

After identifying

- $\mathbb{C}$  with  $\mathbb{C}^2 \xrightarrow{(b \ a)} \mathbb{C}$  via  $\begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ (b \ a) \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow{(bc^{-1} \ 0)} & \mathbb{C} \end{array}$
- $\mathbb{C} \xrightarrow{\times a} \mathbb{C}$ ,  $\mathbb{C} \xrightarrow{\times b} \mathbb{C}$ ,  $\mathbb{C} \xrightarrow{\times c} \mathbb{C}$  with 0

the above sheaf is quasi-isomorphic to

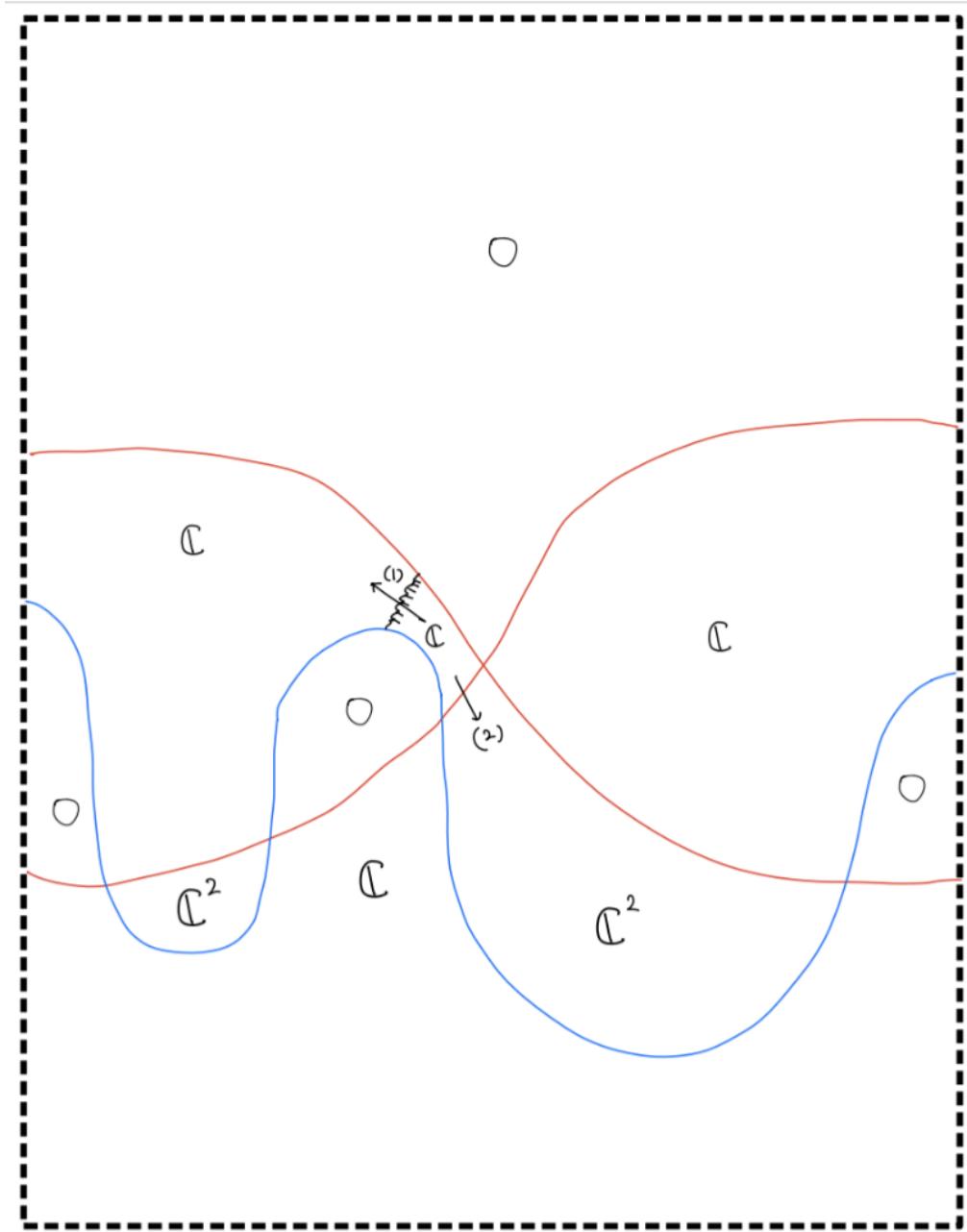


Figure 3.301

**Generalizations maps:**

$$(1) \times bc^{-1}$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

which is quasi-isomorphic to

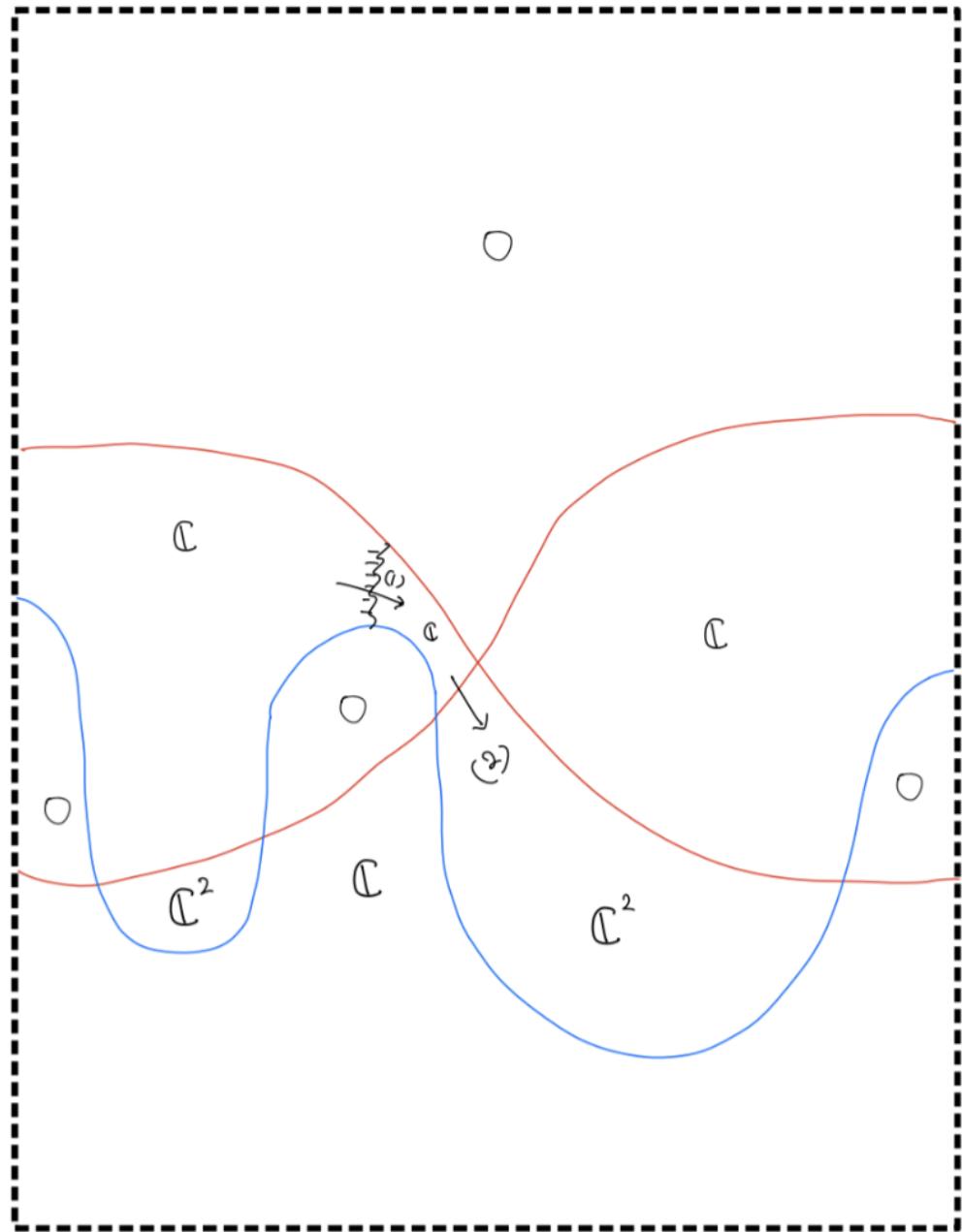


Figure 3.302

**Generalizations maps:**

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

(Step 4) apply  $cobord_3$  to the region surrounded by a purple dotted line

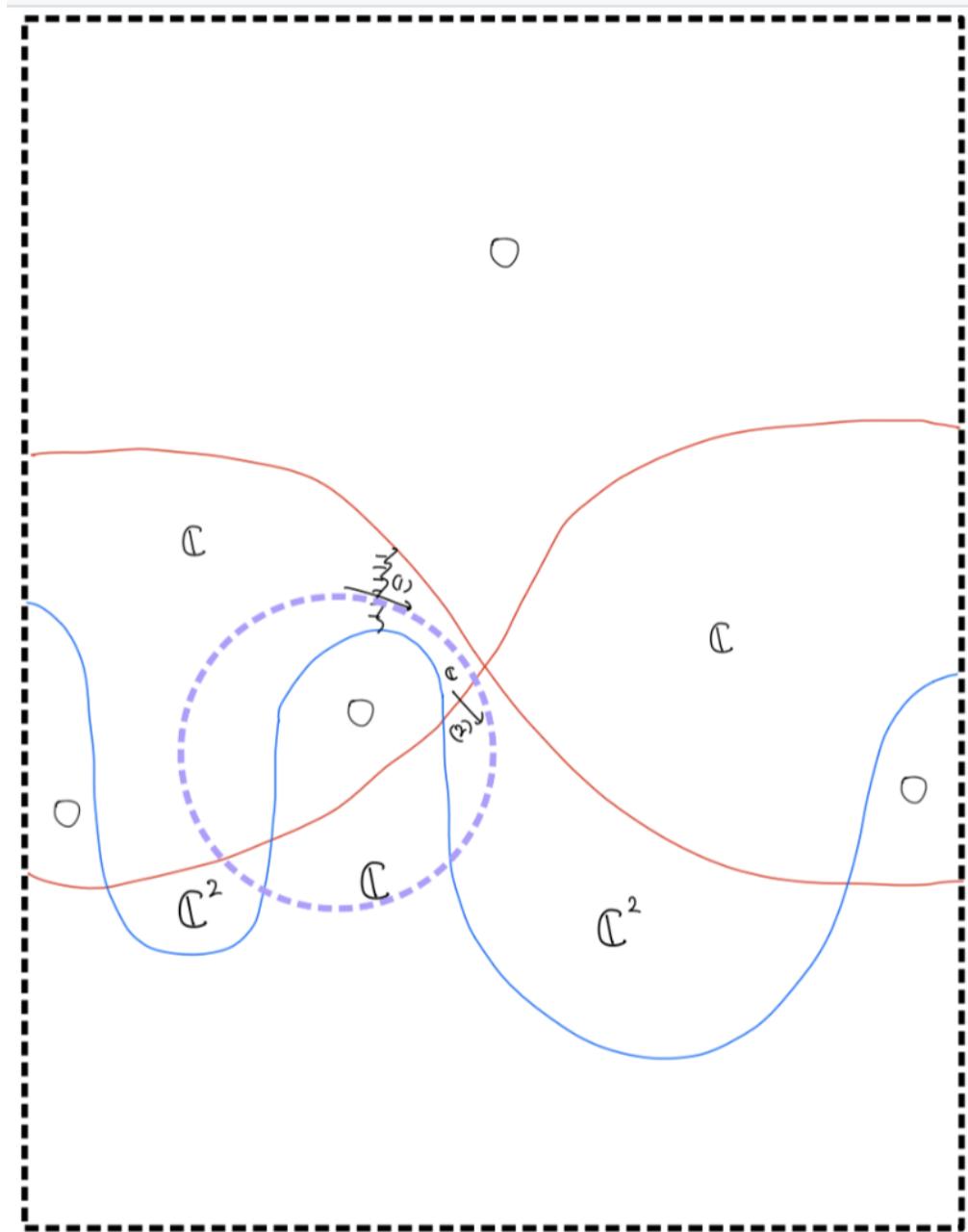


Figure 3.303

we get

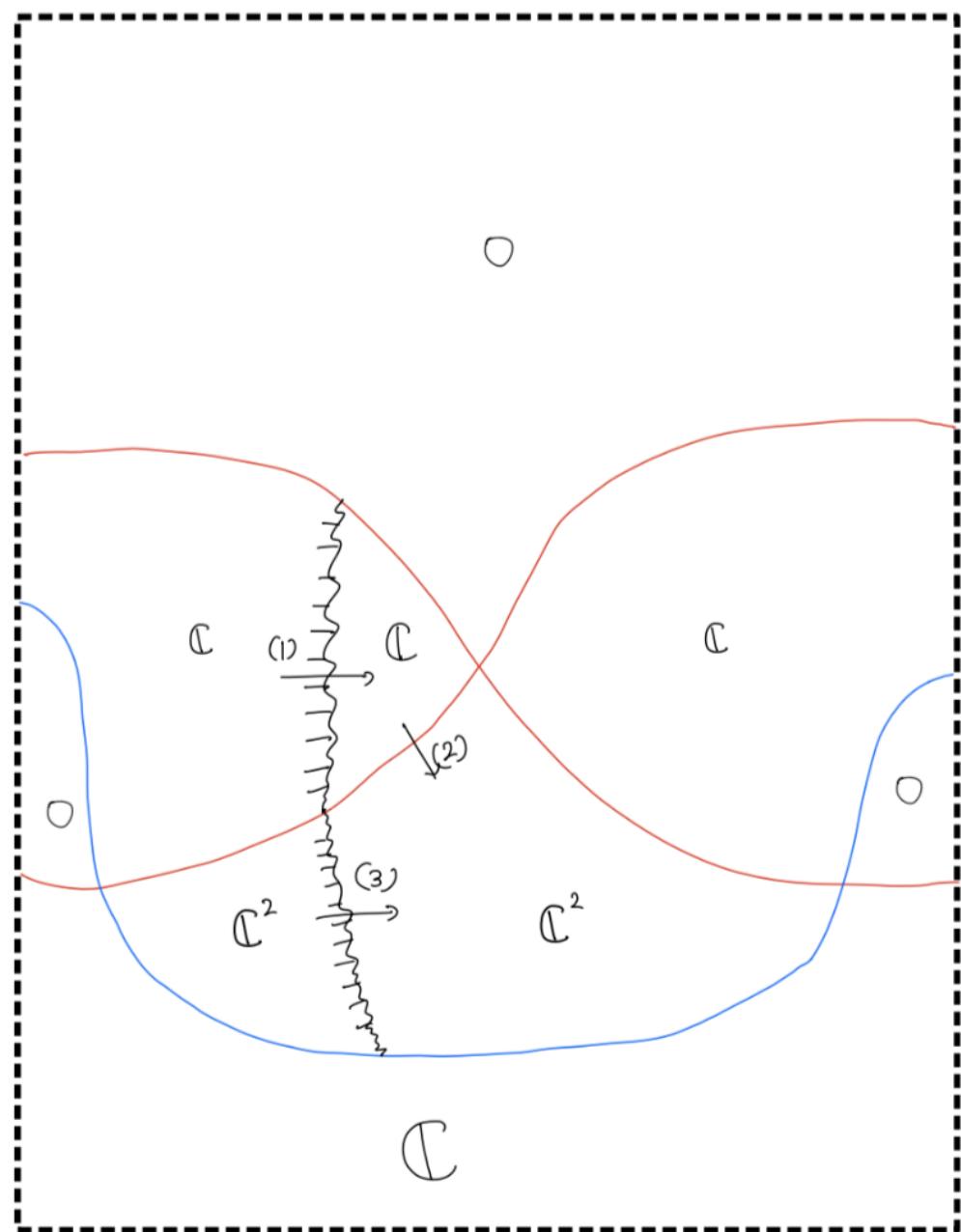


Figure 3.304

**Generalizations maps:**

$$(1) \times b^{-1}c$$

$$(2) \begin{pmatrix} 1 \\ -a^{-1}b \end{pmatrix}$$

$$(3) \begin{pmatrix} b^{-1}c & 0 \\ -a^{-1}c & 1 \end{pmatrix}$$

## 3.14 8th Sheaf Cobordism'

**Definition 205.** Smooth transition function consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0$ ,  $\Lambda_0^\infty$ ,  $\Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-2, 2)_x \times (-1, 1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to  $\{(x, z) \in R \mid z = \Psi(z_{lo} = 0, z_{hi} = 1)(x)\}$ , co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to  $\{(x, z) \in R \mid z = \tau(\frac{x+1}{2}) - \frac{1}{2}\} \cup \{(x, z) \in R \mid z = -\tau(\frac{x+1}{2}) + \frac{1}{2}\}$ , co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to  $\{(x, z) \in R \mid x = -c - \epsilon\}$  where  $c > 0$ ,  $\Psi(0, 1)(c) = \tau(\frac{c+1}{2}) - \frac{1}{2}$  and  $\epsilon > 0$  is a small number such that  $c > \epsilon$ .

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

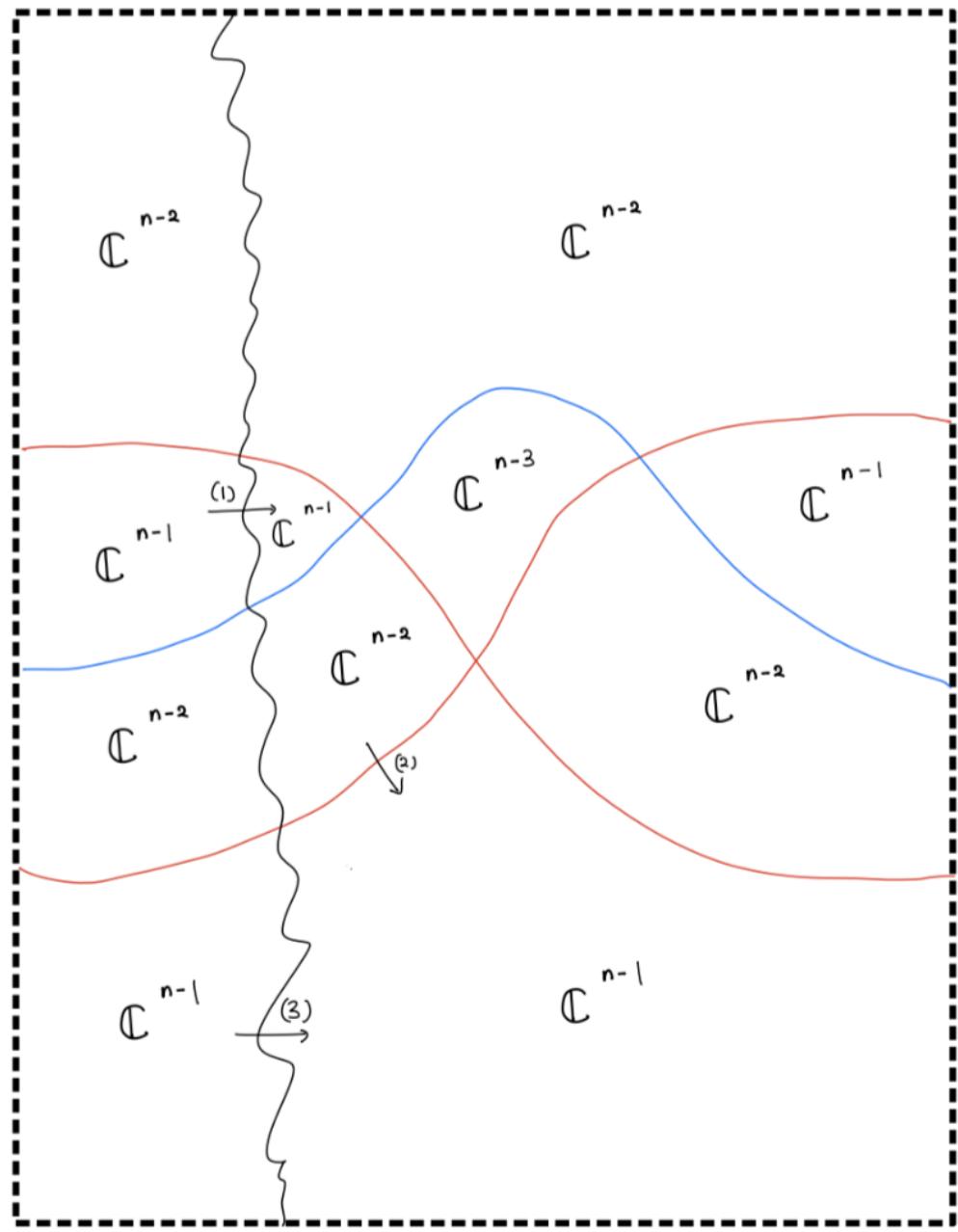


Figure 3.305

**Generalizations maps:**

- (1)  $\text{diag}(d_n, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1, n-2}$
- (3)  $\text{diag}(d_{n-1}, \dots, d_1) + eI_{n-1, n-2}$

where  $e' = d_2^{-1}e$ . Then we define a cobordism starting from the above sheaf, say  $cobord'_8$  supported on  $U$ . At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

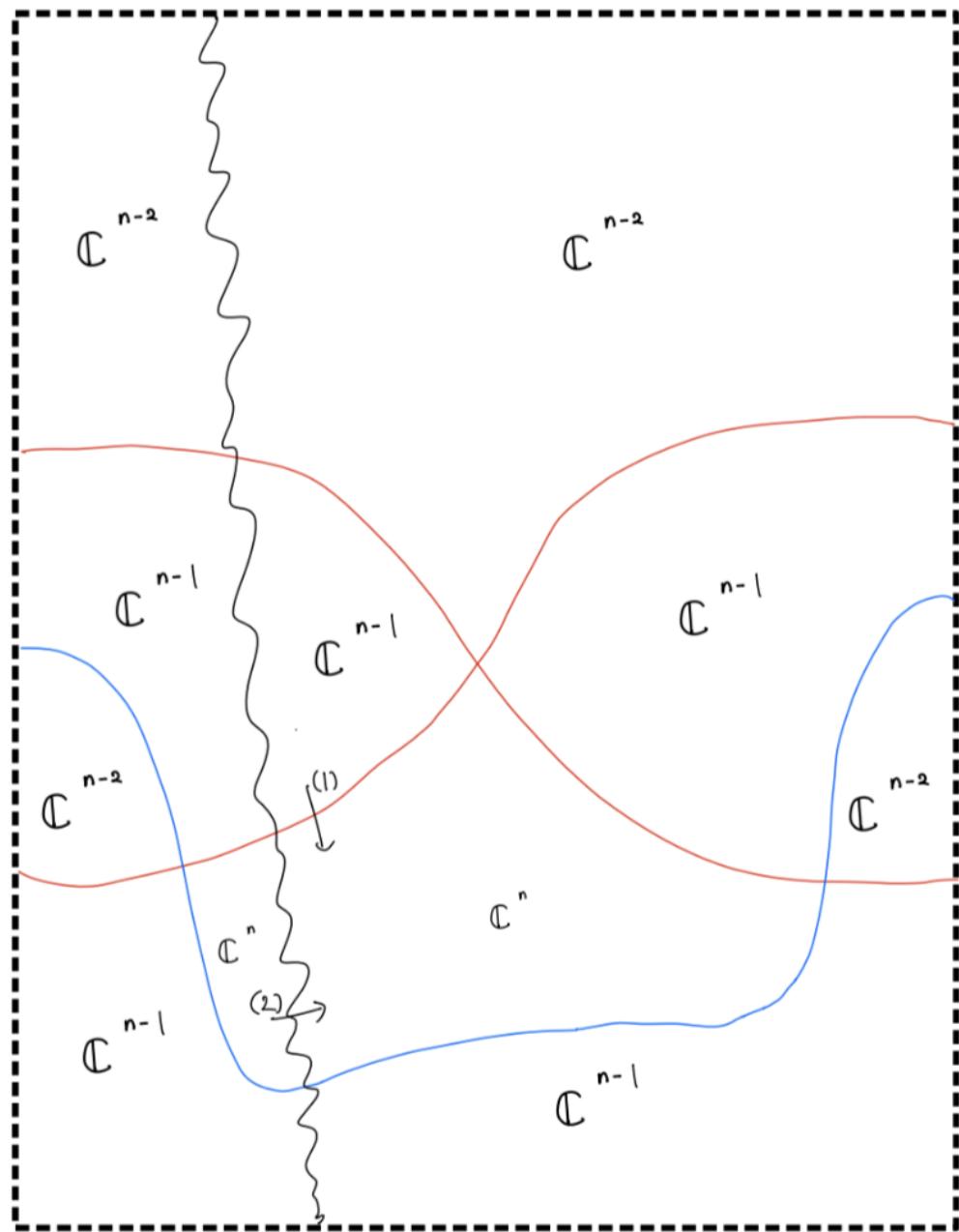


Figure 3.306

**Generalizations maps:**

$$(1) \quad \iota_0 \circ diag(1, \dots, 1) + e'I_{n,n-1}$$

$$(2) \ diag(d_n, \dots, d_1) + eI_{n,n-1}$$

We define  $cobord'_8$  as follows.

(Step 1) we apply  $cobord_1$  to the square regions surrounded by purple dotted lines.

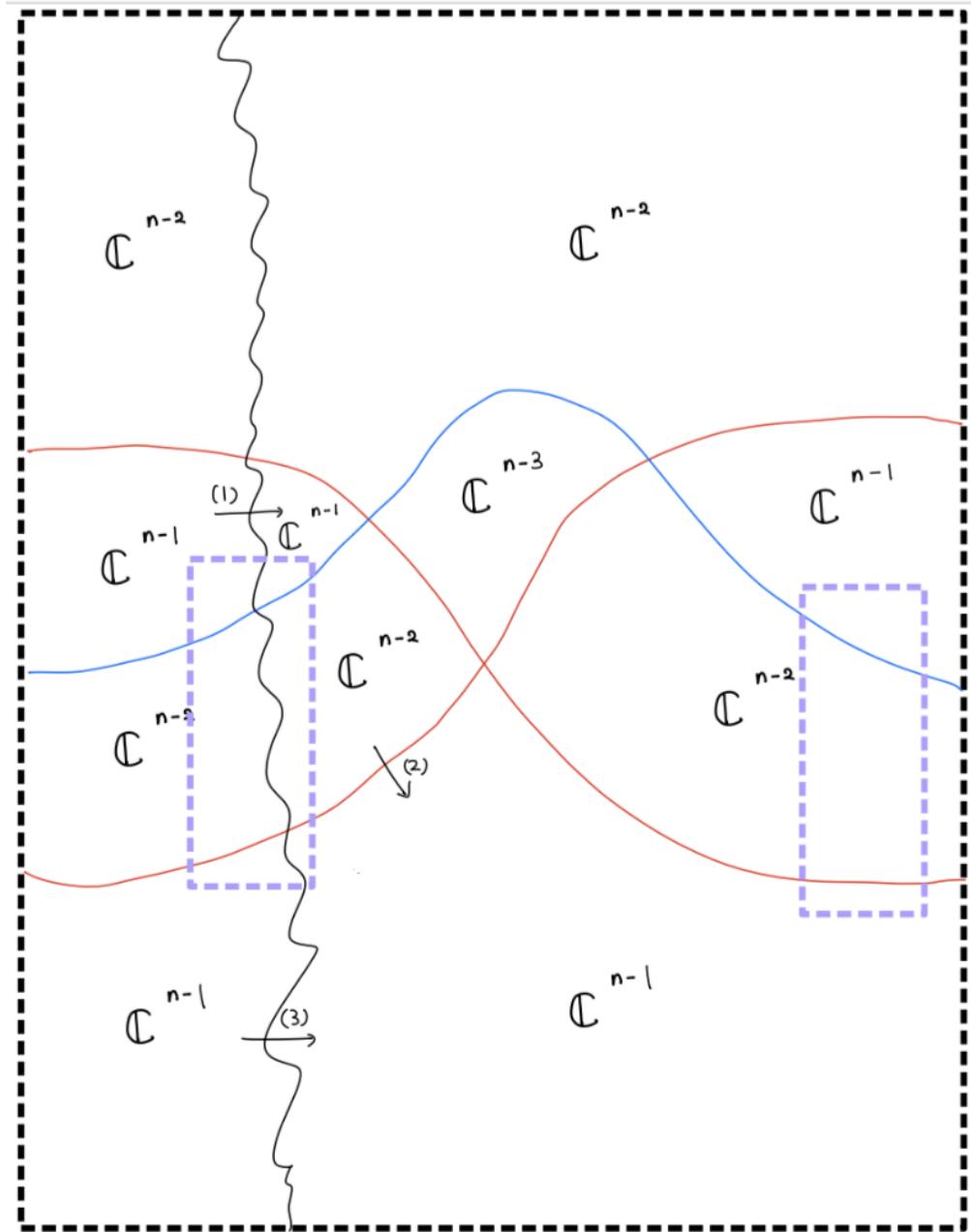


Figure 3.307

we get

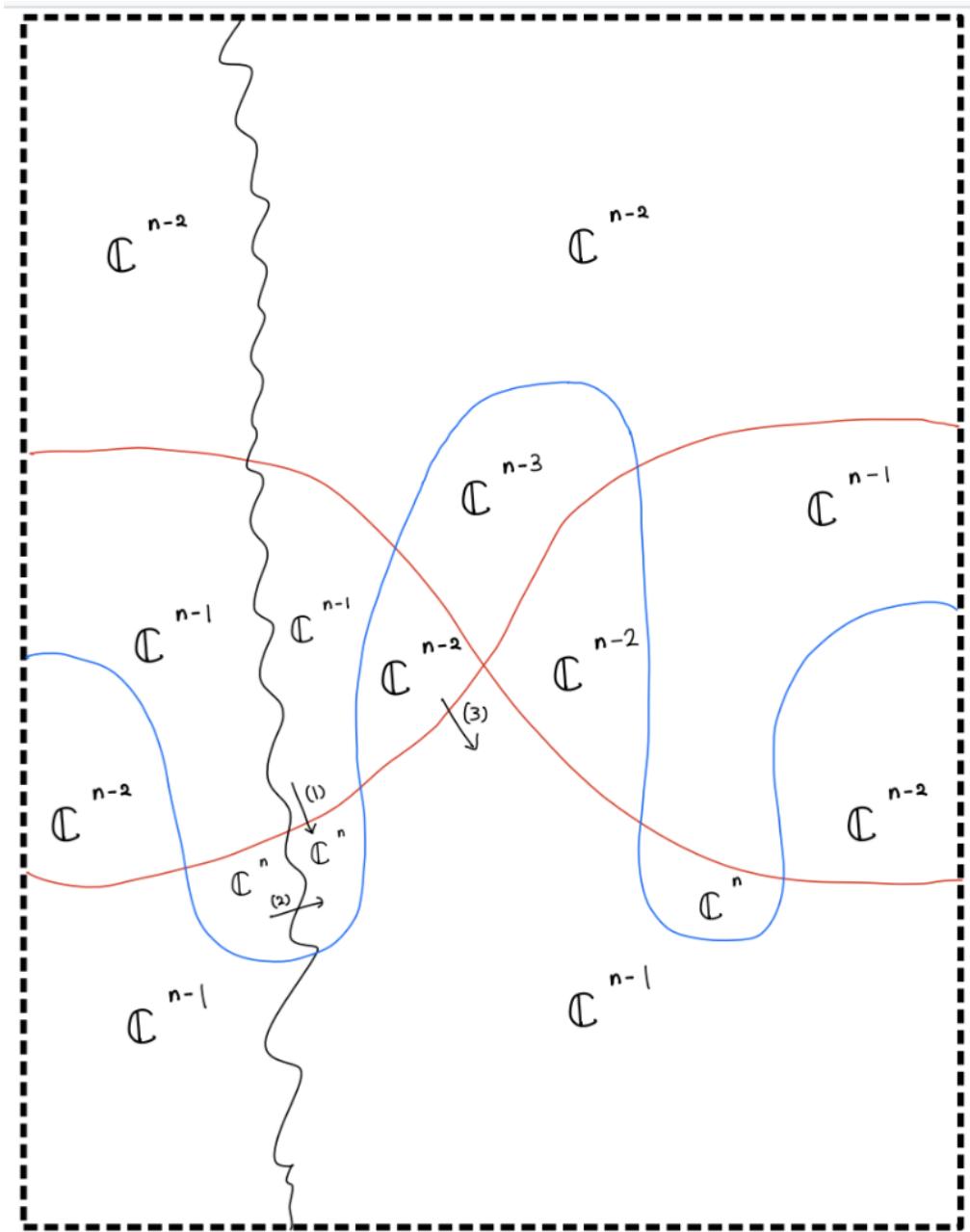


Figure 3.308

**Generalizations maps:**

- (1)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$
- (2)  $\text{diag}(d_n, \dots, d_1) + e I_{n,n-1}$
- (3)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1,n-2}$

(Step 2) apply  $cobord'_4$  to the region surrounded by a purple dotted line.

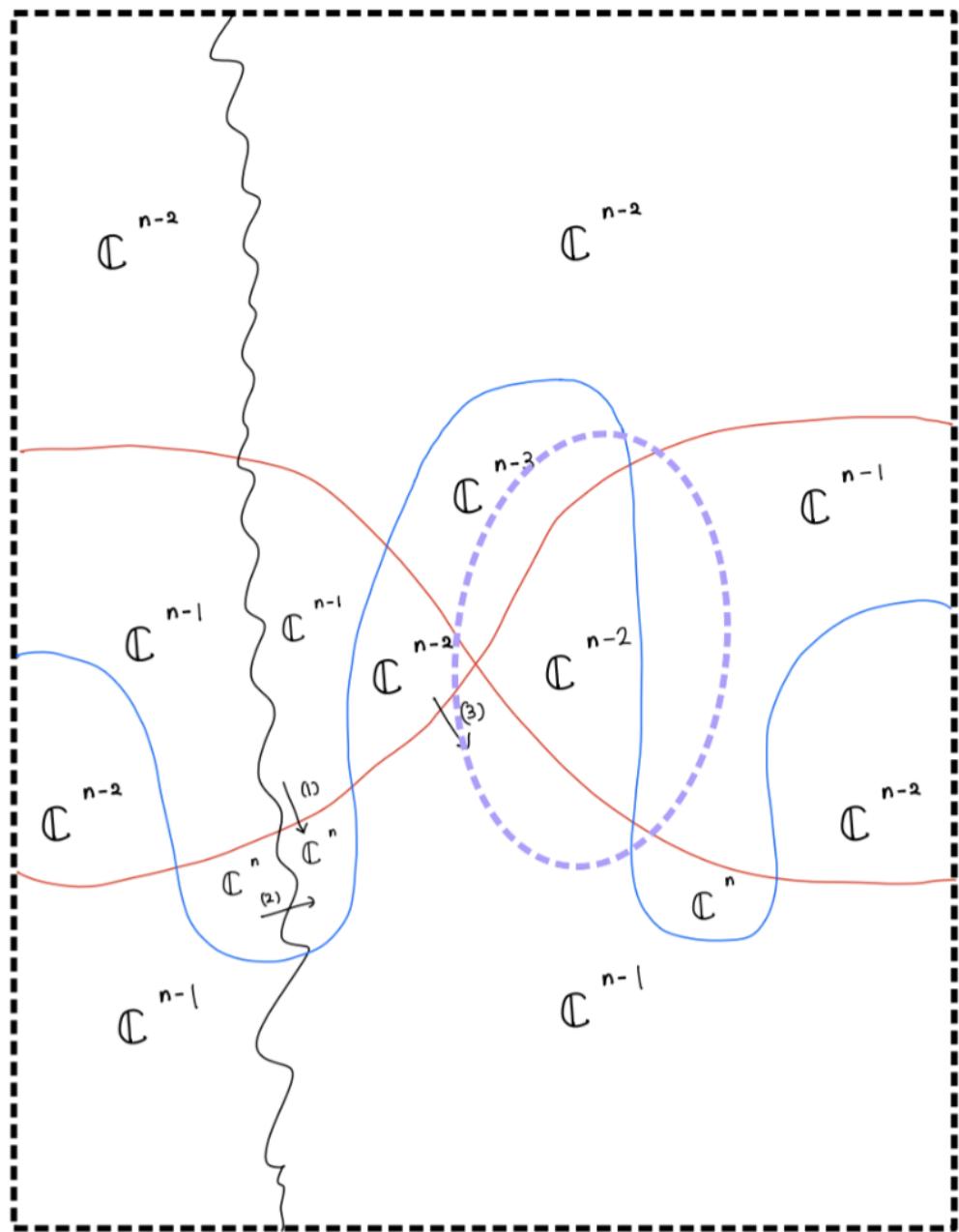


Figure 3.309

we get

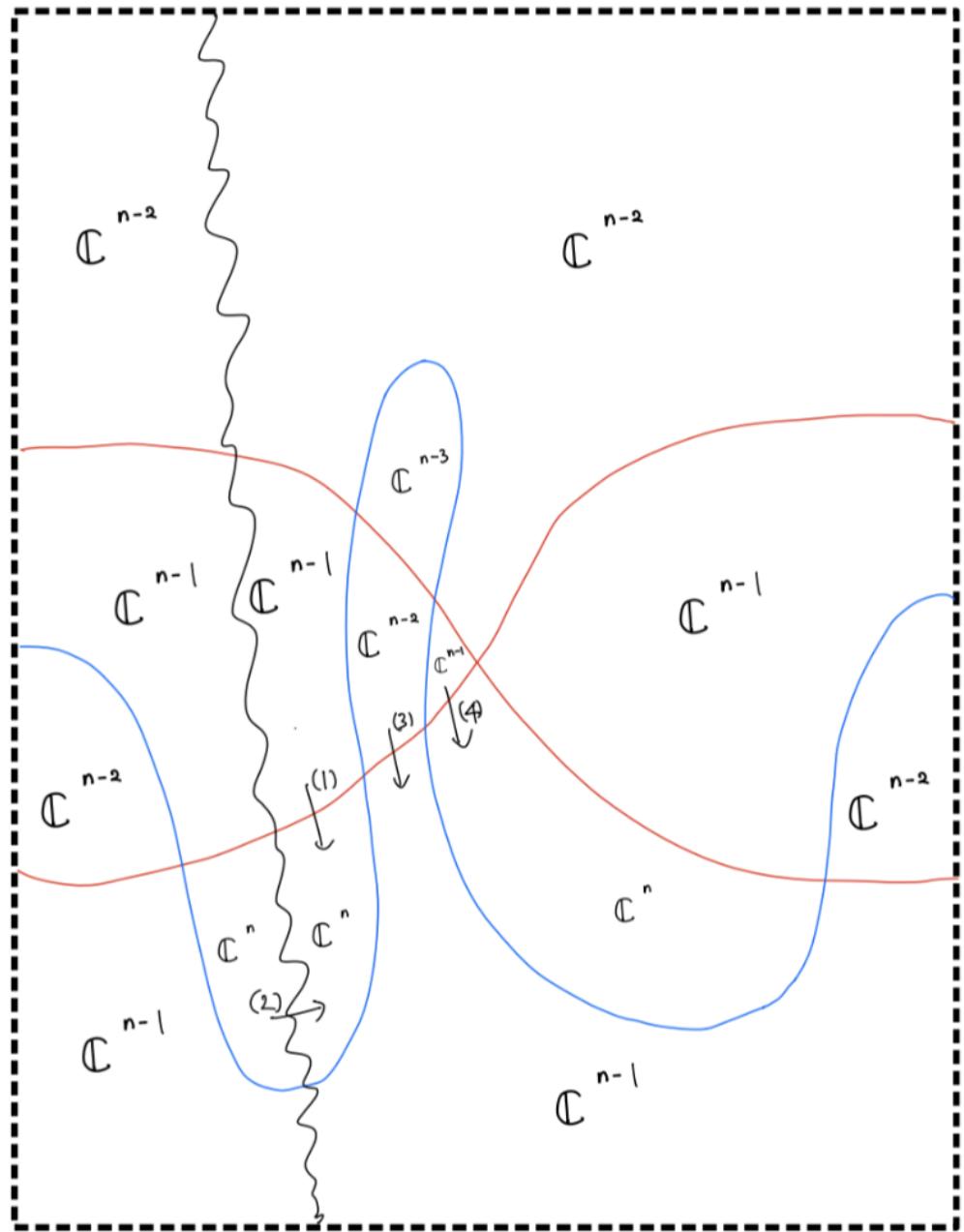


Figure 3.310

**Generalizations maps:**

- (1)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$
- (2)  $\text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$
- (3)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1,n-2}$

$$(4) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$$

(Step 3) apply *cobord*<sub>3</sub> to the region surrounded by a purple dotted line.

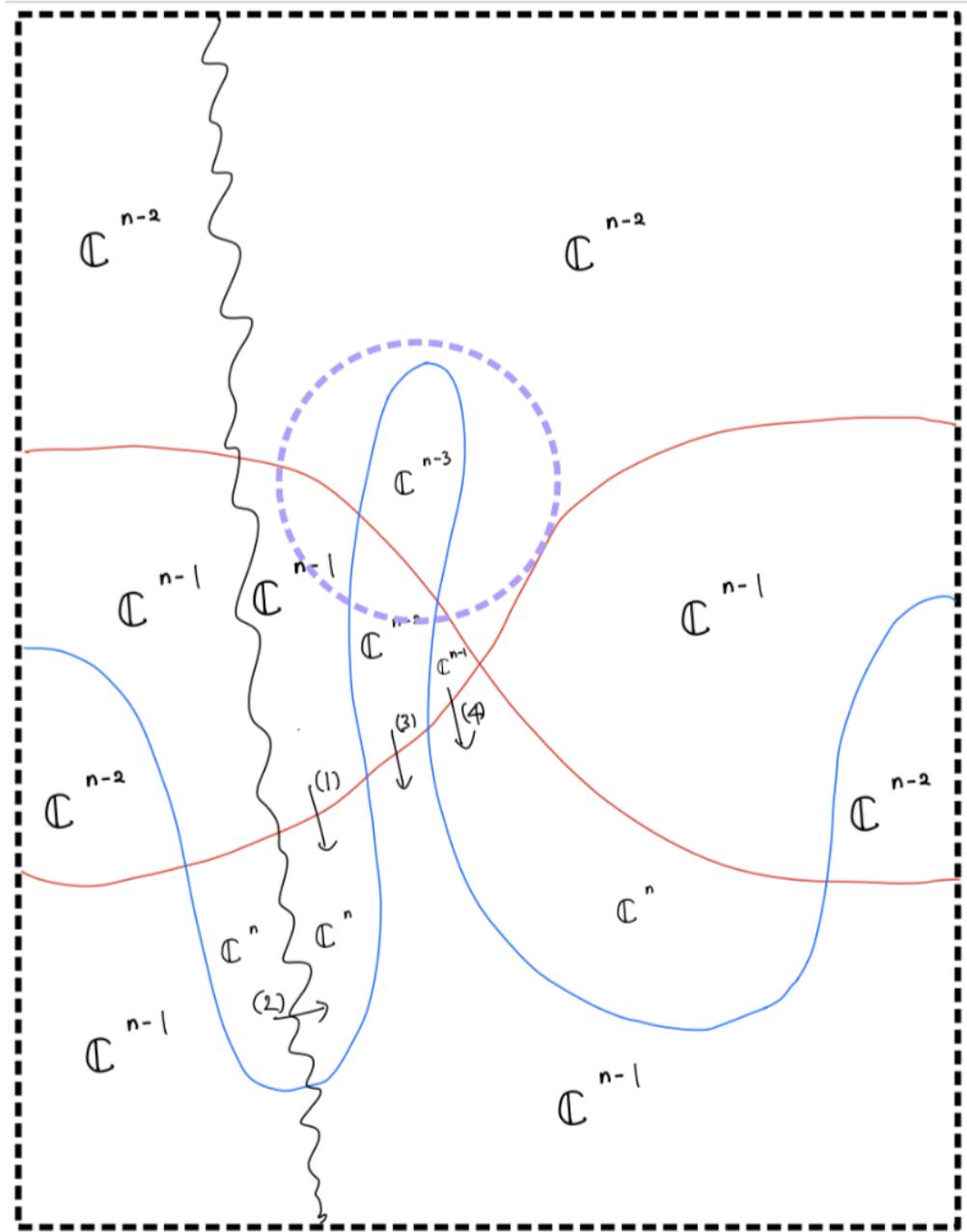


Figure 3.311

we get

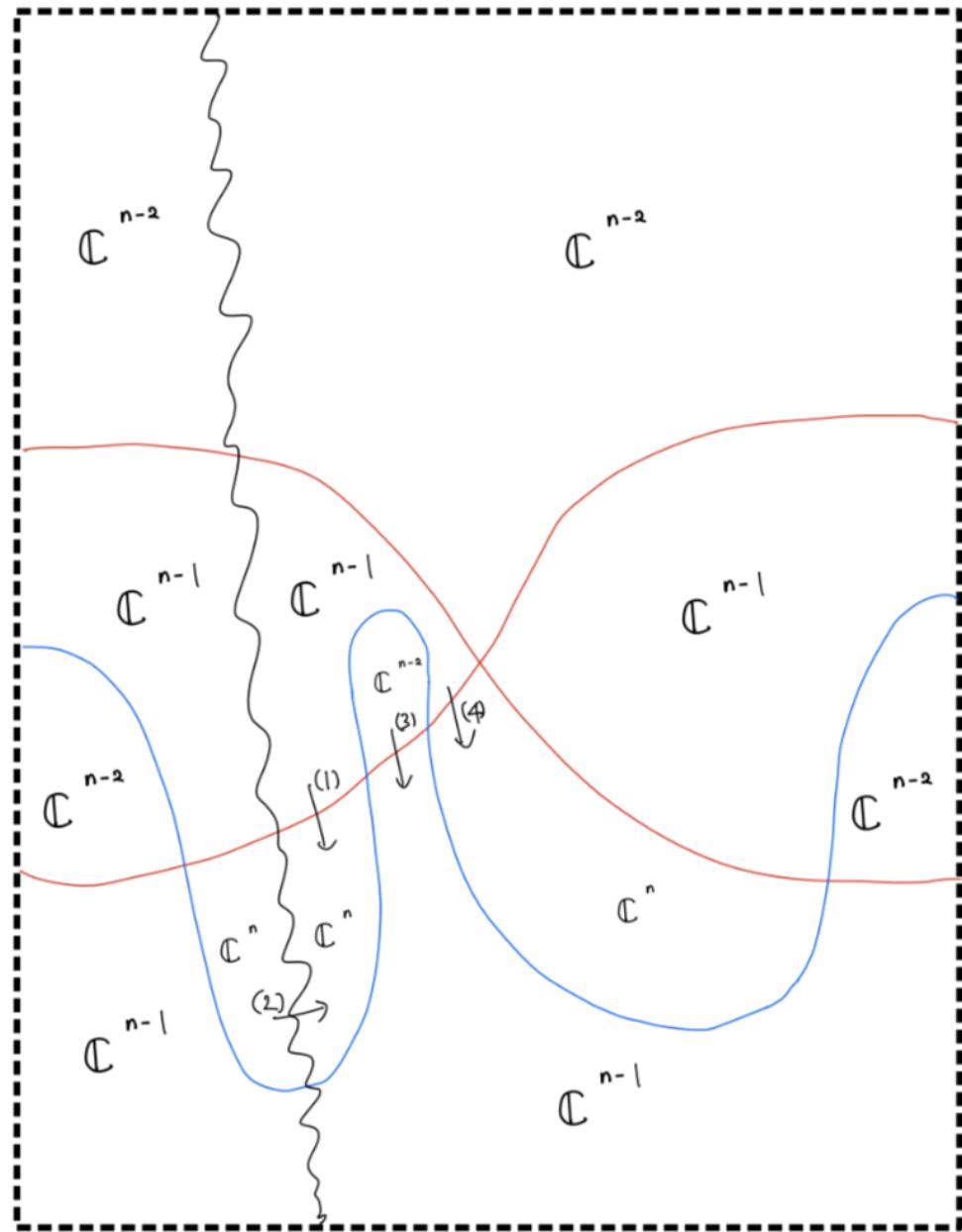


Figure 3.312

**Generalizations maps:**

- (1)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$
- (2)  $\text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$
- (3)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1,n-2}$
- (4)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$

(Step 4) apply  $cobord_3$  to the region surrounded by a purple dotted line

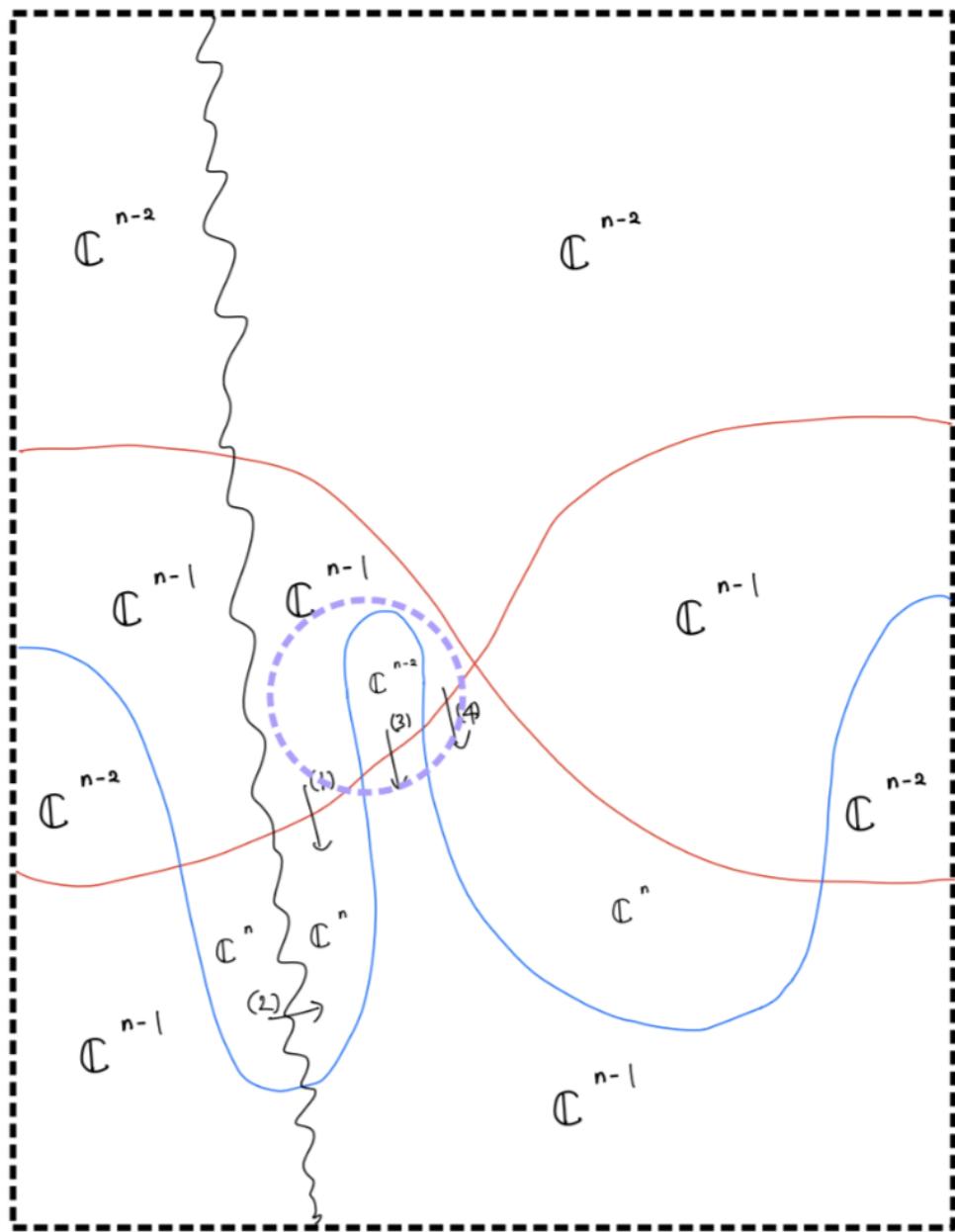


Figure 3.313

we get the final sheaf

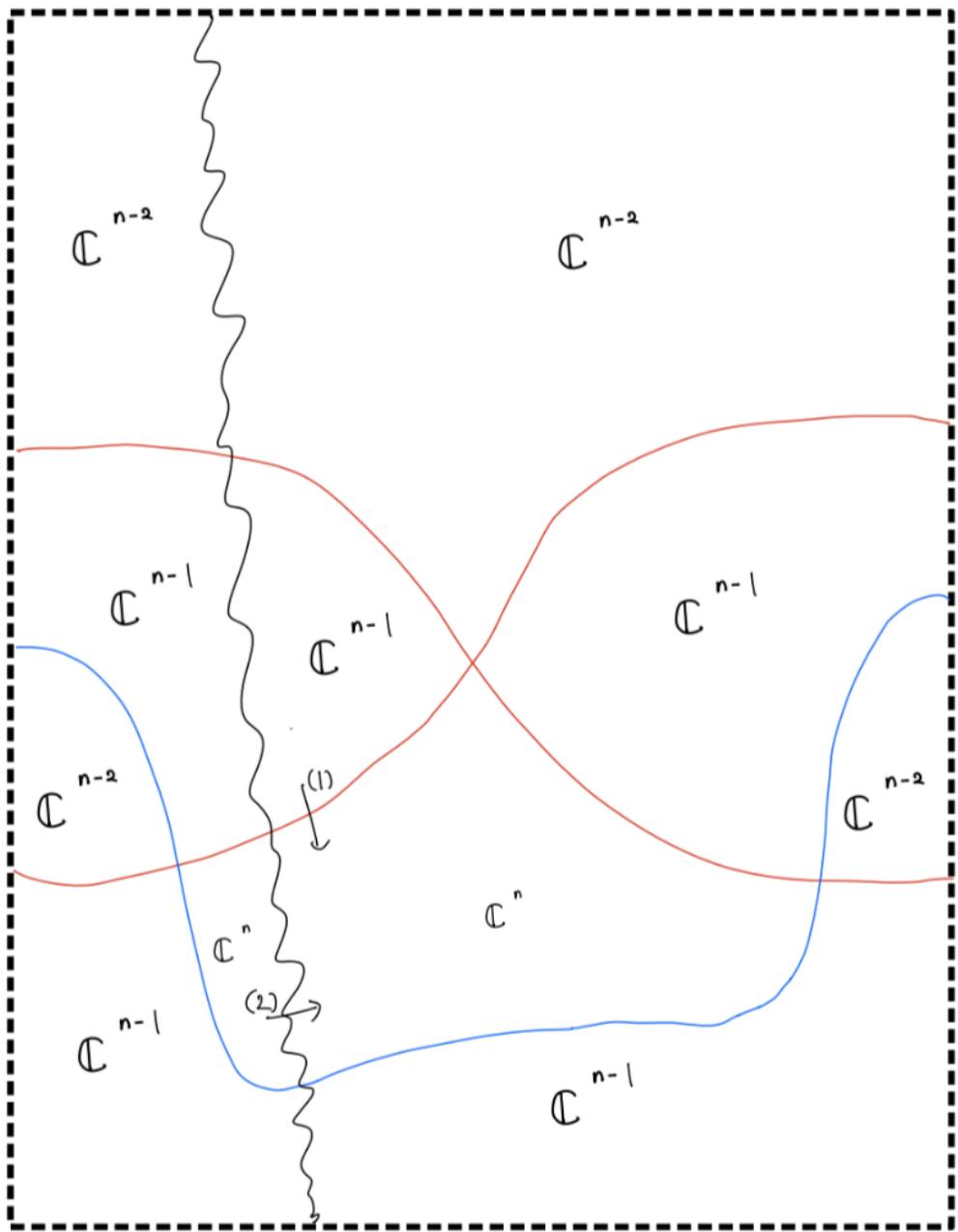


Figure 3.314

**Generalizations maps:**

$$(1) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n,n-1}$$

$$(2) \quad \text{diag}(d_n, \dots, d_1) + eI_{n,n-1}$$

## 3.15 Sheaf Cobordism on Generator Regions

**Definition 206.** Smooth transition function consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

we define

$$\tau(x) = \frac{f(x)}{f(x) + f(1-x)}$$

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0$ ,  $\Lambda_0^\infty$ ,  $\Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-1, 1)_x \times (-n-1, n+1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to the red strands in the figure below, co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to the blue strands in the figure below, co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to the squiggly lines with co-orientations given in the figure below.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

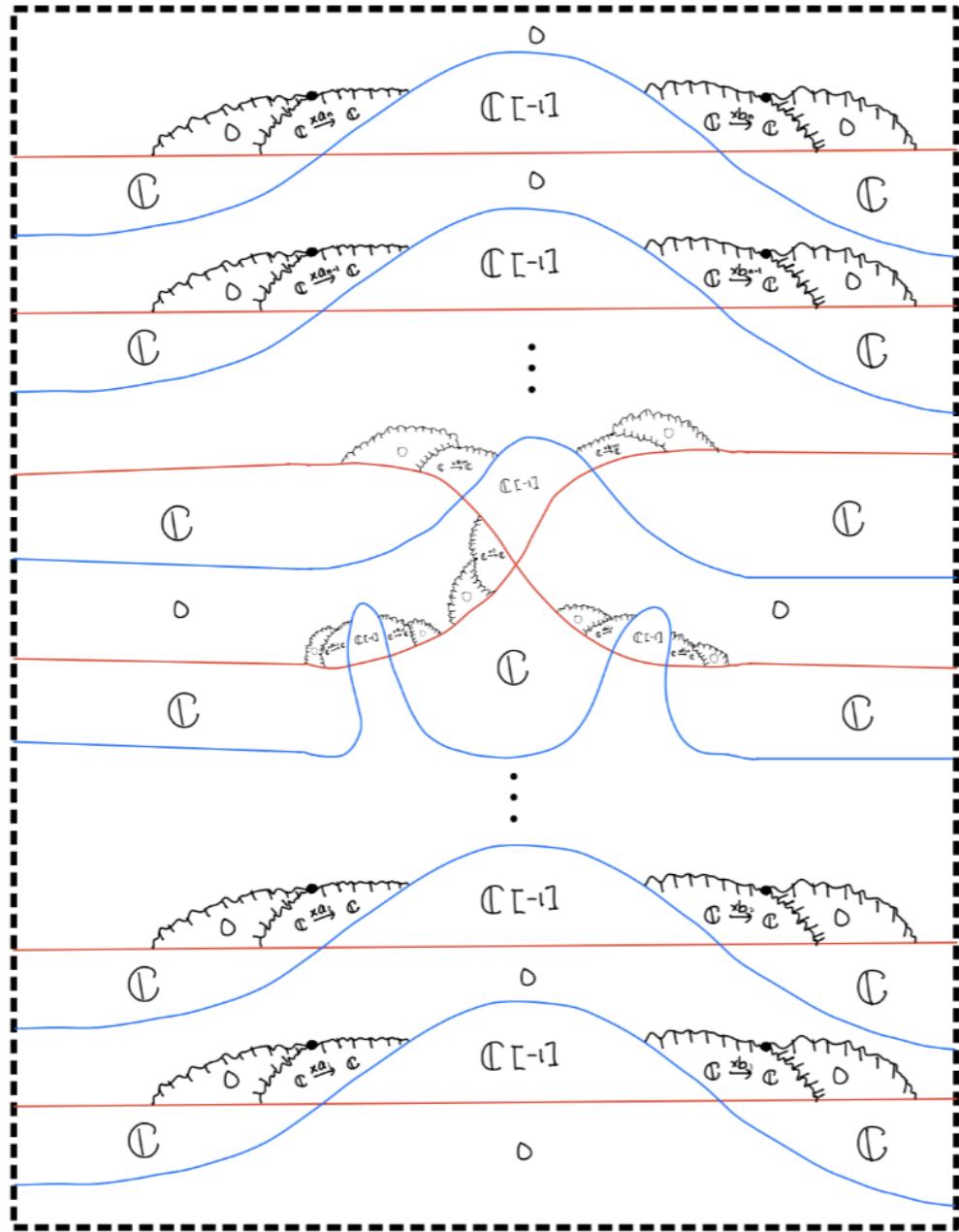


Figure 3.315

Then we define a cobordism starting from the above sheaf, say  $cobord_{gen}(n)$  supported on  $U$ , where  $n$  is the number of blue strands (equivalently, red strands). At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

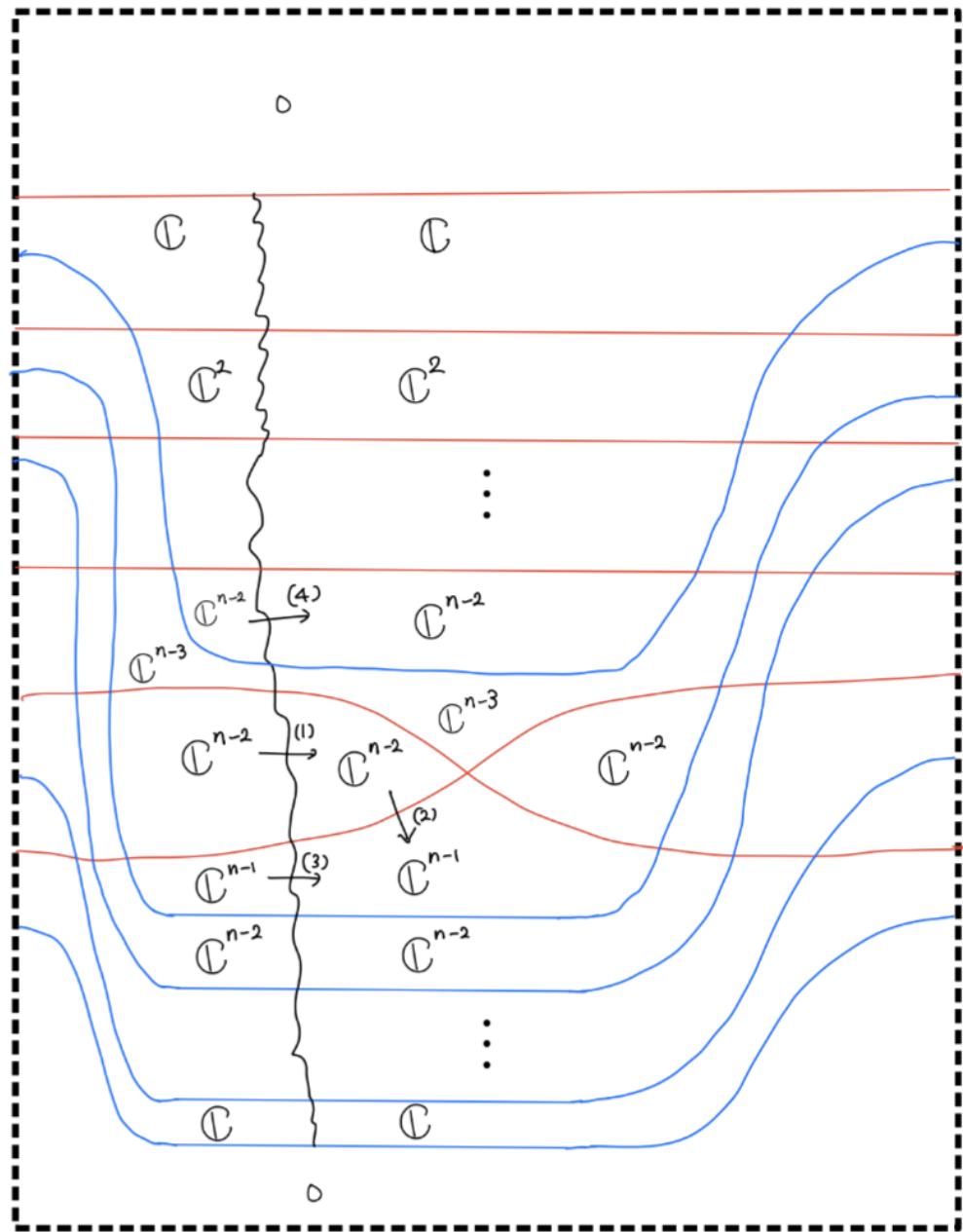


Figure 3.316

### Generalization maps

- (1)  $\text{diag}(d_n, \dots, d_{i+1})$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3)  $\text{diag}(d_n, \dots, d_{i+1}) + e I_{n-i+1, n-i}$

where

- $d_r = a_r b_r^{-1}$
- $e = -a_{i+1} b_i^{-1} c$
- $e' = d_{i+1}^{-1} e$

We define  $cobord_{gen}(n)$  inductively as follows.

- (i) For  $n = 2$ , we define  $cobord_{gen}$  starting from the sheaf below

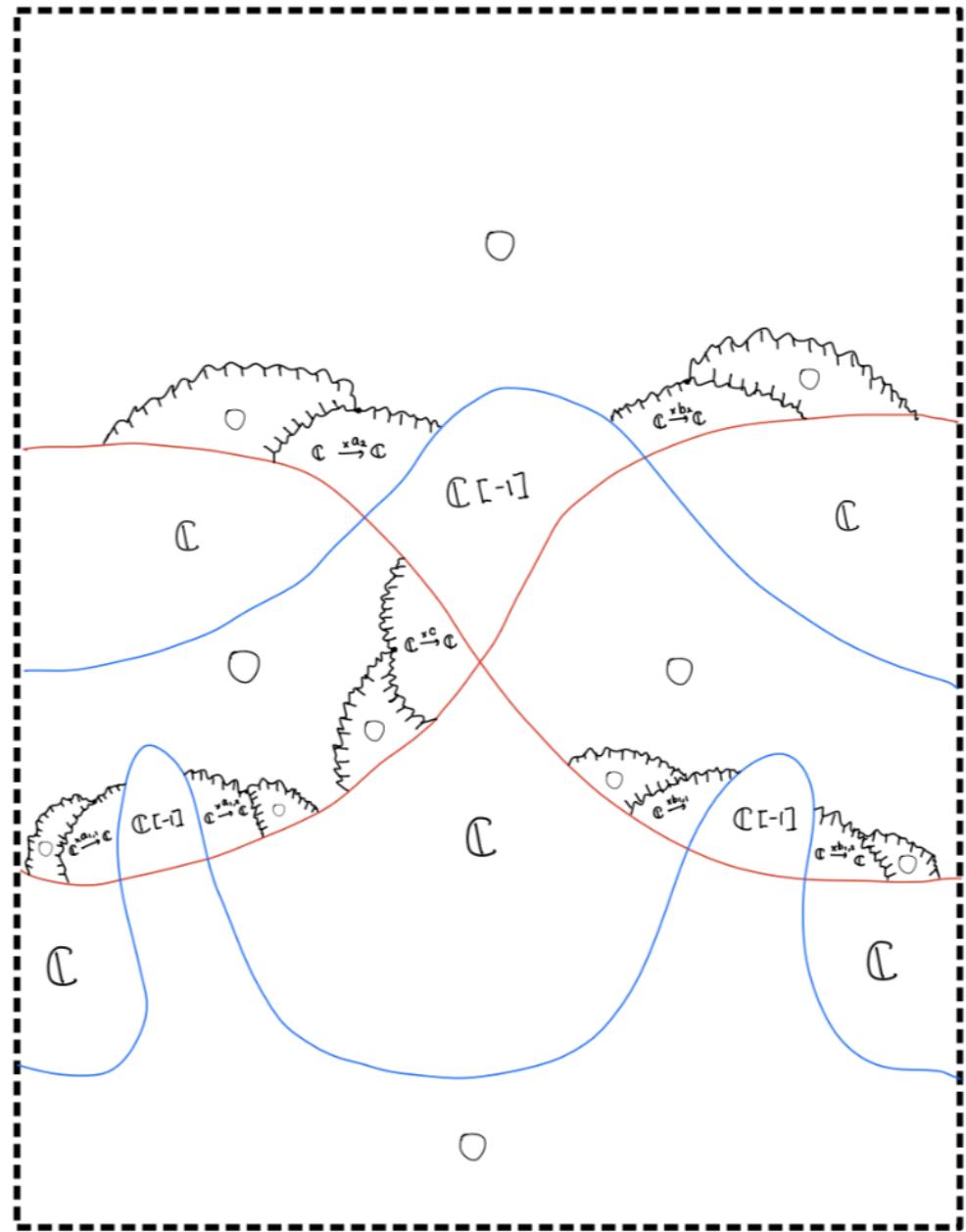


Figure 3.317

as follows.

(Step 1) we apply  $cobord_1$  to the square regions surrounded by purple dotted lines.

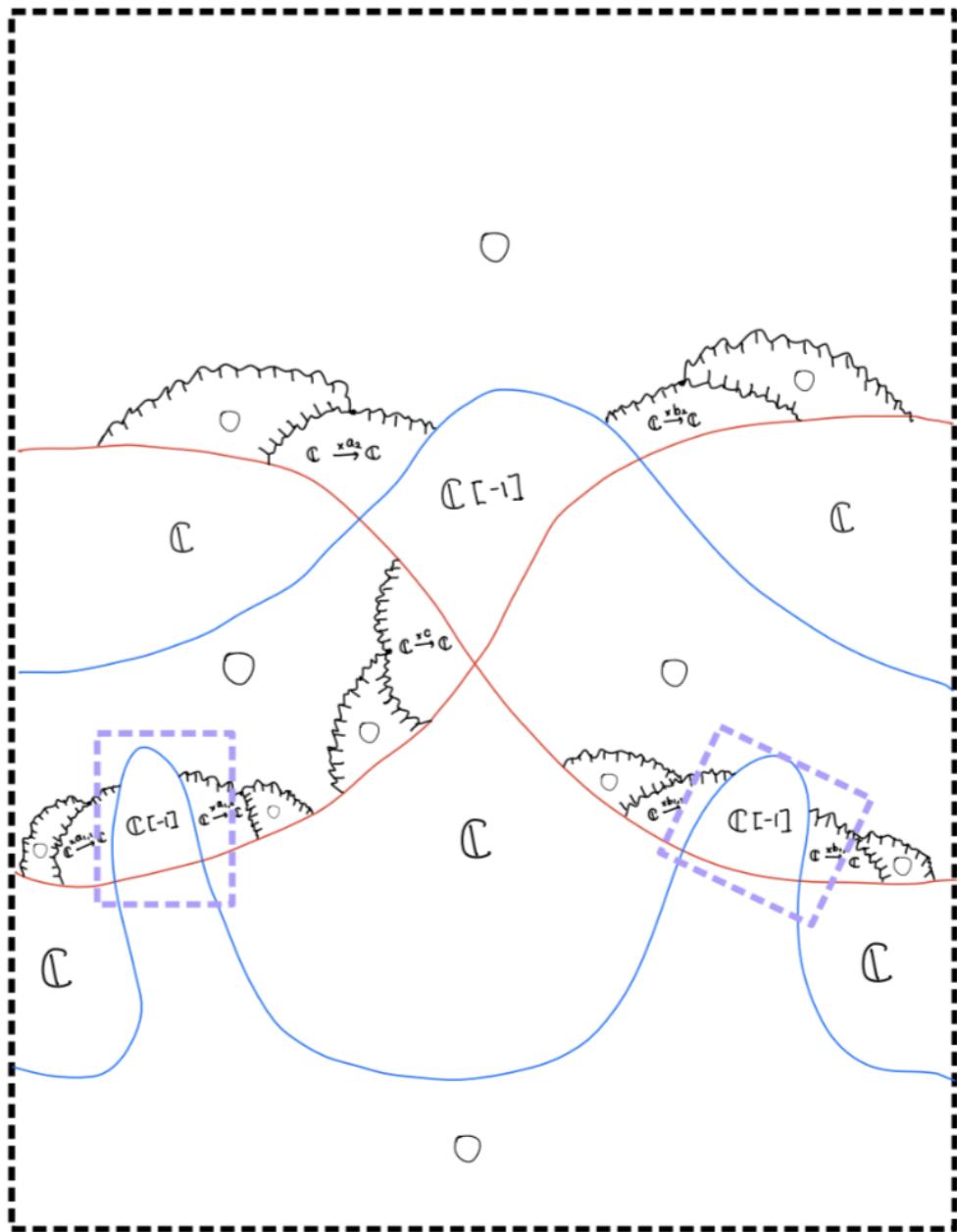


Figure 3.318

we get

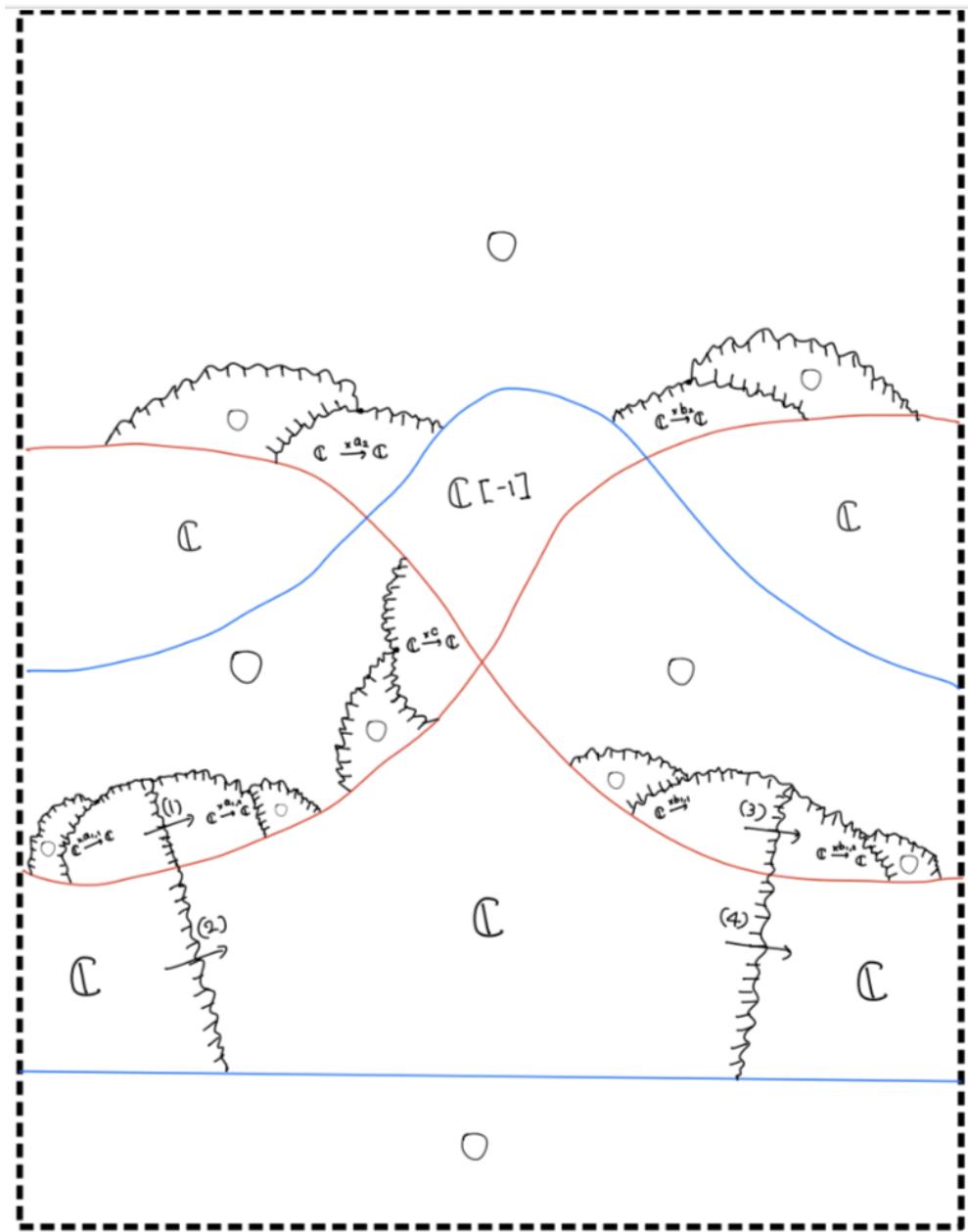


Figure 3.319

### Generalization maps

$$(1) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a_1 \uparrow & & \uparrow \times a_{1,2} \\ \mathbb{C} & \xrightarrow{\times a_{1,1}} & \mathbb{C} \end{array}$$

(2)  $\times a_1$

$$(3) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times b_1^{-1} \uparrow & & \uparrow \times b_{1,2} \\ \mathbb{C} & \xrightarrow{\times b_{1,1}} & \mathbb{C} \end{array}$$

$$(4) \quad \times b_1^{-1}$$

where

- $a_1 = a_{1,1}a_{1,2}^{-1}$
- $b_1 = b_{1,1}^{-1}b_{1,2}$

which is quasi-isomorphic to

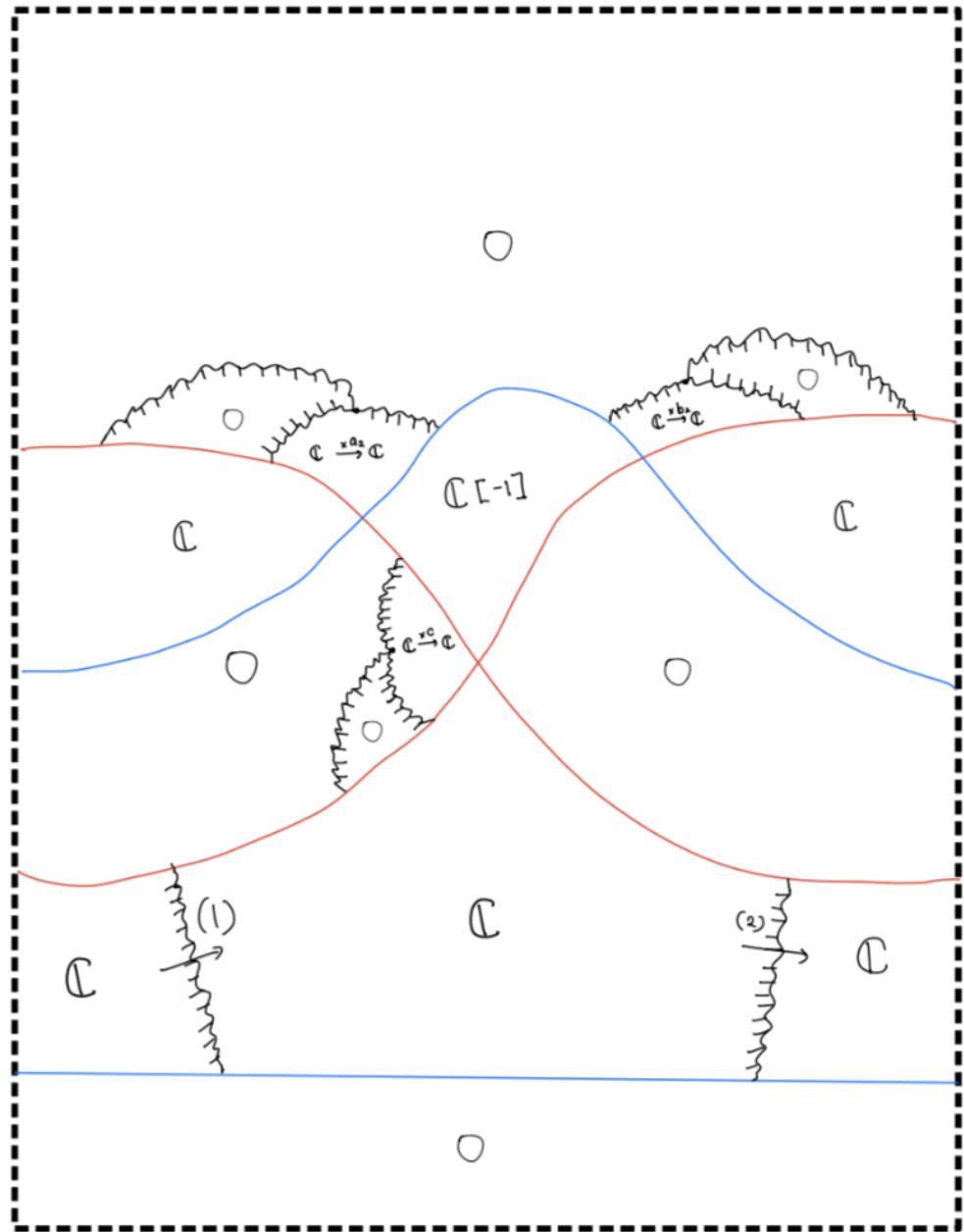


Figure 3.320

**Generalization maps**(1)  $\times a_1$ (2)  $\times b_1^{-1}$ (Step 2) apply *cobord<sub>8</sub>* to the region surrounded by a purple dotted line

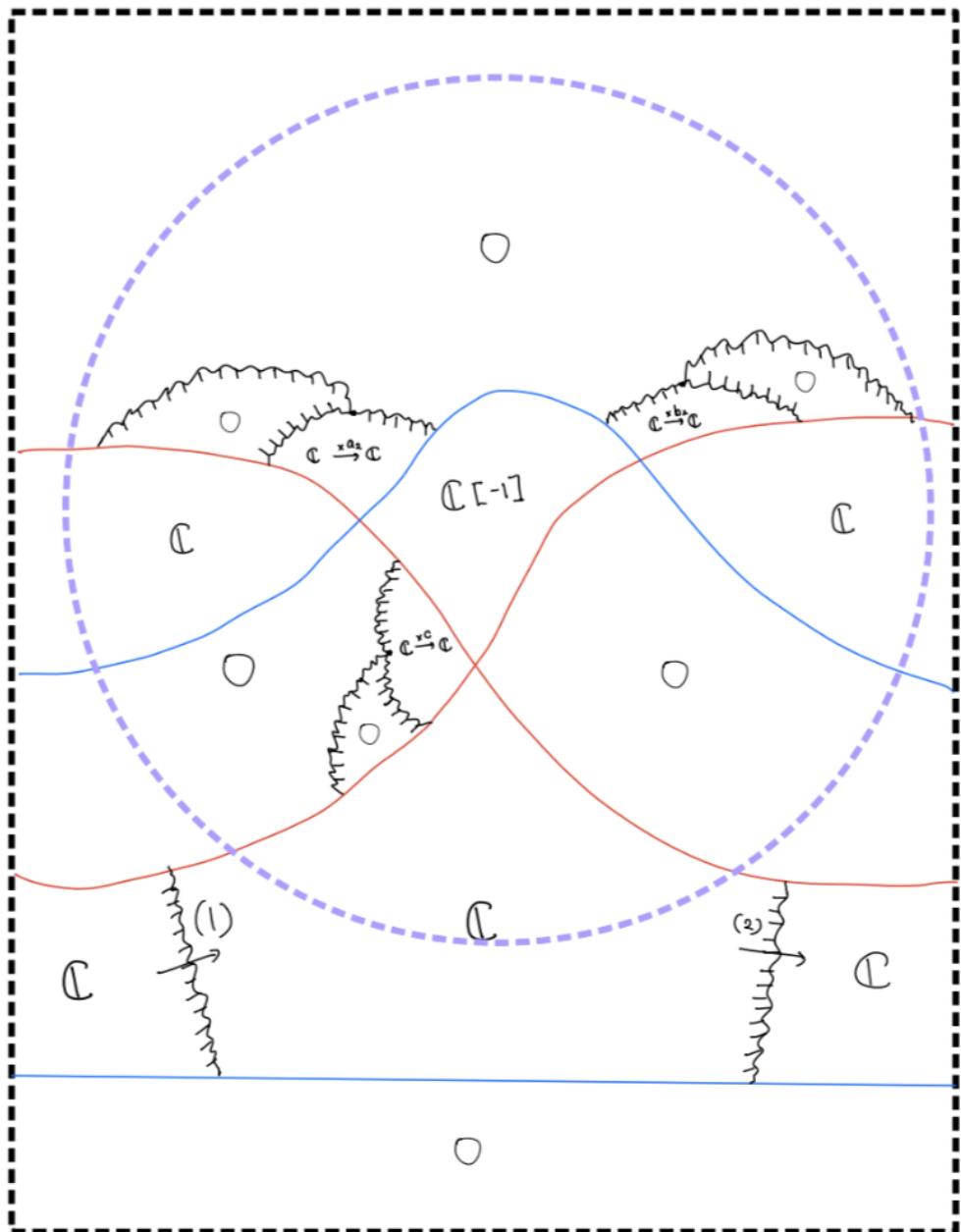


Figure 3.321

we get

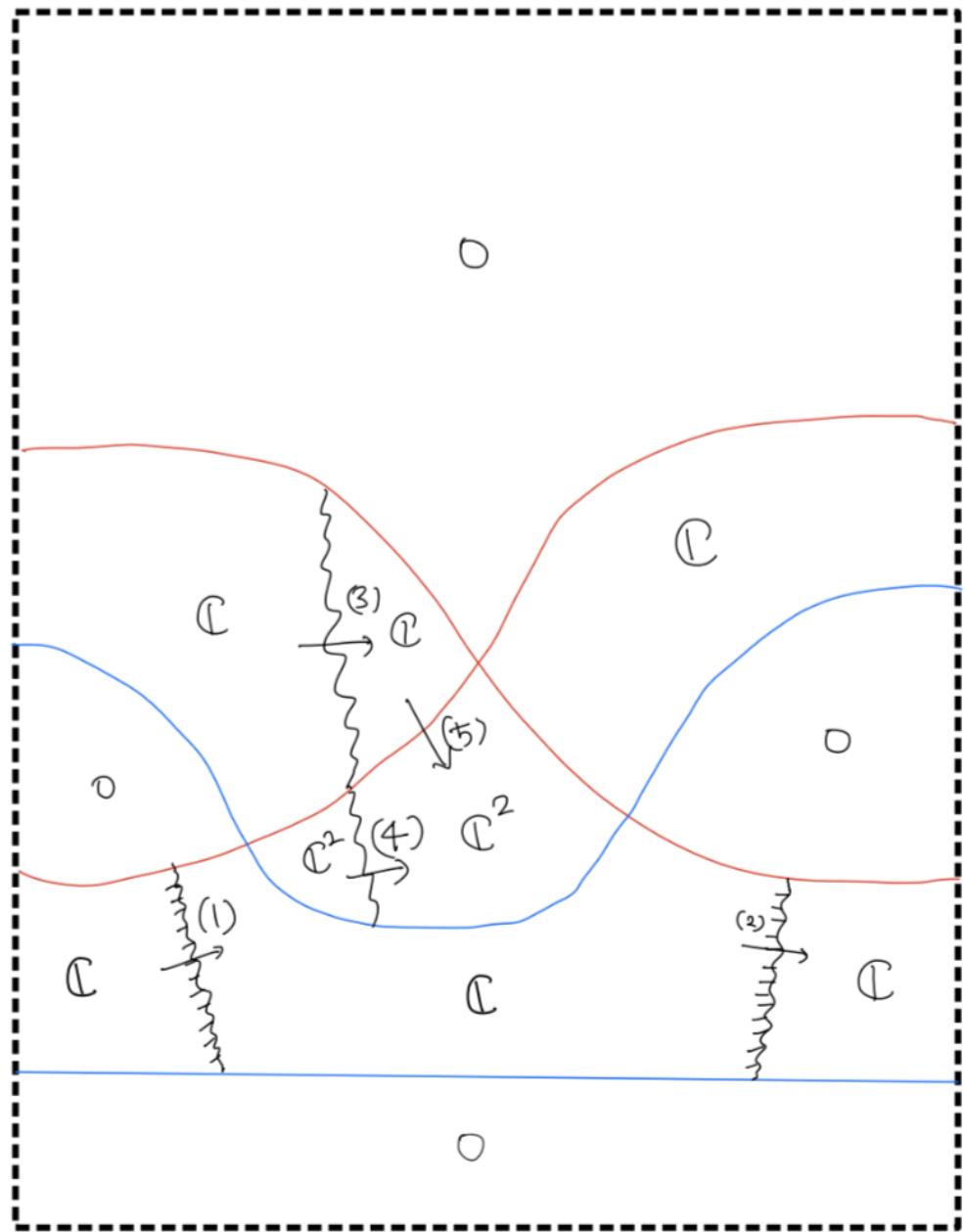


Figure 3.322

### Generalization maps

(1)  $\times a_1$

(2)  $\times b_1^{-1}$

(3)  $\times a_2 b_2^{-1}$

$$(4) \begin{pmatrix} a_2 b_2^{-1} & 0 \\ -a_2 c^{-1} & 1 \end{pmatrix}$$

$$(5) \begin{pmatrix} 1 \\ -b_2 c^{-1} \end{pmatrix}$$

which is quasi-isomorphic to

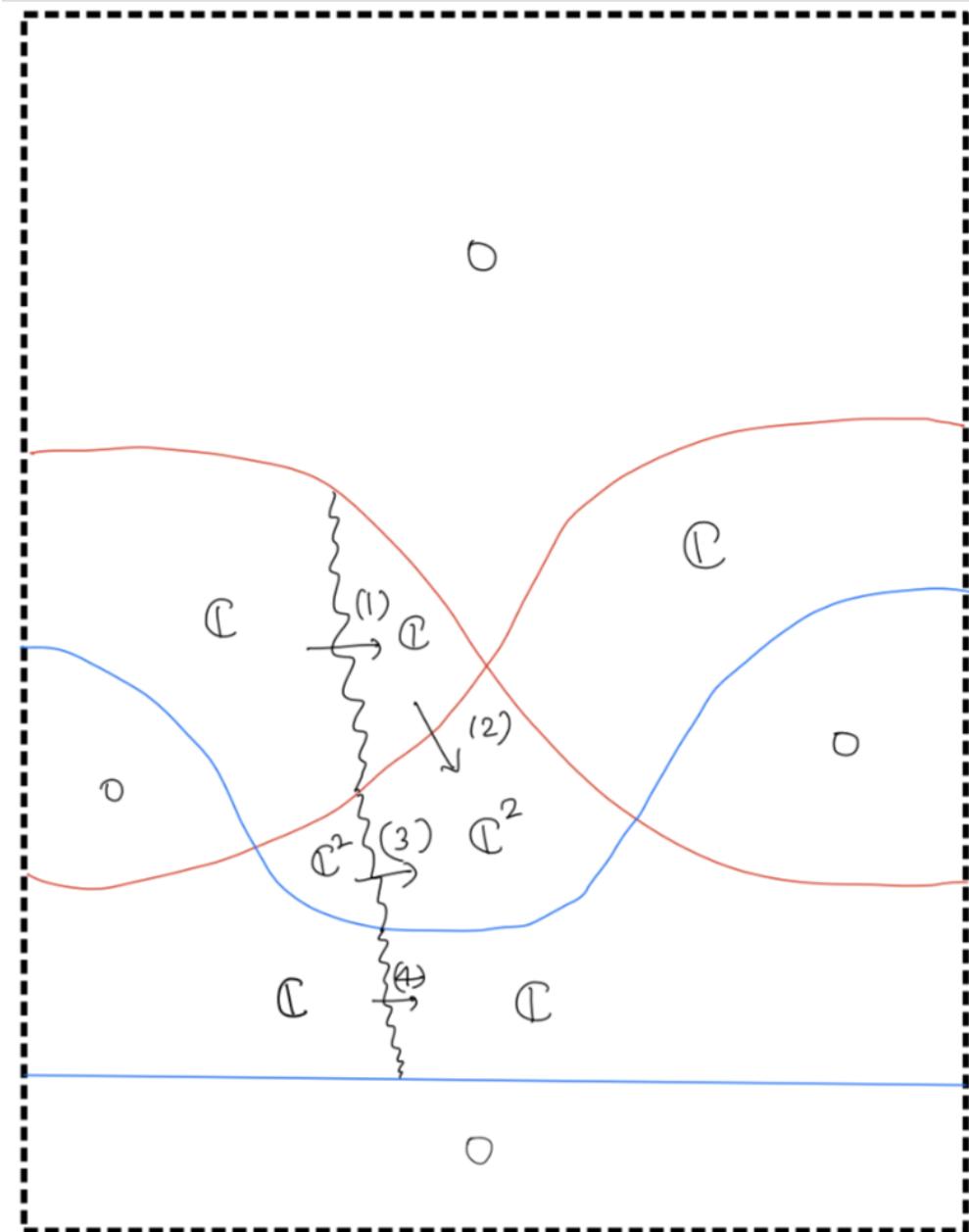


Figure 3.323

### Generalization maps

$$\begin{aligned}
 (1) & \times d_2 \\
 (2) & \begin{pmatrix} 1 \\ e' \end{pmatrix} \\
 (3) & \begin{pmatrix} d_2 & 0 \\ e & d_1 \end{pmatrix} \\
 (4) & \times d_1
 \end{aligned}$$

where

- $d_r = a_r b_r^{-1}$
- $e = -a_2 b_1^{-1} c$
- $e' = d_2^{-1} e$

(ii) For  $n > 2$ ,

(Case 1) if the generator  $s_i$  is  $i \neq 1$ ,

(Step 1) we apply  $cobord_{gen}(n-1)$  to the square region surrounded by purple dotted lines.

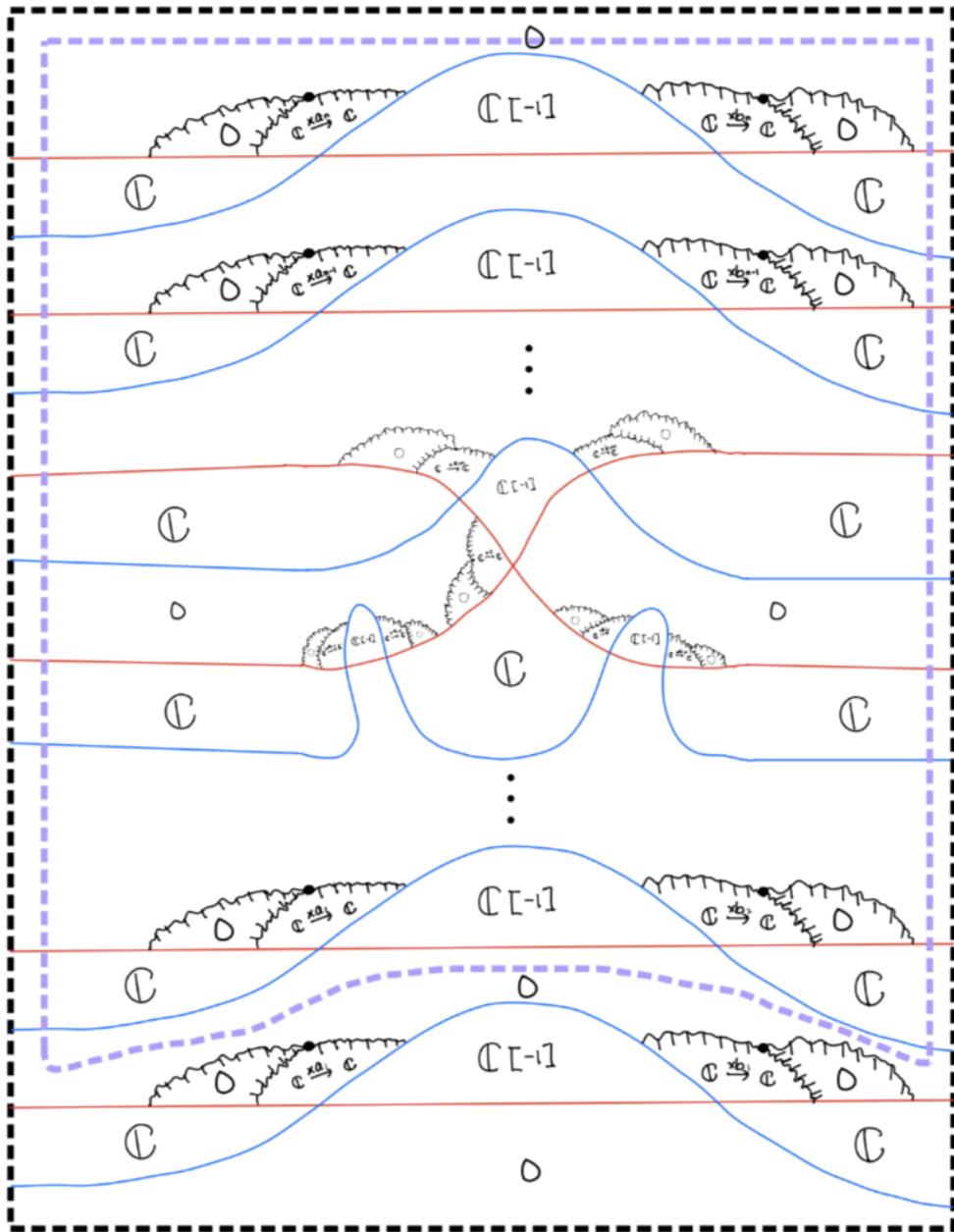


Figure 3.324

by induction hypothesis, we get

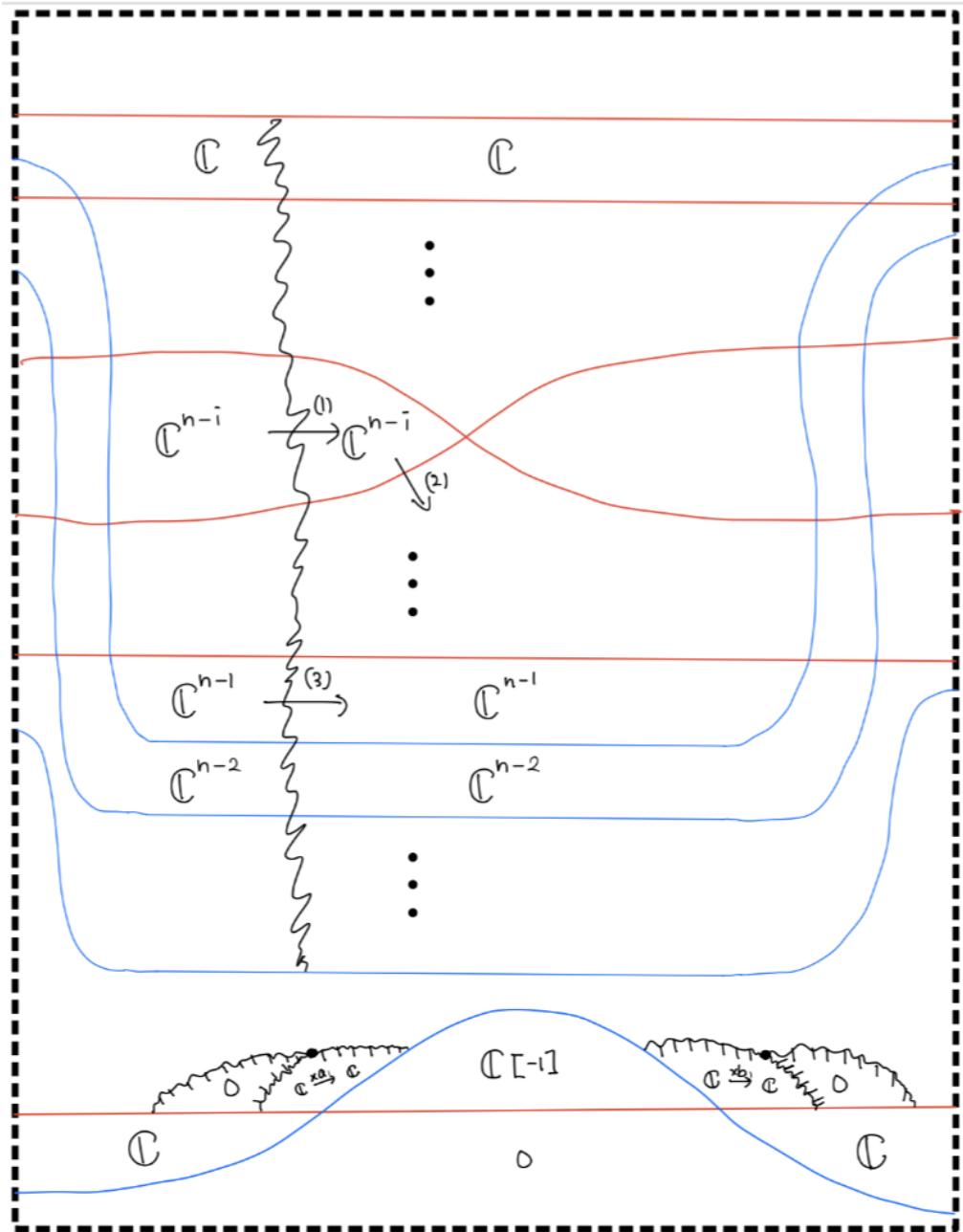


Figure 3.325

### Generalization maps

- (1)  $\text{diag}(d_n, \dots, d_{i+1})$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3)  $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$

(Step 2) apply  $\text{cobord}_2$  to the region surrounded by a purple dotted line.

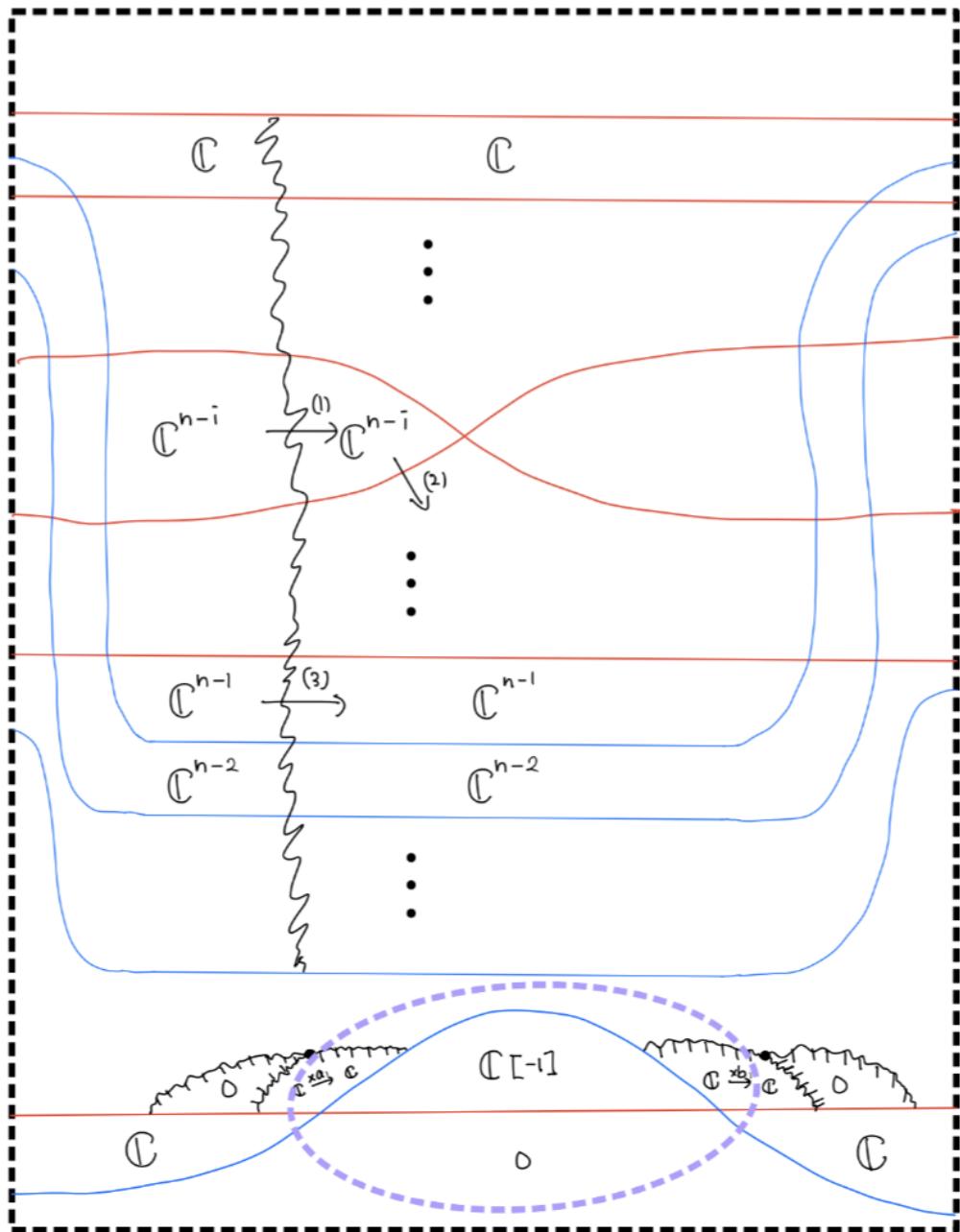


Figure 3.326

we get

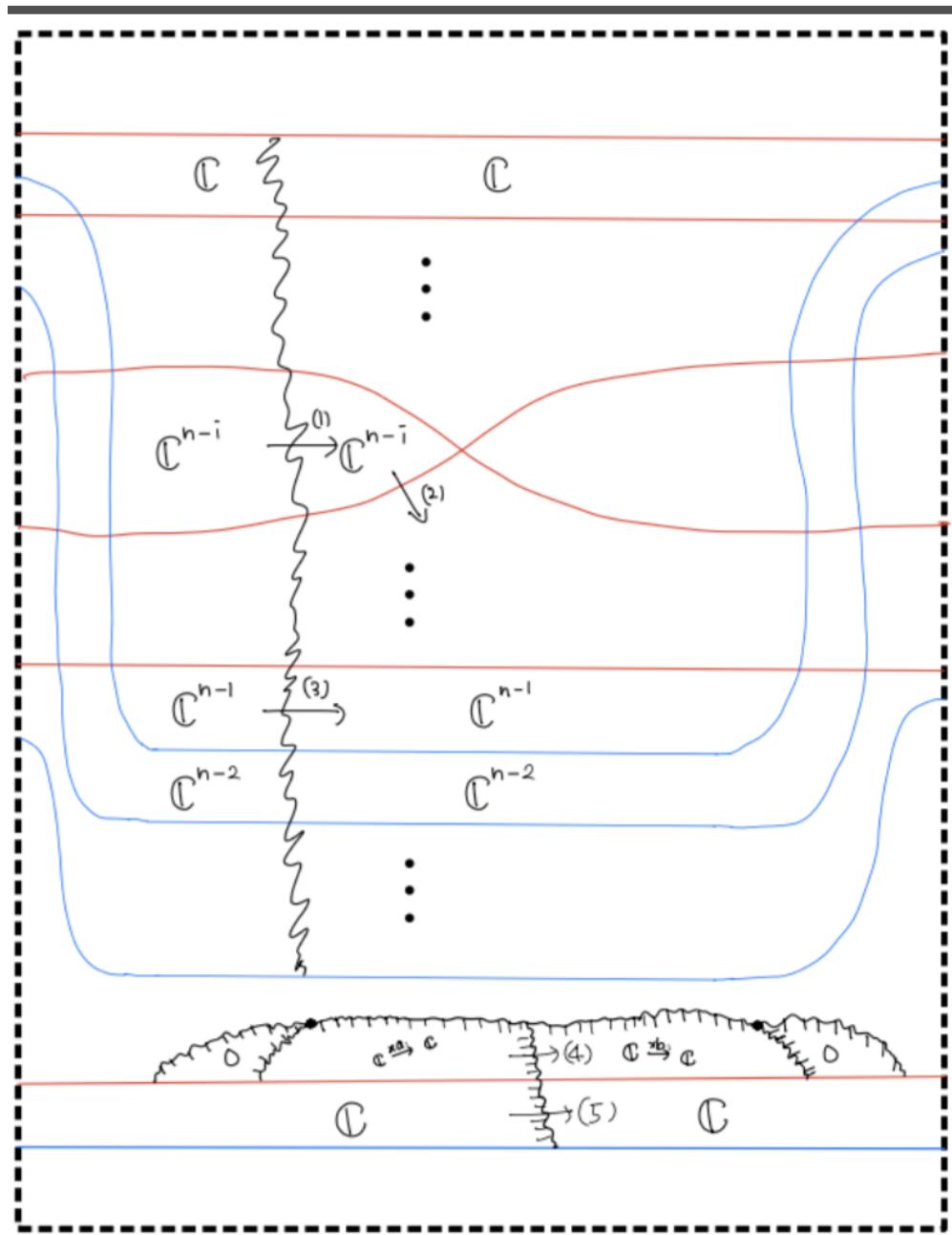


Figure 3.327

### Generalization maps

- (1)  $\text{diag}(d_n, \dots, d_{i+1})$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3)  $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$

$$(4) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times a_1 \uparrow & & \uparrow \times b_1 \\ \mathbb{C} & \xrightarrow{\times d_1} & \mathbb{C} \end{array}$$

(5)  $\times d_1$

which is quasi-isomorphic to

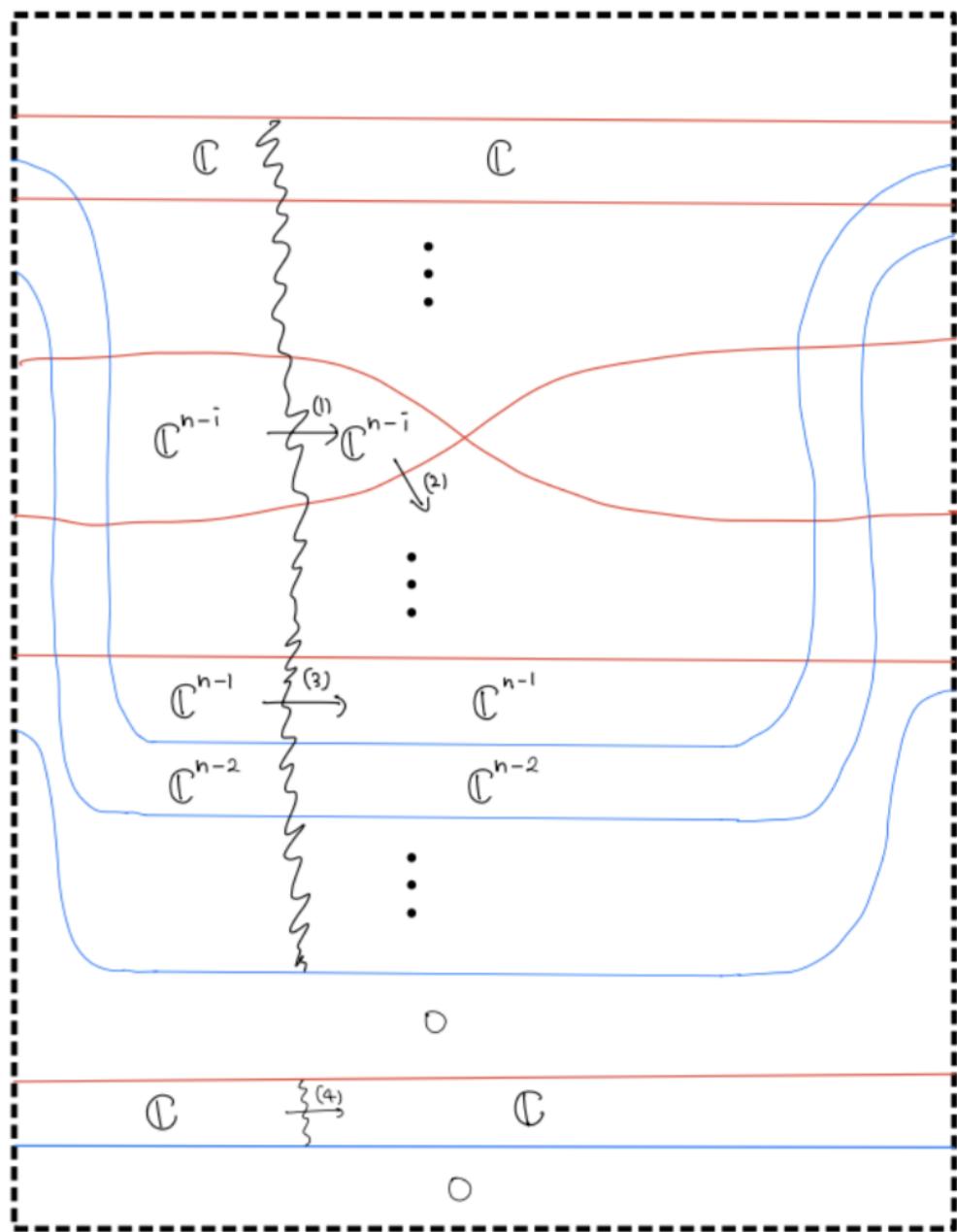


Figure 3.328

**Generalization maps**

- (1)  $\text{diag}(d_n, \dots, d_{i+1})$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3)  $\text{diag}(d_n, \dots, d_2) + e I_{n-i+1, n-i}$
- (4)  $\times d_1$

(Step 3) apply  $\text{cobord}_6$  to the square region surrounded by purple dotted lines.

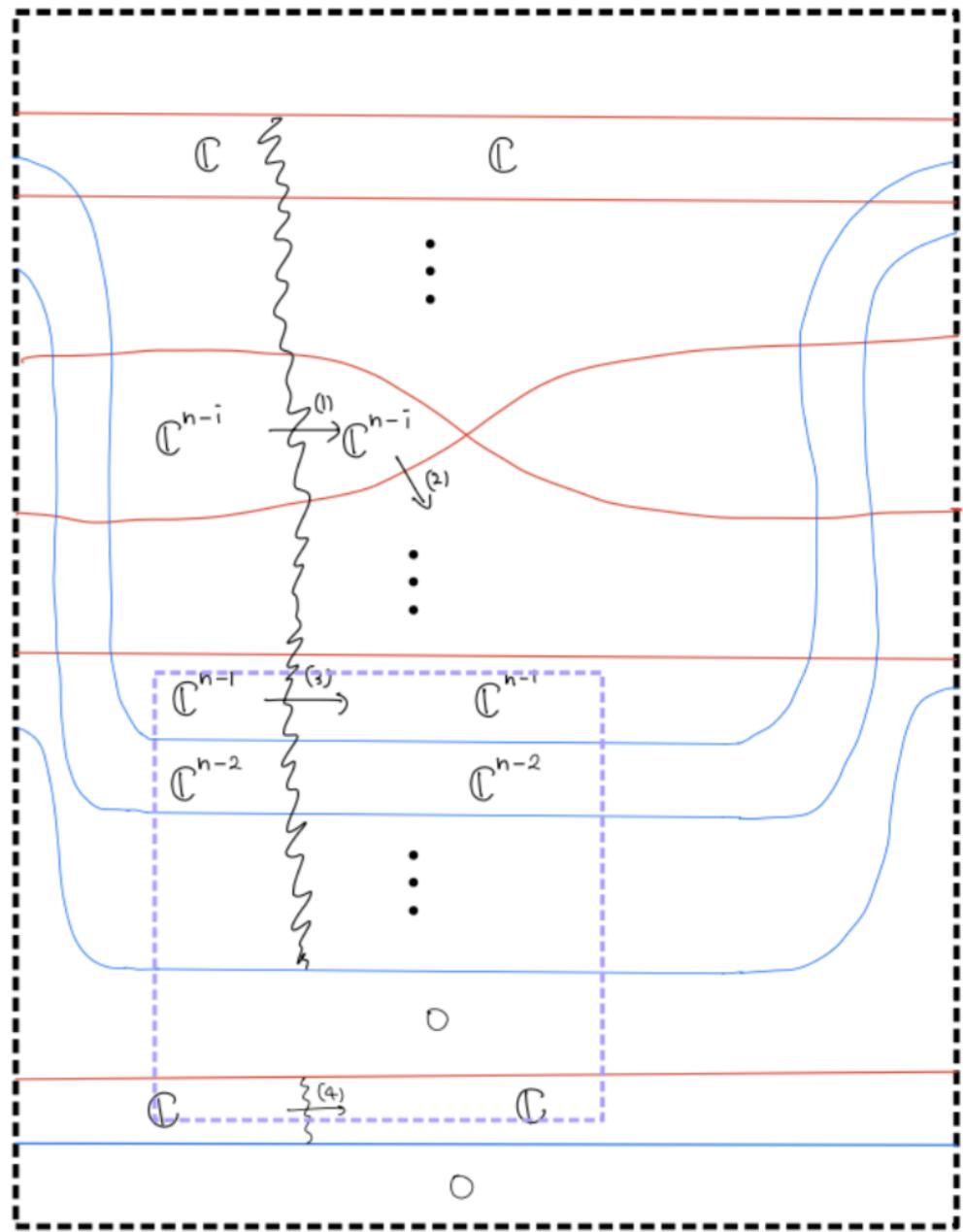


Figure 3.329

we get the final sheaf

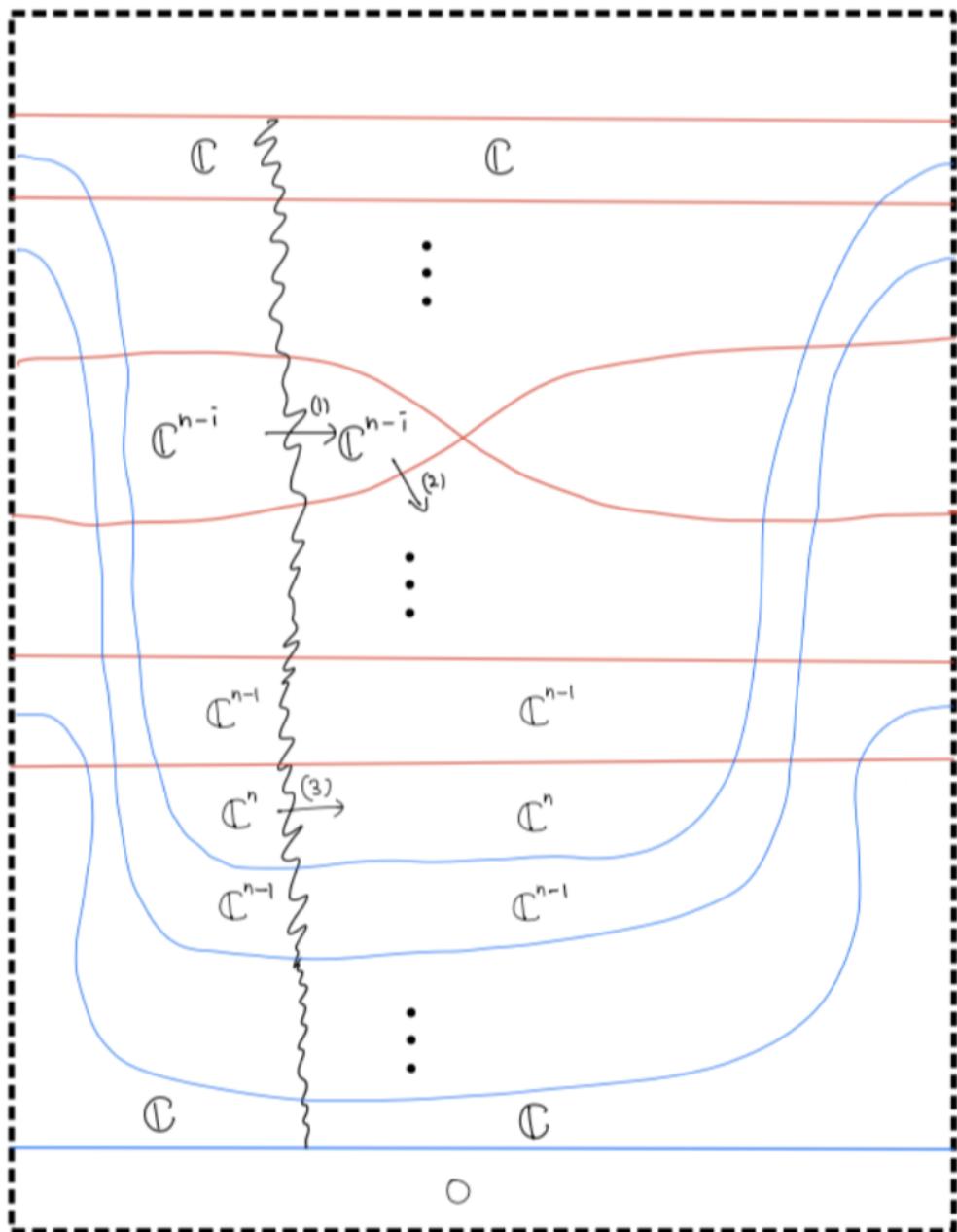


Figure 3.330

### Generalization maps

- (1)  $\text{diag}(d_n, \dots, d_{i+1})$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (3)  $\text{diag}(d_n, \dots, d_1) + e I_{n-i+1, n-i}$

(Case 2) If the generator  $s_i$  is  $i = 1$ ,

(Step 1) apply  $cobord_{gen}(n - 1)$  to the square region surrounded by purple dotted lines

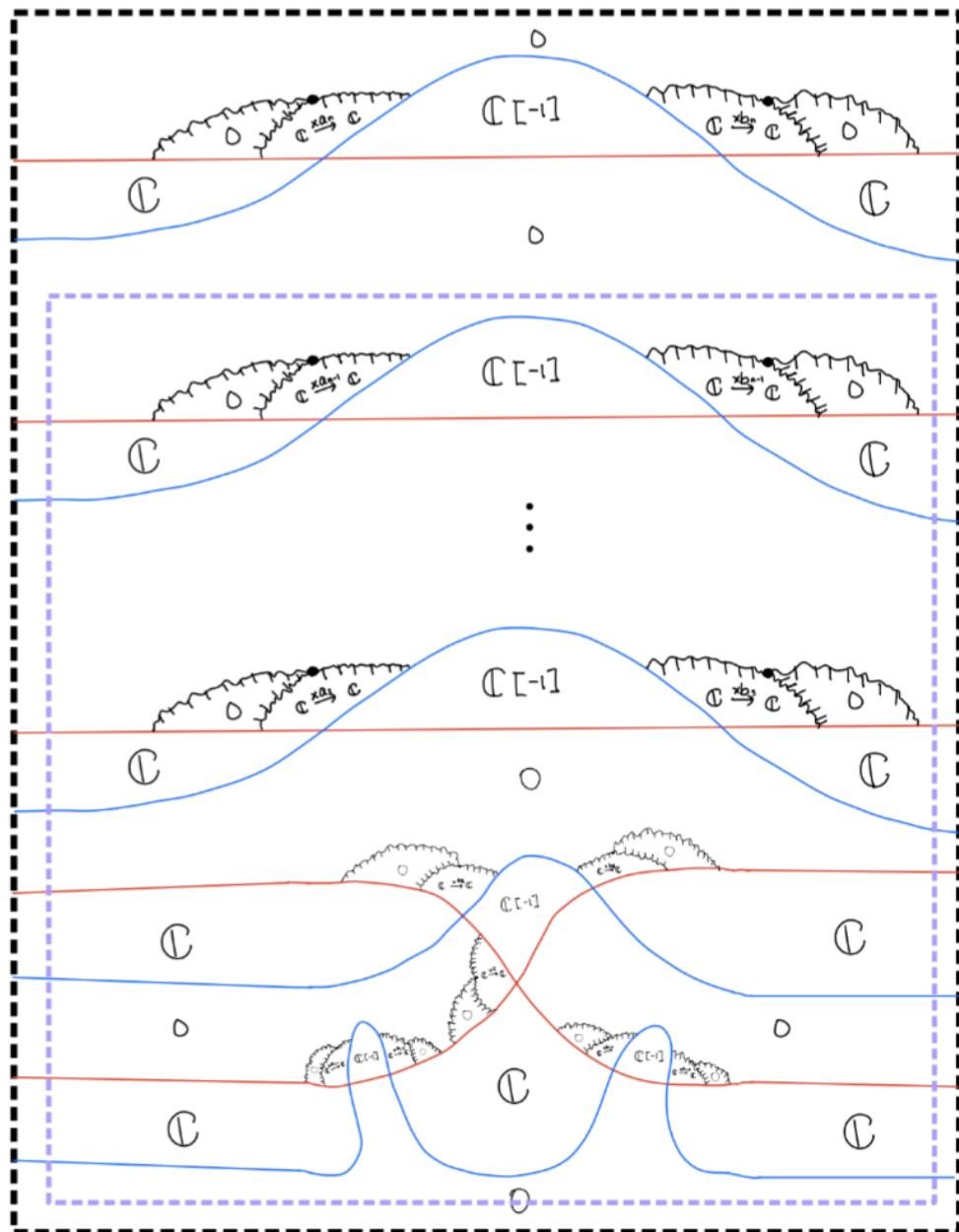


Figure 3.331

we get

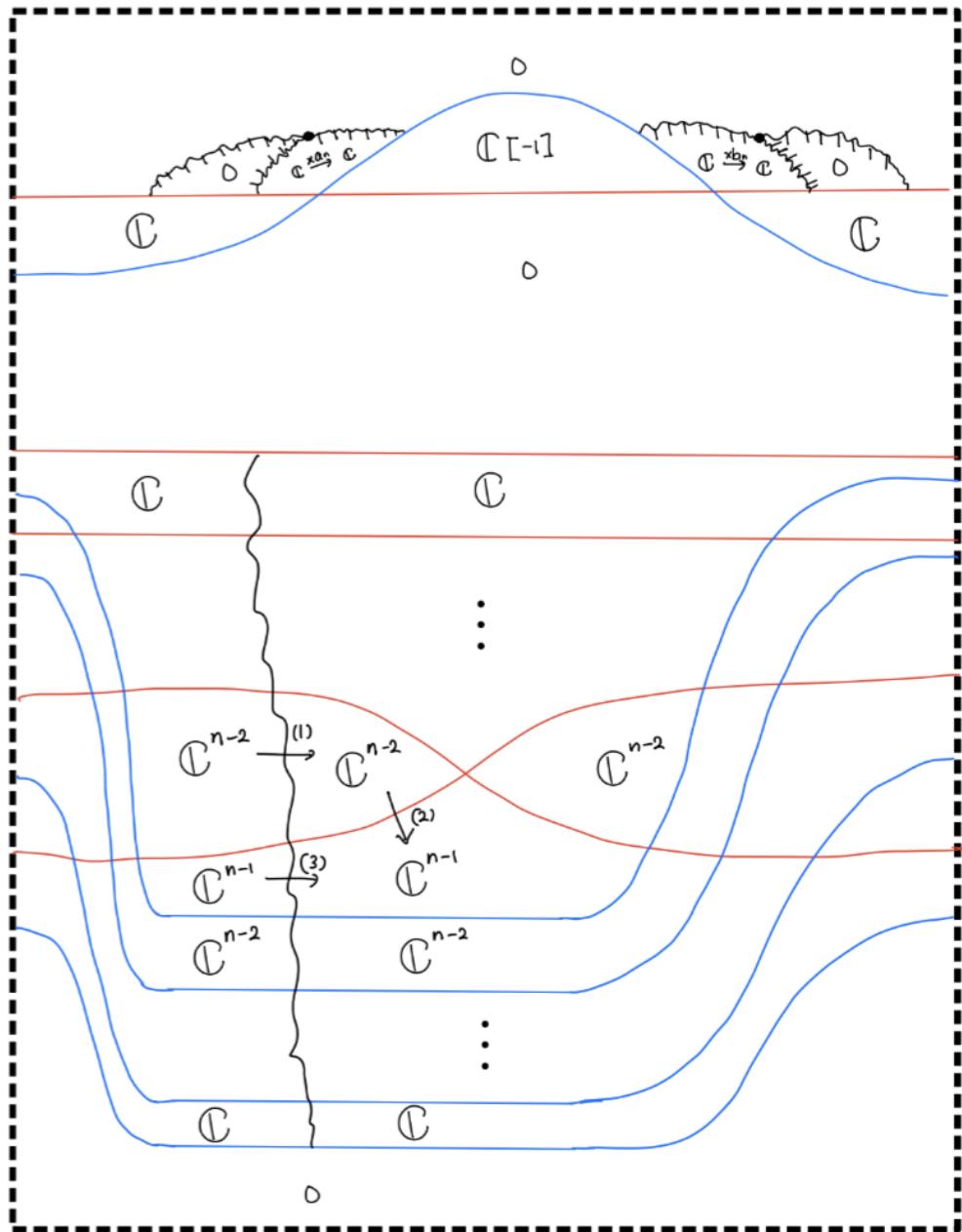


Figure 3.332

### Generalization maps

- (1)  $\text{diag}(d_{n-1}, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3)  $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$

(Step 2) apply  $\text{cobord}_2$  to the region surrounded by a purple dotted line

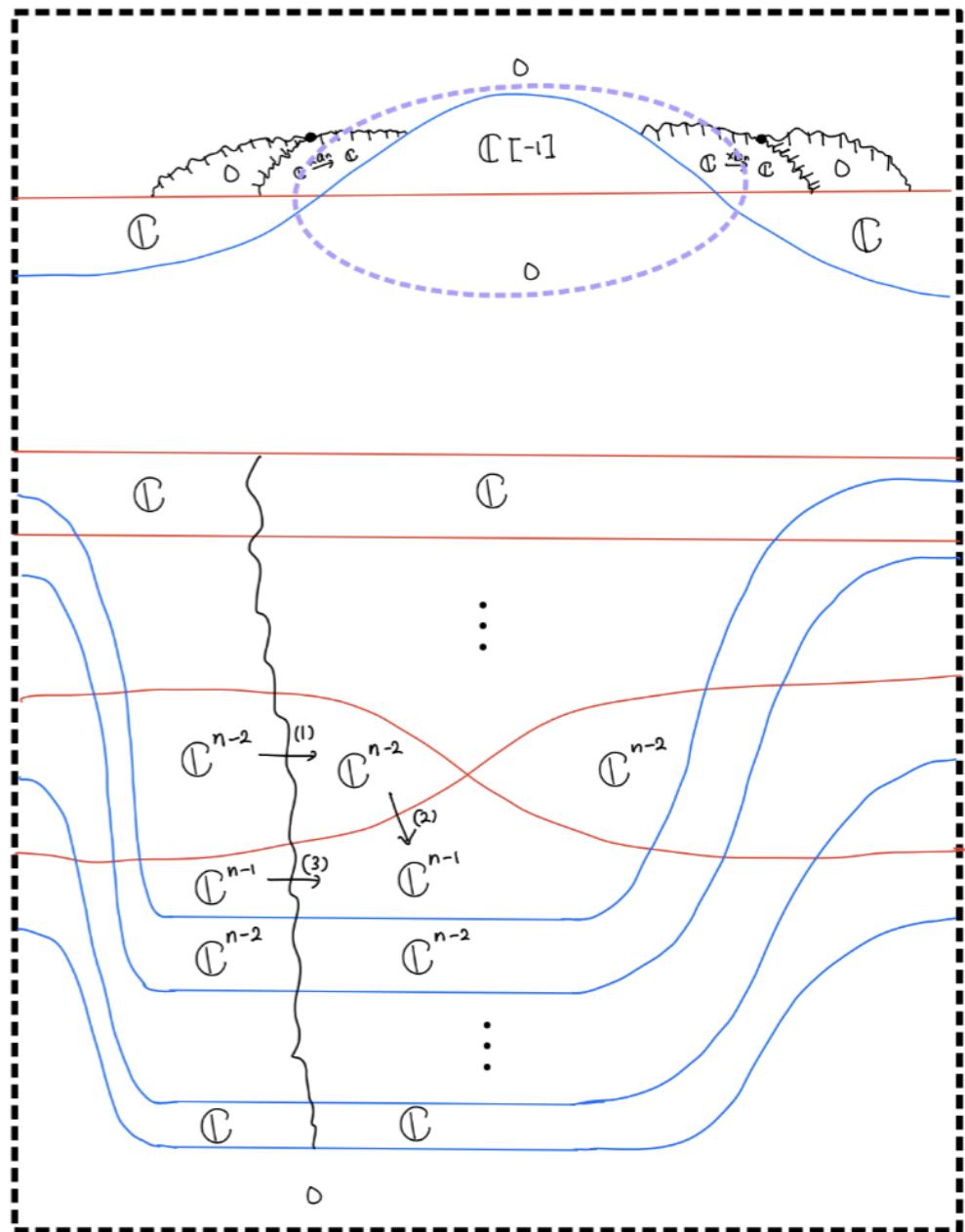


Figure 3.333

we get

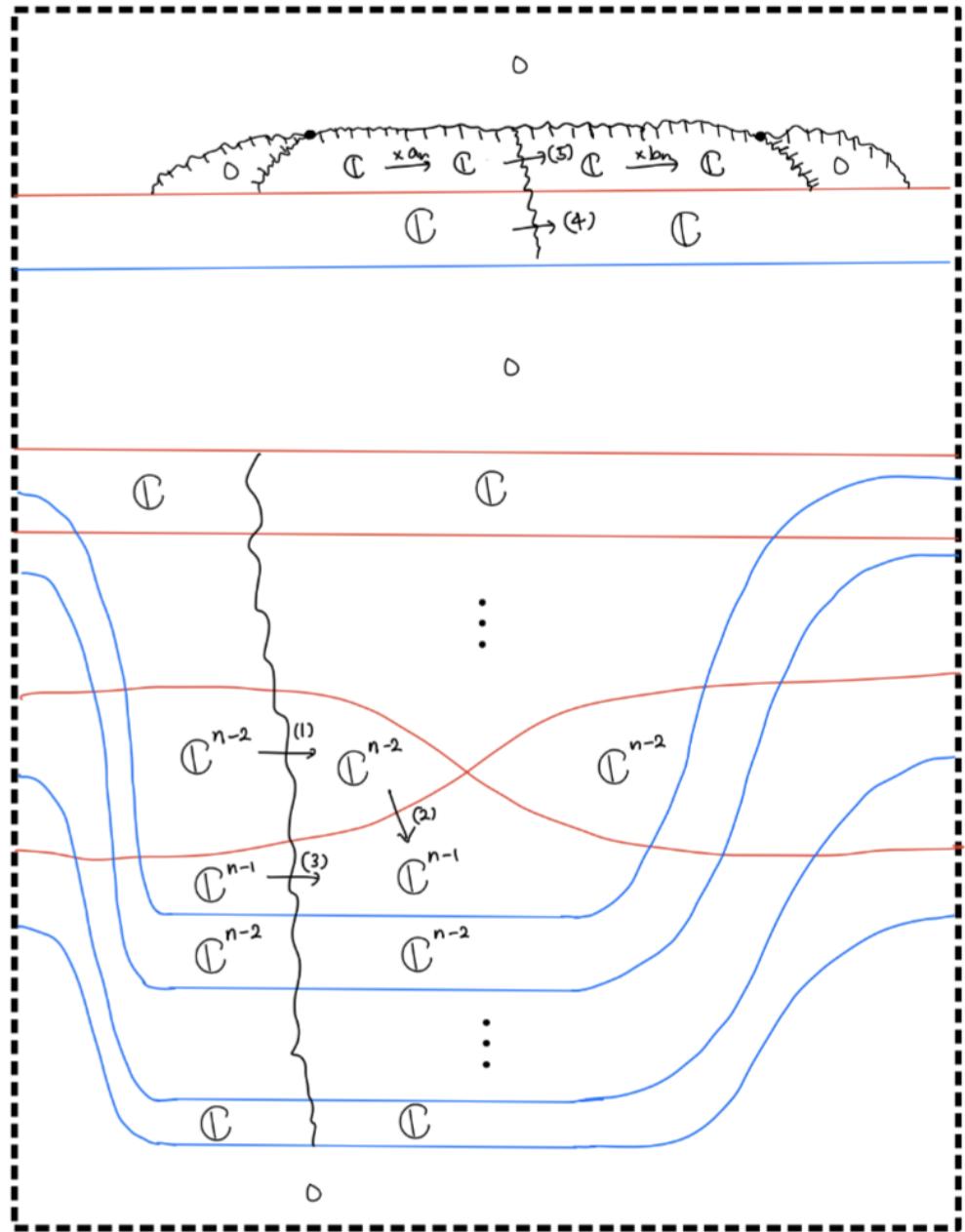


Figure 3.334

### Generalization maps

- (1)  $\text{diag}(d_{n-1}, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3)  $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$
- (4)  $\times d_n$

$$(5) \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\times 1} & \mathbb{C} \\ \times d_n \uparrow & & \uparrow \times b_n \\ \mathbb{C} & \xrightarrow{\times d_n} & \mathbb{C} \end{array}$$

which is quasi-isomorphic to

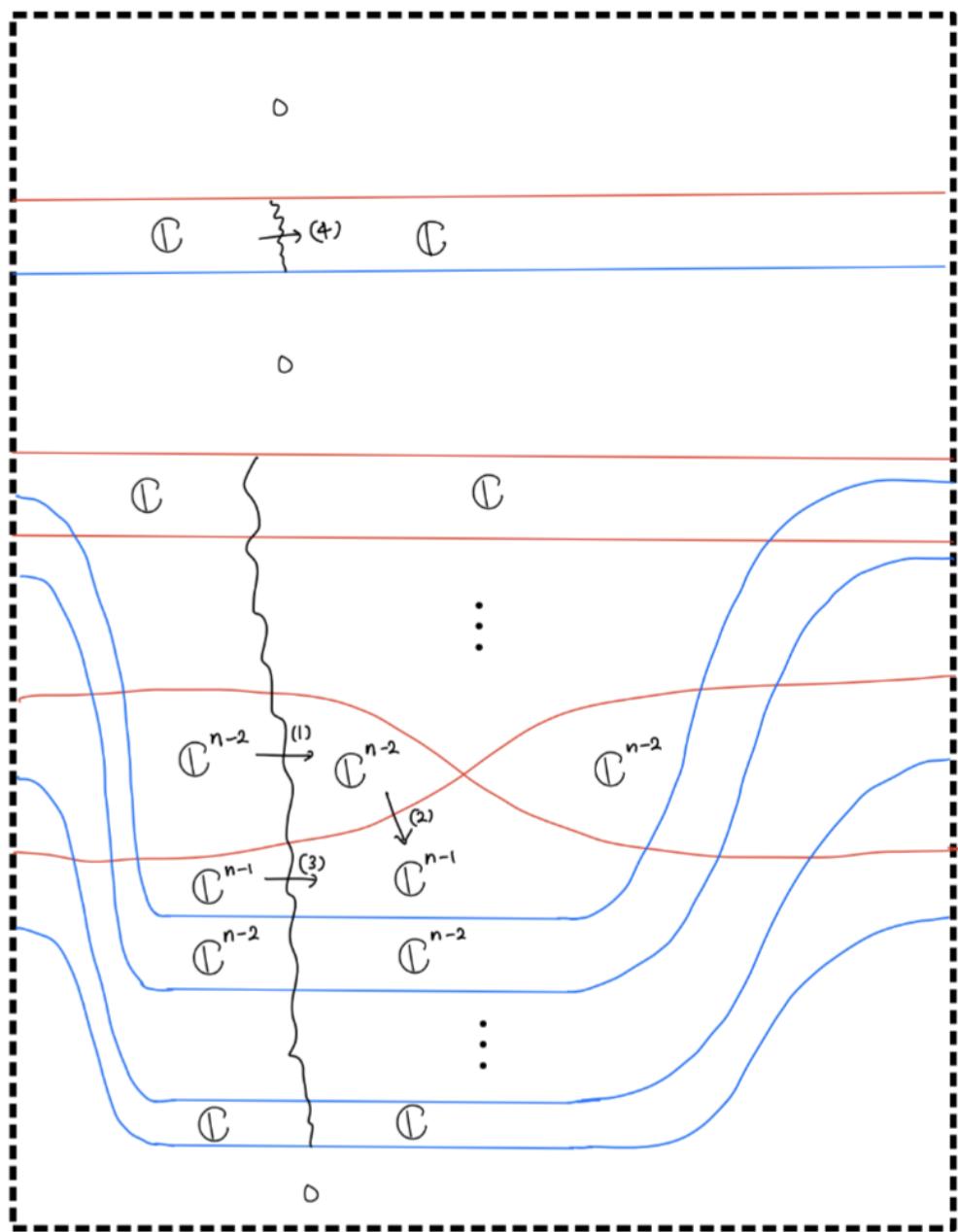


Figure 3.335

### Generalization maps

(1)  $\text{diag}(d_{n-1}, \dots, d_2)$

$$(2) \quad \iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$$

$$(3) \quad \text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$$

$$(4) \quad \times d_n$$

(Step 3) apply  $\text{cobord}_5$  to the square region surrounded by purple dotted lines

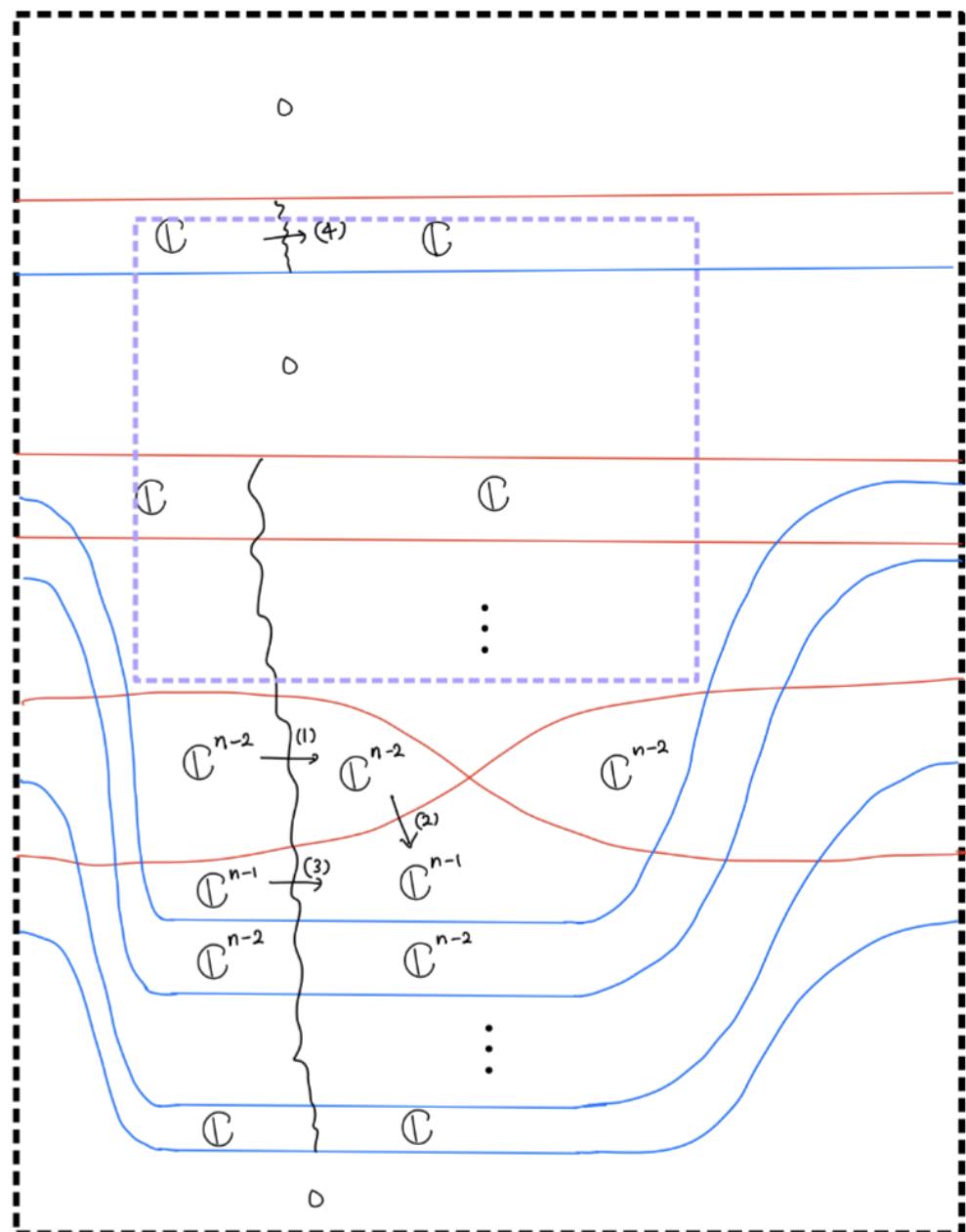


Figure 3.336

we get

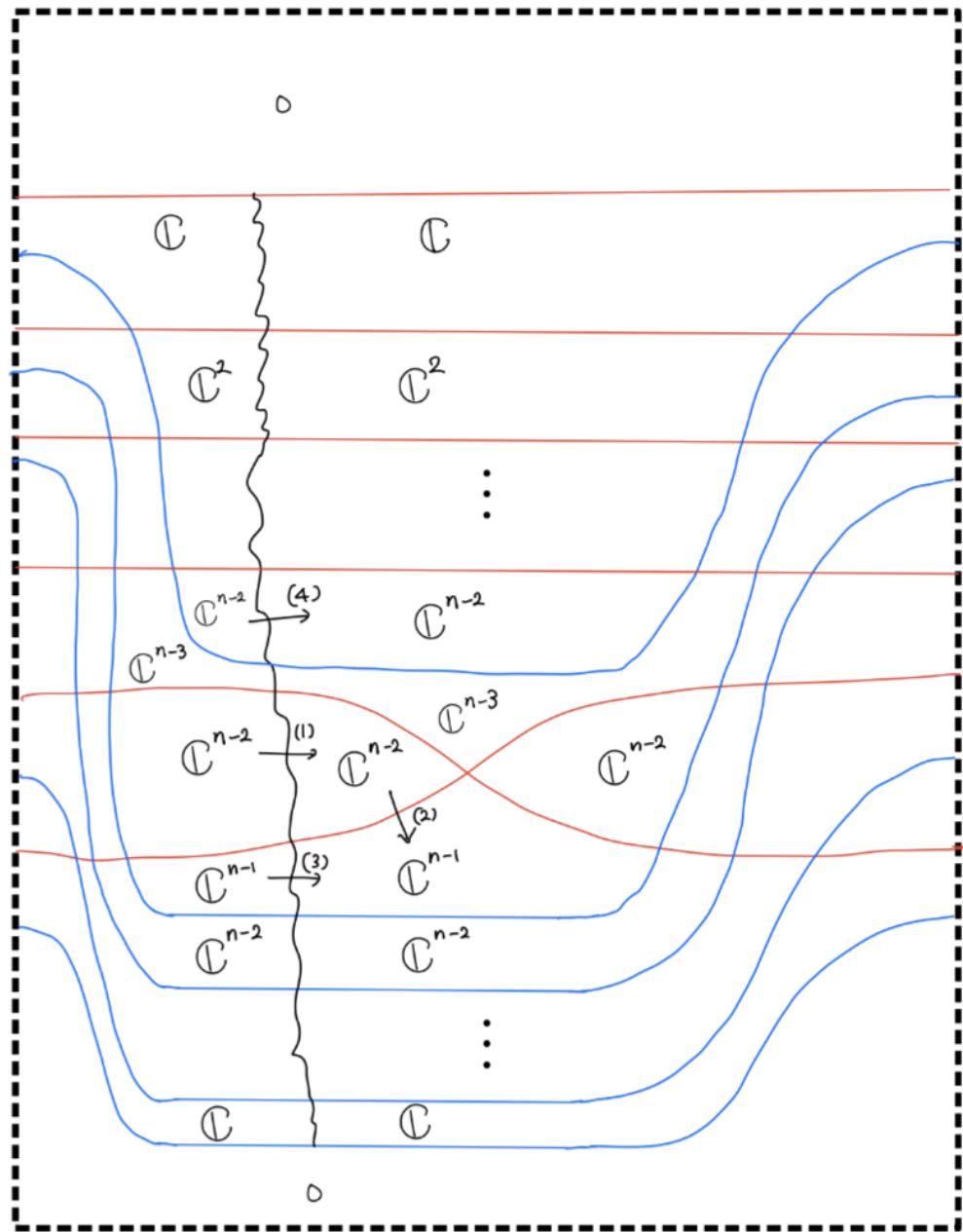


Figure 3.337

### Generalization maps

- (1)  $\text{diag}(d_{n-1}, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e'I_{n-1, n-2}$
- (3)  $\text{diag}(d_{n-1}, \dots, d_1) + eI_{n-1, n-2}$
- (4)  $\text{diag}(d_n, \dots, d_3)$

(Step 4) apply  $cobord_1$  to the regions surrounded by purple dotted lines

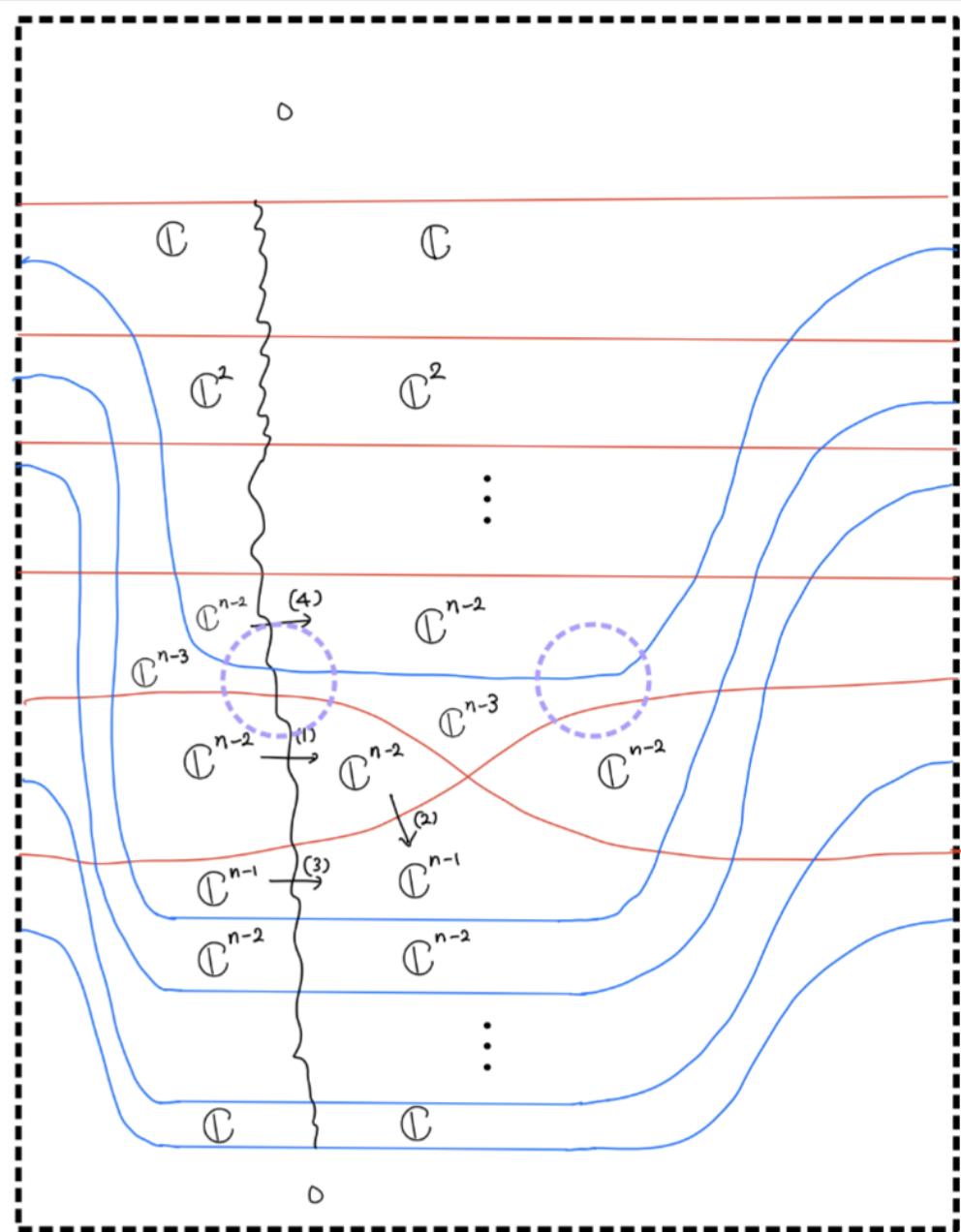


Figure 3.338

we get

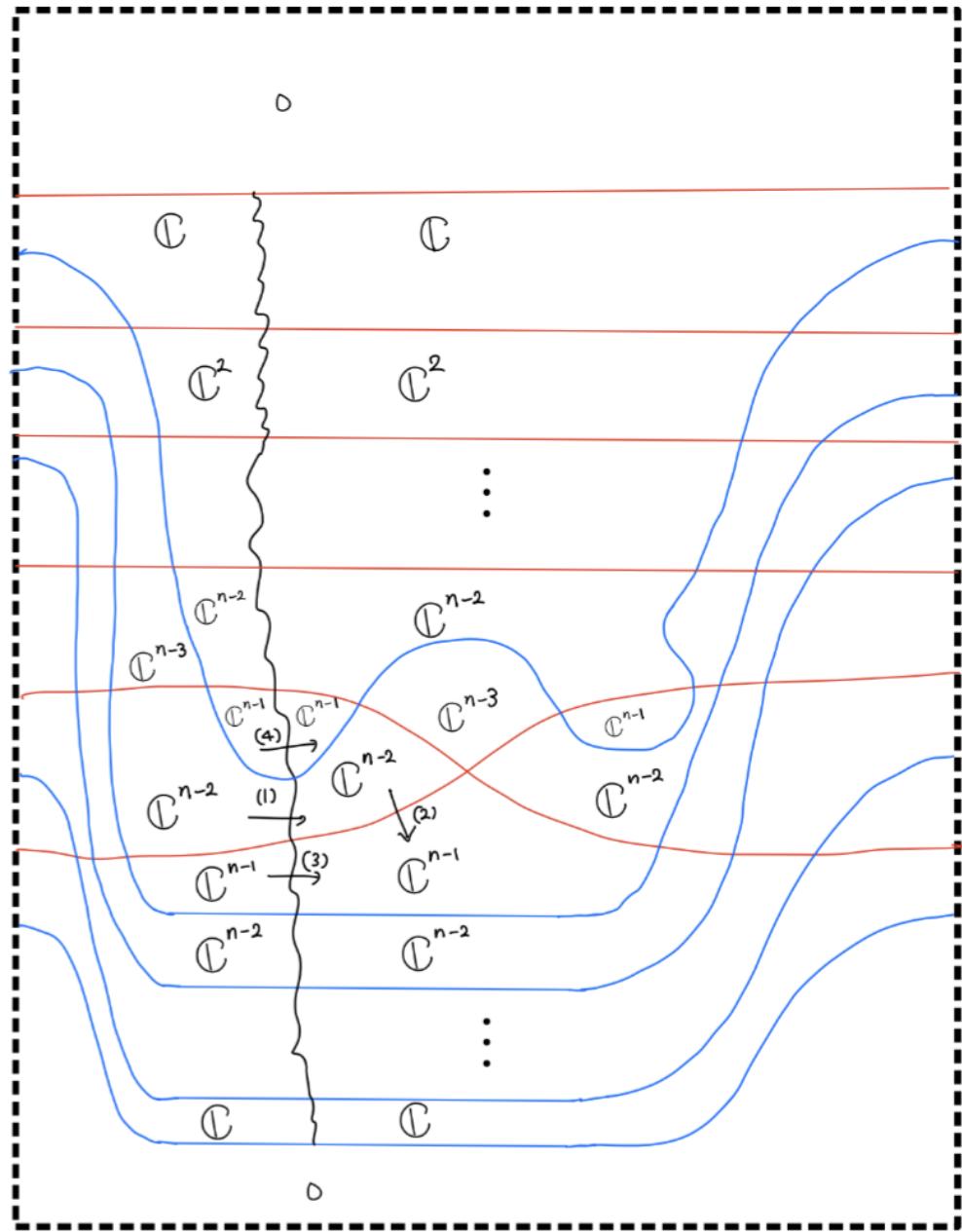


Figure 3.339

### Generalization maps

- (1)  $\text{diag}(d_{n-1}, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-1, n-2}$
- (3)  $\text{diag}(d_{n-1}, \dots, d_1) + e I_{n-1, n-2}$
- (4)  $\text{diag}(d_n, \dots, d_2)$

(Step 5) apply  $cobord'_8$  to the region surrounded by purple dotted lines

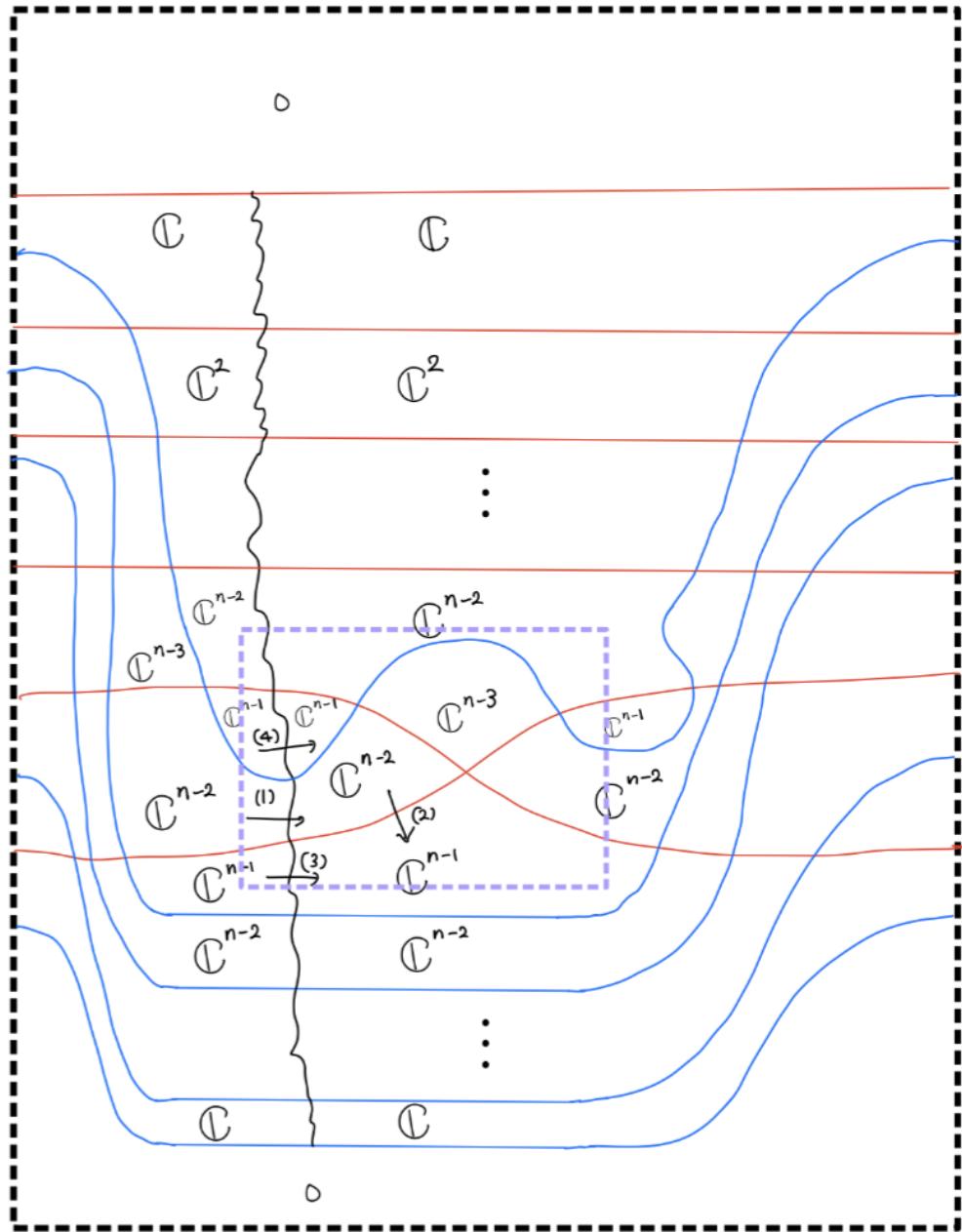


Figure 3.340

we get the final sheaf

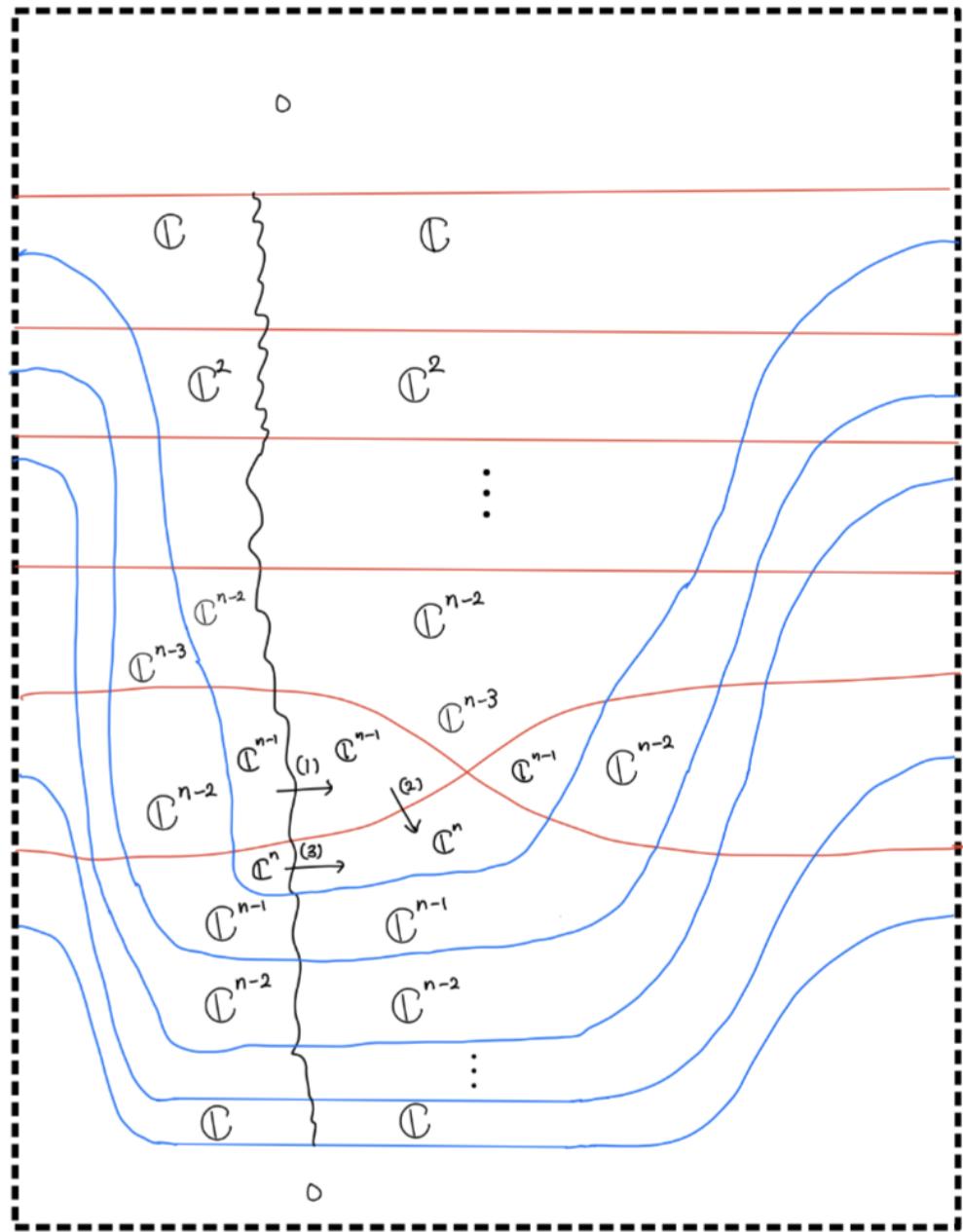


Figure 3.341

### Generalization maps

- (1)  $\text{diag}(d_n, \dots, d_2)$
- (2)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n,n-1}$
- (3)  $\text{diag}(d_n, \dots, d_1) + e I_{n,n-1}$

## 3.16 Sheaf Cobordism on Inter-generator Regions

**Definition 207.** a bump function about  $x = x_0$  of radius  $r_0$ , with  $z$  range from  $z_{lo}$  to  $z_{hi}$  is a bump function

- supported on  $[x_0 - r_0, x_0 + r_0]$
- peaks at  $(x_0, z_{hi})$
- away from support, has value  $z_{lo}$

We denote the bump function about  $x = 0$  of radius 1 by  $\Psi(z_{lo}, z_{hi})$

Suppose we have a punctured Riemann sphere  $M$  and  $\Lambda_0^0$ ,  $\Lambda_0^\infty$ ,  $\Lambda_0^{squig}$ , a nested regions  $U \subset U' \subset M$ , and a chart  $f : U \rightarrow \mathbb{R}^2$  such that  $U'$  maps to  $R := (-1, 1)_x \times (-n-1, n+1)_z$  under  $f$

- $\Lambda_0^0$  gets mapped to  $\bigcup_{k=1}^n \{(x, z) \in R \mid z = \Psi(-k, n-k+\frac{1}{2})(x)\}$ , co-oriented upward.
- $\Lambda_0^\infty$  gets mapped to  $\bigcup_{k=1}^n \{(x, z) \in R \mid z = k\}$ , co-oriented downward.
- $\Lambda_0^{squig}$  gets mapped to  $\{(x, z) \in R \mid x = 0\}$ , co-oriented toward the left.

and a sheaf defined by the following squiggly legible diagram. All the maps corresponding to blue strands are  $\iota_1$  and the red strands  $\iota_0$  otherwise stated. I have omitted these maps from the diagram.

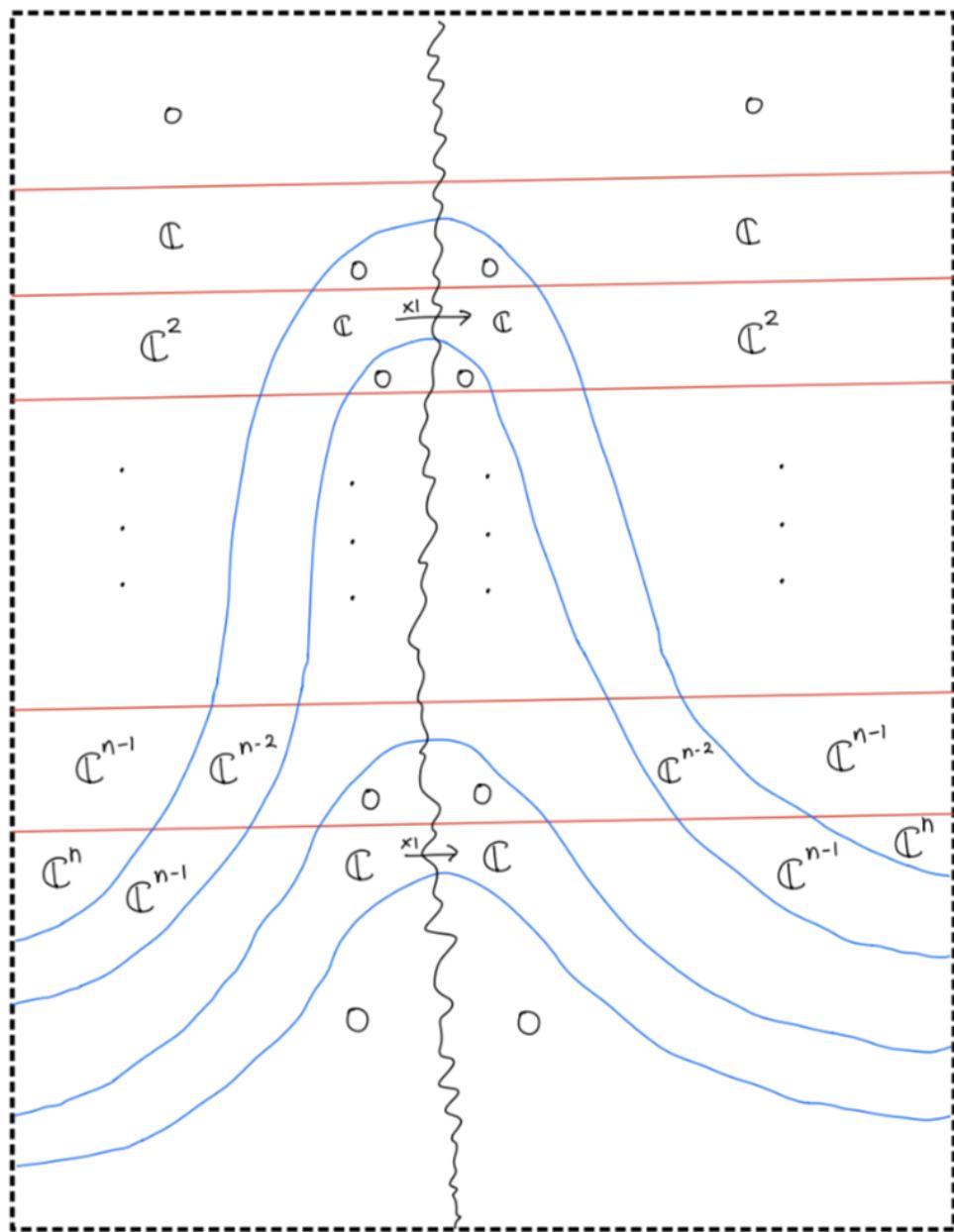


Figure 3.342

which is quasi-isomorphic to

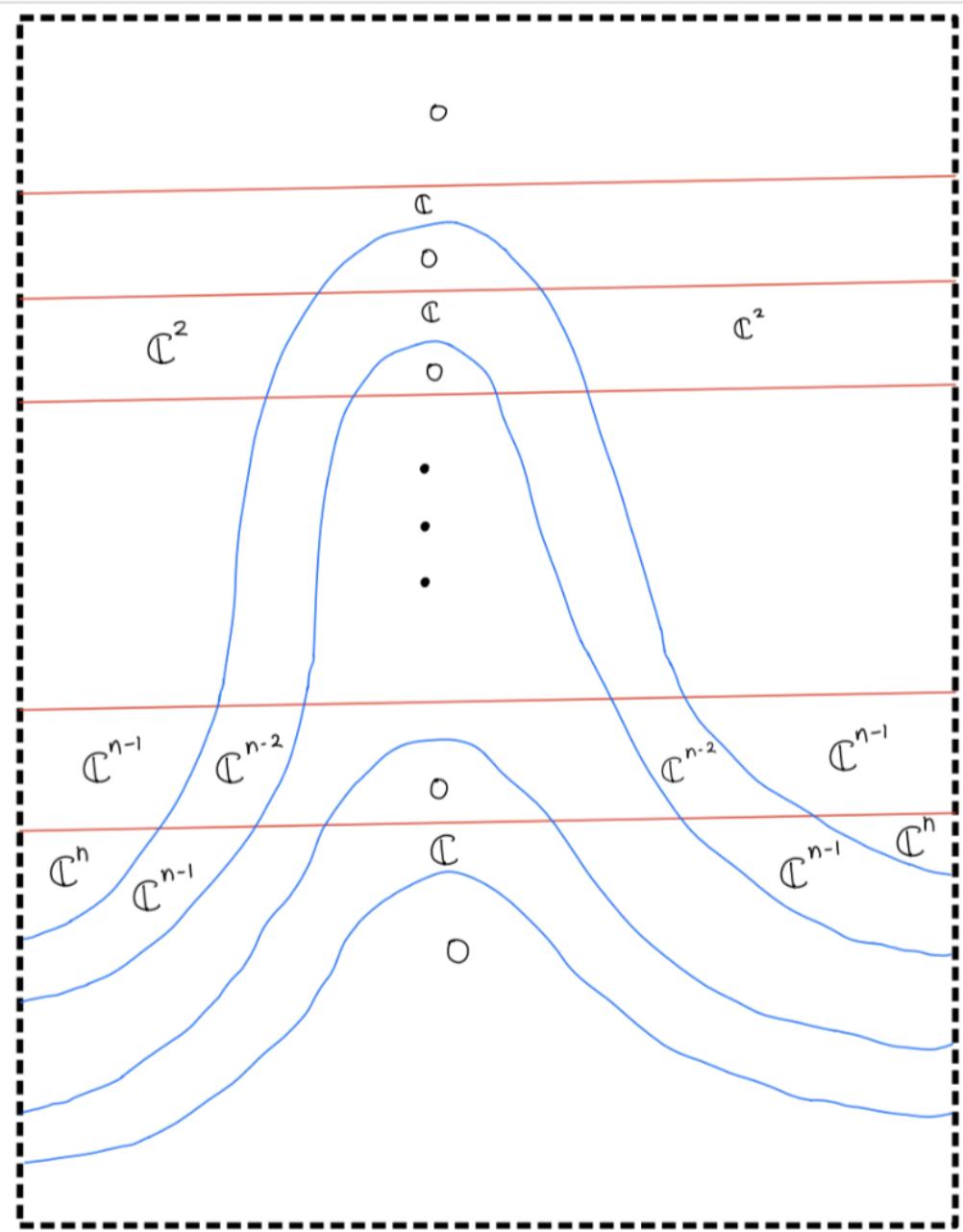


Figure 3.343

Then we define a cobordism starting from the above sheaf, say  $cobord_{inter}(n)$  supported on  $U$ , where  $n$  is the number of blue strands (equivalently red strands). At the end of the cobordism, the sheaf, under the same chart  $f$ , is described as the following squiggly legible diagram.

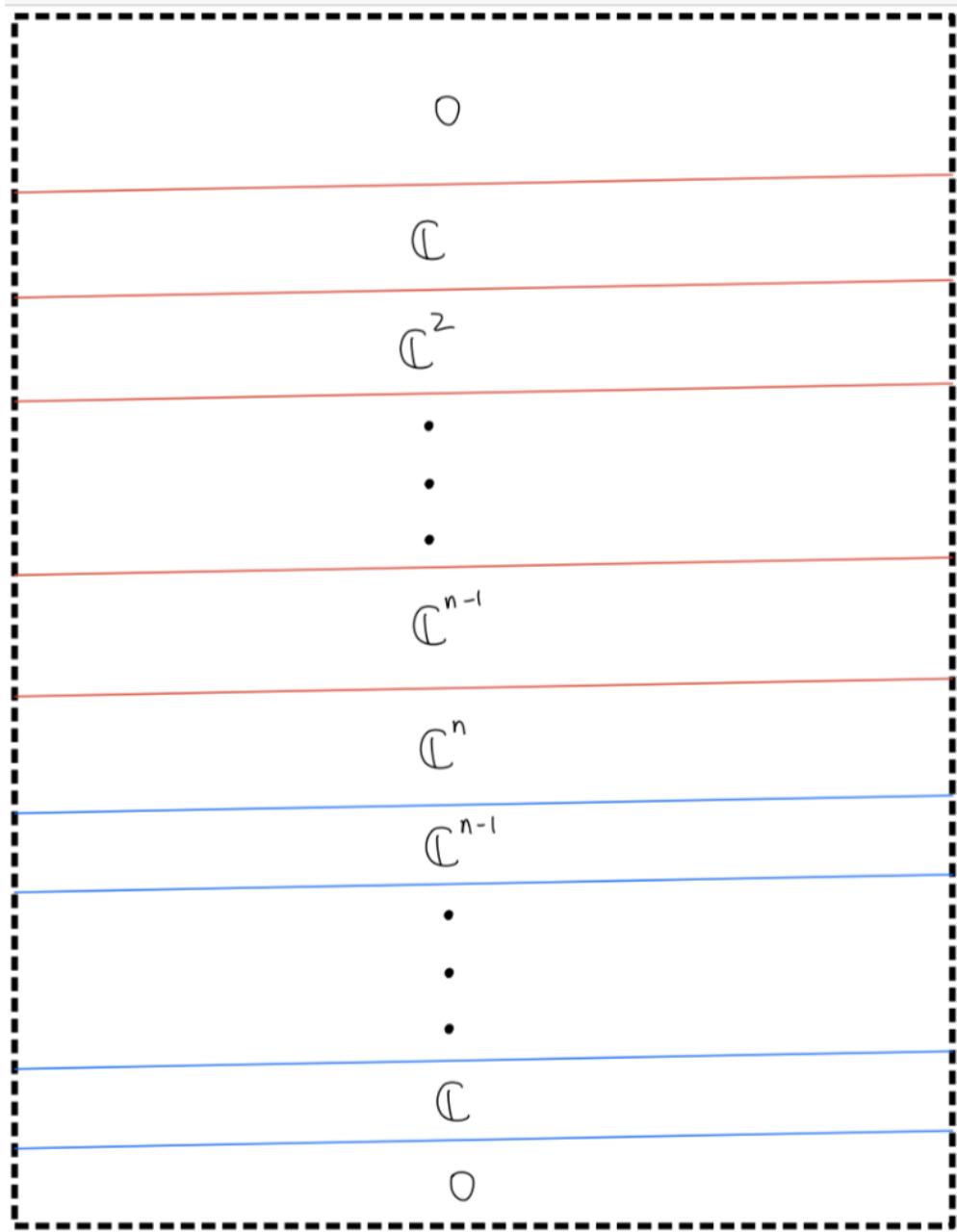


Figure 3.344

We define  $cobord_{inter}(n)$  inductively as follows.

- (i) For  $n = 1$ , we define  $cobord_{inter}(1)$  to be the null cobordism from

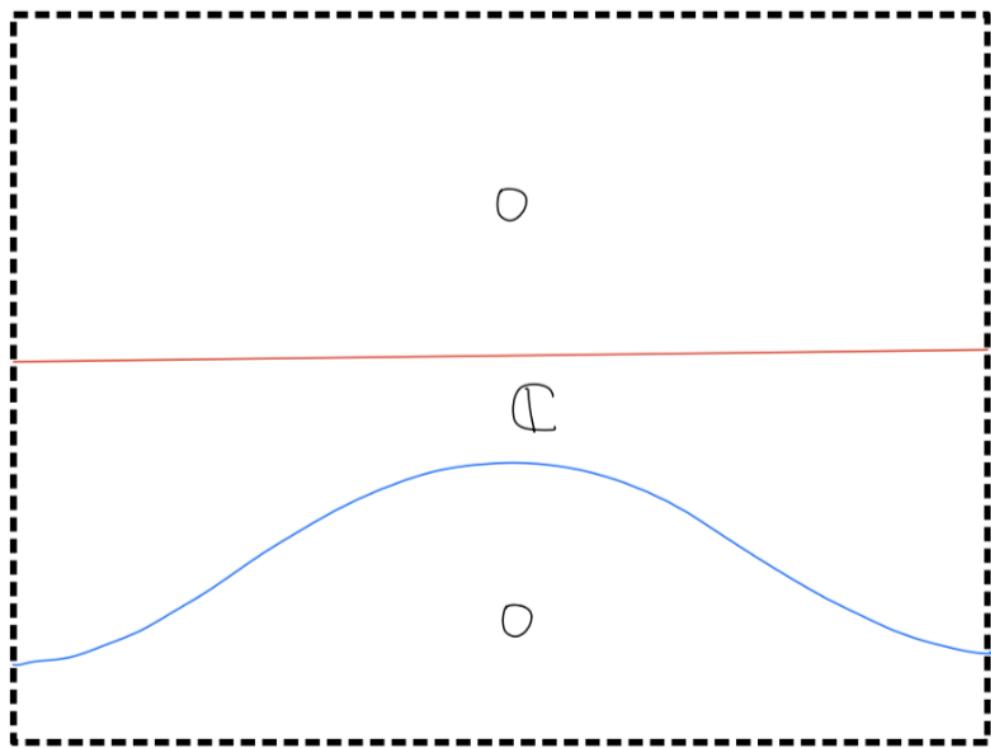


Figure 3.345

to

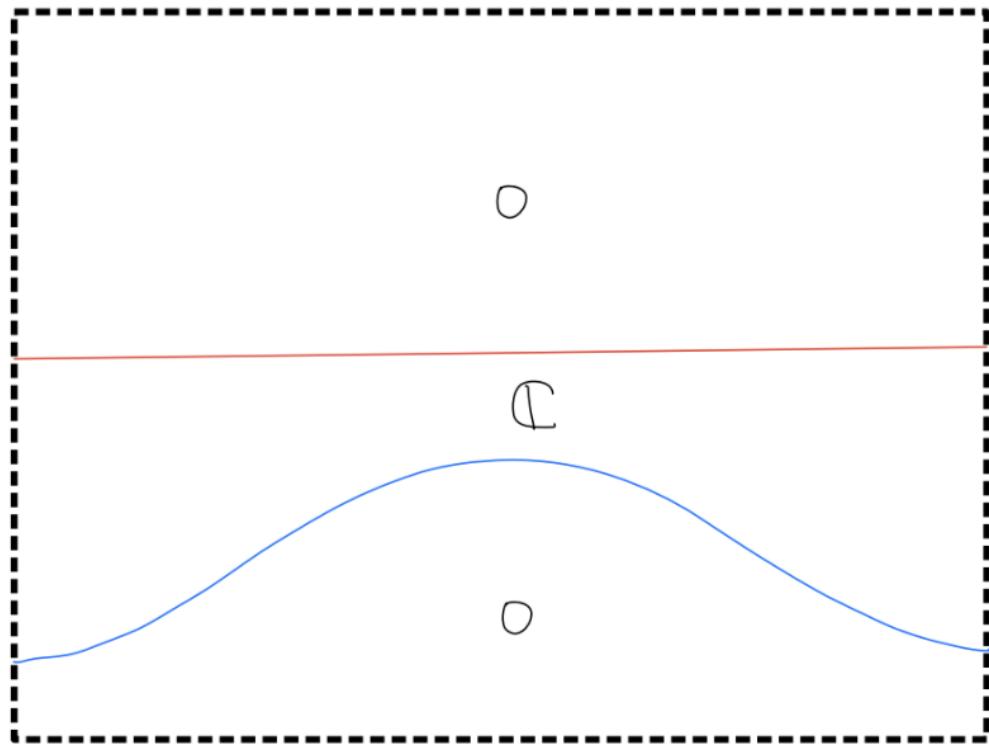


Figure 3.346

(ii) For  $n > 1$ ,

(Step 1) we first apply  $cobord_{inter}(n - 1)$  to the square region surrounded by purple dotted lines.

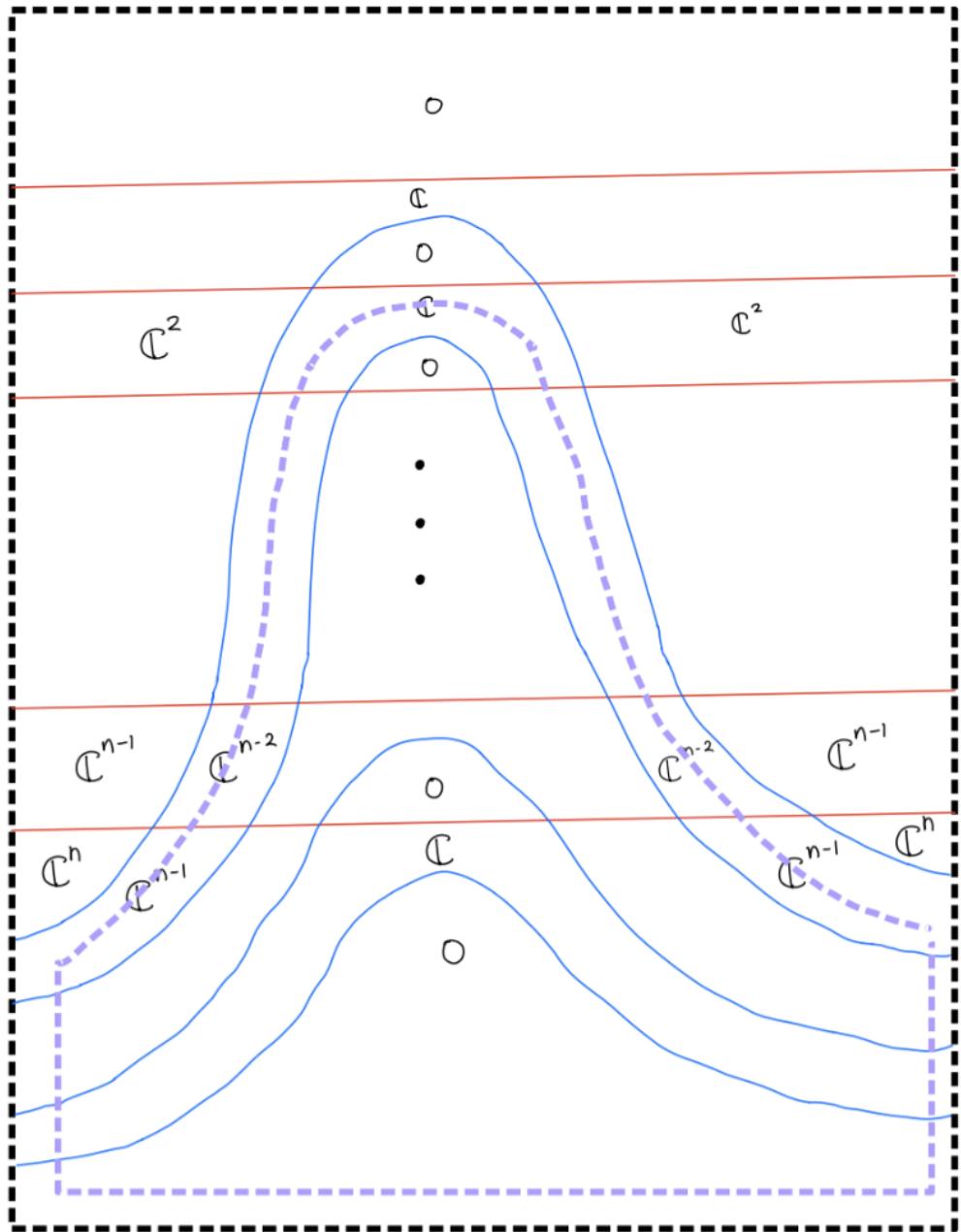


Figure 3.347

by induction hypothesis, we get

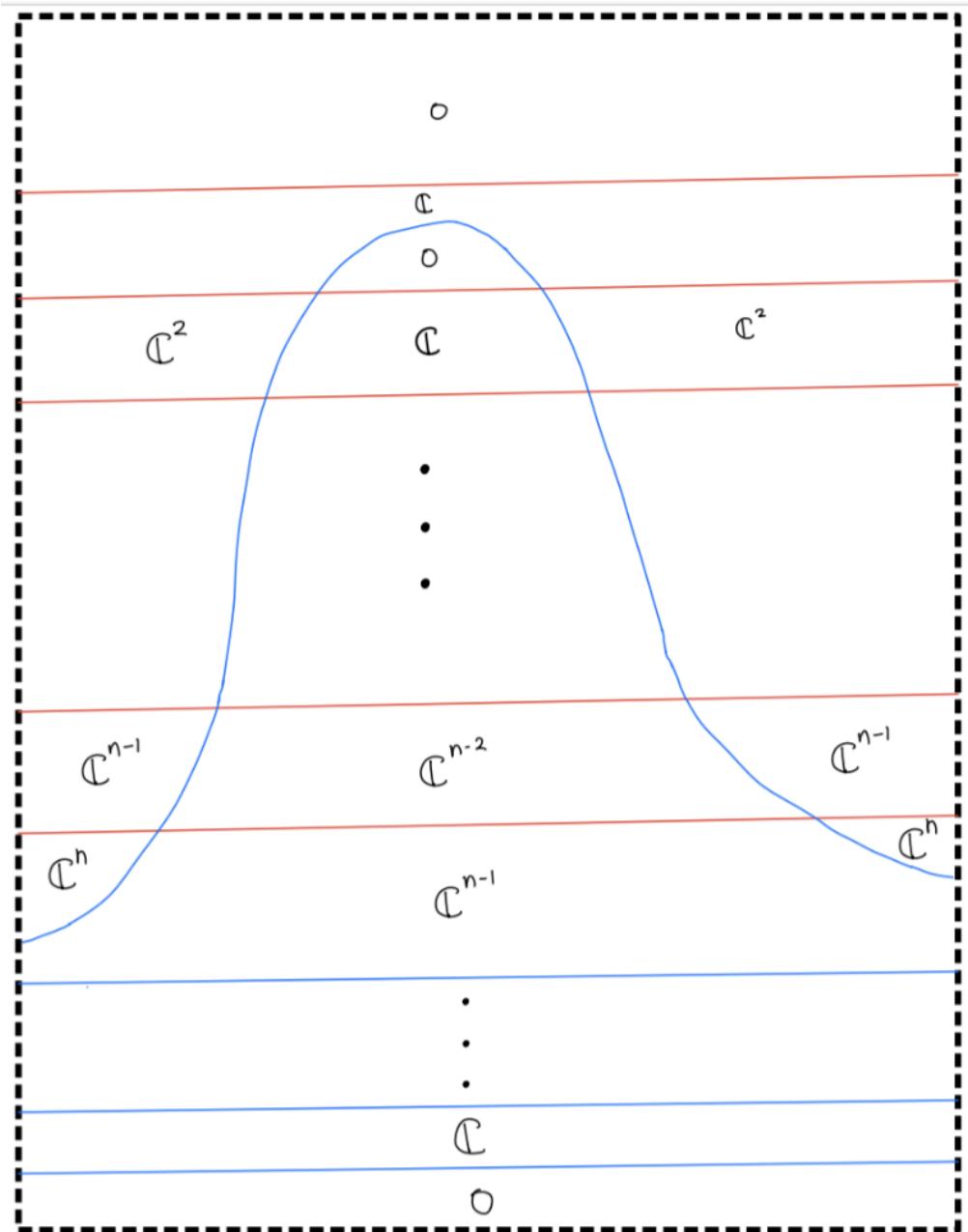


Figure 3.348

(Step 2) apply  $cobord_7$  to the square region surrounded by purple dotted lines.

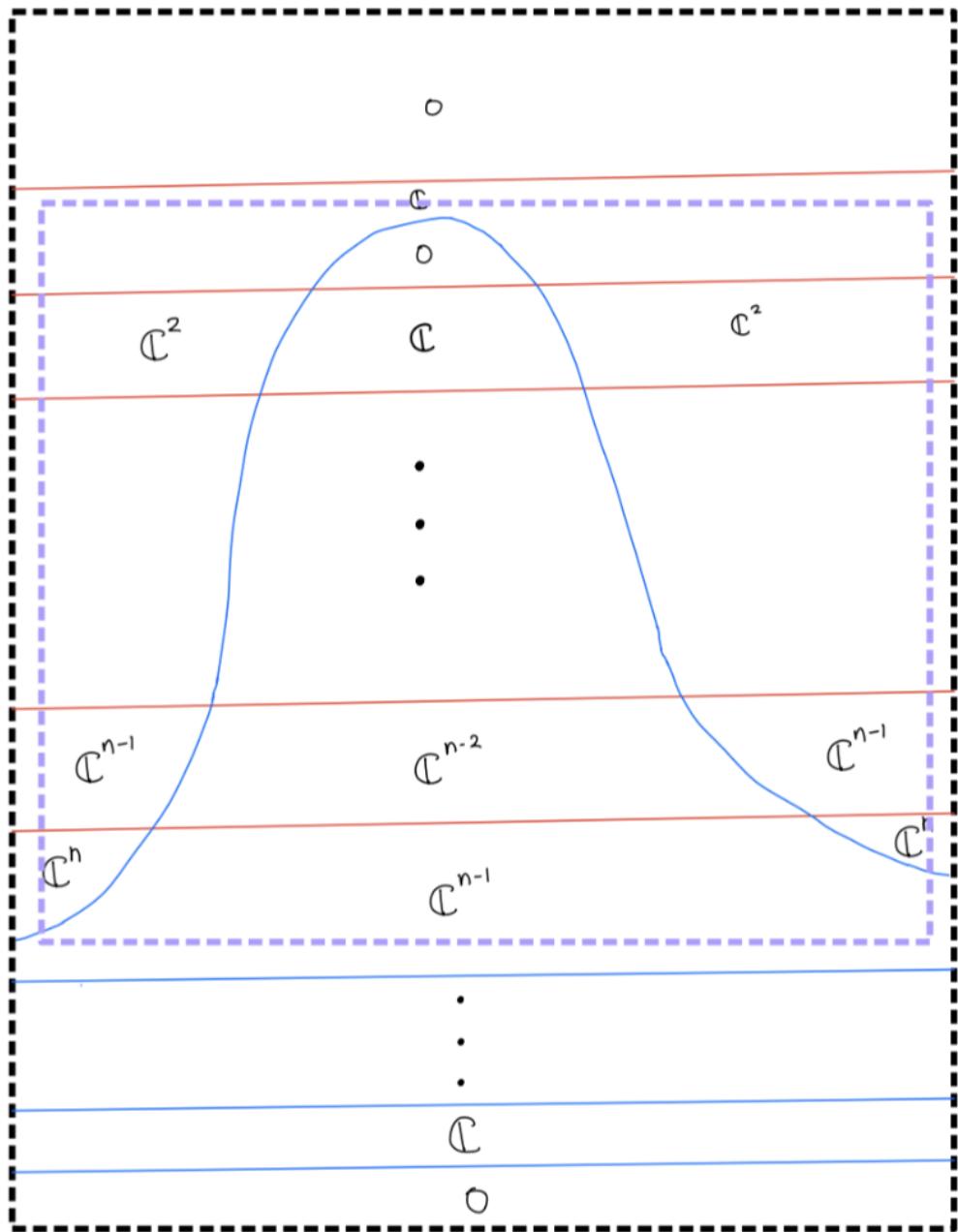


Figure 3.349

we get

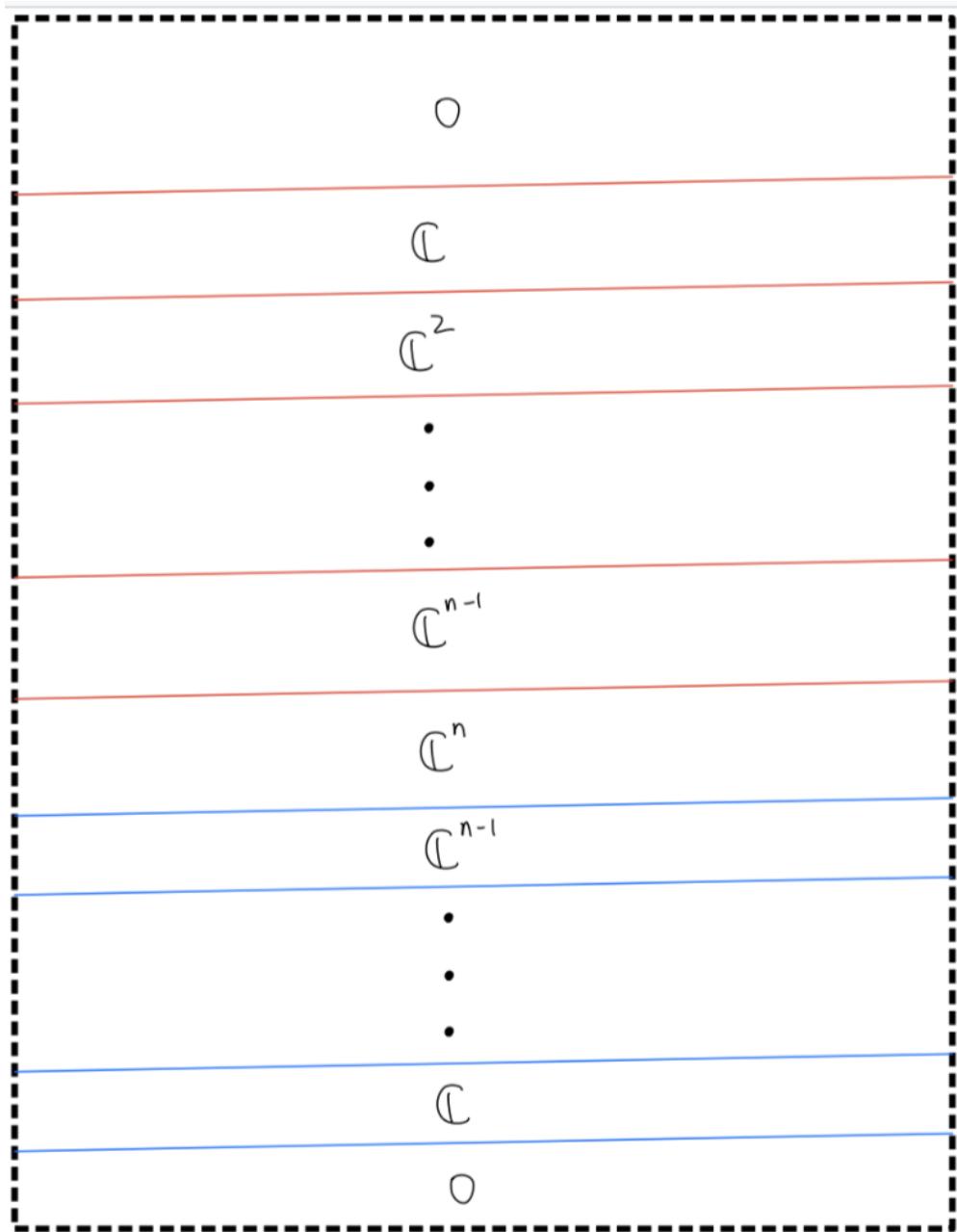


Figure 3.350

which is isomorphic to the final sheaf

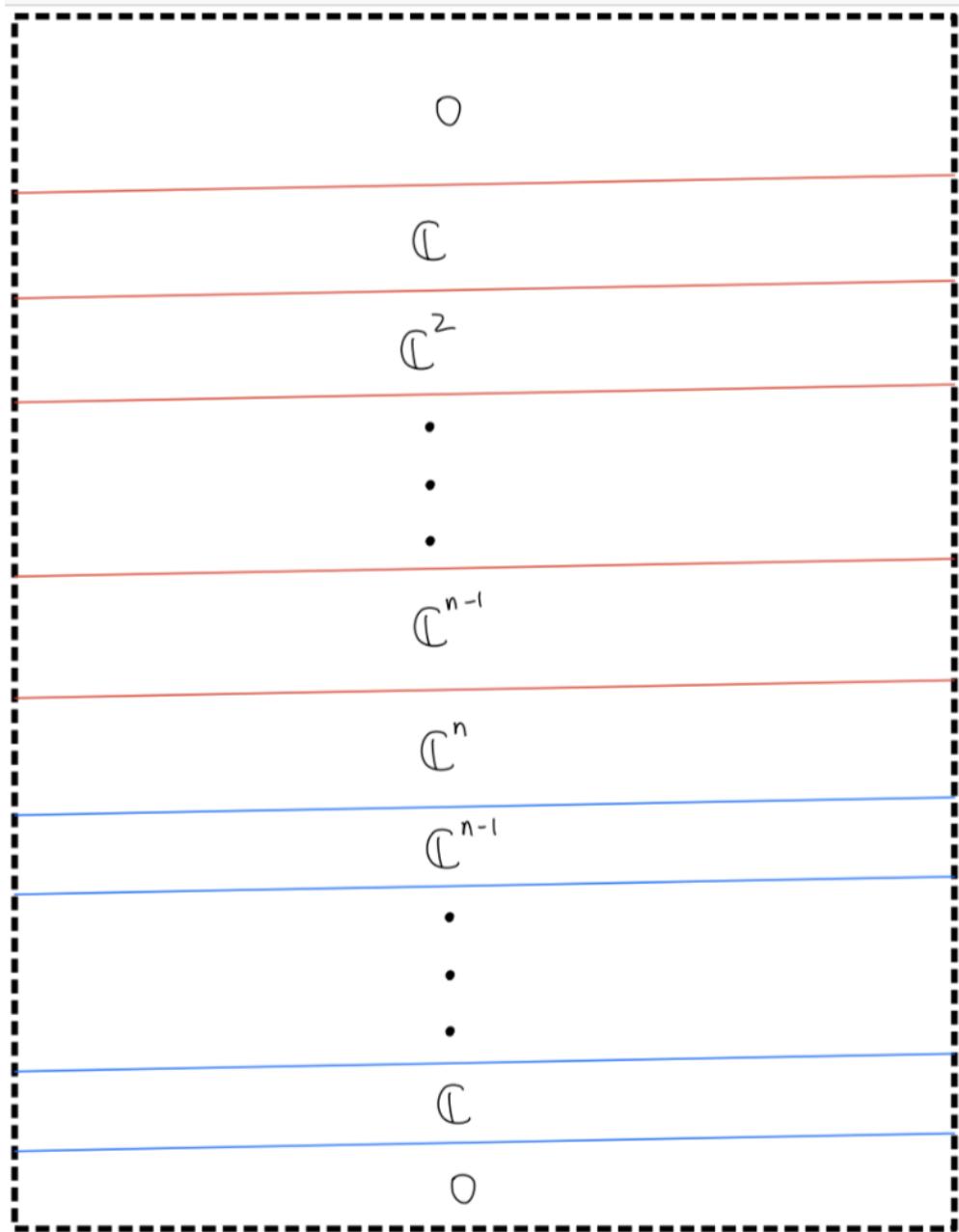


Figure 3.351

which is isomorphic to

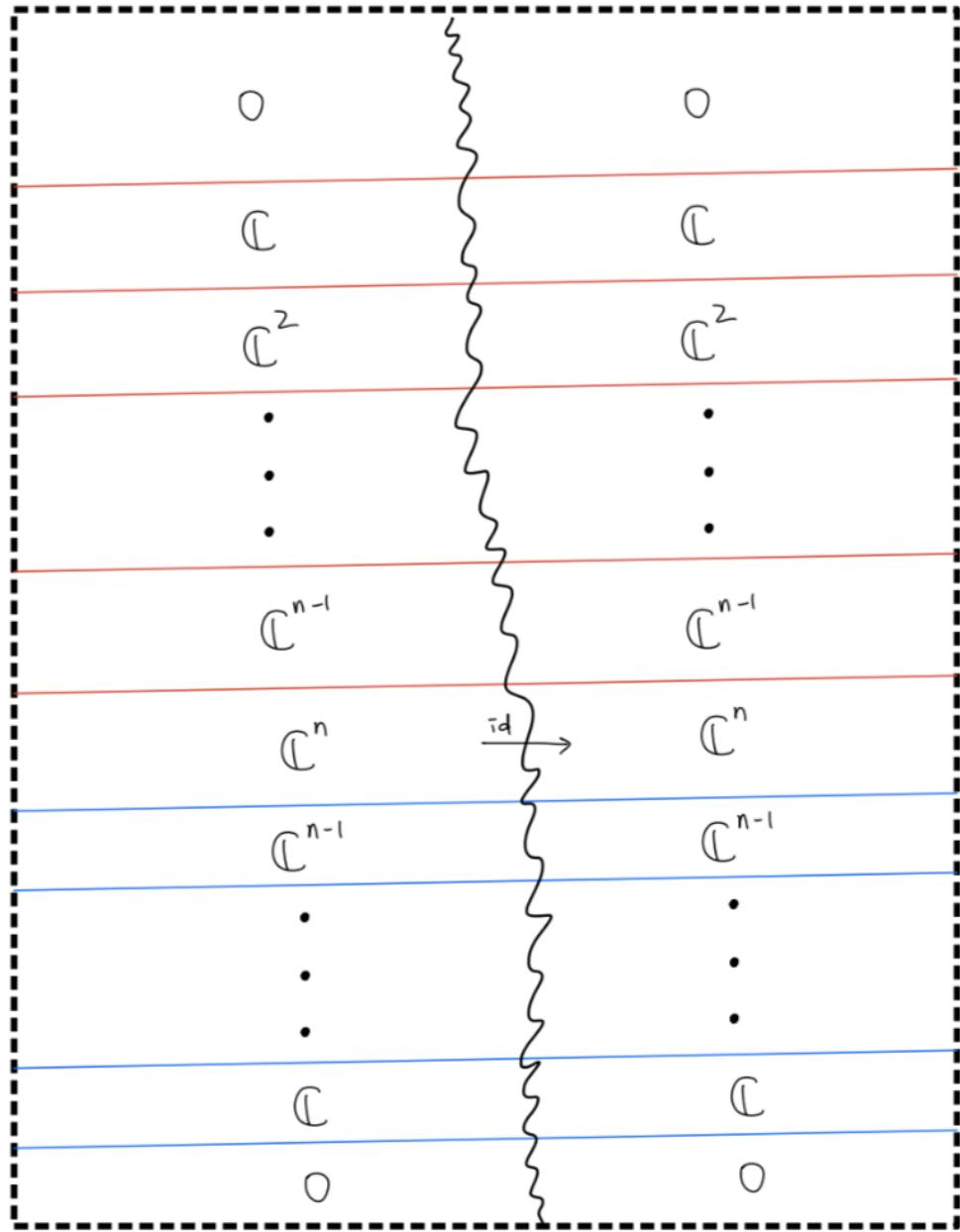


Figure 3.352

### 3.17 Full Sheaf Cobordism

Suppose we have a positive braid word  $\omega = s_{i_1} \cdots s_{i_k}$  on  $n$  strands, then we have an embedding of the cylindrical closures of  $\omega$  and the trivial braid  $\omega_\emptyset$  in a Riemann sphere with punctures at  $0, \infty$  described in section ???. Suppose we have an alternating

sheaf on the natural alternating diagram associated to  $\omega \coprod \omega_\emptyset$  when restricted to the  $j^{th}$  generator region is described by the following legible diagram

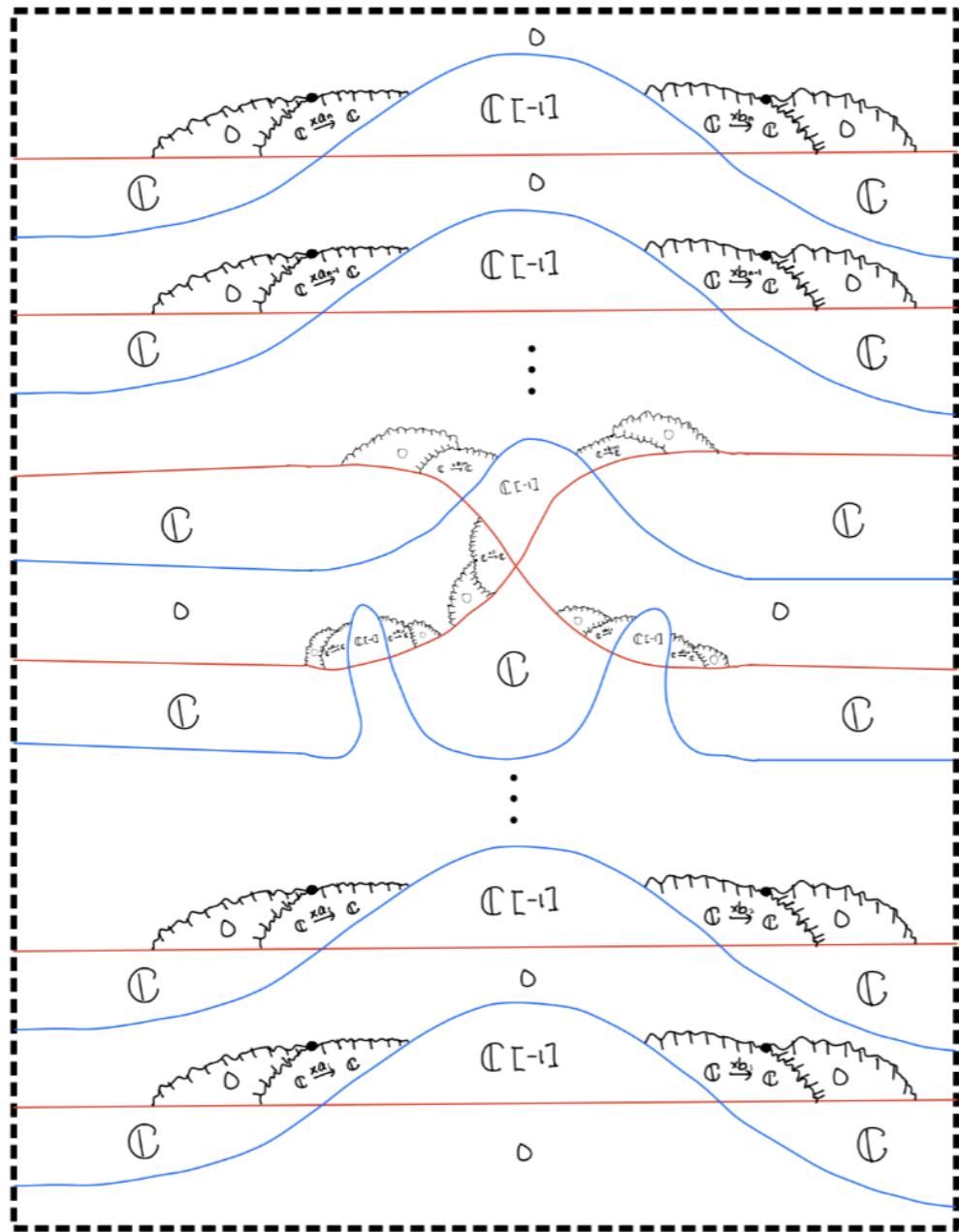


Figure 3.353

and when restricted to inter-generator regions is described by the following squiggle legible diagram

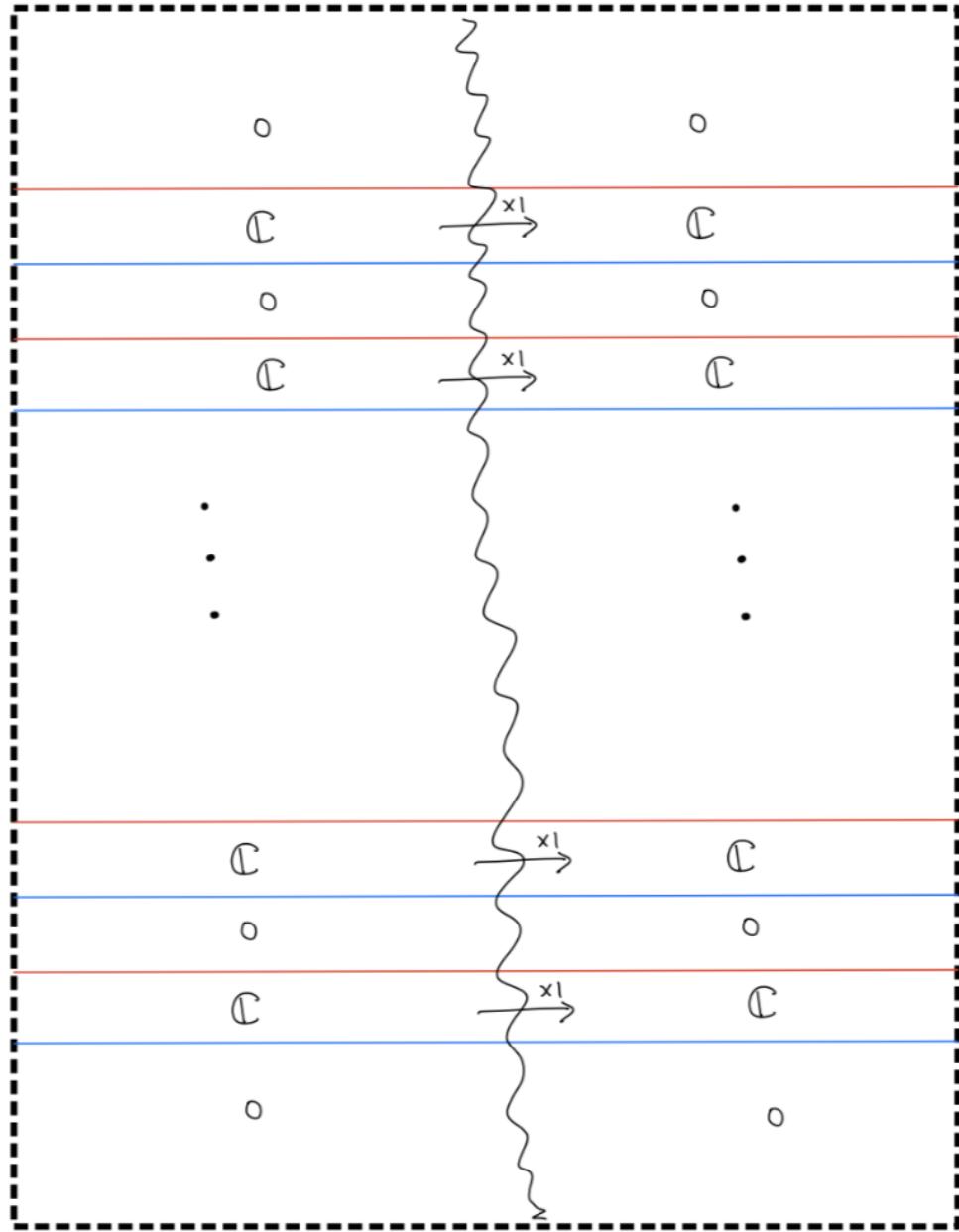


Figure 3.354

then we define a sheaf cobordism from the above alternating sheaf whose underlying Legendrian isotopy and at the separated diagram of  $\omega \coprod \omega_\emptyset$ , thereby describing a cluster coordinate.

(Step 1) to the  $j^{th}$  generator regions ( $j = 1, \dots, k$ ), apply  $cobord_{gen(n)}$ , we get on the  $j^{th}$  generator region

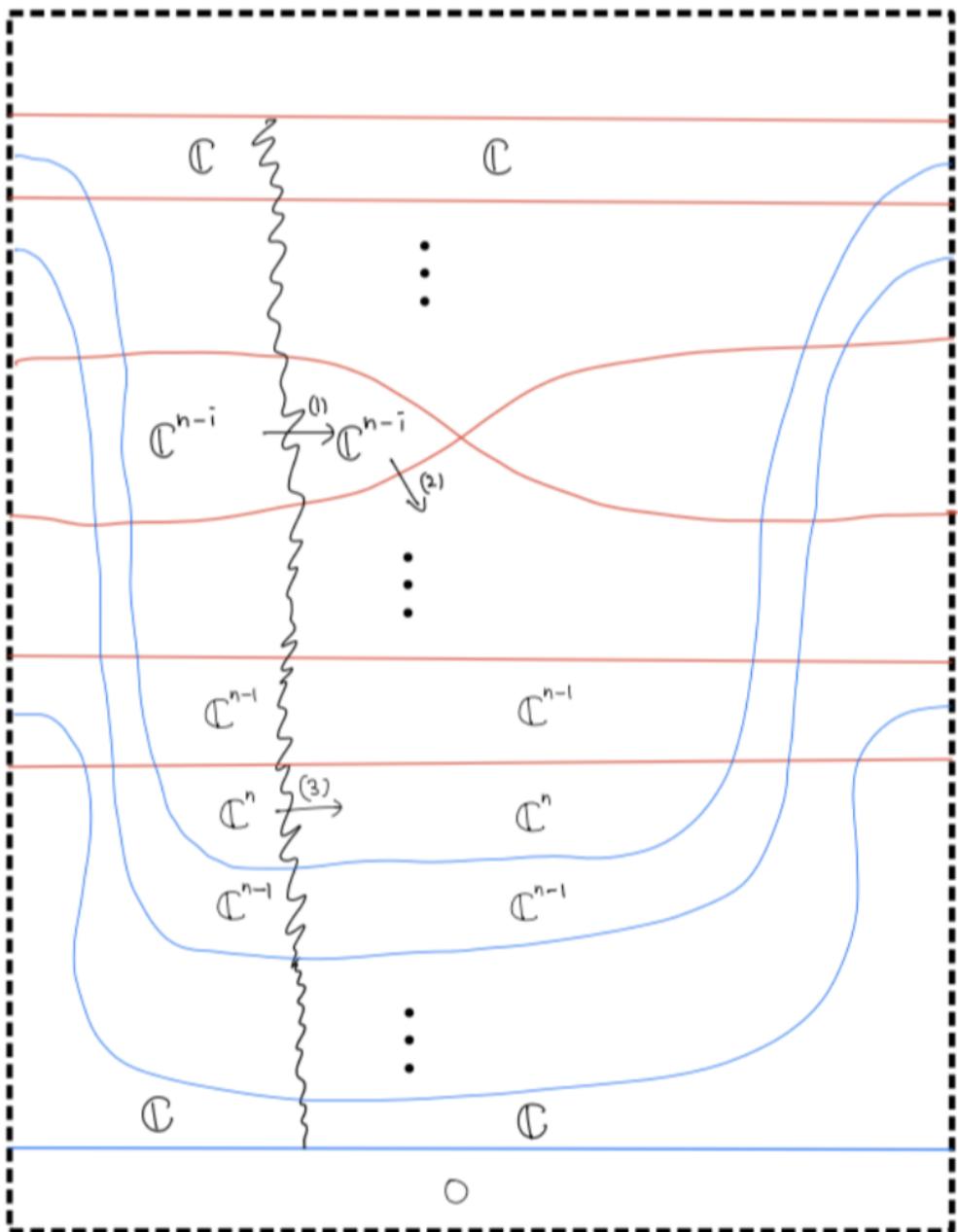


Figure 3.355

### Generalization maps

- (a)  $\text{diag}(d_n^j, \dots, d_{i+1}^j)$
- (b)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (c)  $\text{diag}(d_n^j, \dots, d_1^j) + e I_{n-i+1, n-i}$

where

- $a_i = a_{i,1}a_{i,2}^{-1}$  and  $b_i = b_{i,1}^{-1}b_{i,2}$
- $d_r = a_r b_r^{-1}$
- $e = -a_{i+1}b_i^{-1}c$
- $e' = d_{i+1}^{-1}e$

and on the inter-generator regions we get

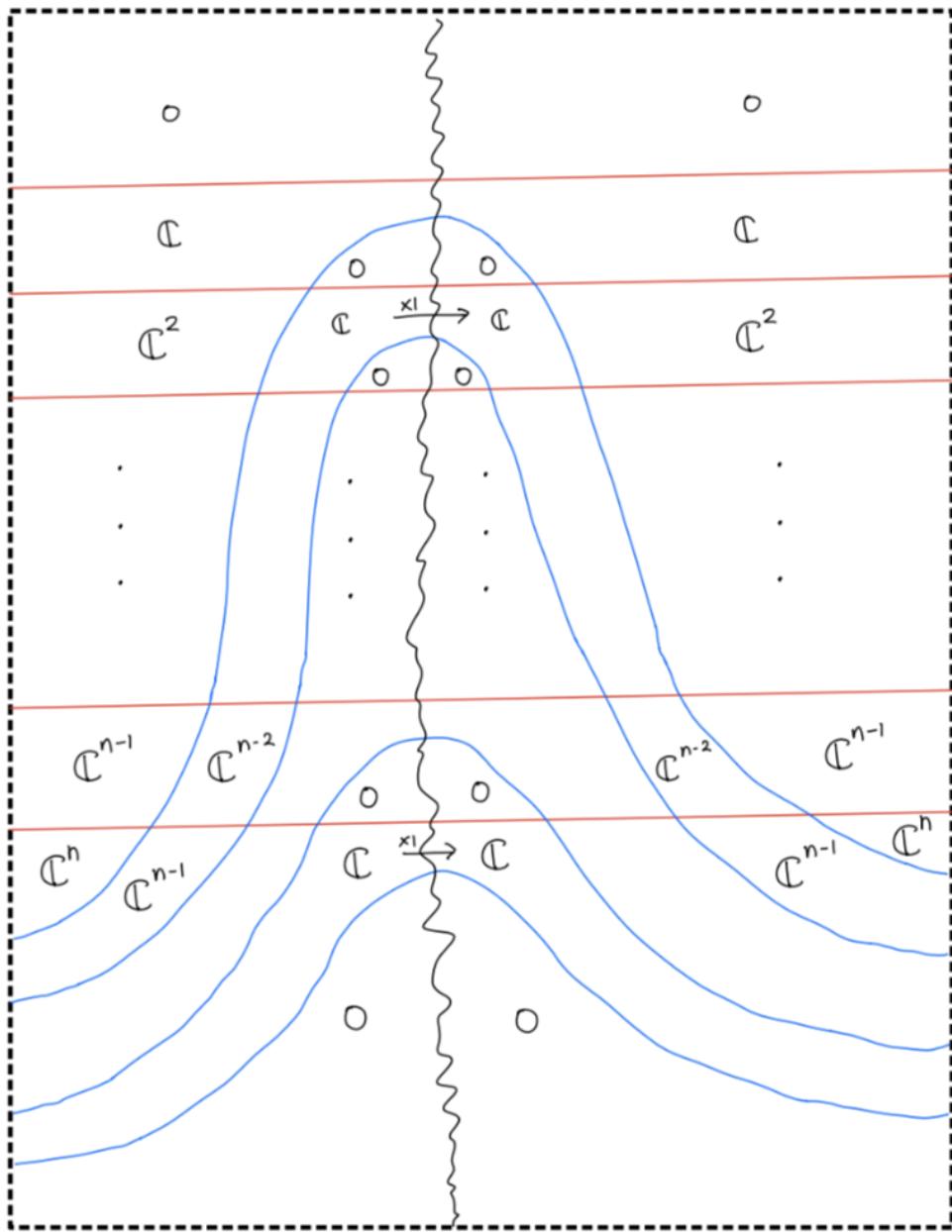


Figure 3.356

(Step 2) to the inter-generator regions, we apply  $cobord_{inter}(n)$ , on the  $j^{th}$  generator region, we get

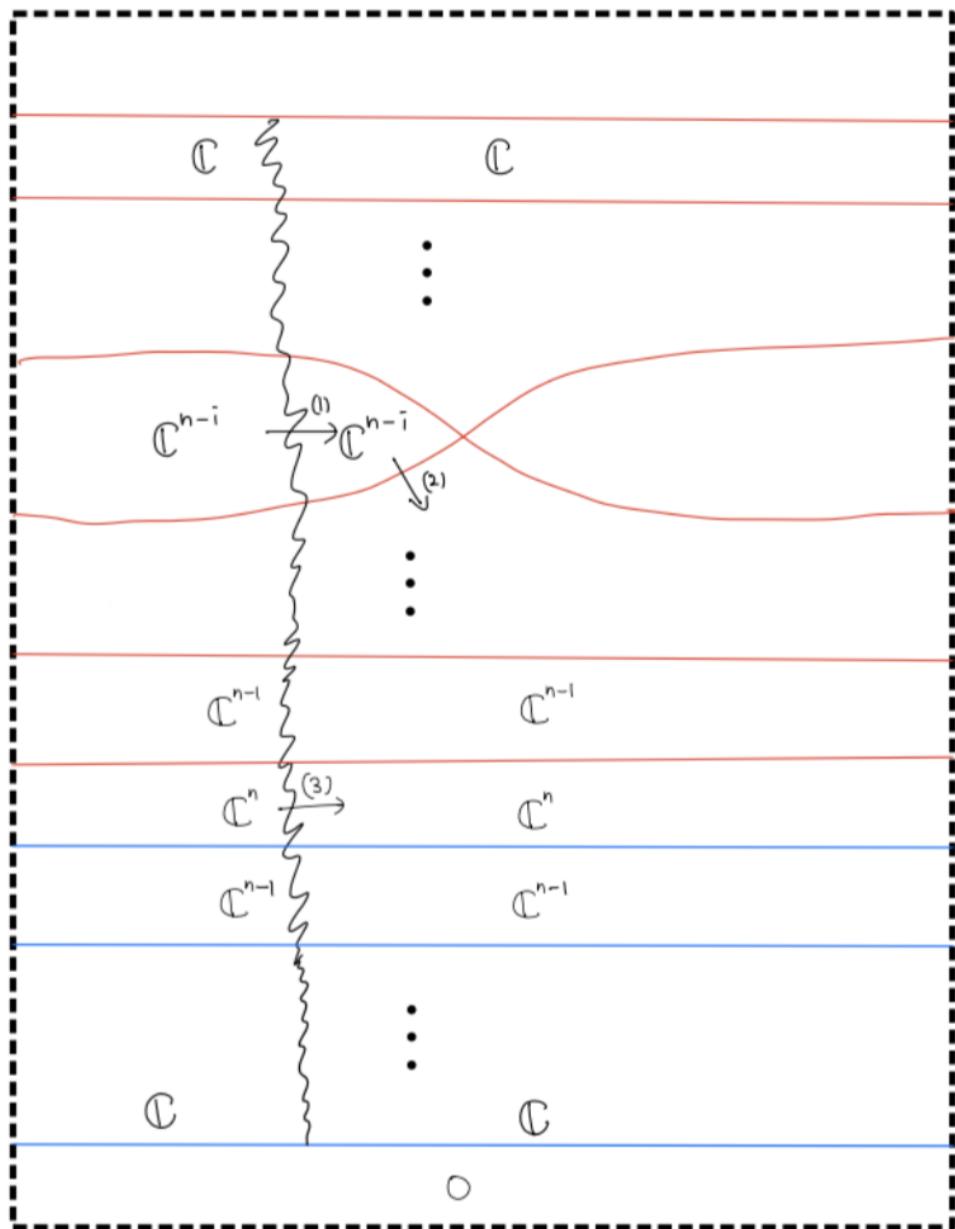


Figure 3.357

### Generalization maps

- (a)  $\text{diag}(d_n^j, \dots, d_{i+1}^j)$
- (b)  $\iota_0 \circ \text{diag}(1, \dots, 1) + e' I_{n-i+1, n-i}$
- (c)  $\text{diag}(d_n^j, \dots, d_1^j) + e I_{n-i+1, n-i}$

and on the inter-generator regions, we get

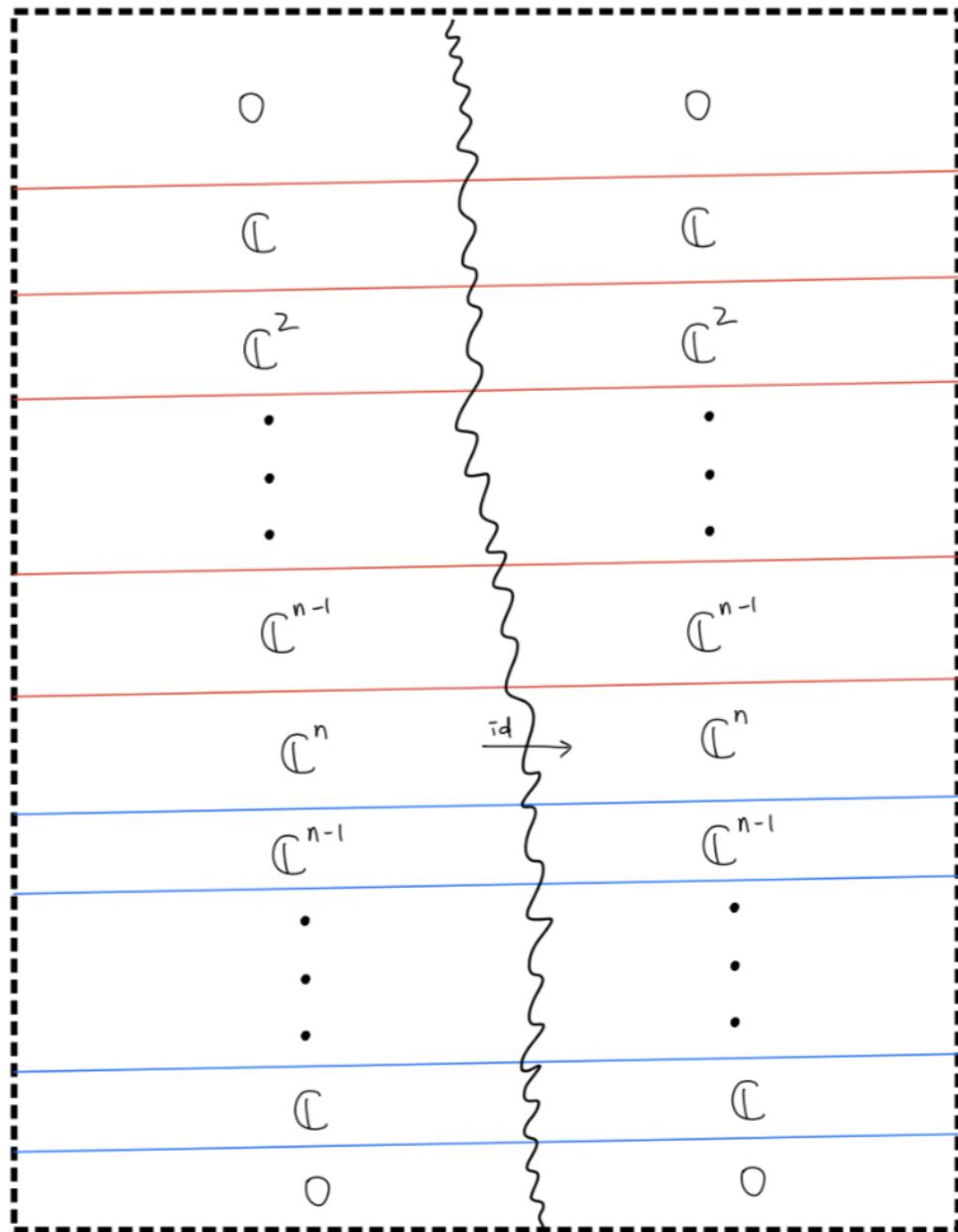


Figure 3.358

# Bibliography

- [Boa15] Philip Boalch. Wild character varieties, points on the riemann sphere and calabi's examples. *arXiv preprint arXiv:1501.00930*, 2015.
- [Dri04] Vladimir Drinfeld. Dg quotients of dg categories. *Journal of Algebra*, 272(2):643–691, 2004.
- [Etn05] John B Etnyre. Legendrian and transversal knots. In *Handbook of knot theory*, pages 105–185. Elsevier, 2005.
- [Gei08] Hansjörg Geiges. *An introduction to contact topology*, volume 109. Cambridge University Press, 2008.
- [GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira. Sheaf quantization of hamiltonian isotopies and applications to nondisplaceability problems. 2012.
- [GM83] Mark Goresky and Robert MacPherson. Stratified morse theory. *Singularities*, pages 517–533, 1983.
- [Kas84] Masaki Kashiwara. The riemann-hilbert problem for holonomic systems. *Publications of the Research Institute for Mathematical Sciences*, 20(2):319–365, 1984.

- [KS13] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds: With a Short History. Les débuts de la théorie des faisceaux*, volume 292. Springer Science & Business Media, 2013.
- [Lur04] Jacob Lurie. *Derived algebraic geometry*. PhD thesis, Massachusetts Institute of Technology, 2004.
- [Nad09] David Nadler. Microlocal branes are constructible sheaves. *Selecta Mathematica*, 15(4):563–619, 2009.
- [Sch12] Jörg Schürmann. *Topology of singular spaces and constructible sheaves*, volume 63. Birkhäuser, 2012.
- [She85] Allen Dudley Shepard. *A cellular description of the derived category of a stratified space*. Brown University, 1985.
- [Sib75] Yasutaka Sibuya. *Global theory of a second order linear ordinary differential equation with a polynomial coefficient*, volume 18. Elsevier, 1975.
- [STWZ19] Vivek Shende, David Treumann, Harold Williams, and Eric Zaslow. Cluster varieties from legendrian knots. 2019.
- [STZ17] Vivek Shende, David Treumann, and Eric Zaslow. Legendrian knots and constructible sheaves. *Inventiones mathematicae*, 207(3):1031–1133, 2017.
- [Toë09] Bertrand Toën. Higher and derived stacks: a global overview. *Algebraic geometry—Seattle 2005. Part 1*:435–487, 2009.
- [Toë14] Bertrand Toën. Derived algebraic geometry. *EMS Surveys in Mathematical Sciences*, 1(2):153–240, 2014.
- [TV04] Bertrand Toën and Gabriele Vezzosi. From hag to dag: derived moduli stacks. In *Axiomatic, enriched and motivic homotopy theory*, pages 173–216. Springer, 2004.

- [TV05] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry i: Topos theory. *Advances in mathematics*, 193(2):257–372, 2005.
- [TV08] Bertrand Toën and Gabriele Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications: Geometric Stacks and Applications*, volume 2. American Mathematical Soc., 2008.